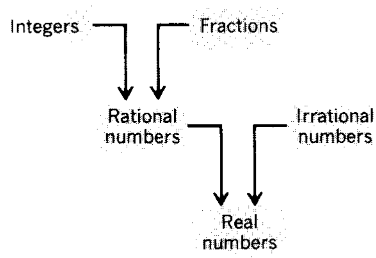

Figure 2.1

Real numbers are all we need for the first 14 chapters of this book, but they are not the only numbers used in mathematics. In fact, the réason for the term "real" is that there are also "imaginary" numbers, which have to do with the square roots of negative numbers. That concept will be discussed later, in Chap. 15.

2.3 THE CONCEPT OF SETS

We have already employed the word "set" several times. Inasmuch as the concept of sets underlies every branch of modern mathematics, it is desirable to familiarize ourselves at least with its more basic aspects.

Set Notation

A *set* is simply a collection of distinct objects. These objects may be a group of (distinct) numbers, or something else. Thus, all the students enrolled in a particular economics course can be considered a set, just as the three integers 2, 3, and 4 can form a set. The objects in a set are called the *elements* of the set.

There are two alternative ways of writing a set: by *enumeration* and by *description*. If we let S represent the set of three numbers 2, 3, and 4, we can write, by enumeration of the elements,

$$S = \{2, 3, 4\}$$

But if we let I denote the set of *all* positive integers, enumeration becomes difficult, and we may instead simply describe the elements and write

$$I = \{x \mid x \text{ a positive integer}\}$$

which is read as follows: "I is the set of all (numbers) x, such that x is a positive integer." Note that braces are used to enclose the set in both cases. In the descriptive approach, a vertical bar (or a colon) is always inserted to separate the general symbol for the elements from the description of the elements. As another example, the set of all real numbers greater than 2 but less than 5 (call it J) can be expressed symbolically as

$$J = \{x \mid 2 < x < 5\}$$

Here, even the descriptive statement is symbolically expressed.

A set with a finite number of elements, exemplified by set S above, is called a *finite set*. Set I and set J, each with an infinite number of elements, are, on the other hand, examples of an *infinite set*. Finite sets are always *denumerable* (or *countable*), i.e., their elements can be counted one by one in the sequence $1, 2, 3, \ldots$. Infinite sets may, however, be either denumerable (set I above), or *nondenumerable* (set J above). In the latter case, there is no way to associate the elements of the set with the natural counting numbers $1, 2, 3, \ldots$, and thus the set is not countable.

Membership in a set is indicated by the symbol $\in$ (a variant of the Greek letter epsilon ϵ for "element"), which is read: "is an element of." Thus, for the two sets S and I defined above, we may write

$$2 \in S \qquad 3 \in S \qquad 8 \in I \qquad 9 \in I \qquad \text{(etc.)}$$

but obviously $8 \notin S$ (read: "8 is not an element of set S"). If we use the symbol R to denote the set of all real numbers, then the statement "x is some real number" can be simply expressed by

$$x \in R$$

Relationships between Sets

When two sets are compared with each other, several possible kinds of relationship may be observed. If two sets S_1 and S_2 happen to contain identical elements,

$$S_1 = \{2, 7, a, f\} \qquad \text{and} \qquad S_2 = \{2, a, 7, f\}$$

then S_1 and S_2 are said to be *equal* ($S_1 = S_2$). Note that the order of appearance of the elements in a set is immaterial. Whenever even one element is different, however, two sets are not equal.

Another kind of relationship is that one set may be a *subset* of another set. If we have two sets

$$S = \{1, 3, 5, 7, 9\} \qquad \text{and} \qquad T = \{3, 7\}$$

then T is a subset of S, because every element of T is also an element of S. A more formal statement of this is: T is a subset of S if and only if "$x \in T$" implies "$x \in S$." Using the set inclusion symbols $\subset$ (is contained in) and $\supset$ (includes), we may then write

$$T \subset S \qquad \text{or} \qquad S \supset T$$

It is possible that two given sets happen to be subsets of each other. When this occurs, however, we can be sure that these two sets are equal. To state this formally: we can have $S_1 \subset S_2$ and $S_2 \subset S_1$ if and only if $S_1 = S_2$.

Note that, whereas the $\in$ symbol relates an individual *element* to a *set*, the $\subset$ symbol relates a *subset* to a *set*. As an application of this idea, we may state on the basis of Fig. 2.1 that the set of all integers is a subset of the set of all rational numbers. Similarly, the set of all rational numbers is a subset of the set of all real numbers.

How many subsets can be formed from the five elements in the set $S = \{1, 3, 5, 7, 9\}$? First of all, each individual element of S can count as a distinct subset of S, such as $\{1\}$, $\{3\}$, etc. But so can any pair, triple, or quadruple of these elements, such as $\{1, 3\}$, $\{1, 5\}, \ldots, \{3, 7, 9\}$, etc. For that matter, the set S itself (with all its five elements) can be considered as one of its own subsets—every element of S is an element of S, and thus the set S itself fulfills the definition of a subset. This is, of course, a limiting case, that from which we get the "largest" possible subset of S, namely, S itself.

At the other extreme, the "smallest" possible subset of S is a set that contains no element at all. Such a set is called the *null set*, or *empty set*, denoted by the symbol $\varnothing$ or $\{\ \}$. The reason for considering the null set as a subset of S is quite interesting: If the null set is not a subset of S ($\varnothing \not\subset S$), then $\varnothing$ must contain at least one element x such that $x \notin S$. But since by definition the null set has no element whatsoever, we cannot say that $\varnothing \not\subset S$; hence the null set is a subset of S.

Counting all the subsets of S, including the two limiting cases S and $\varnothing$, we find a total of $2^5 = 32$ subsets. In general, if a set has n elements, a total of 2^n subsets can be formed from those elements.*

It is extremely important to distinguish the symbol $\varnothing$ or $\{\ \}$ clearly from the notation $\{0\}$; the former is devoid of elements, but the latter does contain an element, zero. The null set is unique; there is only one such set in the whole world, and it is considered a subset of *any* set that can be conceived.

As a third possible type of relationship, two sets may have no elements in common at all. In that case, the two sets are said to be *disjoint*. For example, the set of all positive integers and the set of all negative integers are disjoint sets. A fourth type of relationship occurs when two sets have some elements in common but some elements peculiar to each. In that event, the two sets are neither equal nor disjoint; also, neither set is a subset of the other.

Operations on Sets

When we add, subtract, multiply, divide, or take the square root of some numbers, we are performing mathematical operations. Sets are different from

* Given a set with n elements $\{a, b, c, \ldots, n\}$ we may first classify its subsets into two categories: one with the element a in it, and one without. Each of these two can be further classified into two subcategories: one with the element b in it, and one without. Note that by considering the second element b, we double the number of categories in the classification from 2 to 4 ($= 2^2$). By the same token, the consideration of the element c will increase the total number of categories to 8 ($= 2^3$). When all n elements are considered, the total number of categories will become the total number of subsets, and that number is 2^n.

numbers, but one can similarly perform certain mathematical operations on them. Three principal operations to be discussed here involve the union, intersection, and complement of sets.

To take the *union* of two sets A and B means to form a new set containing those elements (and only those elements) belonging to A, or to B, or to both A and B. The union set is symbolized by $A \cup B$ (read: "A union B").

Example 1 If $A = \{3, 5, 7\}$ and $B = \{2, 3, 4, 8\}$, then

$$A \cup B = \{2, 3, 4, 5, 7, 8\}$$

This example illustrates the case in which two sets A and B are neither equal nor disjoint and in which neither is a subset of the other.

Example 2 Again referring to Fig. 2.1, we see that the union of the set of all integers and the set of all fractions is the set of all rational numbers. Similarly, the union of the rational-number set and the irrational-number set yields the set of all real numbers.

The *intersection* of two sets A and B, on the other hand, is a new set which contains those elements (and only those elements) belonging to *both A and B*. The intersection set is symbolized by $A \cap B$ (read: "A intersection B").

Example 3 From the sets A and B in Example 1, we can write

$$A \cap B = \{3\}$$

Example 4 If $A = \{-3, 6, 10\}$ and $B = \{9, 2, 7, 4\}$, then $A \cap B = \varnothing$. Set A and set B are disjoint; therefore their intersection is the empty set—no element is common to A and B.

It is obvious that intersection is a more restrictive concept than union. In the former, only the elements *common to A and B* are acceptable, whereas in the latter, membership in *either A or B* is sufficient to establish membership in the union set. The operator symbols $\cap$ and $\cup$—which, incidentally, have the same kind of general status as the symbols $\sqrt{\ }$, $+$, $\div$, etc.—therefore have the connotations "and" and "or," respectively. This point can be better appreciated by comparing the following formal definitions of intersection and union:

Intersection:	$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
Union:	$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Before explaining the *complement* of a set, let us first introduce the concept of *universal set*. In a particular context of discussion, if the only numbers used are the set of the first seven positive integers, we may refer to it as the universal set, U. Then, with a given set, say, $A = \{3, 6, 7\}$, we can define another set $\tilde{A}$ (read: "the complement of A") as the set that contains all the numbers in the universal set U which are not in the set A. That is,

$$\tilde{A} = \{x \mid x \in U \text{ and } x \notin A\} = \{1, 2, 4, 5\}$$

Note that, whereas the symbol $\cup$ has the connotation "or" and the symbol $\cap$ means "and," the complement symbol ~ carries the implication of "not."

Example 5 If $U = \{5, 6, 7, 8, 9\}$ and $A = \{5, 6\}$, then $\tilde{A} = \{7, 8, 9\}$.

Example 6 What is the complement of U? Since every object (number) under consideration is included in the universal set, the complement of U must be empty. Thus $\tilde{U} = \varnothing$.

The three types of set operation can be visualized in the three diagrams of Fig. 2.2, known as *Venn diagrams*. In diagram a, the points in the upper circle form a set A, and the points in the lower circle form a set B. The union of A and B then consists of the shaded area covering both circles. In diagram b are shown the same two sets (circles). Since their intersection should comprise only the points common to both sets, only the (shaded) overlapping portion of the two circles satisfies the definition. In diagram c, let the points in the rectangle be the universal set and let A be the set of points in the circle; then the complement set $\tilde{A}$ will be the (shaded) area outside the circle.

Laws of Set Operations

From Fig. 2.2, it may be noted that the shaded area in diagram a represents not only $A \cup B$ but also $B \cup A$. Analogously, in diagram b the small shaded area is the visual representation not only of $A \cap B$ but also of $B \cap A$. When formalized,

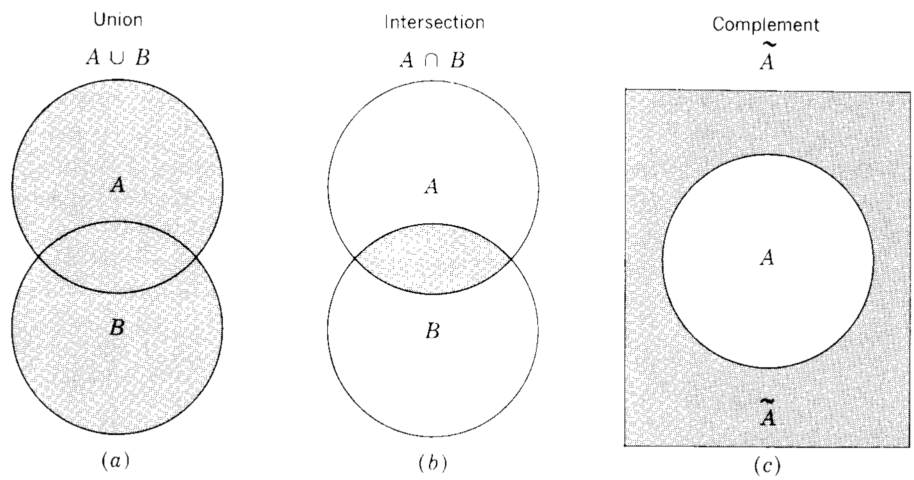

Figure 2.2

this result is known as the *commutative law* (of unions and intersections):

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

These relations are very similar to the algebraic laws $a + b = b + a$ and $a \times b = b \times a$.

To take the union of three sets A, B, and C, we first take the union of any two sets and then "union" the resulting set with the third; a similar procedure is applicable to the intersection operation. The results of such operations are illustrated in Fig. 2.3. It is interesting that the order in which the sets are selected for the operation is immaterial. This fact gives rise to the *associative law* (of unions and intersections):

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

These equations are strongly reminiscent of the algebraic laws $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$.

There is also a law of operation that applies when unions and intersections are used in combination. This is the *distributive law* (of unions and intersections):

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

These resemble the algebraic law $a \times (b + c) = (a \times b) + (a \times c)$.

Example 7 Verify the distributive law, given $A = \{4, 5\}$, $B = \{3, 6, 7\}$, and $C = \{2, 3\}$. To verify the first part of the law, we find the left- and right-hand expressions separately:

Left: $\quad A \cup (B \cap C) = \{4, 5\} \cup \{3\} = \{3, 4, 5\}$

Right: $\quad (A \cup B) \cap (A \cup C) = \{3, 4, 5, 6, 7\} \cap \{2, 3, 4, 5\} = \{3, 4, 5\}$

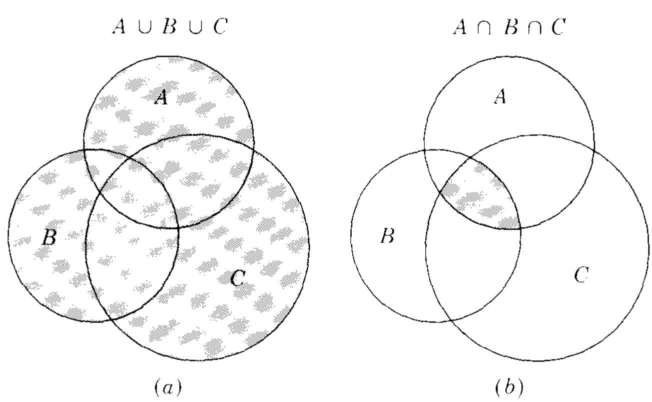

Figure 2.3