陈建琳 2022153527. A+

9. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a random variable—X for the right tire and Y for the left tire, with joint pdf

$$f(x, y) = \begin{cases} K(x^2 + y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the value of K?
b. What is the probability that both tires are underfilled?
c. What is the probability that the difference in air pressure between the two tires is at most 2 psi?
d. Determine the (marginal) distribution of air pressure in the right tire alone.
e. Are X and Y independent rv's?

a)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$
$$1 = \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy$$
$$1 = K\int_{20}^{30}\int_{20}^{30} x^2\,dx\,dy + K\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$$
$$1 = 10K\int_{20}^{30} x^2\,dx + 10K\int_{20}^{30} y^2\,dy$$
$$1 = 20K\left(\frac{19000}{3}\right)$$
$$20K = \frac{3}{19000}$$
$$K = \frac{3}{38000}$$

b. $P(X<26 \text{ and } Y<26) = \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\,dx\,dy$
$$= K\int_{20}^{26}\left[x^2y + \frac{y^3}{3}\right]_{20}^{26} dx$$
$$= K\int_{20}^{26} (6x^2 + 3192)\,dx$$
$$= K(38304)$$
$$= 0.3024$$

c. $P(|X-Y|\le 2) = 1-\int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx$

$= 0.3593$

d $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad (20 \le x \le 30)$

$= \int_{20}^{30} K(x^2+y^2)\,dy$

$= 10Kx^2 + K\frac{y^3}{3}\Big|_{20}^{30}$

$= 10kx^2 + 0.05$

e. $f_Y(y)$ can be obtained by substituting y for x in (d).

$\because f(x,y) \neq f_X(x)\cdot f_Y(y)$

$\therefore X$ and Y are not independent.

12. Two components of a minicomputer have the following joint pdf for their useful lifetimes X and Y:

$$f(x, y) = \begin{cases} xe^{-x(1+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the probability that the lifetime X of the first component exceeds 3?

b. What are the marginal pdf's of X and Y? Are the two lifetimes independent? Explain.

c. What is the probability that the lifetime of at least one component exceeds 3?

a. $P(X>3) = \int_3^{\infty}\int_0^{\infty} xe^{-x(1+y)}\,dy\,dx$
$= \int_0^{\infty} e^{-x}\,dx$
$= 0.050$

b. $f_X(x) = \int_0^{\infty} xe^{-x(1+y)}\,dy = e^{-x} \quad (x \geq 0)$

$f_Y(y) = \int_3^{\infty} xe^{-x(1+y)}\,dx = \frac{1}{(1+y)^2} \quad (y \geq 0)$

$\because f(x,y) \neq f_X(x) \cdot f_Y(y)$

$\therefore$ they are not independent.

c. $P(\text{at least one exceeds three}) = P(X>3 \text{ or } Y>3)$
$= 1 - P(X \leq 3 \text{ and } Y \leq 3)$
$= 1 - \int_0^3\int_0^3 xe^{-x(1-y)}\,dy\,dx$
$= 1 - \int_0^3\int_0^3 xe^{-x} e^{-xy}\,dy$
$= 1 - \int_0^3 e^{-x}(1-e^{-3x})\,dx$
$= e^{-3} + 0.25 - 0.25e^{-12}$
$= 0.300$

Refer to Exercise 1 and answer the following questions:

a. Given that $X = 1$, determine the conditional pmf of Y—i.e., $p_{Y|X}(0 \mid 1)$, $p_{Y|X}(1 \mid 1)$, and $p_{Y|X}(2 \mid 1)$.

b. Given that two hoses are in use at the self-service island, what is the conditional pmf of the number of hoses in use on the full-service island?

c. Use the result of part (b) to calculate the conditional probability $P(Y \le 1 \mid X = 2)$.

d. Given that two hoses are in use at the full-service island, what is the conditional pmf of the number in use at the self-service island?

a. Since $P_X(1) = 0.34$,

$P_{Y|X}(0|1) = \frac{.08}{0.34} = 0.2353$

$P_{Y|X}(1|1) = \frac{0.20}{0.34} = 0.5882$

$P_{Y|X}(2|1) = \frac{0.06}{0.34} = 0.1765$

b. Since $P_X(2) = 0.50$,

$P_{Y|X}(0|2) = 0.12$

$P_{Y|X}(1|2) = 0.28$

$P_{Y|X}(2|2) = 0.60$

c. $P(Y \le 1 | X=2) = P_{Y|X}(0|2) + P_{Y|X}(1|2)$

$= 0.12 + 0.28$

$= 0.40$

d. Since $P_Y(2) = 0.38$,

$P_{X|Y}(0|2) = 0.0526$

$P_{X|Y}(1|2) = 0.1579$

$P_{X|Y}(2|2) = 0.7895$

12. The joint pdf of pressures for right and left front tires is given in Exercise 9.

a. Determine the conditional pdf of Y given that $X = x$ and the conditional pdf of X given that $Y = y$.

b. If the pressure in the right tire is found to be 22 psi, what is the probability that the left tire has a pressure of at least 25 psi? Compare this to $P(Y \geq 25)$.

c. If the pressure in the right tire is found to be 22 psi, what is the expected pressure in the left tire, and what is the standard deviation of pressure in this tire?

a. $$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{k(x^2+y^2)}{10kx^2+0.05} \quad (20 \leq y \leq 30)$$

$$f_{X|Y}(x|y) = \frac{k(x^2+y^2)}{10ky^2+0.05} \quad (20 \leq x \leq 30)$$

Refer to Ex 9, $k = \frac{3}{380000}$.

b. $$P(Y \geq 25 | X = 22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy$$
$$= \int_{25}^{30} \frac{k(22^2+y^2)}{10k(22^2)+0.5}\,dy$$
$$= 0.556$$

$$P(Y \geq 25) = \int_{25}^{30} f_Y(y)\,dy$$
$$= \int_{25}^{30} (10ky^2+0.05)\,dy$$
$$= 0.549.$$

c. $$E(Y|X=22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|22)\,dy$$
$$= \int_{20}^{30} y \frac{k(22^2+y^2)}{10k(22^2)+0.5}\,dy$$
$$= 25.372912$$

$$E(Y^2|X=22) = \int_{20}^{30} y^2 \cdot \frac{k(22^2+y^2)}{10k(22^2)+0.05}\,dy = 652.028640.$$

$$V(Y|X=22) = E(Y^2|X=22) - (E(Y|X=22))^2 = 8.243976.$$

$$\sigma = \sqrt{V(Y|X=22)} = 2.87$$

24. ...dividuals, including A and B, take seats around a circular table in a completely random fashion. Suppose the seats are numbered 1, . . . , 6. Let X = A's seat number and Y = B's seat number. If A sends a written message around the table to B in the direction in which they are closest, how many individuals (including A and B) would you expect to handle the message?

y

	$h(x,y)$	1	2	3	4	5	6
	1	—	2	3	4	3	2
	2	2	—	2	3	4	3
x	3	3	2	—	2	3	4
	4	4	3	2	—	2	3
	5	3	4	3	2	—	2
	6	2	3	4	3	2	—

$p(x,y) = \frac{1}{30}$

$$E(h(X,Y)) = \sum_x \sum_y h(x,y) \cdot p(x,y)$$
$$= \frac{84}{30}$$
$$= 2.80.$$

26. Consider a small ferry that can accommodate cars and buses. The toll for cars is \$3, and the toll for buses is \$10. Let X and Y denote the number of cars and buses, respectively, carried on a single trip. Suppose the joint distribution of X and Y is as given in the table of Exercise 7. Compute the expected revenue from a single trip.

$$\text{Revenue} = 3X + 10Y$$

$$E(\text{Revenue}) = E(3X + 10Y)$$

$$= \sum_{x=0}^{5} \sum_{y=0}^{2} (3x + 10y) \cdot p(x, y)$$

$$= 0 \cdot p(0,0) + \dots + 35 \cdot p(5,2)$$

$$= 15.40$$

33. Use the result of Exercise 28 to show that when X and Y are independent, $\text{Cov}(X, Y) = \text{Corr}(X, Y) = 0$.

28. Show that if X and Y are independent rv's, then $E(XY) = E(X) \cdot E(Y)$. Then apply this in Exercise 25. [*Hint*: Consider the continuous case with $f(x, y) = f_X(x) \cdot f_Y(y)$.]

$E(XY) = E(X) \cdot E(Y)$

$\text{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$= E(X) \cdot E(Y) - E(X) \cdot E(Y)$

$= 0.$

$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = 0.$

35. **a.** Use the rules of expected value to show that Cov(aX + b, cY + d) = ac Cov(X, Y).

b. Use part (a) along with the rules of variance and standard deviation to show that Corr(aX + b, cY + d) = Corr(X, Y) when a and c have the same sign.

c. What happens if a and c have opposite signs?

a. $\mathrm{Cov}(aX+b, cY+d)$

$= E((aX+b)(cY+d)) - E(aX+b)\cdot E(cY+d)$

$= E(acXY + adX + bcY + bd) - (aE(X)+b)(cE(Y)+d)$

$= acE(XY) + adE(X) + bcE(Y) + bd - [acE(X)E(Y) + adE(X) + bcE(Y) + bd]$

$= acE(XY) - acE(X)E(Y)$

$= ac[E(XY) - E(X)E(Y)]$

$= ac\,\mathrm{Cov}(X,Y)$

b. $\mathrm{Corr}(aX+b, cY+d) = \dfrac{\mathrm{Cov}(aX+b, cY+d)}{SD(aX+b)\,SD(cY+d)}$

$= \dfrac{ac\,\mathrm{Cov}(X,Y)}{|a|\cdot|c|\cdot SD(X)\cdot SD(Y)}$

$= \dfrac{ac}{|a\cdot c|}\mathrm{Corr}(X,Y)$

$\because$ when a and c have the same sign, $ac = |ac|$.

$\therefore \mathrm{Corr}(aX+b, cY+d) = \mathrm{Corr}(X,Y)$

c. $\because$ When a and c have different sign, $|ac| = -ac$.

$\therefore \mathrm{Corr}(aX+b, cY+d) = -\mathrm{Corr}(X,Y)$