2.5 Independence

- **Definition**

Two events A and B are independence if P(A | B)=P(A) and are dependent otherwise.

Note:

1. Since $P(A\cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$

   if $P(A \mid B)=P(A)$, then we have

   $P(A)\,P(B) = P(B|A)P(A) \rightarrow P(B|A) = P(B)$ (if $P(A) > 0$)

2. If A and B are independence, so are the following pairs of events:

   a. A' and B b. A and B' c. A' and B'

2.5 Independence

- **Example**

Consider tossing a fair six-sided die once and define events A={2,4,6}, B={1,2,3}, and C={1,2,3,4}. Events A and B are dependent,? Events A and C are independent? Why?

Solution:

We then have P(A)=1/2, P(A | B)=1/3 and P(A | C)=1/2.

That is, events A and B are dependent, whereas events A and C are independent.

2.5 Independence

- **Example 2.33**

Let A and B be any two mutually exclusive events with $P(A)>0$. For example, for a randomly chosen automobile, let A={the car has four cylinders} and B={the car has six cylinders}.

Since the events are mutually exclusive, if B occurs, then A cannot possibly have occurred, so $P(A|B) = 0 \neq P(A)$. The message here is that if ***two events are mutually exclusive, they cannot be independent.*** **(Here: P(A) & P(B) are not zero!)**

2.5 Independence

- **Proposition #1**

A and B are independent *if and only if*

$$P(A \cap B) = P(A)\,P(B)$$

Proof:

1. If A and B are independent, then

   $P(A|B) = P(A)$, and thus

   $P(A \cap B) = P(A|B)P(B) = P(A)\,P(B)$

2. If $P(A \cap B) = P(A)\,P(B)$, then

   $P(A \cap B) = P(A|B)P(B) = P(A)\,P(B)$

   $P(A|B) = P(A)$ $(P(B)>0)$, A and B are independent

2.5 Independence

Example

Testing for Independence In two tosses of a single fair coin show that the events "A head on the first toss" and "A head on the second toss" are independent.

2.5 Independence

Solution:

Consider the sample space of equally likely outcomes for the tossing of a fair coin twice,

$$S = (\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT})$$

and the two events,

A = A head on the first toss = (HH, HT)

B = A head on the second toss = (HH, TH)

Then

$$P(A) = \tfrac{2}{4} = \tfrac{1}{2} \qquad P(B) = \tfrac{2}{4} = \tfrac{1}{2} \qquad P(A \cap B) = \tfrac{1}{4}$$

Thus,

$$P(A \cap B) = \tfrac{1}{4} = \tfrac{1}{2} \cdot \tfrac{1}{2} = P(A)P(B)$$

and the two events are independent. (The theory agrees with our intuition—a coin has no memory.)

2.5 Independence

- **Example 2.34**

It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service. If someone purchases both a washer and a dryer made by this company, what is the probability that both machines need warranty service?

2.5 Independence

Solution:

Let A be the event that washer needs service while under warranty, and B be defined analogously for the dryer, then

$P(A) = 0.3$, $P(B) = 0.1$.

Assuming that the two machines function independently of one another, the desired probability is

$$P(A \cap B) = P(A)\,P(B) = 0.3 \times 0.1 = 0.03.$$

The probability that neither machine needs service is

$$P(A' \cap B') = P(A')\,P(B') = (1-0.3)\,(1-0.1) = 0.63$$

2.5 Independence

Independence of More Than Two Events

The notion of independence of two events can be extended to collections of more than two events.

- **Mutually Independent**

Events A_1, A_2, …A_n are mutually independent if for every k (k=2,3…n) and every subset of indices $i_1, i_2 \ldots i_k$

$$P(A_{i1} \cap A_{i2} \ldots \cap Ai_k) = P(A_{i1})P(A_{i2})\ldots P(A_{ik})$$

2.5 Independence

Example 2.35

Consider first the system illustrated in Figure 2.14(a).

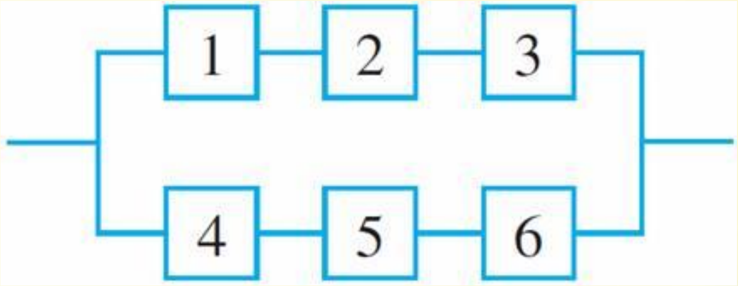

Figure 2.14(a)

There are two subsystems connected in parallel, each one containing three cells. In order for the system to function, at least one of the two parallel subsystems must work.

Within each subsystem, the three cells are connected in series, so a subsystem will work only if all cells in the subsystem work.

- Let A_i denote the event that the lifetime of cell i exceeds a particular lifetime value t_0 (i = 1, 2, . . . , 6).

- We assume that the $A_i's$ are independent events (whether any particular cell lasts more than t_0 hours has no bearing on whether or not any other cell does) and that $P(A_i) = .9$ for every i since the cells are identical.

- What is the probability that the system lifetime exceeds t_0 ?

Solution:

P (system lifetime exceeds t_0)

$= P[(A1 \cap A2 \cap A3) \cup (A4 \cap A5 \cap A6)]$

$= P(A1 \cap A2 \cap A3) + P(A4 \cap A5 \cap A6)$
$\quad - P[(A1 \cap A2 \cap A3) \cap (A4 \cap A5 \cap A6)]$

$= (.9)(.9)(.9) + (.9)(.9)(.9) - (.9)(.9)(.9)(.9)(.9)(.9)$

$= .927$

Alternatively,

P (system lifetime exceeds t_0)

$= 1 - P$ (both subsystem lives are $\leq t_0$)

$= 1 - [P$ (subsystem life is $\leq t_0)]^2$

$= 1 - [1 - P$ (subsystem life is $> t_0)]^2$

$= 1 - [1 - (.9)^3]^2$

$= .927$

Next consider the total-cross-tied system shown in Figure 2.14(b), obtained from the series-parallel array by connecting ties across each column of junctions.

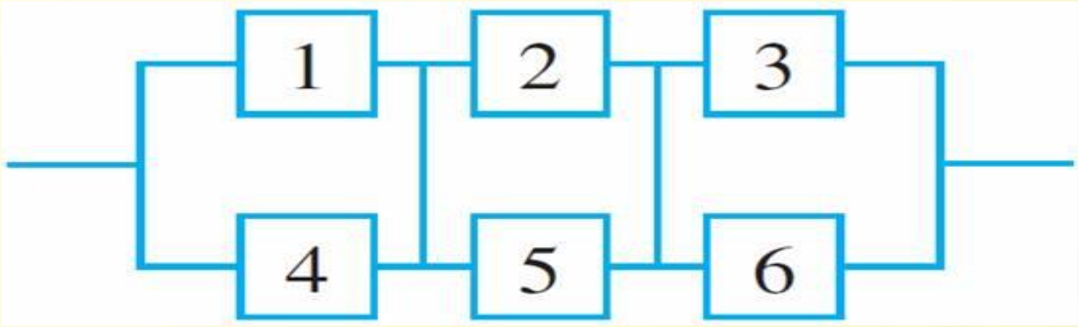

Figure 2.14(b)

Now the system fails as soon as an entire column fails, and system lifetime exceeds t_0 only if the life of every column does so.

For this configuration,

P (system lifetime is at least t_0)

$= [P$ (column lifetime exceeds t_0)$]^3$

$= [1 - P$ (column lifetime $\leq t_0$)$]^3$

$= [1 - P$ (both cells in a column have lifetime $\leq t_0$)$]^3$

$= [1 - (1 - .9)2]^3$

$= .970$