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A Curious Convergent Series

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Treatise on the Pentagon and Decagon. The algebra terminates with the first eight lines on fol. 93^v, and is followed by the work on the pentagon and decagon by the same author. Although this treatise on the pentagon and decagon by Abu Kamil is geometrical in its nature yet the treatment and the solutions are algebraical, including a fourth degree equation ($x^4 = 8000x^2 - \sqrt{51\,200\,000}$) as well as mixed quadratics with irrational coefficients. The twelfth problem is in the Latin: "Et si dicemus tibi trianguli equilateri et equianguli mensura est cum perpendiculari ipsius est 10 ex numero, quanta sit perpendicularis?"[1] This suggests the similar problems in Greek of unknown date and author presented by Heiberg and Zeuthen,[2] as well as the similar problems given by Diophantos. In these Greek problems also lines and areas are summed quite contrary to ancient Greek usage. A further point of interest is that in the equation $x^2 + 75 = 75x$, to which the solution of the thirteenth problem leads, Abu Kamil gives only one solution whereas in the algebra he recognizes that this equation has two positive roots. The opening sentence of this treatise on the pentagon and decagon makes reference to the algebra as immediately preceding it, which indeed is the fact in the Hebrew and Latin manuscripts that are preserved.

Conclusion. The algebra terminates with a general statement to the effect that by the methods taught in this book many more problems can be easily solved. In true Arabic fashion, the closing words are: " Whence praise and glory be to the only Creator."

Let us summarize the results of our study. The most important conclusion of this investigation of Abu Kamil's algebra is that Al-Karkhi and Leonard of Pisa drew extensively from this Arabic writer. Through them this man, though himself comparatively unknown to modern writers, exerted a powerful influence on the early development of algebra. Abu Kamil deserves somewhat the same recognition from modern mathematicians and historians of science as that which Leonard of Pisa and Al-Khowarizmi have received. We may hope that the future will be more just than the past in according to Abu Kamil a prominent place among the mathematicians of the middle ages.

A CURIOUS CONVERGENT SERIES.

By A. J. KEMPNER, University of Illinois.

It is well known that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. The object of this Note is to prove that if the denominators do not

[1] "If we say to you that an equilateral and equiangular triangle, together with its altitude, is measured by 10, what is the altitude?"

[2] *Einige griechische Aufgaben der unbestimmten Analytik*, in *Biblioth. mathem.*, VIII, third series, 118–134.

include all natural numbers 1, 2, 3, $\cdots$, but only those numbers which do not contain any figure 9, the series converges. The method of proof holds unchanged if, instead of 9, any other figure 1, 2, $\cdots$, 8 is excluded, but not for the figure 0.

Proof: The series with which we deal is

$$\text{I.}\quad \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\frac{1}{30}+\cdots$$
$$+\frac{1}{38}+\frac{1}{40}+\cdots+\frac{1}{48}+\frac{1}{50}+\cdots+\frac{1}{58}+\frac{1}{60}+\cdots+\frac{1}{68}+\frac{1}{70}+\cdots$$
$$+\frac{1}{78}+\frac{1}{80}+\cdots+\frac{1}{88}+\frac{1}{100}+\cdots.$$

We form a new series

$$\text{II.}\qquad s = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

by the following rule: a_n is the sum of all terms in I of denominator d, where $10^{n-1} \leqq d < 10^n$. We have then

$$a_1 = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8},$$

$$a_2 = \frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots, \text{ the last term being } \frac{1}{88}, \text{ etc.}$$

Each term of I which forms part of a_1, is $\leqq 1$,

each term of I which forms part of a_2, is $\leqq 1/10$, and

each term of I which forms part of a_n, is $\leqq 1/10^{n-1}$.

We now count the number of terms of I which are contained in a_1, in a_2, $\cdots$, in a_n. Evidently a_1 consists of 8 terms, and $a_1 < 8 \cdot 1 < 9$. In a_2 there are, as is easily seen, less than 9^2 terms of I, and $a_2 < (9^2/10)$. Altogether there are in I less than $9^2 + 9$ terms with denominators under 100. Assume now that we know the number of terms in I which are contained in a_n to be less than 9^n, for $n = 1, 2, 3, \cdots, n$. Then, because each term of I which is contained in a_n is not greater than $1/10^{n-1}$, we have $a_n < (9^n/10^{n-1})$, and the total number of terms in I with denominators under 10^n is less than $9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. For $n = 1$ and $n = 2$ we have verified all this, and we will now show that if it is true for n, then $a_{n+1} < (9^{n+1}/10^n)$. a_{n+1} contains all terms in I of denominator d, $10^n \leqq d < 10^{n+1}$. This interval for d can be broken up into the nine intervals $\alpha \cdot 10^n \leqq d < (\alpha + 1)10^n$, $\alpha = 1, 2, \cdots, 9$. The last interval does not contribute any term to I, the eight remaining intervals contribute each the same number of terms to I, and this is the same as the number of terms contributed by the whole interval $0 < d < 10^n$, that is, by assumption, less than

$9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. Altogether, therefore, a_{n+1} contains less than $8(9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9) < 9^{n+1}$ terms of I, and, as each of these terms is not greater than $1/10^n$, we have $a_{n+1} < (9^{n+1}/10^n)$.

Our series is therefore

$$s = a_1 + a_2 + a_3 + \cdots < 9 + \frac{9^2}{10} + \frac{9^3}{10^2} + \cdots + \frac{9^{n+1}}{10^n} + \cdots = 90,$$

and I is certainly convergent.

The preceding proof also contains all material necessary to show the following: Let M be any positive integer, and N_1 the number of positive integers $< M$ containing no figure 9, N_2 the number of positive integers $< M$ containing at least one figure 9, then

$$\lim_{M=\infty} N_1/N_2 = 0.$$

See problem 207 proposed on page 55 of this issue.

THE PERFECT MAGIC SQUARES FOR 1914.

By V. M. SPUNAR, Chicago, Ill.

Using Zerr's formula*

$$\frac{1914 - (n^3 + n)/2}{n} = \text{Integer},$$

the only values of n for 1914 are 3, 11, 29. (For 1913 there are no values for n whatever.) The least and greatest integers permissible for forming the squares are represented by another formula of Zerr's, namely,

$$\frac{3828 \pm (n^3 - n)}{2n}.$$

For $n = 3$, the least integer = 634, the greatest integer = 642. For $n = 11$, the least integer = 114, the greatest integer = 234. For $n = 29$, the least integer = − 354, the greatest integer = 486.

Hence the magic squares for 1914 with only positive numbers are as follows:

$n = 3$

641	634	639
636	638	640
637	642	635

* American Mathematical Monthly, Vol. XVI, no. 1, page 2.