HW 6.1

A+

1) $\sum x_i = 219.8$; $n=27$

a) point estimate of the mean value is sample mean, which is the sum of all values divided by the number of values: $\bar{x} : \frac{\sum x_i}{n} = \frac{219.8}{27} \approx 8.1407$

b) The point estimate that separates weakest 50% from strongest 50% is the median. Sorted values: 5.9, 6.3, 6.3, 6.5, 6.8, 6.8, 7.0, 7.0, 7.2, 7.3, 7.4, 7.6, 7.7, 7.7, 7.8, 7.8, 7.9, 8.1, 8.2, 8.7, 9.0, 9.7, 9.7, 10.7, 11.3, 11.6, 11.8

Since there are 27 values in the set, the median is the middle 14^{th}, $M=7.7$

c) $\sum x_i^2 = 1860.94$ $\sum x_i = 219.8$ $n=27$
The point estimate of the population S.D. is the sample S.D.

$$S = \sqrt{\frac{\sum x^2 - (\sum x)^2/n}{n-1}} = 1.6595$$

d) The point estimate of the proportion is the sample proportion. The sample proportion is the number of successes (values exceeding 10) divided by sample size $n=27$; $\hat{p} = \frac{4}{27} \approx 0.1481 = 14.81\%$

e) Point estimate of population coefficient of variation $\frac{\sigma}{\mu}$ is the sample coefficient of variation $\frac{s}{\bar{x}}$: $CV = \frac{s}{\bar{x}} = \frac{1.6595}{8.1407} = 0.2039$

8) a) Point estimate of the proportion of all such combounds that are not defective is $\hat{p}$ $\frac{68}{80} = 0.85$

b) Both components have to work: $P(\text{system works}) = 0.85^2 = 0.723$

9) Proposition: For a random variable X with Poisson Distribution with parameter $\mu > 0$, $E(X) = V(X) = \mu$

The unbiased estimator of μ is $\bar{X}$.

$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\cdot n\cdot E(X_i) = \mu$

For $n = 150$, the estimate $\bar{x} = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$

$= \frac{1}{150}(0.13 + 1.37 + \dots + 7.1)$

$= \frac{317}{150}$

$= 2.11$

Variance of the estimator where all X_i have the same distribution and are independent. $V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{1}{n^2}\sum_{i=1}^{n} (X_i)$

$= \frac{1}{n^2}\cdot n \cdot V(X)$

$= \frac{\mu}{n}$

S.D. of the estimator is $\sigma_{\bar{X}} = \sqrt{\frac{\mu}{n}}$

estimated standard error is $\sqrt{\frac{\hat{\mu}}{n}} = \frac{\sqrt{2.11}}{\sqrt{150}} = 0.119$

13) Expected value of random variable X_i (or any other) is

$\mu = E(X) = \int_{-1}^{1} x \cdot 0.5(1 + \theta x)\, dx = 0.5\, \frac{x^2}{2}\Big|_{-1} + 0.5\,\theta\left(\frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{3}\theta$

If the expected values of estimator $\hat{\theta}$ is θ, then the estimulator is unbaised

The expected value is $E(\hat{\theta}) = E(3\bar{X}) = 3\cdot E(\bar{X}) = 3\cdot E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{3}{n}\sum_{i=1}^{n} E(X_i) = \frac{3}{n}\cdot n\cdot E(X_i) = 3\mu = 3\cdot\frac{1}{3}\theta = \theta$

20) a) In order to obtain maximum likelihood estimator, find p which maximize pmf. To do that, look at natural logarithm of pmf. By finding max of $\ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right]$ one also find the maximum of $p^x(1-p)^{n-x}$ because ln won't change max value. To find max of $\ln(x; n, p)$, first take derivative and set it equal to be 0, and solve for p

$$\frac{d}{dp}\left(\ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right]\right) = \frac{d}{dp}\left(\ln\binom{n}{x} + x\ln p + (n-x)\ln(1-p)\right)$$

$$= 0 + x\cdot\frac{1}{p} + (n-x)\cdot\frac{1}{1-p}\cdot(-1)$$

$$= \frac{x}{p} - \frac{n-x}{1-p}$$

$$\frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow \frac{x}{\hat p} - \frac{n-x}{1-\hat p} = 0 \Rightarrow \frac{x}{\hat p} = \frac{n-x}{1-\hat p} \Rightarrow \frac{1-\hat p}{\hat p} = \frac{n-x}{x} \Rightarrow \frac{1}{\hat p} - 1 = \frac{n}{x} - 1$$

estimator is $\hat p = \frac{x}{n} = \frac{3}{20} = 0.15$

b) The estimator is unbiased if the expected value of the estimator is p. The following holds: $E(\hat p) = E\left(\frac{x}{n}\right) = \frac{1}{n}\cdot E(x) = \frac{1}{n}\cdot n\cdot p = p$

For a binomial random variable x with parameters n, p, and $q = 1-p$,
$E(x) = n\cdot p$; $V(x) = np(1-p) = npq$; $\sigma_n = \sqrt{npq}$
Since $E(\hat p) = p$, the estimator is unbiased

c) Let $\hat\theta_i$, $i = 1, 2, \cdots, n$ be max likelihood estimates of parameters θ_i, $i = 1, 2, \cdots, n$. The mle of any function of parameters θ_i is the function of mle's $\hat\theta_i$

The mle of function $h(p) = (1-p)^5$ is $h(\hat p) = (1-\hat p)^5 = (1-0.15)^5$
$= 0.4437$

21) a) Sample moment of first order: $\bar{x} = \frac{1}{n}(x_i + x_2 + \dots + x_n)$

population " " " " : $E(x) = \beta \cdot \Gamma(1+\frac{1}{\alpha})$

Sample " " 2^{nd} " : $\frac{1}{n}(x_1^2 + x_2^2 + \dots + x_n^2)$

population " " 2^{nd} " : $E(x^2) = V(x) + [E(x)]^2$

$= \beta^2 \{ \Gamma(1+\frac{2}{\alpha}) - [\Gamma(1+\frac{1}{\alpha})]^2 \} - \{\beta . \Gamma(1+\frac{1}{\alpha})\}^2 = \beta^2 . \Gamma(1+\frac{2}{\alpha})$

The 2^{nd} equation in the system of equation from which the moment estimators are obtained is $\frac{1}{n}\sum_{i=1}^{n} x_i^2 = E(x^2)$

There, System of equations which needs to be solved for $\hat{\alpha}$ and $\hat{\beta}$ is $\bar{x} = \hat{\beta} \cdot \Gamma(1+\frac{1}{\hat{\alpha}})$, $\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \hat{\beta}^2 \Gamma(1+\frac{2}{\hat{\alpha}})$

$\hat{\beta} = \frac{\bar{x}}{\Gamma(1+\frac{1}{\hat{\alpha}})}$, $\hat{\beta}$ can be computed from 1^{st} equation

From 1^{st} equation, Squaring both sides : $\bar{x}^2 = \hat{\beta}^2 \Gamma^2(1+\frac{1}{\hat{\alpha}})$ or $\hat{\beta}^2 = \frac{\bar{x}^2}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \frac{\bar{x}^2}{\Gamma^2(1+\frac{1}{\hat{\alpha}})} \cdot \Gamma(1+\frac{2}{\hat{\alpha}})$ or $\frac{1}{n}\frac{\sum_{i=1}^{n} x_i^2}{\bar{x}^2} = \frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$

b) $n = 20$, $\bar{x} = 28$, $\sum x_i^2 = 16500$

$\frac{1}{20} \cdot (\frac{16500}{28^2}) = 1.05$, therefore $\frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma(1+\frac{1}{\hat{\alpha}})} = 1.05$

$\frac{[\Gamma(1.2)]^2}{\Gamma(1.4)} = 0.95$ or $\frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)} = \frac{1}{0.95}$ and $\frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)} = 1.05$

which means that $\frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})} = 1.05 = \frac{\Gamma(1+0.4)}{\Gamma^2(1+0.2)}$

$\frac{2}{\hat{\alpha}} = 0.4 \Rightarrow \hat{\alpha} = 5$

estimator $\hat{\beta} = \frac{\bar{x}}{\Gamma(1+\frac{1}{\hat{\alpha}})} = \frac{28}{\Gamma(1.2)}$

29) a) $\ln f(x_1, x_2, x_3, \dots, x_n; \lambda, \theta) = \ln\left[\lambda^n e^{-\lambda \sum_{i=1}^n (x_i - \theta)}\right]$

$= n \ln \lambda - \lambda \sum_{i=1}^n (x_i - \theta)$

$\frac{d}{d\theta} f(x_1, x_2, \dots, x_n; \lambda, \theta) = \frac{d}{d\lambda}\left[n \ln \lambda - \lambda \sum_{i=1}^n (x_i - \theta)\right] = n\frac{1}{\lambda} - \sum_{i=1}^n (x_i - \theta)$

$\Rightarrow \frac{1}{\hat\lambda} = \frac{1}{n}\sum_{i=1}^n (x_i - \hat\theta) \Rightarrow \hat\lambda = \frac{n}{\sum_{x_i=1}^n (x_i - \hat\theta)}$

b) $\hat\theta = 0.64, \quad n = 10$

$\sum_{i=1}^n x_i = 3.11 + 0.64 + \cdots + 1.3 = 55.8$

$\hat\lambda = \frac{n}{\sum_{i=1}^n (x_i - \hat\theta)} = \frac{n}{\sum_{i=1}^n x_i - 10\hat\theta} = \frac{10}{55.8 - 6.4} = 0.202$

32) a) The cdf of a random variable Y can be can be computed as follows

$F_Y(y) = P(Y \le y) = P(\max(X_i \le y)$

$= P(X_1 \le y, X_2 \le y, \dots, X_n \le y)$

$= P(X_1 \le y) \cdot P(X_2 \le y) \cdots P(X_n \le y)$

$= \left(\frac{y}{\theta}\right)^n, \quad 0 \le y \le \theta$

Having cdf it's easy to obtain pdf as derivative of cdf

$f_Y(y) = F'_Y(y) = \frac{n y^{n-1}}{\theta^n}, \quad 0 \le y \le \theta$

It is zero, otherwise

b) If $E(Y) = \theta$, the estimator is unbiased, however

$E(Y) = \int_0^\theta y \cdot \frac{n y^{n-1}}{\theta^n}\, dy = \frac{n}{\theta^n} = \frac{y^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta \ne \theta$

which means estimator is not unbiased. However, estimator $\tilde{y}$

$\tilde{y} = \frac{n+1}{n}$ is unbiased because $E(\tilde{y}) = E\left(\frac{n+1}{n}\right) = \frac{n+1}{n} E(Y)$

$= \frac{n+1}{n} \cdot \frac{n}{n+1} \cdot \theta = \theta$

hence $\hat{Y}$ is unbiased