Physics CST (2023-24) Homework 3

Please send the completed file to my mailbox yy.lam@qq.com by November 13th, with using the filename format:

student_number-name-cst-hw3

Please answer the questions by filling on these sheets. Or alternatively, do the homework as usual by using papers, then take the pictures and paste them onto these question sheets.

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1. A large diameter tank without lid placing on ground filled with water to a height h. The tank is punctured with a small hole at a height 0.5 m above the bottom. What is the water level h being able to eject a stream of water through the hole landing 3 m from the tank?

   Solution. *Pressures are supposed to be the same at the height h and at 0.5 m height, the Bernoulli equation gives*

   $$\frac{1}{2}\rho v_t^2 + \rho g h = \frac{1}{2}\rho v^2 + 0.5\rho g$$

   where v_t and v are the velocities of water dropping at the top and ejecting at the hole, respectively. Since $v \gg v_t$, we neglect v_t and put it to zero. Thus, the velocity of water ejecting from the hole is $v = \sqrt{2g(h-0.5)}$. In the vertical direction, the time taken for landing

   $$0.5 = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{1}{g}}.$$

   As the horizontal range is 3 m, $3 = \sqrt{2g(h-0.5)} \cdot \sqrt{\frac{1}{g}} = \sqrt{2(h-0.5)}$ gives $h = 5$ m.

2. A wind turbine is rotating counterclockwise at 0.5 rev/s and slows to a stop in 10 s. Its blades are 20 m in length. (a) What is the angular acceleration of the turbine? (b) What is the centripetal acceleration of the tip of the blades at $t = 0$ s? (c) What is the magnitude and direction of the total linear acceleration of the tip of the blades at $t = 0$ s?

   Solution. *(a) The angular acceleration is*

   $$\ddot{\theta} = \frac{0 - 0.5(2\pi)}{10} = -0.1\pi \ rad\ s^{-2}.$$

   (b) The centripetal acceleration is

   $$a_c = r\dot{\theta}^2 = 20 \times (0.5(2\pi))^2 = 20\pi^2 \ ms^{-2}.$$

   (c) The tangential acceleration is

   $$a_t = r\ddot{\theta} = 20 \times (-0.1\pi) = -2\pi \ ms^{-2}.$$

   The magnitude of the acceleration is

   $$a = \sqrt{a_c^2 + a_t^2} = \sqrt{(20\pi^2)^2 + (2\pi)^2} = 197.5 \ ms^{-2}.$$

   The angle of direction is

   $$\theta = \tan^{-1}\frac{-2\pi}{20\pi^2} = -1.82^\circ$$

3. To develop muscle tone, a man lifts a 8.0 kg weight held in his hand. He uses his biceps muscle to flex the lower arm through an angle of 60°. (a) What is the angular acceleration if the weight is 28 cm from the elbow joint, here forearm has a moment of inertia of 0.27 kgm^2, and the net force he exerts is 1230 N at an effective perpendicular lever arm of 2.5 cm? (b) What is the angular velocity at 60°? (c) How much work does she do?

Solution. *(a) Since*

$$net\ torque = distance\ of\ the\ perpendicular\ lever\ arm\ \times\ net\ force$$

and also equals to $I\ddot{\theta}$ where I is the ***total*** *moment of inertia of the system, thus*

$$(0.27 + 8 \times 0.28^2)\ddot{\theta} = 0.025 \times 1230 \quad \Rightarrow \quad \ddot{\theta} = 34.3\ rad\ s^{-2}$$

(b) Assuming the angular acceleration $\ddot{\theta}$ being constant, the rotational kinematic equation $\dot{\theta}^2 = 2\ddot{\theta}\theta$ with the initial angular velocity zero gives

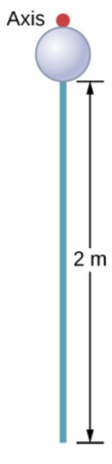
$$\dot{\theta} = \sqrt{2 \times 34.3 \times (60 \times 2\pi/360)} = 8.5\ rad\ s^{-1}$$

(c) As rotational energy can be given by $\tau\theta$, we get

$$0.025 \times 1230 \times \frac{60 \times 2\pi}{360} = 32.2\ J$$

Note. *Using K.E.* $= \frac{1}{2}I\dot{\theta}^2$ *will give the same answer.*

4. A pendulum consists of a rod of length 2 m and mass 2.4 kg with a solid sphere of mass 1.8 kg and radius 0.3 m attached at one end. (i) Find the moment of inertia of the pendulum about the axis. (ii) Find the centre of mass of it from the axis. (iii) What is the angular velocity of the pendulum at its lowest point if it is released from rest at an angle of 25°?



Solution. *(i) Use the parallel axis theorem to find the corresponding moments of inertia of the bodies about the axis. For the rod it is*

$$I_r = \frac{1}{12} \times 2.4 \times 2^2 + 2.4(1 + 0.6)^2 = 6.944\ kgm^2$$

For the sphere it is

$$I_s = \frac{2}{5} \times 1.8 \times 0.3^2 + 1.8 \times 0.3^2 = 0.227\ kgm^2$$

The total moment of inertia of the pendulum about the axis is the sum

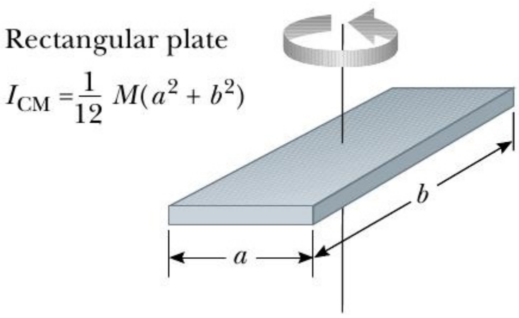
$$I = I_r + I_s = 7.171\ kgm^2.$$

(ii) The position of the centre of mass is at $\dfrac{0.3 \times 1.8 + (1 + 0.6) \times 2.4}{1.8 + 2.4} = 1.043\ m$ *from the axis vertically. (iii) As raising the pendulum inclined at* $25°$ *vertically, the centre of mass is raised by the vertical distance* $0.955(1 - \cos 25°)$. *Using the conservation of energy, the potential energy with the angle of inclination equals to the rotational kinetic energy at the bottom, we get*

$$(1.8 + 2.4) \times 9.8 \times 1.043(1 - \cos 25°) = \frac{1}{2} I\omega^2$$

Inserting the I *in (i) into the expression we obtain* $\omega = 1.060\ rad\ s^{-1}$.

5. (a) Drive the formula for $I_{\rm CM}$ as shown in the figure. (b) Find the new moment of inertia if the rotating axis is shifted to one of the corners.



Solution. *(a) The formula of moment of inertia is given by* $\int r^2 \mathrm{d}m$. *Written in terms of Cartesian coordinate system that* $r^2 = x^2 + y^2$ *and* $\mathrm{d}m = \rho(\mathrm{d}x\mathrm{d}y)$ *where* ρ *is the mass per unit area. Thus,*

$$\begin{aligned}
I &= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho(x^2 + y^2)\mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho x^2 \mathrm{d}x\mathrm{d}y + \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho y^2 \mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2} \rho \frac{x^3}{3}\Big|_{-a/2}^{a/2} \mathrm{d}y + \int_{-b/2}^{b/2} \rho x y^2 \Big|_{-a/2}^{a/2} \mathrm{d}y \\
&= \frac{\rho}{12} a^3 y \Big|_{-b/2}^{b/2} + \rho a \frac{y^3}{3}\Big|_{-b/2}^{b/2} \\
&= \frac{\rho}{12} a^3 b + \frac{\rho}{12} a b^3
\end{aligned}$$

Since $M = \rho ab$, *the moment of inertia is* $I_{CM} = \dfrac{1}{12} M(a^2 + b^2)$. □

(b) The distance between the centre of mass to one of the corners is $\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}$. *Using the parallel axis theorem, the new moment of inertia is*

$$I = \frac{1}{12}M(a^2+b^2) + M\left(\frac{a^2}{4}+\frac{b^2}{4}\right) = \frac{1}{3}M(a^2+b^2).$$

6. Oil of density 730 kgm^{-3} is poured on top of a tank of water, and it floats on the water without mixing. A block of plastic of density 860 kgm^{-3} is placed in the tank, and it is about floating at the interface of the two liquids (completely immerses in the liquids). What fraction of the block's volume is immersed in water?

   Solution. *Let x be the fraction of the volume of the wooden block in water. Thus the volume xV is in water, and volume $(1-x)V$ is in oil, where V is the volume of the block. We have,*

   $$\rho V g = \rho_w x V g + \rho_o(1-x)Vg$$

   where ρ, ρ_w, ρ_o are the densities of the block, water and oil. Therefore,

   $$x = \frac{\rho - \rho_o}{\rho_w - \rho_o} = \frac{860-730}{1000-730} = 0.48$$

7. (a) Using the law of conservation of energy, derive and calculate the escape velocity from the surface of earth.

   (b) What initial vertical speed is necessary to shoot a satellite to 300 km above the earth? (Given $G = 6.7\times10^{-11}$Nm^{2}kg^2, mass and radius of the earth 6.0×10^{24}kg and 6.4×10^6m, respectively.)

   Solution. *(a) The law of conservation of energy implies that the kinetic plus potential energy on the surface of the earth equals the kinetic plus potential energy at some point r:*

   $$\frac{1}{2}mv_e^2 - \frac{GMm}{r_e} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

   where m is a test mass, r_e and r are the radius of earth and an arbitrary distance from the centre of the earth. M is mass of the earth, v_e and v are the vertical launching velocity on the surface of earth and the later velocity depending on r, respectively. To leave off earth gravity completely, we require $r\to\infty$ and $v\to 0$. Setting $v=0$ here is due to the fact that the object has no need to have relative velocity greater than zero under no earth gravity. Thus

   $$\frac{1}{2}mv_e^2 = \frac{GMm}{r_e} \quad\Rightarrow\quad v_e = \sqrt{\frac{2GM}{r_e}}.$$

   The escape velocity is

   $$v_e = \sqrt{\frac{2\times6.7\times10^{-11}\times6\times10^{24}}{6.4\times10^6}} = 11208\ ms^{-1}$$

   (b) By using the same equation of conservation of energy in (a), this time we set $r = r_e + 3\times10^5$ and $v\to0$ since theoretically we only require the minimum launching velocity of the missile to hit the object. Therefore, we have

   $$\frac{1}{2}mv_e^2 = GMm\left(\frac{1}{r_e} - \frac{1}{r_e+3\times10^5}\right).$$

   Inserting the values into the equation for v_e, we obtain $v_e = 2.4\times10^3\ ms^{-1}$.

8. Assuming the density of Earth is constant, the gravitational acceleration $g(r)$ is a function of the radial distance. (a) Derive the equation of average value of the acceleration $\langle g(r)\rangle$ along the radial direction in terms of the G, M, R and r $(r < R)$, being the gravitational constant, mass of Earth, radius of Earth and the radial distance from the centre. (b) Estimate the pressure at the centre of the earth.

Solution. *(a) The average value is*

$$\langle g(r)\rangle = \frac{1}{R}\int_0^R \frac{GMr}{R^3}\mathrm{d}r = \frac{1}{R^4}GM\frac{r^2}{2}\bigg|_0^R = \frac{GM}{2R^2}.$$

(b) Gravitational force at the centre of the the earth is zero, however, the pressure is the maximum due to the non-directional property of pressure. We may use the familiar formula $P = \rho\langle g\rangle R$ to estimate the pressure at the centre where R is the height which is just the radius of the earth. We use the average value $\langle g\rangle$ obtained in (b) over the radial interval. Thus the pressure at the centre of the earth is

$$P = \rho\langle g\rangle R = \left(\frac{M}{\frac{4}{3}R^3\pi}\right)\left(\frac{GM}{2R^2}\right)R = \frac{3GM^2}{8\pi R^3}.$$

9. **(Angular momentum).** A small ball of mass 0.50 kg is attached by a massless string to a vertical rod that is spinning as shown in Figure 1. When the rod has an angular velocity of 6.0 rad s^{-1}, the string makes an angle of 30° with respect to the vertical. (a) If the angular velocity is increased to 10.0 rad s^{-1}, what is the new angle of the string? (b) Calculate the initial and final angular momenta of the ball. (c) Can the rod spin fast enough so that the ball is horizontal?

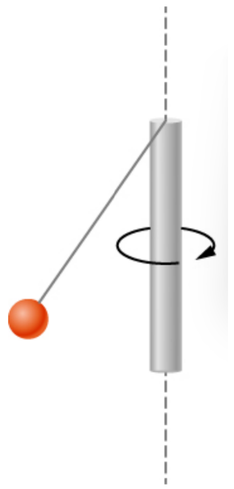


Figure 1:

Solution. *(a) Let r be the varying horizontal radius, ℓ be the length of the string. We write the vertical and horizontal components as*

$$T\cos\theta = mg, \quad T\sin\theta = mr\omega^2$$

where T is the tension along the string. Eliminating T we have

$$\tan\theta = \frac{r\omega^2}{g}.$$

Since $r = \ell \sin\theta$, *we also get*

$$\tan\theta = \frac{\ell \sin\theta\omega^2}{g} \quad \Rightarrow \quad \cos\theta = \frac{g}{\ell\omega^2}$$

or just $\cos\theta \propto \dfrac{1}{\omega^2}$. *Thus the new angle can be obtained by*

$$\cos\theta = \frac{6^2 \cos 30}{10^2} \quad \Rightarrow \quad \theta = 71.8°.$$

(b) The angular momentum is given by

$$L = mr^2\omega = \left(\frac{g\tan\theta}{\omega^2}\right)^2 m\omega = \frac{mg^2\tan^2\theta}{\omega^3}.$$

Thus the initial and the final angular momenta are

$$L_i = \frac{0.5 \times 9.8^2 \tan^2 30}{6^3} = 0.074 \; kg \; m^2 \; s^{-1}$$
$$L_f = \frac{0.5 \times 9.8^2 \tan^2 71.8}{10^3} = 0.44 \; kg \; m^2 \; s^{-1}$$

From (a) we know that $\cos\theta$ *approaches to zero as* $\omega \to \infty$. *Thus the rod becomes horizontal as the angular speed being infinitely fast.*