Physics CST (2023-24) Homework 1

Please send the completed file to my mailbox yy.lam@qq.com by the 25th of September, with using the filename format:

student_number_your_name_cst-hw1

Please answer the questions by filling on these sheets. Or alternatively, do the homework as usual by using papers, then take the pictures and paste them onto these question sheets.

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1. Find the orders of magnitudes of the number (i) 2730 and (ii) 4.2×10^{-8}.

   Solution. *(i)* $2730 = 10^{\log 2730} = 10^{3.44} \sim 10^3$. *(ii)* $4.2 \times 10^{-8} \sim 10^1 \times 10^{-8} \sim 10^{-7}$.

2. Given the vectors $\mathbf{u} = \mathbf{e}_r - 3\mathbf{e}_\theta$ and $\mathbf{v} = 2\mathbf{e}_r + \mathbf{e}_\theta$ in the orthonormal basis, (a) find the magnitude of $\mathbf{u}$ in $\mathbf{v}$ direction. (b) Find the dot product of the time derivatives of the vectors.

   Solution. *(a) It implies to find* $\mathbf{u} \cdot \hat{\mathbf{v}}$. *Since*

   $$\hat{\mathbf{v}} = \frac{2\mathbf{e}_r + \mathbf{e}_\theta}{\sqrt{2^2 + 1^2}} = \frac{2}{\sqrt{5}}\mathbf{e}_r + \frac{1}{\sqrt{5}}\mathbf{e}_\theta$$

   The magnitude is

   $$\mathbf{u} \cdot \hat{\mathbf{v}} = 1\left(\frac{2}{\sqrt{5}}\right) + (-3)\left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{\sqrt{5}}$$

   (b) The derivatives with respect to time of the vectors are

   $$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \dot{\theta}\mathbf{e}_\theta + 3\dot{\theta}\mathbf{e}_r, \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 2\dot{\theta}\mathbf{e}_\theta - \dot{\theta}\mathbf{e}_r.$$

   The dot product of them is

   $$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 2\dot{\theta}^2 - 3\dot{\theta}^2 = -\dot{\theta}^2.$$

3. Write the position vector $\mathbf{r} = 3\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}$ in polar coordinate system with the corresponding orthonormal bases $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Find the velocity vector in terms of these bases.

   Solution. *The position vector in the polar coordinate system is simply*

   $$\mathbf{r} = r\mathbf{e}_r = \sqrt{3^2 + (t^2)^2}\,\mathbf{e}_r = \sqrt{9 + t^4}\,\mathbf{e}_r.$$

   The velocity vector is the time derivative of it

   $$\begin{aligned}\mathbf{v} &= \frac{1}{2}(9 + t^4)^{-1/2}(36t^3)\mathbf{e}_r + \sqrt{9 + t^4}\,\dot{\theta}\mathbf{e}_\theta \\ &= \frac{18t^3}{\sqrt{9 + t^4}}\mathbf{e}_r + \sqrt{9 + t^4}\,\dot{\theta}\mathbf{e}_\theta\end{aligned}$$

4. Suppose that the gravitational acceleration g on the surface of Earth only relates to the gravitational constant G, the mass of Earth M, and the radius of Earth R_E, with no relation to other dimensional quantities. Use dimensional analysis to find the mathematics expression for g.

Solution. *The gravitational acceleration g is related by $g \propto G^a M^b R_E^c$ with some unknown numbers a, b and c. The expression of the dimensions is written as*

$$\begin{aligned}[g] &= [G]^a[M]^b[R_E]^c \\ LT^{-2} &= \frac{L^{3a}}{M^aT^{2a}} \cdot M^b \cdot L^c \\ 1 &= L^{3a+c-1}M^{b-a}T^{-2a+2}\end{aligned}$$

Solving for the unknown numbers from the equations $\begin{cases} 3a+c-1 &= 0 \\ b-a &= 0 \\ -2a+2 &= 0 \end{cases}$, *we got* $a = b = 1$, *and* $c = -2$. *Thus the expression is*

$$g \propto \frac{GM}{R_E^2}$$

5. A sport car (S) is traveling west at 110 km/h. At an intersection 25 km ahead, a truck (T) is traveling north at 75 km/h. (i) Write the position vector of the truck relative to the sport car $\mathbf{r}_{ST}$ in terms of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. (ii) How long after this moment will the sport car and the truck be closest to each other? (iii) How far apart will they be at that point?

Solution. *(i) The position vector of the truck relative to the sport car is*

$$\mathbf{r}_{ST} = \mathbf{r}_{SO} + \mathbf{r}_{OT} = (25 - 110t)\hat{\mathbf{i}} + 75t\hat{\mathbf{j}} \ km$$

where O is the point of intersection of the truck and the sport car. (b) Pythagorean theorem gives the distance between the vehicles

$$|\mathbf{r}_{TC}|^2 = (25 - 110t)^2 + (75t)^2.$$

For the minimum distance, differentiating the distance gives

$$2r\frac{\mathrm{d}r}{\mathrm{d}t} = 2(25 - 110t)(-110) + 2(75t)75 \quad \Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{r}[(25 - 110t)(-110) + 75^2t]$$

Setting it to zero we get minimum time taken

$$(25 - 110t)(-110) + 75^2t = 0 \quad \Rightarrow \quad t = 0.155 \ hr = 9.30 \ min$$

(c) Inserting into the distance function r_{TC}, we obtain the minimum distance $r = 14.08$ km.

6. The position of a particle for $t > 0$ is given by $\mathbf{r}(t) = 2t\hat{\mathbf{i}} - t^3\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$ m. (i) Find the velocity and acceleration. (ii) What is the particle's instantaneous velocity and instantaneous acceleration at $t = 2.0$ s? (iii) What is the average velocity between $t = 1.0$ s and $t = 2.0$ s?

Solution. *(i) Differentiating the position vector with respect to time gives the velocity*

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}(2t\hat{\mathbf{i}} - t^3\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) = 2\hat{\mathbf{i}} - 3t^2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}} \ ms^{-1}.$$

Differentiating it again we got the acceleration

$$\mathbf{a} = -6t\hat{\mathbf{j}} + 6\hat{\mathbf{k}}\ ms^{-2}.$$

(ii) Inserting $t = 2$ into the velocity and acceleration equations we get

$$\begin{aligned}\mathbf{v}(2) &= 2\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + 12\hat{\mathbf{k}}\ ms^{-1},\\ \mathbf{a}(2) &= -12\hat{\mathbf{j}} + 6\hat{\mathbf{k}}\ ms^{-2}.\end{aligned}$$

(iii) The average velocity in the interval is

$$\langle \mathbf{v} \rangle = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2-1} = \frac{2(2-1)\hat{\mathbf{i}} - (2^3 - 1^3)\hat{\mathbf{j}} + 3(2^2 - 1^2)\hat{\mathbf{k}}}{2-1} = 2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\ ms^{-1}.$$

7. The velocity of a particle in the reference frame A is $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$ ms^{-1}. The velocity of reference frame A with respect to the reference frame B is $2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}$ ms^{-1}, and the velocity of reference frame B with respect to the frame C is $-2\hat{\mathbf{j}}$ ms^{-1}. What is the velocity of the particle measured in the reference frame C?

   Solution. *Let P be the particle, whose velocity relative to C is*

$$\begin{aligned}v_{PC} &= v_{PA} + v_{AC}\\ &= v_{PA} + v_{AB} + v_{BC}\\ &= (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) + (2\hat{\mathbf{i}} - 3\hat{\mathbf{k}}) + (-2\hat{\mathbf{j}})\\ &= 4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}\end{aligned}$$

8. A small rock is thrown horizontally from the top of a 65 m building and lands 120 m from the base of the building. Ignore air resistance. (i) How long is the rock in the air? (ii) What must have been the initial horizontal component of the velocity? (iii) What is the vertical component of the velocity just before the rock hits the ground? (iv) What is the speed and direction of the rock just before it hits the ground?

   Solution. *(i) We only have to consider the vertical component, the initial verticle velocity $u = 0$, the time taken is obtained by*

$$-65 = \frac{1}{2}(-9.8)t^2 \quad \Rightarrow \quad t = 3.64\ s$$

(ii) The horizontal motion is just an inertial motion, the velocity is

$$\frac{120}{3.64} = 32.97\ ms^{-1}.$$

(iii) Using $v^2 = u^2 + 2as$ we get the vertical velocity $v = \sqrt{2(-9.8)(-65)} = 35.70\ ms^{-1}$.
(iv) The speed of the rock just before hitting is

$$\sqrt{32.97^2 + 35.70^2} = 48.60\ ms^{-1}.$$

The horizontal inclined angle at the moment it hitting the ground is

$$\tan^{-1}\frac{35.70}{32.97} = 47.3°$$

9. A football player punts the ball at a 60.0° angle. Without an effect from the wind, the ball would travel 40.0 m horizontally. (i) What is the initial speed of the ball? (ii) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.20 m/s. What distance does the ball travel horizontally?

   Solution. *(i) Using the equation $R = v^2 \sin 2\theta / g$, the launching speed is*

$$\sqrt{\frac{40 \times 9.8}{\sin 120^\circ}} = 21.28\ ms^{-1}.$$

   (ii) The horizontal velocity is supposed being $21.28 \cos 60 = 10.64\ ms^{-1}$ without wind blowing, it becomes $10.64 - 1.20 = 9.44\ ms^{-1}$ in its second half journey. The actual distance travel is

$$\frac{40}{2} + \frac{9.44}{10.64} \times \frac{40}{2} = 37.74\ m.$$

   We have used the simple ratio for the distance, as the inertial horizontal motion is linear.

10. (a) Calculate the height of a cliff if it takes 2.80 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 9.0 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

   Solution. *(a) As usual, positive for upward, negative for downward we have the height of the cliff*

$$s = 9 \times 2.8 + \frac{1}{2}(-9.8) \times 2.8^2 = -13.22\ m.$$

   The negative result obviously implies the downward displacement. (b) Throwing downward instead, the time taken is

$$-13.22 = -9t + \frac{1}{2}(-9.8)t^2$$

   gives $2.80\ s$ or $0.96\ s$, we reject the former which does not look sensible, the correct time taken should be $0.96\ s$.