1 Piman-Yor Diffusion Trees

1.1 Modelling of PYDT

$$
\begin{aligned}
c &\sim \mathrm{G}(c|a_c, b_c) &(1)\\
\frac{1}{\sigma^2} &\sim \mathrm{G}(\sigma^2|a_{\sigma^2}, b_{\sigma^2}) &(2)\\
\alpha &\sim \mathrm{Beta}(\alpha|a_\alpha, b_\alpha) &(3)\\
\theta &\sim \mathrm{G}(\theta|a_\theta, b_\theta) &(4)\\
p(t_v|c, \mathcal{T}) &= c(1-t_v)^{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1} &(5)\\
p(\mathbf{z}_v|\mathbf{z}_u, \sigma^2(t_v-t_u)\mathbf{I}) &= (2\pi\sigma^2(t_v-t_u))^{-\frac{D}{2}} \exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{\sigma^2(t_v-t_u)}\Big) &(6)\\
p(\mathbf{z}_k|\mathbf{z}_v, \sigma^2(t_k-t_v)\mathbf{I}) &= (2\pi\sigma^2(t_k-t_v))^{-\frac{D}{2}} \exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{\sigma^2(t_k-t_v)}\Big) &(7)\\
& &(8)
\end{aligned}
$$

1.2 Posterior of $p(t_v|\mathcal{T})$

$$
\begin{aligned}
p(t_v, \mathbf{z}_{\{u,v,k\}}, \sigma^2|c,\mathcal{T}) &= c(2\pi\sigma^2)^{-\frac{D(K+1)}{2}} \exp\Big\{(cJ^{\theta,\alpha}_{\mathbf{n}_v} - 1)\ln(1-t_v) \\
&\quad -\frac{D}{2}\big(\ln(t_v - t_u) + \sum_k \ln(t_k - t_v)\big) \\
&\quad -\frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{2\sigma^2}\frac{1}{t_v - t_u} - \sum_k \frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{2\sigma^2}\frac{1}{t_k - t_v}\Big\} \\
&= C_{\sigma^2}\exp\big\{u(t_v)\big\} \qquad (9) \\
\frac{du}{dt_v} &= -\frac{cJ^{\theta,\alpha}_{\mathbf{n}_v} - 1}{1 - t_v} - \frac{D}{2}\big(\frac{1}{t_v - t_u} - \sum_k \frac{1}{t_k - t_v}\big) \\
&\quad + \frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{2\sigma^2}\frac{1}{(t_v - t_u)^2} - \sum_k \frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{2\sigma^2}\frac{1}{(t_k - t_v)^2} \\
&= -\frac{cJ^{\theta,\alpha}_{\mathbf{n}_v} - 1}{1 - t_v} - \frac{1}{2(t_v - t_u)^2}\big(D(t_v - t_u) - \frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{\sigma^2}\big) \\
&\quad + \sum_k \frac{1}{2(t_k - t_v)^2}\big(D(t_k - t_v) - \frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{\sigma^2}\big) \\
&= A(t_v) \qquad (10) \\
C_{\sigma^2}A(t_v)^{-1}\int \exp\big\{u(t_v)\big\}du &= Z,\ \text{but this integral is intractable!} \qquad (11)
\end{aligned}
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$\Rightarrow$ The range of integration w.r.t. t_v is $0 < t_u \leq t_v \leq \min(t_k) < 1$, obtaining Z by simply summing pdf by the interval $dt = 1e^{-2}$ is computationally tractable for modern hardwares. A few t points usually give sufficient accuracy for Z.

$$
\begin{aligned}
\mathbb{E}_{p(t_v|\mathbf{z}_{\{u,v,k\}},\sigma^2,c,\mathcal{T})}[t_v] &= \frac{1}{Z}\int t_v p(t_v, \mathbf{z}_{\{u,v,k\}}, \sigma^2|c,\mathcal{T})dt_v \\
&= \frac{1}{Z}C_{\sigma^2}A_\mu(t_v)^{-1}\int \exp\big\{u_\mu(t_v)\big\}du \qquad (12)
\end{aligned}
$$