Homework (9, 12, 18, 19)
5.2 (24, 26, 33, 35)

A+

9. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable—X for the right tire and Y for the left tire, with joint pdf

$$f(x, y) = \begin{cases} K(x^2 + y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the value of K?
b. What is the probability that both tires are underfilled?
c. What is the probability that the difference in air pressure between the two tires is at most 2 psi?
d. Determine the (marginal) distribution of air pressure in the right tire alone.
e. Are X and Y independent rv's?

a) since $f(x,y)$ is a joint pdf, it satisfied $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$

Since $x \in [20,30]$, $y \in [20,30]$, that $\int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy = 1$

$$= K\int_{20}^{30}\int_{20}^{30} x^2\,dy\,dx + K\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$$

$$= 10K\int_{20}^{30} x^2\,dx + 10K\int_{20}^{30} y^2\,dy$$

$$= 10K \cdot \tfrac{1}{3}x^3\Big|_{20}^{30} + 10K \cdot \tfrac{1}{3}y^3\Big|_{20}^{30}$$

$$= \frac{20K}{3} \cdot (27000 - 8000) = 1$$

So that $K = \dfrac{3}{380\,000}$ ✓

b) both tires are underfilled.

$$P(X<26 \text{ and } Y<26) = \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\,dx\,dy$$

$$= K\int_{20}^{26}\left(x^2y + \tfrac{1}{3}y^3\right)\Big|_{20}^{26} dx$$

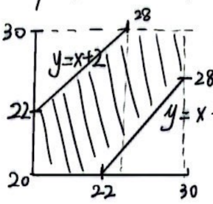
$$= K\int_{20}^{26}(6x^2 + 3192)\,dx$$

$$= K\,(2x^3 + 3192x)\Big|_{20}^{26}$$

$$= 38304 \cdot K = 0.3024$$ ✓

c) that is

$$P(|X-Y| \le 2) = P(X-2 \le Y \le 2+X)$$



that is the find the area of the region.

$$P(|X-Y| \le 2) = 1 - \int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx$$

$$= 1 - \int_{20}^{28}\int_{x+2}^{30} K(x^2+y^2)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} K(x^2+y^2)\,dy\,dx$$

$$= 1 - K\int_{20}^{28}\left(x^2y + \tfrac{1}{3}y^3\right)\Big|_{x+2}^{30} dx - K\int_{22}^{30}\left(x^2y + \tfrac{1}{3}y^3\right)\Big|_{20}^{x-2} dx$$

$$= 0.3593$$ ✓

d) right tire alone

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{20}^{30} K(x^2+y^2)\,dy = 10Kx^2 + 0.05$$ ✓

e) $f_X(x) = 10Kx^2 + 0.05 \quad (x \in [20,30])$

by symmetry, $f_Y(y) = 10Ky^2 + 0.05 \quad (y \in [20,30])$

Since $f(x,y) \ne f_X(x) \cdot f_Y(y)$, X, Y are not independent. ✓

12. Two components of a minicomputer have the following joint pdf for their useful lifetimes X and Y:

$$f(x, y) = \begin{cases} xe^{-x(1+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the probability that the lifetime X of the first component exceeds 3?

b. What are the marginal pdf's of X and Y? Are the two lifetimes independent? Explain.

c. What is the probability that the lifetime of at least one component exceeds 3?

a) $P(X>3) = P(X>3, Y\geq 0) = \int_3^{+\infty}\int_0^{+\infty} xe^{-x(1+y)}dy\,dx = \int_3^{+\infty}\int_0^{+\infty} e^{-x(1+y)}d\,x(1+y)\,dx$

$(xe^x)' = e^x + xe^x$

$\int xe^x dx = \int x\,de^x = xe^x - \int e^x dx$ 分部积分.

$= \int_3^{+\infty} e^{-x}dx = 0.050$

b) $f_X(x) = \int_0^{+\infty} f(x,y)dy = e^{-x} \quad (x \geq 0)$

$f_Y(y) = \int_0^{+\infty} f(x,y)dx = \int_0^{+\infty} xe^{-x(1+y)}dx$

$= \frac{-1}{1+y}\int_0^{+\infty} xe^{-x(1+y)}d(-x)(1+y)$

$= -\frac{1}{(1+y)}\int_0^{+\infty} x\,de^{-x(1+y)}$

$= -\frac{1}{1+y}\left[xe^{-x(1+y)}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-x(1+y)}dx\right]$

$= \frac{1}{(1+y)^2}$

since $f_X(x)\cdot f_Y(y) \neq f(x,y)$. two rvs are not independent.

c) that is

$P(X>3 \text{ or } Y>3) = 1 - P(X\leq 3 \text{ and } Y \leq 3)$

$= 1 - \int_0^3\int_0^3 xe^{-x(1+y)}dy\,dx = 1 - \int_0^3 e^{-x}(1-e^{-3x})dx = e^{-3} + 0.25 - 0.25e^{-12} \approx 0.300$

18. Refer to Exercise 1 and answer the following questions:
 a. Given that $X = 1$, determine the conditional pmf of Y—i.e., $p_{Y|X}(0 \mid 1)$, $p_{Y|X}(1 \mid 1)$, and $p_{Y|X}(2 \mid 1)$.
 b. Given that two hoses are in use at the self-service island, what is the conditional pmf of the number of hoses in use on the full-service island?
 c. Use the result of part (b) to calculate the conditional probability $P(Y \le 1 \mid X = 2)$.
 d. Given that two hoses are in use at the full-service island, what is the conditional pmf of the number in use at the self-service island?

1. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

$p(x, y)$		y = 0	1	2
x	0	.10	.04	.02
	1	.08	.20	.06
	2	.06	.14	.30

 a. What is $P(X = 1 \text{ and } Y = 1)$?
 b. Compute $P(X \le 1 \text{ and } Y \le 1)$.
 c. Give a word description of the event $\{X \ne 0 \text{ and } Y \ne 0\}$, and compute the probability of this event.
 d. Compute the marginal pmf of X and of Y. Using $p_X(x)$, what is $P(X \le 1)$?
 e. Are X and Y independent rv's? Explain.

a) X=1 it is discrete.

$$P_{Y|X}(0|1) = \frac{P(x,y)}{P_X(x)} = \frac{0.08}{0.34} = 0.2353$$

$$P_{Y|X}(1|1) = \frac{0.20}{0.34} = 0.5882$$

$$P_{Y|X}(2|1) = \frac{0.06}{0.34} = 0.1765$$

b) it means that

$P_{Y|X}(x|2)$

$$\begin{cases} P_{Y|X}(0|2) = 0.12 \\ P_{Y|X}(1|2) = 0.28 \\ P_{Y|X}(2|2) = 0.60 \end{cases}$$

c) $P(Y \le 1 \mid X=2) = P_{Y|X}(0|2) + P_{Y|X}(1|2) = 0.12 + 0.28 = 0.40$

d) $P_{X|Y}(x|2) = \dfrac{P(X,Y)}{P_Y(y)}$ $\quad P_{X|Y}(0|2) = 0.0526$ $\quad P_{X|Y}(1|2) = 0.1579$ $\quad P_{X|Y}(2|2) = 0.7895$

19. The joint pdf of pressures for right and left front tires is given in Exercise 9.
 a. Determine the conditional pdf of Y given that $X = x$ and the conditional pdf of X given that $Y = y$.
 b. If the pressure in the right tire is found to be 22 psi, what is the probability that the left tire has a pressure of at least 25 psi? Compare this to $P(Y \ge 25)$.
 c. If the pressure in the right tire is found to be 22 psi, what is the expected pressure in the left tire, and what is the standard deviation of pressure in this tire?

9. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable—X for the right tire and Y for the left tire, with joint pdf

$$f(x, y) = \begin{cases} K(x^2 + y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

2

a) $P_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{k(x^2+y^2)}{10ky^2+0.05}$ $\quad (20 \le x \le 30)$ here $k = \dfrac{3}{380000}$

$P_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{k(x^2+y^2)}{10kx^2+0.05}$ $\quad (20 \le y \le 30)$

b) this means.

$$P(Y \ge 25 \mid X=22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy = \int_{25}^{30} \frac{k((22)^2+y^2)\,dy}{10k(22)^2+0.05} = 0.556$$

$$P(Y \ge 25) = \int_{25}^{30} f_Y(y)\,dy = \int_{25}^{30} (10ky^2 + 0.05)\,dy = 0.549$$

x is fix, we need to know y.

c) $$E(Y \mid X=22) = \int_{-\infty}^{+\infty} y \cdot f_{Y|X}(y|22)\,dy$$

$$= \int_{20}^{30} y \cdot \frac{k((22)^2+y^2)}{10k(22)^2+0.05}\,dy = \int_{20}^{30} \left[\frac{k}{10k(22)^2+0.05}\right] y[(22)^2+y^2]\,dy$$

$$= 25.372912$$

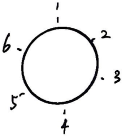
$$E(Y^2 \mid X=22) = \int_{-\infty}^{+\infty} y^2 f_{Y|X}(y|22)\,dy = 652.028640$$

by the formula

$$V(Y \mid X=22) = E(Y^2 \mid X=22) - [E(Y \mid X=22)]^2 = 8.243976$$

$$\sigma = \sqrt{V(Y \mid X=22)} = 2.87$$

24. Six in[illegible]uding A and B, take seats around a cir-cu[illegible]mpletely random fashion. Suppose the seat[illegible]ed 1, . . . , 6. Let X = A's seat number and Y = B's seat number. If A sends a written message around the table to B in the direction in which they are closest, how many individuals (including A and B) would you expect to handle the message?



Let $h(x,y)$ be the number of the individual who handle the message.

$h(x,y)$	x = 1	2	3	4	5	6
y = 1	-	2	3	4	3	2
2	2	-	2	3	4	3
3	3	2	-	2	3	4
4	4	3	2	-	2	3
5	3	4	3	2	-	2
6	2	3	4	3	2	-

$p(x,y) = \frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$ for each pair of (x,y)

$$E[h(x,y)] = \sum_x \sum_y h(x,y) \cdot p(x,y) = \frac{84}{30} = 2.80$$

26. Consider a small ferry that can accommodate cars and buses. The toll for cars is \$3, and the toll for buses is \$10. Let X and Y denote the number of cars and buses, respectively, carried on a single trip. Suppose the joint distribution of X and Y is as given in the table of Exercise 7. Compute the expected revenue from a single trip.

revenue $= 3X + 10Y$

that $h(x,y) = 3X + 10Y$

$p(x, y)$		y = 0	1	2
	0	.025	.015	.010
	1	.050	.030	.020
	2	.125	.075	.050
x	3	.150	.090	.060
	4	.100	.060	.040
	5	.050	.030	.020

$$E[h(x,y)] = \sum_x \sum_y h(x,y) \cdot p(x,y)$$

$$= \sum_{x=0}^{5} \sum_{y=0}^{2} (3x+10y) \cdot p(x,y) = 0 \cdot p(0,0) + 10 \cdot p(0,1) + \dots + 35 \cdot p(5,2)$$

$= 15.40$ so the expected revenue of a single trip is \$15.40

25. A surveyor wishes to lay out a square region with each side having length L. However, because of a measurement error, he lays out a rectangle in which the north–south sides both have length X and the east–west sides both have length Y. Suppose that X and Y are independent and that each is uniformly distributed on the interval $[L - A, L + A]$ (where $0 < A < L$). What is the expected area of the resulting rectangle?

28. Show that if X and Y are independent rv's, then $E(XY) = E(X) \cdot E(Y)$. Then apply this in Exercise 25. [*Hint*: Consider the continuous case with $f(x, y) = f_X(x) \cdot f_Y(y)$.]

33. Use the result of Exercise 28 to show that when X and Y are independent, $\text{Cov}(X, Y) = \text{Corr}(X, Y) = 0$.

the result in 28: if X and Y are independent rv's

then $E(XY) = E(X) \cdot E(Y)$

since $\quad Cov(X,Y) = E(XY) - E(X)E(Y) = E(X) \cdot E(Y) - E(X) \cdot E(Y) = 0.$

$Corr(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$ so that $Corr(X,Y) = 0$

that is, when X and Y are independent, $Cov(X,Y) = Corr(X,Y) = 0$

35. **a.** Use the rules of expected value to show that $\text{Cov}(aX + b, cY + d) = ac\,\text{Cov}(X, Y)$.

b. Use part (a) along with the rules of variance and standard deviation to show that $\text{Corr}(aX + b, cY + d) = \text{Corr}(X, Y)$ when a and c have the same sign.

c. What happens if a and c have opposite signs?

a) since

$$cov(aX+b, cY+d) = E[(aX+b)(cY+d)] - E(aX+b) \cdot E(cY+d)$$

$$= E[acXY + adX + bcY + bd] - [aE(X)+b][cE(Y)+d]$$

$$= acE(XY) + \cancel{adE(X)} + \cancel{bcE(Y)} + \cancel{bd} - (acE(X)E(Y) + \cancel{adE(X)} + \cancel{bcE(Y)} + \cancel{bd}]$$

$$= acE(XY) - acE(X)E(Y) = ac[E(XY) - E(X)E(Y)]$$

$$= ac\,Cov(X,Y)$$

b) the rules of variance and standard deviation.

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Corr(aX+b, cY+d) = \frac{Cov(aX+b, cY+d)}{\sigma(aX+b)\,\sigma(cY+d)} = \frac{ac\,Cov(X,Y)}{|a||c|\sigma_X \sigma_Y}$$

$$= \frac{ac}{|ac|} Corr(X,Y)$$

since a and c have the same sign, $|ac| = ac$. that $Corr(aX+b, cY+d) = Corr(X,Y)$

c) if a and c have opposite signs

$|ac| = -ac$, that $Corr(aX+b, cY+d) = -Corr(X,Y)$