Physics CST (2023-24) Homework 2

Please send the completed file to my mailbox yy.lam@qq.com by October 23rd, with using the filename format:

student_number-name-cst-hw2

Please answer the questions by filling on these sheets. Or alternatively, do the homework as usual by using papers, then take the pictures and paste them onto these question sheets.

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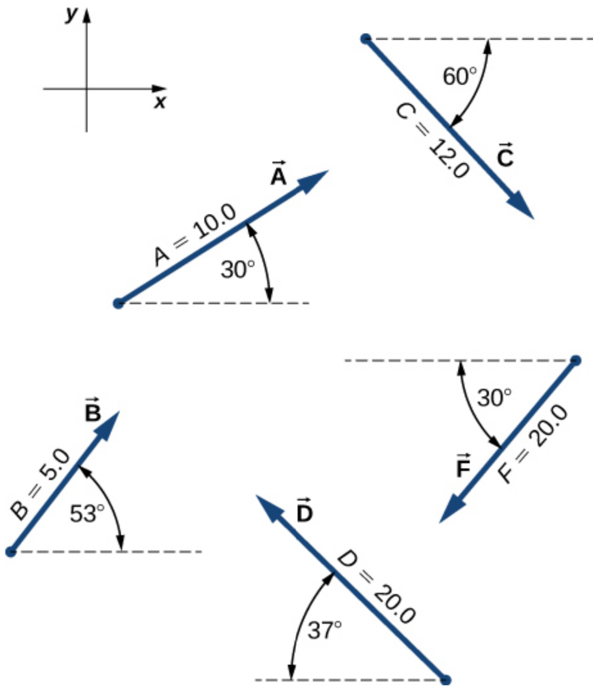
1. Find the following vector products: (a) $\mathbf{A}\times\mathbf{C}$, (b) $\mathbf{A}\times\mathbf{F}$, (c) $\mathbf{D}\times\mathbf{C}$, (d) $\mathbf{A}\times(\mathbf{F}+2\mathbf{C})$, (e) $\hat{\mathbf{i}}\times\mathbf{B}$, (f) $\hat{\mathbf{j}}\times\mathbf{B}$, (g) $(3\hat{\mathbf{i}}-\hat{\mathbf{j}})\times\mathbf{B}$, and (h) $\hat{\mathbf{B}}\times\mathbf{B}$.



Solution. *According to the right hand rule, $\hat{\mathbf{k}}$ is perpendicular out from the paper.*

(a) $\mathbf{A}\times\mathbf{B} = (10)(12)\sin(360-(60+30))\hat{\mathbf{k}} = -120\hat{\mathbf{k}}$

(b) $\mathbf{A}\times\mathbf{F} = (10)(20)\sin 180 = \mathbf{0}$

(c) $\mathbf{D}\times\mathbf{C} = (20)(12)\sin(37+(180-60)) = 93.78\hat{\mathbf{k}}$

(d) *As the problem involves a vector sum, we may write the vectors in component forms to avoid to look for the resultant angle.*

$$\begin{aligned}\mathbf{F}+2\mathbf{C} &= 20\cos(180+30)\hat{\mathbf{i}} + 20\sin(180+30)\hat{\mathbf{j}} + 2(12\cos(360-60)\hat{\mathbf{i}} + 12\sin(360-60)\hat{\mathbf{j}})\\ &= -20\sqrt{3}/2\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + (24/2\hat{\mathbf{i}} - 24\sqrt{3}/2\hat{\mathbf{j}})\\ &= (12-10\sqrt{3})\hat{\mathbf{i}} + (12\sqrt{3}-10)\hat{\mathbf{j}}\end{aligned}$$

Since $\mathbf{A} = 10\cos 30\hat{\mathbf{i}} + 10\sin 30\hat{\mathbf{j}} = 5\sqrt{3}\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, *the cross product is*

$$\begin{aligned}\mathbf{A}\times(\mathbf{F}+2\mathbf{C}) &= (5\sqrt{3}\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \times [(12-10\sqrt{3})\hat{\mathbf{i}} + (12\sqrt{3}-10)\hat{\mathbf{j}}]\\ &= (180-50\sqrt{3})\hat{\mathbf{k}} - (60-50\sqrt{3})\hat{\mathbf{k}}\\ &= -120\hat{\mathbf{k}}\end{aligned}$$

(e) $\hat{\mathbf{i}} \times \mathbf{B} = 10 \sin 53 \hat{\mathbf{k}} = 8.0\hat{\mathbf{k}}$
(f) $\hat{\mathbf{j}} \times \mathbf{B} = -10 \sin(90 - 53)\hat{\mathbf{k}} = -6.0\hat{\mathbf{k}}$
(g) As $(3\hat{\mathbf{i}} - \hat{\mathbf{j}}) \times \mathbf{B} = 3\hat{\mathbf{i}} \times \mathbf{B} - \hat{\mathbf{j}} \times \mathbf{B}$, *using (e) and (f) we get*

$$3 \times 8\hat{\mathbf{k}} - (-6\hat{\mathbf{k}}) = 30\hat{\mathbf{k}}.$$

(e) $\hat{\mathbf{B}} \times \mathbf{B} = \dfrac{\mathbf{B}}{|\mathbf{B}|} \times \mathbf{B} = \mathbf{0}$

2. The velocity of a particle moving along the x axis varies in time according to the expression $v_x = (40 - 5t^2)\ ms^{-1}$, where t is in seconds. (a) Find the average acceleration in the time interval $t = 0$ to $t = 2$ s along the direction $30°$ from the x-axis. (b) Determine the acceleration at $t = 2$ s. (c) What is the acceleration along the y-axis?

   Solution. *(a) The average acceleration is the change of the velocity within a given time interval. Along the x-axis it is*

$$\text{Average acceleration} = \frac{40 - 5 \cdot 2^2 - (40 - 0)}{2 - 0} = -10\ ms^{-2}.$$

   The required acceleration is

$$-10 \cos 30 = -8.66\ ms^{-2}.$$

   (b) The instantaneous acceleration at $t = 2$ *is*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}(40 - 5t^2)\right]_{t=2} = -10t\Big|_{t=2} = -20\ ms^{-2}.$$

   (c) Because the directions of x- and y-axes in Cartesian coordinate system are defined being independent of each other. In other words, $\cos 90 = 0$ *gives the zero acceleration along the y-axis.*

3. Assuming earth is a rigid body with a constant density. Find the gravitational acceleration at a point p inside earth. Express your results in terms of the radius of earth R, the distance r of the point p from the centre, and M the mass of earth.

   Solution. *We simply treat the gravitational acceleration as being created by the centre of mass of the earth as usual. The acceleration* g_p *depends on the position p for taking values however,*

$$g_p = \frac{GM_p}{r^2}$$

   where M_p, r *are the corresponding mass and the radius of the measuring sphere. Thus,*

$$g_p = \frac{GM_p}{r^2} = \frac{G4\pi r^3 \rho}{3r^2} = \frac{G4\pi R^3 r\rho}{3R^3} = \frac{GMr}{R^3}$$

   where M *and* R *are the mass and radius of earth.*

4. A person driving a car traveling 30 m/s passes a stationary motorcycle police officer. 2.5 s after the car passes, the police starts to move and accelerates in pursuit of the speeding car. The motorcycle has constant acceleration of 3.7 m/s^2. (a) How fast will the police officer be traveling when he overtakes the car? (b) Draw curves of x versus t for both the motorcycle and the car, taking $t = 0$ at the moment the car passes the stationary police officer.

Solution. *(a) Let x_c, v_c be the displacment and uniform velocity of the car over the time interval t, we have*

$$x_c = v_c t.$$

The displacement x_m that the motocycle traveled with a constant acceleration a is

$$x_m = \frac{1}{2}a(t-2.5)^2$$

as there is 2.5 s delay of the response. When the motorcycle overtakes the car, $x_c = x_m$, i.e. $\frac{1}{2}a(t-2.5)^2 = v_c t$. Inserting the values into the equation we get

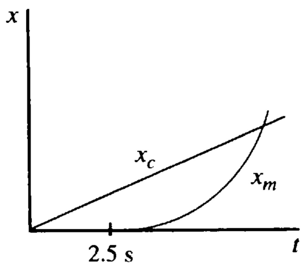
$$\frac{1}{2}(3.7)(t-2.5)^2 = 30t \quad \Rightarrow \quad t^2 - 21.22t + 6.25 = 0$$

Solving the quadratic equation gives

$$t = \frac{21.22 \pm \sqrt{21.22^2 - 4(6.25)}}{2}$$

Thus, $t = 20.9$ s or $t = 0.3$ s. We reject the latter as the policeman starts to move 2.5 s after the car passing. The velocity of the motorcycle when overtaking the car is

$$u + at = 3.7 \times 20.9 = 77.3 \ ms^{-1}.$$



5. A projectile is fired in such a way that its horizontal range is equal to three times its maximum height. What is the angle of projection?

Solution. *The horizontal and vertical components as usual are*

$$R = vt\cos\theta, \quad s = vt\sin\theta - \frac{1}{2}gt^2$$

At the maximum height, $s = h$ and $v\sin\theta = 0$ (the reverse it becomes the initial velocity) at time $t/2$ where t is the whole time interval. When time is t, $s = 0$ and $R = 3h$. Thus the components become

$$3h = vt\cos\theta, \quad -h = 0 - \frac{1}{2}g\left(\frac{h}{2}\right)^2 = -\frac{1}{8}gt^2$$

Eliminating h we get $\frac{3}{8}gt = v\cos\theta$. In the full range the vertical component with vanishing displacement is

$$0 = vt\sin\theta - \frac{1}{2}gt^2 \quad \Rightarrow \quad v\sin\theta = \frac{1}{2}gt$$

Combining these two reduced equations, eliminating v, g and t simultaneously we get

$$\tan\theta = \frac{4}{3} \quad \Rightarrow \quad \theta = 53.1^\circ$$

6. Given the distance of the centre to centre distance of Earth and the Moon 3.84×10^5 km, the time interval for a month 27.3 days, (a) find the acceleration due to Earth's gravity at the distance of the moon. (b) Given the radius of Earth 6370 km, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

   Solution. *Suppose that we do not know the numerical values of the gravitational constant G and the mass of Earth M, we only use the given information for the computation. (a) The gravitational acceleration is*

$$a = r\omega^2 = 3.84 \times 10^8 \times \left(\frac{2\pi}{27.3 \times 24 \times 60 \times 60}\right)^2 = 2.72 \times 10^{-3} \ ms^{-2}$$

   (b) Since the gravitational acceleration (centripetal acceleration) obeys the inverse square law

$$a \propto \frac{1}{r^2} \quad \Rightarrow \quad r\omega^2 \propto \frac{1}{r^2}$$

   with $T = 2\pi/\omega$ gives

$$\frac{T^2}{r^3} = constant$$

   Let r_m, T_m be the given distance of the moon and the period, r_s the average altitude of the satellite, thus the period of the satellite is

$$T = T_m\left(\frac{R_E + r_s}{r_m}\right)^{3/2} = 27.3 \times 24 \times 60 \times 60 \times \left(\frac{(6370 + 1500) \times 10^3}{3.84 \times 10^8}\right)^{3/2} = 6920 \ s$$

   or 115 minutes 20 *seconds.*

7. A hockey puck of mass 0.18 kg is shot across a rough floor with the roughness different at different places, which can be described by a position-dependent coefficient of kinetic friction. For a puck moving along the x-axis the coefficient of kinetic friction is the following function of x, where x is in m: $\mu(x) = 0.1 + 0.05x$. Find the work done by the kinetic friction force on the hockey p when it has moved (a) from $x = 0$ to $x = 1$ m, and (b) from $x = 1$ m to $x = 3$ m.

   Solution. *The frictional force also becomes a function of x as*

$$f = -9.8 \times 0.18(0.1 + 0.05x) = -0.088(2 + x).$$

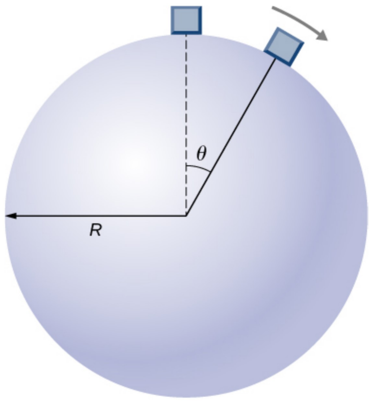
   (a) The work done by the kinetic friction is

$$-\int_0^1 0.088(2 + x)\mathrm{d}x = -0.088\left[2x + \frac{x^2}{2}\right]_0^1 = -0.22 \ J.$$

   (b) The work done for a different interval,

$$-\int_1^3 0.088(2 + x)\mathrm{d}x = -0.088\left[2x + \frac{x^2}{2}\right]_1^3 = -0.70 \ J.$$

8. A body of mass m and negligible size starts from rest and slides down the surface of a frictionless solid sphere of radius R as shown. Find the angle θ while the body leaves the sphere.



Solution. *Before the mass sliding out from the sphere, it keeps a circular motion in*

$$\frac{mv^2}{R} = mg\cos\theta.$$

As the angle θ increases as well as the increasing of v due to gravity, above equation at some point no longer holds. The tangential velocity v becomes the velocity for ***linear*** *kinetic energy which equals to the potential difference between the top and the breaking position, i.e.*

$$\frac{1}{2}mv^2 = mgR(1-\cos\theta)$$

Inserting the first equation while at the threshold condition into the second equation we get

$$\frac{1}{2}(Rmg\cos\theta) = mgR(1-\cos\theta)$$
$$\frac{1}{2}\cos\theta = 1-\cos\theta$$
$$\theta = \cos^{-1}\frac{2}{3}$$

9. A 75 kg crate rests on the bed of a truck. The coefficients of kinetic and static friction between the surfaces are $\mu_k = 0.3$ and $\mu_s = 0.4$, respectively. Find the frictional force on the crate and describe the state of motion of it when the truck is accelerating forward relative to the ground at (a) 2.0 ms^{-2}, and (b) 5.0 ms^{-2}.

Solution. *Let F and f be the force due to the acceleration and the frictional force between the crate and the bed of the truck. Note that F is opposite to the direction of the movement of the truck and f is the same direction to the truck's motion. In equilibrium,*

$$F = f \quad \Rightarrow ma = \mu(mg) \quad \Rightarrow \quad a = \mu g$$

where $\mu \leq \mu_s$. (a) When $a = 2$ ms^{-1} it is smaller than the maximum static frictional force per unit mass $0.4 \times 9.8 = 3.92$ N/kg. The crate maintains at rest relative to the truck. The frictional force is just $F = ma = 75 \times 2 = 150$ N. (b) When $a = 5$ ms^{-1}, $F = 75 \times 5 = 375$ N which is bigger than the maximum static friction

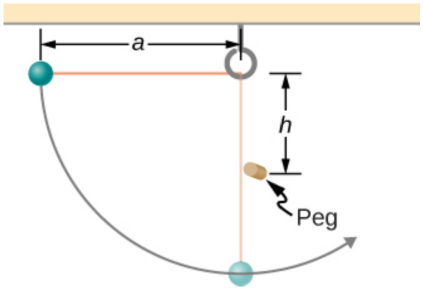
$$f_s = 0.4 \times 75 \times 9.8 = 294\ N$$

in this case the static friction no longer holds. The frictional force becomes kinetic

$$f_k = 0.3 \times 75 \times 9.8 = 220.5\ N$$

As $F > f_s > f_k$, we expect the crate will slide along the backward direction of the truck.

10. A small ball of mass m attached to a string of length a. A small peg is located a distance h below the point where the string is supported. If the ball is released when the string is horizontal, show that h must be greater than $3a/5$ if the ball is to swing completely around the peg. (*Hint:* Set up equations for the centripetal force and conservation law of energy)



Solution. *When the mass reaches to the top of the peg we have the centripetal force equation*

$$T + mg = \frac{mv^2}{r}$$

where T is the tension in the string, v is the tangential velocity at there, and $r = a - h$. The energy equation due to conservation law is

$$mg(h - r) = \frac{1}{2}mv^2$$

Eliminating the mv^2 of the equations we get

$$r(T + mg) = 2mg(h - r) \quad \Rightarrow \quad T = \frac{2mg(h - r) - rmg}{r} = \frac{2mgh - 3mgr}{r}$$

T needs to be a bit bigger than zero for performing a complete revolution, thus

$$2h - 3r > 0 \quad \Rightarrow \quad 2h - 3(a - h) > 0$$

It gives $h > \dfrac{3a}{5}$.