University of BRISTOL

Limit laws and continuous random variables

Henry W J Reeve

henry.reeve@bristol.ac.uk

Statistical Computing & Empirical Methods (EMATM0061)

MSc in Data Science, Teaching block 1, 2021.

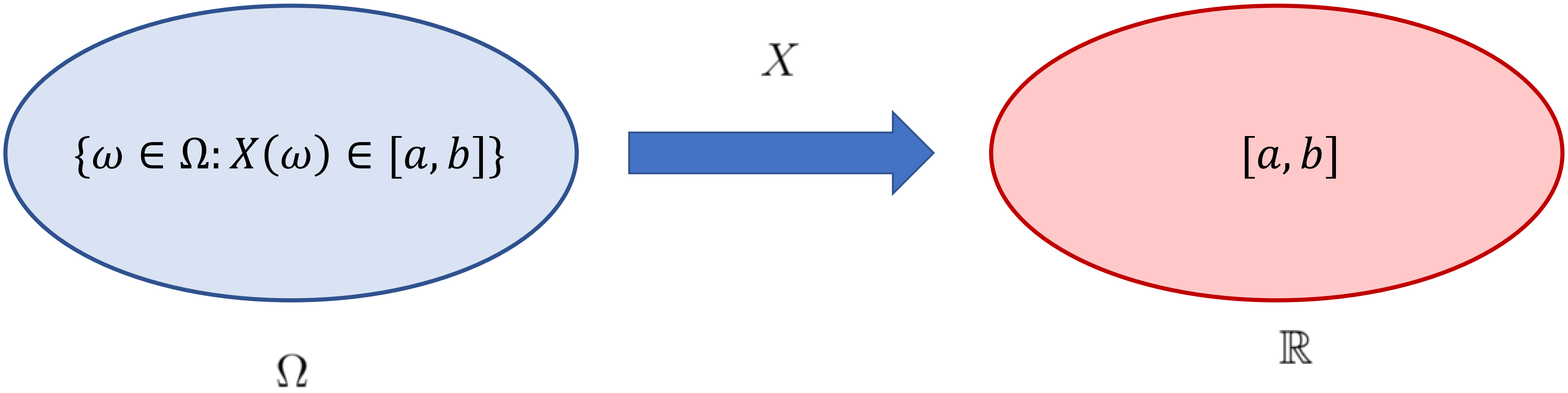
What will we cover today?

- We will introduce the concept of a continuous random variable.
- We will see continuous random variables can be understood via the probability density function.
- We will discuss expectation, variance, standard deviation, covariance and correlation in this context.
- We introduce the important example of Gaussian random variable.
- We will discuss the law of large numbers and the central limit theorem.

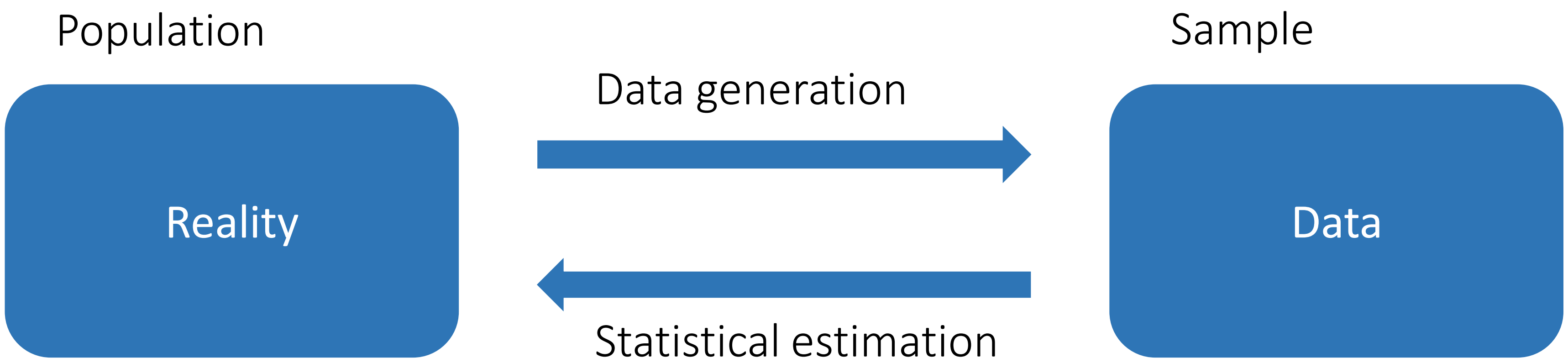
Random variables

Suppose we have a probability space $(\Omega, \mathcal{E}, \mathbb{P})$.

A **random variable** is a mapping $X : \Omega \to \mathbb{R}$ such that for every a, $b \in \mathbb{R}$, $\{\omega \in \Omega \;:\; X(\omega) \in [a, b]\} \in \mathcal{E}$ is an event.



Statistical estimation and probability



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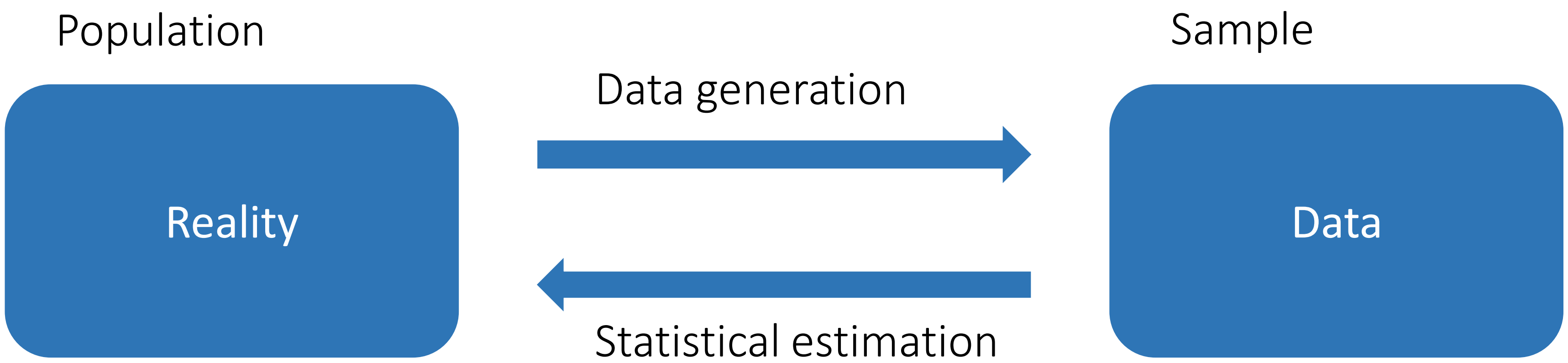

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A sample of independent copies $X_1, \ldots, X_n \sim P_X$.

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If $P_{X_1} = P_{X_2} = \ldots = P_{X_n} = P_X$ then we refer to $X_1, \ldots, X_n$ as **independent copies** of X.

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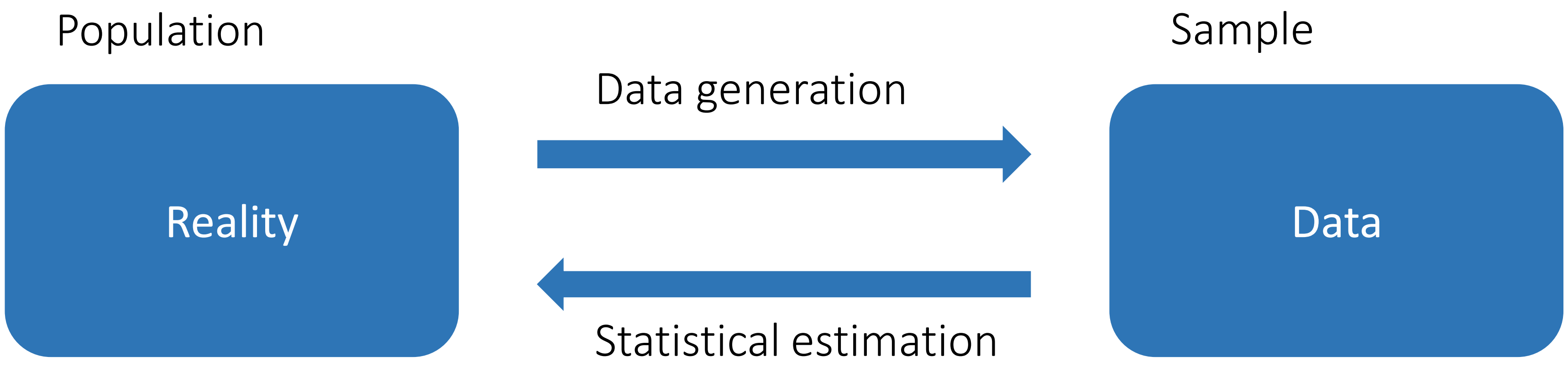

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This is an example of a law of large numbers.

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# plot a histogram to display the results
dice_sample_average_simulation%>%
  ggplot(aes(x=sample_avg))+
  geom_histogram(aes(y=..count../sum(..count..)),
                 binwidth=1/sample_size,fill="blue",color="blue")+
  theme_bw()+
  xlim(c(1,6))+
  xlab("Sample average")+ylab("Proportion")
```

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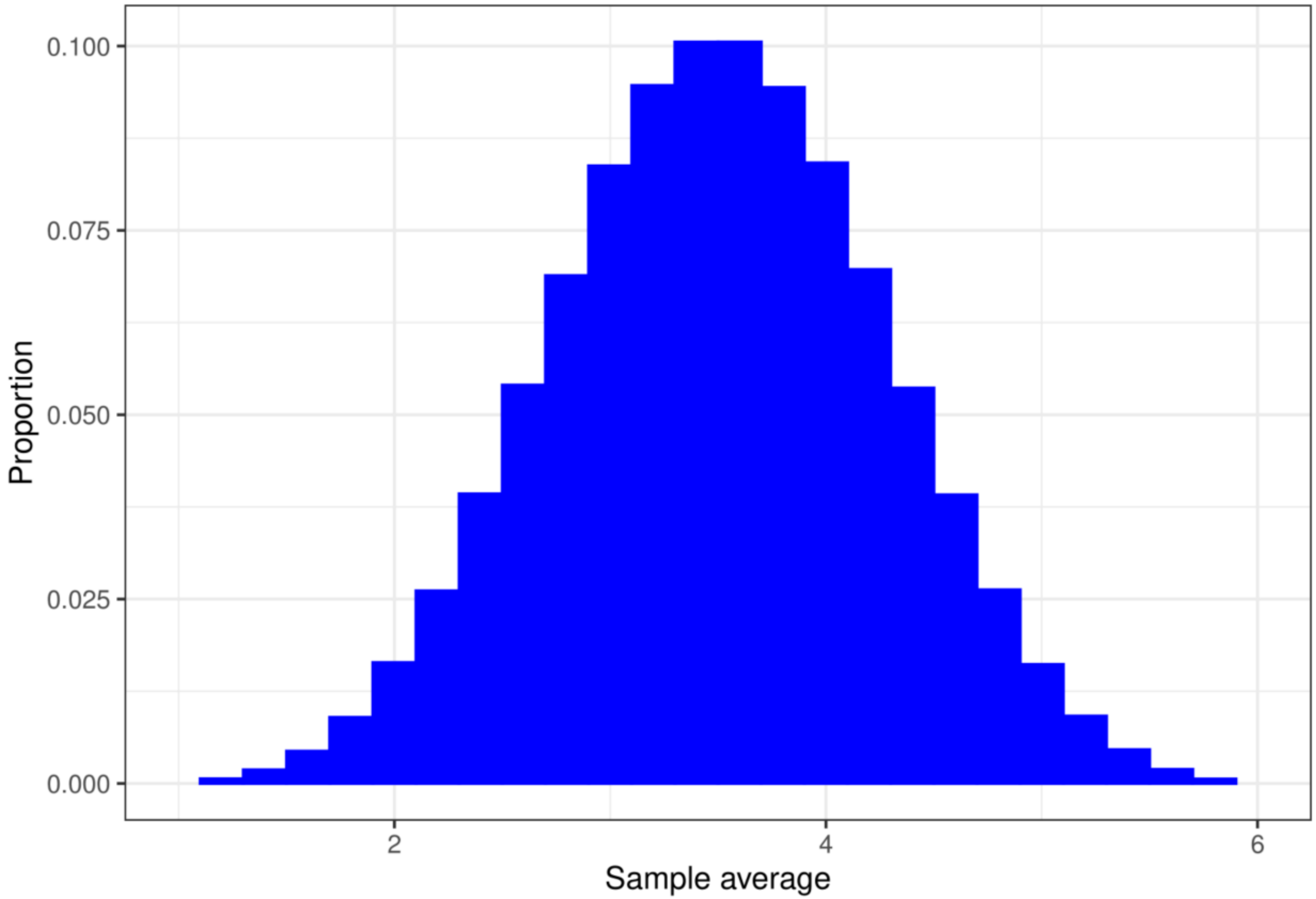

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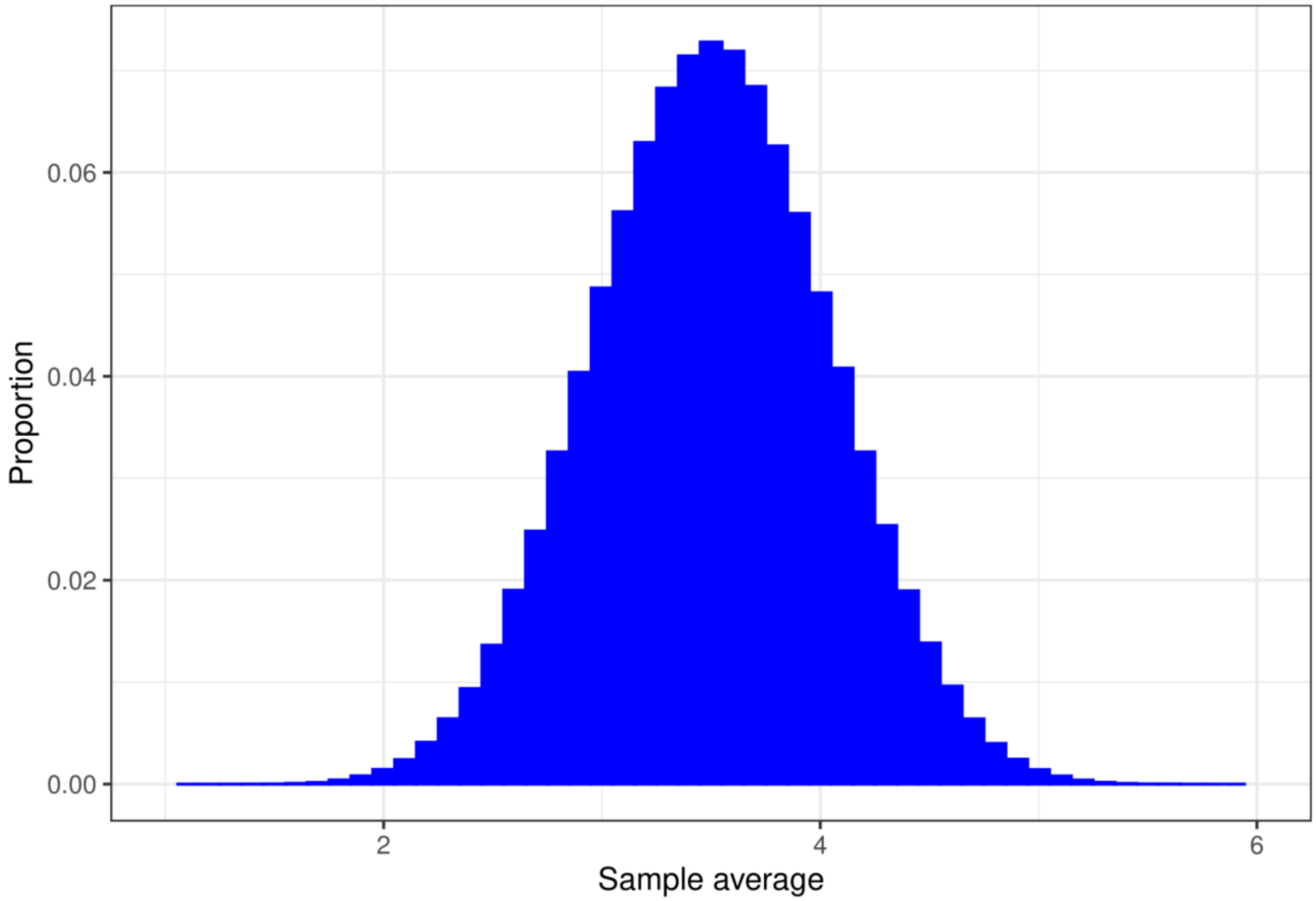

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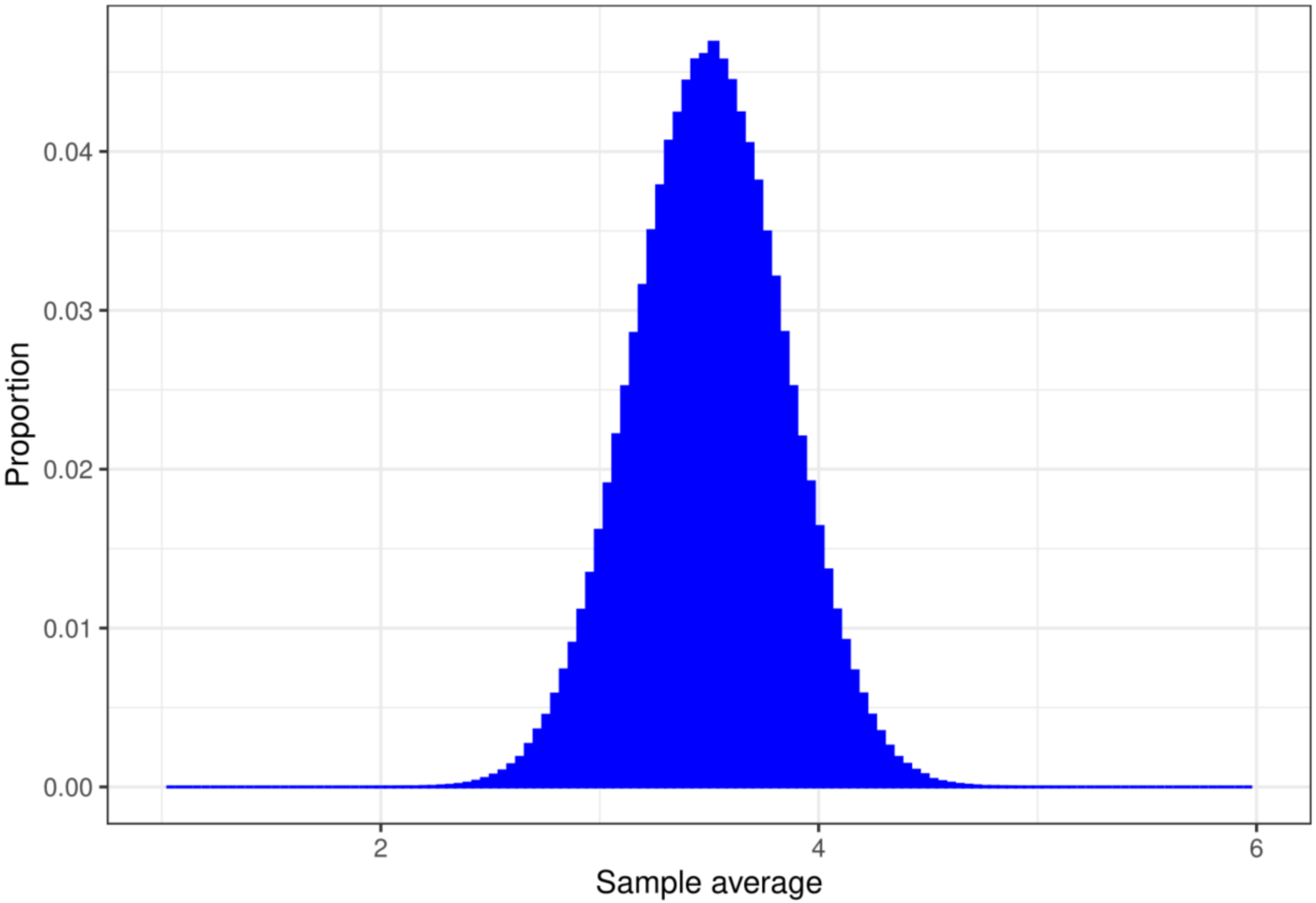

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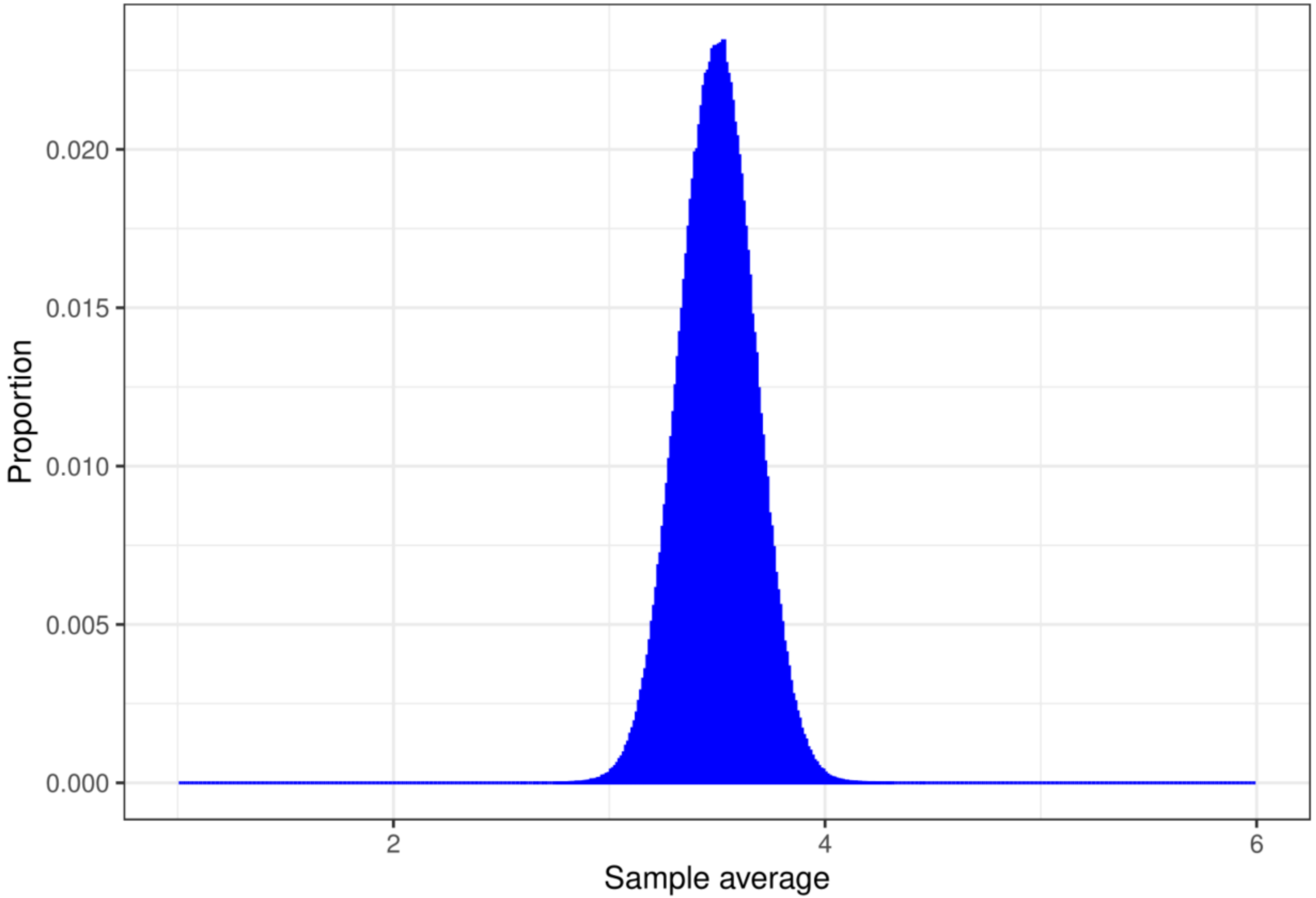

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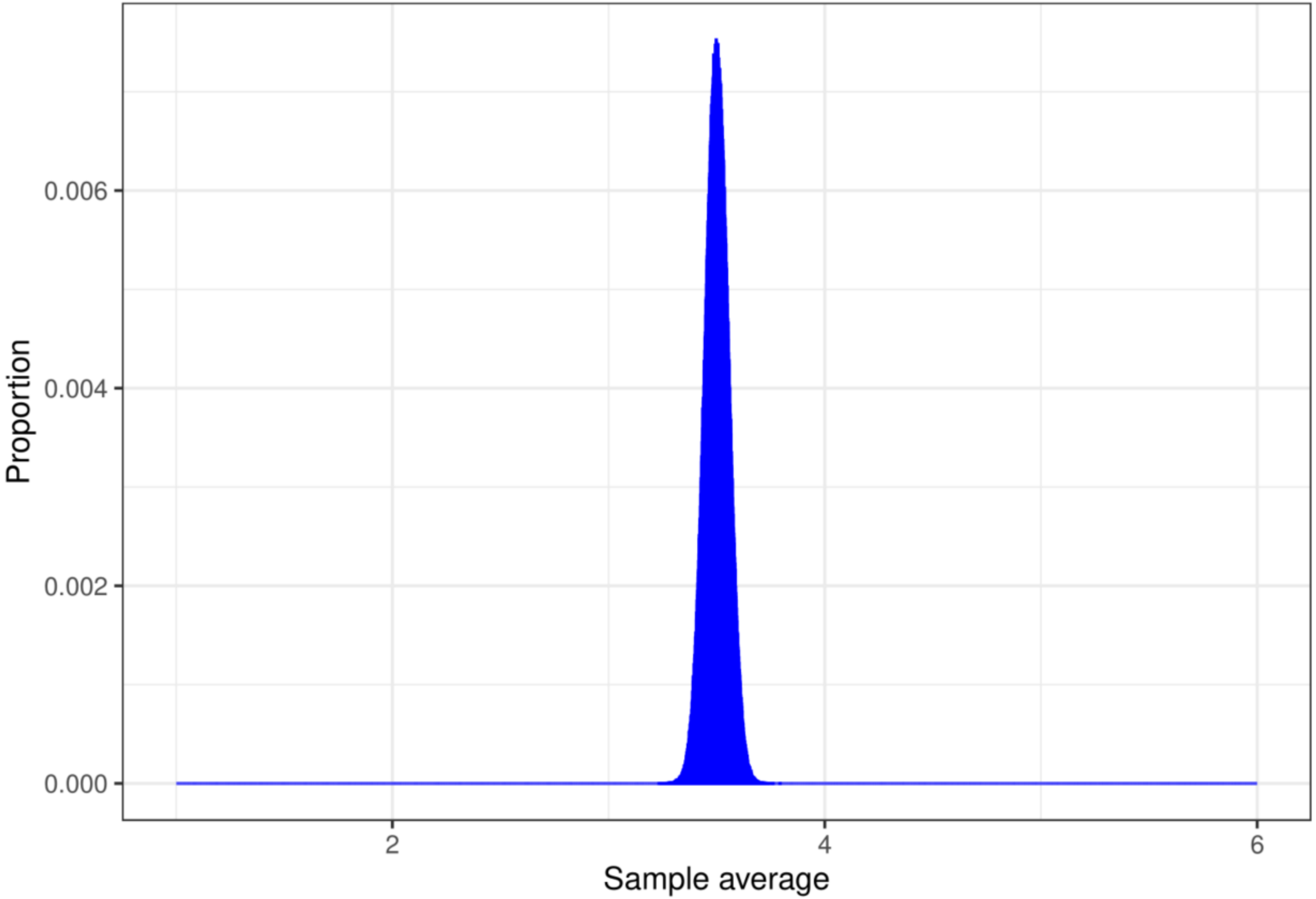

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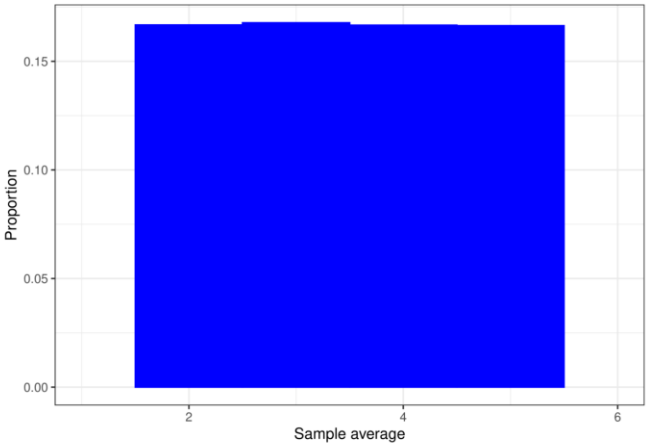

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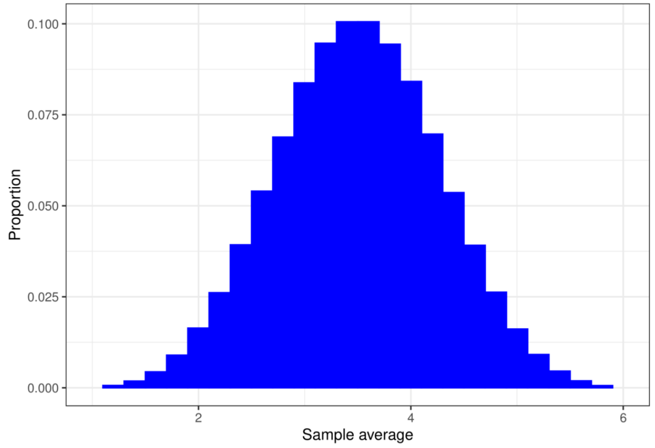

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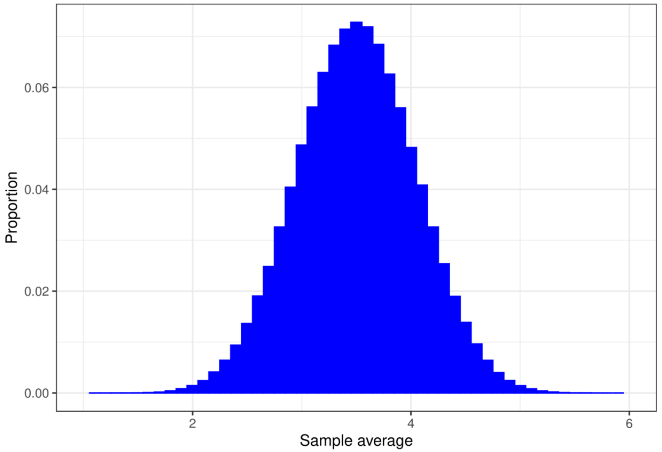

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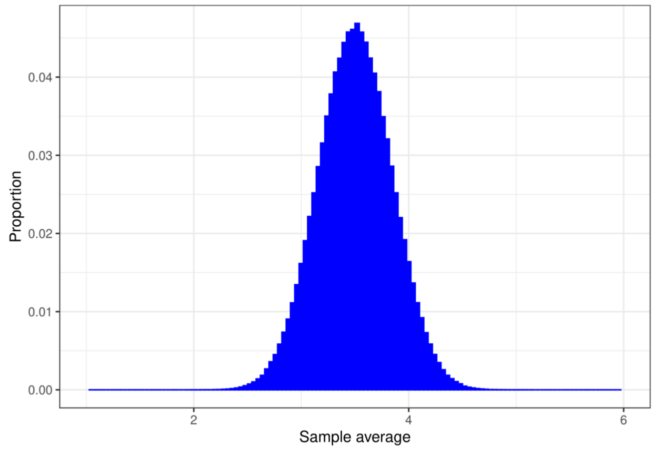

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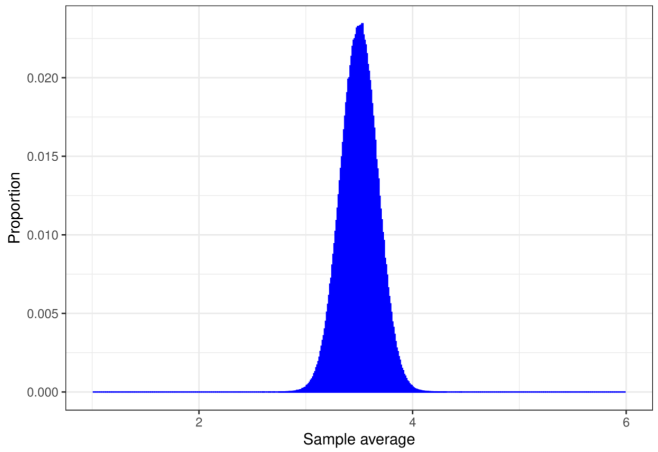

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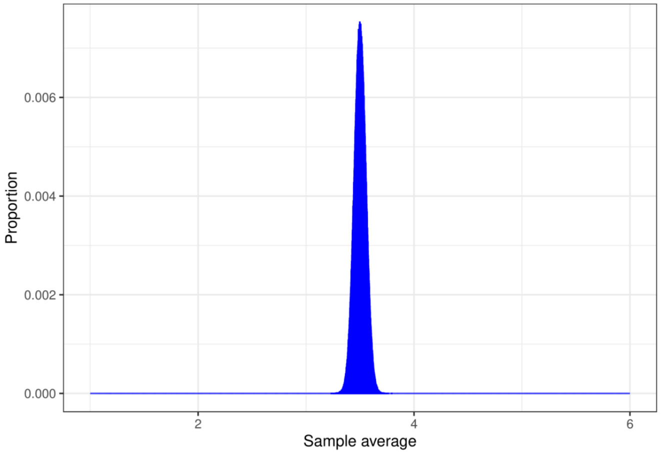

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This result is known as **the weak law of large numbers**.

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This suggests $\frac{1}{n}\sum_{i=1}^{n} X_i - \mu$ is a random variable which shrinks to zero as n goes to infinity.

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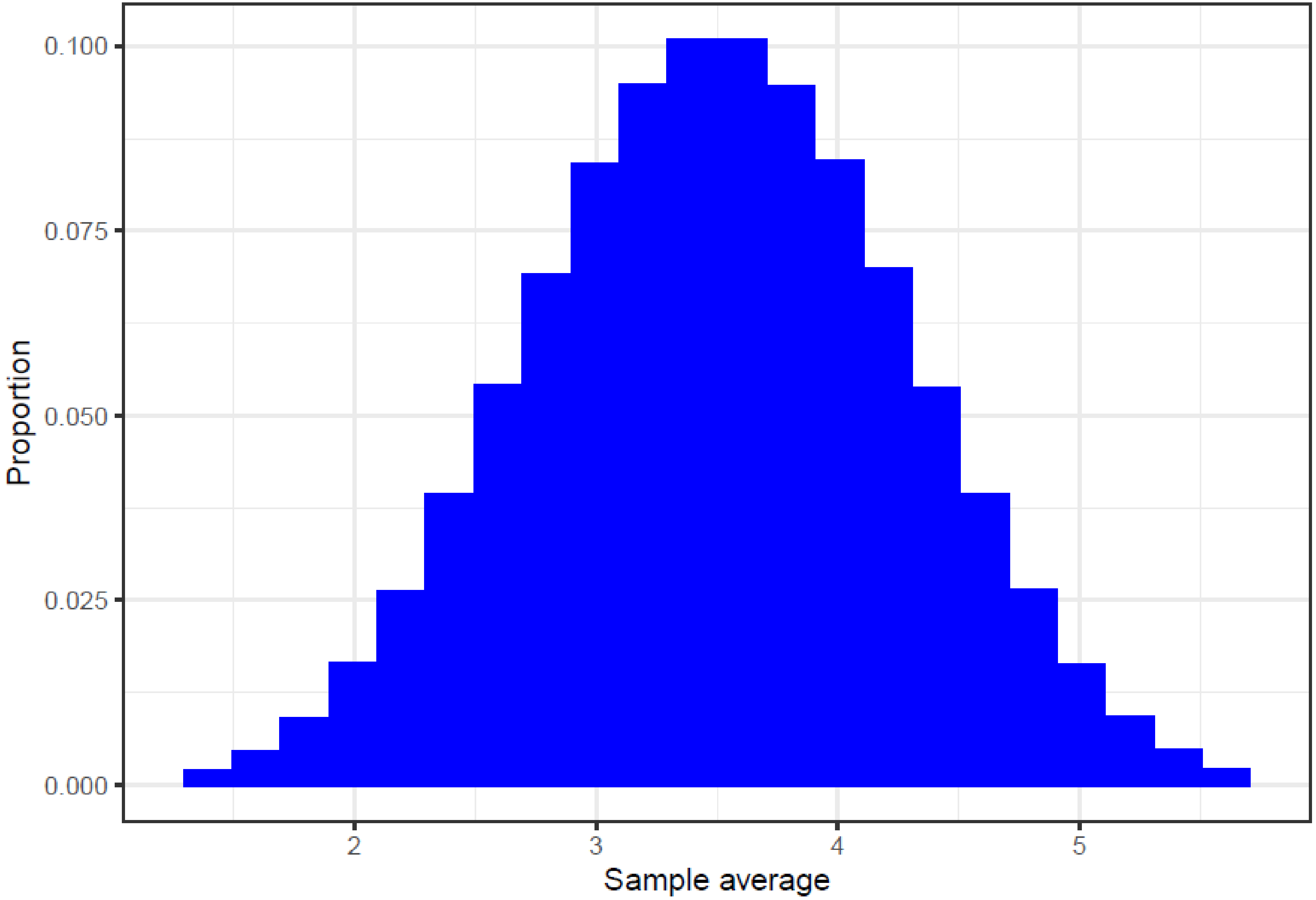
The sample average of dice rolls (renormalized)



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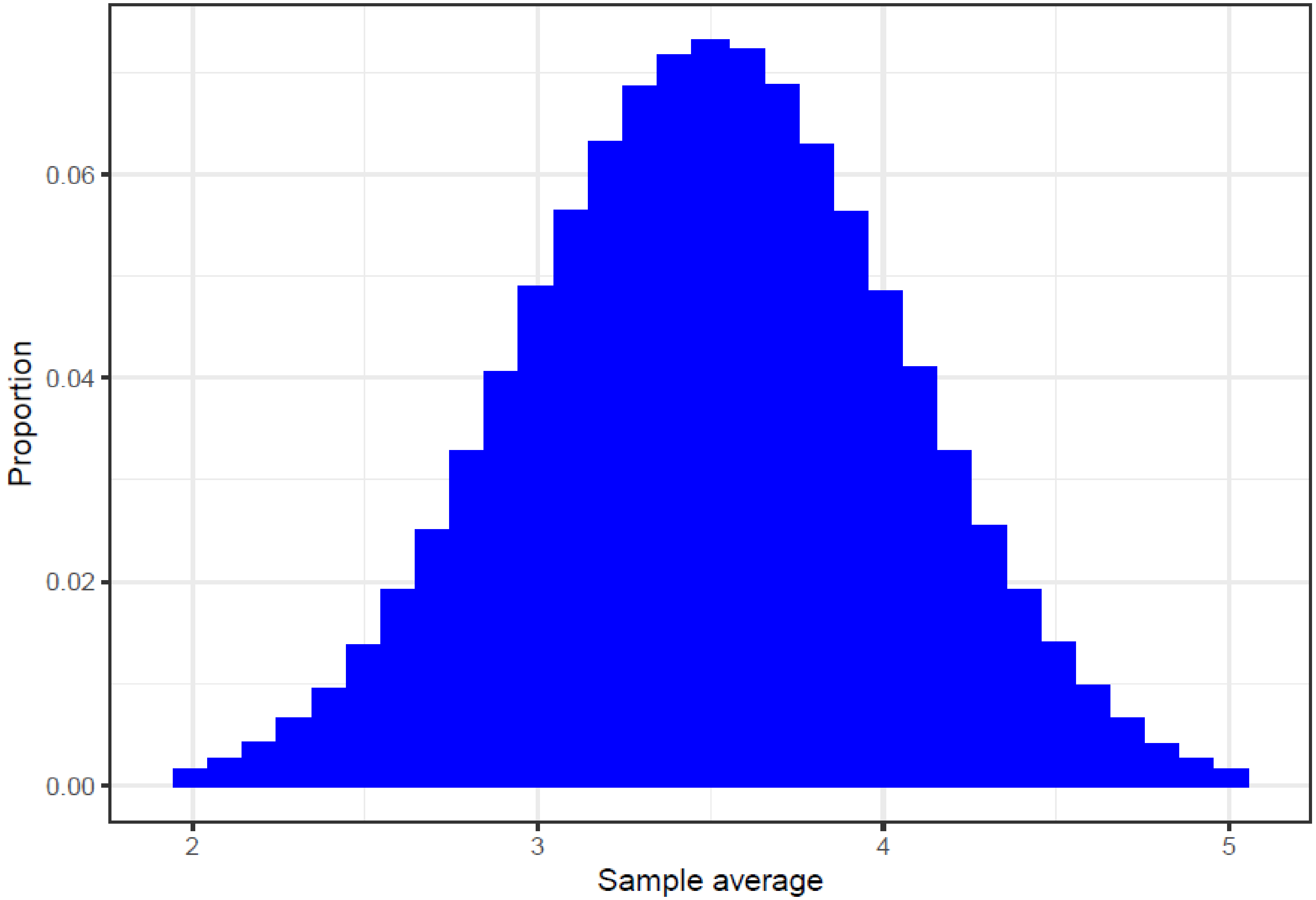

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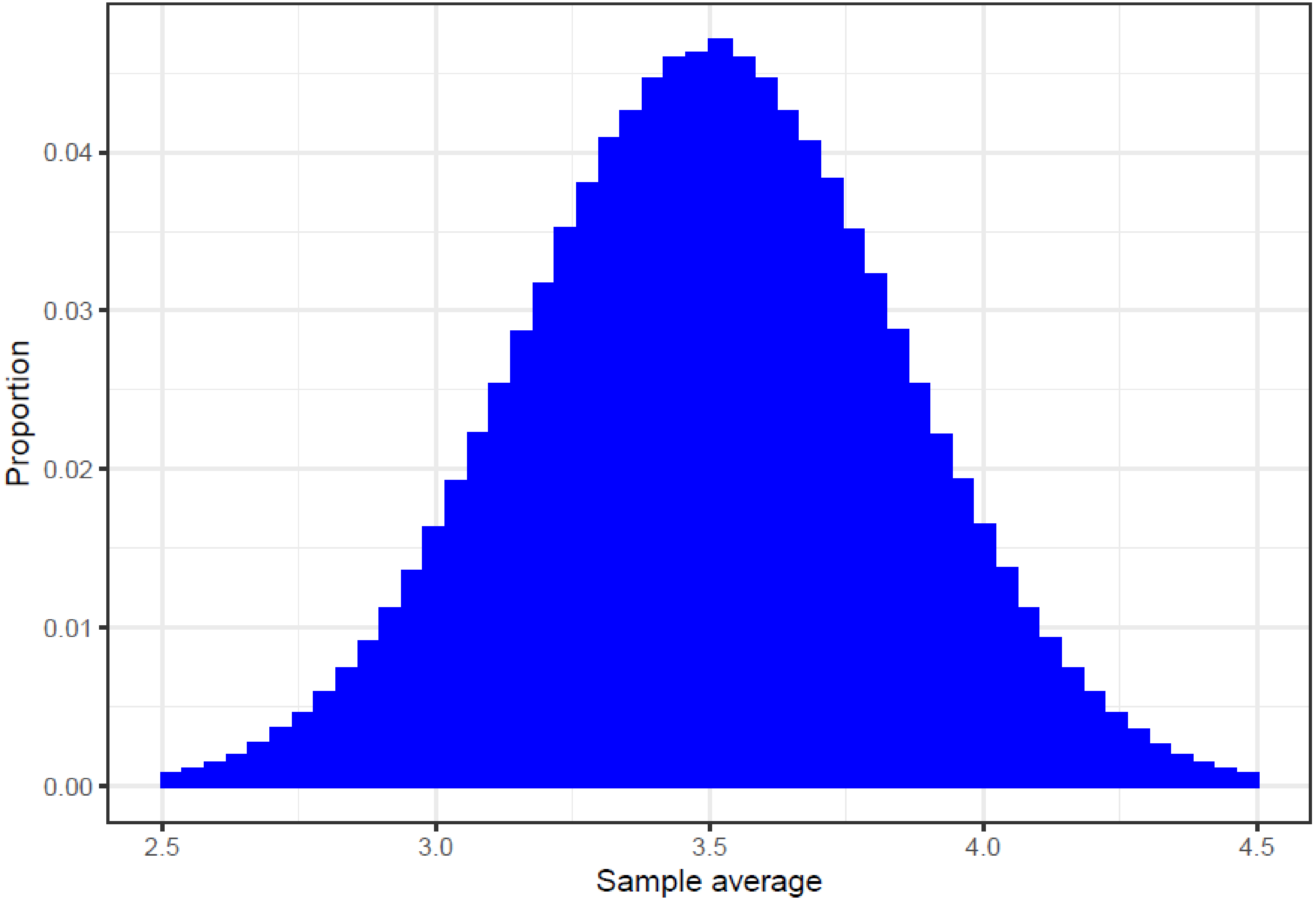

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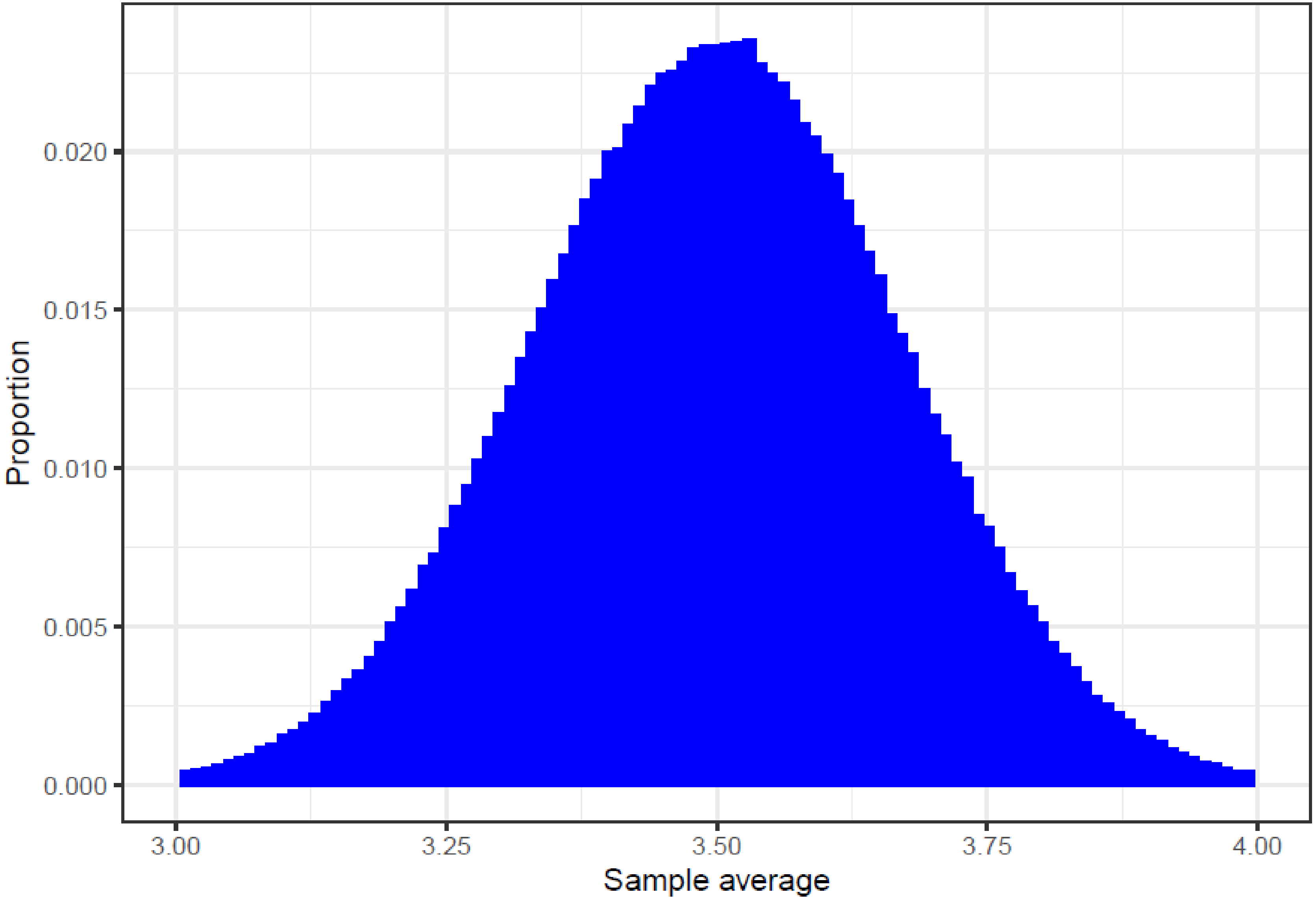

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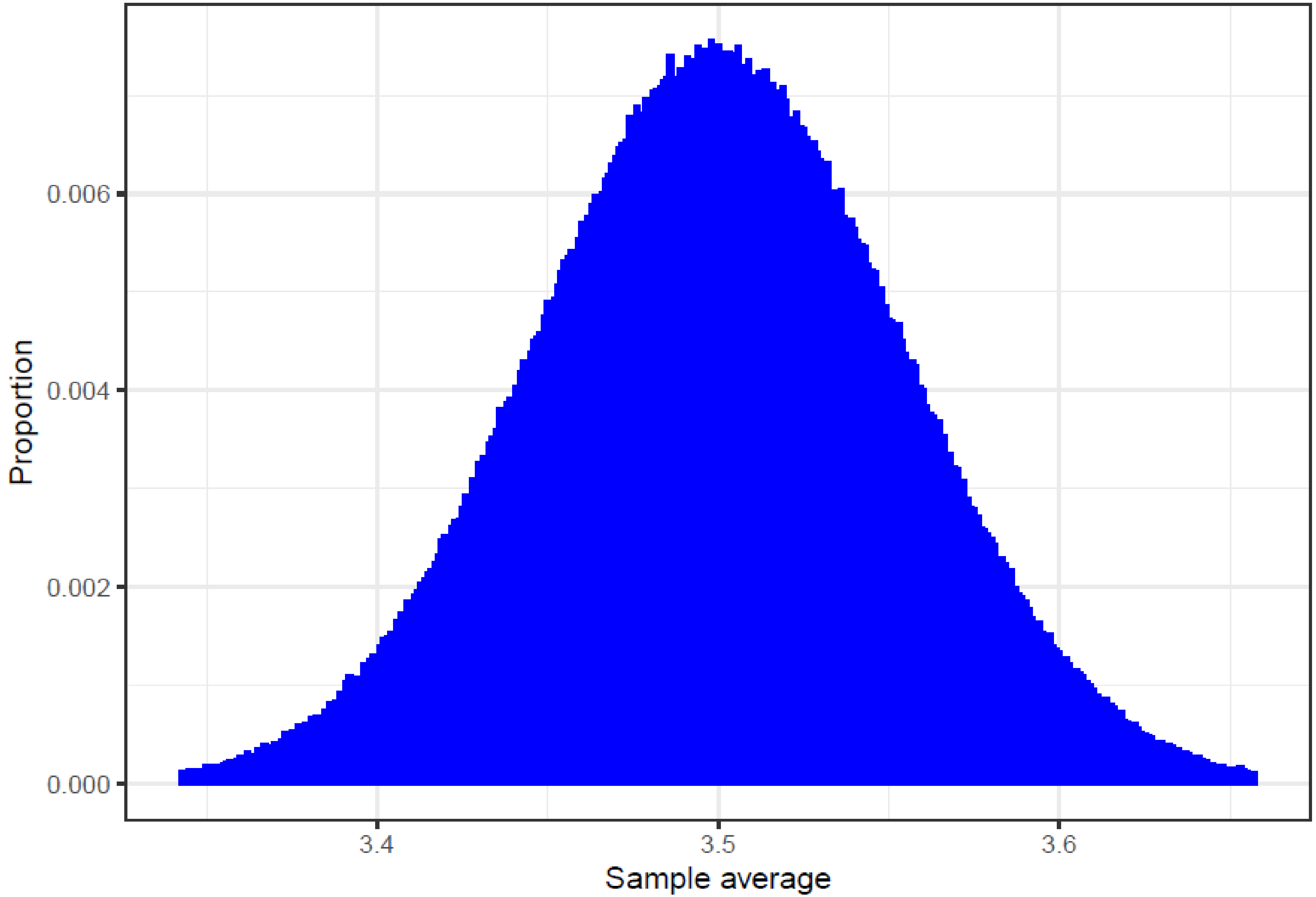

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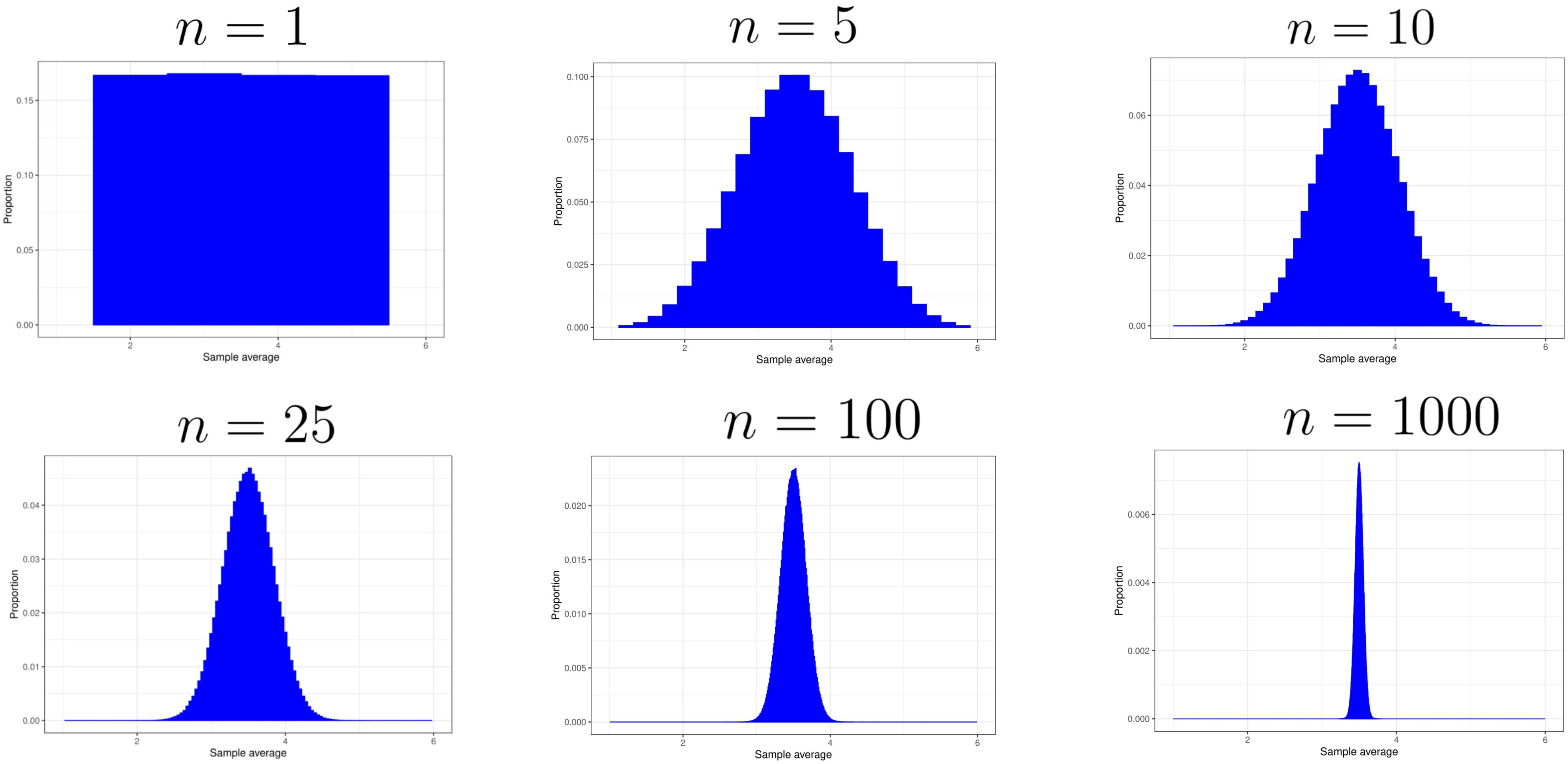

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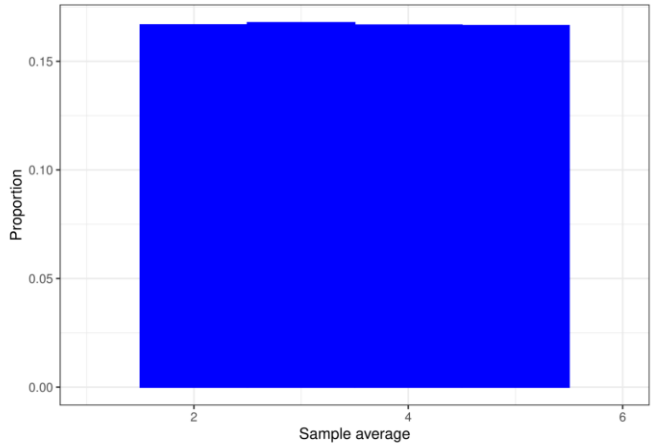

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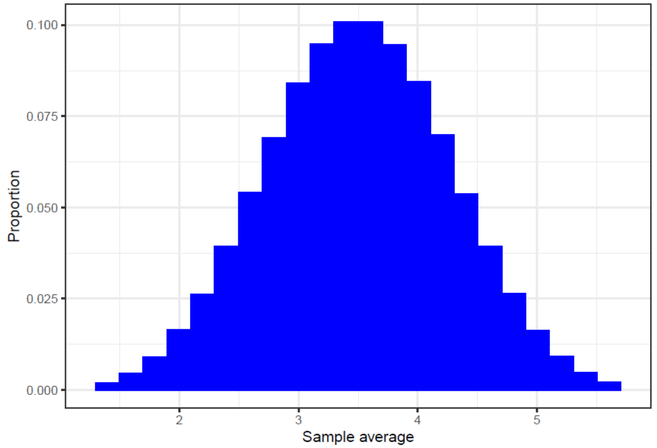

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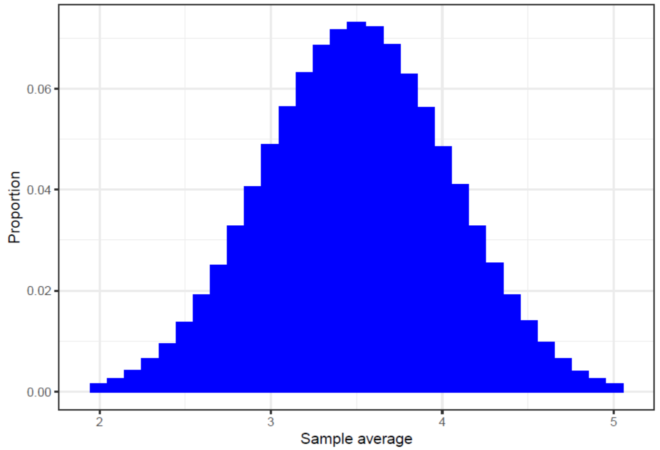
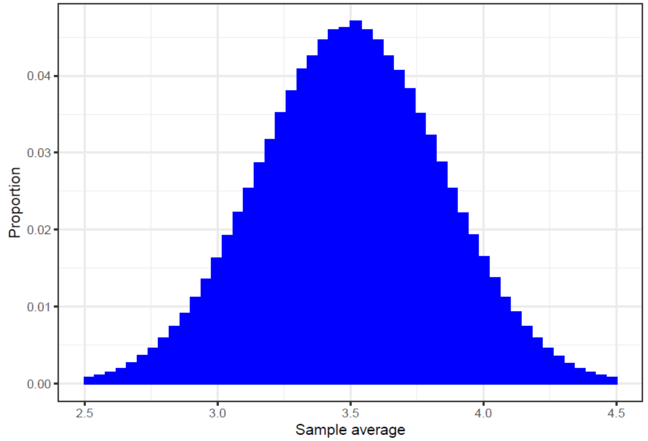
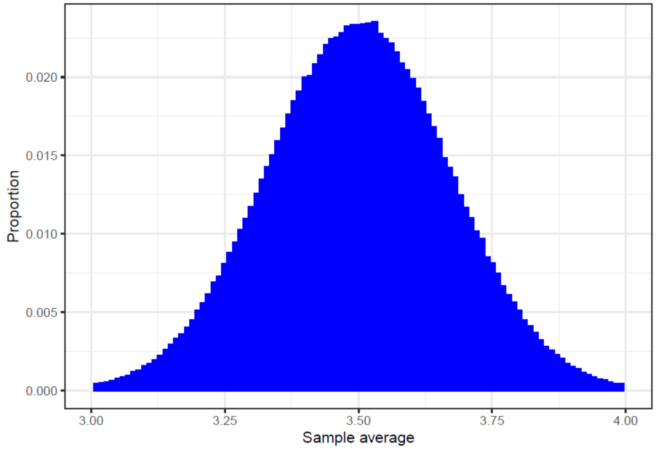


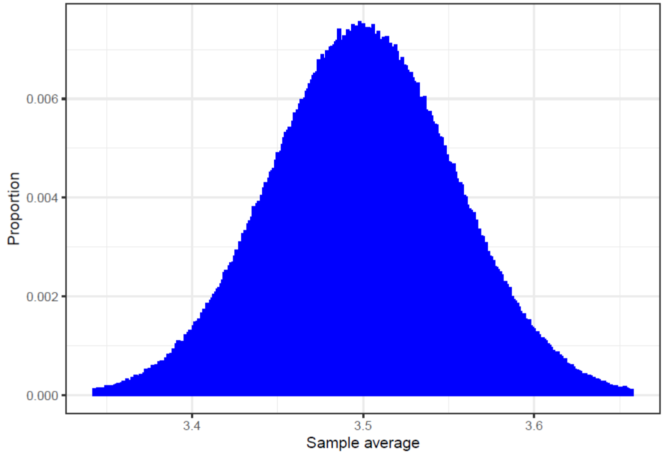
$n = 25$



$n = 100$



$n = 1000$



Now take a break!

Continuous random variables

We often want to model stochastic quantities which can take on a continuum of possible values.

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- The level of rainfall in a given location on a particular day of the year.

Probability density functions

Suppose we have a probability space $(\Omega, \mathcal{E}, \mathbb{P})$.

A **random variable** is a mapping $X : \Omega \to \mathbb{R}$ such that for every a, $b \in \mathbb{R}$, $\{\omega \in \Omega \ : \ X(\omega) \in [a, b]\} \in \mathcal{E}$ is an event.

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If this holds, we say that $f_X : \mathbb{R} \to [0, \infty)$ is the p.d.f. of the random variable $X : \Omega \to \mathbb{R}$.

It follows that the for all $x \in \mathbb{R}$, $F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(z)dz$.

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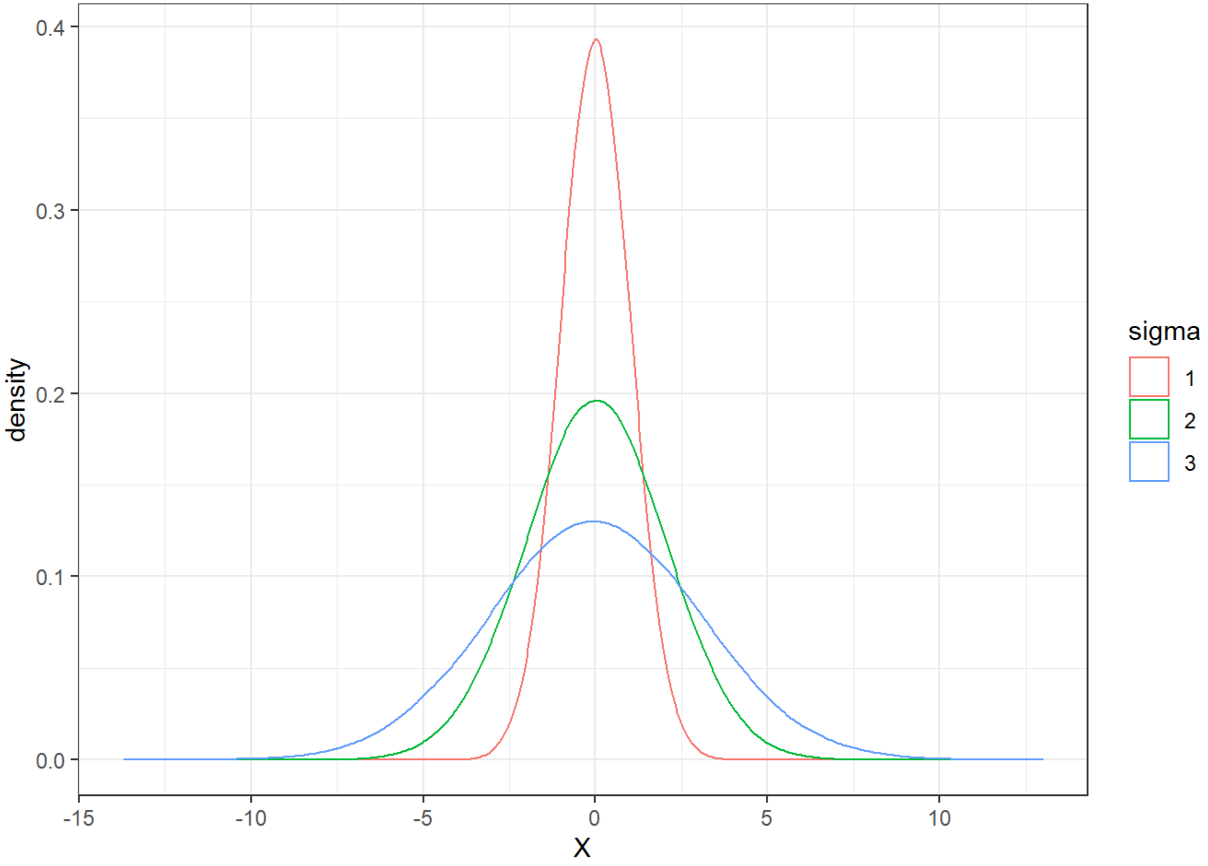

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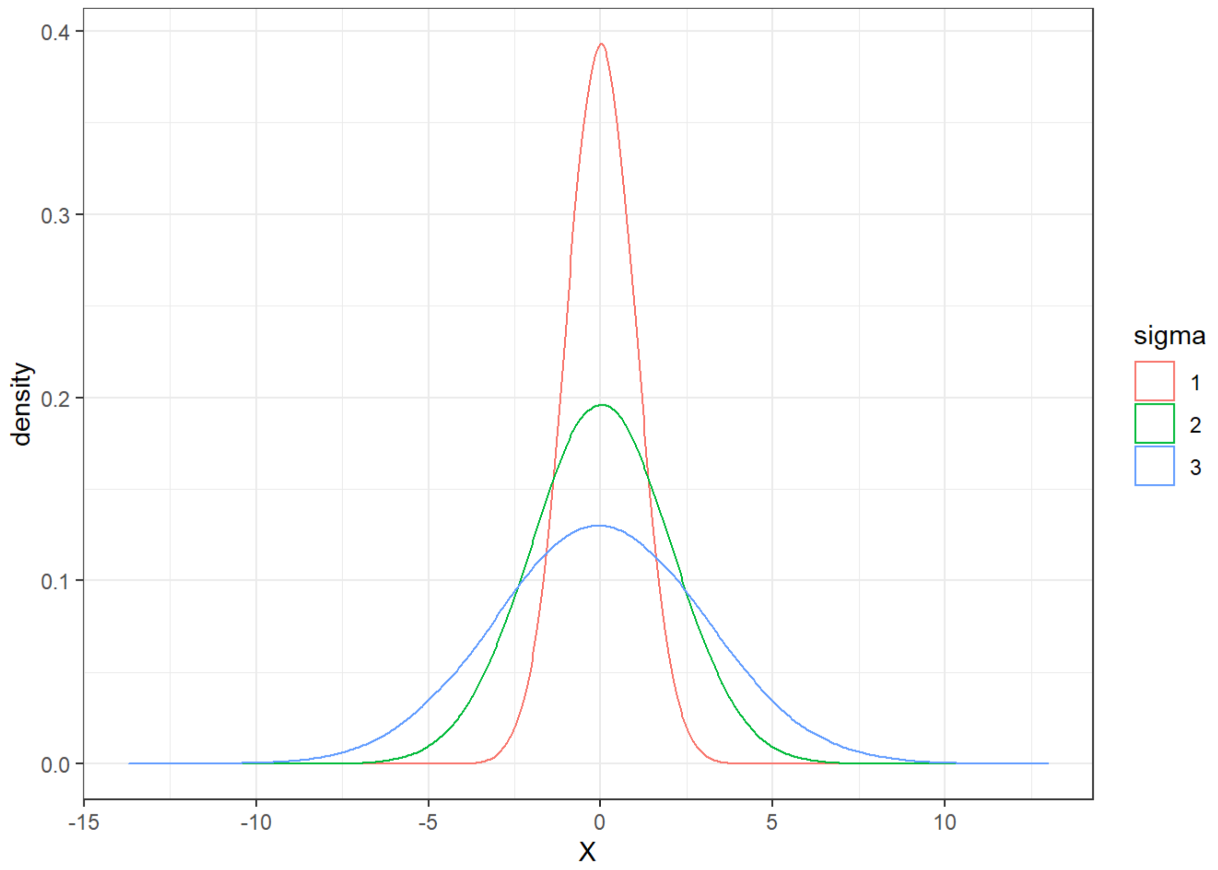

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Now take a break!

Expectation for continuous random variables

A **continuous random variable** is a random variable $X : \Omega \to \mathbb{R}$ with a probability density function $f_X : \mathbb{R} \to [0, \infty)$ such that for all $a, b \in \mathbb{R}$ we have

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Expectation is *linear* in the following sense:

Given random variables $X_1, \ldots, X_K : \Omega \to \mathbb{R}$ and numbers $\alpha_1, \ldots, \alpha_K \in \mathbb{R}$ we have

$$\mathbb{E}\left(\sum_{i=1}^{K} \alpha_i X_i\right) = \sum_{i=1}^{K} \alpha_i \mathbb{E}\left(X_i\right).$$

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&= \frac{\mu}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty} e^{-z^2} dz + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} z \cdot e^{-z^2} dz = \mu.
\end{aligned}$$

Variance for continuous random variables

A **continuous random variable** is a random variable $X : \Omega \to \mathbb{R}$ with a probability density function $f_X : \mathbb{R} \to [0, \infty)$ such that for all $a, b \in \mathbb{R}$ we have

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Standard deviation for continuous random variables

A **continuous random variable** is a random variable $X : \Omega \to \mathbb{R}$ with a probability density function $f_X : \mathbb{R} \to [0, \infty)$ such that for all $a, b \in \mathbb{R}$ we have

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We define the **variance** of a continuous random variable $\mathrm{Var}(X) := \mathbb{E}\left[\{X - \mathbb{E}(X)\}^2\right]$.

In addition, the standard deviation of a random variable is defined by $\sigma(X) := \sqrt{\mathrm{Var}(X)}$.

Variance and covariance for continuous random variables

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Given a pair of random variables $X, Y : \Omega \to \mathbb{R}$ we define the covariance and correlation by

$$\mathrm{Cov}(X,Y) := \mathbb{E}\left[\{X - \mathbb{E}(X)\}\{Y - \mathbb{E}(Y)\}\right] = \mathbb{E}(X \cdot Y) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

$$\mathrm{Corr}(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}}.$$

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Lemma *Given random variables* $X_1, \dots, X_K : \Omega \to \mathbb{R}$ *and* $\alpha_1, \dots, \alpha_K \in \mathbb{R}$ *we have*

$$\mathrm{Var}\left(\sum_{i=1}^{K} \alpha_i X_i\right) = \sum_{i=1}^{K} \alpha_i^2 \mathrm{Var}(X_i) + 2 \sum_{1 \le i < j \le k} \alpha_i \cdot \alpha_j \mathrm{Cov}(X_i, X_j).$$

In particular, if $X_1, \dots, X_k$ *are independent then* $\mathrm{Var}\left(\sum_{i=1}^{K} \alpha_i X_i\right) = \sum_{i=1}^{K} \alpha_i^2 \mathrm{Var}(X_i)$.

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$$u = e^{-z^2} \qquad v = -z$$

$$\frac{du}{dz} = -2ze^{-z^2} \qquad \frac{dv}{dz} = -1$$

By integration by parts, $2\int_{-\infty}^{\infty} z^2 \cdot e^{-z^2} dz = \int_{-\infty}^{\infty} v\frac{du}{dz} dz = [uv]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\frac{dv}{dz} dz$

$$= \left[-z \cdot e^{-z^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}.$$

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with the change of variables $z = \frac{x-\mu}{\sigma\sqrt{2}}$ so that $\frac{\partial z}{\partial x} = \frac{1}{\sigma\sqrt{2}}$

The variance of Gaussian random variables

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$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[(X-\mu)^2\right]$$

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$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[(X-\mu)^2\right] \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f_{\mu,\sigma}(x) dx = \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} \exp\left(-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2\right) dx\end{aligned}$$

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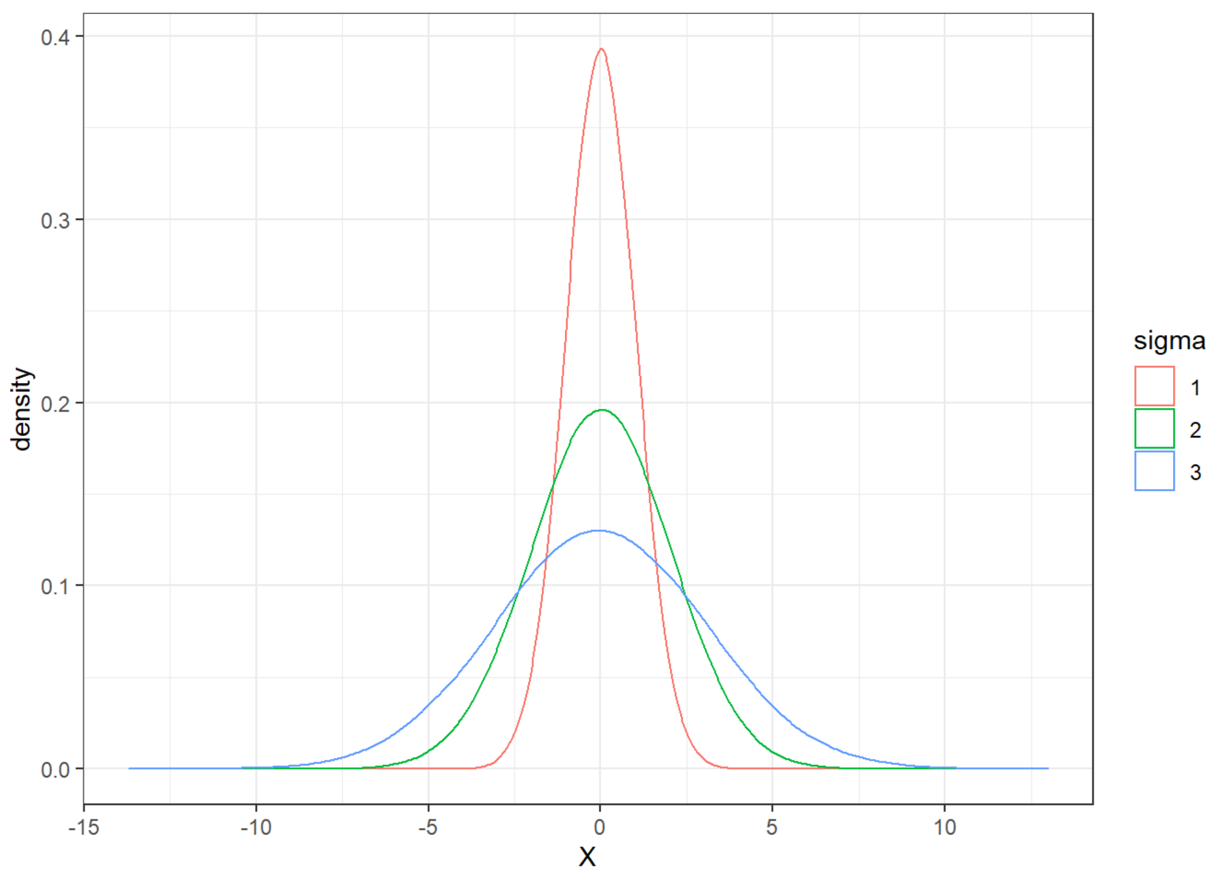
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{2} \cdot z)^2 \cdot e^{-z^2} \left(\frac{\partial z}{\partial x}\right)^{-1} dz = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-z^2} dz = \sigma^2.
\end{aligned}
$$

Gaussian random variables

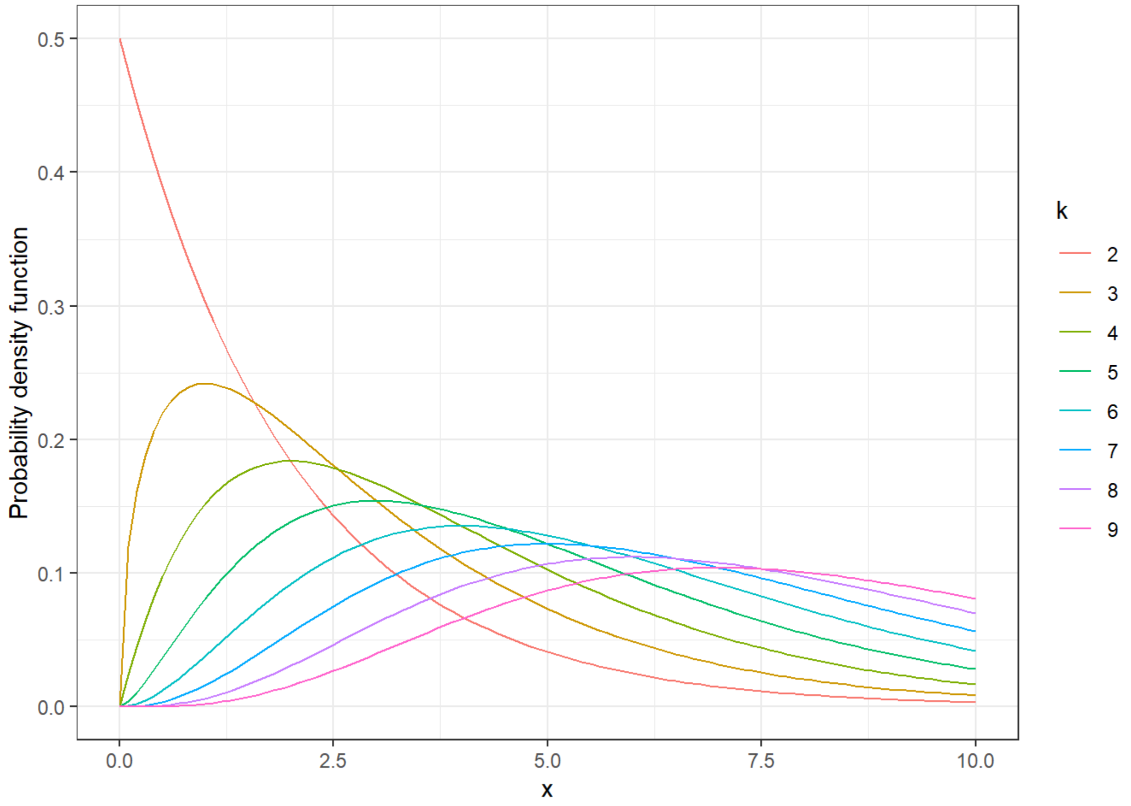


Given $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ we have $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$

Chi squared distribution

A random variable Q is said to be chi-squared with k degrees of freedom $Q \sim \chi^2(k)$ if

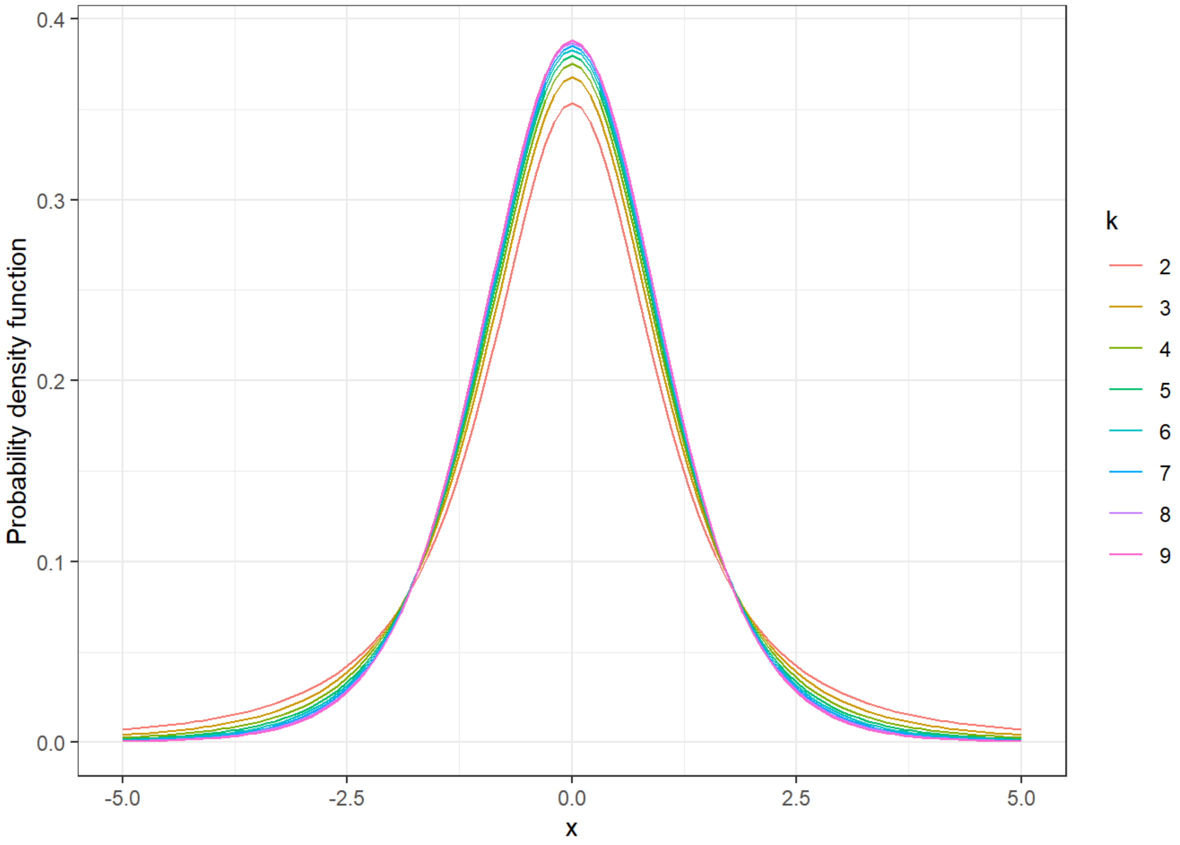
$Q = \sum_{i=1}^{k} Z_i^2$ with $Z_1, \cdots, Z_k \sim \mathcal{N}(0,1)$ independent and identically distributed.



Student's t distribution

A random variable T is said to be t distributed with k degrees of freedom if $T = \dfrac{Z}{\sqrt{Q/k}}$

for two independent random variables $Z \sim \mathcal{N}(0,1)$ and $Q \sim \chi^2(k)$.



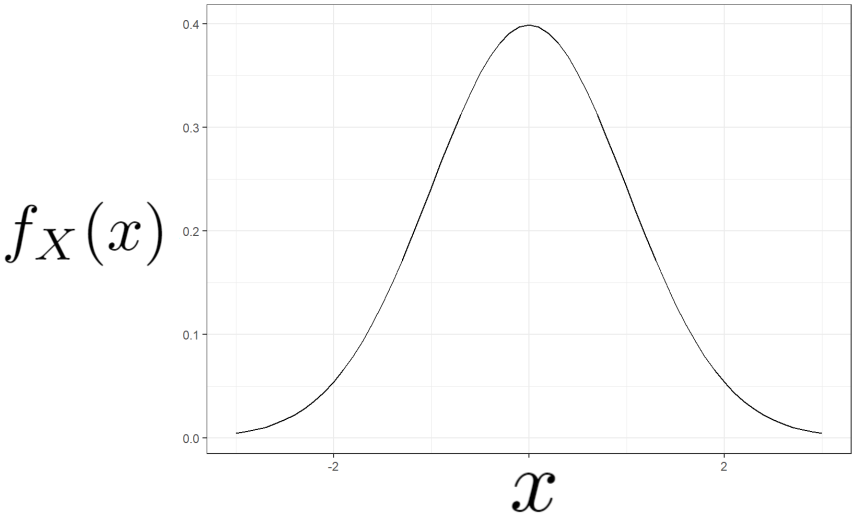
Now take a break!

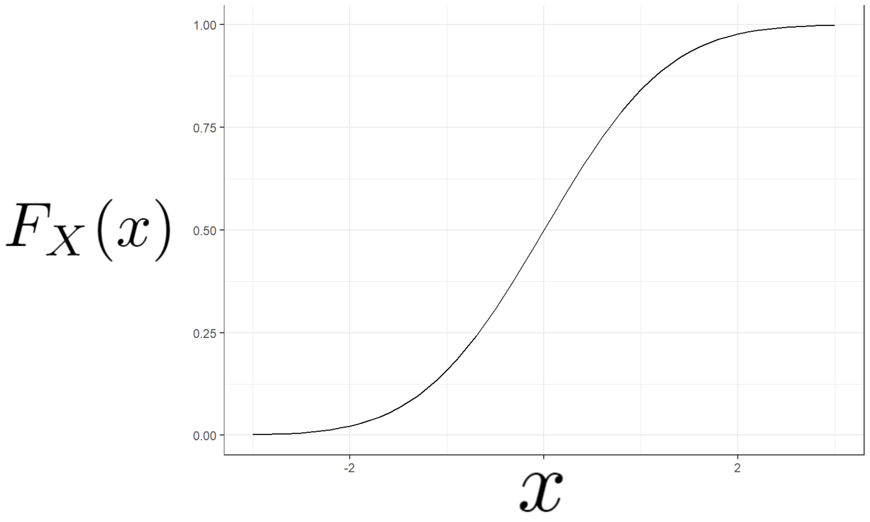
The cumulative distribution function and the density

A **continuous random variable** is a random variable $X : \Omega \to \mathbb{R}$ with a probability density function $f_X : \mathbb{R} \to [0, \infty)$ such that for all $a, b \in \mathbb{R}$ we have $\mathbb{P}\left(X \in [a, b]\right) = \int_a^b f_X(x)dx$.

The **cumulative distribution function** satisfies $F_X(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^x f(z)dz$.

Equivalently, for all $z \in \mathbb{R}$ we have $f_X(z) = \frac{dF_X(x)}{dx}\Big|_z$.





The quantile function

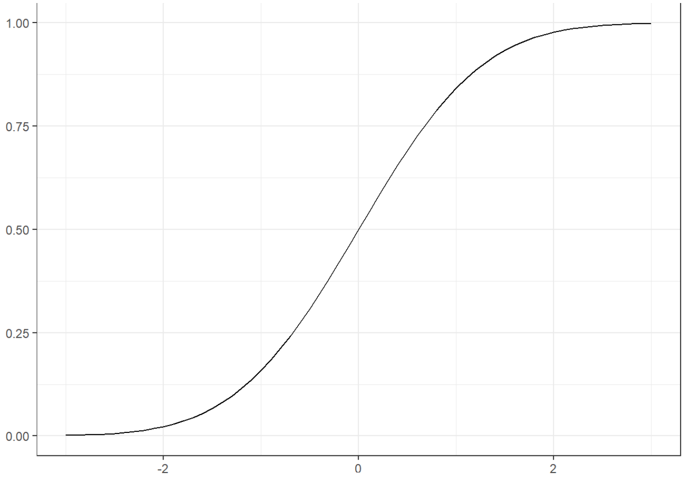
A **continuous random variable** is a random variable $X : \Omega \to \mathbb{R}$ with a probability density function $f_X : \mathbb{R} \to [0, \infty)$ such that for all $a, b \in \mathbb{R}$ we have $\mathbb{P}\left(X \in [a, b]\right) = \int_a^b f_X(x)dx$.

The distribution function $F_X : \mathbb{R} \to [0, 1]$ satisfies $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(z)dz$.

The **quantile function** $F_X^{-1} : [0, 1] \to \mathbb{R}$ is defined by

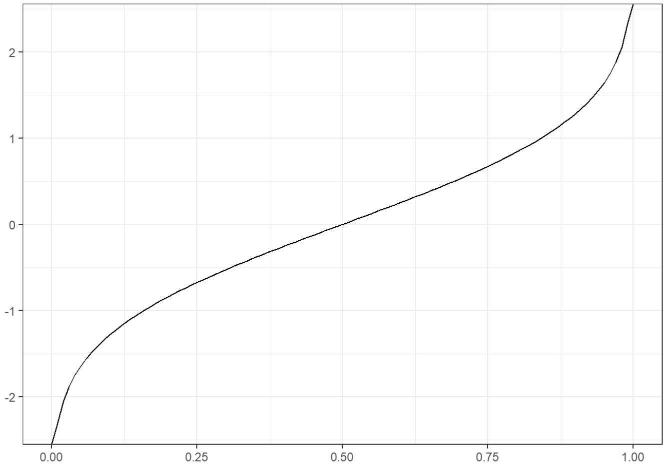
$$F_X^{-1}(p) := \inf\left\{x \in \mathbb{R} : F_X(x) = \mathbb{P}\left(X \leq x\right) \geq p\right\}.$$

$F_X(x)$



x

$F_X^{-1}(p)$



p

The quantile function

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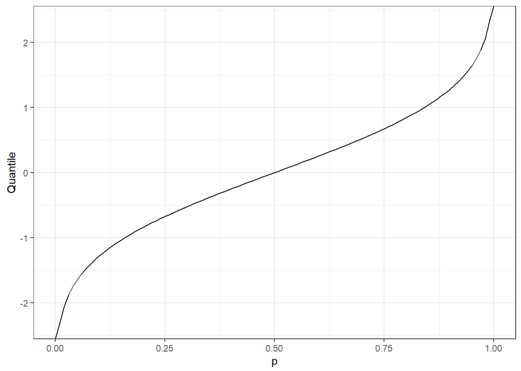
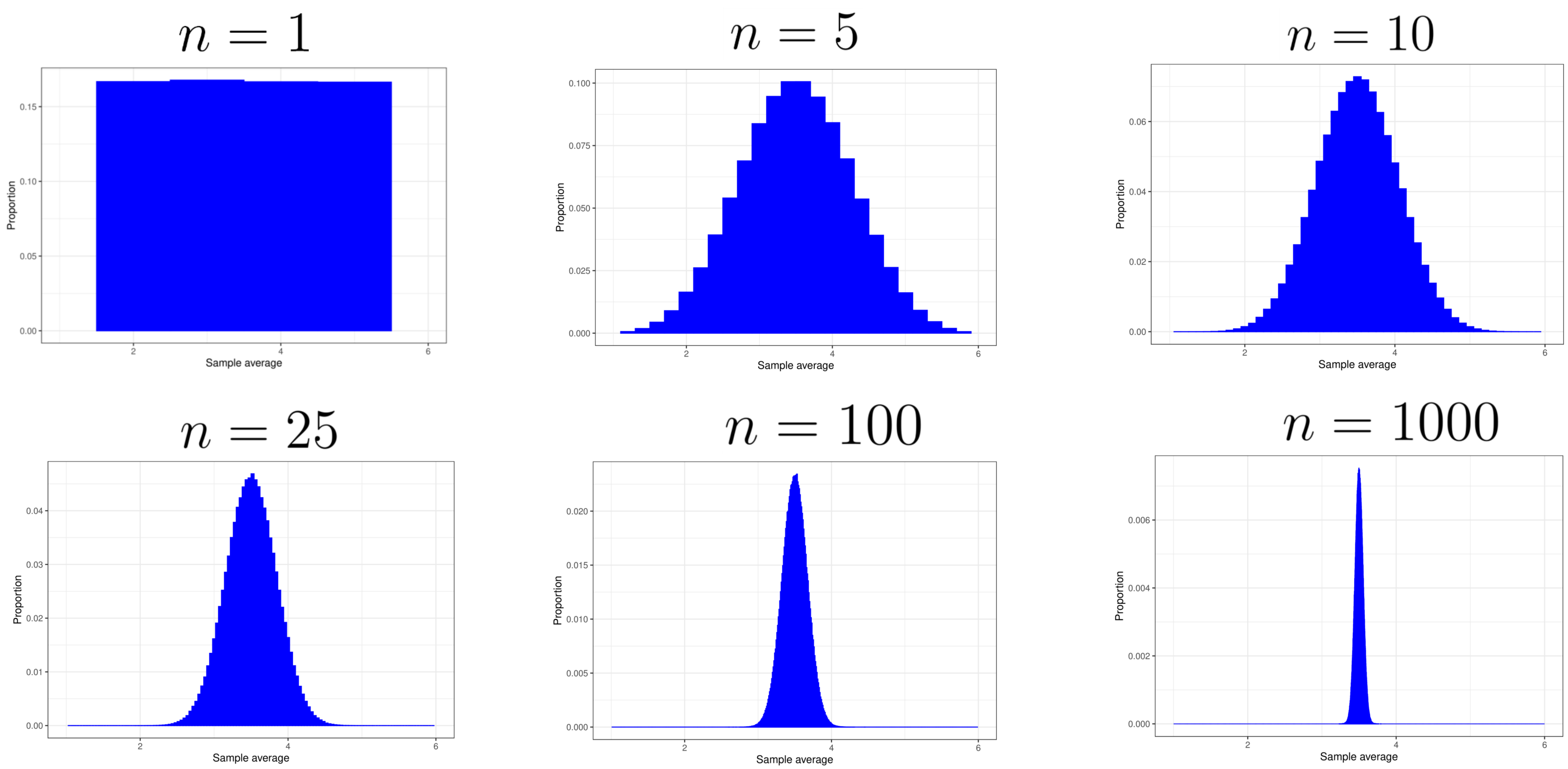
We refer to the values $F_X^{-1}(p)$ for given p as population quantiles.

In particular, $F_X^{-1}(0.5)$ is the population median of the distribution of X.



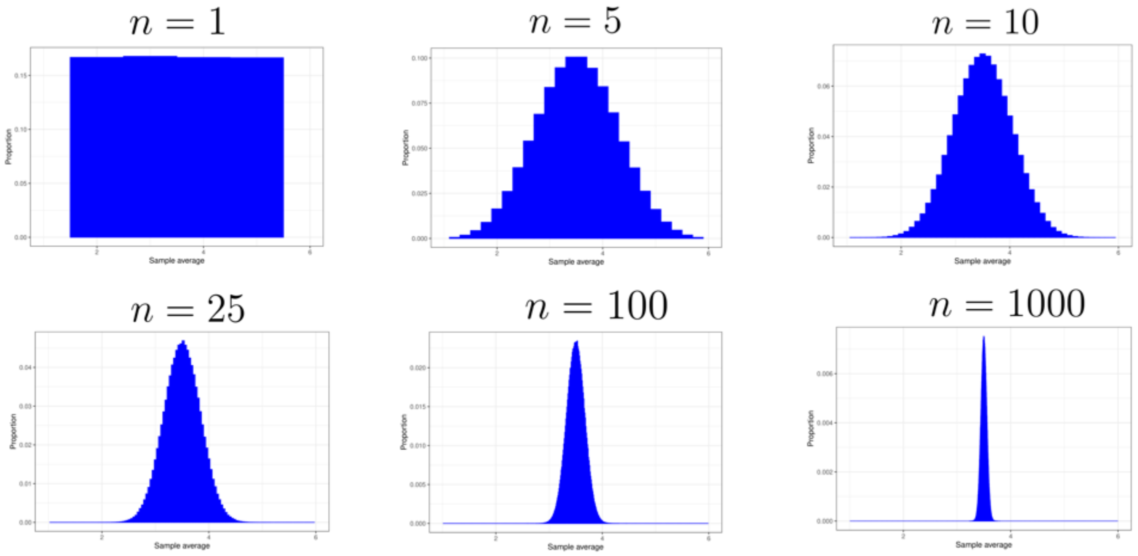
The law of large numbers



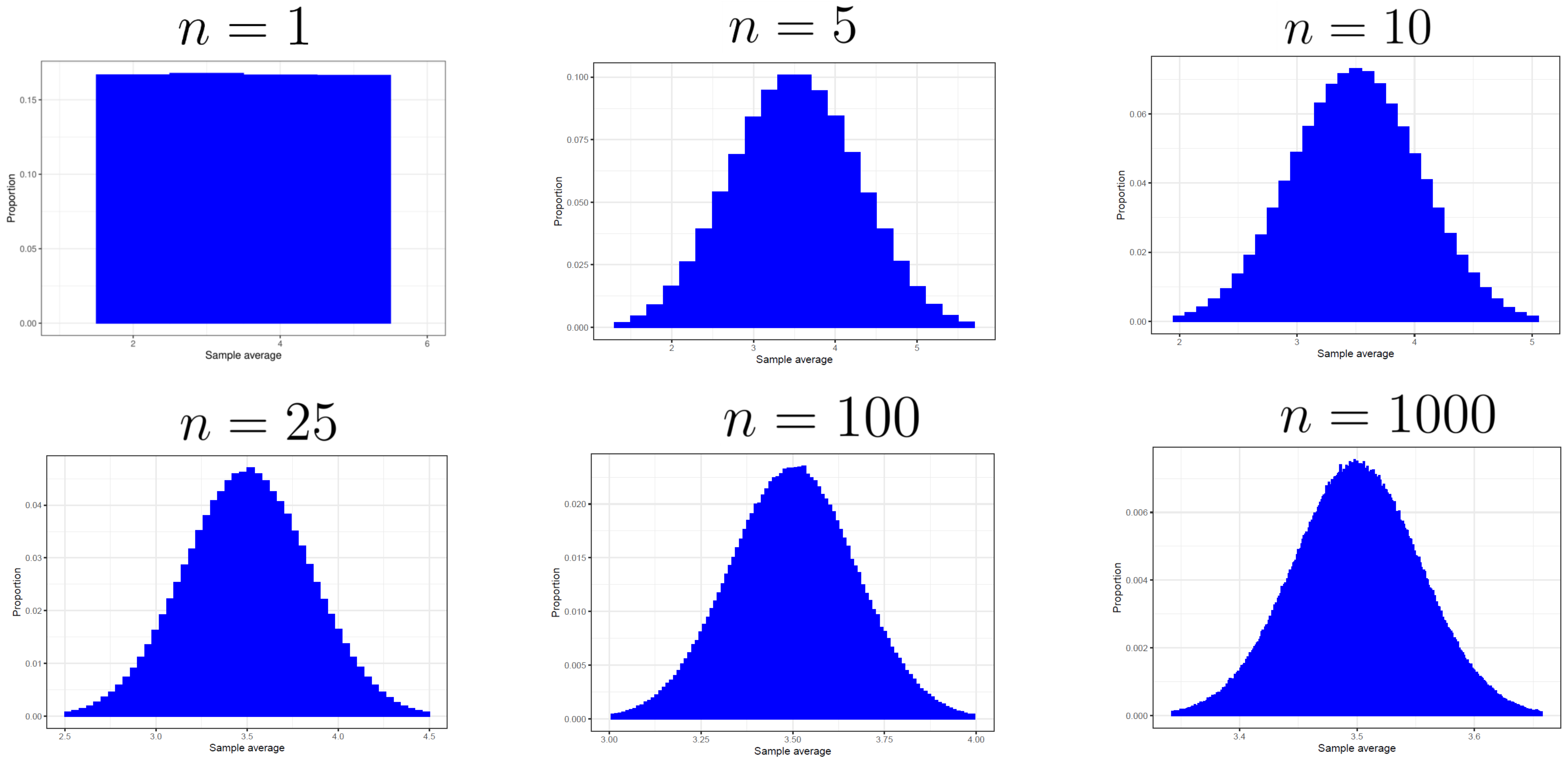
The law of large numbers

Theorem (Bernoulli, circa. 1700). *Let* $X : \Omega \to \mathbb{R}$ *be a random variable with a well-defined expectation* $\mu = \mathbb{E}(X)$. *Let* $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ *be a sequence of independent copies of* X. *Then for all* $\epsilon > 0$,

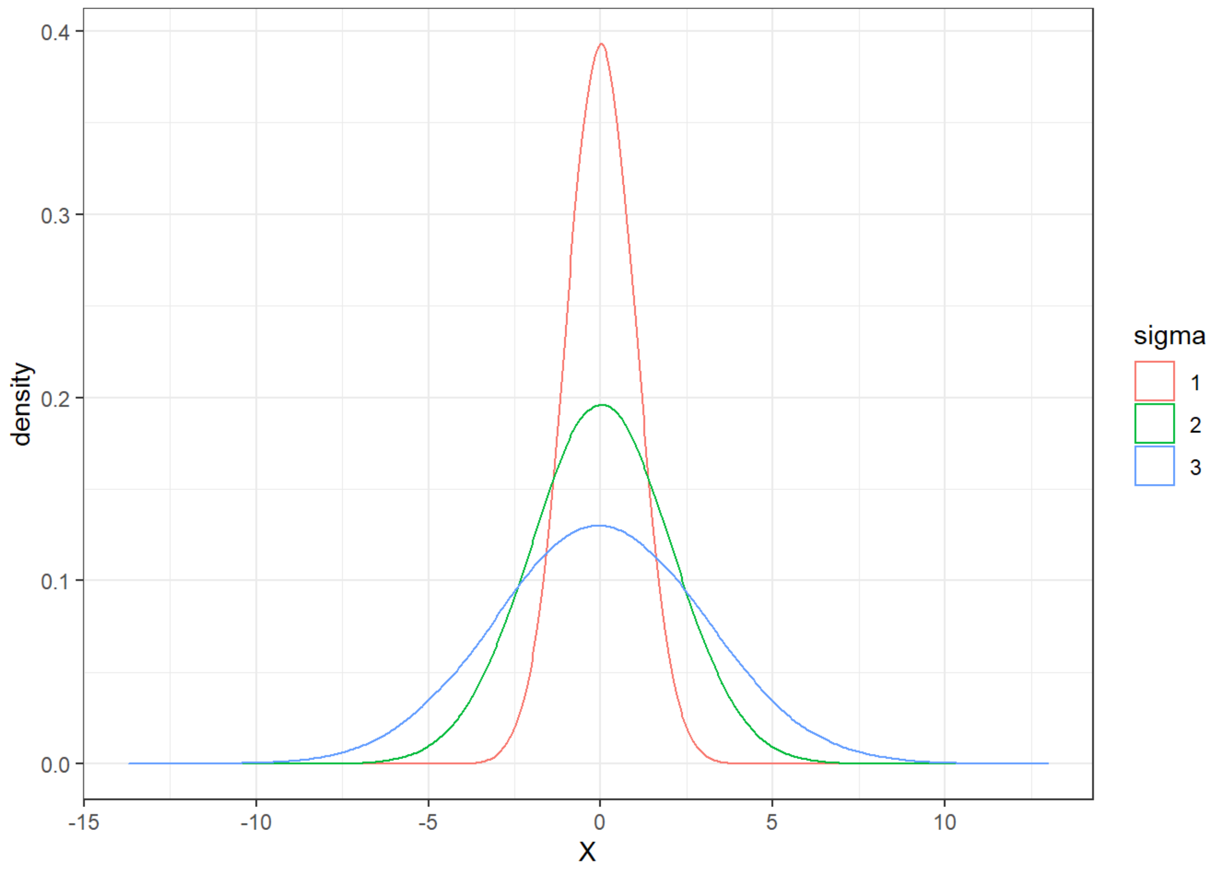
$$\lim_{n \to \infty} \mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right| \geq \epsilon \right) = 0.$$



The central limit theorem



Gaussian random variables

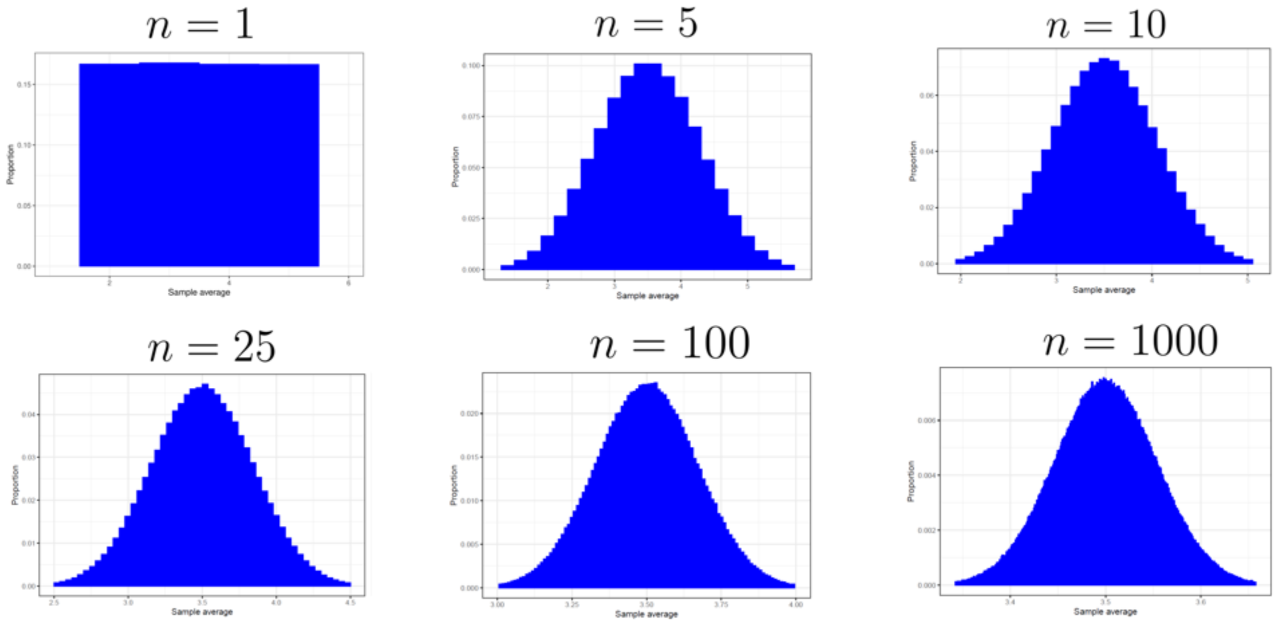


Given $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ we have $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$

The central limit theorem

Theorem (Lindeberg–Lévy). *Let $X : \Omega \to \mathbb{R}$ be a random variable with expectation $\mu = \mathbb{E}(X)$ and variance $\sigma^2 = \mathrm{Var}(X)$. Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ be a sequence of independent copies of X. Let $Z \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable. Then for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{\sqrt{\frac{n}{\sigma^2}} \cdot \left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \le x\right\} = \mathbb{P}\left(Z \le x\right).$$



What have we covered?

- We introduced the concept of a continuous random variable.
- We saw how continuous random variables can be understood via the probability density function.
- We looked at expectation, variance, standard deviation, covariance and correlation in this context.
- We discussed Gaussian random variables and briefly looked at chi-square & Student's t distributions.
- We saw how the law of large numbers describes the limiting behavior of the average.
- We saw how the central limit theorem gives us greater insight as we zoom in around the mean.

University of BRISTOL

Thanks for listening!

Henry W J Reeve

henry.reeve@bristol.ac.uk

Include EMATM0061 in the subject of your email.

Statistical Computing & Empirical Methods (EMATM0061)

MSc in Data Science, Teaching block 1, 2021.