Equivariant Geometric Algebra

Course: Deep Learning

Degree: Msci. In AI & Robotics

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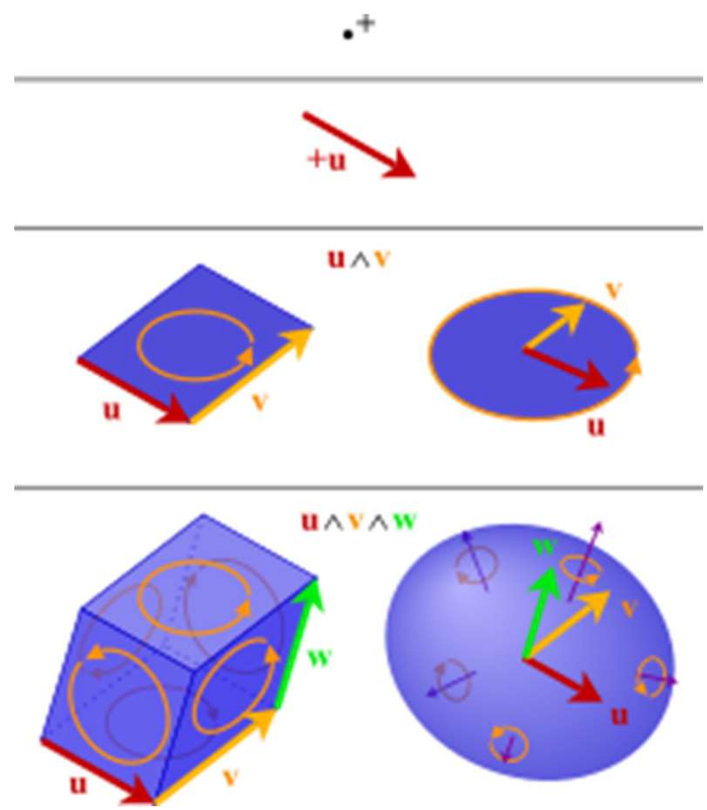
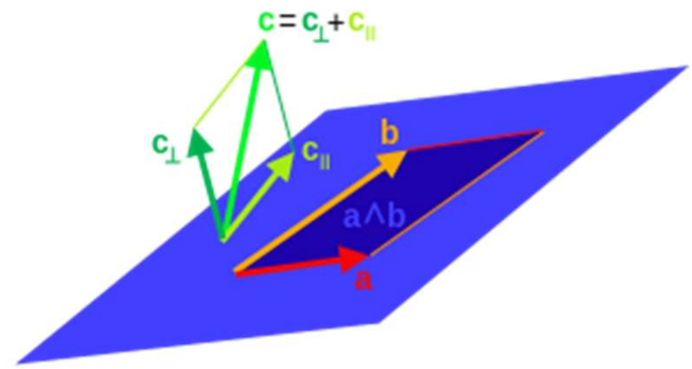
A.Y. 2023/24

- In recent years, two very important and popular research areas in Deep Learning have been:
  - How can symmetries be encoded using equivariant deep learning?
  - How can geometric objects be represented through algebraic structures?





Projective Geometric Algebra

- Denoted as $G_{3,0,1}$, is a 16-dimensional algebra
- Provide a powerful framework for representing and manipulating geometric objects and transformations in 3D space.
- Extend traditional vector algebra to include:
  - **Geometric Primitives** like points, lines, planes, and volumes
  - **Geometric operations** such as rotations, translations, and reflection
- Incorporate 4th homogeneous coordinate e_0
  - Resulting in a vector space spanned by the basis $\{e_0, e_1, e_2, e_3\}$.

Representation of Geometric Objects

- PGA can represent scalars, points, lines, planes, and pseudoscalars in 3D space as multivectors of different grades.
- The 16-dimensional basis vectors' space summary is as follows:
  - Scalar {1} – Grade 0
  - Vectors (Planes) $\{e_0, e_1, e_2, e_3\}$ – Grade 1
  - Bivector (Lines) $\{e_{01}, e_{02}, e_{03}, e_{12}, e_{13}, e_{23}\}$ – Grade 2
  - Trivectors (Points) $\{e_{012}, e_{013}, e_{023}, e_{123}\}$ – Grade 3
  - Pseudoscalar $\{e_{0123}\}$: – Grade 4

1/3 - Operations

- Geometric Product
  - Combines **Inner** and **Outer** products of vectors -> composition of geometric objects and transformations.

$$xy = \langle x, y \rangle + x \wedge y$$

- Reverse
  - Reverse the sequence of the basis vectors in each blade
  - For a k-blade $A = a_1 \wedge a_2 \wedge \cdots \wedge a_k$ where $\wedge$ represents the outer product, the reverse $\tilde{A}$ is given by:

$$\tilde{A} = a_k \wedge a_{k-1} \wedge \cdots \wedge a_1 = (-1)^{\frac{k(k-1)}{2}} (a_1 \wedge a_2 \wedge \cdots \wedge a_k)$$

- Norm
  - Computed as the square root of the inner product of the multivector with itself
    - $||x|| = \sqrt(\langle x, x \rangle)$

2/3 - Operations

- Dual
  - Reflect geometric elements across the duality relationship between the primal and dual spaces.
  - Act on basis elements by swapping “**empty**” and “**full**” dimensions.

Table 1: Duality table

$1^* = e_{0123}$	$e^*_{0123} = 1$
$e^*_0 = e_{123}$	$e^*_{123} = -e_0$
$e^*_1 = -e_{023}$	$e^*_{023} = e_1$
$e^*_2 = e_{013}$	$e^*_{013} = -e_2$
$e^*_3 = -e_{012}$	$e^*_{012} = e_3$
$e^*_{01} = e_{23}$	$e^*_{23} = e_{01}$
$e^*_{02} = -e_{13}$	$e^*_{13} = -e_{02}$
$e^*_{03} = e_{12}$	$e^*_{12} = e_{03}$

- Join
  - Designed to be **dual** to the **meet**.
  - In PGA, where $\wedge$ (meet) is an intersection, it is customary to denote the join by a $\vee$:

$$A \vee B = (A^* \wedge B^*)^*$$

3/3 - Operations

- Grade involution $\hat{x}$
  - Flips the sign of odd-grade elements such as vectors and trivectors
  - Leave even-grade elements unchanged.
- Sandwich Product
  - To apply a versor u to any element x, the sandwich product is used and it is **grade-preserving**:

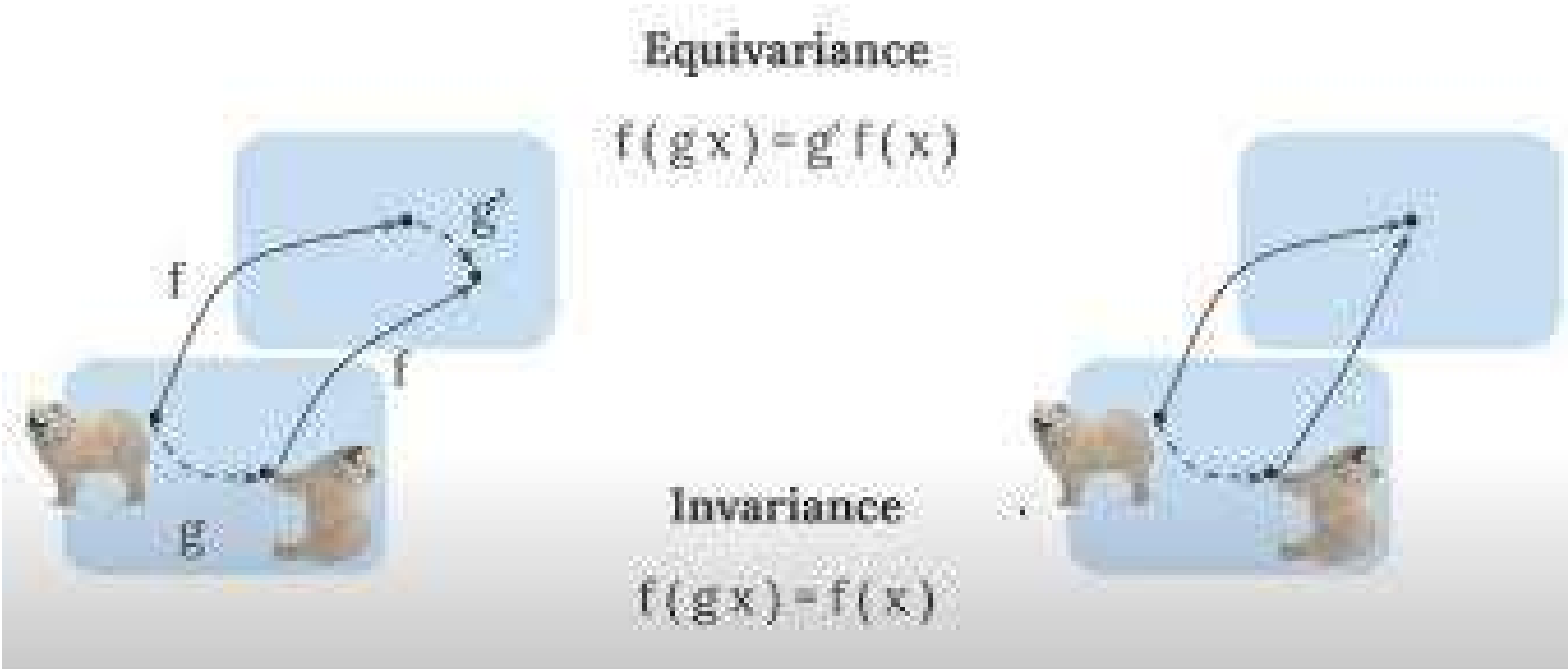
$$\rho_u(x) = \begin{cases} uxu^{-1} & \text{if } u \text{ is even} \\ u\hat{x}u^{-1} & \text{if } u \text{ is odd} \end{cases}$$

Transformations

- **Spin transformations**:
  - Closely related to rotations in space & represented by elements called spinors in geometric algebra.
  - A spinor is a multivector that can be used to represent rotations in any dimension.
  - Spin transformations preserve the orientation and magnitude of vectors.
  - They are associated with even-grade multivectors in geometric algebra.
- **Pin transformations:**
  - Include both rotations and reflections and are represented by pinors.
  - A pinor is a multivector that can represent both rotations and reflections.
  - Pin transformations include both spin transformations (rotations) and reflections.
  - They can be associated with both even and odd-grade multivectors in geometric algebra.

Equivariance Check

- ✓ Equivariance, in this context, means that applying a transformation before the operation yields the same result as applying the operation and then the transformation.



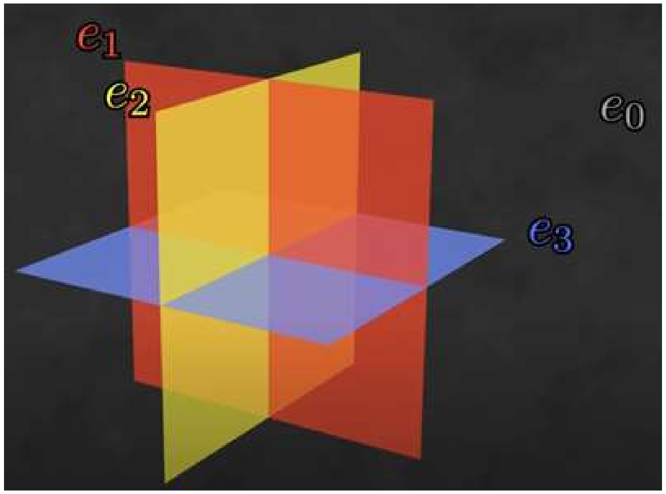
Embedding Table

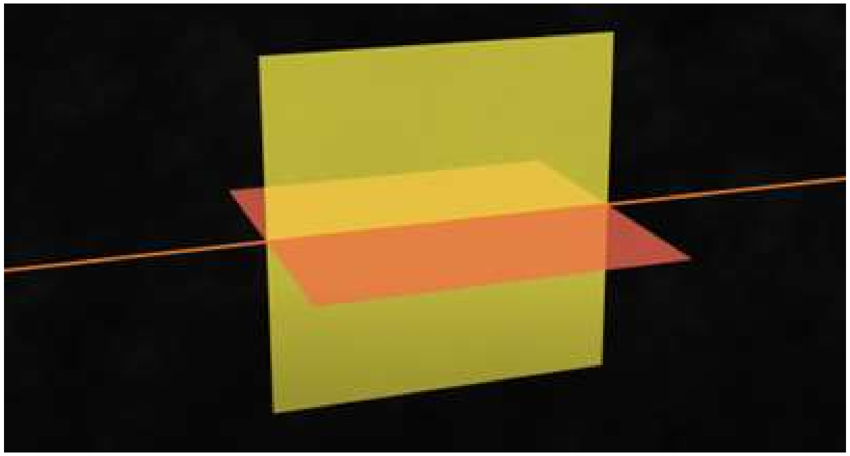
Object / operator	Scalar 1	Vector e_0	e_i	Bivector e_{0i}	e_{ij}	Trivector e_{0ij}	e_{123}	PS e_{0123}
Scalar $\lambda \in \mathbb{R}$	λ	0	0	0	0	0	0	0
Plane w/ normal $n \in \mathbb{R}^3$, origin shift $d \in \mathbb{R}$	0	d	n	0	0	0	0	0
Line w/ direction $n \in \mathbb{R}^3$, orthogonal shift $s \in \mathbb{R}^3$	0	0	0	s	n	0	0	0
Point $p \in \mathbb{R}^3$	0	0	0	0	0	p	1	0
Pseudoscalar $\mu \in \mathbb{R}$	0	0	0	0	0	0	0	μ
Reflection through plane w/ normal $n \in \mathbb{R}^3$, origin shift $d \in \mathbb{R}$	0	d	n	0	0	0	0	0
Translation $t \in \mathbb{R}^3$	1	0	0	$\frac{1}{2}t$	0	0	0	0
Rotation expressed as quaternion $q \in \mathbb{R}^4$	q_0	0	0	0	q_i	0	0	0
Point reflection through $p \in \mathbb{R}^3$	0	0	0	0	0	p	1	0

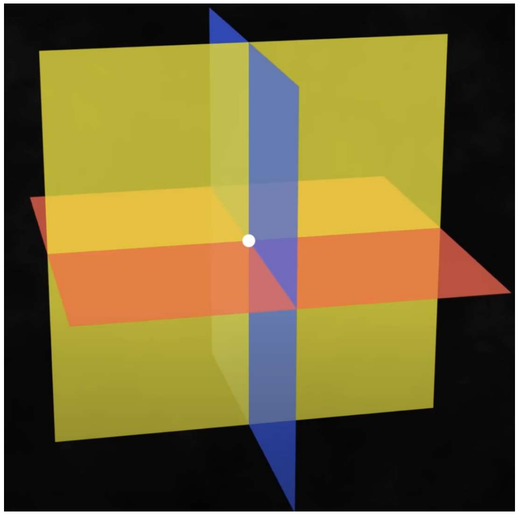
Table 1: Embeddings of common geometric objects and transformations into the projective geometric algebra $\mathbb{G}_{3,0,1}$. The columns show different components of the multivectors with the corresponding basis elements, with $i, j \in \{1, 2, 3\}, j \neq i$, i.e. $ij \in \{12, 13, 23\}$. For simplicity, we fix gauge ambiguities (the weight of the multivectors) and leave out signs (which depend on the ordering of indices in the basis elements).

Embedding Figures

- 3D PGA environment which is equal to the linear space of planes.
- Bivectors are considered as lines
- Trivectors are points







Geometric Algebra Transformer (GATr) Architecture

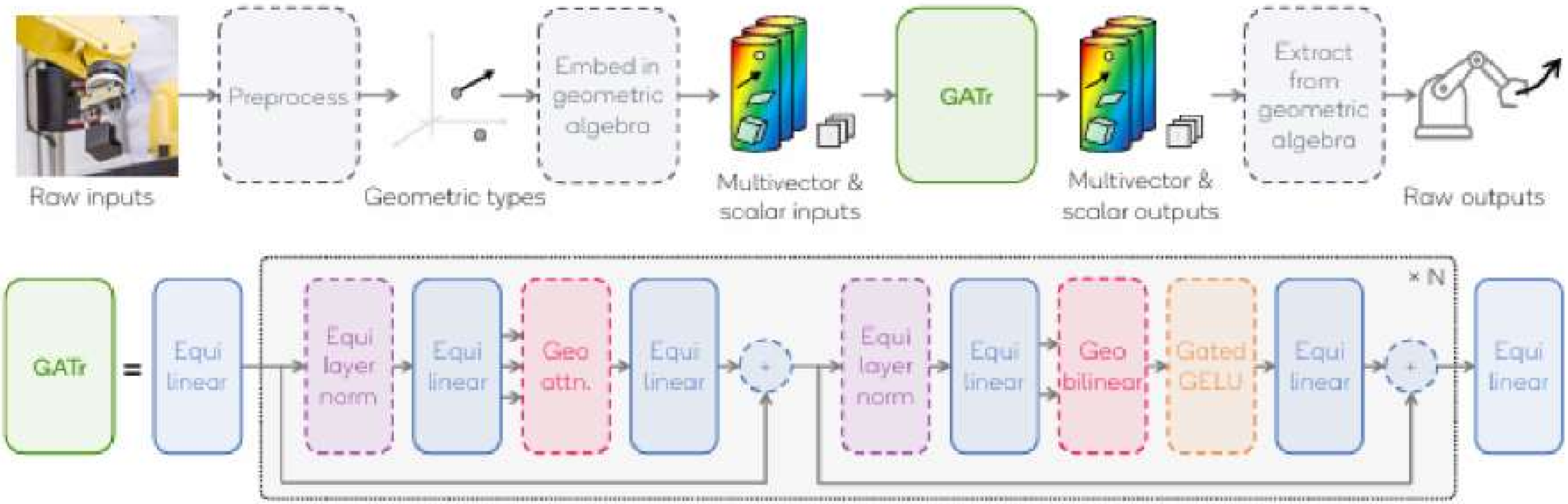


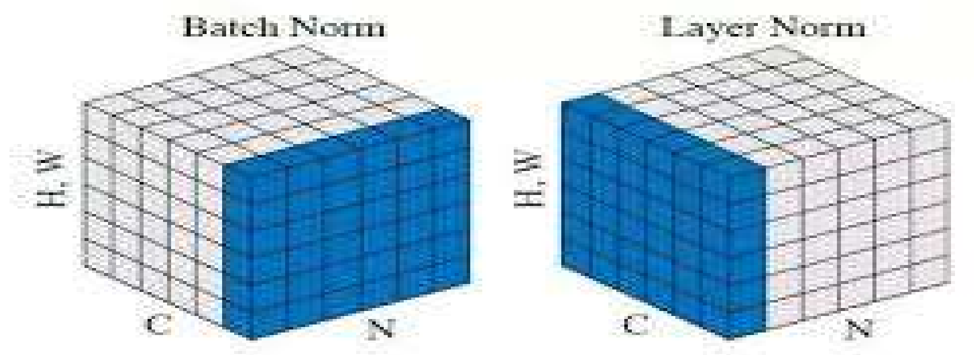
Figure 1: Overview over the GATr architecture. Boxes with solid lines are learnable components, those with dashed lines are fixed.

Components of GATr Architecture

1/5 - Equivariant Layer Norm

- Normalize each multivector separately using the invariant inner product of $\boldsymbol{G}_{3,0,1}$:

$$\frac{x}{\sqrt{\left(E_c \left\langle x, x \right\rangle\right)}}$$



2/5 – Gated Non-linearity

- Apply gated GELU to the scalar component of the multivector outputs to control the activation of the entire multivector:

$$\boldsymbol{GatedGELU}(x) = \boldsymbol{GELU}(x_1)\, x$$

- where x_1 is the scalar component of the multivector x.

- We can use the same **gated nonlinearity** properties for other activations, such as **ReLU**, **Sigmoid**, and **Tanh**.
- For exmple, for **ReLU**:

$$\textbf{GatedReLU}(x) = \textbf{ReLU}(x_1)\, x$$

3/5 - Equivariant Linear Transformation

- Perform equivariant linear transformation on multivector inputs

$$\phi(x) = \sum_{k=0}^{d} w_k \langle x \rangle_k + \sum_{k=0}^{d+1} v_k \, e_0 \, \langle x \rangle_k$$

- where x is the input multivector, $\langle x \rangle_k$ is the blade projection of x to grade k, w_k and v_k are learnable parameters, and e_0 is the homogeneous basis vector.

- Additionally, if auxilary scalars are provided, such as **pressure** in our case, then performs another linear transformation on it, not necessarily equivariant.

4/5 - Geometric Bilinear Transformation

- Perform geometric bilinear operations, that is two operations: **geometric product** and **join**:

$$\boldsymbol{Geometric}(x, y;\ z) = \boldsymbol{Concatenate}_{\boldsymbol{channel}}(xy, \boldsymbol{EquiJoin}(x, y;\ z))$$

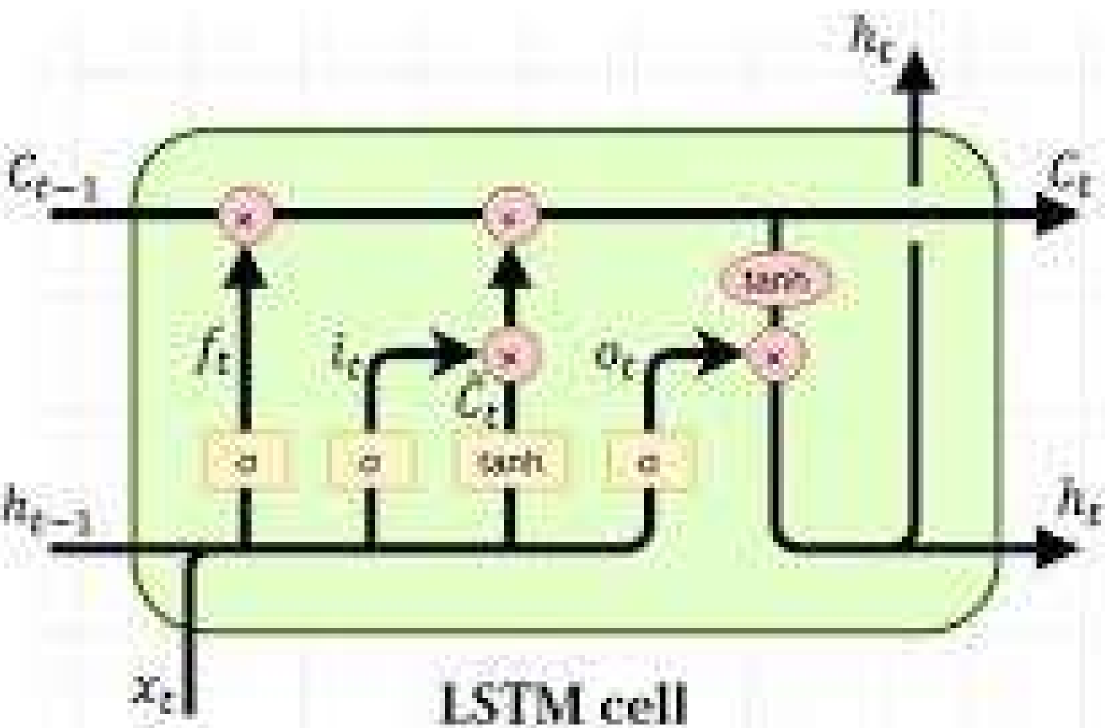
- where $\boldsymbol{EquiJoin}(x, y;\ z) = z_{0123}(x^* \wedge y^*)^*$, and z_{0123} is the **pseudoscalar** component of a reference multivector z.

5/5 - Geometric Attention

- Represent the geometric attention mechanism, a crucial part of the GATr architecture
- Responsible for capturing geometric relationships between multivectors.

$$\boldsymbol{Attention}(q,k,v)\} = \sum_i Softmax_i \frac{\sum_c \langle q_i, k_i \rangle}{\sqrt{(8n_c)}} v_i$$

Equivariant LSTM



$$i_t = \sigma(x_t U^i + h_{t-1} W^i)$$
$$f_t = \sigma(x_t U^f + h_{t-1} W^f)$$
$$o_t = \sigma(x_t U^o + h_{t-1} W^o)$$
$$\tilde{C}_t = \tanh\left(x_t U^g + h_{t-1} W^g\right)$$
$$C_t = \sigma(f_t * C_{t-1} + i_t * \tilde{C}_t)$$
$$h_t = \tanh(C_t) * o_t$$

Results

Model	Non-norm data F1 score	Norm data F1 score
SVM	100%	98.625%
Log Reg	100%	98%

Model	F1 Score	Mem Footprint (MB)	Num Params (K)
GATr	100%	0.1	25.1
EquiLinear	98.855	0.008	2.1
BiEquiLinear	96.47%	0.074	18.4
EquiLSTM	100%	0.009	2.3
BiEquiLSTM	$\approx$ 100%	0.014	3.4


THANK
YOU!