Exercise 1.13

Golden ratio and Fibonacci numbers

Using the definition of Fibonacci numbers:

$$\mathrm{Fib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) & \text{otherwise} \end{cases}$$

and mathematical induction, we will prove that

$$\mathrm{Fib}(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

First, we take $n = 0$ and $n = 1$ as induction base and show that $\mathrm{Fib}(n) = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$ is valid in these cases.

$$\begin{aligned} \mathrm{Fib}(0) &= \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \\ \mathrm{Fib}(1) &= \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \end{aligned}$$

Yes, they agree with the definition.

Next, we presume that the following is true for some $k < n$:

$$\mathrm{Fib}(k) = \frac{\varphi^k - \psi^k}{\sqrt{5}}.$$

We will show that the truth of last statement implies the truth of

$$\mathrm{Fib}(k+1) = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}.$$

By definition of the Fibonacci sequence, and using the equations $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$ we have:

$$\begin{aligned}
\mathrm{Fib}(k+1) &= \mathrm{Fib}(k) + \mathrm{Fib}(k-1) = \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}} \\
&= \frac{\varphi^{k-1}(\varphi+1) - \psi^{k-1}(\psi+1)}{\sqrt{5}} = \frac{\varphi^{k-1}\varphi^2 - \psi^{k-1}\psi^2}{\sqrt{5}} \\
&= \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}. \quad \text{QED.}
\end{aligned}$$

We have just established that the induction step is valid. This can now be used to show that if a statement with $n = k$ is true, then the one with $n = k + 1$ is also true. We have already demonstrated that the base cases with $n = 0$ and $n = 1$ are true. These imply that the case with $n = 2$ must also be true. Step by step, this leads to any n. Therefore, we can safely assert the truth in general, with all n.