The initially surprising fact that the series, obtained from the (divergent) harmonic series by omitting all terms in which the denominator contains the digit 9, actually converges has been known for a long time (several references have been compiled in [3]). However, it is also clear that the series converges extremely slowly, and moreover, the partial sums behave so irregularly that a straightforward extrapolation of the series' value is not feasible. This is likely the reason why the literature only contains more or less rough estimates for a well-defined (and therefore computable) numerical value. Interestingly, the sum of the series can be calculated very precisely.

1 A function equation

Let us denote the set of natural numbers whose decimal representation does not contain the digit 9 by M. This set has a simple structure: a number in M is either a single-digit number, one of 1, 2, ..., 8, or it is formed by appending one of the digits 0, 1, ..., 8 to a number in M with fewer digits. Thus, we have:

$$M = \{1, 2, 3, 4, 5, 6, 7, 8\} \cup \bigcup_{z=0}^{8} \{10m + z : m \in M\} \tag{1}$$

From the above consideration, it follows immediately that there are exactly $8 \cdot 9^{n-1}$ n-digit numbers in M — there are 8 possibilities for the first digit and 9 for each subsequent one. Since the n-digit numbers lie in the range from 10^{n-1} to $10^n - 1$, we have the rough estimate:

$$\sum_{m \in M} \frac{1}{m} \leq \sum_{n=1}^{\infty} 8 \cdot 9^{n-1} \cdot 10^{-n+1} = 80$$

This estimate is, of course, only suitable for a simple convergence proof. For the precise calculation of the series of interest, $\sum_{m \in M} \frac{1}{m}$, we consider the function...

$$s(x) = \sum_{m \in M} \frac{1}{m + x}, x \geq 0,$$

which, due to the uniform convergence of the series for $x > 0$ (with the majorant $\sum_{m \in M} \frac{1}{m}$), is well-defined and continuous. From equation (1), it immediately follows that...

$$
\begin{aligned}
s(x) &= \frac{1}{1+x} + \ldots + \frac{1}{8+x} + \sum_{z=0}^{8} \sum_{m \in M} \frac{1}{10m+z+x} \\
&= \frac{1}{1+x} + \ldots + \frac{1}{8+x} + \frac{1}{10} \sum_{z=0}^{8} \sum_{m \in M} \frac{1}{m+\frac{z+x}{10}} \\
&= \frac{1}{1+x} + \ldots + \frac{1}{8+x} + \frac{1}{10} \sum_{z=0}^{8} s(\frac{z+x}{10})
\end{aligned}
$$

and thus the equation:

$$
s(x) - \frac{1}{10} \sum_{z=0}^{8} s(\frac{z+x}{10}) = \frac{1}{1+x} + \ldots + \frac{1}{8+x} \tag{2}
$$

Introducing the notation $r(x)$ as an abbreviation for the right-hand side of equation (2), we can proceed with the expression defined by...

$$
(Af)(x) = \frac{1}{10} \sum_{z=0}^{8} f(\frac{z+x}{10})
$$

the operator defined on the space $C[0,1]$, allowing equation (2) to be rewritten in the form...

$$
s - As = r \tag{3}
$$

2 Properties of the Operator Equation

The following properties of the operator A are easy to verify:
1. A is linear.
2. $\|A\| = \frac{9}{10} < 1$.
3. If f is non-negative on $[0,1]$, then the same holds for Af.
4. If p_n is a polynomial of degree n, then Ap_n is also a polynomial of degree n.

From this, it immediately follows that the operator $I - A$ is invertible (where, as usual, I denotes the identity operator).

$$
(I-A)^{-1} = I + A + A^2 + \ldots \tag{4}
$$

Thus, the equation

$$
||(I-A)^{-1}|| \leq (1 - ||A||)^{-1} = 10
$$

is uniquely solvable

$$
(I-A)f = g
$$

for any $g \in C[0,1]$ in $C[0,1]$, and the function $s(x)$ we sought is uniquely determined by (3) or (2). Let l now be the function defined by...

$$l(g) = (I - A)^{-1} g(0) \text{for} g \in C[0,1]$$

the defined functional; it is obviously linear. From property 3 and equation (4), it follows that if $g_1(x) < g_2(x)$ for all $x \in [0,1]$, then $l(g_1) < l(g_2)$ also holds. Moreover, it follows that...

$$||l|| \leq ||(I - A)^{-1}|| \leq 10 \tag{5}$$

Due to...

$$\sum_{m \in M} \frac{1}{m} = s(0) = (I - A)^{-1} r(0) = l(r)$$

Our task thus reduces to calculating $l(r)$. The basis for this is the fact that, due to property 4, the functional $l(p)$ for a polynomial p is easy to compute, and it is well-known that functions in $C[0,1]$ can be approximated arbitrarily closely by polynomials.

3 Two Recursions

Of course, for a given polynomial p, the uniquely determined polynomial $q = (I - A)^{-1} p$ can be found by substituting into the equation $q - Aq = p$ and comparing coefficients. However, since we are only interested in $l(p)$, and $p(x)$ can be written as $a_0 + a_1 x + \cdots + a_n x^n$ or equivalently as $b_0 + b_1(1-x) + \cdots + b_n(1-x)^n$, it suffices to derive recursive relationships for the sequences...

$$\alpha_n = l(x^n) \& \beta_n = l((1-x)^n)$$

By definition, we have $l((I - A)g) = g(0)$ for any $g \in C[0,1]$. Let us now set $g(x) = (e^{t/10} - 1)e^{tx}$, which gives us initially...

$$Ag(x) = \frac{1}{10}(e^{t/10} - 1)\left(e^{tx/10} + \ldots + e^{t(x+8)/10}\right) = \frac{1}{10}(e^{9t/10} - 1)e^{tx/10}$$

and from this:

$$l\left((e^{t/10} - 1)e^{tx} - \frac{1}{10}(e^{9t/10} - 1)e^{tx/10}\right) = e^{t/10} - 1$$

Expanding both sides in powers of t and comparing coefficients gives...

$$\sum_{k=1}^{n} \frac{1}{k!(n-k!)} \left(\frac{1}{10^k} - \frac{9^k}{10^{n+1}} \alpha_{n-k}\right) = \frac{1}{n!} \frac{1}{10^n}$$

This equation can be written in the form...

$$\sum_{k=1}^{n} \binom{n}{k} \left(10^{n-k+1} - 9^k\right) \alpha_{n-k} = 10 \tag{6}$$

From equation (6), we can successively obtain $\alpha_0 = 10$, $\alpha_1 = \frac{360}{91}$, and so on. In exactly the same way, by using $g(x) = (e^{t/10} - 1)e^{tx}$, we can obtain...

$$\sum_{k=1}^{n} \binom{n}{k} \left(10^{n-k+1} - 10^k + 1\right) \beta_{n-k} = 10(11^n - 10^n) \tag{7}$$

From this, we obtain $\beta_0 = 10$, $\beta_1 = \frac{550}{91}$, and so on. It should be noted here that, because $x^{n+1} < x^n$ and $(1-x)^{n+1} < (1-x)^n$ for $x \in [0,1]$, the sequences (α_n) and (β_n) are monotonically decreasing.

To apply these results to equation (2), we need an accurate approximation of $r(x)$ by a polynomial, for example, using a truncated power series expansion. In principle, the Taylor series of $r(x)$ (of course, around $x = 1$) is suitable for this, as it converges uniformly on $[0,1]$ — although unfortunately not very quickly (with truncation after n terms, the error is 2^{-n}). The "guilt" lies in the pole of $r(x)$ at $x = -1$. Fortunately, equation (2) can be transformed into a similar equation with a "better" right-hand side.

4 An Analytical Method

At this point, we must note that equation (2) is uniquely solvable only in $C[0,1]$ (or in another space of bounded functions) — there also exist unbounded solutions. The simplest one is $-\frac{1}{x}$, as can be immediately verified by substitution. This means that the (unbounded!) function $s_1(x) = s(x) - \frac{1}{x}$ satisfies the equation $s_1 - As_1 = 0$. However, it is known (see [1, 6.4.8] for $n = 0, m = 10$) that the logarithmic derivative of the Gamma function, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, satisfies...

$$\psi(x) = \ln(10) + \frac{1}{10} \sum_{k=0}^{9} \psi\left(\frac{x+k}{10}\right)$$

From this, it immediately follows that the function $s_2(x) = s_1(x) + \psi(x) + \gamma$ satisfies the equation,

$$s_2(x) - As_2(x) = \ln(10) + \frac{1}{10}\left[\psi\frac{x+9}{10} + \gamma\right] = r_1(x)$$

According to [1, 6.3.5],

$$s_2(x) = s(x) + \frac{1}{x} + \psi(x) + \gamma = s(x) + \psi(x+1) + \gamma$$

This function is therefore in $C[0,1]$, and it holds that $s(0) = s_2(0) = l(r_1)$. However, due to [1, 6.3.14],...

$$r_1(x) = \ln(10) + \frac{1}{10}\left[\psi\left(1 - \frac{1-x}{10}\right) + \gamma\right] = \ln(10) - \frac{1}{10}\sum_{n=2}^{\infty}\zeta(n)\left(\frac{1-x}{10}\right)^{n-1}$$

is true, we obtain the following interesting

Theorem 1. *Result 1: We have*

$$\sum_{m\in M}\frac{1}{m} = \beta_0\ln(10) - \sum_{n=2}^{\infty}10^{-n}\beta_{n-1}\zeta(n), \tag{8}$$

where the β_n are determined by equation (7).

The truncation error of the series (8) can be easily estimated: since $\zeta(n)$ and β_n are monotonically decreasing, the ratio of two consecutive terms is $\leq \frac{1}{10}$. Therefore, the error is $\leq 10^{-1} + 10^{-2} + \ldots = \frac{1}{9}$ of the last included term. The values of $\ln 10$ and $\zeta(n)$ can be found in some standard tables, making (8) very suitable for manual calculation (or rather with a calculator). Including terms up to and including $n = 7$, one finds...

$$\sum_{m\in M}\frac{1}{m} = 22.9206766\ldots$$

However, if we wish to calculate $s(0)$ with significantly higher accuracy, (8) is not the best method, as $\ln 10$ and $\zeta(n)$ would first need to be computed.

5 Chebyshev Approximation

The Taylor series is not a suitable tool for approximating the function $r(x)$ on $[0,1]$ using polynomials. Instead, we make use of some simple properties of the Chebyshev polynomials. These are well-known (see [2, 4.9]) and are defined by...

$$T_n(y) = cos(narccosy)$$

Hence, it holds that...

$$|T_n(y)| \leq 1 \quad \text{for} \quad y \in [-1,1]$$

therefore also...

$$|T_n(1-2x)| \leq 1 \quad \text{for} \quad x \in [0,1]$$

the polynomial $T_{n+1}(3) - T_{n+1}(1-2x)$ vanishes at $x = -1$, and is therefore divisible by $1 + x$ without remainder. Let us denote the quotient by $Q_n(x)$. Since $|T_{n+1}(1-2x)| \leq 1$, it follows that...

$$\frac{T_{n+1}(3)-1}{1+x} \leq \frac{T_{n+1}(3)-T_{n+1}(1-2x)}{1+x} = Q_n(x) \leq \frac{T_{n+1}(3)+1}{1+x}$$

If we further divide by $T_{n+1}(3)$ and introduce the notation

$$\frac{Q_n(x)}{T_{n+1}(3)} = \sum_{k=0}^{n} q_{n_k} x^k$$

..., it then follows that

$$\left(1-\frac{1}{T_{n+1}(3)}\right)\frac{1}{1+x} \leq \sum_{k=0}^{n} q_{n_k} x^k \leq \left(1+\frac{1}{T_{n+1}(3)}\right)\frac{1}{1+x} \quad \text{for} \quad x \in [0,1]$$

and due to

$$\frac{1}{m+x} = \frac{1}{m}\frac{1}{1+x/m} \quad \text{for} \quad m = 1, 2, ..., 8$$

also

$$\left(1-\frac{1}{T_{n+1}(3)}\right)\frac{1}{m+x} \leq \sum_{k=0}^{n} q_{n_k} \frac{1}{m^{k+1}} x^k \leq \left(1+\frac{1}{T_{n+1}(3)}\right)\frac{1}{m+x} \quad \text{for} \quad x \in [0,1]$$

Adding these inequalities for $m = 1, 2, \ldots, 8$ and then applying the functional l, we directly obtain...

Theorem 2. *Result 2: With the above notations (the sequence (α_k) is defined by (6)), and*

$$H_8^{(k)} = 1 + \frac{1}{2^k} + ... + \frac{1}{8^k}$$

it holds that...

$$\left(1-\frac{1}{T_{n+1}(3)}\right)\sum_{m\in M}\frac{1}{m} \leq \sum_{k=0}^{n} q_{n_k}\alpha_k H_8^{(k+1)} \leq \left(1+\frac{1}{T_{n+1}(3)}\right)\sum_{m\in M}\frac{1}{m} \tag{9}$$

The Chebyshev polynomials can be easily computed for large n using their recursion formula, and the division by $1 + x$ is also not a problem when using Horner's method. The degree of the polynomial required for a given accuracy can be easily estimated: According to ([2, 4.13]), it holds that...

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]$$

and therefore...

$$T_n(3) \le \frac{1}{2}(3+2\sqrt{2})^n$$

With a polynomial of degree 135 and a 100-digit calculation, we can derive from (9) obtain,

Result in 100 digits here.