E. B. Vinberg (Ed.)

Geometry II

Spaces of Constant Curvature

With 87 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

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Foreword to the English Edition

The authors have introduced some improvements into the English edition. The literature list has been enlarged (mainly that to Part 2) and a number of short additions have been included in the text referring, in particular, to recent works which appeared after the publication of the Russian edition. We have also made every effort to correct misprints and occasional inaccuracies in the Russian version.

The process of translating the book into English has revealed that the use of geometric terminology in the English language literature on the subject is rather haphazard (which, incidentally, is also the case in Russian). In each case, after consulting many relevant English language publications, we and the translator have made our choice. However, in the footnotes we provide alternative terms also used in the literature.

The authors are extremely grateful to Dr. Minachin who performed the laborious work of translating the book with a sense of responsibility and interest.

E. B. Vinberg

I. Geometry of Spaces of Constant Curvature

D.V. Alekseevskij, E.B. Vinberg,
A.S. Solodovnikov

Translated from the Russian
by V. Minachin

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Preface

Spaces of constant curvature, i.e. Euclidean space, the sphere, and Lobachevskij space, occupy a special place in geometry. They are most accessible to our geometric intuition, making it possible to develop elementary geometry in a way very similar to that used to create the geometry we learned at school. However, since its basic notions can be interpreted in different ways, this geometry can be applied to objects other than the conventional physical space, the original source of our geometric intuition.

Euclidean geometry has for a long time been deeply rooted in the human mind. The same is true of spherical geometry, since a sphere can naturally be embedded into a Euclidean space. Lobachevskij geometry, which in the first fifty years after its discovery had been regarded only as a logically feasible by-product appearing in the investigation of the foundations of geometry, has even now, despite the fact that it has found its use in numerous applications, preserved a kind of exotic and even romantic element. This may probably be explained by the permanent cultural and historical impact which the proof of the independence of the Fifth Postulate had on human thought.

Nowadays modern research trends call for much more businesslike use of Lobachevskij geometry. The traditional way of introducing Lobachevskij geometry, based on a kind of Euclid-Hilbert axiomatics, is ill suited for this purpose because it does not enable one to introduce the necessary analytical tools from the very beginning. On the other hand, introducing Lobachevskij geometry starting with some specific model also leads to inconveniences since different problems require different models. The most reasonable approach should, in our view, start with an axiomatic definition, but it should be based on a well-advanced system of notions and make it possible either to refer to any model or do without any model at all.

Their name itself provides the description of the property by which spaces of constant curvature are singled out among Riemannian manifolds. However, another characteristic property is more important and natural for them — the property of maximum mobility. This is the property on which our exposition is based.

The reader should realize that our use of the term "space of constant curvature" does not quite coincide with the conventional one. Usually one understands it as describing any Riemannian manifold of constant curvature. Under our definition (see Chap. 1, Sect. 1) any space of constant curvature turns out to be one of the three spaces listed at the beginning of the Preface.

Although Euclidean space is, of course, included in our exposition as a special case, we have no intention of introducing the reader to Euclidean geometry. On the contrary, we make free use of its basic facts and theorems. We also assume that the reader is familiar with the basics of linear algebra and affine geometry, the notion of a smooth manifold and Lie group, and the elements of Riemannian geometry.

For the history of non-Euclidean geometry and the development of its ideas the reader is referred to relevant chapters in the books of Klein [1928], Kagan [1949, 1956], Coxeter [1957], and Efimov [1978].

Chapter 1
Basic Structures

§ 1. Definition of Spaces of Constant Curvature

This chapter provides the definition of spaces of constant curvature and of their basic structures, and describes their place among homogeneous spaces on the one hand and Riemannian manifolds on the other. If the reader's main aim is just to study Lobachevskij geometry, no great damage will be done if he skips Theorems 1.2 and 1.3 and the proof of Theorem 2.1.

1.1. Lie Groups of Transformations. We assume that the reader is familiar with the notions of a (real) smooth manifold and of a (real) Lie group. The word "smooth" (manifold, function, map etc.) always means that the corresponding structure is C^∞. All smooth manifolds are assumed to have a countable base of open subsets. By $T_x(X)$ we denote the tangent space to a manifold X at a point x, and by $d_x g$ the differential of the map g at a point x. If no indication of the point is necessary the subscript is omitted.

We now recall some basic definitions of Lie group theory. (For more details see, e.g. Vinberg and Onishchik [1988].)

A group G of transformations[1] of a smooth manifold X endowed with a Lie group structure is said to be a *Lie group of transformations* of the manifold X if the map

$$G \times X \to X, \qquad (g, x) \mapsto gx,$$

is smooth, which means that the (local) coordinates of the point gx are smooth functions of the coordinates of the element g and the point x. Then the stabilizer

$$G_x = \{g \in G : gx = x\}$$

of any point $x \in X$ is a (closed) Lie subgroup of the group G. Its linear representation $g \mapsto d_x g$ in the space $T_x(X)$ is called the *isotropy representation* and the linear group $d_x G_x$ is called the *isotropy group* at the point x.

The stabilizers of equivalent points x and $y = gx \quad (g \in G)$ are conjugate in G, i.e.

$$G_y = gG_xg^{-1}.$$

The corresponding isotropy groups are related in the following way:

$$d_x G_y = (d_x g)(d_x G_x)(d_x g)^{-1}.$$

In other words, if tangent spaces $T_x(X)$ and $T_y(Y)$ are identified by the isomorphism $d_x g$, then the group $d_x G_x$ coincides with the group $d_y G_y$.

[1] By a group of transformations we understand an effective group of transformations, i.e. we assume that different transformations correspond to different elements of the group.

If G is a transitive Lie group of transformations of a manifold X, then for each point $x \in X$ the map

$$G/G_x \to X, \qquad gG_x \mapsto gx$$

is a diffeomorphism commuting with the action of the group G. (The group G acts on the manifold G/G_x of left cosets by left shifts.) In this case the manifold X together with the action of G on it can be reconstructed from the pair (G, G_x).

Definition 1.1. A smooth manifold X together with a given transitive Lie group G of its transformations is said to be a *homogeneous space.*

We denote a homogeneous space by (X, G), or simply X.

A homogeneous space (X, G) is said to be connected or simply-connected[2] if the manifold X has this property.

1.2. Group of Motions of a Riemannian Manifold. A Riemannian metric is said to be defined on a smooth manifold X if a Euclidean metric is defined in each tangent space $T_x(X)$, and if the coefficients of this metric are smooth functions in the coordinates of x. A diffeomorphism g of a Riemannian manifold X is called a *motion* (or an *isometry*) if for each point $x \in X$ the linear map

$$d_x g : T_x(X) \to T_{gx}(X)$$

is an isometry. The set of all motions is evidently a group.

Each motion g takes a geodesic into a geodesic, and therefore commutes with the exponential map, i.e.

$$g(\exp \xi) = \exp dg(\xi)$$

for all $\xi \in T(X)$. Hence each motion g of a connected manifold X is uniquely defined by the image gx of some point $x \in X$ and the differential $d_x g$ at that point. This enables us to introduce coordinates into the group of motions, turning it into a Lie group. To be more precise, the following theorem holds.

Theorem 1.2 (Kobayashi and Nomizu [1981]). *The group of motions of a Riemannian manifold X is uniquely endowed with a differentiable structure, which turns it into a Lie group of transformations of the manifold X.*

If the group of motions of a Riemannian manifold X is transitive, then X is complete. Indeed, in this case there exists $\varepsilon > 0$, which does not depend on x, such that for any point $x \in X$ and for any direction at that point there exists a geodesic segment of length ε issuing from x in that direction. This implies that each geodesic can be continued indefinitely in any direction.

A Riemannian manifold X is said to have *constant curvature* c if at each point its sectional curvature along any plane section equals c.

[2] We assume that any simply-connected space is, by definition, connected.

Simply-connected complete Riemannian manifolds of constant curvature admit a convenient characterization in terms of the group of motions.

Theorem 1.3 (Wolf [1972]). *A simply-connected complete Riemannian manifold is of constant curvature if and only if for any pair of points $x, y \in X$ and for any isometry $\varphi : T_x(X) \to T_y(X)$ there exists a (unique) motion g such that $gx = y$ and $d_x g = \varphi$.*

The first part of the statement follows immediately from the fact that motions preserve curvature and that any given two-dimensional subspace of the space $T_x(X)$ can, by an appropriate isometry, be taken into any given two-dimensional subspace of the space $T_y(X)$. For the proof of the converse statement see Chap. 8, Sect. 1.3.

1.3. Invariant Riemannian Metrics on Homogeneous Spaces. Let (X, G) be a homogeneous space. A Riemannian metric on X is said to be *invariant* (with respect to G) if all transformations in G are motions with respect to that metric. An invariant Riemannian metric can be reconstructed from the Euclidean metric it defines on any tangent space $T_x(X)$. This Euclidean metric is invariant under the isotropy group $d_x G_x$. Conversely, if a Euclidean metric is defined in the space $T_x(X)$ and is invariant under the isotropy group, then it can be moved around by the action of the group G thus yielding an invariant Riemannian metric on X. Thus, an invariant Riemannian metric on X exists if and only if there is a Euclidean metric in the tangent space invariant under the isotropy group.

We now consider the question of when such a metric is unique.

A linear group H acting in a vector space V is said to be irreducible if there is no non-trivial subspace $U \subset V$ invariant under H.

Lemma 1.4. *Let H be a linear group acting in a real vector space V. If H is irreducible, then up to a (positive) scalar multiple there is at most one Euclidean metric in the space V invariant under H.*

Proof. Consider any invariant Euclidean metric (if such a metric exists) turning V into a Euclidean space. Then each invariant Euclidean metric q on V is of the form $q(x) = (Ax, x)$, where A is a positive definite symmetric operator commuting with all operators in H. Let c be any eigenvalue of A. The corresponding eigenspace is invariant under H, and consequently coincides with V. This implies that $A = cE$, i.e. $q(x) = c(x, x)$. □

The Lemma implies that if the isotropy group of a homogeneous space is irreducible, then there exists, up to a (positive) scalar multiple, at most one invariant Riemannian metric.

If a homogeneous space X is connected and admits an invariant Riemannian metric, then the isotropy representation is faithful at each point $x \in X$, since each element of the stabilizer of x, being a motion, is uniquely defined by its differential at that point.

1.4. Spaces of Constant Curvature

Definition 1.5. A simply-connected homogeneous space is said to be a *space of constant curvature* if its isotropy group (at each point) is the group of all orthogonal transformations with respect to some Euclidean metric.

The last condition is called the *maximum mobility axiom*. For the possibility of giving up the condition that the space is simply connected see Sect. 2.5.

Let (X, G) be a space of constant curvature. The maximum mobility axiom immediately implies that there is a unique (up to a scalar multiple) invariant Riemannian metric on X. With respect to this metric X is a Riemannian manifold of constant curvature (the trivial part of Theorem 1.3). The fact that G is a transitive group implies that X is a complete Riemannian manifold. Note also that G is the group of *all* its motions. Indeed, for each motion g and for each point $x \in X$, there exist an element $g_1 \in G$ such that $gx = g_1 x$, i.e. $(g_1^{-1} g)x = x$, and an element $g_2 \in G_x$ such that $d_x(g_1^{-1} g) = d_x g_2$. But then $g_1^{-1} g = g_2$ and $g = g_1 g_2 \in G$.

Conversely, by Theorem 1.3, any simply-connected complete Riemannian manifold X of constant curvature satisfies the conditions of Definition 1.5 if one takes for G the group of all its motions.

Thus, spaces of constant curvature (in the sense of the above definition) are simply-connected complete Riemannian manifolds of constant curvature considered up to change of scale, which explains their name. However, below in presenting the geometry of these spaces the fact that they are of constant Riemannian curvature is never used directly, and it is quite sufficient for the reader to be familiar with the simplest facts of Riemannian geometry (including the notion of a geodesic but excluding that of parallel translation or curvature).

Let (X, G) be a space of constant curvature. Since the manifold X is simply-connected, firstly it is orientable, and secondly each connected component of the group G contains exactly one connected component of the stabilizer (of each point). Since the isotropy representation is faithful, the stabilizer is isomorphic to the orthogonal group. The orthogonal group consists of two connected components, one including all orthogonal transformations with determinant 1, the other including all orthogonal transformations with determinant -1. Hence the group of motions of a space of constant curvature consists of two connected components, one of which includes all the motions preserving orientation (*proper motions*) and the other includes all motions reversing it (*improper motions*).

1.5. Three Spaces. For each $n \geq 2$ there are at least three n-dimensional spaces of constant curvature.

1. *Euclidean Space E^n.* Denoting the coordinates in the space $\mathbb{R}^n$ by $x_1, \dots, x_n$, we define the scalar product by the formula

$$(x, y) = x_1 y_1 + \ldots + x_n y_n,$$

thus turning $\mathbb{R}^n$ into a Euclidean vector space.

Let

$$X = \mathbb{R}^n, \quad G = T_n \leftthreetimes O_n \qquad \text{(semidirect product)},$$

where T_n is the group of parallel translations (isomorphic to $\mathbb{R}^n$) and O_n is the group of orthogonal transformations of $\mathbb{R}^n$.

For any parallel translation t_a along a vector $a \in \mathbb{R}^n$ and any orthogonal transformation $\varphi \in O_n$ one has

$$\varphi t_a \varphi^{-1} = t_{\varphi(a)},$$

which shows that G is indeed a group. Evidently, G acts transitively on X.

For each $x \in X$ the tangent space $T_x(X)$ is naturally identified with the space $\mathbb{R}^n$. The isotropy group then coincides with the group O_n.

Thus (X, G) is a space of constant curvature. It is called the n-dimensional *Euclidean space*, and denoted by E^n.

The Riemannian metric on the space E^n is induced by the Euclidean metric on the space $\mathbb{R}^n$, i.e. it is of the form

$$ds^2 = dx_1^2 + \ldots + dx_n^2.$$

Its curvature is 0.

2. *Sphere* S^n. Denoting the coordinates in the space $\mathbb{R}^{n+1}$ by $x_0, x_1, \ldots, x_n$, we introduce a scalar product in $\mathbb{R}^{n+1}$ by the formula

$$(x, y) = x_0 y_0 + x_1 y_1 + \ldots + x_n y_n,$$

which turns $\mathbb{R}^{n+1}$ into a Euclidean vector space.

Let

$$X = \{x \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}, \quad G = O_{n+1}.$$

For each $x \in X$ the tangent space $T_x(X)$ is naturally identified with the orthogonal complement to the vector x in the space $\mathbb{R}^{n+1}$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis in the space $T_x(X)$, then $\{x, e_1, \ldots, e_n\}$ is an orthonormal basis in the space $\mathbb{R}^{n+1}$.

Since each orthonormal basis in the space $\mathbb{R}^{n+1}$ can, by an appropriate orthogonal transformation, be taken into any other orthonormal basis, the group G acts transitively on X, and the isotropy group at each point x coincides with the group of all orthogonal transformations of the space $T_x(X)$.

For $n \geq 2$ the manifold X is simply-connected and hence (X, G) is a space of constant curvature. It is called the n-dimensional *sphere* and denoted by S^n.

The Riemannian metric on the space S^n is induced by the Euclidean metric on $\mathbb{R}^{n+1}$, i.e. it is of the form

$$ds^2 = dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

(Remember that the coordinate functions $x_0, x_1, \ldots, x_n$ are not independent on S^n.) The curvature of this metric is 1.

3. *Lobachevskij Space* $Л^n$. Denoting the coordinates in the space $\mathbb{R}^{n+1}$ by $x_0, x_1, \ldots, x_n$, we introduce a scalar product in $\mathbb{R}^{n+1}$ by the formula

$$(x, y) = -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n,$$

which turns $\mathbb{R}^{n+1}$ into a pseudo-Euclidean vector space, denoted by $\mathbb{R}^{n,1}$.

Each pseudo-orthogonal (i.e. preserving the above scalar product) transformation of $\mathbb{R}^{n,1}$ takes an open cone of time-like vectors

$$C = \{x \in \mathbb{R}^{n,1} : (x, x) < 0\}$$

consisting of two connected components

$$C^+ = \{x \in C : x_0 > 0\}, \qquad C^- = \{x \in C : x_0 < 0\}$$

onto itself.

Denote by $O_{n,1}$ the group of all pseudo-orthogonal transformations of the space $\mathbb{R}^{n,1}$, and by $O'_{n,1}$ its subgroup of index 2 consisting of those pseudo-orthogonal transformations which map each connected component of the cone C onto itself. Let

$$X = \{x \in \mathbb{R}^{n,1} : -x_0^2 + x_1^2 + \ldots + x_n^2 = -1, x_0 > 0\}, \quad G = O'_{n,1}.$$

A basis $\{e_0, e_1, \ldots, e_n\}$ is said to be orthonormal if $(e_0, e_0) = -1, (e_i, e_i) = 1$ for $i \neq 0$ and $(e_i, e_j) = 0$ for $i \neq j$. For example, the standard basis is orthonormal.

For any $x \in X$ the tangent space $T_x(X)$ is naturally identified with the orthogonal complement of the vector x in the space $\mathbb{R}^{n,1}$, which is an n-dimensional Euclidean space (with respect to the same scalar product). If $\{e_1, \ldots, e_n\}$ is an orthonormal basis in it, then $\{x, e_1, \ldots, e_n\}$ is an orthonormal basis in the space $\mathbb{R}^{n,1}$. It follows then (in the same way as for the sphere) that the group G acts transitively on X and the isotropy group coincides with the group of all orthogonal transformations of the tangent space.

The manifold X has a diffeomorphic projection onto the subspace $x_0 = 0$, and is therefore simply-connected. Hence (X, G) is a space of constant curvature. It is called the n-dimensional *Lobachevskij space* (or hyperbolic space) and denoted by $Л^n$.

The Riemannian metric on the space $Л^n$ is induced by the pseudo-Euclidean metric on the space $\mathbb{R}^{n,1}$, i.e. it is of the form

$$ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

Its curvature is -1.

Remark 1. For the sake of uniformity the procedure of embedding in $\mathbb{R}^{n+1}$ can also be applied to the Euclidean space E^n. Its motions are then induced by linear transformations in the following way. Let $x_0, x_1, \dots, x_n$ be coordinates in $\mathbb{R}^{n+1}$, and let $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ be the subspace defined by the equation $x_0 = 0$. The space E^n can then be identified with the hyperplane $x_1 = 1$, its motions being induced by linear transformations of the space $\mathbb{R}^{n+1}$ preserving x_0 and inducing orthogonal transformations in $\mathbb{R}^n$ (with respect to the standard Euclidean metric). Note that under this interpretation the subspace $\mathbb{R}^n$ may naturally be regarded as a tangent space of E^n (at any point).

Remark 2. All the above constructions can also be carried out in a coordinate-free form. For example, the sphere S^n can be defined as the set of vectors of square 1 in the $(n+1)$-dimensional Euclidean vector space, and its group of motions as the group of orthogonal transformations of that space. The space $Л^n$ can be defined as the connected component of the set of vectors of square -1 in the $(n+1)$-dimensional pseudo-Euclidean vector space of signature $(n, 1)$, and its group of motions as the index 2 subgroup of the group of pseudo-orthogonal transformations of that space which consists of transformations preserving each connected component of the cone of time-like vectors. Another approach is to consider a coordinate system in which the scalar product is not written in the standard way (but has the right signature).

Remark 3. For $n = 1$ and 0 the above constructions define the following homogeneous spaces:

$E^1 \simeq Л^1$ (Euclidean line),
S^1 (circle),
$E^0 \simeq Л^0$ (point),
S^0 (double point).

The spaces $E^1 \simeq Л^1$ and $E^0 \simeq Л^0$ are spaces of constant curvature while, under our definition, the spaces S^1 and S^0 do not belong to this class as they are not simply connected.

The models of S^n and $Л^n$ constructed above together with the model of E^n given in Remark 1 will be called *vector models*, while the model of E^n described at the beginning of this section will be referred to as the *affine model*. When there is a reason to indicate that a model under discussion is related to a coordinate system in the above manner we will refer to it as a *standard vector (affine) model*.

Unless otherwise stated we will assume that the Riemannian metric in S^n and $Л^n$ is normalized as in this section. If the Riemannian metric is divided by $k > 0$, the curvature is multiplied by k^2.

1.6. Subspaces of the Space $\mathbb{R}^{n,1}$. In view of the extensive use below of the vector model of the Lobachevskij space we now present the classification of subspaces of the pseudo-Euclidean vector space $\mathbb{R}^{n,1}$. A subspace $U \subset \mathbb{R}^{n,1}$ is said to be *elliptic* (respectively, *parabolic, hyperbolic*) if the restriction of the scalar product in $\mathbb{R}^{n,1}$ to U is positive definite (respectively, positive semi-definite and degenerate, indefinite). Subspaces of each type can

be characterized by their position with respect to the cone C of time-like vectors (Fig. 1).

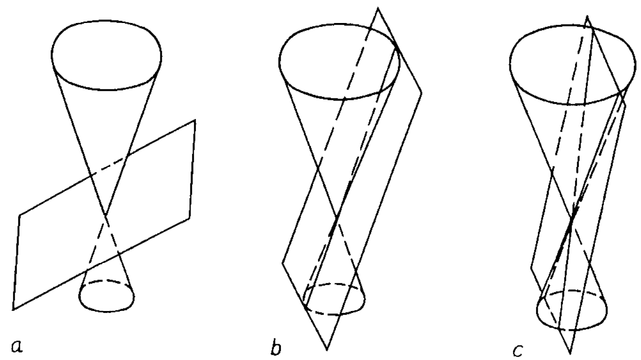


Fig. 1

Any two subspaces of the same type and dimension can, by the Witt Theorem (see, e.g., Berger [1984]), be mapped onto one another by a pseudo-orthogonal transformation.

The orthogonal complement $U^\perp$ to an elliptic (respectively, parabolic, hyperbolic) subspace U is a hyperbolic (respectively, parabolic, elliptic) subspace.

§ 2. The Classification Theorem

2.1. Statement of the Theorem. Two homogeneous spaces (X_1, G_2) and (X_2, G_2) are said to be *isomorphic* if there exist a diffeomorphism $f : X_1 \to X_2$, and an isomorphism of Lie groups $\varphi : G_1 \to G_2$ such that

$$f(gx) = \varphi(g) f(x)$$

for all $x \in X_1, g \in G_1$.

Theorem 2.1. *Any space of constant curvature is isomorphic to one of the spaces E^n, S^n, $Л^n$.*

(These spaces have been described in Sect. 1.5.)

Note that for $n \geq 2$ the spaces E^n, S^n, and $Л^n$ are not isomorphic, since they have different curvature. Incidentally, this is also a consequence of numerous differences in their elementary geometry, which will be considered below.

The proof of Theorem 2.1 will be given in Sect. 2.2–2.4.

2.2. Reduction to Lie Algebras. A homogeneous space (X, G) is uniquely (up to an isomorphism) defined by the pair (G, K), where $K = G_x$ is the stabilizer of a point $x \in X$ (see Sect. 1.1).

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the tangent algebras of the Lie groups G and K, respectively. If the group G is connected and the manifold X is simply-connected, then the pair (G, K) is uniquely defined by the pair $(\mathfrak{g}, \mathfrak{k})$ (see, e.g. Gorbatsevich and Onishchik [1988]). However, the groups of motions of spaces of constant curvature are not connected, so this argument cannot be applied directly. Nevertheless, one can prove the following lemma.

Lemma 2.2. *Any space (X, G) of constant curvature is uniquely (up to an isomorphism) defined by the pair $(\mathfrak{g}, \mathfrak{k})$.*

Proof. The fact that the manifold X is connected implies that the connected component G_+ of the group G acts transitively on X, which means that the above reasoning can be applied to the homogeneous space (X, G_+). What remains to be shown is that the group G (as a group of transformations of the manifold X) can be reconstructed from G_+.

With that goal in mind, note that the isotropy group of the homogeneous space (X, G_+) at a point $x \in X$ coincides with the connected component of the group $d_x G_x$, which is the special orthogonal group of the space $T_x(X)$. One can easily see that the action of the group SO_n in $\mathbb{R}^n$, as well as that of the larger group O_n, is irreducible. Lemma 1.4 now implies that the G-invariant Riemannian metric on X is the unique (up to a scalar multiple) G_+-invariant Riemannian metric, and is therefore uniquely defined by the group G_+. The group G is then reconstructed from G_+ as the group of all motions with respect to this metric. □

The proof of the theorem will be completed if we show that for each n there are at most three non-isomorphic pairs $(\mathfrak{g}, \mathfrak{k})$ corresponding to n-dimensional spaces of constant curvature.

2.3. The Symmetry. Let (X, G) be a space of constant curvature.

The maximum mobility axiom implies that for each $x \in X$ there is a (unique) motion $\sigma_x \in G$ satisfying the condition

$$\sigma_x x = x, \qquad d_x \sigma = -\mathrm{id}\,.$$

This motion is called the *symmetry* about the point x.

The existence of symmetry leads to the following consequences.

First, it is evident that

$$g \sigma_x g^{-1} = \sigma_{gx}$$

for all $g \in G$. In particular, all elements of G_x commute with σ_x.

In geodesic coordinates in a neighbourhood of each point x the symmetry σ_x coincides with multiplication by -1, which implies that in this neighbourhood x is the only fixed point. Hence there is a neighbourhood of unity in the group G in which all elements commuting with σ_x belong to the subgroup G_x.

Therefore the centralizer of the symmetry σ_x in G contains the subgroup G_x as an open subgroup, so G_x may differ from it only by other connected components.

Fix a point $x \in X$ and let $G_x = K, \sigma_x = \sigma$. Denote by s the inner automorphism of G defined by the element σ, and by S the corresponding automorphism $d_e s = \mathrm{Ad}\,\sigma$ (where Ad is the adjoint representation of the group G) of the tangent algebra $\mathfrak{g}$ of the group G.

The subalgebra of fixed points of the automorphism S is the tangent algebra of the subgroup of fixed points of the automorphism s and, as shown above, coincides with the tangent algebra $\mathfrak{k}$ of the subgroup K. Now, since $S^2 = \mathrm{id}$, the algebra $\mathfrak{g}$ is decomposed, as a vector space, into the direct sum of eigenspaces of the automorphism S corresponding to the eigenvalues 1 and -1.

The first of them is $\mathfrak{k}$, and the second we denote by $\mathfrak{m}$. Since S is an automorphism of the algebra $\mathfrak{g}$, the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{1}$$

has the following properties:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{2}$$

Now, since σ commutes with all elements in K, the element $S = \mathrm{Ad}\,\sigma$ commutes with all elements in $\mathrm{Ad}\,K$. Therefore, both terms in the decomposition (1) are invariant under $\mathrm{Ad}\,K$.

2.4. Structure of the Tangent Algebra of the Group of Motions. Under the assumptions and notation of the preceding section we fix an invariant Riemannian metric on X. (Recall that it is defined up to a scalar multiple.)

This defines a Euclidean metric in the space $V = T_x(X)$. Denote by $O(V)$ the group of orthogonal transformations of the space V and by $\mathfrak{so}(V)$ its tangent algebra consisting, as is well known, of all skew-symmetric transformations.

The definition of a space of constant curvature implies that the mapping $k \mapsto d_x k$ is an isomorphism of the group K onto the group $O(V)$. Being smooth, it is an isomorphism of Lie groups, and consequently its differential is a Lie algebra isomorphism between $\mathfrak{k}$ and $\mathfrak{so}(V)$. These isomorphisms identify the group K with the group $O(V)$, and its tangent algebra $\mathfrak{k}$ with the Lie algebra $\mathfrak{so}(V)$. This defines the first term in the decomposition (1) and the adjoint action of the group K on it.

The second term in (1) is naturally identified with the space V. Namely, consider the map

$$\alpha : G \to X, \quad g \mapsto gx.$$

Since the group G acts transitively on X, the differential $d_e\alpha$ maps $\mathfrak{g}$ onto V in such a way that its kernel is the tangent algebra to the stabilizer of the point

x, i.e. the subalgebra $\mathfrak{k}$ (see, e.g. Vinberg and Onishchik [1988]). Therefore, $d_e\alpha$ maps $\mathfrak{m}$ isomorphically onto V. This isomorphism identifies $\mathfrak{m}$ with V.

One has, therefore,

$$K = O(V), \quad \mathfrak{k} = \mathfrak{so}(V), \quad \mathfrak{m} = V. \tag{3}$$

Assuming that K acts on G by inner automorphisms, one can easily see that α commutes with this action. Therefore, its differential $d_e\alpha$ also commutes with the action of K (we assume that K acts on $\mathfrak{g}$ by differentials of inner automorphisms, i.e. by the adjoint representation). This implies that under our identification the adjoint action of the group K on $\mathfrak{m}$ coincides with the natural action of the group $O(V)$ on V :

$$(\operatorname{Ad} C)v = Cv \qquad (C \in O(V) = K, \quad v \in V = \mathfrak{m}). \tag{4}$$

Hence it follows that the adjoint action of the algebra $\mathfrak{k}$ on $\mathfrak{m}$ coincides with the natural action of the algebra $\mathfrak{so}(V)$ on V:

$$[A, v] = Av \qquad (A \in \mathfrak{so}(V) = \mathfrak{k}, \quad v \in V = \mathfrak{m}). \tag{5}$$

This defines commutators of elements of $\mathfrak{k}$ with elements of $\mathfrak{m}$.

To complete the definition of the algebra $\mathfrak{g}$ one has to specify the commutation law for any two elements in $\mathfrak{m}$. This law can be written in the form

$$[u, v] = T(u, v) \qquad (u, v \in V = \mathfrak{m}, \quad T(u, v) \in \mathfrak{so}(V) = \mathfrak{k}) \tag{6}$$

where $T : V \times V \to \mathfrak{so}(V)$ is a skew-symmetric bilinear map.

Applying the operator $\operatorname{Ad} C$, where $C \in O(V) = K$, to (6) and using the fact that $\operatorname{Ad} C$ is an automorphism of the algebra $\mathfrak{g}$ and the above description of its action on $\mathfrak{k}$ and $\mathfrak{m}$, one has

$$T(Cu, Cv) = CT(u, v)C^{-1}. \tag{7}$$

The last relation holds for each $C \in O(V)$ and makes it possible to define the map T almost uniquely. Indeed, let w be a vector orthogonal to u and v. Taking for C the reflection in the hyperplane orthogonal to w and applying relation (7) to w we have $T(u, v)w = -CT(u, v)w$, i.e. $T(u, v)w = aw$, where $a \in \mathbb{R}$. Since $T(u, v)$ is a skew-symmetric operator, one has $a = 0$, and consequently $T(u, v)w = 0$. Thus the operator $T(u, v)$ acts non-trivially only in the subspace spanned by the vectors u and v.

Now let u and v be orthogonal unit vectors. One has

$$T(u, v)u = \rho v, \quad T(u, v)v = -\rho u,$$

where ρ is a real number. Formula (7) implies that ρ does not depend on u and v since any pair of orthogonal unit vectors can be taken into any other such pair by an appropriate orthogonal transformation.

Under the same assumptions on u and v, this implies that for each vector w

$$T(u, v)w = \rho[(u, w)v - (v, w)u]. \tag{8}$$

Since both sides of this formula are linear and skew-symmetric in u and v, relation (8) holds for all u and v. Formulae (6) and (8) define a commutation law in $\mathfrak{m}$ (for a given ρ).

Thus, the pair $(\mathfrak{g}, \mathfrak{k})$ is completely determined by ρ.

If $\rho \neq 0$, then by an appropriate choice of scale one can get $\rho = \pm 1$. Therefore, there are at most three possibilities for the pair $(\mathfrak{g}, \mathfrak{k})$, which implies the statement of the Theorem.

A direct verification shows that for $\rho = 0$, 1, and -1 one obtains the spaces E^n, S^n, and $Л^n$, respectively. For $n = 1$ there is only one such space, as in this case ρ is not defined.

2.5. Riemann Space. Let us now replace the condition included in Definition 1.5 that a space of constant curvature must be simply-connected by the weaker condition that it should be connected. New possibilities arise which, however, can be described without much difficulty. They include firstly the circle S^1 and secondly the n-dimensional *Riemann space* (or the *elliptic space*) R^n obtained from the sphere S^n by identifying antipodal points. The space R^n is the n-dimensional real projective space endowed with the Riemannian metric induced by the metric on the sphere.

Proof. Let (X, G) be a connected homogeneous space satisfying the maximum mobility axiom.

Consider a simply-connected covering $p : \tilde{X} \to X$ of the manifold X. It is known that for each diffeomorphism $f : X \to X$ and each pair of points $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ satisfying the condition $f(p(\tilde{x}_1)) = p(\tilde{x}_2)$ there exists a unique diffeomorphism $\tilde{f} : \tilde{X} \to \tilde{X}$ satisfying the conditions $f(\tilde{x}_1) = \tilde{x}_2$ and $p(\tilde{f}(\tilde{x})) = f(p(\tilde{x}))$ for all $\tilde{x} \in \tilde{X}$. The diffeomorphism $\tilde{f}$ is said to cover the diffeomorphism f. The set of all diffeomorphisms of the manifold $\tilde{X}$ covering diffeomorphisms in G is a group. Denote it by $\tilde{G}$. The map $\varphi : \tilde{G} \to G$ associating with each diffeomorphism in $\tilde{G}$ the diffeomorphism in G which it covers, is a group epimorphism. The kernel Γ of this epimorphism is nothing else but the covering group of the covering p.

The group $\tilde{G}$ acquires a Lie group structure under which its action on $\tilde{X}$ and the homomorphism φ are smooth and the subgroup Γ is discrete. As a discrete normal subgroup of a Lie group, the group Γ is contained in the centralizer $Z(\tilde{G}_+)$ of the connected component $\tilde{G}_+$ of the group $\tilde{G}$.

One can easily see that the group $\tilde{G}$ acts transitively on $\tilde{X}$ and that the homogeneous space $(\tilde{X}, \tilde{G})$ satisfies the maximum mobility axiom. Since, by assumption, this space is simply-connected, it is isomorphic to one of the spaces E^n, S^n, $Л^n$.

Suppose that the covering p is non-trivial, i.e. $\Gamma \neq e$. A detailed analysis of groups of motions of the spaces of constant curvature shows that $Z(\tilde{G}_+) \neq e$ only if $\tilde{X} \simeq E^1$ or S^n. In the first case $Z(\tilde{G}_+) = \tilde{G}_+ \simeq \mathbb{R}$, $\Gamma \simeq Z$, and

$X = \tilde{X}/\Gamma \simeq S^1$. In the second case $\Gamma = Z(\tilde{G}_+) = \pm\mathrm{id}$ and $X = S^n/\pm\mathrm{id}$ is the Riemann space for which $G = O_{n+1}/\pm\mathrm{id} = PO_{n+1}$. □

§ 3. Subspaces and Convexity

In the rest of this chapter we consider a fixed n-dimensional space (X, G) of constant curvature. The word "point" then means "point of the space X", the word "motion" means "motion of the space X", etc.

3.1. Involutions. A remarkable feature of the spaces of constant curvature is that they have many "subspaces" of the same type (the meaning of the word "many" is explained in Sect. 3.2). From the point of view of Riemannian geometry subspaces are totally geodesic submanifolds. It is, however, much more convenient to define them in terms of the group of motions, namely, as the sets of fixed points of involutions.

The fact that motions commute with the exponential map implies that for each motion g the set X^g of its stable points is a totally geodesic submanifold and that for all $x \in X^g$ the tangent space $T_x(X^g)$ coincides with the subspace of invariant vectors of the linear operator $d_x g$.

A motion σ is said to be an *involution* if $\sigma^2 = \mathrm{id}$. The differential of an involution σ at a point $x \in X^\sigma$ is the orthogonal reflection in the subspace $T_x(X^\sigma)$. This implies that any involution σ is uniquely defined by the submanifold X^σ provided this submanifold is not empty. It is also evident that

$$gX^\sigma = X^{g\sigma g^{-1}} \tag{9}$$

for any motion g. In particular, the submanifold X^σ is invariant under g if and only if g commutes with σ (provided X^σ is not empty).

For each involution σ, denote by G^σ the group of motions of the manifold X^σ induced by those transformations in G that commute with σ. This group is naturally isomorphic to the quotient group of the centralizer of the involution σ in the group G with respect to the subgroup of motions acting identically on X^σ.

Theorem 3.1. *The submanifold X^σ is not empty for any involution $\sigma \in G$ with the exception of the case $X = S^n$, $\sigma = -\mathrm{id}$. For each $k = 0, 1, \dots, n$ there is an involution $\sigma = G$ for which $\dim X^\sigma = k$. Moreover,*

(1) *all such involutions are conjugate in the group G ;*
(2) *the group G^σ acts transitively on X^σ;*
(3) *the homogeneous space (X^σ, G^σ) is the space of constant curvature of the same type as (X, G) except for the cases when $X = S^n$ and $k = 1$ or 0; in these cases (X^σ, G^σ) is either the circle S^1 or the double point S^0, respectively.*

Proof. The proof can be obtained with the use of the explicit description of involutions in the vector model of the space X (see Sect. 1.5). In this model, each involution $\sigma \in G$ is induced by an involutory linear transformation of the space $\mathbb{R}^{n+1}$, which will also be denoted by σ. A well-known statement of linear algebra yields

$$\mathbb{R}^{n+1} = V^+(\sigma) \oplus V^-(\sigma), \tag{10}$$

where $V^+(\sigma)$ and $V^-(\sigma)$ are the eigenspaces of σ corresponding to the eigenvalues 1 and -1, respectively.

The condition $\sigma \in G$ means that

for $X = E^n$: $V^+(\sigma) \subset \mathbb{R}^n$ and $V^-(\sigma) \cap \mathbb{R}^n$ is orthogonal to $V^-(\sigma)$;
for $X = S^n$: the spaces $V^+(\sigma)$ and $V^-(\sigma)$ are orthogonal with respect to the Euclidean metric in the space $\mathbb{R}^{n+1}$;
for $X = Л^n$: the spaces $V^+(\sigma)$ and $V^-(\sigma)$ are orthogonal with respect to the pseudo-Euclidean metric in the space $\mathbb{R}^{n+1}$, and $V^-(\sigma)$ is an elliptic subspace.

Under this notation, one has $X^\sigma = X \cap V^+(\sigma)$, so that if $V^+(\sigma) \neq 0$, then $X^\sigma \neq \emptyset$ and $\dim X^\sigma = \dim V^+(\sigma) - 1$. Evidently, $V^+(\sigma) = 0$ only if $X = S^n$ and $\sigma = -\mathrm{id}$.

The properties of the decomposition (10) imply that it is completely determined by the subspace $V^+(\sigma)$. For involutions σ_1 and σ_2 such that $\dim V^+(\sigma_1) = \dim V^+(\sigma_2)$ there exists a linear transformation $g \in G$ such that $gV^+(\sigma_1) = V^+(\sigma_2)$, and consequently $g\sigma_1 g^{-1} = \sigma_2$.

The subspace $V^+(\sigma)$ inherits the type of its structure from the space $\mathbb{R}^{n+1}$, i.e.

for $X = E^n$: a distinguished subspace of codimension 1 ($= V^+(\sigma) \cap \mathbb{R}^n$) and a Euclidean metric on it;
for $X = S^n$: a Euclidean metric;
for $X = Л^n$: a pseudo-Euclidean metric together with a distinguished connected component ($= V^+(\sigma) \cap C^+$) of the time-like cone.

These structures are invariant under all linear transformations $g \in G$ commuting with σ. Conversely, each linear transformation of the space $V^+(\sigma)$ preserving the corresponding structure can be extended to a linear transformation $g \in G$ commuting with σ: one simply sets $g|_{V_-(\sigma)} = \mathrm{id}$.

This implies the statement of the theorem. □

3.2. Planes

Definition 3.2. A non-empty set $Y \subset X$ is said to be a *plane* if it is the set of fixed points for an involution $\sigma \in G$. The homogeneous space (Y, G^σ) is then called a *subspace* of the space (X, G) and the involution σ is called the *reflection* in the plane Y.

The reflection in the plane Y will be denoted by σ_Y.

In the vector model, a k-dimensional plane Y in the space X is a non-empty intersection of a $(k+1)$-dimensional subspace U of the space $\mathbb{R}^{n+1}$ with X.

The condition that it is non-empty means, for $X = E^n$, that the subspace U is not contained in $\mathbb{R}^n$, and for $X = Л^n$, that U is hyperbolic. The subspace U is called the *defining subspace* for the plane Y. It is evident that $U = \langle Y \rangle$.

For $X = S^n$ and $Л^n$ the *tangent space* of the plane Y at a point y is identified with the orthogonal subspace of $\langle y \rangle$ in U, and for $X = E^n$ with $U \cap \mathbb{R}^n$. For $X = S^n$ or $Л^n$, its orthogonal subspace in $T_y(X)$ is naturally identified with $U^\perp$, and for $X = E^n$ with the orthogonal subspace of $U \cap \mathbb{R}^n$ in $\mathbb{R}^n$. This subspace of $\mathbb{R}^{n+1}$ (which does not depend on the point $y \in Y$) is said to be the *normal space* of the plane Y and is denoted by $N(Y)$.

For any motion the set of its fixed points is a plane (provided it is non-empty). A non-empty intersection of any family of planes is also a plane.

Relation (9) implies that any motion takes a plane into a plane.

A 0-dimensional plane is a point for $X = E^n$ or $Л^n$, and a double point for $X = S^n$ (i.e. a pair of antipodal points on the sphere). A one-dimensional plane is called a (*straight*) *line*, and an $(n-1)$-dimensional plane is called a *hyperplane*.

Note that our notion of a plane, as we have just defined it in the affine model of the Euclidean space, coincides with that in the sense of affine geometry.

The next two theorems provide a description of the set of planes. For the Euclidean space, they are well-known facts of affine geometry.

Theorem 3.3. *For any point and any k-dimensional direction there is exactly one k-dimensional plane passing through the point in the given direction.*

Proof. It follows from the maximum mobility axiom that there is a (unique) involution for which the differential at the given point is the reflection in the given k-dimensional subspace of the tangent space. The submanifold of fixed points of this involution is the required plane. □

Corollary. (*Straight*) *lines are geodesics in the sense of Riemannian geometry.*

Proof. Any line is a geodesic as a subset of fixed points for an involution. Conversely, since for any point and any direction there is exactly one geodesic passing through the point in the given direction, any geodesic is a (straight) line. □

Theorem 3.4. *For any set of $k+1$ points there exists a plane of dimension $\leq k$ passing through all of them.*

Proof. Let $x_0, x_1, \dots, x_k$ denote the given points. In the vector model of the space X the required plane is the intersection of X with the subspace of $\mathbb{R}^{n+1}$ spanned by the vectors $x_0, x_1, \dots, x_k$. □

The points $x_0, x_1, \dots, x_k$ are said to be *in general position* if they are not contained in any plane of dimension $< k$. The theorem we have just proved implies that for any set of $k+1$ points in general position there is exactly one k-dimensional plane passing through them. Indeed, if the points lie in two different k-dimensional planes, then they also lie in their intersection, which is a plane of dimension $< k$.

3.3. Half-Spaces and Convex Sets. According to Theorem 3.4, for any pair of distinct points x, y (which, if $X = S^n$, are not antipodal) there exists a unique line l passing through them. The *segment* joining x to y (denoted by xy) is the segment of the line l with end-points x and y if $l \simeq E^1$, or the shorter of the two arcs with end-points x and y if $l \simeq S^1$. From the point of view of Riemannian geometry the segment xy is the unique shortest curve joining x to y. If $x = y$, the segment consists, by definition, of a single point x.

Definition 3.5. A set $P \subset X$ is said to be *convex* if for any pair of points $x, y \subset P$ (which, if $X = S^n$, are not antipodal) it contains the segment xy.

Any intersection of convex sets is evidently also a convex set.

A plane is a convex set, since together with any pair of its points (which, if $X = S^n$, are not antipodal) it contains not just the segment but the whole line passing through them.

In the vector model the set $P \subset X$ is convex if and only if the set $\mathbb{R}_+ P = \{ax : x \in P, \quad a \geq 0\}$ is a convex cone in the vector space $\mathbb{R}^{n+1}$.

In the affine model convex sets of a Euclidean space are convex sets in the sense of affine geometry.

Theorem 3.6. *Let P be a $(k-1)$-dimensional plane and Q a k-dimensional plane $(k > 0)$ containing P. The set $Q \setminus P$ consists of two connected components both of which, as well as their unions with the plane P, are convex sets.*

Proof. Let L and M be the defining subspaces for the planes P and Q respectively. One has $L \subset M$. Take a linear function f on $\mathbb{R}^{n+1}$ which vanishes on L but does not vanish on M. Then

$$Q \setminus P = \{x \in Q : f(x) > 0\} \cup \{x \in Q : f(x) < 0\}$$

provides the decomposition of $Q \setminus P$ into two non-intersecting convex subsets, which are its connected components. Their unions with the plane P are defined by the inequalities $f(x) \geq 0$ and $f(x) \leq 0$, respectively, and are evidently also convex. □

Definition 3.7. In the notation of Theorem 3.6 the connected components of the set $Q \setminus P$ (respectively, their unions with the plane P) are said to be *open* (respectively, *closed*) *half-planes* into which the plane P divides the plane Q.

Below, unless otherwise stated, "half-plane" always means a closed half-plane.

In particular, each hyperplane H divides the whole space X into two *half-spaces*, which the reflection σ_H takes into each other. They are said to be bounded by the hyperplane H, and are denoted by H^+ and H^-. A choice of one of them (the "positive" one, denoted by H^+) is an *orientation of the hyperplane* H.

Theorem 3.8. *Any closed convex set is an intersection of half-spaces.*

Proof. Denote the set in question by P. The proof is obtained in the vector model by applying the analogous theorem of affine geometry (see, e.g., Berger [1978]) to the closure of the convex cone $\mathbb{R}_+P$ in the ambient vector space of the model. □

It seems quite natural, in the light of this theorem, to consider those convex sets which are intersections of finitely many half-spaces.

Definition 3.9. A *convex polyhedron* is an intersection of finitely many half-spaces, having a non-empty interior[3].

3.4. Orthogonal Planes

Definition 3.10. Two planes Y and Z are said to be *orthogonal* if their intersection is a 0-dimensional plane and if at each point where they intersect their tangent spaces are orthogonal.

In the vector model of the spaces $X = S^n$ and $Л^n$, the orthogonality of the planes Y and Z means that

(1) the intersection of their defining subspaces $\langle Y\rangle$ and $\langle Z\rangle$ is a one-dimensional subspace of the form $\langle y\rangle$, where $y \in X$ (i.e. any one-dimensional subspace if $X = S^n$, and a hyperbolic one-dimensional subspace if $X = Л^n$);

(2) the sections of the subspaces $\langle Y\rangle$ and $\langle Z\rangle$ by the orthogonal subspace of $\langle y\rangle$ (which, as we remember, is identified with the tangent space $T_y(X)$) are orthogonal to each other.

For the statement of the next theorem we introduce a notion concerning planes on a sphere. Let $Y \subset S^n$ be any plane. The set of points mapped by the reflection σ_Y into the points antipodal to them coincides, in the vector model, with the intersection of the sphere with the orthogonal subspace $\langle Y\rangle^\perp$ of the subspace $\langle Y\rangle$, and is therefore a plane of dimension $n - 1 - \dim Y$. This plane is said to be the *polar plane* of the plane Y and denoted by Y^*. Evidently, $(Y^*)^* = Y$. The polar plane of a point is a hyperplane, and vice versa.

[3] The alternative term is *convex polytope*. It is however more appropriate to apply that name to a convex hull of finitely many points and not to an intersection of finitely many half-spaces. By the Weyl-Minkowski theorem, a convex hull of finitely many points is a *bounded* convex polyhedron. In the case of Lobachevskij space the notion of a convex polytope may also include convex polyhedra of finite volume in view of the fact that they are convex hulls of finitely many points, either ordinary or at infinity.

Theorem 3.11. *Let Y be a k-dimensional plane and let x be a point. If (in the case of the sphere) $x \notin Y^*$, then there is a unique $(n-k)$-dimensional plane L passing through x and orthogonal to Y. If the extra condition $x \notin Y$ is satisfied, then there is a unique line l passing through x and orthogonal to Y; moreover $l \subset L$.*

Proof. Leaving out the well-known case $X = E^n$, consider the vector model of the space $X = S^n$ or $Л^n$. Note that $x \notin \langle Y\rangle^\perp$ (for $X = S^n$ it is implied by the hypothesis, and for $X = Л^n$ it follows from the fact that $\langle Y\rangle^\perp$ is an elliptic subspace).

There is a unique decomposition

$$x = y' + z' \qquad (y' \in \langle Y\rangle, \quad z' \in \langle Y\rangle^\perp, \quad y' \neq 0).$$

According to the above description of orthogonal planes in the vector model, the defining subspace for each plane $Z \subset X$ passing through x and orthogonal to Y is of the form

$$\langle Z\rangle = \langle y'\rangle \oplus U,$$

where $U \subset \langle Y\rangle^\perp$ is a subspace of dimension equal to $\dim Z$. If $\dim Z = n-k$, one has $U = \langle Y\rangle^\perp$. If $\dim Z = 1$ and $x \notin Y$, one has $z' \neq 0$, and therefore $U = \langle z'\rangle$. This implies the statement of the theorem. □

If $X = S^n$ and $x \in Y^*$, each line passing through x is orthogonal to Y.

Definition 3.12. The *projection* (more precisely, the *orthogonal projection*) of a point x on a plane Y is the point y at which the line l passing through x and orthogonal to Y intersects the plane Y. For $X = S^n$ one takes the intersection point nearest to x, and if $x \in Y^*$ the projection is not defined. If $x \in Y$, then its projection is defined to be the point x itself.

The segment xy is called the *perpendicular* dropped from the point x on the plane Y.

The proof of Theorem 3.10 implies that, for $X = S^n$ or $Л^n$ in the vector model, the projection y of a point x on the plane Y is represented by a vector which is a scalar multiple of the orthogonal projection y' of the vector x on the subspace $\langle Y\rangle$, the scalar being a positive real number. For $X = S^n$ it is a consequence of the inequalities $(x,y) > 0$ and $(x,y') = (x,x) > 0$, and for $X = Л^n$ it follows from the inequalities $(x,y) < 0$ and $(x,y') = (x,x) < 0$.

The map associating with each point its projection on the plane Y (provided it is defined) is called the *projection* (more precisely, the *orthogonal projection*) on the plane Y, denoted by π_Y.

This description of the projection on a plane in the vector model easily yields the following "theorem on three perpendiculars" for the spaces S^n and $Л^n$.

Theorem 3.13. *If a plane Z is contained in a plane Y, then*

$$\pi_Z\pi_Y = \pi_Z.$$

§ 4. Metric

4.1. General Properties. In a space of constant curvature X, as in any complete Riemannian manifold, for any pair of points x, y there is a shortest curve joining them (which is automatically a geodesic). Its length is said to be the *distance* between the points x and y, denoted by $\rho(x, y)$. As noted in Sect. 3.3, if (in the case of the sphere) the points x and y are not antipodal, such a curve is unique and coincides with the segment xy of the line passing through x and y. If $X = S^n$, and the points x and y are antipodal, then one can take for such a curve any half-line joining x to y. In this case $\rho(x, y) = \pi$ (the largest possible distance between any two points on the sphere).

A space X equipped with a distance ρ is a metric space. In particular, the triangle inequality holds:

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

The equality holds if and only if the segments xy and yz lie on the same line so that each of them extends the other and (in the case of the sphere) their total length does not exceed half of this line.

The distance ρ agrees with the topology of the space X and, in particular, is continuous in both arguments. Any bounded closed subset in X is compact.

The distance $\rho(A, B)$ between two sets $A, B \subset X$ is defined as the infimum of distances between their points. If the sets A and B are closed, and at least one of them is compact, then $\rho(A, B) = \rho(x, y)$ for some points $x \in A$, $y \in B$.

If B is a submanifold and $\rho(A, B) = \rho(x, y)$ for some points $x \in A$, $y \in B$, then a simple argument of Riemannian geometry shows that the segment xy is orthogonal to B (at the point y). In particular, this implies that the distance from the point x to the plane Y is equal to the length of the perpendicular from x to the plane Y, with the exception of the case when $X = S^n$ and $x \in Y^*$. In the latter case all the segments connecting x with the points of the plane Y are orthogonal to Y and have the same length equal to $\pi/2$, so that $\rho(x, Y) = \pi/2$ (the largest possible distance from a point to a plane on the sphere).

4.2. Formulae for Distance in the Vector Model. In the vector model of the space $X = S^n$ or $Л^n$, the distance between points and the distance from a point to a hyperplane can be computed by the following simple formulae:

$X = S^n$	$X = Л^n$
$\cos \rho(x, y) = (x, y)$ $\sin \rho(x, H_e) = \|(x, e)\|$	$\cosh \rho(x, y) = -(x, y)$ $\sinh \rho(x, H_e) = \|(x, e)\|$

Here e is a vector satisfying the condition $(e, e) = 1$, and H_e is the hyperplane which it defines, i.e.

$$H_e = \{x \in X : (x, e) = 0\}. \tag{11}$$

(The vector e is defined by the hyperplane H_e up to multiplication by -1.)

Proof. Let us prove the first formula for $X = Л^n$. Since both sides of the formula are invariant under motions, we can assume that in the standard coordinate system in $\mathbb{R}^{n,1}$ the line passing through the points x and y is defined by the parametric equations

$$x_0 = \cosh t, \quad x_1 = \sinh t, \quad x_i = 0 \quad (i = 2, \ldots, n),$$

where the point x corresponds to $t = 0$ and the point y to some $t = T > 0$. Then on the one hand $(x, y) = -\cosh T$, and on the other hand one has

$$ds^2 = -dx_0^2 + dx_1^2 = (-\sinh^2 t + \cosh^2 t)dt^2 = dt^2$$

along this line, so that $ds = dt$, and the length of the segment xy equals T.

The second formula is proved as follows. One has

$$\rho(x, H_e) = \rho(x, y),$$

where y is the projection of the point x on the hyperplane H_e. According to Sect. 3.4, $y = cy'$, where y' is the orthogonal projection of the vector x on the subspace $\langle H_e \rangle = \langle e \rangle^\perp$ and c is a positive real number. Evidently,

$$y' = x - (x, e)e.$$

The coefficient c is defined by the condition $(y, y) = -1$. The distance is then found from the first formula. □

For a similar formula giving the distance between disjoint hyperplanes in Lobachevskij space see Chap. 4, Sect. 1.9.

4.3. Convexity of Distance. We recall that a continuous function f on the real line is said to be *convex* if

$$f\left(\frac{s+t}{2}\right) \leq \frac{f(s) + f(t)}{2} \tag{12}$$

for all $s, t \in \mathbb{R}$. This condition is known to imply a more general inequality

$$f(ps + qt) \leq pf(s) + qf(t) \quad \text{for} \quad p, q \geq 1, \quad p + q = 1. \tag{13}$$

If the function f is twice differentiable, then its convexity is equivalent to the fact that $f''(t) \geq 0$ for all $t \in \mathbb{R}$. A function f is said to be strictly convex if (12) is a strict inequality for all $s \neq t$. In this case (13) is also a strict inequality for all $s \neq t$ and $p, q > 0$.

Definition 4.1. A function f defined on the space $X = E^n$ or $Л^n$ is said to be *convex* if its restriction to any line in the space X is a convex function of the natural parameter on that line (i.e. the parameter providing its isomorphism with E^1).

Theorem 4.2. *The function $f(y) = \rho(x, y)$ in the space $X = E^n$, or Л^n is convex for all $x \in X$, and its restriction to any line not passing through x is strictly convex.*

Proof. According to the definition of a convex function, the statement of the theorem means that the length of any median in a triangle is less than half the sum of its sides meeting at the vertex from which it issues. For Lobachevskij space it is proved in the same manner as in Euclidean geometry. Namely, the symmetry about the midpoint of the base of the triangle reduces the problem to the triangle inequality (Fig. 2). □

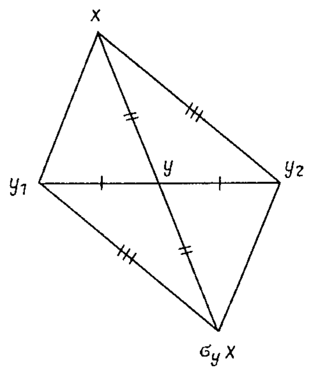


Fig. 2

Moreover, one can show that the distance in the space $X = E^n$ or Л^n is a convex function in both variables. This means that if the distance is considered as a function defined on the direct product $X \times X$, then its restriction to the direct product of any two lines is a convex function on this product (as for a Euclidean plane). The special cases of that theorem are the theorem on the median in a triangle (which is equivalent to Theorem 4.2) and the theorem stating that a midline in a triangle does not exceed half of its base.

For the case $X = S^n$, a direct computation yields the following substitute for Theorem 4.2.

Theorem 4.3. *Let $x \in S^n$ be a point and $l \subset S^n$ a line that does not lie in the polar x^* of the point x. Let $l_+ \subset l$ be the half-line whose midpoint coincides with the projection of x on l. Then the restriction of the function $f(y) = \rho(x, y)$ to l_+ is convex, and if $x \notin l$, then it is strictly convex.*

(If $l \subset x^*$, then f is constant on l.)

Chapter 2
Models of Lobachevskij Space

§ 1. Projective Models

1.1. Homogeneous Domains. Along with the *vector* model introduced in Chap. 1 there are other models of the Lobachevskij space Л^n. The most interesting among them are covered by the notion of a *homogeneous domain.*

Let (X, G) be a homogeneous space. We introduce the following notation:

(1) If $M \subset X$ and $H \subset G$ are such that $HM \subset M$, then $H|_M$ denotes the set of transformations of M induced by the transformations from H.

(2) If $M \subset X$, then G_M denotes the subgroup of G consisting of all $g \subset G$ such that $gM = M$. Evidently the group G_M takes each of the sets ∂M and $\overline{M}$ onto itself.

(3) If $M \subset X$, then $G(M)$ denotes the group $G_M|_M$ of transformations of the set M.

Now let M be an open subset in X. The subgroup G_M is closed in G. Indeed, if $gM \neq M$ for some $g \in \overline{G}_M$, then there is a point $x \notin M$ such that either $gx \in M$ or $g^{-1}x \in M$, which means that the same holds for some $g' \in G_M$. Thus for any open subset M the subgroup G_M is a *Lie subgroup.*

Definition 1.1. If M is a domain in X, and the group $G(M)$ acts transitively on M, then the homogeneous space $(M, G(M))$ (or the domain M itself) is said to be a *homogeneous domain* in the homogeneous space (X, G).

Let M be a homogeneous domain.

Definition 1.2. The pair $(\partial M, G_M|_{\partial M})$ is called the *boundary* or the *absolute* of M, and the pair $(\overline{M}, G_M|_{\overline{M}})$ is called its *closure.* The same names are also given to the sets ∂M and $\overline{M}$.

Hereafter we consider only those cases where $G_M|_{\partial M} = G(\partial M)$ and $G_M|_{\overline{M}} = G(\overline{M})$.

It seems quite natural to construct and study models of the Lobachevskij space as homogeneous domains in some well-known homogeneous spaces. For such an ambient space X one can take, e.g., the real projective space P^n, whose points are one-dimensional subspaces in $\mathbb{R}^{n+1}$. Its transformation group is $PGL_{n+1}(\mathbb{R})$, the quotient of the full linear group $GL_{n+1}(\mathbb{R})$ with respect to the subgroup $\{\lambda E | \lambda \in \mathbb{R} \setminus \{0\}\}$. Models of this type are called *projective.*

1.2. Projective Model of Lobachevskij Space. We now recall some facts and notions of projective geometry (in the form convenient for what follows).

Let U and U' be two domains in the projective space $P^n, n > 2$. A diffeomorphism f of U onto U' is said to be *a collineation* if it takes any subset of the form $U \cap l$, where l is a line in P^n, onto a subset $U' \cap l'$, where l' is also a line. Any collineation from U onto U' can be extended to a collineation of the entire space P^n.

Let Q_0 be the set of generatrices of the cone

$$-x_0^2 + x_1^2 + \ldots + x_n^2 = 0 \tag{1}$$

in $\mathbb{R}^{n+1}$. The set Q_0 is a hypersurface in P^n. The set $P^n \backslash Q_0$ evidently consists of two connected components such that one and only one of them contains no lines. This component is the set of generatrices of the cone

$$-x_0^2 + x_1^2 + \ldots + x_n^2 < 0 \tag{2}$$

(or, more precisely, the cone obtained by adding the origin to the set defined by (2)). It is said to be the *inner domain* of Q_0, and denoted by $\operatorname{int} Q_0$. It is clear that any collineation of the space P^n preserving Q_0 takes $\operatorname{int} Q_0$ onto $\operatorname{int} Q_0$.

Definition 1.3. An *oval quadric* in the space P^n is any hypersurface projectively equivalent to Q_0.

Following the notation of Sect. 1.1, we denote the group of all projective transformations of the space P^n by G.

Theorem 1.4. *The homogeneous space* $(\operatorname{int} Q, G(\operatorname{int} Q))$, *where* Q *is an oval quadric in* P^n, *is isomorphic to the Lobachevskij space* $Л^n$.

Proof. We begin with the standard vector model of the Lobachevskij space $Л^n$, realized as the hypersurface

$$-x_0^2 + x_1^2 + \ldots + x_n^2 = -1, \quad x_0 > 0 \tag{3}$$

in the pseudo-Euclidean vector space $\mathbb{R}^{n,1}$ equipped with the metric $-dx_0^2 + dx_1^2 + \ldots + dx_n^2$. Associating with each point x of the model (3) the one-dimensional subspace $\{\lambda x | \lambda \in \mathbb{R}\}$ in $\mathbb{R}^{n,1}$, one obtains a bijection of the hyperspace (3) onto the domain $\operatorname{int} Q_0$. Its group of motions then goes into $G(\operatorname{int} Q_0)$. Indeed, in the vector model (3) motions are the transformations induced by the group $O'_{n,1}$. This group can be described as the group of all linear transformations in $\mathbb{R}^{n,1}$ with determinant ± 1 taking the cone (2) into itself and preserving the condition $x_0 > 0$. This means that the group of transformations in P^n induced by that group is the group of all collineations of the space P^n preserving Q_0. □

Note that planes in the $\operatorname{int} Q_0$ model are images of planes in the vector model (3). Since planes in the vector model are non-empty intersections of the hypersurface (3) with subspaces in $\mathbb{R}^{n,1}$, planes in the $\operatorname{int} Q_0$ model are non-empty intersections of $\operatorname{int} Q_0$ with planes in P^n. In particular, straight lines in the $\operatorname{int} Q_0$ are non-empty intersections of $\operatorname{int} Q_0$ with lines in P^n.

Corollary. *Any diffeomorphism of the Lobachevskij space* $Л^n$ *that takes lines into lines is a motion.*

Definition 1.5. A homogeneous domain $(\operatorname{int} Q,\ G(\operatorname{int} Q)\ )$, where Q is an oval quadric in P^n, is called a *projective model* of the space $Л^n$.

The absolute for the projective model is $(Q, G(Q))$. Note that the group $G(Q)$ acts transitively on Q. For example, if Q is a sphere in a Euclidean space whose completion is P^n, then $G(Q)$ includes all rotations of the sphere.

1.3. Projective Euclidean Models. The Klein Model. Denote by PE^n the model of the projective space P^n obtained by adding points at infinity to the Euclidean space E^n.

Let Q be an oval quadric in PE^n. As we have just proved, the homogeneous space $(\operatorname{int} Q, G(\operatorname{int} Q))$ can be considered as a model of the Lobachevskij space $Л^n$. Models that can be represented in this form will be called *projective Euclidean* models of the space $Л^n$ (in the space PE^n). Of course, all of them are projectively equivalent. However, from the affine point of view these models fall into three classes depending on the type of the set $Q \cap E^n$. A model of that kind can be an ellipsoid, an elliptic paraboloid, or a hyperboloid of two sheets. The difference between the three cases can also be characterized by the position of the surface Q with respect to the improper hyperplane H of the space PE^n. In the first case, the set $Q \cap E^n$ is empty. In the second, it consists of a single point (the point at which H is tangent to Q), in the third, it is an oval quadric in H. Representatives of each of the three classes are called projective models of the space $Л^n$ in an *ellipsoid, paraboloid*, or *hyperboloid*, respectively.

Now we note some general properties of such models. Let us agree that if Γ is a Л-plane (i.e., a plane in the sense of Lobachevskij geometry) in such a model, then Γ^* denotes the corresponding plane in PE^n (i.e., the plane for which $\Gamma = \Gamma^* \cap \operatorname{int} Q$).

(1) Let Γ be a Л-plane, and L a Л-line in the model. Then L and Γ are orthogonal (in the sense of Lobachevskij geometry) if and only if L^* contains the pole p_{Γ^*} of the plane Γ^* with respect to the absolute (Fig. 3).

Proof. Consider a collineation g of the space PE^n which in our model induces the reflection in Γ. Since $gQ = Q$ and $g\Gamma = \Gamma$, one has $gp_{\Gamma^*} = p$, and consequently $gL^* = L^*$, where L^* is any line in PE^n passing through p_{Γ^*}. □

(2) If two points p_1 and p_2 are Л-symmetric with respect to the hyperplane Γ, and p is the Л-midpoint of the segment p_1p_2, then the cross-ratio of the (ordered) 4-tuple p_1, p_2, p_{Γ^*}, p equals -1.

Proof. Denote the cross-ratio $[p_1, p_2; p_{\Gamma^*}, p]$ by λ. If g is a collineation of the space PE^n, which in our model induces the reflection in the hyperplane passing through p perpendicular to p_1p_2, then g takes the 4-tuple $p_1, p_2, p_{\Gamma^*}, p$ into the 4-tuple $p_2, p_1, p_{\Gamma^*}, p$, whence $[p_1, p_2; p_{\Gamma^*}, p] = [p_2, p_1; p_{\Gamma^*}, p]$ and therefore

$\lambda^2 = 1$. Since, evidently, each of the pairs p_1, p_2 and p_{Γ^*}, p separates the other one (p_{Γ^*} lies outside Q), we have $\lambda = -1$. □

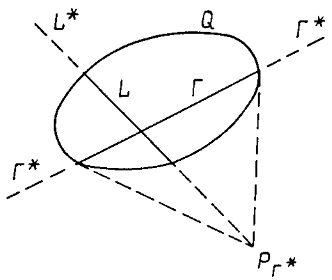


Fig. 3

The most convenient *projective Euclidean model* is that in the *unit ball*, known as the *Klein model*, and denoted by K^n. Thus, the Klein model is the ball in the Euclidean space E^n in which the motions are defined as those collineations of the space PE^n that take the ball onto itself. The absolute in the Klein model is the (homogeneous) space $(S^{n-1}, G(S^{n-1}))$.

1.4. "Affine" Subgroup of the Group of Automorphisms of a Quadric. Let $(\operatorname{int} Q, G(\operatorname{int} Q))$ be a projective Euclidean model of the space $Л^n$ (in PE^n). Consider collineations of the space PE^n preserving not only Q but also the improper hyperplane H of the space PE^n, or, equivalently, preserving both Q and the point p_H, the pole of H with respect to Q. Such collineations constitute a group of affine transformations in PE^n. The restriction of this group to $\operatorname{int} Q$ is naturally called the "affine" subgroup of the group $G(\operatorname{int} Q)$. Denote this subgroup by $AG(\operatorname{int} Q)$. There are three possibilities.

(1) *Q is an ellipsoid.* Then $H \subset \operatorname{ext} Q$ (where $\operatorname{ext} Q$ is the exterior of Q), and $p_H \in \operatorname{int} Q$, more precisely, p_H is the centre O of Q. Therefore $AG(\operatorname{int} Q)$ is the group of all Л-motions that preserve the point O, i.e., the *group of all Л-rotations about O.*

Note that if Q is a *sphere* then affine transformations preserving Q are ordinary Euclidean rotations about its centre O. Thus, for the Klein model, Л-rotations about the centre O are represented by ordinary Euclidean rotations about O. This implies, in particular, that if in the Klein model Γ is a hyperplane passing through the centre of Q, then Л-reflection in Γ is an ordinary Euclidean reflection.

(2) *$Q \cap E^n$ is a hyperboloid of two sheets.* Then H intersects $\operatorname{int} Q$ along a Л-hyperplane. One has $p_H \in \operatorname{ext} Q$, more precisely, p_H is the vertex of the asymptotic cone for Q. The group $AG(\operatorname{int} Q)$ consists of all Л-motions preserving the Л-hyperplane $H \cap \operatorname{int} Q$.

(3) *$Q \cap E^n$ is a paraboloid.* Then H is tangent to Q at the point p_H (the improper point of the paraboloid). In that case (unlike the two preceding cases) the group $AG(\operatorname{int} Q)$ acts *transitively* on $\operatorname{int} Q$. Moreover, it also contains a subgroup whose action on $\operatorname{int} Q$ is *simply-transitive.*

Proof. Let $\operatorname{int} Q$ be defined by the inequality

$$y_1 > y_2^2 + \ldots + y_n^2.$$

Consider the set of affine transformations $f_{C,\lambda}$ of the form

$$\begin{aligned} y_1' &= \lambda^2 y_1 + 2\lambda(C, Y) + (C, C), \\ Y' &= \lambda Y + C, \end{aligned}$$

where $Y = (y_2, \ldots, y_n)$, $C = (c_2, \ldots, c_n)$, $(C, Y) = c_2 y_2 + \ldots + c_n y_n$ and λ is any nonzero real number. One can easily check that each of the transformations $f_{C,\lambda}$ preserves $\operatorname{int} Q$ and that they constitute a group with simply-transitive action on $\operatorname{int} Q$. Note also that for $\lambda = 1$, $C \neq 0$ the transformation $f_{C,\lambda}$ has only one fixed point in PE^n, while for $\lambda \neq 1$ it has two fixed points (p_H and one more point on the paraboloid). □

The fact that for a paraboloid the group $AG(\operatorname{int} Q)$ acts transitively has the following invariant meaning: the group of all collineations of the projective space P^n preserving a given oval quadric Q and a given point $q \in Q$ (or, equivalently, preserving Q and the tangent hyperplane to Q at the point q) acts transitively on $\operatorname{int} Q$.

1.5. Riemannian Metric and Distance Between Points in the Projective Model

Proposition 1.6. *Let $\varphi = 0$ be the equation of an oval quadric Q in PE^n, where φ is a polynomial of degree 2 in Cartesian coordinates. Then, in the projective model, the Lobachevskij metric in $\operatorname{int} Q$ is given by the formula*

$$ds^2 = -\frac{1}{2\varphi}\left(d^2\varphi - \frac{1}{2\varphi}(d\varphi)^2\right). \tag{4}$$

Proof. We first consider the case of the Klein model.

Fig. 4 shows the vector model (3), the cone (2) corresponding to it, and its section by the plane $E^n : x_0 = 1$, i.e. the ball

$$K^n : x_1^2 + \ldots + x_n^2 < 1.$$

Taking into account that $-y_0^2 + y_1^2 + \ldots + y_n^2 = -1$, one can easily deduce from the relation $y = \lambda x$, which in the coordinate notation means

$$y_0 = \lambda,\ y_1 = \lambda x_1,\ \ldots,\ y_n = \lambda x_n, \tag{5}$$

that $\lambda = \varphi^{-1/2}$, where

$$\varphi = 1 - \sum_1^n x_i^2.$$

Substituting expressions (5) for $y_0, y_1, \ldots, y_n$ into $ds^2 = -dy_0^2 + dy_1^2 + \ldots + dy_n^2$, one obtains the induced metric in the ball K^n. A simple computation yields

$$ds^2 = \left(1 - \sum_1^n x_i^2\right)^{-2} \left[\left(1 - \sum_1^n x_i^2\right) \sum_1^n dx_i^2 + \left(\sum_1^n x_i dx_i\right)^2\right], \tag{6}$$

which is equivalent to (4). Since the ball K^n with the group of transformations induced in it by the group of motions of the model (3) is the Klein model of the space $Л^n$, this proves Proposition 1.6 in the case of the Klein model. Note that the metric ds^2 written in the form (4) is evidently invariant under any affine transformations, whereby Proposition 1.6 holds for the model in an arbitrary ellipsoid.

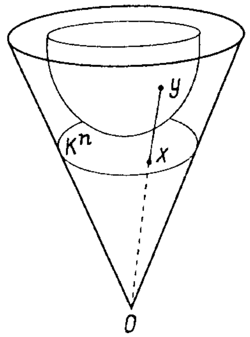


Fig. 4

Now consider any *projective* transformation in PE^n, i.e. a transformation $x \mapsto y$ defined in the coordinate notation by the formulae

$$x_1 = \frac{\alpha_1}{\beta}, \ldots, \quad x_n = \frac{\alpha_n}{\beta}, \tag{7}$$

where $\alpha_1, \ldots, \alpha_n$ and β are polynomials of degree 1 in $y_1, \ldots, y_n$, and the corresponding determinant of order $n+1$ is equal to $\Delta \neq 0$. Under this transformation the surface $1 - \sum_1^n x_i^2 = 0$ goes into $\psi = 0$, where

$$\psi = \beta^2 - \sum_1^n \alpha_i^2. \tag{8}$$

A direct computation shows that the transformation (7) takes the metric (6) into

$$ds^2 = -\frac{1}{2\psi}\left(d^2\psi - \frac{1}{2\psi}(d\psi)^2\right). \square \tag{9}$$

Remark. The determinant of the metric form (6) is equal to $(1 - \sum_1^n x_i^2)^{-n-1}$; hence, in the Klein model, the volume element is of the form

$$\left(1 - \sum_1^n x_i^2\right)^{-\frac{n+1}{2}} dx_1 \ldots dx_n.$$

For the metric (9), where ψ is defined by (8), the volume element is given by the expression $\psi^{-(n+1)/2} dy_1 \ldots dy_n$.

Proposition 1.7. *The distance between points x and y in the projective model* $(\operatorname{int} Q, G(\operatorname{int} Q))$ *is given by the formula*

$$\rho(x, y) = \frac{1}{2} |\ln[x, y; p, q]|, \tag{10}$$

where p and q are the points at which the line xy intersects the absolute Q (in PE^n), and $[x, y; p, q]$ denotes the cross-ratio of the 4-tuple.

Proof. Since the right-hand side in (10) has an invariant projective meaning, one need consider only the case of the Klein model.

Let

$$u = (u_0, u_1, 0, \ldots, 0), \quad v = (v_0, v_1, 0, \ldots, 0) \tag{11}$$

be two points in the vector model (3), and L the Л-line passing through them and defined by the equations $x_2 = \ldots = x_n = 0$. The parametric representation of the line L can be written in the form

$$x_0 = \cosh t, \quad x_1 = \sinh t, \quad x_2 = \ldots = x_n = 0.$$

Therefore, one has $ds^2 = dt^2$ (along L), and consequently

$$\rho(u, v) = |t_u - t_v|.$$

Taking the projections of the points u and v from the origin on the plane $x_0 = 1$, we get the points

$$u' = \left(1, \frac{u_1}{u_0}, 0 \ldots, 0\right), \quad v' = \left(1, \frac{v_1}{v_0}, 0, \ldots, 0\right)$$

in the Klein model. The line $u'v'$ intersects the absolute S^{n-1} at the points

$$p = (1, 1, 0, \ldots, 0), \quad q = (1, -1, 0, \ldots, 0).$$

We have

$$|[u',v';p,q]| = \frac{|u'p|}{|u'q|} : \frac{|v'p|}{|v'q|} = \frac{|\frac{u_1}{u_0}-1|}{|\frac{u_1}{u_0}+1|} : \frac{|\frac{v_1}{v_0}-1|}{|\frac{v_1}{v_0}+1|}$$
$$= \frac{|\tanh t_u - 1|}{|\tanh t_u + 1|} : \frac{|\tanh t_v - 1|}{|\tanh t_v + 1|} = e^{2|t_u - t_v|},$$

and hence

$$|t_u - t_v| = \frac{1}{2}|\ln[u',v';p,q]| \quad \text{or} \quad \rho(u',v') = \frac{1}{2}|\ln[u',v';p,q]|.$$

Since in the vector model (3) an appropriate motion takes any two points into points u and v of the form (11), the resulting equation holds for *any* pair of points u' and v' in the Klein model. □

§ 2. Conformal Models

2.1. Conformal Space

Definition 2.1. The *conformal space* C^n is a homogeneous space $(Q, G(Q))$, where Q is an oval quadric in P^{n+1}.

(Here G denotes the set of all collineations of the space P^{n+1}.)

This definition immediately implies that the absolute of the projective model $(\operatorname{int} Q, G(\operatorname{int} Q))$ of the space $Л^n$ is the conformal space C^{n-1}.

Remark. The space C^1 is isomorphic to P^1. For an explanation of this fact see Sect. 2.4.

Any subset of the form $Q \cap H$, where H is any plane in P^{n+1} having a nonempty intersection with Q, is called a *sphere* in the space C^n. In particular, if Q is a Euclidean sphere S^n in PE^{n+1}, then spheres in the conformal space S^n are Euclidean spheres of various dimensions contained in S^n.

Any diffeomorphism of the sphere S^n that takes any circle (i.e. a section of the sphere by a 2-plane) into a circle will be called *a circular transformation* of the sphere S^n.

Theorem 2.2. *The group $G(S^n)$ coincides with the group* $\operatorname{Circ} S^n$ *of all circular transformations of the sphere S^n.*

Proof. Let $g \in G(S^n)$. Then g can be extended to a collineation g^* of the space PE^{n+1} preserving S^n. The collineation g^* evidently takes a plane section of the sphere S^n into a plane section, whereby $g \in \operatorname{Circ} S^n$. Conversely, let $g \in \operatorname{Circ} S^n$. Let us associate with each hypersphere $S \subset S^n$ (i.e. a subsphere of dimension $n-1$) its pole in PE^{n+1} (Fig. 5). The transformation g takes a subsphere of any dimension again into a subsphere. In particular, it takes S again into a hypersphere. Hereby g induces a transformation g' in the set of poles, i.e. in the exterior domain of S^n. One can easily see that g' is a collineation of this domain. But then there is a collineation g^* of the space PE^{n+1} that is an extension of g. Hence, $g \in G(S^n)$. □

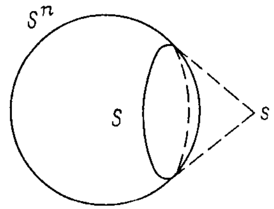


Fig. 5

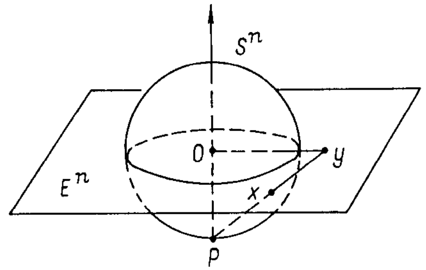


Fig. 6

Corollary. *The space C^n is isomorphic to $(S^n, \operatorname{Circ} S^n)$.*

An important model of the space C^n is constructed by the *stereographic projection*. Let us project the sphere

$$x_0^2 + x_1^2 + \ldots + x_n^2 = 1$$

in the space E^{n+1} from its "South Pole" $p = (-1, 0, \ldots, 0)$ onto the hyperplane $E^n\colon x_0 = 0$ (Fig. 6).

Denoting this projection by φ and letting $\varphi : x \mapsto y$, one has

$$y_i = \frac{x_i}{x_0 + 1} \quad (i = 1, \ldots, n).$$

These formulae make it possible to find the metric in E^n induced by φ from the metric $ds^2 = dx_0^2 + dx_1^2 + \ldots + dx_n^2$ on the sphere S^n. The resulting formula is

$$ds^2 = \left(1 + \sum_1^n y_i^2\right)^{-2} \sum_1^n dy_i^2. \tag{12}$$

Formula (12) shows that φ is a conformal mapping (i.e. it preserves angles between curves). Incidentally, it is also a consequence of the fact that φ coincides with the restriction to S^n of an inversion in the space E^{n+1}. The point p is the centre of the inversion, while the set S of its fixed points is the sphere of radius $\sqrt{2}$ with centre p (see Fig. 7: one has $|Op| = |Ox|$ in the right-angled triangle pOy, hence $|px| \cdot |py| = 2|Op|^2$). This implies another property of φ, namely that it takes circles on the sphere S^n into circles or lines in E^n.[1]

[1] For the definition of inversion and its properties of preserving angles between curves and mapping circles and lines into circles and lines see Rozenfel'd [1966].

It is obvious that φ maps $S^n \setminus \{p\}$ bijectively onto E^n. In order to extend φ to the entire sphere S^n we complete E^n with a new point ∞, considering it as the image of the pole p. Let $\hat{E}^n = E^n \cup \{\infty\}$. Define a smooth structure in a neighbourhood of the point ∞ on $\hat{E}^n$ by means of the mapping φ. The set $\hat{E}^n$ then becomes a smooth manifold and φ a conformal mapping of S^n onto $\hat{E}^n$. The group $\operatorname{Circ} \hat{E}^n$ (obtained by transferring $\operatorname{Circ} \hat{S}^n$ to $\hat{E}^n$ by means of φ) is a transitive group of conformal diffeomorphisms of $\hat{E}^n$. The space $(\hat{E}^n, \operatorname{Circ} \hat{E}^n)$ is isomorphic to $(S^n, \operatorname{Circ} S^n)$, i.e. isomorphic to C^n.

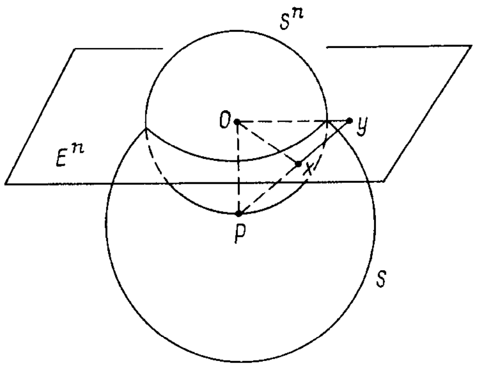


Fig. 7

Definition 2.3. The homogeneous space $(\hat{E}^n, \operatorname{Circ} \hat{E}^n)$ is called the *flat model* of the conformal space C^n.

In the $\hat{E}^n$ model, the role of spheres is played by the images of spheres in S^n, i.e. by spheres and planes in E^n (the planes are completed by the point ∞). The group $\operatorname{Circ} \hat{E}^n$ consists of all diffeomorphisms that take (conformal) spheres into spheres.

Theorem 2.4. *The group $G(S^n)$ coincides with the group* $\operatorname{Conf} S$ *of all conformal diffeomorphisms of the sphere S^n.*

Proof. The inclusion

$$\operatorname{Circ} S^n \subset \operatorname{Conf} S^n \tag{13}$$

is easily proved using the flat model $\hat{E}^n$ of the conformal sphere S^n. Let $g \in \operatorname{Circ} \hat{E}^n$. Suppose first that $g(\infty) = \infty$. Then $gE^n = E^n$, and lines in E^n are mapped into lines. Hence $g|_{E^n}$ is an affine transformation. This condition together with the condition that circles are taken into circles implies that $g|_{E^n}$ is either a Euclidean motion or a similarity transformation, and consequently $g \in \operatorname{Conf} \hat{E}^n$. In the case when $g(\infty) = v$, where $v \in E^n$, one takes any inversion h for which $hv = \infty$. Then $hg(\infty) = \infty$, whereby $hg \in \operatorname{Conf} \hat{E}^n$ and consequently $g \in \operatorname{Conf} \hat{E}^n$.

The reverse inclusion

$$\operatorname{Conf} S^n \subset \operatorname{Circ} S^n \tag{14}$$

for $n = 2$ is a well-known fact of complex analysis (any conformal diffeomorphism of the extended complex plane is either a linear fractional map or a composition of a linear fractional map with complex conjugation). For $n > 2$ the proof of (14) is more complicated. We refer the reader to Berger [1978], Vol. 1, Sect. 9.5.4, where the following version of the Liouville theorem is proved: for $n > 2$ any diffeomorphism of the complex plane of the class C^k, $k \geq 4$, that maps a domain $U \subset E^n$ conformally onto a domain $U' \in E^n$, is a composition of inversions and reflections (which does not hold for E^2). The fact that any conformal diffeomorphism $U \to U'$ belongs to the class C^4 is a very subtle one (see references therein). □

We should mention that only the first of inclusions (13) and (14) will be used below.

Corollary. *The space C^n is isomorphic to $(S^n, \operatorname{Conf} S^n)$.*

This explains the term "conformal space".

Let M be a smooth manifold and g a Riemannian metric on M. Its *conformal class* $\{g\}$ is the set of all metrics of the form φg where φ is any positive smooth function on M. We say that an angular measure is defined on M if a conformal class of metrics is chosen on M; then one can associate with each pair $\xi, \eta \in T_p(M)$ $(p \in M)$ the number $\cos \widehat{(\xi, \eta)}$, defined (in the usual way) by any metric belonging to this class.

If M is the sphere S^n in E^{n+1}, then the unique conformal class invariant under the action of the group of motions of S^n is $\{g_0\}$, where g_0 is the Riemannian metric induced by the inclusion of S^n in E^{n+1}. This class is also invariant under $\operatorname{Circ} S^n$, since $\operatorname{Circ} S^n \subset \operatorname{Conf} S^n$. Thus, for the group $\operatorname{Circ} S^n$ (for which, evidently, there are no invariant Riemannian metrics on S^n) there exists a unique invariant conformal class on S^n. This implies that *the conformal space has a unique angular measure.*

2.2. Conformal Model of the Lobachevskij Space. Consider again the conformal space $C^n = (Q, G(Q))$, where Q is an oval quadric in P^{n+1}. Choose a subspace $P^n \subset P^{n+1}$ passing through a point lying in the interior of Q. The intersection $Q \cap P^n$ is a hypersurface in Q. From the point of view of conformal geometry of the space C^n this hypersurface is a hypersphere.

The set $Q \setminus (Q \cap P^n)$ evidently consists of two connected components. Denote one of them by Q_+ (a "half-space" in Q) and consider the group $G(Q_+)$.

Theorem 2.5. *The group $G(Q_+)$ acts transitively on Q_+. The homogeneous domain $(Q_+, G(Q_+))$ in the space C^n is isomorphic to the space* $Л^n$.

Proof. Suppose that Q is the sphere $S^n \subset E^{n+1}$. Consider the Klein model of the space $Л^{n+1}$ in the unit ball K^{n+1} with boundary S^n. Consider a section of this ball by a hyperplane $E^n \subset E^{n+1}$ passing through the centre of the ball and denote this section by K^n. The ball K^n is the Klein model of the space Л^n in E^n.

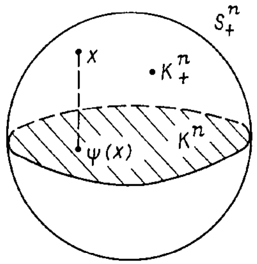


Fig. 8

The set $K^{n+1} \setminus K^n$ falls into two (open) Л-half-spaces. Let K^{n+1}_+ be one of them. Denote the corresponding half-sphere of the sphere S^n by S^n_+ (Fig. 8).

Let g be any collineation of the space PE^{n+1} preserving K^{n+1}_+; let $g_1 = g|_{S^n_+}$ and $g_2 = g|_{K^n}$. The mapping $g \mapsto g_1$ defines the isomorphism between $G(K^{n+1}_+)$ and $G(S^n_+)$, while the mapping $g \mapsto g_2$ defines the isomorphism between $G(K^{n+1}_+)$ and $G(K^n)$. This results in the isomorphism

$$\varphi: g_1 \mapsto g_2$$

between $G(S^{n+1}_+)$ and $G(K^n)$. If ψ is the orthogonal projection from S^n_+ on K^n, then

$$g_2\psi = \psi g_1,$$

since g is a motion in Л^{n+1}, and consequently takes lines perpendicular to K^n (in the sense of Lobachevskij geometry, which in this case also means perpendicularity in the Euclidean sense) into lines perpendicular to K^n. The isomorphism $\varphi: G(S^n_+) \to G(K^n)$ together with the bijection $\psi: S^n_+ \to K^n$ and the properties just stated imply that there is an isomorphism between the spaces $(S^n_+, G(S^n_+))$ and $(K^n, G(K^n))$. □

Remark 1. The homogeneous domains $(S^n_+, G(S^n_+))$ and $(K^n, G(K^n))$ (the first of them in the conformal space S^n and the second in the projective space PE^n) have common absolute, and the isomorphism ψ between them constructed above can be extended to an isomorphism of their closures.

Remark 2. The role of planes in the model $(S^n_+, G(S^n_+))$ of the space Л^n is played by the images (under the mapping ψ^{-1}) of the planes in the Klein model. This implies that the set of planes in the S^n_+ model coincides with

the set of all Euclidean hemispheres contained in S^n_+ and orthogonal to ∂S^n_+, or, equivalently, intersections of S^n_+ with the spheres of the conformal space S^n orthogonal to ∂S^n_+. In particular, lines in the S^n_+ model are Euclidean semicircles lying in S^n_+ and orthogonal to ∂S^n_+.

As we have shown in Sect. 1.2, any collineation of the space Л^n is a motion (see Corollary to Theorem 1.4). The same result holds for conformal transformations, namely, the group $\operatorname{Conf} \text{Л}^n$ coincides with the group $\operatorname{Isom} \text{Л}^n$.

Proof. The case $n = 2$ will be considered later (see Theorem 2.12 of Sect. 2.4). For $n > 2$, take the realization of Л^n in the form of a hemisphere S^n_+ used in the proof of Theorem 2.5. Any transformation $g \in \operatorname{Conf} S^n_+$ can be extended to $g^* \in \operatorname{Conf} S^n_+$ (see the end of the proof of Theorem 2.5) which implies that $g^* \in G(S^n)$, and consequently $g \in G(S^n_+) = \operatorname{Isom} \text{Л}^n$. □

Definition 2.6. The homogeneous domain $(Q_+, G(Q_+))$ in the conformal space C^n is called the *conformal model* of the Lobachevskij space Л^n.

The distinctive feature of this model, which explains its name, is the fact that if the space C^n is realized in the form $(S^n, G(S^n))$, then the group of motions of the model is a subgroup of the group $\operatorname{Conf} S^n$, and consequently angular measures on the model coincide with Euclidean angular measures in S^n (i.e., one can "see" the angles on the model).

Proposition 2.7. *Any isomorphism between the projective and the conformal models of the space* Л^n *can be extended to their closures. In particular, the absolutes of both models are isomorphic.*

Proof. It is evident that any two conformal models of the space Л^n are isomorphic and that this isomorphism can be extended to their closures. The statement of the Proposition is then implied by the Remark to Theorem 2.5. □

Proposition 2.8. *If* Γ *is a hyperplane in* Л^n, *then* $\partial\Gamma$ *is a hypersphere in the conformal space* $\partial\text{Л}^n$. *The angle between two hyperplanes* Γ_1 *and* Γ_2 *in the space* Л^n *coincides with the angle between the hyperspheres* $\partial\Gamma_1$ *and* $\partial\Gamma_2$ *in the space* $\partial\text{Л}^n$.

Proof. One has $\Gamma = T \cap Q_+$, where T is a hyperplane in P^{n+1} (Fig. 9). This implies that $\partial\Gamma = T \cap \partial Q_+$, i.e. $\partial\Gamma$ is a hypersphere in C^{n-1}. The second part of the proposition follows from the fact that if a motion g of the space Л^n takes a pair Γ_1, Γ_2 into a pair Γ_1', Γ_2', then the motion g^* of the space C^{n-1} takes the pair $\partial\Gamma_1, \partial\Gamma_2$ into the pair $\partial\Gamma_1', \partial\Gamma_2'$. (The motions g and g^* are restrictions of one and the same collineation of P^{n+1} to Q_+ and ∂Q_+, respectively.) □

This proposition is especially useful if Л^n is realized as the Klein model, since in that case $\partial\text{Л}^n$ is a Euclidean sphere with the ordinary angular measure.

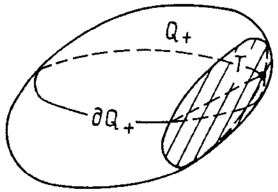


Fig. 9

Note that the vector model of the space Л^n in $\mathbb{R}^{n,1}$ (considered earlier) is, in fact, one of its conformal models. It is obtained if one takes for P^{n+1} the space PE^{n+1}, for Q the surface $-x_0^2+x_1^2+\cdots+x_n^2=-1$ (more precisely, its closure in PE^{n+1}), and for Q_+ one of the "halves" (defined by the condition $x_0>0$) into which Q is divided by the improper hyperplane H of the space PE^{n+1} (the second "half" is defined by the condition $x_0<0$). The motions in the model are collineations preserving both Q and H, i.e. transformations of the group $O'_{n,1}$. The absolute of this model is $Q\cap H$.

2.3. Conformal Euclidean Models. Consider the flat model ($\hat{E}^n$, Circ $\hat{E}^n$) of the conformal space C^n. Any hypersphere in this model is either a hypersphere in E^n, or a hyperplane in E^n completed by the point ∞. Therefore, up to a Euclidean similarity, there are exactly three conformal models of the space Л^n in $\hat{E}^n$:

(1) the unit ball $B^n\subset E^n$:

$$y_1^2+\cdots+y_2^2<1 \tag{15}$$

with the group $C_n(B^n)$, where[2] $C_n=\operatorname{Circ}\hat{E}^n$;

(2) the exterior domain of the sphere, i.e. the domain

$$y_1^2+\cdots+y_n^2>1,$$

completed by the point ∞, with a similar group;

(3) the half-space H^n :

$$y_n>0, \tag{16}$$

with the group $C_n(H^n)$.

Models (1) and (2) are taken into one another by inversion with respect to the unit sphere ∂B^n, so we consider only the models 1 and 3. The model $(B^n, C_n(B^n))$ is called the *conformal model* of the space Л^n in the *unit ball*, and the model $(H^n, C_n(H^n))$ is called the model in the *half-space* or the

[2] In Sect. 1 the unit ball has been denoted by K^n. The two kinds of notation (K^n or B^n) are used depending on which of the models of the space Л^n is considered (the Klein model, or the conformal model). We also remind the reader that $C_n(B^n)$ denotes the restriction to B^n of the subgroup of C^n preserving B^n.

Poincaré model. Below we refer to them simply as the B^n model and the H^n model, although following the lines of Sect. 1.2 they should be called "conformal Euclidean" models of the space Л^n.

We now find the Riemannian metric for each of these models,naturally taking the Cartesian coordinates $y_1,\ldots,y_n$ in E^n.

Proposition 2.9. *The Riemannian metric of the model B^n is given by the formula*

$$ds^2=4(1-|y|^2)^{-2}\sum_1^n dy_i^2, \tag{17}$$

where $|y|=(\sum_1^n y_i^2)^{1/2}$.

Proof. We take the conformal model S_+^n used in the proof of Theorem 2.5, i.e. a hemisphere in E^{n+1}. Under the stereographic projection φ described in Sect. 2.1 the model S_+^n goes into the model B^n. To each point $x=(x_0,x_1,\ldots,x_n)\in S_+^n$ there correspond two points in the plane E^n ($x_0=0$): the point $\psi(x)$ with coordinates $x_1,\ldots,x_n$ and the point $\varphi(x)$ with coordinates $y_1,\ldots,y_n$ (Fig. 10). It is easily computed that they are related by the formulae

$$x_i=\frac{2y_i}{1+|y|^2}\quad (i=1,\ldots,n), \tag{18}$$

where $|y|=(y_1^2+\cdots+y_n^2)^{1/2}$. The Lobachevskij metric in S_+^n, expressed in coordinates $x_1,\ldots,x_n$, is of the form (6). Substituting (18) for $x_1,\ldots,x_n$ one gets the expression of the metric in terms of $y_1,\ldots,y_n$, which coincides with (17). □

Note the following obvious property of the B^n model: any rotation of the model about the point $O: y_1=\ldots=y_n=0$ (the centre of the ball B^n) is a usual Euclidean rotation about O.

Proposition 2.10. *The Riemannian metric of the model H^n is of the form*

$$ds^2=y_n^{-2}\sum_1^n dy_i^2. \tag{19}$$

Proof. We use the inversion α with respect to the hypersphere $S\subset E^n$ with centre $s=(0,\ldots,0,-1)$ and radius $\sqrt{2}$, applying it to the model B^n (as defined by formulae (15) and (17)) (Fig. 11). Under this transformation, the ball B^n goes into the half-space $y_n>0$. Letting $\alpha: y\mapsto z$, one can write

$$z_i=\frac{2y_i}{\rho^2}\quad (i=1,\ldots,n-1),\quad z_n+1=\frac{2(y_n+1)}{\rho^2},$$

where $\rho^2=y^2+\ldots+y_{n-1}^2+(y_n+1)^2$. The metric (17) then goes into the metric

$$ds^2 = z_n^{-2} \sum_1^n dz_i^2$$

defined on the half-space $z_n > 0$. □

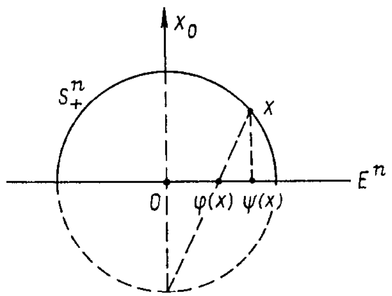


Fig. 10

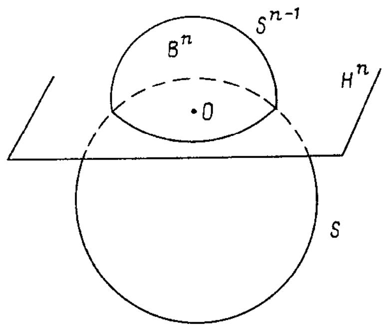


Fig. 11

For the reader's convenience we now collect the formulae related to the B^n and H^n models into a single table:

Model	Domain in E^n	Metric	Volume element
B^n	$\sum_1^n y_i^2 < 1$	$ds^2 = 4(1-\lvert y\rvert^2)^{-2} \sum_1^n dy_i^2$	$dv = 2^n(1-\lvert y\rvert^2)^{-n} dy_1 \dots dy_n$
H^n	$y_n > 0$	$ds^2 = y_n^{-2} \sum_1^n dy_i^2$	$dv = y_n^{-n} dy_1 \dots dy_n$

We now present some other facts concerning conformal Euclidean models of the Lobachevskij space $Л^n$. In the statements of Properties 1 and 2 below, X denotes any of the models B^n or H^n.

(1) A plane in the model X is a non-empty intersection of X with a sphere in C^n (i.e. either with a Euclidean sphere or a Euclidean plane) orthogonal to ∂X. This follows from the characterization of planes in the model S^n_+ given in Remark 2 to Theorem 2.5 and from the fact the isomorphism between the models S^n_+ and B^n is established by the mapping φ. In particular, lines in the model B^n are either diameters of the ball B^n or Euclidean circular arcs orthogonal to ∂B^n. Lines in H^n are either Euclidean semicircles or half-lines orthogonal to ∂H^n.

(2) Let Γ be a hyperplane in the model X and Γ^* the corresponding Euclidean sphere or hyperplane (so that Γ is $\Gamma^* \cap X$). Let g be the Euclidean inversion with respect to Γ^* if Γ^* is a sphere, or the Euclidean reflection in Γ^* if Γ^* is a hyperplane. Then in the model X one has $gX = X$ and $g|_X$ is the reflection in Γ.

Since $Л$-reflections in hyperplanes generate the entire group Isom $Л^n$, any motion of the model x can be described as a *composition of finitely many inversions with respect to Euclidean hyperspheres orthogonal to* ∂X.

There is a simple formula for *distances* in each of the conformal Euclidean models. First, note that the "cross-ratio" of any four points u, v, p, q in $\hat{E}^n$

$$[[u, v; p, q]] = \frac{|up|}{|uq|} : \frac{|vp|}{|vq|}, \tag{20}$$

(where $|up|$ denotes the Euclidean distance between u and p) is invariant under inversions and reflections in $\hat{E}^n$ (see Berger [1978], Vol. 2, Sect. 18.10.7).

Proposition 2.11. *In the conformal Euclidean model (i.e. either B^n or H^n) the $Л$-distance between points u and v is given by the formula*

$$\rho(u, v) = |\ln[[u, v; p, q]]|, \tag{21}$$

where p and q are boundary points of the $Л$-line uv.

Proof. Let u and v be two arbitrary points belonging to the hemisphere S^n_+ used in the proof of Theorem 2.5 (Fig. 12), and let p and q be boundary points of the $Л$-line uv. Let $u' = \psi(u)$, $v' = \psi(v)$. According to Proposition 1.7, $\rho(u, v) = \frac{1}{2}|\ln[u, v; p, q]|$. One can easily check (see Fig. 12) that

$$|[u', v', p, q]| = [[u, v; p, q]]^2,$$

whence

$$\rho(u, v) = |\ln[[u, v; p, q]]|.$$

As we have already mentioned, the stereographic projection φ taking S^n_+ into B^n is a restriction to S^n_+ of an inversion in the space E^{n+1}. But an inversion does not change the "cross-ratio" (21). This implies formula (21) for the B^n model. Since H^n is obtained from B^n by an inversion, the same formula holds for H^n as well. □

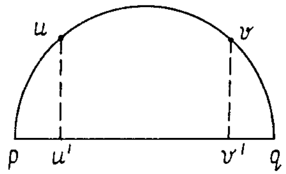


Fig. 12

In the H^n model a special case should be noted, namely when the points u and v lie on the same Euclidean perpendicular to ∂H^n. Then one of the points p and q (say q) is ∞, and formula (21) takes the form

$$\rho(u,v) = \left|\ln \frac{|up|}{|vp|}\right|.$$

2.4. Complex Structure of the Lobachevskij Plane. Consider the Poincaré model H^2 of the plane $Л^2$. We shall interpret H^2 as the upper half-plane $\operatorname{Im} z > 0$ of the complex plane. It is well known that $\operatorname{Hol} H^2$ (the group of all holomorphic diffeomorphisms of H^2) consists of all transformations of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, $d \in \mathbb{R}$, $ad - bc > 0$ (Shabat [1985], Sect. 11). For each pair of tangent directions to H^2 (a tangent direction is determined by a point $p \in H^2$ and a vector $\xi \in T_p(H^2)$ defined up to a positive scalar multiple) there exists a unique mapping $g \in \operatorname{Hol} H^2$ taking the first direction into the second.

This implies that any antiholomorphic diffeomorphism of H^2 is of the form

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d},$$

where a, b, c, $d \in \mathbb{R}$, $ad - bc < 0$.

Theorem 2.12. *The group of motions of the model H^2 coincides with the group of all holomorphic and antiholomorphic diffeomorphisms of the half-plane H^2.*

Proof. Denote by $\operatorname{Isom}_+ H^2$ the group of all proper (i.e. preserving the orientation) isometries of the model H^2. Since each isometry is a conformal mapping, one has $\operatorname{Isom}_+ H^2 \subset \operatorname{Hol} H^2$. But the group $\operatorname{Isom}_+ H^2$ is transitive on the set of tangent directions. Therefore $\operatorname{Isom}_+ H^2 = \operatorname{Hol} H^2$. It only remains to note that the mapping $z \to 1/\bar{z}$ is a motion of H^2. □

Theorem 2.12 makes it possible to describe the plane $Л^2$ as a Riemann surface (i.e., a one-dimensional complex manifold) which is conformally equivalent to a half-plane. The corresponding group of motions consists of all its holomorphic and antiholomorphic diffeomorphisms.

Two other two-dimensional spaces of constant curvature (S^2 and E^2) can also be considered as Riemann surfaces whose groups of motions are realized as sets of holomorphic and antiholomorphic diffeomorphisms. However, unlike $Л^2$, for both of them the group Isom_+ is strictly less than the corresponding Hol group. For example, the group $\operatorname{Hol} E^2$ consists of diffeomorphisms of the form $z \to \alpha z + \beta$, where α and β are any complex numbers such that $\alpha \neq 0$, while the group $\operatorname{Isom}_+ E^2$ is defined by the additional condition $|\alpha| = 1$.

Note that by the Poincaré-Koebe uniformization theorem the Riemann surfaces $(Л^2, S^2, E^2)$ are the only possible conformally non-equivalent simply-connected Riemann surfaces.

Remark. The absolute of the model H^2 is the one-dimensional manifold $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with the induced transformation group of the form

$$x \mapsto \frac{ax+b}{cx+d} \quad (a,b,c,d \in \mathbb{R}, \quad ad - bc \neq 0).$$

This implies that the absolute is isomorphic to the projective line P^1. On the other hand, the absolute of the conformal model for $Л^2$ is isomorphic to the absolute of the projective model (Proposition 2.7), i.e. to the one-dimensional conformal space C^1. Thus the *spaces C^1 and P^1 are isomorphic* (as homogeneous spaces).

§ 3. Matrix Models of the Spaces $\mathbf{Л}^2$ and $\mathbf{Л}^3$

Models of two- and three-dimensional Lobachevskij spaces are especially important due to their numerous applications in algebra, complex analysis, differential equations, number theory, etc. In particular, for the problems related to quadratic forms, the *matrix* models are useful.

3.1. Matrix Model of the Space $\mathbf{Л}^2$. Let $S^*L_2(\mathbb{R})$ be the group of all real 2×2-matrices with determinant ± 1, and M_2 the vector space of all real symmetric 2×2-matrices. A representation of the group $S^*L_2(\mathbb{R})$ can be defined in M_2 by associating with each $A \in S^*L_2(\mathbb{R})$ the linear transformation $\varphi(A)\colon M_2 \to M_2$ by the formula

$$\varphi(A)V = AVA^T,$$

where

$$V = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \in M_2.$$

In fact, $\varphi(A)$ is the transformation law for the quadratic form

$$px^2 + 2qxy + ry^2$$

when the variables x and y undergo the linear transformation defined by the matrix A.

Obviously, any $\varphi(A)$ preserves the form

$$\begin{vmatrix} p & q \\ q & r \end{vmatrix} = pr - q^2. \tag{22}$$

Introducing (instead of p and q) new coordinates s and t into M_2 such that $p = s + t$, $r = s - t$, one obtains the following form for the right hand side of (22):

$$s^2 - t^2 - q^2. \tag{23}$$

Thus we have constructed a representation of the group $S^*L_2(\mathbb{R})$ in the vector space M_2 expressed in the coordinates s, t, q and preserving the form (23), i.e.

a representation by matrices from $O_{2,1}$. Note that for some A the determinant of $\varphi(A)$ is equal to -1, e.g.

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This implies that $\varphi(S^*L_2(\mathbb{R}))$ consists (like the group $S^*L_2(\mathbb{R})$ itself) of two components. Both of them belong to $O'_{2,1}$, since they preserve the cone of positive definite matrices.

Since the dimension of the group $S^*L_2(\mathbb{R})$ coincides with that of $O'_{2,1}$ and the kernel of the mapping $A \mapsto \varphi(A)$ consists of the matrices E and $-E$, we conclude that $A \mapsto \varphi(A)$ is an isomorphism of the group $S^*L_2(\mathbb{R})/\{E, -E\}$ onto the group $O'_{2,1}$.

Consider the orbit of the point

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2$$

(for this point one has $p = r = 1$, $q = 0$ or $s = 1$, $t = 0$, $q = 0$) with respect to the group $\varphi(S^*L_2(\mathbb{R}))$, i.e. with respect to the group $O'_{2,1}$. It is the "upper" sheet of the hyperboloid

$$s^2 - t^2 - q^2 = 1,$$

which provides a model for $Л^2$. We can therefore conclude that the homogeneous space of positive definite real symmetric 2×2-matrices with determinant 1 considered with the action of the group $\varphi(S^*L_2(\mathbb{R}))$ is isomorphic to the Lobachevskij plane $Л^2$.

Let us also find the stabilizer of the point V_0. The quadratic form corresponding to V_0 is $x^2 + y^2$. Therefore $\varphi(A)V_0 = V_0$ if and only if $A \in O_2$. Taking into account that both of the matrices E and $-E$ belong to O_2, one gets the following result: the space $Л^2$ is isomorphic to $S^*L_2(\mathbb{R})/O_2$.

3.2. Matrix Model of the Space $Л^3$. This model is constructed in the same way as the matrix model for $Л^2$. Denote by T_2 the set of all Hermitian 2×2-matrices, i.e. matrices of the form

$$V = \begin{pmatrix} p & q \\ \bar{q} & r \end{pmatrix},$$

where $p, r \in \mathbb{R}$, $q \in \mathbb{C}$. The elements of T^2 can be interpreted as binary Hermitian forms

$$px\bar{x} + qx\bar{y} + \bar{q}y\bar{x} + ry\bar{y}.$$

The formula

$$\psi(A)V = AV\overline{A}^T,$$

where $A \in SL_2(\mathbb{C})$, $V \in T^2$, defines a representation ψ of the group $SL_2(\mathbb{C})$ in the space T^2. All transformations $\psi(A)$ as well as the complex conjugation $V \mapsto \overline{V}$ preserve the form

$$\begin{vmatrix} p & q \\ \bar{q} & r \end{vmatrix} = pr - q\bar{q}. \tag{24}$$

Introducing the (real) coordinates s, t, u, v into T^2, where

$$p = s + t, \quad r = s - t, \quad q = u + iv,$$

one can write the form (24) as

$$s^2 - t^2 - u^2 - v^2.$$

Now, in complete analogy with the preceding section, one obtains the following result.

The homogeneous space of positive definite Hermitian 2×2-matrices with determinant 1, considered with the action of the group $\psi(SL_2(\mathbb{C}))$ extended by the operation of complex conjugation, is isomorphic to the Lobachevskij space $Л^3$. The group $\psi(SL_2(\mathbb{C}))$ then corresponds to the group $\mathrm{Isom}_+ Л^3$ of proper motions of the space $Л^3$.

Consider, as in the preceding case, the point

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T_2.$$

The corresponding Hermitian form is $x\bar{x}+y\bar{y}$, and consequently $\psi(A)(V_0) = V_0$ if and only if $A \in U_2$. Therefore the space $(Л^3, \mathrm{Isom}_+ Л^3)$ is isomorphic to $SL_2(\mathbb{C})/SU_2$.

The same result can be obtained from different considerations using the Poincaré model H^3. Here is a brief outline of the argument.

The boundary of the half-space H^3 is the plane E^2 defined by the equation $y_3 = 0$. Consider it as the complex plane $z = y_1 + iy_2$. There is a unique isomorphism φ between the groups $\mathrm{Isom}\, H^3$ and $\mathrm{Conf}\, \hat{E}^2$ (roughly speaking, φ is the natural isomorphism between $G(\mathrm{int}\, Q)$ and $G(Q)$, see Sect. 2.1). Thus there is an isomorphism $\varphi\colon \mathrm{Isom}_+ H^3 \simeq SL_2(\mathbb{C})/\{E, -E\}$. One can show that under this homomorphism the stabilizer of the point $(0, 0, 1) \in H^3$ corresponds to the group $SU_2/\{E, -E\}$.

Chapter 3
Plane Geometry

This chapter considers (from the same point of view whenever possible) geometric properties of *two-dimensional* spaces of constant curvature, i.e. the sphere S^2, the Euclidean plane E^2, and the Lobachevskij plane $Л^2$. For the realizations of S^2 and E^2 we take the vector model and the affine model, respectively. For $Л^2$ four models will be used. First, we use the *vector model*, i.e. the realization of $Л^2$ in the form of the surface

$$-x_0^2 + x_1^2 + x_2^2 = -1, \quad x_0 > 0 \tag{1}$$

in the pseudo-Euclidean space $\mathbb{R}^{2,1}$ with the pseudo-Riemannian metric $-dx_0^2 + dx_1^2 + dx_2^2$. The three other models of $Л^2$ are:

1. *The Poincaré model*

$$H^2 = \{(x, y) \mid y > 0\}$$

with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \tag{2}$$

which below, for brevity sake, is referred to as the "H^2 model".

2. *The conformal model in the unit disc*

$$B^2 = \{(x, y) \mid x^2 + y^2 < 1\}$$

with the metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2} \tag{3}$$

called below the "B^2 model".

3. *The Klein model,* i.e. the unit disc with the metric

$$ds^2 = \frac{(1 - y^2)\,dx^2 + 2xydxdy + (1 - x^2)dy^2}{(1 - x^2 - y^2)^2}, \tag{4}$$

called the "K^2 model".

A description of motions and lines for each of these models has been given in Chap. 2. We reiterate that the H^2 and B^2 models are considered as subsets of the conformal plane $\widehat{E}^2 = E^2 \cap \{\infty\}$ and the K^2 model as a subset of the projective plane PE^2.

Most of our attention will be directed to the Lobachevskij plane, since it has a "richer" geometry compared to that of the sphere or the Euclidean plane and is (of all the three geometries S^2, E^2, and $Л^2$) the least known among the general audience.

The isomorphism between the H^2 and B^2 models is established by the mapping

$$z \mapsto \frac{z - i}{z + i},$$

which takes the upper half-plane onto the unit disc and the metric (2) into (3). The isomorphism between B^2 and K^2 is achieved by the mapping

$$z \mapsto \frac{2z}{1 + |z|^2},$$

which takes the unit disc onto itself and the metric (3) into (4).

For each of the models H^2, B^2, and K^2 (see Sect. 1.1 and Proposition 2.7 in Chap. 2) its closure is called the *extended Lobachevskij plane* and sometimes denoted by $\overline{Л^2}$. From the topological point of view $\overline{Л^2}$ is a disc. If $Л^2$ is realized by the Poincaré model, the boundary of the disc is the real axis completed by the point ∞, while in the B^2 and K^2 models the boundary is the unit circle. As a homogeneous space, the boundary is isomorphic to C^1 or P^1, and the transformation induced by an appropriate motion of the model can take any three points lying on it into any other three points.

For a more detailed exposition (as compared to this Chapter) of the two-dimensional Lobachevskij geometry we refer the reader to Beardon [1983], Chap. 7 and the references therein. The spherical geometry is considered in more detail in Berger [1978]. For the axiomatic construction of Lobachevskij geometry see, e.g. Efimov [1978].

§ 1. Lines

1.1. Divergent and Parallel Lines on the Lobachevskij Plane. Any pair of lines on the sphere S^2 has exactly two common points. Two lines on the Euclidean plane E^2 or the Lobachevskij plane $Л^2$ may have either one common point or none. This, of course, makes a sharp difference between the geometry of S^2 on the one hand, and the geometries of E^2 and $Л^2$ on the other. At the same time, there are great differences between the geometries of E^2 and $Л^2$.

For any given line L in the space E^2 and any point p not lying on L there is a unique line L' through p that does not intersect L (i.e. is parallel to it). The position of L' relative to L is completely determined by the distance from p to L .

In the space $Л^2$ the situation is more complicated (and more interesting). Denote by $\mathcal{P}$ the pencil of lines with centre p. One can easily see (e.g. by taking the Klein model) that there are two possibilities:

1. L' does not intersect L, and all lines in $\mathcal{P}$ that are close to L' also do not intersect L (Fig. 13). In that case the lines L and L' are said to be *divergent*[1]. Two divergent lines have a unique common perpendicular (see below) the length of which evidently characterizes the relative position of L and L'.

[1] The terms *ultra-parallel* and *disjoint* are also in use.

2. L' does not intersect L but there exist lines in $\mathcal{P}$ arbitrarily close to L that do intersect L (Fig. 14). In this case L and L' are said to be *parallel* (in the sense of Lobachevskij). If the lines L and L' are parallel, the *direction of parallelism* can be defined on each of them (shown in Fig. 14 by the arrow).

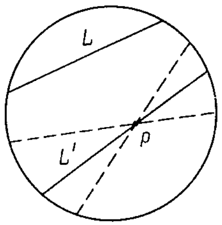


Fig. 13

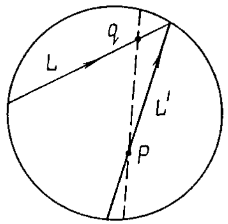


Fig. 14

The direction of parallelism on L is the direction characterized by the following condition: if a point $q \in L$ moves along L in this direction, then the line pq approaches L'. The direction of parallelism on L' is defined in the same way (with L and L' changing places). Thus "to be parallel" is a relation between *oriented (directed)* lines. Making a convention that each oriented line is, by definition, parallel to itself, we can now say that parallelism is an *equivalence relation.*

Figure 15 shows parallel lines in the Poincaré model, and Fig. 16 in the B^2 model. Another definition of parallel lines can be given in terms of *points at infinity.* Let L be a line. The two points of the absolute that are the limit points of L are said to be its points at infinity. One can easily see that the lines L and L' are parallel if and only if they have a common point at infinity.

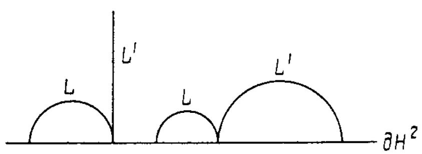


Fig. 15

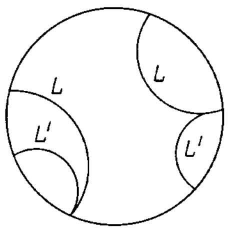


Fig. 16

The *angle* between two parallel lines L and L' is, by definition, equal to zero. This definition is motivated by the fact that if a point $p \in L'$ is fixed and a point $q \in L$ moves to infinity in the direction of parallelism, then the angle φ between the lines L and pq tends to zero. (See Fig. 17, where the situation is shown in the B^2 model).

Proposition 1.1. *If a line L_1 is parallel to L_1', L_2 is parallel to L_2', and $L_1 \neq L_1'$, $L_2 \neq L_2'$, then there is a motion taking L_1 into L_2, and L_1' into L_2'.*

Proof. Figure 18 shows the lines L_1, L_1', L_2, L_2' in the B^2 model. The motion under which $z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3$ is the desired one. □

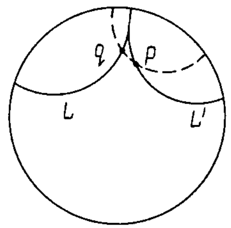


Fig. 17

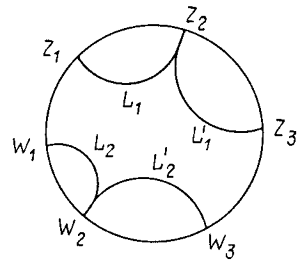


Fig. 18

Proposition 1.2. *Two lines L and L' are divergent if and only if they have a common perpendicular. Such a perpendicular is unique.*

Proof. Take the model $K^2 \subset PE^2$. The lines in PE^2 extending the chords L and L' intersect at some point u (Fig. 19). The lines uv and uw are tangent to the absolute. The chord T connecting v with w is the common Л-perpendicular to L and L' (see Chap. 2, Sect. 1.3). The common perpendicular is unique, since the extension of the chord representing it in the model passes through the poles of the lines L and L'. □

The existence of the two types of pairs of non-intersecting lines on the Lobachevskij plane (i.e. divergent and parallel) is responsible for some facts on the Lobachevskij plane that look paradoxical. For example, any angle contains an entire line inside it (Fig. 20). Another apparent paradox is that there are triangles that cannot be inscribed in a circle, etc. Some of the most important and significant of these facts will be considered below. One cannot hope to master Lobachevskij geometry without getting used to these facts and developing a proper intuition different from the Euclidean one.

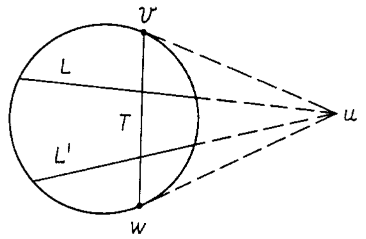


Fig. 19

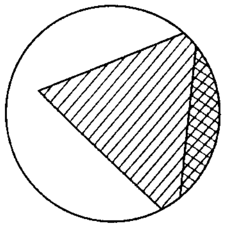


Fig. 20

1.2. Distance Between Points of Different Lines. Let L be a line and p a point on the Lobachevskij plane. We now derive a useful formula for $\rho(p, L)$ in the H^2 model.

Taking for L the positive half of the ordinate axis, one can write

$$\rho(p, L) = \rho(p, q),$$

where $q \in L, |Op| = |Oq|$ (Fig. 21). Hence

$$\rho(p, L) = \int \frac{\sqrt{dx^2 + dy^2}}{y},$$

where the integral is taken along the Euclidean circular arc connecting p with q and orthogonal to L . Letting $\theta = \angle pOq$, we have

$$\rho(p, L) = \int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \frac{|Op|d\varphi}{|Op|\sin\varphi} = -\frac{1}{2}\ln\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{2}\ln\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}. \tag{5}$$

Thus, $\rho(p, L)$ is defined merely by the angle θ. The distance increases from 0 to ∞ when θ changes from 0 to $\pi/2$.

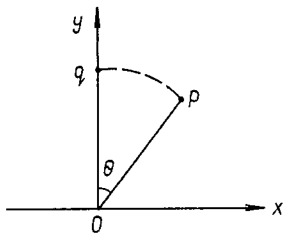


Fig. 21

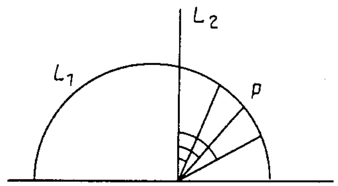


Fig. 22

Corollary 1. *If L_1 and L_2 are two intersecting lines, then the distance between the point $p \in L_1$ and the line L_2 increases monotonically from 0 to ∞ when p moves away from the intersection point of L_1 and L_2.* (See Fig. 22.)

Corollary 2. *If L_1 and L_2 are two parallel lines and $p \in L_1$, then $\rho(p, L_2)$ increases from 0 to ∞ when p moves along L_1 in the direction opposite to the direction of parallelism.* (See Fig. 23.)

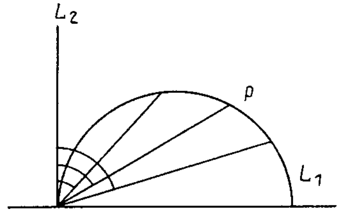


Fig. 23

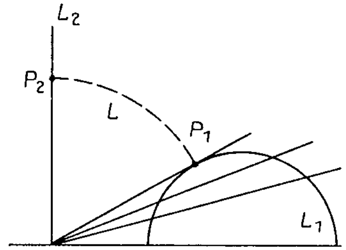


Fig. 24

Corollary 3. *Let L_1 and L_2 be two divergent lines and p_1p_2 their common perpendicular ($p_1 \in L_1, p_2 \in L_2$). Then $\rho(p, L_2)$ increases monotonically from $\rho(p_1, p_2)$ to ∞ when p moves along either of the two rays of the line L_1 issuing from p_1.* (See Fig. 24.)

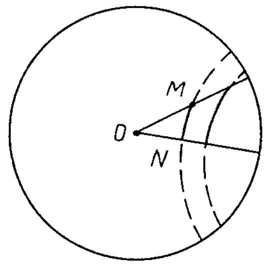


Fig. 25

Let us mention one more useful fact of Lobachevskij plane geometry. Consider an acute angle with the apex O. Take a point M on one side of the angle and drop the perpendicular MN onto the other side. It turns out that if the point M moves away from O to infinity, then the angle OMN tends to zero. This is easily seen in the B^2 model. Assuming that the apex of the angle is at the centre of the disc B^2, and the sides of the angle are two radii of B^2, the statement is evident from Fig. 25.

§ 2. Polygons

2.1. Definitions. Conditions of Convexity

Definition 2.1. A *polygon* in a two-dimensional space X of constant curvature is a simply-connected closed domain $M \subset X$ such that ∂M is the union of finitely many segments, rays or lines.

A point $p \in \partial M$ is a *vertex* of a polygon M if the intersection of ∂M with some disc with centre p consists of two radii of the disc that are not extensions of one another. The *angle* of a polygon at a vertex p is defined in the usual way.

Definition 2.2. A polygon M is said to be *convex* if it is a convex set in the space X.

There is an equivalent definition.

Definition 2.3. A *convex polygon* is an intersection of finitely many half-planes having a non-empty interior.

The equivalence of Definitions 2.2 and 2.3 is checked without difficulty.

This definition evidently implies that any angle α of a convex polygon satisfies the condition

$$0 < \alpha < \pi. \tag{6}$$

The converse statement is also true: if M is a polygon in the sense of Definition 2.1 and each of its angles satisfies condition (6), then M is convex.

Proof. In fact, a more general statement holds: any connected locally convex closed domain M is convex. Indeed, consider the shortest of the curves joining a pair of interior points x, y of M, which in the case $X = S^n$ are not antipodal. The local convexity of M implies that it is a geodesic and that a neighborhood of it is contained in M. Then it can be nothing else but the segment xy. □

Definition 2.4. A polygon is said to be *proper* if
for $X = E^2$ it is a bounded set;
for $X = S^2$ it contains no antipodal points;
for $X = \text{Л}^2$ it contains no half-planes.

Figure 26 illustrates the difference between proper and improper polygons on the Lobachevskij plane (in the Klein model). In the improper case the double shaded area shows the half-plane belonging to M.

If M is a proper polygon in Л^2, then its closure in Л^2 may include the points of the absolute $\partial\text{Л}^2$. These points will also be regarded as vertices of M and called *vertices at infinity*. The angle of a polygon at its vertex at infinity is naturally defined to be zero.

A proper polygon having n vertices (including those at infinity) is called an n-gon.

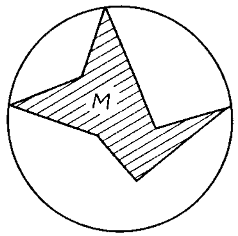


Fig. 26a. Proper polygon

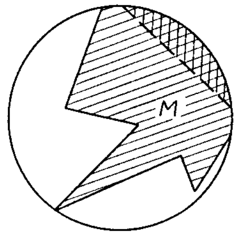


Fig. 26b. Improper polygon

Again let X be a two-dimensional space of constant curvature and suppose that $p_1, \ldots, p_k$ is a finite set of points belonging to the space X itself (if X is either S^2 or E^2) or to its closure $\overline{X}$ (if X is Л^2). The *convex hull* of this set of points is

for $X = S^2$ or $X = E^2$ the smallest convex set in X containing the points $p_1, \ldots, p_k$;

for $X = \text{Л}^2$ the smallest convex set in Л^2 whose closure in $\overline{\text{Л}^2}$ contains the points $p_1, \ldots, p_k$.

The convex hull is denoted by conv $\{p_1, \ldots, p_k\}$.

Proposition 2.5. *A convex polygon is proper if and only if it coincides with the convex hull of the set of its vertices.*

See Berger [1978], Vol. 1, Sect. 12.1.15.

2.2. Elementary Properties of Triangles.

Let X be one of the spaces S^2, E^2, Л^2. Let p_1, p_2, p_3 be three points belonging to X or $\overline{X}$ (the latter in the case $X = \text{Л}^2$) not all lying on the same line (or in the closure of the same line). The set conv $\{p_1, p_2, p_3\}$ is called the *triangle with vertices* p_1, p_2, p_3 and denoted $\langle p_1, p_2, p_3 \rangle$. Evidently, $\langle p_1, p_2, p_3 \rangle = T_1 \cap T_2 \cap T_3$, where T_i is the closed half-plane bounded by the line $p_j p_k$ (i, j, k are all different) and containing p_i.

Note that in the case of the Lobachevskij plane some or even all of the points p_1, p_2, p_3 can be at infinity, cf. Fig. 27 (in the Klein model).

The angles of the triangle $\langle p_1, p_2, p_3 \rangle$ will be denoted by $\alpha_1, \alpha_2, \alpha_3$ (where α_i is the angle at the vertex p_i).

We now present some facts from the basic course of geometry that hold not only for the Euclidean plane E^2 but also for any two-dimensional space of constant curvature.

1. *In an isosceles triangle the angles at the base are equal, and the altitude drawn to the base coincides with the median and bisector issuing from the same vertex.*

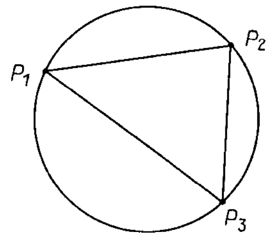


Fig. 27

2. *Congruence criteria*[2] *for (ordinary) triangles:*
(SAS) *"by two sides and the included angle";*
(ASA) *"by a side and the two angles adjacent to it";*
(SSS) *"by the three sides".*

Below we establish one more congruence criterion for triangles which holds only in the spaces S^2 and $Л^2$:
(AAA) *"by the three angles".*

There is no such criterion in Euclidean geometry, since triangles there may be similar but non-congruent.

3. *In any triangle the greatest angle lies opposite the greatest side.*

A well-known proof of this proposition invokes the theorem about an exterior angle of a triangle, which does not hold for S^2. However, a weaker assumption is sufficient, namely

$$\pi - \alpha_1 > \alpha_2 - \alpha_3. \tag{7}$$

This means that an exterior angle is greater than the difference of the two interior angles not adjacent to it. The desired statement is now easily deduced by applying (7) to the triangle shown by braces in Fig. 28. In the case of the sphere, inequality (7) will be proved in the next section.

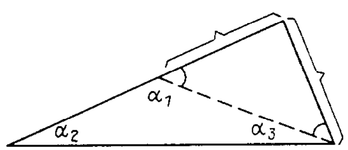


Fig. 28

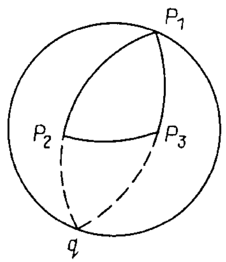


Fig. 29

[2] As usual, two figures are said to be congruent if they can be taken into one another by a motion.

4. *The length of any side of a triangle is less than the total length of its other two sides.*

Note that a corollary to this fact is the following statement: *the total length of all three sides of a spherical triangle is less than* 2π. Indeed, for any triangle $\langle p_1, p_2, p_3\rangle$ on the sphere S^2 we have

$$\rho(p_1,p_2)+\rho(p_2,p_3)+\rho(p_3,p_1) < \rho(p_1,p_2)+\rho(p_2,q)+\rho(q,p_3)+\rho(p_3,p_1) = 2\pi,$$

where q is the point of the sphere antipodal to p_1 (Fig. 29).

2.3. Polar Triangles on the Sphere. To each triangle $\triangle$ on a sphere there corresponds a triangle $\triangle'$ *polar* to $\triangle$, which is defined as follows. If $\triangle = \langle p_1, p_2, p_3\rangle$, then $\triangle'$ is the intersection of three hemispheres S_1, S_2, S_3, where S_i $(i = 1,2,3)$ is the hemisphere bounded by the plane passing through O perpendicular to the vector $\overline{Op_i}$, and containing p_i (Fig. 30). Denoting the vertices of the triangle $\triangle'$ by p'_1, p'_2, p'_3 and setting $x_i = \overline{Op_i}$, $x'_i = \overline{Op'_i}$, one can define $\triangle'$ by the following set of conditions:

$$(x_i, x'_i) > 0, \quad (x_i, x'_j) = 0, \quad \text{where} \quad i,j = 1,2,3, \quad i \neq j.$$

Hence it follows that if $\triangle'$ is polar to $\triangle$, then $\triangle$ is polar to $\triangle'$, i.e. the relation of polarity between triangles on a sphere is symmetric.

Proposition 2.6. *The lengths a'_1, a'_2, a'_3 of the sides of the triangle $\triangle'$ are related to the angles $\alpha_1, \alpha_2, \alpha_3$ of the triangle $\triangle$ by the following formulae:*

$$a'_i + \alpha_i = \pi \quad (i = 1,2,3),$$

Proof. The side a'_i corresponds to the angle between the vectors $\overline{Op'_j}$, and $\overline{Op'_k}$ (where i,j,k are all different), which is equal to the difference between π and the angle between the planes (O, p_i, p_j) and (O, p_i, p_k) (orthogonal to these vectors). Hence $a'_i = \pi - \alpha$. □

Corollary. *Inequality* (7) *holds for the angles $\alpha_1, \alpha_2, \alpha_3$ of any spherical triangle $\triangle$.*

Proof. Write inequality $a'_1 + a'_2 > a'_3$ for the polar triangle $\triangle'$. Then one gets (7) by replacing each of the numbers a'_i $(i = 1,2,3)$ by $\pi - \alpha_i$. □

2.4. The Sum of the Angles in a Triangle. Again let X be a two-dimensional space of constant curvature. How can one determine the sign of its curvature? One way is to take any triangle and compare the sum of its angles with π.

Theorem 2.7. *The sum of the angles of any triangle on the sphere is greater than π, on the Euclidean plane it is equal to π, and on the Lobachevskij plane it is less than π.*

Proof. The cases to be considered are those of the sphere and the Lobachevskij plane.

For S^2 the statement of the theorem follows from Proposition 2.6 and the fact that the sum of the sides of a spherical triangle is less than 2π (see the end of Sect. 2.2).

In the case of $Л^2$ let us use the conformal B^2 model. One can assume that one of the vertices of the triangle is the centre of the disc B^2, so the two sides meeting at this point are actually parts of two radii. The rest is evident from Fig. 31. □

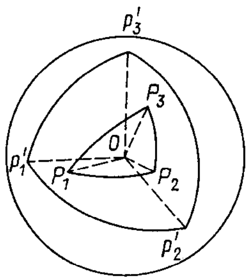


Fig. 30

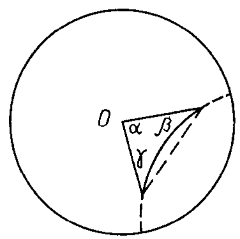


Fig. 31

2.5. Existence of a Convex Polygon with Given Angles. The theorem on the sum of the angles in a triangle implies that the sum of the angles in any convex n-gon on a Lobachevskij plane is less than $(n-2)\pi$, on a Euclidean plane it is equal to this value, and on a sphere it is greater. This is easily proved by cutting the n-gon into $n-2$ triangles with diagonals issuing from the same vertex.

Theorem 2.8. *For any numbers $\alpha_1, \dots, \alpha_n$ satisfying the conditions*

$$0 < \alpha_i < \pi \quad (i = 1, \dots, n), \tag{8}$$

$$\alpha_1 + \ldots + \alpha_n \le (n-2)\pi, \tag{9}$$

there exists a (bounded) convex n-gon with the angles $\alpha_1, \dots, \alpha_n$ in $Л^2$ or E^2.

Proof. Denote the sum $\alpha_1 + \cdots + \alpha_n$ by σ. The following two cases are possible:

$$\sigma < (n-2)\pi; \tag{10}$$

$$\sigma = (n-2)\pi; \tag{11}$$

Consider the case (10) first. Then, naturally, the desired polygon M is to be constructed in $Л^2$. We shall look for M in the class of polygons admitting an inscribed circle.

Fix a positive number d and construct for each angle α_i the quadrilateral Q_i shown in Fig. 32. If we can choose d in such a way that

$$\sum_1^n \varphi_i = \pi, \tag{12}$$

the problem is solved: one can simply lay the quadrilaterals $Q_1, \dots, Q_n$ one beside the other, successively joining them along the sides equal to d, the resulting n-gon being the desired one.

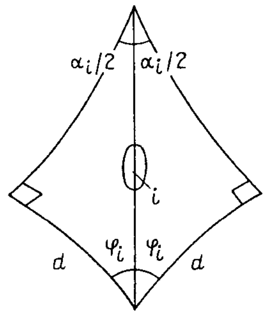


Fig. 32

For small d the difference between φ_i and $\frac{\pi}{2} - \frac{\alpha_i}{2}$ is also small, so the difference between $\sum_1^n \varphi_i$ and

$$\sum_1^n \left(\frac{\pi}{2} - \frac{\alpha_i}{2}\right) = \frac{1}{2}\left(n\pi - \sum_1^n \alpha_i\right)$$

is small as well. By condition (10), the last value is greater than π. On the other hand, each of the angles φ_i tends to zero as $d \to \infty$ (see the end of Sect. 1), so $\sum \varphi_i$ also tends to zero. Since the function $\sum \varphi_i$ (which depends on d) is continuous and among the values it assumes are those greater than π as well as those less than π, there is an intermediate value of d for which $\sum \varphi_i$ is equal to π.

In the case (11) one looks for the desired polygon M in E^2. Then the construction we have just described yields (12) for all values of d, thus solving the problem. □

Remark 1. For the Lobachevskij plane, condition (8) can be replaced by a more general condition:

$$0 \le \alpha_i < \pi \quad (i = 1, \dots, n), \tag{8'}$$

The same argument as above proves that there is a convex n-gon in $Л^2$ with the angles $\alpha_1, \dots, \alpha_n$ (provided, of course, that $\sigma < (n-2)\pi$). The vertices of the polygon for which $\alpha_i = 0$ are at infinity.

Remark 2. One can show that the n-gon with n given angles $\alpha_1, \dots, \alpha_n$ depends, up to a motion, on $n-3$ parameters for $Л^2$, and on $n-2$ parameters for E^2.

Corollary to Theorem 2.8. *For any three angles with the sum less than π there is a triangle on the Lobachevskij plane having precisely these angles.*

As we shall see below, such a triangle is unique up to a motion of the plane.

One can show that for the existence of a spherical triangle with the angles α_1, α_2, α_3 $(0 < \alpha_i < \pi,\ i = 1, 2, 3)$ the following conditions are necessary and sufficient:

$$\alpha_1 + \alpha_2 + \alpha_3 > \pi,$$
$$-\alpha_1 + \alpha_2 + \alpha_3 < \pi,$$
$$\alpha_1 - \alpha_2 + \alpha_3 < \pi,$$
$$\alpha_1 + \alpha_2 - \alpha_3 < \pi.$$

(For necessity see (7).) As will be shown below, such a spherical triangle is unique up to a motion.

2.6. The Angular Excess and the Area of a Polygon. Denote by $\sigma(M)$ the sum of the angles in a proper polygon M in a two-dimensional space of constant curvature.

Definition 2.9. The *angular excess* of a proper n-gon M is the number

$$\Delta(M) = \sigma(M) - (n-2)\pi.$$

In particular, the angular excess of a triangle is the difference between the sum of its angles and π.

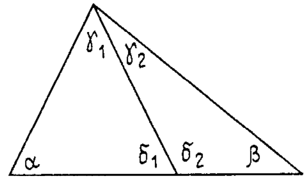
Theorem 2.10. *In each of the spaces S^2 and $Л^2$ the area $S(M)$ of a proper polygon M equals the absolute value of its angular excess:*

$$S(M) = |\Delta(M)|. \qquad (13).$$

Proof. The function $\Delta(M)$ is evidently invariant under any motion. One can show without much difficulty that $\Delta(M)$ is additive: if $M = \cup_1^n M_i$, where no two polygons M_i and M_j have common interior points, then $\Delta(M)$ equals the sum of $\Delta(M_1), \dots, \Delta(M_n)$ (see Fig. 33 for an illustration). Since $\Delta(M)$ is both invariant and additive, one has $S(M) = \lambda|\Delta(M)|$, where λ =const. To show that $\lambda = 1$ let us find the area of some specific polygon. In the case of S^2 we can take, for example, the triangle having three right angles. Its area is 1/8 of the total area of the sphere. In the case of $Л^2$ one can take the triangle shown in Fig. 34 (in the H^2 model). It has $|\Delta(M)| = \pi$ and

$$S(M) = \int\int_M \frac{dx\,dy}{y^2} = \pi.$$

For details see, e.g. Berger [1978], Sect. 18.3 and Sect. 19.5. □



$$\alpha + \beta + (\gamma_1 + \gamma_2) - \pi =$$
$$= (\alpha + \gamma_1 + \delta_1 - \pi) + (\beta + \gamma_2 + \delta_2 - \pi)$$

Fig. 33

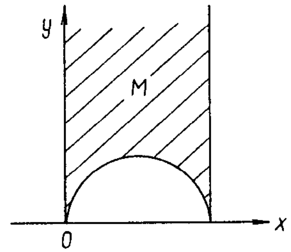


Fig. 34

Formula (13) implies, in particular, that the area of a triangle on the Lobachevskij plane does not exceed π.

Remark. Formula (13) is a particular case of the more general Gauss-Bonnet formula

$$\int_M K d\mu = \Delta(M)$$

(see Rashevskij [1967], Sect. 112) where M is any simply-connected closed domain in a two-dimensional Riemannian manifold bounded by geodesic arcs, K is the Gaussian curvature (which, in general, is not constant), and $d\mu$ is the area element.

§ 3. Metric Relations

3.1. The Cosine Law for a Triangle. Naturally, we skip the case E^2 and consider the remaining two cases, i.e. S^2 and $Л^2$ simultaneously, since the metric relations in S^2 and $Л^2$ are of a very similar form. This similarity is only one manifestation of the fact that both spaces S^2 and $Л^2$ can be defined as surfaces

$$(x, x) = \varepsilon \qquad (14)$$

in the space $\mathbb{R}^3$ with the scalar product

$$(x, y) = \varepsilon x_0 y_0 + x_1 y_1 + x_2 y_2,$$

where $\varepsilon = 1$ for S^2, and $\varepsilon = -1$ for $Л^2$ (in the last case there is an extra condition $x_0 > 0$).

Figure 35 shows a triangle ABC on the surface (14), where

$$\overline{OA} = x, \quad \overline{OB} = y, \quad \overline{OC} = z.$$

The vector

$$y' = y - \varepsilon(y, x)x$$

issuing from A lies in the plane defined by the vectors x, y, and is orthogonal to x. One can easily see that the direction of the vector y' coincides (at the point A) with the direction of the side AB. Similarly, the vector

$$z' = z - (z, x)x$$

issuing from A is tangent to the side AC. Denoting the angle between the sides AB and AC by α, we can therefore write

$$\cos\alpha = \frac{(y', z')}{\sqrt{(y', y')}\sqrt{(z', z')}}$$

(in any plane tangent to (14) the "Euclidean" formula for the cosine holds, since the scalar square of any vector in the tangent plane is defined by a positive definite quadratic form). Thus

$$\cos\alpha = \frac{(y, z) - \varepsilon(y, x)(z, x)}{\sqrt{\varepsilon(1 - (y, x)^2)}\sqrt{\varepsilon(1 - (z, x)^2)}}.$$

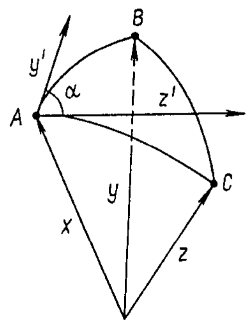


Fig. 35

In the case of S^2 we have

$$(x, y) = \cos c, \quad (x, z) = \cos b, \quad (y, z) = \cos a,$$

where a, b, c are the side lengths of the triangle ABC on the sphere. Hence

$$\cos\alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{15}$$

In the case of $Л^2$

$$(x, y) = -\cosh c, \quad (x, z) = -\cosh b, \quad (y, z) = -\cosh a$$

(see Chap. 1, Sect. 4.2), whereby

$$\cos\alpha = \frac{-\cosh a + \cosh b \cosh c}{\sinh b \sinh c}. \tag{16}$$

Formulae (15) and (16) provide the so-called Cosine Law. Let us rewrite it in the following way.

The Cosine Law for S^2

$$\cos a = \cos b \cos c + \sin b \sin c \cdot \cos\alpha. \tag{C_+}$$

The Cosine Law for $Л^2$

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cdot \cos\alpha. \tag{C_-}$$

Corollary. *If an angle α in a triangle increases, while the lengths of the sides b and c do not change, then the side a also increases.*

Proof. For S^2 and $Л^2$ the corollary follows from (C_+) and (C_-), while for E^2 it is a consequence of the usual (Euclidean) Cosine Law. □

3.2. Other Relations in a Triangle. Formulae (C_+) and (C_-) imply a number of other important formulae relating the sides and angles in a triangle. The most important are the Sine Law and the Dual Cosine Law.

The Sine Law for S^2:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}. \tag{S_+}$$

The Sine Law for $Л^2$:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}. \tag{S_-}$$

The Dual Cosine Law for S^2:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma \cdot \cos a. \tag{C'_+}$$

The Dual Cosine Law for $Л^2$:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma \cdot \cosh a. \tag{C'_-}$$

Formula (S_+) is proved very easily. It follows from (C_+) that

$$\frac{\sin^2\alpha}{\sin^2 a} = \frac{\sin^2 b \sin^2 c - (\cos a + \cos b \cos c)^2}{\sin^2 a \sin^2 b \sin^2 c},$$

and one has only to note that the expression

$$1 - \cos^2 a - \cos^2 b - \cos^2 c - 2\cos a \cos b \cos c$$

appearing in the numerator is symmetric in a, b, c. Formula (S_-) is proved in the same way.

Formulae (C'_+), and (C'_-) can also be obtained from (C_+) and (C_-) by simple transformations. Incidentally, formula (C'_+) is also a direct consequence of (C_+) and Proposition 2.6.

A special note should be made of the laws (C'_+) and (C'_-). They have no analogues in Euclidean geometry, since they imply that the sides of a triangle on the sphere or the Lobachevskij plane are uniquely determined by its angles! This yields the (AAA) congruence criterion "by the three angles" for triangles in S^2 and Л^2 (see Sect. 2.2 of this chapter).

In the particular case of a right triangle ($\gamma = \frac{\pi}{2}$), the laws (C_+) and (C_-) imply that

$$\cos c = \cos a \cos b \quad \text{for } S^2, \tag{17}$$

$$\cosh c = \cosh a \cos b \quad \text{for } \text{Л}^2. \tag{18}$$

These are non-Euclidean analogues of the Pythagoras theorem.

The laws given above make it possible to derive all the relations between the sides and the angles in a triangle. The following table provides a list of formulae which, for a right triangle, given any two of the elements α, β, a, b, c $(\gamma = \frac{\pi}{2})$, enables one to find any third element:

for S^2	for Л^2
$\cos c = \cos a \cos b$	$\cosh c = \cosh a \cosh b$
$\sin b = \sin c \sin \beta$	$\sinh b = \sinh c \sin \beta$
$\tan a = \tan c \cos \beta$	$\tanh a = \tanh c \cos \beta$
$\cos c = \cot \alpha \cot \beta$	$\cosh c = \cot \alpha \cot \beta$
$\cos \alpha = \cos a \sin \beta$	$\cos \alpha = \cosh a \sin \beta$
$\tan a = \sin b \tan \alpha$	$\tan a = \sinh b \tan \alpha$

In addition, we also give formulae expressing the area S of a triangle on the sphere or the Lobachevskij plane in terms of the sides of the triangle (a formula expressing S in terms of the angles is provided by Theorem 2.10):

$$\tan^2 \frac{S}{4} = \tan \frac{p}{2} \tan \frac{p-a}{2} \tan \frac{p-b}{2} \tan \frac{p-c}{2} \quad \text{for} \quad S^2,$$

$$\tanh^2 \frac{S}{4} = \tanh \frac{p}{2} \tanh \frac{p-a}{2} \tanh \frac{p-b}{2} \tanh \frac{p-c}{2} \quad \text{for} \quad \text{Л}^2,$$

where $p = \frac{1}{2}(a+b+c)$. For E^2 there is the well-known Heron formula

$$S^2 = p(p-a)(p-b)(p-c).$$

Remark 1. The reader has probably already noted that the relations between the sides and angles in a spherical triangle turn into their analogues on the Lobachevskij plane if one replaces a, b, c with ia, ib, ic (since $\cos iz = \cosh z$, $\sin iz = i \sinh z$). It comes as no great surprise if one recalls that the spaces S^2 and Л^2 can be defined in a similar way (see the beginning

of the present section). The fact the trigonometric formulae in Л^2 look like those in spherical trigonometry with every linear expression x replaced by ix (as if they were trigonometric formulae on a sphere with an imaginary radius) was already noted by Lobachevskij himself (Lambert mentioned an imaginary sphere back in the 18th century). Lobachevskij even considered this fact as a kind of evidence — unfortunately, it was not enough to be a conclusive one — supporting the claim that his geometry was non-contradictory.

Remark 2. All the formulae given above refer to the "canonical" metrics in S^2 and Л^2, i.e. the invariant Riemannian metrics with curvature 1 and -1, respectively. If the "canonical" metric is multiplied by any scalar $k^{-1} > 0$, one gets an invariant metric with the curvature k^2 and $-k^2$, respectively. This would incur only one change for the above formulae — all the linear expressions must be multiplied by k. For example, formula (18) turns into

$$\cosh kc = \cosh ka \cosh kb. \tag{19}$$

Suppose that $k \to 0$ and the values a, b, c depend on k in such a way that

$$\lim a = a_0, \quad \lim b = b_0, \quad \lim c = c_0$$

as $k \to 0$.

As follows from (19),

$$1 + \frac{1}{2!}(kc)^2 + \ldots = \left(1 + \frac{1}{2!}(ka)^2 + \ldots\right)\left(1 + \frac{1}{2!}(kb)^2 + \ldots\right)$$

or

$$k^2c^2 + \ldots = k^2(a^2 + b^2) + \ldots,$$

where dots replace the terms of degree greater than 2 in k. Dividing the last equality by k^2 and taking the limit as $k \to 0$, one gets

$$c_0^2 = a_0^2 + b_0^2,$$

which is the ordinary Pythagoras theorem.

A similar argument can be applied to any formula of Lobachevskij or spherical geometry. The general conclusion is that Euclidean geometry is the limit of Lobachevskij geometry or spherical geometry as its curvature tends to zero.

3.3. The Angle of Parallelism. Consider a triangle in Л^2 with one angle equal to $\frac{\pi}{2}$ and another angle equal to 0. Up to a motion, such a triangle is uniquely defined by its side x opposite to the zero angle (Fig. 36).

In particular, the third angle is a function of x. It is denoted by $\Pi(x)$ and called the *angle of parallelism* (corresponding to the side x).

The formula $\cos \alpha = \cosh a \sin \beta$, which we have already mentioned above, implies that

$$1 = \cosh x \sin \Pi(x),$$

and a simple calculation yields

$$\Pi(x) = 2 \arctan e^{-x}.$$

The trigonometric functions of the angle of parallelism $\Pi(x)$ are expressed in terms of the hyperbolic functions of x as follows:

$$\sin \Pi(x) = \frac{1}{\cosh x}, \quad \cos \Pi(x) = \tanh x, \quad \Pi(x) = \frac{1}{\sinh x}.$$

If x increases from zero to infinity, the function $\Pi(x)$ decreases from $\frac{\pi}{2}$ to 0.

3.4. Relations in Quadrilaterals, Pentagons, and Hexagons. Figure 37 shows a quadrilateral in Л^2 with the largest possible number (three) of right angles. A set of independent relations between a_1, a_2, a_3 is provided.

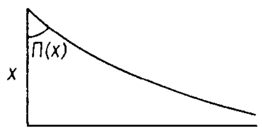
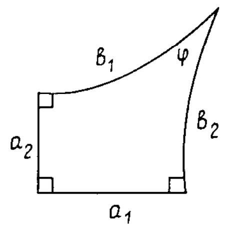


Fig. 36



$\sinh a_1 \sinh a_2 = \cos \varphi$
$\cosh a_1 = \cosh b_1 \sin \varphi$

Fig. 37

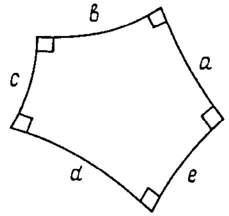
Figures 38 and 39 depict a pentagon and a hexagon all the angles of which are right angles. For a derivation of the formulae see Beardon [1983], Sect. 7.17–7.19. Note that these relations can be considered as the analytical continuations of metric relations in a triangle when one, two, or all three of its vertices go beyond the absolute. In that case the role of the cosine of the angle between the corresponding sides is played by the hyperbolic cosine of the distance between them.

Keeping the remaining notation of Fig. 37, we now consider the case when the angle between the sides a_1 and a_2 is not necessarily a right angle denoting it by ψ. Then

$$\cos \varphi = \sinh a_1 \sinh a_2 - \cos \psi \cosh a_1 \cosh a_2.$$

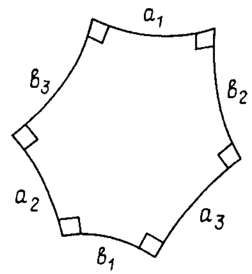
The length of the diagonal l joining the vertices of the angles φ and ψ can be found by the formula

$$\sinh^2 l = \frac{\sinh^2 a_1 + \sinh^2 a_2 + 2 \cos \varphi \sinh a_1 \sinh a_2}{\sin^2 \varphi}.$$



$\tanh a \cosh b t h c = 1$
$\sinh a \sinh b = \cosh d$

Fig. 38



$$\frac{\sinh a_1}{\sinh b_1} = \frac{\sinh a_2}{\sinh b_2} = \frac{\sinh a_3}{\sinh b_3}$$

$\cosh b_1 \sinh a_2 \sinh a_3 =$
$\cosh a_1 + \cosh a_2 \cosh a_3$

Fig. 39

3.5. The Circumference and the Area of a Disc. Taking the B^2 model, consider a Euclidean circle C about the origin with the (Euclidean) radius $r < 1$. The curve C is also a Л-circle, the radius of which is

$$\rho = \int_0^r \frac{2dx}{1 - x^2} = \ln \frac{1 + r}{1 - r}. \tag{20}$$

Denote by $L(\rho)$ the Л-length of the curve C. As the formula for the Л-metrics in B^2 shows, $L(\rho)$ equals the Euclidean length of C multiplied by $\frac{2}{1-r^2}$:

$$L(\rho) = \frac{4\pi r}{1 - r^2}.$$

It remains to express the right-hand side in terms of ρ. It follows from (20) that

$$r = \frac{e^\rho - 1}{e^\rho + 1} = \tanh \frac{\rho}{2},$$

whence $L(\rho) = 4\pi \frac{\tanh \frac{\rho}{2}}{1 - \tanh^2 \frac{\rho}{2}} = 2\pi \sinh \rho$. Thus the circumference of a circle of radius ρ on the Lobachevskij plane equals $2\pi \sinh \rho$.

The formula for the area $S(\rho)$ of the disc of radius ρ follows immediately:

$$S(\rho) = \int_0^\rho L(\sigma) d\sigma = 2\pi(\cosh \rho - 1) = 4\pi \sinh^2 \frac{\rho}{2}.$$

In the case of the sphere S^2 the circumference and the area of the disc are given by the formulae

$$L(\rho) = 2\pi \sin \rho, \quad S(\rho) = 4\pi \sin^2 \frac{\rho}{2},$$

respectively.

§ 4. Motions and Homogeneous Lines

4.1. Classification of Motions of Two-Dimensional Spaces of Constant Curvature. A proper (i.e. preserving orientation) motion g is said to be

(a) a *rotation about a point P through an angle α* if $gp = p$ and the angle between the lines pq and $p(gq)$ (where q is an arbitrary point) equals α;

(b) a *parallel displacement by d along the line L* if $gL = L$ and $\rho(p, gp) = d$ for all $p \in L$.

The existence and uniqueness of the rotation through any angle α and of the parallel displacement by any distance $d > 0$ follows from the definition of a space of constant curvature.

One can easily see that in the case of the sphere S^2 any parallel displacement along any line L is also a rotation about some point (namely, either of the poles of the line L in S^2).

A well-known fact of Euclidean geometry is that any proper motion of the sphere or the Euclidean plane is either a rotation or a parallel displacement. On the contrary, on the Lobachevskij plane there are proper motions that have neither fixed points nor invariant lines.

In order to verify this we use the Poincaré model H^2. Then any proper motion g is of the form

$$gz = \frac{az + b}{cz + d}, \tag{21}$$

where $a, b, c, d \in \mathbb{R}, \quad ad - bc = 1$. Consider the fixed points of the transformation g on the extended complex plane $\hat{E}^2 = E^2 \cup \{\infty\}$. There can be at most two such points, and if $gz = z$, then $g\bar{z} = \bar{z}$. Thus there are only three possibilities:

(1) g has two fixed points $u + iv$ and $u - iv$, where $v \neq 0$. Let us assume that $v > 0$. Then g is a Л-rotation about the point $u + iv$;

(2) g has two fixed points u_1 and u_2, where $u_1, u_2 \in \mathbb{R} \cup \{\infty\}$, $u_1 \neq u_2$. Denote by L the Л-line with the points u_1 and u_2 at infinity. Clearly, $gL = L$, so g is a parallel displacement along the Л-line L. Note that g has no other invariant lines;

(3) g has a single fixed point $u \in \mathbb{R} \cup \{\infty\}$. Replacing g by a conjugate transformation (in the group of all motions) if necessary, one can assume that $u = \infty$. Then

$$g(z) = z + b, \quad b \in \mathbb{R}. \tag{22}$$

Evidently the transformation g has neither fixed points, nor invariant lines in H^2.

A motion g having a single fixed point u (belonging to $\mathbb{R} \cup \{\infty\}$) in $\hat{E}^2$ is called a *rotation about the point u at infinity* or a *parabolic rotation*. For the motion written in the form (22) we have $u = \infty$.

The set of Euclidean half-lines of the form $\{x_0 + iy \mid y \in \mathbb{R},\ y > 0\}$, oriented in the direction of the increase of y, is a pencil of parallel lines in the H^2 model denoted by $\mathcal{P}$. Under the motion (22) this pencil goes into itself, and

(1) for any pair of lines L_1 and L_2 in $\mathcal{P}$ there exists a unique motion of the form (22) sending L_1 into L_2;

(2) if $p_1 \in L, \quad p_2 \in L_2$ and $g(p_1) = p_2$, then the segment p_1p_2 forms equal interior angles with the lines L_1 and L_2 (Fig. 40) and is called a *secant of equal slope* to the lines L_1 and L_2.

The above considerations imply the following theorem.

Theorem 4.1. *Any proper motion g of the Lobachevskij plane is a rotation about a point, or a a parallel displacement along a line, or a parabolic rotation. For any two oriented parallel lines L_1 and L_2 there is a unique parabolic rotation sending L_1 into L_2.*

Now let g be an improper motion. If g has an invariant line L, then g is a composition of two transformations: a reflection in L and a parallel displacement along L. Such a motion is called a *glide-reflection* in L. A particular case of a glide-reflection is the ordinary reflection in a line.

Theorem 4.2. *In a two-dimensional space of constant curvature any improper motion g is a glide-reflection.*

Proof. In the cases of the sphere and the Euclidean plane the statement is a well-known fact of elementary geometry. The Lobachevskij plane is best treated in this case in the Poincaré model. One has

$$g(z) = \frac{a\bar{z} + b}{c\bar{z} + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = -1$. The equation $g(z) = z$ defines two distinct points in $\mathbb{R} \cup \{\infty\}$ for $a + d \neq 0$, and either a circle or a half-line orthogonal to ∂H^2 for $a + d = 0$. In the first case g is a glide-reflection, and in the second case it is an ordinary reflection in a line. □

4.2. Characterization of Motions of the Lobachevskij Plane in the Poincaré Model in Terms of Traces of Matrices. Any proper motion g of the model H^2 is of the form (21), and is therefore defined by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

Theorem 4.3. *In the model H^2 a matrix $A \in SL_2(\mathbb{R})$, $A \neq \pm E$, corresponds to*

a rotation about a point if $|\operatorname{tr} A| < 2$;
a parallel displacement along a line if $|\operatorname{tr} A| > 2$;
a parabolic rotation if $|\operatorname{tr} A| = 2$.

Proof Any rotation g is conjugate to a rotation about the point i, which, in turn, is described by the pair of matrices

$$\pm \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

with the traces $2\cos\varphi$ and $-2\cos\varphi$. In this case one has $|\operatorname{tr} A| < 2$ (provided that $g \neq id$).

A parallel displacement along any line is conjugate to a parallel displacement along the line $x = 0$, $y > 0$, i.e. to a transformation of the form $z \mapsto a^2 z$ $(a > 0)$. The (two) corresponding matrices have the traces $a + \frac{1}{a}$ and $-(a + \frac{1}{a})$. Hence (provided that $g \neq id$) one has $|\operatorname{tr} A| > 2$.

Finally, any parabolic rotation is conjugate to a transformation of the form $z \mapsto z + b$, which is described by (two) matrices with the traces 2 and -2. □

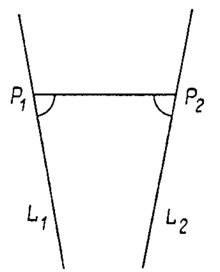


Fig. 40

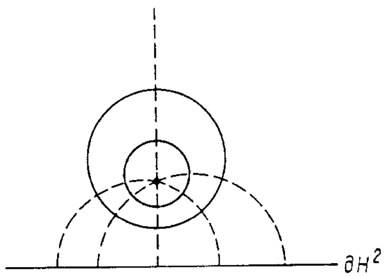


Fig. 41

4.3. One-Parameter Groups of Motions of the Lobachevskij Plane and Their Orbits. Let G be a one-parameter group of motions in Л^2. Consider any motion $g_0 \in G$ different from id. One has $g_0 g = g g_0$ for any $g \in G$. There are three possibilities.

(1) g_0 is a rotation about a point p. Then $gp = g_0 gp$, and since p is the only fixed point for g_0 we have $gp = p$. Hence G is the group of rotations about p. Its orbits are circles with centre p, while the orthogonal trajectories to the orbits form a pencil of lines with centre p. A set of lines in L^2 having a common point p is said to be an *elliptic pencil* with centre p.

The dotted lines in Fig. 41 show an elliptic pencil of lines in the H^2 model. The solid lines depict the orthogonal trajectories of the pencil, i.e. Л^2-circles with common centre p. Since the H^2 model is a conformal one, these circles are also circles in the Euclidean sense.

(2) g_0 is a parallel displacement along a line L. Then $gL = g_0 gL$, and since L is the only line invariant under g_0 one has $gL = L$. Therefore G is the group of parallel displacements along L. Its orbits are curves described by the condition that all points of such a curve are positioned at the same distance from L. They are said to be *equidistant curves* with the base line L (i.e. the curves "equally distanced" from L). The line L itself is evidently also an equidistant curve. The orthogonal trajectories to orbits are Л-lines perpendicular to L. The set of lines in Л^2 perpendicular to a given line L is called a *hyperbolic pencil.*

Figure 42 shows equidistant curves in the H^2 model. The left-hand side of the figure depicts the case when L is represented by a Euclidean ray orthogonal to L, and the drawing on the right shows the case when L is represented by a Euclidean semicircle orthogonal to ∂H^2. In the first case equidistant curves are represented (see formula (5) in Sect. 1.2) by the Euclidean rays lying in H^2 and issuing from the same point as the ray L. In the second case equidistant curves are Euclidean circular arcs having the same end-points as L. The corresponding hyperbolic pencil is shown by dotted lines.

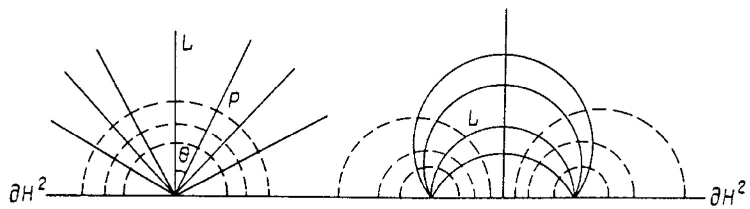


Fig. 42

(3) g_0 is a parabolic rotation. Since g_0 has neither fixed points nor invariant lines, the same holds for any $g \in G, g \neq id$. Hence each $g \in G$ is a parabolic rotation. The fact that the fixed point (at infinity) of the motion g_0 is preserved by any $g \in G$ implies that G is the group of all parabolic rotations about the common point at infinity.

The left-hand side of Fig. 43 shows orbits (solid lines) and their orthogonal trajectories (dotted lines) for the one-parameter group

$$g_t z = z + t \quad (t \in \mathbb{R})$$

of motions in the H^2 model. The orbits are represented by the Euclidean lines parallel to ∂H^2, while the orthogonal trajectories to them form a pencil of Л-lines with centre ∞. The right-hand side of the figure shows the orbits and their orthogonal trajectories for the group of rotations about the point p at infinity lying on ∂H^2.

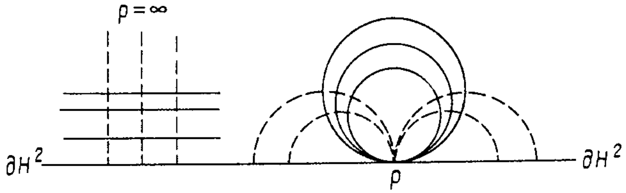


Fig. 43

The set of all oriented lines in $Л^2$ parallel to a given oriented line is said to be a *parabolic pencil.* The orthogonal trajectories of a parabolic pencil are called *horocycles*. In the Poincaré model, horocycles are represented by Euclidean lines parallel to ∂H^2, or Euclidean circles tangent to ∂H^2.

Note that a horocycle is often called a Л-circle with centre at infinity. Fixing a line L on the plane $Л^2$ and a point $p_0 \in L$, consider the circle with centre $p \in L$ passing through p_0 (Fig. 44). When the point p moves along the line L to infinity in either direction (the chosen direction is marked by the arrow in the figure) the circle will approach some limiting line Q which is the horocycle with "centre" at one of the two points at infinity of the line L.

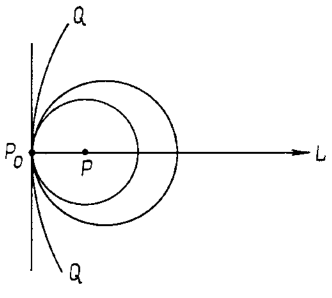


Fig. 44

For brevity, orbits of one-parameter groups will be called *homogeneous lines*. The above statements about circles, equidistant curves, and horocycles in the Poincaré model lead to a unified characterization of these lines in the conformal model as curves of the form $H^2 \cap \mathcal{L}$, where $\mathcal{L}$ is any Euclidean line or circle having a non-empty intersection with H^2. Incidentally, this also

implies that any three distinct points on the Lobachevskij plane belong to a unique homogeneous curve.

In the vector model (1) homogeneous curves are represented by plane sections of the surface (1). A section is a circle, equidistant curve, or horocycle, depending on whether the directing subspace of the plane giving the section is an elliptic, hyperbolic, or parabolic one.

4.4. The Form of the Metric in Coordinates Related to a Pencil. Let X be a two-dimensional space of constant curvature and $\mathcal{P}$ the elliptic pencil of lines with centre p_0. In the corresponding polar coordinates r, φ the metric of the space X is of the form

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad \text{for} \quad X = E^2, \tag{23}$$
$$ds^2 = dr^2 + \sin^2 r d\varphi^2 \quad \text{for} \quad X = S^2, \tag{24}$$
$$ds^2 = dr^2 + \sinh^2 r d\varphi^2 \quad \text{for} \quad X = Л^2. \tag{25}$$

These formulae follow from the fact that the circumference of the circle of radius r in these spaces is equal to 2π, $2\pi \sin r$, and $2\pi \sinh r$, respectively.

Now let $\mathcal{P}$ be a hyperbolic pencil, i.e. the set of lines perpendicular to a given line L. Let us introduce the linear coordinate t on L defined as the signed distance to a fixed point $p_0 \in L$. Labelling one of the half-planes bounded by L with +, and the other with −, we introduce the coordinates r, t in the space X as follows (Fig. 45): r is the distance from the point $p \in X$ to L taken with the appropriate sign, and t is the linear coordinate of the projection of p on L.[3]

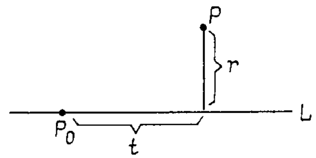


Fig. 45

For the Euclidean plane one evidently has

$$ds^2 = dr^2 + dt^2.$$

For the sphere

$$ds^2 = dr^2 + \cos^2 r dt^2, \tag{26}$$

[3] In the case when $X = S^2$ the coordinates r, t should be considered as defined on the set $S^2 \setminus T$, where T is the half-line $t = \pm\pi$. In this domain one has $-\frac{\pi}{2} < r < \frac{\pi}{2}$, $-\pi < t < \pi$.

since the circumference of the circle equidistant to L and passing through p (Fig. 46) is equal to $2\pi \cos r$.

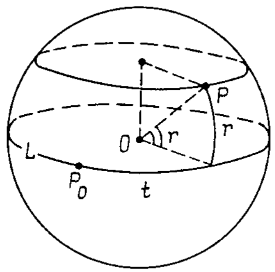


Fig. 46

By a formal substitution $s \to is$, $r \to ir$, $t \to it$ (see Remark 1 at the end of Sect. 3.2) one can express the metric on the Lobachevskij plane in the coordinates r, t:

$$ds^2 = dr^2 + \cosh^2 r dt^2. \tag{27}$$

(Incidentally, it also shows that the length of an equidistant arc with the "base" a (Fig. 47) equals $a \cosh r$, where r is the distance to the "base".)

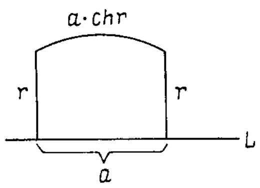


Fig. 47

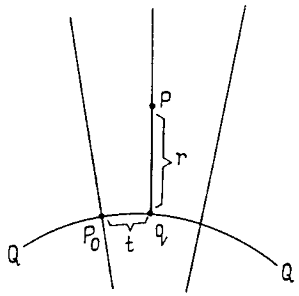


Fig. 48

Finally, let $\mathcal{P}$ be a parabolic pencil on the Lobachevskij plane $Л^2$. Choose one of the horocycles corresponding to the pencil $\mathcal{P}$ and denote this horocycle by Q. Introduce a linear coordinate t on Q equal, up to the sign, to the distance from a fixed point $p_0 \in Q$ measured along Q. The curve Q divides the entire plane into two parts. Let us label one of them with "+" and the other with "−". For any point $p \in Л^2$ its coordinates are (Fig. 48): the signed distance r

from p to Q, i.e. $r = \pm\rho(p, q)$, where q is the point at which Q intersects the line of the pencil passing through p, and the linear coordinate t of the point q on Q. Clearly,

$$ds^2 = dr^2 + \varphi(r)dt^2.$$

In order to find the function φ let us consider, in the Poincaré model H^2, the parabolic pencil with centre ∞ (Fig. 48). Take for Q the line $y = 1$. The metric in H^2 is of the form

$$ds^2 = y^{-2}(dx^2 + dy^2),$$

which implies that $dt = dx$, whence $x = t$, and $dr = \frac{dy}{y}$, whereby $y = e^r$. Thus

$$ds^2 = dr^2 + e^{-2r}dt^2. \tag{28}$$

Since the result is invariant, formula (28) clearly holds for any parabolic pencil $\mathcal{P}$.

Chapter 4
Planes, Spheres, Horospheres and Equidistant Surfaces

§ 1. Relative Position of Planes

1.1. Pairs of Subspaces of a Euclidean Vector Space. Let V be a Euclidean vector space, and $O(V)$ the group of its orthogonal transformations. Any subspace U of the space V is the set of fixed points of an involution $\sigma_U \in O(V)$, the orthogonal reflection in U. The correspondence $U \leftrightarrow \sigma_U$ between subspaces and involutions is bijective.

Let $\sigma, \tau \in O(V)$ be two involutions and $\alpha = \tau\sigma$. Then the following relations are satisfied:

$$\sigma^2 = id, \quad \sigma\alpha\sigma^{-1} = \alpha^{-1}.$$

The mapping $R: a \to \alpha$, $b \to \sigma$ defines an orthogonal representation of the *infinite dihedral group*

$$\mathcal{D} = \langle a, b \mid b^2 = e, bab^{-1} = a^{-1} \rangle.$$

Conversely, any such representation R defines a pair of involutions $\sigma = R(b)$, $\tau = R(ab)$. Thus the classification of pairs of subspaces of the space V is equivalent to the classification of orthogonal representations of the group $\mathcal{D}$ in the space V.

Any orthogonal representation (of any group) is decomposed into an orthogonal sum of irreducible representations. Any irreducible orthogonal representation of the group $\mathcal{D}$ is either one-dimensional or two-dimensional. In the latter case it is of the form

$$R_\varphi(a) = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}, \quad R_\varphi(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 0 < \varphi < \frac{\pi}{2},$$

and corresponds to a pair of lines in the Euclidean plane forming the angle φ.

An orthogonal decomposition $V = V_1 \oplus \ldots \oplus V_k$ is said to be *compatible with a subspace* U if $U = (U \cap V_1) \oplus \ldots \oplus (U \cap V_k)$. This is evidently equivalent to the fact that all the subspaces V_i are invariant under the involution σ_U. Therefore, if R is an orthogonal representation of the group $\mathcal{D}$ corresponding to a pair of subspaces $U, W \subset V$, then the orthogonal decomposition of the space V into a sum of $R(\mathcal{D})$-invariant subspaces is none other than the orthogonal decomposition compatible with the spaces U and W.

Thus for any pair of subspaces there is an orthogonal decomposition of the space V into a sum of one-dimensional and two-dimensional subspaces compatible with them. This leads to the following classification of pairs of subspaces.

Definition 1.1. Two subspaces U and W are said to be *perpendicular* to each other if either of the following equivalent conditions holds:

(1) the involutions σ_U and σ_W commute;
(2) there is an orthogonal decomposition of the space V

$$V = V_{11} \oplus V_{12} \oplus V_{21} \oplus V_{22} \text{ such that } U = V_{11} \oplus V_{12}, \quad W = V_{11} \oplus V_{21}.$$

(Note that the perpendicularity in the sense of this definition is a weaker condition than the orthogonality. Two perpendicular subspaces are orthogonal if and only if their intersection is zero.)

Definition 1.2. Two subspaces U and W are said to be *isoclinic*, and the angle φ $(0 < \varphi < \frac{\pi}{2})$ is said to be the *angle* between them, if either of the following equivalent conditions holds:

(1) the angle between any non-zero vector of one of the subspaces and the other subspace is equal to φ;
(2) the space V has an orthogonal decomposition $V = V_0 \oplus V_1 \oplus \ldots \oplus V_k$ compatible with U and W such that $U \cap V_0 = W \cap V_0 = 0$, $\dim V_i = 2$, $\dim (U \cap V_i) = \dim (W \cap V_i) = 1$, and the angle between the lines $U \cap V_i$ and $W \cap V_i$ equals φ for all $i = 1, \ldots, k$.

If $U + W = V$, then the latter condition is equivalent to the existence of a complex structure in the space V such that

(1) the operator I of the complex structure is orthogonal (and simultaneously skew-symmetric);

(2) $(U, IU) = 0$;
(3) $W = (E \cos\varphi + I \sin\varphi)U$.

Theorem 1.3. *For any two subspaces U and W of a Euclidean vector space V there is a unique orthogonal decomposition*

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_k$$

compatible with them such that

(1) *the subspaces $U_0 = U \cap V_0$ and $W_0 = W \cap V_0$ are perpendicular;*
(2) *for $i = 1, \ldots, k$ the subspaces $U_i = U \cap V_i$ and $W_i = W \cap V_i$ are isoclinic, generate the entire space V_i and form the angle φ_i;*
(3) *the angles $\varphi_1, \ldots, \varphi_k$ are different.*

The angles $\varphi_1, \ldots, \varphi_k$ are called *stationary angles* of the pair $\{U, W\}$.

The number $\dim U_i = \dim W_i$ is called the *multiplicity* of the stationary angle φ_i. The angles $\varphi_1, \ldots, \varphi_k$ are critical values of the angular function $\varphi_{U,W}: U \times W \to \mathbb{R}$ associating with each pair of non-zero vectors $u \in U$, $w \in W$ the angle between them. (Other critical values of this function are $\pi - \varphi_1, \ldots, \pi - \varphi_k$, and 0 if $U \cap W \neq 0$, or $\frac{\pi}{2}$ if $U_0, W_0 \neq U \cap W$.)

A practical way of finding stationary angles is the following. Let $\{e_1, \ldots, e_l\}$ and $\{f_1, \ldots, f_l\}$ be orthonormal bases in the subspaces U and W respectively, and $G = ((e_i, f_j))$ their Gram matrix. Then the numbers $\cos^2 \varphi_1, \ldots, \cos^2 \varphi_k$ taken with their respective multiplicities, and 1 taken with the multiplicity equal to $\dim (U \cap W)$, are non-zero eigenvalues of the symmetric matrix GG^T (or $G^T G$).

1.2. Some General Notions. We now introduce some general notions describing the relative position of a pair of planes in a space of constant curvature.

Definition 1.4. Two planes Y and Z are said to be *perpendicular* if the reflections σ_Y and σ_Z in these planes commute.

(This is a weaker condition than the orthogonality.)

Both in Euclidean space and Lobachevskij space any two perpendicular planes Y and Z intersect. Indeed, the midpoint of the interval connecting any point $z \in Z$ with the point $\sigma_Y(z)$ belongs to $Y \cap Z$. Two intersecting planes are perpendicular if and only if the tangent spaces to them at each common point are perpendicular.

Definition 1.5. Two sets Y and Z (e.g. planes) are said to be *mutually equidistant* if the distance of any point of each of these sets to the other set is constant (and equal to $\rho(Y, Z)$).

1.3. Pairs of Planes on the Sphere. The description of the relative position of a pair of planes Y, Z in the sphere S^n is reduced to the description of the relative position of their defining subspaces $\langle Y \rangle$, $\langle Z \rangle$ in the Euclidean vector space $\mathbb{R}^{n+1}$. The planes Y and Z are perpendicular if and only if the subspaces $\langle Y\rangle$ and $\langle Z\rangle$ are perpendicular, and are mutually equidistant if and only if the subspaces $\langle Y\rangle$ and $\langle Z\rangle$ are isoclinic or orthogonal. In particular, any two mutually equidistant planes the distance between which is less than $\frac{\pi}{2}$ are of the same dimension.

1.4. Pairs of Planes in the Euclidean Space. A plane $Y \subset E^n$ is defined by any of its points and the directing (i.e. the tangent) subspace U of the Euclidean vector space V associated with E^n.

Any two planes $Y, Z \subset E^n$ with the directing subspaces U, W have a common perpendicular, which is defined uniquely up to a parallel displacement along a vector from $U \cap W$. (If the planes Y and Z intersect, then their common perpendicular is defined as the segment xx, where x is any common point of Y and Z.) The length of a common perpendicular is equal to the distance $\rho(Y,Z)$ between Y and Z. The relative position of the planes Y and Z is characterized by the distance between them and the relative position of their directing subspaces U and W.

The planes Y and Z are mutually equidistant if and only if $U = W$.

1.5. Pseudo-orthogonal Transformations. In order to develop a classification of pairs of planes (as well as motions) in Lobachevskij space we need a description of pseudo-orthogonal transformations of a pseudo-Euclidean vector space V of signature $(n,1)$.

If A is a pseudo-orthogonal transformation and H the hyperbolic subspace invariant under it, then $H^\perp$ is an invariant elliptic subspace, and $V = H \oplus H^\perp$. The restriction of the transformation A to $H^\perp$ is the usual orthogonal transformation. Thus in order to describe pseudo-orthogonal transformations it is sufficient to describe their restrictions to a minimal invariant hyperbolic subspace.

Theorem 1.6. *For any pseudo-orthogonal transformation A of a pseudo-Euclidean vector space V of signature $(n,1)$, any minimal invariant hyperbolic subspace H has dimension ≤ 3. It is defined uniquely up to a pseudo-orthogonal transformation commuting with A. The restriction of A to H is given in the following table.*

dim H	Basis in H	Gram matrix of this basis	Matrix of $A\vert_H$
1	$\{h\}$	(-1)	$\pm(1)$
2	$\{u,v\}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\pm\begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \lambda > 0$
3	$\{u,e,v\}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(The "+" sign corresponds to transformations preserving connected components of the cone of time-like vectors, the "−" sign corresponds to transformations permuting them.)

Proof. It is achieved with the use of the following facts:

(1) the eigenvectors (including the imaginary ones) of the transformation A, corresponding to the eigenvalues λ and μ for $\lambda\mu \neq 1$, are orthogonal;
(2) $(\operatorname{Im}(A-E)^k, \operatorname{Ker}(A-E)^k) = 0$;
(3) the space V contains no two-dimensional totally isotropic subspaces.

1.6. Pairs of Hyperbolic Subspaces of the Space $\mathbb{R}^{n,1}$. Let V be a pseudo-Euclidean vector space of signature $(n,1)$, and $O(V)$ the group of its pseudo-orthogonal transformations. An involution $\sigma \in O(V)$ is said to be hyperbolic if the set of its fixed vectors is a hyperbolic subspace of V. An argument similar to that given in Sect. 1.1 shows that the problem of classifying pairs of hyperbolic subspaces of V is equivalent to the problem of classifying pseudo-orthogonal representations $R: \mathcal{D} \to O(V)$ of the infinite dihedral group $\mathcal{D} = \langle a,b\rangle$ which takes the elements b and ab into hyperbolic involutions.

One can easily see that a minimal hyperbolic subspace H invariant under the pseudo-orthogonal transformation $R(a)$ (see Theorem 1.6) can be chosen in such a way that it is also invariant under $R(b)$. One can then assume that the matrix of the restriction of $R(b)$ to H is of the form given in the following table.

dim H	Matrix of $R(a)\vert_H$	Matrix of $R(b)\vert_H$	$U \cap H$	$W \cap H$
1	(1)	(1)	$\langle h\rangle$	$\langle h\rangle$
2	$\begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \lambda > 0$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\langle u+v\rangle$	$\langle e^\lambda u + v\rangle$
3	$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\langle u,v\rangle$	$\langle u, e+2v\rangle$

Finding the subspaces of fixed vectors of the involutions $R(b)|_H$ and $R(ab)|_H$, we arrive at the following theorem.

Theorem 1.7. *For any two hyperbolic subspaces U and W of a pseudo-Euclidean vector space V of signature $(n,1)$ there is a hyperbolic subspace H of dimension ≤ 3, perpendicular to both of them, such that the intersections $U \cap H$ and $W \cap H$ are of the form given in the preceding table. (For the notation see the table of Theorem 1.6.) The subspace H is defined uniquely up to a pseudo-orthogonal transformation preserving both U and W.*

The perpendicularity of H to both U and W means that the decomposition $V = H \oplus H^\perp$ is compatible with these subspaces. Their intersections with the elliptic subspace $H^{\perp}$ are described by Theorem 1.3. Note that the subspace $U \cap W$ is hyperbolic for $\dim H = 1$, elliptic for $\dim H = 2$, and parabolic for $\dim H = 3$. In the second case the subspace H is the sum of eigenspaces of the operator $\sigma_W \sigma_U$, and is therefore defined uniquely.

1.7. Pairs of Planes in the Lobachevskij Space. The classification of pairs of planes in the Lobachevskij space Л^n is equivalent to the classification of pairs of hyperbolic subspaces of the space $\mathbb{R}^{n,1}$ given in the preceding section. In the context of Lobachevskij geometry, however, one can make its geometric meaning more clear, and this is what we are now about to do.

Definition 1.8. A *perpendicular secant* of a pair of planes Y and Z is a minimal plane Π perpendicular to both Y and Z.

It follows from the results of the preceding section that $\dim \Pi \leq 2$, and that the plane Π is defined uniquely up to a motion preserving both Y and Z, and if $\dim \Pi = 1$, then it is defined uniquely.

Consider all the three possible cases separately.

Condition $\dim \Pi = 0$ means that the planes Y and Z intersect.

Condition $\dim \Pi = 1$ means that the planes Y and Z have a unique common perpendicular $y_0 z_0$ $(y_0 \in Y,\ z_0 \in Z,\ y_0 \neq z_0)$. In that case the planes are said to be *divergent*. The length of the common perpendicular is equal to the length $\rho(Y, Z)$ between Y and Z. In the notation of the table given in Sect. 1.6 one has

$$\rho(Y, Z) = \frac{\lambda}{2}.$$

Condition $\dim \Pi = 2$ means that the planes Y and Z do not intersect, but have a common point p at infinity. In this case they are said to be *parallel* (*in the sense of Lobachevskij*). The plane Π intersects Y and Z in parallel lines passing through the point p.

In each of the three cases consider the subspaces

$$N_Y(\Pi) = \langle Y \rangle \cap N(\Pi) = T_y(Y) \cap N(\Pi) \quad (y \in Y \cap \Pi),$$

$$N_Z(\Pi) = \langle Z \rangle \cap N(\Pi) = T_z(Z) \cap N(\Pi) \quad (z \in Z \cap \Pi).$$

in the normal subspace $N(\Pi)$ of the plane Π (see Chap. 1, Sect. 3.2). The relative position of the planes Y and Z is characterized by the number $\dim \Pi$, i.e. by whether these planes intersect, are divergent, or parallel; by the distance $\rho(Y, Z)$ if they are divergent; and by the relative position of the subspaces $N_Y(\Pi)$ and $N_Z(\Pi)$ in the Euclidean vector space $N(\Pi)$.

1.8. Pairs of Lines in the Lobachevskij Space. A line l in the space Л^n is defined by its points p, q at infinity. Let $u, v \in \partial C^+$ be isotropic vectors representing these points. Under the condition that $(u, v) = -2$, the formula

$$x(t) = \frac{1}{2}(e^t u + e^{-t} v) \tag{1}$$

defines a line l parametrized by the arc length. The vectors u and v are defined up to a transformation of the form $u \mapsto \lambda u,\ v \mapsto \lambda^{-1} v \quad (\lambda > 0)$.

The orthogonal projection of the point x on the line l is equal to $x(\lambda)$, where

$$\lambda = \frac{1}{2} \ln \frac{(x, v)}{(x, u)},$$

and the distance $\rho(x, l)$ is found by the formula

$$\cosh \rho(x, l) = \sqrt{(x, u)(x, v)}.$$

Let us now use these terms to characterize the relative position of two lines. Let l' be another line, and u', v' the corresponding isotropic vectors. The non-negative numbers

$$\alpha = (u, u')(v, v'), \quad \beta = (u, v')(v, u'), \tag{2}$$

defined up to permutation, are invariants of the pair of lines, and satisfy the inequalities

$$2 - \sqrt{\alpha\beta} \leq \frac{\alpha + \beta}{2} \leq 2 + \sqrt{\alpha\beta}, \tag{3}$$

where equality on the left corresponds to the case of intersecting (and therefore lying in the same plane) lines, while equality on the right corresponds to the case of divergent lines lying in the same plane. The two equalities hold simultaneously if and only if the lines are parallel.

The distance ρ between the lines l and l' is found by the formula

$$2\cosh^2 \rho = \frac{\alpha + \beta}{2} + \sqrt{\alpha\beta}, \tag{4}$$

and the angle φ between them by the formula

$$2\cos^2 \varphi = \frac{\alpha + \beta}{2} - \sqrt{\alpha\beta}. \tag{5}$$

1.9. Pairs of Hyperplanes. In the vector model, any hyperplane H of the space $X = S^n$ or Л^n is of the form

$$H_e = \{x \in X \colon (x, e) = 0\}, \tag{6}$$

where $e \in \mathbb{R}^{n+1}$, $(e, e) = 1$. The vector e is defined by the hyperplane H uniquely up to the sign. Its choice is equivalent to the choice of an orientation of the hyperplane H, i.e. to specifying which of the half-spaces bounded by it is the "positive" one. We shall assume that

$$H_e^+ = \{x \in X \colon (x, e) > 0\}, \quad H_e^- = \{x \in X \colon (x, e) < 0\}. \tag{7}$$

We now describe the relative position of two oriented hyperplanes H_e and H_f in terms of the vectors e and f.

The hyperplanes H_e and H_f intersect if $|(e,f)| < 1$ (which is always true if $X = S^2$). The value of the dihedral angle $H_e^- \cap H_f^-$ can then be found by the formula

$$\cos \widehat{H_e H_f} = -(e,f). \tag{8}$$

The hyperplanes H_e and H_f are parallel if $|(e,f)| = 1$.

The hyperplanes H_e and H_f are divergent if $|(e,f)| > 1$. The distance between them can be found by the formula

$$\cosh \rho(H_e, H_f) = |(e,f)|. \tag{9}$$

In the last two cases the condition $(e,f) < 0$ means that $H_e^- \cap H_f^- = \emptyset$ or $H_e^+ \cap H_f^+ = \emptyset$ (see Fig. 49).

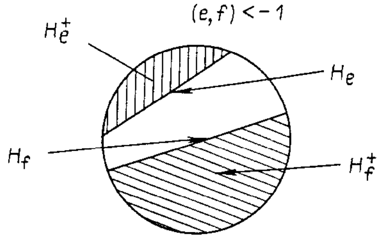


Fig. 49

Let us prove formula (9), for example. Note that in this case the subspace $\langle e,f\rangle$ is a hyperbolic one. The corresponding line in the space $Л^n$ is orthogonal to H_e and H_f. Its common points with H_e and H_f are represented by vectors proportional to the vectors

$$x = f - (e,f)e, \quad y = e - (e,f)f.$$

Hence (see Chap. 1, Sect. 4.2)

$$\cosh \rho(H_e, H_f) = \frac{|(x,y)|}{\sqrt{(x,x)(y,y)}} = |(e,f)|.$$

□

§ 2. Standard Surfaces

2.1. Definitions and Basic Facts. We will single out some subgroups in the group of motions of the space $X = S^n$ or $Л^n$. Considering the space X in its vector model, take any non-zero subspace $U \subset \mathbb{R}^{n+1}$ and denote by $G(U)$ the group of motions arising from pseudo-orthogonal transformations of the space $\mathbb{R}^{n+1}$ that are the identity on $U^\perp$ (and consequently preserve U).

Proposition 2.1. *The orbits of the group $G(U)$ are the intersections of X with linear manifolds of the form $x + U$. If $x \notin U^\perp$, then the tangent space to the orbit $G(U)x$ at the point x is*

$$U_x = \{u \in U : (u,x) = 0\} = U \cap T_x(X)$$

and the homogeneous space $(G(U)x, G(U))$ satisfies the maximum mobility axiom (see Chap. 1, Sect. 1.4) with respect to the induced Riemannian metric. (If $x \in U^\perp$, then the orbit $G(U)x$ consists of a single point.)

Proof. Evidently $G(U)x \subset x + U$. Conversely, if $u \in U$ and $y = x + u \in X$, then the linear mapping $\langle x, U^\perp\rangle \to \langle y, U^\perp\rangle$ that takes x into y and is the identity on $U^\perp$ is an isometric one. By Witt's theorem it can be extended to a pseudo-orthogonal transformation of the space $\mathbb{R}^{n+1}$. Hence $y \in G(U)x$.

Now one has

$$G(U)x = (x+U) \cap X = x + Q,$$

where $Q \subset U$ is a submanifold defined by the equation

$$(u,u) + 2(u,x) = 0 \tag{10}$$

and if $X = Л^n$ by the condition $x + u \in C^+$. If $x \notin U^\perp$, equation (10) defines a quadric with the tangent space U_x at the point 0.

In order to verify the maximum mobility axiom, note that any orthogonal transformation of the space U_x can be extended to a pseudo-orthogonal transformation of the space $\mathbb{R}^{n+1}$ that is the identity on $U_x^\perp = \langle x, U^\perp\rangle$. The last transformation defines a motion belonging to the group $G(U)$ and preserving the point x. □

Definition 2.2. The subgroups of the form $G(U)$ are said to be the *standard subgroups* of the group of motions of the space X. The orbits of the group $G(U)$ that do not lie in $U^\perp$ are called the *standard surfaces* associated with the subspace U.

The subspace U is uniquely defined by any standard surface associated with it, as the tangent subspace of its affine hull in $\mathbb{R}^{n+1}$. If the intersection of (arbitrary) standard surfaces associated with the subspaces $U_1, \dots, U_k$ is not empty and does not consist of a single point, then it is a standard surface associated with the subspace $U_1 \cap \ldots \cap U_k$.

For $x \in U$ the standard surface $(x+U) \cap X$ coincides with the plane $U \cap X$, and for $x \notin U$ it is a *standard hypersurface* (a standard surface of codimension 1) in the plane $\langle x, U\rangle \cap X$. Therefore the study of standard surfaces can be reduced to the study of standard hypersurfaces.

If $Y = U \cap X \neq \emptyset$, then the standard subgroup $G(U)$ has an intrinsic description as the group of motions preserving the plane Y and acting trivially on its normal space. In this case one of its orbits is the plane Y. If $Z =$

$U^\perp \cap X \neq \emptyset$, then $G(U)$ consists of motions that are the identity on the plane Z. In this case, from the point of view of the intrinsic geometry of the space X, those of its orbits that do not lie in Z are spheres (with centres in Z) belonging to the planes of complementary dimension orthogonal to Z.

For $X = S^n$ one always has $Y \neq \emptyset$, and provided that $U \neq \mathbb{R}^{n+1}$ we also have $Z \neq \emptyset$.

For $X = Л^n$ the subspace U may be a hyperbolic, an elliptic, or a parabolic one. In the first case $Y \neq \emptyset$, in the second $Y = \emptyset$, $Z \neq \emptyset$, and in the third $Y = Z = \emptyset$.

For $X = E^n$ one can naturally define two types of *standard subgroups*. The first consist of motions preserving some plane Y and acting trivially on its normal space, and the second consists of motions that are the identity on some plane Z. The orbits of the standard groups of the first type are planes parallel to Y, while those of the standard groups of the second type are spheres in the planes of complementary dimension orthogonal to Z. These surfaces are naturally called *standard surfaces in the Euclidean space*.

2.2. Standard Hypersurfaces. The standard hypersurfaces in the space $X = S^n$ or $Л^n$ are associated with n-dimensional subspaces of $\mathbb{R}^{n+1}$ and are obtained as the sections of the space X by hyperplanes of $\mathbb{R}^{n+1}$.

Any n-dimensional subspace U of $\mathbb{R}^{n+1}$ is defined by a non-zero vector e orthogonal to it. The standard hypersurfaces associated with U are of the form

$$H_e^c = \{x \in X : (x, e) = c\}. \tag{11}$$

The vector e is chosen up to a scalar multiple. Evidently $H_e^c = H_{\lambda e}^{\lambda c}$.

The homogeneity considerations imply that for a fixed e and different c the hypersurfaces H_e^c are mutually equidistant (see Definition 1.5). The distance between them can be measured along any curve orthogonal to them. These curves turn out to be straight lines. Indeed, the hypersurface H_e^c is orthogonal to all lines whose defining subspaces contain the vector e. There is exactly one such line passing through any point of the space X not contained in $\langle e \rangle$ (i.e. with the exception of at most two points). The set of all such lines is called the *pencil of lines* associated with the vector e and is denoted below by B_e.

Pencils of lines, and consequently standard hypersurfaces can be described in terms of the geometry of the space X.

If the intersection $\langle e \rangle \cap X$ contains at least one point x, the pencil B_e is said to be *elliptic with centre* x, and is simply the set of all lines passing through the point x. In this case the hypersurfaces H_e^c are spheres with centre x.

If the intersection $\langle e \rangle^\perp \cap X$ is not empty, it is a hypersurface H of the space X. The pencil B_e is then said to be *hyperbolic with base* H, and is the set of all lines orthogonal to H. The hypersurfaces H_e^c in this case are called *equidistant hypersurfaces with base* H. Each of them is a connected component of the set of all points lying at a fixed (but depending on c) distance from H. (The entire set is the union of H_e^c and H_e^{-c}.)

For $X = S^n$ any pencil B_e is both elliptic and hyperbolic, and each hyperplane H_e^c is both a sphere and an equidistant hypersurface.

For $X = Л^n$ the pencil B_e is elliptic if $(e, e) < 0$, and hyperbolic if $(e, e) > 0$. In the case $(e, e) = 0$, the pencil B_e is said to be *parabolic with centre at the point* $p = \langle e \rangle$ *at infinity*, and is the set of all lines passing through that point. In this case the hypersurfaces H_e^c are said to be *horospheres with centre* p.

Theorem 2.3. *Any standard hypersurface of the sphere S^n is isomorphic, as a homogeneous space, to S^{n-1}. In the Lobachevskij space $Л^n$ any sphere is isomorphic to S^{n-1}, any horosphere to E^{n-1}, and any equidistant hypersurface to $Л^{n-1}$.*

Thus any standard hypersurface, and thereby any standard surface in a space of constant curvature, is a space of constant curvature with the trivial exceptions of the circle S^1 and the double point S^0.

The fact that a horosphere in a Lobachevskij space is isomorphic to a Euclidean space is justly considered as one of the most remarkable facts of Lobachevskij geometry. It was established independently by N. I. Lobachevskij and J. Bolyai.

Proof. For a sphere with centre x the required isomorphism is established by associating with each of its points y the unit tangent vector of the segment xy at the point x, and for an equidistant hypersurface with base H by projecting it on the hyperplane H.

In the case of the horosphere $H_e^c \subset Л^n$ with centre at the point $p = \langle e \rangle$ at infinity, its isomorphism with the space E^{n-1} is established in the following way. Let $U = \langle e \rangle^\perp$, $\mathbb{R}^n = \mathbb{R}^{n+1}/\langle e \rangle$, $\mathbb{R}^{n-1} = U/\langle e \rangle$. The group $G(U)$, which acts naturally in the space $\mathbb{R}^n$, preserves the subspace $\mathbb{R}^{n-1}$ and the Euclidean metric induced on it. The horosphere H_e^c is projected diffeomorphically on the hyperplane of the space $\mathbb{R}^n$ parallel to $\mathbb{R}^{n-1}$, so this hyperplane can be considered as a vector model of the Euclidean space E^{n-1}. □

In the conformal model of the space $Л^n$, any sphere, horosphere, or equidistant hypersurface is an intersection of a sphere of the conformal space with $Л^n$.

Proof. Indeed, the vector model of the space $Л^n$ can be considered as its conformal model on the hyperboloid (Chap. 2, Sect. 2.2), and the statement follows from the fact that any intersection of the hyperboloid with a hyperplane is a sphere in the conformal space. □

2.3. Similarity of Standard Hypersurfaces. Let $e \in \mathbb{R}^{n+1}$ $(e \neq 0)$, $U = \langle e \rangle^\perp$, $G_e = G(U)$. Projections along lines of the pencil B_e define diffeomorphisms

$$f_b^a : H_e^a \to H_e^b, \tag{12}$$

commuting with the action of the group G_e. However, these diffeomorphisms do not preserve the Riemannian metric, but multiply it by a scalar factor, i.e. they are similarity transformations. (Nothing else is possible in view of Chap. 1, Sect. 1.3.) In order to find the homothetic ratios we now give an analytical description of the pencil B_e in the vector model.

First, we parameterize the pencil B_e by points of the $(n-1)$-dimensional manifold Y_e. If B_e is an elliptic pencil with centre x, we take for Y_e the unit sphere in the space $U = T_x(X)$, and associate with each oriented line in the pencil its tangent vector at x. (In this case it is convenient to consider the elements of the pencil to be oriented lines, so that each line occurs in the pencil twice — with opposite orientations.) If B_e is a parabolic pencil, we take for Y_e the horosphere H_e^1, and associate with each line in the pencil its point of intersection with this horosphere. (The horosphere H_e^1 can, as a matter of fact, be chosen arbitrarily as it depends on the normalization of the vector e.) Finally, if B_e is a hyperbolic pencil with base H, we take for Y_e the hyperplane H, and associate with each line in the pencil its point of intersection with this hyperplane.

Then the parametric equation of the line in the pencil B_e corresponding to a point $y \in Y_e$ is of the form

$$f(t,y) = y\varphi(t) + e\psi(t), \tag{13}$$

where the form of the function f is given explicitly in the following table. The parameter t is the arc length, and the zero value of the parameter is chosen at the centre of the pencil B_e if it is an elliptic one, on its base if it is hyperbolic, and on the horosphere H_e^1 if it is parabolic. In the case $X = S^n$, the same pencil is considered as elliptic in the first line of the table, and as hyperbolic in the second. For $X = Л^n$ and $(e,e) \le 0$, it is assumed that $e \in C^-$.

x	(e,e)	(y,e)	(y,y)	$f(t,y)$	c	I	$\varkappa$
S^n	1	0	1	$y\sin t + e\cos t$	$\cos t$	$(0,\pi)$	$(1-c^2)^{-1}$
	1	0	1	$y\cos t + e\sin t$	$\sin t$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$(1-c^2)^{-1}$
$Л^n$	-1	0	1	$y\sinh t - e\cosh t$	$\cosh t$	$\mathbb{R}_+$	$(c^2-1)^{-1}$
	0	1	-1	$ye^t + e\sinh t$	e^t	$\mathbb{R}$	0
	1	0	-1	$y\cosh t + e\sinh t$	$\sinh t$	$\mathbb{R}$	$-(c^2+1)^{-1}$

For all $t \in I$, where I is the interval given in the table, the mapping $y \mapsto f(t,y)$ defines a diffeomorphism

$$f_c : Y_e \to H_e^c, \tag{14}$$

commuting with the action of the group G_e.

The table also shows the dependence of c on t. Formula (13) implies that

$$df_c(y) = \varphi(t)dy.$$

Hence f_c is a similarity transformation with the similarity factor $\varphi(t)$. The homothetic ratio of the diffeomorphism f_b^a is found from the relation $f_b^a = f_b f_a^{-1}$.

The curvature $\varkappa$ of the hyperplane H_e^c equals $\varphi(t)^{-2}\varepsilon$, where $\varepsilon = \pm 1$ or 0 is the curvature of the manifold Y_e. Its value, expressed in terms of c, is given in the last column of the table.

2.4. Intersection of Standard Hypersurfaces. Let $H_{e_1}^{c_1}, \dots, H_{e_k}^{c_k}$ be standard hypersurfaces in the space $X = S^n$ or $Л^n$. Suppose that the vectors $e_1, \dots, e_k \in \mathbb{R}^{n+1}$ are linearly independent. Let $G = (g_{ij})$ be their Gram matrix, $G^{-1} = (g^{ij})$ its inverse (if it exists), and

$$\sigma = \sum_{i,j} g^{ij} c_i c_j.$$

The relative position of the hypersurfaces $H_{e_1}^{c_1}, \dots, H_{e_k}^{c_k}$ is partly determined by their intersection, which may be empty, or consist of a single point, or be an $(n-k)$-dimensional standard hypersurface.

A necessary and sufficient condition for the intersection to be non-empty is given in the following table. (If the inequality for σ is in fact an equality, then the intersection consists of a single point.)

X	$\det G$	Non-emptiness condition	Curvature
S^n	>0	$\sigma \le 1$	$(1-\sigma)^{-1}$
$Л^n$	>0		$-(1+\sigma)^{-1}$
	$=0$		0
	<0	$\sigma \le -1$	$-(1+\sigma)^{-1}$

Proof. Let $\det D \ne 0$ and let $\{e_1^*, \dots, e_k^*\}$ be the basis of the subspace $W = \langle e_1, \dots, e_k \rangle$ dual to the basis $\{e_1, \dots, e_k\}$. Let $f = \sum c_j e_j^*$. Then $(f, e_i) = c_i$ for all $i = 1, \dots, k$. Hence

$$\bigcap_{i=1}^{k} H_{e_i}^{c_j} = (f + W^\perp) \cap X.$$

One can easily see that $(f,f) = \sigma$. This yields the non-emptiness condition given in the table. If the intersection is neither empty nor a single point, it is a standard hypersurface in the $(n-k+1)$-dimensional plane $Y = \langle f, W^\perp \rangle \cap X$ and is defined in it by the equation $(y,f) = \sigma$. Its curvature is found from the table of the preceding section. $\square$

§ 3. Decompositions into Semi-direct Products

One can obtain different decompositions of the spaces of constant curvature into semi-direct products of Riemannian manifolds of lesser dimension (which are also spaces of constant curvature) by considering orbits of standard subgroups of the groups of motions and their direct products.

3.1. Spherical, Horospherical, and Equidistant Decompositions. The orthogonal grid formed by the lines of the pencil B_e and the hypersurfaces H_e^c defines a decomposition of the Riemannian manifold $X = S^n$ or Л^n into a semi-direct product $I \leftthreetimes Y$ (see the notation in Sect. 2.3).

More precisely, the mapping

$$(f, y) \mapsto f(t, y)$$

is a diffeomorphism of the direct product $I \times Y$ onto $X \setminus (\langle e \rangle \cap X)$, while the Riemannian metric of the manifold X is expressed in terms of the Riemannian metrics of the manifolds I and Y_e according to the following table:

X	Name	Decomposition	Riemannian metric	Volume element
S^n	Spherical	$(0, \pi) \leftthreetimes S^{n-1}$	$dt^2 + \sin^2 t\, dy^2$	$\sin^{n-1} t\, dt\, dy$
	Equidistant	$(-\frac{\pi}{2}, \frac{\pi}{2}) \leftthreetimes S^{n-1}$	$dt^2 + \cos^2 t\, dy^2$	$\cos^{n-1} t\, dt\, dy$
Л^n	Spherical	$\mathbb{R}_+ \leftthreetimes S^{n-1}$	$dt^2 + \sinh^2 t\, dy^2$	$\sinh^{n-1} t\, dt\, dy$
	Horospherical	$\mathbb{R} \leftthreetimes E^{n-1}$	$dt^2 + e^{2t} dy^2$	$e^{(n-1)t}\, dt\, dy$
	Equidistant	$\mathbb{R} \leftthreetimes \text{Л}^{n-1}$	$dt^2 + \cosh^2 t\, dy^2$	$\cosh^{n-1} t\, dt\, dy$

(In the last column dy denotes the volume element of the manifold Y_e.)

3.2. Spherical-equidistant Decomposition. The following construction is a combination of the spherical and equidistant decompositions.

Let $Y \subset X$ be a plane of arbitrary dimension k, and Z the unit sphere in its normal space. A point $x \in X$ is defined by its distance t to the plane Y, its projection y on that plane, and the tangent vector z of the segment yx at the point y (see Fig. 50). The correspondence $x \leftrightarrow (t, y, z)$ defines a diffeomorphism between the manifolds $X \setminus Y$ and $I \times Y \times Z$, where $I = \left(0, \frac{\pi}{2}\right)$ for $X = S^n$, and $I = \mathbb{R}_+$ for $X = \text{Л}^n$.

The lines orthogonal to the plane Y and the submanifolds $t = \text{const}$ form an orthogonal grid. Each of these submanifolds is an orbit of the group $G(U) \times G(U^\perp)$, where $U = \langle Y \rangle$ is the defining subspace of the plane Y. As a Riemannian manifold it is decomposed into the direct product of orbits of the groups $G(U)$ and $G(U^\perp)$, which are similar to Y and Z respectively. The homothetic ratios can be found from the table given in Sect. 2.3. This defines the decomposition of the Riemannian manifold $X \setminus Y$ into a semi-direct product $I \leftthreetimes (Y \times Z)$.

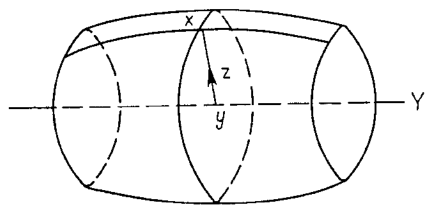


Fig. 50

For $X = S^n$ this gives the decomposition

$$S^n \setminus S^k = \left(0, \frac{\pi}{2}\right) \leftthreetimes (S^k \times S^{n-k-1}). \tag{15}$$

The Riemannian metric is of the form

$$dx^2 = dt^2 + \cos^2 t\, dy^2 + \sin^2 t\, dz^2, \tag{16}$$

with the volume element

$$dx = \cos^k t\, \sin^{n-k-1} t\, dt\, dy\, dz. \tag{17}$$

For $X = \text{Л}^n$ one gets the decomposition

$$\text{Л}^n \setminus \text{Л}^k = \mathbb{R}_+ \leftthreetimes (\text{Л}^k \times S^{n-k-1}). \tag{18}$$

The Riemannian metric is of the form

$$dx^2 = dt^2 + \cosh^2 t\, dy^2 + \sinh^2 t\, dz^2, \tag{19}$$

with the volume element

$$dx = \cosh^k t\, \sinh^{n-k-1} t\, dt\, dy\, dz. \tag{20}$$

Chapter 5
Motions

In this chapter we establish the main properties of motions in spaces of constant curvature. We also give the classification of motions and describe the structure of the groups of motions.

We denote by G the entire group of motions of a space of constant curvature X, and by G_+ its connected component.

§ 1. General Properties of Motions

1.1. Description of Motions. Let x, y be two points, and $e = \{e_1, \ldots, e_n\}$, $f = \{f_1, \ldots, f_n\}$ two bases in the tangent spaces at these points having the same Gram matrix, i.e. $(e_i, e_j) = (f_i, f_j)$. The definition of a space of constant curvature implies that there is a unique motion $g \in G$ taking the point x into the point y, and the basis e into the basis f. The following proposition is an analogue of this statement for the "reference frames" consisting of directed intervals of the space X.

Proposition 1.1. *Let $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ be two sets of points in general position (see Chap. 1, Sect. 3.2). If $\rho(x_i, x_j) = \rho(y_i, y_j)$ for all $i, j = 0, 1, \ldots, n$, then there is a unique motion taking the first set into the second.*

Proof. We consider only the case of $X = S^n$ or $Л^n$. In terms of the vector model both sets can be considered as two bases in the ambient vector space $\mathbb{R}^{n+1}$ having the same Gram matrices. Such bases can evidently be taken into one another by a (pseudo)-orthogonal transformation. □

Note that the position of any point $x \in X$ is uniquely defined by its distances to $n+1$ points $x_0, x_1, \ldots, x_n$ in general position. Indeed, in the vector model of the space $X = S^n$ or $Л^n$ these distances define the scalar products of the vector $x \in X \subset \mathbb{R}^{n+1}$ with the vectors $x_0, x_1, \ldots, x_n$ forming a basis in the space $\mathbb{R}^{n+1}$.

1.2. Continuation of a Plane Motion. Let Y be a k-dimensional plane, and $N(Y)$ its normal space.

Definition 1.2. A motion $g \in G$ is said to be a *glide* along the plane Y if it leaves the plane Y invariant and acts identically on its normal space $N(Y)$.

The group of all glides along the plane Y will be denoted by $G(Y)$, and its connected component containing unity by $G_+(Y)$.

Since $\mathbb{R}^{n+1} = \langle Y \rangle \oplus N(Y)$, any motion g of the plane Y can be uniquely continued to a glide $\hat{g}$ along Y. This glide is called the *canonical continuation of the motion g*. The glide $\hat{g}$ leaves any plane containing Y invariant, and takes any point $x \in X$ into the point $\hat{g}x$ described by the following conditions:

(1) the point $\hat{g}x$ lies in the $(k+1)$-dimensional plane spanned by x and Y on the same side of Y as the point x;
(2) the projection of the point $\hat{g}x$ on Y coincides with gy, where y is the projection of the point x on Y.

Definition 1.3. A motion preserving all the points of the plane Y is called a *rotation about the plane Y*.

The group of all rotations about Y will be denoted by G^Y. Evidently $G^Y \simeq O(N(Y))$.

Proposition 1.4. *The group G_Y of all motions of the space X preserving the plane Y is decomposed into the direct product*

$$G_Y = G(Y) \times G^Y.$$

1.3. Displacement Function

Definition 1.5. The function $\rho_g(x) = \rho(x, gx)$ on X is said to be the *displacement function* of the motion g.

The formula for the distance between points implies that the displacement function is differentiable at any point x such that $gx \neq x$, and in the case of the sphere, $gx \neq -x$, where $-x$ is the point antipodal to x.

Denote by M_g^λ the set of critical points of the displacement function with the critical value λ. (Recall that critical points for a function are those points where the differential of the function either equals zero or does not exist.)

In particular, M_g^0 is the set of fixed points of the motion g, and in the case of the sphere M_g^π is the set of points mapped into their antipodes.

Theorem 1.6. *Let g be a motion of the space X.*

(1) *Any non-empty critical set M_g^λ is a g-invariant plane. If $\lambda > 0$, and in the case of the sphere, if $\lambda < \pi$, then for any point $x \in M_g^\lambda$ there is a g-invariant line lying in M_g^λ and passing through x.*
(2) *For $X = E^n$ the displacement function ρ_g has exactly one critical value λ, which is the absolute minimum. The motion g induces a parallel translation in M_g^λ.*
(3) *For $X = Л^n$ the displacement function ρ_g has at most one critical value λ, which is necessarily the absolute minimum. For $\lambda > 0$ the corresponding critical set M_g^λ is a line.*

Proof. In terms of the vector model of the space $X = S^n$ or $Л^n$ the critical set M_g^λ is the set of critical points of the quadratic form $F(x) = (x, gx)$ on the surface $(x, x) = \text{const} = \pm 1$. This set is defined by the equation

$$(dx,\, gx) + (x,\, g\,dx) = 2\mu(x,\, dx),$$

i.e. it is the intersection of X with the eigenspace of a symmetric linear transformation $g+g^{-1}$. Hence M_g^λ is a plane for $X = S^n$ or $Л^n$. The case $X = E^n$ is treated in the same way. If $x \in M_g^\lambda$, then the subspace $\langle x, gx\rangle$ is invariant under g because $g^2x = 2\mu gx - x$. This proves (1). To prove (2) and (3) it is enough to note that the distance between any two g-invariant lines is bounded. There are no such lines in the Lobachevskij space, while in the Euclidean space such lines are parallel. □

Definition 1.7. The non-empty critical set M_g^λ of the displacement function ρ_g is said to be *the axis of the motion g*.

The preceding results imply the following corollaries.

Corollary 1.8. (1) *Any motion of the Euclidean space has a unique axis.*

(2) *Any motion of the sphere S^n has an axis, the number of axes does not exceed $[n/2]$.*

(3) *Any motion of the Lobachevskij space has at most one axis. If a motion has no fixed points and has an axis, then its axis is a line.*

Definition 1.9. A motion g having a constant displacement function $\rho_g(x) = \lambda =$ const is called a *Clifford translation*. The number λ is said to be *the displacement value*.

It follows from Theorem 1.6 that Clifford translations in E^n are parallel displacements, and that there are no non-identical Clifford translations in $Л^n$. Clifford translations on the sphere are described in Section 2.1.

§ 2. Classification of Motions

2.1. Motions of the Sphere. It is a known fact of linear algebra that any orthogonal transformation *of a Euclidean vector space* can, in an appropriate orthonormal basis, be expressed in the form

$$g = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1, R(\varphi_1), \ldots, R(\varphi_s)),$$

where $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is the matrix of rotation through an angle φ, $0 < \varphi < \pi$.

Denote by V_ε the eigenspace of g corresponding to the eigenvalue $\varepsilon = 1, -1$, respectively, and by $V(\varphi)$ the direct sum of the subspaces corresponding to the diagonal blocks $R(\varphi_i)$ with $\varphi_i = \varphi$. The transformation g induces in each subspace $V(\varphi)$ a complex structure operator I acting in each g-invariant two-dimensional subspace $U \subset V(\varphi)$ as the rotation through $\frac{\pi}{2}$ in the same direction as $g|_U$. The restriction of g to the subspace $V(\varphi)$ is of the form

$$g|_{V(\varphi)} = E\cos\varphi + I\sin\varphi$$

and induces a Clifford translation in the plane $V(\varphi) \cap S^n$ of the sphere with the displacement value φ. The subspaces V_ε, $V(\varphi_i)$ are defined uniquely, and yield the orthogonal decomposition

$$\mathbb{R}^{n+1} = V_1 \oplus V_{-1} \oplus V(\varphi_1) \oplus \ldots \oplus V(\varphi_k).$$

The planes $V_\varepsilon \cap S^n$, $V(\varphi_i) \cap S^n$ are the axes of g, and the respective critical values of the displacement function equal 0 for $V_1 \cap S^n$, π for $V_{-1} \cap S^n$, and φ_i for $V(\varphi_i) \cap S^n$. This leads to the following result.

Theorem 2.1. *Let g be a motion of the sphere, and $M_1, \ldots, M_l$ its axes.*

(1) *The motion g induces a Clifford translation $g_i = g|_{M_i}$ on each axis, and is decomposed into a product of commuting glides along the axes M_i induced by these translations.*

(2) *A motion is defined by the critical values of the displacement function and the dimension of the corresponding axis uniquely up to conjugacy in the group of motions.*

Corollary 2.2. *Clifford translations of the sphere S^n with the displacement value λ, $0 < \lambda < \pi$, are in one-to-one correspondence with orthogonal complex structures in the ambient space $\mathbb{R}^{n+1}$. A complex structure I corresponds to the Clifford translation $g_I^\lambda = E\cos\lambda + I\sin\lambda$. In particular, on an even-dimensional sphere there are just two Clifford translations, namely* $\pm$id.

Since any Clifford translation g_I^λ, $\lambda < \pi$, is contained in a one-parameter group of Clifford translations g_I^t, $t \in \mathbb{R}$, whose trajectories are g_I^t-invariant lines, one has another corollary.

Corollary 2.3. *The Clifford translation g_I^λ, $0 < \lambda < \pi$, defines a foliation of the sphere into mutually equidistant g_I^λ-invariant lines $\{g_I^t x\}$.*

2.2. Motions of the Euclidean Space

Theorem 2.4. *Any motion g of the Euclidean space has the unique axis M (corresponding to the absolute minimum of the displacement function), and is the product of a rotation about M and a parallel translation along a vector lying in M.*

Note that parallel translations can be described as those motions (or Clifford translations) the axes of which coincide with the entire space, and rotations as those motions for which the critical value of the displacement function vanishes.

2.3. Motions of the Lobachevskij Space

Definition 2.5. A motion of the Lobachevskij space $Л^n$ is said to be *elliptic* (or a rotation) if it has a fixed point in $Л^n$, *hyperbolic* if it has no fixed points in $Л^n$ but has an invariant line, and *parabolic* if it has neither fixed points nor invariant lines but has a fixed point at infinity.

An example of a hyperbolic motion is a *parallel displacement along a line* l, the canonical continuation of a proper motion of the line. An example of a parabolic motion is given by a *parabolic translation* along a two-dimensional plane, the canonical continuation of a parabolic rotation of the two-dimensional plane.

In the vector model these motions are described as follows.

Let l be an oriented line with the defining subspace $U = \langle u, v\rangle$, where u, v are isotropic vectors representing points at infinity of the line l and normalized by the condition $(u, v) = -1$. With respect to the direct decomposition $\mathbb{R}^{n,1} = \mathbb{R}u \oplus \mathbb{R}v \oplus U^{\perp}$, a parallel displacement along the line l is defined by a diagonal transformation

$$\operatorname{diag}(e^{-\lambda}, e^{\lambda}, \mathrm{id}).$$

The line l is the axis of this parallel displacement, and the corresponding critical value of the displacement function equals λ.

Let Y be a two-dimensional plane, and $\{u, e, v\}$ a basis of its defining subspace U with the Gram matrix

$$G = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

This basis defines a parabolic translation S along the plane Y. The restriction of the corresponding pseudo-orthogonal transformation to U, in the basis $\{u, e, v\}$, is given by the matrix

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The isotropic vector u is the unique (up to a scalar multiple) fixed isotropic vector of this transformation. The S-invariant point $p = \langle u\rangle$ at infinity corresponding to it is referred to as the *centre of the parabolic translation* S. The displacement function of a parabolic translation has no critical points.

Theorem 2.6. (1) *Any motion of a Lobachevskij space is an elliptic, hyperbolic or parabolic one.*

(2) *Any hyperbolic motion g is uniquely decomposed into the product of a parallel displacement along a line l and a rotation about this line. The line l is the unique axis of the motion g.*

(3) *Any parabolic motion g is uniquely decomposed into the product of a parabolic translation S along a two-dimensional plane Y and a rotation R about this plane. A parabolic motion has no axes.*

Remark. In the last case, if the rotation axis R does not coincide with Y a plane Y is not defined uniquely.

Proof. The proof follows from Theorem 1.6 of Chapter 4, which describes the canonical form of a pseudo-orthogonal transformation in the space $\mathbb{R}^{n,1}$. A pseudo-orthogonal transformation A induces an elliptic, hyperbolic, or parabolic motion if the dimension of the minimal A-invariant hyperbolic subspace H equals 1, 2, or 3, respectively. □

2.4. One-parameter Groups of Motions. The above classification of motions implies that any proper motion of a space of constant curvature can be included in a one-parameter group of motions.

Let us now describe the form of one-parameter groups of parallel displacements along lines and parabolic translations in a Lobachevskij space.

Denote by $a \wedge b$ the skew-symmetric transformation of the space $\mathbb{R}^{n,1}$ given by the formula $x \mapsto (b, x)a - (a, x)b$, $x \in \mathbb{R}^{n,1}$, $a, b \in \mathbb{R}^{n,1}$. The transformation $a \wedge b$ generates a one-parameter group $\exp t(a \wedge b)$ of pseudo-orthogonal transformations, and therefore of motions of the space $Л^n$. In the notation of Sect. 2.3 the one-parameter group of motions along a line L with the defining space $U = \langle u, v\rangle$ is of the form $\exp t(u \wedge v)$. Its restriction to U is given in the basis $\{u, v\}$ by the matrix

$$\operatorname{diag}(e^{-t}, e^{t}).$$

The one-parameter group of parabolic translations along a plane Y with the defining subspace $U = \langle u, e, v\rangle$ is of the form $\exp t(u \wedge e)$. Its restriction to U is given in the basis $\{u, e, v\}$ by the matrix

$$\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\{g_t\}$, $\{h_t\}$ be two one-parameter groups of motions the elements of which commute. Then their product $\{g_t h_t\}$ is also a one-parameter group of motions.

The results of Sect. 2.1 imply that any one-parameter group of motions of the sphere can be decomposed into a product of commuting one-parameter groups, which are the canonical continuations of the one-parameter groups $E\cos t + I\sin t$ of Clifford translations of some mutually polar planes. This provides a description of one-parameter groups of rotations in any space of constant curvature.

The structure of one-parameter groups in E^n and $Л^n$ with no fixed points is described by the following theorem.

Theorem 2.7. (1) *Any one-parameter group of motions of the Euclidean space E^n having no fixed points is either the group of parallel translations along vectors of the form tv, $t \in \mathbb{R}$, or the product of such group by a one-parameter group of rotations about a line with the direction vector v.*

(2) *Any one-parameter group of hyperbolic motions of the Lobachevskij space is either the group of parallel displacements along a certain line l, or*

the product of this group by a one-parameter group of rotations about the line l.

(3) *Any one-parameter group of parabolic motions of the Lobachevskij space is either the one-parameter group of parabolic translations along a two-dimensional plane* Y, *or the product of this group by a one-parameter group of rotations about the plane* Y.

§ 3. Groups of Motions and Similarities

Throughout this section we mean by a "group of motions" a closed subgroup of the full group of motions G of a space X of constant curvature. We take the vector model and consider the group G as a linear group.

3.1. Some Basic Notions

Definition 3.1. A group of motions H is said to be *reducible* if it has an invariant plane Y. Otherwise H is said to be *irreducible*.

Note that this definition does not conform to the usual definition of an irreducible linear group.

According to the definition, a reducible group is contained in a group G_Y which is the direct product of the group $G(Y)$ of glides along the plane Y and the group $G^Y \simeq O(N(Y))$ of rotations about Y.

A linear group H is said to be *algebraic* if it can be singled out in the full linear group by a system of polynomial equations in the matrix elements. A subgroup of finite index in an algebraic group is said to be *quasi-algebraic*.

The groups of all motions of the sphere and the Euclidean space are algebraic, while the group of all motions of the Lobachevskij space is quasi-algebraic. (It is a subgroup of index 2 of the algebraic group $O_{n,1}$.)

A linear group H is said to be *reductive* (respectively, *diagonal, unipotent*) if it is completely reducible (can be written, in some basis, by diagonal or unitriangular matrices, respectively).

Definition 3.2. A decomposition $G = K \cdot A \cdot N$ of a linear Lie group G into a product of Lie subgroups K, A, N is said to be an *Iwasawa decomposition* if

(1) $G = K \times A \times N$ as a manifold;
(2) K is a compact subgroup, N is a unipotent subgroup, and A is a connected diagonal subgroup normalizing N.

In this case K, A, N are automatically maximal compact, connected diagonal and unipotent subgroups, respectively.

It is a known fact that any quasi-algebraic linear group has an Iwasawa decomposition and that any two such decompositions are conjugate by an inner automorphism. Any quasi-algebraic subgroup G_1 of a quasi-algebraic group G has an Iwasawa decomposition $G_1 = K_1 \cdot A_1 \cdot N_1$ compatible with some Iwasawa decomposition $G = K \cdot A \cdot N$ of the group G in the sense that

$$K_1 = K \cap G_1, \quad A_1 = A \cap G_1, \quad N_1 = N \cap G_1.$$

3.2. Criterion for the Existence of a Fixed Point

Theorem 3.3. *Suppose that a group of motions* H *of the space* X *has a compact orbit* K. *If* X *is the sphere, suppose also that the orbit* K *lies in an open half-space. Then the group* H *has a fixed point (and is consequently compact).*

Corollary 1. *Any compact group of motions of the space* $X = E^n$ *or* Л^n *has a fixed point, and is therefore reducible.*

Corollary 2. *All maximal compact subgroups of the full group of motions are conjugate.*

The theorem is implied by the following lemma.

Lemma 3.4. *Let* $K \subset X$ *be a compact set contained in an open half-space. Then there is a unique closed ball of minimal radius containing* K.

The centre of such a minimal ball containing the compact orbit K of the group H is evidently a fixed point of the group H.

Proof. In order to prove the lemma it is enough to check that the intersection of two balls of radius r with centres at points $A, B \in X$ is contained in the ball of radius $r' < r$ with centre at the point O which is the midpoint of the segment AB. In the case of the sphere one can also assume that $r < \frac{\pi}{2}$. Let a point C belong to the intersection of the balls. Then in the triangle ABC the sides AC and BC are not greater than r. Hence it follows (Theorems 4.2 and 4.3 of Chap. 1) that the median CO is less than r. □

Remark. Theorem 3.3 and Lemma 3.4 can be generalized to the case when X is an arbitrary complete Riemannian manifold if one modifies the statement by replacing an open half-space by a geodesically convex domain.

3.3. Groups of Motions of the Sphere. The full group $G = O_{n+1}$ of motions of the sphere S^n is compact. This fact is responsible for a number of special features of the group of motions of the sphere as compared with the groups of motions of other spaces of constant curvature. Note that there are many irreducible groups of motions of the sphere: any irreducible orthogonal representation $\varphi : H \to O_{n+1}$ of an arbitrary compact group H defines an irreducible group $\varphi(H)$ of motions of S^n. On the other hand, there are, up to conjugacy in the group G, only finitely many connected transitive groups H of motions of the sphere S^n (Gorbatsevich and Onishchik, 1988). All of them are listed in the following table which also includes the stabilizer H_x of a point of the sphere in the group H and the decomposition of its isotropy representation into irreducible components. The trivial H_x-module of dimension k is denoted by U^k, and the irreducible H_x-module of dimension k by V^k.

n	H	Stabilizer	Isotropy representation
$m-1$	SO_m	SO_{m-1}	V^{m-1}
$2m-1$	U_m	U_{m-1}	$U^1 \oplus V^{2(m-1)}$
$2m-1$	SU_m	SU_{m-1}	$U^1 \oplus V^{2(m-1)}$
$4m-1$	$Sp_1 \cdot Sp_m$	$Sp_1 \cdot Sp_{m-1}$	$V^3 \oplus V^{4(m-1)}$
$4m-1$	$SO_2 \cdot Sp_m$	$SO_2 \cdot Sp_{m-1}$	$U^1 \oplus V^2 \oplus V^{4(m-1)}$
$4m-1$	Sp_m	Sp_{m-1}	$U^3 \oplus V^{4(m-1)}$
15	$Spin_9$	$Spin_7$	$V^7 \oplus V^8$
7	$Spin_7$	G_2	V^7
6	G_2	SU_3	V^6

Corollary 1. *The groups SO_m, $Spin_7$, G_2 are the only transitive groups of motions of the sphere with the irreducible isotropy representation.*

Corollary 2. *The group SO_2 of motions of S^1 and the group SU_2 of motions of S^3 are the only simply transitive groups of motions of the sphere.*

3.4. Groups of Motions of the Euclidean Space. The full group G of motions of the Euclidean space E^n has a normal subgroup of parallel displacements, which may be considered as the vector group of the Euclidean vector space V associated with E^n. Each point $x \in E^n$ defines the isomorphism $\varphi_x : O(V) \to G_x$ of the orthogonal group $O(V)$ with the stabilizer G_x of the point x under which $\varphi(A)$ is a motion with the linear part A. The group G_x is a maximal compact subgroup of the group G, and V is its (unique) maximal unipotent subgroup. The Iwasawa decomposition of the group G is of the form $G = G_x V$. This implies (see Sect. 3.1) that any algebraic subgroup H of the group G is of the form $H = K \cdot U$, where K is a subgroup of some stabilizer G_x and $U \subset V$ is a K-invariant subspace. For non-algebraic subgroups this is not necessarily the case, but, taking the algebraic closures, one obtains the following results.

Theorem 3.5 (see, e.g. Alekseevskij [1975]). *Let H be a connected group of motions of the Euclidean space E^n.*

(1) *If the group H is irreducible, then it acts transitively in E^n. In the general case it has an orbit which is a plane.*
(2) *If the group H is reductive, then it preserves some point, and is therefore compact.*
(3) *The group H has a decomposition $H = K \cdot F$, where K is a compact subgroup, and F is a metabelian normal subgroup which acts freely. In particular, if H is transitive, then F is simply transitive.*

The structure of simply transitive groups of motions is described as follows. Let $V = U \oplus W$ be an orthogonal decomposition of the space V. Any homomorphism $\psi : U \to O(W)$ defines a commutative group $U^\psi = \{\varphi_x(\psi(u))u : u \in U\}$ of helical motions, and a simply transitive group $F = U^\psi W$. Any simply transitive group can be constructed in this way. Arbitrary groups of motions acting freely are described in a similar manner.

3.5. Groups of Similarities. A peculiar feature of the Euclidean space, which makes its position so special not only among other spaces of constant curvature but among all complete Riemannian manifolds, is that it has essential similarities, i.e. transformations changing the distances between points by a factor λ, where $\lambda = \text{const} \neq 1$. Any essential similarity g of the Euclidean space has a unique fixed point x, and can be decomposed into the product of a homothetic transformation with centre x and a rotation about x.

A group of similarities which is not a group of motions is said to be *essential*. A connected essential group of similarities P admits a decomposition $P = A \cdot H$, where A is a one-parameter group of essential similarities, and H a normal subgroup consisting of motions.

Theorem 3.6. *A connected essential group of similarities P of the Euclidean space admits a decomposition*

$$P = A \cdot K \cdot F,$$

where A is a one-parameter group of essential similarities preserving some point x, K is a group of rotations about x commuting with A, and F is a normal subgroup acting freely. The normal subgroup $H = K \cdot F$ consists of all motions of the group P, and the orbit Px of the point x is the plane on which the group F acts simply transitively.

Proof. The desired decomposition is constructed from the decomposition $P = A \cdot H$ of the group P given above and the decomposition $H = K \cdot F$ of the subgroup of motions described in Section 3.5. The possibility of choosing the group A in such a way that it commutes with K follows from the fact that the compact group K is completely reducible. □

3.6. Groups of Motions of the Lobachevskij Space. Let $G \simeq O'_{n,1}$ be the group of motions of the Lobachevskij space Л^n considered in the vector model, and G_a the stabilizer of an arbitrary vector $a \neq 0$ of the space $\mathbb{R}^{n,1}$. The group G_a is isomorphic to O_n, $O_{n-1} \cdot \mathbb{R}^n$, or $O'_{n-1,1}$ if $(a,a) > 0$, $(a,a) = 0$, or $(a,a) < 0$, respectively. The orbits of the group G_a in the space Л^n that are not points are standard hypersurfaces, i.e. hyperplanes, spheres, horospheres, or equidistant hypersurfaces (see Chap. 4). If the vector a is not isotropic, then the group $G_a^+ = G_a \cap \text{Isom}_+ \text{Л}^n$ is a maximal connected subgroup of G.

For an isotropic vector u the group G_u^+ is contained in a unique maximal connected subgroup of G. This subgroup is of the form $G_p^+ = G_p \cap \text{Isom}_+ \text{Л}^n$, where the group G_p consists of all motions $g \in G$ preserving the point at infinity $p = \langle u \rangle \in \partial Л^n$. The group G_p acts transitively in the space $Л^n$, and the orbits of its normal subgroup G_u are horospheres H_u^c on which the group G_u acts as the full group of motions (with respect to the induced Euclidean metric). Denote by N_p the normal subgroup of the group G_p acting on horospheres as the group of parallel displacements. The group N_p is called the *horospherical subgroup* of G associated with the point p at infinity.

The Iwasawa decomposition of the group G is defined by a choice of an oriented line l and a point $x \in l$ and is of the form

$$G = G_x \cdot A_l \cdot N_p,$$

where $p \in \partial Л^n$ is the point at infinity of the line l, and A_l the one-parameter group of parallel displacements along l. This decomposition induces an Iwasawa decomposition of the maximal subgroup G_p:

$$G_p = G^l \cdot A_l \cdot N_p,$$

where $G^l \simeq O_{n-1}$ is the group of rotations about the line l. The group G^l coincides with the centralizer of the group A_l in the group G_x, and the horospherical subgroup $N_p \simeq \mathbb{R}^{n-1}$ is a normal subgroup of the group G_p.

The tangent Lie algebras $L(H)$ of the Lie groups H defined above are described as follows.

Let u, v be isotropic vectors representing points at infinity of the line l such that $(u, v) = -1$, and $U = \langle u, v \rangle^\perp$. Then, in the notation of Sect. 2.4, one has

$$L(N_p) = u \wedge U, \quad L(A_l) = \langle u \wedge v \rangle,$$

$$L(G^l) = \mathfrak{so}(U), \quad L(G_p) = L(G^l) + L(A_l) + L(N_p).$$

Let us describe the action of the group G and of its maximal subgroups on the absolute. Recall (see Chap. 2) that a conformal structure (i.e. a conformal class of Riemannian metrics) is defined on the absolute, and the group G acts on it as the full group of conformal transformations. Since the stabilizer of a point $p \in \partial Л^n$ is the group G_p, the absolute is identified with the quotient space G/G_p. The complement E_p of the point p in $\partial Л^n$ can be considered as a Euclidean space. The Euclidean metric in E_p (defined up to a scalar multiple) is induced by the metric on the horosphere H_u^1, $p = \langle u \rangle$ under the projection $f\colon H_u^1 \to E_p$ along the lines of the parabolic pencil B_u (see Fig. 43). The group G_p (respectively, G_u) acts transitively in the Euclidean space E_p as the full group of similarities (respectively, motions), and the horospherical group N_p acts as the group of parallel displacements.

Now consider the group G_e preserving a vector e with $(e, e) = 1$. It has three orbits on the absolute $\partial Л^n$: a closed orbit S of codimension one, and open orbits M^+, M^-. The orbit S is a hypersphere in the conformal space $\partial Л^n$ and consists of all points at infinity of each of the equidistants H_e^c, $c \in \mathbb{R}$. The group G_e acts on S as the full group of conformal transformations. The orbits M^+, M^- are hemispheres bounded by S. The projection of the hyperplane H_e along lines of the hyperbolic pencil B_e defines diffeomorphisms $F_\pm\colon H_e \to M^\pm$ commuting with the action of the group G_e. Therefore, these orbits (as homogeneous spaces of the group G_e) are spaces of constant negative curvature.

Finally, the stabilizer G_x of the point $x \in Л^n$ acts transitively on the absolute $\partial Л^n$. The central projection f of a sphere with centre at the point x on the absolute $\partial Л^n$ along the lines of the pencil B_x commutes with the action of the group G_x. Therefore, as a homogeneous space of the group G_x, the absolute is a space of constant positive curvature.

Remark. Denote by $\eta(x)$ the G_x-invariant metric of constant curvature 1 on the absolute induced by the metric of the sphere $H_x^{\sqrt{2}}$ under the diffeomorphism f. The mapping $x \mapsto \eta(x)$ defines a one-to-one correspondence between points of the Lobachevskij space and Riemannian metrics of constant curvature 1 on the absolute compatible with the conformal structure.

We now describe the structure of arbitrary connected groups of motions of the Lobachevskij space $Л^n$.

Theorem 3.7. (1) *Any connected reductive group H of motions of the space $Л^n$ is of the form $H = G_+(Y) \times K$, where $G_+(Y)$ is the group of proper glides along some plane Y, and K is a connected group of rotations about Y.*

(2) *Any non-reductive group H of motions of the space $Л^n$ preserves some point p at infinity and is consequently contained in the group G_p isomorphic to the group of similarities of the Euclidean space E^{n-1}.*

Corollary. *Let H be a proper connected subgroup of the group G.*

(1) *The group H is irreducible if and only if it preserves a point p at infinity and acts transitively on the Euclidean space $E_p = \partial Л^n \setminus \{p\}$.*

(2) *The group H is transitive if and only if it preserves a point p at infinity and acts on the Euclidean space E_p as the transitive essential group of similarities. In particular, any transitive group of motions is irreducible.*

Proof. Statement (1) follows from a general theorem proved by Karpelevich [1953]. Now let H be a non-reductive group. The group H leaves invariant a parabolic subspace in $\mathbb{R}^{n,1}$, and consequently its kernel, which is a one-dimensional isotropic subspace $p = \langle u \rangle$. This proves (2). □

Chapter 6
Acute-angled Polyhedra

Acute-angled polyhedra have been the object of study in connection with discrete groups generated by reflections (see Part 2, Chap. 6 of the present volume). However, the completeness of results obtained in this area makes it possible to organize them into a separate theory in which the peculiar character of Lobachevskij geometry manifests itself very brightly.

§ 1. Basic Properties of Acute-angled Polyhedra

1.1. General Information on Convex Polyhedra. According to the definition (see Definition 3.8 of Chap. 1), each convex polyhedron P is an intersection of finitely many half-spaces:

$$P = \bigcap_{i=1}^{s} H_i^- . \tag{1}$$

(Here H_i^- is a half-space bounded by the hyperplane H_i.) It may always be assumed that none of the half-spaces H_i^- contains the intersection of all the others. In what follows we assume this without special mention. Under this condition the half-spaces H_i^- are uniquely determined by the polyhedron P. Each of the hyperplanes H_i is said to *bound* the polyhedron P.

In the Klein model K^n of the Lobachevskij space $Л^n$ any convex polyhedron $P \subset Л^n$ is the intersection of a convex polyhedron $\hat{P}$, lying in the ambient Euclidean space and bounded by the same number of hyperplanes, with the ball K^n. A polyhedron P is bounded (respectively, has finite volume) in the space $Л^n$ if and only if $\hat{P} \subset K^n$ (respectively, $\hat{P} \subset \overline{K^n}$). These considerations make it possible to automatically transfer a number of theorems on convex polyhedra in Euclidean space to convex polyhedra in Lobachevskij space.

In particular, by the Weyl-Minkowski theorem, any bounded convex polyhedron in a Euclidean space is the convex hull of finitely many points and vice versa. Hence it follows that bounded convex polyhedra (respectively, convex polyhedra of finite volume) in the Lobachevskij space are convex hulls of finite sets of points (respectively, finite sets of points, either ordinary or at infinity).

A convex polyhedron is said to be *degenerate* if it is a cone with the apex at an ordinary point or (in the case $X = Л^n$) a point at infinity, or if the bounding hyperplanes are perpendicular to the same hyperplane. Non-degeneracy of a spherical polyhedron means that it contains no antipodes. Any convex polyhedron of finite volume in Euclidean or Lobachevskij space is non-degenerate.

1.2. The Gram Matrix of a Convex Polyhedron. One can associate a matrix with each convex polyhedron P in an n-dimensional space X of constant curvature, which for $X = S^n$ or $Л^n$, as a rule, characterizes the polyhedron P uniquely up to a motion, thus providing a useful algebraic tool for studying convex polyhedra.

Denote by V one of the following vector spaces:
for $X = E^n$ the Euclidean space $\mathbb{R}^n$ of the affine model of E^n;
for $X = S^n$ the Euclidean space $\mathbb{R}^{n+1}$ of the vector model of S^n;
for $X = Л^n$ the pseudo-Euclidean space $\mathbb{R}^{n,1}$ of the vector model of $Л^n$.

Let us assume that the polyhedron P is represented in the form (1), and consider for each i the unit vector e_i of the space V orthogonal to the hyperplane H_i and directed away from P. For $X = S^n$ or $Л^n$ this means that the polyhedron P is the intersection in the space V of the convex polyhedral cone

$$K(P) = \{x \in V : (x, e_i) \leq 0,\ i = 1, \dots, s\} \tag{2}$$

with X.

Definition 1.1. The Gram matrix of the system of vectors $\{e_1, \dots, e_s\}$ is said to be the *Gram matrix of the polyhedron* P, denoted by $G(P)$.

The diagonal elements of the matrix $G(P)$ are equal to 1, while those off the main diagonal are, in accordance with Sect. 1.9 of Chap. 4, defined by the relative position of the half-spaces $H_1^-, \dots, H_s^-$.

For $X = E^n$ the matrix $G(P)$ is positive semi-definite of rank $\leq n$. A polyhedron P is non-degenerate if and only if $\operatorname{rk} G(P) = n$. The matrix $G(P)$ defines the system of vectors $\{e_1, \dots, e_s\}$ up to an orthogonal transformation, and therefore the polyhedron P up to a motion and parallel displacement of the bounding hyperplanes.

For $X = S^n$ the matrix $G(P)$ is positive semi-definite of rank $\leq n + 1$. A polyhedron P is non-degenerate if and only if $\operatorname{rk} G(P) = n + 1$. The matrix $G(P)$ defines the system of vectors $\{e_1, \dots, e_s\}$ up to an orthogonal transformation, and therefore the polyhedron P up to a motion.

For $X = Л^n$ the matrix $G(P)$ is either indefinite of rank $\leq n+1$ (with the negative index of inertia 1), or positive semi-definite of rank $\leq n$. A polyhedron P is non-degenerate if and only if $\operatorname{rk} G(P) = n+1$. With the exception of the case when the matrix $G(P)$ is positive semi-definite and degenerate, it defines the system of vectors $\{e_1, \dots, e_s\}$ up to a pseudo-orthogonal transformation, and therefore the polyhedron P up to a motion or replacing the polyhedron with another one bounded by the same hyperplanes but contained in the intersection of the opposite half-spaces, provided such a polyhedron exists. (The last alternative is due to the fact that the group of motions of $Л^n$ does not coincide with the group of all pseudo-orthogonal transformations of the space V but with its subgroup of index 2.) In particular, if a polyhedron P has finite volume, the Gram matrix defines it (in the class of all convex polyhedra) unambiguously up to a motion.

1.3. Acute-angled Families of Half-spaces and Acute-angled Polyhedra

Definition 1.2. A family of half-spaces $\{H_1^-, \dots, H_s^-\}$ is said to be *acute-angled* if for any distinct i, j either the hyperplanes H_i and H_j intersect and the dihedral angle $H_i^- \cap H_j^-$ does not exceed $\frac{\pi}{2}$, or $H_i^+ \cap H_j^+ = \emptyset$. A convex polyhedron (1) is said to be *acute-angled* if $\{H_1^-, \dots, H_s^-\}$ is an acute-angled family of half-spaces.

The following "inheritance" properties hold.

Theorem 1.3. *If $\{H_1^-, \dots, H_s^-\}$ is an acute-angled family of half-spaces, then for all $i_1, \dots, i_t$ the intersections of the half-spaces $H_1^-, \dots, H_s^-$ with the plane $Y = H_{i_1} \cap \dots \cap H_{i_t}$ that are different from Y form an acute-angled family of half-spaces in the space Y such that the angle between any two intersecting hyperplanes $H_j \cap Y$ and $H_k \cap Y$ of the space Y does not exceed the angle between H_j and H_k. If $P = \cap_{i=1}^s H_i^-$ has a non-empty interior in X, then $P \cap Y$ has a non-empty interior in Y.*

Proof. An inductive reasoning reduces the proof to the case $t = 1$. Let $Y = H_i$, and let $H_j \cap H_i$ and $H_k \cap H_i$ be intersecting hyperplanes of the space H_i. We now prove that the angle between them (in the space H_i) does not exceed the angle between H_j and H_k. With that in mind, consider the section of H_i by the three-dimensional plane orthogonal to $H_i \cap H_j \cap H_k$. The desired statement is equivalent to the fact that a plane angle of an acute-angled trihedral angle in a three-dimensional space does not exceed the dihedral angle opposite to it, which is a simple consequence of formulae of spherical trigonometry.

The last statement of the theorem in the case $Y = H_i$ is implied by the evident fact that the projection of P on H_i is contained in P (it is sufficient to verify this for $s = 2$). □

By definition, a convex polyhedron (1) is acute-angled if for any distinct indices i, j for which the hyperplanes H_i and H_j intersect, the dihedral angle $H_i^- \cap H_j^-$ does not exceed $\frac{\pi}{2}$. This does not necessarily refer just to the dihedral angles at $(n-2)$-dimensional faces of the polyhedron P, as one cannot *a priori* exclude the situation when the hyperplanes H_i and H_j intersect but the $(n-1)$-dimensional faces $F_i = H_i \cap P$ and $F_j = H_j \cap P$ are not adjacent. However, the preceding theorem shows that for acute-angled polyhedra such a situation is in fact impossible.

A question arises, whether it is enough to mention just the dihedral angles at $(n-2)$-dimensional faces in the definition of acute-angled polyhedra. The affirmative answer is provided by the following theorem, which the reader can, however, ignore since it is not essential for what follows.

Theorem 1.4 (Andreev [1970c]). *If the dihedral angles at all $(n-2)$-dimensional faces of a convex polyhedron P do not exceed $\frac{\pi}{2}$, then the hyperplanes of its non-adjacent $(n-1)$-dimensional faces do not intersect, and consequently the polyhedron P is an acute-angled one.*

Theorem 1.3 shows that each face of an acute-angled polyhedron P is also an acute-angled polyhedron, the dihedral angles of which do not exceed the respective dihedral angles of the polyhedron P.

1.4. Acute-angled Polyhedra on the Sphere and in the Euclidean Space

Theorem 1.5. *Any non-degenerate acute-angled polyhedron P on the sphere S^n (respectively, in the Euclidean space E^n) is a simplex (respectively, a direct product of a number of simplices and a simplicial cone).*

The last statement means that the space E^n can be decomposed into the direct product $E^{n_0} \times E^{n_1} \times \dots E^{n_k}$ $(n_0 \geq 0,\ n_1, \dots, n_k > 0,\ n_0 + n_1 + \dots + n_k = n)$ in such a way that $P = P_0 \times P_1 \times \dots \times P_k$, where P_0 is a simplicial cone in the space E^{n_0}, and $P_1, \dots, P_k$ are simplices in the spaces $E^{n_1}, \dots, E^{n_k}$, respectively.

This theorem is implied by the following simple lemma (for the proof see, e.g. Bourbaki [1969]).

Lemma 1.6. *Any system of vectors $\{e_1, \dots, e_s\}$ in a Euclidean vector space satisfying the condition*

$$(e_i, e_j) \leq 0 \qquad \text{for} \quad i \neq j$$

is a union of finitely many subsystems which are orthogonal to each other, and each of them is either linearly independent or satisfies a unique (up to a scalar multiple) linear condition with positive coefficients.

A simplicial cone in E^n cuts on the sphere S^{n-1} (with centre at the apex of the cone) a simplex having the same dihedral angles. Therefore a complete description of all acute-angled polyhedra on the sphere and in a Euclidean space can also be achieved by describing acute-angled simplices. A convenient way to do that is in terms of their Gram matrices.

A square matrix A is said to be *decomposable* if by some permutation of the rows and the same permutation of the columns it can be brought to the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 and A_2 are square matrices. One can easily see that the Gram matrix of a Euclidean simplex is not decomposable.

Theorem 1.7. *Any positive definite (respectively, degenerate indecomposable positive semi-definite) symmetric matrix with 1's on the diagonal and non-positive entries off it is the Gram matrix of a spherical (respectively, Euclidean) simplex which is defined uniquely up to a motion (respectively, up to a similarity transformation).*

Proof. The proof follows immediately from Lemma 1.6 and the fact that any positive semi-definite symmetric matrix is the Gram matrix for some system of vectors in a Euclidean vector space. □

1.5. Simplicity of Acute-angled Polyhedra. The fact that any non-degenerate acute-angled spherical polyhedron is a simplex implies a restriction on the combinatorial structure of an acute-angled polyhedron in any space of constant curvature.

A convex polyhedron P is said to be *simple in its* $(n-k)$*-dimensional face* F if this face is contained in exactly k (i.e. the least possible number of) $(n-1)$-dimensional faces. Then it is also simple in any face containing F. A polyhedron P is said to be *simple* if it is simple in each of its faces. Evidently, for a bounded polyhedron, simplicity in its vertices is sufficient.

Theorem 1.8. *Any acute-angled polyhedron P is simple.*

Proof. Let, e.g., p be a vertex of the polyhedron P. Consider the intersection of P with a sufficiently small sphere with centre p. This intersection is a non-degenerate acute-angled spherical polyhedron, and hence a simplex. The last statement means that the polyhedron P is simple at the vertex p. □

§ 2. Acute-angled Polyhedra in Lobachevskij Space

2.1. Description in Terms of Gram Matrices. The combinatorial structure of an acute-angled polyhedron in Lobachevskij space can be rather complicated, so its description in terms of Gram matrices requires a separate study of systems of linear inequalities of a special kind.

A system of hyperplanes is said to be *indecomposable* if it cannot be decomposed into two non-empty subsystems perpendicular to one another, and *non-degenerate* if these hyperplanes have no point in common (either ordinary or at infinity), and are not perpendicular to a single hyperplane.

Theorem 2.1. *Let $\{H_1^-,\dots,H_s^-\}$ be an acute-angled family of half-spaces of the space $Л^n$ such that the family of hyperplanes $\{H_1,\dots,H_s\}$ is indecomposable and non-degenerate. Let*

$$P^- = \bigcap_{i=1}^{s} H_i^-, \quad P^+ = \bigcap_{i=1}^{s} H_i^+.$$

Then one of the following statements holds:

(1) *P^- has a non-empty interior, and P^+ is empty;*
(2) *P^+ has a non-empty interior, and P^- is empty.*

Proof. Taking the vector model, consider the unit vectors $e_1,\dots,e_s \in \mathbb{R}^{n,1}$ such that

$$H_i^- = \{x \in Л^n : (x,e_i) \le 0\}.$$

Let $G = (g_{ij})$ be their Gram matrix. By hypothesis, it is indecomposable and of rank $n+1$, and consequently has signature $(n,1)$. Let $\lambda > 0$ be its smallest eigenvalue. Then $1-\lambda$ is the greatest eigenvalue of the matrix $E-G$, which has only non-negative entries. By the Perron-Frobenius theorem (see, e.g. Gantmakher [1967]) this eigenvalue corresponds to an eigenvector $c = (c_1,\dots c_s)$ with positive components. Consider the vector

$$v = \sum_{j=1}^{s} c_j e_j \in \mathbb{R}^{n,1}.$$

One has

$$(v,e_i) = \sum_{j=1}^{s} g_{ij} c_j = \lambda c_i < 0 \quad \text{for all } i,$$

$$(v,v) = \sum_{j=1}^{s} c_j (v,e_j) < 0,$$

whereby either $v \in C^+$ and $v_0 = \frac{v}{\sqrt{|(v,v)|}}$ is an interior point of the set P^-, or $v \in C^-$ and v_0 is an interior point of the set P^+. Suppose the former case holds, and let $x \in P^+$. Then

$$(v,x) = \sum_{j=1}^{s} c_j (e_j,x) > 0,$$

which is impossible since $v,x \in C^+$. Hence $P^+ = \emptyset$. □

We say that a convex polyhedron in the space $Л^n$ is *indecomposable* if the family of hyperplanes bounding it is indecomposable, or, which is the same, that its Gram matrix is indecomposable. For a non-degenerate polyhedron, this is equivalent to the fact that none of its faces is perpendicular to all its other $(n-1)$-dimensional faces not containing it. Any convex polyhedron of finite volume is indecomposable.

Theorem 2.1 implies (see Sect. 1.2) the following theorem.

Theorem 2.2. *Any indecomposable symmetric matrix of signature $(n,1)$ with 1's along the main diagonal and non-positive entries off it is the Gram matrix for some convex polyhedron in the space $Л^n$. This polyhedron is defined uniquely up to a motion.*

2.2. Combinatorial Structure. Let $P \subset Л^n$ be a convex polyhedron. We associate with each face F of P the submatrix G^F of the matrix $G(P)$ formed by the rows and columns corresponding to the $(n-1)$-dimensional faces of the polyhedron P containing F. The last statement of Theorem 1.3 implies the following theorem.

Theorem 2.3. *If P is a convex polyhedron, then the mapping $F \mapsto G^F$ establishes a one-to-one correspondence between the set of all faces of the* *polyhedron P and the set of all positive definite principal submatrices of the matrix $G(P)$.*

Thus, the principal submatrices of $G(P)$ corresponding to the faces of the polyhedron P are characterized by their intrinsic properties. A similar characterization can be obtained for the principal submatrices corresponding (in the same sense) to vertices at infinity.

Definition 2.4. A point at infinity $p \in \partial Л^n$ is a *vertex at infinity* of a convex polyhedron $P \subset Л^n$ if $p \in \overline{P}$ and the intersection of P with a sufficiently small horosphere S_p with centre p is a bounded subset of this horosphere regarded as an $(n-1)$-dimensional Euclidean space.

The polyhedron $P \cap S_p$ is evidently convex, and its dihedral angles are equal to the respective dihedral angles of the polyhedron P. In particular, if P is an acute-angled polyhedron, then so is $P \cap S_p$, and hence it is a direct product of simplices. Thus the combinatorial structure of an acute-angled polyhedron in a neighbourhood of any of its vertices at infinity is that of a cone over a direct product of simplices.

A symmetric matrix with non-positive elements off the diagonal is said to be *parabolic* if by some permutation of the rows and the same permutation of the columns it can be brought to the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix},$$

where $A_1, \ldots, A_k$ are degenerate indecomposable positive semi-definite matrices.

The preceding statements imply that if P is an acute-angled polyhedron and p is its vertex at infinity, then the matrix G^p is a parabolic one.

Theorem 2.5. *If P is an acute-angled polyhedron, then the mapping $p \mapsto G^p$ establishes a one-to-one correspondence between the set of all vertices at infinity of the polyhedron P and the set of all parabolic principal submatrices of rank $n-1$ of the matrix $G(P)$.*

For the proof see, e.g., Vinberg [1985].

2.3. Description in Terms of Dihedral Angles. A convex k-gon with the angles $\alpha_1, \ldots, \alpha_k$ (in particular, an acute-angled one) exists on the Lobachevskij plane if and only if

$$\alpha_1 + \ldots + \alpha_k < \pi(k-2)$$

(see Chap. 3, Sect. 2.5), and depends (up to a motion) on $k-3$ parameters. The situation is completely different in spaces of larger dimension.

A *combinatorial type* of convex polyhedra of finite volume in the space $Л^n$ is the set of all polyhedra whose closures in $Л^n$ are combinatorially isomorphic to a given bounded convex polyhedron Q in the space E^n. The combinatorial type is said to be *simple* if the polyhedron Q is simple, and *almost simple* if the combinatorial structure of Q in a neighbourhood of each of its vertices is that of a cone over a direct product of simplices.

Theorem 2.6 (Andreev [1970a]). *A bounded acute-angled polyhedron in the space $Л^n$ for $n \geq 3$ is defined by its combinatorial type and dihedral angles uniquely up to a motion.*

The proof of this theorem is easily reduced to the three-dimensional case, where it is proved in the same way as the Cauchy theorem about the uniqueness of a bounded Euclidean polyhedron with a given development. Theorem 2.6 can also be considered as the statement dual (for acute-angled polyhedra) to the hyperbolic counterpart of the Cauchy theorem. It is quite remarkable that there is no Euclidean counterpart of Theorem 2.6.

Theorem 2.6 probably also holds for unbounded polyhedra of finite volume.

We now consider the problem of the existence of an acute-angled polyhedron with given dihedral angles. A satisfactory solution is known only in the two- and three-dimensional cases, where the existence conditions are expressed by simple inequalities.

Theorem 2.7 (Andreev [1970a]). *A bounded acute-angled polyhedron of a given simple combinatorial type, different from that of a tetrahedron or a triangular prism, and having given dihedral angles exists in the space $Л^3$ if and only if the following conditions are satisfied:*

(1) *if three faces meet at a vertex, then the sum of dihedral angles between them is greater than π;*
(2) *if three faces are pairwise adjacent, but not concurrent, then the sum of the dihedral angles between them is less than π;*
(3) *if four faces are "cyclically" adjacent (like the lateral faces of a quadrilateral prism), then at least one of the dihedral angles between them is different from $\frac{\pi}{2}$.*

For a triangular prism an extra condition is necessary, namely that at least one of the angles formed by the lateral faces with the bases must be different from $\frac{\pi}{2}$, and for a tetrahedron that the determinant of the Gram matrix (which in this case is completely determined by the dihedral angles) must be negative.

The necessity of these conditions is almost evident. If a desired polyhedron exists, then the dihedral angles referred to in the first (respectively, second) condition are equal to the angles of the triangle cut by the planes of the three faces on the sphere (respectively, plane) orthogonal to all three of them, and therefore the sum of these angles is greater (respectively, less) than π. If all the dihedral angles referred to in the third condition equal $\frac{\pi}{2}$, then the principal submatrix of the Gram matrix corresponding to the four faces under consideration is of the form

$$\begin{pmatrix} 1 & -a & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & 0 & 1 & -b \\ 0 & 0 & -b & 1 \end{pmatrix}, \quad a,b>1,$$

and its negative index of inertia equals 2, which is impossible.

The sufficiency of the conditions can be considered as a theorem dual (in the case of acute-angled polyhedra) to the hyperbolic counterpart of the A.D.Aleksandrov [1950] theorem on the existence of a bounded Euclidean polyhedron with a given development. The proof utilizes the basic ideas of the proof of Aleksandrov's theorem as well as some additional considerations.

Theorem 2.8 (Andreev [1970b]). *An acute-angled polyhedron of finite volume having a given almost simple combinatorial type other than that of a tetrahedron or a triangular prism and having given dihedral angles exists in the space $Л^3$ if and only if the following conditions are satisfied:*

(1) *if three faces meet at a vertex, then the sum of the dihedral angles between them is at least π;*
(2) *if four faces meet at a vertex, then all dihedral angles between them equal $\frac{\pi}{2}$;*
(3) *if three faces are pairwise adjacent but not concurrent at a vertex, then the sum of the dihedral angles between them is less than π;*
(4) *if a face F_i is adjacent to faces F_j and F_k, while F_j and F_k are not adjacent but have a common vertex not in F_i, then at least one of the angles formed by F_i with F_j and with F_k is different from $\frac{\pi}{2}$;*
(5) *if four faces are "cyclically" adjacent but do not meet at a vertex, then at least one of the dihedral angles between them is different from $\frac{\pi}{2}$.*

The extra conditions for the cases of a tetrahedron and a triangular prism are the same as those for bounded polyhedra.

A generalization of these results to arbitrary (not necessarily acute-angled) polyhedra in $Л^3$ see in Hodgson, Rivin, and Smith [1992].

Chapter 7 Volumes

All the results now known on the volumes of solids on the sphere and in Lobachevskij space have either already appeared in the classical works of Lobachevskij (see Collected Works [1949a, 1949b]) and Schläfli [1858] or are based on their ideas. (Of the more recent works on this subject we note the papers by Coxeter [1935] and Milnor [1982].)

In the present chapter our intention is to present these results in their final form and to provide a choice of proofs the ideas of which are the simplest.

Following the lines of the preceding chapters we assume, unless otherwise stated, that the action is unfolding in an n-dimensional space of constant curvature denoted by X. The parameter ε used in a number of formulae is set, by definition, to be 1 for $X = S^n$ and -1 for $X = Л^n$.

In this chapter the *director cone* of a set M at a point p is defined as the cone $C_p(M) \subset T_p(M)$ formed by the tangent rays to segments px for all $x \in M$.

The *angular measure* of a cone in a Euclidean vector space V is the volume of its intersection with the unit sphere. For a cone C its angular measure is denoted by $\sigma(C)$. In particular, $\sigma(V)$ is the volume of the unit sphere in the space V. We remind the reader that the volume of the unit sphere in the $(n+1)$-dimensional Euclidean space is given by

$$\sigma_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} = \begin{cases} \frac{2^{k+1}\pi^k}{(2k-1)!!} & \text{if } n = 2k; \\ \frac{2\pi^{k+1}}{k!} & \text{if } n = 2k+1. \end{cases} \tag{1}$$

For a convex cone $C \subset V$ the *dual* cone C^* is the convex cone

$$C^* = \{x \in V : (x,y) \geq 0 \quad \text{for all} \quad y \in C\}.$$

§ 1. Volumes of Sectors and Wedges

1.1. Volumes of Sectors. Let B be a ball of radius r. A *spherical sector* is a subset Q of B containing with each of its points the whole radius of B passing through this point. The *central angle* of a spherical sector Q is the cone $C(Q) = C_p(Q) \subset T_p(X)$, where p is the centre of the ball B (for notation see the introduction to the present chapter).

With the use of the spherical decomposition of the space X (Chap. 4, Sect. 3.1) one easily obtains the following formula for the volume of a spherical sector:

$$\operatorname{vol} Q = \varphi(r)\sigma(C(Q)), \tag{2}$$

where

$$\varphi(r) = \begin{cases} \int_0^r \sin^{n-1}\xi d\xi & \text{if } X = S^n; \\ \frac{1}{n}r^n & \text{if } X = E^n; \\ \int_0^r \sinh^{n-1}\xi d\xi & \text{if } X = Л^n. \end{cases}$$

For the Lobachevskij space $Л^n$ a *horospherical sector* is a subset of the horoball B containing with each of its points the radius of the horoball passing through this point. Let $P(Q)$ be the intersection of the horospherical sector Q with the horosphere $S = \partial B$. The horospherical decomposition of the space X yields the formula

$$\operatorname{vol} Q = \frac{1}{n-1} \operatorname{vol} P(Q). \tag{3}$$

Finally, the *equidistant sector* of height h is the set Q formed by perpendiculars of length h to the hyperplane H starting at the points of a set $B(Q) \subset H$ and lying on the same side of the hyperplane H. The set $B(Q)$ is said to be the *base* of the sector Q. The equidistant decomposition of the space X gives the formula

$$\operatorname{vol} Q = \psi(h) \operatorname{vol} B(Q), \tag{4}$$

where

$$\psi(h) = \begin{cases} \int_0^h \cos^{n-1} \xi d\xi & \text{if } X = S^n; \\ h & \text{if } X = E^n; \\ \int_0^h \cosh^{n-1} \xi d\xi & \text{if } X = \text{Л}^n. \end{cases}$$

1.2. Volume of a Hyperbolic Wedge. Let p be a point in a hyperplane H, and P an $(n-2)$-dimensional plane in H that does not contain p. Let F be a subset of P, and $M \subset H$ the set formed by the segments joining p to the points of F. Denote by g_t the parallel displacement by t along the line l passing through p and orthogonal to H. The set W spanned by M for all parallel displacements g_t, $0 \le t \le a$, is called the *hyperbolic wedge* of thickness a; the set F is said to be its *edge* and the point p its *vertex* (Fig. 51).

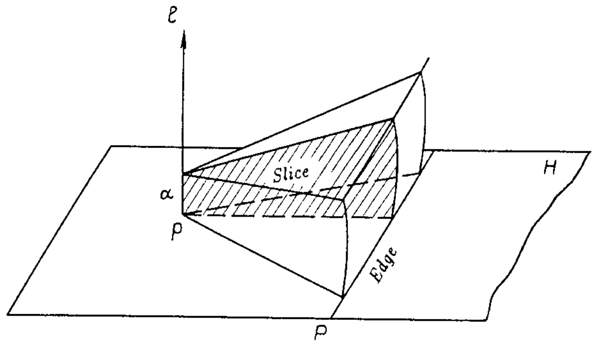


Fig. 51

Let Π be the 2-plane passing through the point p and orthogonal to the plane P. The *slice* of the wedge W is the part of the plane Π spanned by the perpendicular pq from the point p to the plane P under the parallel displacements g_t, $0 \le t \le a$. (Note that the slice may lie outside the wedge.)

Theorem 1.1. *The volume of a hyperbolic wedge is equal to*

$$\frac{1}{n-1}(\textit{Edge volume})(\textit{Slice area}).$$

(The statement of this theorem is due to E.B.Vinberg, but a very close statement for $X = S^n$ was proved by Schläfli [1858].)

Proof. It is sufficient to prove the statement for an infinitesimal wedge of thickness dt the edge of which is an infinitesimal element dP of the plane P (Fig. 52). Consider the decomposition of the space X associated with the line l. For $X = \text{Л}^n$ this decomposition is of the form $\text{Л}^n = \mathbb{R}_+ \leftthreetimes (S^{n-2} \times \text{Л}^1)$ and the volume of the wedge is equal to

$$dV = \left(\int_0^r \sinh^{n-2} \xi \cosh \xi \, d\xi \right) \sigma(dC) \, dt = \frac{1}{n-1} \sinh^{n-1} r \cdot \sigma(dC) \, dt,$$

where r is the distance from the point p to the edge dP and $dC = C_p(dP) \subset T_p(H)$ is the director cone of the set dP. However, $\sinh^{n-2} r \, \sigma(dC)$ is nothing else but the volume of the projection of the edge dP on the sphere of radius r with centre at the point p in the hyperplane P. This volume equals $\sin \alpha$ vol dP, where α is the angle between the plane P and the segment px, $x \in dP$ (see Fig. 52). Hence

$$dV = \frac{1}{n-1} \sinh r \sin \alpha \operatorname{vol} dP \, dt = \frac{1}{n-1} \sinh a \operatorname{vol} dP \, dt,$$

where a is the length of the perpendicular pq. It only remains to note that $\sinh a \, dt$ is the area of the slice, which is an equidistant sector (shaded in Fig. 52) of the plane Π.

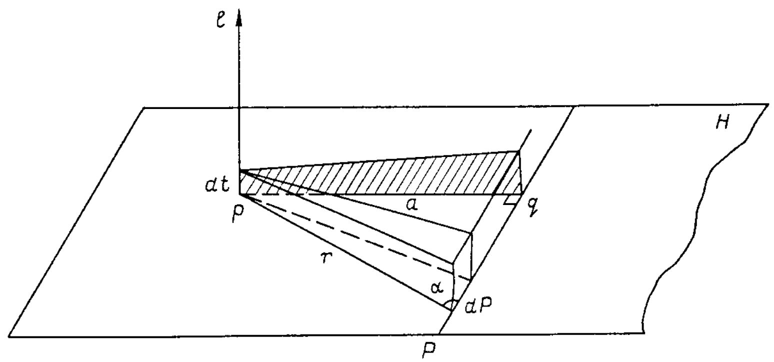


Fig. 52

In the case $X = S^n$ the computation is carried out in a similar way with the hyperbolic functions replaced by their trigonometric counterparts. For $X = E^n$ any wedge is a right prism and the statement of the theorem is easily verified in a straightforward manner. □

1.3. Volume of a Parabolic Wedge. For $X = \text{Л}^n$ one can modify the definition of a hyperbolic wedge by taking for p a point at infinity and for $\{g_t\}$ the one-parameter group of the canonical continuations of parabolic rotations of the plane Π about p. The resulting set W (Fig. 53) is naturally called a *parabolic wedge*.

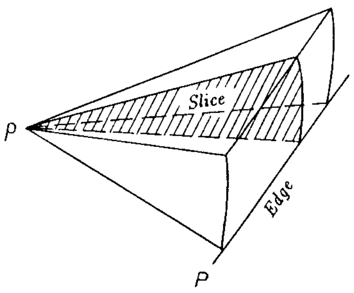


Fig. 53

Theorem 1.1 holds for any parabolic wedge, as one can easily show by passing to the limit. The slice is then a horospherical sector with its area, according to (3), equal to the length of its boundary arc.

1.4. Volume of an Elliptic Wedge. Let P_0 and P be $(n-2)$-dimensional planes[1] in the hyperplane H. Let $F \subset P$ be a set in H lying on the same side of P_0 and let $M \subset H$ be the set spanned by perpendiculars dropped from the points of the set F onto P_0. Denote by g_t the rotation about the plane P_0 through the angle t. The set W spanned by the set M under the rotations g_t, $0 \leq t \leq \alpha$, is said to be an *elliptic wedge* with the angle α (Fig. 54).

Denote by F_0 the projection of the set F on P_0. Using the decomposition associated with the plane P_0, and applying the same kind of reasoning as in the proof of Theorem 1.1, one obtains the following formula for the volume of an elliptic wedge in the spaces $X = S^n$ and Л^n:

$$\varepsilon \operatorname{vol} W = \frac{1}{n-1}\alpha(\operatorname{vol} F_0 - c \operatorname{vol} F), \tag{5}$$

where c equals the cosine of the angle between the planes P and P_0 if they intersect, 1 if they are parallel, and the hyperbolic cosine of the distance between them if they are divergent.

In particular, for the volume of a right circular cone K in the spaces $X = S^3$ and Л^3 one has

$$\varepsilon \operatorname{vol} K = \pi(h - l\cos\theta),$$

[1] For $X = S^n$ we require that the plane P does not pass through the pole of the plane P_0 on H.

where h is the height of the cone, l is its generatrice and θ is the angle between them. (For $X = \text{Л}^3$ this formula was found by Lobachevskij (see Collected Works [1949a]).)

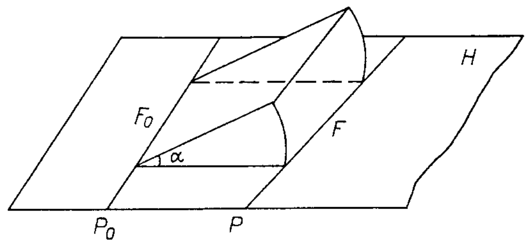


Fig. 54

One can also deduce from formula (5) a rule for computing the volume of an elliptic wedge, similar to Theorem 1.1. The statement, however, depends on the type of the space X and on the mutual position of the planes P and P_0.

§ 2. Volumes of Polyhedra

2.1. Volume of a Simplex as an Analytic Function of the Dihedral Angles. Each simplex in the space $X = S^n$ or Л^n is defined, up to a motion, by its Gram matrix, which, in turn, is defined by the dihedral angles of the simplex. The volume, therefore, is a function of the dihedral angles.

Let $N = \frac{n(n+1)}{2}$ and agree to number the coordinates of vectors in $\mathbb{C}^N$ by unordered pairs of integers i, j, where $i, j = 0, 1, \ldots, n$, $i \neq j$. For each vector $\alpha \in (0, \pi)^N$ denote by $G(\alpha)$ the symmetric matrix of order $n+1$ with 1's on the main diagonal and $(-\cos\alpha_{ij})$ off it. Thus, if α is the set of dihedral angles for some simplex, then $G(\alpha)$ is its Gram matrix.

Denote by M_+ (respectively, M_0, M_-) the family of all sets of dihedral angles for simplexes in S^n (respectively, E^n, Л^n). One can see without difficulty that a vector $\alpha \in (0, \pi)^N$ belongs to one of the sets M_+, M_0 and M_- if and only if the matrix $G(\alpha)$ is of an appropriate signature, and, in the case of M_0 and M_-, if the cofactors of all entries are positive.

Let $M = M_+ \cup M_0 \cup M_-$. For each $\alpha \in M$ denote by $T(\alpha)$ the simplex having α for its set of dihedral angles.

Theorem 2.1 (cf. Coxeter [1935], Aomoto [1977]). *For each even integer n there exists an analytic function v which is defined on an open subset of the space $\mathbb{C}^N$ containing the set M and assumes the following values on M:*

$$v(\alpha) = \begin{cases} \operatorname{vol} T(\alpha) & \textit{if} \quad \alpha \in M_+; \\ 0 & \textit{if} \quad \alpha \in M_0; \\ (-1)^{\frac{n}{2}} \operatorname{vol} T(\alpha) & \textit{if} \quad \alpha \in M_-. \end{cases}$$

For each odd integer n there exists a two-valued analytic function v which is defined on an open subset of the space $\mathbb{C}^N$ containing the set M, ramifies on the set $\det G(\alpha) = 0$, and assumes the following values on M:

$$v(\alpha) = \begin{cases} \pm\operatorname{vol} T(\alpha) & \textit{if} \quad \alpha \in M_+; \\ 0 & \textit{if} \quad \alpha \in M_0; \\ \pm i \operatorname{vol} T(\alpha) & \textit{if} \quad \alpha \in M_-. \end{cases}$$

Proof. For each $\alpha \in M_+ \cup M_-$ we compute the volume of the simplex $T(\alpha)$ using the vector model for the spaces $X = S^n$ and $Л^n$. Let $K(\alpha)$ be the simplicial cone in the ambient (pseudo) Euclidean vector space $\mathbb{R}^{n+1}$ corresponding to the simplex $T(\alpha)$, and let μ be the canonical measure on the space $\mathbb{R}^{n+1}$. Then

$$\int_{K(\alpha)} e^{-|(x,x)|} \mu(dx) = c \operatorname{vol} T(\alpha), \tag{6}$$

where

$$c = \int_0^\infty e^{-t^2} t^n dt \left(= \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right)\right).$$

Let $e_0, e_1, \ldots, e_n$ be the unit vectors orthogonal to the n-dimensional faces of the cone $K(\alpha)$ and directed outside. Their Gram matrix is $G(\alpha)$. The edges of the cone $K(\alpha)$ go along vectors $e_0^*, e_1^*, \ldots, e_n^*$, defined by the conditions $(e_i, e_j^*) = -\delta_{ij}$. Their Gram matrix is the inverse of the matrix $G(\alpha)$. Denote by $g_{ij}^*(\alpha)$ the cofactors of the elements of the matrix $G(\alpha)$. In the coordinate system $\xi_0, \xi_1, \ldots, \xi_n$, defined by the vectors $e_0^*, e_1^*, \ldots e_n^*$, the integral (6) takes the form

$$\int_{\xi_0, \xi_1, \ldots, \xi_n \geq 0} e^{-|\det G(\alpha)|^{-1} \sum g_{ij}^*(\alpha) \xi_i \xi_j} |\det G(\alpha)|^{-\frac{1}{2}} d\xi_0 d\xi_1 \ldots d\xi_n =$$

$$= |\det G(\alpha)|^{\frac{n}{2}} \int_{\xi_0, \xi_1, \ldots, \xi_n \geq 0} e^{-\sum g_{ij}^*(\alpha) \xi_i \xi_j} d\xi_0 d\xi_1 \ldots d\xi_n.$$

For the desired analytic function v one can take

$$v(\alpha) = c^{-1} (\det G(\alpha))^{\frac{n}{2}} \int_{\xi_0, \xi_1, \ldots, \xi_n \geq 0} e^{-\sum g_{ij}^*(\alpha) \xi_i \xi_j} d\xi_0 d\xi_1 \ldots d\xi_n. \qquad \square$$

2.2. Volume Differential. Let a convex polyhedron in the space $X = S^n$ or $Л^n$ be deformed in such a way that its combinatorial structure is preserved

while its dihedral angles vary in a differentiable manner. Under these conditions its volume is also varying in a differentiable manner, and its differential is given by the Schläfli formula

$$\varepsilon\, d(\operatorname{vol} P) = \frac{1}{n-1} \sum_{\dim F = n-2} \operatorname{vol} F \, d\alpha_F, \tag{7}$$

where the sum is taken over all $(n-2)$-dimensional faces F of the polyhedron P and α_F denotes the dihedral angle at the face F. (In Schläfli [1858] this formula is proved for spherical polygons.)

Proof. Let a polyhedron P be triangulated so that the triangulation is deformed together with the polyhedron. The right-hand side of formula (7) then equals the sum of similar expressions for individual simplices of the triangulation. It is therefore sufficient to prove the formula for a simplex.

Let a simplex T be bounded by the hyperplanes $H_0, H_1, \ldots, H_n$. Consider its deformation under which the hyperplane H_0 undergoes a parallel displacement along the line m which is the intersection line of the hyperplanes $H_2, \ldots, H_n$. Under such a deformation the only dihedral angle that changes is the angle α at the $(n-2)$-dimensional face $F = T \cap H_0 \cap H_1$.

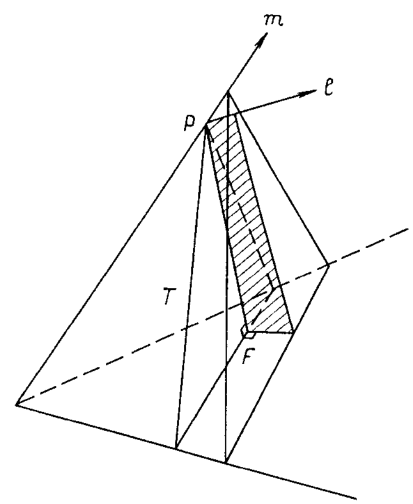


Fig. 55

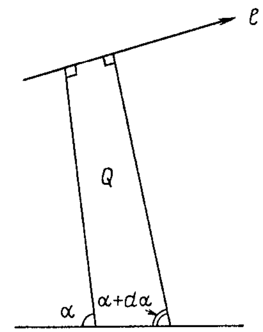


Fig. 56

Let p be the vertex of the simplex T not contained in H_1, and let l be the line passing through p and orthogonal to H_0 (see Fig. 55). An infinitesimal displacement along the line m moves the hyperplane H_0 into the same position as the appropriate (infinitesimal) displacement along the line l. Hence the volume differential for the simplex T equals the volume of an infinitesimal hyperbolic wedge having the edge F and the vertex p and lying between the original position of the hyperplane H_0 and its new position. The area of the slice of this wedge equals the area of the quadrilateral Q shaded in Fig. 55 and shown separately in Fig. 56. (The quadrilateral lies in the 2-plane passing through the point p and orthogonal to the plane of the face F.)

As shown in Fig. 56 the angular excess of the quadrilateral Q equals $d\alpha$, so its area is $\varepsilon d\alpha$ and, by Theorem 1.1, one has

$$\varepsilon \, d \operatorname{vol} T = \frac{1}{n-1} \operatorname{vol} F d\alpha. \qquad \square$$

2.3. Volume of an Even-dimensional Polyhedron. The Poincaré and Schläfli Formulae. For an even integer n the volume of a polyhedron in the space $X = S^n$ or Π^n can be expressed in terms of the volumes of polyhedra of lesser dimension.

We define the necessary notation.

For a cone C in a Euclidean vector space V its *relative angular measure* $\mu(C)$ is defined as the portion of the full angle it occupies, i.e.

$$\mu(C) = \frac{\sigma(C)}{\sigma_{n-1}} \quad (n = \dim V \geq 2).$$

For $n = 1$ and 0 we use the following convention: the relative angular measure of the full space is 1 (as for $n \geq 2$), and the relative angular measure for a ray in the 1-dimensional space is $\frac{1}{2}$.

Now let P be a convex polyhedron. The angle of the polyhedron P at a k-dimensional face F is defined as follows. Let x be an interior point of the face F, and let Π be the $(n-k)$-dimensional plane passing through p and orthogonal to the plane Y of the face F. Now recall that the tangent space $T_x(\Pi)$ is nothing else but the normal space $N(Y)$ of the plane Y (see Chap. 1, Sect. 3.2). The *angle of the polyhedron P at the face F* is the director cone of the section $P \cap \Pi$ at the point x. It is a convex polyhedral cone in the space $N(Y)$, which does not depend on the choice of the point x. This cone will be denoted by P/F.

Under this notation the volume of a (bounded) convex polyhedron P in the space $X = S^n$ or Π^n for even n may be computed by the formula

$$\varepsilon^{\frac{n}{2}} \operatorname{vol} P = \frac{1}{2}\sigma_n \sum_F (-1)^{\dim F} \mu(P/F), \tag{8}$$

where the sum is taken over all faces F of the polyhedron P (including the polyhedron itself). Formula (8) is also known as the Poincaré formula.

Proof. One can easily see that if the polyhedron P is triangulated, the right-hand side of formula (8) equals the sum of similar expressions for individual simplices of the triangulation. Hence it is sufficient to verify the formula for a simplex. Each simplex is defined by its dihedral angles, and Theorem 2.1 implies that the relative angular measure of the angle at each of its faces is an analytic function of the dihedral angles. Hence, by the same theorem, it is sufficient to verify formula (8) for a spherical simplex.

Now let the simplex $T \subset S^n$ be formed by the intersection of half-spaces $H_0^-, H_1^-, \ldots, H_n^-$, and let $\varphi_0, \varphi_1, \ldots, \varphi_n$ be the characteristic functions of these subspaces. One has

$$\operatorname{vol} T = I(\varphi_0 \varphi_1 \ldots \varphi_n),$$

where $I(\varphi)$ denotes the integral of the function φ over the sphere S^n. Since the intersection of the half-spaces $H_0^+, H_1^+, \ldots, H_n^+$ is the simplex antipodal to T, one also has

$$\operatorname{vol} T = I((1-\varphi_0)(1-\varphi_1)\ldots(1-\varphi_n)) =$$

$$= \sum_{k=0}^{n} (-1)^k \sum_{i_1 < \ldots < i_k} I(\varphi_{i_1} \ldots \varphi_{i_k}) - I(\varphi_0 \varphi_1 \ldots \varphi_n),$$

and consequently

$$\operatorname{vol} T = \frac{1}{2} \sum_{k=0}^{n} (-1)^k \sum_{i_1 < \ldots < i_k} I(\varphi_{i_1} \ldots \varphi_{i_k}).$$

It only remains to note that

$$I(\varphi_{i_1} \ldots \varphi_{i_k}) = \operatorname{vol}(H_{i_1}^- \cap \ldots \cap H_{i_k}^-) = \sigma_n \mu(T/F_{i_1 \ldots i_k}),$$

where $F_{i_1 \ldots i_k}$ is the $(n-k)$-dimensional face of the simplex T, cut out by the hyperplanes $H_{i_1}, \ldots, H_{i_k}$. $\square$

An application of Theorem 2.1 also shows that for a Euclidean polyhedron

$$\sum_F (-1)^{\dim F} \mu(P/F) = 0. \tag{9}$$

Similar considerations for odd n prove formula (9) for all three spaces.

The summands in the formula (8) corresponding to the faces of odd codimension are expressible in terms of the volumes of even-dimensional spherical polyhedra and may be dropped out by means of similar formulae. This yields the formula

$$\varepsilon^{\frac{n}{2}} \operatorname{vol} P = \frac{1}{2}\sigma_n \sum_{F : \dim F \text{ even}} \kappa(F) \mu(P/F), \tag{10}$$

where $\kappa(F)$ is a rational coefficient depending only on the combinatorial structure of the face F. (For spherical polyhedra this formula was found and investigated by Schläfli in his 1858 paper.)

For some even-dimensional polyhedra we can give the values of this combinatorial invariant. For the n-simplex T_n one has

$$\kappa(T_n) = A_n = \frac{4(2^{n+2}-1)}{n+2} B_{n+2},$$

where $B_2, B_4, \dots$ are the Bernoulli numbers. In particular,

$$A_0 = 1, \quad A_2 = -\frac{1}{2}, \quad A_4 = 1, \quad A_6 = -\frac{17}{4}, \quad A_8 = 31.$$

For an n-dimensional simple polyhedron P (see Chap. 6, Sect. 1.5)

$$\kappa(P) = \sum_{k=0}^{\frac{n}{2}} A_{2k} a_{n-2k}(P),$$

where $a_i(P)$ is the number of i-dimensional faces of the polyhedron P. (In particular, for an m-gon one has $\kappa = 1 - \frac{m}{2}$.)

2.4. Volume of an Even-dimensional Polyhedron. The Gauss-Bonnet Formula. A different approach to the problem of calculating volumes of even-dimensional polyhedra involves the use of the generalized Gauss-Bonnet formula obtained for arbitrary curved polyhedra on Riemannian manifolds by Allendoerfer and Weil [1943]. (See also Chern [1944]). For a convex polyhedron P in the space $X = S^n$ or $Л^n$ this formula takes the form

$$\sum_{k=0}^{[\frac{n}{2}]} \varepsilon^k \sum_{F:\dim F=2k} \frac{2\operatorname{vol} F}{\sigma_{2k}} \mu((P/F)^*) = 1, \tag{11}$$

where the sum is taken over all faces F of even dimension. For even n it includes the volume of the polyhedron P itself, making it possible to express it in terms of the volumes of polyhedra of lesser dimension. The cone $(P/F)^*$ dual to the angle P/F of the polyhedron P at the face F can be viewed as a generalization of the exterior angle of a polygon.

For a polyhedron P in the Euclidean space formula (11) is reduced to the equality

$$\sum_{F:\,\dim F=0} \mu((P/F)^*) = 1, \tag{12}$$

generalizing the well-known fact that the sum of all exterior angles of a Euclidean polygon equals 2π.

§ 3. Volumes of 3-dimensional Polyhedra

3.1. The Lobachevskij Function. Volumes of some polyhedra in the space $Л^3$ are functions of dihedral angles expressed in terms of elementary functions and the so-called Lobachevskij function. Following Milnor [1982] we define the *Lobachevskij function* denoted by $Л$ by the formula

$$Л(x) = -\int_0^x \ln|2\sin\xi|\,d\xi. \tag{13}$$

This function is related to the function

$$L(x) = -\int_0^x \ln\cos\xi\,d\xi,$$

which is traditionally called the Lobachevskij function, by the equation

$$L(x) = Л(x - \frac{\pi}{2}) + x\ln 2.$$

The function $Л$ is odd, periodic with period π, vanishes at points $\frac{n\pi}{2}$, reaches its maximum equal to

$$M = 0.5074708\dots$$

at the points $n\pi + \frac{\pi}{6}$, and the minimum equal to $-M$ at the points $n\pi - \frac{\pi}{6}$. It is analytic everywhere except at the points $n\pi$ where its derivative tends to $+\infty$ (see Fig. 57). For each natural number n there is the identity

$$Л(nx) = n\sum_{k=0}^{n-1} Л\left(x + \frac{k\pi}{n}\right). \tag{14}$$

In particular,

$$Л(2x) = 2Л(x) + 2Л\left(x + \frac{\pi}{2}\right). \tag{15}$$

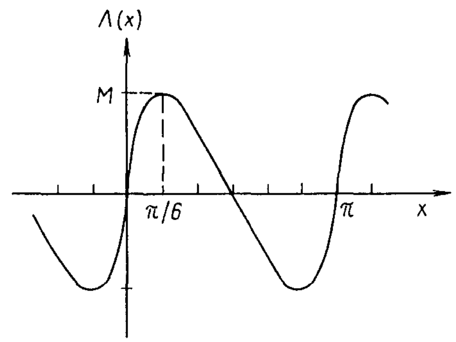


Fig. 57

For the proof of the identity (14) and some other properties of the function Л see Milnor [1982]. For its computation see Zagier [1986].

3.2. Double-rectangular Tetrahedra. A tetrahedron is said to be *double-rectangular* if its edge AB is orthogonal to the plane BCD and its edge CD is orthogonal to the plane DAB (Fig. 58).

Tetrahedra of this kind are of special interest since any tetrahedron can be decomposed into double-rectangular ones by dropping perpendiculars from one of its vertices onto the opposite face and onto the lines bounding that face (Fig. 59). [2] Three out of the six dihedral angles of a double-rectangular tetrahedron T are right angles. Denote the remaining ones by α, β, γ (as shown on Fig. 58). The Gram matrix for the tetrahedron T is of the form

$$G = \begin{pmatrix} 1 & -\cos\alpha & 0 & 0 \\ -\cos\alpha & 1 & -\cos\beta & 0 \\ 0 & -\cos\beta & 1 & -\cos\gamma \\ 0 & 0 & -\cos\gamma & 1 \end{pmatrix}.$$

with the determinant

$$\Delta = \sin^2\alpha\sin^2\gamma - \cos^2\beta.$$

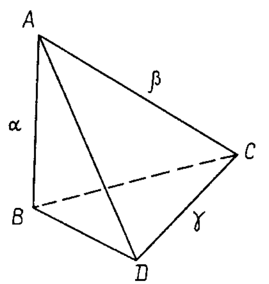


Fig. 58

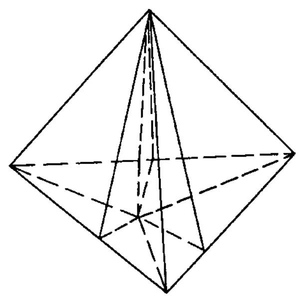


Fig. 59

For $X = S^3$ or $Л^3$ each double-rectangular tetrahedron T is defined by its angles α, β, γ uniquely up to a motion. In the vector model its vertices correspond to one-dimensional subspaces such that the Gram matrix for their basis vectors is the inverse of G (see the proof of Theorem 2.1). This enables one to find the edges of the tetrahedron T.

[2] For the sake of simplicity we say "decomposed" although, since some of the perpendiculars may fall outside the tetrahedron, one may obtain not a decomposition, but an "algebraic sum", which includes some of its terms with minus signs. The same applies to all the subsequent use of the term "decomposition" in the rest of this section.

In particular, denote by a, b, c the edges for which the corresponding dihedral angles are α, β, γ respectively. Then for $X = S^3$ one has

$$\tan a\,\tan\alpha = \tan b\,\tan\left(\frac{\pi}{2} - \beta\right) = \tan c\,\tan\gamma = \frac{\sqrt{\Delta}}{\cos\alpha\,\cos\gamma}, \tag{16}$$

and for $X = Л^3$

$$\tanh a\,\tan\alpha = \tanh b\,\tan\left(\frac{\pi}{2} - \beta\right) = \tanh c\,\tan\gamma = \frac{\sqrt{-\Delta}}{\cos\alpha\,\cos\gamma}. \tag{17}$$

3.3. Volume of a Double-rectangular Tetrahedron. The Lobachevskij Formula. Let $X = Л^3$. With the notation above one can find the (acute) angle δ from the equation

$$\tan\delta = \frac{\sqrt{-\Delta}}{\cos\alpha\,\cos\gamma}. \tag{18}$$

Then the volume of the tetrahedron T may be calculated from the following formula due to Lobachevskij (which he derived in one of the papers included in his Collected Works [1949b]):

$$\operatorname{vol} T = \frac{1}{4}\Big[Л(\alpha+\delta) - Л(\alpha-\delta) - Л\left(\frac{\pi}{2} - \beta + \delta\right) + Л\left(\frac{\pi}{2} - \beta - \delta\right) +$$
$$Л(\gamma+\delta) - Л(\gamma-\delta) + 2Л\left(\frac{\pi}{2} - \delta\right)\Big]. \tag{19}$$

(In Lobachevskij's work β is expressed in terms of other elements of the tetrahedron. The fact that it actually represents a dihedral angle of the tetrahedron has been realized by Coxeter [1935].) For the geometrical meaning of the angle δ see Maier [1954].

Proof. Relations (17) and (18) easily imply that

$$a = \frac{1}{2}\ln\frac{\sin(\alpha+\delta)}{\sin(\alpha-\delta)}, \quad c = \frac{1}{2}\ln\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)},$$

$$b = \frac{1}{2}\ln\frac{\sin\left(\frac{\pi}{2} - \beta + \delta\right)}{\sin\left(\frac{\pi}{2} - \beta - \delta\right)}. \tag{20}$$

Let us now contract the tetrahedron into a point in such a way that it remains double-rectangular and the value of δ does not change. Formula (17) shows that each of the angles α, $\frac{\pi}{2} - \beta$, and γ then tends to $\frac{\pi}{2}$. According to (7)

$$d(\operatorname{vol} T) = \frac{1}{2}(a\,d\alpha + b\,d\beta + c\,d\gamma).$$

Substituting (20) into the last expression and integrating, one obtains (19) (following simple transformations involving formula (15)). □

One can only admire Lobachevskij who found this truly remarkable formula alone, making his way from the very beginnings of non-Euclidean geometry!

If the vertex A of a double-rectangular tetrahedron $T \subset \text{Л}^3$ goes beyond the absolute, one can truncate it by a plane orthogonal to all its faces passing through A. The same applies to the vertex C. Interestingly, the volume of the resulting "truncated double-rectangular tetrahedron" is also given by the Lobachevskij formula (Kellerhals [1989]). This formula also holds if both vertices are truncated but the edge AC contains points of the space Л^3. If the entire edge AC goes beyond the absolute, the volume of the truncated polyhedron, called the Lambert cube, is given by a somewhat different formula (similar to Lobachevskij's) which can also be found in Kellerhals [1989].

Analytic continuation makes it possible to apply the Lobachevskij formula to the computation of volumes of spherical double-rectangular tetrahedra. With that problem in mind, continue the Lobachevskij function from the interval $(0, \pi)$ to the strip $0 < \operatorname{Re} z < \pi$ on the complex plane using the formula

$$\text{Л}(z) = -\int_0^z \ln(2 \sin \zeta) d\zeta, \tag{21}$$

where the integral is taken along an arbitrary path within that strip. For a sufficiently small spherical double-rectangular tetrahedron for which

$$\cos^2 \alpha + cos^2 \beta + cos^2 \gamma > 1,$$

there exists a purely imaginary δ satisfying equation (18). Then the right-hand side of formula (19) makes sense, and by Theorem 2.1 equals $\pm i \operatorname{vol} T$.

The volumes of non-Euclidean polyhedra of odd dimension $n \geq 5$ can apparently be expressed in terms of the Lobachevskij functions of higher order connected to polylogarithms in the same way as the ordinary Lobachevskij function is connected to dilogarithms. At any rate, this is the case for $n = 5$, see Böhm and Hertel [1981] and Kellerhals [1991].

3.4. Volumes of Tetrahedra with Vertices at Infinity. The volume formula for a tetrahedron in Л^3 is greatly simplified if the latter has one or more of its vertices at infinity.

First we note that for a vertex at infinity the sum of the dihedral angles between the faces meeting at this vertex equals π (as the sum of angles of the Euclidean triangle cut out by the tetrahedron on the horosphere having its centre at this vertex).

For a double-rectangular tetrahedron $T = ABCD$ (Fig. 58) having two vertices at infinity A and C one has $\alpha = \frac{\pi}{2} - \beta = \gamma$. The volume of this tetrahedron is easily computed in the Poincaré model (Milnor [1982]) by straightforward integration and turns out to be

$$\operatorname{vol} T = \frac{1}{2} \text{Л}(\alpha). \tag{22}$$

This provides a direct geometric interpretation for the Lobachevskij function. (The formula can, of course, be also obtained from the Lobachevskij formula.)

For a tetrahedron T that has all its vertices at infinity, the relations between dihedral angles imply that the dihedral angles corresponding to opposite edges are equal. Denoting the different dihedral angles of such a tetrahedron by α, β, γ (Fig. 60) one also has $\alpha + \beta + \gamma = \pi$. The volume of such a tetrahedron is now easily computed by decomposing it into six double-rectangular tetrahedra (as in Fig. 59) each having two vertices at infinity (Milnor [1982]). It turns out to be

$$\operatorname{vol} T = \text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma). \tag{23}$$

The expression has its maximum value for $\alpha = \beta = \gamma \quad (= \frac{\pi}{3})$, i.e. if the tetrahedron T is a regular one (Milnor [1982]). (For the generalization of that formula to the n-dimensional case see Haagerup and Munkholm [1981].)

It is shown in Rivin [1992] that the function (23) is concave in (α, β, γ). This immediately implies that it reaches its maximum value at $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$. In fact Rivin proved that for any combinatorial type of polyhedra in Л^3 with all vertices at infinity the ordered sets of their dihedral angles form a convex polyhedron in a suitable space $\mathbb{R}^N$ and the volume is a concave function on it.

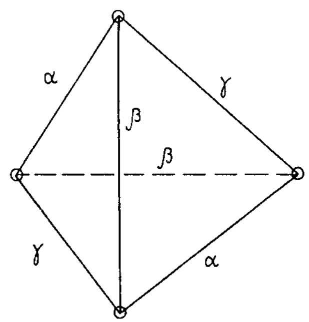


Fig. 60

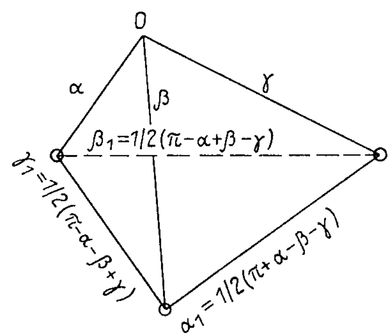


Fig. 61

Finally, for a tetrahedron T with three vertices at infinity, all its dihedral angles are expressed in terms of the dihedral angles α, β, γ at the finite vertex O. Their values are shown in Fig. 61 (on which vertices at infinity are surrounded by small circles). The volume of such a tetrahedron is

$$\operatorname{vol} T = \frac{1}{2}\Big[\text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma) + \text{Л}(\alpha_1) + \text{Л}(\beta_1) + \text{Л}(\gamma_1) - \text{Л}\left(\frac{1}{2}(\alpha + \beta + \gamma - \pi)\right)\Big]. \tag{24}$$

Proof. Indeed, by continuing the edges of the tetrahedron T to infinity one obtains six points at infinity, which can be considered as the vertices of an octahedron. Continuing the faces of the tetrahedron T one cuts this octahedron into eight tetrahedra such that for each of them there is another one symmetric to it with respect to their common vertex O (Fig. 62). One of them is T itself (in Fig. 62 it may, for example, be represented by the tetrahedron $OABC$). Each of these tetrahedra has three vertices at infinity and its dihedral angles at the vertex O coincide either with the dihedral angles of the tetrahedron T or with the angles adjacent to them. Hence, all its dihedral angles can be expressed in terms of the dihedral angles of the tetrahedron T.

Any two adjacent tetrahedra of the decomposition form a tetrahedron having all its vertices at infinity. Their volumes may be computed by formula (23). On the other hand, one can, without great difficulty, find their linear combination equal to T. For example, let T_1 and T_2 be two tetrahedra of the decomposition adjacent to T, and let T_2' be the tetrahedron symmetric to T_2. Then

$$2\operatorname{vol} T = \operatorname{vol}(T \cup T_1) + \operatorname{vol}(T \cup T_2') - \operatorname{vol}(T_1 \cup T_2'). \qquad \square$$

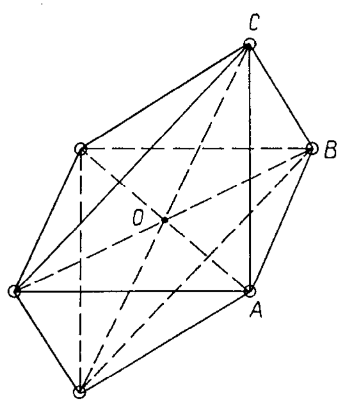


Fig. 62

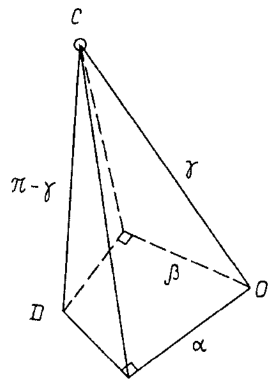


Fig. 63

3.5. Volume of a Pyramid with the Apex at Infinity. The formulae of the preceding section are, in fact, specializations of the formula for the volume of a pyramid with the apex at infinity.

We begin by considering a quadrilateral pyramid P with the apex C at infinity, the side edge CD orthogonal to the base, and the sides of the base issuing from D forming right angles with two other sides. Denote the dihedral angles at the vertex O of the base opposite to D by α, β, γ as shown in Fig. 63. Considering the Euclidean quadrilateral cut out by the pyramid P on the horosphere with centre C, one finds that the dihedral angle at the edge

CD is equal to $\pi - \gamma$ and that all the other dihedral angles are right angles. The volume of the pyramid is

$$\operatorname{vol} P = \frac{1}{2}\left[Л(\gamma) + Л\left(\frac{1}{2}(\pi + \alpha + \beta - \gamma)\right) + Л\left(\frac{1}{2}(\pi + \alpha - \beta - \gamma)\right) + Л\left(\frac{1}{2}(\pi - \alpha + \beta - \gamma)\right) + Л\left(\frac{1}{2}(\alpha + \beta + \gamma - \pi)\right)\right]. \qquad (25)$$

Proof. Continuing to infinity the two sides of the base of the pyramid P issuing from the vertex O, one can include P in a tetrahedron T having three vertices A, B, C at infinity (Fig. 64). The complement to P in this tetrahedron is naturally decomposed into two double-rectangular tetrahedra T_1 and T_2 adjacent to P, each having two vertices at infinity, and the tetrahedron $T_0 = ABCD$. One has

$$\operatorname{vol} P = \operatorname{vol} T - \operatorname{vol} T_0 - \operatorname{vol} T_1 - \operatorname{vol} T_2.$$

The dihedral angles of the tetrahedra T_1 and T_2 at the edge CD equal their dihedral angles at the opposite edges, i.e. α and β respectively. By formula (22)

$$\operatorname{vol} T_1 = \frac{1}{2}Л(\alpha), \quad \operatorname{vol} T_2 = \frac{1}{2}Л(\beta).$$

The volumes of the tetrahedra T and T_0 can be found by formula (24) provided their dihedral angles at ordinary vertices are known. But the dihedral angles of the tetrahedron T at the vertex O are the angles of the pyramid P, i.e. α, β, and γ. Of the dihedral angles of the tetrahedron T_0 at the vertex D two are right angles, while the third is found from the condition that the sum of the dihedral angles at the edge CD belonging to the tetrahedra T_0, T_1, T_2 and the pyramid P equals 2π. The value of the third angle is $\pi - \alpha - \beta + \gamma$. $\square$

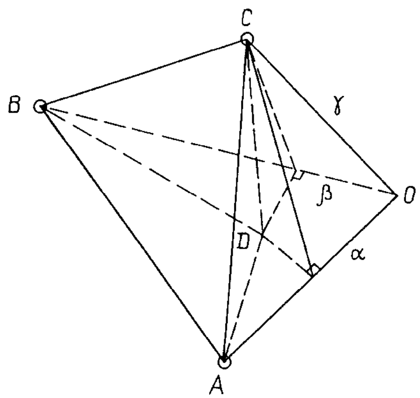


Fig. 64

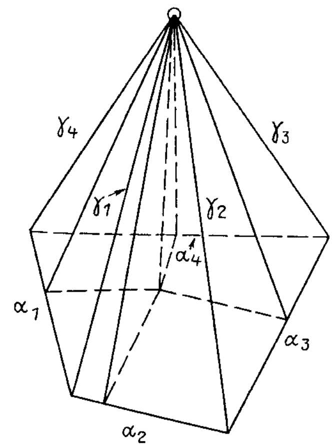


Fig. 65

An arbitrary n-sided pyramid P with the apex at infinity can be decomposed into n quadrilateral pyramids of the above type by dropping perpendiculars from its apex onto its base and onto the lines bounding the base (Fig. 65). Hence the volume of this pyramid is

$$\operatorname{vol} P = \frac{1}{2}\sum_{i=1}^{n}\left[\text{Л}(\gamma_i) + \text{Л}\left(\frac{1}{2}(\pi + \alpha_i + \alpha_{i+1} - \gamma_i)\right)\right.$$
$$+\text{Л}\left(\frac{1}{2}(\pi + \alpha_i - \alpha_{i+1} - \gamma_i)\right) + \text{Л}\left(\frac{1}{2}(\pi - \alpha_i + \alpha_{i+1} - \gamma_i)\right)$$
$$\left.+\text{Л}\left(\frac{1}{2}(\alpha_i + \alpha_{i+1} + \gamma_i - \pi)\right)\right], \tag{26}$$

where $\alpha_1, \ldots, \alpha_n$ are the dihedral angles at the base, $\alpha_{n+1} = \alpha_1$, and $\gamma_1, \ldots, \gamma_n$ are the dihedral angles at the side edges. (The formulae (24), (25) and (26) and the proofs given above are due to E.B.Vinberg [1991]. A partial case of formula (25) for $\gamma = \frac{\pi}{2}$ can be found in Lobachevskij's works (see his Collected Works [1949b]).)

Chapter 8
Spaces of Constant Curvature as Riemannian Manifolds

In this chapter we assume that the reader has a good command of the foundations of Riemannian geometry.

1.1. Exponential Mapping. In the vector model of the space $X = S^n$ or Л^n the exponential mapping $\exp_x : T_x(X) \to X$ at a point $x \in X$ is defined by the formula

$$\exp_x \xi = xc(|\xi|) + \xi s(|\xi|), \tag{1}$$

where the functions c and s are given by the following table:

X	$c(t)$	$s(t)$
S^n	$\cos t$	$\frac{\sin t}{t}$
Л^n	$\cosh t$	$\frac{\sinh t}{t}$

In the case $X = \text{Л}^n$ the mapping $\exp_x$ is a diffeomorphism of the tangent space $T_x(\text{Л}^n)$ onto the space Л^n, and in the case $X = S^n$ it is a diffeomorphism of the ball of radius π in $T_x(S^n)$ onto the punctured sphere $S^n \setminus \{-x\}$.

On the orthogonal subspace of ξ the differential of the exponential mapping is defined by the formula

$$(d_\xi \exp_x)(\nu) = s(|\xi|)\nu \quad \text{for} \quad (\xi, \nu) = 0. \tag{2}$$

Since $\frac{\sin t}{t} \le 1$ (respectively, $\frac{\sinh t}{t} \ge 1$) for all t, the exponential mapping in S^n (respectively, Л^n) does not increase (respectively, does not decrease) the length of tangent vectors, and consequently does not increase (respectively, does not decrease) the distance between points. Hence, for example, given two sides and the included angle in a spherical (respectively, hyperbolic) triangle, the length of the third side is less (respectively, greater) than in the Euclidean one. Incidentally, these properties of the exponential mapping are not related to the fact that the sectional curvature is constant, but only to the fact that its sign is constant. The same results hold for any Riemannian manifolds of non-negative (respectively, non-positive) curvature.

The mapping $\exp_x$ transfers the Cartesian coordinates from the space $T_x(X)$ into the space X yielding the so-called *normal* or *geodesic* coordinates in X. Expressed in these coordinates, the Riemannian metric is of the form

$$ds^2 = s(r)^2 \sum_i dx_i^2 + \frac{1 - s(r)^2}{r^2}\left(\sum_i x_i\, dx_i\right)^2, \tag{3}$$

where $r = \sqrt{\sum x_i^2}$, and the volume element is of the form

$$dx = s(r)^{n-1} dx_1 \ldots dx_n. \tag{4}$$

1.2. Parallel Translation. A parallel translation along a (geodesic) line l in a space of constant curvature X is the differential of a displacement along l (see Chap. 5, Sect. 1.2, 2.3). Thus, under the canonical identification of the orthogonal subspace of $T_x(l)$ in $T_x(X)$ for $x \in l$ with the normal space $N(l)$ of the line l (Chap. 1, Sect. 3.2), a parallel translation along l induces the identical transformation of the space $N(l)$. Hence a parallel translation along any curve lying in a plane Y induces the identical transformation of the space $N(Y)$. This defines an absolute parallelism in the normal bundle over any plane Y.

The holonomy groups of S^n and Л^n coincide with SO_n. This, together with the theory of invariants of the group SO_n, implies that all parallel tensor fields in these spaces are obtained from the metric tensor, its inverse, and the discriminant tensor by the operations of tensor multiplication and contraction.

1.3. Curvature. It is known that if a Riemannian manifold X has constant sectional curvature $\varkappa$ at some point, then its curvature tensor at this point is of the form

$$R_{ij,kl} = \varkappa(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{5}$$

In other words, if the curvature tensor is considered as a linear operator in the space of bivectors, then this operator is a homothetic transformation with the homothetic ratio $\varkappa$.

We now give an outline of the proof that any simply-connected complete Riemannian manifold X of constant curvature is isometric to one of the spaces E^n, S^n, Л^n (cf. Chap. 1, Theorem 1.3).

Let $x^1, \dots, x^n$ be the normal coordinates with centre at a point $x \in X$, Consider the restrictions of the components of the metric tensor $g = (g_{ij})$ to the geodesic $x^i = t\xi^i$, where $\xi = (\xi^i)$ is a unit vector in the space $T_x(X)$. They satisfy the differential equations of Herglotz:

$$g''_{ij} + 2t^{-1}g'_{ij} - \frac{1}{2}g^{kl}g'_{ik}g'_{jl} + 2R_{ik,jl}\xi^k\xi^l = 0, \tag{6}$$

where (g^{ij}) is the inverse of the metric tensor, and g' denotes the derivative of g with respect to t. If the manifold X has a constant curvature $\varkappa$, then, substituting the components of the curvature tensor (5) into (6), one gets the system of ordinary differential equations with respect to the functions g_{ij} in the variable t. Together with the initial conditions

$$g_{ij} = \delta_{ij}, \quad g'_{ij} = 0 \quad \text{for} \quad t = 0,$$

this system defines the metric tensor uniquely.

Since all values $\varkappa$ of the curvature can be realized in the spaces E^n, S^n, Л^n, this argument implies that a Riemannian manifold of constant curvature is locally isometric to one of these spaces. If it is also complete and simply-connected, a local isometry can be extended to a global one.

1.4. Totally Geodesic Submanifolds. Since the planes in a space of constant curvature X are totally geodesic submanifolds, for any k-dimensional direction at a point x of the space X there is a k-dimensional totally geodesic submanifold passing through x in this direction. The following theorem shows that this is a characteristic property of spaces of constant curvature.

Theorem 1.1 (Cartan [1928]). *Let X be a complete Riemannian manifold of dimension $n \geq 3$ and $x \in X$. The following conditions are equivalent:*

(1) *there is an integer k, $2 \leq k < n$, such that for any k-dimensional direction at the point x a totally geodesic submanifold passes through x in this direction;*

(2) *for any k-dimensional (where $k \geq 2$) direction at the point x a totally geodesic submanifold passes through x in this direction;*

(3) *any orthogonal transformation of the tangent space $T_x(X)$ is the differential of a motion preserving the point x.*

Corollary. *Let X be a simply-connected complete Riemannian manifold of dimension $n \geq 3$, and k a fixed integer, $2 \leq k < n$. If for any k-dimensional direction at any point x of the space X a totally geodesic submanifold passes through x in this direction, then X is a space of constant curvature.*

In fact, a weaker condition is sufficient, namely that this property holds for a pair of non-conjugate points of the manifold X (Cartan [1928]).

1.5. Hypersurfaces. Let Y be a hypersurface in a space of constant curvature X. The second fundamental form of the hypersurface Y describing the principal part of its deviation from the tangent plane at the point $y \in Y$ defines a symmetric operator A in the space $T_y(Y)$ called *the Peterson operator*. The curvature tensor S of the hypersurface Y, considered as a linear operator in the space of bivectors, is expressed in terms of the Peterson operator by the Gauss formula (Kobayashi and Nomizu [1963 and 1969])

$$S = \wedge^2 A + \varkappa \, \mathrm{id}, \tag{7}$$

where $\varkappa$ is the curvature of the space X and $(\wedge^2 A)(\xi \wedge \eta) = A\xi \wedge A\eta$ for $\xi, \eta, \in T_y(Y)$.

Hence the curvature $\varkappa(\xi, \mu)$ of the hypersurface Y along the two-dimensional direction defined by a pair of orthogonal unit vectors ξ, μ is

$$\varkappa(\xi, \mu) = \varkappa + (A\xi, \xi)(A\mu, \mu) - (A\xi, \mu)^2.$$

In particular, if the hypersurface Y is convex, then the operator A is either positive or negative definite, and consequently $\varkappa(\xi, \mu) \geq \varkappa$.

A hypersurface is said to be *umbilical* if at each point its Peterson operator is a scalar one. The isotropy considerations show that each standard hypersurface is umbilical.

Theorem 1.2 (Kobayashi and Nomizu [1963 and 1969]). *Any connected umbilical hypersurface in a space of constant curvature of dimension $n \geq 3$ is an open subset of a standard hypersurface.*

As follows from formula (8), any umbilical hypersurface has constant curvature. If $\dim X = n \geq 4$ and $\operatorname{rk} A > 1$, the reverse statement is also true. For all $n \geq 3$ any hypersurface of constant curvature for which $\operatorname{rk} A = 1$ is *developable*, i.e. can be obtained by bending a domain D of some hyperplane Y along a one-parameter family of $(n-2)$-dimensional planes $Z_t \subset Y$, no two of which have common points inside the domain D. Any complete developable hypersurface in S^n is a hyperplane (Borisenko [1981]), and in E^n it is a cylinder (Kobayashi and Nomizu [1963 and 1969]). Complete developable hypersurfaces in Л^n are obtained in the Klein model K^n as intersections of K^n with developable hypersurfaces in the ambient space E^n having their singularities outside K^n.

In a three-dimensional space of constant curvature there are non-developable non-umbilical surfaces of constant curvature. Examples of such surfaces are given by pseudo-spheres in E^3 and analogous surfaces in Л^3, as well as by Clifford surfaces (i.e. orbits of a maximal torus of the group of motions) in S^3. The classification of connected complete surfaces of constant curvature in three-dimensional spaces of constant curvature is a difficult problem of geometry in the large. It is known (see, e.g., Kobayashi and Nomizu [1963 and 1969]) that any such surface in the Euclidean space is either a sphere or a cylinder (which means, in particular, that there are no complete surfaces of constant negative curvature). Any complete non-umbilical surface of constant curvature in S^3 is a Clifford surface (Borisenko [1974]). In the space $Л^3$ there are no complete non-umbilical surfaces of constant positive or null curvature (Volkov and Vladimirova [1971]).

1.6. Projective Properties. A Riemannian manifold is said to be *projectively flat* if any of its points has a neighborhood projectively isomorphic to a domain in the Euclidean space. The Klein model of the Lobachevskij space and the central projection of a hemisphere on the tangent space show that the spaces of constant curvature are projectively flat. The converse statement is also true.

Theorem 1.3 (Rashevskij [1967]). *Any simply-connected complete projectively flat Riemannian manifold is a space of constant curvature.*

Since any projective isomorphism between domains in the projective space can be extended to a global projective transformation (see Sect. 1.2, Chap. 2), any projective automorphism of the Lobachevskij space is a motion. Projective automorphisms of the Euclidean space are affine transformations. Finally, from the projective point of view, the sphere is nothing else but a two-sheeted covering of the projective space, and its group of projective automorphisms is a two-sheeted covering of the projective group. Thus, the spaces E^n and S^n admit projective automorphisms that are not motions. The following theorem shows that this is largely an exclusive property of these two spaces.

Theorem 1.4 (Solodovnikov [1969]). *A simply connected complete analytic Riemannian manifold X admitting a one-parameter group H of projective automorphisms not contained in the group of motions is either $X = S^n$ or $X = E^n \times Y$, where Y is a Riemannian manifold. In the latter case the group H consists of affine automorphisms.*

1.7. Conformal Properties. A Riemannian manifold is said to be *conformally flat* if each of its points has a neighbourhood conformally isomorphic to a domain in Euclidean space. All spaces of constant curvature are conformally flat. There are many other simply-connected complete conformally flat Riemannian manifolds, but the following theorem holds.

Theorem 1.5 (Kuiper [1949]). *Any simply connected compact conformally flat Riemannian manifold is conformally isomorphic to the sphere.*

Any conformal automorphism of Lobachevskij space is a motion (Chap. 2, Sect. 2.2). However, this holds neither for Euclidean space nor the sphere. The group of conformal automorphisms of the space E^n is the group of similarities, while the group of conformal automorphisms of the sphere S^n is the conformal group isomorphic to $O'_{n+1,1}$ (Chap. 2, Sect. 2.1).

Theorem 1.6 (Alekseevskij [1973]). *Any connected Riemannian manifold X admitting a conformal automorphism that is not a motion of a Riemannian manifold conformally isomorphic to X is conformally isomorphic to either the Euclidean space or the sphere.*

1.8. Pseudo-Riemannian Spaces of Constant Curvature. The notion of a space of constant curvature can be transferred to the pseudo-Riemannian case. A pseudo-Riemannian space of constant curvature is defined in group terms as a simply-connected homogeneous space with the isotropy group $O_{k,l}$, and in the framework of pseudo-Riemannian geometry as a simply-connected complete pseudo-Riemannian manifold of signature (k,l) having constant sectional curvature. The equivalence of the two definitions is proved in the same way as in the Riemannian case.

An argument similar to that of Chap. 1, Sect. 2 leads to a classification of pseudo-Riemannian spaces of constant curvature of arbitrary signature (k,l). For $k \geq l > 0$ they are the spaces $(\mathbb{R}^{k+l} \rtimes O_{k,l})/O_{k,l}$, $O_{k+1,l}/O_{k,l}$, $O_{k,l+1}/O_{k,l}$, the curvature of which equals 0, 1, and -1, respectively. The only reservation is that, instead of the space $O_{k,2}/O_{k,1}$ whose fundamental group is isomorphic to $\mathbb{Z}$, one has to take its simply-connected covering $\widetilde{O_{k,2}/O_{k,1}}$. (The other spaces are simply-connected.)

In particular, the pseudo-Riemannian spaces of constant curvature with signature $(3,1)$ are the flat Minkowski space, which provides the geometric basis of special relativity, and the two de Sitter spaces $O_{4,1}/O_{3,1}$ and $\widetilde{O_{3,2}/O_{3,1}}$, used as cosmological models in general relativity.

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II. Discrete Groups of Motions of Spaces of Constant Curvature

E.B. Vinberg, O.V. Shvartsman

Translated from the Russian
by V. Minachin

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Preface

Discrete groups of motions of spaces of constant curvature, as well as other groups that can be regarded as such (although they may be defined differently), arise naturally in different areas of mathematics and its applications. Examples are the symmetry groups of regular polyhedra, symmetry groups of ornaments and crystallographic structures, discrete groups of holomorphic transformations arising in the uniformization theory of Riemannian surfaces, fundamental groups of space forms, groups of integer Lorentz transformations etc. (see Chapter 1). Their study fills a brilliant page in the development of geometry.

The theory of discrete groups of motions of the spaces of constant curvature is, formally, a part of the general theory of discrete subgroups of Lie groups as it formed in the 50s and the 60s. This theory is the subject of Vinberg, Gorbatsevich and Shvartsman [1988][1], an article in another volume of the present series. In the sequel we often refer to this paper when we consider general matters. However, discrete groups of motions of spaces of constant curvature merit separate consideration for a number of reasons.

Firstly, their geometric treatment has been most successful: in particular, in the spaces of constant curvature there is a unique geometric construction of discrete groups generated by reflections (see Chapter 5). Secondly, the group of motions of Euclidean space has a transitive abelian normal subgroup (of parallel translations) and this is especially fortunate for the study of its discrete subgroups (see Chapter 3). Thirdly, there is an argument of an opposite nature, i.e. that discrete groups of motions of Lobachevskij spaces (as well as of its complex counterpart) constitute an exceptional case in the theorem on the arithmeticity of lattices in semisimple Lie groups which precludes any arithmetic solution for the problem of their classification (see Chapter 6). Finally, the fourth aspect is that for discrete groups of motions of the Lobachevskij plane there is, despite that difficulty, a unique theory solving in a sufficiently exhaustive manner the problem of their classification (see Chapter 4).

For the subjects related to the geometry of the spaces of constant curvature we refer to the article "Geometry of Spaces of Constant Curvature", Part 1 of the present volume.

[1] Hereafter referred to as VGS.

Chapter 1
Introduction

§ 1. Basic Notions

1.1. Definition of Discrete Groups of Motions. Let X be an n-dimensional space of constant curvature, i.e. the Euclidean space E^n, the sphere S^n, or the Lobachevskij space $Л^n$, and let $\operatorname{Isom} X$ be the group of its motions.

Definition 1.1. A *family* of subsets $\{M_\alpha : \alpha \in A\}$ of the space X numbered by elements of a set A is said to be *locally finite* if for each point there is a neighbourhood intersecting only finitely many subsets of this family.

This condition is equivalent to the fact that each compact set intersects only finitely many subsets of the family. (We emphasize that some of the subsets M_α may coincide, but are nevertheless considered to be different elements of the family.)

Definition 1.2. A subgroup $\Gamma \subset \operatorname{Isom} X$ is a *discrete group of motions* if for each point $x \in X$ the family $\{\gamma x : \gamma \in \Gamma\}$ is locally finite.

The family $\{\gamma x : \gamma \in \Gamma\}$ contains each point with multiplicity equal to the order of the stabilizer Γ_x of the point x. Hence it is locally finite if and only if the orbit Γx is discrete and the stabilizer Γ_x is finite. (In fact, one can show that the group is discrete provided its orbits are discrete.)

The last definition is equivalent to the general definition of a discrete group of homeomorphisms of a topological space given in VGS. On the other hand, it is equivalent to the condition that Γ is a discrete subgroup of the Lie group $\operatorname{Isom} X$ (for a more general theorem see VGS).

Any finite group of motions is discrete. Any finite group of motions of the space E^n or $Л^n$ has a fixed point (Part 1, Chap. 5, Theorem 3.3) and may be considered as a discrete group of motions of a sphere with its center at that point.

Since the sphere is compact, any discrete group of motions of it is finite. We now give some examples of infinite discrete groups of motions.

Example 1. The group of parallel translations of the space E^n along vectors with integer coordinates in a fixed basis is a discrete group of motions.

Example 2. The group of transformations of the form

$$z \mapsto \frac{az+b}{cz+b}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

is a discrete group of motions of the Lobachevskij plane in the Poincaré model. This is the so-called *Kleinian modular group* arising in the description of the moduli space of elliptic curves (see VGS, Chap. 1, Example 2.6).

1.2. Quotient Spaces. Let Γ be a discrete group of motions of an n-dimensional space X of constant curvature.

Denote by X/Γ the quotient space of X with respect to the equivalence relation defined by the group action, and by p the canonical mapping $X \to X/\Gamma$ taking each point $x \in X$ into its equivalence class, i.e. the orbit Γx.

The set X/Γ is endowed with the quotient topology. A set $U \subset X/\Gamma$ is said to be open if and only if the subset $p^{-1}(U)$ is open in X. It is a Hausdorff topology. The mapping p is continuous and open (Beardon [1983], Chapter 1).

One says that the *group Γ acts freely* (or without fixed points) if the stabilizer of each point is trivial, i.e. if $\gamma x = x$ $(\gamma \in \Gamma,\ x \in X)$ implies $\gamma = e$. In that case p is a covering map, and, in particular, a local homeomorphism. Since the differentiable structure and Riemannian metric on the space X are invariant under the group Γ, the mapping p carries both of them to the quotient space X/Γ.

In the general case, the mapping p is a covering map only over a dense open subspace of the quotient space X/Γ.

More precisely, for each motion $\gamma \in \Gamma$, $\gamma \neq e$, consider the subset X^γ of its fixed points. It is either empty or a plane different from X. If $X^\gamma \neq \emptyset$, then γ preserves the projection of each point x onto the plane X^γ and its distance to that plane; in particular

$$\rho(x, \gamma x) \leq \rho(x, X^\gamma) + \rho(\gamma x, X^\gamma) = 2\rho(x, X^\gamma).$$

This (together with the definition of discrete groups of motions) implies that the family $\{X^\gamma : \gamma \in \Gamma\}$ is locally finite.

Let $X' = X \setminus (\cup_{\gamma \in \Gamma, \gamma \neq e} X^\gamma)$. The preceding considerations imply that it is a dense open subset of the space X. It is invariant under Γ. The quotient space X'/Γ is naturally identified with a dense open subset $p(X')$ of the space X/Γ. Since Γ acts freely on X', the map $p: X' \to X'/\Gamma$ is a covering. It carries both the differentiable structure and Riemannian metric from X' to X'/Γ although in general one cannot extend them to the entire space X/Γ.

Nevertheless, the above construction makes it possible to define, in a unique way, a Borel measure on the quotient space X/Γ, namely, the measure defined by the Riemannian metric on X'/Γ, while the volume of the complement is defined to be zero. Hereafter we always consider this measure on X/Γ.

For each measurable subset $M \subset X$ the subset $p(M) \subset X/\Gamma$ is also measurable and $\operatorname{vol} p(M) \leq \operatorname{vol} M$.

Definition 1.3. A discrete group Γ of motions of the space X, for which $\operatorname{vol} X/\Gamma < \infty$ (respectively for which X/Γ is compact) is called a *crystallographic group of motions* (respectively, *uniform discrete group of motions*).

From the standpoint of the theory of discrete subgroups of Lie groups, crystallographic (respectively, uniform discrete) groups of motions of the space X are lattices (respectively, uniform discrete subgroups) in the Lie group $\operatorname{Isom} X$, see VGS.

A way of finding the quotient space X/Γ is given in the next section.

If Γ is a discrete (and therefore finite) group of motions of the sphere S^n, then

$$\operatorname{vol} S^n/\Gamma = \frac{\operatorname{vol} S^n}{|\Gamma|}. \tag{1}$$

1.3. Fundamental Domains. We keep the notation and assumptions of the preceding section.

Definition 1.4. A locally finite covering of the space X by closed domains[2], no two of which have common interior points, is said to be a *decomposition* of the space X.

Definition 1.5. A closed domain $D \subset X$ such that the subsets γD, $\gamma \in \Gamma$, constitute a decomposition of the space X is a *fundamental domain* of the group Γ.

A fundamental domain is, roughly speaking, a subset that is sufficiently good from the topological point of view and contains representatives of each orbit of the group Γ and, for almost all orbits, exactly one such representative.

For a given discrete group of motions a fundamental domain is not defined uniquely (unlike the quotient space). As a rule, its choice depends on an arbitrary continuous function (see Example 2 below).

The quotient space X/Γ is obtained from a fundamental domain of the group Γ by "pasting together" certain sections of its boundary. We now give a more precise description of that procedure.

The equivalence relation on X defined by the action of the group Γ induces an equivalence relation on any (not necessarily invariant) subset $D \subset X$. The quotient space of D with respect to that equivalence relation will be denoted by D/Γ. It is endowed with the quotient topology. The mapping

$$j: D/\Gamma \to X/\Gamma, \tag{2}$$

associating with each equivalence class in D the orbit of the group Γ that contains it, is a continuous embedding.

Proposition 1.6. *If D is a fundamental domain of the group Γ, then the mapping* (2) *is a homeomorphism. The quotient space X/Γ is compact if and only if D is compact.*

For the proof see, e.g., Beardon [1983].

Note that two points of a fundamental domain can be equivalent only if both of them are boundary points. Pasting such points together one obtains the quotient space X/Γ.

Usually one considers fundamental domains bounded by piecewise differentiable hypersurfaces. Such domains satisfy the condition

[2] By a closed domain we mean the closure of some domain (a non-empty open subset).

$$\operatorname{vol} D = \operatorname{vol} D^0, \tag{3}$$

where D^0 denotes the open interior of the domain D.

Proposition 1.7. *If a fundamental domain D of the group Γ satisfies condition* (3), *then*

$$\operatorname{vol} X/\Gamma = \operatorname{vol} D.$$

In particular, $\operatorname{vol} X/\Gamma < \infty$ *if and only if* $\operatorname{vol} D < \infty$.

Proof. Consider the canonical map $p: X \to X/\Gamma$. Since $p(D) = X/\Gamma$, one has $\operatorname{vol} X/\Gamma \leq \operatorname{vol} D$, but since the map p is injective on D^0, one has $\operatorname{vol} X/\Gamma \geq \operatorname{vol} D^0 = \operatorname{vol} D$. □

Corollary. *The volumes of all fundamental domains of the group Γ satisfying condition* (3) *are equal.*

Example 1. Let $\Gamma = S_n$ be the symmetric group acting in the Euclidean space E^n by permutations of the Cartesian coordinates $x_1, x_2, \ldots, x_n$. For a fundamental domain of the group Γ one can take the cone C defined by the linear inequalities

$$x_1 \geq x_2 \geq \ldots \geq x_n. \tag{4}$$

No two distinct points of that cone are equivalent under Γ, and therefore the quotient space E^n/Γ may be identified with C. Considering this group as the group of motions of the sphere S^{n-1} with centre at the origin, one can take for the fundamental domain $D = C \cap S^{n-1}$. Hence

$$\operatorname{vol} D = \frac{\operatorname{vol} S^{n-1}}{n!}.$$

Example 2. Let Γ be the group of parallel translations of the Euclidean space E^n along vectors with integer coordinates in the basis $\{e_1, \ldots, e_n\}$. Any parallelepiped D having $e_1, \ldots, e_n$ for its edges (and a vertex at an arbitrary point) is a fundamental domain of the group Γ. The quotient space E^n/Γ is obtained by identifying its opposite sides and is an n-dimensional torus. One has

$$\operatorname{vol} E^n/\Gamma = \operatorname{vol} D = |\det\{e_1, \ldots, e_n\}|, \tag{5}$$

where $\det\{e_1, \ldots, e_n\}$ denotes the determinant of the matrix of coordinates of the vectors $e_1, \ldots, e_n$ in an orthonormal basis.

For $n = 2$ the decomposition corresponding to the above fundamental domain is given in Fig. 1a. Figures 1b and 1c show other fundamental domains of the same group. In the coordinate system related to the basis $\{e_1, e_2\}$ the first of them is defined by the inequalities

$$0 \leq x_1 \leq 1, \quad \varphi(x_1) \leq x_2 \leq \varphi(x_1) + 1,$$

where φ is a continuous function. One can take for φ any continuous function defined on the closed interval [0,1].

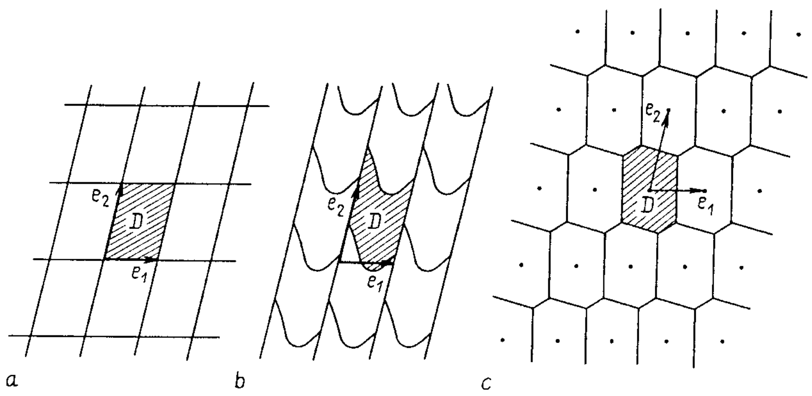


Fig. 1

Example 3. Let Γ be the Kleinian modular group (Example 2 of Section 1.1) acting on the Lobachevskij plane Л^2. As will be shown in Chapter 2 (Example of Section 1.5) one can take for a fundamental domain of that group the triangle D with one vertex at infinity described in the Poincaré model by the inequalities

$$|\operatorname{Re} z| \le \frac{1}{2}, \quad |z| \ge 1. \tag{6}$$

The corresponding decomposition is shown in Fig. 2, where γ_1 and γ_2 denote the motions

$$\gamma_1 : z \mapsto z+1, \quad \gamma_2 : z \mapsto -\frac{1}{z} \tag{7}$$

of the group Γ. The first of them is a parabolic rotation around the vertex of triangle D at infinity, while the second motion is the rotation through π about the midpoint of the opposite side. The motions γ_1 and γ_2 realize the equivalence of boundary points of the triangle D. The quotient space $\text{Л}^2/\Gamma$ is obtained by identifying the two sides of the triangle meeting at the vertex at infinity and the two halves of the third side. It is homeomorphic to the plane $\mathbb{R}^2$ and is, in particular, non-compact. Nevertheless its volume, which is equal to the volume of the triangle D, is finite. More precisely, since the angles of D at ordinary vertices are equal to $\frac{\pi}{3}$, one has

$$\operatorname{vol} \text{Л}^2/\Gamma = \operatorname{vol} D = \pi - \left(\frac{\pi}{3} + \frac{\pi}{3} + 0\right) = \frac{\pi}{3}.$$

1.4. The Dirichlet Domain. The existence of a fundamental domain for any discrete group Γ of motions can be proved by the following construction.

Definition 1.8. The *Dirichlet domain* of the group Γ with centre at the point x_0 is the domain

$$D(\Gamma, x_0) = D(x_0) = \{x \in X : \rho(x, x_0) \le \rho(x, \gamma x_0) \quad \text{for all} \quad \gamma \in \Gamma\}. \tag{8}$$

The domain $D(x_0)$ can be described as follows. Let Γ_0 be the stabilizer of x_0 in the group Γ. For each $\gamma \in \Gamma \setminus \Gamma_0$ denote by H_γ the hyperplane that passes through the midpoint of the segment joining the points x_0 and γx_0 and is orthogonal to it. It is the set of points at equal distance from x_0 and γx_0. The inequality $\rho(x, x_0) \le \rho(x, \gamma x_0)$ defines a half-space bounded by the hyperplane H_γ (which contains x_0). Denote this half-space by H_γ^-. By definition

$$D(x_0) = \cap_{\gamma \in \Gamma \setminus \Gamma_0} H_\gamma^-,$$

implying that $D(x_0)$ is a closed convex subset. Actually it is a generalized convex polyhedron.

We remind the reader that a convex polyhedron in a space X is the intersection of finitely many (closed) half-spaces having a non-empty interior.

Definition 1.9. A *generalized convex polyhedron* is a closed domain such that its intersection with any bounded convex polyhedron containing at least one of its interior points is a convex polyhedron.

One can easily see that any decomposition of the space into convex subsets is a decomposition into generalized convex polyhedra. In particular, any convex fundamental domain of a discrete group of motions is a generalized convex polyhedron.

Proposition 1.10. *The Dirichlet domain $D(x_0)$ of the group Γ is a generalized convex polyhedron invariant with respect to the stabilizer Γ_0 of the point x_0. Subsets of the form $\gamma D(x_0)$, where γ runs over representatives of left cosets of Γ_0 in Γ, form a decomposition of the space X. In particular, if $\Gamma_0 = \{e\}$, then $D(x_0)$ is a fundamental domain of the group Γ.*

Proof. Since $\rho(x_0, \gamma x_0) = 2\rho(x_0, H_\gamma)$ for all $\gamma \in \Gamma \setminus \Gamma_0$ and the orbit Γx_0 is discrete, any ball with centre x_0 intersects only finitely many hyperplanes H_γ. Hence the family $\{H_\gamma : \gamma \in \Gamma \setminus \Gamma_0\}$ is locally finite. Hence it follows that $D(x_0)$ is a generalized convex polyhedron and that

$$D(x_0)^0 = \{x \in X : \rho(x, x_0) < \rho(x, \gamma x_0) \quad \text{for all} \quad \gamma \in \Gamma \setminus \Gamma_0\}. \tag{9}$$

Now it is evident that

$$\gamma D(x_0) = D(\gamma x_0), \tag{10}$$

and therefore $\gamma D(x_0)$ (respectively, $\gamma D(x_0)^0$) consists of those points x for which γx_0 is a point of the orbit Γx_0 nearest to x (respectively, the unique point nearest to x). It follows that subsets of the form $\gamma D(x_0)$, where γ runs over representatives of left cosets of Γ_0 in Γ, cover the entire space X, no two of them have interior points in common, and each point belongs to only finitely many such subsets. This covering is locally finite because such is the covering $\{H_\gamma : \gamma \in \Gamma \setminus \Gamma_0\}$. □

Occasionally it is useful to write the definition of the Dirichlet domain in the following way:

$$D(x_0) = \{x \in X \colon \rho(x, x_0) \leq \rho(\gamma x, x_0) \quad \text{for all} \quad \gamma \in \Gamma\}. \tag{11}$$

If Γ is the group of orthogonal transformations of the space $\mathbb{R}^n$ (considered as a Euclidean space), then a simple computation shows that

$$D(x_0) = \{x \in \mathbb{R}^n \colon (x, x_0) \geq (x, \gamma x_0) \quad \text{for all} \quad \gamma \in \Gamma\}. \tag{12}$$

Example 1. Suppose that the group $\Gamma = S^n$ acts in the space $\mathbb{R}^n$ by permutations of coordinates. For each point $x_0 = (x_{01}, \ldots, x_{0n})$ denote by $S(x_0)$ the set of ordered pairs (i, j) such that $x_{0i} \geq x_{0j}$. Then

$$D(x_0) = \{x = (x_1, \ldots, x_n) \colon x_i \geq x_j \quad \text{for all} \quad (i, j) \in S(x_0)\}.$$

In particular, for $x_{01} > x_{02} > \ldots > x_{0n}$ the domain $D(x_0)$ coincides with the fundamental domain of the group Γ of Sect. 1.3, Example 1.

Example 2. Let Γ be the group of parallel translations of the Euclidean plane E^2 along vectors with integer coordinates in the basis $\{e_1, e_2\}$. A different choice of the basis yields the same group Γ, provided the transition matrix is an invertible integer matrix. Therefore one can assume that

$$0 \leq 2(e_1, e_2) \leq (e_1, e_1) \leq (e_2, e_2)$$

(see VGS, Chap. 3, Sect. 2). Under such conditions the system of inequalities (8) defining the Dirichlet domain of the group Γ is equivalent to the subsystem of six inequalities corresponding to parallel translations along the vectors $\pm e_1, \pm e_2$ and $\pm(e_1 - e_2)$. The domain $D(x_0)$ is either a hexagon or a rectangle (if $(e_1, e_2) = 0$). The point x_0 is its centre of symmetry. Since the group Γ acts freely, $D(x_0)$ is a fundamental domain of the group Γ for any choice of the point x_0. It is this fundamental domain that is shown in Fig. 1c.

Without the constructive part (whose role is not especially important), Proposition 1.10 takes the form of the following statement of principle.

Theorem 1.11. *Any discrete group of motions of a space of constant curvature has a convex fundamental domain.*

Corollary. *Any crystallographic group of motions of Euclidean space is uniform.*

Proof. Indeed, for such a group any convex fundamental domain must be bounded, since in Euclidean space any unbounded convex domain has infinite volume. □

1.5. Commensurable Groups

Definition 1.12. Two subgroups of a group are said to be *commensurable* if their intersection has a finite index in each of them.

Commensurability is an equivalence relation in the set of all subgroups of a given group.

Let X be a space of constant curvature.

Proposition 1.13. *Let one of two commensurable subgroups of the group* $\operatorname{Isom} X$ *be a discrete (respectively, uniform discrete, crystallographic) group of motions. Then the other subgroup is of the same type.*

For the proof (in a more general setting) see VGS, Chap. 1, Sect. 1. The basic argument is that for any subgroup Δ of finite index in a discrete group of motions Γ and for any fundamental domain D of the group Γ one can take for a fundamental domain of Δ the set $\cup_{i=1}^{s} \gamma_i D$, where $\gamma_1, \ldots, \gamma_s$ are representatives of right cosets of Δ in Γ. This implies, in particular, that

$$\operatorname{vol} X/\Delta = [\Gamma \colon \Delta] \cdot \operatorname{vol} X/\Gamma. \tag{13}$$

Going over to a commensurable group often results in major simplifications. For example, by the Schoenflies-Bieberbach theorem (see Chap. 3, Sect. 1), any crystallographic group of motions of the Euclidean space has an (abelian) subgroup of finite index consisting of parallel translations.

For spaces of negative curvature there is the following weak counterpart to the Bieberbach theorem.

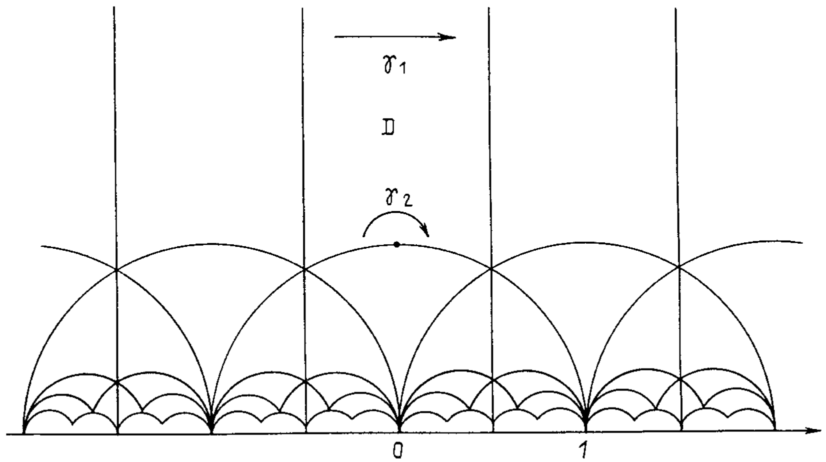


Fig. 2

Theorem 1.14. *Any crystallographic group of motions of Lobachevskij space has a torsion-free subgroup of finite index, and consequently acts freely.*

This theorem is a very special case of the Selberg lemma which states that each finitely generated linear group over a field of characteristic 0 has a torsion-free subgroup of finite index (see VGS, Chap. 1, Theorem 3.2). The fact that any crystallographic group is finitely generated will be proved in Chapter 2 (Theorem 2.5).

On the other hand, it is often convenient to enlarge a crystallographic group instead of diminishing it by adding motions that have many fixed points. For example, extending the Kleinian modular group $PSL_2(\mathbb{Z})$ (Example 2 of Section 1.1) to the group[3] $PGL_2(\mathbb{Z})$, generated by reflections, one can easily see that its fundamental domain is of the form shown in Fig. 2.

§ 2. Origins of Discrete Groups of Motions

2.1. Symmetry Groups. A symmetry of a geometric figure F is defined by the group of motions taking the figure into itself. This group is called the *symmetry group of the figure* F and denoted by $\mathrm{Sym}\, F$. Its subgroup (of index 1 or 2) consisting of proper motions taking F into itself is said to be the proper symmetry group of F, denoted by $\mathrm{Sym}_+ F$.

The word "figure" in the preceding paragraph may mean a subset, or a family of subsets that are allowed to change places, or even a "coloured" family of subsets, i.e. a family divided into subfamilies in such a way that a subset is allowed to change places only with subsets of the same family (i.e. with subsets "colored in the same color"). Each of these concepts includes the preceding one as a special case.

Under certain conditions the symmetry group can be guaranteed to be discrete or even finite.

Proposition 2.1. *Let $F \subset X$ be a finite (respectively, discrete) subset not lying in the same hyperplane. Then* $\mathrm{Sym}\, F$ *is a finite (respectively, discrete) group of motions.*

Classical examples of figures having non-trivial symmetry groups are regular polygons and polyhedra, ornaments, and crystal structures.

The symmetry group of any bounded n-dimensional convex polyhedron (for $n = 2$ a polygon) in Euclidean space coincides with the symmetry group of the set of its vertices, and is therefore finite, by Proposition 2.1. In most cases it is trivial, but some polyhedra possess quite an extensive symmetry groups. The most symmetric of them are regular polyhedra (see Chap. 5, Sect. 3 for a precise definition). Regular polygons and three-dimensional regular polyhedra ("the Platonic solids", see Fig. 3) were already known in antiquity. The multi-dimensional regular polyhedra were enumerated by Schläfli in 1850.

An *ornament* is a surface pattern having a discrete group of symmetries. The most interesting from the mathematical standpoint are plane ornaments

[3] If G a linear group, then PG denotes the projective group defined by it, which is isomorphic to the quotient group of G by the subgroup of all its scalar transformations.

whose symmetry group is a uniform discrete group of motions. Authentic examples of such ornaments taken from the book of Weyl [1952] are given in Fig. 4. One of them represents a window of a mosque in Cairo, and the one next to it shows the lattice work the Chinese use for the support of their paper windows.

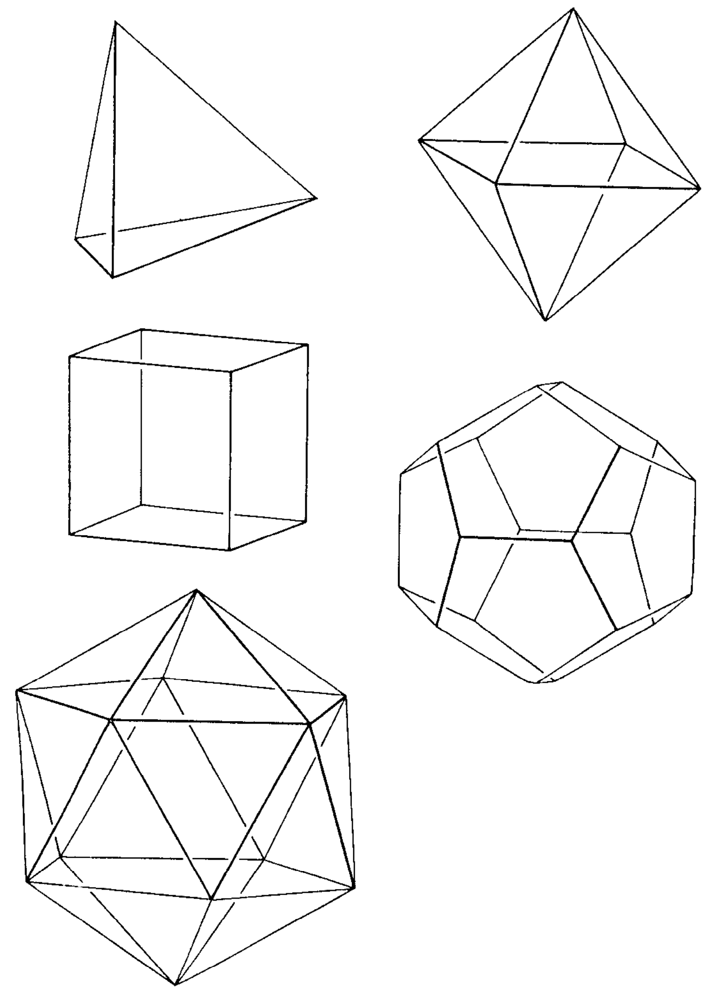


Fig. 3

Crystal structures, i.e. atomic arrays in ideal crystals, are kinds of three-dimensional ornaments. For example, the crystal structure of salt (Fig. 5) consists of atoms of sodium and chlorine alternating at the nodes of a cubic lattice. Figure 6 represents the crystal structures of diamond and graphite, consisting of carbon atoms only.

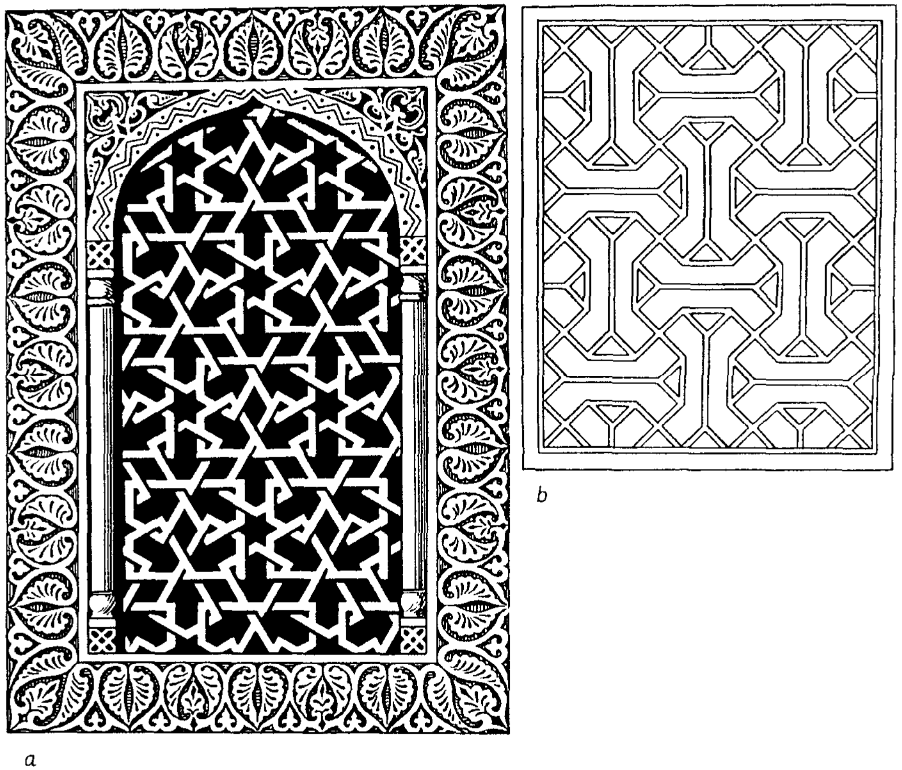


Fig. 4

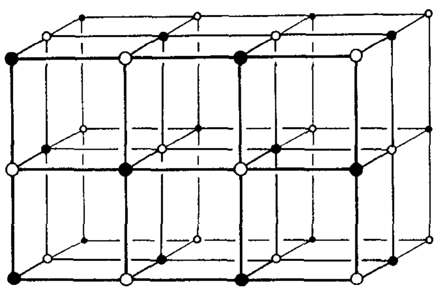


Fig. 5

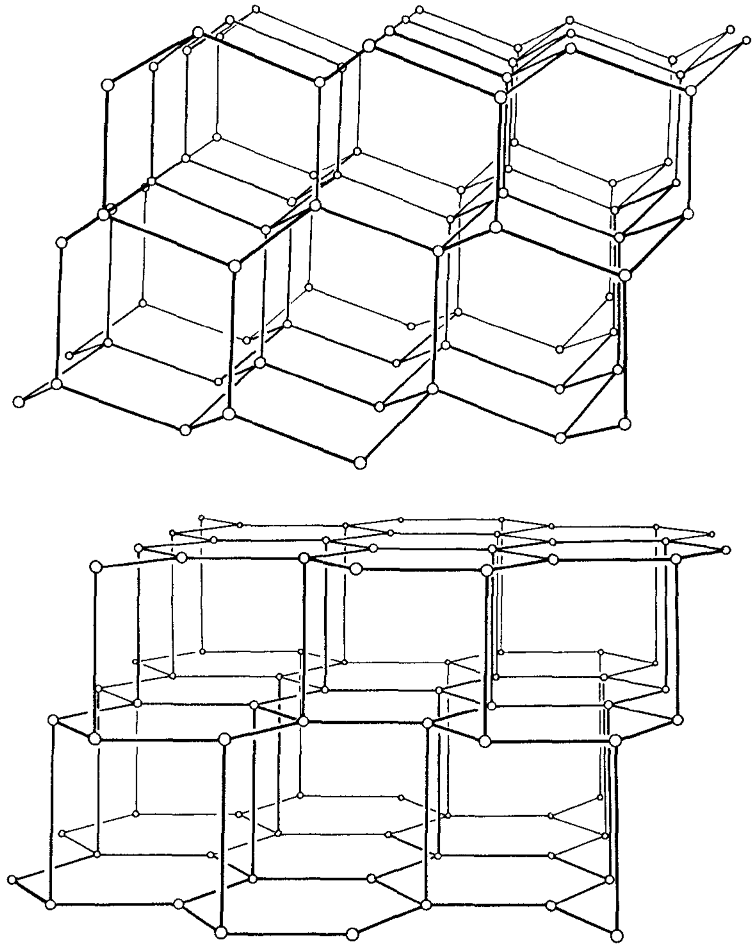


Fig. 6

The symmetry groups of crystal structures (i.e. crystallographic groups in the narrow sense) are uniform discrete groups of motions of three-dimensional Euclidean space. This explains the origin of the general term "crystallographic group" applied to the groups of motions of any space of constant curvature. (Recall that, by the Corollary to Theorem 1.11, crystallographic groups of motions of Euclidean space are uniform discrete groups of motions.)

All crystallographic groups of motions of the Euclidean plane and the three-dimensional Euclidean space were found by E.S.Fedorov around 1890 (that is why they are also called "Fedorov groups"). To within a conjugacy in the group of all affine transformations, there are 17 plane groups, and 219 three-dimensional ones. Independently of Fedorov, and a little later, the crystallographic groups of motions of three-dimensional Euclidean space were enumerated by Schoenflies. Quite remarkably, all of them turned out to be the symmetry groups of actual crystal structures. For more about these groups see in Chap. 3.

2.2. Arithmetic Groups. Consider a rational quadratic form

$$f(x) = \sum_{i,j=0}^{n} a_{ij}x_ix_j \quad (a_{ij} = a_{ji} \in \mathbb{Q})$$

of signature $(n, 1)$ defined in the space $\mathbb{R}^{n+1}$ $(n \geq 2)$ with the coordinates $x_0, x_1, \ldots, x_n$. Denote by $O(f)$ (respectively, $O(f, \mathbb{Z})$) the group of all linear transformations (respectively, integer linear transformations) of the space $\mathbb{R}^{n+1}$ preserving f, and by $O'(f)$ (respectively, $O'(f, \mathbb{Z})$) its subgroup of index 2, consisting of the transformations preserving each of the two connected components of the cone $f(x) < 0$. The group $O'(f)$ is the group of all motions of the Lobachevskij space $Л^n$ in the vector model (see Part 1, Chap. 1), and the group $O'(f, \mathbb{Z})$ is its discrete subgroup, i.e. a discrete group of motions of the space $Л^n$.

For $n = 2$ groups of this type were investigated by Fricke in 1891. For a number of specific groups he explicitly found fundamental domains, and showed, in particular, that each of them has finite area, i.e. that the corresponding group is a crystallographic one. These results were included in the book of Fricke and Klein [1897]. Venkov [1937] proved that the group $O'(f, \mathbb{Z})$ is always a crystallographic group of motions of the space $Л^n$, and that it is uniform if and only if f does not represent zero over $\mathbb{Q}$.

One can easily show that the groups $O'(f_1, \mathbb{Z})$ and $O'(f_2, \mathbb{Z})$ are commensurable (under an appropriate isomorphism of the models of the space $Л^n$ corresponding to the forms f_1 and f_2) if and only if the forms f_1 and f_2 are similar (i.e. equivalent up to a scalar multiple) over $\mathbb{Q}$. Hence this construction enables one to obtain, for any n, infinitely many non-conjugate classes of commensurable crystallographic groups of motions of the space $Л^n$.

A generalization of the preceding construction is possible in which the role of $\mathbb{Q}$ is played by a totally real field of algebraic numbers (see Chap. 6). The resulting groups of motions of the Lobachevskij space are also crystallographic.

The Kleinian modular group $PSL_2(\mathbb{Z})$ (Sect. 1.1, Example 2) is also an arithmetically defined group of motions, although its definition does not fit into the above approach. However, under an appropriate isomorphism of the Lobachevskij plane models, this group turns out to be the subgroup of index 2 of the group $O'(x_0x_1 + x_2^2, \mathbb{Z})$.

2.3. Fundamental Groups of Space Forms. In 1890, inspired by a number of examples due to Clifford, Klein formulated the problem of describing all connected compact Riemannian manifolds of constant curvature. Then Killing showed that these manifolds are Riemannian manifolds of the form

X/Γ, where X is one of the spaces of constant curvature, and Γ is a uniform discrete group of its motions acting freely. He called them *Clifford-Klein space forms*. At present this term is usually applied to all complete connected Riemannian manifolds of constant curvature. They are described in a similar way, the only difference being that the group Γ is not required to be uniform. A space form is said to be *Euclidean, spherical,* or *hyperbolic* depending on the sign of the curvature.

Example 1. A torus (endowed with the flat metric) is a Euclidean space form. It is obtained as a quotient of the Euclidean space with respect to the discrete group consisting of parallel translations along vectors with integer coordinates in a fixed basis. Since the basis can be chosen arbitrarily, this yields a continuum of non-isometric tori.

Example 2. The three-dimensional sphere S^3 can be viewed as the group of unit quaternions so that left and right shifts by the elements of this group are motions of the sphere. For any finite subgroup $\Gamma \subset S^3$ the coset manifold S^3/Γ is a spherical space form. In particular, taking for Γ the binary icosahedral group, i.e. the full inverse image of the proper icosahedral group under the epimorphism $S^3 \to SO_3$ (which identifies antipodes), one gets the so-called *spherical dodecahedral space*[4]. This space was discovered by Poincaré in 1904. Quite remarkably, it has the same homology as the sphere.

Example 3. According to the uniformization theorem, each Riemann surface (i.e. a connected one-dimensional complex manifold) is obtained as a quotient of the Riemann sphere, complex plane, or upper half-plane with respect to a discrete group of holomorphic transformations acting freely. These three complex manifolds can be considered as two-dimensional spaces of constant curvature (see Part 1, Chap. 2). The holomorphic transformations without fixed points are their (proper) motions. Hence Riemann surfaces are orientable two-dimensional space forms and vice versa. In particular, any compact Riemann surface of genus ≥ 2 provides an example of a hyperbolic space form.

The classification problem of spherical space forms is a problem of the theory of finite groups. Its complete solution was published by Wolf [1972]. The problem of classifying compact Euclidean space forms is, by Bieberbach's theory, also reduced to the topics related to finite groups (see Chap. 3, Sect. 2.7).

By virtue of Theorem 1.14 the classification problem for compact hyperbolic space forms is only slightly different from the classification problem of all uniform discrete groups of motions of the Lobachevskij space. At the time of writing the solution of either of these problems seems quite improbable in any dimension greater than three. As for the two- and three-dimensional

[4] The term "dodecahedral space" does not stem from the fact that the corresponding group Γ originates from the dodecahedral group (which is the same as the icosahedral one) but is due to the fact that the fundamental domain of the action of the group Γ on the sphere S^3 can be chosen in the form of a spherical dodecahedron yielding this space if its opposite faces are identified.

cases, each of them is the subject of a separate theory. The two-dimensional theory is virtually completed (see Chap. 4). Its peculiar feature is a close relation to the theory of Riemann surfaces (see Example 3 above). The main achievement of the three-dimensional theory is the establishment of the connection between the classification of three-dimensional space forms and that of three-dimensional topological manifolds.

Compact hyperbolic space forms of dimension ≥ 3 are known to be isometric if and only if they are homeomorphic (see Chap. 7). Thus the classification problem for three-dimensional compact hyperbolic space forms can be reformulated as follows: which of the compact three-dimensional manifolds admit a "hyperbolic structure", i.e. a Riemannian metric of constant negative curvature? There is a conjecture of Thurston [1982] characterizing these manifolds in topological terms. (See also McMullen [1992].) He has proved it in many (in some sense, almost all) particular cases. If it is indeed true in all cases, this means that any compact three-dimensional manifold consists, in a certain sense, of manifolds admitting a hyperbolic structure, and of those admitting *geometric structures of other types. There are 8 types of geometric structures* that must be taken into account (Euclidean, spherical, and hyperbolic among them). The manifolds admitting geometric structures other than hyperbolic are classified in Scott [1983].

The first example of a three-dimensional compact hyperbolic manifold was constructed by Löbell [1931]. Interesting generalizations of this example can be found in Mednykh [1985], and Vesnin [1987].

The ideology we have just described can be extended to discrete groups whose action is not necessarily free, if one considers so-called orbifolds instead of manifolds (see, e.g. Scott [1983]). A classification of simply-connected orientable three-dimensional compact orbifolds admitting geometric structures other than hyperbolic can be found in Dunbar [1988]. In this connection, the following quite amazing result of Hilden, Lozano, Montesinos and Whitten [1987] should be mentioned: there are discrete groups Γ of motions of the space $Л^3$ (called universal) such that any orientable three-dimensional compact manifold is homeomorphic to $Л^3/\Delta$, where Δ is an appropriate subgroup of finite index in Γ (whose action is not necessarily free).

Chapter 2
Fundamental Domains

§ 1. Description of a Discrete Group in Terms of a Fundamental Domain

Fundamental domains and related combinatorial structures form the basis of the combinatorial geometric approach to the study of discrete groups of motions going back to the memoirs of Poincaré about Fuchsian [1882] and Kleinian [1883] groups. Poincaré's approach was subsequently rediscovered, rigorously substantiated and generalized in various ways in numerous later works. See, e.g. A.D.Aleksandrov [1954], Abels [1966], Maskit [1971], Seifert [1975]. In the present exposition our attention is restricted to the spaces of constant curvature and fundamental domains that are convex polyhedra. The reader should, however, be aware of the fact that the main theorems can, without great difficulty, be extended to arbitrary simply-connected Riemannian manifolds and curved fundamental domains.

Below we assume that the action takes place in an n-dimensional space of constant curvature X. In Sect. 1.1–1.3 the term "convex polyhedron" can be understood as referring to a generalized convex polyhedron (see Definition 1.9 of Chap. 1).

1.1. Normalization of a Fundamental Domain. In some cases it is convenient to take for faces of a convex polyhedron some subsets of its true faces. We will say that a locally finite covering of the boundary of a convex polyhedron with closed convex sets (called (false) faces) is a *false faceting* if it satisfies the following conditions:

(F1) no face contains a face of the same dimension;
(F2) the intersection of any two faces is a union of faces.

Properties (F1) and (F2) imply that the boundary of any face is a union of faces.

A decomposition of the space into convex polyhedra is said to be *normal* if the intersection of any two adjacent polyhedra of the decomposition is a face of each of them. A "bricklaying" (Fig. 7) is an example of a decomposition that is not normal.

Any decomposition into convex polyhedra can be canonically normalized by introducing false faces, namely all non-empty intersections of polyhedra of the decomposition that do not contain other such intersections of the same dimension. Thus a "bricklaying" becomes normalized if the constituent rectangles are viewed as hexagons.

A convex fundamental polyhedron P of a discrete group of motions Γ is said to be *normal* if the decomposition $\{\gamma P : \gamma \in \Gamma\}$ it defines is normal. As we have seen above, any convex fundamental polyhedron can be canonically normalized by introducing false faces. Rectangles of the "bricklaying" provide an example.

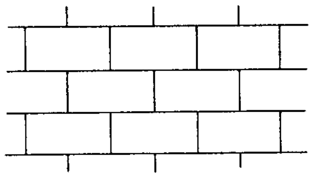


Fig. 7

1.2. Generators. Let P be a convex fundamental polyhedron of a discrete group Γ normalized, if necessary, as in Sect. 1.1.

The polyhedra γP, $\gamma \in \Gamma$, are said to be *chambers*. A sequence of chambers $P_0, P_1, \dots, P_k$ is said to be a *chain of chambers* if $P_0 = P$ and each chamber P_i is adjacent to P_{i-1} for all $i = 1, \dots, k$.

For any $(n-1)$-dimensional face F of the polyhedron P we denote by $s(F)$ the motion of the group Γ taking the polyhedron P into the polyhedron adjacent to P along the face F, and by F' the inverse image of the face F under this motion. Evidently one has

$$s(F) \cdot s(F') = e. \tag{1}$$

Motions of the form $s(F)$ are called *adjacency transformations*. Denote the set of all adjacency transformations by S. There is a one-to-one correspondence between finite sequences of elements of the set S and the chains of chambers, namely, each sequence $(s_1, \dots, s_k)$, where $s_i = s(F_i)$ corresponds to the chain of chambers

$$P_0 = P,\ P_1 = s_1 P,\ P_2 = s_1 s_2 P, \dots, P_k = s_1 s_2 \dots s_k P, \tag{2}$$

in which the chambers P_i and P_{i-1} are adjacent along the face $s_1 s_2 \dots s_{i-1} F_i$ (Fig. 8).

Evidently, any chamber can be joined to P by a chain. This implies the following theorem.

Theorem 1.1. *The group Γ is generated by adjacency transformations.*

Corollary. *Any uniform discrete group of motions has a finite system of generators.*

Similarly, since any chamber containing a point $x \in P$ can be joined to P by a chain of chambers containing this point, one arrives at the following theorem providing the intrinsic description of Γ-equivalence of the points of the polyhedron P.

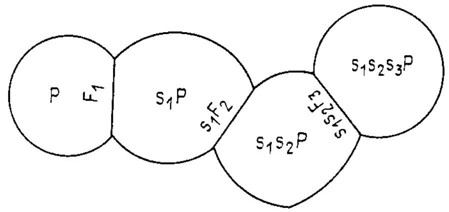


Fig. 8

Theorem 1.2. *If $x, y \in P$ and $x = \gamma y$ for some $\gamma \in \Gamma$, then there exist adjacency transformations $s_i = s(F_i)$ $(i = 1, \dots, k)$ such that $\gamma = s_1 s_2 \dots s_k$ and $s_{i+1} \dots s_k y \in F_i'$ for all $i = 1, \dots, k$.*

1.3. Defining Relations. Relations between the generators of the system S, i.e. equalities of the form

$$s_1 s_2 \dots s_k = e \quad (s_1, \dots, s_k \in S), \tag{3}$$

correspond to the chains of chambers (2) for which $P_k = P$. Such chains are called *cycles of chambers*.

Each $(n-1)$-dimensional face F of the polyhedron P defines the cycle of chambers $(P, s(F)P, P)$ corresponding to relation (1). With each $(n-2)$-dimensional face there is associated a cycle consisting of all chambers containing this face in the order in which they are encountered while circling this face. The corresponding relation is called the *Poincaré relation*.

Theorem 1.3. *The Poincaré relations together with relations* (1) *form a set of defining relations of the group Γ.*

Proof. Since the space X is simply-connected, each cycle of chambers can be transformed into a trivial one (i.e. consisting of a single chamber P) using a sequence of transformations of the following two types:

(1) a segment of the sequence of the form (P_0, P_1, P_0) is replaced by the segment (P_0), or vice versa;
(2) a segment of the form $(P_0, P_1, \dots, P_{k-1}, P_0)$, where $P_0, P_1, \dots, P_{k-1}$ are the chambers surrounding an $(n-2)$-dimensional face of the chamber P_0, is replaced by the segment (P_0), or vice versa.

Transformations of the first type correspond to the application of relations (1), transformations of the second type to the application of the Poincaré relations. □

Consider the Poincaré relations in more detail. Suppose that the circuit around an $(n-2)$-dimensional face f of P corresponds to relation (3). Then

$$f_0 = f,\ f_1 = s_1^{-1} f,\ f_2 = (s_1 s_2)^{-1} f, \dots, f_{k-1} = (s_1 s_2 \dots s_{k-1})^{-1} f \tag{4}$$

are all (possibly repeated) $(n-2)$-dimensional faces of P equivalent to f with respect to the action of the group Γ. The dihedral angle of the polyhedron P at the face f_i equals the dihedral angle of the polyhedron $s_1 s_2 \ldots s_i P$ at the face f. Hence the sum of the dihedral angles of the polyhedron P at the faces $f_0(=f), f_1, \ldots, f_{k-1}$ equals 2π.

The reversed circuit around the face f corresponds (see (1)) to the relation $s_k^{-1} \ldots s_2^{-1} s_1^{-1} = e$ and the circuit around the face f_i to the relation $s_{i+1} s_{i+2} \ldots s_k s_1 s_2 \ldots s_{i-1} = e$. All these relations are equivalent to (3).

Example 1. Suppose that the group S_n acts in the space E^n by permuting the coordinates (Chap. 1, Sect. 1.3, Example 1). Its fundamental polyhedron C is bounded by the hyperplanes $x_i = x_{i+1}$ $(i = 1, \ldots, n-1)$. The adjacency transformations are reflections in hyperplanes. They correspond to transpositions $s_i = (i, i+1)$. Relations (1) are of the form $s_i^2 = e$. The Poincaré relations are of the form $(s_i s_j)^{m_{ij}} = e$ where m_{ij} is half the number of chambers containing the $(n-2)$-dimensional face cut on C by the equations $x_i = x_{i+1}$, $x_j = x_{j+1}$. One can easily see that $m_{ij} = 2$ for $|i-j| > 1$, and $m_{ij} = 3$ for $|i-j| = 1$. Thus

$$S_n = \langle s_1, \ldots, s_{n-1} | s_i^2 = e, \quad s_i s_j = s_j s_i \quad \text{for} \quad |i-j| > 1, \quad (s_i s_{i+1})^3 = e \rangle.$$

Example 2. Let Γ be the group of parallel translations of the space E^n along vectors with integer coordinates (Example 2, Sect. 1.3, Chap. 1). Its fundamental parallelepiped D can be defined in the basis $\{e_1, \ldots, e_n\}$ by the inequalities $0 \le x_i \le 1$. The adjacency transformations are parallel translations $s_1, \ldots, s_n$ along the vectors of the basis $e_1, \ldots, e_n$ and their inverses, which take opposite $(n-1)$-dimensional faces into each other. The Poincaré relations are of the form $s_i s_j s_i^{-1} s_j^{-1} = e$. Theorem 1.3 actually says that Γ is a free abelian group with the generators $s_1, s_2, \ldots, s_n$.

1.4. Points of a Fundamental Domain at Infinity. Investigation of non-uniform discrete groups of motions of Lobachevskij space calls for a closer look at the behaviour of fundamental domains at infinity.

Let P be a convex fundamental domain of a discrete group Γ of motions of the space $Л^n$.

Proposition 1.4. *No point at infinity of the domain P can be a fixed point of a hyperbolic motion belonging to the group Γ.*

Proof. Suppose that the point at infinity q of the domain P is a fixed point of a hyperbolic motion $\gamma \in \Gamma$ with axis l. One can assume, without loss of generality, that q is a repulsing point. Let p be an ordinary point of the domain P. Then the entire ray pq lies in P. At the same time, there is a sequence of points $x_m \in pq$ such that $\lim_{m\to\infty} \gamma^m x_m = x_0 \in l$, which contradicts the fact that the family $\{\gamma^m P \colon m \in Z\}$ is locally finite. □

Corollary. *With each point x at infinity of the domain P one can associate a horosphere $S(x)$ with centre at this point such that if $x_2 = \gamma x_1$ for some $\gamma \in \Gamma$ then $S(x_2) = \gamma S(x_1)$.*

We now consider fixed points of parabolic motions.

Definition 1.5. A point q at infinity is said to be *quasi-parabolic* if its stabilizer Γ_q contains parabolic motions.

Since any infinite discrete group of motions of Euclidean space has a motion without fixed points (see Chap. 3, Theorem 1.3), a point q at infinity of a fundamental domain P is quasi-parabolic if and only if $|\Gamma_q| = \infty$.

Note that the stabilizer of a quasi-parabolic point cannot contain any hyperbolic motions. Indeed, if $\alpha \in \Gamma_q$ is a parabolic motion, and $\gamma \in \Gamma_q$ is a hyperbolic motion for which q is the repulsing point, then $\lim_{m\to\infty} \gamma^m \alpha \gamma^{-m}$ is an elliptic motion, which contradicts the discreteness of the group Γ.

Definition 1.6. A quasi-parabolic point q is said to be *parabolic* if the group Γ_q acts on any horosphere with centre q as a crystallographic group.

Suppose that P is a normal convex fundamental polyhedron with finitely many (maybe false) faces. Its point x at infinity is said to be isolated if the $(n-1)$-dimensional faces containing it have no ordinary points in common (and therefore have no common points at infinity other than x). Evidently there are only finitely many such points.

Now, for each point x at infinity of the polyhedron P we denote by $P(x)$ the intersection of P with a small horosphere with centre x. It is a convex Euclidean polyhedron defined uniquely up to a similarity. If x is not an isolated point, then $P(x)$ is a cone defined by the set of $(n-1)$-dimensional faces containing x. In particular, there are only finitely many such cones up to congruence.

Proposition 1.7. *Each point x at infinity of the polyhedron P is equivalent to just finitely many of its points at infinity. For these points the analogue of Theorem 1.2 holds. Denote them by $x_1, \ldots, x_r$. The point x is quasi-parabolic (respectively, parabolic) if and only if none of the polyhedra $P(x_i)$ contains an open convex cone (respectively, all the polyhedra $P(x_i)$ are bounded).*

Proof. Let $x_i = \gamma_i x$ $(i = 1, \ldots, r)$ be the points at infinity of P equivalent to x (possibly not all of them). Let $S(x)$ be a sufficiently small horosphere with centre x, and $S(x_i) = \gamma_i S(x)$. Then $P_i = \gamma_i^{-1} P(x_i)$ are convex polyhedra on the horosphere $S(x)$ no two of which have common interior points. Only finitely many of these polyhedra are not cones, and those that are cones are congruent to a cone in a fixed finite set of cones. There can be only finitely many congruent non-intersecting open convex cones in Euclidean space, so r is bounded above by a predetermined number. Thus the first statement of the theorem is proved.

Suppose next that $x_1, \dots, x_r$ are all points of P at infinity equivalent to x. Then $P_1 \cup \ldots \cup P_r$ is a fundamental domain for the group Γ_x acting on the horosphere $S(x)$. This implies the remaining statements of the theorem. □

Corollary. *There can be only finitely many quasi-parabolic points among the points at infinity of the fundamental polyhedron P.*

For more on quasi-parabolic points see Sect. 2.

1.5. The Existence of a Discrete Group with a Given Fundamental Polyhedron. So far we have been dealing with the situation when the discrete group was assumed to be defined from the very beginning. Poincaré's method, however, makes it possible to construct a discrete group with a given fundamental domain.

Let P be a convex polyhedron with finitely many (maybe, false) faces, and let an involutory permutation $F \mapsto F'$ be defined on the set of its $(n-1)$-dimensional faces. Suppose that for each such face F an adjacency transformation is given, i.e. a motion $s(F)$ is defined taking the face F' into the face F, and the polyhedron P into a polyhedron lying on the other side of the hyperplane of the face F (actually, it is enough to define the restriction of $s(F)$ to F'). Suppose also that $s(F)s(F') = \mathrm{id}$.

Suppose next that for each $(n-2)$-dimensional face f of P there exist (uniquely determined) adjacency transformations $s_1, s_2, \dots, s_k$ such that $s_1 s_2 \dots s_k = \mathrm{id}$ (the identical motion), and the polyhedra (2) form a "circuit" around f. For an appropriately constructed sequence $s_1, s_2, \dots$ these conditions are equivalent to the following three conditions (the notation is the same as in (4)):

(R1) the motion $s_1 s_2 \dots s_k$ preserves the orientation;

(R2) the sum of the dihedral angles at the faces $f_0 (= f), f_1, \dots, f_{k-1}$ equals 2π;

(R3) the composition of mappings

$$f = f_k \xrightarrow{s_k} f_{k-1} \xrightarrow{s_{k-1}} \dots \xrightarrow{s_2} f_1 \xrightarrow{s_1} f_0 = f$$

is the identical mapping of the face f onto itself.

(Note that for $n = 2$, and, in general, when f has no non-trivial symmetry, there is no need to check the last condition.)

Finally, if $X = Л^n$, and the polyhedron P is not bounded, there are some extra conditions. In order to formulate them, we first introduce the notion of S-equivalence of points (either ordinary or at infinity) of the polyhedron P. Two points x and y are said to be S-equivalent if there are adjacency transformations $s_i = s(F_i)$ $(i = 1, \dots, k)$ such that $x = s_1 s_2 \dots s_k y$ and $s_{i+1} \dots s_k y \in F_i'$ for $i = 1, \dots, k$ (see Theorem 1.2). Defining isolated points at infinity of the polyhedron P as in Sect. 1.4 we now require that they satisfy the following two conditions:

(P1) the S-equivalence class of any isolated point at infinity of the polyhedron P is finite;

(P2) for each such class consisting of points $x_1, \dots, x_r$ these points can be surrounded by horospheres $S_1, \dots, S_r$ in such a way that if for some $(n-1)$-dimensional face F of P one has $x_i \in F'$ and $s(F)x_i = x_j$, then $s(F)S_i = S_j$.

Note that for a polyhedron of finite volume its isolated points at infinity are its vertices at infinity. Condition (P1) is then satisfied automatically.

Theorem 1.8. *Under the above assumptions the group Γ generated by the adjacency transformations is discrete, and the polyhedron P is its fundamental domain.*

Proof. The outline of the proof is as follows:

(1) an auxiliary metric space $\tilde{X}$ is pasted together from copies of the polyhedron P. The group $\tilde{\Gamma}$ of its isometries is defined in such a way that the pair $(\tilde{X}, \tilde{\Gamma})$ satisfies the conclusion of the theorem. Simultaneously, a continuous mapping $p : \tilde{X} \to X$ and an epimorphism $\pi : \tilde{\Gamma} \to \Gamma$ are defined in such a way that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\gamma}} & \tilde{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\pi(\tilde{\gamma})]{} & X \end{array} \qquad (5)$$

commutes for all $\tilde{\gamma} \in \tilde{\Gamma}$;

(2) a number $\varepsilon > 0$ is proved to exist such that the mapping p isometrically maps the ε-neighborhood of any point $\tilde{x} \in \tilde{X}$ onto the ε-neighborhood of the point $p(\tilde{x})$; this implies that p is a covering and, since the space X is simply-connected, p is an isometry and π is an isomorphism.

We now give the precise definitions. With each $(n-1)$-dimensional face F of the polyhedron P we associate the symbol $\tilde{s}(F)$ and denote by $\tilde{\Gamma}$ the group generated by these symbols with defining relations of the form $\tilde{s}(F) \cdot \tilde{s}(F') = e$ and $\tilde{s}(F_1) \dots \tilde{s}(F_k) = e$, where $(s(F_1), \dots, s(F_k))$ is the sequence of adjacency transformations corresponding to a circuit around an $(n-2)$-dimensional face. Denote by π the epimorphism $\tilde{\Gamma} \to \Gamma$ mapping $\tilde{s}(F)$ into $s(F)$.

Consider the equivalence relation R in the direct product $\tilde{\Gamma} \times P$ generated by the equivalencies of the form

$$(\tilde{\gamma}\tilde{s}(F), x) \sim (\tilde{\gamma}, s(F)x) \quad (x \in F').$$

Let $\tilde{X} = (\tilde{\Gamma} \times P)/R$, and, for each pair $(\tilde{\gamma}, x)$, denote its class of equivalence by $[\tilde{\gamma}, x]$. Denote by p the mapping $\tilde{X} \to X$ under which $[\tilde{\gamma}, x]$ goes into $\pi(\tilde{\gamma})x$. Define the action $\tilde{\Gamma} : \tilde{X}$ induced by the action $\tilde{\Gamma} : \tilde{\Gamma} \times P$ by left shifts on the first factor. Then the diagram (5) commutes.

Each of the polyhedra $[\tilde{\gamma}, P]$ from which the space $\tilde{X}$ is pasted is mapped bijectively onto the polyhedron $\pi(\tilde{\gamma})P$ in the space X. Define the metric on $[\tilde{\gamma}, P]$ in such a way that this mapping is an isometry, and then define the distance between any two points of the space $\tilde{X}$ as the length of a shortest "broken line" joining these points such that each straight segment of the line lies in one of the polyhedra $[\tilde{\gamma}, P]$. Then the mapping p does not increase the distance, and is therefore continuous.

The main difficulty lies in proving statement (2). It is enough to prove it for points of the form $\tilde{x} = [e, x]$, $x \in P$. First, one considers the points x lying inside the polyhedron P far enough from its boundary, then the points lying inside some $(n-1)$-dimensional face far enough from its boundary, and so on. If $X = Л^n$ and the polyhedron P is not bounded, then using conditions (P1) and (P2) one starts by deleting sufficiently small horospherical neighbourhoods of isolated points and the corresponding points at infinity. The proof uses induction on n. □

Example. Consider a triangle ABC on the Lobachevskij plane with the vertex C at infinity, and the angles A and B both equal to $\frac{\pi}{m}$ (Fig. 9). For adjacency transformations take the parabolic rotation s_1 about the point C that takes BC into AC, and the symmetry s_2 about the midpoint O of the side AB.

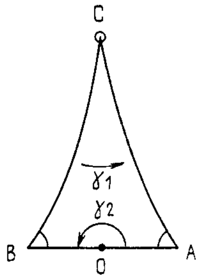


Fig. 9

Condition (P2) is evidently satisfied. To the cyclic path around the vertex A there corresponds the sequence $(s_1, s_2, \ldots, s_1, s_2)$ of length $2m$. Conditions (R1) and (R2) are satisfied. Therefore the motions s_1 and s_2 generate a discrete group Γ_m with the defining relations $s_1^2 = s_2^2 = e$, $(s_1 s_2)^m = e$, and the triangle ABC is its fundamental domain.

In particular, for $m = 3$ in the Poincaré model, one gets the triangle D of Chap. 1, Sect. 1.3, Example 3 and

$$s_1 = \gamma_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s_2 = \gamma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where the square brackets denote matrices considered up to the sign. Since any unimodular integer matrix can be represented as a product of elementary integer matrices, $\Gamma_3 = \langle \gamma_1, \gamma_2 \rangle = PSL_2(\mathbb{Z})$ is the Kleinian modular group. This proves that triangle D is a fundamental domain of the Kleinian modular group.

1.6. Limitations of Poincaré's Method. A practical application of Poincaré's method is generally confronted with overwhelming combinatorial and geometric difficulties. However, for some special classes of groups it proves quite effective. First of all, such are discrete groups in two-dimensional spaces of constant curvature (see Chap. 4), for which the method was actually devised, and discrete groups generated by reflections in hyperplanes (see Chap. 5).

The method was also used to construct certain subgroups of finite index in groups generated by reflections (see, e.g. Gutsul [1980], Makarov [1983], Vinberg [1985]). Zhuk [1983] has found all tetrahedra of finite volume that are fundamental domains of discrete groups in three-dimensional spaces of constant curvature. Many examples of fundamental polyhedra in $Л^3$ have been generated on a computer (Grayson [1983], Riley [1983]).

One of the examples of successful implementation of Poincaré's method in the three-dimensional case that is not connected with groups generated by reflections is the realization of the Fibonacci groups

$$F(2, m) = \langle x_1, x_2, \ldots, x_m | x_i x_{i+1} = x_{i+2}, \quad i \bmod m \rangle$$

for $m = 2n$, $n \geq 4$ as uniform discrete groups of motions of the space $Л^3$ obtained by Helling, Kim, and Mennicke (Mennicke [1988]).

Limitations of Poincaré's method in dimension ≥ 3 are due mainly to the complexity of the combinatorial structure of fundamental polyhedra. This is why the following result of Jørgensen and Marden [1986] is so interesting.

Let Γ be a uniform discrete group of motions of the space $Л^3$. Then there is a simple (see Part 1, Chap. 6, Sect. 1.5) convex fundamental polyhedron of the group Γ for which all the Poincaré relations are of length 3. Moreover, such is the Dirichlet domain $D(x_0)$ for any point x_0 in a dense open subset. A similar but more complicated result has been obtained in the same paper for non-uniform crystallographic groups.

1.7. Homogeneous Decompositions. Poincaré's method can also be applied in a more general situation, when the polyhedron P is not necessarily a fundamental domain of the group Γ.

Definition 1.9. A *homogeneous decomposition of a space* is a pair $(\mathcal{T}, \Gamma)$, where $\mathcal{T}$ is a decomposition, and Γ is a subgroup of the group $\operatorname{Sym} \mathcal{T}$ acting transitively on the decomposition $\mathcal{T}$.

One often understands by a homogeneous decomposition just the first member in the pair $(\mathcal{T}, \Gamma)$, assuming that $\Gamma = \operatorname{Sym} \mathcal{T}$. Homogeneous decompositions are also called "regular"[1], although the latter term is sometimes used for different notions.

If P is a fundamental domain of a discrete group Γ, then $\{\gamma P: \gamma \in \Gamma\}$ is a homogeneous decomposition with the group Γ (although its symmetry group may be greater). In this case the stabilizer of any element of the decomposition in the group Γ is trivial. Examples of homogeneous decompositions with non-trivial stabilizers are decompositions into equal regular polyhedra ("honeycombs", see Chap. 5, Sect. 3).

For homogeneous decompositions the analogues of Theorems 1.1, 1.3, and 1.8 hold.

Let $\{\mathcal{T}, \Gamma\}$ be a homogeneous decomposition into convex polyhedra, P one of the polyhedra of the decomposition, Δ its stabilizer in the group Γ. One can define adjacency transformations as in Sect. 1.2, albeit not uniquely. Taken together with the group Δ they generate the group Γ. Relations (1) and the Poincaré relations are replaced by the relations whose right-hand sides are some elements of the group Δ. Taken together with the relations in the group Δ and the relations of the form $\delta s(F) = s(\delta F)\delta'$, they constitute a set of defining relations in the group Γ. Theorem 1.8 is modified accordingly.

A convex polyhedron that is a part of some homogeneous decomposition is said to be a *stereohedron*. An algorithm proposed by Delone, Galiulin, Dolbilin, and Shtogrin [1976] and Dolbilin [1976] can be used to decide whether a given convex bounded polyhedron is a stereohedron or not.

§ 2. Geometrically Finite Groups of Motions of Lobachevskij Space

A discrete group of motions of the space $Л^n$ is said to be *geometrically finite* if it admits a normal convex fundamental polyhedron P with finitely many (maybe false) faces.

The results of Sect. 1 imply, in particular, that any geometrically finite group is finitely generated. Note also that to be geometrically finite is a property of a class of commensurable discrete groups of motions. Thus, studying this property, one can, if necessary, always consider discrete groups having no fixed points.

Any finitely generated discrete subgroup $\Gamma \subset \mathrm{Isom}\, Л^2$ is geometrically finite (Chap. 4, Sect. 2), but already in $Л^3$ there are examples of finitely generated discrete groups that are not geometrically finite (Greenberg [1977]). However, the class of geometrically finite discrete groups is the best studied.

In order to formulate the basic results in this area we need to introduce a number of new notions. Hereafter we assume that the action takes place in the space $Л^n$. For each set $M \subset Л^n$ we denote its closure in $\overline{Л^n}$ by $\overline{M}$.

[1] Alternative terms are *tesselations*, or *tilings*, although these words are also used with different meanings.

2.1. The Limit Set of a Discrete Group of Motions. Let Γ be a discrete group of motions. If it is infinite, its action in $\overline{Л^n}$ cannot be discrete.

The set $\Lambda(\Gamma) = \overline{\Gamma x} \cap \partial Л^n$, $x \in Л^n$, is said to be the *limit set* of the group Γ. The set $\Lambda(\Gamma)$ does not depend on the choice of the point x. It is evidently closed and Γ-invariant, and unless it consists of two points it is the smallest subset of the absolute having these two properties. Furthermore, if the groups Γ_1 and Γ_2 are commensurable, then $\Lambda(\Gamma_1) = \Lambda(\Gamma_2)$.

The set $\Lambda(\Gamma)$ is either finite (and then $|\Lambda(\Gamma)| \leq 2$), or perfect (Ahlfors [1981]). In the first case the group Γ is said to be an *elementary* one. Elementary groups admit an algebraic characterization.

Proposition 2.1. *A discrete group of motions is elementary if and only if it contains an abelian normal subgroup of finite index.*

The results of Sect. 1.4 and Theorem 1.1 of Chap. 3 imply that the stabilizer of a quasi-parabolic point is elementary.

The group Γ evidently acts discretely on $\overline{Л^n} - \Lambda(\Gamma)$. Denote the quotient space $(\overline{Л^n} - \Lambda(\Gamma))/\Gamma$ by S_Γ. A group Γ is said to be of *compact type* if the space S_Γ is compact. For example, discrete groups of compact type on the Lobachevskij plane are nothing else but finitely generated discrete groups containing no parabolic elements (see Chap. 4, Sect. 2.2).

Note also that if the stabilizer Γ_x of a point $x \in \partial Л^n$ is infinite, then $x \in \Lambda(\Gamma)$. In particular, quasi-parabolic points (see Sect. 1.4) are limit points of the group Γ.

2.2. Statement of the Main Results

Theorem 2.2 (Apanasov [1982], Tukia [1985b]). *Any discrete group of motions of compact type is geometrically finite.*

Not every geometrically finite discrete group of motions is of compact type. There is, however, the following theorem.

Theorem 2.3 (Apanasov [1982, 1984], Tukia [1985b]). *Let Γ be a geometrically finite discrete group of motions. Then the quotient space $S_\Gamma = (\overline{Л^n} - \Lambda(\Gamma))/\Gamma$ can be compactified by adding to it finitely many equivalence classes of quasi-parabolic points.*

There is a crucial step in the proof of this theorem which deserves a separate formulation.

Theorem 2.4 (Apanasov [1984]). *Let Γ be a geometrically finite discrete group of motions, and P its normal convex fundamental polyhedron. Then any quasi-parabolic point of the group Γ is equivalent to a point of the set $\overline{P} \cap \partial Л^n$.*

There is no doubt that uniform discrete groups of motions are geometrically finite. What about crystallographic groups?

Theorem 2.5 (Garland and Raghunathan [1970], Wielenberg [1977]). *Any crystallographic group of motions of the space $Л^n$ is geometrically finite.*

We will now formulate the result used, in one way or another, in all the proofs of Theorem 2.5 known to us.

Theorem 2.6 (Kazhdan and Margulis [1968]). *Any quasi-parabolic point of a crystallographic group of motions is parabolic.*

This is a particular case of a fundamental result in the theory of discrete subgroups of Lie groups due to these authors (see VGS, Chap. 3, Sect. 4). A number of facts of general nature (which have by now become standard) is used to derive Theorem 2.5 from Theorem 2.6. Many of them are of interest in their own right, and merit a separate consideration.

2.3. Some General Properties of Discrete Groups of Motions of Lobachevskij Space. The key fact in many topics in the theory of discrete groups of motions of the space $Л^n$ is the Margulis Lemma.

Theorem 2.7 (The Margulis Lemma) (Margulis [1975], Apanasov [1983]). *For any discrete group Γ of motions of the space $Л^n$, any point x, and any $\varepsilon > 0$, denote by $\Gamma_\varepsilon(x)$ the subgroup in Γ generated by the set $\{\gamma \in \Gamma \mid \rho(\gamma x, x) \le \varepsilon\}$. Then there is a constant ε (depending on the dimension only) such that the group $\Gamma_\varepsilon(x)$ contains an abelian normal subgroup whose index does not exceed some constant N, depending on the dimension only.*

The proof of Theorem 2.7 is similar to that of the Schoenflies-Bieberbach theorem (see Chap. 3, Sect. 1).

Consider two elements γ_1 and γ_2 of a discrete group Γ and denote by $\langle\gamma_1, \gamma_2\rangle$ the subgroup that they generate. As follows from Theorem 2.7, if the subgroup $\langle\gamma_1, \gamma_2\rangle$ is not elementary, then γ_1 and γ_2 cannot both be "too close" to the identical transformation. In the three-dimensional case an explicit estimate of this "closeness" is provided by the following theorem.

Theorem 2.8 (Jørgensen's inequality) (Jørgensen [1976]). *If two elements γ_1 and γ_2 of a discrete group Γ of proper motions of the space $Л^3$ generate a non-elementary subgroup, then*

$$|\mathrm{tr}^2\gamma_1 - 4| + |\mathrm{tr}(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) - 2| \ge 1$$

(where Γ is considered as a subgroup of $PSL_2(\mathbb{C})$).

A generalization of this result to arbitrary dimension has been obtained by Martin [1989].

Jørgensen's inequality certainly does not ensure that the group of motions generated by γ_1 and γ_2 is discrete. The question of when two given motions of the space $Л^3$ generate a discrete group is very difficult and, in its general form, unsolved. For rotations having perpendicular skew axes it was solved by Klimenko [1989].

We now consider a useful corollary to Theorem 2.7.

Lemma 2.9 (Greenberg [1977]). *Let Γ be a discrete group of motions of the space $Л^n$. If the stabilizer Γ_x of a point x of the absolute contains a parabolic translation t, then there exists a horoball B_x with centre x such that $B_x \cap \gamma(B_x) = \emptyset$ for any $\gamma \in \Gamma - \Gamma_x$ (a horoball satisfying this property will be called independent).*

Proof. Let $B_x = \{y \in Л^n \mid \rho(ty, y) \le \varepsilon\}$, where ε is the constant from Theorem 2.7. Suppose that there is a $\gamma \in \Gamma - \Gamma_x$ such that $\gamma(B_x) \cap B_x \neq \emptyset$, i.e. $\gamma y \in B_x$ for some point $y \in B_x$. Then $\rho(\gamma^{-1}t\gamma y, y) = \rho(t\gamma y, \gamma y) \le \varepsilon$, and by Theorem 2.7 certain powers of the parabolic translations $\gamma^{-1}t\gamma$ and t commute. Hence the elements $\gamma^{-1}t\gamma$ and t have a common fixed point, i.e. $\gamma x = x$, contradicting the assumption. □

The statement of the preceding lemma can be made more precise.

Lemma 2.10 (Apanasov [1983]). *If the stabilizer Γ_x of a point $x \in \partial Л^n$ contains a parabolic translation t, then the horoball $B_x = \{y \in Л^n \mid \rho(ty, y) \le 1\}$ is independent.*

We conclude this subsection with two more useful results, the first of which follows from the general theory of discrete subgroups of Lie groups (VGS, Chap. 1, Theorem 1.3) and plays an important role in the proof of Theorem 2.6.

Proposition 2.11 (Raghunathan [1972]). *Let Γ be a discrete group of motions of the space $Л^n$. Let π be the natural projection $Л^n \to Л^n/\Gamma$, and $\{x_m\}$ a sequence of points converging to infinity. A sufficient condition, which for crystallographic groups is also a necessary one, for the sequence $\pi(x_m)$ to be discrete is that there exist elements $\gamma_1, \dots, \gamma_m, \dots$ in Γ different from e such that*

$$\lim_{m\to\infty} \rho(\gamma_m x_m, x_m) = 0.$$

Lemma 2.12. *No point $x \in \partial Л^n$ is a fixed point of two hyperbolic motions $\gamma_1, \gamma_2 \in \Gamma$ having different axes.*

Proof. Let l_1 and l_2 be the axes of γ_1 and γ_2. One can assume, without loss of generality, that x is a repulsing point for γ_1. Then $\lim_{m\to\infty} \gamma_1^m l_2 = l_1$, so some subsequence of the sequence $\{\gamma_1^m\gamma_2\gamma_1^{-m}\}$ converges to a hyperbolic motion with axis l_1, which for $l_1 \neq l_2$ contradicts the discreteness of the group Γ. □

2.4. Compactification of the Quotient Space of a Crystallographic Group. Let Γ be a discrete group of proper motions of the space $Л^n$. For simplicity, we assume that the action of Γ has no fixed points. If Γ is uniform, then the quotient space $S_\Gamma = Л^n/\Gamma$ is compact. If the quotient space is not compact but has finite volume, one can construct its natural compactification $\hat{S}_\Gamma$ (it is the compactification referred to in Theorem 2.3), which is a special case of the *Satake compactification* (see VGS, Chap. 3). The construction is achieved as follows. Consider the action of Γ on the union $\text{Л}^n \cup \Pi_\Gamma$, where Π_Γ is the set of parabolic points of Γ. By Theorem 2.11 there are only finitely many classes of Γ-equivalent parabolic points in the set $\hat{S}_\Gamma = (\text{Л}^n/\Gamma) \cup (\Pi_\Gamma/\Gamma) = S_\Gamma \cup \{\hat{Z}_1, \ldots, \hat{Z}_s\}$. We will endow this set with a topology by keeping the topology on S_Γ and taking for a fundamental system of neighbourhoods of the point $\hat{Z}_i$ the system of neighbourhoods of the form $U_{\hat{Z}_i} = B_{z_i}/\Gamma_{z_i}$, where B_{z_i} runs over the set of independent horoballs with centre at a point $z_i \in \hat{Z}_i$ (any representative of the class $\hat{Z}_i$). One can verify that this topology turns $\hat{S}_\Gamma$ into a topological space, which is not necessarily a manifold, but a compact orientable pseudo-manifold in the sense of Seifert and Threlfall [1934].

Topologists prefer to deal with compact manifolds with boundary homotopically equivalent to the quotient space S_Γ. In order to construct such a manifold, note that an independent horoball B_z can be associated with each parabolic point z of a crystallographic group Γ in such a way that

(a) $B_{z_1} \cap B_{z_2} = \emptyset$ if $z_1 \neq z_2$;
(b) $B_{\gamma z} = \gamma B_z$ for any parabolic point z and any $\gamma \in \Gamma$.

This is an immediate consequence of the fact that there are only finitely many classes of Γ-equivalent parabolic points.

Denote by Л^n_Γ the complement to the family of horoballs chosen in this way. One can easily see that the quotient space $S^*_\Gamma = \text{Л}^n_\Gamma/\Gamma$ is a compact manifold with boundary homotopically equivalent to the quotient space $S_\Gamma = \text{Л}^n/\Gamma$. The boundary of the manifold S^*_Γ is a union of finitely many flat Riemannian manifolds.

2.5. A Criterion of Geometrical Finiteness. For $n \leq 3$ there is a *criterion of geometrical finiteness* of a discrete group Γ stated exclusively in terms of the set $\Lambda(\Gamma)$.

First we introduce a new type of limit points of a discrete group Γ.

Definition 2.13. A point q at infinity is said to be a *conical* limit point (in old mathematical papers it used to be called an "approximation point") if there is a sequence of elements $\{\gamma_m\}$ in Γ such that for any point $x \in \text{Л}^n$ and any line l passing through the point q there is a positive constant M such that $\lim \gamma_m x = q$ and $\rho(\gamma_m x, l) < M$ for all m.

Remark. One can easily see that if the conditions of the definition are satisfied for some point x and some line l, then they are satisfied for all points x and lines l (possibly with different values of M). Figure 10 shows a limit conical point 0 in the Poincaré model of the Lobachevskij plane. The shaded cone is the locus of points whose distance from l is less than M.

Lemma 2.14. *A fixed point of a hyperbolic motion is a conical limit point.*

Proof. If $\gamma \in \Gamma$ is a hyperbolic motion, then any point on its axis can be taken for x in the definition of a conical point, and either of the sequences $\langle\gamma^m\rangle$ or $\langle\gamma^{-m}\rangle$, $m \in \mathbb{N}$, can be taken for $\{\gamma_m\}$. □

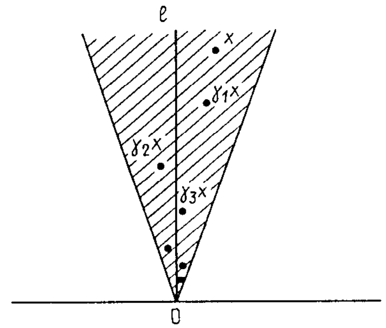


Fig. 10

Example. Let Γ be the Kleinian modular group acting in the upper half-plane H^2. Then $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\}$, and all the irrational numbers are conical points, while all rational numbers (including ∞) are parabolic.

Definition 2.15. A quasi-parabolic point x of a discrete group Γ is said to be *cuspidal* if there is a Γ_x-invariant subset $M \subset \partial\text{Л}^n - \{x\}$ closed in $\partial\text{Л}^n - \{x\}$ such that (a) the quotient space M/Γ_x is compact; (b) $\gamma(\partial\text{Л}^n - M) \subset M$ for any $\gamma \notin \Gamma_x$.

Evidently, each parabolic point is a cuspidal one with $M = \partial\text{Л}^n - \{x\}$. Thus, for $n = 2$ any quasi-parabolic point is cuspidal.

Theorem 2.16 (Beardon and Maskit [1974]). *A discrete group Γ of motions of the space Л^n, $n = 2, 3$, is geometrically finite if and only if all its limit points are either conical or cuspidal quasi-parabolic, and in this case any convex fundamental polyhedron has finitely many faces.*

For an arbitrary n the statement that holds in all cases is that the limit set of a geometrically finite discrete group of motions consists of conical and cuspidal quasi-parabolic points only (Apanasov [1983], Tukia [1985b]).

However, a simple example due to Apanasov [1982] shows that for $n \geq 4$ there are geometrically finite discrete groups of motions of the space Л^n, some fundamental polyhedra of which have infinitely many faces. The example originates in the Euclidean space, namely if Γ is a cyclic discrete group of motions of the space E^3 generated by a helical motion γ with the rotation angle non-commensurable with π, then the Dirichlet domain of the group Γ has finitely many faces if its centre lies on the axis of γ, and infinitely many otherwise. A similar construction in Л^3 does not yield such a result (all Dirichlet domains have finitely many faces), but one obtains a desired example in the space Л^4 (as well as in any space Л^n, $n > 4$) by assuming that E^3 is embedded in Л^4 as a horosphere and extending the action of the above group Γ to Л^4.

Chapter 3 Crystallographic Groups

In this chapter the term "crystallographic group" is understood in a narrow sense, i.e. as referring to a uniform discrete group of motions of the Euclidean space (see Corollary to Theorem 1.11 of Chap. 1 in that respect).

§ 1. The Schoenflies-Bieberbach Theorem

1.1. Statement of the Main Theorem. The cornerstone in the theory of crystallographic groups is the following theorem, first proved by Schoenflies in 1891 in three dimensions and then by Bieberbach in 1911 in the n-dimensional case. (For modern versions of the proof see Auslander [1960], Vinberg [1975], Wolf [1977], Buser [1985], Abels and Dress [1986]).

Theorem 1.1. *Any crystallographic group of motions contains a subgroup of finite index consisting of parallel translations.*

The group of all parallel translations contained in a crystallographic group Γ is the kernel of the homomorphism of the group Γ into the orthogonal group associating with each motion $\gamma \in \Gamma$ its differential (the linear part) $d\gamma$. Thus the statement of the theorem is equivalent to the fact that the group $d\Gamma$ is finite.

Any discrete group of parallel translations of the space E^n is a free abelian group of rank $k \leq n$ generated by parallel translations along k linearly independent vectors. The group of all parallel translations contained in a crystallographic group Γ is, by Theorem 1.1, also a crystallographic group (see Chap. 1, Proposition 1.13), and is therefore generated by parallel translations along vectors of some basis in $\mathbb{R}^n$. In this basis, transformations of the group $d\Gamma$ are expressed by integer matrices. Hence not every finite linear group can play the role of $d\Gamma$. (For more details see Sect. 2.3.)

An outline of the proof of Theorem 1.1 is given in Sect. 1.2 and 1.3.

1.2. Commutators of Orthogonal Transformations and Motions. Let us introduce a norm into the algebra $L_n(\mathbb{R})$ of all linear transformations of the space $\mathbb{R}^n$ as usual by the formula

$$||A|| = \max_{|x|=1} |Ax|,$$

where $|x|$ is the length of the vector x. A simple computation yields the following estimate of how close the commutator $(A,B) = ABA^{-1}B^{-1}$ of two orthogonal transformations is to the identity operator:

$$||(A,B) - E|| \leq 2||A-E|| \; ||B-E|| \quad (A, B \in O_n). \tag{1}$$

In particular, if $||A-E||$, $||B-E|| < \frac{1}{2}$, then $||(A,B)-E|| < \frac{1}{2}$.

Lemma 1.2. *If $A, B \in O_n$ and $||B - E|| < \sqrt{2}$, then $(A,(A,B)) = E$ implies that $(A,B) = E$.*

Proof. Consider both A and B as unitary transformations of the space $\mathbb{C}^n$, and let $V_1, \ldots, V_k$ be the eigenspaces of A. If $(A,(A,B)) = E$, then BAB^{-1} commutes with A and hence its eigenspaces $BV_1, \ldots, BV_k$ are the sums of subspaces contained in $V_1, \ldots, V_k$. If $BV_i \neq V_i$ for some i, then there is a unit vector $x \in V_i$ such that $Bx \in V_j$, $j \neq i$, but then $|Bx - x| = \sqrt{2}$, contradicting the hypothesis. □

Corollary. *If $A, B \in O_n$ and $||A-E||$, $||B-E|| < \frac{1}{2}$, then*

$$(\underbrace{A, \ldots, (A}_{k \text{ times}}, B) \ldots) = E$$

implies that $(A,B) = E$.

Any motion α of the space E^n is of the form

$$\alpha(x) = Ax + a \quad (A \in O_n, \; a \in \mathbb{R}^n), \tag{2}$$

where $A = d\alpha$ is its linear part and a is the "translation" part (depending, of course, on the origin). We shall write $\alpha = (A, a)$.

Let $\alpha = (A,a)$ and $\beta = (B,b)$ be two motions, and $\gamma = (C,c)$ their commutator. One can easily see that

$$|c| \leq ||B-E|| \; |a| + ||A-E|| \, |b|. \tag{3}$$

In particular, if $||A-E||$, $||B-E|| < \frac{1}{2}$, then $|c| < \frac{1}{2}(|a| + |b|)$.

1.3. Proof of the Main Theorem. Let $U \subset \mathbb{R}^n$ be the linear space spanned by all vectors the translations along which belong to a given crystallographic group Γ. Then Γ acts in the quotient space E^n/U as a crystallographic group having no parallel translations. We can therefore assume below that the group Γ itself contains no parallel translations.

Now the compactness of the group O_n implies that for any group $G \subset O_n$ its subgroup generated by all transformations $A \in G$ for which $||A-E|| < \frac{1}{2}$ is of finite index. In particular, taking $G = d\Gamma$, we see that the subgroup of Γ generated by all motions $\alpha \in \Gamma$ for which $||d\alpha - E|| < \frac{1}{2}$ has finite index in Γ. We can therefore assume that the group Γ itself is generated by such motions.

Let $\alpha = (A,a)$, $\beta = (B,b)$ be two motions in the group Γ such that $||A-E||$, $||B-E|| < \frac{1}{2}$. Consider their iterated commutator $\gamma_k = (\underbrace{\alpha, \ldots, (\alpha}_{k}, \beta) \ldots)$.

Its differential equals $C_k = (\underbrace{A, \dots, (A}_{k}, B) \dots)$. It follows from formula (1) that $C_k \to E$ as $k \to \infty$. At the same time formula (3) implies that the translation part of the motions γ_k is bounded. Thus there is a subsequence of motions γ_k converging to a parallel translation, but since there are no parallel translations in Γ, it must converge to e, and since the group Γ is discrete, one has $\gamma_k = e$ for some k. By the Corollary to Lemma 1.2 one has $\gamma_1 = (\alpha, \beta) = e$. Thus the generators of the group Γ commute, and therefore the group Γ itself is commutative.

Now take any non-identical motion $\gamma_0 \in \Gamma$. Its axis (see Part 1, Chap. 5, Sect. 2) is invariant under all motions commuting with it, and is consequently invariant under the group Γ, which contradicts the assumption that it is crystallographic. Hence $\Gamma = \{e\}$ and $n = 0$. □

1.4. Arbitrary Discrete Groups of Motions of the Euclidean Space. The preceding argument can be developed without difficulty, yielding the following description of arbitrary (not necessarily uniform) discrete groups of motions. (Cf. the description of connected groups of motions given in Part 1, Chap. 5, Sect. 3)

Theorem 1.3. *Any discrete group Γ of motions of Euclidean space has an invariant plane Π on which it acts as a crystallographic group. There is an abelian subgroup of finite index in Γ such that the restriction of its action to Π is its isomorphism onto a (crystallographic) group of parallel translations of the plane Π.*

Any minimal invariant plane can be taken for Π. It is defined uniquely up to a parallel translation along a vector orthogonal to it and invariant under the group $d\Gamma$.

§ 2. Classification of Crystallographic Groups

2.1. Cohomological Description. According to Theorem 1.1 each crystallographic group Γ is an extension of a group of parallel translations contained in it by means of the finite group $G = d\Gamma \subset O_n$. Denote by L the lattice in the space $\mathbb{R}^n$ consisting of vectors of parallel translations belonging to Γ. In view of the equality $\gamma t_a \gamma^{-1} = t_{d\gamma(a)}$ this lattice is invariant under the group G.

For any $g \in G$ there is a motion $\gamma \in \Gamma$ of the form

$$\gamma(x) = gx + a(g), \tag{4}$$

where the vector $a(g) \in \mathbb{R}^n$ is defined up to a vector from L. Thus the mapping

$$\bar{a}\colon G \to \mathbb{R}^n/L, \quad g \mapsto a(g) + L. \tag{5}$$

is well-defined. Since the composition of two motions from Γ also belongs to Γ, one has

$$\bar{a}(g_1 g_2) = \bar{a}(g_1) + g_1 \bar{a}(g_2). \tag{6}$$

This means that $\bar{a}$ is a 1-*cocycle* on the group G with values in the group $\mathbb{R}^n/L$.

Conversely, for any finite group $G \subset O_n$, a lattice $L \subset \mathbb{R}^n$ invariant under G, and a cocycle $\bar{a}$ on G with values in $\mathbb{R}^n/L$, the motions of the form (4) constitute a crystallographic group.

The parallel translation of the origin along a vector b replaces the cocycle $\bar{a}$ by the cocycle

$$\bar{a}'(g) = \bar{a}(g) + g\bar{b} - \bar{b}, \tag{7}$$

where $\bar{b} = b + L$. Such cocycles are said to be *cohomological* to the cocycle $\bar{a}$.

1-cocycles on G with values in $\mathbb{R}^n/L$ form an abelian group (with respect to addition). Its quotient group with respect to the cohomology relation is said to be the *first cohomology group* of G with values in $\mathbb{R}^n/L$ and is denoted by $H^1(G, \mathbb{R}^n/L)$.

Let $\alpha \in H^1(G, \mathbb{R}^n/L)$ be the cohomology class containing the cocycle $\bar{a}$. The triple (G, L, α) defines the group Γ unambiguously.

Thus there is a one-to-one correspondence between crystallographic groups Γ and triples (G, L, α). In particular, triples with $\alpha = 0$ correspond to *split* (or *symmorphic*) crystallographic groups which, under an appropriate choice of the origin, contain for each motion both its linear and translation parts.

The classification problem for all crystallographic groups includes the classification problem for crystallographic groups of parallel translations, i.e., in fact, the problem of classifying lattices in the Euclidean vector space. The latter problem, which is far from trivial, is the subject of the reduction theory (see, for example, Cassels [1978] or VGS). It becomes trivial if one ignores the metric structure: all the lattices in a vector space are the same. That is why it is natural to classify crystallographic groups up to affine equivalence, i.e. up to conjugacy in the group of all affine transformations. Under this interpretation, crystallographic groups can also be described by triples (G, L, α), but the space $\mathbb{R}^n$ should be considered just as a vector space (with no metric on it).

2.2. Abstract Structure. If a crystallographic group is considered as an abstract one, then its subgroup of parallel translations can be characterized as the greatest abelian normal subgroup. Then the abstract description of a crystallographic group Γ is reduced to specifying a finite group G of automorphisms of the lattice L, and a mapping

$$b\colon G \times G \to L,$$

$$(g_1, g_2) \mapsto a(g_1 g_2) - a(g_1) - g_1 a(g_2), \tag{8}$$

defining the multiplication law for elements of the group Γ given in the form (4).

The mapping b is a 2-cocycle on the group G with values in the lattice L. If vectors $c(g) \in L$ are added to vectors $a(g)$, the cocycle b is replaced by the cocycle

$$b'(g_1, g_2) = b(g_1, g_2) + c(g_1 g_2) - c(g_1) - g_1 c(g_2). \tag{9}$$

Thus, a cohomology class $\beta \in H^2(g, L)$ is well-defined. The triple (G, L, β) describes the abstract structure of the group Γ.

Going over from a cocycle $\alpha \in H^1(G, \mathbb{R}^n/L)$ to a cocycle $\beta \in H^2(G, L)$ is nothing else but the connecting homomorphism δ of the exact cohomological sequence

$$\ldots \to H^1(G, \mathbb{R}^n) \to H^1(G, \mathbb{R}^n/L) \xrightarrow{\delta} H^2(G, L) \to H^2(G, \mathbb{R}^n) \to \ldots. \tag{10}$$

Since all but the zero-dimensional cohomologies of a finite group with values in a vector space are trivial, the above segment of the exact cohomological sequence implies that δ is an isomorphism. The first of the theorems stated below is an interpretation of the fact that δ is injective, and the second that it is surjective.

Theorem 2.1 (Bieberbach [1911]). *Two crystallographic groups are isomorphic if and only if they are affinely equivalent.*

Theorem 2.2 (Zassenhaus [1948]). *For any extension of a free abelian group Γ of rank n by a finite group G acting faithfully on L there exists a crystallographic group isomorphic to Γ.*

2.3. Classification Steps. The preceding argument shows that the classification of crystallographic groups can, in principle, be accomplished in the following way:

(1) find all finite linear groups $G \subset O_n$ admitting an invariant lattice;

(2) find all invariant lattices L for each such group G;

(3) compute the cohomology group $H^1(G, \mathbb{R}^n/L)$ for each pair (G, L);

(4) determine the cohomology classes corresponding to affinely equivalent crystallographic groups. (Crystallographic groups defined by the triples (G, L, α) and (G, L, β) are affinely equivalent if and only if the classes α and β are equivalent with respect to the natural action in $H^1(G, \mathbb{R}^n/L)$ of the normalizer of the group G in the group of automorphisms of the lattice L.)

If Γ is a crystallographic group defined by the triple (G, L, α), then G is said to be the *class* of the group Γ, and the pair (G, L) its *arithmetic class*. From the point of view of matrices, the class is the group G considered up to conjugacy in $GL_n(\mathbb{R})$ (or in O_n, which is the same), and the *arithmetic class* is the same group considered up to conjugacy in $GL_n(\mathbb{Z})$. (The embedding $G \hookrightarrow GL_n(\mathbb{Z})$ is defined by writing the group G in a basis of the lattice L.)

The term "(arithmetic) class" is also understood as "the set of all crystallographic groups of a given (arithmetic) class".

The following observation can help in finding the classes of crystallographic groups. A linear transformation preserving a lattice has a characteristic polynomial with integer coefficients. If, moreover, the transformation is of prime order p, then its characteristic polynomial must be divisible by the cyclotomic polynomial $\Phi_p(t) = t^{p-1} + t^{p-2} + \ldots + t + 1$, and therefore $p \leq n + 1$.

Let us illustrate the situation in the case $n = 2$. Any finite group of orthogonal transformations of $\mathbb{R}^2$ is either the cyclic group C_m consisting of rotations through angles equal to integer multiples of $2\pi/m$, or the dihedral group D_m, which is the extension of C_m by a reflection. The above argument shows that either of the groups C_m or D_m preserves the lattice only if $m = 1, 2, 3, 4$, or 6, thus yielding 10 classes of crystallographic groups.

For each of these groups G except D_1, D_2, and D_3 there is the unique (up to the action of the normalizer of G in $GL_2(\mathbb{R})$) invariant lattice. For each of the groups D_1, D_2, and D_3 there are two such lattices. Their basis vectors and reflection axes are shown in Fig. 11. Thus one has 13 arithmetic classes.

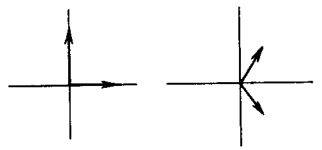


Fig. 11a. $G = D_1$ or D_2

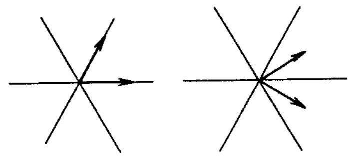


Fig. 11b. $G = D_3$

Any crystallographic group Γ of the class C_m is split: it suffices to take for the origin the fixed point of a motion from Γ the linear part of which generates the group $G = C_m$. Of the other groups, only the groups of arithmetic classes (D_m, L) where $m = 1, 2$, or 4 and L is a rectangular lattice can be non-split. For $m = 1$ and 4 there is one non-split group, and two for $m = 2$.

Consider, e.g., the case $m = 2$. Let s_1 and s_2 be reflections in orthogonal lines generating the group $G = D_2$, and let a_1 and a_2 be the vectors lying on these lines and generating the lattice L. The cocycle $\overline{a}$ on G with values in $\mathbb{R}^2/L$ is defined by the vectors $a(s_1)$ and $a(s_2)$, which must satisfy the conditions

$$a(s_1) + s_1 a(s_1) \in L, \quad a(s_2) + s_2 a(s_2) \in L, \tag{11}$$

following from the relations $s_1^2 = s_2^2 = e$. By means of the transformations (7) one can obtain $a(s_1) = \lambda_1 a_1$ and $a(s_2) = \lambda_2 a_2$. Then, as follows from (11), one has $2\lambda_1, 2\lambda_2 \in \mathbb{Z}$. Hence $H_1(G, \mathbb{R}^2/L) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since the permutation of vectors a_1 and a_2, which normalizes the group Γ, permutes the corresponding cohomology classes, we conclude that this arithmetic class contains 3 non-equivalent crystallographic subgroups. Thus there are in all 17 crystallographic groups of motions of the plane.

2.4. The Finiteness Theorem. The first two classification steps given in the preceding section are the most difficult. One can combine them into a single step, namely the classification of finite subgroups of the group $GL_n(\mathbb{Z})$ up to conjugacy. Symmetry groups of lattices in Euclidean vector spaces (written in the bases of the lattices) can be singled out among such subgroups. They are known as *Bravais subgroups*, after the crystallographer who in 1848 enumerated all possible symmetry groups of three-dimensional lattices. Bravais subgroups can also be interpreted as stabilizers of the natural action of the group $GL_n(\mathbb{Z})$ on the set of positive definite quadratic forms in n variables. Thus the reduction theory can be used for their determination, implying, in particular, that for any n there are (up to conjugacy) only finitely many Bravais subgroups in $GL_n(\mathbb{Z})$ (see Cassels [1978] or VGS). Any finite subgroup of $GL_n(\mathbb{Z})$ is evidently contained in a Bravais subgroup. Hence, up to conjugacy, there are only finitely many finite subgroups. This means that for a fixed n there are only finitely many arithmetic classes of crystallographic groups.

On the other hand, for any arithmetic class (G, L), the group

$$H^1(G, \mathbb{R}^n/L) \simeq H^2(G, L)$$

is compact (as its first representation shows), and discrete (which is implied by the second representation), and therefore finite. This implies the following theorem.

Theorem 2.3 (Bieberbach [1911]). *For any dimension there are only finitely many crystallographic groups considered up to affine equivalence.*

The proof of this theorem provided a solution for the first part of Hilbert's 18th Problem.

2.5. Bravais Types. Considering Bravais subgroups as stabilizers of quadratic forms, one can easily see that an intersection of Bravais subgroups is also of this type. Therefore for any finite subgroup $G \subset GL_n(\mathbb{Z})$ there is the smallest Bravais subgroup containing G. Denote this subgroup by $\hat{G}$.

If Γ is a crystallographic group defined by the triple (G, L, α), then the linear group $\hat{G}$ is said to be the *syngony* (or *holohedry*) of the group Γ, and the pair $(\hat{G}, L)$ its *Bravais type* (or *arithmetic holohedry*). From the matrix point of view syngony is the group $\hat{G}$ considered up to conjugacy in $\hat{G}L_n(\mathbb{R})$, while the Bravais type is the same group considered up to conjugacy in $\hat{G}L_n(\mathbb{Z})$. Under a metric realization of "general position" of the group Γ, when the group $\operatorname{Sym} L$ is minimal, the group $\hat{G}$ coincides with $\operatorname{Sym} L$.

The term "syngony (Bravais type)" is also understood as the "set of all crystallographic groups of a given syngony (Bravais type)".

Example. For $n = 2$ there are 4 syngonies: C_2, D_2, D_4 and D_6. As the group D_2 allows two essentially different invariant lattices (see Fig. 11), the syngony D_2 falls into two Bravais types. All other syngonies consist of a single Bravais type. Thus there are in all 5 Bravais types.

2.6. Some Classification Results. Crystallographic groups in E^3 were classified at the end of the last century by E.S.Fedorov and Schoenflies. A

detailed description of these groups can, be found, for example, in Shubnikov [1946] and Lyubarskij [1958]. Crystallographic groups in E^4 have been classified only recently (Brown, Bülow, Neubüser, Wondratschek, and Zassenhaus [1978]). Being unable to give all these classifications here, we can only reveal some "statistics".

The first three rows of the following table present the numbers of subgroups of the groups $GL_n(\mathbb{Z})$, up to conjugacy in $GL_n(\mathbb{R})$ in the left column, and up to conjugacy in $GL_n(\mathbb{Z})$ in the right. Thus the number of syngonies and Bravais types are in the second row, and the number of classes and arithmetic classes in the third.

	n=2		n=3		n=4	
Number of maximal finite subgroups		2		4		9
Number of Bravais subgroups	4	5	7	14	33	64
Total number of finite subgroups	10	13	32	73	227	710
Number of crystallographic groups	17		219		4783	

In larger dimension, all maximal finite subgroups of the group $GL_n(\mathbb{Z})$ have been found for $5 \le n \le 10$ (Ryshkov [1972], Plesken and Pohst [1977, 1980], Ryshkov and Lomakina [1980]).

2.7. Euclidean Space Forms. *Euclidean space forms* or complete connected *locally flat Riemannian manifolds* are manifolds of the form E^n/Γ, where Γ is a discrete group of motions acting freely. Hereafter we shall call them, for short, flat manifolds.

In view of Theorem 1.3 any flat manifold is a vector bundle (with flat fibres) over a compact flat manifold, while Theorem 1.1 implies that any compact flat manifold can be covered by a (flat) torus.

The absence of fixed points for a discrete group Γ imposes some conditions on the group G of its linear parts. In particular, it must contain no linear transformations having no fixed vectors.

Example. For $n = 2$ the group G is either trivial or generated by a single reflection. Hence the group Γ is either trivial, or generated by a single translation, or a glide-reflection, or is generated by two translations, or a translation and a glide-reflection. Thus, up to affine isomorphism, there are only 5 two-dimensional flat manifolds. From the topological point of view, they are the plane, cylinder, Möbius strip, torus, and Klein bottle, respectively.

All three-dimensional flat manifolds are also known (Wolf [1977]), as well as all four-dimensional compact flat manifolds (Charlap [1986]). There are, up to affine isomorphism, 10 three-dimensional compact flat manifolds, and 75 four-dimensional ones.

There is the following obvious cohomological interpretation of the absence of fixed points for crystallographic groups.

Proposition 2.4. *A crystallographic group Γ defined by the triple (G, L, α) has no fixed points if and only if the restriction of the cohomology class α to any prime order cyclic subgroup of the group G is not zero.*

Charlap [1986] has obtained a classification of all crystallographic groups acting freely (in any dimension) whose group of linear parts is a cyclic group of prime order.

§ 3. Homogeneous Decompositions of Euclidean Space

3.1. The Finiteness Theorem. The main theorem on homogeneous decompositions of Euclidean spaces states that in some sense there are (for a fixed dimension) only finitely many such decompositions.

For a precise formulation of this result we need the following definition. Let $(\mathcal{T}_1, \Gamma_1)$ and $(\mathcal{T}_2, \Gamma_2)$ be homogeneous decompositions into convex polyhedra, normalized by a false faceting if necessary, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sets of their faces (i.e. the faces of all constituent polyhedra). *Decompositions* $(\mathcal{T}_1, \Gamma_1)$ and $(\mathcal{T}_2, \Gamma_2)$ are said to be *combinatorially equivalent* if there is a bijection $\alpha: \mathcal{F}_1 \to \mathcal{F}_2$ preserving the incidence relation, and an isomorphism $\varphi: \Gamma_1 \to \Gamma_2$ such that $\alpha(\gamma F) = \varphi(\gamma)\alpha(F)$ for all $F \in \mathcal{F}_1$, $\gamma \in \Gamma_1$.

Theorem 3.1 (Delone and Sandakova [1961]). *For any n there are only finitely many combinatorially non-equivalent normal homogeneous decompositions of the space E^n into bounded polyhedra.*

To be more precise, Delone and Sandakova [1961] proved that the number of faces of polyhedra appearing in homogeneous decompositions is bounded, but this easily implies the statement of the theorem.

A full classification of homogeneous decompositions of the Euclidean plane into bounded convex polygons was obtained in Delone [1959], Dolbilin [1976], and Delaunay, Dolbilin, and Shtogrin [1978]. There are 97 combinatorial types of such decompositions including 47 normal ones.

3.2. Parallelohedra. Normalized fundamental polyhedra of crystallographic groups of parallel translations in E^n are called *parallelohedra*. They form homogeneous decompositions whose elements can be obtained from each other by parallel translations. There is the following description of parallelohedra (Delaunay [1929], Aleksandrov [1934], Venkov [1954]).

Theorem 3.2. *A bounded convex polyhedron (possibly provided with a false faceting) is a parallelohedron if and only if*

(1) *it is centrally symmetric;*
(2) *each of its $(n-1)$-dimensional faces is centrally symmetric;*
(3) *its projection on a plane along any $(n-2)$-dimensional face f is either a quadrilateral, or a hexagon, and each vertex of this projection is a projection of some $(n-2)$-dimensional face (parallel to f).*

A Poincaré relation of the form $s_1 s_2 s_1^{-1} s_2^{-1} = e$ corresponds to each 4-tuple of parallel $(n-2)$-dimensional faces, and the pair of Poincaré relations $s_1 s_2 s_3 = e$, $s_1 s_3 s_2 = e$ corresponds to each 6-tuple of such faces.

Chapter 4
Fuchsian Groups

A discrete group of motions of the Lobachevskij plane is said to be *Fuchsian* if it consists of motions preserving the orientation. For any discrete group Γ of motions of the Lobachevskij plane its subgroup Γ^+ of index ≤ 2 consisting of motions preserving orientation is a Fuchsian group. In this way the study of arbitrary discrete groups of motions of the Lobachevskij plane is to a large extent reduced to the study of Fuchsian groups.

§ 1. Fuchsian Groups from the Topological Point of View

1.1. Ramified Coverings. By a (topological) *surface* we mean a connected two-dimensional orientable topological manifold. Any simply-connected surface is known to be isomorphic either to the sphere or to the plane. Any surface S can be covered with a simply-connected surface X (the universal covering) homeomorphic to the plane, unless S is a sphere. If $p: X \to S$ is the universal covering, then the covering group Γ (which is isomorphic to the fundamental group of the surface S) acts on X discretely, and without fixed points. The surface S is homeomorphic to X/Γ.

Consider the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ (respectively, the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$) on the complex plane, and the mapping $\varphi_k: \mathbb{D} \to \mathbb{D}$, $\varphi_k(z) = z^k$ ($k \in \mathbb{N}$) (respectively, $\varphi_\infty: \mathbb{H} \to \mathbb{D}$, $\varphi_\infty(z) = e^{2\pi i z}$).

A continuous mapping p of a surface X into a surface S is said to be a *(generalized) ramified covering* if

(a) the complement $S - p(X)$ is discrete;
(b) for each point $s \in S$ there is a neighbourhood Q such that $p^{-1}(Q)$ is the union of disjoint open sets U_i such that for any i the mapping $p: U_i \to Q$ is topologically isomorphic to a standard mapping φ_{k_i} (where $k_i \in \mathbb{N}$ or $k_i = \infty$) assuming that s corresponds to the zero point of the disc $\mathbb{D}$.

The numbers k_i are said to be the *ramification indices* of the mapping p over the point s.[1] If $k_i > 1$ for at least one value of i, then s is called a *ramification point* of the mapping p.

If Γ is a discrete group of orientation-preserving homeomorphisms of a surface X, then the mapping $\pi: X \to X/\Gamma$ is a ramified covering (Kerekjarto [1923]).

Example 1. The mapping $z \mapsto z^k$ of the Riemann sphere $\hat{\mathbb{C}}$ into itself is a topological ramified covering $p: S^2 \to S^2$ with the ramification points 0, ∞ and the ramification index k over each of them.

Example 2. The mapping $z \mapsto e^z$ of the complex plane $\mathbb{C}$ into the Riemann sphere provides an example of a topological ramified covering of a sphere by a plane with the ramification points 0, ∞ and the ramification index ∞ over each of them.

Example 3. Let X be one of the spaces S^2, E^2, or $Л^2$. Consider a triangle in X with the angles π/k_1, π/k_2, π/k_3 (where k_i is either a positive integer or ∞) (see Part 1). Denote by Γ the group generated by reflections in the sides of the triangle, and by Γ^+ its subgroup of index 2 consisting of proper motions. Then $p: X \to X/\Gamma^+ \simeq S^2$ is a ramified covering of the sphere which has three ramification points with the ramification indices k_1, k_2, and k_3 respectively.

Example 4. Consider a quadrilateral on the plane $Л^2$ with angles equal to $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2k}$, and the group generated by the reflections R_1, R_2, R_3, R_4 in the sides of this quadrilateral (the generators are listed clockwise). Denoting by Γ the subgroup generated by the motions R_1R_3 and R_2R_4, one can easily check that the mapping $p: Л^2 \to Л^2/\Gamma \simeq T^2$ is a ramified covering of a torus having a single ramification point of index k.

A ramified covering $p: X \to S$ is said to be a *Galois covering* if the mapping $p: X \to p(X)$ is isomorphic to the mapping of X onto the quotient space of X with respect to a discrete group Γ of homeomorphisms, called the *Galois group of the covering p*. All ramification indices of a Galois covering over a point $s \in S$ are equal. If $s = p(x)$, they coincide with the order of the stabilizer $\Gamma_x = \{\gamma \in \Gamma: \gamma x = x\}$. If $s \notin p(X)$ they are equal to ∞.

1.2. Signature and Uniformization of a Surface with Signature. Planar Groups. A *signature* on the surface S is a mapping $\tau: S \to \mathbb{N} \cup \{\infty\}$ such that the set $\operatorname{Supp}\tau = \{s \in S \mid \tau(s) \neq 1\}$, called the *support of the signature* τ, is discrete. Points of the support are said to be *distinguished*. A signature is said to be *finite* if the set $\operatorname{Supp}\tau$ is finite. A signature on a compact surface is automatically finite. It is sometimes convenient to consider a surface without signature as the surface with the trivial signature ($\operatorname{Supp}\tau = \emptyset$).

[1] Alternative terms also in use: *branched covering, branch index* (or *branch number*), *branch point*.

A *uniformization of a surface S with signature* τ is a pair (X, p), where X is a simply-connected surface, and $p: X \to S$ a ramified Galois covering whose the ramification index over any point $s \in S$ equals $\tau(s)$. The Galois group of the covering p is said to be the *uniformization group* of the surface S.

Theorem 1.1 (on the topological uniformization) (Wong [1971], Zieschang, Vogt, and Coldewey [1980]). *For any surface S with signature, with the exception of the cases listed below, a uniformization exists and is unique up to isomorphism. The exceptional cases for which there is no uniformization are*

(1) *a sphere with a single distinguished point;*
(2) *a sphere with two distinguished points of different indices.*

Proof. Denote the surface $S - \operatorname{Supp}\tau$ by S', and let $p': R \to S'$ be the universal covering with the Galois group Γ'. Then $p': R \to S$ is a ramified Galois covering with all the ramification indices equal to ∞ (unless S is a sphere with one distinguished point). For any point $s \in \operatorname{Supp}\tau$ consider its neighbourhood Q such that if U_s is a connected component of the inverse image $(p')^{-1}(Q)$, then the mapping $p': U_s \to Q$ is topologically isomorphic to the standard mapping $\varphi_\infty: \mathbb{H} \to \mathbb{D}$ and is therefore the quotient mapping with respect to an infinite cyclic group $\Gamma'(U_s)$.

Now denote by $\overline{R}$ a partial compactification of the space R obtained, for all $s \in \operatorname{Supp}\tau$ such that $\tau(s) \neq \infty$, by adding to each component U_s a "point at infinity" in such a manner that deleted neighbourhoods of this point correspond under the homeomorphism $U_s \to \mathbb{H}$ to the half-planes $\operatorname{Im} z > c$. Clearly, the space $\overline{R}$ is simply-connected, and the group action of Γ' can be naturally extended to $\overline{R}$. The stabilizer of the point at infinity for the component U_s is $\Gamma'(U_s)$.

Let γ'_s be a generator of the group $\Gamma'(U_s)$, and N the normal subgroup of the group Γ' generated by the elements $(\gamma'_s)^{\tau(s)}$ for all $s \in \operatorname{Supp}\tau$ such that $\tau(s) \neq \infty$. Then $X = \overline{R}/N$ is a simply-connected space (the space $\overline{R}$ is simply-connected, and the group N is generated by elements whose action on $\overline{R}$ has fixed points (Armstrong [1968])), and the natural mapping $p: X \to X/(\Gamma'/N)$ is the desired covering with the Galois group $\Gamma = \Gamma'/N$. To prove this, one has to verify that the image γ_s (in the group Γ) of the element γ'_s has order $\tau(s)$ (*a priori* its order can be a divisor of $\tau(s)$ other than $\tau(s)$). The conditions of the theorem imply that one of the following two statements holds:

(1) the point s can be included in a topological disc containing either one more ramification point of the same index, or two more ramification points;
(2) the point s can be included in a "handle" containing no other ramification points.

By contracting the complement to the disc, or the handle into a point, one obtains the surface S'' with signature, the uniformization for which has been constructed in the Examples of Sect. 1.1. The group Γ is mapped epimorphically onto the uniformization group Γ'' of the surface S''. Since the order of the image of the element γ_s in Γ'' is $\tau(s)$, the element γ_s has the same order in Γ. □

The uniformization $p : X \to S$ of a surface S with signature possesses the following property of universality.

Theorem 1.2. *For any ramified covering $q : Y \to S$ whose indices over any point $s \in S$ divide $\tau(s)$ (we assume that any integer divides ∞), there is a ramified Galois covering $r : X \to Y$ such that the diagram*

$$\begin{tikzcd} X \arrow[d, "r"'] \arrow[dr, "p"] & \\ Y \arrow[r, "q"'] & S \end{tikzcd}$$

commutes.

Uniformization groups of surfaces with signature, considered as discrete groups of homeomorphisms of simply-connected surfaces, are said to be *planar groups*.

1.3. Planar Groups of Finite Type. This will be the name given to the planar groups corresponding to compact surfaces with signature. Consider the uniformization group Γ of a compact surface S of genus g with signature τ, $\operatorname{Supp}\tau = \{s_1, \ldots, s_r\}$, $\tau(s_i) = k_1$. The set $(g; k_1, \ldots, k_r)$ will be called the signature of the group Γ, which is written as $\Gamma = \Gamma(g; k_1, \ldots, k_r)$. The signature provides a complete description of the group Γ as a discrete group of homeomorphisms.

Following the lines of the proof of Theorem 1.1, and using van Kampen's theorem (Massey [1967]), the group $\Gamma(g; k_1, \ldots, k_r)$ can be defined by its generators and relations.

Theorem 1.3 (Zieschang, Vogt, and Coldewey [1980]). *A planar group $\Gamma = \Gamma(g; k_1, \ldots, k_r)$ can be defined by the following set of generators and relations*

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g,\ \gamma_1, \ldots, \gamma_r \Big| \prod_{i=1}^{g} (\alpha_i, \beta_i) \prod_{k=1}^{r} \gamma_i = 1, \gamma_j^{k_j} = 1 \text{ for } k_j \neq \infty \rangle$$

(as usual, $(\alpha, \beta) = \alpha\beta\alpha^{-1}\beta^{-1}$).

The system of generators given in Theorem 1.3 is said to be a *canonical system of generators* of a planar group $\Gamma(g; k_1, \ldots, k_r)$.

The following result ensures the existence of a "convenient" fundamental domain for the action of a planar group $\Gamma(g; k_1, \ldots, k_r)$ on a simply-connected surface X.

Theorem 1.4 (Zieschang, Vogt, and Coldewey [1980]). *One can choose a fundamental domain for the action of the group $\Gamma = \Gamma(g; k_1, \ldots, k_r)$ on X in the form of a topological polygon P with sides $a_1, b_1', a_1', b_1, \ldots, a_g, b_g', a_g', b_g$, $c_1, c_1', \ldots, c_r, c_r'$ (the sides on the boundary of P are enumerated clockwise; if $k_i = \infty$,then the sides c_i and c_i' have no vertex in common) in such a way that $a_i = \alpha_i(a_i')$, $b_i = \beta_i(b_i')$, $c_i = \gamma_i(c_i')$, where $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r)$ is a canonical system of generators of the group Γ, and the sides are identified with the reversal of the orientation induced by the direction of the circuit on the boundary of P.*

The fundamental polygon chosen in Theorem 1.4 is said to be a *canonical fundamental polygon* of the planar group Γ.

The elements conjugate to the powers of generators γ_j, where $k_j = \infty$, are said to be *boundary elements* of the group Γ.

1.4. Algebraic Structure of Planar Groups. From now on by a "planar group" we always mean a planar group of finite type.

A planar group $\Gamma(g; k_1, \ldots, k_r)$ is said to be of *compact type* if all k_i are finite, and of *non-compact type* otherwise.

Theorem 1.3 easily implies that a planar group of non-compact type is a free product of cyclic subgroups. No group of compact type is of this kind with the exception of the group $\Gamma(0; k, k)$, which is cyclic itself. The number $\chi(\Gamma) = -2g + 2 - \sum_{i=1}^{r} (1 - 1/k_i)$ is said to be the *(generalized) Euler characteristic of the group Γ.*

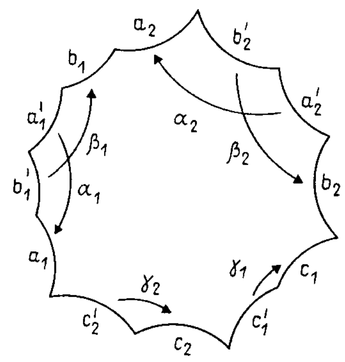


Fig. 12

Theorem 1.5 (Singerman [1970]). *Let Γ' be a subgroup of finite index of a planar group $\Gamma = \Gamma(g; k_1, \ldots, k_r)$. Then*

(a) *Γ' is a planar group, and if Γ is of non-compact (compact) type then Γ' is also of non-compact (respectively, compact) type;*
(b) *if $[\Gamma:\Gamma'] = n$, then the signature $(g'; k'_1, \ldots, k'_p)$ of the group Γ' is determined according to the following rule: let the permutation of the set Γ/Γ' defined by the left shift by the canonical generator γ_i of the group Γ be decomposed into a product of independent cycles of lengths $l_{i1}, \ldots, l_{iq_i}$ $\left(\sum_j l_{ij} = n\right)$. Then the indices k'_i of the group Γ' are precisely the numbers k_i/l_{ij} different from 1;*
(c) *$\chi(\Gamma') = n\chi(\Gamma)$ (the Riemann-Hurwitz formula).*

Proof. Consider the canonical ramified covering $p: X/\Gamma' \to X/\Gamma$. Then q_i is the number of inverse images of the i-th distinguished point of the surface $S = X/\Gamma$, and $l_{i1}, \ldots, l_{iq_i}$ are the ramification indices of p over this point. Statement (c) follows from standard combinatorial considerations concerning the computation of Euler's characteristic of the surface $S' = X/\Gamma'$ (see Singerman [1970] for details.) □

Remark. The intersection of all subgroups of finite index in a planar group is $\{1\}$. Moreover, any subgroup of infinite index in a planar group is a free product of cyclic subgroups (Hoare, Karrass, and Solitar [1972]

Any planar group contains a torsion-free subgroup of finite index (a consequence of Theorem 1.14 of Chap.1 and Theorem 2.1). The type and signature of a subgroup of finite index of a planar group satisfies the necessary conditions (a) and (b) of Theorem 1.5. At first glance, these conditions seem rather rough, and this makes the following theorem even more remarkable.

Theorem 1.6 (Edmonds, Ewing, and Kulkarni [1982]). *Let Γ' be an infinite planar torsion-free group. The group Γ' can be embedded in a given planar group Γ as a subgroup of index n if and only if both Γ and Γ' are of compact or non-compact type simultaneously, and $\chi(\Gamma') = n\chi(\Gamma)$.*

Corollary 1.7 (Edmonds, Ewing, and Kulkarni [1982]). *Let $\Gamma(g; k_1, \ldots, k_r)$ be an infinite planar group, and $r \geq 3$. Then the minimal index of a torsion-free subgroup contained in it is $n = 2^\varepsilon l$, where l is the least common multiple of the finite indices $k_1, \ldots, k_s$ of the group Γ, $\varepsilon = 1$ if l is even and the numbers l/k_q are odd for an odd number of indices q, $1 \leq q \leq s$, and $\varepsilon = 0$ otherwise.*

Example. The minimal value of the index of a torsion-free subgroup of the group $\Gamma(0; 2, 3, 7)$ is 84.

§ 2. Geometry of Fuchsian Groups

2.1. Introducing a Metric

Theorem 2.1. *Let $\Gamma = \Gamma(g; k_1, \ldots, k_r)$ be a planar group of finite type. Then Γ is isomorphic (as a group of homeomorphisms) to a discrete group of motions of one of the three spaces of constant curvature: S^2, E^2, or $Л^2$. The sign of the curvature of the space X coincides with the sign of $\chi(\Gamma)$. In particular, if $\chi(\Gamma) < 0$, then Γ is isomorphic to a Fuchsian group. The signatures of planar groups for which $\chi(\Gamma) \geq 0$ are listed in the following table.*

$\chi(\Gamma) > 0$	$\chi(\Gamma) = 0$
$(0; k, k)$ $(k < \infty)$	$(0; \infty, \infty)$
$(0; 2, 2, k)$ $(k < \infty)$	$(0; 2, 2, \infty)$
$(0; 2, 3, 3)$	$(0; 2, 3, 6)$
$(0; 2, 3, 4)$	$(0; 2, 4, 4)$
$(0; 2, 3, 5)$	$(0; 3, 3, 3)$
	$(0; 2, 2, 2, 2)$
	(1)

Proof. We shall show that the canonical fundamental domain of Theorem 1.4 can be chosen in the form of a convex polygon in one of the spaces of constant curvature. First, consider the case $\chi(\Gamma) < 0$. There is (Part 1, Chap. 3, Sect. 2) a convex circumscribed $(4g + 2r)$-gon on the Lobachevskij plane with the sides $a_1, b'_1, a'_1, b_1, \ldots, a_g, b'_g, a'_g, b_g, c_1, c'_1, \ldots, c_r, c'_r$, all the angles of which equal $\frac{2\pi}{4g+r}$ with the exception of the angle between the sides c_j and c'_j, which equals $\frac{2\pi}{k_j}$ $(j = 1, \ldots, r)$. The congruence law for right hyperbolic triangles implies that the sides a_i, b_i, a'_i, b'_i $(i = 1, \ldots, g)$ in this polygon are equal. For the same reason, the sides c_j and c'_j $(j = 1, \ldots, r)$ are also pairwise equal. Therefore, for the homeomorphisms α_i, β_i, and γ_i one can take the motions of the plane $Л^2$ from the canonical system of generators. Note that the resulting fundamental polygon is the Dirichlet polygon $D(x_0)$, where x_0 is the centre of the inscribed circle. There is no difficulty in giving an example of a canonical convex fundamental polygon either on the sphere S^2 or on the Euclidean plane E^2 for any other signature in the table. □

Since any finite group of motions of E^2 or $Л^2$ has a fixed point (see Part 1, Chap. 5, Sect. 3), Theorem 2.1 implies, in particular, that the cyclic subgroups $\langle \gamma_j \rangle$, $k_j < \infty$, form a complete system of representatives of classes of conjugate maximal finite cyclic subgroups in the planar group $\Gamma(g; k_1, \ldots, k_r)$.

It follows from Theorem 1.3 that any planar group of finite type is finitely generated. The converse statement is also true.

Theorem 2.2 (Greenberg [1977]). *Let Γ be a finitely generated discrete group of motions of the space $X = S^2$, E^2, or $Л^2$. Then Γ is a planar group of finite type.*

The statement of the theorem is evident for $X = S^2$, almost evident for $X = E^2$, and far from evident for $X = Л^2$. In the last case it follows from the following theorem.

Theorem 2.3 (Greenberg [1977]). *Any finitely generated Fuchsian group is geometrically finite.*

Note that Theorem 2.1 deals with the realization of an admissible signature by a Fuchsian group. A stronger statement is also true.

Theorem 2.4 (Zieschang, Vogt, and Coldewey [1980]). *Let Γ be a Fuchsian group with signature $\tau = (g; k_1, \ldots k_r)$. Then Γ has a canonical convex fundamental polygon.*

We observe in conclusion that a similar theory (including the notions of signature, canonical system of generators, etc.) has also been developed for discrete groups containing orientation-reversing motions (Macbeath [1967], Zieschang, Vogt, and Coldewey [1980]).

2.2. Nielsen's Domain for a Fuchsian Group. The Euler characteristic $\chi(\Gamma)$ of a planar group Γ has an interesting geometric interpretation. In order to reveal it, we introduce an important notion of *Nielsen's domain* $K(\Gamma)$ of a discrete group Γ of motions of the space $Л^n$. By definition, $K(\Gamma)$ is the convex hull of the set of limit points of the group Γ (see Chap. 2, Sect. 2).

For a Fuchsian group Γ, the structure of the domain $K(\Gamma)$ is simple, namely, the complement $\partial Л^2 - \Lambda(\Gamma)$ is the union of countably many disjoint arcs, such that each of them is its connected component. Join the end-points of each of these arcs c by a straight line in $Л^2$, and consider the closed half-plane in $Л^2$, whose boundary contains the arc of the absolute complementary to c. Nielsen's domain $K(\Gamma)$ is the intersection of all such half-planes. If Γ is a non-elementary Fuchsian group (see Chap. 2, Sect. 2), then the open kernel $K(\Gamma)^0$ of Nielsen's domain is an open convex Γ-invariant set. It can be characterized as the smallest of all such sets.

A Fuchsian group Γ is said to be a *group of the first kind* if the set $\Lambda(\Gamma)$ of its limit points coincides with $\partial Л^2$. Otherwise it is said to be a *group of the second kind*. For a Fuchsian group of the first kind one has $K(\Gamma) = Л^2$.

Theorem 2.5 (Greenberg [1977]). *Let Γ be a non-elementary Fuchsian group of signature $\tau = (g; k_1, \ldots, k_r)$. Then the volume of the quotient space $K(\Gamma)/\Gamma$ is finite and equals $-2\pi\chi(\Gamma)$.*

Proof. Taking the projective model of $Л^2$, consider the canonical convex fundamental polygon P of the group Γ, and construct the common perpendicular to each pair of its sides c_j, c'_j intersecting at a point beyond the absolute. This perpendicular is an invariant line for the hyperbolic boundary element γ_j. The resulting "truncated" convex polygon is the fundamental domain for the action of Γ in $K(\Gamma)$. Its area is finite and equals $-2\pi\chi(\Gamma)$ (see Part 1, Chap. 3, Sect. 2). □

The converse statement is also true.

Theorem 2.6 (Greenberg [1977]). *If the volume of the quotient space $K(\Gamma)/\Gamma$ is finite, then the Fuchsian group Γ is geometrically finite, or, equivalently, finitely generated.*

§ 3. The Teichmüller Space of a Fuchsian Group

The notion of signature makes it possible to distinguish Fuchsian groups from the topological but not from the metric point of view. Two Fuchsian groups of the same signature are not necessarily conjugate in the group of motions of the Lobachevskij plane. This gives rise to the problem of describing classes of equivalent realizations of a planar group with given signature in the form of a discrete group of motions.

3.1. The Teichmüller Space and the Moduli Space of an Abstract Group. Let G be a connected Lie group, and Γ an abstract group. Denote by $\mathrm{Hom}\,(\Gamma, G)$ the set of homomorphisms of the group Γ into the group G with the pointwise convergence topology, i.e. a sequence $h_n \in \mathrm{Hom}\,(\Gamma, G)$ is said to converge to a homomorphism h if $\lim\limits_{n\to\infty} h_n(\gamma) = h(\gamma)$ for each $\gamma \in \Gamma$.

Consider the subset $W(\Gamma, G)$ of homomorphisms h in $\mathrm{Hom}\,(\Gamma, G)$ such that

(1) h is a monomorphism;
(2) the group $h(\Gamma)$ is discrete in G;
(3) the quotient space $G/h(\Gamma)$ is compact.

The fundamental result of Weil says that $W(\Gamma, G)$ is open in $\mathrm{Hom}\,(\Gamma, G)$ (see VGS, Chap. 1, Sect. 6).

Consider, on the other hand, the set $S(G)$ of all discrete subgroups of the Lie group G endowed with the *Chabauty topology*: if Γ_n is a sequence of groups, then $\lim\limits_{n\to\infty} \Gamma_n = \Gamma$ means that

(a) each element $\gamma \in \Gamma$ is the limit of a sequence of elements $\gamma_n \in \Gamma_n$, $n = 1, 2, \ldots$;
(b) if $\gamma_n \in \Gamma_n$, and $\lim\limits_{n\to\infty} \gamma_n = \gamma$, then $\gamma \in \Gamma$.

(Conditions (a) and (b) mean that two discrete groups are close to each other if their intersections with any compact $K \subset G$ are close.)

Consider the subset $S(\Gamma, G)$ consisting of uniform discrete subgroups isomorphic to the group Γ. There is a natural mapping $\tau : W(\Gamma, G) \to S(\Gamma, G)$ associating with each homomorphism $h \in W(\Gamma, G)$ the uniform discrete subgroup $h(\Gamma) \in S(\Gamma, G)$.

The group $\mathrm{Aut}\, G$ acts naturally on the spaces $\mathrm{Hom}\,(\Gamma, G)$ and $W(\Gamma, G)$ by the formula $\sigma(h) = \sigma \circ h$ for any $\sigma \in \mathrm{Aut}\, G$, and the group $\mathrm{Aut}\, \Gamma$ by the formula $\varphi(h) = h \circ \varphi^{-1}$ for any $\varphi \in \mathrm{Aut}\, \Gamma$. The group G acts by conjugations:

$g(h)(\gamma) = gh(\gamma)g^{-1}$, $h \in \operatorname{Hom}(\Gamma, G)$, $\gamma \in \Gamma$. Furthermore, the action of $\operatorname{Aut} G$ commutes with that of $\operatorname{Aut} \Gamma$.

Theorem 3.1 (Macbeath and Singermann [1975]). *The group* $\operatorname{Aut}\Gamma$ *acts freely on the space* $W(\Gamma, G)$ *as a discrete group of homeomorphisms of this space. The natural mapping* τ *induces a homeomorphism of the quotient space* $W(\Gamma, G)/\operatorname{Aut}\Gamma$ *and the space* $S(\Gamma, G)$ *(the former is endowed with the quotient topology, the latter with that of Chabauty).*

The quotient space $T(\Gamma, G)$ (or simply $T(\Gamma)$ if there can be no misunderstanding about the Lie group)$= W(\Gamma, G)/\operatorname{Aut} G$ is said to be the *Teichmüller space* of the group Γ.

The quotient space $M(\Gamma, G)$ (or simply $M(\Gamma)$)$= S(\Gamma, G)/\operatorname{Aut} G$ is said to be the *moduli space* of the group Γ.

The following commutative diagram portrays the relationship between the spaces and groups we have just introduced.

$$\begin{tikzcd} W(\Gamma,G) \arrow[r, "\operatorname{Aut}\Gamma"] \arrow[d, "\operatorname{Aut}G"'] \arrow[dr, "\operatorname{Aut}G\times\operatorname{Aut}\Gamma"] & S(\Gamma,G) \arrow[d, "\operatorname{Aut}G"] \\ T(\Gamma)=T(\Gamma,G) \arrow[r] & M(\Gamma,G)=M(\Gamma) \end{tikzcd}$$

(each arrow represents the quotient mapping with respect to the corresponding group).

The commutativity of the diagram implies that $M(\Gamma) = T(\Gamma)/\operatorname{Aut}\Gamma$. The group $\operatorname{Aut}\Gamma$ acts non-effectively on $T(\Gamma)$, and the kernel of non-effectivity evidently contains the group $\operatorname{Int}\Gamma$ of inner automorphisms. The quotient group $\operatorname{Mod}\Gamma = \operatorname{Aut}\Gamma/\operatorname{Int}\Gamma$ is a discrete group of transformations of the space $T(\Gamma)$, called the *modular group.*

Finally, we observe that if the group Γ is finitely presentable, and G is a real algebraic group, then the topological space $\operatorname{Hom}(\Gamma, G)$ can be equipped with the structure of a real algebraic variety (see VGS, Chap. 1, Sect. 6).

3.2. The Teichmüller Space of the Fundamental Group of a Closed Surface. Let Γ be the fundamental group of a closed surface of genus g. Then the planar group $\Gamma = \Gamma(g)$ can be realized as a discrete group of motions of one of the three spaces: S^2 if $g = 0$, E^2 if $g = 1$, and $\mathrm{Л}^2$ if $g > 1$. Let G be the group of proper motions of the corresponding space. In the first case the group Γ is trivial, and the space $T(\Gamma)$ evidently consists of a single point.

In the second case Γ is a free abelian group with two generators, and the space $T(\Gamma)$ is naturally identified with the upper half-plane.

Proof. The homeomorphism $h \in W(\Gamma)$ is defined by an ordered pair of linearly independent vectors on the plane, i.e. a pair of complex numbers (p, q) such that $p/q = \tau$, $\operatorname{Im}\tau \neq 0$. The group $\operatorname{Aut} G$ is isomorphic to the group of similarities of the plane that do not necessarily preserve the orientation. Therefore two pairs (p, q) and (p', q') define the same point of the Teichmüller space if and only if $\tau = \tau'$ or $\tau = \overline{\tau'}$, so the space $T(\Gamma)$ can be identified with the upper half-plane. □

Theorem 3.2. *If* $\Gamma = \Gamma(g)$, $g > 1$, $G = \operatorname{Isom}_+ \mathrm{Л}^2 \simeq PSL_2(\mathbb{R})$ *then*

(a) *the space* $W(\Gamma)$ *is a real analytic manifold of dimension* $6g - 3$;

(b) *the space* $T(\Gamma)$ *is a real analytic manifold of dimension* $-3\chi(\Gamma) = 6g - 6$.

Proof. Choose a canonical system of generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ in the group Γ satisfying the relation $\prod_{i=1}^{g}(\alpha_i, \beta_i) = 1$. Then each element $h \in \operatorname{Hom}(\Gamma, G)$ is completely defined by the set $\{h(\alpha_i) = A_i,\ h(\beta_i) = B_i\}$ of $2g$ elements of the group G satisfying the relation $\prod_{i=1}^{g}(A_i, B_i) = 1$.

Note that the elements A_i and B_i $(i = 1, \dots, g)$ are non-commuting hyperbolic motions in the group G. Consider the mapping

$$\psi\colon G^{2g} \to G,\ \psi(X_1, Y_1, \dots, X_g, Y_g) = \prod_{i=1}^{g}(X_i, Y_i).$$

To prove (a) it is sufficient to verify that the rank of ψ at the point $(A_1, B_1, \dots, A_g, B_g)$ equals 3. This is implied by the following lemma.

Lemma 3.3. *Let* A *and* B *be non-commuting hyperbolic motions in the group* G. *Then the rank of the mapping* $\psi\colon G^2 \to G$, $\psi(X, Y) = XYX^{-1}Y^{-1}$ *at the point* (A, B) *equals 3.*

Proof. Let us identify (by left shifts) the tangent spaces at all points of the group G with its tangent Lie algebra $\mathfrak{g}$. Then $d\psi(A, B)(\xi, \eta) = BA[(B^{-1}\xi B - \xi) + (\eta - A^{-1}\eta A)]A^{-1}B^{-1}$, where $\xi, \eta \in \mathfrak{g}$. Denote by $\mathcal{L}_A$ (respectively, $\mathcal{L}_B$) the subspaces in $\mathfrak{g}$ spanned by the vectors of the form $A^{-1}\eta A - \eta$ (respectively, $(B^{-1}\xi B - \xi)$). Clearly, $\dim \mathcal{L}_A = \dim \mathcal{L}_B = 2$, and a direct verification shows that if the hyperbolic motions A and B do not commute, then $\mathcal{L}_A \neq \mathcal{L}_B$. □

Recommencing the proof of Theorem 3.2, we observe that the group $\operatorname{Aut} G \simeq PGL_2(\mathbb{R})$ acts freely on the real analytic manifold $W(\Gamma)$. Hence $T(\Gamma)$ is a real analytic manifold of dimension $(6g - 3) - 3 = 6g - 6$. □

Theorem 3.4 (Fricke and Klein [1897]). *Let* $\Gamma = \Gamma(g)$, $g > 1$. *Then the space* $T(\Gamma)$ *is homeomorphic to* $\mathbb{R}^{6g-6}$.

We now follow Douady [1979] and Seppälä and Sorvali [1985] in outlining the proof of this classical theorem recently suggested by Thurston. His approach employs another model of the Teichmüller space (incidentally, an older one).

Let S be a compact surface (possibly with boundary) of genus $g > 1$. Denote by $\operatorname{Diff} S$ the group of diffeomorphisms of S, and by $\operatorname{Diff}^0 S$ its connected component consisting of diffeomorphisms isotopic to the identity. Let $M(S)$ be the set of complete hyperbolic metrics of constant negative curvature -1 on S equipped with the natural topology. Each diffeomorphism $q: S \to S$ induces the mapping $q^*: M(S) \to M(S)$ defined by the condition that the diffeomorphism q is an isometry $(S, q^*(m)) \to (S, m)$, $m \in M(S)$. Thereby the group $\mathrm{Diff}^0 S$ acts naturally on $M(S)$. The quotient space $T(S) = M(S)/\mathrm{Diff}^0 S$ is said to be the *Teichmüller space* of the surface S.

We now observe that the choice of a hyperbolic metric $m \in M(S)$ on S defines the homomorphism $h: \Gamma \to \mathrm{Isom}_+ Л^2$ up to a motion of the space $Л^2$, i.e. defines the element $[h] \in T(\Gamma)$. The mapping $[m] \to [h]$ is well-defined, and establishes a homeomorphism between the spaces $T(S)$ and $T(\Gamma)$ (this makes use of following topological fact (known as the Baer theorem): a diffeomorphism of a surface S inducing an inner automorphism of the fundamental group is isotopic to the identity diffeomorphism, see Abikoff [1980]).

We now proceed with the proof of the theorem.

Proof of Theorem 3.4 The basic element in Thurston's constructions is a "pair of pants". A pair of pants $\mathcal{P}$ is a compact surface with boundary topologically isomorphic to a sphere with three holes. Let m be a complete hyperbolic metric on $\mathcal{P}$. Denote by $\gamma_1, \gamma_2, \gamma_3$ the components of the boundary ∂P, and by $l_m(\gamma_i)$ the length of the geodesic homotopic to the curve $\gamma_j \in \partial P$, $j = 1, 2, 3$ (each homotopy class of simple closed curves on a surface with a complete hyperbolic metric contains a unique geodesic, see Abikoff [1980]).

Proposition 3.5. *The mapping*

$$l : T(\mathcal{P}) \to (\mathbb{R}_+^*)^3, \; l([m]) = (l_m(\gamma_1), l_m(\gamma_2), l_m(\gamma_3))$$

is a homeomorphism.

Proof. Any three pairwise divergent lines in $Л^2$ form an ideal triangle. Each ideal triangle defines a pair of pants $(\mathcal{P}, m)$ obtained as the quotient space $Л^2/\Gamma^+$, where Γ is the group generated by reflections in the sides of the triangle and Γ^+ its subgroup of proper motions. The length $l_m(\gamma_j)$ equals twice the distance between the corresponding sides of the triangle. The pair of pants $(\mathcal{P}, m)$ defines the ideal triangle uniquely (up to a proper motion of $Л^2$), and is in turn defined by the triangle up to isometry.

Indeed, consider the inverse image of geodesics homotopic to the pants' boundary. It consists of pairwise divergent lines dividing the plane into ideal triangles (Natanzon [1972]). The triangle bounded by the common perpendiculars to the sides of any one of them is the desired one.

To complete the proof of Proposition 3.5 it remains to observe that an ideal triangle is defined by distances between its sides, and that these distances can assume any positive values (Part 1, Chap. 3, Sect. 3). □

Any surface S of genus $g > 1$ can be decomposed into $(2g - 2)$ pairs of pants. An example of such a decomposition is given in Fig. 13.

Choose a decomposition of S into $(2g - 2)$ pairs of pants $\mathcal{P}_1, \ldots, \mathcal{P}_{2g-2}$, and denote the $(3g - 3)$ boundary components of the decomposition by

$\gamma_1, \ldots, \gamma_{3g-3}$. Clearly, each class $[m] \in T(S)$ can be represented by a hyperbolic metric for which γ_j is a geodesic, $j = 1, \ldots, 3g - 3$. Proposition 3.5 easily implies the following statement.

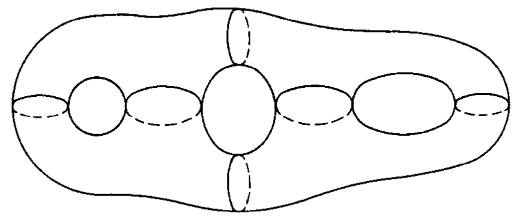


Fig. 13

Proposition 3.6. *The mapping*

$$l : T(S) \to (\mathbb{R}_+^*)^{3g-3}, \; [m] \mapsto (l_m(\gamma_1), \ldots, l_m(\gamma_{3g-3}))$$

is well-defined and surjective.

We now describe a natural action of the group $\mathbb{R}^{3g-3}$ on the space $T(S)$. Choose orientations on the curves γ_j, and construct a "collar" $\gamma_j \times [0, 1]$ to the left of each boundary component γ_j. The collars can be assumed to be disjoint. If $m \in M(S)$ is such that γ_j is a geodesic, then for any $t \in \mathbb{R}$ we define a diffeomorphism $\varphi_j(m, t)$ of the collar $\gamma_j \times [0, 1]$ into itself (*Dehn's twist*) as follows:

(a) $\varphi_j(m, t) = \mathrm{id}$ in a neighbourhood of $\gamma_j \times \{1\}$;
(b) $\varphi_j(m, t)$ is an isometry on γ_j such that the lift of $\varphi_j(m, t)$ to the universal covering $\mathbb{R} \times [0, 1]$ of the collar is the translation by $tl_m(\gamma_j)$ on $\mathbb{R} \times \{0\}$.

Dehn's twist yields a new metric $\varphi_j^*(m, t)(m)$ on the surface, which coincides with the old one outside the collar $\gamma_j \times [0, 1]$.

For any $(t_1, \ldots, t_{3g-3}) \in \mathbb{R}^{3g-3}$ denote by $\Phi(m, t_1, \ldots, t_{3g-3})$ the metric on S obtained after this procedure is applied to a neighbourhood of each geodesic γ_j, $j = 1, \ldots, 3g - 3$.

Theorem 3.7 (Seppälä and Sorvali [1985]). *The mapping* $l : T(S) \to (\mathbb{R}_+^*)^{3g-3}$ *defines a principal* $\mathbb{R}^{3g-3}$*-bundle.*

In particular, the mapping l has a continuous global section

$$s : (\mathbb{R}_+^*)^{3g-3} \to T(S),$$

and for each $[m] \in T(S)$ there is a unique point $(t_1^{(m)}, \ldots, t_{3g-3}^{(m)}) \in \mathbb{R}^{3g-3}$ such that $[m] = [\Phi(s \circ l[m]), t_1^{(m)}, \ldots, t_{3g-3}^{(m)}]$. Then the mapping

$$T(S) \to (\mathbb{R}_+^*)^{3g-3} \times \mathbb{R}^{3g-3},\ [m] \to (l([m]),\ t_1^{(m)}, \ldots, t_{3g-3}^{(m)})$$

is a homeomorphism (depending on s). This proves Theorem 3.4. □

Besides the natural real analytic structure, a number of other remarkable structures can be introduced into the Teichmüller space of the fundamental group of a closed surface of genus $g \geq 1$, among them the complex, kählerian, symplectic, and Finsler structures. Each of them plays its role in analytic (algebraic) geometry and analysis (see Macbeath and Singermann [1975], Abikoff [1980], Farkas and Kra [1980]).

Consider now a planar group $\Gamma = \Gamma(g; k_1, \ldots, k_r)$ with $\chi(\Gamma) < 0$ and distinguish a number (say t) of classes of conjugate boundary elements in it. Denote by $W_t(\Gamma)$ the set of homeomorphisms $h : \Gamma \to G = PSL_2(\mathbb{R})$ such that

(a) h is a monomorphism;
(b) the group $h(\Gamma)$ is discrete;
(c) h maps the distinguished elements into hyperbolic motions, and the remaining boundary elements into parabolic motions.

Theorem 3.8 (Abikoff [1980]). *The Teichmüller space $T_t(\Gamma) = W_t(\Gamma)/\operatorname{Aut} G$ is a real analytic manifold of dimension $n = 6g - 6 + t + 2r$.*

3.3. Fuchsian Groups and Riemann Surfaces. Let S be a compact Riemann surface with signature. Ignoring its complex structure, and applying (in non-exceptional cases) Theorem 1.1 one gets a planar uniformization group Γ acting on a simply connected surface X. There is, however, a unique complex structure on X under which the homeomorphism $p: X \to X/\Gamma = S$ is a holomorphic mapping. By the Poincaré-Koebe theorem, a simply connected Riemann surface X is isomorphic to the Riemann sphere $\hat{\mathbb{C}}$, or the complex plane $\mathbb{C}$, or the upper half-plane $\mathbb{H}$. We thus arrive at the following *analytic uniformization theorem.*

Theorem 3.9 (Zieschang, Vogt, and Coldewey [1980]). *Let S be a non-exceptional Riemann surface of genus g with signature τ. There is a unique (up to conjugacy) discrete group $\Gamma = \Gamma(g; k_1, \ldots, k_r)$ of analytic automorphisms of a simply-connected Riemann surface X such that $X/\Gamma = S$, and $X = \hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}$ if $\chi(\Gamma) > 0$, $= 0$, or < 0 respectively.*

Note that, conversely, if Γ is a discrete group of proper motions of one of the three spaces $X = S^2$, E^2, or Л^2, then the elements of Γ are automorphisms of standard complex structures on these spaces. Hence the quotient space $S = X/\Gamma$ is a one-dimensional complex manifold, i.e. a Riemann surface (with signature).

Indeed, the stabilizer Γ_x of any point $x \in X$ is a cyclic group of finite order. Hence, in a neighbourhood of the fixed point x the covering $p: X \to S$ is isomorphic to the mapping $\varphi_k: z \mapsto z^k$, and therefore $p(x)$ is a non-singular point on S. □

If the arrows in the statement of Theorem 1.2 represent analytic mappings, the resulting statement yields *the universality property of analytic uniformization* in the category of Riemann surfaces and their (ramified) analytic coverings.

3.4. Extensions of Fuchsian Groups. Maximal Fuchsian Groups. The following theorem was proved by Kerckhoff in 1982 using the original technique developed by Thurston in his study of Teichmüller spaces.

Theorem 3.10 (Kerckhoff [1983]). *Let Δ be an abstract group containing a Fuchsian group Γ of compact type as a normal subgroup of finite index. If the natural homomorphism $\Delta \to \operatorname{Aut}\Gamma$ is a monomorphism, then the group Δ is isomorphic to a Fuchsian group.*

This statement was for a long time known as *Nielsen's conjecture*, and had been confirmed in numerous particular cases. Nielsen proved it under the condition that Δ/Γ is a cyclic group (for the history of the problem see Zieschang [1981]).

One can assume, without loss of generality, that the group Γ is torsion-free. Then, in geometrical language, Theorem 3.10 means that any finite subgroup of the modular group $\operatorname{Mod}\Gamma$ has a fixed point in the space $T(\Gamma)$ (Zieschang [1981]).

A finitely generated Fuchsian group is said to be *maximal* if it is not contained in any larger Fuchsian group. An admissible signature τ is said to be maximal if there is a maximal Fuchsian group Γ with signature τ. One can show that in this case almost all Fuchsian groups of signature τ are maximal (non-maximal groups forming a submanifold of lesser dimension in the space $T(\Gamma)$).

Example. Since any compact surface of genus 2 admits a hyperelliptic involution, the signature (2) is not maximal. Any Fuchsian group with this signature is contained in a Fuchsian group of signature $(0; 2, 2, 2, 2, 2, 2)$ as a normal subgroup of index 2.

All non-maximal signatures of Fuchsian groups are enumerated in Singermann [1972]. The list is compiled by comparing dimensions of the Teichmüller spaces of a Fuchsian group Γ and those of its possible (abstract) extension Γ'.

Chapter 5
Reflection Groups

Discrete reflection groups are distinguished among all discrete groups of motions by the simplicity of their geometric description, and, as a consequence, by the simplicity of their defining relations. On the other hand, they are naturally related to numerous geometric and algebraic objects (regular polyhedra, decompositions into equal regular polyhedra, semisimple Lie groups, algebraic $K3$ surfaces etc.).

In this chapter we make extensive use of the theory of acute-angled polyhedra given in Part 1, Chap. 6. In what follows, "reflection" always means "reflection in a hyperplane", and "reflection group" means "group generated by reflections".

§ 1. Basic Notions and Theorems

1.1. Coxeter Polyhedra. The simplest type of fundamental polyhedron for a discrete group of motions Γ (see Chap. 2, Sect. 1) is obtained if for each i the adjacency transformation s_i is the reflection in the hyperplane of the face F_i. In this case the Poincaré word corresponding to the circuit around the $(n-2)$-dimensional face $F_{ij} = F_i \cap F_j$ is of the form $(s_i s_j)^{m_{ij}}$ (see Fig. 14), the angle between F_i and F_j equals $\frac{\pi}{m_{ij}}$, and $s_i s_j$ is the rotation through $\frac{2\pi}{m_{ij}}$ about the plane of the face F_{ij}. The defining relations of the group Γ are of the form

$$s_i^2 = e, \quad (s_i s_j)^{m_{ij}} = e.$$

If the faces F_i and F_j are not adjacent, we set $m_{ij} = \infty$.

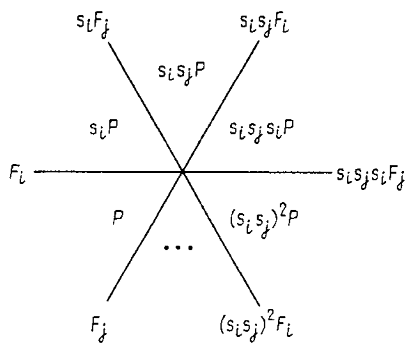


Fig. 14

Definition 1.1. A (generalized) convex polyhedron

$$P = \bigcap_{i \in I} H_i^-$$

is said to be a *(generalized) Coxeter polyhedron* if for all i, j, $i \neq j$, such that the hyperplanes H_i and H_j intersect, the dihedral angle $H_i^- \cap H_j^-$ is a submultiple of π.

Any Coxeter polyhedron is acute-angled (see Part 1, Chap. 6). Therefore the hyperplanes of its non-adjacent faces do not intersect.

The hyperplanes of $(n-1)$-dimensional faces of a convex polyhedron are said to be its *walls.*

Theorem 1.2. *Let P be a (generalized) Coxeter polyhedron, and $\Gamma(P)$ the group generated by reflections in its walls. Then $\Gamma(P)$ is a discrete group of motions, and P is its fundamental polyhedron. The stabilizer of any point $x \in P$ in $\Gamma(P)$ (including a point at infinity for $X = Л^n$) is generated by reflections in the walls of P containing x.*

If P is an ordinary polyhedron, then this theorem is a special case of Theorem 1.8, Theorem 1.2, and Proposition 1.7 of Chap. 2. In the general case, it can either be proved in the same way as the corresponding theorem for linear reflection groups (Bourbaki [1968], Vinberg [1971]), or derived from it with the help of the vector model. The only part of this approach that requires a separate proof is the statement that the polyhedra γP, $\gamma \in \Gamma(P)$ cover the space X, and this is provided for by the following lemma.

Lemma 1.3. *Let P be an arbitrary (generalized) convex polyhedron, and $\Gamma(P)$ the group generated by reflections in its walls. Then the polyhedra γP, $\gamma \in \Gamma(P)$, cover the space X.*

Proof. First we prove that $X' = \bigcup_{\gamma \in \Gamma(P)} \gamma P$ is an open set. This is verified by proving that each point $x_0 \in P$ has a neighbourhood contained in X'. Let $\Gamma_0(P)$ be the group generated by reflections in the walls of P containing x_0, and r the minimum of the distances from x_0 to the faces of P not containing x_0. The lemma can be proved by induction on $\dim X$. Applying the inductive hypothesis to the intersection of the polyhedron P with the sphere of radius r and centre at x_0, we see that this sphere, and consequently all its interior points, are covered by the polyhedra γP, $\gamma \in \Gamma_0(P)$, and is therefore contained in X'.

Suppose now that $X' \neq X$. Let x_0 be an interior point of the polyhedron P, and x the point of the closed set $X \setminus X'$ nearest to it. Since $x \notin P$, there is a wall of the polyhedron P separating x_0 from x. Let y be the point symmetric to x with respect to this wall. Evidently, $y \notin X'$ and $\rho(x_0, y) < \rho(x_0, x)$, contradicting the choice of x. $\square$

1.2. Discrete Reflection Groups. Let Γ be a discrete group of motions generated by reflections (*discrete reflection group*). The family of mirrors for all reflections belonging to the group Γ is evidently locally finite and invariant under Γ. Therefore these mirrors decompose the space into generalized convex polyhedra permuted by the group Γ. We shall call them the *chambers* of Γ.

Chambers having a common $(n-1)$-dimensional face are said to be *adjacent*. Adjacent chambers are permuted by the reflection in the hyperplane of their common face. This implies that any chamber can be taken into any other chamber by a transformation from the group Γ.

Proposition 1.4. *Let Γ be a discrete reflection group, and P its chamber. Then P is a generalized Coxeter polyhedron, and $\Gamma = \Gamma(P)$. In particular, P is a fundamental polyhedron of the group Γ.*

Proof. Suppose that the walls H_i and H_j of the chamber P intersect, and that the dihedral angle $H_i^- \cap H_j^-$ (containing P) is not a submultiple of π. Then there exists a number k such that the hyperplane $(s_i s_j)^k H_i$ or $(s_i s_j)^k H_j$ (either of which is the mirror for some reflection in Γ) intersects the interior of the angle $H_i^- \cap H_j^-$, and therefore intersects P^0, which contradicts the definition of a chamber. Thus P is a Coxeter polyhedron. Clearly, $\Gamma(P) \subset \Gamma$.

If a chamber is obtained from P by a transformation $\gamma \in \Gamma(P)$, then the reflections in its walls, being conjugate to reflections in the walls of P by the element γ, belong to $\Gamma(P)$. This implies in turn that all chambers adjacent to γP can be obtained from P by transformations from the group $\Gamma(P)$. A step-by-step application of this argument shows that all chambers can be obtained from P by transformations from the group $\Gamma(P)$, and that all reflections in Γ belong to $\Gamma(P)$. Since, by hypothesis, Γ is generated by reflections, one has $\Gamma = \Gamma(P)$. □

Thus any discrete reflection group is generated by reflections in the walls of a Coxeter polyhedron.

The same argument is also helpful in the investigation of discrete groups of motions containing reflections, but not necessarily generated by them.

For a discrete group of motions Γ we denote by Γ_r the subgroup generated by all reflections in Γ. The subgroup Γ_r will be called the *reflection subgroup*.

Proposition 1.5 (Vinberg [1975]). *Let Γ be an arbitrary discrete group of motions, Γ_r its reflection subgroup, and P a chamber of the group Γ_r. Then Γ_r is a normal subgroup of Γ, and*

$$\Gamma = \Gamma_r \rtimes H,$$

where $H = \Gamma \cap \operatorname{Sym} P$.

1.3. Coxeter Schemes. Coxeter polyhedra can be described conveniently in the language of Coxeter schemes.

A one-dimensional simplicial complex is called a *graph*. A *subgraph* of any graph is a subcomplex that contains together with any two adjacent vertices the edge joining them. Further, we define a *scheme* as a graph in which to every edge a positive weight is attached. A *subscheme* of a scheme is a subgraph in which every edge carries the same weight as in the whole scheme. The number of vertices of a scheme S is called its *order* and denoted by $|S|$.

Schemes can be used to describe symmetric matrices with units on the main diagonal and non-positive elements otherwise, namely, to each scheme S with the vertices $v_1, \dots, v_n$ there corresponds the symmetric matrix $A = (a_{ij})$ in which $a_{ii} = 1$ for all i, and a_{ij} for $i \neq j$ is the negative of the weight of the edge $v_i v_j$ if the vertices v_i and v_j are adjacent and 0 otherwise. The matrix A is defined up to the same permutation of rows and columns. Subschemes of the scheme S correspond to principal submatrices of the matrix A. The matrix A is indecomposable if and only if the scheme S is connected.

The *determinant* (*rank, inertia index*) of a scheme is the determinant (rank, inertia index) of the corresponding symmetric matrix.

Definition 1.6. The *scheme of an acute-angled polyhedron* P is the scheme corresponding to its Gram matrix $G(P)$.

In other words, it is the scheme whose vertices correspond to the walls of the polyhedron P; two vertices are joined by an edge if the corresponding walls are not orthogonal; the weight of an edge is the negative of the cosine of the angle between the walls if they intersect, -1 if they are parallel, and the negative of the hyperbolic cosine of the distance between them if they are divergent (in the case of Lobachevskij space).

Definition 1.7. A *Coxeter scheme* is a scheme in which the weight of each edge is either at least 1, or has the form $\cos \frac{\pi}{m}$, where m is an integer no less than 3. (We may assume that the weight 1 corresponds to the value $m = \infty$.)

The definitions imply that an acute-angled polyhedron is a Coxeter polyhedron if and only if its scheme is a Coxeter scheme.

Graphically an edge of a Coxeter scheme is depicted as follows:

if the weight is $\cos \frac{\pi}{m}$, by an $(m-2)$-fold line or a simple line with the label m;
if the weight is 1, by a heavy line or a simple line with the label ∞;
if the weight is greater than 1, by a dotted line with a label equal to the weight (often this label is omitted).

The number $m-2$ is called the multiplicity of an edge of weight $\cos \frac{\pi}{m}$. The multiplicity of an edge of weight at least 1 is taken to be infinite.

1.4. Reflection Groups on the Sphere and in Euclidean Space. A connected Coxeter scheme is said to be *elliptic* (respectively, *parabolic*) if the corresponding symmetric matrix is positive definite (respectively, semidefinite and degenerate). An arbitrary Coxeter scheme is said to be elliptic or parabolic if all its connected components are.

According to Part 1, Chap. 6, Theorem 1.7, non-degenerate n-dimensional spherical (respectively, Euclidean) Coxeter polyhedra are described by Coxeter schemes of rank $n+1$ (respectively, of rank n) all connected components of which are elliptic (respectively, elliptic or parabolic). Non-degenerate spherical Coxeter polyhedra are simplices, while Euclidean polyhedra are direct products of simplices and simplicial cones. In particular, bounded Euclidean Coxeter polyhedra are direct products of simplices and are described by parabolic Coxeter schemes.

The classification of connected elliptic and parabolic Coxeter schemes is, in principle, reduced to a calculation of determinants. It was obtained by Coxeter [1934]. These schemes are listed in Tables 1 and 2 respectively. The subscript is the rank of the scheme which in Table 1 equals its order, and in Table 2 is one less than the order of the scheme. The last column in Table 1 gives the orders of the corresponding (finite) reflection groups.

The group Γ of motions of the space E^n corresponding to the scheme $\tilde{L}_n$ in Table 2 is a semidirect product of the finite group Γ_0 corresponding to the scheme L_n of Table 1 and a commutative group generated by parallel translations along vectors orthogonal to the walls of the chamber of the group Γ_0.

Table 1. Connected elliptic Coxeter schemes

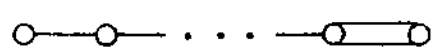
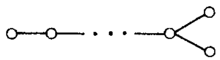
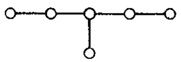
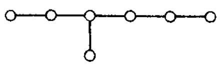
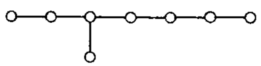
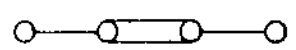
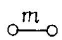
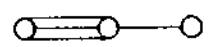
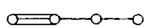
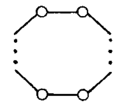
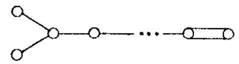
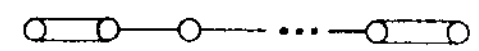
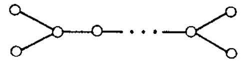
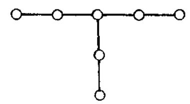
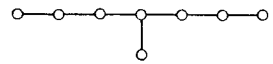
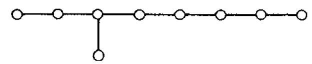
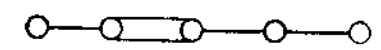
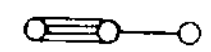
$A_n (n \geq 1)$		$(n+1)!$
B_n or C_n $(n \geq 2)$		$2^n \cdot n!$
$D_n (n \geq 4)$		$2^{n-1} \cdot n!$
E_6		$2^7 \cdot 3^4 \cdot 5$
E_7		$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8		$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
F_4		1152
$G_2^{(m)}$		$2m$
H_3		120
H_4		14400

Table 2. Connected parabolic Coxeter schemes

$\tilde{A}_1$	
$\tilde{A}_n (n \geq 2)$	
$\tilde{B}_n (n \geq 3)$	
$\tilde{C}_n (n \geq 2)$	
$\tilde{D}_n (n \geq 4)$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	
$\tilde{F}_4$	
$\tilde{G}_2$	

§2. Reflection Groups in Lobachevskij Space

2.1. General Properties. A Coxeter scheme is said to be *hyperbolic* if its negative inertia index is 1.

According to Part 1, Chap. 6, Theorem 2.2, indecomposable non-degenerate Coxeter polyhedra in the space Л^n are described by connected hyperbolic Coxeter schemes of rank $n+1$, and, by Part 1, Chap. 6, Theorems 2.3. and 2.5, the faces of such a polyhedron correspond to the elliptic subschemes of its Coxeter scheme (in particular, its vertices correspond to the elliptic subschemes of rank n), and its vertices at infinity to parabolic subschemes of rank $n-1$.

Note that any Coxeter polyhedron, like any acute-angled polyhedron, is simple in its ordinary faces (Part 1, Chap. 6, Theorem 1.8) and has the combinatorial structure of a cone over a direct product of simplices in a neighbourhood of any vertex at infinity (Part 1, Chap. 6, Sect. 2.2).

By Andreev's theorem (Part 1, Chap. 5, Theorem 2.6), a bounded Coxeter polyhedron in Л^n for $n \geq 3$ is uniquely (up to a motion) determined by its combinatorial structure and dihedral angles. The strong rigidity theorem (see Chap. 7, Sect. 1) implies a similar statement for Coxeter polyhedra of finite volume. This means that one can omit labels on the dotted edges of the Coxeter schemes describing Coxeter polyhedra of finite volume in the space Л^n, $n \geq 3$.

Therefore there are at most countably many Coxeter polyhedra of finite volume in the space Л^n for any $n \geq 3$. No classification is known at the time of writing, although it is apparently possible. The main results concerning this problem (which is equivalent to classifying crystallographic reflection groups in the Lobachevskij spaces) are given in Sect. 2.2-2.5.

2.2. Crystallographic Reflection Groups in the Lobachevskij Plane and in the Three-dimensional Lobachevskij Space. The classification of crystallographic reflection groups in the Lobachevskij plane was essentially obtained by Poincaré and Dyck in 1882. The fundamental polygon of such a group may have an arbitrary number k of sides ($k \geq 3$) and angles $\frac{\pi}{m_1}, \cdots, \frac{\pi}{m_k}$ (where m_i is either ∞ or an integer ≥ 2), provided only that

$$\frac{1}{m_1} + \cdots + \frac{1}{m_k} < k - 2$$

(see Part 1, Chap. 3, Theorem 2.8). Such a polyhedron depends on $k-3$ real parameters.

A classification of crystallographic reflection groups in the three-dimensional Lobachevskij space follows, in principle, from the description of acute-angled polyhedra given by Andreev (Part 1, Chap. 5, Theorems 2.7 and 2.8).

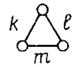
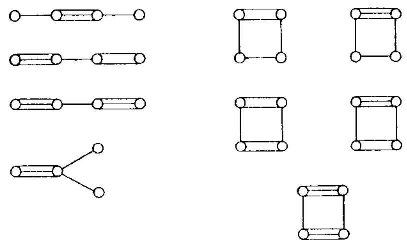
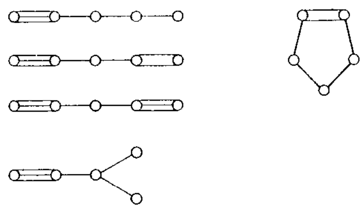
This description implies, in particular, that there are bounded Coxeter polyhedra in Л^3 whose combinatorial structure is, in a sense, arbitrarily complex. For example, let Q be a simple bounded convex polyhedron in E^3 that is not a tetrahedron and has neither triples of pairwise adjacent but non-concurrent faces, nor quadruples of cyclically adjacent faces. Then there is a bounded convex polyhedron in Л^3 of the same combinatorial type as Q such that all its dihedral angles are right angles. Obviously it is a Coxeter polyhedron. (The first examples of this kind were constructed by Löbell [1931].)

In Л^3 there are also bounded Coxeter polyhedra with arbitrarily small dihedral angles. For example, for any $m \geq 7$ there is a triangular prism whose dihedral angles between lateral faces are $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{m}$, one of the bases is perpendicular to all lateral faces, while the other is perpendicular to the two of them making the angle $\frac{\pi}{m}$ and forms the angle $\frac{\pi}{3}$ with the third one. (Several series of examples of this kind were first constructed by Makarov [1965].)

2.3. Lannér and Quasi-Lannér Groups. It is not difficult to list all bounded Coxeter simplices in the Lobachevskij space Л^n for any n. They are described by connected hyperbolic Coxeter schemes of order $n+1$ whose proper subschemes are elliptic. We shall call the corresponding groups *Lannér groups*, and the schemes *Lannér schemes*, after F.Lannér who first enumerated them in 1950.

Lannér groups exist in Л^n only for $n \leq 4$. Their schemes are listed in Table 3.

Table 3. Lannér schemes

n	Schemes
2	 $(2 \leq k,\ l, m < \infty,\ \frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1)$
3	
4	

Unbounded simplexes of finite volume in the Lobachevskij spaces can be enumerated in a similar manner. They are described by connected hyperbolic Coxeter schemes, whose proper subschemes are either elliptic or connected parabolic. We shall call them *quasi-Lannér schemes* and the corresponding groups *quasi-Lannér groups*.

Quasi-Lannér groups exist in the spaces Л^n only for $n \leq 9$. Their schemes are given in Table 4.

Table 4. Quasi-Lannér schemes

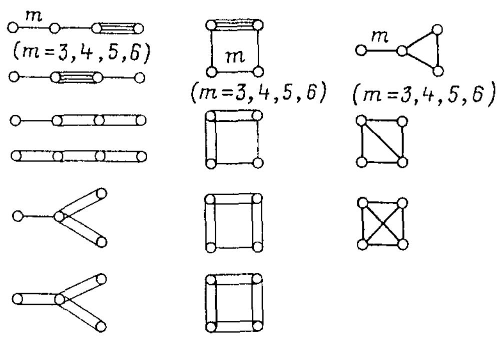
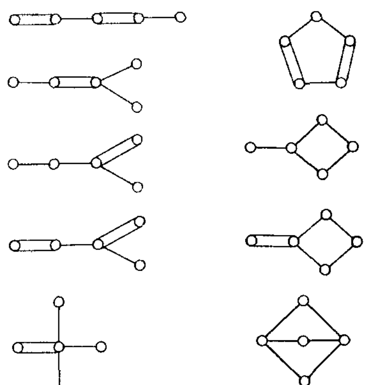
n	Schemes
2	 $(2 \le k,\ l \le \infty,\ \frac{1}{k} + \frac{1}{l} < 1)$
3	
4	

Table 4 (continued)

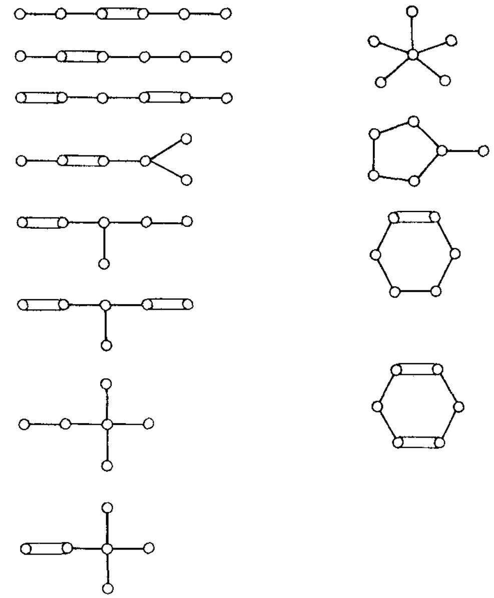
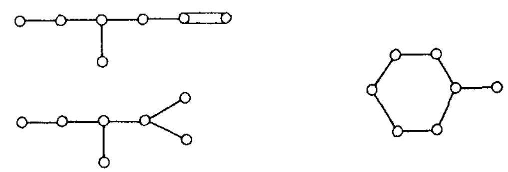
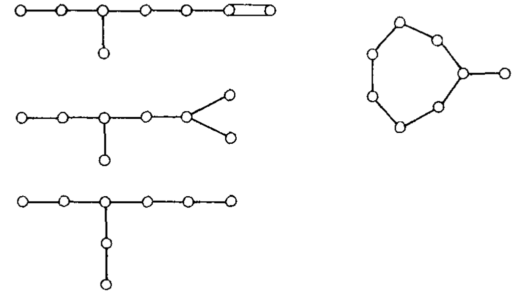
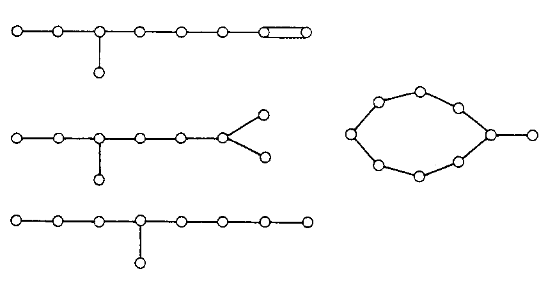
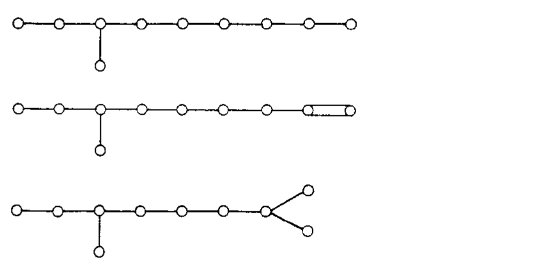
n	Schemes
5	
6	
7	

Table 4 (continued)

n	Schemes
8	
9	

2.4. Some Other Examples. A large number of different Coxeter polyhedra of finite volume (including bounded ones) in the space Л^n can be obtained by "truncating ideal vertices"[1], or "pasting together along congruent faces". We now describe these constructions and give some examples.

In the vector model, any convex polyhedron $P \subset \text{Л}^n$ is the intersection of Л^n with a convex polyhedral cone $K \subset \mathbb{R}^{n,1}$ bounded by the same number of hyperplanes as P. Let P be an indecomposable (see Part 1, Chap. 6, Sect. 2.1) non-degenerate acute-angled polyhedron. Then the cone K contains no lines and does not intersect the "past cone" C^- (see Part 1, Chap. 6, Theorem 2.1), and therefore lies strictly on one side of an n-dimensional elliptic subspace U. The hyperplane $E^n = x_0 + U$ $(x_0 \in C^+)$ intersects the "future cone" C^+ by a ball K^n which can be considered as the Klein model of the space Л^n. In this model one has

$$P = \hat{P} \cap K^n,$$

where $\hat{P} = K \cap E^n$ is a bounded convex polyhedron in the space E^n.

An *ideal vertex* of the polyhedron P is a vertex of the polyhedron $\hat{P}$ that is not in $\overline{K}$ but every edge of $\hat{P}$ issuing from it intersects K^n. In the vector

[1] In the English translation of Vinberg [1985] published in *Russian Mathematical Surveys* **40**, No. 1, 31–75 (1985) this process is referred to as "excising ideal vertices".

model, each ideal vertex of the polyhedron P corresponds to a "space-like" edge of the cone K.

Proposition 2.1 (Vinberg [1985]). *Let $P \subset \text{Л}^n$ be an indecomposable non-degenerate acute-angled polyhedron, q its ideal vertex, and e a vector lying on the edge of the cone K corresponding to it. Then the hyperplane H_e of Л^n intersects orthogonally all edges of P whose continuations pass through q, and $P' = P \cap H_e^-$ is also an indecomposable acute-angled polyhedron.*

We refer to the transition from P to P' as *truncating the ideal vertex* q. The polyhedron P' has an $(n-1)$-dimensional face lying in the hyperplane H_e and perpendicular to all $(n-1)$- dimensional faces adjacent to it. If P is a Coxeter polyhedron, so is P'.

We define an *acute-angled simplex with k ideal vertices* as an indecomposable non-degenerate acute-angled $(n+1)$-hedron P such that all edges of its continuation $\hat{P}$ intersect K^n, and exactly k of the vertices of $\hat{P}$ do not lie in $\overline{K^n}$ (and are thus ideal vertices of P). Truncating all ideal vertices of such a polyhedron yields an $(n+k+1)$-hedron of finite volume having the same vertices at infinity as P.

Coxeter simplices with ideal vertices in the space Л^n exist only for $n \leq 5$, because truncating each ideal vertex must yield an $(n-1)$-dimensional bounded Coxeter simplex in the section. They are described by hyperbolic Coxeter schemes of order $n+1$ whose subschemes of order $n-1$ are all elliptic. (See examples of such schemes below.)

The notion of an ideal vertex can be generalized if we require only that the edges of the polyhedron $\hat{P}$ issuing from it intersect $\overline{K^n}$. On the classification of Coxeter simplices with ideal vertices in this generalized sense see Maxwell [1982]. They exist only for $n \leq 10$.

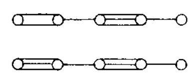
If two Coxeter polyhedra P_1 and P_2 in the space Л^n have congruent $(n-1)$-dimensional faces F_1 and F_2 respectively, then pasting them together along these faces may also result in a Coxeter polyhedron. This is, for example, the case if either of the following conditions holds:

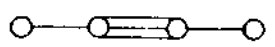
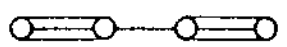
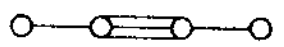
(1) the faces F_1 and F_2 are perpendicular to all $(n-1)$-dimensional faces adjacent to them;

(2) the polyhedra P_1 and P_2 are symmetric with respect to the hyperplane of their common face $F(= F_1 = F_2)$, and this face forms the angles $\frac{\pi}{2m}$, where $m \in \mathbb{Z}$ or $m = \infty$ with all $(n-1)$-dimensional faces adjacent to it.

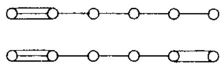
Using the processes of truncating and pasting together Makarov [1968] constructed infinite series of bounded Coxeter polyhedra in Л^4 and Л^5. These polyhedra can be described as follows.

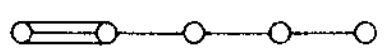
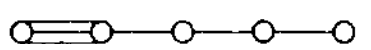
The Coxeter schemes



give simplices in Л^4 with two ideal vertices. Truncating these vertices yields bounded Coxeter polyhedra P_1 and P_2 each having two three-dimensional simplicial faces perpendicular to all adjacent faces. The schemes of these faces for the polyhedron P_1 are  and  and for the polyhedron P_2  and  . On forming garlands out of any collection of copies of P_1 and P_2 pasted together along congruent simplicial faces, one can obtain infinitely many different bounded Coxeter polyhedra in $Л^4$.

Similarly, the Coxeter scheme



give simplices in $Л^5$ with a single ideal vertex. Truncating these vertices yields bounded simplicial prisms P_1 and P_2. Their bases perpendicular to the lateral faces have the scheme  . The second base of the prism P_2 forms an angle of $\frac{\pi}{4}$ with one of the lateral faces and is perpendicular to the remaining ones. By pasting a mirror image of the prism P_2 to this face, we obtain a Coxeter polyhedron P having two simplicial faces with the scheme  , perpendicular to all adjacent faces. By forming garland of any collection of copies of P and adding copies of P_1 or P_2 to its ends, or adding nothing, we can obtain infinitely many Coxeter polyhedra in $Л^5$.

For further examples see Vinberg [1985] and Chap. 2, Sect. 2 of this article. Some classification results for Coxeter polyhedra of finite volume in $Л^n$ under various restrictions on their combinatorial structure or dihedral angles were obtained by Kaplinskaya [1974], Im Hof [1990], and Prokhorov [1988].

2.5. Restrictions on Dimension. Despite the abundance of examples in lesser dimensions, there are no Coxeter polyhedra of finite volume whatsoever in Lobachevskij spaces of sufficiently large dimension (and thus no crystallographic reflection groups).

Theorem 2.2 (Vinberg [1984, 1985]). *There are no bounded Coxeter polyhedra in the space* $Л^n$ *for* $n \geq 30$.

Theorem 2.3 (Prokhorov [1986], Khovanskij [1986]). *There are no Coxeter polyhedra of finite volume in the space* $Л^n$ *for* $n \geq 996$.

The bounds given by these theorems are apparently far from exact. In any case, examples of bounded Coxeter polyhedra in the space $Л^n$ are known only for $n \leq 8$, and examples of Coxeter polyhedra of finite volume only for $n \leq 21$ (see Chap. 6, Sect. 2).

§ 3. Regular Polyhedra and Honeycombs

3.1. The Symmetry Group and Schläfli Symbol of a Regular Polyhedron. In what follows, we understand by a convex polyhedron in an n-dimensional space of constant curvature a convex polyhedron of finite volume.

In the case of Lobachevskij space we add to a polyhedron all its vertices at infinity (if there are any).

Regular polyhedra are the most symmetric of all polyhedra. A convenient way to couch this intuitively clear definition into precise terms is by using the notion of a flag.

Definition 3.1. A *flag* of an n-dimensional convex polyhedron P is a sequence

$$\mathcal{F} = \{F_0, F_1, \dots, F_{n-1}\}, \tag{1}$$

where F_i is an i-dimensional (closed) face of P, and $F_{i-1} \subset F_i$ for $i = 1, 2, \dots, n-1$.

Two flags differing by a single element are said to be *adjacent*.

For any two flags $\mathcal{F}'$, $\mathcal{F}''$ of P there is at most one motion $g \in \operatorname{Sym} P$ taking $\mathcal{F}'$ into $\mathcal{F}''$. If the flags $\mathcal{F}'$ and $\mathcal{F}''$ are adjacent, then the motion g (provided it exists) is a reflection.

Definition 3.2. A convex polyhedron P is said to be *regular* if for any two of its flags $\mathcal{F}'$, $\mathcal{F}''$ there is a (unique) motion $g \in \operatorname{Sym} P$ taking $\mathcal{F}'$ into $\mathcal{F}''$.

Thus the order of the symmetry group of a regular polyhedron equals the number of its flags. The definition also implies that any face of a regular polyhedron is again a regular polyhedron. The symmetry group of a regular polyhedron has a unique fixed point, called the *centre* of the polyhedron.

Let P be a regular polyhedron. For any $i = 0, 1, 2, \dots, n-1$ the flag (1) has exactly one adjacent flag of the form

$$\mathcal{F}_i' = \{F_0, F_1, \dots, F_{i-1}, F_i', F_{i+1}, \dots, F_{n-1}\}. \tag{2}$$

The motion $s_i \in \operatorname{Sym} P$ taking $\mathcal{F}$ into $\mathcal{F}_i'$ is the reflection in the hyperplane H_i passing through the centre of P and the centres of the faces $F_0, F_1, \dots F_{i-1}, F_{i+1}, \dots, F_{n-1}$. Denote by H_i^- the half-space bounded by H_i containing the centre of the face F_i.

The simplicial cone

$$K = \bigcap_{i=0}^{n-1} H_i^- \tag{3}$$

is said to be the *fundamental cone* of the polyhedron P (associated with the flag $\mathcal{F}$). The vertex of the cone K is at the centre c_n of the polyhedron P, and its edges pass through the centres $c_0, c_1, \dots, c_{n-1}$ of the faces $F_0, F_1, \dots, F_{n-1}$ forming the flag $\mathcal{F}$ (see Fig. 15).

One can easily see that the cone K is a fundamental domain for the group $\operatorname{Sym} P$. Therefore the group $\operatorname{Sym} P$ is generated by reflections $s_0, s_1, \dots, s_{n-1}$ in its walls.

Now let p_i ($i = 1, \dots, n-1$) be the number of i-dimensional (or $(i-1)$-dimensional, which makes no difference) faces of the polyhedron P containing the face F_{i-2} and contained in the face F_{i+1} (we assume that $F_{-1} = \emptyset$ and $F_n = P$). The ordered set $\{p_1, \dots p_{n-1}\}$ is called the *Schläfli symbol of the polyhedron* P.

The number of images of the cone K containing its $(n-2)$-dimensional face $K \cap H_i \cap H_j$ equals the number of flags of P differing from $\mathcal{F}$ just in its i-th and j-th elements. This number equals 4 for $|i-j| > 1$ and $2p_i$ for $|i-j| = 1$. Thus the Coxeter scheme of the cone K is of the form

$$\circ\overset{p_1}{—}\circ\overset{p_2}{—}\circ— \cdots —\circ\overset{p_{n-1}}{—}\circ\,. \tag{4}$$

In particular, it is completely determined by the Schläfli symbol.

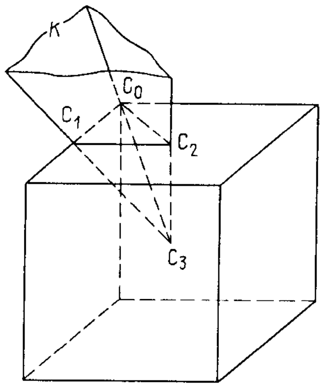


Fig. 15

For each k-dimensional face F of P its Schläfli symbol is the initial segment $\{p_1, p_2, \dots p_{k-1}\}$ of the Schläfli symbol of the polyhedron itself. On the other hand, in a neighbourhood of any interior point x of the face F the structure of the polyhedron P is determined by a tail segment of the Schläfli symbol of P. Namely, let Y be the $(n-k)$-dimensional plane orthogonally intersecting F at x, and S a sufficiently small sphere of the space Y with centre x. Then $P \cap S$ is a regular spherical polyhedron with the Schläfli symbol $\{p_{k+2}, \dots, p_{n-1}\}$. The same is true if $F = \{x\}$ is a vertex at infinity, the only difference being that the sphere is replaced by a horosphere in the space $Л^n$.

3.2. Classification of Regular Polyhedra. The simplex $T = K \cap P$ with the vertices at the points $c_0, c_1, \dots, c_n$ (see Fig. 15) is said to be the *fundamental simplex* of the regular polyhedron P. It is a fundamental domain for the action of the group $\operatorname{Sym} P$ on P. Its $(n-1)$-dimensional face opposite to the vertex c_n is orthogonal to the edge $c_{n-1}c_n$. Therefore the simplex T together with the polyhedron P can be uniquely reconstructed from the

oriented[2] scheme (4) and the length of the edge $c_{n-1}c_n$ (the radius of the sphere inscribed in P). This implies that a regular polyhedron in the Euclidean space is defined by its Schläfli symbol up to similarity. A regular polyhedron on the sphere or in Lobachevskij space is determined (up to a motion) by its Schläfli symbol and one continuous parameter for which one can take, for example, the radius of the inscribed sphere, or the dihedral angle.

Any ordered set $\{p_1, p_2, \dots, p_{n-1}\}$ for which the scheme (4) is elliptic is the Schläfli symbol of some regular polyhedron P. This polyhedron is constructed by taking a Coxeter polyhedral cone K with the scheme (4), and truncating it by a hyperplane orthogonal to the edge corresponding to the extreme right vertex of the scheme and (in the case of Lobachevskij space) sufficiently close to the apex of the cone. This yields a simplex T. The desired polyhedron is

$$P = \bigcup_{\gamma \in \Gamma} \gamma T,$$

where Γ is the group generated by reflections in the walls of the cone K.

Thus the classification of regular polyhedra is reduced to the classification of oriented linear (i.e. connected and having neither cycles nor branching vertices) elliptic Coxeter schemes, and the latter can be extracted from Table 1.

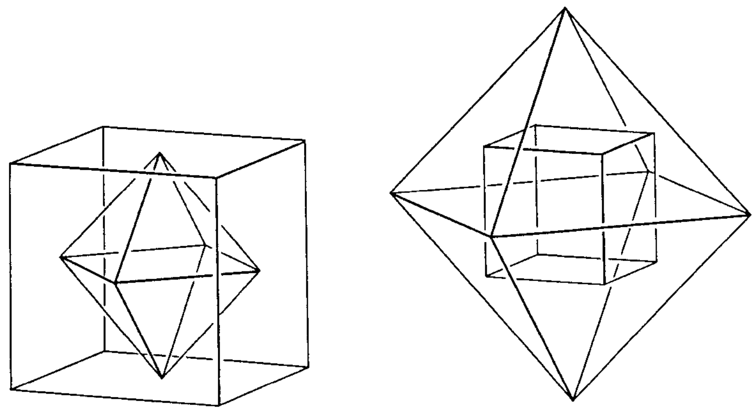


Fig. 16

Reversing the orientation of the scheme yields the polyhedron of so-called dual type. (For symmetric schemes the dual types of regular polyhedra coincide.) The regular polyhedron P^* dual to a polyhedron P can be obtained as

[2] Here "oriented" means an orientation of the entire scheme (and not that of separate edges).

the convex hull of the set of centres of all $(n-1)$-dimensional faces of P (see Fig. 16). Each i-dimensional face of the polyhedron P^* is the convex hull of the set of all $(n-1)$-dimensional faces of P containing an $(n-i-1)$-dimensional face. The symmetry groups of P and P^* coincide.

Table 5 lists the Schläfli symbols of regular polyhedra. For each symbol the dihedral angle α_{Euc} of the corresponding regular Euclidean polyhedron is given, as well as the minimal possible angle $\alpha_{\min}$ of the corresponding polyhedron in Lobachevskij space. (The angle $\alpha_{\min}$ is realized in the polyhedron with vertices at infinity.) The dihedral angle α in a regular spherical polyhedron of a given type can take any value satisfying the inequality $\alpha_{\mathrm{Euc}} < \alpha < \pi$, and the dihedral angle α in a regular polyhedron in Lobachevskij space can take any value such that $\alpha_{\min} \le \alpha < \alpha_{\mathrm{Euc}}$.

Table 5. Regular polyhedra

Dimension	Schläfli symbol	Name	Number of faces of codimension 1	α_{Euc}	$\alpha_{\min}$
n	$\{3,3,\dots,3,3\}$	regular simplex	$n+1$	$\arccos\frac{1}{n}$	$\arccos\frac{1}{n-1}$
n	$\{4,3,\dots,3,3\}$	cube	$2n$	$\frac{\pi}{2}$	$\arccos\frac{1}{n-1}$
n	$\{3,3,\dots,3,4\}$	cocube	2^n	$2\arccos\frac{1}{\sqrt{n}}$	$2\arccos\frac{1}{\sqrt{n-1}}$
4	$\{3,4,3\}$	regular 24-hedron	24	$\frac{2\pi}{3}$	$\frac{\pi}{2}$
4	$\{5,3,3\}$	regular 120-hedron	120	$\frac{4\pi}{5}$	$\arccos\frac{1}{3}$
4	$\{3,3,5\}$	regular 600-hedron	600	$\pi-\arccos\frac{3\sqrt{5}+1}{8}$	$\pi-\arccos\frac{\sqrt{5}}{3}$
3	$\{5,3\}$	dodecahedron	12	$\pi-\arccos\frac{1}{\sqrt{5}}$	$\frac{\pi}{3}$
3	$\{3,5\}$	icosahedron	20	$\pi-\arccos\frac{\sqrt{5}}{3}$	$\frac{3\pi}{5}$
2	$\{m\}$	regular m-gon	m	$\pi\left(1-\frac{2}{m}\right)$	0

Remark. In the three-dimensional case a regular simplex is called a tetrahedron, and a cocube an octahedron.

The classification of regular polyhedra was obtained by Schläfli in 1850. The proofs of the above assertions and further information on regular polyhedra can be found in the book by Coxeter [1973] (see also Berger [1978]).

3.3. Honeycombs. A decomposition of a space into congruent regular polyhedra is called a (regular) *honeycomb*, and the constituent polyhedra are

called its *cells*. Evidently any honeycomb is a homogeneous decomposition in the sense of Chap. 2, Sect. 1.7.

A *honeycomb* with cells congruent to a given regular polyhedron P exists if and only if the dihedral angle of P is a submultiple of 2π. (For the Lobachevskij plane zero angle is also possible.) One can easily find all such honeycombs using Table 5. All honeycombs with bounded cells were first found by Schlegel in 1883, those with unbounded cells by Coxeter in 1954 (Coxeter [1955]).

Another approach to describing honeycombs involves the analysis of their symmetry groups. If $\mathcal{P}$ is such a honeycomb, then any motion taking one cell into another takes the whole honeycomb into itself, i.e. belongs to the group $\operatorname{Sym}\mathcal{P}$. Therefore the fundamental simplex T of any cell $P \in \mathcal{P}$ is a fundamental domain of the group $\operatorname{Sym}\mathcal{P}$ generated by reflections in its walls.

Let $\{p_1,\dots,p_{n-1}\}$ be the Schläfli symbol of the polyhedron P, and p_n the number of cells having a common $(n-2)$-dimensional face. Then the dihedral angle of P equals $2\pi/p_n$, and the simplex T is the Coxeter polyhedron with the scheme

$$\circ\overset{p_1}{-}\circ\overset{p_2}{-}\circ-\cdots-\circ\overset{p_{n-1}}{-}\circ\overset{p_n}{-}\circ\,. \qquad (5)$$

Of all the vertices of the simplex T only one, namely the vertex of the polyhedron P, can be at infinity. Thus the scheme (5) satisfies the following condition:

(F) The subscheme obtained by deleting any vertex with the exception of the extreme left one is elliptic and the subscheme obtained by deleting the extreme left vertex is either elliptic, or parabolic.

The ordered set $\{p_1,\dots,p_n\}$ is said to be *the Schläfli symbol of the honeycomb* $\mathcal{P}$. It defines the honeycomb uniquely up to similarity in the case of Euclidean space, and up to a motion in the other two cases. Clearly, each ordered set $\{p_1,\dots,p_n\}$ for which the scheme (5) satisfies condition (F) is the Schläfli symbol of some honeycomb.

Thus the classification of honeycombs is reduced to the classification of oriented Coxeter schemes satisfying condition (F), and the latter can be extracted from Tables 1–4.

Reversing the orientation of the scheme (5) (if possible) yields the so-called dual honeycomb. (For symmetric schemes dual honeycombs are congruent.) For the vertices of the honeycomb $\mathcal{P}^*$ dual to a honeycomb $\mathcal{P}$ one can take the centres of the cells of $\mathcal{P}$. Then each i-dimensional face of $\mathcal{P}^*$ is the convex hull of the set of centres of all cells of $\mathcal{P}$ containing an $(n-i)$-dimensional face of $\mathcal{P}$. The symmetry groups of $\mathcal{P}$ and $\mathcal{P}^*$ coincide.

For example, the scheme $\circ\!\equiv\!\circ\!-\!\circ$ defines a honeycomb in the Euclidean plane with hexagonal cells (the classic honeycomb), and the dual honeycomb with triangular cells whose vertices are positioned at the centres of these hexagons. This pair of honeycombs is depicted in Fig. 17; the shaded area is the fundamental simplex of both honeycombs.

Honeycombs on the sphere S^n are defined by linear elliptic Coxeter schemes of rank n (see Table 1), and are in natural one-to-one correspondence with regular polyhedra in E^{n+1}. Geometrically this correspondence can be described as follows: the convex hull in E^{n+1} of the set of vertices of such a honeycomb is a regular polyhedron inscribed in S^n, and conversely, if $P \subset E^{n+1}$ is a regular polyhedron inscribed in S^n, then the central projections of its n-dimensional faces onto S^n form a honeycomb on S^n.

Honeycombs in the Euclidean spaces are defined by linear parabolic Coxeter schemes, i.e. the schemes $\tilde{C}_n$, $\tilde{F}_4$, and $\tilde{G}_2$ (see Table 2). The scheme $\tilde{C}_n$ defines the trivial cubic honeycomb in E^n, the scheme $\tilde{F}_4$ the pair of dual honeycombs in E^4 whose cells are cocubes and regular 24-hedra, respectively, and the scheme $\tilde{G}_2$ the pair of dual honeycombs in the Euclidean plane shown in Fig. 17.

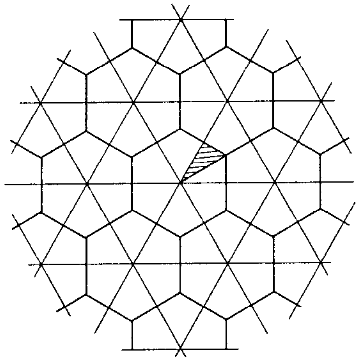


Fig. 17

Finally, honeycombs in the Lobachevskij space $Л^n$ are defined by the linear Lannér and quasi-Lannér schemes (see Tables 3 and 4). The latter, however, qualify only if the deletion of an extreme vertex yields an elliptic scheme. Such honeycombs exist only for $n \leq 5$. Their Schläfli symbols are listed in Table 6 in which the symbols of honeycombs with unbounded cells are marked with an asterisk.

Let us recall that the honeycomb $\mathcal{P}$ with the Schläfli symbol $\{p_1, \dots, p_n\}$ is, by definition, formed by regular polyhedra with the dihedral angle $2\pi/p_n$ and the Schläfli symbol $\{p_1, \dots, p_{n-1}\}$. In a neighbourhood of an interior point x of a k-dimensional face F the structure of such a honeycomb is determined by a tail segment of its Schläfli symbol. Indeed, let Y be the $(n-k)$-dimensional plane orthogonally intersecting the face F at the point x, and S a sufficiently small sphere of the space Y with centre x. Then the intersections of the cells of $\mathcal{P}$ containing x with S are equal spherical polyhedra constituting a honeycomb on the sphere S with the Schläfli symbol $\{p_{k+2}, \dots, p_n\}$. The same holds if $F = \{x\}$ is a vertex at infinity, the only difference being that the sphere is replaced by a horosphere (of the space $Л^n$).

Table 6. Honeycombs in the Lobachevskij spaces

Dimension	Schläfli symbol	Dimension	Schläfli symbol
2	$\{p,q\}$ $(3 \leq p,q < \infty,$ $\frac{1}{p}+\frac{1}{q}<\frac{1}{2})$	4	$\{5,3,3,3\}$
			$\{3,3,3,5\}$
			$\{5,3,3,4\}$
			$\{4,3,3,5\}$
	$\{p,\infty\}^*$ $(3 \leq p < \infty)$		$\{5,3,3,5\}$
			$\{3,4,3,4\}^*$
3	$\{3,5,3\}$	5	$\{3,3,3,4,3\}^*$
	$\{5,3,4\}$		
	$\{4,3,5\}$		
	$\{5,3,5\}$		
	$\{3,3,6\}^*$		
	$\{4,3,6\}^*$		
	$\{5,3,6\}^*$		
	$\{3,4,4\}^*$		

Chapter 6
Arithmetic Groups

The general method of constructing arithmetic discrete subgroups of semi-simple Lie groups (see Margulis [1991] or VGS) can be used, in particular, to obtain crystallographic groups of motions in Lobachevskij space. In the first section we give an explicit description of this class of crystallographic groups independent of the general theory.

§ 1. Description of Arithmetic Discrete Groups of Motions of Lobachevskij Space

1.1. Arithmetic Groups of the Simplest Type. Let $K \subset \mathbb{R}$ be a totally real algebraic number field, and A the ring of its integers. A non-degenerate quadratic form

$$f(x) = \sum_{i,j=0}^{n} a_{ij} x_i x_j \quad (a_{ij} = a_{ji} \in K)$$

is said to be *admissible* if its negative inertia index equals 1, and for any non-identity embedding $\sigma: K \to \mathbb{R}$ the quadratic form

$$f^\sigma(x) = \sum_{i,j=0}^{n} a_{ij}^\sigma x_i x_j$$

is positive definite. Then the group $O'(f, A)$ of linear transformations with coefficients in A preserving the form f and mapping each connected component of the cone $C = \{x \in \mathbb{R}^{n+1} : f(x) < 0\}$ onto itself is a discrete group of motions of the space $Л^n$ (in the vector model defined by the form f). According to the general theory of arithmetic discrete groups (see VGS), if $K = \mathbb{Q}$ and the form represents zero in $\mathbb{Q}$, then the quotient space $Л^n/O'(f, A)$ is noncompact but has finite volume, in all other cases it is compact. For $K = \mathbb{Q}$ the proof of these statements was first given in a paper of Venkov [1937], which can still be recommended to the reader.

The preceding construction can be generalized as follows. Instead of considering linear transformations with coefficients in A, i.e. preserving the standard lattice $A^{n+1} \subset K^{n+1}$, one can consider linear transformations preserving an arbitrary lattice $L \subset K^{n+1}$. (A lattice in a finite-dimensional vector space over the field K is a finitely generated A-submodule whose linear span over K coincides with the entire space.) The resulting group is commensurable with the group $O'(f, A)$. Considering the lattice L as a quadratic A-module, with the scalar product[1] defined by the form f, we denote this group by $O'(L)$.

However, if A is a principal ideal ring, this generalization produces no new groups. Indeed, in this case any lattice $L \subset K^{n+1}$ is a free A-module, and $O'(L) = O'(g, A)$, where g is a quadratic form obtained from f by replacing the standard coordinate system in the space K^{n+1} by the coordinate system related to the basis of the lattice L.

The discrete groups of motions of the space $Л^n$ obtained in the above manner are said to be *arithmetic discrete groups of motions of the simplest type*. The field K is then said to be the *field of definition* of the corresponding group.

Two arithmetic discrete groups defined by admissible quadratic forms f_1 and f_2 are commensurable (under an appropriate isomorphism of the corresponding vector models of the space $Л^n$) if and only if they have the same field of definition K, and the form f_1 is equivalent (over K) to a form λf_2 ($\lambda \in K$).

1.2. Quaternion Algebras. In the following constructions we shall be using (generalized) quaternion algebras. Here is some essential information on them.

A *quaternion algebra* D over a field K of characteristic $\neq 2$ is a simple central algebra of dimension 4. Any quaternion algebra is isomorphic to the algebra $D(a, b)$ with the basis $\{1, i, j, k\}$ and defining relations $i^2 = a$, $j^2 = b$, $ij = -ji = k$, where $a, b \in K^*$. In particular, the ordinary (Hamilton) quaternion algebra $\mathbb{H}$ over the field $\mathbb{R}$ is isomorphic to $D(-1, -1)$. The matrix algebra $M_2(K)$ is isomorphic to $D(1, 1)$.

[1] We can assume, without loss of generality, that $(x, y) \in A$ for all $x, y \in L$.

Any quaternion algebra over an algebraically closed field is isomorphic to a matrix algebra. A quaternion algebra that is not isomorphic to a matrix algebra is a division ring. By the Frobenius theorem, any quaternion algebra over $\mathbb{R}$ is isomorphic either to $\mathbb{H}$ or to $M_2(\mathbb{R})$.

In any quaternion algebra D there is an involutory antiautomorphism (the *standard involution*) $q \mapsto \bar{q}$ such that the set of its fixed elements coincides with the ground field K. In matrix algebra it is of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. In the algebra $D(a, b)$ the standard involution multiplies i, j, k by -1.

For any $q \in D$ the elements $\operatorname{Tr}(q) = q + \bar{q}$ and $N(q) = q\bar{q}$ belong to the field K. They are said to be the *trace* and the *norm* of the quaternion q. In matrix algebra they are the trace and the determinant of the matrix, respectively. The set of quaternions having the norm 1 forms a multiplicative group denoted by $SL_1(D)$. In particular,

$$SL_1(M_2(K)) = SL_2(K).$$

Quaternions with the trace 0 are said to be *pure imaginary*. They form a three-dimensional vector space denoted by D_0. One usually defines a scalar product in it by the formula

$$(x, x) = -N(x) \, (= x^2). \tag{1}$$

For $a \notin (K^*)^2$ the mapping

$$t + xi + yj + zk \mapsto \begin{pmatrix} t + x\sqrt{a} & y + z\sqrt{a} \\ b(y - z\sqrt{a}) & t - x\sqrt{a} \end{pmatrix} \tag{2}$$

is an isomorphism of the algebra $D(a, b)$ onto the K-subalgebra of the algebra $M_2(K(\sqrt{a}))$.

If L is an extension of the field K, and D a quaternion algebra over K, then $D \otimes L$ is a quaternion algebra over L.

For any quaternion algebra D over a field K and any homomorphism $\sigma : K \to F$, $x \mapsto x^\sigma$, where F is another field, denote by D^σ the quaternion algebra over the field $K^\sigma \subset F$ whose structure constants are obtained from those of D by applying the homomorphism σ. In particular, $D(a, b)^\sigma = D(a^\sigma, b^\sigma)$.

Let K be a field of algebraic numbers, A the ring of its integers, and D a quaternion algebra over K. An *order* of the algebra D is any subring of it containing 1 that is a lattice in the vector space D.

Example 1. The subring $M_2(A)$ is an order of the algebra $M_2(K)$.

Example 2. If $a, b \in A$, then the quaternions with integer coefficients form an order of the algebra $D(a, b)$.

The *group of units* of an order $Q \subset D$ is the group $SL_1(Q) = SL_1(D) \cap Q$. The groups of units of any two orders of the algebra D are commensurable.

1.3. Arithmetic Fuchsian Groups. Let $K \subset \mathbb{R}$ be a totally real field of algebraic numbers, and D a quaternion algebra over K such that $D \otimes$

$\mathbb{R} \simeq M_2(\mathbb{R})$, and suppose that $D^\sigma \otimes \mathbb{R} \simeq \mathbb{H}$ for all non-identity embeddings $\sigma : K \to \mathbb{R}$. Then for any order $\mathcal{Q}$ in the algebra D the group $PSL_1(\mathcal{Q}) = SL_1(\mathcal{Q})/\{+1,-1\}$ is a discrete subgroup of the group $PSL_2(\mathbb{R})$, and hence a Fuchsian group. If $K = \mathbb{Q}$ and $D \simeq M_2(\mathbb{Q})$ the quotient space $Л^2/PSL_1(\mathcal{Q})$ is non-compact but has a finite volume, in all other cases it is compact. Any arithmetic Fuchsian group is commensurable with a group thus obtained.

However, this construction yields no groups substantially different from the arithmetic groups of the simplest type, since under the above assumptions the group $PSL_1(\mathcal{Q})$ can be embedded in the group $O'(L)$, where $L = \mathcal{Q} \cap D_0$, as a subgroup of finite index.

Example. Let $p \equiv -1 \pmod 4$ be a prime number. Then the algebra $D(-1,p)$ over the field $\mathbb{Q}$ is a division ring. Let $\mathcal{Q}$ be its order consisting of quaternions with integer coefficients. The isomorphism (2) represents the group $SL_1(\mathcal{Q})$ by matrices of the form

$$\begin{pmatrix} t+x\sqrt{p} & y+z\sqrt{p} \\ -y+z\sqrt{p} & t-x\sqrt{p} \end{pmatrix}, \quad t,x,y \in \mathbb{Z},$$

with determinant 1, so the group $SL_1(\mathcal{Q})$ can be embedded in the group $SL_2(\mathbb{R})$. The group $PSL_1(\mathcal{Q}) \subset PSL_2(\mathbb{R})$ is a uniform Fuchsian group. According to the above argument, it can be embedded in the group $O'(-x^2 + py^2 + pz^2, \mathbb{Z})$ as a subgroup of finite index. Fuchsian groups obtained in this way for different p are not commensurable.

1.4. Arithmetic Groups of Motions of the Space $Л^3$. Now let $K \subset \mathbb{C}$ be an imaginary field of algebraic numbers, whose embeddings in $\mathbb{C}$ with the exception of the identity embedding and the complex conjugation are real. Let D be a quaternion algebra over K such that $D^\sigma \otimes \mathbb{R} \simeq \mathbb{H}$ for all embeddings $\sigma: K \to \mathbb{R}$. Then for any order $\mathcal{Q}$ in the algebra D the group $PSL_1(\mathcal{Q})$ is a discrete subgroup of the group $PSL_2(\mathbb{C})$ and hence a discrete group of motions of the space $Л^3$. If K is an imaginary quadratic field and $D \simeq M_2(K)$, then the quotient space $Л^3/PSL_1(\mathcal{Q})$ is non-compact but has finite volume, in all other cases it is compact. Any arithmetic discrete group of motions of the space $Л^3$ is commensurable with a group thus obtained.

The group $PSL_1(\mathcal{Q})$ is commensurable with an arithmetic group of the simplest type if and only if K is an imaginary quadratic extension of a real algebraic number field.

Example. Let $K = \mathbb{Q}(\sqrt{-m})$, where m is a square-free positive integer, $D = M_2(K)$, $\mathcal{Q} = M_2(A)$, where A is the ring of integers of the field K. The group

$$Bi(m) = PSL_1(\mathcal{Q}) = PSL_2(A) \subset PSL_2(\mathbb{C})$$

is a crystallographic (but not uniform) group of motions of the space $Л^3$. These groups are called the *Bianchi groups* after the mathematician who at the end of the last century found fundamental domains for some of them (Bianchi [1892]). (See also Swan [1971], Vinberg [1987], Shaikheev [1987], Shvartsman [1987].) The group $Bi(m)$ is commensurable with the group $O'(x_1x_2 + x_3^2 + mx_4^2, \mathbb{Z})$.

1.5. Arithmetic Groups of Motions of the Space $Л^n$ for $n \geq 4$. For n even, any arithmetic discrete group of motions of the space $Л^n$ is commensurable with an arithmetic group of the simplest type. For $n = 2m-1$ there is another construction of arithmetic discrete groups related to a quaternion algebra.

Let D be a quaternion algebra over a field K and let

$$F(x,y) = \sum_{i,j} \overline{x}_i a_{ij} y_j \quad (a_{ij} \in D,\, a_{ji} = -\overline{a}_{ij})$$

be a non-degenerate skew-Hermitian form on the right D-module D^m. Denote by $U(F,D)$ the group of automorphisms of this D-module preserving the form F. If K is a field of algebraic numbers, and $\mathcal{Q}$ is an order of the algebra D, then denote by $U(F,\mathcal{Q})$ the subgroup of the group $U(F,D)$ consisting of automorphisms preserving the "lattice" $\mathcal{Q}^m$.

If $D \simeq M_2(K)$, then the group $U(F,D)$ is isomorphic to the orthogonal group of the $2m$-dimensional metric vector space over K. Indeed, consider D^m as a ($4m$-dimensional) vector space over K. Assuming that $D = D(1,1)$, let

$$D^m_\pm = \{x \in D^m : xi = \pm x\}.$$

Then $D^m = D^m_+ \oplus D^m_-$ and $D^m_- = D^m_+ j$, so $\dim D^m_+ = \dim D^m_- = 2m$. One can easily see that the restriction of F to D^m_+ is of the form $F(x,y) = f(x,y)(i-1)j$ $(x,y \in D^m_+)$, where f is a non-degenerate symmetric bilinear form on D^m_+ (which takes values in K). The mapping associating with each transformation $A \in U(F,D)$ its restriction to D^m_+ is an isomorphism of the group $U(F,D)$ onto the group $O(f,K)$. The form f is defined by F uniquely up to similarity. For $K = \mathbb{R}$ its signature, defined up to a permutation of indices, is called the signature of the form F.

Now let $K \subset \mathbb{R}$ be a totally real algebraic number field, and D a quaternion algebra over K such that $D \not\simeq M_2(K)$ but $D^\sigma \otimes \mathbb{R} \simeq M_2(\mathbb{R})$ for all embeddings $\sigma: K \to \mathbb{R}$. Next, let F be a non-degenerate skew-Hermitian form on D^m. Suppose that F has the signature $(2m-1,1)$, and that for any non-identity embedding $\sigma: K \to \mathbb{R}$ the form F^σ (on $(D^\sigma)^m$) has the signature $(2m,0)$.

According to the preceding argument, one can identify the group $U(F, D \otimes \mathbb{R})$ with the group $O(f,\mathbb{R})$, where f is a symmetric bilinear form of signature $(2m-1,1)$, and thus identify the group $U(F,D)$ with a subgroup of $O(f,\mathbb{R})$. Then for any order $\mathcal{Q}$ of algebra D the group $U'(F,\mathcal{Q}) = U(F,\mathcal{Q}) \cap O'(f,\mathbb{R})$ is a discrete group of motions of the space $Л^{2m-1}$ (in the vector model defined by the form f). If $K = \mathbb{Q}$ and the form F represents zero in D, then the quotient space $Л^{2m-1}/U'(F,\mathcal{Q})$ is non-compact but has finite volume, in all other cases it is compact.

For any odd $n \neq 3, 7$, each arithmetic discrete group of motions of the space $Л^n$ is commensurable with either an arithmetic group of the simplest type or a group of the above type. For $n = 7$ there is a special type of arithmetic discrete groups of motions connected with the Cayley algebra.

§ 2. Reflective Arithmetic Groups

2.1. Application of Reflection Groups to the Study of Arithmetic Groups. A crystallographic group Γ of motions of the space $Л^n$ is said to be *reflective* if its reflection subgroup (see Chap. 5, Sect. 1.2) is of finite index.

Among arithmetic discrete groups of motions of the space $Л^n$, only those of the simplest type and their subgroups of finite index can contain reflections, or, what is more, be reflective (Vinberg [1967], Shvartsman [1968]). There is an effective algorithm for finding a fundamental polyhedron of the reflection subgroup for these groups (Vinberg [1972, 1975]). This enables one, if it is desirable, to find a fundamental polyhedron, generators, and defining relations of the arithmetic group itself.

We now list some results obtained in this way. First, let

$$f_n(x) = -x_0^2 + x_1^2 + \ldots + x_n^2.$$

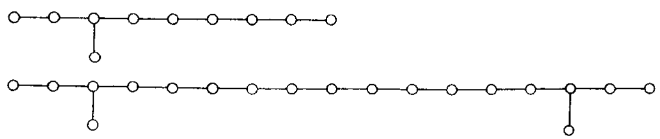
It is proved by Vinberg [1972] and Vinberg and Kaplinskaya [1978] that the group $O'(f_n, \mathbb{Z})$ is reflective for $n \leq 19$. The schemes Σ_n of the fundamental polyhedra P_n of the reflection subgroups $O_r(f_n, \mathbb{Z})$ of the groups $O'(f_n, \mathbb{Z})$ for $n \leq 17$ are given in the following table. The schemes Σ_{18} and Σ_{19} are described in Vinberg and Kaplinskaya [1978] and include 37 and 50 vertices, respectively. Their symmetry is rather high: the symmetry group of the scheme Σ_{18} is isomorphic to the group S_4, and that of Σ_{19} to S_5.

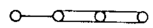
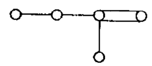
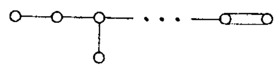
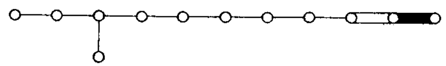
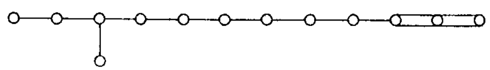
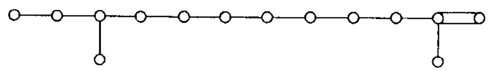
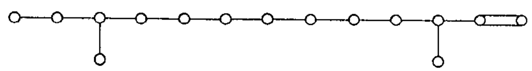
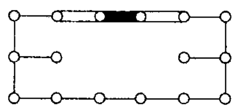
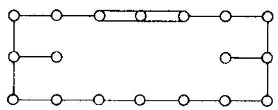
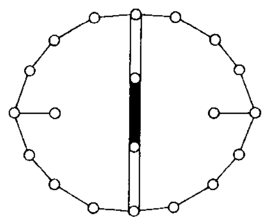
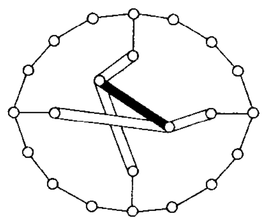
In all these cases one has

$$O'(f_n, \mathbb{Z}) = O_r(f_n, \mathbb{Z}) \rightthreetimes \operatorname{Sym} P_n$$

and the group $\operatorname{Sym} P_n$ is naturally isomorphic to the symmetry group of the scheme Σ_n.

The form f_n is known to be the unique (up to integral equivalence) odd unimodular integral quadratic form of signature $(n, 1)$. For $n \equiv 1 \pmod 8$ there is, in addition, an even unimodular integral quadratic form g_n of signature $(n, 1)$ which is also unique up to integral equivalence. For $n = 9, 17$ the group $O'(g_n, \mathbb{Z})$ is reflective. The schemes of the fundamental polyhedra of the groups $O_r(g_9, \mathbb{Z})$, $O_r(g_{17}, \mathbb{Z})$ are of the form



n	Σ_n
2	
3	
4	
5–9	
10	
11	
12	
13	
14	
15	
16	
17	

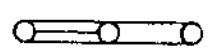
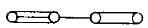
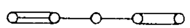
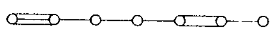
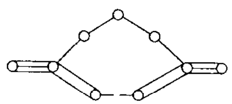
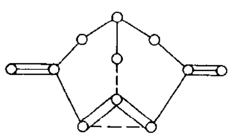
The reflection subgroup $O_r(g_{25}, \mathbb{Z})$ of the group $O'(g_{25}, \mathbb{Z})$ was studied by Conway [1983]. It is of infinite index but is, in a sense, sufficiently large. Its fundamental polyhedron P is circumscribed about a horosphere S so that the tangency points form a lattice in S (as in the 24-dimensional Euclidean space) isomorphic to the famous *Leech lattice* divided by $\sqrt{2}$, and all the points of the polyhedron P at infinity other than the centre of the horosphere S are its vertices at infinity. The group $O'(g_{25}, \mathbb{Z})$ is the semidirect product of the group $O'(g_{25}, \mathbb{Z})$ and the symmetry group of the polyhedron P naturally isomorphic to the group of motions of E^{24} preserving the Leech lattice.

This description of the group $O'(g_{25}, \mathbb{Z})$ was used in the investigation of groups of automorphisms (including unimodular ones) of integral quadratic forms of signature $(n, 1)$ for $n < 25$ (Borcherds [1987]). A quadratic form g with signature (21,1) and discriminant 4 was thus found such that the corresponding group $O'(g, \mathbb{Z})$ is reflective. The fundamental polyhedron of the group $O_r(g, \mathbb{Z})$ in the space $Л^{21}$ has 210 faces, and has the highest dimension among the known Coxeter polyhedra of finite volume in the Lobachevskij spaces (cf. Chap. 5, Theorem 2.3).

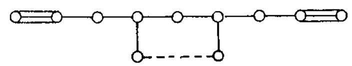
The fundamental domains of the groups $O'(f_n, \mathbb{Z})$ and $O'(g_n, \mathbb{Z})$ (and the more so of their reflection subgroups) are unbounded for any n. Now let

$$h_n(x) = -\varphi x_0^2 + x_1^2 + \ldots + x_n^2, \quad \varphi = \frac{1+\sqrt{5}}{2}.$$

Then $O'(h_n, \mathbb{Z}[\varphi])$ is a uniform discrete group of motions of the space $Л^n$. It is reflective for $n \le 7$ (Bugaenko [1984]). The schemes T_n of the fundamental polyhedra of the groups $O_r(h_n, \mathbb{Z}[\varphi])$ for $n \le 7$ are given in the following table.

n	T_n
2	
3	
4	
5	
6	
7	

For $n > 7$ the group $O'(h_n, \mathbb{Z}[\varphi])$ is not reflective. There is, however, an admissible quadratic form h of signature (8,1) with discriminant $-(\sqrt{5}+1)$ over the ring $\mathbb{Z}[\varphi]$ for which the group $O'(h, \mathbb{Z}[\varphi])$ is reflective (Bugaenko [1992]). The fundamental polyhedron of the group $O_r(h, \mathbb{Z}[\varphi])$ in the space $Л^8$ with the Coxeter scheme



has the highest dimension among the known bounded Coxeter polyhedra in the Lobachevskij spaces (cf. Chap. 5, Theorem 2.3).

For more examples of this approach to the investigation of arithmetic groups see Vinberg [1972a], Bugaenko [1987], Shaikheev [1987], Nikulin [1981], Vinberg [1983]). In the last two papers these results are applied to the study of algebraic surfaces of type $K3$.

2.2. Classification Problem for Reflective Quadratic Forms. In the notation of Sect. 1.1, we say that an admissible quadratic form f over a field K is reflective if the group $O'(f, A)$ is reflective. Similarly, an admissible[2] quadratic *A-module* L is said to be *reflective* if the group $O'(L)$ is reflective. The classification of all reflective quadratic forms or, more generally, of all reflective quadratic modules is an interesting problem. The following results give hope that such a classification is possible.

Theorem 2.1 (Nikulin [1980, 1981]). *For any fixed $n \ge 2$ and fixed degree of the field K there is only finitely many (up to similarity) reflective quadratic A-modules with signature $(n, 1)$.*

(Two quadratic A-modules L_1 and L_2 are said to be similar if there are an isomorphism of modules $\varphi : L_1 \to L_2$ and a number $c \in K$ such that $(\varphi(x), \varphi(y)) = c(x, y)$ for any $x, y \in L_1$.)

Theorem 2.2 (Vinberg [1984]). *There are no reflective quadratic A-modules with signature $(n, 1)$ for any $n \ge 30$ and any field K.*

(For $K \ne \mathbb{Q}$ it follows, of course, from Theorem 2.2 of Chap. 5.)

Better bounds for the dimension can be obtained under appropriate conditions on the field K. For example, it is proved in Vinberg [1984] that if $K \ne \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})$, or $\mathbb{Q}(\cos\frac{2\pi}{7})$, then there are no reflective quadratic A-modules of signature $(n, 1)$ for $n \ge 22$.

The bounds given above are apparently not exact. At the time of writing the final results have been obtained only for unimodular quadratic modules over certain rings. Thus a unimodular quadratic $\mathbb{Z}$-module with signature $(n, 1)$ is reflective if and only if $n \le 19$ (Vinberg [1972, 1975]). An admissible unimodular quadratic $\mathbb{Z}[\frac{\sqrt{5}+1}{2}]$-module with signature $(n, 1)$ is reflective if and only if $n \le 7$ (Bugaenko [1984]).

[2] That is, equipped with a scalar product defined by an admissible quadratic form over K.

§ 3. Existence of Non-Arithmetic Groups

Evidently there are only countably many arithmetic discrete groups of motions in the Lobachevskij space of any dimension. On the other hand, there is a continuum of crystallographic (and also uniform) groups of motions of the Lobachevskij plane (see Chap. 4). Thus almost all of them are non-arithmetic. However, as follows from the strong rigidity theorem (see Chap. 7), there are only countably many crystallographic groups of motions of the space $Л^n$ for $n > 2$. A question arises whether or not all of them are arithmetic. This is all the more interesting because according to the result of Margulis [1975, 1991] all crystallographic groups of motions in symmetric spaces of rank ≥ 2 are arithmetic (while the Lobachevskij space of any dimension is a symmetric space of rank 1).

3.1. Arithmeticity Criterion for Reflection Groups. Let $C = (c_{ij})$ be a square matrix. We define a *cyclic product* of entries of C as a product of the form $c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_{k-1} i_k} c_{i_k i_1}$. A cyclic product is said to be irreducible if the indices $i_1, \dots i_k$ are distinct. It is evident that any cyclic product can be decomposed into a product of irreducible ones, and that there are only finitely many irreducible cyclic products.

Theorem 3.1 (Vinberg [1967]). *Let Γ be a crystallographic reflection group in the Lobachevskij space, $G = (g_{ij})$ the Gram matrix of its fundamental polyhedron, and K (respectively, $\tilde{K}$) the field generated by the cyclic products (respectively, entries) of the matrix G. The group Γ is arithmetic if and only if*

(A1) *the field $\tilde{K}$ is a totally real algebraic number field;*
(A2) *for any embedding $\sigma : \tilde{K} \to \mathbb{R}$ that is not the identity on K, the matrix $G^\sigma = (g_{ij}^\sigma)$ is positive semi-definite;*
(A3) *the cyclic products of the matrix $2G$ are integers of the field K.*

Under these conditions, the field of definition of the group Γ is K.

We now make some observations on the applications of this theorem.

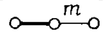
1. The condition that the matrix G^σ is positive semi-definite is equivalent to the fact that all its principal minors are non-negative. Since principal minors of the matrix G are algebraic sums of its cyclic products, and thus belong to the field K, condition (A2) can be stated in terms of the field K; namely, under all non-identity embeddings $K \to \mathbb{R}$ the principal minors of the matrix G must become non-negative. One can show that condition (A1) is then satisfied automatically.

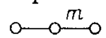
2. One can easily show that for all embeddings $K \to \mathbb{R}$ the principal minors of the matrix G corresponding to elliptic subschemes of the Coxeter scheme remain positive. Thus, condition (A2) holds if at least one negative minor M of order $(n+1)$ of the matrix G (and such a minor always exists) becomes positive under all non-identity embeddings $K \to \mathbb{R}$. In turn, this holds if and only if the number M generates the field K and its conjugates are all positive.

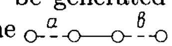
3. Since the numbers of the form $2\cos\frac{\pi}{m}$ are algebraic integers, condition (A3) holds automatically if the Coxeter scheme of the polyhedron P contains no dotted edges.

4. If the polyhedron P is unbounded (but has finite volume), then the field of definition of the group Γ (in case it is arithmetic) can only be the field $\mathbb{Q}$. In this situation, the arithmeticity criterion can be stated as follows: all cyclic products of the matrix $2G$ must be rational integers.

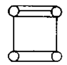
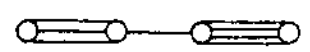
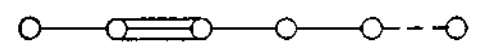
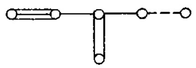
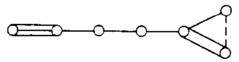
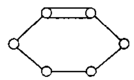
We now give some examples of this approach to the investigation of arithmeticity of reflection groups in the Lobachevskij plane.

Example 1. Let the group Γ be generated by reflections in the sides of the triangle with the scheme  . (It is a right triangle with a vertex at infinity, and the angle $\frac{\pi}{m}$ at an ordinary vertex.) We have $K = \mathbb{Q}(\cos^2\frac{\pi}{m}) = \mathbb{Q}(\cos\frac{2\pi}{m})$. The arithmeticity condition reduces to $K = \mathbb{Q}$, i.e. $m = 3, 4$, or 6.

Example 2. Let the group Γ be generated by reflections in the sides of the triangle with the scheme  . In this case one also has $K = \mathbb{Q}(\cos\frac{2\pi}{m})$. The arithmeticity condition means that the determinant of the Gram matrix (equal to $\frac{1}{4}(1-\cos\frac{2\pi}{m})$) must become positive under any non-identity embedding $K \to \mathbb{R}$, i.e. $\cos\frac{2k\pi}{m} < \frac{1}{2}$ for $(k,m) = 1$, $1 < k < \frac{m}{2}$. One can easily see that this holds only for $m = 7, 8, 9, 10, 11, 12, 14, 16, 18, 24$ and 30.

Example 3. Let the group Γ be generated by reflections in the sides of the quadrilateral with the scheme  . (It is a quadrilateral having three right angles and the fourth angle equal to $\frac{\pi}{3}$.) Since the determinant of the Gram matrix must vanish, one has $(a^2-1)(b^2-1) = \frac{1}{4}$, and, consequently, $K = \mathbb{Q}(a^2) = \mathbb{Q}(b^2)$. The group Γ is arithmetic if and only if the numbers $A = 4a^2$ and $B = 4b^2$ related by $(A-4)(B-4) = 4$ are totally real algebraic numbers and their conjugates are less than 4. For $A, B \in \mathbb{Q}$, for example, this holds only if $\{A, B\} = \{8, 5\}$ or $\{6, 6\}$.

3.2. Existence of Non-Arithmetic Reflection Groups. The possibility of finding examples of non-arithmetic crystallographic groups of motions of the spaces $Л^n$ for $n \geq 3$ among reflection groups was discovered by Makarov [1966]. Using Theorem 3.1 one can easily find such examples already among Lannér and quasi-Lannér groups (see Tables 3 and 4 in Chap. 5, Sect. 3). Now non-arithmetic uniform reflection groups in $Л^n$ are known for $n \leq 5$ and non-arithmetic crystallographic reflection groups for $n \leq 10$. Examples of such groups for $n = 3, 4, 5$ are given in the table below. In three of these examples the fundamental polyhedron is a simplex, and in the remaining three cases it is a simplicial prism.

n	Uniform groups	Non-uniform groups
3		
4		
5		

Examples of non-arithmetic non-uniform crystallographic reflection groups in Л^n for $6 \le n \le 10$ are constructed in Ruzmanov [1989].

3.3. Existence of Non-Arithmetic Crystallographic Groups in Any Dimension. Gromov and Piatetski-Shapiro [1988] suggested a general construction which enables one to obtain (both uniform and non-uniform) non-arithmetic crystallographic groups of motions of the space Л^n for any n. We now present a slightly modified version of this construction. (See also Margulis [1991].)

We first consider the following general situation. Let $\Gamma \subset \operatorname{Isom}_+ \text{Л}^n$ be a crystallographic group acting without fixed points, and σ a reflection normalizing it. The group Γ can be embedded as a (normal) subgroup of index 2 into the group $\hat{\Gamma} = \Gamma \cup \Gamma\sigma$ containing, together with the reflection σ, reflections of the form $\gamma\sigma\gamma^{-1}$, $\gamma \in \Gamma$.

Let H be the mirror of the reflection σ. Then γH is the mirror for $\gamma\sigma\gamma^{-1}$. Since the product of any two such reflections belongs to Γ and thereby has no fixed points, the hyperplanes γH, $\gamma \in H$ do not intersect and divide the space into convex polyhedra which we shall call *chambers*. This decomposition is invariant under the group $\hat{\Gamma}$. In particular, any two adjacent chambers are symmetric with respect to their common face.

For any chamber there is a transformation in the group Γ taking it into one of the chambers C^+ and $C^- = \sigma(C^+)$ adjacent to H. Let Γ^+ and $\Gamma^- = \sigma\Gamma^+\sigma^{-1}$ be the stabilizers of these chambers in the group Γ.

Lemma 3.2. *If the group $\Gamma_0 = \Gamma^+ \cap \Gamma^-$ is dense in the Zariski topology in* $\operatorname{Isom}_+ H$, *then the group Γ^+ is dense in the Zariski topology in* $\operatorname{Isom}_+ \text{Л}^n$.

Proof. It follows from the description of algebraic subgroups of the group $\operatorname{Isom}_+ \text{Л}^n$ (Part 1, Chap. 5, Sect. 3) that if the group Γ^+ is not dense in $\operatorname{Isom}_+ \text{Л}^n$, then it preserves either a proper plane, or a point at infinity. Suppose, for example, that it preserves a plane $\Pi \neq \text{Л}^n$. One can easily see that $\Pi \neq H$. If $\Pi \cap H \neq \emptyset$, then Γ_0 preserves the plane $\Pi \cap H$. If the planes Π and H are parallel, then Γ_0 preserves their common point at infinity. If they are divergent, then Γ_0 preserves the end point of their common perpendicular lying on H. In any of these three cases the group Γ_0 is not dense in $\operatorname{Isom}_+ H$. The case when Γ^+ preserves a point at infinity is treated in a similar manner. □

Consider the hyperbolic manifold $M = \text{Л}^n/\Gamma$. The reflection σ induces an involution of this manifold whose set of fixed points contains (as a connected component) the hypersurface S which is the image of the hyperplane H.

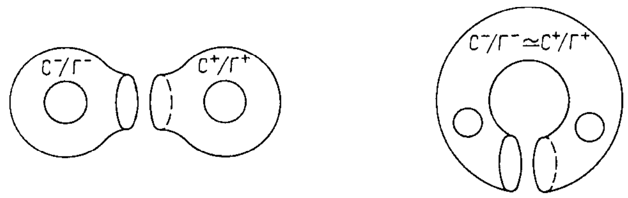
Suppose that the following condition holds.

(S) No motion in the group Γ preserves the hyperplane H and permutes the chambers C^+ and C^-.

Then $S = H/\Gamma_0$. Two cases are now possible.

First case. The group Γ has two orbits on the set of chambers containing C^+ and C^- respectively. Then cutting the manifold M along S yields two hyperbolic manifolds with boundary isometric to C^+/Γ^+ and C^-/Γ^-, respectively (Fig. 18a).

Second case. The group Γ acts transitively on the set of chambers. Then cutting the manifold M along S yields a connected hyperbolic manifold with boundary isometric to $C^+/\Gamma^+ \simeq C^-/\Gamma^-$ (Fig. 18b).



a b

Fig. 18

Now let $\Gamma_1, \Gamma_2 \subset \operatorname{Isom}_+ \text{Л}^n$ be two crystallographic groups acting without fixed points and normalized by a same reflection σ. Suppose that both Γ_1 and Γ_2 satisfy condition (S) and induce the same group Γ_0 on the mirror H of the reflection σ. Then, cutting the hyperbolic manifolds $M_1 = \text{Л}^n/\Gamma_1$ and $M_2 = \text{Л}^n/\Gamma_2$ along the images of H, one obtains hyperbolic manifolds with isometric boundaries which can be pasted together into a new hyperbolic manifold M of finite volume using one of the options shown symbolically in Fig. 19. The fundamental group Γ of this manifold contains the groups Γ_1^- and Γ_2^+ intersecting in the subgroup Γ_0.

We now proceed to the direct construction of examples of non-arithmetic groups. Let K be a totally real algebraic number field, A the ring of its integers, and L_0 an admissible quadratic A-module of signature $(n-1, 1)$. The pseudo-Euclidean space $L_0 \otimes \mathbb{R}$ can be embedded in the space $\mathbb{R}^{n,1}$, where it defines a hyperplane H of the space $Л^n$. Let σ be the reflection in this hyperplane, and e a unit vector orthogonal to $L_0 \otimes \mathbb{R}$.

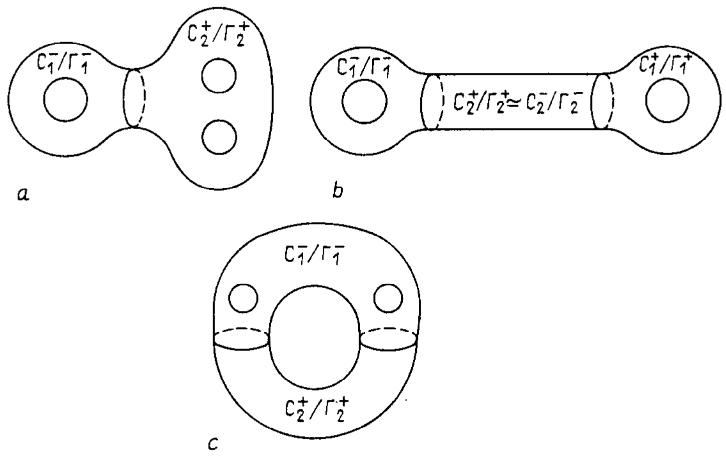


Fig. 19

Choose two positive numbers $a_1, a_2 \in A$ such that their conjugates are also positive. The quadratic A-modules

$$L_1 = L_0 \oplus \sqrt{a_1}Ae, \quad L_2 = L_0 \oplus \sqrt{a_2}Ae$$

(embedded in $\mathbb{R}^{n,1}$) are admissible.

Note the following fact (Cassels [1978]): if p is an odd prime number that does not ramify in the field K, then the congruence subgroup

$$\mathrm{Aut}\,(L,p) = \{\varphi \in \mathrm{Aut}\,L : \varphi(x) - x \in pL\ \forall x \in L\}$$

in the group $\mathrm{Aut}\,L$ of automorphisms of any torsion-free A-module L has no elements of finite order.

In particular, if p is such a number, then the corresponding congruence subgroups Γ_1, Γ_2, and Γ_0 of the groups $O'(L_1)$, $O'(L_2)$, and $O'(L_0)$ have no elements of finite order, and therefore their action in the Lobachevskij space is fixed-point-free. The groups Γ_1 and Γ_2 are normalized by the reflection σ and induce the group Γ_0 on H. Furthermore, since $p \neq 2$, they also preserve the orientation and satisfy condition (S). The group Γ_0, like any arithmetic group, is dense in the Zariski topology in Isom_+H (VGS, Chap. 1, Sect. 8).

We now use the above procedure and paste the hyperbolic manifolds $M_1 = Л^n/\Gamma_1$ and $M_2 = Л^n/\Gamma_2$ together into another hyperbolic manifold $M = Л^n/\Gamma$. To check whether or not the group Γ is arithmetic, one can use the following proposition, which follows easily from the results contained in Vinberg [1971a].

Proposition 3.3. *Let Δ_1 and Δ_2 be arithmetic discrete subgroups of a connected semi-simple Lie group G having no compact factors. If the intersection $\Delta_1 \cap \Delta_2$ is dense in the Zariski topology in G, then the subgroups Δ_1 and Δ_2 are commensurable.*

Since $\Gamma \supset \Gamma_1^- \cup \Gamma_2^+$, this proposition and Lemma 3.2 imply that if the group Γ is arithmetic, then the groups Γ_1 and Γ_2 are commensurable, but this holds only if $KL_1 = KL_2$, i.e. if $a_1/a_2 \in (K^*)^2$.

The numbers a_1, a_2 can be chosen in such a way that the latter condition is not satisfied, and then the group Γ is not arithmetic.

Chapter 7
Sociology of Discrete Groups in the Lobachevskij Spaces

The results presented in this chapter deal mainly with relations between different crystallographic groups of motions in the Lobachevskij space, and concern their "society" in general rather than individual groups.

§ 1. Rigidity and Deformation

1.1. Theorem on Strong Rigidity. The fundamental group of a compact surface of genus $g > 1$ admits a continuum of geometrically different realizations in the form of a uniform discrete group of motions of the Lobachevskij plane (Chap. 4, Sect. 3). In other words, on a compact surface of genus $g > 1$ there is a continuum of non-equivalent metrics of constant curvature equal to -1.

Starting with dimension 3 the situation changes drastically, namely the following "strong rigidity" property comes into play.

Theorem 1.1. *Any isomorphism of crystallographic groups of motions of the space $Л^n$, $n \geq 3$, is induced by a conjugation in the group* $\mathrm{Isom}\,Л^n$.

Corollary. *If two complete hyperbolic (of curvature -1) manifolds of finite volume are homeomorphic, then they are isometric.*

The proof of Theorem 1.1 contained in the works of Mostow [1968], Margulis [1970], and Prasad [1973] is based on the theorem of Efremovich and Tikhomirova [1964] on equimorphisms of the Lobachevskij spaces. A proof based on different ideas was suggested by Gromov (see Munkholm [1980]).

The following theorem is a simple corollary to Theorem 1.1. (Cf. Chap. 4, Sect. 3.1).

Theorem 1.2 (Mostow [1968]). *For any crystallographic group Γ of motions of the space* $Л^n$, $n \geq 3$, *the group* $\operatorname{Aut}\Gamma$ *is naturally isomorphic to the normalizer* $N(\Gamma)$ *of* Γ *in the group* $\operatorname{Isom} Л^n$. *The group* $N(\Gamma)$ *is discrete, and therefore the group* $\operatorname{Aut}\Gamma/\operatorname{Int}\Gamma$ *is finite.*

The class of discrete groups for which the strong rigidity theorem holds can be widened. A discrete group Γ of motions of the space $Л^n$ is said to be of *divergent type* if, in the conformal model in the unit ball B^n, the series $\sum_{\gamma\in\Gamma}(1-|\gamma(x)|)^{n-1}$ diverges for some (and consequently any) $x \in B^n$. Any crystallographic group is a group of divergent type.

Theorem 1.3 (Ahlfors [1981], Agard [1983]). *For $n \geq 3$ any isomorphism of discrete groups of divergent type is induced by a conjugation in the group* $\operatorname{Isom} Л^n$.

1.2. Deformations. Let $G = \operatorname{Isom} Л^n$ and let $\Gamma \subset G$ be a crystallographic group. Denote by h_0 the identity embedding of the group Γ in G. The group Γ is said to be *rigid* if all deformations of the homomorphism h_0 are trivial, i.e. if any homomorphism $h : \Gamma \to G$ sufficiently close to h_0 is of the form $h = \sigma \cdot h_0$, where σ is an (inner) automorphism of the group G.

Theorem 1.4. *Any crystallographic group of motions of the space $Л^n$, $n > 3$, as well as any uniform discrete group of motions of the space $Л^3$, is rigid.*

For uniform discrete groups this theorem is a special case of the theorem of Weil on discrete subgroups of arbitrary semi-simple Lie groups, while for non-uniform subgroups it was proved by Garland and Raghunathan (see VGS).

As a rule, non-uniform crystallographic groups of motions of the space $Л^3$ are not rigid. This surprising fact was established by Thurston. More precisely, he proved the following theorem.

Theorem 1.5 (see Gromov [1981]). *Let Γ be a non-uniform crystallographic group of proper motions of the space $Л^3$ acting without fixed points, and let* $G = \operatorname{Isom} Л^3$. *Then in a neighborhood of the identity embedding* $h_0 \colon \Gamma \to G$ *the dimension of the real analytic manifold* $\operatorname{Def}(\Gamma, G) = \operatorname{Hom}(\Gamma, G)/\operatorname{Aut} G$ *equals 2τ, where τ is the number of equivalence classes of parabolic points of the group Γ.*

What follows contains no proof but rather an explanation of the theorem. We first illustrate the approach by explaining why a uniform discrete group Γ of motions of the space $Л^3$ is rigid.

Let P be a fundamental polyhedron of a uniform discrete group Γ. A false faceting can make all its faces triangular. A deformation of the polyhedron P preserving its combinatorial structure is said to be *admissible* if for any class of equivalent edges the deformation does not destroy the equality of their lengths and preserves the sum of dihedral angles at them. To any admissible deformation of the polyhedron P there corresponds a deformation of the homomorphism h_0, and any deformation of h_0 can be obtained in this way.

Let us calculate the number of parameters on which an admissible deformation of the polyhedron P depends. Let a_0 be the number of its vertices, a_1 the number of its edges, a_2 the number of its faces, and b_0 the number of classes of equivalent vertices. An arbitrary deformation of the polyhedron P preserving its combinatorial structure depends on $3a_0 - 6$ parameters. The admissibility requirement imposes a_1 conditions. They are, however, related. The relations can be discovered by covering a small neighbourhood of a vertex of the deformed polyhedron P with equivalent polyhedra. Such a covering results, in view of the admissibility conditions, in a tiling of a sphere (with centre at the vertex under consideration) with polygons. The angles of each of these polygons equal the corresponding dihedral angles of the deformed polyhedron P, and the lengths of their sides equal the plane angles of the corresponding triangular faces; these angles (like those in a non-Euclidean triangle) are uniquely defined by the lengths of the edges of P. Now observe that the angles and sides of any polygon of the tiling are uniquely determined by the angles and sides of the others. Since there are three relations between the lengths of the sides and the angles in a spherical polygon, each class of equivalent vertices contributes three relations between the admissibility conditions. Thus, in all, one obtains $3b_0$ relations.

Now one has to take into account that some admissible deformations produce fundamental polyhedra of the same group. Indeed, taking the fundamental polyhedron of any group one can arbitrarily move one of its vertices, and correspondingly the vertices equivalent to it, to obtain the fundamental polyhedron of the same group. Thus, the fundamental polyhedron of the given group depends on $3b_0$ parameters.

As a result, any deformation of the homomorphism h_0 depends on

$$3a_0 - 6 - a_1 + 3b_0 - 3b_0 = 3a_0 - a_1 - 6 = 0$$

parameters. (Here we have used the Euler identity $a_0 - a_1 + a_2 = 2$, and the equality $2a_1 = 3a_2$.)

Now let Γ be a non-uniform crystallographic group. Suppose, for simplicity, that its fundamental polyhedron P has only one vertex q at infinity, and that only four faces meet at that point (the latter can always be attained). Let α and β be commuting parabolic translations realizing the pairwise equivalence of these faces. Then the vertices adjacent to q are obtained from any of them by the motions α, β, and $\alpha\beta$ (see Fig. 20).

For any deformation of the polyhedron P we shall disregard the faces passing through the vertex q at infinity, and instead consider two commuting (in general, helical) motions $h(\alpha)$ and $h(\beta)$ into which the deformation turns α and β. Note that such a pair of motions depends on 8 parameters, since they are defined by the position of their common axis, displacement distances, and the rotation angles.

The above procedure for calculating the number of parameters should be modified as follows:

(a) one has to disregard the vertex q and the edges passing through it, as well as dihedral angles at these edges;
(b) one has to take into account only one of the four vertices adjacent to q, since the others are obtained from it by the motions $h(\alpha), h(\beta)$, and $h(\alpha)h(\beta)$; the pairwise congruence of the edges joining these vertices need not be included in the admissibility conditions because it is satisfied automatically.

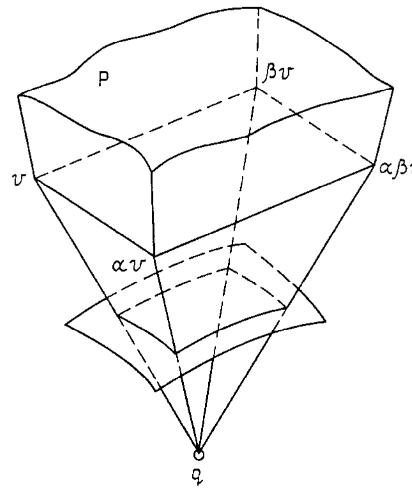


Fig. 20

Thus the deformation of the homomorphism depends on

$$3(a_0-1-3)+8-6-(a_1-4-2)+3(b_0-1)-3(b_0-1)=3a_0-a_1-4=2$$

parameters, as asserted. □

We now give an explicit parametrization of the manifold $\operatorname{Def}(\Gamma, G)$ in a neighbourhood of the point h_0.

Let q be a parabolic point of the group Γ, and α and β parabolic translations generating the group Γ_q. Consider a homomorphism $h: \Gamma \to G$ taking α and β into (commuting) helical motions $h(\alpha)$ and $h(\beta)$. Denote by l the common axis of these motions, by a and b the signed values of their displacement distances, and by φ and ψ their signed rotation angles. One can easily see that if the homomorphism h is sufficiently close to the identity embedding h_0, then $\left|\begin{smallmatrix} a & b \\ \varphi & \psi \end{smallmatrix}\right| \neq 0$. Thus there are uniquely defined real numbers r and s such that

$$ra+sb=0, \quad r\varphi+s\psi=2\pi.$$

Denote by $\widehat{\mathbb{R}^2}$ the plane $\mathbb{R}^2$ compactified by the point ∞, and associate with each homomorphism $h: \Gamma \to G$ sufficiently close to h_0 the point $(r,s) \in \widehat{\mathbb{R}^2}$ defined in the above manner if $h(\alpha)$ and $h(\beta)$ are helical motions, and the point ∞ if $h(\alpha)$ and $h(\beta)$ are parabolic translations (no other cases are possible). Thus, in a neighbourhood of h_0 a mapping $f: \operatorname{Def}(\Gamma, G) \to \widehat{\mathbb{R}^2}$ is defined.

Now let $q_1, \dots, q_\tau$ be representatives of all classes of equivalent parabolic points of the group Γ, and $f_i: \operatorname{Def}(\Gamma, G) \to \widehat{\mathbb{R}^2}$ the mapping defined in a neighbourhood of the point h_0 in the above manner for $q = q_i$.

Theorem 1.6 (Gromov [1981], Neumann, and Zagier [1985]). *The mapping*

$$f: h \mapsto (f_1(h), \dots, f_\tau(h))$$

defines a homeomorphism of a neighbourhood of the point h_0 of the manifold $\operatorname{Def}(\Gamma, G)$ onto a neighbourhood of the point $(\infty, \dots, \infty)$ of the manifold $\widehat{\mathbb{R}^2} \times \dots \times \widehat{\mathbb{R}^2}$.

The meaning of this parametrization is partly explained in the following section.

1.3. Dehn-Thurston Surgery. The deformations discussed in the preceding section do not necessarily result in discrete groups. It turns out, however, that discrete groups can be obtained for some special values of the deformation parameters.

A point $(u_1, \dots, u_\tau) \in \widehat{\mathbb{R}^2} \times \dots \times \widehat{\mathbb{R}^2}$ is said to be *exceptional* if for any i either $u_i = \infty$ or $u_i = (r_i, s_i)$, where r_i and s_i are coprime integers.

Theorem 1.7 (Gromov [1981], Neumann, and Zagier [1985]). *Under the conditions of Theorem 1.5, if the homomorphism $h: \Gamma \to G$ is sufficiently close to h_0 and $f(h)$ is an exceptional point, then $h(\Gamma)$ is a fixed-point-free crystallographic group with the number of classes of equivalent parabolic points equal to the number of indices i for which $f_i(h) = \infty$.*

In particular, if $\tau = 1$ and $h \neq \sigma \cdot h_0$, where σ is an automorphism of the group G, then $h(\Gamma)$ is a uniform discrete group.

Proof. As in the proof of Theorem 1.5, suppose for simplicity that the fundamental polyhedron P of the group Γ has a unique vertex q at infinity. Let $h: \Gamma \to G$ be a homomorphism sufficiently close to h_0 and taking α and β into helical motions $h(\alpha)$ and $h(\beta)$ with the same axis l. Consider an admissible deformation of the polyhedron P corresponding to the homomorphism h. The deformed part of the boundary of the polyhedron P without the faces passing through q can be closed by adding four curvilinear films pairwise equivalent by the motions $h(\alpha)$ and $h(\beta)$ (a part of the boundary of each of the films lies on the line l, see Fig. 21). Let Q be the closed domain bounded by this surface. We claim that it is a fundamental domain for the group $h(\Gamma)$.

In view of the admissibility conditions, we have only to check that domains equivalent to Q cover in an appropriate way a small neighbourhood of the intersection $Q \cap l$. This, in turn, will be achieved if we verify that the intersection D of Q with a sufficiently small equidistant surface S of the line l (see Fig. 21) is a fundamental domain for the action of the abelian group $h(\Gamma_q) = \langle h(\alpha), h(\beta) \rangle$ on S.

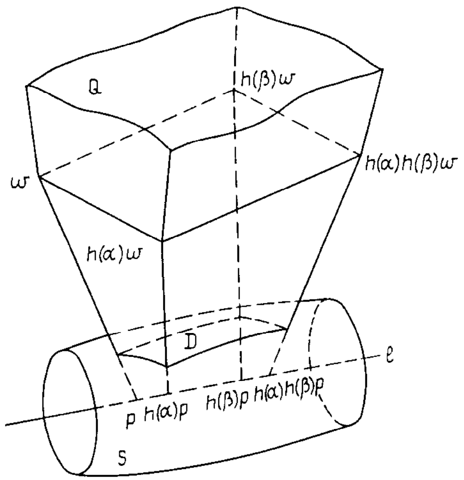


Fig. 21

Note that the equidistant surface S, as a Riemannian manifold, is a direct product of a circle and a line (Part 1, Chap. 4, Sect. 3.2), i.e. a Euclidean cylinder, and its universal covering S is a Euclidean plane. The connected inverse image $\tilde{D}$ of D lying in this plane is the fundamental domain for the group generated by parallel translations covering the motions $h(\alpha)$ and $h(\beta)$ of S. In the notation of the preceding section, they are the translations along the vectors $(a \cosh t, \varphi \sinh t)$ and $(b \cosh t, \psi \sinh t)$, where t is the distance from S to the line l. This implies, firstly that the images of D under the transformations from $h(\Gamma_q)$ cover S, and secondly that

$$\text{area } D = \text{area } \tilde{D} = \left\| \begin{matrix} a \cosh t & b \cosh t \\ \varphi \sinh t & \psi \sinh t \end{matrix} \right\| = \left\| \begin{matrix} a & b \\ \varphi & \psi \end{matrix} \right\| \cosh t \sinh t.$$

To prove that D is a fundamental domain of the group $h(\Gamma_q)$, we have only to check that this group acts discretely on S, and that the area of its fundamental domain equals the area of D.

Let $f(h) = (r, s)$ be an exceptional point. It follows from (1) that the group $h(\Gamma_q)$ contains no pure rotations about l and that the smallest in absolute value (for non-trivial transformations in this group) displacement distance along l equals $c = \frac{a}{s} = -\frac{b}{r}$. This means that the group $h(\Gamma_q)$ is generated by a helical motion with the translation distance c. Thus it acts discretely on S, and one can take for its fundamental domain a "belt" of S projected on the segment of length $|c|$ of the line l. This belt has area $2\pi|c| \cosh t \sinh t$ which, according to (1), equals the area of D. □

Note that the homomorphism h considered in the proof of the theorem is not injective: its kernel contains the element $\alpha^r \beta^s$ (and is generated by it as a normal subgroup of the group Γ).

From the topological point of view, the transition from the manifold $Л^3/\Gamma$ to the manifold $Л^3/h(\Gamma)$ is the so-called *Dehn surgery* with the parameters r, s. Indeed, by deleting from the manifold $Л^3/\Gamma$ the image of a sufficiently small horospherical neighbourhood of the point q, one obtains a manifold with the boundary homeomorphic to a torus. The generators A and B of its fundamental group correspond to the elements α and β of the group Γ. The transition to the manifold $Л^3/h(\Gamma)$ means that this boundary is pasted over by a solid torus (the image of the equidistant neighbourhood of the line l) in which the element $A^r B^s$ of the fundamental group bounds a disc.

Corollary. *Under the conditions of Theorem 1.5 there is a sequence of homomorphisms $h_k \colon \Gamma \to G$ converging to h_0 such that for any k the group $h_k(\Gamma)$ is a crystallographic group with the number of classes of equivalent parabolic points equal to $\tau - 1$.*

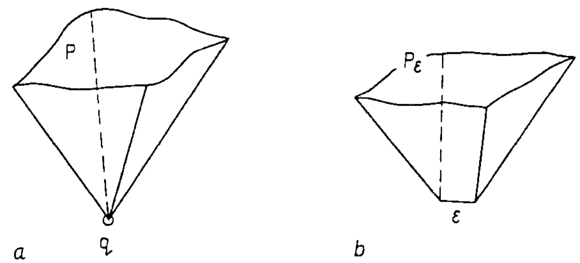


Fig. 22

Example. Although the following example refers to groups having fixed points, it serves very well to clarify the situation. Let P be a Coxeter polyhedron of finite volume with the vertex q at infinity at which four faces (forming right dihedral angles) meet, and let Γ be the group generated by reflections in the walls of this polyhedron. For any sufficiently small $\varepsilon > 0$ consider the convex polyhedron P_ε having the same combinatorial structure and the same dihedral angles as P, the only difference being that the four faces which met at the vertex q now form a "wedge" with the dihedral angle ε (Fig. 22).

The existence of such a polyhedron for a sufficiently small ε is guaranteed by the theorem of Andreev (Part 1, Chap. 6, Theorem 2.8). Let $h_\varepsilon : \Gamma \to G$ be the homomorphism taking the reflection in each wall of the polyhedron P into the reflection in the corresponding wall of the polyhedron P_ε. As a rule, the group $h_\varepsilon(\Gamma)$ generated by reflections in the walls of the polyhedron P_ε is not discrete. However, for $\varepsilon = \frac{\pi}{k}$, $k \in \mathbb{Z}$, the polyhedron P_ε is a Coxeter polyhedron and $h_\varepsilon(\Gamma)$ is a discrete reflection group. Its defining relations include, together with those of the group Γ, the relation $(r_1 r_2)^k = e$, where r_1 and r_2 are reflections in the planes of the faces forming the edge of the wedge.

§ 2. Commensurable Groups

2.1. Commensurator. For an arbitrary subgroup Γ of an arbitrary group G the set $\operatorname{Comm}\Gamma = \{g \in G : g\Gamma g^{-1}$ is commensurable with $\Gamma\} \subset G$ is a subgroup. It is called the *commensurator* of the group Γ, and contains all subgroups commensurable with Γ.

For example, if $G = PGL_n(\mathbb{C})$, $\Gamma = PGL_n(\mathbb{Z})$, then $\operatorname{Comm}\Gamma = PGL_n(\mathbb{Q})$

Theorem 2.1. *Let $\Gamma \subset G = \operatorname{Isom} Л^n$ be a crystallographic group. Then*

(1) *if the group Γ is arithmetic, then the group $\operatorname{Comm}\Gamma$ is dense in G;*
(2) *if the group Γ is not arithmetic, then the group $\operatorname{Comm}\Gamma$ is discrete (and thus commensurable with Γ).*

The first part of the theorem is trivial. For example, if $\Gamma = O'(f, A)$ is an arithmetic subgroup of the simplest type (Chap. 6, Sect. 1.1) then $\operatorname{Comm}\Gamma = O'(f, K)$. The second part is a special case of a result of Margulis on discrete subgroups of arbitrary semi-simple Lie groups (see VGS, Chap. 3, Sect. 6.4). □

One can now pose the problem of finding maximal subgroups in the class $C(\Gamma)$ of subgroups commensurable with Γ. If Γ is not arithmetic, then in view of Theorem 2.1 the only such subgroup is $\operatorname{Comm}\Gamma$. If it is arithmetic, then there are infinitely many pairwise non-conjugate maximal subgroups in $C(\Gamma)$. For their description (in a more general situation) see (VGS, Chap. 3, Sect. 6.3).

2.2. Millson's Property. A group Γ is said to have Millson's property if it has a subgroup of finite index whose quotient group by the commutator subgroup is infinite. One can easily see that commensurable subgroups of any group either possess or do not possess Millson's property simultaneously.

If Γ is a uniform discrete group of motions of the space $Л^n$ acting without fixed points, then Millson's property means that the manifold $Л^n/\Gamma$ admits a finite covering whose first Betti number does not vanish. In contrast to the case of symmetric spaces of rank > 1 (see VGS, Chap. 3, Sect. 7.1) any crystallographic group Γ of motions of the Lobachevskij space $Л^n$ apparently has Millson's property. It is known to be the case for certain if

(1) $n = 2$ (evident);
(2) $n = 3$ and Γ is a uniform discrete reflection group (Hempel [1982]);
(3) Γ is an arithmetic discrete group of the simplest type (Millson [1976]).

§ 3. Covolumes

3.1. The Set of Covolumes. An important numerical invariant of a discrete group Γ of motions of the Lobachevskij space $Л^n$ is its *covolume*

$$v(\Gamma) = \operatorname{vol} Л^n/\Gamma.$$

A good idea of the variety of crystallographic groups is given by the following results on the set of their covolumes.

We denote:

by V_n the set of covolumes of all crystallographic groups in the space $Л^n$;

by W_n the set of covolumes of all torsion-free crystallographic groups (i.e. acting without fixed points);

by AV_n the set of covolumes of arithmetic discrete groups.

Each point of these sets will be considered with the multiplicity equal to the number of non-conjugate crystallographic groups of the corresponding type having a given covolume.

Theorem 3.1 (Wang [1972]). *For $n \geq 4$ the set V_n is discrete.*

Theorem 3.2. *The set AV_n is discrete.*

For $n \geq 4$ it follows from the preceding theorem, and for $n = 2, 3$ it is proved in Borel [1981], Chinburg [1983].

Theorem 3.3 (Margulis and Rohlfs [1986]). *The set of covolumes of all crystallographic groups commensurable with a given one is contained in a cyclic group $c\mathbb{Z}$, $c > 0$.*

The loss of rigidity in dimension 3 is responsible for the fact that the sets V_3 and W_3 look unusual.

Theorem 3.4. *The sets V_3 and W_3 are closed well-ordered sets of order-type ω^ω. Each of their points is of finite multiplicity.*

The limit points of the set W_3 are precisely covolumes of non-uniform crystallographic torsion-free groups. This follows firstly from the rigidity of uniform groups, and secondly from the following proposition.

Proposition 3.5 (Neumann and Zagier [1985]). *Under the conditions of Theorem 1.7 the function* $u \mapsto v(f^{-1}(u))$ *is continuous on the set of exceptional points* $u \in \widehat{\mathbb{R}^2} \times \ldots \times \widehat{\mathbb{R}^2}$ *sufficiently close to the point* $(\infty, \ldots, \infty)$, *and the values it assumes (with the exception of the point* $(\infty, \ldots, \infty)$) *are less than* $v(\Gamma)$.

(For the asymptotic behaviour of this function see Neumann and Zagier [1985].)

In particular, $v(\Gamma)$ is approximated from below by covolumes of torsion-free crystallographic groups having one class of parabolic points less than Γ.

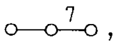
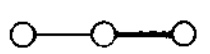
The limit points of the set V_3 are also the covolumes of non-uniform crystallographic groups, but the converse statement does not hold: it is not true that the covolume of any such group is a limit point for the set V_3, because some of these groups are rigid.

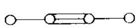
For the sets V_2 and W_2 the formulae for the area of the fundamental polygon (Chap. 4, Sect. 2.2) easily imply that $W_2 = 4\pi\mathbb{N}$, and V_2, like V_3, is a well-ordered set of order-type ω^ω.

3.2. Discrete Groups of Minimal Covolume. The covolumes of crystallographic groups in the space $Л^n$ are bounded from below by a positive constant depending only on n. This is a consequence of the following geometric fact, which is easily proved with the help of the Margulis lemma (Chap. 2, Theorem 2.7).

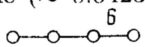
Theorem 3.6 (Kazhdan and Margulis [1968], Apanasov [1983]). *Any fundamental Dirichlet domain of any crystallographic group* $\Gamma \subset \operatorname{Isom} Л^n$ *contains a ball whose radius depends only on* n.

Since the set of covolumes is well-ordered, there is a group of minimal covolume in any class of crystallographic groups in the space $Л^n$. It is of interest to find these groups and their covolumes for some natural classes of groups. The facts known at the time of writing are the following.

The minimal covolume among uniform discrete groups of motions of the Lobachevskij plane belongs to the reflection group with the scheme , and that among non-uniform groups to the reflection group with the scheme  ($= PGL_2(\mathbb{Z})$). Both of them are arithmetic, and their covolumes equal $\frac{\pi}{42}$ and $\frac{\pi}{6}$, respectively.

Among arithmetic discrete groups of motions of the Lobachevskij space $Л^3$ the minimal covolume (≈ 0.0195) belongs to the normalizer of the reflection group with the scheme  which is its semi-direct product with the group of order 2 (Chinburg and Friedman [1986]). This group apparently has the minimal covolume among all discrete groups of motions of the space $Л^3$.[3]

[3] As the authors were told by G.J. Martin, this was recently proved by F.W. Gehring and himself.

Among non-uniform discrete groups of motions of the space $Л^3$ the minimal covolume (≈ 0.0423) belongs to the (arithmetic) reflection group with the scheme  (Meyerhoff [1986]).

Covolumes of the crystallographic groups acting freely in the space $Л^3$ are of special interest, since they provide an important topological invariant of the corresponding hyperbolic manifolds. At the time of writing, volumes of a large number of hyperbolic manifolds have been computed (see, e.g., Adams, Hildebrand, and Weeks [1991]). The two smallest among them are ≈ 0.9814 and ≈ 0.9427. The corresponding manifolds are constructed in Thurston [1982] and Matveev and Fomenko [1988].

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