Sums of Reciprocals of Integers Missing a Given Digit
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MATHEMATICAL NOTES

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SUMS OF RECIPROCALS OF INTEGERS MISSING A GIVEN DIGIT

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The harmonic series $\Sigma_{n=1}^{\infty} 1/n$ diverges. If we omit those terms for which n has, say, at least one "9" in its base 10 representation, then the remaining series converges **[6]**. In fact, this result holds for any base $b \geqslant 2$ and any digit m, $0 \leqslant m \leqslant b-1$. (See **[4**, Theorem 144, p. 120**]**.) Various estimates for these sums are obtained in [1]–[7], although nowhere is a method discussed that easily produces accurate values of the sums.

This note gives a simple technique for accurately summing these series. We also give, to twenty decimal places, the values of the ten sums where the "missing digit" ranges from 0 through 9, base 10. (See Table 1.) This method easily generalizes to other bases.

Choose the missing digit m, $0 \leqslant m \leqslant 9$. Let S be the set of positive integers that have no digit equal to m, and let S_i be the set of integers in S which have exactly i digits. Define

$$s(i,j) = \sum_{x \in S_i} 1/x^j \qquad (j \geqslant 1). \tag{1}$$

The sum we are interested in is

$$t_m = \sum_{i=1}^{\infty} s(i,1), \tag{2}$$

the sum of the reciprocals of all positive integers missing the digit m.

Now t_m cannot be computed from (2) because the convergence is much too slow: the remainder after summing 10^{27} terms still exceeds one! We can, however, derive a recursion formula which gives $s(i+1,j)$ from $s(i,j+n)$, $n \geqslant 0$, and for small i, $s(i,j+n)$ can be computed directly.

The recursion formula is based on the observation that

$$S_{i+1} = \bigcup_{x \in S_i} [\{10x, 10x+1, \ldots, 10x+9\} \setminus \{10x+m\}].$$

Therefore,

$$s(i+1,j) = \sum_x \left[\sum_k{}' (10x+k)^{-j} \right] \tag{3}$$

where the outer sum extends over all x in S_i, and Σ' is over $k = 0, 1, \ldots, 9, k \neq m$.

Let us make these definitions:

$$c(j,n) = (-1)^n \binom{j+n-1}{n} \qquad (j > 0, n \geqslant 0)$$

$$b_n = 0^n + 1^n + \cdots + 9^n - m^n \qquad (n > 0), b_0 = 9$$

$$a(j,n) = b_n c(j,n)/10^{j+n} \ .$$

Next, rearrange (3) using the binomial theorem to expand $(10x+k)^{-j}$:

$$\begin{aligned} s(i+1,j) &= \sum_x \left\{ (10x)^{-j} \sum_k{}' \left[\sum_{n=0}^{\infty} c(j,n)(k/10x)^n \right] \right\} \\ &= \sum_x \left\{ (10x)^{-j} \left[\sum_{n=0}^{\infty} b_n c(j,n)(10x)^{-n} \right] \right\} \\ &= \sum_x \left[\sum_{n=0}^{\infty} a(j,n) x^{-(j+n)} \right]. \end{aligned}$$

According to (1), we can write this last sum as

$$s(i+1,j) = \sum_{n=0}^{\infty} a(j,n) s(i,j+n). \tag{4}$$

The needed $s(i,j)$ for $i \leqslant 4$ can be computed explicitly. (4) is then used with at most 10 terms to get $s(i,1)$ for $5 \leqslant i \leqslant 30$. For $i \geqslant 31$, we use the estimate

$$\sum_{i=31}^{\infty} s(i,1) \sim 9 \cdot s(30,1).$$

This comes from using only the first term of (4). It can be shown that the round-off and truncation errors will not affect the twentieth decimal place of t_m.

It seems plausible that the t_m are irrational.

TABLE 1.

m	t_m			
0	23.10344	79094	20541	61603
1	16.17696	95281	23444	26657
2	19.25735	65328	08072	22453
3	20.56987	79509	61230	37107
4	21.32746	57995	90036	68663
5	21.83460	08122	96918	16340
6	22.20559	81595	56091	88416
7	22.49347	53117	05945	39817
8	22.72636	54026	79370	60283
9	22.92067	66192	64150	34816

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A GENERALIZATION OF ZOLOTAREV'S THEOREM

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In 1872 Zolotarev [4] published the following theorem:

THEOREM 1. *Let p be an odd prime and let a be an integer not divisible by p. Let π_a be the permutation on the group of reduced residues* mod p *given by*

$$\pi_a(k) \equiv ak (\mathrm{mod}\, p),\ k = 1, \ldots, p-1.$$

Then π_a is an even or an odd permutation according as a is a quadratic residue or nonresidue mod p; *in symbols*

$$\operatorname{sgn} \pi_a = \left(\frac{a}{p}\right),$$

where sgn *is the signum of the permutation and $\left(\frac{a}{p}\right)$ is the Legendre symbol.*

Several proofs of this theorem have been given [1], [2], [3]. A group-theoretic proof can be constructed by considering the map $a \to \operatorname{sgn} \pi_a$ as a homomorphism from the group of reduced residues onto the group $\{\pm 1\}$ and showing that the kernel is exactly the subgroup of squares mod p. This argument leads to the following generalization.

THEOREM 2. *Let G be a finite group, and let G^+ be the subgroup of G generated by the squares in G. For a in G let π_a be the permutation on G given by*

$$\pi_a(k) = ka,\ k \text{ in } G.$$

Then if the order of G is even and its 2-Sylow subgroups are cyclic,

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