HW 5-1 A+

9) a) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

$= \int_{20}^{30}\int_{20}^{30} k(x^2+y^2)\,dx\,dy$

$= k\int_{20}^{30}\int_{20}^{30} x^2\,dy\,dx + k\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$

$= 10k\int_{20}^{30} x^2\,dx + 10k\int_{20}^{30} y^2\,dy$

$= 20k\cdot\left(\frac{19000}{3}\right)$

$k = \frac{3}{380000}$

b) $P(X<26 \text{ and } Y<26)$

$= \int_{20}^{26}\int_{20}^{26} k(x^2+y^2)\,dx\,dy$

$= k\int_{20}^{26}\left[x^2y + \frac{y^3}{3}\right]_{20}^{26} dx$

$= k\int_{20}^{26}(6x^2+3192)\,dx$

$= k(38304)$

$= 0.3024$

c) $P(|x-y|\le 2) = \iint_{III} f(x,y)\,dx\,dy$

$= 1 - \iint_{I} f(x,y)\,dx\,dy - \iint_{II} f(x,y)\,dx\,dy$

$= 1 - \int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx$

$= 0.3593$

d) $f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{20}^{30} k(x^2+y^2)\,dy$

$= 10kx^2 + k\frac{y^3}{3}\Big|_{20}^{30}$

$= 10kx^2 + 0.05 \quad , \quad 20\le x\le 30$

e $f_Y(y)$ can be obtained by substituting y for x in (d)
$f(x,y) \neq f_x(x) \cdot f_Y(y)$, so X and Y are not independent

12) a) $P(X>3) = \int_3^\infty \int_0^\infty x e^{-x(1+y)}\,dy\,dx$
$= \int_3^\infty e^{-x}\,dx = 0.050$

b) The marginal pdf of x is $f_x(x) = \int_0^\infty x e^{-x(1+y)}\,dy = e^{-x}$ for $x \geq 0$.
The marginal pdf of Y is $f_Y(y) = \int_3^\infty x e^{-x(1+y)}\,dy = \frac{1}{(1+y)^2}$ for $y \geq 0$
$f(x,y)$ is not the product of the marginal pdf, so the two rvs are not independent

c) $P(\text{at least one exceeds } 3) = P(X>3 \text{ or } Y>3)$
$= 1 - P(X \leq 3 \text{ and } Y \leq 3)$
$= 1 - \int_0^3 \int_0^3 x e^{-x(1+y)}\,dy\,dx$
$= 1 - \int_0^3 \int_0^3 x e^{-x} e^{-xy}\,dy$
$= 1 - \int_0^3 e^{-x}(1-e^{-3x})\,dx$
$= e^{-3} + 0.25 - 0.25e^{-12}$
$= 0.300$

18) a) $P_{Y|X}(y|1)$ results from dividing each entry in $x=1$ row of the joint probability table by $P_X(1) = 0.34$

$P_{Y|X}(0|1) = \frac{0.08}{0.34} = 0.2353$

$P_{Y|X}(1|1) = \frac{0.20}{0.34} = 0.5882$

$P_{Y|X}(2|1) = \frac{0.06}{0.34} = 0.1765$

b) $P_{Y|X}(x|2)$ is requested
divide each entry in the $y=2$ row by $P_X(2) = 0.50$

y	0	1	2
$P_{Y\|X}(y\|2)$	0.12	0.28	0.60

c) $P(Y \le 1 \mid x=2) = P_{Y|X}(0|2) + P_{Y|X}(1|2)$

$= 0.12 + 0.28$

$= 0.40$

d) $P_{X|Y}(x|2)$ results from dividing each entry in the $y=2$ column by $p_Y(2) = 0.38$

x	0	1	2
$P_{X|Y}(x|2)$	0.0526	0.1579	0.7895

19) a) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{k(x^2+y^2)}{10kx^2+0.05}$, $\quad 20 \le y \le 30$

$f_{X|Y}(x|y) = \dfrac{k(x^2+y^2)}{10ky^2+0.05}$, $\quad 20 \le x \le 30$, $\quad k = \dfrac{3}{380000}$

b) $P(Y \ge 25) \mid x=22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy$

$= \int_{25}^{30} \dfrac{k((20)^2+y^2)}{10k(22)^2+0.05}\,dy = 0.556$

$P(Y \ge 25) = \int_{25}^{30} f_Y(y)\,dy$

$= \int_{25}^{30} (10ky^2 + 0.05)\,dy$

$= 0.549$

c) $E(Y|X=22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|22)\,dy$

$= \int_{20}^{30} y \cdot \frac{k((22)^2 + y^2)}{10k(22)^2 + 0.05}\,dy$

$= 25.372912$

$E(Y^2|X=22) = \int_{20}^{30} y^2 \cdot \frac{k((22)^2 + y^2)}{10k(22)^2 + 0.05}\,dy$

$= 652.028640$

$V(Y|X=22) = E(Y^2|X=22) - [E(Y|X=22)]^2$

$= 8.243976$

$\sigma = \sqrt{V(Y|X=22)} = 2.87$

HW 5-2

24) $h(x,y)$ = the number of individuals who handle the message x, y, and $h(x,y)$ can be

$h(x,y)$	1	2	3	4	5	6
1	–	2	3	4	3	2
2	2	–	2	3	4	3
3	3	2	–	2	3	4
4	4	3	2	–	2	3
5	3	4	3	2	–	2
6	2	3	4	3	2	–

$P(x,y) = \frac{1}{30}$ for each (x,y)

$$E[h(X,Y)] = \sum_x \sum_y h(x,y) \cdot P(x,y)$$
$$= \sum_x \sum_y h(x,y) \cdot \frac{1}{30}$$
$$= \frac{84}{30}$$

26) Revenue $= 3X + 10Y$

$$E(\text{revenue}) = E(3X + 10Y)$$
$$= \sum_{x=0}^{5} \sum_{y=0}^{2} (3x + 10y) \cdot P(x,y)$$
$$= 0 \cdot P(0,0) + \cdots + 35 \cdot P(5,2)$$
$$= \$15.4$$

33) $E(XY) = E(X) \cdot E(Y)$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$
$$= E(X) \cdot E(Y) - E(X) \cdot E(Y)$$
$$= 0$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} \Rightarrow Corr(X \cdot Y) = 0$$

35) a) $Cov(aX+b,\ cY+d)$

$= E[(aX+b)(cY+d)] - E(aX+b)\cdot E(cY+d)$

$= E[acXY + adX + bcY + bd] - (aE(X)+b)(cE(Y)+d)$

$= acE(XY) + adE(X) + bcE(Y) + bd - [acE(X)E(Y) + adE(X) + bcE(Y) + bd]$

$= acE(XY) - acE(X)E(Y)$

$= ac[E(XY) - E(X)E(Y)$

$= ac[E(XY) - E(X)E(Y)]$

$= ac\,Cov(X,Y)$

b) $Corr(aX+b,\ cY+d)$

$= \dfrac{Cov(aX+b,\ cY+d)}{SD(aX+b)SD(cY+d)}$

$= \dfrac{ac\,Cov(X,Y)}{|a|\cdot|c|SD(X)SD(Y)}$

$= \dfrac{ac}{|ac|}\ corr(X,Y)$

if $ac = |ac|$, $Corr(aX+b,\ cY+d) = Corr(X,Y)$

c) if $-ac = |ac|$, $Corr(aX+b, cY+d) = -Corr(X,Y)$