Section 5.1

Ex 9

a. $\int_{20}^{30}\int_{20}^{30} k(x^2+y^2)\,dx\,dy = 1$ (by ~~definition~~ definition),

$= k\int_{20}^{30}\int_{20}^{30} x^2\,dx\,dy + k\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$

$= k\left(\int_{20}^{30} \frac{19000}{3}\,dy + \int_{20}^{30} 10y^2\,dy\right)$

$= k\left(\frac{190000}{3} + \frac{190000}{3}\right)$, so $k = \frac{3}{380000}$

b. we know $20 \le x \le 26$, $20 \le y \le 26$, so $p(x \le 26, y \le 26) = \int_{20}^{26}\int_{20}^{26} \frac{3}{380000}(x^2+y^2)\,dx\,dy = \frac{189}{625}$

~~c. we know that $|x-y| \le 2$, the graph is~~

~~$p(x,y) = 1 - \int_{22}^{30}\int_{20}^{28} \frac{3}{380000}(x^2+y^2)\,dx\,dy - \int_{20}^{28}\int_{18}^{30} \frac{3}{38000}(x^2+y^2)\,dx\,dy = 1 - \frac{7576}{11875} = \frac{10124}{11875}$~~

c. we know that $|x-y| \le 2$, so

$p(x,y) = \int_{20}^{30}\int_{20}^{30} \frac{3}{380000}(x^2+y^2) - \int_{x+2}^{30}\int_{20}^{28} \frac{3}{380000}(x^2+y^2)\,dx\,dy - \int_{20}^{x-2}\int_{22}^{30} \frac{3}{380000}(x^2+y^2)\,dx\,dy$

$= 1 - \frac{6}{380000}\int_{x+2}^{30}\int_{20}^{28} (x^2+y^2)\,dx\,dy = 1 - \frac{-x^3-6x^2-1756x+17824}{23750}$ $(20 \le x \le 30)$

d. $f_X(x) = \int_{20}^{30} \frac{3}{38000}(x^2+y^2)\,dy = \frac{3x^2+1900}{380000}$

e. $f_Y(y) = \frac{3y^2+1900}{38000}$ (by contrast), $f_X \cdot f_Y = \frac{3}{38000}(x^2+y^2)+0.1 \ne f(x,y)$,

So it is dependent

B+

Ex12.

a. $f_X(x) = \int_0^{\infty} x e^{-x(1+y)} dy = e^{-x}$, $p = \int_2^{\infty} f_X(x)\, dx = \frac{1}{e^3}$

b. $f_X(x) = e^{-x}$; $f_Y(y) = \int_0^{\infty} x e^{-x(1+y)} dx =$ ~~which can not be calculated, so~~

x and y are depedent

c. $p(X \text{ or } Y \geq 3) = 1 - p(X \leq 3 \text{ and } Y \leq 3) = 1 - \int_0^3 \int_0^3 x e^{-x(1+y)} dx dy = \frac{1}{4} + \frac{1}{e^3}$

Ex12

a. $p(X > 3) = \int_3^{\infty} \int_0^{\infty} x e^{-x(1+y)} dy dx = \int_3^{\infty} e^{-x} dx = 0.5$

b. $f_X(x) = \int_0^{\infty} x e^{-x(1+y)} dy = e^{-x}$, $f_Y(y) = \int_3^{\infty} x e^{-x(1+y)} dx = \frac{1}{(1+y)^2}$, $f_X(x) \cdot f_Y(y) \neq f(x,y)$

So X and y are dependent

c. p(at least one exceeds 3) $= 1 - p(X \leq 3 \text{ and } Y \leq 3) = 1 - \int_0^3 \int_0^3 x e^{-x(1+y)} dy dx = e^{-3} + \frac{1}{4} - \frac{1}{4} e^{-12}$

Ex18. (by the table)

a. $P_{Y|X}(0|1) = \frac{f(0,1)}{f_X(1)} = \frac{0.08}{0.34} = 0.2353$

$P_{Y|X}(1|1) = \frac{0.2}{0.34} = 0.5882$

$P_{Y|X}(2|1) = \frac{0.6}{0.34} = 0.1765$

b. $P_{Y|X}(y|2)$ is requested, the diagram below is:

y	0	1	2
$P_{Y\|X}(y\|2)$	0.12	0.28	0.6

c. $P(Y \leq 1 \mid X = 2) = P_{Y|X}(0|2) + P_{Y|X}(1|2) = 0.12 + 0.28 = 0.4$

d. $P_Y(2) = 0.38$

X	0	1	2
$P_{X\|Y}(x\|2)$	0.0526	~~0.1776~~ 0.1579	0.7895

Ex19.

a. $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{k(x^2+y^2)}{10kx^2+0.05}$ $20 \le y \le 30$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{k(x^2+y^2)}{10ky^2+0.05}$ $20 \le x \le 30$ $(k = \frac{3}{380000})$

b. $P(Y \ge 25 | X=22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy = \int_{25}^{30} \frac{k(22^2+y^2)}{10k\,22^2+0.05}\,dy = 0.783$

$P(Y \ge 25) = \int_{25}^{30} f_Y(y)\,dy = \int_{25}^{30} (10ky^2+0.05)\,dy = 0.75$

c. $E(Y|X=22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|22)\,dy = \int_{20}^{30} y \cdot \frac{k(22^2+y^2)}{10k\,22^2+0.05}\,dy = 25.37$

$E(Y^2|X=22) = \int_{20}^{30} y^2 \frac{k(22^2+y^2)}{10k\,22^2+0.05}\,dy = 652.02$

$V(Y|X=22) = E(Y^2|X=22) - E(Y|X=22)^2 = 8.24$

Ex 24.

$p(x,y) = \frac{1}{30}$, $E[h(x,y)] = h(x,y) \cdot p(x,y) = \frac{84}{30} = 2.8$

Ex26.

Revenue $= 3X + 10Y$, $E(\text{revenue}) = E(3X+10Y) = \sum_{x=0}^{5}\sum_{y=0}^{2} (3x+10y) \cdot p(x,y) = 0 \cdot p(0,0) + \dots 35 \cdot p(5,2) = 15.4$

Ex33.

Since $E(XY) = E(X) \cdot E(Y)$, $Cov(X,Y) = E(XY) - E(X) \cdot E(Y) = E(X) \cdot E(Y) - E(X) \cdot E(Y) = 0$

$Corr_{(X,Y)} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0$

Ex 35

a. $Cov(aX+b, cY+d) = E[(aX+b)(cY+d)] - E(aX+b)(cY+d) = E[acXY + adX + bcY + bd] - E(aE(X)+b)(cE(Y)+d)$

$= ac[E(XY)] - acE(X)E(Y) = ac\,Cov(X,Y)$

b. $Corr(aX+b, cY+d) = \frac{Cov(aX+b, cY+d)}{\sqrt{V(aX+b)}\sqrt{V(cY+d)}} = \frac{ac\,Cov(X,Y)}{|a||c|\sqrt{V(X)}\sqrt{V(Y)}} = Cov(X,Y)$

c. $Corr(aX+b, cY+d) = -Corr(X,Y)$