Chapter 3. Discrete Random Variables and Probability Distributions

Chapter 3:
Discrete Random Variables and Probability Distributions

- 3.1 Random Variables
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The Geometric Distributions(Supplementary)

(1) Definition of the Geometric Distribution

(2) Examples of the Geometric Distributions

Introduction to Geometric Distribution

Consider a sequence of independent Bernoulli trials with a constant success probability p.

Whereas the binomial distribution is the distribution of the number of successes occurring in a fixed number of trials n, it is sometimes of interest to count instead the number of trials performed until the first success occurs. Such a random variable is said to have a geometric distribution with parameter p

Example 3.12

Starting at a fixed time, we observe the gender of each newborn child at a certain hospital until a boy (B) is born. Let p=P(B), assume that successive births are independent, and define the rv X by **X=number of births observed**. Then

$p(1) = P(X=1) = P(B) = p$

$p(2) = P(X=2) = P(GB) = P(G)\ P(B) = (1-p)p$

$p(3) = P(X=3) = P(GCB) = P(G)P(G)\ P(B) = (1-p)^2p$

$\ldots$

$\mathbf{p(k) = P(X=k) = P(G\ldots GB) = (1-p)^{k-1}p}$

Example 3.14 (Ex. 3.12 Cont')

$$p(x) = \begin{cases} (1-p)^{x-1} p, & x = 1,2,3,\ldots \\ 0 & otherwise \end{cases}$$

Find F(x)

Solution: For a positive integer x,

$$F(x) = \sum_{y \le x} p(y) = \sum_{y=1}^{x} (1-p)^{y-1} p = p \sum_{y=1}^{x} (1-p)^{y-1}$$

$$F(x) = p \cdot \frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x$$

The Geometric Distribution

The number of trials up to and including the *first* success in a sequence of independent Bernoulli trials with a constant success probability p has a **geometric** distribution with parameter p. The probability mass function is

$$P(X = x) = (1 - p)^{x-1}\, p$$

for $x = 1, 2, 3, 4 \ldots,$ and the cumulative distribution function is

$$P(X \leq x) = 1 - (1 - p)^{x}$$

The geometric distribution with parameter p has an expected value and a variance of

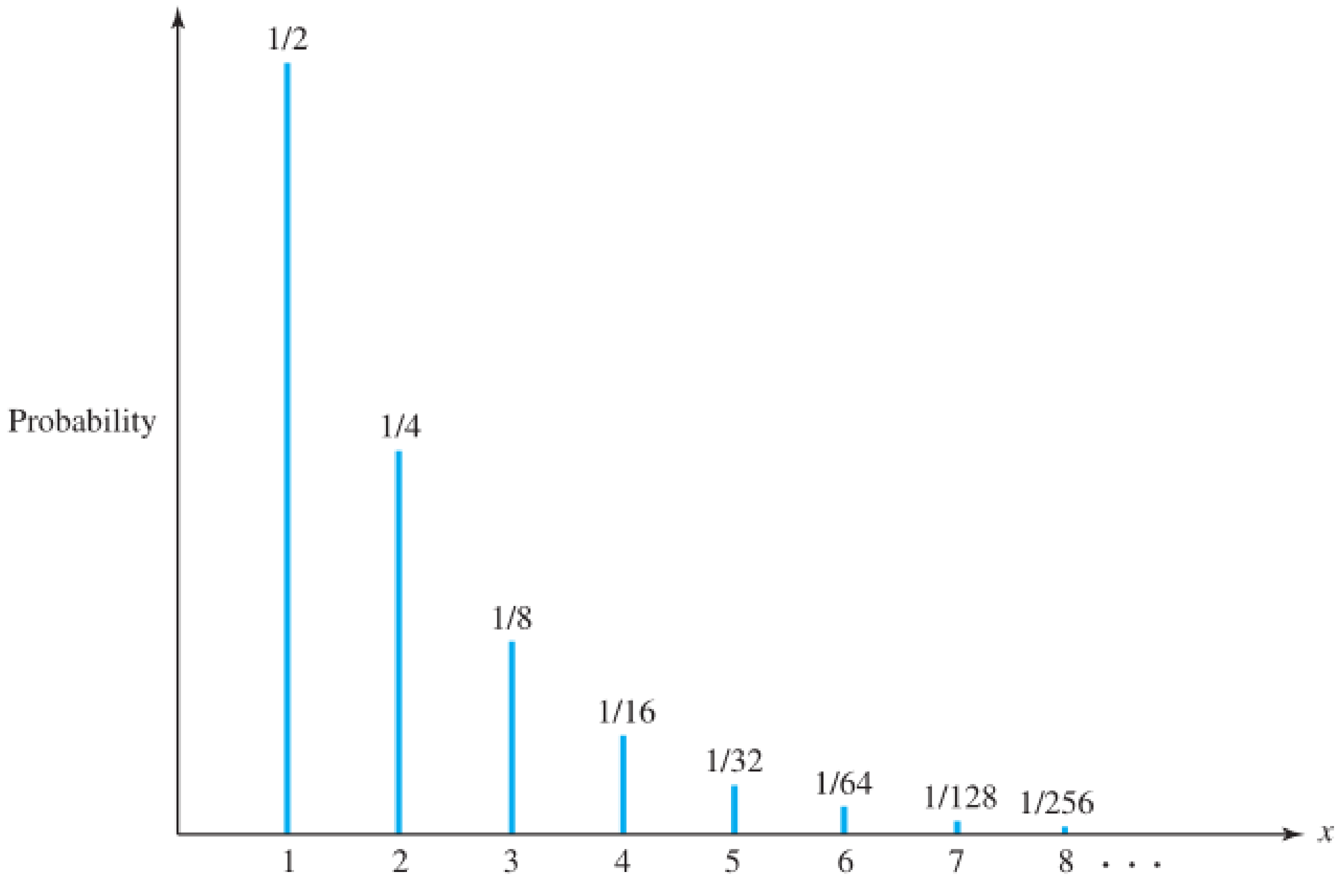
$$E(X) = \frac{1}{p} \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{1-p}{p^2}$$

The distribution with p = 1/2 is appropriate for modeling the number of tosses of a fair coin made until a head is obtained for the first time, since in this case the“success” probability (the probability of obtaining a head) is p = 1/2.

The probability that a head is obtained for the first time on the fourth coin toss is

$$P(X=4)=(1-p)^{4-1}\,p=\left(\frac{1}{2}\right)^3\times\frac{1}{2}=\frac{1}{16}$$

which is simply the probability of obtaining three tails followed by a head.



x	1	2	3	4	5	6	7	8	. . .
$P(X \le x)$	1/2	3/4	7/8	15/16	31/32	63/64	127/128	255/256	. . .

FIGURE 3.11 Probability mass function and cumulative distribution function of a geometric distribution with parameter p = 1/2

Example Telephone Ticket Sales

Telephone ticket sales for a popular event are handled by a bank of telephone salespersons who start accepting calls at a specified time. In order to get through to an operator, a caller has to be lucky enough to place a call at just the time when a salesperson has become free from a previous client. Suppose that the chance of this is 0.1. What is the distribution of the number of calls that a person needs to make until a salesperson is reached?

In this problem, the placing of a call represents a Bernoulli trial with a “success” probability, that is, the probability of reaching a salesperson, of p = 0.1, as illustrated in Figure 3.14.The geometric distribution is appropriate since the quantity of interest is the number of calls made *until the first success.*

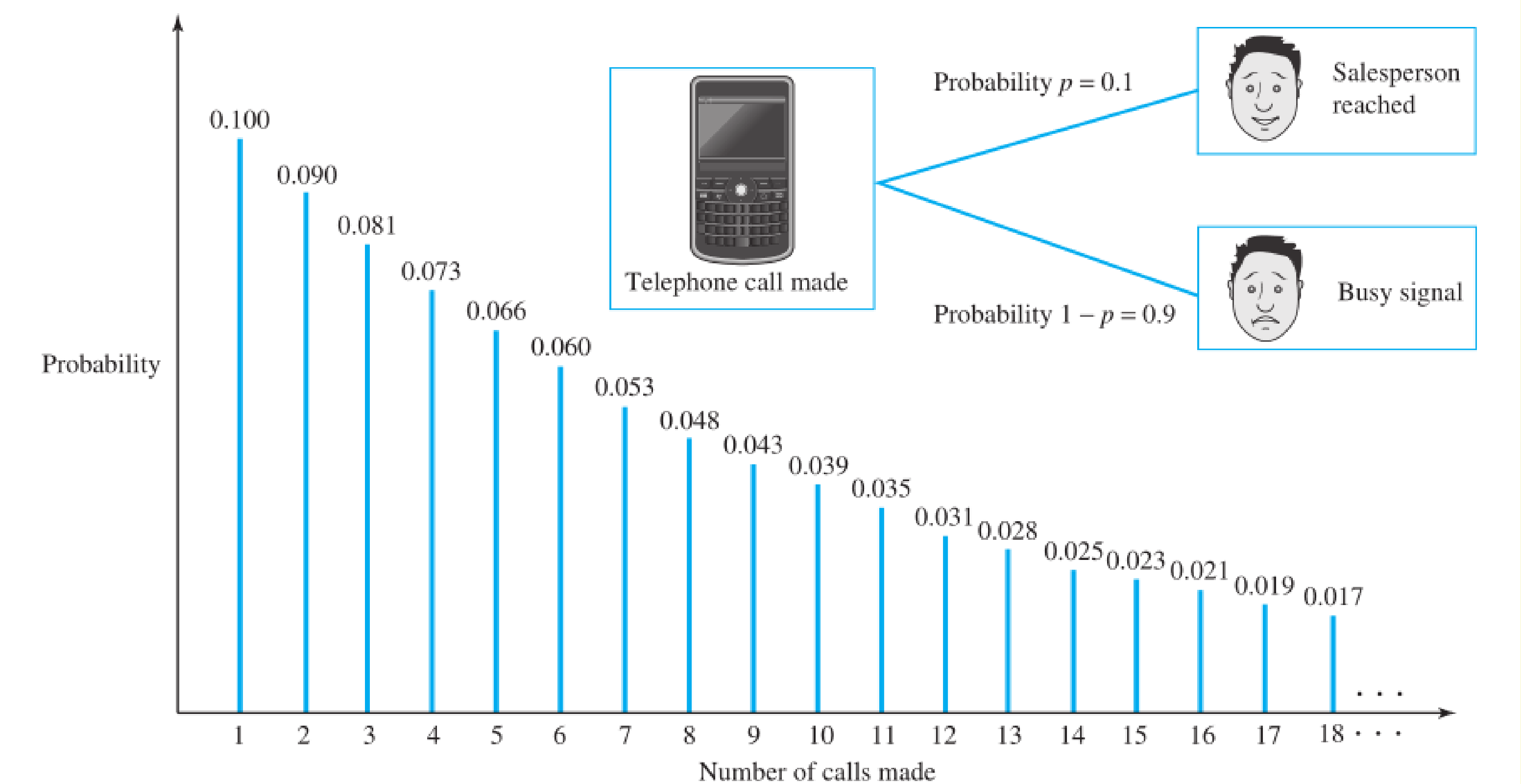


FIGURE 3.14 Probability mass function of a geometric distribution with parameter p = 0.1, the distribution of the number of calls made until a ticket salesperson is reached

For example, the probability that a caller gets through on the fifth attempt, say, is therefore

$$P(X = 5) = 0.9^4 \times 0.1 = 0.066$$

The expected number of calls needed to get through to a salesperson is

$$E(X) = \frac{1}{p} = \frac{1}{0.1} = 10$$

and the probability that 15 or more calls are needed is

$$P(X \geq 15) = 1 - P(X \leq 14) = 1 - \left(1 - 0.9^{14}\right) = 0.9^{14} = 0.229$$

which is simply the probability that the first 14 calls are unsuccessful.

3.5 Hypergeometric and Negative Binomial Distributions

3.5 Hypergeometric and Negative Binomial Distributions

- The assumptions leading to the hypergeometric distribution are as follows:

1. The population or set to be sampled consists of ***N* individuals**, objects, or elements (a finite population).
2. Each individual can be characterized as a success (S) or a failure (F), and there are ***M* successes** in the population.
3. A sample of ***n*** individuals is selected without replacement in such a way that each subset of size *n* is equally likely to be chosen.

Consider X = the number of S's in the sample,

the probability distribution of X depends on the parameters ***n*, *M*** **and *N*, P(X=x) = h(x; n,M,N)**

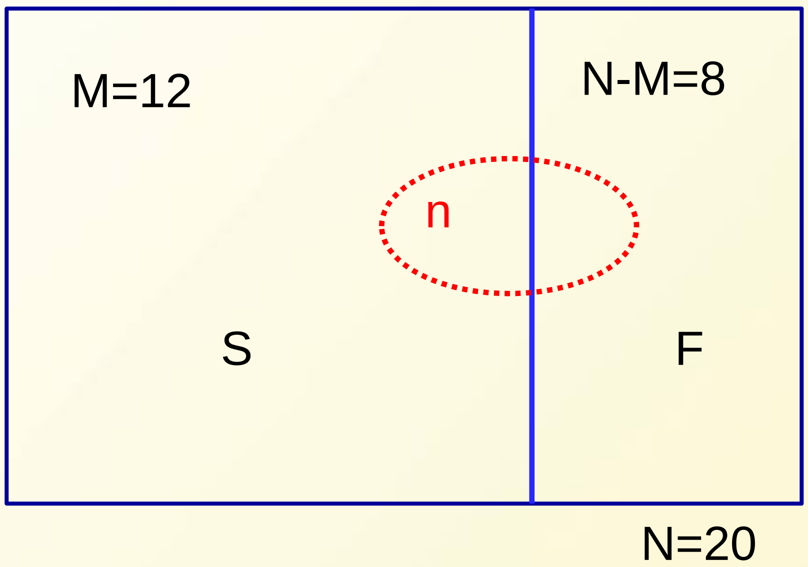
- **Example 3.35**

A office received 20 service orders for problems with printers, of which 8 were laser printers and 12 were inkjet models. A sample of 5 of there service orders is to be selected for inclusion in a customer satisfaction survey. Suppose that the 5 are selected randomly, what is the probability that exactly x of the selected service orders were for inkjet printers?

In this example, N = 20, M = 12, n = 5

$$P(X = x) = h(x; 5, 12, 20) = \frac{\# of outcomes X = x}{\# of possible outcomes}$$

3.5 Hypergeometric and Negative Binomial Distributions



# of Possible outcomes: $\binom{N}{n} = \binom{20}{5}$

$$P(X = x) = \frac{\binom{12}{x}\binom{8}{5-x}}{\binom{20}{5}}$$

# of outcomes having X=x

Step 1: Choosing x elements from subset S $\binom{M}{x} = \binom{12}{x}$

Step 2: Choosing 5-x elements from subset F $\binom{N-M}{n-x} = \binom{8}{5-x}$

3.5 Hypergeometric and Negative Binomial Distributions

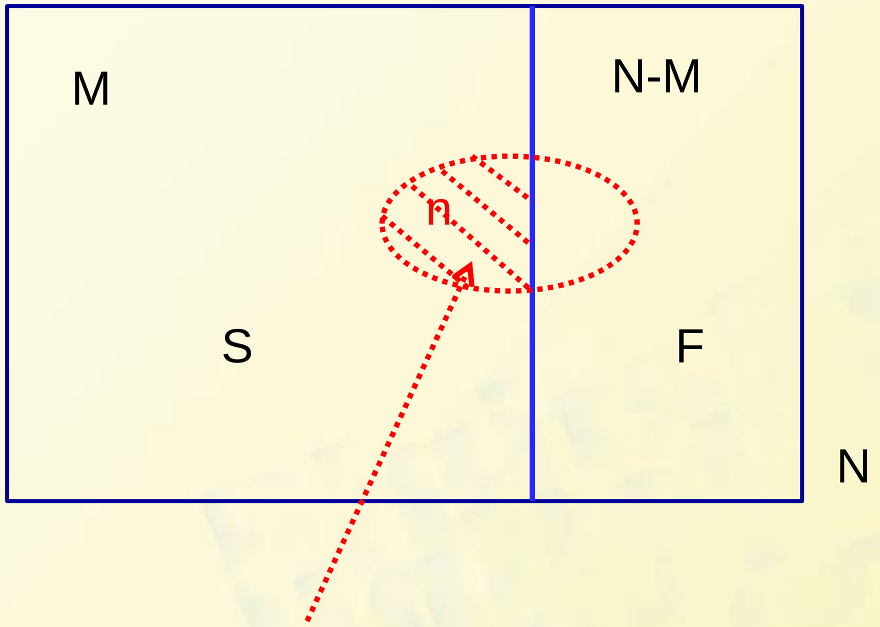
- Proposition:

If X is the number of S's in a completely random sample of size *n* drawn from a population consisting of *M* S's and *(N-M)* F's, then the probability distribution of X, called the hypergeometric distribution, is given by

$$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

The range of rv X

In general, the sample size n is smaller than the number of successes in population (M)



X = the number of S's in a randomly selected sample of size n

$$\text{Max}(0, n-(N-M)) \leq X \leq \text{Min}(n, M)$$

- **Example 3.36**

Five individuals from an animal population thought to be near extinction in a certain region have been caught, tagged, and released to mix into the population. After they have had an opportunity to mix, a random sample of 10 of these animals is selected .

Let X=the number of tagged animals in the second sample. If there are actually 25 animals of this type in the region, what is the probability that (a) $X=2$? (b) $X \leq 2$?

Solution:

In this example, N=25, M=5, n =10

$$P(X = x) = \frac{\binom{5}{x}\binom{20}{10-x}}{\binom{25}{10}}, x = 0,1,2,3,4,5$$

For part (a),

$$P(X = 2) = h(2;10,5,25) = \frac{\binom{5}{x}\binom{20}{8}}{\binom{25}{10}} = 0.385$$

For part (b),

$$P(X \leq 2) = P(X = 0, 1, or\ 2) = \sum_{x=0}^{2} h(x;10,5,25)$$

$$= 0.057 + 0.257 + 0.385 = 0.699$$

- **Proposition**

The mean and variance of the hypergeometric rv X having pmf h(x;n,M,N) are

$$E(X) = np; V(X) = \left(\frac{N-n}{N-1}\right) \cdot n \cdot p \cdot (1-p)$$

≤ 1

where p=M/N

Note: the means of the binomial and hypergeometric rv's are equal, while the variances of the two rv's differ by the factor (N-n)/(N-1) (called ***finite population correction factor***)

Example 3.37 (Ex. 3.36 Cont')

In the animal-tagging example, n=10, M=5, and N=25, so p=5/25=0.2 and what are E(X) and V(X)?

Solution:

$E(X) = 10(0.2)=2$

$V(X) = (15/24)\ (10)(0.2)(0.8) = 1$

If the sampling was carried out with replacement, V(X)=1.6 (Binomial Distribution)

The Negative Binomial Distribution

The negative binomial rv and distribution are based on an experiment satisfying the following conditions:

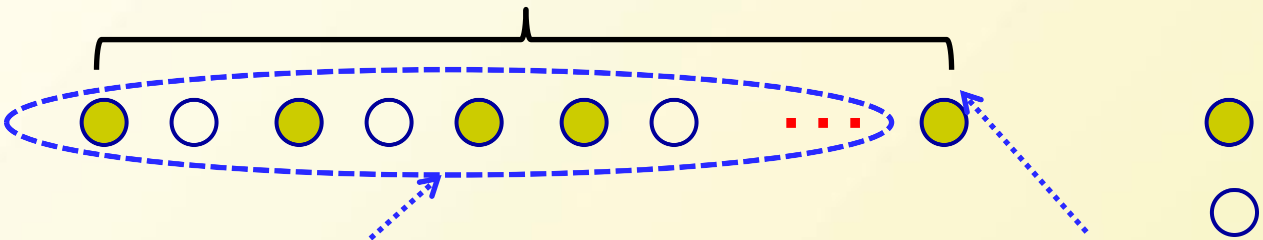
1. The experiment consists of a sequence of independent trials.
2. Each trial can result in either a success (S) or a failure (F).
3. The probability of success is constant from trial to trial, so P(S on trial i)=p for i=1,2,3…
4. The experiment continues **until a total of r successes have been observed**, where **r is a specified positive integer**.

The random variable of interest is X= the number of failures that precede the rth success. X is called a negative binomial variable (Here: the number of success is fixed, while the number of trials is random).

3.5 Hypergeometric and Negative Binomial Distributions

Fixed **Random**

Total number: r (S) + x (F)



Step #1:

Arrage (r-1) S in the first r+x-1 trails

$$\binom{x+r-1}{r-1} p^{r-1}(1-p)^{x}$$

Binomial probability

Step #2:

Fixed the final S

$$p$$

$$nb(x;r,p) = \binom{x+r-1}{r-1} p^{r}(1-p)^{x}, x = 0,1,2,\ldots$$

- **Example 3.38**

A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. Let $p = P$(a randomly selected couple agrees to participate). If $p = 0.2$, what is the probability that 15 couples must be asked before 5 are found who agree to participate? That is, with S={agrees to participate}, (A) what is the probability that 10 F's occur before the fifth S?

(B) The probability that at most 10 F's are observed (at most 15 couples are asked) is?

3.5 Hypergeometric and Negative Binomial Distributions

Solution:

Substituting r=5, p=0.2, and x=10 into $nb(x;r,p)$ gives

$$nb(10;5,.2) = \binom{14}{4}(0.2)^5(0.8)^{10} = 0.034$$

The probability that at most 10 F's are observed (at most 15 couples are asked) is

$$p(X \le 10) = \sum_{x=0}^{10} nb(x;5,0.2) = (0.2)^5 \sum_{x=0}^{10} \binom{x+4}{4}(0.8)^x = 0.164$$

3.5 Hypergeometric and Negative Binomial Distributions

In some sources, the negative binomial rv is taken to be the number of trials X+r rather than the number of failures.

In the special case r=1, the pmf is

$$nb(x;\,1,\,p) = (1-p)^x\, p \qquad\qquad x = 0,1,2,\ldots \qquad (3.17)$$

Both X=number of F's and Y=number of trials (=1+X) are called geometric random variables, and the pmf in (3.17) is called the geometric distribution

- Proposition

If X is a negative binomial rv with pmf bn(x;r,p), then

$$E(X) = \frac{r(1-p)}{p}\,;\ V(X) = \frac{r(1-p)}{p^2}$$

Difference between hypergeometric and binomial distribution

The Hypergeometric and Negative Binomial Distributions are both closely related to the binomial distribution

Whereas the binomial distribution is the approximate probability model for sampleing without replacement form a finite dichotomous (S-F) population.

The hypergeometric distributions is the exact probability model for the number of S's in the sample.

The binomial rv X is the number of S's when the number *n* of trials is fixed, whereas the negative binomial distribution arises form fixing the number of S's desired and letting the number of trials be random.

3.6 The Poisson Probability Distribution

The **Poisson distribution provides a useful model for many random phenomena.**

It can be used as a limiting form of the binomial distribution

3.6 The Poisson Probability Distribution

- Poisson Distribution

A random variable X is said to have a **Poisson distribution** with parameter λ (λ>0) if the pmf of X is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0,1,2,3,....$$

The value of λ is frequency a rate per unit time or per unit area. The constant e is the base of the natural logarithm system.

3.6 The Poisson Probability Distribution

- The Maclaurin infinite series expansion of e^{λ}

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \qquad (3.19)$$

if the two extreme terms in Expression (3.19) are multiplied by $e^{-\lambda}$ and then $e^{-\lambda}$ is placed inside the summation, the result is

$$1 = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!}$$

Which shows that $p(x;\lambda)$ **Fulfills the second condition necessary for specifying a pmf**

- **Proposition**

If X has a Poisson distribution with parameter λ, then
E(X)=V(X)= λ.

Proof:

$$E(X)=\sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{x!}=\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{(x-1)!}=\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^{y+1}}{y!}=\lambda e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^{y}}{y!}=\lambda$$

$$E(X^2)=\sum_{x=0}^{\infty} x^2\frac{e^{-\lambda}\lambda^{x}}{x!}=\lambda e^{-\lambda}\sum_{x=1}^{\infty} x\frac{\lambda^{x-1}}{(x-1)!}$$

$$=\lambda e^{-\lambda}\{\sum_{x=1}^{\infty}[(x-1)\frac{\lambda^{x-1}}{(x-1)!}]+[\frac{\lambda^{x-1}}{(x-1)!}]\}=\lambda e^{-\lambda}[\lambda\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}+\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}]$$

$$=\lambda e^{-\lambda}[\lambda e^{\lambda}+e^{\lambda}]=\lambda^2+\lambda$$

$$V(X)=E(X^2)-E(X)^2=\lambda^2+\lambda-\lambda^2=\lambda$$

3.6 The Poisson Probability Distribution

- **Example 3.39**

Let X denote the number of creatures of a particular type captured in a trap during a given time period. Suppose that X has a Poisson distribution with λ=4.5, so on average traps will contain 4.5 creatures. The probability that a trap contains exactly five creatures is

$$P(X=5)=\frac{e^{-4.5}(4.5)^5}{5!}=0.1708$$

The probability that a trap has at most five creatures is

$$P(X\le 5)=\sum_{x=0}^{5}\frac{e^{-4.5}(4.5)^x}{x!}=e^{-4.5}\left[1+4.5+\frac{(4.5)^2}{2!}+\ldots+\frac{(4.5)^5}{5!}\right]=0.7029$$

3.6 The Poisson Probability Distribution

- **Example (Ex. 3.39 Cont')**

  Both the expected number of creatures trapped and the variance of the number trapped equal 4.5, and $\delta_x=(4.5)^{1/2}=2.12$

3.6 The Poisson Probability Distribution

The Poisson Distribution as a Limit

The rationale for using the Poisson distribution in many situations is provided by the following proposition.

Proposition:

Suppose that in the binomial pmf b(x;n,p), we let $n \to \infty$ and $p \to 0$ in such a way that *np* approaches a value $\lambda > 0$. Then $b(x;n,p) \to p(x;\lambda)$

According to this proposition, in any binomial experiment in which *n* is large and *p* is small, $b(x;n,p) \approx p(x;\lambda)$

As a rule, this approximation can safely be applied if $n \geq 100$, and $p \leq 0.01$ and $np \leq 20$

3.6 The Poisson Probability Distribution

- **Example 3.40**

If a publisher of nontechnical books takes great pains to ensure that its books are free of typographical errors, so that the probability of any given page containing at least one such error is 0.005 and errors are independent from page to page, what is the probability that one of its 400-page novels will contain exactly one page with errors? At most three pages with errors?

3.6 The Poisson Probability Distribution

Solution:

With S denoting a page containing at least one error and F an error-free page, the number X of pages containing at least one error is a binomial rv with n = 400 and p = 0.005, so np=2. We wish

$$P(X=1)=b(1;400,0.005)\approx p(1;2)=\frac{e^{-2}(2)^{1}}{1!}=0.271$$

$$\&P(X\le 3)\approx\sum_{x=0}^{3}p(x;2)=\sum_{x=0}^{3}e^{-2}\frac{2^{x}}{x!}=0.135+0.271+0.271+0.180=0.857$$