Chapter 3. Discrete Random Variables and Probability Distributions

Chapter 3:
Discrete Random Variables and Probability Distributions

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3.3 Expected Values of Discrete Random Variables

- **The Expected Value of X**

Let X be a discrete rv with set of possible values *D* and pmf p(x). The expected value or mean value of X, denoted by E(X) or μ_X (or μ for short), is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

Note: When the sum does not exist, we say the expectation of *X* does not exist. (finite or infinite case?)

3.3 Expected Values of Discrete Random Variables

- **Example**

Consider selecting at random a student who is among the 15,000 registered for the current term at Mega University. Let **X= the number of course for which the selected student is registered**, and suppose that X has the pmf as following table

X	1	2	3	4	5	6	7
$P(x)$	0.01	0.03	0.13	0.25	0.39	0.17	0.02
Number registered	150	450	1950	3750	5850	2550	300

$$\mu_x = ?$$

3.3 Expected Values of Discrete Random Variables

- **Example 3.14**

$$\mu_X = 1 \cdot p(1) + 2 \cdot p(2) + ... + 7 \cdot p(7)$$
$$= (1)(.01) + 2(.03) + ... + (7)(.02)$$
$$= .01 + .06 + ... + .14 = 4.57$$

- **Example 3.18**

  Let X=1 if a randomly selected component needs warranty service and 0 otherwise. Then X is a Bernoulli rv with pmf

$$p(x)=\begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & x\neq 0 \end{cases}$$

E(x)=?

Example 3.18

Solution:

$E(x) = 0\times p(0) + 1\times p(1) = p(1) = p.$

Note: the expected value of X is just the probability that X takes on the value 1.

- **Example 3.19**

  The general form for the pmf of X=number of children born up to and including the first boy is

$$p(x)=\begin{cases} p(1-p)^{x-1} & x=1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

E(x)=?

- **Example 3.19**

Solution:

$$E(x)=\sum_{D} x\cdot p(x)=\sum_{x=1}^{\infty} xp(1-p)^{x-1}=p\sum_{x=1}^{\infty}\left[-\frac{d}{dp}(1-p)^{x}\right]$$

$$=-p\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^{x}=-p\frac{d\left[\frac{1-p}{1-(1-p)}\right]}{dp}=-p\left[\frac{1-p}{p}\right]'=\frac{1}{p}$$

- **Example 3.20**

Let X, the number of interviewers a student has prior to getting a job, have pmf

$$p(x) = \begin{cases} k/x^2 & x=1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Where k is chosen so that $\sum_{X=1}^{\infty}(k/x^2)=1$. (In a mathematics course on infinite series, it is shown that $\sum_{x=1}^{\infty}(1/x^2)<\infty$, which implies that such a k exists, but its exact value need not concern us). The expected value of X is

$$\mu = E(X) = \sum_{X=1}^{\infty} x \cdot \frac{k}{x^2} = k \sum_{x=1}^{\infty} \frac{1}{x}$$

Harmonic Series!

- **Example 3.21**

 Suppose a bookstore purchases ten copies of a book at $ 6.00 each, to sell at $12.00 with the understanding that at the end of a 3-month period any unsold copy can be redeemed for $2.00. If X=the number of copies sold, **then what is the net revenue?**

3.3 Expected Values of Discrete Random Variables

Solution:

Net revenue=$h(X)$=12X+2(10-X)-60=10X-40.

Here, we are interested in the expected value of the net revenue (h(X)) rather than X itself.

3.3 Expected Values of Discrete Random Variables

- The Expected Value of a **function**

Let X be a discrete rv with set of possible values D and pmf p(x). Then the expected values or mean value of any function $h(X)$, denoted by $\boldsymbol{E[h(X)]}$ or $\boldsymbol{\mu_{h(X)}}$, is computed by

$$E[h(X)] = \sum_{x \in D} h(x) \cdot p(x)$$

- **Example 3.23**

A computer store has purchased three computers of a certain type at \$500 apiece. It will sell them for \$1000 apiece. The manufacturer has agree to repurchase any computers still unsold after specified period at \$200 apiece.

Let X denote the number of computers sold, and suppose that $p(0)=0.1$, $p(1)=0.2$, $p(2)=0.3$ and $p(3)=0.4$. The $h(x)$ denoting the profit associated with selling X units.

What is the profit and expected profit?

3.3 Expected Values of Discrete Random Variables

Solution:

h(X) =revenue- cost =1000X+200(3-X)-1500 =800X-900.

The expected profit is

$$E(h(X)) = h(0)p(0)+h(1)p(1)+h(2)p(2)+h(3)p(3)$$
$$= (-900)(0.1)+(-100)(0.2)+(700)(0.3)+(1500)(0.4)$$
$$= 700$$

- Rule of Expected Value

$$E(aX+b) = a\,E(X) + b$$

Proof:

$$E(aX+b) = \sum_D (ax+b)p(x)$$
$$= a\sum_D xp(x) + b\sum_D p(x)$$
$$= aE(x) + b$$

1. For any constant a, E(aX)=aE(X) (b=0)
2. For any constant b, E(X+b)=E(X)+b (a=1)

The Variance of X

Let X have pmf p(x) and the expected value μ. Then the variance of X, denoted by V(X) or σ^2_X, or just σ^2, is

$$V(X) = \sum_{x \in D} (x-\mu)^2 \cdot p(x) = E[(X-\mu)^2]$$

The standard deviation (SD) of X is

$$\sigma_X = \sqrt{\sigma_X^2}$$

- **Example**

  If X is the number of cylinders on the next car to be tuned at a service facility, with pmf as given, then what is the V(X) ?

x	4	6	8
$p(x)$	0.5	0.3	0.2

3.3 Expected Values of Discrete Random Variables

Solution:

$$V(X) = \sigma^2 = \sum_{x=4}^{8} (x - 5.4)^2 \cdot p(x)$$

$$= (4 - 5.4)^2(.5) + (6 - 5.4)^2(.3) + (8 - 5.4)^2(.2) = 2.44$$

3.3 Expected Values of Discrete Random Variables

- **A short formula for σ^2**

$$V(X) = \sigma^2 = [\sum_D x^2 p(x)] - \mu^2 = E(X^2) - [E(X)]^2$$

Proof:

$$V(X) = \sum_{x \in D} (x-\mu)^2 \cdot p(x) = E[(X-\mu)^2]$$

$$= \sum_{x \in D} x^2 \cdot p(x) - 2\mu \sum_D xp(x) + \mu^2 \sum_D p(x)$$

$$= E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

3.3 Expected Values of Discrete Random Variables

- **Example 3.25**

The pmf of the number of cylinders X on the next car to be turned at a certain facility was given in Example 3.24 as p(4)=0.5, p(6)=0.3 and p(8)=0.2, from which μ=5.4, and

$$E(X^2) = (4^2)(0.5) + (6^2)(0.3) + (8^2)(0.2) = 31.6$$

$$\delta^2 = E(X^2) - E(X)^2 = 31.6 - (5.4)^2 = 2.44$$

- **Rules of Variance**

$$V(aX+b)=\sigma^2_{aX+b}=a^2\sigma^2_X \quad \& \quad \sigma_{aX+b}=|a|\sigma_X$$

1. $\sigma^2_{aX}=a^2\sigma^2_X,\sigma_{aX}=|a|\sigma_X$

2. $\sigma^2_{X+b}=\sigma^2_X$

3.3 Expected Values of Discrete Random Variables

- **Example 3.26**

In the computer sales problem of Example 3.23, $E(X)=2$ and

$$E(X^2)=(0)^2(0.1)+(1)^2(0.2)+(2)^2(0.3)+(3)^2(0.4)=5$$

so $V(X)=5-(2)^2=1$.

What is the variance and deviation of profit function $h(X)=800X-900$?

Solution:

The variance: $(800)^2V(X)=(640{,}000)(1)=640{,}000$

standard deviation 800.

3.4 The Binomial Probability Distribution

- The requirements for a **binomial experiment**

1. The experiment consists of a sequence of *n* smaller experiments called trials, where *n* is fixed in advance of the experiment.
2. Each trail can result in one of the same two possible outcomes (dichotomous trials), which we denote by success (S) or failure (F).
3. The trails are independent, so that the outcome on any particular trail does not influence the outcome on any other trail.
4. The probability of success is constant from trail to trail; we denote this probability by *p*.

3.4 The Binomial Probability Distribution

- **Example 3.27**

The same coin is tossed successively and independently *n* times. We arbitrarily use S to denote the outcome H(heads) and F to denote the outcome T(tails). **Then this experiment satisfies Condition 1-4.**

Tossing a thumbtack *n* times, with S=point up and F=point down, also results in a **binomial experiment.**

3.4 The Binomial Probability Distribution

- **Example 3.29**

  Suppose a certain city has 50 licensed restaurants, of which 15 currently have at least one serious health code violation and the other 35 have no serious violations. There are five inspectors, each of whom will inspect one restaurant during the coming week. The name of each restaurant is written on a different slip of paper, and after the slips are thoroughly mixed, each inspector in turn draws one of the slips **without replacement**. Label the ith trail as success if the ith restaurant selected (i=1,…5) has no serious violations. Then

  P(S on first trail) = 35/50 = 0.7 &

  P(S on second trial) = P(SS) + P(FS)

  = P(second S | first S) P(first S) + P(second S| first F) P(first F)

  =(34/49)(35/50) +(35/49)(15/50)= (35/50)(34/49+15/49) = 0.7

  Similarly, P(S on ith trail) = 0.7 for i=3,4,5.

- **Example 3.29 (Cont')**

P(S on fifth trail | SSSS)

= (35-4) / (50-4) = 31/46

$\neq$

P(S on fifth trail | FFFF)

= 35 / (50-4) = 35/46

Thus the experiment is not binomial because the trials are not independent. In general, if sampling is **without replacement**, the experiment will not yield independent trials.

- **Example 3.30**

Suppose a certain state has 500,000 licensed drivers, of whom 400,000 are insured. A sample of 10 drivers is chosen without replacement. The ith trial is labeled S if the ith driver chosen is insured.

Although this situation would seem identical to that of Example 3.28, the important difference is that the **size of the population being sampled is very large relative** to the sample size. In this case

$P(S \text{ on } 2 \mid S \text{ on } 1) = 3999{,}999/4999{,}999 = 0.8$ &

$P(S \text{ on } 10 \mid S \text{ on first } 9) = 399{,}991/499{,}991 = 0.799996 \approx 0.8$

These calculations suggest that although the trials are not exactly independent, the conditional probabilities differ so slightly from one another that for practical purposes **the trials can be regarded as independent with constant P(S)=0.8. Thus, to a very good approximation, the experiment is binomial with n =10 and p=0.8.**

3.4 The Binomial Probability Distribution

- **Rule**

Consider sampling without replacement from a dichotomous population of size N. **If the sample size (number of trials) *n* is at most 5% of the population size**, the experiment can be analyzed as though it were exactly a binomial experiment.

In Ex. 3.30, the sample size n is 10, and the population size N is 500,000, $10/500000<0.05$.

However, in Ex. 3.29, the sample size n =5, and the population size N is 50, $5/50 > 0.05$.

3.4 The Binomial Probability Distribution

- **Binomial random variable**

Definition:

Given a binomial experiment consisting of *n* trails, the binomial random variable X associated with this experiment is defined as

X = the number of S's among the *n* trials

Suppose, for instance, that n=3. Then there are eight possible outcomes for the experiment:

SSS SSF SFS SFF FSS FSF FFS FFF

X(SSS) = 3, X(SSF) =2, … X(FFF)=0

3.4 The Binomial Probability Distribution

- **X~ Bin(n,p)**

Possible values for X in an n-trial experiment are x = 0,1,2,…,n. we will often write X~Bin(n,p) to indicate that X is a binomial rv based on n trials with success probability p.

Notation:

Because the pmf of a binomial rv depends on the two parameters n and p, we denote the pmf by b(x;n,p)

3.4 The Binomial Probability Distribution

- Theorem:

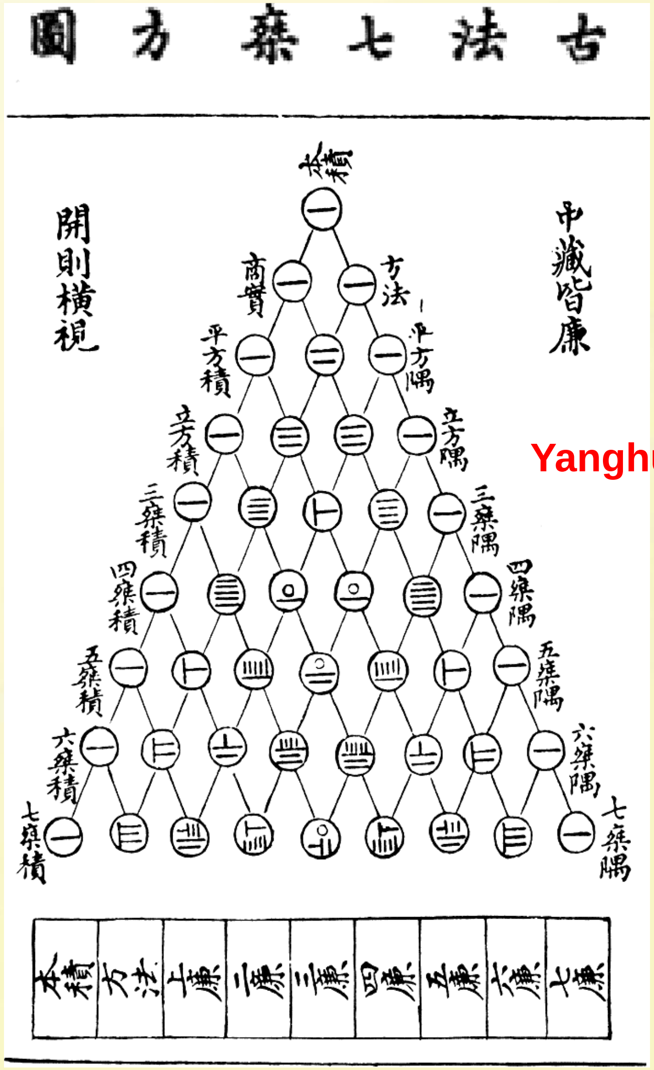
$$b(x;n,p)=\begin{cases}\binom{n}{x}p^{x}(1-p)^{n-x}, & x=0,1,2,\ldots,n\\ 0, & otherwise\end{cases}$$

3.4 The Binomial Probability Distribution

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

$$\sum_{x=0}^{n} b(x;n,p) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= [p+(1-p)]^n = 1$$



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3.4 The Binomial Probability Distribution

- The outcomes and probabilities for a binomial experiment with 3 trails

Outcomes	x	Probability	Outcomes	X	Probability
SSS	3	p^3	FSS	2	$p^2(1-p)$
SSF	2	$p^2(1-p)$	FSF	1	$p(1-p)^2$
SFS	2	$p^2(1-p)$	FFS	1	$p(1-p)^2$
SFF	1	$p(1-p)^2$	FFF	0	$(1-p)^3$

$$b(2;3,p) = P(SSF) + P(SFS) + P(FSS)$$

$$= \binom{3}{2} p^2 (1-p)^{3-2} = 3p^2(1-p),$$

- **Example 3.31**

Each of six randomly selected cola drinkers is given a glass containing cola S and one containing cola F. The glasses are identical in appearance except for a code on the bottom to identify the cola. Suppose there is actually no tendency among cola drinkers to prefer one cola to the other. Then p=P(a selected individual prefers S) =0.5, so with X=the number among the six who prefer S, X~Bin(6,0.5).

$$P(X=3)=b(3;6,0.5)=\binom{6}{3}(0.5)^3(0.5)^3=20(0.5)^6=0.313$$

$$P(3\le X)=\sum_{x=3}^{6}b(x;6,0.5)=\sum_{x=3}^{6}\binom{6}{x}(0.5)^x(0.5)^{6-x}=0.656$$

3.4 The Binomial Probability Distribution

- **Notation**

  For X~Bin(n,p), the cdf will be denoted by

$$P(X \le x) = B(x; n, p) = \sum_{y=0}^{x} b(y; n, p), x = 0, 1, \ldots, n$$

- Binomial Table

  Refer to Appendix Table A.1

3.4 The Binomial Probability Distribution

- B(n,p) with n=5, p=0.1,0.3,0.5,0.7 and 0.9

	p					
x		0.1	0.3	0.5	0.7	0.9
	0	0.590	0.168	0.031	0.002	0.000
	1	0.919	0.528	0.188	0.031	0.000
	2	0.991	0.837	0.500	0.163	0.009
	3	1.000	0.969	0.812	0.472	0.081
	4	1.000	0.998	0.969	0.832	0.410
	5	**1.000**	**1.000**	**1.000**	**1.000**	**1.000**

B(3; 5, 0.5)

B(2; 5, 0.7)

- **Example 3.32**

Suppose that 20% of all copies of a particular textbook fail a certain binding strength test. Let X denote the number among 15 randomly selected copies that fail the test. Then X has a binomial distribution with n=15 and p=0.2.

1. The probability that at most 8 fail the test is

$$P(X \le 8) = \sum_{y=0}^{8} b(y;15,0.2) = B(8;15,0.2) = 0.999$$

2. The probability that exactly 8 fail is

$$P(X = 8) = P(X \le 8) - P(X \le 7) = B(8;15,0.2) - B(7;15,0.2) = 0.999 - 0.996 = 0.003$$

3. The probability that at least 8 fail is

$$P(X \ge 8) = 1 - P(X \le 7) = 1 - B(7;15,0.2) = 1 - 0.996 = 0.004$$

4. The probability that between 4 and 7

$$P(4 \le X \ge 7) = P(X \le 7) - P(X \le 3) = B(7;15,0.2) - B(3;15,0.2) = 0.996 - 0.648 = 0.348$$

3.4 The Binomial Probability Distribution

- The Mean and Variance of X

For $n=1$, the binomial distribution became the Bernoulli distribution. The mean value of Bernoulli variables is $\mu = p$

(see example 3.18)

So the expected number of S's on any single trial is p.

Example 3.18

- Let X=1 if a randomly selected component needs warranty service and 0 otherwise. **Then X is a Bernoulli rv with pmf**

$$p(x)=\begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & x\neq 0 \end{cases}$$

E(x)=?

Solution:

$E(x) = 0\times p(0) + 1\times p(1) = p(1) = p$.

Note: the expected value of X is just the probability that X takes on the value 1.

3.4 The Binomial Probability Distribution

▪The Mean and Variance of X

Since a binomial experiment consists of *n* trials, intuition suggests that for

$$X \sim Bin(n, p) \quad , \qquad E(X) = np$$

But the expression for V(X) is not so intuitive

3.4 The Binomial Probability Distribution

- Proposition

  If $X \sim Bin(n,p)$, then $E(X)=np$, $V(X)=np(1-p)=npq$, and

$$\sigma_X = \sqrt{npq}$$

where $q=1-p$

3.4 The Binomial Probability Distribution

- **Example 3.34**

  If 75% of all purchases at a certain store are made with a credit card and X is the number among ten randomly selected purchases made with a credit card.

  E(X)=?   V(X)=?

3.4 The Binomial Probability Distribution

Solution:

Since $X \sim Bin(10, 0.75)$. Thus

$E(X) = np = (10)(0.75) = 7.5$,

$V(X) = npq = 10(0.75)(0.25) = 1.875$.

If we perform a large number of independent binomial experiments, each with $n=10$ trails and $p=0.75$, then the average number of S's per experiment will be close to 7.5.