1 Piman-Yor Diffusion Trees

1.1 Modelling of PYDT

$$
\begin{aligned}
c &\sim \mathrm{G}(c|a_c, b_c) && (1)\\
\frac{1}{\sigma^2} &\sim \mathrm{G}(\sigma^2|a_{\sigma^2}, b_{\sigma^2}) && (2)\\
\alpha &\sim \mathrm{Beta}(\alpha|a_\alpha, b_\alpha) && (3)\\
\theta &\sim \mathrm{G}(\theta|a_\theta, b_\theta) && (4)
\end{aligned}
$$

1.2 Posterior of $p(t_v|\mathcal{T})$

$$
\begin{aligned}
p(t_v|c,\mathcal{T}) &= c(1-t_v)^{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1} && (5)\\
p(\mathbf{z}_v|\mathbf{z}_u,\sigma^2(t_v-t_u)\mathbf{I}) &= (2\pi\sigma^2(t_v-t_u))^{-\frac{D}{2}}\exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_v-\mathbf{z}_u\parallel^2}{\sigma^2(t_v-t_u)}\Big) && (6)\\
p(\mathbf{z}_k|\mathbf{z}_v,\sigma^2(t_k-t_v)\mathbf{I}) &= (2\pi\sigma^2(t_k-t_v))^{-\frac{D}{2}}\exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_k-\mathbf{z}_v\parallel^2}{\sigma^2(t_k-t_v)}\Big) && (7)\\
p(t_v,\mathbf{z}_{\{u,v,k\}},\sigma^2|c,\mathcal{T}) &= c(2\pi\sigma^2)^{-\frac{D(K+1)}{2}}\exp\Big\{(cJ^{\theta,\alpha}_{\mathbf{n}_v}-1)\ln(1-t_v)\\
&\quad -\frac{D}{2}\big(\ln(t_v-t_u)+\sum_k \ln(t_k-t_v)\big)\\
&\quad -\frac{\parallel \mathbf{z}_v-\mathbf{z}_u\parallel^2}{2\sigma^2}\frac{1}{t_v-t_u}-\sum_k\frac{\parallel \mathbf{z}_k-\mathbf{z}_v\parallel^2}{2\sigma^2}\frac{1}{t_k-t_v}\Big\}\\
&= C_{\sigma^2}\exp\big\{u(t_v)\big\} && (8)\\
\frac{du}{dt_v} &= -\frac{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1}{1-t_v}-\frac{D}{2}\Big(\frac{1}{t_v-t_u}-\sum_k\frac{1}{t_k-t_v}\Big)\\
&\quad +\frac{\parallel \mathbf{z}_v-\mathbf{z}_u\parallel^2}{2\sigma^2}\frac{1}{(t_v-t_u)^2}-\sum_k\frac{\parallel \mathbf{z}_k-\mathbf{z}_v\parallel^2}{2\sigma^2}\frac{1}{(t_k-t_v)^2}\\
&= -\frac{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1}{1-t_v}-\frac{1}{2(t_v-t_u)^2}\big(D(t_v-t_u)-\frac{\parallel \mathbf{z}_v-\mathbf{z}_u\parallel^2}{\sigma^2}\big)\\
&\quad +\sum_k\frac{1}{2(t_k-t_v)^2}\big(D(t_k-t_v)-\frac{\parallel \mathbf{z}_k-\mathbf{z}_v\parallel^2}{\sigma^2}\big)\\
&= A(t_v) && (9)\\
C_{\sigma^2}A(t_v)^{-1}\int\exp\big\{u(t_v)\big\}du &= C_{\sigma^2}A(t_v)^{-1}\exp\big\{u(t_v)\big\}\Big|^{t_v=t_{k'}}_{t_v=t_u},\ \text{where } t_{k'}=\min(t_k)\\
&= A(t_{k'})^{-1}p(t_{k'},\mathbf{z}_{\{u,v,k\}},\sigma^2|c,\mathcal{T})-A(t_u)^{-1}p(t_u,\mathbf{z}_{\{u,v,k\}},\sigma^2|c,\mathcal{T})\\
&= Z && (10)\\
\mathbb{E}_{p(t_v|\mathbf{z}_{\{u,v,k\}},\sigma^2,c,\mathcal{T})}[t_v] &= \frac{1}{Z}\int t_v p(t_v,\mathbf{z}_{\{u,v,k\}},\sigma^2|c,\mathcal{T})dt_v\\
&= \frac{1}{Z}C_{\sigma^2}A_\mu(t_v)^{-1}\int\exp\big\{u_\mu(t_v)\big\}du && (11)
\end{aligned}
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