Chapter 2. Probability

What is probability?

- **Probability**

The term probability refers to the study of **randomness** and **uncertainty**.

e.g. a card is drawn form a deck of 52 playing card, it is uncertain which card will be drawn.

In any situation in which one of **a number of possible outcomes may occur**, the theory of probability provides methods for **quantifying** the changes, or likelihoods, associated with the various outcomes.

Chapter 2: Probability

- 2.1 Sample Spaces and Events
- 2.2 Axioms, Interpretations, and Properties of Probability
- 2. 3 Counting Techniques
- 2.4 Conditional Probability
- 2.5 Independence

2.1 Sample Spaces and Events

- **Experiment**

  An experiment is any action or process whose outcome is subject to uncertainty, ***e.g.***

  tossing a coin once or several times

  selecting a card or cards from a deck, *etc*.

2.1 Sample Spaces and Events

- **Sample Space**

The sample space of an experiment, denoted by S , is the set of all possible outcomes of that experiment, ***e.g.***

Examining whether a single fuse is defective or not.

The two possible outcomes: D (defective) & N(not defective)

Two fuses in sequence:

S={DD DN ND NN}

2.1 Sample Spaces and Events

- Example 2.3

Two gas stations are located at a certain intersection. **Each one has six gas pumps**. Consider the experiment in which **the number of pumps in use** at a particular time of day is determined for each of the stations. The possible outcomes:

	0	1	2	3	4	5	6
0	(0 0)	(0 1)	(0 2)	(0 3)	(0 4)	(0 5)	(0 6)
1	(1 0)	(1 1)	(1 2)	(1 3)	(1 4)	(1 5)	(1 6)
2	(2 0)	(2 1)	(2 2)	(2 3)	(2 4)	(2 5)	(2 6)
3	(3 0)	(3 1)	(3 2)	(3 3)	(3 4)	(3 5)	(3 6)
4	(4 0)	(4 1)	(4 2)	(4 3)	(4 4)	(4 5)	(4 6)
5	(5 0)	(5 1)	(5 2)	(5 3)	(5 4)	(5 5)	(5 6)
6	(6 0)	(6 1)	(6 2)	(6 3)	(6 4)	(6 5)	(6 6)

2.1 Sample Spaces and Events

- Example 2.4

If a new type-D flashlight battery has a voltage that is outside certain limits, that battery is characterized as a failure(F); otherwise, it is a success(S).

Suppose an experiment consists of testing each battery as it comes off an assembly line until we first observe a success. The sample space is

$$\boldsymbol{S} = \{S, FS, FFS, FFFS, \ldots\}$$

which contains an **infinite number** of possible outcomes.

Events

In our study of probability, we will be interested not only in the individual outcomes of S but also in various collections of outcomes from S .

Definition

An **event** is any collection (subset) of outcomes contained in the sample space S .

Simple Event (elementary event)

An event consists of exactly one outcome

Compound Event

An event consists of more than one outcome

Example: Simple and Compound Event

Consider the a experiment, “A wheel with 18 numbers on the perimeter is spun and allowed to come to rest so that a pointer points within a numbered sector”, the sample space is

$$S = \{1,2,3,\ldots,17,18\}$$

what is the event E (subset of the sample space S) that corresponds to each of the following outcomes? Indicate whether the event is a simple event or a compound event.

(A) The outcome is a prime number. (B) The outcome is the square of 4.

Solution:

(A) The outcome is a prime number if any of the simple events 2, 3, 5, 7, 11, 13, or 17 occurs.* Thus, to say the event "A prime number occurs" is the same as saying the experiment has an outcome in the set

$$E = \{2, 3, 5, 7, 11, 13, 17\}$$

Since event E has more than one element, it is a compound event.

(B) The outcome is the square of 4 if 16 occurs. Thus, to say the event "The square of 4 occurs" is the same as saying the experiment has an outcome in the set

$$E = \{16\}$$

Since E has only one element, it is a simple event.

- **Example 2.5**

Consider an experiment in which each of three vehicles taking a particular freeway exit turns left(L) or right(R) at the end of the exit ramp.

Sample space (the 8 possible outcomes):

{LLL, RLL, LRL, LLR, LRR, RLR, RRL RRR }

Thus there are **8 simple events:**

{LLL},{RLL},{LRL},{LLR},{LRR},{RLR},{RRL},{RRR}

- **Example 2.5(Cont)**

Some compound events include

- the event that exactly one of the three vehicles turns right :

  {RLL, LRL, LLR}

- the event that all three vehicles turns in the same direction:

  {LLL, RRR}

Example 2.6 (Ex.2.3 continued)

- When the number of pumps in use at each of two six-pump gas station is observed, there are 49 possible outcomes, so there are **49 simple events:**
- $E_1=\{(0,0)\}, E_2=\{(0,1)\}, \ldots, E_{49}=\{(6,6)\}$.

- **Example of compound event are:**
- The event that the number of pumps in use is the same for both stations:

  $\{(0,0),\{1,1\},\{2,2\},\{3,3\}, \{4,4\},\{5,5\},\{6,6\}\}$

- The event that the total number of pumps in use is four

  $\{(0,4), \{1,3\}, \{2,2\}, \{3,1\}, \{4,0\}\}$

2.1 Sample Spaces and Events

- Example 2.7 (Ex. 2.4 continued)

➢ the event that at most three batteries are examined:

{S, FS, FFS}

➢ the event that an even number of batteries are examined

{FS, FFFS, FFFFFS, ...}

2.1 Some Relations from Set Theory

An event is just a set, so relationships and results from elementary set theory can be used to study events.

The following operations will be used to create new events from given events.

- **Union** of two events A and B, denoted by $A \cup B$, and read "A or B", that is, all outcomes in at least one of the events A and B.

●DEFINITION Union

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

Here we use the word **or** in the way it is always used in mathematics; that is, x may be an element of set A or set B or both.

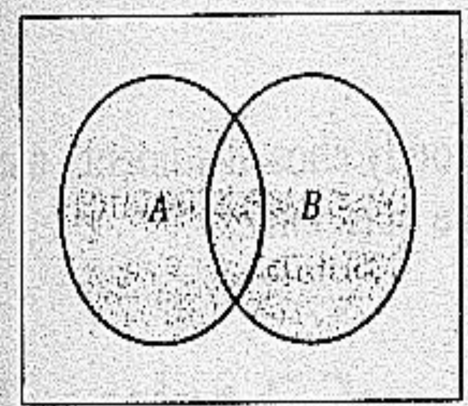


FIGURE 1 $A \cup B$ is the shaded region.

- **Intersection** of two events A and B, denoted by A ∩ B and read A and B is the event consisting of all outcomes that are in both A and B.

- **DEFINITION Intersection**

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

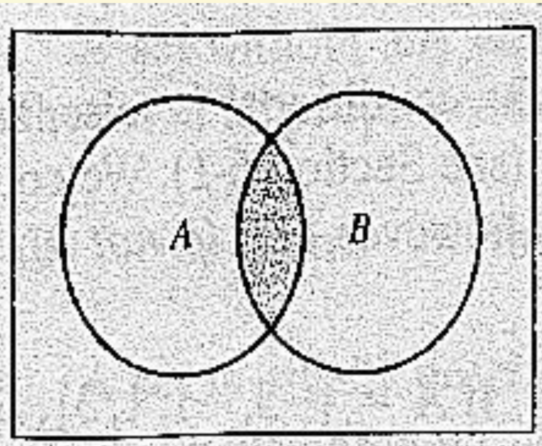


FIGURE 2 $A \cap B$ is the shaded region.

If $A \cap B = \varnothing$, the sets A and B are said to be **disjoint;** this is illustrated in Figure 3.

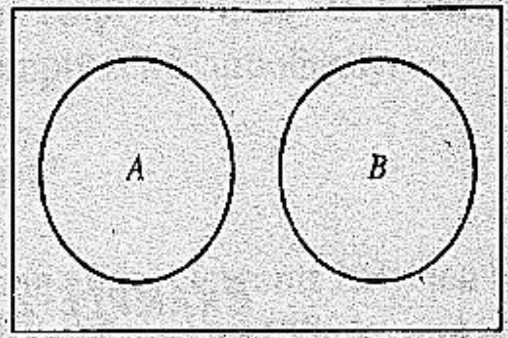


FIGURE 3 $A \cap B = \varnothing$; A and B are disjoint.

Definition:

When A and B have no outcomes in common, they are said to be mutually exclusive or disjoint events.

➢**Complement** of an event A, denoted by A', is the set of all outcomes in S that are no contained in A

Definition: Complement

$$A' = \{x \in U \mid x \notin A\}$$

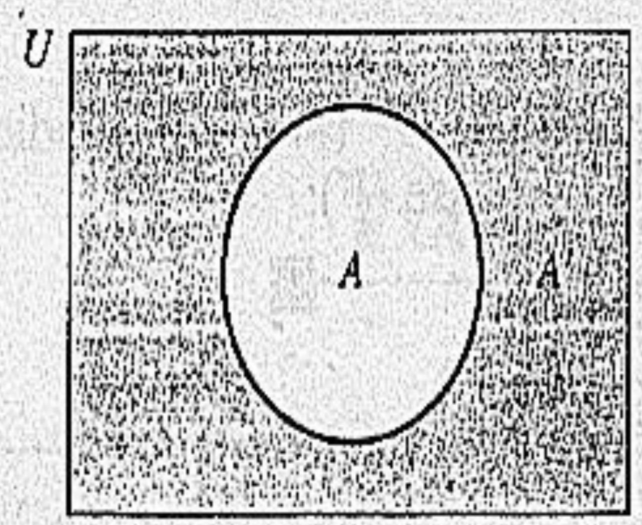


FIGURE 4 The complement of A is A'.

- **Example 2.8**

Let $A=\{0,1,2,3,4\}$, $B=\{3,4,5,6\}$ and $C=\{1,3,5\}$. Then

$A \cup B = \{0,1,2,3,4,5,6\} =$ ***S*** ,

$A \cup C = \{0,1,2,3,4,5\}$

$A \cap B = \{3,4\}$

$A \cap C=\{1,3\}$

$A'=\{5,6\}$

$(A \cup C)'=\{6\}$

Some Relations from Set Theory

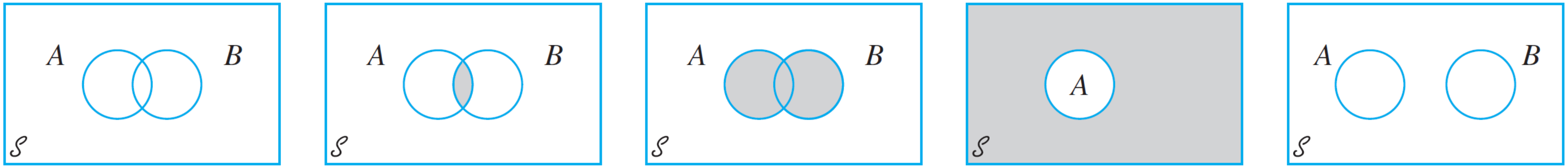
The operations of union and intersection can be extended to more than two events.

For any three events A, B, and C, the event $A \cup B \cup C$ is the set of outcomes contained in at least one of the three events, whereas $A \cap B \cap C$ is the set of outcomes contained in all three events.

Given events A_1, A_2, A_3 ,..., these events are said to be **mutually exclusive** (or pairwise disjoint) if **no two events have any outcomes in common.**

Some Relations from Set Theory

Figure 2.1 shows examples of Venn diagrams.



(a) Venn diagram of events A and B

(b) Shaded region is $A \cap B$

(c) Shaded region is $A \cup B$

(d) Shaded region is A'

(e) Mutually exclusive events

Figure 2.1 Venn diagrams

- **2.2 Axioms, Interpretations, and Properties of Probability**

2.2 Axioms, Interpretations, and Properties of Probability

- Given an experiment and a sample space ***S*** , the objective of probability is to assign to each event ***A*,** a number ***P*(*A*)**, called the probability of the event ***A***, which will give a precise measure of the chance that ***A*** will occur. All assignments should satisfy the three following axioms of probability.

➢ **Axiom 1: for any event A, P(A)>=0**

Chance of *A* occurring should be nonnegative.

➢ **Axiom 2: P(S)=1**

The maximum possible probability of 1 is assigned to ***S*** .

Axioms, Interpretations, and Properties of Probability

Axiom 3

If A_1, A_2, A_3,… is an finite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{k} P(A_i)$$

If A_1, A_2, A_3,… is an infinite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Proposition

where $\varnothing$ is the null event (the event containing no outcomes whatsoever).

- This in turn implies that the property contained in Axiom 3 is valid for *a finite* collection of disjoint events.

$$P(\varnothing) = 0$$

- Example 2.11

In the experiment in which a single coin is tossed, the sample space is $S=\{H,T\}$. Then

$$P(S) = P(H) + P(T) = 1$$

since $H \cup T = S$ and $H \cap T = \phi$

Let $P(H) = p$, where p is any fixed number between 0 and 1, then $P(T) = 1- p$ is an assignment consistent with the axioms.

- Example 2.12 (Ex. 2.4 continued)

$E_1=\{S\}$, $E_2=\{FS\}$, $E_3=\{FFS\}$, $E_4=\{FFFS\}$…

Support the probability of any particular battery being satisfactory is 0.99, then

$$P(E_1) = 0.99$$

$$P(E_2) = 0.01 \times 0.99$$

$$P(E_3) = (0.01)^2 \times 0.99 \ \ldots$$

Note: $\boldsymbol{S} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots$

and $E_i \cap E_j = \phi$ (i is not j)

$$P(S) = 1 = P(E_1) + P(E_2) + P(E_3) + \ldots$$

- **Two Special Events**

➢ **Impossible event**

The event contains no simple event

$P(A)=0$

➢ **Certain event**

The event contains all simple events

$P(B)=1$

- Interpreting Probability

➢ Axioms #1-3 serve only to rule out assignments inconsistent with our intuitive notions of probability.

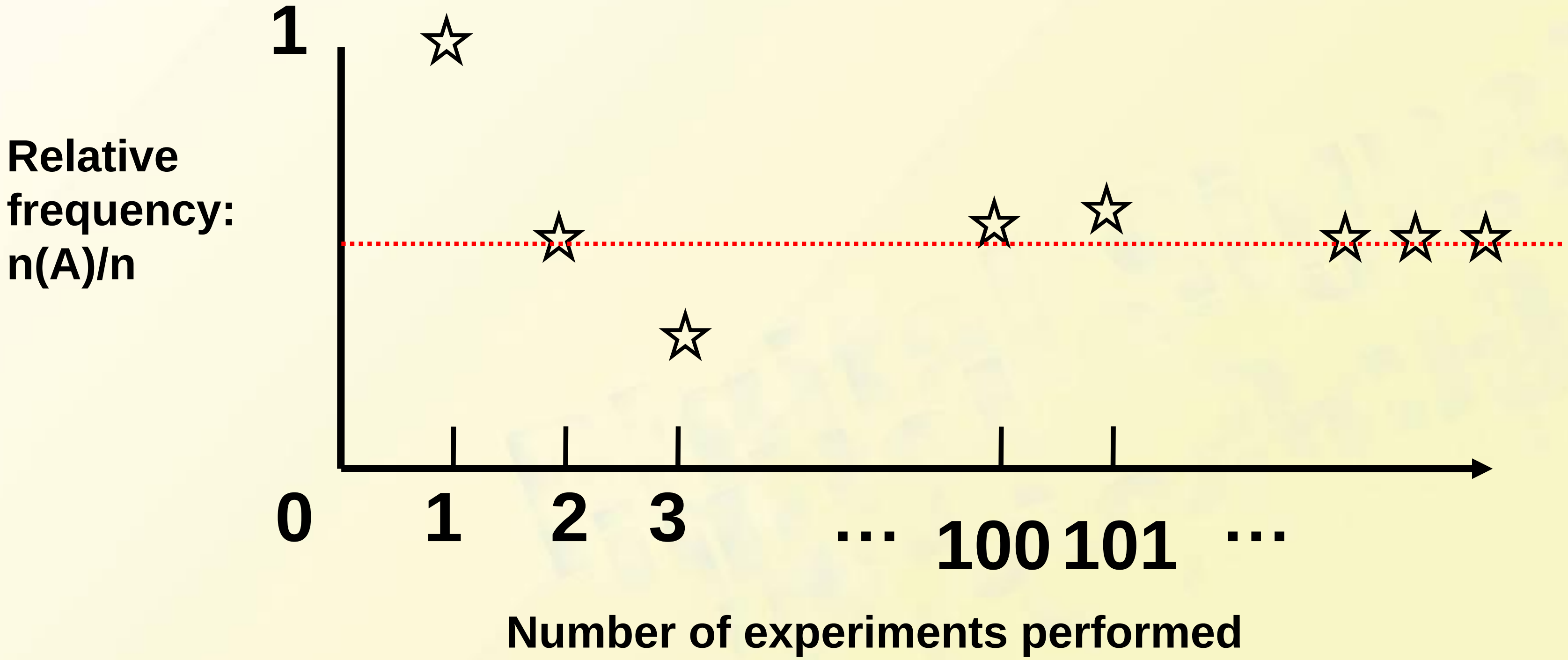
➢ Methods for assigning appropriate/correct probability

1. Based on **repeatedly experiments (objective)**, ***e.g.*** coin-tossing

2. Based on some **reasonable assumption** or **prior information (subjective)**, ***e.g.*** a *fair* die

Note: May be different for different observers.

- Relative frequency vs. Probability



As *n* gets arbitrarily large, *n*(*A*)/*n* approaches a limiting value, the limit value is *P*(*A*).

- **Property #1**

  For any event A, $P(A)=1 - P(A')$

Proof:

By the definition of A', we have

$$\boldsymbol{S} = A \cup A', \; A \cap A' = \phi$$

Since

$$1=P(S) = P(A \cup A') = P(A) + P(A')$$

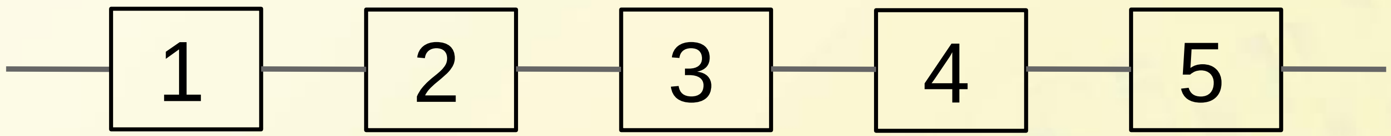
then

$$P(A) = 1- P(A')$$

2.2 Axioms, Interpretations, and Properties of Probability

- **Example 2.13**

Consider a system of five identical components connected in series, as illustrated in the following figure



Denote a component that fails by F and one that doesn’t fail by S.

Suppose that if 90% of all these components do not fail and different components fail independently of one anther.

Let A be the event that the system fails, what is P(A)?

Solution:

Let A be the event that the system fails.

A={FSSSS, SFSSS, …} there are 31 different outcomes in A.

However, A' the event that the system works, consists of the single outcome SSSSS.

$P(A')=P(SSSSS)=0.9^5$

Thus, **$P(A) = 1- P(A') = 1-0.9^5=0.41$**

So, among a large number of such systems, roughly 41% will fail.

- **Property #2:**

If A and B are mutually exclusive, then $P(A \cap B) = 0$

Proof:

Because $A \cap B$ contains no outcomes, $(A \cap B)' = \boldsymbol{S}$.

Thus we have that

$$1 = P[(A \cap B)'] = 1 - P(A \cap B)$$

which implies

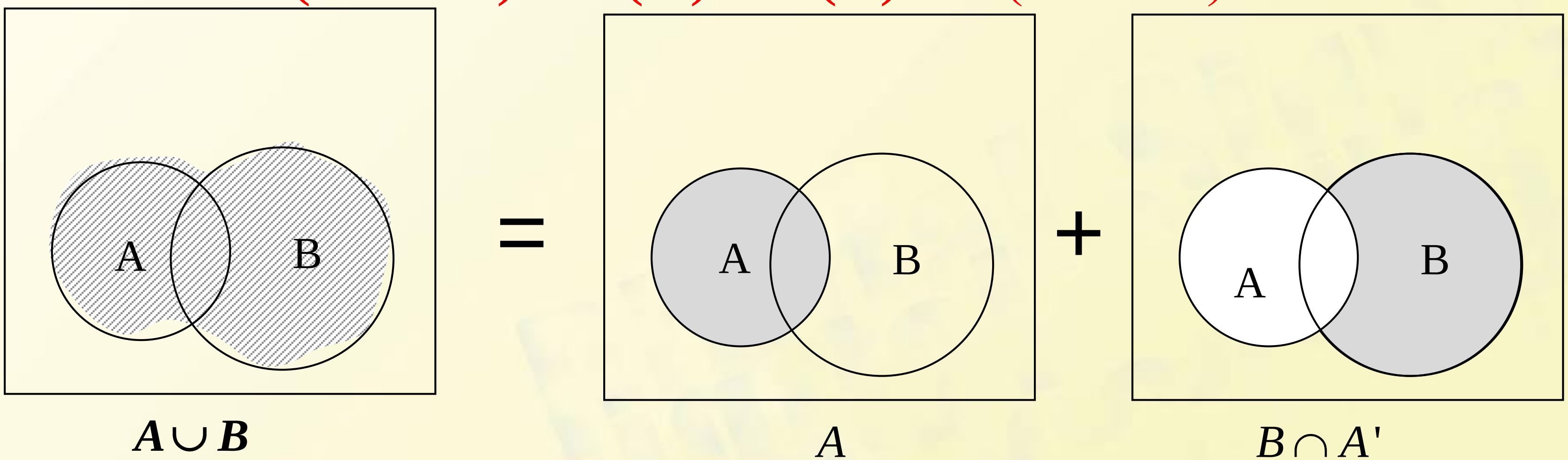
$$P(A \cap B) = 0$$

2.2 Axioms, Interpretations, and Properties of Probability

- **Property #3:**

For any two events A and B (**For events that are not mutually exclusive**),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Proof:

$P(A \cup B) = P(A) + P(B \cap A')$

$= P(A) + [P(B) - P(A \cap B)]$

Note: $B = (B \cap A') \cup (A \cap B)$

& $(B \cap A') \cap (A \cap B) = \phi$

- **Example 2.14** (Page.60)

  A = {subscribes to the metropolitan paper}

  B ={subscribes to the local paper}

  $P(A) = 0.6$, $P(B) = 0.8$, $P(A \cap B) = 0.5$

  P(subscribes to at least one of the two newspapers)

$= P(A \cup B) = P(A) + P(B) – P(A \cap B) = 0.6+0.8-0.5=0.9$

  P(exactly one)

$= P(A \cap B') + P(A' \cap B) =0.1+0.3 =0.4$

2.2 Axioms, Interpretations, and Properties of Probability

- Determining Probabilities Systematically

When the number of possible outcomes (simple events) is large, there will be many compound events. A simple way to determine probabilities for these events is that

- First determine probability $P(E_i)$ for all simple events.

  Note: $P(E_i)>=0$ and $\sum_{\text{all } i} P(E_i)=1$

- The probability of any compound event A is computed by adding together the $P(E_i)$'s for all E_i's in A

$$P(A) = \sum_{\text{all Ei's in A}} P(E_i)$$

Note: $Ei \cap Ej = \phi$, i is not j

- **Example 2.15**

Denote the six elementary events {1}, {2}, …{6} associated with tossing a six-sided die once by $E_1, E_2, \ldots E_6$.

If the die is constructed so that any of the **three even outcomes is twice** as likely to occur as any of the **three odd outcomes** (**unfair die**).

(A) What's the probability of elementary events?

(B) Let A be the event that outcome is even, what's P(A)?

(C) Let B be the event that outcome <=3, what's P(A)?

Solution:

(A)The probabilities of **elementary events is**

$P(E_1)=P(E_3)=P(E_5) = 1/9$ and $P(E_2)=P(E_4)=P(E_6)=2/9$,

(B)The event A={**outcome is even**}$= E_2 \cup E_4 \cup E_6$

$$P(A) = P(E_2)+P(E_4)+P(E_6)=2/3$$

(C) The event B={**outcome <=3**}$= E_1 \cup E_2 \cup E_3$

$$P(B) = P(E_1)+P(E_2)+P(E_3)=4/9$$

- **Equally Likely Outcomes**

In many experiments consisting of ***N*** outcomes, it is reasonable to assign equal probabilities to all ***N*** simple events.

e.g. tossing a fair coin or fair die, selecting cards from a well-shuffled deck of 52.

With p=P(E_i) for every i, then

$$1 = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} p = P \cdot N$$

➔ **p= 1/N**

Consider an event A, with N(A) denoting the number of outcomes containing in A, then

$$P(A) = \sum_{E_i \ in\ A} P(E_i) = \sum_{E_i \ in\ A} \frac{1}{N} = \frac{N(A)}{N}$$

Equally Likely Outcomes

- **Example 2.16**

When two dice are rolled separately, there are N=36 outcomes. If both the dice are fair, all 36 outcomes are equally likely.

- (A) What's the probability of elementary events?

(B) Let A be the event that sum of two number equal to 7, what's P(A)?

Solution:

(A) P(Ei) = 1/36.

(B) The event A={sum of two number =7} consists of the six outcomes (1,6) (2,5) (3,4), (4,3), (5,2) and (6,1),

So

P(A) = N(A)/N = 6/36 = 1/6

2.3 Counting Techniques

Addition principle

Multiplication principle

Permutation

Combination

2.3 Counting Techniques

Addition principle

(I) From land A to land B, you can by train or car. There are three trains and two buses in one day. How many different transport ways moves from land A to land B?

Solution:

Form land A to Land B, there 3 ways by train and 2 ways by bus. Therefore, there are total 3+2=5 different ways.

Addition principle:

Generally, there are the following principle:

Addition principle: to do one thing, and completed it can have n-class way, in a first class way there are m_1 different ways, in the second class way there are m_2 different ways, ... in n class there are m_n different ways. So there are total $N = m_1 + m_2 + \ldots + m_n$ different ways.

Addition principle

Addition Principle (for Counting)

For any two sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \qquad (3)$$

If A and B are disjoint, then

$$n(A \cup B) = n(A) + n(B) \qquad (4)$$

Example 1

Employee Benefits According to a survey of business firms in a certain city, 750 firms offer their employees health insurance, 640 offer dental insurance, and 280 offer health insurance and dental insurance. How many firms offer their employees health insurance or dental insurance?

Solution

If H is the set of firms that offer their employees health insurance and D is the set that offer dental insurance, then

$H \cap D$ = set of firms that offer health insurance **and** dental insurance

$H \cup D$ = set of firms that offer health insurance **or** dental insurance

Thus,

$$n(H) = 750 \qquad n(D) = 640 \qquad n(H \cap D) = 280$$

and

$$\begin{aligned} n(H \cup D) &= n(H) + n(D) - n(H \cap D) \\ &= 750 + 640 - 280 = 1{,}110 \end{aligned}$$

Thus, 1,110 firms offer their employees health insurance or dental insurance. ∎

Multiplication principle

Three are three roads from village A to village B and two roads from village B to C. How many total different way form m village A to village C?

Solution:

From Village A Village B to C Village 3X2 = 6 different moves.

Multiplication principle

Generally, there is the following principle:

Multiplication principle: to do one thing, and completed it needs to be divided into n steps, In the first step there are m_1 different ways. In second step are m_2 different ways. ......In N step there are different m_n methods. Therefore ,there are total $N = m_1 m_{2 \ldots} m_n$ different ways.

2.3 Counting Techniques

- Ordered Pair

By an ordered pair, we mean that, if O_1 and O_2 are ***different*** objects, then the pair (O_1,O_2) is different from the pair (O_2,O_1).

- Counting the number of ordered pair

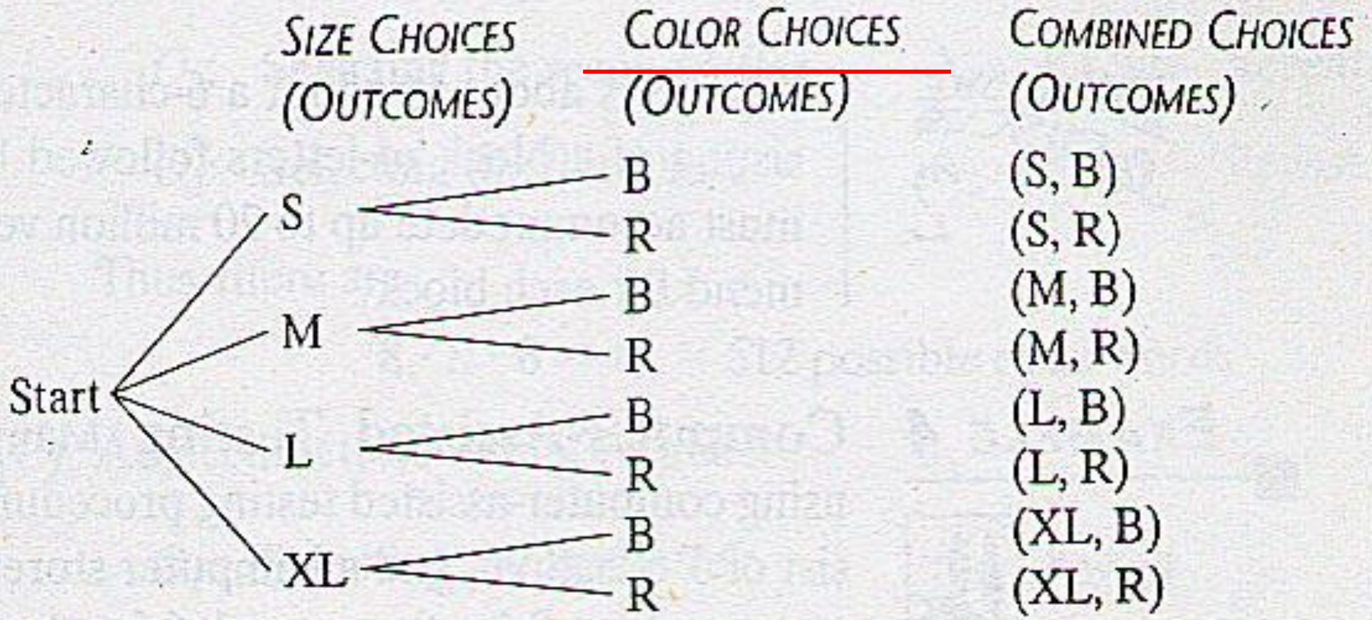
If the first element of object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is n_1n_2

Example 3

Product Mix A retail store stocks windbreaker jackets in small, medium, large, and extra large, and all are available in blue or red. What are the combined choices, and how many combined choices are there?

Solution

To solve the problem we use a tree diagram:



Thus, there are 8 possible combined choices (outcomes). There are 4 ways that a size can be chosen and 2 ways that a color can be chosen. The first element in the ordered pair represents a size choice, and the second element represents a color choice.

Multiplication Principle (for Counting)

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

Example 4

Computer-Assisted Testing Many colleges and universities are now using computer-assisted testing procedures. Suppose a screening test is to consist of 5 questions, and a computer stores 5 comparable questions for the first test question, 8 for the second, 6 for the third, 5 for the fourth, and 10 for the fifth. How many different 5-question tests can the computer select? (Two tests are considered different if they differ in one or more questions.)

Solution

O_1:	Selecting the first question	N_1:	5 ways
O_2:	Selecting the second question	N_2:	8 ways
O_3:	Selecting the third question	N_3:	6 ways
O_4:	Selecting the fourth question	N_4:	5 ways
O_5:	Selecting the fifth question	N_5:	10 ways

Thus, the computer can generate

$$5 \cdot 8 \cdot 6 \cdot 5 \cdot 10 = 12{,}000 \text{ different tests}$$

Permutations

DEFINITION

Permutation of a Set of Objects

A **permutation** of a set of distinct objects is an arrangement of the objects in a specific order without repetition.

DEFINITION

Factorial

For n a natural number,

$$n! = n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1 \quad 4! = 4\cdot 3\cdot$$

$$0! = 1$$

$$n! = n\cdot(n-1)!$$

Note: Many calculators have an $\boxed{n!}$ key or its equivalent.

Number of Permutations of n Objects

The number of permutations of n distinct objects without repetition, denoted by $P_{n,n}$, is

$$P_{n,n} = n(n-1) \cdot \cdots \cdot 2 \cdot 1 = n! \quad n \text{ factors}$$

Example: The number of permutations of 7 objects is

$$P_{7,7} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! \quad 7 \text{ factors}$$

- **DEFINITION**

Permutation of n Objects Taken r at a Time

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.

Number of Permutations of n Objects Taken r at a Time

The number of permutations of n distinct objects taken r at a time without repetition is given by*

$$P_{n,r} = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1) \quad r \text{ factors}$$

$$P_{5,2} = 5\cdot 4 \quad 2 \text{ factors}$$

or

$$P_{n,r} = \frac{n!}{(n-r)!} \qquad 0 \le r \le n \qquad P_{5,2} = \frac{5!}{(5-2)!} = \frac{5!}{3!}$$

Note: $P_{n,n} = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ permutations of n objects taken n at a time.

Remember, by definition, $0! = 1$.

Example 2

Permutations Given the set $\{A, B, C\}$, how many permutations are there of this set of 3 objects taken 2 at a time? Answer the question

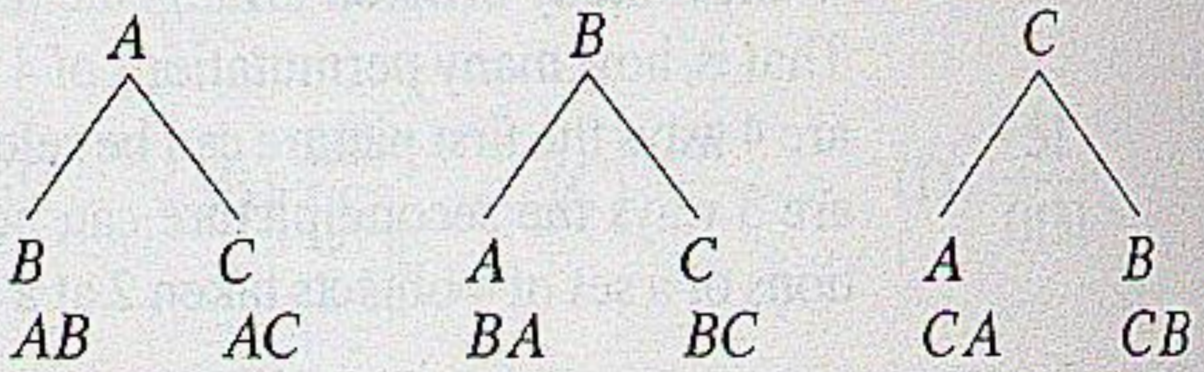
(A) Using a tree diagram

(B) Using the multiplication principle

(C) Using the two formulas for $P_{n,r}$

Solution:

(A) Using a tree diagram:



There are 6 permutations of 3 objects taken 2 at a time.

(B) Using the multiplication principle:

O_1: Fill the first position $\quad N_1$: 3 ways

O_2: Fill the second position $\quad N_2$: 2 ways

Thus, there are

$$3 \cdot 2 = 6 \text{ permutations of 3 objects taken 2 at a time}$$

(C) Using the two formulas for $P_{n,r}$:

$$P_{3,2} = \overset{\text{2 factors}}{\overbrace{3 \cdot 2}} = 6 \qquad \text{or} \qquad P_{3,2} = \frac{3!}{(3-2)!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$$

Thus, there are 6 permutations of 3 objects taken 2 at a time. Of course, all three methods produce the same answer. ■

In example 2 you probably found the multiplication principle the easiest method to use. But for large values of *n* and *r* you will find that the factorial formula is more convenient.

Example 3

Permutations Find the number of permutations of 13 objects taken 8 at time. Compute the answer using a calculator.

Solution:

We use the factorial formula for $P_{n,r}$:

$$P_{13,8} = \frac{13!}{(13-8)!} = \frac{13!}{5!} = 51{,}891{,}840$$

Using a tree diagram to solve this problem would involve a monumental effort. Using the multiplication principle would involve multiplying $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ (8 factors), which is not too bad. A calculator can provide instant results (see Fig. 1). ■

Combination

DEFINITION

Combination of n Objects Taken r at a Time

A **combination** of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.

Number of Combinations of n Objects Taken r at a Time

The number of combinations of n distinct objects taken r at a time withou repetition is given by*

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

$$C_{52,5} = \binom{52}{5} = \frac{P_{52,5}}{5!} = \frac{52!}{5!(52-5)!}$$

Example 4

Permutations and Combinations From a committee of 10 people,

(A) In how many ways can we choose a chairperson, a vice-chairperson, and a secretary, assuming that one person cannot hold more than one position?

(B) In how many ways can we choose a subcommittee of 3 people?

Solution:

Note how parts (A) and (B) differ. In part (A), order of choice makes a difference in the selection of the officers. In part (B), the ordering does not matter in choosing a 3-person subcommittee. Thus, in part (A), we are interested in the number of *permutations* of 10 objects taken 3 at a time; and in part (B), we are interested in the number of *combinations* of 10 objects taken 3 at a time. These quantities are computed as follows (and since the numbers are not large, we do not need to use a calculator):

(A) $P_{10,3} = \dfrac{10!}{(10-3)!} = \dfrac{10!}{7!} \boxed{= \dfrac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}} = 720$ ways

(B) $C_{10,3} = \dfrac{10!}{3!(10-3)!} = \dfrac{10!}{3!7!} \boxed{= \dfrac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3 \cdot 2 \cdot 1 \cdot \cancel{7!}}} = 120$ ways ■

Example 5

Combinations Find the number of combinations of 13 objects taken 8 at a time. Compute the answer exactly, using a calculator.

Solution

$$C_{13,8} = \binom{13}{8} = \frac{13!}{8!(13-8)!} = \frac{13!}{8!5!} = 1{,}287$$

Compare the result in Example 5 with that obtained in Example 3, and note that $C_{13,8}$ is substantially smaller than $P_{13,8}$ (see Fig. 3).

Permutations and combinations are similar in that both are selections in which repetition is *not* allowed. But there is a crucial distinction between the two:

In a permutation, order is vital.

In a combination, order is irrelevant.

Example 8

Counting Techniques A company has 7 senior and 5 junior officers. An ad hoc legislative committee is to be formed. In how many ways can a 4-officer committee be formed so that it is composed of

(A) Any 4 officers?

(B) 4 senior officers?

(C) 3 senior officers and 1 junior officer?

(D) 2 senior and 2 junior officers?

(E) At least 2 senior officers?

Solution

(A) Since there are a total of 12 officers in the company, the number of different 4-member committees is

$$C_{12,4} = \frac{12!}{4!(12-4)!} = \frac{12!}{4!8!} = 495$$

(B) If only senior officers can be on the committee, the number of different committees is

$$C_{7,4} = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!} = 35$$

(C) The 3 senior officers can be selected in $C_{7,3}$ ways, and the 1 junior officer can be selected in $C_{5,1}$ ways. Applying the multiplication principle, the number of ways that 3 senior officers and 1 junior officer can be selected is

$$C_{7,3} \cdot C_{5,1} = \frac{7!}{3!(7-3)!} \cdot \frac{5!}{1!(5-1)!} = \frac{7!5!}{3!4!1!4!} = 175$$

(D) $C_{7,2} \cdot C_{5,2} = \dfrac{7!}{2!(7-2)!} \cdot \dfrac{5!}{2!(5-2)!} = \dfrac{7!5!}{2!5!2!3!} = 210$

(E) The committees with *at least* 2 senior officers can be divided into three disjoint collections:

1. Committees with 4 senior officers and 0 junior officers
2. Committees with 3 senior officers and 1 junior officer
3. Committees with 2 senior officers and 2 junior officers

The number of committees of types 1, 2, and 3 was computed in parts (B), (C), and (D), respectively. The total number of committees of all three types is the sum of these quantities:

$$\underset{\text{Type 1}}{C_{7,4}} + \underset{\text{Type 2}}{C_{7,3} \cdot C_{5,1}} + \underset{\text{Type 3}}{C_{7,2} \cdot C_{5,2}} = 35 + 175 + 210 = 420$$

■