1 Piman-Yor Diffusion Trees

1.1 Modelling of PYDT

$$
\begin{aligned}
c &\sim \mathrm{G}(c|a_c, b_c) && (1)\\
\frac{1}{\sigma^2} &\sim \mathrm{G}(\sigma^2|a_{\sigma^2}, b_{\sigma^2}) && (2)\\
\alpha &\sim \mathrm{Beta}(\alpha|a_\alpha, b_\alpha) && (3)\\
\theta &\sim \mathrm{G}(\theta|a_\theta, b_\theta) && (4)
\end{aligned}
$$

1.2 Posterior of $p(t_v|\mathcal{T})$

$$
\begin{aligned}
p(t_v|\mathcal{T}) &= c(1-t_v)^{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1} && (5)\\
p(\mathbf{z}_v|\mathbf{z}_u, \sigma^2(t_v-t_u)\mathbf{I}) &= (2\pi\sigma^2(t_v-t_u))^{-\frac{D}{2}} \exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{\sigma^2(t_v-t_u)}\Big) && (6)\\
p(\mathbf{z}_k|\mathbf{z}_v, \sigma^2(t_k-t_v)\mathbf{I}) &= (2\pi\sigma^2(t_k-t_v))^{-\frac{D}{2}} \exp\Big(-\frac{1}{2}\frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{\sigma^2(t_k-t_v)}\Big) && (7)\\
p(t_v, \mathbf{z}_{\{u,v,k\}}, \sigma^2|\mathcal{T}) &= c(2\pi\sigma^2)^{-D} \exp\Big\{(cJ^{\theta,\alpha}_{\mathbf{n}_v}-1)\ln(1-t_v) \\
&- \frac{D}{2}\big(\ln(t_v-t_u) + \sum_k \ln(t_k-t_v)\big) \\
&- \frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{2\sigma^2}\frac{1}{t_v-t_u} - \sum_k \frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{2\sigma^2}\frac{1}{t_k-t_v}\Big\} \\
&= C_{\sigma^2}\exp\big\{u(t_v)\big\} && (8)\\
\frac{du}{dt_v} &= -\frac{cJ^{\theta,\alpha}_{\mathbf{n}_v}-1}{1-t_v} - \frac{D}{2}\big(\frac{1}{t_v-t_u} - \sum_k \frac{1}{t_k-t_v}\big) \\
&+ \frac{\parallel \mathbf{z}_v - \mathbf{z}_u \parallel^2}{2\sigma^2}\frac{1}{(t_v-t_u)^2} - \sum_k \frac{\parallel \mathbf{z}_k - \mathbf{z}_v \parallel^2}{2\sigma^2}\frac{1}{(t_k-t_v)^2} \\
&= A(t_v) && (9)\\
C_{\sigma^2}A(t_v)^{-1}\int \exp\big\{u(t_v)\big\}du &= C_{\sigma^2}A(t_v)^{-1}\exp\big\{u(t_v)\big\}\Big|^{t_v=t_{k'}}_{t_v=t_u},\ \text{where } t_{k'} = \min(t_k) \\
&= A(t_{k'})^{-1}p(t_{k'}, \mathbf{z}_{\{u,v,k\}}, \sigma^2|\mathcal{T}) - A(t_u)^{-1}p(t_u, \mathbf{z}_{\{u,v,k\}}, \sigma^2|\mathcal{T}) \\
&= Z && (10)\\
\mathbb{E}_{p(t_v|\mathbf{z}_{\{u,v,k\}}, \sigma^2, \mathcal{T})}[t_v] &= \frac{1}{Z}\int t_v p(t_v, \mathbf{z}_{\{u,v,k\}}, \sigma^2|\mathcal{T})dt_v \\
&= \frac{1}{Z}\int \exp\big\{u_\mu(t_v)\big\}A_\mu(t_v)^{-1}du && (11)
\end{aligned}
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