Regular Series Expansions of π and $\ln 2$ Using the NeCo Function and the Dirichlet η Function

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1 Overview of the Present Study

In this study, we introduce a new function, the NeCo function, defined based on Dirichlet's eta function, and construct a novel holomorphic series representation using the values of the eta function at negative integers. In particular, we derive new expansion formulas for mathematical constants such as π and $\ln 2$ through integral structures arising from the NeCo function.

2 Research Motivation and Background

I have long been fascinated by the intricate structure of the Riemann zeta function, $\zeta(s)$, particularly its analytic continuation and functional equation. Through my own independent investigations into its properties, I have gone through numerous trials and errors. However, I came to feel that simply following conventional methodologies was insufficient for making new discoveries, and I began to sense the need for a shift in perspective.

In this process, I turned my attention to Dirichlet's eta function, $\eta(s)$, which is closely related to $\zeta(s)$ but often receives less attention. $\eta(s)$ has a well-behaved series representation, and I noticed that its values at negative integers possess an intriguing structure. This led me to consider that a new approach might be possible.

By viewing the values $\eta(-k)$ as a sequence and using them as a core to define a new holomorphic series, I began to uncover natural connections with constants such as π and $\ln 2$, as well as with various other functions. This eventually led me to the definition of a new function, which I call the NeCo function.

I believe that this research is not the result of accidental insight, but rather a necessary outcome of trying to ask new questions about the world of the zeta function from a different angle, while striving to follow a function-theoretically legitimate path.

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Theorem

Theorem 1.1

2.1. Let $a \in \mathbb{C}$ be a complex number satisfying $\Re(a) \geq 1$. Define the following functions:

$$N(a,x) := \int \frac{dx}{1+x^a}, \quad N'(a,x) := \frac{1}{1+x^a}.$$

Then, the following identity holds:

$$\int_0^\infty N'(a,e^{-t})\,dt = -1 + \sum_{k=0}^{\infty} (-a)^k \eta(-k),$$

where $\eta(s)$ denotes the Dirichlet eta function.

Connection to the NeCo Function

The NeCo function $C(a)$ is defined as the exponential generating function whose coefficients are the special values of the Dirichlet eta function at non-positive integers:

$$C(a) := \sum_{k=0}^{\infty} (-a)^k \eta(-k).$$

Therefore, the above integral relation simplifies to:

$$\int_0^\infty \frac{dt}{1+e^{-at}} = -1 + C(a).$$

$$C(a) = 1 - \int_0^\infty \frac{e^{-t}}{1+e^{-at}}\,dt = \sum_{k=0}^{\infty} (-a)^k \eta(-k)$$

On the Radius of Convergence of Series Expansions

To discuss the convergence of series expansions, the following two points must be carefully considered:

1. **Convergence Condition of the Geometric Series Expansion:**
   When expanding the function
   $$f(t) = \frac{1}{1+e^{-at}}$$
   into a geometric series, the condition for convergence is
   $$|e^{-at}| < 1.$$
   Since this condition is always satisfied in the domain $\Re(a) \geq 1$, the resulting series **converges absolutely**.

2. **Radius of Convergence of the Maclaurin Expansion:**
   The Maclaurin series of e^{-ant} has an **infinite radius of convergence**. That is, each term in the expansion **converges absolutely**, and therefore the full series **converges globally**.

From these observations, it follows that the series expansion within the integral is convergent, and the interchange of summation and integration is justified.

2.1. Theorem 1.1

Let $a \in \mathbb{C}$ satisfy $\Re(a) \geq 1$. Define

$$N(a,x) \;=\; \int \frac{dx}{1+x^a}, \qquad N'(a,x) \;=\; \frac{1}{1+x^a}.$$

Then the following relation holds:

$$\int_0^\infty N'\big(a,e^{-t}\big)\,dt \;=\; -1+\sum_{k=0}^{\infty}(-a)^k\,\eta(-k),$$

or equivalently,

$$\int_0^\infty \frac{e^{-t}}{1+e^{-at}}\,dt \;=\; 1-\sum_{k=0}^{\infty}(-a)^k\,\eta(-k).$$

Proof

Define

$$f(t) \;=\; \frac{1}{1+e^{-at}},$$

with $t>0$ and $\Re(a) \geq 1$.

1. By the geometric series expansion,

$$f(t) \;=\; \frac{1}{1+e^{-at}} \;=\; 1+\sum_{n=1}^{\infty}(-1)^n e^{-ant}.$$

2. Expand each term e^{-ant} in its Maclaurin series:

$$e^{-ant} \;=\; \sum_{k=0}^{\infty}\frac{(-ant)^k}{k!}.$$

3. Substitute into the previous sum and interchange summation (justified by absolute convergence):

$$f(t) = 1+\sum_{k=0}^{\infty}\frac{(-at)^k}{k!}\sum_{n=1}^{\infty}(-1)^n n^k.$$

4. Note the identity
$$\sum_{n=1}^{\infty}(-1)^n n^k \;=\; -\,\eta(-k).$$
Hence
$$f(t) = 1 \;-\; \sum_{k=0}^{\infty}\frac{(-at)^k}{k!}\,\eta(-k).$$

5. Multiply both sides by e^{-t} and integrate from 0 to ∞:
$$\int_0^{\infty}\frac{e^{-t}}{1+e^{-at}}\,dt = 1 - \sum_{k=0}^{\infty}(-a)^k\,\eta(-k).$$

This completes the proof.

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Proposition Theorem Definition

Theorem 1.1

Let $a \in \mathbb{C}$ with $\Re(a) \geq 1$, and define
$$N(a,x) = \int \frac{dx}{1+x^a}, \quad N'(a,x) = \frac{1}{1+x^a}.$$
Then the following identities hold:
$$\int_0^{\infty} N'(a,e^{-t})\,dt = -1 + \sum_{k=0}^{\infty}(-a)^k \eta(-k),$$
or equivalently,
$$\int_0^{\infty}\frac{e^{-t}}{1+e^{-at}}\,dt = 1 - \sum_{k=0}^{\infty}(-a)^k \eta(-k).$$

Proof

Define the function $f(t) = \frac{1}{1+e^{-at}}$ for $t > 0$, $\Re(a) \geq 1$. Then we expand using the geometric series:
$$f(t) = \frac{1}{1+e^{-at}} = 1 + \sum_{n=1}^{\infty}(-1)^n e^{-ant}.$$
Next, expand e^{-ant} via the Maclaurin series:
$$e^{-ant} = \sum_{k=0}^{\infty}\frac{(-ant)^k}{k!}.$$

Substituting and interchanging the order of summation (justified by absolute convergence),

$$f(t) = 1 + \sum_{k=0}^{\infty} \frac{(-at)^k}{k!} \sum_{n=1}^{\infty} (-1)^n n^k.$$

Noting that

$$\sum_{n=1}^{\infty} (-1)^n n^k = -\eta(-k),$$

we obtain

$$f(t) = 1 - \sum_{k=0}^{\infty} \frac{(-at)^k}{k!} \eta(-k).$$

Multiplying both sides by e^{-t} and integrating from 0 to ∞,

$$\int_0^\infty \frac{e^{-t}}{1+e^{-at}}\, dt = 1 - \sum_{k=0}^{\infty} (-a)^k \eta(-k). \quad \square$$

2.2. For $a > 0$, the following identity holds:

$$\int_0^\infty \frac{e^{-t}}{1+e^{-at}}\, dt = \frac{1}{2a}\left(\psi\left(\frac{1+a}{2a}\right) - \psi\left(\frac{1}{2a}\right)\right),$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function.

Proof. Expand:

$$\int_0^\infty \frac{e^{-t}}{1+e^{-at}}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-(1+an)t}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+an}.$$

Using the identity:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} = \frac{1}{2}\left(\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right) \quad (\Re(z) > 0),$$

setting $z = \frac{1}{a}$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+an} = \frac{1}{2a}\left(\psi\left(\frac{1+a}{2a}\right) - \psi\left(\frac{1}{2a}\right)\right). \quad \square$$

$\square$

2.3 (NeCo Function and Polylogarithms)**.** For $a > 0$, the following identity holds:

$$\sum_{k=0}^{\infty} (-a)^k \eta(-k) = \sum_{k=0}^{\infty} (-a)^k \operatorname{Li}_{-k}(-1).$$

Proof. Using the identity

$$\mathrm{Li}_{-k}(-1) = -\sum_{n=1}^{\infty}(-1)^n n^k = \eta(-k),$$

each term matches, and we obtain

$$\sum_{k=0}^{\infty}(-a)^k\eta(-k) = -\sum_{k=0}^{\infty}(-a)^k\,\mathrm{Li}_{-k}(-1). \quad \square$$

$\square$

3. Evaluation of Constants Using the NeCo Function

From the structure of the NeCo function, several mathematical constants can be expressed as:

$$\sum_{k=0}^{\infty}\eta(-k) = \ln 2,$$
$$\sum_{k=0}^{\infty}2^k\eta(-k) = 2 - \frac{\pi}{4} \quad \Rightarrow \quad \pi = 8 - 4\sum_{k=0}^{\infty}2^k\eta(-k).$$

4. Trigonometric Form and Reflection Formula of the NeCo Function

4.1 (NeCo Function)**.** The NeCo function is defined for $a > 0$ as

$$C(a) = 1 - \frac{1}{2a}\left[\psi\left(\frac{1+a}{2a}\right) - \psi\left(\frac{1}{2a}\right)\right].$$

4.1. The NeCo function satisfies the identity:

$$C(a) = 1 - \frac{\pi}{a}\csc\left(\frac{\pi}{a}\right).$$

Proof. Let $u = \frac{1}{2a}$, so that

$$\frac{1+a}{2a} = u + \frac{1}{2}, \quad \frac{1}{2a} = u.$$

Hence,

$$C(a) = 1 - \frac{1}{2a}\left[\psi(u+\tfrac{1}{2}) - \psi(u)\right].$$

Using the known identity:

$$\psi(u+\tfrac{1}{2}) - \psi(u) = \pi\tan(\pi u),$$

we find:

$$C(a) = 1 - \frac{\pi}{a}\csc\left(\frac{\pi}{a}\right). \quad \square$$

□

4.1 (Half-Integer Tangent Identity). For any $u \in \mathbb{C}$,

$$\psi\left(u + \tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2} - u\right) = \pi \tan(\pi u).$$

4.2 (Reflection Formula for the Digamma Function). For all $z \notin \mathbb{Z}$,

$$\psi(1 - z) - \psi(z) = \pi \cot(\pi z).$$

Proof of the Trigonometric Identity for $C(a)$. We start by decomposing the digamma difference:

$$\begin{aligned}
\psi\left(u + \tfrac{1}{2}\right) - \psi(u) &= \left[\psi\left(u + \tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2} - u\right)\right] + \left[\psi\left(\tfrac{1}{2} - u\right) - \psi(u)\right] \\
&= \pi \tan(\pi u) + \pi \cot(\pi u) \\
&= \pi \left[\tan(\pi u) + \cot(\pi u)\right].
\end{aligned}$$

Using the trigonometric identity

$$\tan x + \cot x = 2 \csc(2x),$$

we obtain

$$\tan(\pi u) + \cot(\pi u) = 2 \csc(2\pi u),$$

so that

$$\psi\left(u + \tfrac{1}{2}\right) - \psi(u) = 2\pi \csc(2\pi u).$$

Substituting $u = \frac{1}{2a}$, we get:

$$2\pi \csc(2\pi u) = 2\pi \csc\left(\frac{\pi}{a}\right),$$

and hence,

$$\begin{aligned}
C(a) &= 1 - \frac{1}{2a}\left[\psi\left(u + \tfrac{1}{2}\right) - \psi(u)\right] \\
&= 1 - \frac{1}{2a} \cdot 2\pi \csc\left(\frac{\pi}{a}\right) \\
&= 1 - \frac{\pi}{a} \csc\left(\frac{\pi}{a}\right).
\end{aligned}$$

This proves the trigonometric expression:

$$C(a) = 1 - \frac{\pi}{a} \csc\left(\frac{\pi}{a}\right),$$

which immediately implies the **reciprocal identity**:

$$C\left(\frac{1}{a}\right) = 1 - \pi a \, \csc(\pi a). \quad \square$$

$\square$

4.3 (Reflection Formula of the NeCo Function)**.** For the NeCo function $C(a)$, the difference

$$C(a) - C\left(\tfrac{1}{a}\right)$$

can be expressed using trigonometric functions as follows:

$$C(a) - C\left(\tfrac{1}{a}\right) = -\pi\left(\tfrac{1}{a}\csc\left(\tfrac{\pi}{a}\right) - a\csc(\pi a)\right).$$

4.4 (Connection to Dirichlet L-functions)**.** Let $\chi(n) = (-1)^{n-1}$ be the non-principal Dirichlet character modulo 2. Then the Dirichlet L-function satisfies

$$L(s, \chi^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s).$$

Using this identity, the NeCo function can be expressed as a generating function for the special values of the Dirichlet L-function at non-positive integers:

$$\boxed{C(a) = \sum_{k=0}^{\infty} (-a)^k \cdot L(-k, \chi^{-1})}$$

This representation shows that the NeCo function naturally reconstructs the sequence of special values of $L(s, \chi^{-1})$ at negative integers.

article amsmath,amssymb,amsthm enumitem Proposition

4.1 (Functional Equation of the NeCo Function)**.** For all real $a \neq 0$, the NeCo function

$$C(a) = 1 - \frac{1}{2a}\left[\psi\left(\tfrac{1+a}{2a}\right) - \psi\left(\tfrac{1}{2a}\right)\right]$$

satisfies

$$C(-a) = C(a) - 2a + \pi\Big(\cot\tfrac{\pi}{2a} - \tan\tfrac{\pi}{2a}\Big).$$

Proof. We use the digamma identity

$$\psi(-x) = \psi(x) + \frac{1}{x} + \pi\cot(\pi x).$$

[label=0.]**Definition & substitution.**

From the definition,

$$C(a) = 1 - \frac{1}{2a}\Big[\psi\big(\tfrac{1+a}{2a}\big) - \psi\big(\tfrac{1}{2a}\big)\Big],$$

substituting $a \mapsto -a$ gives

$$C(-a) = 1 + \frac{1}{2a}\Big[\psi\big(\tfrac{a-1}{2a}\big) - \psi\big(-\tfrac{1}{2a}\big)\Big],$$

since $\frac{1-a}{-2a} = \frac{a-1}{2a} = \frac{1}{2} - \frac{1}{2a}$. **Half-integer shift.**
With $u = \frac{1}{2a}$, the identity $\psi(u + \frac{1}{2}) - \psi(\frac{1}{2} - u) = \pi\tan(\pi u)$ yields

$$\psi\big(\tfrac{1}{2} - \tfrac{1}{2a}\big) = \psi\big(\tfrac{1}{2} + \tfrac{1}{2a}\big) - \pi\tan\tfrac{\pi}{2a}.$$

Reflection–shift for $\psi(-\frac{1}{2a})$.
Applying $\psi(-x) = \psi(x) + \frac{1}{x} + \pi\cot(\pi x)$ with $x = \frac{1}{2a}$ gives

$$\psi\big(-\tfrac{1}{2a}\big) = \psi\big(\tfrac{1}{2a}\big) + 2a + \pi\cot\tfrac{\pi}{2a}.$$

Combine in $C(-a)$.
Substitute steps 2–3 into $C(-a)$:

$$C(-a) = 1 + \frac{1}{2a}\Big[\psi(\tfrac{1}{2} + \tfrac{1}{2a}) - \psi(\tfrac{1}{2a}) - 2a - \pi\big(\tan\tfrac{\pi}{2a} + \cot\tfrac{\pi}{2a}\big)\Big].$$

Subtract from $C(a)$.
Since $C(a) = 1 - \dfrac{1}{2a}[\psi(\frac{1}{2} + \frac{1}{2a}) - \psi(\frac{1}{2a})]$, one finds

$$C(a) - C(-a) = 2a - \pi\Big(\cot\tfrac{\pi}{2a} - \tan\tfrac{\pi}{2a}\Big).$$

Rearrange.
Therefore

$$C(-a) = C(a) - 2a + \pi\Big(\cot\tfrac{\pi}{2a} - \tan\tfrac{\pi}{2a}\Big),$$

as claimed.

$\square$