A+

5.1 : 9, 12, 18, 19

5.2 : 24, 26, 33, 35.

9) The joint probability density function:

Let X and Y be contiguous random variables. f(x,y) is a function that is non-negative and for which $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

For every adequate set A the following holds: $P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$.

$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{20}^{30}\int_{20}^{30} k(x^2+y^2)\,dx\,dy = k\int_{20}^{30}\int_{20}^{30} x^2\,dy\,dx + k\int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy$

$= k\int_{20}^{30} x^2 (y|_{20}^{30})\,dx + k\int_{20}^{30} y^2 (x|_{20}^{30})\,dy = 10k \cdot \frac{x^3}{3}\Big|_{20}^{30} + 10k\,\frac{y^3}{3}\Big|_{20}^{30}$

$= 2\cdot 10k\left(\frac{30^3}{3} - \frac{20^3}{3}\right) = 20k \cdot \frac{19000}{3} = \frac{380000}{3}\cdot k$

$1 = \frac{380000}{3} k \Rightarrow k = \frac{3}{380000}$

b) Under-fill limit is 26 psi

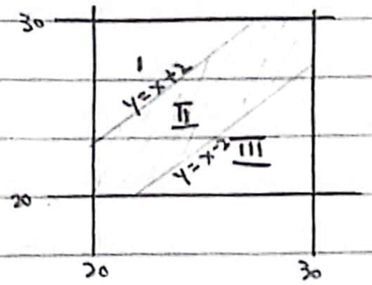
$P(X < 26 \text{ and } Y < 26) = P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy = \int_{20}^{26}\int_{20}^{26} k(x^2+y^2)\,dx\,dy$

$= k\int_{20}^{26}\int_{20}^{26} x^2\,dy\,dx + k\int_{20}^{26}\int_{20}^{26} y^2\,dx = k\int_{20}^{26} x^2 (y|_{20}^{26})\,dx + k\int_{20}^{26} y^2 (x|_{20}^{26})\,dy$

$= 6k\cdot\frac{x^3}{3}\Big|_{20}^{26} + 6k\cdot\frac{y^3}{3}\Big|_{20}^{26} = 2\times 6k\left(\frac{26^3}{3} - \frac{20^3}{3}\right) = 0.3024 = 2\times 6\times\frac{3}{380000}\left(\frac{26^3}{3} - \frac{20^3}{3}\right)$

c) Find subset of $20 \le x \le 30$, $20 \le y \le 30$ for which the difference is at most 2.

We can represent the distance between 2 points x and y as $|x-y|$. There for, find probability of event $\{|X-Y| \le 2\}$




$P(|X-Y| \le 2) = \iint_{II} f(x,y)\,dx\,dy$

$= 1 - \iint_{I} f(x,y)\,dx\,dy - \iint_{III} f(x,y)\,dx\,dy$

$= 1 - \int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx$

$= 1 - I_1 - I_2 = 1 - 0.3203 - 3203 = 0.3594$

(After much algebra)

⋮

d) Marginal probability mass function of a cont random var X $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$, $-\infty<x<\infty$

" " " " " " " " Y $f_Y(x)=$ " , $-\infty<y<\infty$

The following holds $f_X(x)=\int_{20}^{30} k(x^2+y^2)\,dy = 10kx^2+k\frac{y^3}{3}\Big|_{20}^{30} = 10kx^2+\frac{3}{380000}\cdot\left(\frac{30^3}{3}-\frac{20^3}{3}\right) = 10kx^2+0.05$

$f_X(x)=10kx^2+0.05$, $\quad 20\le x\le 30$

$f_X(x)=0$, $\quad x\notin[20,30]$

We would get the same marginal distribution for Y if we substitute x with y.

e) 2 random variables X and Y are independent iff

1) $p(x,y)=p_X(x)\cdot p_Y(y)$, for every $(x,y)$ and when X and Y discrete r.v's.

2) $f(x,y)=f_X(x)\cdot f_Y(y)$, " " " " " " " continuous "

Obviously, the random variables aren't independent, This is because functions $f(x,y)=k(x^2+y^2)$

and $f_X(x)\cdot f_Y(y)=(10kx^2+0.05)(10\cdot ky^2+0.05)$ are not equal.

$f_Y(y)$ can be obtained by substituting y for x in (d); clearly $f(x,y)\neq f_X(x)\cdot f_Y(y)$

12 a)

For every adequate set A: $P[(X,Y)\in A]=\iint_A f(x,y)\,dx\,dy$

$P(X>3)=\int_3^\infty \overbrace{\int_0^\infty x e^{-x(1+y)}\,dy}^{\text{marginal pdf of } y}\,dx = \int_3^\infty\int_0^\infty x e^{-x}\cdot e^{-xy}\,dy\,dx=\int_3 x\cdot e^{-x}\left(-\frac{1}{x}\right)e^{-xy}\Big|_0^\infty$

$=\int_3^\infty e^{-x}\,dx=-e^{-x}\Big|_3^\infty=e^{-3}=0.05$

b) The marginal pdf of X is

$f_X(x)=\int_{-\infty}^\infty f(x,y)\,dy=\int_0^\infty x e^{-x(1+y)}\,dy=\int_0^\infty x e^{-x}\cdot e^{-xy}\,dy = x e^{-x}\left(-\frac{1}{x}\right)e^{-xy}\Big|_0^\infty$

$=-e^{-x}\cdot\left(\lim_{y\to\infty}e^{-xy}-1\right)=e^{-x}$

$f_X(x)=\begin{cases}e^{-x}, & x\ge 0,\\ 0, & \text{otherwise.}\end{cases}$

The marginal pdf of Y is

$f_Y(y)=\int_{-\infty}^\infty f(x,y)\,dx=\int_0^\infty x e^{-x(1+y)}\,dx$ $\qquad \lim_{x\to\infty}\frac{x}{e^{x(1+y)}}=\lim_{x\to\infty}\frac{1}{(1+y)e^{x(1+y)}}=0$

$=\left|\begin{array}{l} u=x \Rightarrow du=dx \\ e^{-x(1+y)}dx=dv \Rightarrow v=-\frac{1}{1+y}e^{-x(1+y)}\end{array}\right|$

$=x\cdot\left(-\frac{1}{1+y}e^{-x(1+y)}\right)\Big|_0^\infty+\int_0^\infty\frac{1}{1+y}e^{-x(1+y)}\,dx \overset{L'H}{=} 0+\frac{1}{1+y}\int_0^\infty e^{-x(1+y)}\,dx$

$=\frac{1}{1+y}\cdot\left(-\frac{1}{1+y}e^{-x(1+y)}\right)\Big|_0^\infty=\frac{1}{(1+y)^2}$

marginal pdf of Y is $f_Y(y)=\begin{cases}\frac{1}{(1+y)^2}, & y\ge 0\\ 0, & \text{otherwise.}\end{cases}$

It is obvious that the joint pdf is not the product of marginal pdf's for every $(x,y)$, hence random variables are dependent.

c) $\{X>3\} \cup \{Y>3\} = 1 - P(\{X \le 3\} \cap \{Y \le 3\}) = 1 - \int_0^3\int_0^3 x \cdot e^{-x(1+y)}\, dy\, dx$

$= 1 - \int_0^3\int_0^3 x e^{-x} e^{-xy}\, dy\, dx = 1 - \int_0^3 x e^{-x}\left(-\frac{1}{x}\left(e^{-3x} - e^{-x \cdot 0}\right)\right) dx$

$= 1 - \int_0^3 e^{-x}(1 - e^{-3x})\, dx = 1 - \left(-e^{-x}\Big|_0^3\right) + \left(-\frac{1}{4}e^{-4x}\Big|_0^3\right) = 0.3$

(8)

from exercise 1:

| $p(x,y)$ | | $y=0$ | $1$ | $2$ |
|---|---|---|---|---|
| $x$ | 0 | 0.1 | 0.04 | 0.02 |
| | 1 | 0.08 | 0.2 | 0.06 |
| | 2 | 0.06 | 0.14 | 0.3 |

The marginal probability mass function of discrete random variable $X$ is

$p_X(x) = \sum_y p(x,y)$, for every $x$,

By definition, for $x \in \{0,1,2\}$

$p_X(x) = \sum_{y \in \{0,1,2\}} p(x,y) = p(x,0) + p(x,1) + p(x,2)$

For every $x \in \{0,1,2\}$ the marginal probability is the sum of a particular row. Therefore, we have

$$\left.\begin{aligned} P_X(0) &= 0.1 + 0.04 + 0.02 = 0.16 \\ P_X(1) &= 0.08 + 0.2 + 0.06 = 0.34 \\ P_X(2) &= 0.06 + 0.14 + 0.3 = 0.5 \end{aligned}\right\} \text{marginal pmf of } X$$

a) $P_X(1) = 0.34$. The conditional pmf of $Y$ given $X=1$ is

$P_{Y|X}(0|1) = \frac{p(1,0)}{p_X(1)} = \frac{0.08}{0.34} = 0.2353$

$P_{Y|X}(1|1) = \frac{p(1,1)}{p_X(1)} = \frac{0.2}{0.34} = 0.5882$

$P_{Y|X}(2|1) = \frac{p(1,2)}{p_X(1)} = \frac{0.06}{0.34} = 0.1765$

b) $P_X(2) = 0.5$, because we need to determine conditional pmf of $Y$ given $X=2$. Similarly as in (a),

$P_{Y|X}(0|2) = \frac{p(1,0)}{P_X(2)} = \frac{0.06}{0.5} = 0.12$

$P_{Y|X}(1|2) = \frac{p(1,1)}{P_X(2)} = \frac{0.14}{0.5} = 0.28$

$P_{Y|X}(2|2) = \frac{p(1,2)}{P_X(2)} = \frac{0.3}{0.5} = 0.6$

c) $P(Y \le 1 | X=2) = P(Y=0|X=2) + P(Y=1|X=2)$
$= p_{Y|X}(0|2) + p_{Y|X}(1|2) = 0.12 + 0.28 = 0.4$

d) We need to compute conditional pmf of X given $Y=2$, so we only need $p_Y(2)$. $p_Y(2) = 0.02 + 0.06 + 0.3 = 0.38$

The conditional pmf of X given $Y=2$ is

$p_{X|Y}(0|2) = \frac{p(0,2)}{p_Y(2)} = \frac{0.2}{0.38} = 0.0526$

$p_{X|Y}(1|2) = \frac{p(1,2)}{p_Y(2)} = \frac{0.06}{0.38} = 0.1579$

$p_{X|Y}(2|2) = \frac{p(2,2)}{p_Y(2)} = \frac{0.3}{0.38} = 0.7895$

19) According exercise 9, $K = \frac{3}{380000}$

a) The conditional probability density function:

The marginal " mass " " of X

" " " " " of Y

From 9d, we get $f_X(x) = 10Kx^2 + 0.05$, $20 \le x \le 30$
$f_X(x) = 0$, $x \notin [20, 30]$

We would get the same marginal distribution for Y if we substitute x with y

So, conditional pdf of Y given that $X=x$ is $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{K(x^2+y^2)}{10Kx^2+0.05}, 20 \le y \le 30$

Similarly, " " " " " " " " $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{K(x^2+y^2)}{10Ky^2+0.05}, 20 \le x \le 30$

b) Probability of the first event, $\{Y \ge 25 | X = 22\}$

$P(Y \ge 25 | X = 22) = \int_{25}^{30} f_{Y|X}(y|22)\,dy = \int_{25}^{30} \frac{K(22^2+y^2)}{10K\cdot 22^2 + 0.05}\,dy = \int_{25}^{30} \frac{K(22^2)}{10K\cdot 22^2+0.05}\,dy +$
$\int_{25}^{30} \frac{Ky^2}{10K\cdot 22^2 + 0.05}\,dy = \frac{K\cdot 22^2}{10K\cdot 22^2+0.05}(30-25) + \frac{K}{10K\cdot 22^2 + 0.05} \frac{y^3}{3}\Big|_{25}^{30} = 0.56$

Probability of $2^{nd}$ event, $\{Y \ge 25\}$ is

$P(Y \ge 25) = \int_{25}^{30} f_Y(y)\,dy = \int_{25}^{30} (10Ky^2 + 0.05)\,dy = \int_{25}^{30} 10Ky^2\,dy + \int_{25}^{30} 0.05\,dy$
$= 10K \frac{y^3}{3}\Big|_{25}^{30} + 0.05(30-25) = 0.549$

Probability of $1^{st}$ event is greater than the $2^{nd}$

c) Expected value (mean value of a continuous random variable X with pdf $f(x)$ is $E(X) = \mu_X = \int_{-\infty}^{\infty} x f(x)\,dx$.

$E(Y|X=22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|22)\,dy = \int_{20}^{30} y \frac{K(22^2+y^2)}{10K\cdot 22^2 + 0.05}\,dy$

$$= \int_{20} y \cdot \frac{k \cdot 22^2}{10k \cdot 22^2 + 0.05}\, dy + \int_{20}^{30} \frac{ky^3}{10k \cdot 22^2 + 0.05}\, dy = \frac{k \cdot 22^2}{10k \cdot 22^2 + 0.05}(30-20) + \frac{k}{10k \cdot 22^2 + 0.05} \cdot \frac{y^3}{3}\Big|_{20}^{30}$$

$= 25.37$ psi let tire

$$E(Y^2 \mid X=22) = \int_{-\infty}^{\infty} y^2 \cdot f_{Y|X}(y \mid 22)\, dy = \int_{20}^{30} y^2 \cdot \frac{k(22^2+y^2)}{10k \cdot 22^2 + 0.05}\, dy$$

$$= \int_{20}^{30} y^2 \cdot \frac{k \cdot 22^2}{10k \cdot 22^2 + 0.05}\, dy + \int_{20}^{30} \frac{ky^4}{10k \cdot 22^2 + 0.05}\, dy$$

$$= \frac{k \cdot 22^2}{10k \cdot 22^2 + 0.05} \cdot \frac{y^3}{3}\Big|_{20}^{30} + \frac{k}{10k \cdot 22^2 + 0.05} \frac{y^5}{5}\Big|_{20}^{30} = 652{,}03.$$

The variance $V(Y \mid X=22) = \sigma_{Y|X=22} = E(Y^2 \mid X=22) - [E(Y \mid X=22)]^2$

$= 652{,}03 - 25.37^2 = 8{,}3931$

Standard dev: $\sigma_{Y|X=22} = \sqrt{V(Y \mid X=22)} = \sqrt{8{,}3931} = 2{,}8971$

24) Expected value (Mean value) of a random variable $g(X, Y)$, where $g(\cdot)$ is a function, denoted as $E[g(X,Y)]$ is given by:

$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y) \cdot p(x,y) & , X \text{ and } Y \text{ discrete}, \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx\, dy & , X \text{ and } Y \text{ continuous}. \end{cases}$$

where $p(x,y)$ is pmf and $f(x,y)$ pdf

So determine $g(X, Y)$ (number of individuals (including A & B) who handle the message. A and B cannot sit on the same spot.

Minimum number of individuals handling message is 2 (only A and B)

Maximum " " " " " " 4 (when A " " sit across)

The seats are numbered: X represent A'seats number, Y for B'seats number.

(2,1) means A sit on seat 2, B sit on seat 1

$g(2,1) = 2$ (only 2 indovaals (A & B) handles the message).

| $g(x,y)$ | $y$ = 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $x$ = 1 | – | 2 | 3 | 4 | 3 | 2 |
| 2 | 2 | – | 2 | 3 | 4 | 3 |
| 3 | 3 | 2 | – | 2 | 3 | 4 |
| 4 | 4 | 3 | 2 | – | 2 | 3 |
| 5 | 3 | 4 | 3 | 2 | – | 2 |
| 6 | 2 | 3 | 4 | 3 | 2 | – |

Number of ways A an B can sit, $6 \times 5 = 30$

So $p(x,y) = \frac{1}{30}$

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p(x,y)$$

$$= \frac{1}{30} \sum_x \sum_y g(x,y)$$

$$= \frac{1}{30} \cdot 84 = 2.8$$

$6 \times 14$?

26) $g(X,Y) = 3X + 10Y$

$E(3X+10Y) = \sum_x \sum_y (3x+10y)\cdot p(x,y) = (3\cdot 0 + 10\cdot 0)\cdot p(0,0) + (3\cdot 0 + 10\cdot 1)\cdot p(0,1)$
$+ (3\cdot 0 + 10\cdot 2)\cdot p(0,2) + \cdots + (3\cdot 5 + 10\cdot 2)\cdot p(5,2) = 0\cdot 0.025 + 10\cdot 0.015 +$
$20\cdot 0.01 + \cdots + 35\cdot 0.02 = 15.4$

33) Proposition: $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

The correlation coefficient of X and Y is $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y}$

Since $E(XY) = E(X)\cdot E(Y)$, $Cov(X,Y) = E(XY) - E(X)\cdot E(Y) =$
$E(X)\cdot E(Y) - E(X)\cdot E(Y) = 0$ and since $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$, then $Corr(X,Y) = 0$

35) $Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$

a) $Cov(aX+b, cY+d) = E([aX+b]\cdot[cY+d]) - E(aX+b)\cdot E(cY+d) =$
$E(acXY + adX + bcY + bd) - (aE(X)\cdot(cE(Y)+d)$
$= ac\,E(XY) + ad\,E(X) + bc\,E(Y) + bd - (ac\,E(X)E(Y)) + ad\,E(X) + bc\,E(Y) + bd)$
$= acE(XY) - acE(X)E(Y) = ac(E(XY) - E(X)E(Y)) = ac\,Cov(X,Y)$

b) $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y}$

(a) + properties of $\sigma$

$Corr(aX+b, cY+d) = \frac{Cov(aX+b, cY+d)}{\sigma_{aX+b}\cdot\sigma_{cY+d}} = \frac{ac\,Cov(X,Y)}{|a|\sigma_x \cdot |c|\sigma_y} = \frac{ac}{|ac|}$ ~~uma~~ $\times Corr(X,Y)$

When a and c have the same sign, ~~we have~~ $ac = |ac|$, and
$Corr(aX+b, cY+d) = \frac{ac}{ac} Corr(X,Y) = Corr(X,Y)$.

c) If a and c have opposite signs, than $|ac| = -ac$,
$Corr(aX+b, cY+d) = \frac{ac}{-ac} Corr(X,Y) = -Corr(X,Y)$.