THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXIII | MAY, 1916 | NUMBER 5

A CURIOUS CONVERGENT SERIES.

By FRANK IRWIN, University of California.

1. Under the above title Dr. Kempner has shown, in the issue of the MONTHLY for February, 1914, that the series derived from the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, by striking out those terms whose denominators contain the digit 9 is a convergent series. It has seemed to me that it would be not without interest to attempt to extend this result, to see if one could not considerably cut down the class of numbers omitted from the harmonic series, and still get a convergent series. The result reached is as follows:

Proposition. If we strike out from the harmonic series those terms whose denominators contain the digit 9 at least a times, and, *at the same time*, the digit 8 at least b times, the digit 7 at least c times, and so on, to the digit 0 at least j times ($a, b, c, \cdots j$ being any given integers), the series so obtained will converge.

It should be explicitly pointed out that each condition added, or, to put the same thing in another way, each increase in any of the numbers $a, b, c, \cdots$, makes the class of numbers stricken out *smaller*. For instance, with $a = 4$, $b = 2$, $c = 1$, we should leave out the number 1/899979897, but not if $a = 4$, $b = 3$, $c = 1$.

Special Case. We first prove that the series derived from the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, by striking out those terms whose denominators contain the digit 9 at least a times converges, and that this is true for any other digit, including 0, equally well with 9. That this holds for $a = 1$ is what Dr. Kempner has proved (the case of the digit 0 will be treated specially at the end). We have, then, merely to prove that, if it holds for any given a, it will hold for $a + 1$.

Let us compare the two series:

Series I. The series obtained from the harmonic series by striking out terms containing at least a 9's in the denominators.

Series II. The series obtained by striking out terms containing at least $(a+1)$ 9's in the denominators.

What we wish to prove is that, if series I converges, then will also series II. The principle we shall employ is that, if the series formed of the terms that do not occur in I, but have been introduced into II, converges, then series II, as the sum of two convergent series, will itself converge. Now, these terms are evidently the terms containing *exactly* a 9's.

How many such terms are there? That is, how many are there among those whose denominators consist of n digits in all? This may be readily treated as a problem in permutations. Two cases occur. First, if the *first* digit be a 9, we can pick out among the remaining $n-1$ digits $a-1$ places in which to put the remaining $a-1$ 9's in $\binom{n-1}{a-1}$ ways; then for each one of these choices we can fill each of the remaining $n-a$ places with any one of the digits $0, 1, 2, \cdots 8$, that is, we can fill them in 9 ways each, or 9^{n-a} ways altogether; there are, then, $\binom{n-1}{a-1} 9^{n-a}$ numbers of n digits beginning with a 9 and containing exactly $a-1$ other 9's. In the same way, secondly, we may determine that there are $\binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}$ numbers of n digits not beginning with a 9 but containing exactly a 9's (the factor 8 arises from the fact that we can put one of the digits $1, 2, \cdots 8$ only in the first place, *not* the digit 0).

Each of the terms of the harmonic series having one of these numbers for its denominator is less than $1/10^{n-1}$; together, then, their sum is less than

$$\binom{n-1}{a-1} \cdot 9^{n-a}/10^{n-1} + \binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}/10^{n-1};$$

so that the series formed of these terms with exactly a 9's will certainly converge if each of the series,

$$\sum_n \binom{n-1}{a-1} \cdot 9^{n-a}/10^{n-1} \qquad \text{and} \qquad \sum_n \binom{n-1}{a} \cdot 8 \cdot 9^{n-a-1}/10^{n-1},$$

converges. But each of these series may be shown to converge by applying the Cauchy ratio test.

Our general proposition may now be proved from this special case by mathematical induction. For, what this special case tells us is that the proposition is true when $b = c = \cdots = j = 0$. From this we can prove the proposition for the case $c = d = \cdots = j = 0$, $d = e = \cdots = j = 0$, and so on. As an example, then, take the following.

Suppose we are given that the series obtained from the harmonic series by striking out the terms in which the denominators contain the digit 9 at least a times and, at the same time, the digit 8 at least b times converges; this is the case $c = d = \cdots = j = 0$; call this series III. We shall prove that then the same will be true of Series IV, obtained by striking out the terms containing at least a 9's and b 8's and c 7's; this is the case $d = e = \cdots = j = 0$.

It will be sufficient, as before, to show that the terms not occurring in series III that have been introduced into series IV form, of themselves, a convergent series. To this end it is merely necessary to notice that none of these terms contains more than $(c - 1)$ 7's; for since, by our special case, the series consisting of *all* terms that do not contain more than $c - 1$ 7's is convergent, *à fortiori* will any series made up of a selection of such terms be convergent, since all these terms are positive.

This completes the proof except for the special treatment demanded by the digit 0. It will, perhaps, suffice to indicate how Dr. Kempner's proposition would be extended to cover this case, leaving to the reader the proof of our "special case" for the digit 0. We are to show, then, that the series obtained from the harmonic by striking out terms containing the digit 0 converges. There are 9^n numbers of n digits not containing 0; the sum of their reciprocals is less than $9^n/10^{n-1}$; and our series is less than $\Sigma 9^n/10^{n-1}$, *i. e.*, 90.

2. Let us return to Dr. Kempner's series, namely that obtained from the harmonic series by striking out terms whose denominators contain the digit 9. This series converges; what is its sum? Dr. Kempner merely shows that it is less than 90. Its value actually lies between 22.4 and 23.3. This result may be obtained with no great amount of labor, and a considerably closer approximation might be reached, if desired, by the methods here employed.

Consider those terms of our series that have two digits in the denominator:

$$a_2 = \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{18} + \frac{1}{20} + \frac{1}{21} + \cdots + \frac{1}{28} + \cdots + \frac{1}{80} + \frac{1}{81} + \cdots + \frac{1}{88}.$$

Compare with these the terms with three digits in the denominator. The first nine of them, $1/100 + 1/101 + \cdots + 1/108$, are each less than or equal to $1/100$, and their sum is less than $9/100$, that is, less than $9/10$ of $1/10$, the first number of a_2. In the same way the next set of nine, $1/110 + 1/111 + \cdots + 1/118$, is less than $9/10$ of $1/11$, the second number of a_2 and so on: the sum of all the terms with three digits is less than $9/10$ of a_2. Similarly, the sum of all terms with four digits is less than $(9/10)^2$ of a_2; and in general, the sum of all terms with n digits is less than $(9/10)^{n-2}$ of a_2. Therefore the sum of our series is less than

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \left[1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots\right] a_2 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} + 10a_2.$$

The value of a_2 may be quickly computed with the help of a table of reciprocals. We have:

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} < 2.72$$

$$a_2 < 2.058; \quad 10a_2 < 20.58$$

The sum of the series is less than 23.3

To obtain an inferior limit for the sum of the series, we show in the same way that the sum of the terms with n digits in the denominator is greater than $(9/10)^{n-2} \cdot a_2'$, where

$$a_2' = \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \frac{1}{22} + \cdots + \frac{1}{29} + \cdots + \frac{1}{81} + \frac{1}{82} + \cdots + \frac{1}{89};$$

and, therefore, the series is greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + [9/10 + (9/10)^2 + \cdots]a_2',$$

that is, greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + 9a_2',$$

which turns out to be greater than 22.4.

A still closer approximation may be found by starting with a_3 and a_3', that is, with the terms having three digits in the denominator and with

$$a_2' = \frac{1}{101} + \cdots + \frac{1}{109} + \frac{1}{111} + \cdots + \frac{1}{119} + \cdots + \frac{1}{881} + \cdots + \frac{1}{889}.$$

ON THE MATRIX EQUATION $BX = C$.[1]

By H. T. BURGESS, University of Wisconsin.

Section 1. To Find the Matrix X. The problem is to calculate the elements of the matrix X to satisfy the matrix equation $BX = C$:

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{Vmatrix} = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}.$$

If we compute the matrix product BX, we get

$$BX = \begin{Vmatrix} \Sigma b_{1\epsilon}x_{\epsilon 1} & \Sigma b_{1\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{1\epsilon}x_{\epsilon n} \\ \Sigma b_{2\epsilon}x_{\epsilon 1} & \Sigma b_{2\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{2\epsilon}x_{\epsilon n} \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma b_{n\epsilon}x_{\epsilon 1} & \Sigma b_{n\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{n\epsilon}x_{\epsilon n} \end{Vmatrix},$$

where the summation runs for $\epsilon = 1, 2, \cdots, n$.

The conditions to be fulfilled are obtained by setting the elements of the product BX equal to the corresponding elements of C. Taking these by columns we get the following n-sets of simultaneous linear equations:

---

[1] For the elementary properties of matrices the reader may conveniently consult Bôcher's *Introduction to Higher Algebra*, using the index to find the appropriate sections.