A+

6.2 20, 21, 29, 32

1) $\Sigma x_i = 219.8$; $n = 27$

a) Point estimate of the mean value is sample mean, which is the sum of all values divided by the number of values: $\bar{x} = \frac{\Sigma x_i}{n} = \frac{219.8}{27} \approx 8.1407$

b) The point estimate that seperates weakest 50% from strongest 50% is the median. Sorted values: 5.9, 6.3, 6.3, 6.5, 6.8, 6.8, 7.0, 7.0, 7.2, 7.3, 7.4, 7.6, 7.7, 7.7, 7.8, 7.8, 7.9, 8.1, 8.2, 8.7, 9.0, 9.7, 9.7, 10.7, 11.3, 11.6, 11.8

Since there are 27 values in the set, the median is the middle, 14 th, $M = 7.7$

c) $\Sigma x_i^2 = 1860.94$; $\Sigma x_i = 219.8$, $n = 27$

Point estimate of the population standard deviation is the sample standard deviation

$$s = \sqrt{\frac{\Sigma x^2 - (\Sigma x)^2/n}{n-1}} = \sqrt{\frac{1860.94 - (219.8)^2/27}{27-1}} = 1.6595$$

d) The point estimate of the proportion is the sample proportion. The sample proportion is the number of successes (values exceeding 10) divided by the sample size $n = 27$; $\hat{p} = \frac{4}{27} \approx 0.1481 = 14.81\%$

e) Point estimate of population coefficient of variation $\frac{\sigma}{\mu}$ is the sample coefficient of variation $\frac{s}{\bar{x}}$: $CV = \frac{s}{\bar{x}} = \frac{1.6595}{8.1407} = 0.2039$

8)

a) Point estimate of the proportion of all such components that are not defective is $\hat{p} = \frac{68}{80} = 0.85$

b) Both components have to work : $P(\text{system works}) = 0.85^2 = 0.723$

9) Proposition: For a random variable X with Poisson Distribution with parameter $\mu > 0$, $E(X) = V(X) = \mu$

The unbiased estimator of μ is $\bar{X}$. Proof follows

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} \cdot n \cdot E(X_i) = \mu$$

For $n = 150$, the estimate $\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n) = \frac{1}{150}(0.18 + 1.37 + \dots + 7.1)$

$= \frac{317}{150} = 2.11$

Variance of the estimator where all X_i have the same distribution and are independent. $V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) = \frac{1}{n^2} \cdot n \cdot V(X_i) = \frac{\mu}{n}$

standard dev of the estimator is $\sigma_{\bar{x}} = \sqrt{\frac{\mu}{n}}$

estimated standard error is $\sqrt{\frac{\hat{\mu}}{n}} = \frac{\sqrt{2.11}}{\sqrt{150}} = 0.119$

19) Expected value of random variable X_1 (or any other) is

$\mu = E(X) = \int_{-1}^{1} x \cdot 0.5(1+\theta x)\,dx = 0.5 \frac{x^2}{2}\Big|_{-1}^{1} + 0.5 \cdot \theta \cdot \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3}\theta$

If the expected values of estimator $\hat{\theta}$ is θ, then the estimulator is unbiased.

The expected value is $E(\hat{\theta}) = E(3\bar{X}) = 3E(\bar{X}) = 3 \cdot E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{3}{n}\sum_{i=1}^{n} E(X_i) = \frac{3}{n} \cdot n \cdot E(X_1) = 3\mu = 3 \cdot \frac{1}{3} \cdot \theta = \theta$

20) In order to obtain maximum likelihood estimator, find p which maximize pmf. To do that, look at natural logarithm of pmf. By finding max of $\ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right]$ one also finds the maximum of $p^x(1-p)^{n-x}$ because ln won't change max value. To find max of $\ln(x;n,p)$, first take derivative and set it equal to be 0, and solve for p.

$\frac{d}{dp}\left(\ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right]\right) = \frac{d}{dp}\left(\ln\binom{n}{x} + x\ln p + (n-x)\ln(1-p)\right)$

$= 0 + x \cdot \frac{1}{p} + (n-x) \cdot \frac{1}{1-p} \cdot (-1) = \frac{x}{p} - \frac{n-x}{1-p}$.

$\frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow \frac{x}{\hat{p}} - \frac{n-x}{1-\hat{p}} = 0 \Rightarrow \frac{x}{\hat{p}} = \frac{n-x}{1-\hat{p}} \Rightarrow \frac{1-\hat{p}}{\hat{p}} = \frac{n-x}{x} \Rightarrow \frac{1}{\hat{p}} - 1 = \frac{n}{x} - 1$

estimator is $\hat{p} = \frac{x}{n} = \frac{3}{20} = 0.15$

b) The estimator is unbiased if the expected value of the estimator is p. The following holds: $E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n} \cdot E(X) = \frac{1}{n} \cdot n \cdot p = p$

Proposition: For a binomial random variable X with parameters n, p, and $q = 1-p$, $E(X) = n \cdot p$; $V(X) = np(1-p) = npq$; $\sigma_X = \sqrt{npq}$

Since $E(\hat{p}) = p$, the estimator is unbiased.

c) Invariance Principle: Let $\hat{\theta}_i, i = 1, 2, \dots, n$ be maximum likelihood estimates of parameters $\theta_i, i = 1, 2, \dots, n$. The mle of any function of parameters θ_i is the function of the mle's $\hat{\theta}_i$.

The mle of function $h(p) = (1-p)^5$ is $h(\hat{p}) = (1-\hat{p})^5 = (1-0.15)^5 = 0.4437$

21)

a) Sample moment of first order: $\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$

population " " " ": $E(X) = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right)$.

Sample " " 2^{nd} ": $\frac{1}{n}(X_1^2 + X_2^2 + \dots + X_n^2)$

population " " 2^{nd} ": $E(X^2) = V(X) + [E(X)]^2$

$= \beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\} - \left\{\beta \cdot \Gamma\left(1+\frac{1}{\alpha}\right)\right\}^2 = \beta^2 \cdot \Gamma\left(1+\frac{2}{\alpha}\right)$

The 2^{nd} equation in the system of equations from which the moment estimators are obtained is $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E(X^2)$

Therefor, system of equations which needs to be solved for $\hat{\alpha}$ and $\hat{\beta}$ is

$\bar{X} = \hat{B}\cdot\Gamma(1+\frac{1}{\hat{\alpha}})$, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \hat{\beta}^2\Gamma(1+\frac{2}{\hat{\alpha}})$.

$\hat{\beta} = \frac{\bar{X}}{\Gamma(1+\frac{1}{\hat{\alpha}})}$, $\hat{B}$ can be computed from 1st equation.

From 1st equation, squaring both sides $\therefore$ $\bar{X}^2 = \hat{\beta}^2\Gamma^2(1+\frac{1}{\hat{\alpha}})$ or $\hat{\beta}^2 = \frac{\bar{X}^2}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$

$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\bar{X}^2}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}\cdot\Gamma(1+\frac{2}{\hat{\alpha}})$ or $\frac{1}{n}\frac{\sum_{i=1}^{n} X_i^2}{\bar{X}^2} = \frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$

b) $n=20$, $\bar{x}=28$, $\sum x_i^2 = 16,500$

$\frac{1}{20}\cdot\left(\frac{16500}{28^2}\right) = 1.05$, therefor $\frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma(1+\frac{1}{\hat{\alpha}})} = 1.05$

$\frac{[\Gamma(1.2)]^2}{\Gamma(1.4)} = 0.95$ or $\frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)} = \frac{1}{0.95}$ and $\frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)} = 1.05$

which means that $\frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})} = 1.05 = \frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)}$

$\frac{2}{\hat{\alpha}} = 0.4 \Rightarrow \hat{\alpha} = 5$

estimator $\hat{\beta} = \frac{\bar{X}}{\Gamma(1+\frac{1}{\hat{\alpha}})} = \frac{28}{\Gamma(1.2)}$

29)

a) $\ln f(x_1, x_2, x_3, \dots, x_n; \lambda, \theta) = \ln\left[\lambda^n e^{-\lambda\sum_{i=1}^{n}(x_i-\theta)}\right]$

$= n\ln\lambda - \lambda\sum_{i=1}^{n}(x_i-\theta)$

$\frac{d}{d\theta} f(x_1, x_2, \dots, x_n; \lambda, \theta) = \frac{d}{d\lambda}\left[n\ln\lambda - \lambda\sum_{i=1}^{n}(x_i-\theta)\right] = n\frac{1}{\lambda} - \sum_{i=1}^{n}(x_i-\theta)$

$\Rightarrow \frac{1}{\hat{\lambda}} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\theta}) \Rightarrow \hat{\lambda} = \frac{n}{\sum_{x=1}^{n}(x_i-\hat{\theta})}$

b) $\hat{\theta} = 0.64$, $n=10$

$\sum_{i=1}^{n} x_i = 3.11 + 0.64 + \dots + 1.3 = 55.8$

$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}(x_i-\hat{\theta})} = \frac{n}{\sum_{i=1}^{n} x_i - 10\hat{\theta}} = \frac{10}{55.8-6.4} = 0.202$

32) The cdf of a random variable Y can be computed as follows

$F_Y(y) = P(Y \le y) = P(\max(X_i \le y))$ → All X_i are smaller

$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$

$= P(X_1 \le y)\cdot P(X_2 \le y)\cdots P(X_n \le y)$ (independence)

$= \left(\frac{y}{\theta}\right)^n$, $0 \le y \le \theta$.

Having cdf, it's easy to obtain pdf as derivative of cdf

$f_Y(y) = F'_Y(y) = \frac{ny^{n-1}}{\theta^n}$, $0 \le y \le \theta$

It is zero, otherwise.

b) If $E(Y) = \theta$, than estimator is unbiased, however

$E(Y) = \int_0^\theta y\cdot\frac{ny^{n-1}}{\theta^n}dy = \frac{n}{\theta^n}\,\cancel{\frac{y^{n+1}}{n+1}} = \frac{y^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta \ne \theta$

which means estimator is not unbiased. However, estimator $\hat{Y} = \frac{n+1}{n}$ is unbiased because $E(\hat{Y}) = E(\frac{n+1}{n}) = \frac{n+1}{n} E(Y) = \frac{n+1}{n} \frac{n}{n+1} \theta = \theta$, hence $\hat{Y}$ is unbiased.