Physics CST (2022-23) Homework 1

Please send the completed file to my mailbox yy.lam@qq.com by the 14th of September, with using the filename format:

student_number-name-cst-hw1

Please answer the questions by filling on these sheets. Or alternatively, do the homework as usual by using papers, then take the pictures and paste them onto these question sheets.

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1. Given the vectors $\mathbf{u} = 2\mathbf{e}_r + \mathbf{e}_\theta$ and $\mathbf{v} = 3\mathbf{e}_r - 2\mathbf{e}_\theta$ in the orthonormal basis, find the magnitude of $\mathbf{u}$ in $\mathbf{v}$ direction.

   Solution. *It implies to find* $\mathbf{u} \cdot \hat{\mathbf{v}}$. *Since*

$$\hat{\mathbf{v}} = \frac{3\mathbf{e}_r - 2\mathbf{e}_\theta}{\sqrt{3^2 + 2^2}} = \frac{3}{\sqrt{13}}\mathbf{e}_r - \frac{2}{\sqrt{13}}\mathbf{e}_\theta$$

   The magnitude is $\mathbf{u} \cdot \hat{\mathbf{v}} = 2\left(\frac{3}{\sqrt{13}}\right) + 1\left(\frac{-2}{\sqrt{13}}\right) = \frac{4}{\sqrt{13}}$

2. (a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

   Solution. *(a) As usual, positive for upward, negative for downward we have the height of the cliff*

$$s = 8 \times 2.35 + \frac{1}{2}(-9.8) \times 2.35^2 = -8.26 \ m.$$

   (b) The time taken is

$$-8.26 = -8t + \frac{1}{2}(-9.8)t^2 \quad \Rightarrow \quad t = 0.42 \ s.$$

3. The gravitational acceleration g on the surface of Earth relates to the gravitational constant G, the mass of Earth M, and the radius of Earth R_E. Use dimensional analysis to find the mathematics expression for g.

Solution. *The gravitational acceleration g is related by $g \propto G^a M^b R_E^c$ with some unknown numbers a, b and c. The expression of the dimensions is written as*

$$\begin{aligned} [g] &= [G]^a [M]^b [R_E]^c \\ LT^{-2} &= \frac{L^{3a}}{M^a T^{2a}} \cdot M^b \cdot L^c \\ 1 &= L^{3a+c-1} M^{b-a} T^{-2a+2} \end{aligned}$$

Solving for the unknown numbers from the equations $\begin{cases} 3a + c - 1 &= 0 \\ b - a &= 0 \\ -2a + 2 &= 0 \end{cases}$, *we got* $a = b = 1$, *and* $c = -2$. *Thus the expression is*

$$g \propto \frac{GM}{R_E^2}$$

4. A car is traveling east at 70 km/h. At an intersection 26 km ahead, a truck is traveling north at 50 km/h. (a) Write the position vector of the truck relative to the car $\mathbf{r}_{CT}$ in terms of the unit vector $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. (b) How long after this moment will the vehicles be closest to each other? (c) How far apart will they be at that point?

Solution. *(a) The position vector of the truck relative to the car is*

$$\mathbf{r}_{CT} = \mathbf{r}_{CO} + \mathbf{r}_{OT} = (-26 + 70t)\hat{\mathbf{i}} + 50\hat{\mathbf{j}}$$

where O is the road intersection. (b) Pythagorean theorem gives the distance between the vehicles

$$|\mathbf{r}_{TC}|^2 = (-26 + 70t)^2 + (50t)^2.$$

For the minimum distance, differentiating the distance gives

$$2r\frac{\mathrm{d}r}{\mathrm{d}t} = 2(-26 + 70t)(70) + 2(50)^2 t \quad \Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r}[(-26 + 70t)(70) + 50^2 t]$$

Setting it to zero we get minimum time taken

$$70(-26 + 70t) + 50^2 t = 0 \quad \Rightarrow \quad t = 0.246 \ hr = 14.76 \ min$$

(c) Inserting into the distance function, we obtain the minimum distance $r = 15.11$ km.

5. Find the unit vector of the cross product of the vectors $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$.

   Solution. *The cross product of them is*

$$(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) = -12\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

   The magnitude is $\sqrt{(-12)^2 + 8^2 + 1^2} = \sqrt{209}$. *Thus the unit vector is*

$$\frac{1}{\sqrt{209}}(-12\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

6. A vector $\mathbf{v}$ in 2-dimensional plane in polar coordinates is

$$\mathbf{v} = 3t\mathbf{e}_r - t^3\mathbf{e}_\theta.$$

   Find the derivative of $\mathbf{v}$ with respect to t.

   Solution.

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= 3\mathbf{e}_r + 3t\dot{\mathbf{e}}_r - 3t^2\mathbf{e}_\theta - t^3\dot{\mathbf{e}}_\theta \\ &= 3\mathbf{e}_r + 3t\dot{\theta}\mathbf{e}_\theta - 3t^2\mathbf{e}_\theta + t^3\dot{\theta}\mathbf{e}_r \\ &= (3 + t^3\dot{\theta})\mathbf{e}_r + 3t(\dot{\theta} - t)\mathbf{e}_\theta\end{aligned}$$

7. The position of a particle for $t > 0$ is given by $\mathbf{r}(t) = 3t^2\hat{\mathbf{i}} - 7t^3\hat{\mathbf{j}} - 5t^{-2}\hat{\mathbf{k}}$ m. (a) What is the velocity as a function of time? (b) What is the acceleration as a function of time? (c) What is the particle's velocity at $t = 2.0$ s? (d) What is its speed at $t = 1.0$ s and $t = 3.0$ s? (e) What is the average velocity between $t = 1.0$ s and $t = 2.0$ s?

Solution. *(a) Differentiating the position vector with respect to time gives the velocity*

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}(3t^2\hat{\mathbf{i}} - 7t^3\hat{\mathbf{j}} - 5t^{-2}\hat{\mathbf{k}}) = 6t\hat{\mathbf{i}} - 21t^2\hat{\mathbf{j}} + 10t\hat{\mathbf{k}}\ ms^{-1}.$$

(b) Differentiating it again we got the acceleration

$$\mathbf{a} = 6\hat{\mathbf{i}} - 42t\hat{\mathbf{j}} + 10\hat{\mathbf{k}}\ ms^{-2}.$$

(c) Inserting $t = 2$ into the velocity we get

$$\mathbf{v}(2) = 12\hat{\mathbf{i}} - 84\hat{\mathbf{j}} + 20\hat{\mathbf{k}}\ ms^{-1}.$$

(d) The velocities of the particle are $\mathbf{v}(1) = 6\hat{\mathbf{i}} - 21\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$ and the previous $\mathbf{v}(2)$. The speeds are

$$\begin{aligned} |\mathbf{v}(1)| &= \sqrt{6^2 + 21^2 + 10^2} = 24.0\ ms^{-1}, \\ |\mathbf{v}(2)| &= \sqrt{12^2 + 84^2 + 20^2} = 87.2\ ms^{-1}. \end{aligned}$$

(e) The average velocity in the interval is

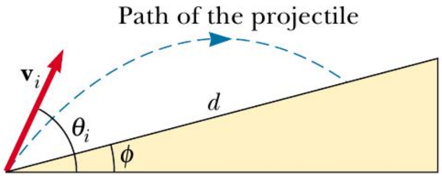
$$\langle \mathbf{v} \rangle = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = 9\hat{\mathbf{i}} - 49\hat{\mathbf{j}} + 3.75\hat{\mathbf{k}}\ ms^{-1}.$$

8. The velocity of a particle in reference frame A is $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ ms^{-1}. The velocity of reference frame A with respect to reference frame B is $\hat{\mathbf{i}} - 4\hat{\mathbf{k}}$ ms^{-1}, and the velocity of reference frame B with respect to C is $2\hat{\mathbf{j}}$ ms^{-1}. What is the velocity of the particle in reference frame C?

Solution.

$$\begin{aligned} v_{CP} &= v_{CB} + v_{BP} \\ &= v_{CB} + v_{BA} + v_{AP} \\ &= 2\hat{\mathbf{j}} + (\hat{\mathbf{i}} - 4\hat{\mathbf{k}}) + (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \\ &= 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 4\hat{\mathbf{k}} \end{aligned}$$

9. A projectile is fired up an inclined plane with an initial speed v_i at an angle θ_i as shown. Show that the maximum distance of d is $\dfrac{2v_i^2\cos(45^\circ+\phi/2)\sin(45^\circ-\phi/2)}{g\cos^2\phi}$.



Solution. *The horizontal and vertical components of the trajectory are*

$$\begin{aligned} d\cos\phi &= v_i\cos\theta_i\, t, \\ d\sin\phi &= v_i\sin\theta_i\, t + \frac{1}{2}(-g)t^2. \end{aligned}$$

Eliminating t to solve for d we get

$$\begin{aligned} d\sin\phi &= v_i\sin\theta_i\frac{d\cos\phi}{v_i\cos\theta_i} - \frac{g}{2}\frac{d^2\cos^2\phi}{v_i^2\cos^2\theta_i} \\ d &= \frac{\sin\theta_i\cos\phi - \sin\phi\cos\theta_i}{\cos\theta_i}\cdot\frac{2v_i^2\cos^2\theta_i}{g\cos^2\phi} \\ &= \frac{2v_i^2\cos\theta_i\sin(\theta_i-\phi)}{g\cos^2\phi} \end{aligned}$$

We may differentiate the range d with respect to the launching angle θ_i, then set the derivative to zero for the extrema (maximum or minimum):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta_i}d &= \frac{2v_i^2[(-\sin\theta_i)\sin(\theta_i-\phi)+\cos\theta_i\cos(\theta_i\phi)]}{g\cos^2\phi} \\ &= \frac{2v_i^2\cos(2\theta_i-\phi)}{g\cos^2\phi} \end{aligned}$$

The second derivative of d is

$$d'' = -\frac{4v_i^2\sin(2\theta_i-\phi)}{g\cos^2\phi} < 0$$

since the sine function is supposed to be positive here. The extremum is the maximum when $\mathrm{d}d/\mathrm{d}\theta_i = 0$, i.e.,

$$\cos(2\theta_i-\phi)=0 \quad\Rightarrow\quad \theta_i = 45^\circ + \frac{\phi}{2}$$

Substituting into the range function we obtain the maximum range:

$$d = \frac{2v_i^2\cos(45^\circ+\phi/2)\sin(45^\circ-\phi/2)}{g\cos^2\phi}$$