2.4 Conditional Probability

In this section, we examine how the information "an event *B* has occurred" affects the probability assigned to *A*.

We will use the notation $P(A \mid B)$ to represent the **conditional probability of *A* given that the event *B* has occurred.** *B* is the "conditioning event."

Example 2.24:

Complex components are assembled in a plant that uses two different assembly lines, A and A’. Line A uses older equipment than A’, so it is somewhat slower and less reliable. Suppose on a given day line A has assembled 8 components, of which 2 have been identified as defective (B) and 6 as nondefective (B’), whereas A’ has produced 1 defective and 9 nondefective components. This information is summarized in the accompanying table.

		Condition	
		B	B’
Line	A	2	6
	A’	1	9

Event A: Line A component selected
Event B: the chosen component turns out to be defective

What’s P(A) and P(A|B) ?

2.4 Conditional Probability

The sales manager randomly selects 1 of these 18 components for a demonstration. Prior to the demonstration

$$P(\text{line } A \text{ component selected}) = P(A) = \frac{N(A)}{N} = \frac{8}{18}$$

However, if the chosen component turns out to be defective, then the event B has occurred, so the component must have been 1 of the 3 in the B column of the table. Since these 3 components are equally likely among themselves after B has occurred,

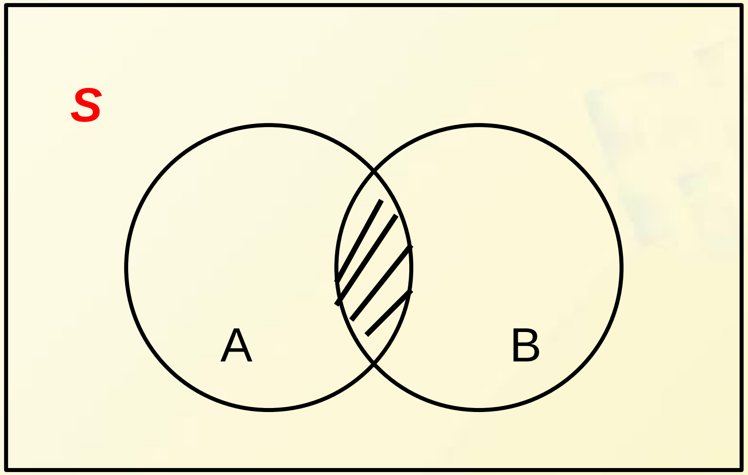
$$P(A \mid B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)}$$

2.4 Conditional Probability

- Definition of Conditional Probability

For any two events A and B with P(B)>0, the conditional probability of A given that B has occurred is defined by

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$



Note:

1. Given that B has occurred, the relevant sample space is **no longer S** but consists of **outcomes in B**;

2. A has occurred **if and only if** one of the outcomes in the **intersection** occurred.

2.4 Conditional Probability

- Example 2.25

Consider randomly selecting a buyer and let A={memory card purchased} and B={battery purchased}. Then P(A)=0.6, P(B)=0.4 and P(both purchased) = $P(A \cap B)$ = 0.3. Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

That is, of all those purchasing an extra battery, 75% purchased an optional memory card. Similarly,

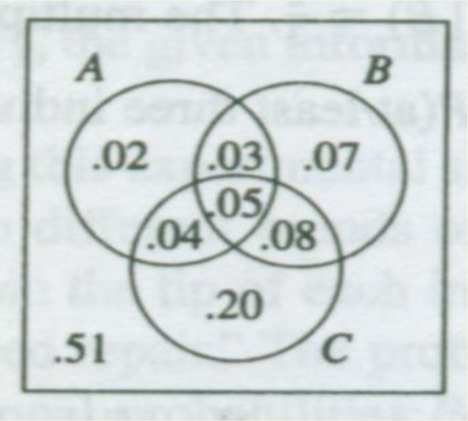
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5 \quad \neq P(A|B)$$

2.4 Conditional Probability

- Example 2.26

A news magazine publishes three columns entitled "Art"(A), "Books"(B), and "Cinema"(C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05



P(A|B) =0.348

P(A|B ∪C) =0.255

P(A|A ∪B∪C) =0.286

P(A ∪B|C) =0.459

2.4 Conditional Probability

- The Multiplication Rule

$$P(A \cap B) = P(A \mid B)\, P(B)$$

This rule is important because it is often the case that $P(A \cap B)$ is desired, whereas that both $P(B)$ and $P(A|B)$ can be specified from the problem description.

Note :

1. $P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$
2. $P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_1 \cap A_2)\, P(A_1 \cap A_2)$

$$= P(A_3 \mid A_1 \cap A_2)\, P(A_2 \mid A_1)\, P(A_1)$$

3. $P(A_1 \cap A_2 \ldots \cap A_n) = P(A_n \mid A_1 \cap A_2 \ldots A_{n-1})\, P(A_{n-1} \mid A_1 \cap A_2 \ldots A_{n-2}) \ldots P(A_2 \mid A_1)\, P(A_1)$

2.4 Conditional Probability

- **Example 2.27**

Four individuals have responded to request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?

Solution:

Let B={first type not O+}, A = {second type not O+} ,so P(B)=3/4

Given that the first type is not O+, two of the three individuals left are not O+, so P(A|B)=2/3

The multiplication rule now gives:

P(at least three individuals must be typed) = $P(A \cap B)$

$$= P(A \mid B)\, P(B)$$
$$= (2/3)*(3/4)$$
$$= 0.5$$

2.4 Conditional Probability

- **Example 2.28**

For the blood typing experiment of Example 2.27,

What is the probability of third type is O+?

2.4 Conditional Probability

- **Example 2.28**

Solution:

P(third type is O+)

= P(third is ∩ first isn't ∩ second isn't)

= P(third is | first isn't ∩ second isn't) P(first isn't ∩ second isn't)

= P(third is | first isn't ∩ second isn't) P(second isn't | first isn't) P(first isn't)

= $1/2 \times 2/3 \times 3/4 = 0.25$

- **Example 2.29**

A chain of video stores sells three different brands of VCRs. Of its VCR sales, 50% are brand 1, 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's VCRs require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 VCR that will need repair while under warranty?
2. What is the probability that a randomly selected purchaser has a VCR that will need repair while under warranty?
3. If a customer returns to the store with a VCR that needs warranty repair work, what is the probability that it is a brand 1 VCR? A brand 2 VCR? A brand 3 VCR?

2.4 Conditional Probability

- **Solution:**

First Stage:

a customer selecting one of the three brands of VCR

Let $P(A_i)$ = {brand *i* is purchased}, where i= 1, 2, 3

then $P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$

Second Stage:

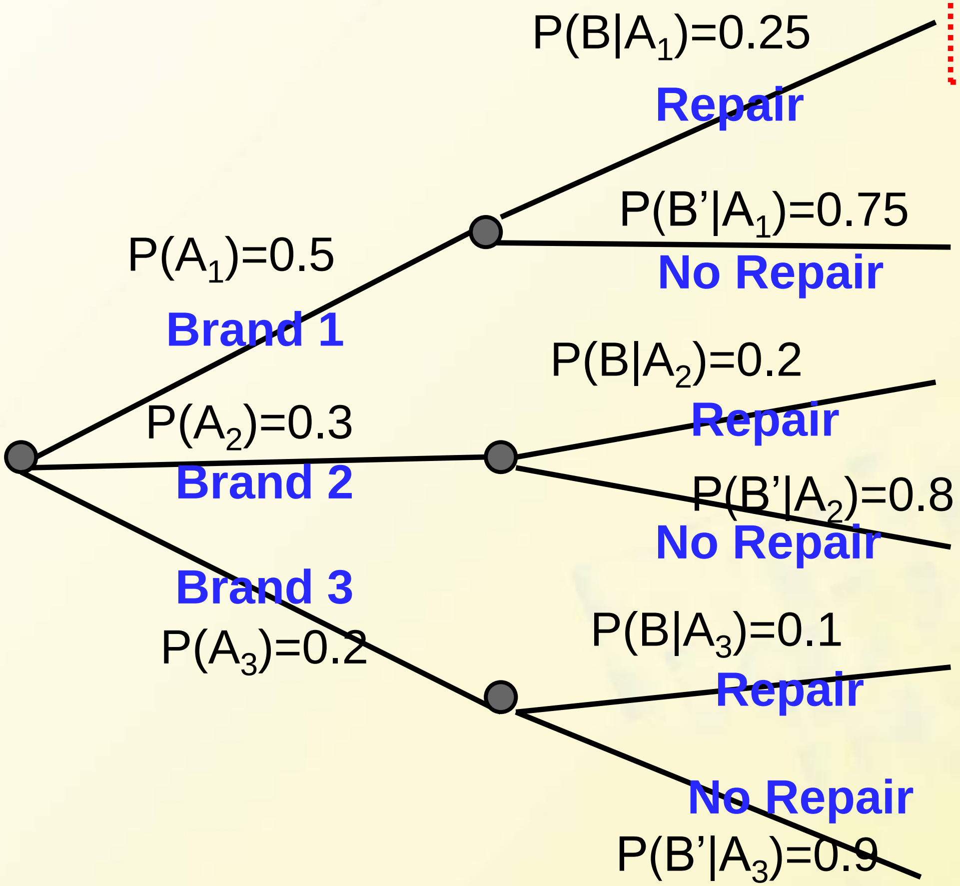
observing whether the selected VCR needs warranty repair

Let B = {needs repair}   B'={doesn't need repair}

then $P(B|A_1) =0.25$, $P(B|A_2) =0.20$, $P(B|A_3) =0.10$

2.4 Conditional Probability

- Example 2.29 (Cont')



Q1

$P(A_1 \cap B) = P(A_1)\ P(B|A_1) = 0.125$

$P(A_2 \cap B) = P(A_2)\ P(B|A_2) = 0.06$

$P(A_3 \cap B) = P(A_3)\ P(B|A_3) = 0.02$

Q2

$P(B) = 0.125+0.06+0.02=0.205$

2.4 Conditional Probability

- Example 2.29 (Cont')

Q3

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.125}{0.205} = 0.61$$

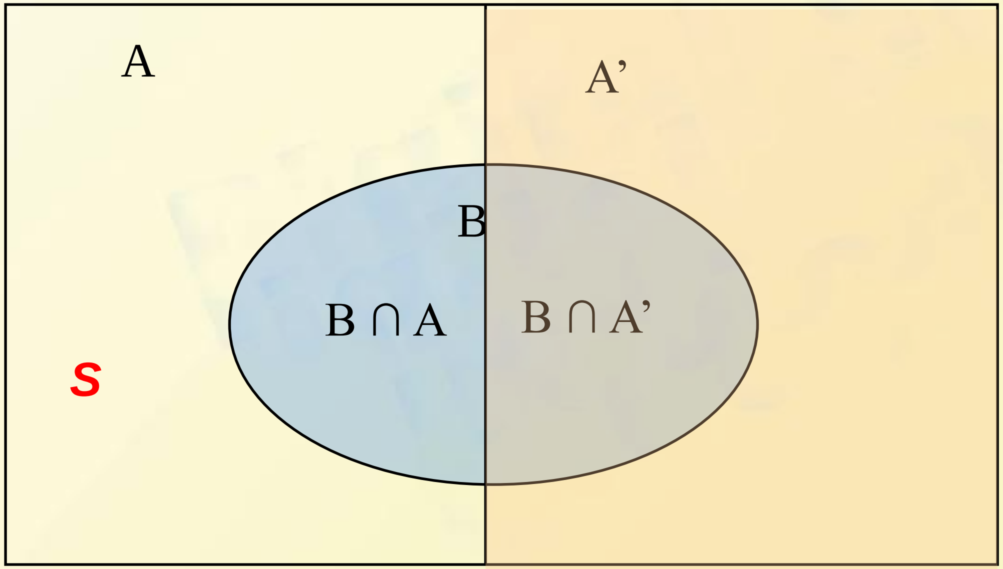
$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.06}{0.205} = 0.29$$

$$P(A_3 \mid B) = 1 - P(A_2 \mid B) - P(A_1 \mid B) = 0.1$$

2.4 Conditional Probability

- **The Law of Total Probability (2-D case)**

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B \mid A)P(A) + P(B \mid A')P(A')$$



Note:
A U A’ =S
A ∩ A’ = φ

Partition of ***S***

Definition:

We say that event $A_1, A_2, \ldots A_3$ represent a partition of the sample space S if

(1) $A_i \cap A_j = \Phi$ for all $i \neq j$

(2) $\bigcup_{i=1}^{n} A_i = S$

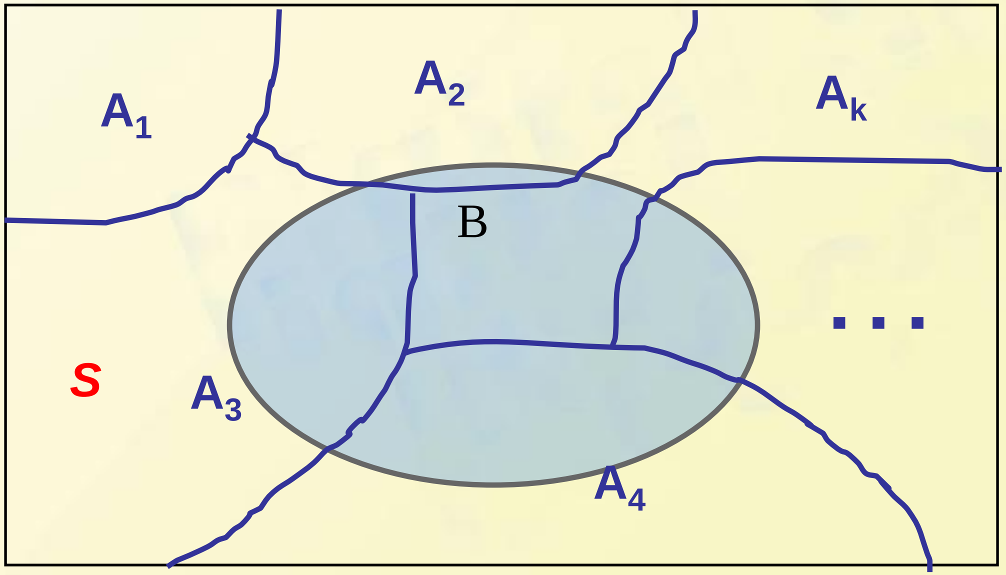
(3) $P(A_i) > 0$ for all $i = 1, 2, \ldots, n$

2.4 Conditional Probability

- **The Law of Total Probability (general cases)**

Let $A_1, \ldots A_k$ be mutually exclusive and exhaustive events (Partition of **S**). Then for any other event B

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(A_i)P(B \mid A_i)$$



2.4 Conditional Probability

- **Bayes' Theorem**

Let $A_1, A_2, \ldots A_k$ be a collection of k mutually exclusive and exhaustive events with P(A)>0 for i=1,…k, then for any other event B for which P(B)>0.

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j)P(B \mid A_j)}{P(B)}$$

$$= \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)} \qquad j = 1, 2, \ldots, k$$

2.4 Conditional Probability

- **Example 2.30**

Incidence of a rare disease. Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Solution:

Let: A_1={individual has the disease}

A_2={individual does not have the disease},

and B ={positive test result}.

Then $P(A_1)=0.001$;

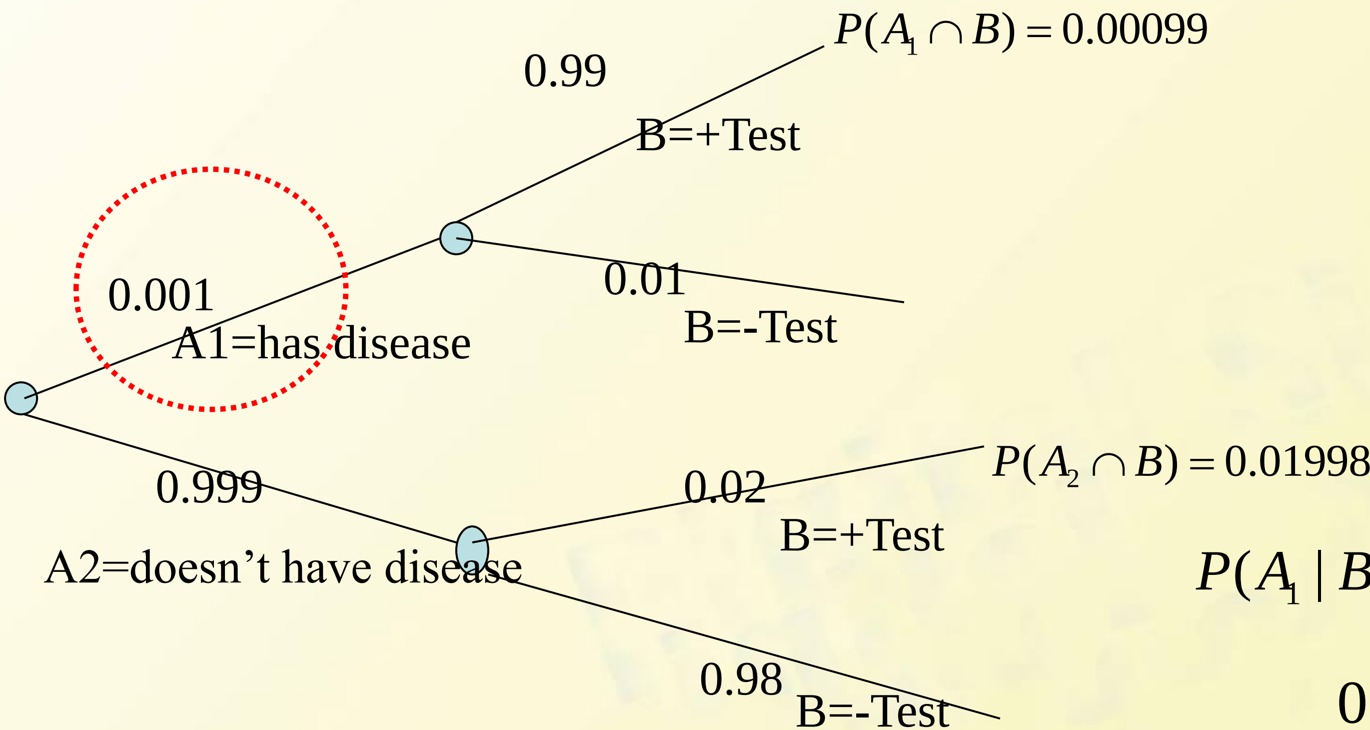
$P(A_2)=0.999$,

$P(B|A_1)=0.99$ and $P(B|A_2)=0.02$.

The three diagram for this problem is in Fig.2.12

2.4 Conditional Probability

- Example 2.30 Cont'



$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)}$$

$$= \frac{0.00099}{0.02097} = 0.047$$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B) = 0.2097$