3.4 : 46, 47, 48, 54

3.5 : 68, 69, 72, 75

3.6 : 79, 84, 86, 87

A

46)

Theorem :

$$b(x;n,p)\begin{cases}\binom{n}{x}p^{x}(1-p)^{n-x} & , x=0,1,2,\dots,n\\ 0 & , \text{otherwise}\end{cases}$$

a) $n=8$, $x=3$ and $p=0.35$

$b(3;8,0.35)=\binom{8}{3}\cdot 0.35^{3}(1-0.35)^{8-3}=\binom{8}{3}\cdot 0.35^{3}\cdot 0.65^{5}=\frac{8!}{3!5!}\cdot 0.35^{3}\cdot 0.65^{5}=0.2786$

b) $b(5;8,0.6)=\binom{8}{5}\cdot 0.6^{5}(1-0.6)^{8-5}=\binom{8}{5}\cdot 0.6^{5}\cdot 0.4^{3}=\frac{8!}{3!5!}\cdot 0.6^{5}\cdot 0.4^{3}=0.2786$

c) when $3\le X\le 5$, Binomial random variable X can only values 3, 4, 5

when $n=7$, $p=0.6$

$P(3\le X\le 5)=b(3;7,0.6)+b(4;7,0.6)+b(5;7,0.6)=\binom{7}{3}\cdot 0.6^{3}(1-0.6)^{7-3}+\binom{7}{4}\cdot 0.6^{4}(1-0.6)^{7-4}$
$+\binom{7}{5}\cdot 0.6^{5}(1-0.6)^{7-5}=\frac{7!}{3!4!}\cdot 0.6^{3}\cdot 0.4^{4}+\frac{7!}{4!3!}\cdot 0.6^{4}\cdot 0.4^{3}+\frac{7!}{5!2!}\cdot 0.6^{5}\cdot 0.4^{2}=0.7451$

d) Complement of event $\{1\le X\}$ is $\{X=0\}$

for $n=9$ and $p=0.1$

$P(1\le X)=1-P(X=0)=1-\binom{9}{0}\cdot 0.1^{0}\cdot(1-0.1)^{9-0}=1-\frac{9!}{9!}\cdot(0.9)^{9}=0.6126$

47)

a) According to Appendix Table A.1 Cumulative Binomial Probabilities :

row $x=4$, column $p=0.30$ and table $n=15$

$B(4;15,0.3)=0.515$

b) $B(3;15,0.3)=0.297$

$b(4;15,0.3)=B(4;15,0.3)-B(3;15,0.3)=0.515-0.297=0.219$

c) $B(6;15,0.7)=0.015$ & $B(5;15,0.7)=0.004$

$b(6;15,0.7)=B(6;15,0.7)-B(5;15,0.7)=0.015-0.004=0.011$

d) $P(X\le 4)$ when $X\sim Bin(15,0.3)$, is the value in row $x=4$ and in column $p=0.3$ of table $n=15$, $P(X\le 4)=0.515$

$P(X\le 1)$ when $X\sim Bin(15,0.3)$, is the value in row 1 and in column $p=0.3$ of table $n=15$, $P(x\le 1)=0.035$

$P(2\le X\le 4)=P(X\le 4)-P(X\le 1)=0.515-0.035=0.48$

e) $P(X\le 1)$ when $X\sim Bin(15,0.3)$, the value in row 1, column $p=0.3$, table $n=15$, $P(x\le 1)=0.035$

$P(X \geq 2) = 1 - P(X \leq 1) = 1 - 0.035 = 0.965$

f) $P(X \leq 1)$ when $X \sim Bin(15, 0.7)$, $P(X \leq 1) = 0.000$

g) $P(x < 6)$ when $X \sim Bin(15, 0.3)$, row $x=5$, column $p = 0.3$, table $n = 15$

$P(x \leq 6) = P(x \leq 5) = 0.722$

$P(x < 2)$ when $X \sim Bin(15, 0.3)$, row $x = 2$, column $p = 0.3$, table $n = 15$

$P(x \leq 2) = 0.127$

$P(2 < X < 6) = P(x \leq 5) - P(x \leq 2) = 0.722 - 0.127 = 0.595$

48) $P(x \leq 2)$ when $X \sim Bin(25, 0.05)$, row $x = 2$, column $p = 0{,}05$, table $n = 25$

a) $P(x \leq 2) = 0.873$ (according to Table A.1)

OR $P(x \leq 2) = B(2; 25, 0.05) = \sum_{y=0}^{2} b(y; 25, 0.05) = \binom{25}{0} p^0 (1-p)^{25-0} + \binom{25}{1} p^1 (1-p)^{25-1}$
$+ \binom{25}{2} p^2 (1-p)^{25-2} = \left(\frac{25!}{25!} \times 0.05^0 \times 0.95^{25}\right) + \left(\frac{25!}{1! \cdot 24!} \times 0.05^1 \times 0.95^{24}\right) + \left(\frac{25!}{2! 23!} \cdot 0.05^2 \times 0.95^{23}\right)$
$= 0{,}277 + 0.365 + 0{,}231 = 0{,}873$

b) $P(x \geq 5)$ when $X \sim Bin(25, 0.05)$
$P(x < 5)$

$P(x \geq 5) = 1 - P(x \leq 4) = 1 - B(4; 25, 0.05) = 1 - 0.993 = 0.007$

OR $P(X \geq 5) = 1 - P(x < 5) = 1 - B(4; 25, 0.05) = \sum_{y=0}^{4} b(y; 25, 0.05) = 1 - \left(\binom{25}{0} p^0 (1-p)^{25-0}\right.$
$\left. + \binom{25}{1} p^1 (1-p)^{25-1} + \binom{25}{2} p^2 (1-p)^{25-2} + \binom{25}{3} p^3 (1-p)^{25-3} + \binom{25}{4} p^4 (1-p)^{25-4}\right)$
$= 1 - \left(\frac{25!}{25!} \cdot 0.05^0 \cdot (0.95)^{25} + \frac{25!}{24! 1!} \cdot 0.05^1 \cdot 0.95^{24} + \frac{25!}{23! 2!} \cdot 0.05^2 \cdot 0.95^{23} + \frac{25!}{3! 22!} \cdot 0.05^3 \cdot 0.95^{22}\right.$
$\left. + \frac{25!}{21! 4!} \cdot 0.05^4 \cdot 0.95^{21}\right) = 1 - (0.28 + 0.36 + 0.23 + 0.09 + 0.03) = 0.007$

c) $P(1 \leq x \leq 4) = P(x \leq 4) - P(x \leq 0) = 0.993 - 0.277 = 0.716$

OR $P(1 \leq x \leq 4) = P(x=1, x=2, x=3, x=4) = B(4; 25, 0.05) - P(x=0)$
$= \sum_{y=0}^{4} b(y; 25, 0.05) - P(x=0) = 0.28 + 0.36 + 0.23 + 0.09 + 0.03 - 0.28 = 0.72$

d) $P(X=0) = P(x \leq 0) = 0.277$

e) For a Binomial random variable X with parameters $n, p, q = 1-p$:

$E(X) = n.p$

$V(X) = npq = np(1-p)$

$\sigma_X = \sqrt{npq}$

The expected value $E(X) = n.p = 25 . 0.05 = 1.25$

Standard deviation of x $\sigma_x = \sqrt{npq} = \sqrt{25 . 0.05 (1 - 0.05)} = 1.09$

54) X = number of customers that want the oversize version

a) $p = 0.6$, 10 randomly selected customers $\Rightarrow n = 10$, $X \sim Bin(10, 0.6)$

$P(x \geq 6) = 1 - (x \leq 5) = 1 - B(5; 10, 0.6) = 1 - 0.367 = 0.633$

b) $E(X) = np = 10 \cdot 0.6 = 6$

Variance $V(X) = np(1-p) = 10 \cdot 0.6(0.4) = 2.4$

Standard deviation $\sigma_x = \sqrt{V(X)} = \sqrt{2.4} = 1.5492$

Probability of event $E(X) - \sigma_x \leq X \leq E(X) + \sigma_x$ or $6 - 1.5492 \leq x \leq 6 + 1.5492$

is $P(4.4508 \leq X \leq 7.5492) = P(5 \leq X \leq 7) = B(7; 10, 0.6) - B(4; 10, 0.6) = 0.833 - 0.166$

$= 0.667$

c) Only possible values are between 3 and 7 (inclusively), otherwise 1 type will run out.

$P(3 \leq x \leq 7) = \sum_{y=3}^{7} b(y; 10, 0.6) = B(7; 10, 0.6) - B(2; 10, 0.6) = 0.833 - 0.012 = 0.821$

68) Assume that population has M successes (S), N-M failures (F). If random variable

a) X = number of succeses in a random sample size n, then it has the probability mass function: $h(x; n, M, N) = P(X = x) = \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ for all integers x for which $\max\{0, n-N+M\} \leq x \leq \min\{n, M\}$. The probability distribution is called hypergeometric distribution.

The parameters of the hypergeometric distribution are $n = 6$, $M = 12$ and $N = 20$.

b) $P(X = 2) = h(2; 6, 12, 20) = \dfrac{\binom{12}{2}\binom{20-12}{6-2}}{\binom{20}{6}} = \dfrac{\binom{12}{2}\binom{8}{4}}{\binom{20}{6}} = \left(\dfrac{12!}{2!10!} \cdot \dfrac{8!}{4!4!}\right) / \dfrac{20!}{6!14!} = 0.119$

c) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = h(0; 6, 12, 20) + h(1; 6, 12, 20) + h(2; 6, 12, 20)$

$= \left(\binom{12}{0}\binom{20-12}{6-0} / \binom{20}{0}\right) + \left(\binom{12}{1}\binom{20-12}{6-1} / \binom{20}{1}\right) + \left(\binom{12}{2}\binom{20-12}{6-2} / \binom{20}{2}\right) = 0.1373$

$P(x \geq 2) = 1 - (P < 2) = 1 - P(X \leq 1) = 1 - [P(X=0) + P(X=1)] = 1 - (0.0007 + 0.0174)]$

$= 0.9819$

c) for a random variable X with hypergeometric distribution and pmf $h(x; n, M, N)$:

mean value $E(X) = n \cdot \frac{M}{N} = 6 \cdot \frac{12}{20} = 3.6$

variance $V(X) = \left(\frac{N-n}{N-1}\right) \cdot n \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right) = \left(\frac{20-6}{20-1}\right) \cdot 6 \cdot \frac{12}{20} \cdot \left(1 - \frac{12}{20}\right) = \frac{14}{19} \cdot 6 \cdot 0.6 \cdot 0.4 = 1.061$

standard deviation $\sigma_X = \sqrt{V(X)} = \sqrt{1.061} = 1.03$

69) X is hypergeometric with $n = 6$, $N = 12$ and $M = 7$

a) $P(X = 5) = \left(\binom{M}{x}\binom{N-M}{n-x}\right) / \binom{N}{n} = \left(\binom{7}{5}\binom{12-7}{6-5}\right) / \binom{12}{6} = \left(\frac{7!}{5!2!}\right)\left(\frac{5!}{1!4!}\right) / \frac{12!}{6!6!} = \frac{5}{44} \approx 0.114$

b) $P(X \leq 4) = 1 - P(x > 4) = 1 - [P(X=5) + P(X=6)] = 1 - \left[\dfrac{\binom{7}{5}\binom{5}{1}}{\binom{12}{6}} + \dfrac{\binom{7}{6}\binom{5}{0}}{\binom{12}{6}}\right]$

$+ 0.007) = 1 - 0.121 = 0.879$

OR

$p(0) = 0$, because this event is impossible (when there are no successes, then there need to be 6 failures, but we only have 5 failures in the population)

Addition rule for disjoint / mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$

$P(x \le 4) = p(0) + p(1) + p(2) + p(3) + p(4) = 0 + \frac{\binom{7}{1}\binom{12-7}{6-1}}{\binom{12}{6}} + \frac{\binom{7}{2}\binom{12-7}{6-2}}{\binom{12}{6}} + \frac{\binom{7}{3}\binom{12-7}{6-3}}{\binom{12}{6}} + \frac{\binom{7}{4}\binom{12-7}{6-4}}{\binom{12}{6}} =$

$\frac{\binom{7}{1}\binom{5}{5}}{\binom{12}{6}} + \frac{\binom{7}{2}\binom{5}{4}}{\binom{12}{6}} + \frac{\binom{7}{3}\binom{5}{3}}{\binom{12}{6}} + \frac{\binom{7}{4}\binom{5}{2}}{\binom{12}{6}} = \frac{1}{132} + \frac{5}{44} + \frac{25}{66} + \frac{25}{66} = \frac{116}{132} = \frac{29}{33} = 0.879$

c) $E(X) = \mu = n \cdot \frac{M}{N} = 6 \cdot \frac{7}{12} = 3.5$

$V(X) = \left(\frac{N-n}{N-1}\right) \cdot n \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right) = \left(\frac{12-6}{12-1}\right) \cdot 6 \cdot \frac{7}{12} \cdot \left(1 - \frac{7}{12}\right) = 0.795$

$\sigma_X = \sqrt{V(X)} = \sqrt{0.795} = 0.892$

Determine the values that are 1 standard deviation from the mean:

$\mu - \sigma = 3.5 - 0.892 = 2,608$

$\mu + \sigma = 3.5 + 0.892 = 4,392$

We need to determine the probability that X is more than 1 standard deviation above the mean and thus the probability that $X > 4$

$P(x > \mu + \sigma) = P(x > 4) = 1 - P(X \le 4) = 1 - 0.879 = 0,121$

OR $P(X > \mu + \sigma) = P(x > 3.5 + 0.892) = P(X > 4.392) = P(x = 5 \text{ or } X = 6) = P(X=5)$
$+ P(X=6) = 0.114 + 0.007 = 0.121$

d) $n = 15$, $\frac{M}{N} = \frac{40}{400} = 0.1$

$h(x; 15, 40, 400) \approx b(x; 15, 0.1)$. Using this approximation, $P(X \le 5) \approx B(5; 15, 0.1) = 0.998$ from the binomial table.

OR $(X = k)$, $C_k^n \cdot p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k}$

$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + \dots + P(X=5) = \left(\frac{15!}{0!\,15!} \cdot 0.1^0 \cdot (1-0.1)^{15-0}\right)$
$+ \left(\frac{15!}{1!\,14!} \cdot 0.1^1 \cdot (1-0.1)^{15-1}\right) + \dots + \left(\frac{15!}{5!(15-5)!} \cdot 0.1^5 \cdot (1-0.1)^{15-5}\right) = 0.2059 + 0.3432 + \dots + 0.065$
$= 0,998$

72)

a) There are $N = 11$ candidates, $M = 4$ (top four), $n = 6$ for first day.

probability x of top 4 interviewed on first day equals $h(x; 6, 4, 11)$
$= \frac{\binom{4}{x}\binom{11-4}{6-x}}{\binom{11}{6}}$ for $x \in \{0, 1, 2, 3, 4\}$

$\frac{M}{N} = 6 \cdot \frac{4}{11} = 2.18$, we expect 2 candidates to be interviewed the first day of the top 4 candidates.

75)

a) Negative binomial distribution: $nb(x; r, p) = \binom{x+r-1}{r-1} p^r (1-p)^x$, where p = probability and r = number of successes.

Negative binomial distribution is the probability distribution of a variable that measures the number of failures needed to obtain the rth succes (with independent trials and constant probability of success), thus X has a negative binomial distribution with $r = 2$ and $p = P(\text{female}) = 1 - P(\text{male}) = 1 - 0.5 = 0.5$

As in, let X = number of males before the 2^{nd} female.

$P(X = x) = nb(x; 2, 0.5) = \binom{x+2-1}{2-1} \cdot 0.5^2 \cdot (1-0.5)^x = (x+1)(0.5)^{x+2}$

b) 4 children = 2 males + 2 females

$P(\text{exactly 4 children}) = P(\text{exactly 2 males}) = P(X=2) = nb(2; 2, 0.5)$
$= (2+1) \cdot (0.5)^4 = 0.188$

c) $P(\text{at most 4 children}) = P(X \le 2) = \sum_{x=0}^{2} nb(x; 2, 0.5) = P(X=0) + P(X=1)$
$+ P(X=2) = nb(0; 2, 0.5) + nb(1; 2, 0.5) + nb(2; 2, 0.5) = (0+1)(0.5)^{0+2} + (1+1)(0.5)^{1+2}$
$+ (2+1)(0.5)^{2+2} = 0.25 + 0.25 + 0.188 = 0.688$

Having atmost 4 children means having: 2F + 0M or 2F + 1M or 2F + 2M

d) Mean of negative binomial distribution $\mu = \frac{r(1-p)}{p}$, where r = number of succeses, $(1-p)$ = probability of failures and p = probability of succeses

$\mu = \frac{r(1-p)}{p} = \frac{2(1-0.5)}{0.5} = 2$, is the expected number of males.

Since the female need to have exact 2 females, we expect them to have $2 + 2 = 4$ children in total.

79)

A random variable X with probability mass function $p(x; \mu) = e^{-\mu} \frac{\mu^x}{x!}$ for $x = 0, 1, \dots$ is said to have Poisson distribution with parameter $\mu > 0$.

$\mu = 5 > 0$, so find the probabilities from the table in Appendix.

The probabilities are given as $F(x; \mu) = F(x, 5) = \sum_{y=0}^{x} e^{-5} \frac{5^y}{y!}$.

a) $P(X \le 8) \xrightarrow{\text{row } 8} = F(8; 5) = \sum_{x=0}^{8} P(X = x) = \sum_{x=0}^{8} e^{-5} \frac{5^x}{x!} = 0.932$

b) $P(X = 8) = e^{-5} \frac{5^x}{x!} = P(X \le 8) - P(X \le 7) = F(8; 5) - F(7; 5)$
$= 0.932 - 0.876 = 0.065$

c) $P(9 \le X) = P(X \ge 9) = 1 - P(X < 9) = 1 - P(X \le 8) = 1 - 0.932 = 0.068$

d) $P(5 \le X \le 8) = P(X \le 8) - P(X < 5) = P(X \le 8) - P(X \le 4) = F(8;5) - F(4;5)$

$= 0.932 - 0.44 = 0.492$

e) $P(5 < X < 8) = P(X < 8) - P(X < 5) = P(X \le 7) - P(X \le 5) = 0.867 - 0.616 = 0.251$

84)

a) Formula Poisson probability. $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$

Addition Rule for disjoint or mutually exclusive events: $P(A \cup B) = P(A \text{ or } B) = P(A) + P(B)$

n = sample size = 10000

$p = 0.1\% = 0.001$

The number of successes among a fixed number of independent trials with a constant probability of success follows a binomial distribution.

mean $\mu = n.p = 10000 \times 0.001 = 10$

Standard deviation $\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{n \cdot p(p+1)} = \sqrt{10.000 \cdot 0.001 \cdot (1 - 0.001)} \approx 3.16$

b) X has approximately a Poisson distribution with $\mu = 10$, so $(P > 10) \approx 1 - F(10;10)$

$= 1 - 0.583 = 0.417$

OR $P(X \le 10) = P(X = 0) + P(X = 1) + \dots + P(X = 10) = \frac{10^0 e^{-10}}{0!} + \frac{10^1 e^{-10}}{1!} + \dots + \frac{10^{10} e^{-10}}{10!}$

$= 0.000 + 0.000 + \dots + 0.1251 = 0.5826$

$P(X > 10) = 1 - P(X \le 10) = 1 - 0.5826 = 0.4174$

c) When $n > 50$ and $np < 5$, use Poisson distribution.

$n = 10000 > 50$

$np = 10.000 \times 0.001 = 10 > 5$

$\lambda = \mu = 10$, $k = 0$

$P(X = 0) = \frac{10^0 e^{-10}}{0!} = 0.0000454$

86) $\lambda = 5$ per hour, $k = 4$

a) $P(X = 4) = \frac{5^4 e^{-5}}{4!} = 0.1755$

b) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=4) = \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} + \frac{5^2 e^{-5}}{2!} + \frac{5^3 e^{-5}}{3!}$

$= 0.0067 + 0.0337 + 0.0842 + 0.1404 = 0.265$

$P(X \ge 4) = 1 - P(X \le 3) = 1 - 0,265 = 0,735$

c) $\lambda = \frac{3}{4} \times 5 = 3.75$ per 45 minutes

$\mu_X = \lambda = 3.75$, So expect 3.75 people to arrive during a 45 min period.

$\lambda t = 4 \cdot 2 = 8$

a) So $P(X=10) = F(10;8) - F(9;8) = 0.816 - 0.717 = 0.099$

OR $P(X=10) = P_{10}(8) = e^{-8} \cdot \left(\frac{8^{10}}{10!}\right) = 0.099$

b) for 30-min period: $\lambda t = 4 \cdot \frac{1}{2} = 2$, So $P(X=0) = F(0,2) = 0.135$

OR $P(X=0) = P_0(0.5) = e^{-2} \cdot \frac{2^0}{0!} = 0.135$

c) $E(X) = \mu = 4 \times \frac{1}{2} = 2$.