Chapter 3 Interpolation

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Motivation

- In the nature, we obtain only finite data via taking samples.
- Usually, we need the value which does not appear in the samples.

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Example

Time	7:00	8:00	9:00	10:00	11:00
Temperature	25°C	26°C	27°C	28°C	29°C

Questions:

1. What is the temperature at 9:30?
2. How about 8:20 or other time points (may be infinite)?

A solution

1. Generate a function $f(x)$ on the real numbers approximating the temperature at every time-point x;
2. Compute $f(x)$, which is the temperature at x.

Example

- $f(x) = x + 18$;
- $f(9.5) = 27.5$°C;
- $f(8.33...) = 26.33...$°C.

Two remaining questions

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2. Answer 1: it exactly matches each data of sample dataset.

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1. Question 1: which properties $f(x)$ satisfies?
2. Answer 1: it exactly matches each data of sample dataset.
3. Question 2: how to generate a desired $f(x)$?
4. Answer 2: the content of this chapter.

Outline

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Polynomial

Definition (Polynomial)

A function $f(x)$ is a **polynomial**, if

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \text{ for some } n \in \mathbb{N}.$$

where $c_n \neq 0$ is the leading coefficient of f and n is the degree of f.

- Summation notation: $\sum_{i=0}^{n} c_i x^i$;
- $P(x)$: a polynomial;
- $P_n(x)$: a polynomial with degree n.

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Expressions	Polynomial	Leading coefficients	Degrees
$x^3 + 3x^2 + 4x + 1$	✓	1	3

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$2x + 3$	✓: Linear	2	1
60	✓: Constant	60	0
$\sin x + \cos x$	✗	-	-

Polynomial interpolation

Definition (Polynomial interpolation)

A polynomial $y = P(x)$ **interpolates** the data points $(x_1, y_1), \ldots, (x_n, y_n)$ if $P(x_i) = y_i$ for $1 \leq i \leq n$.

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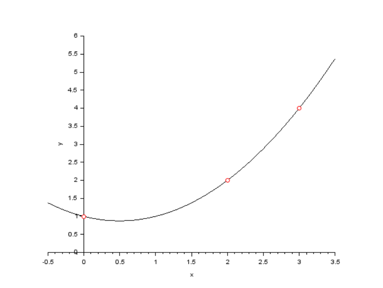
Example

Let $(0, 1), (2, 2)$ and $(3, 4)$ be the points.
$P(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 1$ interpolates the above points.



Lagrange interpolation

- Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$,
- we use the template $y_1 L_1(x) + \cdots y_n L_n(x)$ to interpolate the n points where
  1. $L_k(x)$ is a polynomial;
  2. $L_k(x_k) = 1$;
  3. $L_k(x_j) = 0$ for $j \neq k$.

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Definition

$$L_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})\cancel{(x-x_k)}(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})\cancel{(x_k-x_k)}(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

Summation notation: $\dfrac{\prod_{j=1,j\neq k}^{n}(x-x_j)}{\prod_{j=1,j\neq k}^{n}(x_k-x_j)}$.

Lagrange interpolation

Example

Suppose that we are given 3 points $(0,1),(2,2),(3,4)$.

- $L_1(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)} = \frac{x^2-5x+6}{6}$;
- $L_2(x) = \frac{(x-0)(x-3)}{(2-0)(2-3)} = -\frac{x^2-3x}{2}$;
- $L_3(x) = \frac{(x-0)(x-2)}{(3-0)(3-2)} = -\frac{x^2-2x}{3}$.

Lagrange interpolation

Definition (Lagrange interpolation)

The **Lagrange interpolating polynomial** for these points is

$$P(x) = y_1 L_1(x) + \cdots y_n L_n(x).$$

Summation notation: $\sum_{k=1}^{n} y_k \frac{\prod_{j=1, j \neq k}^{n} (x - x_j)}{\prod_{j=1, j \neq k}^{n} (x_k - x_j)}$

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Proposition

Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n points with distinct x_i. Let $P(x)$ be the Lagrange interpolating polynomial for these points. Then, $P(x)$ interpolates the points, i.e., $P(x_i) = y_i$ for $1 \leq i \leq n$.

Lagrange interpolation

Example

Suppose that we are given 3 points $(0,1),(2,2),(3,4)$.

$$\begin{aligned} P(x) &= 1L_1(x)+2L_2(x)+4L_3(x) \\ &= \frac{x^2-5x+6}{6}+2(-\frac{x^2-3x}{2})+4(-\frac{x^2-2x}{3}) \\ &= \frac{x^2}{2}-\frac{x}{2}+1. \end{aligned}$$

Main theorem

Theorem

Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n points with distinct x_i. Then there exists ***one and only one*** *polynomial $P(x)$ of degree $n-1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \cdots, n$.*

Proof of Existence.

Lagrange interpolation.

Main theorem

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A degree n polynomial $P_n(x)$ have at most n zeros, unless $P_n(x) \equiv 0$.

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Proof of Uniqueness.

1. $P(x)$ and $Q(x)$: any two polynomials
   - whose degrees are $\leq n - 1$;
   - interpolate all n points.

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A degree n polynomial $P_n(x)$ have at most n zeros, unless $P_n(x) \equiv 0$.

Proof of Uniqueness.

1. $P(x)$ and $Q(x)$: any two polynomials
   - whose degrees are $\leq n-1$;
   - interpolate all n points.
2. $H(x) = P(x) - Q(x)$: a polynomial
   - whose degree is also $\leq n-1$;
   - $H(x_i) = 0$ for $i = 1, \cdots, n$, *i.e.*, $H(x)$ have n zeros.

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A degree n polynomial $P_n(x)$ have at most n zeros, unless $P_n(x) \equiv 0$.

Proof of Uniqueness.

1. $P(x)$ and $Q(x)$: any two polynomials
   - whose degrees are $\leq n-1$;
   - interpolate all n points.
2. $H(x) = P(x) - Q(x)$: a polynomial
   - whose degree is also $\leq n-1$;
   - $H(x_i) = 0$ for $i = 1, \cdots, n$, *i.e.*, $H(x)$ have n zeros.
3. So $H(x) \equiv 0$, *i.e.*, $P(x) = Q(x)$.

Outline

A more efficient method than Lagrange interpolation

- Lagrange interpolation: needs many computations and does not support incremental computation;
- An alternative method (Newton's divided differences): more manageable, less computations and support incremental computation.
  - Template: $c_0 + c_1(x - x_1) + \ldots + c_{n-1}(x - x_1)\cdots(x - x_{n-1})$.

Coefficient

Definition

$f[x_1 \dots x_n]$: the degree $n-1$ coefficient of the polynomial that interpolates $(x_1, y_1), \dots, (x_n, y_n)$.

- $f[x_5 x_1 x_3]$: the degree 2 coefficient of the polynomial that interpolates $(x_5, y_5), (x_1, y_1), (x_3, y_3)$.

Proposition

Base case $f[x_j] = y_j$ *for* $j = 1, \cdots, n$*;*

Inductive step $f[x_i \cdots x_j] = \frac{f[x_{i+1} \cdots x_j] - f[x_i \cdots x_{j-1}]}{x_i - x_j}$.

Coefficient

Example

$$
\begin{array}{c|ccc}
x_1 & f[x_1] & & \\
 & & f[x_1\, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1} & \\
x_2 & f[x_2] & & f[x_1\, x_2\, x_3] = \dfrac{f[x_2\, x_3] - f[x_1\, x_2]}{x_3 - x_1} \\
 & & f[x_2\, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2} & \\
x_3 & f[x_3]. & &
\end{array}
$$

Coefficient

Example

The data points are $(0,1),(2,2),(3,4)$. The coefficients are as follows:

$$
\begin{array}{c|ccc}
0 & 1 & & \\
 & & \dfrac{2-1}{2-0}=\dfrac{1}{2} & \\
2 & 2 & & \dfrac{2-\frac{1}{2}}{3-0}=\dfrac{1}{2} \\
 & & \dfrac{4-2}{3-2}=2 & \\
3 & 4 & &
\end{array}
$$

Newton's divided difference formula

Definition

The **Newton's divided difference formula** is

$$\begin{aligned} P(x) = f[x_1] \quad &+ \quad f[x_1 x_2](x - x_1) \\ &+ \quad f[x_1 x_2 x_3](x - x_1)(x - x_2) \\ &+ \quad \cdots \\ &+ \quad f[x_1 \dots x_n](x - x_1) \cdots (x - x_{n-1}). \end{aligned}$$

Summation notation: $P(x) = f[x_1] + \sum_{i=2}^{n} f[x_1 \dots x_i] \prod_{j=1}^{i-1} (x - x_j).$

Example

The data points are $(0,1), (2,2), (3,4)$. Then, the Newton's divided difference formula for these points is

$$P(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2) = \frac{x^2}{2} - \frac{x}{2} + 1.$$

Incremental computation

Example

The data points are $(0,1),(2,2),(3,4)$, together with $(1,0)$. The coefficients are as follows:

$$
\begin{array}{c|cccc}
0 & 1 & & & \\
 & & \frac{1}{2} & & \\
2 & 2 & & \frac{1}{2} & \\
 & & 2 & & -\frac{1}{2} \\
3 & 4 & & 0 & \\
 & & 2 & & \\
1 & 0 & & &
\end{array}
$$

Then, the Newton's divided difference formula for these points is

$$P(x) = 1 + \frac{1}{2}(x-0) + \frac{1}{2}(x-0)(x-2) - \frac{1}{2}(x-0)(x-2)(x-3).$$

Outline

How to compute the value of a complicated function

- Suppose that $f(x) = \sin(x)$.
- Question: what is $f(x)$ when $x = 1, 2, 3, \ldots$?

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- Suppose that $f(x) = \sin(x)$.
- Question: what is $f(x)$ when $x = 1, 2, 3, \ldots$?
- Answer:
  1. compute a polynomial $P(x)$, which is approximate to $\sin(x)$, via polynomial interpolation;
  2. compute an approximate value of $f(x)$ via the polynomial.

How to compute the value of a complicated function

- The fundamental domain for sine: $[0, \frac{\pi}{2}]$.
  1. $\sin(x) = \sin(\pi - x)$ when $\frac{\pi}{2} < x \leq \pi$;
  2. $\sin(x) = -\sin(2\pi - x)$ when $\pi < x \leq 2\pi$.

4 data points:

Accurate	$(0,0)$	$(\frac{\pi}{6}, 0.5)$	$(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$	$(\frac{\pi}{2}, 1)$
Approximate	$(0,0)$	$(0.5236, 0.5)$	$(1.0472, 0.8660)$	$(1.5708, 1)$

The coefficients are as follows:

$$
\begin{array}{c|llll}
0 & 0 & & & \\
 & & 0.9549 & & \\
0.5236 & 0.5 & & -0.2443 & \\
 & & 0.6990 & & -0.1139 \\
1.0472 & 0.8660 & & -0.4232 & \\
 & & 0.2559 & & \\
1.5708 & 1 & & &
\end{array}
$$

How to compute the value of a complicated function

The interpolating polynomial $P(x)$:
$0.9549x - 0.2443x(x - 0.5236) - 0.1139x(x - 0.5236)(x - 1.0472)$.

x	Accurate value	Approximate value	error
1	0.8415	0.8411	0.0004
2	0.9093	0.9102	0.0009
3	0.1411	0.1428	0.0017
4	-0.7568	-0.7557	0.0011

Outline

Interpolation error formula

Question: How much error is made when a function $y = f(x)$ is replaced by an interpolating polynomial?

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Theorem

- *$f(x)$: $n+1$ times continuously differentiable function;*
- *$P(x)$: the interpolating polynomial fitting the n points*
  $$(x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

The interpolation error is

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c),$$

where $\min\{x, x_1, \ldots, x_n\} \leq c \leq \max\{x, x_1, \ldots, x_n\}$.

Interpolation error formula

Example

- Four points: $(0,0)$, $(\frac{\pi}{6}, 0.5)$, $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ and $(\frac{\pi}{2}, 1)$;
- $\sin(x) - P(x) = \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} f''''(c)$;
- $|\sin(x) - P(x)| \leq \frac{|(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})|}{24}$;

Interpolation error formula

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- $\sin(x) - P(x) = \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} f''''(c)$;
- $|\sin(x) - P(x)| \leq \frac{|(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})|}{24}$;
- $|\sin(1) - P(1)| \leq \frac{|(1-0)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})|}{24} \approx 0.0005348$;
- $|\sin(1) - P(1)| \approx 0.0004$;

Interpolation error formula

Example

- Four points: $(0,0)$, $(\frac{\pi}{6}, 0.5)$, $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$ and $(\frac{\pi}{2}, 1)$;
- $\sin(x) - P(x) = \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} f''''(c)$;
- $|\sin(x) - P(x)| \leq \frac{|(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})|}{24}$;
- $|\sin(1) - P(1)| \leq \frac{|(1-0)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})|}{24} \approx 0.0005348$;
- $|\sin(1) - P(1)| \approx 0.0004$;
- Smaller errors when x is closer to the middle of the interval of x_i's than when it is near one of the ends.
- $|\sin(0.2) - P(0.2)| \leq \frac{|(0.2-0)(0.2-\frac{\pi}{6})(0.2-\frac{\pi}{3})(0.2-\frac{\pi}{2})|}{24} \approx 0.00313$ $> 5\frac{|(1-0)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})|}{24}$;
- $|\sin(0.2) - P(0.2)| \approx 0.00189 > |\sin(1) - P(1)|$.

Outline

Proof of Newton form

- Given n points: $(x_1, y_1), \ldots, (x_n, y_n)$;
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- $P(x)$: the polynomial that interpolates the above points;
- $f[x_1 \ldots x_k]$: the degree $k-1$ coefficient of the polynomial that interpolates the first k points;

Proof of Newton form

Lemma

$f[x_1 \dots x_k] = f[\sigma(x_1) \dots \sigma(x_k)]$ *for any permutation* σ *of the* x_i.

Example

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\begin{array}{c|ccc}
x_1 & f[x_1] & & \\
 & & f[x_1 x_2] & \\
x_2 & f[x_2] & & f[x_1 x_2 x_3] \\
 & & f[x_2 x_3] & \\
x_3 & f[x_3] & &
\end{array}
\qquad
\begin{array}{c|ccc}
x_3 & f[x_3] & & \\
 & & f[x_3 x_2] & \\
x_2 & f[x_2] & & f[x_3 x_2 x_1] \\
 & & f[x_2 x_1] & \\
x_1 & f[x_1] & &
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\end{array}
\qquad
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x_3 & f[x_3] & & \\
 & & f[x_3 x_2] & \\
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 & & f[x_2 x_1] & \\
x_1 & f[x_1] & &
\end{array}
$$

- $f[x_1 x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{f[x_1]-f[x_2]}{x_1-x_2} = f[x_2 x_1]$;

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\begin{array}{c|cccc}
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\end{array}
$$

- $f[x_1 x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{f[x_1]-f[x_2]}{x_1-x_2} = f[x_2 x_1]$;
- $f[x_2 x_3] = f[x_3 x_2]$ and $f[x_1 x_2 x_3] = f[x_3 x_2 x_1] = f[x_2 x_3 x_1]$.

Proof of Newton form

Lemma

$f[x_1 \dots x_k] = f[\sigma(x_1) \dots \sigma(x_k)]$ *for any permutation* σ *of the* x_i.

Example

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\begin{array}{c|ccc}
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x_3 & f[x_3] & &
\end{array}
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\end{array}
$$

- $f[x_1 x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{f[x_1]-f[x_2]}{x_1-x_2} = f[x_2 x_1]$;
- $f[x_2 x_3] = f[x_3 x_2]$ and $f[x_1 x_2 x_3] = f[x_3 x_2 x_1] = f[x_2 x_3 x_1]$.

Proof.

By uniqueness of the interpolating polynomial.

Proof of Newton form

Lemma

$P(x)$ can be written in the form

$$c_0 + c_1(x - x_1) + \ldots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}).$$

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$P(x)$ can be written in the form

$$c_0 + c_1(x - x_1) + \ldots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}).$$

Proof.

1. $c_{n-1} = f[x_1 x_2 \cdots x_n]$.

Proof of Newton form

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$P(x)$ can be written in the form

$$c_0 + c_1(x - x_1) + \ldots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}).$$

Proof.

1. $c_{n-1} = f[x_1 x_2 \cdots x_n]$.
2. From $n-2$ to 0: let c_k be the degree k coefficient of the polynomial

$$P_k(x) : P(x) - c_{n-1}(x - x_1)\cdots(x - x_{n-1}) - \cdots - c_{k+1}(x - x_1)\cdots(x - x_{k+1}).$$

Proof of Newton form

Lemma

$P(x)$ can be written in the form

$$c_0 + c_1(x - x_1) + \ldots + c_{n-1}(x - x_1)\cdots(x - x_{n-1}).$$

Proof.

1. $c_{n-1} = f[x_1 x_2 \cdots x_n]$.
2. From $n - 2$ to 0: let c_k be the degree k coefficient of the polynomial

$$P_k(x) : P(x) - c_{n-1}(x - x_1)\cdots(x - x_{n-1}) - \cdots - c_{k+1}(x - x_1)\cdots(x - x_{k+1}).$$

- Actually, each $P_k(x)$ is the polynomial interpolating the first $k + 1$ points;
- The degree k coefficient of the polynomial $P_k(x)$ is $f[x_1 x_2 \cdots x_{k+1}]$.

Proof of Newton form

Theorem

1. $P(x) = f[x_1] + f[x_1 x_2](x - x_1) + \ldots + f[x_1 \ldots x_n] \prod_{j=1}^{n} (x - x_j)$;
2. for $k > 1$, $f[x_1 \ldots x_k] = \frac{f[x_2 \ldots x_k] - f[x_1 \ldots x_{k-1}]}{x_k - x_1}$.

Proof of Newton form

Proof of (1).

It suffices to prove that $c_{k-1} = f[x_1 x_2 \cdots x_k]$ for $1 \leq k \leq n$.

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$c_{n-1} = f[x_1 x_2 \cdots x_n]$ is proved in the above lemma.
Now, we consider the case where $1 \leq k < n$.
When $x \in \{x_1, \ldots, x_k\}$,

- The first k terms of $P(x)$ are non-zero;
- The last $n-k$ terms of $P(x)$ are zero.

$$P(x) = c_0 + c_1(x - x_1) + \ldots + c_{k-1} \prod_{j=1}^{k-1} (x - x_j) \not\Rightarrow 0 \quad (P_{k-1}(x))$$
$$+ c_k \prod_{j=1}^{k} (x - x_j) + \ldots + c_{n-1} \prod_{j=1}^{n-1} (x - x_j) \Rightarrow 0.$$

Proof of Newton form

Proof of (1).

It suffices to prove that $c_{k-1} = f[x_1 x_2 \cdots x_k]$ for $1 \leq k \leq n$.
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$$
\begin{aligned}
P(x) = {} & c_0 + c_1(x - x_1) + \ldots + c_{k-1} \prod_{j=1}^{k-1} (x - x_j) \not\Rightarrow 0 \ \ (P_{k-1}(x)) \\
& + c_k \prod_{j=1}^{k} (x - x_j) + \ldots + c_{n-1} \prod_{j=1}^{n-1} (x - x_j) \Rightarrow 0.
\end{aligned}
$$

Since the magenta part is with degree at most $k-1$ and interpolates the first k points, $c_{k-1} = f[x_1 \ldots x_k]$.

Proof of Newton form

Proof of (2).

The interpolating polynomial of $x_2, x_3, \ldots, x_{k-1}, x_1, x_k$ is

$$\begin{aligned} P_1(x) &= f[x_2] + f[x_2 x_3](x - x_2) + \ldots \\ &\quad f[x_2 \ldots x_{k-1} x_1](x - x_2) \cdots (x - x_{k-1}) + \\ &\quad f[x_2 \ldots x_{k-1} x_1 x_k](x - x_2) \cdots (x - x_{k-1})(x - x_1) \end{aligned}$$

Proof of Newton form

Proof of (2).

The interpolating polynomial of $x_2, x_3, \ldots, x_{k-1}, x_1, x_k$ is

$$\begin{aligned} P_1(x) \quad = \quad & f[x_2] + f[x_2 x_3](x - x_2) + \ldots \\ & f[x_2 \ldots x_{k-1} x_1](x - x_2) \cdots (x - x_{k-1}) + \\ & f[x_2 \ldots x_{k-1} x_1 x_k](x - x_2) \cdots (x - x_{k-1})(x - x_1) \end{aligned}$$

The interpolating polynomial of $x_2, x_3, \ldots, x_{k-1}, x_k, x_1$ is

$$\begin{aligned} P_2(x) \quad = \quad & f[x_2] + f[x_2 x_3](x - x_2) + \ldots \\ & f[x_2 \ldots x_{k-1} x_k](x - x_2) \cdots (x - x_{k-1}) + \\ & f[x_2 \ldots x_{k-1} x_k x_1](x - x_2) \cdots (x - x_{k-1})(x - x_k) \end{aligned}$$

Proof of Newton form

Proof of (2).

Setting $P_1(x_k) = P_2(x_k)$ and canceling terms yields

$$\begin{aligned} & f[x_2 \dots x_{k-1} x_1](x_k - x_2) \cdots (x_k - x_{k-1}) + \\ & f[x_2 \dots x_{k-1} x_1 x_k](x_k - x_2) \cdots (x_k - x_{k-1})(x_k - x_1) \\ = \; & f[x_2 \dots x_{k-1} x_k](x_k - x_2) \cdots (x_k - x_{k-1}) + \\ & f[x_2 \dots x_{k-1} x_k x_1](x_k - x_2) \cdots (x_k - x_{k-1})(x_k - x_k) \end{aligned}$$

So $f[x_2 \dots x_{k-1} x_1] + f[x_2 \dots x_{k-1} x_1 x_k](x_k - x_1) = f[x_2 \dots x_{k-1} x_k]$.
The above can be rearranged to $f[x_1 \dots x_k] = \frac{f[x_2 \dots x_k] - f[x_1 \dots x_{k-1}]}{x_k - x_1}$.

Proof of error formula

Theorem

- $f(x)$: $n+1$ *times continuously differentiable function;*
- $P(x)$: *the interpolating polynomial fitting the* n *points*
  $$(x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

The interpolation error is

$$f(x) - P(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c),$$

where $\min\{x, x_1, \ldots, x_n\} \le c \le \max\{x, x_1, \ldots, x_n\}$.

Proof of error formula

Proof.

- By adding one more point x to the set of points, the new interpolation polynomial:
  $$P_n(t) = P_{n-1}(t) + f[x_1 \dots x_n x](t - x_1) \dots (t - x_n)$$
  where $P_{n-1}(t) = P(t)$.

Proof of error formula

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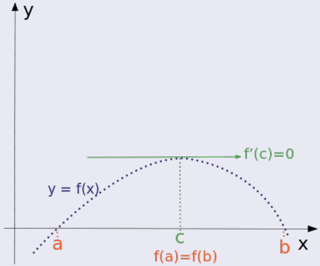
$$P_n(t) = P_{n-1}(t) + f[x_1 \ldots x_n x](t - x_1) \ldots (t - x_n)$$
where $P_{n-1}(t) = P(t)$.
Evaluated at the extra point x, $P_n(x) = f(x)$, so
$$f(x) = P_{n-1}(x) + f[x_1 \ldots x_n x](x - x_1) \ldots (x - x_n).$$
We want to prove that $f[x_1 \ldots x_n x] = \frac{f^{(n)}(c)}{n!}$.

Proof of error formula

Theorem (Rolle's Theorem)

Let f be a continuously differentiable function on the interval $[a, b]$, and $f(a) = f(b)$. Then, there exists a number c between a and b s.t. $f'(c) = 0$.



Proof of error formula

Proof.

- Define

$$h(t) = f(t) - P_{n-1}(t) - f[x_1 \dots x_n x](t - x_1) \dots (t - x_n).$$

Note that $h(x) = h(x_1) = \cdots = h(x_n) = 0.$

Proof of error formula

Proof.

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$$h(t) = f(t) - P_{n-1}(t) - f[x_1 \dots x_n x](t - x_1) \dots (t - x_n).$$
Note that $h(x) = h(x_1) = \cdots = h(x_n) = 0$.
By Rolle's Theorem, there are n points $(x', x'_1, \dots, x'_{n-1})$ between each neighboring pair of $x, x_1, \dots, x_n$ s.t. $h' = 0$.

Proof of error formula

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There are $n - 1$ points $(x'', x''_1, \dots, x''_{n-2})$ between each neighboring pair of $x', x'_1, \dots, x_{n-1}$ s.t. $h'' = 0$.

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$\vdots$

$h^{(n)}(c) = 0$ where $\min\{x, x_1, \dots, x_n\} \le c \le \max\{x, x_1, \dots, x_n\}$.

Proof of error formula

Proof.

- Note that $h^{(n)}(t) = f^{(n)}(t) - n!f[x_1 \ldots x_n x]$.
- $f[x_1 \ldots x_n x] = \frac{f^{(n)}(c)}{n!}$.

Outline

Weierstrass approximation theorem

Every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial.

Theorem (Weierstrass approximation theorem)

Let $f(x)$ be a continuous function defined on an interval $[a, b]$. There exists a polynomial $P_n(x)$ s.t.

$$\lim_{n\to\infty}\left(\max_{a\le x\le b}|f(x) - P_n(x)|\right) = 0.$$

Runge phenomenon

Question: Does using more points lead to a more accurate reconstruction of $f(x)$?

Runge phenomenon

Question: Does using more points lead to a more accurate reconstruction of $f(x)$?

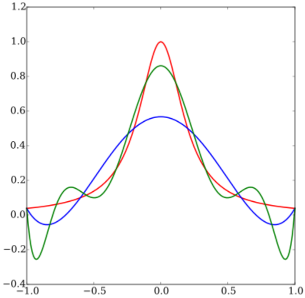
Answer: No!

Example (Runge phenomenon)

- $f(x) = \frac{1}{1+25x^2}$ where $-1 \leq x \leq 1$.
- $P_n(x)$: a polynomial interpolated at equidistant points $(x_i, f(x_i))$ s.t. $x_i = \frac{2i}{n} - 1$ where $i \in \{0, 1, \ldots, n\}$.

$$\lim_{n\to\infty} \left(\max_{-1\leq x\leq 1} |f(x) - P_n(x)| \right) = +\infty.$$

Runge phenomenon



- The red curve is the Runge function.
- The blue curve is a 5th-order interpolating polynomial.
- The green curve is a 9th-order interpolating polynomial.
- Problem: polynomial wiggle near the ends of the interval.
- Solution: move some of the points toward the edges of the interval.

Outline

Motivation of Chebyshev interpolation

- The interpolation error: $\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!}f^{(n)}(c)$;
- The numerator: $(x-x_1)(x-x_2)\cdots(x-x_n)$;

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- Question: Is it possible to find particular $x_1,\ldots,x_n$ s.t. the maximum value of numerator is as small as possible?

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- Fix the interval to be $[-1,1]$.
- Question: Is it possible to find particular $x_1,\ldots,x_n$ s.t. the maximum value of numerator is as small as possible?
- Answer: Chebyshev nodes.

Chebyshev nodes

Definition (Chebyshev nodes)

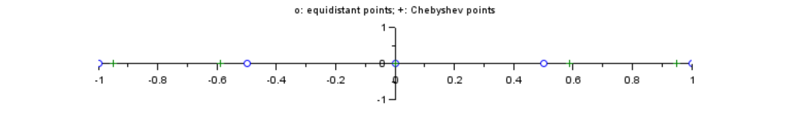
For a given $n \in \mathbb{N}$, **Chebyshev nodes** in the interval $[-1, 1]$ are

$$x_i = \cos \frac{(2i-1)\pi}{2n}, \text{ for } i = 1, \ldots, n.$$

Example

Suppose that $n = 5$. Then, the Chebyshev nodes are as follows:

$$x_1 = \cos \frac{9\pi}{10},\ x_2 = \cos \frac{7\pi}{10},\ x_3 = \cos \frac{5\pi}{10},\ x_4 = \cos \frac{3\pi}{10} \text{ and } x_5 = \cos \frac{\pi}{10}.$$



Chebyshev theorem

Question: What is the maximum value of the interpolation error when we choose the Chebyshev nodes?

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Theorem (Chebyshev theorem)

1. *The minimum value of* $\max\limits_{-1\le x\le 1}|(x-x_1)\cdots(x-x_n)|$ *is* $\frac{1}{2^{n-1}}$.
2. *When* x_i*'s are the Chebyshev nodes, the enumerator achieves the minimum value.*

Chebyshev theorem

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Example

Let $f(x) = e^x$, and $x_1, \ldots, x_5$ the Chebyshev nodes. Then,

$$|(x - x_1) \cdots (x - x_n)| \le \frac{1}{2^4}.$$

The error is $\le \frac{e}{2^4 5!} \approx 0.00142$.

Chebyshev theorem

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Let $f(x)=e^x$, and $x_1,\ldots,x_5$ the Chebyshev nodes. Then,

$$|(x-x_1)\cdots(x-x_n)|\leq\frac{1}{2^4}.$$

The error is $\leq\frac{e}{2^4 5!}\approx 0.00142$.

If $x_1,\ldots,x_5$ are 5 evenly spaced base points, *i.e.*, $\{-1,-\frac{1}{2},0,\frac{1}{2},1\}$,

The error at $x=0.75$ is $\leq\frac{|(x+1)(x+\frac{1}{2})(x)(x-\frac{1}{2})(x-1)|}{5!}e\approx 0.02323$.

Outline

Chebyshev polynomials

Definition (Chebyshev polynomials)

The nth **Chebyshev polynomial** $T_n(x) = \cos(n \arccos x)$.

Chebyshev polynomials

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- $(x - x_1) \cdots (x - x_n) = \frac{T_n(x)}{2^{n-1}}$ if x_i are the Chebyshev nodes and $x \in [-1, 1]$.

Chebyshev polynomials

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- $T_n(x)$ is based on trigonometric functions.
- However, in fact, it is a polynomial.

Chebyshev polynomials

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- $T_n(x)$ is based on trigonometric functions.
- However, in fact, it is a polynomial.

Proof.

Base case:

1. $T_0(x) = \cos 0 = 1$;
2. $T_1(x) = \cos(\arccos x) = x$.

Chebyshev polynomials

Proof.

Inductive step:

- Let $y = \arccos x$, *i.e.*, $\cos y = x$.

Chebyshev polynomials

Proof.

Inductive step:

- Let $y = \arccos x$, *i.e.*, $\cos y = x$.
- $T_{n+1}(x) = \cos(n+1)y = \cos(ny+y) = \cos ny \cos y - \sin ny \sin y$;

Chebyshev polynomials

Proof.

Inductive step:

- Let $y = \arccos x$, *i.e.*, $\cos y = x$.
- $T_{n+1}(x) = \cos(n+1)y = \cos(ny + y) = \cos ny \cos y - \sin ny \sin y$;
- $T_{n-1}(x) = \cos(n-1)y = \cos(ny - y) = \cos ny \cos y + \sin ny \sin y$;

Chebyshev polynomials

Proof.

Inductive step:

- Let $y = \arccos x$, *i.e.*, $\cos y = x$.
- $T_{n+1}(x) = \cos(n+1)y = \cos(ny + y) = \cos ny \cos y - \sin ny \sin y$;
- $T_{n-1}(x) = \cos(n-1)y = \cos(ny - y) = \cos ny \cos y + \sin ny \sin y$;
- $T_{n+1}(x) + T_{n-1}(x) = 2\cos ny \cos y = 2xT_n(x)$;

Chebyshev polynomials

Proof.

Inductive step:

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- $T_{n+1}(x) = \cos(n+1)y = \cos(ny + y) = \cos ny \cos y - \sin ny \sin y$;
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- $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Example

- $T_0(x) = 1$;
- $T_1(x) = x$;

Chebyshev polynomials

Proof.

Inductive step:

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Example

- $T_0(x) = 1$;
- $T_1(x) = x$;
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Chebyshev polynomials

Proof.

Inductive step:

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- $T_{n+1}(x) = \cos(n+1)y = \cos(ny + y) = \cos ny \cos y - \sin ny \sin y$;
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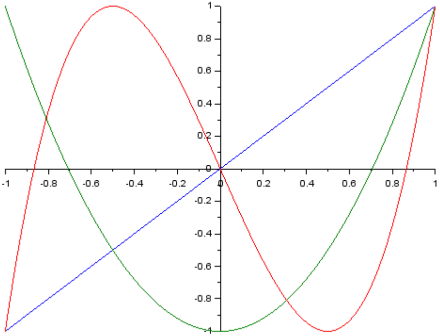
- $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Example

- $T_0(x) = 1$;
- $T_1(x) = x$;
- $T_2(x) = 2x \cdot x - 1 = 2x^2 - 1$;
- $T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$.

Some facts of Chebyshev polynomials



- $T_1(x) = x$
- $T_2(x) = 2x^2 - 1$
- $T_3(x) = 4x^3 - 3x$

Some facts of Chebyshev polynomials

1. $\deg(T_n) = n$;
2. The leading coefficient of T_n is 2^{n-1} for $n \geq 1$;
3. $T_n(1) = 1$ and $T_n(-1) = (-1)^n$;
4. $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$;
5. All zeros of $T_n(x)$ are in $[-1, 1]$ ($x = \cos\frac{(2i-1)\pi}{2n}$ for $1 \leq i \leq n$);
6. $T_n(x)$ alternates between -1 and 1 a total of $n+1$ times.
   For $0 \leq i \leq n$, $T_n(\cos\frac{i\pi}{n}) = \begin{cases} -1, & i \text{ is odd;} \\ 1, & i \text{ is even.} \end{cases}$

Some facts of Chebyshev polynomials

Fact 1.

- Base case ($n = 0$ and $n = 1$): $\deg(T_0(x)) = \deg(1) = 0$ and $\deg(T_1(x)) = \deg(x) = 1$.

Some facts of Chebyshev polynomials

Fact 1.

- Base case ($n = 0$ and $n = 1$): $\deg(T_0(x)) = \deg(1) = 0$ and $\deg(T_1(x)) = \deg(x) = 1$.
- Inductive step ($n > 1$): $\deg(T_n) = \deg(2xT_{n-1}(x) - T_{n-2}(x)) = \deg(T_{n-1}(x)) + 1$.
- By the induction assumption, $\deg(T_{n-1}(x)) = n - 1$.
- Hence, $\deg(T_n) = n - 1 + 1 = n$.

Some facts of Chebyshev polynomials

Fact 2.

- Base case ($n = 1$ and $n = 2$): $\texttt{lc}(T_1(x)) = \texttt{lc}(x) = 1$ and $\texttt{lc}(T_2(x)) = \texttt{lc}(2x^2 - 1) = 2$.

Some facts of Chebyshev polynomials

Fact 2.

- Base case ($n = 1$ and $n = 2$): $\texttt{lc}(T_1(x)) = \texttt{lc}(x) = 1$ and $\texttt{lc}(T_2(x)) = \texttt{lc}(2x^2 - 1) = 2$.
- Inductive step ($n > 1$):
  $\texttt{lc}(T_n) = \texttt{lc}(2xT_{n-1}(x) - T_{n-2}(x)) = 2 \cdot \texttt{lc}(T_{n-1}(x))$.
- By the induction assumption, $\texttt{lc}(T_{n-1}(x)) = 2^{n-2}$.
- Hence, $\texttt{lc}(T_n) = 2 \cdot 2^{n-2} = 2^{n-1}$.

Proof of Chebyshev polynomials

Theorem (Chebyshev theorem)

1. *The minimum value of* $\max\limits_{-1\leq x\leq 1} |(x-x_1)\cdots(x-x_n)|$ *is* $\frac{1}{2^{n-1}}$.
2. *When* x_i*'s are the Chebyshev nodes, the enumerator achieves the minimum value.*

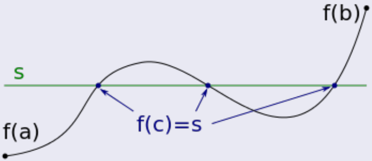
Proof of 2.

$(x-x_1)\cdots(x-x_n) = (x-\cos\frac{\pi}{2n})\cdots(x-\cos\frac{(2n-1)\pi}{2n}) = \frac{T_n(x)}{2^{n-1}}$.
Together with $|T_n(x)| \leq 1$, we get that **2** holds.

Proof of Chebyshev polynomials

Theorem (Intermediate value theorem)

Let f be a continuous function on interval $[a, b]$, and s a value between $f(a)$ and $f(b)$. Then, there exists c with $a \leq c \leq b$ s.t. $f(c) = s$.



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- $f_n(x) = P_n(x) - \frac{T_n(x)}{2^{n-1}}$;
- $f_n(x) < 0$ for $x = \cos\frac{2k\pi}{n}$ where $0 \le 2k \le n$ since $T_n(x) = 1$;
- $f_n(x) > 0$ for $x = \cos\frac{(2k+1)\pi}{n}$ where $0 \le 2k+1 \le n$ since $T_n(x) = -1$;
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- By intermediate value theorem, $f_n(x)$ has at least n roots.
- On the other hand, $\deg(f_n) \le n-1$;
- By fundamental theorem of algebra, $f_n(x)$ has at most $n-1$ roots.
- Contradiction!

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4. $|(x - x_1) \cdots (x - x_n)| \leq \frac{(b-a)^n}{2^{2n-1}}$ holds on $[a, b]$.

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- A more efficient method: Newton's divided differences;
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- The problem of polynomial interpolation using equidistant points: Runge phenomenon;
- A mitigation to the problem: Chebyshev interpolation.

Thank you!