A

1. a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy = K\int_{20}^{30}\int_{20}^{30} x^2\,dy\,dx + K\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$

$= 10K\int_{20}^{30} x^2\,dx + 10K\int_{20}^{30} y^2\,dy$

$= 20K \times \frac{19000}{3} = \frac{380000}{3}K = 1$

$K = \frac{3}{380000}$

b) $P(X<26 \text{ and } Y<26) = \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\,dx\,dy = 12K\int_{20}^{26} x^2\,dx$

$= 38304K = 0.3024$

c) $P(|X-Y| \leq 2) = 1 - \int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx$

$= 1 - K\int_{20}^{28}\int_{x+2}^{30} (x^2+y^2)\,dy\,dx - K\int_{22}^{30}\int_{20}^{x-2} (x^2+y^2)\,dy\,dx$

$= 1 - \left(K\int_{20}^{28} x^2(30-x-2)\,dx + \frac{K}{3}\int_{20}^{28}(30^3-(x+2)^3)\,dx\right) - \left(K\int_{22}^{30} x^2(x-2-20)\,dx + \frac{K}{3}\int_{22}^{30}((x-2)^3-20^3)\,dx\right)$

$= 1 - \left(16554.67K + \frac{72064K}{3}\right) - \left(24021.33K + \frac{49664K}{3}\right)$

$= 1 - 0.3203 - 0.3203 = 0.3594$

d) $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{20}^{30} K(x^2+y^2)\,dy = 10Kx^2 + K\frac{y^3}{3}\Big|_{20}^{30}$

$= 10 \times Kx^2 + 0.05$

$$f(x) \begin{cases} 10 \times Kx^2 + 0.05 & , 20 \leq x \leq 30 \\ 0 & , \text{otherwise} \end{cases}$$

e) Not independent.

12. a) $P(X>3) = \int_{3}^{\infty}\int_{0}^{\infty} xe^{-x(1+y)}\,dx\,dy = \int_{3}^{\infty} e^{-x}\,dx = 0.05$

b) The marginal pdf of X is $\int_{0}^{\infty} xe^{-x(1+y)}\,dy = e^{-x}$, $0 \leq x$

The marginal pdf of Y is $\int_{0}^{\infty} xe^{-x(1+y)}\,dx = \frac{1}{(1+y)^2}$, $0 \leq y$

Not independent

c) P(at least one exceeds 3) $= 1 - P(X \leq 3 \text{ and } Y \leq 3)$

$= 1 - \int_{0}^{3}\int_{0}^{3} xe^{-x(1+y)}\,dy\,dx = 1 - \int_{0}^{3}\int_{0}^{3} xe^{-x}e^{-xy}\,dy$

$= 1 - \int_{0}^{3} e^{-x}(1-e^{-3x})\,dx = e^{-3} + 0.25 - 0.25e^{-12} = 0.3$

18. a) $P_X(1) = 0.34$

$$P_{Y|X}(0|1) = \frac{P(1,0)}{P_X(1)} = \frac{0.08}{0.34} = 0.2353$$

$$P_{Y|X}(1|1) = \frac{P(1,1)}{P_X(1)} = \frac{0.2}{0.34} = 0.5882$$

$$P_{Y|X}(2|1) = \frac{P(1,2)}{P_X(1)} = \frac{0.06}{0.34} = 0.1765$$

b) $P_X(2) = 0.5$

$$P_{Y|X}(0|2) = \frac{P(1,0)}{P_X(2)} = \frac{0.06}{0.5} = 0.12$$

$$P_{Y|X}(1|2) = \frac{P(1,1)}{P_X(2)} = \frac{0.14}{0.5} = 0.28$$

$$P_{Y|X}(2|2) = \frac{P(1,2)}{P_X(2)} = \frac{0.3}{0.5} = 0.6$$

c) $P(Y \leq 1 \mid X = 2) = 0.12 + 0.28 = 0.4$

d) $P_Y(2) = 0.02 + 0.06 + 0.3 = 0.38$

$$P_{X|Y}(0|2) = \frac{P(1,0)}{P_Y(2)} = \frac{0.02}{0.38} = 0.0526$$

$$P_{X|Y}(1|2) = \frac{P(1,1)}{P_Y(2)} = \frac{0.06}{0.38} = 0.1579$$

$$P_{X|Y}(2|2) = \frac{P(1,2)}{P_Y(2)} = \frac{0.3}{0.38} = 0.7895$$

19. a) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{k(x^2+y^2)}{10kx^2+0.05}$

$f_{X|Y}(x|y) = \frac{k(x^2+y^2)}{10ky^2+0.05}$ $\quad 20 \leq x \leq 30$

b) $P(Y \geq 25 \mid X = 22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy$

$$= \int_{25}^{30} \frac{k(22^2+y^2)}{10k \times 22^2 + 0.05}\,dy$$

$$= 0.783$$

$P(Y \geq 25) = \int_{25}^{30} f_Y(y)\,dy = \int_{25}^{30} (10ky^2 + 0.05)\,dy = 0.15$

c) $E(Y \mid X = 22) = \int_{-\infty}^{\infty} y \times f_{Y|X}(y|22)\,dy$

$$= \int_{20}^{30} y \left(\frac{22^2 k + y^2 k}{10k \times 22^2 + 0.05} \right) dy$$

$$= 25.372912\ldots$$

$E(Y^2 \mid X = 22) = \int_{20}^{30} y^2 \frac{22^2 k + y^2 k}{10k \times 22^2 + 0.05}\,dy = 652.0286\ldots$

$V(Y \mid X = 22) = E(Y^2 \mid X = 22) - (E(Y \mid X = 22))^2 = 8.243916\ldots$

24. $p(x,y) = \frac{1}{30}$

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) \times p(x,y)$$
$$= \frac{1}{30} \sum_x \sum_y g(x,y)$$
$$= \frac{1}{30} \times 84 = 2.8$$

26. $g(x,y) = 3X + 10Y$

$$E(3X+10Y) = \sum_x^{5} \sum_y^{2} (3x+10y) \times p(x,y)$$
$$= (3\times 0 + 10\times 0) \times p(0,0) + (3\times 0 + 10\times 1)\times p(0,1) + \dots + (3\times 5 + 10\times 2)\times p(5,2)$$
$$= 15.4$$

33. $$Cov(X,Y) = E(XY) - E(X) \times E(Y)$$
$$= E(X)E(Y) - E(X)E(Y) = 0$$

correlation coefficient of X and Y is $\frac{Cov(X,Y)}{\sigma_x \times \sigma_y}$

35. a) $$Cov(aX+b, cY+d) = E((aX+b)(cY+d)) - E(aX+b) \times E(cY+d)$$
$$= E(acXY + adX + bcY + bd) - (aE(X)+b) \times (cE(Y)+d)$$
$$= acE(XY) + adE(X) + bcE(Y) + bd - (acE(X)E(Y) + adE(X) + bcE(Y) + bd)$$
$$= acE(XY) - acE(X)E(Y)$$
$$= ac\,Cov(X,Y)$$

b) $$Corr(aX+b, cY+d) = \frac{Cov(aX+b, cY+d)}{\sqrt{Var(aX+b)Var(cY+d)}} = \frac{Cov(aX+b, cY+d)}{|a||c|\sqrt{Var(X)Var(Y)}} = Corr(X,Y)$$

c) If a and c have opposite signs, $Corr(aX+b, cY+d) = -Corr(X,Y)$.