Continuous Random Variables and Probability Distributions

- **4.1 Continuous Random Variables and Probability Density Functions**
- **4.2 Cumulative Distribution Functions and Expected Values**
- **4.3 The Normal Distribution**
- **4.4 The Gamma Distribution and Its Relatives**
- **4.5 Other Continuous Distributions**
- **4.6 Probability Plots**

4.5 Other Continuous Distributions

- **The Weibull Distribution**

  A random variable X is said to have a Weibull distribution with parameters α and β ($\alpha > 0, \beta > 0$) if the cdf of X is

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-(x/\beta)^{\alpha}} & x \geq 0\\ 0 & x < 0\end{cases}$$

When α =1, the pdf reduces to the exponential distribution (with λ =1/ β), so the exponential Distribution is a special case of both the gamma and Wellbull distributions.

4.5 Other Continuous Distributions

- Mean and Variance

$$\mu = \beta\Gamma\left(1+\frac{1}{\alpha}\right); \quad \sigma^2 = \beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$$

- The cdf of a Weibull Distribution

$$F(x;\alpha,\beta) = \begin{cases} 0 & x < 0 \\ 1 - e^{-(x/\beta)^\alpha} & x \ge 0 \end{cases}$$

The Lognormal Distribution

A nonnegative rv X is said to have a lognormal distribution if the rv $Y = \ln(X)$ has a normal distribution . The resulting pdf of a lognormal rv when $\ln(X)$ is normally distributed with parameters μ and σ is

$$f(x;\mu,\sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln(x)-\mu)^2/(2\sigma^2)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- **Mean and Variance**

$$E(X)=e^{\mu+\sigma^2/2}\quad ;V(X)=e^{2\mu+\sigma^2}\cdot\left(e^{\sigma^2}-1\right)$$

- **The cdf of Lognormal Distribution**

$$F(x;\mu,\sigma)=P(X\le x)=P[\ln(X)\le\ln(x)]$$
$$=P\left(Z\le\frac{\ln(x)-\mu}{\sigma}\right)=\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)$$

The Beta Distribution

A random variable X is said to have a beta distribution with parameters α, β, A, and B if the pdf of X is

$$f(x;\alpha,\beta,A,B)=\begin{cases}\dfrac{1}{B-A}\cdot\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\left(\dfrac{x-A}{B-A}\right)^{\alpha-1}\left(\dfrac{B-x}{B-A}\right)^{\beta-1}, & A\le x\le B\\ 0 & \text{otherwise}\end{cases}$$

The case A = 0, B =1 gives the standard beta distribution. And the mean and variance are

$$\mu=A+(B-A)\cdot\frac{\alpha}{\alpha+\beta};\quad \sigma^2=\frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

4.6 Probability Plots

- **Probability Plot**

An investigator obtained a numerical sample $x_1, x_2, \ldots, x_n$ and wish to know whether it is plausible that **it came from a population distribution of some particular type** (and/or the corresponding parameters).

An effective way to check a distributional assumption is to construct the so-called Probability plot.

4.6 Probability Plots

- **Sample Percentiles**

Definition:

Order the n sample observations from the smallest to the largest. Then the ith smallest observation in the list is taken to be the $[100(i-.5)/n]$th sample percentile.

Considering the following pairs (as a point on a 2-D coordinate system) in a figure

$$\begin{pmatrix} [100(i-0.5)/n]\text{th percentile,} & i\text{th smallest sample} \\ \text{of the distribution} & \text{observation} \end{pmatrix}$$

Note: If the sample percentiles are close to the corresponding population distribution percentiles, then all points will fall close to a srtaigh line.

4.6 Probability Plots

- **Normal Probability Plot**

Just a special case of the probability plot

$$\begin{pmatrix} [100(i-0.5)/n]\text{th percentile,} & i\text{th smallest sample} \\ \text{of the distribution} & \text{observation} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} [100(i-0.5)/n]\text{th z percentile,} & \begin{matrix} i\text{th smallest sample} \\ \text{observation} \end{matrix} \end{pmatrix}$$

Used to check the Normality of the sample data

- **Example 4.29**

The value of a certain physical constant is known to an experimenter. The experimenter makes $n = 10$ independent measurements of this value using a particular measurement device and records the resulting measurement errors (**error = observed value - true value**). These observations appear in the following table.

Sample	**-1.91**	**-1.25**	**-0.75**	**-0.53**	**0.20**	**0.35**	**0.72**	**0.87**	**1.40**	**1.56**

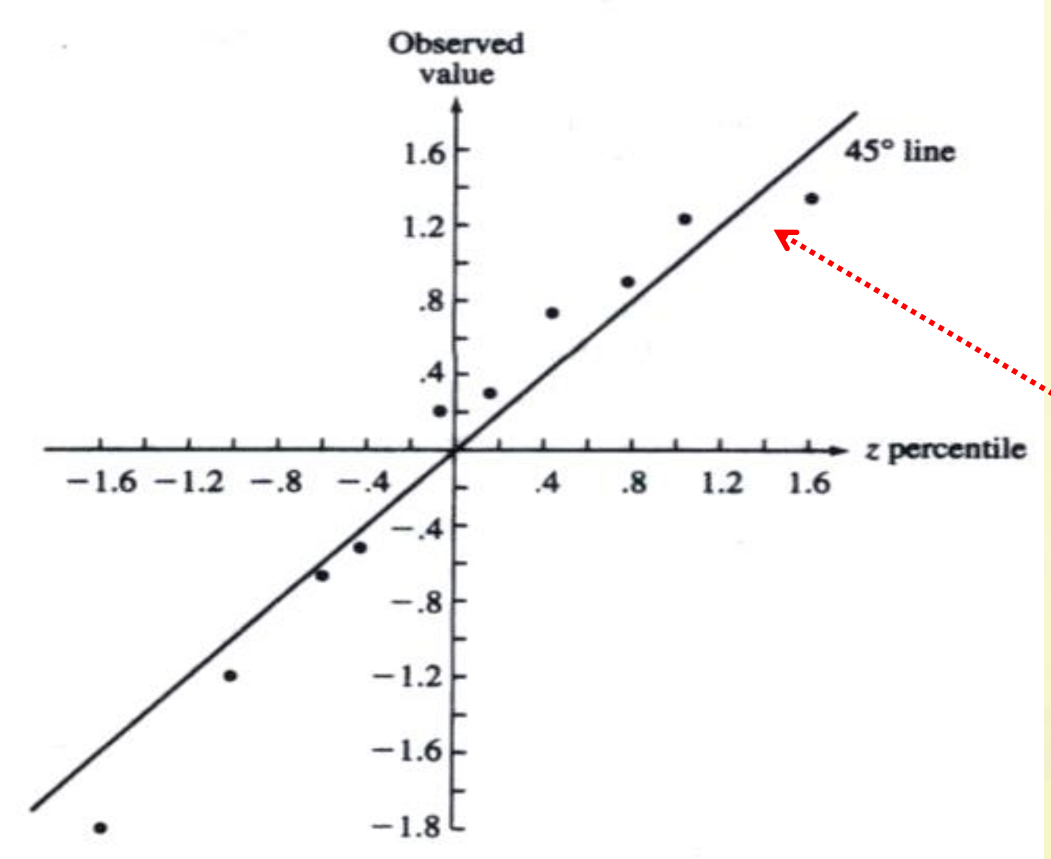
Probability Plots

Percentage	5	15	25	35	45
z percentile	−1.645	−1.037	−.675	−.385	−.126
Sample observation	−1.91	−1.25	−.75	−.53	.20

Percentage	55	65	75	85	95
z percentile	.126	.385	.675	1.037	1.645
Sample observation	.35	.72	.87	1.40	1.56

4.6 Probability Plots

- **Example 4.29 (Cont')**

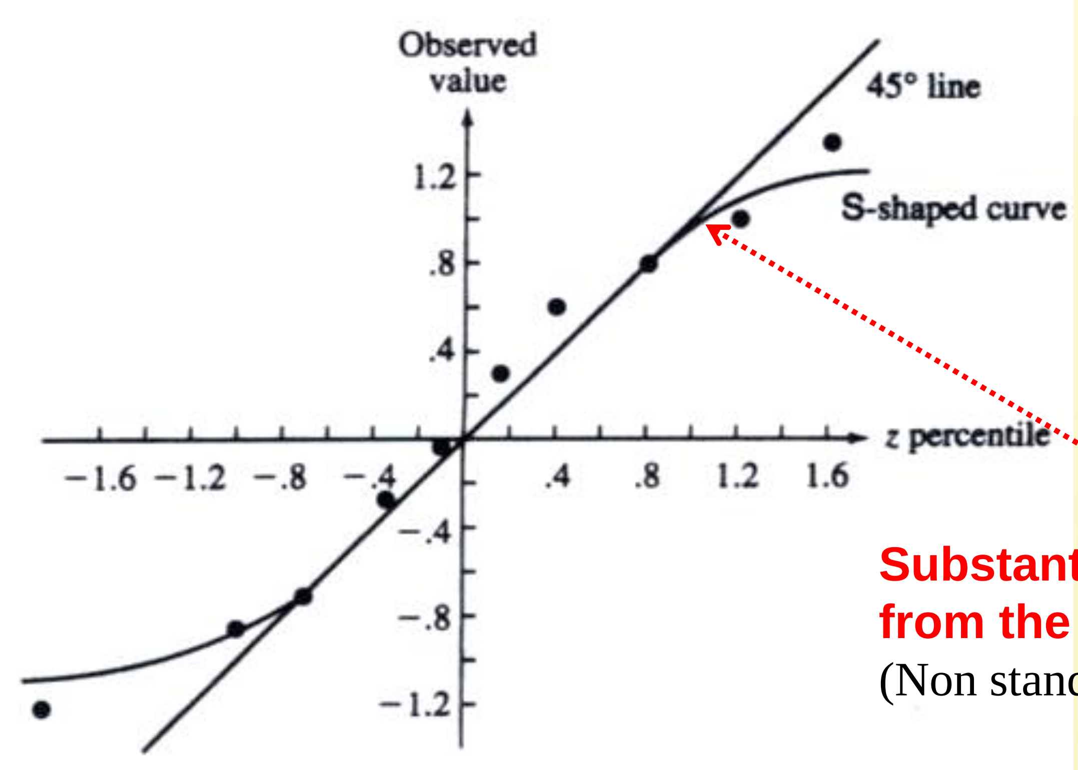


Close the 45^0 line
(Approximately standard normal distribution)

Figure Plots of pairs (*z* percentile, observed value) for the data of Example 4.28:first sample

4.6 Probability Plots

- **Example 4.28 (Cont’)**

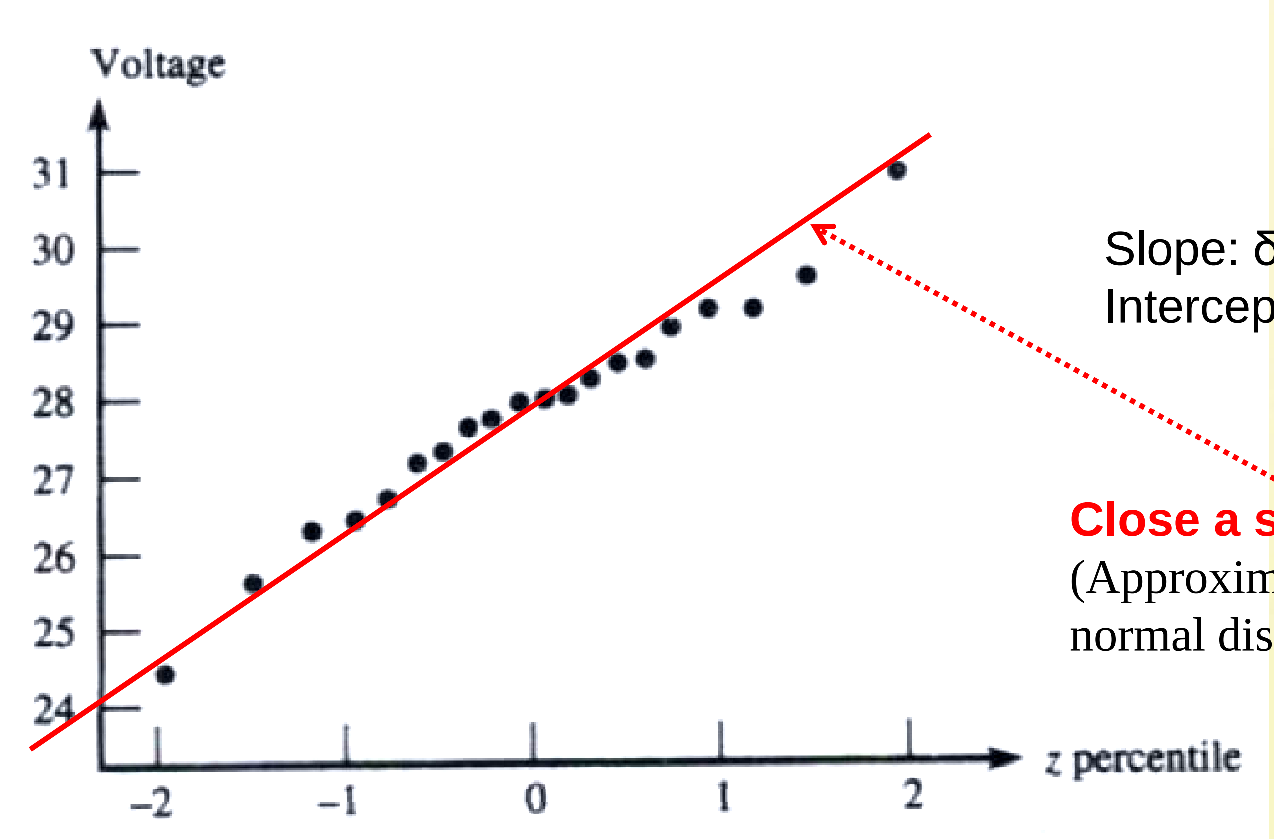


Substantial deviations from the 45^0 line
(Non standard normal distribution)

Figure Plots of pairs (z percentile, observed value) for the data of Example 4.28:second sample

4.6 Probability Plots

- **Example 4.30**



Slope: δ
Intercept : μ

Close a straight line
(Approximately
normal distribution)

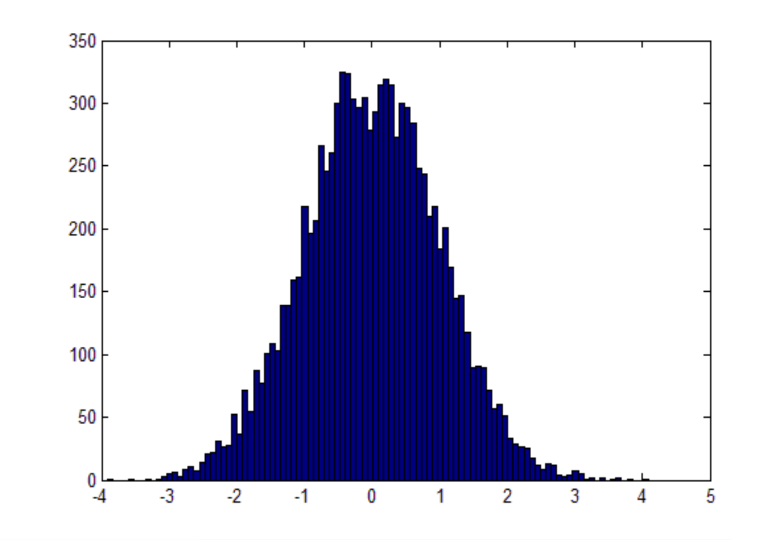
4.6 Probability Plots

- **Categories of a non-normal population distribution**

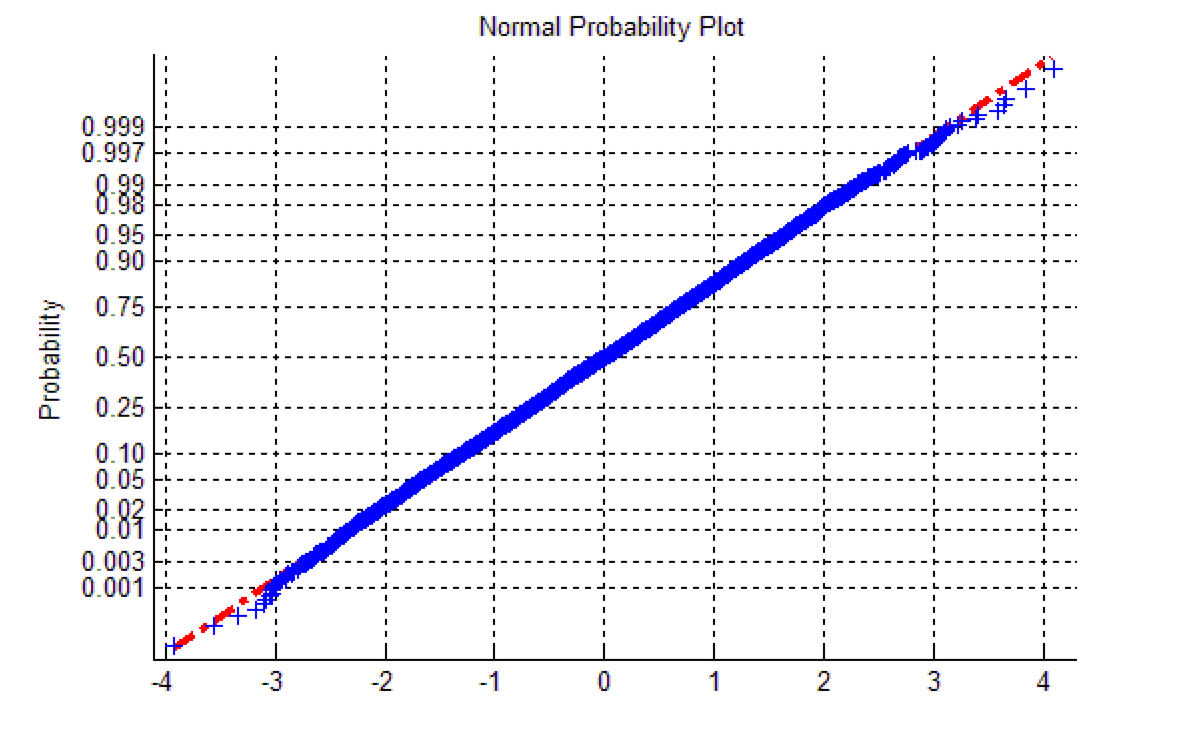
1. It is **symmetric** and has “**lighter tails**” than does a normal distribution; that is, the density curve declines more rapidly out in the tails than does a normal curve.
2. It is **symmetric** and **heavy-tailed** compared to normal distribution.
3. **It is skewed.**

4.6 Probability Plots

- **Normal Probability plot of the normal distribution**

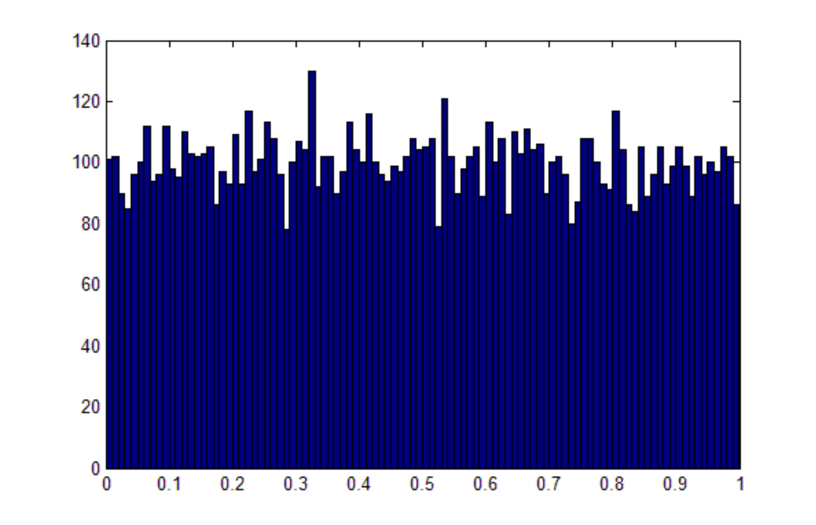


Simulation Data



4.6 Probability Plots

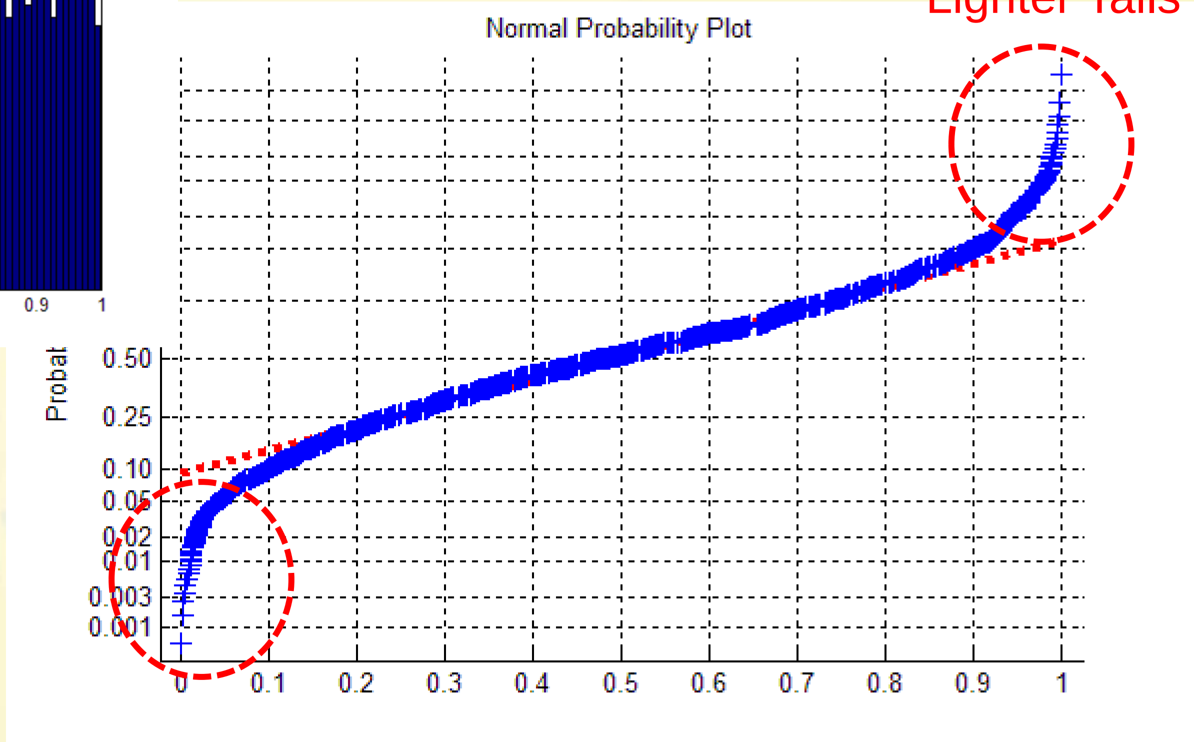
- **Normal Probability plot of the uniform distribution**



Simulation Data

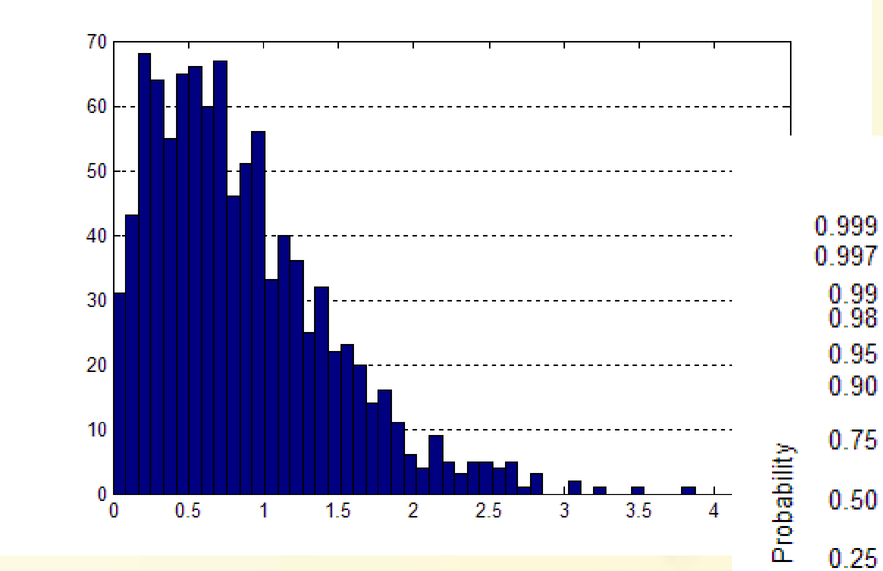
Symmetry

Lighter Tails

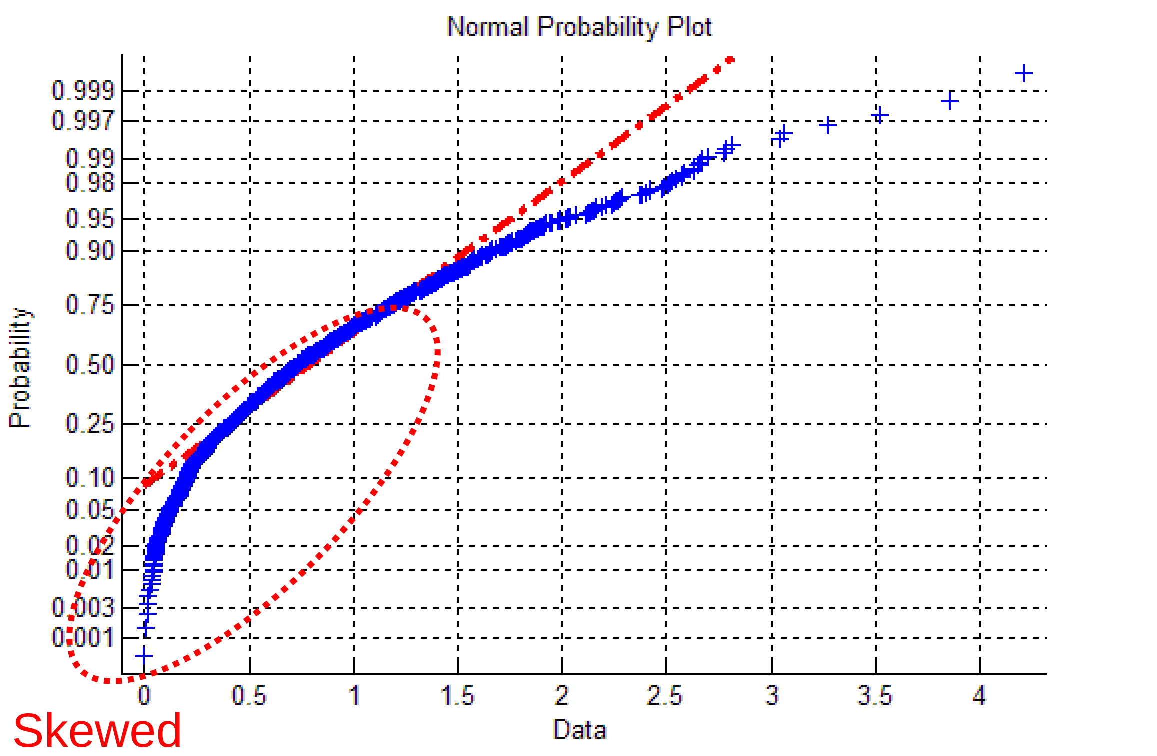


4.6 Probability Plots

- **Normal Probability plot of the Weibull distribution**



Simulation Data



Positively Skewed