Continuous Random Variables and Probability Distributions

- 4.1 Continuous Random Variables and Probability Density Functions
- 4.2 Cumulative Distribution Functions and Expected Values
- 4.3 The Normal Distribution
- 4.4 The Gamma Distribution and Its Relatives
- 4.5 Other Continuous Distributions
- 4.6 Probability Plots

4.4 The Gamma Distribution and Its Relatives

- Gamma Function

For $\alpha>0$, the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}dx$$

The most important **properties** of the gamma function are the following:

1. For any $\alpha>1$, $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$;

2. For any positive integer n, $\Gamma(n)=(n-1)!$

3. $\Gamma(1/2) = \sqrt{\pi}$

4.4 The Gamma Distribution and Its Relatives

- **The Family of Gamma Distributions**

A continuous random variable X is said to have a gamma distribution if the pdf of X is

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}\,x^{\alpha\text{-}1}\mathrm{e}^{-x/\beta} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

where the parameters α and β satisfy α >0, β > 0.

4.4 The Gamma Distribution and Its Relatives

- Mean and Variance

The mean and variance of a random variable X having the gamma distribution $f(x;\alpha,\beta)$ are

$$E(X) = \mu = \alpha\beta$$

$$V(X) = \sigma^2 = \alpha\beta^2$$

4.4 The Gamma Distribution and Its Relatives

- **Standard Gamma Distribution**

The standard gamma distribution has β = 1.

$$f(x;\alpha)=\begin{cases}\dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

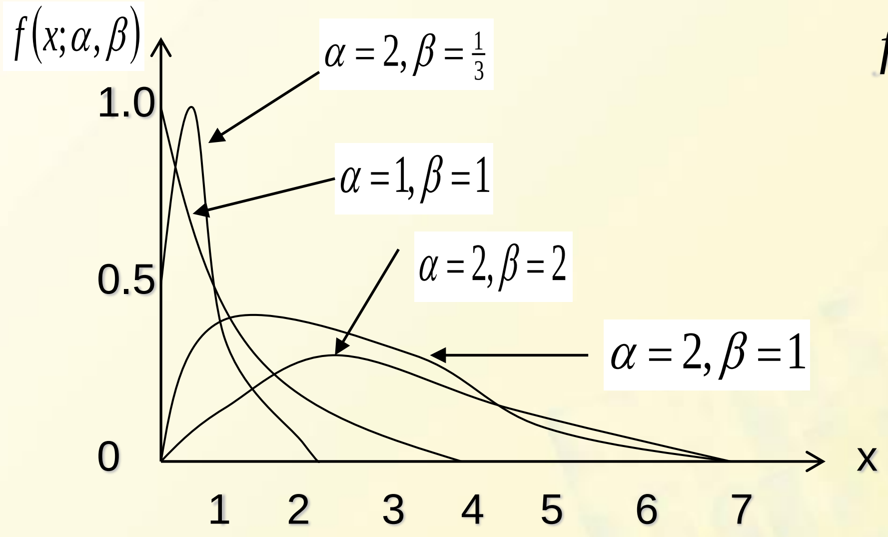
Satisfying the two Basic Properties of a pdf:

$$1: f(x;a)\geq 0$$

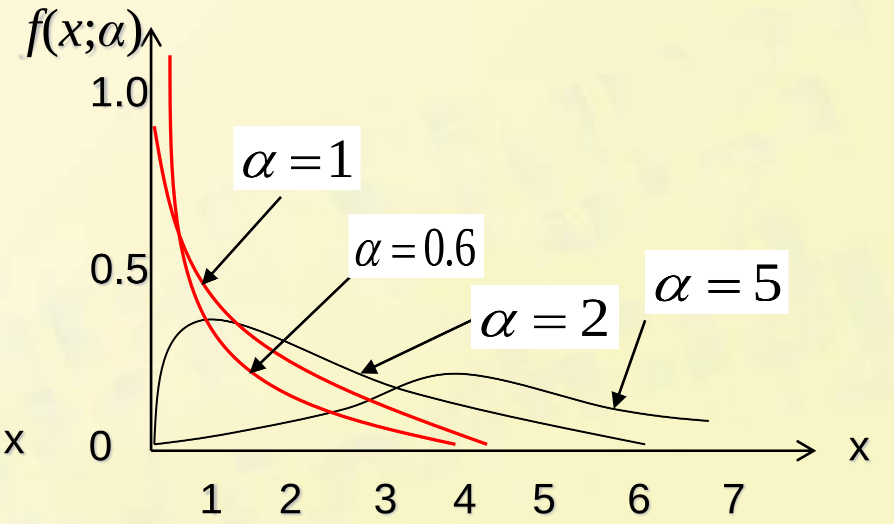
$$2: \int_0^{\infty} f(x;a)dx=\frac{\int_0^{\infty}x^{\alpha-1}e^{-x}dx}{\Gamma(\alpha)}=\frac{\Gamma(\alpha)}{\Gamma(\alpha)}=1$$

4.4 The Gamma Distribution and Its Relatives

- Illustrations of the Gamma pdfs



(a) Gamma density curves



(b) Standard gamma density curves

4.4 The Gamma Distribution and Its Relatives

- The cdf of a standard gamma distribution

$$F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy \qquad x > 0$$

Incomplete gamma function (or without the denominator $\Gamma(\alpha)$ sometimes)

Refer to Appendix Table A.4

4.4 The Gamma Distribution and Its Relatives

- **Example 4.22**

  Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with $\alpha=2$ sec. Then

  $P(3 \le X \le 5)=?$

  $P(X>4)=?$

4.4 The Gamma Distribution and Its Relatives

Solution:

$$P(3 \le X \le 5) = F(5;2) - F(3;2)$$

$$= 0.960 - 0.801 = 0.159$$

$$P(X>4) = 1 - P(X \le 4) = 1 - F(4;2) = 1 - 0.908 = 0.092$$

4.4 The Gamma Distribution and Its Relatives

- Proposition

Let X have a gamma distribution with parameters α and β. Then for any $x > 0$, the **cdf** of X is given by

$$P\left(X \le x\right) = F\left(x;\alpha,\beta\right) = F\left(\frac{x}{\beta};\alpha\right)$$

where F(•; α) is the incomplete gamma function.

4.4 The Gamma Distribution and Its Relatives

- **Example 4.24**

Suppose the survival time X in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha=8$ and $\beta=15$, then

(1) The probability that a mouse survives between 60 and 120 weeks is ?

(2) The probability that a mouse survives at least 30 weeks is

4.4 The Gamma Distribution and Its Relatives

- **The Exponential Distribution**

X is said to have an exponential distribution with parameter λ ($\lambda>0$) if the pdf of X is

$$f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

Just a special case of the general gamma pdf

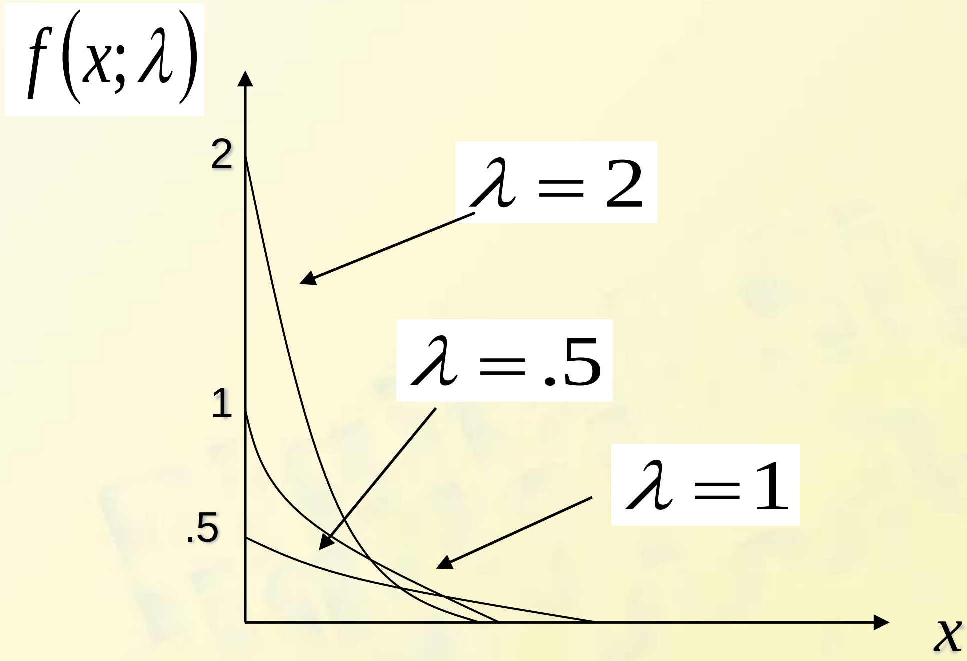
$\alpha=1$ and $\beta = 1/\lambda$

therefore, we have

$E(X) = \alpha\beta = 1/\lambda$; $V(X) = \alpha\beta^2 = 1/\lambda^2$

4.4 The Gamma Distribution and Its Relatives

- Illustrations of the Exponential pdfs



4.4 The Gamma Distribution and Its Relatives

- **The cdf of Exponential Distribution**

  Unlike the general gamma pdf, the exponential pdf can be easily integrated.

$$F(x;\lambda)=\begin{cases} 0 & x<0 \\ 1-e^{-\lambda x} & x\geq 0 \end{cases}$$

4.4 The Gamma Distribution and Its Relatives

- **Example**

Suppose the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to inquiry) has an exponential distribution with expected response time equal to 5 sec. then E(X) =1/ λ =5, so λ=0.2. the probability that the response tine is at most 10 sec is

$$P(X \le 10) = F(10; 0.2) = 1 - e^{-(0.2)(10)} = 0.865$$

The probability that response time is between 5 and 10 sec is

$$P(5 \le X \le 10) = F(10; 0.2) - F(5; 0.2) = 0.233$$

- **The Chi-Squared Distribution**

  Let v be a positive integer. Then a random variable X is said to have a chi-squared distribution with parameter v if the **pdf of X** is the gamma density with $\alpha = v/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x, v) = \begin{cases} \dfrac{1}{2^{v/2}\Gamma(v/2)} x^{(v/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

  The parameter v is called the number of degrees of freedom of X. The symbol χ^2 is often used in place of "chi-squared."