Problems and examples (Physics 2023-24 CST)

1. **(Scaling).** The reasoning of a single cell must be in microscopic size to provide a rigid structure that would seem to contradict the fact that human nerve cells in the spinal cord can be as much as a meter long, although their widths are still very small. Why is this possible?

   Solution. We may let r and L be the radius and length of a tube for the calculation of the area-volume ratio:
   $$\frac{\text{Area}}{\text{Volume}} = \frac{2\pi r L}{\pi r^2 L} = \frac{2}{r}.$$
   The ratio is a constant since r is supposed to be a constant. Therefore, no matter how long a nerve cell is, the area-volume ratio is always the same. It implies that the length of a nerve cell does not affect the strength of the cell body.

2. **(Dimensional analysis).** Please use dimensional analysis to find the expression for centripetal force F. Suppose that the force F of a particle with mass m moving with uniform speed v in a circle of radius r is proportional to some powers of m, say m^a; r as r^b, and the v as v^c.

   Solution. We expression is simply
   $$F \propto m^a r^b v^c$$
   Using the method of dimensional analysis, we get
   $$\begin{aligned} MLT^{-2} &= M^a L^b (LT^{-1})^c \\ &= M^a L^{b+c} T^{-c}. \end{aligned}$$
   Thus, $a = 1$, $b = -1$, $c = 2$. The expression is
   $$F \propto \frac{mv^2}{r}.$$

3. **(Vector).** Find the angle between the vectors $\mathbf{F}_1 = 10\hat{\mathbf{i}} - 20.4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{F}_2 = -15\hat{\mathbf{i}} - 6.2\hat{\mathbf{k}}$.

   Solution. Rearranging the terms of the dot product formula we get
   $$\cos\theta = \frac{\mathbf{F}_1 \cdot \mathbf{F}_2}{|\mathbf{F}_1||\mathbf{F}_2|} = \frac{10(-15) + (-20.4)(0) + 2(-6.2)}{\sqrt{10^2 + 20.4^2 + 2^2}\sqrt{15^2 + 6.2^2}} = -0.439$$
   Thus $\theta = 116°$.

4. **(Vector).** What is the magnitude of the vector $\mathbf{A} = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ in the direction of the vector $\mathbf{B} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$?

   Solution. That is, to find the dot product of $\mathbf{A} \cdot \hat{\mathbf{B}}$ where $\hat{\mathbf{B}}$ is the unit vector of $\mathbf{B}$. It is
   $$(2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \cdot \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{1^2 + (-2)^2 + 3^2}} = -\frac{8}{\sqrt{14}}.$$

5. **(Velocity).** A person is parachute jumping. During the time between when she leaps out of the plane and when she opens her chute, her altitude is given by the equation
   $$y = 10000 - 50\left(t + 5e^{-t/5}\right)$$
   metre. Find her velocity at $t = 7.0$ s.

Solution. Differentiating y with respect to t gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 - 50\frac{\mathrm{d}}{\mathrm{d}t}\left(t + 5\mathrm{e}^{-t/5}\right)\\ &= -50(1 - \mathrm{e}^{-t/5})\end{aligned}$$

When $t = 7$, her velocity is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=7} = -50(1 - \mathrm{e}^{-7/5}) = -37.67\mathrm{ms}^{-1}$$

The negative sign indicates her moving downward.

6. **(Acceleration).** A child throws a marble into the air with an initial speed v_t. Another child drops a ball at the same instant. Compare the accelerations of the two objects while they are in flight.

   Solution. We only need to consider the vertical components of the both cases since the horizontal acceleration of them are zero.

   For the marble, after it is released from the hand there is no other external force (acceleration) to act on it except gravity, no matter how the boy throws the marble to any direction. The second object—the ball obviously falls by gravity. Therefore, the accelerations of the two objects in air are the same—gravity.

7. **(Acceleration).** Alice has a position as a function of time given by $x = a/(b + t^2)$ where a, b are positive constants. Find her maximum speed.

   Solution. Differentiate the function once and twice with respect to time for the velocity and acceleration:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{2at}{(b + t^2)^2}\\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= a\frac{(b + t^2)^2(-2) - (-2t)[2(b + t^2)2t]}{(b + t^2)^4}\end{aligned}$$

   Put $\mathrm{d}^2x/\mathrm{d}t^2 = 0$ for the maximum velocity, we obtain

$$b + t^2 = 0 \quad \text{or} \quad b - 3t^2 = 0$$

   (Actually we need to differentiate the acceleration again evaluated at that time to verify if it is concave upward or download for the minimum or maximum.) Rejecting the first solution, since b is supposed to be positive the first equation will give imaginary time (unphysical). Thus, at

$$t = \sqrt{\frac{b}{3}}$$

   gives her maximum velocity:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-2a\sqrt{b/3}}{(b + b/3)^2} = -a\left(\frac{4b}{3}\right)^{-3/2}.$$

8. **(Average velocity).** Show that average velocity $\langle v\rangle = \dfrac{u + v}{2}$ when the acceleration is constant.

Solution. Setting the initial time $t = 0$ as usual, the average velocity is equal to

$$\begin{aligned}\langle v \rangle &= \frac{ut + at^2/2 - u(0) + a(0)^2/2}{t} \\ &= u + \frac{1}{2}at \\ &= \frac{u + u + at}{2} \\ &= \frac{u + v}{2}\end{aligned}$$

It should be stressed that this equality is only valid in **constant acceleration.**

9. **(Projectile).** A stone is thrown from the top of a building upward at an angle of 30° to the horizontal and with an initial speed of 20 m/s, as shown in Figure 1. If the height of the building is 45 m,

   (a) how long is it before the stone hits the ground?

   (b) What is the speed of the stone just before it strikes the ground?

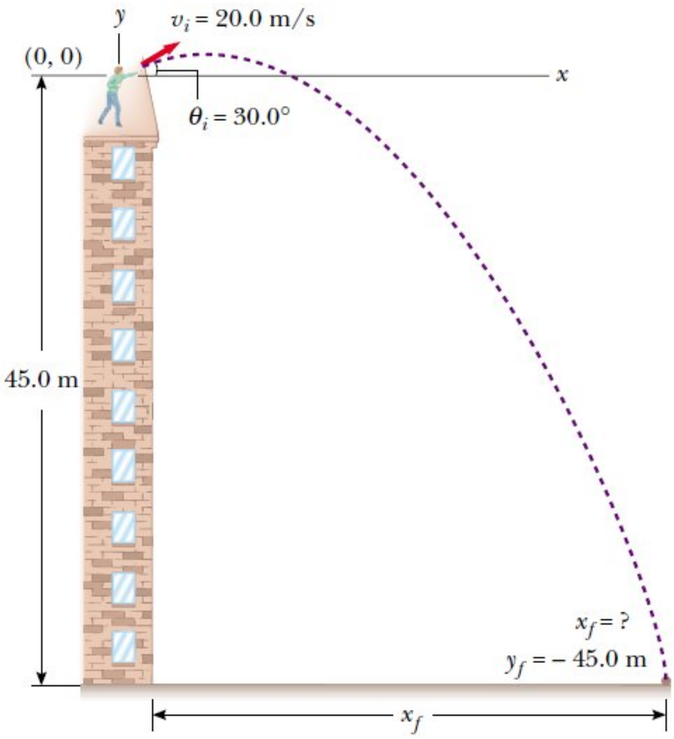


Figure 1:

Solution. (a) The initial x and y components of the stone's velocity are

$$\begin{aligned}v_{xi} &= v_i \cos\theta_i = 20 \times \cos 30 = 17.3 \text{ m/s}, \\ v_{yi} &= v_i \sin\theta_i = 20 \times \sin 30 = 10 \text{ m/s}.\end{aligned}$$

To find the time taken t, we can directly use $y_f = v_{yi}t + \frac{1}{2}a_y t^2$ with $y_f = -45$ m, $a_y = -g$, and $v_{yi} = 10$ m/s:

$$-45 = 10t - \frac{1}{2} \times 9.8 \times t^2$$

Solving the quadratic equation for t gives, for the positive root, $t = 4.22$ second.

(b) We can use $v_{yf} = v_{yi} + a_y t$, with $t = 4.22$ s to obtain the y component of the velocity just before the stone strikes the ground:

$$v_{yf} = 10 - 9.8 \times 4.22 = -31.4 \text{ m/s}.$$

Since $v_{xf} = v_{xi} = 17.3$ m/s, the required speed is

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3)^2 + (-31.4)^2} = 35.9 \text{ m/s}.$$

10. **(Projectile).** A long-jumper leaves the ground at an angle of 20° above the horizontal and at a speed of 11 ms^{-1}. (a) How far does he jump in the horizontal direction? (b) What is the maximum height reached?

Solution. (a) As discussed in the lecture we write down the horizontal $R = 11 \cos 20\, t$ and the vertical $y = 11 \sin 20\, t + (-9.8)t^2/2$ component equations. We obtain the range

$$R = \frac{11^2 \sin(2 \times 20)}{9.8} = 7.94 \text{ m}.$$

The time interval for the maximum height is just half of time of the whole journey by symmetry. We may directly solve for t from the above equations again, or by differentiating the vertical equation with respect to time then setting it to zero:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(11 \sin 20\, t + \frac{1}{2}(-9.8)t^2\right) = 0 \quad \text{gives} \quad t = \frac{11 \sin 20}{9.8} = 0.384 \text{ s}.$$

The maximum height is

$$11 \sin 20(0.384) + \frac{1}{2}(-9.8)(0.384)^2 = 0.722 \text{ m}.$$

11. **(Terminal velocity).** Calculate the radius of a parachute that will slow a 70-kg parachutist to a terminal velocity of 14 m/sec.

Solution. Assuming the parachute has no mass. Let R be the radius of the parachute. The terminal velocity is given by

$$v_t = \sqrt{\frac{W}{CA}} = \sqrt{\frac{W}{CR^2\pi}}$$

where W, C, A are the weight of the parachutist, coefficient of air resistance, and the normal surface area against the falling direction. Rearranging the terms of the equation we have

$$R = \sqrt{\frac{W}{C\pi v_t^2}}.$$

Substituting the data into the equation we obtain

$$R = \sqrt{\frac{70 \times 10}{0.88\pi \times 14^2}} \approx 1.1 \text{ metre}.$$

12. **(Force).** A helicopter of mass m is taking off vertically. The only forces acting on it are the earth's gravitational force and the force, F, of the air pushing up on the propeller blades.

(a) If the helicopter lifts off at $t = 0$, what is its vertical speed at time t?

(b) Plug numbers into your equation from part (a), using $m = 2300$ kg, $F = 27000$ N, and $t = 4.0$ s.

Solution. (a) The net force of the system is

$$F - W = ma \quad \Rightarrow \quad a = \frac{F - W}{m}$$

where W, a are the weight and the acceleration of the helicopter. Setting $u = 0$ we have

$$v = u + at \quad \Rightarrow \quad v = \frac{(F - W)t}{m}.$$

(b)

$$v = \frac{(27000 - 2300 \times 9.81) \times 4}{2300} = 7.7 \text{ ms}^{-1}.$$

13. **(Friction).** A block of mass m_1 on a rough, horizontal surface is connected to a ball of mass m_2 by a lightweight cord over a lightweight, frictionless pulley, as shown in Figure 2. A force of magnitude F at an angle θ with the horizontal is applied to the block as shown. The coefficient of kinetic friction between the block and surface is μ_k. Determine the magnitude of the acceleration of the two objects.

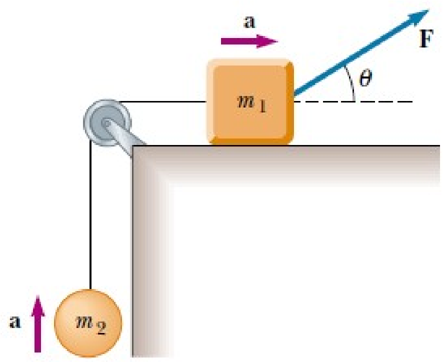


Figure 2:

Solution. The applied force **F** has x and y components $F\cos\theta$ and $F\sin\theta$, respectively. Applying Newton's second law to both objects and assuming the motion of the block is to the right, we obtain the equations of motion of the block:

$$\sum F_x = F\cos\theta - f_k - T = m_1 a_x = m_1 a, \tag{1}$$

$$\sum F_y = n + F\sin\theta - m_1 g = m_1 a_y = 0 \tag{2}$$

where f_k and n are the kinetic friction and the normal reaction forces. The equations of motion of the ball:

$$\sum F_x = m_2 a_x = 0,$$

$$\sum F_y = T - m_2 g = m_2 a_y = m_2 a. \tag{3}$$

Because the two objects are connected, we can equate the magnitudes of the x component of the acceleration of the block and the y component of the acceleration of the ball. Since $f_k = \mu_k n$, and $n = m_1 g - F\sin\theta$, we obtain

$$f_k = \mu_k(m_1 g - F\sin\theta). \tag{4}$$

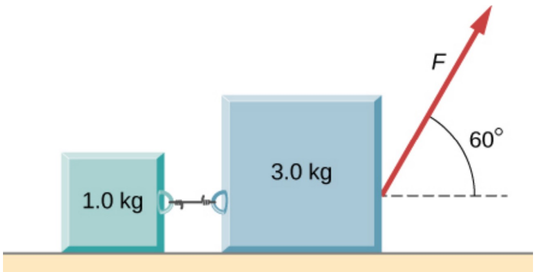
That is, the frictional force is reduced because of the positive y component of $\mathbf{F}$. Substituting (4) and the value of T from (3) into (1) gives

$$F\cos\theta - \mu_k(m_1 g - F\sin\theta) - m_2(a+g) = m_1 a.$$

Solving for a, we obtain

$$a = \frac{F(\cos\theta + \mu_k\sin\theta) - g(m_2 + \mu_k m_1)}{m_1 + m_2}.$$

14. **(Friction).** Two blocks connected by a string are pulled across a horizontal surface by a force applied to one of the blocks, as shown below. The coefficient of kinetic friction between the blocks and the surface is 0.25. If each blcok has an acceleration of 2.0 ms^{-2} to the right, what is the magnitude F of the applied force?



Solution. Let T be the tension along the string between the blocks, hence we have two equations for the blocks:

$$\begin{aligned} T - \mu 1 g &= 1a, \\ F\cos 60 - T - \mu(3g - F\sin 60) &= 3a \end{aligned}$$

Adding the equations together to eliminate T, we got the equation of motion of the system

$$F\cos 60 - \mu(3g - F\sin 60) - \mu g = (1+3)a$$

Inserting the given values F, μ, g into the equation, the force is $F = 24.8$ N.

15. **(Centripetal force).** A pilot of mass m in a jet aircraft executes a loop-thee-loop, as shown in Figure 3. In this manoeuvre, the aircraft moves in a vertical circle of radius 2.7 km at a constant speed of 225 m/s. Determine the force exerted by the seat on the pilot (a) at the bottom of the loop and (b) at the top of the loop. Express your answers in terms of the weight of the pilot mg.

Solution. (a) The net force is the product of the mass and the centripetal acceleration:

$$n_{\text{bot}} - mg = m\frac{v^2}{r}$$

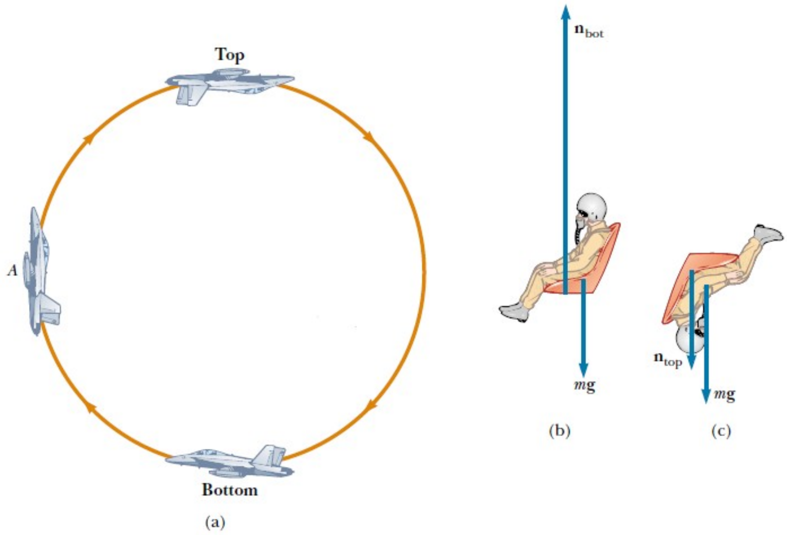


Figure 3:

Substituting the values given for the speed and radius gives

$$n_{\text{bot}} = mg\left(1 + \frac{v^2}{rg}\right) = mg\left(1 + \frac{225}{2.7 \times 10^3 \times 9.8}\right) = 2.91\, mg.$$

Hence, the magnitude of the force $\mathbf{n}_{\text{bot}}$ exerted by the seat on the pilot is greater than the weight of the pilot by a factor of 2.91, i.e., he experiences an apparent weight that is greater than his true weight by a factor of 2.91. (b) On the other hand, the net force equation for the top case is

$$n_{\text{top}} + mg = m\frac{v^2}{r}.$$

Thus,

$$n_{\text{top}} = mg\left(\frac{v^2}{rg} - 1\right) = mg\left(\frac{225}{2.7 \times 10^3 \times 9.8} - 1\right) = 0.913\, mg.$$

The magnitude of the force exerted by the seat on the pilot is less than his true weight by a factor of 0.913, and the pilot feels lighter.

16. **(Centripetal force).** A ball of mass 0.50 kg is attached to the end of a cord 1.50 m long. The ball is whirled in a horizontal circle . If the cord can withstand maximum tension of 50.0 N, what is the maximum speed the ball can attain before the cord breaks? Assume that the string remains horizontal during the motion.

    Solution. The equation for uniform horizontal circular motion is

$$\frac{mv^2}{r} = T.$$

    Now putting $T = 50$ N for the maximum speed, we get

$$\frac{0.5v_{\text{max}}^2}{1.5} = 50 \quad \Rightarrow \quad v_{\text{max}} = 12.25 \text{ ms}^{-1}$$

17. **(Centripetal force).**

(a) Consider a satellite of mass m moving in a circular orbit round the Earth at a constant speed v and at an altitude h above the Earth's surface, as illustrated in Figure 4. Determine the speed of the satellite in terms of G, h, R_E (the radius of the Earth), and M_E (the mass of the Earth).

(b) The satellite is in a circular orbit around the Earth at an altitude of 1000 km. The radius of the Earth is equal to 6.37×10^6 m, and its mass is 5.98×10^{24} kg. Find the speed of the satellite, and then find the period. ($G = 6.673 \times 10^{-11}$ Nm^2/kg^2)

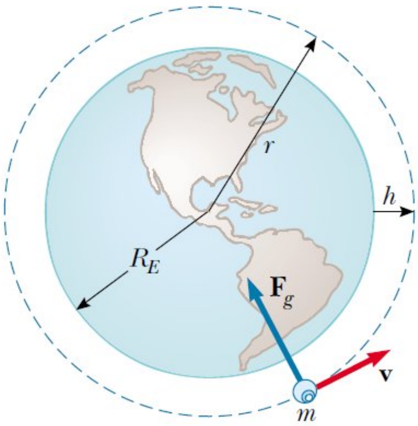


Figure 4:

Solution. (a) The only external force acting on the satellite is gravity, which acts toward the centre of the Earth and keeps the satellite in its circular orbit. Therefore,

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r},$$

gravity is the source of the centripetal force, otherwise, the satellite would move to a straight line and leave away the Earth. Solving for v and because $r = R_E + h$, we obtain

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{GM_E}{R_E + h}}.$$

(b) Directly substituting the data into above equation, it gives

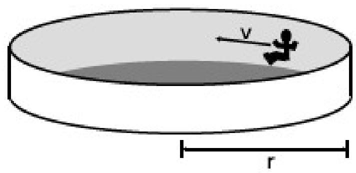
$$v = \sqrt{\frac{6.673 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^6 + 10^6}} = 7.36 \times 10^3 \text{ m/s}.$$

The time taken for one complete revolution (period) is given by

$$\begin{aligned} T = \frac{2\pi(R_E + h)}{v} &= 2\pi\sqrt{\frac{(R_E + h)^3}{GM_E}} \\ &= 2\pi\sqrt{\frac{(6.37 \times 10^6 + 10^6)^3}{6.673 \times 10^{-11} \times 5.98 \times 10^{24}}} \\ &= 6.29 \times 10^3 \text{ sec}. \end{aligned}$$

18. **(Centripetal force).** Consider the carnival ride in which the riders stand against the wall inside a large cylinder. As the cylinder rotates, the floor of the cylinder drops

and the passengers are pressed against the wall by the centrifugal force. Assuming that the coefficient of friction between a rider and the cylinder wall is 0.6 and that the radius of the cylinder is 5 m, what is the minimum angular velocity of the cylinder that will hold the rider firmly against the wall?



Solution. Let N be the normal reaction force perpendicular to the wall toward the centre of the cylinder. The vertical and horizontal equations for equilibrium (the rider does not fall) are:

$$mr\dot{\theta}^2 = N, \quad mg = \mu N$$

where $\dot{\theta}$ is the angular velocity, μ the coefficient of static friction between the rider and the wall. Solving for $\dot{\theta}$ we have

$$\dot{\theta} = \sqrt{\frac{g}{r\mu}}.$$

Putting $g = 9.8$, $r = 5$ and $\mu = 0.6$ into the equation we obtain

$$\dot{\theta} = \sqrt{\frac{9.8}{5 \times 0.6}} = 1.8 \text{ rad/sec.}$$

19. **(Energy).** Compare the work required to accelerate a car of mass 2000 kg from 30.0 to 40.0 km/h with that required for an acceleration from 50.0 to 60.0 km/h.

Solution. Although in both cases the velocity differences are the same, the kinetic energies required to bring them to the desired velocities are different owing to the nonlinear proportionality of increment. The explicit calculation is

$$\frac{1}{2}2000(40^2 - 30^2)\left(\frac{10^3}{60^2}\right)^2 = 54012 \approx 54 \text{ kJ},$$

$$\frac{1}{2}2000(60^2 - 50^2)\left(\frac{10^3}{60^2}\right)^2 = 84876 \approx 85 \text{ kJ}$$

20. **(Gravitational potential).** A particle of mass m is displaced through a small vertical distance h near the earth's surface. Show that in this situation the general expression for the change in gravitational potential energy reduces to the familiar relationship $U = mgh$.

Solution. Since

$$U = -GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GMm\left(\frac{r_f - r_i}{r_i r_f}\right)$$

where r_i, r_f are the initial and final altitudes from the centre of the earth. Because both the initial and final positions of the particle are close to the earth's surface, then

$$r_f - r_i = h \quad \text{and} \quad r_i r_f \approx R_E^2$$

where R_E is the radius of the earth. Since $g = GM/R_E^2$, we obtain

$$U \approx \frac{GMmh}{R_E^2} = mgh.$$

21. **(Energy).** Engineers desire to model the magnitude of the elastic force of a bungee cord using the equation

$$F(x) = a\left[\frac{x+9}{9} - \left(\frac{9}{x+9}\right)^2\right],$$

where x is the stretch of the cord along its length and a is a constant. If it takes 22.0 kJ of work to stretch the cord by 16.7 m, determine the value of the constant a.

Solution. Integrating the given force function over the stretched length gives the energy

$$\begin{aligned} 22 \times 10^3 &= \int_0^{16.7} a\left[\frac{x+9}{9} - \left(\frac{9}{x+9}\right)^2\right] dx \\ &= a\left[\frac{x^2}{18} + x + \frac{81}{x+9}\right]_0^{16.7} \\ &= a\left(\frac{16.7^2}{18} + 16.7 + \frac{81}{25.7} - 9\right) \end{aligned}$$

which gives $a = 835$ N.

22. **(Momentum).** A very massive object with velocity v collides head-on with an object at rest whose mass is very small. No kinetic energy is converted into other forms. Prove that the low-mass object recoils with velocity $2v$. (Hint: Use the centre of mass frame of reference.)

Solution. In one-dimensional elastic collision, we have the expressions of conservation of momentum and conservation of kinetic energy:

$$\begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ \frac{m_1u_1^2}{2} + \frac{m_2u_2^2}{2} &= \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} \end{aligned}$$

where m_1, m_2 are the masses, and u_1, u_2 are the initial velocities before the collision, v_1, v_2 are the final velocities after the collision. Solving these simultaneous equations for the v_1, v_2 we get

$$\begin{aligned} v_1 &= \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2}, \\ v_2 &= \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2}. \end{aligned}$$

In our case we set $m_1 \gg m_2$, and $u_1 = v$, $u_2 = 0$. Using the second formula for the v_2,

$$v_2 = \frac{2m_1v}{m_1 + m_2} \simeq 2v.$$

23. **(Impulse).** A car traveling at 27 m/s collides with a building. The collision with the building causes the car to come to a stop in approximately 1 second. The driver, who weighs 860 N, is protected by a combination of a variable-tension seat-belt and an airbag. The airbag and seat-belt slow his velocity, such that he comes to a stop in approximately 2.5 s.

(a) What average force does the driver experience during the collision?

(b) Without the seat-belt and airbag, his collision time would have been approximately 0.20 s. What force would he experience in this case?

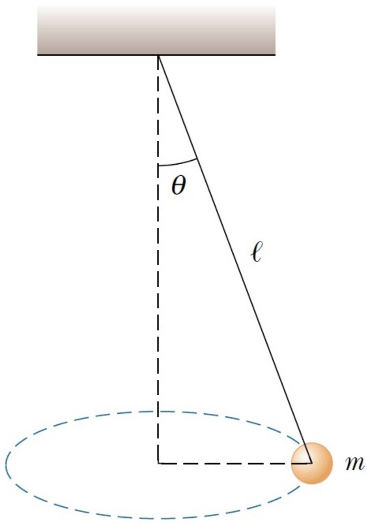
Solution. (a) The average force or the impulsive force is

$$F_{\text{av}} = \frac{860}{9.8} \times (0 - 27) \times \frac{1}{2.5} = -948 \text{ N}.$$

(b) Without the airbag and the seatbelt his collision time becomes $1 + 0.2 = 1.2$ s. The impulsive force is

$$F_{\text{av}} = \frac{860}{9.8} \times (0 - 27) \times \frac{1}{1.2} = -1975 \text{ N}$$

24. **(Centripetal force).** A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane. During the motion, the supporting wire of length l maintains the constant angle θ with the vertical. Show that the magnitude of the



angualr momentum of the mass about the centre of the circle is

$$L = \sqrt{\frac{m^2 g l^3 \sin^4\theta}{\cos\theta}}.$$

Solution. Let T be the tension along the cord, the equation for the plane circular motion is

$$T\sin\theta = \frac{mv^2}{r}.$$

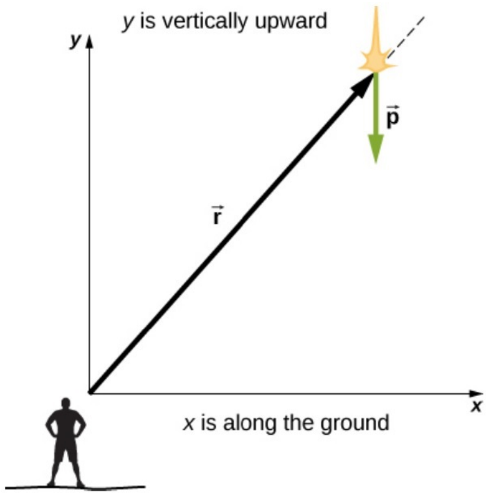
Since the angular momentum $L = mvr$ and $r = l\sin\theta$, above equation may simply be rewritten as

$$T = \frac{L^2}{ml^3\sin^4\theta}.$$

Using the vertical equation $T\cos\theta = mg$, eliminating T for L we get

$$\frac{mg}{\cos\theta} = \frac{L^2}{ml^3\sin^4\theta} \quad\Rightarrow\quad L = \sqrt{\frac{m^2 g l^3 \sin^4\theta}{\cos\theta}}$$

25. **(Angular momentum).** A meteor enters Earth's atmosphere and is observed by someone on the ground by the position vector $\mathbf{r} = 25\hat{\mathbf{i}} + 28\hat{\mathbf{j}}$ km before it burns up in the atmosphere. At the instant the observer sees the meteror, it has linear momentum $\mathbf{p} = -30\hat{\mathbf{j}}$ kg km s^{-1}, with a mass 16 kg and a constant acceleration $-2\hat{\mathbf{j}}$ ms^{-2} along its path. (a) What is the angular momentum of the meteor observed by the person? (c) What is the torque on the meteor about the origin (observer)?



Solution. (a) The angular momentum about the observer is the cross product of position vector by the linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (25\hat{\mathbf{i}} + 28\hat{\mathbf{j}})(10^3) \times (-30\hat{\mathbf{j}})(10^3) = -7.5 \times 10^8\hat{\mathbf{k}} \text{ kg ms}^{-1}$$

(b) The torque about the observer is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (25\hat{\mathbf{i}} + 28\hat{\mathbf{j}})(10^3) \times 16(-2\hat{\mathbf{j}}) = 8 \times 10^5\hat{\mathbf{k}} \text{ Nm}$$

Note. This problem shows that a linear motion is also viewed as a rotation except the observer in the direction (or negative direction) of motion.

26. **(Rotation).** A wheel 1 m in diameter rotates with an angular acceleration of 4 rad s^{-2}. (a) If the wheel's initial angular velocity is 2.0 rad s^{-1}, what is its angular velocity after 10 s? (b) Through what angle does it rotate in the 10 s interval? (c) What are the tangential speed and acceleration of a point on the rim of the wheel at the end of the 10 s interval?

Solution. (a) Using the formula $\dot{\theta}_f = \dot{\theta}_i + \ddot{\theta}t$, we get

$$\dot{\theta}_f = 2 + 4(10) = 42 \text{ rad s}^{-1}.$$

(b) The rotated angle is

$$\Delta\theta = \dot{\theta}_i t + \frac{1}{2}\ddot{\theta}t^2 = 2 \times 10 + \frac{1}{2} \times 4 \times 10^2 = 220 \text{ rad}.$$

(c) The tangential speed is

$$r\dot{\theta}_f = 0.5 \times 42 = 21 \text{ ms}^{-1},$$

and the tangential acceleration is

$$r\ddot{\theta} = 0.5 \times 4 = 2 \text{ ms}^{-2}.$$

27. **(Rigid body motion).** A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is 0.5 kgm^2, what is the rotational kinetic energy of the forearm?

    Solution. Suppose that the ball being light enough does not affect the moment of inertia of the forearm. Since the linear velocity of the ball is the tangential velocity of the rotating forearm about the elbow, the rotational kinetic energy of the forearm is

$$\frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.5 \times \left(\frac{20}{0.48}\right)^2 = 434 \text{ J}$$

28. **(Angular momentum).** A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 5. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

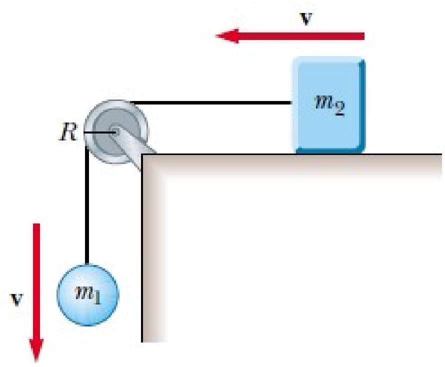


Figure 5:

Solution. At the instant the sphere and block have a common speed v, the angular momentum of the sphere is m_1vR, and that of the block is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Hence, the total angular momentum of the system is

$$L = m_1vR + m_2vR + \frac{Iv}{R}. \tag{5}$$

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the force of gravity m_2g, and so these forces do not contribute to the torque. The force of gravity m_1g acting on the sphere produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about the pulley axle; that is, $\sum\tau = m_1gR$.

Using this result with (5), we find

$$\begin{aligned}\sum \tau &= \frac{\mathrm{d}L}{\mathrm{d}t} \\ m_1 g R &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(m_1+m_2)Rv + \frac{Iv}{R}\right] \\ &= (m_1+m_2)R\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{I}{R}\frac{\mathrm{d}v}{\mathrm{d}t}.\end{aligned}$$

Because $\mathrm{d}v/\mathrm{d}t = a$, thus

$$a = \frac{m_1 g}{(m_1+m_2) + I/R^2}.$$

29. **(Angular momentum).** A disk has radius r and mass M at rest mounted with its axis vertical. A bullet of mass m and velocity v which is fired horizontally and tangential to the disk lodges in the perimeter of the disk. What is the angular velocity of the disk?

    Solution. The problem is similar to some linear inelastic collision problems, here we shall use angular momentum conservation instead. The angular momentum of the bullet relative to the axis of the disk is mvr. After the collision we have

    $$mvr = mr^2\dot{\theta} + I\dot{\theta}.$$

    Putting $I = Mr^2/2$ for a disk we get

    $$\dot{\theta} = \left(\frac{2m}{2m+M}\right)\frac{v}{r}.$$

30. **(Moment of inertia).** A circular hoop of mass m and radius r spins like a wheel while its centre remains at rest. Its period is T. Find the moment of inertia of the hoop. Show that its kinetic energy equals $2\pi^2 mr^2/T^2$.

    Solution. Let ρ be the mass per unit length of the hoop. The moment of inertia of it is

    $$I = \int_0^{2\pi} r^2 \cdot \rho r \mathrm{d}\theta = 2\pi\rho r^3 = mr^2.$$

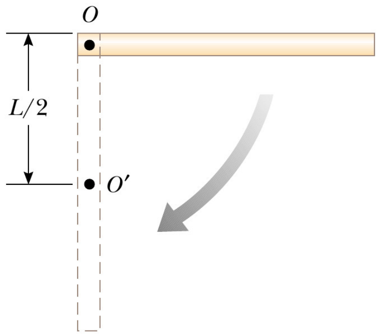
    The rotational kinetic energy is thus

    $$\mathrm{KE}_R = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}mr^2\left(\frac{2\pi}{T}\right)^2 = \frac{2\pi^2 mr^2}{T^2}.$$

31. **(Rigid body motion).** A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end as shown. The rod is released from rest in the horizontal position. (a) What is its angular speed when it reaches its lowest position? (b) Determine the linear speed of the centre of mass v_{cm} and the linear speed of the lowest point on the rod when it is in the vertical position.

    Solution. (a) At the moment when the rod rotates to the shown position the kinetic energy due to rotation is equal to the potential energy of the rod before it released;

    $$\frac{1}{2}MgL = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$



where we have used the fact $I = ML^2/3$. Therefore the angular speed is

$$\omega = \sqrt{\frac{3g}{L}}.$$

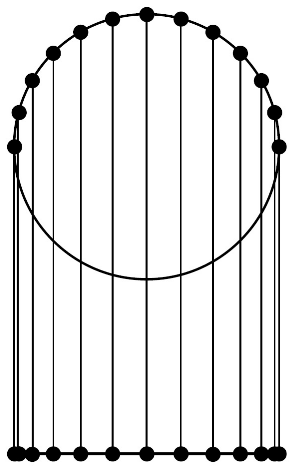
(b) The linear speed of the centre of mass is

$$v_{\mathrm{cm}} = \frac{L}{2}\omega = \frac{1}{2}\sqrt{3gL}.$$

Because the lowest point has the double linear speed of the point CM, it is

$$2v_{\mathrm{cm}} = \sqrt{3gL}.$$

32. **(Centripetal force).** One-dimensional *harmonic motion* of a small particle with mass m is regarded as an uniform circular motion projected onto a line. Please use the



projection to show the period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

of the motion where k is the coefficient of the material defined in Hook's law $F = -kx$.

Solution. The usual uniform circular motion is given by

$$F = -mr\omega^2$$

where F is the centripetal force, ω the angular velocity, the negative sign indicating the acceleration always pointing to the centre. Using the standard projection of r on x-axis, we have $x = r\cos\theta$. Thus,

$$F = -mr\omega^2$$
$$F = -m\frac{x}{\cos\theta}\omega^2$$
$$F_x = F\cos\theta = -mx\omega^2$$

Using Hook's law $F_x = -kx$ above equation simply gives

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

where we have used $\omega = 2\pi/T$.

33. **(Pressure).** In about 1657 Otto von Guericke, inventor of the air pump, evacuated a sphere made of two brass hemispheres (Figure 6). Two teams of eight horses each could pull the hemispheres apart only on some trials, and then "with greatest difficultly," with the resulting sound likened to a cannon firing. (a) Show that the force F required to pull the evacuated hemispheres apart is $\pi R^2(P_0 - P)$, where R is the radius of the hemispheres and P is the pressure inside the hemispheres, which is much less than P_0. (b) Determine the force if $P = 0.1P_0$ and $R = 0.3$m.

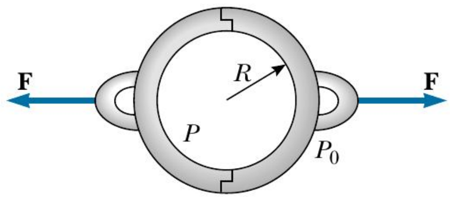


Figure 6:

Solution. The half of the surface area of the interior of the sphere is $2\pi R^2$. The total pressure produced by the forces equals to the difference of atmospheric pressure P_0 and the interior pressure P

$$\frac{F}{2\pi R^2} + \frac{F}{2\pi R^2} = P_0 - P \quad \Rightarrow \quad F = \pi R^2(P_0 - P)$$

Putting all the numbers into the formula one should obtain 2.564×10^4 N.

34. **(Pressure).** You can chew through very tough objects with your incisors because they exert a large force on the small area of a pointed tooth. What pressure can you create by exerting a force of 500 N with your tooth on an area of 1 mm^2?

    Solution. The pressure is $500/(1 \times 10^{-6}) = 5 \times 10^8$ Nm^{-2}.

35. **(Pressure).** A U-tube originally containing mercury has water added to one arm to a depth of 20 cm. What is the pressure at the water-mercury interface? What is the height of the mercury column as measured from the water mercury level? (density of mercury: 14,000 kg/m^3)

Solution. Since we would like to find the gauge value of pressure only, we may simply ignore the atmospheric pressure in our calculation. (Atmospheric pressure P_0 will be canceled out on both sides of the tube.) The gauge pressure at the water-mercury interface is

$$1000 \times 9.8 \times 0.2 = 1960 \text{ Pa}.$$

This pressure is balanced by the pressure due to mercury, i.e., $1960 = \rho_m g h_m$. Therefore,

$$h_m = \frac{1960}{14000 \times 9.8} = 0.14 \text{ m}.$$

36. **(Viscosity).** A small spherical particle falls in a liquid against the buoyant force and the drag force which is assumed to be given by the Stokes law, $F_s = 6\pi r \eta v$. Show that the terminal speed is given by $v = \frac{2R^2 g}{9\eta}(\rho_s - \rho_l)$, where R is the radius of the sphere, ρ_s is its density, and ρ_l is the density of the fluid and η the coefficient of viscosity.

Solution. The equation of motion of the particle is $W - B - F_s = ma$ where W, B, F_s are the weight of the particle, the buoyancy and the drag force against it. The equation becomes

$$\rho_s(4\pi R^3/3)g - \rho_l(4\pi R^3/3)g - 6\pi R\eta v = \rho_s(4\pi R^3/3)a$$

The particle attains the terminal velocity when $a = 0$, thus

$$\frac{2}{9}R^2(\rho_s - \rho_l)g - \eta v = 0 \quad \Rightarrow \quad v = \frac{2R^2 g}{9\eta}(\rho_s - \rho_l)$$

37. **(Buoyancy).** A solid sphere has a diameter of 1.2 cm. It floats in water with 0.4 cm of its diameter above water level (Figure 7). Determine the density of the sphere.

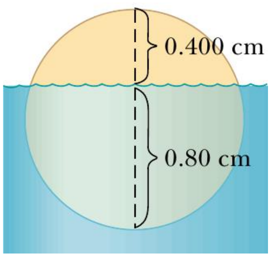


Figure 7:

Solution. We have to find the general formula for a portion of a sphere first. Suppose a circle $x^2 + y^2 = a^2$ with radius a, the volume of a portion of it is

$$\int_{-a}^{x'} y^2 \pi \mathrm{d}x = \int_{-a}^{x'} \pi(a^2 - x^2)\mathrm{d}x = \frac{\pi}{3}(3a^2 x' + 2a^3 - x'^3)$$

where x' is the length from the centre along the x-axis. A simple checking putting $x' = a$ gives $4\pi a^3/3$ for the volume of a sphere. For the volume shown in the diagram, putting $a = 0.6$ and $x' = 0.2$ into the formula gives the volume of the immersed portion of the sphere being 0.67 m^3. Since the whole volume is $(0.6)^3\pi \times 4/3 = 0.9$, the density of the sphere is

$$\frac{0.67}{0.9} \times 1000 = 744 \text{ kgm}^{-3}.$$

38. **(Flow).** How many cubic metre of blood does the heart pump in a 75-year lifetime, assuming the average flow rate is 5 L/min? (*Ans:* 2.0 × 10^5 m^3)

Solution. The product of flow rate and time is the total volume of blood over that interval. Therefore,

$$(5 \times 10^{-3} \times 60^{-1}) \times (75 \times 365.25 \times 24 \times 60^2) = 197235 \approx 2 \times 10^5 \text{ m}^3.$$

39. **(Bernoulli's equation.)** When a wind blows between two large buildings, a significant drop in pressure can be created. The air pressure is normally 1 atm inside the building, so the drop in pressure just outside can cause a plate glass window to pop out of the building and crash to the street below. What pressure difference would result from a 27 m/s wind? What force would be exerted on a $2 \times 3\,\text{m}^2$ plate glass window? The density of air is 1.29 kg/m^3 at 27° and 1 atm.

Solution. The pressures inside and outside of the building are supposed to be the same if no wind blowing at outside. When wind blows, air pressure will drop according to Bernoulli's equation. The net pressure pressing on the glass is

$$\Delta P = \frac{1}{2} \times 1.29 \times 27^2 = 470 \text{ Pa}.$$

And it is equivalent to a force $470 \times 2 \times 3 = 2820$ N acting on the glass.

40. **(Bernoulli's equation).** Fluid is flowing through the horizontal pipe as shown in Figure 8. Please show that

$$v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}.$$

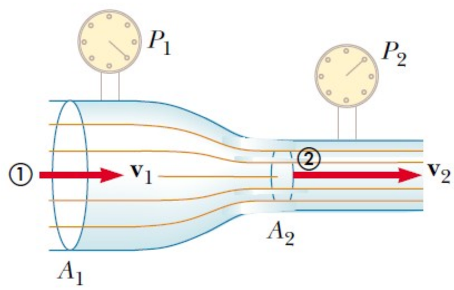


Figure 8: Fluid flows through the pipe.

Solution. From the equation of continuity, $A_1 v_1 = A_2 v_2$, we find that

$$v_1 = \frac{A_2}{A_1} v_2.$$

Substituting this expression into the Bernoulli's equation with the same altitude, i.e., $h = 0$, gives

$$P_1 + \frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

Therefore,

$$v_2 = A_1\sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}.$$

41. **(Heat capacity).** A 20 g piece of aluminium at 90 °C is dropped into a cavity in a large block of ice at 0 °C. How much ice does the aluminium melt? (Given specific heat of aluminium is 0.215 cal/g-°C, latent heat of fusion of water 3.33×10^5 J/kg)

Solution. Assume all the energy going to melt the ice from the piece of aluminium. We have the conservation equation for the energy:

$$20 \times 0.215 \times 90 \times 4.184 = 3.33 \times 10^5 \times m$$

where m is the mass of ice being melted in kilogram, and we have put the conversion 1 cal = 4.184 J. Thus,

$$m = 0.00486 \text{ kg} = 4.86 \text{ g}$$

of ice being melted.

42. **(Heat capacity).** A person fires a silver bullet with a mass of 2 g and with a muzzle speed of 200 m/s into the pine wall of a saloon. Assume that all the internal energy generated by the impact remains with the bullet. What is the temperature change of the bullet (Given: specific heat capacity of silver is 234 J/kg °C).

Solution. The kinetic energy of the bullet is $mv^2/2$. The temperature of the bullet increases because the kinetic energy becomes the extra internal energy. The temperature change is the same as that which would take place if the kinetic energy were transferred by heat from a stove to the bullet. Using 234 J/kg °C as the specific heat of silver, we obtain

$$\Delta T = \frac{Q}{mc} = \frac{\frac{1}{2}mv^2}{mc} = \frac{v^2}{2c} = \frac{200^2}{2 \times 234} = 85.5\,^\circ\text{C}.$$

43. **(Latent heat).** Liquid helium has a very low boiling point, 4.2 K, and a very low latent heat of vaporization, 2.09×10^4 J/kg. If energy is transferred to a container of boiling liquid helium from an immersed electric heater at a rate of 10.0 W, how long does it take to boil away 1.00 kg of the liquid?

Solution. We must supply 2.09×10^4 J of energy to boil away 1.00 kg. Because 10.0 W = 10.0 J/s, 10.0 J of energy is transferred to the helium each second. Therefore, the time it takes to transfer 2.09×10^4 J of energy is

$$t = \frac{2.09 \times 10^4}{10} = 2.09 \times 10^3 \text{ s} \approx 35 \text{ min}.$$

44. **(Heat flows).** The average thermal conductivity of the walls (including the windows) and roof of the house depicted in Figure 9 is 0.480 W/m °C, and their average thickness is 21.0 cm. The house is heated with naturnal gas having a heat of combustion (that is, the energy provided per cubic metre of gas burned) of 9,300 kcal/m^3. How many cubic meters of gas must be burned each day to maintain an inside temperature of 25.0°C if the outside temperature is 0.0°C? Disregard radiation and the energy lost by heat through the ground.

Solution. The energy flow rate P of conduction is given by

$$P = kA\frac{\Delta T}{L} \tag{6}$$

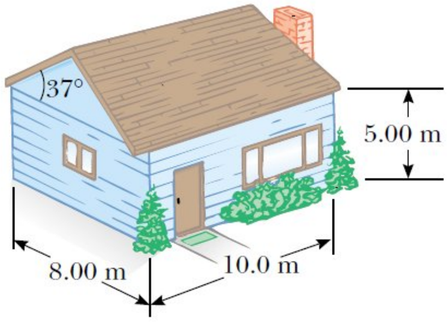


Figure 9:

where k is the thermal conductivity, A the area of the normal direction of heat flow, L the permeable length and ΔT the temperature difference. We have all the numerical data except the area of the house and flow rate that we have got to calculate:

$$A = 2(10 \times 5) + 2(8 \times 5) + \frac{2}{2} \times 8 \times 4 \tan 37 + 2 \times 10 \times \frac{4}{\cos 37} = 304.28 \text{ m}^2$$

and

$$P = \frac{9300 \times 4184 \times V}{24 \times 60 \times 60}$$

where V is the volume of the natural gas required for one day we want to calculate. We have changed the unit from kcal to Joule, and have expressed the 24 hours in term of second. Substituting all these values and $\Delta T = 25$ K, $L = 0.21$ m and $k = 0.48$ into the flow rate equation (6), i.e.,

$$\frac{9300 \times 4184 \times V}{24 \times 60 \times 60} = 0.48 \times 304.28 \times \frac{25}{0.21}$$

we obtain $V = 38.6 \text{ m}^3$.

45. **(Heat flow).** Two slabs of thickness L_1 and L_2 and thermal conductivities k_1 and k_2 are in thermal contact with each other, as shown in Figure 10. The temperature of their outer surface are T_1 and T_2, respectively, and $T_2 > T_1$. Determine the temperature at the interface and the rate of energy transfer per unit area by conduction through the slabs in the steady-state condition.

Solution. If T is the temperature at the interface, then the rates at which energies are transferred through the first and second slabs are

$$H_1 = \frac{k_1 A(T - T_1)}{L_1}, \quad H_2 = \frac{k_2 A(T_2 - T)}{L_2}$$

where A is the area of each interface. When a steady state is reached, these two rates must be equal; hence,

$$\frac{k_1 A(T - T_1)}{L_1} = \frac{k_2 A(T_2 - T)}{L_2}$$

Solving for T gives

$$T = \frac{k_1 L_2 T_1 + k_2 L_1 T_2}{k_1 L_2 + k_2 L_1}.$$

Substituting it into the previous one of the equations we obtain the heat flow per unit area

$$H = \frac{A(T_2 - T_1)}{L_1/k_1 + L_2/k_2}.$$

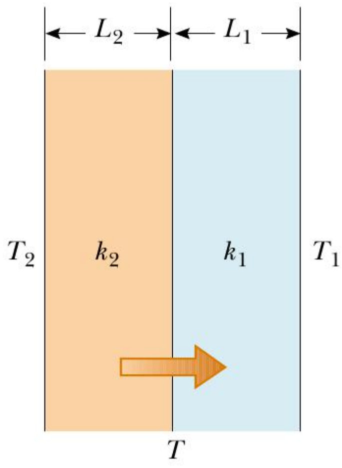


Figure 10:

46. **(Pressure).** A person takes about 20 breaths per minute with 0.5 litre of air in each breath. How much heat is removed per hour by the moisture in the exhaled breath if the incoming air is dry and the exhaled breath is fully saturated? Assume that the water vapour pressure in the saturated exhaled air is 24 torr.

    Solution. We change all the units to SI. The exhaled air pressure in each breath is $1.33 \times 10^2 \times 24$ Pa. The volume of the exhaled air is 0.5×10^{-3} m^3. Because

    $$\Delta PV = \text{Change of thermal energy}$$

    and it is assumed that the exhaled air is fully saturated of water vapour. Therefore, the heat content of the exhaled air in an hour is

    $$(1.33 \times 10^2 \times 24) \times (0.5 \times 10^{-3}) \times (20 \times 60) = 191.52 \text{ J}.$$

47. **(Ideal gas).** How many molecules are in 1 cm^3 of helium gas at $20°C$? (*Ans:* 2.47×10^{19})

    Solution. Assuming the gas obeys ideal gas law under 1 atmospheric pressure, we have

    $$\begin{aligned} PV &= Nk_BT \\ N &= \frac{PV}{k_BT} \\ &= \frac{10^5 \times 10^{-6}}{1.38 \times 10^{-23} \times (273 + 20)} \\ &= 2.47 \times 10^{19} \end{aligned}$$

    number of helium molecules.

48. **(Ideal gas).** (a) What is the average kinetic energy of a gas molecule at 24 °C? (b) Find the rms (root mean square) speed of an oxygen molecule at this temperature. (Given the mass of an oxygen molecule: 5.356×10^{-26} kg)

    Solution. (a) Since the average kinetic energy of a particle is defined by $\frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}k_BT$, it gives

    $$\frac{3}{2} \times 1.38 \times 10^{-23} \times (273 + 24) = 6.1479 \times 10^{-21} \text{ J}.$$

(b) Assume that the rms speed of nitrogen and oxygen molecules are similar. Since the mass of an oxygen molecule is about 5.356×10^{-26} kg, its rms speed is

$$v_{\text{rms}} = \sqrt{6.1479 \times 10^{-21} \times 2/(5.356 \times 10^{-26})} = 479.14 \text{ ms}^{-1}.$$

49. **(Thermodynamics.)** A coal-fired power station uses heat transfer from burning coal to do work to turn turbines for generating electricity. Suppose that 2.2×10^{14} J of heat transfer from coal and 1.35×10^{14} J of heat transfer into the environment. (a) What is the work done by the power station? (b) What is the efficiency of the power station?

Solution. (a) Assume the ideal situation having no energy going into the internal energy, i.e., $\Delta U = 0$, we have the work output

$$W = Q_h - Q_c = (2.2 - 1.35) \times 10^{14} = 0.85 \times 10^{14} \text{ J}.$$

(b) The efficiency of the power station is given by

$$\frac{0.85 \times 10^{14}}{2.2 \times 10^{14}} \times 100\% = 38.6\%$$

50. **(Entropy).** Heat is slowly added to a 0.08 kg chunk of ice at 0 °C until it completely melts into water at the same temperature. What is the entropy change of the ice? (Given the latent heat of fusion of water: 335 Jg^{-1})

Solution. The ice is melted by the addition of heat

$$Q = mL_f = 0.08 \times 10^3 \times 335 = 26800 \text{ J}.$$

As the ice gaining heat it is positive, the entropy increases by

$$\Delta S = \frac{26800}{273} = 98.17 \text{ J/K}$$

51. **(Entropy).** A large, cold object is at −10°C, and a large, hot object is at 130°C. Show that it is impossible for a small amount of energy, for example, 6 J, to be transferred spontaneously from the cold object to the hot one without a decrease in the entropy of the universe and therefore a violation of the second law.

Solution. By definition the change of entropy is the change of heat transfer divided by the temperature. In order to transfer the 6 J energy from the cold object to the hot object, the entropy change of the **system** is

$$\left(\frac{6}{273+130}\right)_{\text{hot object}} + \left(\frac{-6}{273-10}\right)_{\text{cold object}} = -0.007925$$

The main point of the result is that the negative sign indicates the decreasing of the entropy of the system whilst the energy transfer. The process is prohibited by the second law of thermodynamics.

52. **(Entropy).** Suppose you toss 5 coins starting with 4 heads and 1 tail, and you get one the most likely results, 2 heads and 3 tails. What is the change in entropy?

Solution. The change of the entropy is the final entropy minus the initial entropy of the system

$$\Delta S = S_f - S_i = k \ln \Omega_f - k \ln \Omega_i.$$

The number of ways to get 4 heads and 1 tail is ${}^5C_1 = 5$, to get 2 heads and 3 tails is ${}^5C_2 = 10$. Thus,

$$\Delta S = 1.38 \times 10^{-23}(\ln 10 - \ln 5) = 9.57 \times 10^{-24}\ \mathrm{JK}^{-1}.$$

53. **(Oscillation)** An object oscillates with simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation

$$x = 4\cos\left(\pi t + \frac{\pi}{4}\right)$$

where t is in seconds, the amplitude is in metre and the angles are in radians.

(a) Determine the amplitude, frequency, and period.

(b) Calculate the velocity and acceleration of the object at any time t.

(c) Using (b), determine the position, velocity, and acceleration of the object at $t = 1$ s.

(d) Determine the maximum speed and maximum acceleration of the object.

(e) Find the displacement of the object between $t = 0$ and $t = 1$ s.

Solution. (a) We see that the amplitude $A_0 = 4$ m and $\omega = \pi$ rad/s. Thus,

$$\nu = \frac{\omega}{2\pi} = \frac{\pi}{2\pi} = 0.5\ \mathrm{Hz} \quad \text{and} \quad T = \frac{1}{\nu} = 2\ \mathrm{s}.$$

(b) The velocity and the acceleration are

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -4\sin\left(\pi t + \frac{\pi}{4}\right)\ \mathrm{ms}^{-1},$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -4\pi^2\cos\left(\pi t + \frac{\pi}{4}\right)\ \mathrm{ms}^{-2}.$$

(c) Direct computation gives

$$x = 4\cos\left(\pi + \frac{\pi}{4}\right) = -2.83\ \mathrm{m},$$
$$v = -4\sin\left(\frac{5\pi}{4}\right) = 8.89\ \mathrm{ms}^{-1},$$
$$a = -4\pi^2\cos\left(\frac{5\pi}{4}\right) = 27.9\ \mathrm{ms}^{-2}.$$

(d) In (b), the maximum values of the sine and cosine functions are unity. Therefore,

$$v_{\mathrm{max}} = 4\pi = 12.6\ \mathrm{ms}^{-1},$$
$$a_{\mathrm{max}} = 4\pi^2 = 39.5\ \mathrm{ms}^{-2}.$$

(e) The x coordinate at $t = 0$ is

$$x_i = 4\cos\left(0 + \frac{\pi}{4}\right) = 2.83\ \mathrm{m}.$$

In (c), we found that the x coordinate at $t = 1$ s is -2.83 m; therefore, the displacement between the range is

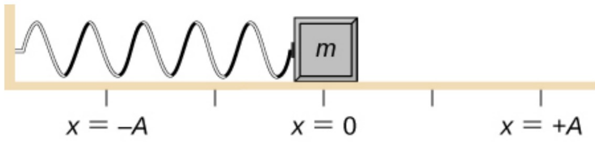
$$\Delta x = x_f - x_i = -2.83 - 2.83 = -5.66\ \mathrm{m}.$$

54. A car with a mass of 1,300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20,000 N/m. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road (Assuming the mass is evenly distributed).

Solution. As supposing the total mass is evenly distributed, each spring supports $\frac{1300+160}{4} = 365$ kg. Applying the formula $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, the frequency of the vibration of the car is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{20000}{365}} = 1.18 \text{ Hz.}$$

55. A block with a mass of 200 g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frinctionless surface. The block is displaced 5.00 cm from equilibrium and released from rest. (a) Find the period of its motion. (b) Express the displacement, speed, and acceleration as functions of time. (c) Determine the maximum speed of the block. (d) What is the maximum acceleration of the block?



Solution. (a) Applying the formula we get

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{200\times10^{-3}}{5}} = 1.26 \text{ s.}$$

(b) Since $\omega = 2\pi/T$, we immediately get 5.00 rad/s. Given the maximum displacement (amplitude) is 5.00 cm, we have the expressions:

$$\begin{aligned} x &= A\cos\omega t = (0.050)\cos 5.00t, \\ v &= \omega A\sin\omega t = -(0.250)\sin 5.00t, \\ a &= \omega^2 A\cos\omega t = -(1.25)\cos 5.00t. \end{aligned}$$

(c) & (d) For the maximum speed and the maximum acceleration, they simply are 0.250 m/s and 1.25 m/s^2 from the expressions of (b).

56. A 0.50 kg cube connected to a light spring for which the force sonstant is 20.0 N/m oscillates on a horizontal, frictionless track. Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

Solution. The energy of the system is

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}\times 20\times 3\times 10^{-2} = 9.0\times10^{-3} \text{ J.}$$

When the speed is maximum, the potential energy vanishes $U = 0$, we therefore have

$$\frac{1}{2}mv_{\max}^2 = 9\times10^{-3} \quad\Rightarrow\quad v_{\max} = \sqrt{\frac{18\times10^{-3}}{0.5}} = 0.190 \text{ m/s.}$$

57. **(Standing waves.)** A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows. (a) Determine the frequencies of the first three harmonics of the culvert if it is open at both ends. Take $v = 343$ m/s as the speed of sound in air. (b) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

Solution. (a) The wavelengths of the first three harmonics are

$$\lambda_1 = 2 \times 1.23 = 2.46 \text{ m},$$
$$\lambda_2 = 1 \times 1.23 = 1.23 \text{ m},$$
$$\lambda_3 = \frac{2}{3} \times 1.23 = 0.82 \text{ m}.$$

The corresponding frequencies are

$$\nu_1 = \frac{343}{2.46} = 139.4 \text{ Hz},$$
$$\nu_2 = \frac{343}{1.23} = 278.9 \text{ Hz},$$
$$\nu_3 = \frac{343}{0.82} = 418.3 \text{ Hz}.$$

(b) The wavelengths of the three lowest frequencies are $4L$, $4L/3$, $4L/5$ where L is the length of the culvert. Using the $\lambda = v/\lambda$ again, we have

$$\nu_1 = \frac{343}{4 \times 1.23} = 69.7 \text{ Hz},$$
$$\nu_2 = \frac{343}{4 \times 1.23/3} = 209.1 \text{ Hz},$$
$$\nu_3 = \frac{343}{4 \times 1.23/5} = 348.6 \text{ Hz}.$$

58. **(Superposition.)** Find the resultant wave of being created by the superposition of two transverse sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium.

Solution.
$$\psi_1 = \psi_0 \sin(kx - \omega t), \quad \psi_2 = \psi_0 \sin(kx + \omega t)$$

where ψ_1 represents a wave traveling to the right and ψ_2 represents one traveling to the left, with the same amplitude ψ_0. Adding these two functions according to superposition principle gives the resultant wave function ψ:

$$\begin{aligned}\psi &= \psi_1 + \psi_2 \\ &= \psi_0 \sin(kx - \omega t) + \psi_0 \sin(kx + \omega t) \\ &= (2\psi_0 \sin kx) \cos \omega t\end{aligned}$$

which is the wave function for a standing wave as there is no wave propagating that the wave varying with respect to time only changing the amplitude without any position translation.

59. **(sound velocity.)** A uniform cord has a mass of 0.3 kg and a length of 6 m (Figure 11). The cord passes over a pulley and supports a 2 kg object. Using the formula $v = \sqrt{T/\mu}$ where T, μ are the tension and mass per unit length, find the speed of a pulse travelling along this cord.

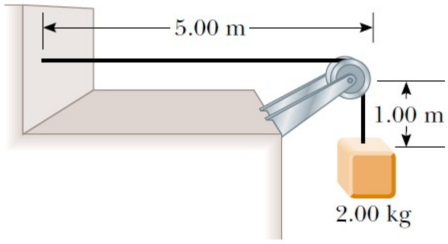


Figure 11: The tension T in the cord is maintained by the suspended object. The speed of any wave travelling along the cord is given by $v = \sqrt{T/\mu}$

Solution. Neglecting the little mass of the vertical potion of the cord, the mass per unit length and tension of the cord are

$$\mu = \frac{0.3}{6} = 0.05 \text{ kgm}^{-1},$$
$$T = 2g = 2 \times 10 = 20 \text{ N}.$$

Directly use the given formula, we obtain

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{20}{0.05}} = 20 \text{ m/s}.$$

60. **(Resonance).** Find the length of an one end open and one end closed air pipe if the change of frequency between two alternate resonances (i.e., every twice) of the standing waves is 100 Hz at 50 °C.

Solution. Let λ_1, λ_2 be the two alternate resonances. We have the relations:

$$\frac{L}{n} = \frac{\lambda_1}{4} \quad \text{and} \quad \frac{L}{n+4} = \frac{\lambda_2}{4}$$

There only exists the odd harmonics, where $n = 1, 3, 5, \ldots$. Eliminating n we get

$$\frac{L}{4L/\lambda_1 + 4} = \frac{\lambda_2}{4} \quad \Rightarrow \quad L = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

Using $\lambda = v/\nu$, we obtain

$$L = \frac{v}{\nu_2 - \nu_1} \quad \Rightarrow \quad L = \frac{v}{\Delta \nu}$$

Knowing the speed of sound at 24 °C, when at 50 °C the speed is given by

$$v = 343\sqrt{\frac{273 + 50}{273 + 24}} = 357.7 \text{ ms}^{-1}.$$

Thus, the length of the air pipe is

$$L = \frac{357.7}{100} = 3.58 \text{ m}.$$

61. **(Intensity.)** The intensity I of a sound wave is given by

$$I = \frac{\Delta P_{\max}^2}{2\rho v}$$

where ΔP_{max} is the variation of the maximum pressure amplitude, $v = 343$ m/s the speed of sound waves in air, and $\rho = 1.2$ kg/m^3 the density of air. Please determine the pressure amplitude associated with the limits of threshold of hearing and threshold of pain.

Solution. The intensity of threshold of hearing and threshold of pain are 10^{-12} W/m^2 and 1 W/m^2, respectively. We directly use the given formula to find the variations of air pressures. For the threshold of hearing,

$$\Delta P_{max} = \sqrt{2 \times 1.2 \times 343 \times 10^{-12}} = 2.870 \times 10^{-4} \text{ Pa}.$$

The variation of air pressure for the threshold of pain is

$$\Delta P_{max} = \sqrt{2 \times 1.2 \times 343 \times 1} = 28.70 \text{ Pa}.$$

It seems that our ears can detect a very small amount of variation of air pressure.

62. **(Loudness.)** The intensity of a sound wave at a distance r from the source is given by

$$I = \frac{P_{av}}{A} = \frac{P_{av}}{4\pi r^2}$$

where P_{av} is the average power emitted by the source. If a point source emits sound waves with an average power output 80 W (a) Find the intensity 3 m from the source. (b) Find the distance at which the sound level is 40 dB.

Solution. (a) Use the given formula, we obtain

$$I = \frac{80}{4\pi \times 3^2} = 0.707 \text{ W/m}^2.$$

(b) The given intensity is in the unit of dB. We must change it to W/m^2 for the given formula:

$$10^{-12} \log^{-1} \frac{40}{10} = 10^{-8} \text{ W/m}^2.$$

Therefore,

$$\begin{aligned} r &= \sqrt{\frac{P_{av}}{4\pi I}} \\ &= \sqrt{\frac{80}{4\pi \times 10^{-8}}} \\ &= 25231 \text{ m}. \end{aligned}$$

This unexpected result seems contradicting to our daily experiences. How could a point source of sound only having 80W propagate to over 25 km long distance far away from the source with sound level 40 dB where we are able to hear? The calculation is assumed that there is no ambient air pressure variation, no wind, etc. However in the situation on earth, sound (air pressure) is always diluted by the surroundings.

63. **(Loudness.)** Two identical machines are positioned the same distance from a worker. The intensity of sound delivered by each machine at the location of the worker is 2×10^{-7} W/m^2. Find the sound level heard by the worker (a) when one machine is operating and (b) when both machines are operating.

Solution. (a) One machine is operating:

$$\text{Logarithmic intensity} = 10\log\left(\frac{2\times10^{-7}}{10^{-12}}\right) = 10\log(2\times10^{5}) = 53\text{ dB}.$$

(b) When both machines are operating, the intensity is double to 4×10^{-7} W/m^2, therefore

$$10\log\left(\frac{4\times10^{-7}}{10^{-12}}\right) = 10\log(4\times10^{5}) = 56\text{ dB}.$$

From these, we see that when the intensity is doubled, the sound level increases by only 3 dB.

64. **(Doppler effect.)** A commuter train passes a passenger platform at a constant speed of 40 m/s. The train horn is sounded at its characteristic frequency of 320 Hz. (a) What wavelength is detected by a person on the platform as the train approaches? (b) What change in frequency is detected by a person on the platform as the train passes?

Solution. (a) The detected wavelength that is shorter than the wavelength measured in the moving system is

$$vT - 40T = \frac{343-40}{320} = 0.95\text{ m}$$

where T is the period and v the speed of sound in air. (b) The changes of the wavelengths in the cases of approaching and leaving away are $vT-40T$ and $vT+40T$, respectively. Thus, the change of frequency while the training passing is

$$\begin{aligned}\Delta\nu &= \frac{v}{vT+40T} - \frac{v}{vT-40T}\\ &= \frac{343\times320}{343+40} - \frac{343\times320}{343-40}\\ &= -75.7\text{ Hz}.\end{aligned}$$

The negative sign indicates the decrease of the frequency.

65. **(Doppler effect.)** As an ambulance travels east down a highway at a speed of 33.5 m/s, its siren emits sound at a frequency of 400 Hz. What frequency is heard by a person in a car traveling west at 24.6 m/s (a) as the car approaches the ambulance and (b) as the car moves away from the ambulance? (Given: speed of sound 343 m/s)

Solution. (a) Let v, v_o, v_s be the speed of sound, the speed of the observer and the speed of the ambulance. Because the cars are moving approaching to each other, the new wavelength of the sound in front of the ambulance created by the siren is $vT - v_sT$ where T is the period. That is, $(v-v_s)/\nu$. The frequency of the sound heard by the person is

$$\begin{aligned}\frac{\text{relative velocity of the observer and sound}}{\text{wavelength of sound in front of the ambulance}} &= \frac{v+v_o}{v-v_s}\nu\\ &= \frac{343+24.6}{343-33.5}\times400\\ &= 475\text{ Hz}.\end{aligned}$$

(b) As the car moves away from the ambulance, the sound wave at the back side of the ambulance is elongated; the wavelength is $vT + v_s T = (v + v_s)/\nu$. The relative velocity of the observer and the sound is also slowed to $v - v_o$. Therefore, the person should hear the new frequency of the sound to be

$$\frac{v - v_o}{v + v_s}\nu = \frac{343 - 24.6}{343 + 33.5} \times 400 = 338 \text{ Hz}.$$

66. **(Doppler effect.)** Write down the equations of the observed frequencies for the four cases of Doppler effect in terms of the speed of sound source v_s, the speed of observer v_o and the speed of sound v with the condition $v_s, v_o < v$.

Solution. (a)

$$v_s \longrightarrow \qquad v_o \longrightarrow$$

As O moving away from S (freq. $\downarrow$) and S approaching to O (freq. $\uparrow$) the observed frequency is $\nu' = \dfrac{v - v_o}{v - v_s}\nu$.

(b)

$$\longleftarrow v_s \qquad \longleftarrow v_o$$

As O approaching to S (freq. $\uparrow$) and S moving away from O (freq. $\downarrow$) the observed frequency is $\nu' = \dfrac{v + v_o}{v + v_s}\nu$.

(c)

$$\longleftarrow v_s \qquad v_o \longrightarrow$$

As O moving away from S (freq. $\downarrow$) and S moving away from O (freq. $\downarrow$) the observed frequency is $\nu' = \dfrac{v - v_o}{v + v_s}\nu$.

(d)

$$v_s \longrightarrow \qquad \longleftarrow v_o$$

As O approaching to S (freq. $\uparrow$) and S approaching to O (freq. $\uparrow$) the observed frequency is $\nu' = \dfrac{v + v_o}{v - v_s}\nu$.

Note. It does not matter either the speed v_s or v_o is larger so long as smaller than v.