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General Theory of Vortices

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ABSTRACT

This book is devoted to the mathematical explication of the analogies between hydrodynamics, geometric optics, and mechanics. It turns out that studying families of trajectories of Hamiltonian systems in fact reduces to studying the multidimensional hydrodynamics of an ideal fluid. In particular, the well-known Hamilton–Jacobi method corresponds to the case of potential flows. Some applications of such an approach are described, in particular, the vortex method for exactly integrating differential equations in dynamics.

The book is addressed to scientists and graduate students interested in mathematical physics, mechanics, and differential equations.

Contents

Introduction

> The English teach mechanics as an experimental science, while on the Continent, it has always been considered a more deductive and a priori science. Unquestionably, the English are right.*
>
> H. Poincaré, *Science and Hypothesis*

Descartes, Leibnitz, and Newton

As is well known, the basic principles of dynamics were stated by Newton in his famous work *Philosophiae Naturalis Principia Mathematica*, whose publication in 1687 was paid for by his friend, the astronomer Halley. In essence, this book was written with a single purpose: to prove the equivalence of Kepler's laws and the assumption, suggested to Newton by Hooke, that the acceleration of a planet is directed toward the center of the Sun and decreases in inverse proportion to the square of the distance between the planet and the Sun. For this, Newton needed to systematize the principles of dynamics (which is how Newton's famous laws appeared) and to state the "theory of fluxes" (analysis of functions of one variable). The principle of the equality of an action and a counteraction and the inverse square law led Newton to the theory of gravitation, the interaction at a distance. In addition, Newton discussed a large number of problems in mechanics and mathematics in his book, such as the laws of similarity, the theory of impact, special variational problems, and algebraicity conditions for Abelian integrals. Almost everything in the *Principia* subsequently became classic. In this connection, A. N. Krylov, who translated the *Principia* into Russian, said that each sentence from Newton's book "was not forgotten but grew into large libraries of manuals, treatises, dissertations, and thousands of journals."

After these words, the modern reader will probably find it strange that Newton's *Principia* at first provoked a rather poor reaction in Continental centers of science. For example, in 1689 (i.e., two years after the *Principia* appeared), Leibnitz published an article explaining the planetary motions in the spirit of Descartes's vortex theory: not only a force directed toward the Sun but also a circular motion of a tenuous matter (ether) influences the planetary motion. He later returned to this question several times to describe the model in more detail. We should not think that his ideas resulted from a lack of knowledge: Newton and Leibnitz had a long correspondence with each other. We must also note that Leibnitz was not an orthodox Cartesian. In addition to a dispute with Newton over the priority in discovering the method of infinitesimals, Leibnitz had a long polemic with people sharing Descartes's ideas. He insisted on the opinion that the true measure of motion was the

*[Translated from the Russian translation.]

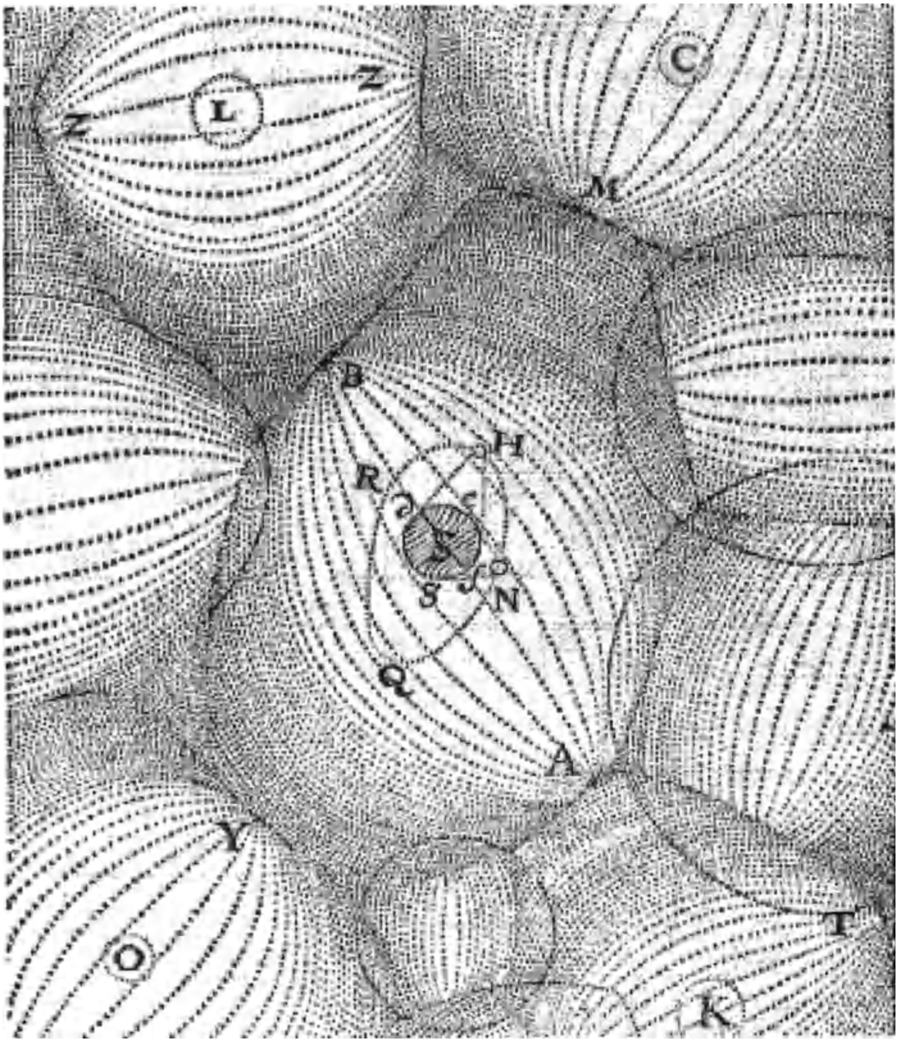

Fig. 1. Detail of a diagram from the 1644 *Principia philosophiae* of Rene Descartes, depicting his conception of the cosmos as an aggregate of contiguous vortices, most with a star at their center. The Sun is denoted by S.

weight of a body multiplied by the square of its velocity (*vis viva* according to Leibnitz) and not the product of weight and velocity, as stated by Descartes.

In 1713, after the second edition of the *Principia*, the situation changed abruptly. The new edition of Newton's book was supplemented with some added parts such as a well-developed foreword written by Cotes (a publisher and a member of the Cambridge Trinity board) and also a general *Precept* added by Newton at the end of the *Principia.* In these materials, the vortex theory was criticized rather sharply. In defining this Cartesian theory, Cotes's derisive style used expressions like "ridiculous invention," "nonsense," etc. Newton's style was more modest, although it is now known that Newton himself carefully edited the foreword written by Cotes. This foreword offended Leibnitz, who shared Descartes's opinion on cosmogony and made a number of essential additions, and it added fuel to the dispute over the priority in discovering differential and integral calculus. The priority dispute is widely known (see, e.g., the fascinating book by Arnold [8]). In contrast, the vortex theory dispute has been almost forgotten today and is only briefly mentioned in books on the history of mechanics. On the Continent, however, Newton's works met strong opposition, which lasted for decades. Leibnitz was not alone in opposing it; there were also such outstanding scientists as Huygens, Varignon, J. and D. Bernoulli, etc. "The German and French scientists are furiously attacking Newton's philosophy and agree with Descartes's," Jones (a supporter of Newton's philosophy) wrotes to Cotes in 1711 (see [34]). Among all the arguments for Newton's theory, there was also a thesis proposing a freedom of opinion: "They [the Cartesians – V. K.] do have the right to differ, but they should be fair enough to others and let them wish the same freedom that they desire to be given. So, let us hold to the philosophy of *Newton*, which we consider to be more correct"* (a passage from the foreword written by Cotes for the second edition of the *Principia*).

For a better understanding of the dispute, we recall some general ideas in Descartes's theory. These ideas were stated in *Discours de la méthode* (1637) and in *Principia philosophiae* (1644). According to Descartes, the understanding of cosmology starts from acceptance of the initial *chaos*, whose moving elements are ordered according to certain fixed laws and form the Cosmos. (As we see, these ideas have very much in common with those in contemporary synergetics!) According to Descartes, the Universe is filled with a tenuous fluid matter (prototype of the *ether*), which is constantly in a vortex motion. This motion moves the largest particles of matter off the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his *Treatise on Light*, "the material of the Heaven must rotate the planets not only about the Sun but also about their own centers... and this will hence form several small Heavens rotating in the same direction as the great Heaven."* The term *vortex* (tourbillon) originated from a comparison with a river current swirling around objects carried by the water.

Huygens used a simple example to explain the main idea of the vortex theory: vortex motions of water in a bucket. Two identical bodies placed at different distances from the vortex axis rotate with different velocities. The closer a body is to the axis, the greater its velocity. This observation qualitatively corresponds to the law of the diminishing velocity of a planet in

*[Translated from the Russian translation.]

accordance with its increasing distance from the Sun. But does it correspond to Kepler's law?

Newton and Bernoulli

Newton himself considered this question in the *Principia* (Chap. 9). According to Newton, if a homogeneous viscous liquid is moved by a cylinder or sphere uniformly rotating about its axis, then in the stationary case, the rotation period of liquid particles is respectively proportional to their distance or squared distance from the axis of rotation. However, according to Kepler's third law, this should result in a semicubical function of distance. Newton concludes the *Precept* for his theorems with the following words: "I would like the philosophers to think of a condition by which it would be possible to explain a phenomenon based on the sesquitripilicate proportion by the vortex theory."*

In his work (1730) awarded a prize by the Paris Academy of Sciences, J. Bernoulli judged Newton's ideas to be flawed. The subject for the competition announced by the Academy was to explain the elliptical shape of planetary orbits. (This occurred 40 years after Newton published his book!) Incidentally, in this work, Bernoulli proposed the analytic method for obtaining the elliptical shape of planetary orbits by applying the gravitation law (which we can now find in mechanics textbooks). Newton used labored geometric expressions in imitation of ancient authors. According to Bernoulli, the dependence of the rotation period of a particle on distance also never corresponded to Kepler's law. Moreover, his conclusions turned out to be not quite correct. The first correct solution of this hydrodynamic problem was obtained by Stokes in 1845: when a cylinder or sphere rotates, the power of the distance is respectively equal to two or three.

In the competition papers of 1732 and 1734 on the reason for the inclination of planetary orbits to the Sun's equator, J. Bernoulli gradually moved away from the vortex theory. The ideas of D. Bernoulli (who shared the Paris Academy prize with his father) evolved similarly.

Newton and Bernoulli both studied the viscous liquid. Incidentally, if we consider Descartes's ether an ideal fluid, we can simply obtain the semicubical relation. For simplicity, we consider a plain-parallel flow and obtain conditions for uniform rotatory motion of flow particles. Let r be the distance from the axis of rotation, v be the particle velocity, ρ be the liquid density, and p be the pressure (v, ρ, and p are functions of r). It is easy to show that the continuity equation

$$\mathrm{div}(\rho v) = 0$$

*[Translated from the Russian translation.]

automatically holds, and two other dynamic equations reduce to the single relation

$$\rho \frac{v^2}{r} = p', \tag{1}$$

where the prime denotes the derivative with respect to r. This equation was actually presented by Huygens in his theory of centrifugal forces. According to Kepler's law, $v = cr^{1/2}$, $c = \text{const}$. Taking, for example, $\rho = \rho_0 = \text{const}$ (a homogeneous liquid), we obtain

$$p = p_0 - \frac{\rho_0 c^2}{r}, \qquad p_0 = \text{const},$$

from (1). However, at small values of r, the pressure is always negative; this does not correspond to the properties of real liquids in normal conditions. We must also note that the curl of the velocity field in rotatory motion is orthogonal to the plane of the flow and equals $[rv']/r$. According to Newton (with Stokes's clarifications), $v = C_1/r$, $C_1 = \text{const}$. Hence, the liquid in this case undergoes a vortex-free motion (with a multivalued potential). If we accept Kepler's law, we obtain the vortex flow (as it should be according to Descartes).

Newton gave a simpler, but stronger, argument against Descartes's theory. According to Newton, the motion of celestial bodies is described by second-order differential equations: to define a trajectory of a body, it is necessary to specify not only its position but also its velocity at a certain instant. If Descartes's theory is in fact correct, bodies are carried by the ether, and the equations of motion are consequently of first order: the velocity of a particle depends only on its position. However, Newton noted that some of the observed comets move in a direction opposite to that of all the planets.

Voltaire, Maupertuis, and Clairaut

It is necessary to say that far from all Continental scientists shared Descartes's ideas about the vortex theory. A number of famous French scientists (Pascal, Fermat, Roberval, etc.) accepted his ideas rather guardedly. The main role in promoting Newton's theory belonged not to scientists but to the writer and philosopher Voltaire. As we would say now, Voltaire was a dissident. His *Lettres philosophiques* were based on comparing and contrasting the situations in England and France. According to Voltaire, England is the homeland of human reason: everything is fine on the blessed island with the citizens enjoying their political freedom and freedom of thought. And all is bad back home in France. The comparison of the London Royal Society and the Paris Academy of Sciences, certainly, does not speak well for the latter. According to Voltaire, the Society is independent, free of charge, and involved in work, while the Academy is isolated from practical affairs and only publishes volumes of compliments. Voltaire compared Newton and Descartes in the same spirit. Newton is a wise man and modest besides, venturing to explain Nature, while Descartes is a dreamer and all his philosophy is a novel. "Arriving in London, a Frenchman finds everything different including philosophy. He left a full Universe and finds emptiness. In Paris, the Universe is viewed as consisting of ether vortices; here, in the same world space, mysterious forces direct the play. We think that pressure from the moon causes the tides, but the English think that the sea is attracted to the moon. In Paris, the Earth presents itself in the form of a melon; in London, it is flattened from two sides"* (Letter 14). Voltaire's ridicule in comparing the two systems was ostensibly directed equally to both sides. However, the reactions in Paris and in London were different: his book was banned in France but met with approval in England.

The Paris Academy of Sciences organized several expeditions to determine the length of arcs of meridian with the objective of clarifying the shape of the Earth. The expedition to Lapland (1735–1742) led by Maupertuis was successful: the Earth appeared flattened at the poles, as predicted by Newton's theory. Maupertuis was a Newtonian and was the one who explained the meaning of his theory to Voltaire. But this did not save Maupertuis from Voltaire's malicious gibes concerning the theological aspects of the variational principle of dynamics, which is named for Maupertuis today.

Gradually, primarily because of the works of Clairaut (incidentally, a participant in the Maupertuis expedition), Newton's theory of gravitation gained a wide acceptance. First was his *Theory of the Shape of the Earth* based on the law of gravitation. Second was Clairaut's prediction of the appearance of Halley's comet in 1759 based on the perturbation theory. It is worth recalling that the French translation of the *Principia* by the Marquise Emily du Chatlé published in 1759 in Paris was edited by Clairaut. The initiator of this edition was the same Voltaire.

According to Poincaré, each Truth has its instant of celebration between the eternity when it is considered untrue and the eternity when it is considered trivial. True, for Newton's theory of gravitation, this instant lasted the life of an entire generation. Moreover, our story does not end here.

Helmholtz and Thomson

The interest in the vortex theory revived in the middle of the 19th century because of the works of Helmholtz and Thomson (Lord Kelvin) on the vortex motion of an ideal fluid. It was proved that the circulation of velocity along a closed contour moving together with the fluid particles is constant, and, as a consequence, the law of the freezing-in of vortex lines was established. (We recall Descartes's ideas about the ether vortex transferring material bodies!)

The theory of vortex motion attracted even greater interest when Kelvin proposed his vortex theory of atoms ("On vortex atoms," *Phil. Mag.*, 1867).

*[Translated from the Russian translation.]

According to Kelvin, the Universe should be considered as a pure fluid (ether) containing separate, indissolubly linked Helmholtz vortices (atoms grouped into molecules). From this standpoint, gravitation should be explained statistically (in the spirit of Lesage's theory of 1764) as the impacts from a large number of small, rapidly moving vortices. Thomson proposed the beautiful name *ichthyodes* for them. As Klein wrote in his book *Development of Mathematics in the 19th Century*, "the theory did not go beyond the bounds of a remark, leading to nothing of substance, but it retains a known charm for the susceptible imagination."* In spite of this, Kelvin's theory stimulated a range of important research in studying the stability and fluctuations of various vortex structures.

The leading idea of Thomson's research program was the desire to find a mechanical model of complex physical phenomena in which action at a distance would be replaced with a direct contact (as in Descartes's theory). At that time, it was very popular to think that mechanics was a basis for physics. For example, Maxwell in his earlier works considered an electromagnetic model in which the induction currents of a magnet are caused by the medium rotating about magnetic force lines and there are small frictional balls between the rotating parts of the medium to avoid friction. Maxwell considered these balls the true state of electricity. Despite significant efforts, Maxwell could not move far in constructing adequate mechanical models of electromagnetism. Consequently, he accepted the now-usual idea of fields.

We also recall one more attempt to solve the problem of "action at a distance" using the theory of "latent motions." The main idea of this theory can be explained with the example of a rotating symmetric top. Because the rotation of the top about its axis of symmetry cannot be observed, we can suppose that the top does not rotate at all and explain its behavior by the influence of additional gyroscopic and potential forces. In the general case, this idea can be realized only within the framework of Routh's theory of the decrease of the order of systems with symmetries. We assume that a mechanical system with $n+1$ degrees of freedom moves by inertia and its Lagrangian, which equals the kinetic energy, has a one-parameter group of symmetries. If we decrease the order of the system by factoring with respect to orbits of this group, we see that the Routh function, which equals the Lagrangian of the reduced system with n degrees of freedom, contains a term independent of velocity. This term can be interpreted as a potential force affecting the reduced system. Helmholtz, W. Thomson (Lord Kelvin), J. J. Thomson, and Hertz insisted on the idea that all the mechanical quantities appearing as "potential energies" were in fact caused by latent "cyclic" motions. This concept of the kinetic theory was most fully detailed by Hertz in his book *The Principles of Mechanics, Presented in a New Form.* It turns out that a system with a compact configuration space can really be obtained from geodesic flows by using Routh's method. However, in the case of a noncompact space (the most

*[Translated from the Russian translation.]

interesting from the standpoint of the theory of gravitation), it is no longer so.

About the Book

In the present book, one more attempt is made to "rehabilitate" Descartes's vortex theory. Certainly, this book does not develop the theory of action at a distance in the spirit of Helmholtz and Thomson. Its main object is to systematize analogies between the usual mechanics of conservative systems and ideal fluid dynamics. It turns out that the family of phase trajectories composing an invariant manifold uniquely projected on the configuration space of a mechanical system admits a natural and convenient description in the terms of multidimensional hydrodynamics. On the other hand, in a number of problems, it is necessary to study not separate trajectories but families of them. For example, in geometric optics, the main object for constructing images is the *ray system*, not the separate light rays. If we also take the deep analogy between optics and mechanics opened by J. Bernoulli and developed by Hamilton into account, then the general theory of vortices stated in this book allows comprehending the basic results in mechanics, geometric optics, and hydrodynamics from a single standpoint. This theory reveals some general mathematical ideas that appeared in mechanics, optics, and hydrodynamics at different times under different names, and this gives a certain aesthetic pleasure. In addition, the general theory of vortices has interesting applications in numerical calculations, stability theory, and the theory of exact integration of dynamic equations.

Paraphrasing Newton, this book could be called

PHILOSOPHIAE CARTESIAN PRINCIPIA MATHEMATICA.

The author dedicates this book to the 400th birthday of Rene Descartes, a great scientist and human being, who, as Voltaire said, taught his contemporaries to reason.

Chapter 1
Hydrodynamics, Geometric Optics, and Classical Mechanics

§1. Vortex Motions of a Continuous Medium

1.1. In the investigation of the general properties of vortex lines, a significant role is played by the equation

$$\frac{\partial u}{\partial t} = \operatorname{rot}(u \times v), \tag{1.1}$$

where $v(x,t)$ is the velocity of the particles of a medium in the three-dimensional Euclidean space $E^3 = \{x\}$ and $u(x,t)$ is a solenoidal vector field, $\operatorname{div} u = 0$. The physical meaning of the field u is determined by the specific problem under investigation. Integral curves of the vector field u (at a fixed instant t) are called *vortex lines.*

For example, in magnetohydrodynamics, which deals with media of infinite conductivity, Eq. (1.1) describes the magnetic field strength; in this case, the vortex lines coincide with the magnetic field lines.

The barotropic flow of an ideal fluid in a potential force field is a more fundamental example. We recall that fluid motion is described by the *Euler equation*

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v\right) = -\frac{\partial p}{\partial x} + \rho F, \qquad F = -\frac{\partial V}{\partial x}, \tag{1.2}$$

where ρ is the fluid density, p is the pressure, F is the external mass force density, and V is the potential energy density. For a barotropic fluid, there exists a pressure function $P(x,t)$ such that

$$dP = \frac{dp}{\rho}.$$

In particular, a homogeneous fluid ($\rho = \text{const}$) is barotropic. To obtain a closed system of equations, we must add the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \tag{1.3}$$

to Euler equation (1.2); this equation expresses the mass conservation of the moving volume.

Under these assumptions, Eq. (1.2) can be transformed into the form of the Lamb equation

$$\frac{\partial v}{\partial t} + \left[\frac{\partial v}{\partial x} - \left(\frac{\partial v}{\partial x}\right)^{\mathrm{T}}\right] v = -\frac{\partial f}{\partial x}, \qquad f = \frac{v^2}{2} + P + V, \tag{1.4}$$

V. V. Kozlov, *Dynamical Systems X*

where the superscript T denotes transposition of the Jacobi matrix

$$\frac{\partial v}{\partial x} = \left\| \frac{\partial v^i}{\partial x^j} \right\|.$$

In hydrodynamics, the function f is usually called the *Bernoulli function.*

As is known, multiplying a skew-symmetric matrix $\partial v/\partial x - (\partial v/\partial x)^{\mathrm{T}}$ by a vector v in the three-dimensional Euclidean space, we obtain the vector cross product $(\operatorname{rot} v) \times v$; therefore, Eq. (1.4) can be rewritten in the equivalent form

$$\frac{\partial v}{\partial t} = v \times \operatorname{rot} v - \frac{\partial f}{\partial x}. \tag{1.5}$$

Taking the curl of both sides and using the relation $\operatorname{rot} \operatorname{grad} f = 0$, we obtain Eq. (1.1) with $u = \operatorname{rot} v$; the field of a curl is always solenoidal because $\operatorname{div} \operatorname{rot} = 0$. Vortex lines are integral curves of the field of the velocity curl (vortex); this explains the choice of the term in the general case.

The motion of fluid particles in E^3 is described by the differential equation

$$\dot{x} = v(x, t), \tag{1.6}$$

where the dot denotes differentiation with respect to t. Let $x(t, x_0)$ be its solution satisfying the initial condition $x(0, x_0) = x_0$. The family of mappings $E^3 \to E^3$ defined by the formula

$$x_0 \to x(t, x_0) \tag{1.7}$$

is called the *flow* of system (1.6). In the stationary case, where v is independent of t, the family of transformation (1.7) is a group. Transformations (1.7) are usually denoted by g_v^t (or simply g^t if this does not lead to confusion).

1.2. Let D be a measurable domain in E^3 and $g^t(D)$ be its image under transformation (1.7). By (1.3), the mass of the moving domain $g^t(D)$ is constant,

$$\int_{g^t(D)} \rho \, d^3x = \text{const}.$$

Now let Σ be a two-dimensional bounded surface and $\Gamma = \partial\Sigma$ be its boundary. In fluid mechanics, the following formula for the flow of a solenoidal field is well known:

$$\frac{d}{dt} \int_{g^t(\Sigma)} (u, n) \, d\sigma = \int_{g^t(\Sigma)} \left(\frac{\partial u}{\partial t} + \operatorname{rot}(v \times u), n \right) d\sigma, \tag{1.8}$$

where $(\cdot\,, \cdot)$ denotes the inner product in the Euclidean space E^3, n is a unit normal vector, and $d\sigma$ is the surface element of the surface Σ. Using (1.1), we obtain the conservation law for the flow of the field u through a moving surface from (1.8):

$$\int_{g^t(\Sigma)} (u, n)\, d\sigma = \text{const}\,. \tag{1.9}$$

This implies the *Helmholtz–Thomson theorem* on the freezing-in of vortex lines: the flow of system (1.5) transforms vortex lines into vortex lines. This result explains the appearance of magnetic storms on the Earth. The Sun is a sphere of turbulent plasma with almost infinite conductivity. From time to time, prominences appear on the Sun: matter is thrown to the surface of the Sun with enormous speed and then dissipates and moves away. By the Helmholtz–Thomson theorem, this matter transfers the magnetic field and creates magnetic storms upon reaching the Earth.

Now let Γ be a closed contour, the boundary of a bounded surface Σ. We consider the 1-form

$$(v, dx) = \sum v_i\, dx^i.$$

The integral of this form over Γ is called the *circulation of the velocity* along the contour Γ (Thomson). Applying the Stokes formula

$$\oint_\Gamma (v, dx) = \int_\Sigma (u, n) d\sigma, \qquad u = \operatorname{rot} v,$$

and taking (1.9) into account, we obtain the theorem on the constancy of the circulation of the velocity along the "fluid" contour:

$$\oint_{g^t(\Sigma)} (v, dx) = \text{const}\,. \tag{1.10}$$

The *Lagrange theorem* on potential flows is an important consequence of this result. We recall that the velocity field $v(x,t)$ is said to be a *potential* field if

$$v = \frac{\partial \varphi}{\partial x}. \tag{1.11}$$

The function $\varphi(x,t)$ is called the *potential.* The Lagrange theorem states that if the velocity field of a barotropic ideal fluid in a potential force field is a potential field at the initial instant (e.g., $t = 0$), then it is a potential field for all t.

Substituting (1.11) in Lamb equation (1.5) and using the obvious identity

$$\operatorname{rot} \frac{\partial \varphi}{\partial x} = 0,$$

we obtain the relation

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t} + f\right) = 0.$$

Therefore, the expression in parentheses is a function depending only on time:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + P + V = g(t). \tag{1.12}$$

This relation is called the *Lagrange–Cauchy integral.* Performing the gauge transformation

$$\varphi \to \varphi - \int g(t)\, dt,$$

which preserves the velocity field, we obtain Eq. (1.12) with $g = 0$.

1.3. We return to the investigation of Eqs. (1.1) and (1.3). We set $w = u/\rho$. It is clear that integral curves of the field w are exactly the vortex lines introduced above.

Theorem 1. *The field $w(x,t)$ satisfies the equation*

$$\frac{\partial w}{\partial t} = [v, w]. \tag{1.13}$$

The bracket $[\cdot\,,\,\cdot]$ is the commutator of vector fields. We recall its definition. Let $a = \{a_i\}$ and $b = \{b_i\}$ be two vector fields. The field $c = \{c_i\}$ defined as

$$c = \frac{\partial a}{\partial x} b - \frac{\partial b}{\partial x} a$$

with the components

$$c_j = \sum_i \left(b_i \frac{\partial a_j}{\partial x^i} - a_i \frac{\partial b_j}{\partial x_i} \right)$$

is called the *commutator* of these fields. If L_a, L_b, and L_c are the differentiation operators along the respective fields a, b, and c, then

$$L_c = L_b L_a - L_a L_b.$$

The expression in the right-hand side of this relation is the commutator of the operators L_a and L_b.

The fields a and b *commute* if $[a, b] = 0$. This property occurs if and only if the phase flows of the fields a and b commute, i.e.,

$$g_a^p g_b^q = g_b^q g_a^p$$

for all $p, q \in \mathbb{R}$.

Theorem 1 was initially obtained by Arnol'd [3] for the case of a homogeneous ideal fluid, where $\rho = \text{const}$; in this case, we can take $w = \operatorname{rot} v$. The general case is considered in [40]. If the medium motion is stationary, then the fields v and u/ρ commute (at first glance, the fields u and ρv, the momentum density, seem to commute). Equation (1.13) is usually called the *Euler* *equation for the moment variation.* It is an infinite-dimensional analogue of the Euler equations describing the rotation of a top (see [4]).

Proof of Theorem 1. We first calculate

$$\frac{\partial}{\partial t}\left(\frac{u}{\rho}\right) = \frac{1}{\rho}\operatorname{rot}(v \times u) + \frac{u}{\rho^2}\operatorname{div}(\rho v). \tag{1.14}$$

On the other hand,

$$\left[v, \frac{u}{\rho}\right] = \frac{1}{\rho}[v, u] + \frac{u}{\rho^2}L_v\rho. \tag{1.15}$$

Using the well-known identity

$$\operatorname{rot}(a \times b) = [a, b] + a\operatorname{div} b - b\operatorname{div} a$$

and the solenoidality of the field u, we transform (1.15) into the form

$$\left[v, \frac{u}{\rho}\right] = \frac{1}{\rho}\operatorname{rot}(v \times u) + \frac{u}{\rho}\operatorname{div} v + \frac{u}{\rho^2}L_v\rho. \tag{1.16}$$

Because $L_v\rho + \rho\operatorname{div} v = \operatorname{div}(\rho v)$, Eqs. (1.14) and (1.16) imply (1.13). □

We now consider stationary motion of a medium in the case where all the characteristics of motion are explicitly independent of time. First, we assume that $u \times v \neq 0$ (so-called vortex motions in the *strong sense*); then vortex lines differ from streamlines (or, equivalently, trajectories of fluid particles or integral curves of the field v). In this case, we can naturally assign the plane $\pi(x)$ generated by linear combinations of independent vectors $v(x)$ and $u(x)$ (or, equivalently, v and $w = u/\rho$) to each point x from the domain of the flow. By Theorem 1, the fields v and w commute, $[v, w] = 0$; therefore, by the *Frobenius theorem*, this *plane distribution* is *involutive* (or *integrable*), i.e., for each x, there exists a unique maximal integral surface M_x of this distribution such that the plane tangent to M_x at any point $z \in M_x$ coincides with $\pi(z)$. Clearly, streamlines and vortex lines lie on integral surfaces. In the general case, the surfaces M can be immersed in E^3 in a very complicated manner; generally speaking, they are not closed.

The integrability property of the plane distribution $\pi(x)$ can be obtained as follows. We consider the vector field $s = u \times v$; at each point $x \in E^3$, the nonzero vector $s(x)$ is orthogonal to $\pi(x)$. The following question arises: Is there a family of surfaces

$$f(x_1, x_2, x_3) = c, \quad c \in \mathbb{R},$$

orthogonal to the field s? The orthogonality condition becomes

$$s = g\frac{\partial f}{\partial x},$$

where $g \neq 0$ is a smooth function. Because

$$\operatorname{rot} s = \frac{\partial g}{\partial x} \times \frac{\partial f}{\partial x},$$

we obtain

$$(\operatorname{rot} s, s) = 0. \tag{1.17}$$

We see that the necessary condition for the integrability of the distribution $\pi(x)$ is exactly Eq. (1.17). It can be easily proved that it is also sufficient.

In the case under consideration, condition (1.17) is fulfilled because $\operatorname{rot} s = \operatorname{rot}(u \times v) = 0$ by virtue of (1.1).

The motion of medium particles on compact integral surfaces is the simplest example. Let M be a compact surface without boundary. Because the fields v and w are tangent to M, are linearly independent at each point, and commute, we see that M is a two-dimensional torus (more precisely, is diffeomorphic to a torus). In a system of angular coordinates $\varphi_1, \varphi_2 \bmod 2\pi$ on the torus, the differential equations of streamlines and vortex lines

$$\dot{x} = v(x) \qquad \text{and} \qquad x' = w(x)$$

become

$$\begin{aligned} \dot{\varphi}_1 &= \omega_1, \quad & \dot{\varphi}_2 &= \omega_2, \\ \varphi_2' &= \Omega_1, \quad & \varphi_2' &= \Omega_2, \end{aligned}$$

where $\omega, \Omega = \text{const}$. (See Chap. 10 in [5] for the proof of this differential-topological theorem; the general case of n-dimensional manifolds that admit n pairwise commuting tangent vector fields is studied in this book.)

Therefore, in the case of compact integral surfaces, streamlines are either closed (and hence the rotation periods of particles at different closed trajectories coincide) if the frequency ratio ω_1/ω_2 is rational or everywhere dense on M if ω_1/ω_2 is irrational.

Obviously, the equation

$$\frac{dx}{ds} = u(x) = \rho(x) w(x)$$

becomes

$$\frac{d\varphi_i}{ds} = \Omega_i \rho, \quad i = 1, 2,$$

in the coordinates φ_1, φ_2, and the integral lines of the vector field u (in magnetic hydrodynamics, the magnetic force lines) are also straight lines. Therefore, all these lines are either closed or everywhere dense on M. But in the case where $\rho \neq \text{const}$, the integral lines of the field u are closed at distinct intervals of the paremeter s, unlike streamlines.

If the tangent fields u and v are complete on M (i.e., their phase flows g_u^p and g_v^q are defined for all $p, q \in \mathbb{R}$), then in the noncompact case, M is

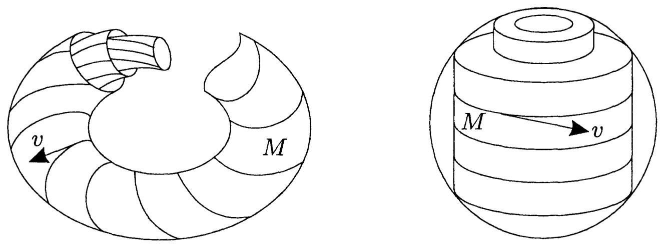

Fig. 2. Streamlines of vortex flows

diffeomorphic to a plane or to a cylinder, and in some coordinates on M, streamlines and vortex lines can be straight in the whole.

1.4. The most interesting application of this construction is to the problem of barotropic flows of an ideal fluid in a potential force field. By the *Bernoulli theorem*, the Bernoulli function f is constant on streamlines and vortex lines. Therefore, integral surfaces M coincide with the level surfaces of the *Bernoulli integral* $f = c$.

Theorem 2. *If the domain of the flow is compact and bounded by a regular analytic surface and the velocity field* v *is analytic and* $v \times \operatorname{rot} v \neq 0$*, then almost all connected Bernoulli surfaces*

$$M = \{x : f(x) = c\}$$

(*excluding, possibly, a finite number*) *are diffeomorphic either to a two-dimensional torus or to an annulus* (*i.e., direct product of an interval and a circle*). *On each of the tori, all streamlines are either closed or everywhere dense*; *on each of the annuli, all streamlines are closed. Rotation periods of fluid particles on different trajectories lying on the same Bernoulli surface coincide.*

This statement is proved for the case of an incompressible fluid in [3] and for the general case in [40].

Proof. Because $v \times \operatorname{rot} v \neq 0$, Lamb equation (1.5) implies that f is a nonconstant analytic function, $df \neq 0$. Therefore, all Bernoulli surfaces $f = c$ (excluding, possibly, a finite number) are regular; this follows from the isolatedness property of critical values of an analytic function defined on a compact set [72]. Obviously, we have $v \times \operatorname{rot} v \neq 0$ on regular Bernoulli surfaces. If the set $\{f = c\}$ has no common point with the boundary, then we can use the construction described above. The case where M intersects the boundary is simpler (see [3]). Streamlines for these two cases are shown in Fig. 2 taken from [3]. □

Now let $u \neq 0$ but $u \times v = 0$ (vortex motion in the *weak sense*). Then streamlines and vortex lines coincide, and $f = 0$. Streamlines can have a complicated structure, and the equations of motion $\dot{x} = v(x)$ can have no nonconstant integrals. As an example, we refer to Arnold's ABC-flow on the three-dimensional torus $\mathbb{T}^3 = \{x, y, z \bmod 2\pi\}$ with the Euclidean metric. The components of the velocity field v are

$$A \sin z + C \cos y, \qquad B \sin x + A \cos z, \qquad C \sin y + B \cos x. \tag{1.18}$$

Here, $\operatorname{rot} v = v$. In [79], it was shown that field (1.18) has no nonconstant analytic integrals for almost all A, B, and C. The numerical computations of Hannon demonstrate that some trajectories fill everywhere-dense three-dimensional domains on the torus $\mathbb{T}^3$; this indicates that the flow becomes chaotic.

§2. Point Vortices on the Plane

2.1. We consider a plane-parallel flow of a homogeneous ideal fluid. Let $a(x, y, t)$ and $b(x, y, t)$ be the components of the velocity v of fluid particles in the Cartesian coordinates x, y. The continuity equation $\operatorname{div} v = 0$ implies that for all t, the 1-form $a\,dy - b\,dx$ is the differential of some function $\Psi(x, y, t)$:

$$a = \frac{\partial \Psi}{\partial y}, \qquad b = -\frac{\partial \Psi}{\partial x}.$$

Hence, the equations of motion for the fluid particles

$$\dot{x} = \frac{\partial \Psi}{\partial y}, \qquad \dot{y} = -\frac{\partial \Psi}{\partial x} \tag{2.1}$$

have the *canonical form* of the *Hamilton differential equations.* The role of the Hamiltonian is played by the function Ψ. In the stationary case, Ψ is constant on streamlines and is therefore usually called the *stream function.*

In hydrodynamics, the flow with the stream function

$$\Psi = -\frac{\varkappa}{2\pi} \ln r, \qquad r^2 = (x - x_0)^2 + (y - y_0)^2, \tag{2.2}$$

plays a significant role. The velocity field in this case is

$$v = \frac{\varkappa}{2\pi}\left(-\frac{y - y_0}{r}, \frac{x - x_0}{r}\right). \tag{2.3}$$

We say that the flow generates the vortex of intensity $\varkappa$ placed at the point (x_0, y_0). Velocity field (2.3) is a potential field: it is the gradient of the multivalued function

$$\Phi = \frac{\varkappa}{2\pi} \arctan \frac{y - y_0}{x - x_0}. \tag{2.4}$$

Functions (2.2) and (2.4) are conjugate harmonic functions,

$$\Delta \Phi = 0, \qquad \Delta \Psi = 0,$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the *Laplace operator.* It is easy to see that the vortex intensity is equal to the velocity circulation

$$\oint a\,dx + b\,dy$$

along an arbitrary closed contour encircling the point (x_0, y_0) one time counterclockwise. Therefore, the vortex motion is concentrated at the singular point (x_0, y_0).

2.2. If n point vortices with the intensities $\varkappa_s$ and coordinates (x_s, y_s) are given, then the stream function

$$\Psi = -\frac{1}{2\pi}\sum_{s=1}^{n} \varkappa_s \ln r_s, \qquad r_s^2 = (x - x_s)^2 + (y - y_s)^2,$$

naturally appears. The motion of a fluid particle in the field of n point vortices is described by Eqs. (2.1). By the Helmholtz–Thomson theorem, vortices are "frozen" into the ideal fluid, and their intensities are constant in time. Therefore, the motion of the vortices themselves can be described by the system of differential equations

$$\begin{gathered} \dot{x}_s = \frac{\partial \Psi_s}{\partial y_s}, \qquad \dot{y}_s = -\frac{\partial \Psi_s}{\partial x_s}, \\ \Psi_s = -\frac{1}{2\pi}\sum_{k\neq s} \varkappa_k \ln r_{ks,}, \quad 1 \le s \le n, \end{gathered} \tag{2.5}$$

where r_{ks} is the distance between the vortices with the intensities $\varkappa_k$ and $\varkappa_s$.

Kirchhoff [36] proposed writing these equations in a more symmetric Hamiltonian form:

$$\begin{gathered} \varkappa_s \dot{x}_s = \frac{\partial H}{\partial y_s}, \qquad \varkappa_s \dot{y}_s = -\frac{\partial H}{\partial x_s}, \\ H = -\frac{1}{2\pi}\sum_{j\neq k} \varkappa_j \varkappa_k \ln r_{jk}, \quad 1 \le s \le n. \end{gathered} \tag{2.6}$$

These equations have the three integrals

$$P_x = \sum \varkappa_s x_s, \qquad P_y = \sum \varkappa_s y_s, \qquad M = \frac{1}{2}\sum \varkappa_s \left(x_s^2 + y_s^2\right)$$

(in addition to the Hamiltonian H). These functions have a clear mechanical meaning: H is the kinetic energy of the interacting vortices, P is the momentum of the plane-parallel flow of the unbounded fluid, and M is half the angular momentum relative to the coordinate origin (see [36]).

Kirchhoff equations (2.6) can be transformed into the form of the usual canonical Hamilton equations by setting

$$\xi_s = \sqrt{\pm\varkappa_s}\, x_s, \qquad \eta_s = \sqrt{\pm\varkappa_s}\, y_s, \quad s = 1, \ldots, n.$$

The plus sign is chosen for $\varkappa > 0$ and the minus sign in the opposite case. We let K denote the function H represented in the coordinates ξ and η. Then Eqs. (2.6) become

$$\dot{\xi}_s = \mp\frac{\partial K}{\partial \eta_s}, \qquad \dot{\eta}_s = \pm\frac{\partial K}{\partial \xi_s}, \quad 1 \le s \le n.$$

The *Poisson bracket*

$$\{f,g\} = \sum_s \frac{1}{\varkappa_s}\left(\frac{\partial f}{\partial y_s}\frac{\partial g}{\partial x_s} - \frac{\partial f}{\partial x_s}\frac{\partial g}{\partial y_s}\right)$$

has the standard form in the coordinates ξ, η. Obviously, the Hamiltonian H commutes with the functions P_x, P_y, and M:

$$\{H, P_x\} = \{H, P_y\} = \{H, M\} = 0.$$

The relations

$$\{P_x, P_y\} = -\sum \varkappa_k = \text{const}, \qquad \{P_x, M\} = -P_y, \qquad \{P_y, M\} = P_x \tag{2.7}$$

can be easily verified. Therefore, the momenta P_x and P_y commute if the sum of the vortex intensities is equal to zero.

2.3. In the case where intensities are positive, Kirchhoff equations (2.6) can be represented in the form of a *gradient* system. Let $(\,\cdot\,,\,\cdot\,)$ be a Riemannian metric on a manifold Σ and Φ be a function on Σ. The differential equations

$$\dot{x} = v(x), \quad x \in \Sigma,$$

are called *gradient* equations (or *evolution* equations) if

$$(v,\,\cdot\,) = d\Phi(\,\cdot\,). \tag{2.8}$$

Obviously, potential fields generate gradient equations. Gradient systems were investigated by Lyapunov in stability theory, by Smale from the standpoint of structural stability, and by Thom and his disciples in catastrophe theory. The function Φ is usually called the *potential function* or *potential.*

Similar to (2.4), we set

$$\Phi = \frac{1}{2\pi}\sum_{s \ne k} \varkappa_s \varkappa_k \varphi_{sk}, \qquad \varphi_{sk} = \arctan\frac{y_s - y_k}{x_s - x_k}$$

(see Fig. 3) and rewrite Kirchhoff equations (2.6) in gradient form (2.8):

$$\varkappa_s \dot{x}_s = \frac{\partial \Phi}{\partial x_s}, \qquad \varkappa_s \dot{y}_s = \frac{\partial \Phi}{\partial y_s}, \quad 1 \le s \le n. \tag{2.9}$$

The Riemannian metric in $\mathbb{R}^{2n} = \{x_1, \dots, x_n, \ y_1, \dots, y_n\}$ is defined by

$$\sum \varkappa_s \left(dx_s^2 + dy_s^2\right).$$

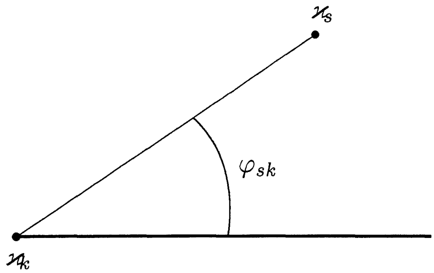

Fig. 3. Potential of the system of point vortices

Equation (2.8) implies

$$\dot{\Phi} = \left| \frac{\partial \Phi}{\partial x} \right|_*^2, \tag{2.10}$$

where $|\cdot|_*$ denotes the length of a covector in the dual space. Therefore, if Φ is a single-valued function, then $\Phi(x(t))$ tends to either $+\infty$ or some constant c as $t \to +\infty$ (if Σ is a compact manifold, then c is a critical value of the function Φ). Hence, as $t \to +\infty$, the solution $x(t)$ either tends to infinity or approaches the set of critical points of the function Φ.

In the case considered, the potential Φ is a multivalued function, and the result for the asymptotic behavior of solutions to system (2.9) is inapplicable. But the continuous branch of the function Φ either infinitely increases or monotonically tends to a constant as $t \to \infty$.

2.4. For $n = 2$, formula (2.10) becomes

$$\dot{\Phi} = \frac{\varkappa_1 \varkappa_2 (\varkappa_1 + \varkappa_2)}{4\pi^2 r_{12}^2}. \tag{2.11}$$

We study the motion of two point vortices in detail. The integral $H = \text{const}$ implies that the distance r_{12} between the vortices is constant. We now calculate the potential Φ using (2.11); it is equal (up to a multiplier) to the rotation angle of the line segment connecting the vortices. By (2.11), this segment rotates with the uniform angular velocity

$$\frac{\varkappa_1 + \varkappa_2}{2\pi r_{12}^2}. \tag{2.12}$$

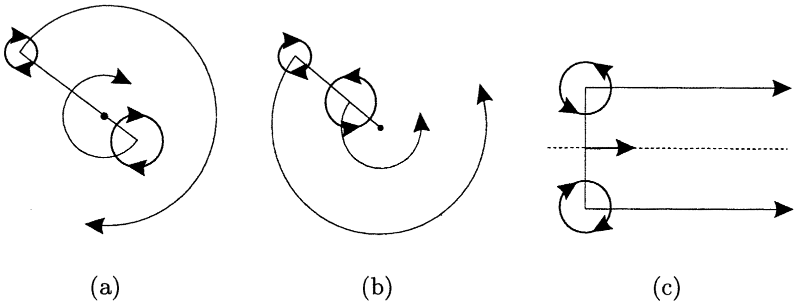

Fig. 4. Dynamics of two point vortices

We distinguish two cases. In the first case, the sum of the intensities $\varkappa_1 + \varkappa_2$ is nonzero. We introduce the *center of vorticity*, the point with the coordinates

$$\frac{\varkappa_1 x_1 + \varkappa_2 x_2}{\varkappa_1 + \varkappa_2} \quad \text{and} \quad \frac{\varkappa_1 y_1 + \varkappa_2 y_2}{\varkappa_1 + \varkappa_2}.$$

This point (an analogue of the center of mass) lies on the straight line connecting the interacting vortices and is fixed (because the momenta P_x and P_y are constant). In this case, therefore, the segment r_{12} uniformly rotates about the center of vorticity, and the trajectories of the vortices are concentric circles with the radii

$$\frac{\varkappa_2 r_{12}}{\varkappa_1 + \varkappa_2} \quad \text{and} \quad \frac{\varkappa_1 r_{12}}{\varkappa_1 + \varkappa_2}. \tag{2.13}$$

In the second case, where $\varkappa_1 + \varkappa_2 = 0$, the segment does not rotate but moves rectilinearly with the velocity $\varkappa/2\pi r$ ($\varkappa = \varkappa_1 = -\varkappa_2$) in the direction orthogonal to the segment. This value for the velocity can be obtained as follows. The velocity of the first vortex equals the product of radius (2.13) and angular velocity (2.12):

$$\frac{\varkappa_2 r_{12}}{\varkappa_1 + \varkappa_2} \frac{\varkappa_1 + \varkappa_2}{2\pi r_{12}^2} = \frac{\varkappa_2}{2\pi r_{12}}.$$

Letting $\varkappa_2$ approach $-\varkappa_1$, we then obtain the velocity of the translational motion of the vortex pair in the case of opposite intensities.

These results concerning the motion of two vortices were obtained by Helmholtz. We mention one more result in [39], which establishes the connection between the problem of two point vortices and the problem of two particles with an attractive force proportional to the inverse cube of the distance between them. The last problem was first studied by Newton in the *Principia* and then in more detail by Jacobi in his *Lectures on Dynamics.*

Theorem 3. *We consider the problem of the motion of two particles with the masses $\varkappa_1$ and $\varkappa_2$. If the potential energy of the attractive force is*

$$V = -\frac{\varkappa_1 \varkappa_2 (\varkappa_1 + \varkappa_2)}{8\pi^2 r_{12}^2}, \tag{2.14}$$

then all motions of the vortex pair with the intensities $\varkappa_1$ and $\varkappa_2$ are among all motions of these particles.

Proof. The particle motion is described by the Hamilton equations with the Hamiltonian

$$H = \frac{X_1^2 + Y_1^2}{2\varkappa_1} + \frac{X_2^2 + Y_2^2}{2\varkappa_2} + V,$$

where X_k and Y_k, $k = 1, 2$, are the canonical momenta conjugate to the coordinates x_k and y_k. This system has four degrees of freedom; its phase space is hence eight-dimensional. It is easy to verify that the Hamilton equations

$$\begin{aligned} \dot{x}_k &= \frac{\partial H}{\partial X_k} = \frac{X_k}{\varkappa_k}, & \dot{y}_k &= \frac{\partial H}{\partial Y_k} = \frac{Y_k}{\varkappa_k}, \\ \dot{X}_k &= -\frac{\partial H}{\partial x_k} = -\frac{\partial V}{\partial x_k}, & \dot{Y}_k &= -\frac{\partial H}{\partial y_k} = -\frac{\partial V}{\partial y_k}, \quad k = 1, 2, \end{aligned} \tag{2.15}$$

have a four-dimensional invariant manifold M^4 defined by the equations

$$\begin{aligned} X_1 &= -\frac{\varkappa_1 \varkappa_2}{2\pi} \frac{y_1 - y_2}{r_{12}^2}, & Y_1 &= \frac{\varkappa_1 \varkappa_2}{2\pi} \frac{x_1 - x_2}{r_{12}^2}, \\ X_2 &= -\frac{\varkappa_1 \varkappa_2}{2\pi} \frac{y_1 - y_2}{r_{12}^2}, & Y_2 &= \frac{\varkappa_1 \varkappa_2}{2\pi} \frac{x_1 - x_2}{r_{12}^2}. \end{aligned} \tag{2.16}$$

This means that if the trajectory of system (2.15) belongs to M^4 at the initial instant, then the whole trajectory also lies on M^4.

Taking the relations $X_k = \varkappa_k \dot{x}_k$ and $Y_k = \varkappa_k \dot{y}_k$ into account, we conclude that Eqs. (2.16) coincide with Kirchhoff equations (2.6) for $n = 2$. □

If potential energy (2.14) decreases proportionally to the inverse distance between the points, then the motion of the vortex pair satisfies the third Kepler law. It is not yet clear whether Theorem 3 can be generalized to the case of three or more vortices.

2.5. Unlike the problem of three gravitating bodies, the problem of three vortices is integrable. Trajectories of vortices were studied in detail by Grebly in 1877 (see [56] for the modern exposition). Unfortunately, the problem of four vortices is not integrable in general, as was shown by Ziglin [80]; the behavior of phase trajectories is not regular but chaotic. In the particular case where $\sum \varkappa_s = 0$ and $P_x = P_y = 0$, the equations of motion of four point vortices can be integrated in quadratures. The method of integration is described in [46]; a detailed analysis for some important cases can be found in [56].

The stationary motion of n point vortices in the case where the distances between vortices are constant (i.e., where motion of the vortex system, like that of a rigid body, is either translational or rotatory about the center of vorticity with a uniform angular velocity) is very interesting. Unfortunately, this algebraic problem is very difficult even in the case where all the vortices have the same intensities. In 1883, Thomson studied the particular case where the vortices are placed at the vertices of a regular polygon and stated that such stationary rotation is stable for $n \leq 6$ and unstable for $n > 7$. In [19], the author proved that stable stationary rotations exist for all n and found all stable equilibrium configurations for $n \leq 50$ using a computer. It turned out that the vortices are placed on concentric circles ("atomic shells" in Thomson's terminology). In [2, 74], immobile stable configurations of n vortices for $n = m^2$ with integer m were found. Unfortunately, this problem is not solved completely. There are important examples (from the application standpoint) of the stationary motion of an infinite number of point vortices (e.g., the Karman chains; see Sec. 156 in [51]).

Kirchhoff equations (2.6) yield an important example of a Hamiltonian system of differential equations that appear not as a result of the Legendre transformation in classical dynamics.

2.6. We emphasize that Hamilton equations (2.1) are a consequence of the incompressibility condition only. Therefore, they are valid for planar flows of a homogeneous viscous liquid; in this case, the motion is governed by the *Navier–Stokes equation*

$$\frac{\partial v}{\partial t} = v \times \operatorname{rot} v - \frac{\partial f}{\partial x} + \nu \Delta v, \tag{2.17}$$

where $\nu > 0$ is the coefficient of viscosity and Δ is the Laplace operator. We consider the particular case of plane-parallel flow where the fluid particles move uniformly in the plane x, y along concentric circles centered at the point $(0,0)$. Clearly, the vector $\operatorname{rot} v$ has the components $(0,0,\zeta)$, and the field $v \times \operatorname{rot} v$ is a potential field. Taking the curl of both sides of (2.17), we obtain the equation for the z-component ζ:

$$\frac{\partial \zeta}{\partial t} = \nu \Delta \zeta, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.18}$$

The same equation describes heat propagation in the plane.

We suppose that initially (for $t = 0$), we have the point vortex of intensity $\varkappa$ at the point $x = y = 0$. The analogy with the heat propagation law for an instantaneous heat point source suggests that the solution of Eq. (2.18) is

$$\zeta = \frac{\varkappa}{4\pi \nu t} \exp\left(-\frac{r^2}{4\nu t}\right), \qquad r^2 = x^2 + y^2 \tag{2.19}$$

(this can be easily verified by direct calculation). Because

$$\iint_{-\infty}^{+\infty} \zeta \, dx \, dy = \varkappa,$$

the total intensity of the vortex is constant. As $t \to +0$, the "density" ζ of the vortex is concentrated at the origin and is a generalized Dirac function. The value of the velocity for solution (2.19) is

$$|v| = \frac{\varkappa}{2\pi r}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right). \tag{2.20}$$

As t increases from zero to infinity, this value decreases from $\varkappa/2\pi r$ to zero. For fixed $r > 0$, the value ζ of the vortex density increases from 0 to its maximum and then asymptotically tends to zero. Therefore, we observe *diffusion* of the vortex.

By the Green formula, the circulation of the velocity along a circle of radius r is

$$2\pi \int_0^r \zeta r \, dr = \varkappa\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right).$$

In contrast to the case of an ideal fluid, it depends on time in this case and, of course, tends to $\varkappa$ as $t \to +0$ and decreases to zero as $t \to \infty$. We note that flows of a viscous fluid have no integral invariants of types (1.9) and (1.10) in the general case (see [44] for the proof) because of the chaotic behavior of particle trajectories for $\nu > 0$.

The stream function Ψ for solution (2.19) depends on r and t and is the antiderivative of function (2.20) (for fixed t).

§3. Systems of Rays, Laws of Reflection and Refraction, and the Malus Theorem

3.1. From the standpoint of geometric optics, propagation of light in the three-dimensional Euclidean space $E^3 = \{x\}$ can be represented as a flow of particles. Trajectories of particles are called *rays*.

It is well known (and is intuitively clear) that in a homogeneous and isotropic transparent medium, light propagates along straight lines with a constant velocity. This property does not hold if the medium is inhomogeneous (i.e., the speed of light depends on the point $x \in E^3$) or anisotropic (i.e., the speed of light depends on the direction of the ray). The optical properties of a medium are defined by the *index of refraction*, which is equal to the inverse speed of light. In the general case, the index of refraction is a function of the point x and the direction of the velocity v:

$$n = f\left(x, \frac{v}{|v|}\right). \tag{3.1}$$

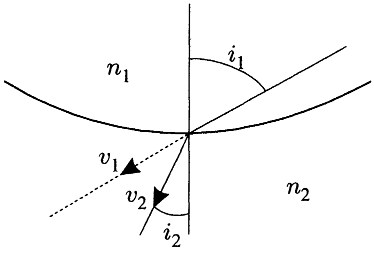

Fig. 5. On the refraction law

The bending of light rays due to variation of the index of refraction is called *refraction.* The simplest case of refraction is the refraction of light at the interface of two homogeneous optical media with different indices of refraction n_1 and n_2 (see Fig. 5). In this case, the following two experimental laws discovered by Descartes hold:

1. the incident and refracted rays lie in a plane that is normal to the interface;
2. if i_1 is the angle of incidence and i_2 is the angle of refraction, then

$$n_1 \sin i_1 = n_2 \sin i_2.$$

In the English literature, the law of refraction is usually associated with the Dutch scientist Snell, who discovered this law before Descartes but did not publish it. This situation is connected with Huygens's assertion that Descartes during his stay in Holland could have seen Snell's manuscript. "However," as Aragot writes, "Huygens does not claim that Descartes *in fact* saw it; consequently, taking away the honor of discovering the law perpetrates an unheard-of scientific predation upon French geometry."

Descartes's laws can be formulated more briefly: the vector

$$n_1 \frac{v_1}{|v_1|} - n_2 \frac{v_2}{|v_2|} = n_1^2 v_1 - n_2^2 v_2, \tag{3.2}$$

where v_1 and v_2 are the light velocity vectors in media with the indices of refraction n_1 and n_2, is orthogonal to the interface at the refraction point. The reflection law (which was possibly known to ancient authors) has the same form with $n_1 = n_2$.

3.2. In constructing optical images, the essential role is played by not an individual light ray but a system of rays. A system of rays is a family of rays such that for any point x (from some domain of E^3), there exists a unique ray passing through x and the direction of a ray smoothly depends on x. Therefore, the system of rays is unambiguously related to the field $v(x)$ of velocities of light particles. Obviously, in a homogeneous isotropic medium, $|v(x)| = \text{const}$, and the integral lines of the field $v(x)$ are straight lines.

Proposition 1. *The vector field v of an arbitrary system of rays in a homogeneous optical medium satisfies the relation*

$$v \times \operatorname{rot} v = 0. \tag{3.3}$$

Proof. Without loss of generality, it is sufficient to prove (3.3) only for $x = 0$. We suppose that the axis x_1 coincides with the light ray, i.e., $v(0) = (c, 0, 0)$, $c \neq 0$. We expand the components of the field v in Taylor series in x_1, x_2, and x_3 up to the second order:

$$\begin{aligned} v_1 &= c + a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots, \\ v_2 &= b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots, \\ v_3 &= c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots . \end{aligned} \tag{3.4}$$

Because the vector $v(0)$ is directed along the axis x_1, we have $v_2 = v_3 = 0$; therefore, $b_1 = c_1 = 0$. Further,

$$|v|^2 = c^2 + 2c\,(a_1 x_1 + a_2 x_2 + a_3 x_3) + \ldots .$$

Because $|v| = \text{const}$, we have $a_1 = a_2 = a_3 = 0$. We now see that field (3.4) has the form

$$v_1 = c + \cdots, \qquad v_2 = b_2 x_2 + b_3 x_3 + \ldots, \qquad v_3 = c_2 x_2 + c_3 x_3 + \ldots .$$

Its curl at the point $x = 0$ has the components

$$c_2 - b_3, \quad 0, \quad 0.$$

Therefore, the vectors $\operatorname{rot} v$ and v are collinear. □

Relation (3.3) seems to appear initially in the paper by Sommerfeld and Runge in 1911 [71].

We now obtain the conditions for the orthogonality of the system of rays to a surface family in E^3. This problem was already considered in Sec. 1. Condition (1.17) is a required criterion:

$$(v, \operatorname{rot} v) = 0. \tag{3.5}$$

Comparing (3.4) and (3.5), we obtain $\operatorname{rot} v = 0$; therefore,

$$v = \frac{\partial \varphi}{\partial x}, \qquad \varphi : E^3 \to \mathbb{R}. \tag{3.6}$$

Such systems of rays are called *Hamilton systems.* Hamilton called the function $W = \varphi/c^2$ the *characteristic function*, and Bruns called it the *eikonal.* This function satisfies the *Hamilton partial differential equation* or the *eikonal equation*

$$\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2 + \left(\frac{\partial W}{\partial x_3}\right)^2 = n^2, \tag{3.7}$$

where $n = 1/c$ is the index of refraction. The rate of change of the function W along the light ray is

$$\frac{dW}{dt} = \sum \frac{\partial W}{\partial x_i} v_i = \sum \frac{\partial W}{\partial x_i} \frac{\partial \varphi}{\partial x_i} = 1;$$

therefore, the time for movement of light particles from the surface $W(x) = s_1$ to the surface $W(x) = s_2$ equals $s_2 - s_1$. This is why the function W is also called the *optical path.*

Systems of rays such that $\operatorname{rot} v \neq 0$ are called *Kummer systems.* Kummer investigated such systems using purely algebraic methods unrelated to optical problems [50].

A simple example of a Kummer system that is not a Hamilton system is

$$v_1 = c \sin x_3, \qquad v_2 = c \cos x_3, \qquad v_3 = 0.$$

We have $\operatorname{rot} v = v$; obviously, integral lines of the field v are straight lines parallel to the coordinate plane x_1x_2.

Plane-parallel systems of rays (where the components of the velocity field v are independent of x_3 and $v_3 = 0$) are Hamilton systems (the curl of the velocity is orthogonal to the plane x_1x_2). In particular, any system of rays in the plane is a Hamilton system.

Proposition 1 states the obvious analogy between optical systems of rays and stationary flows of an ideal fluid (in the particular case where the Bernoulli function is constant). Therefore, systems of rays have invariants (1.9) and (1.10); these invariants allow proving important theorems in geometric optics. Hamilton systems of rays correspond to potential flows, the eikonal (multiplied by the squared index of refraction) corresponds to the velocity potential, and the eikonal equation corresponds to the Lagrange–Cauchy integral. Kummer systems correspond to vortex flows in the weak sense (e.g., the *ABC*-flows of Arnol'd).

3.3. As an example of using the *optical–hydrodynamic analogy*, we prove the *Malus theorem*, which is the foundation of analytic optics. We note that this theorem was the starting point of Hamilton's voluminous investigations.

Theorem 4 (Malus). *If a system of rays is orthogonal to a regular surface, then it is a Hamilton system and remains a Hamilton system after an arbitrary number of reflections and refractions.*

This theorem is contained in Malus's treatise on geometric optics submitted to the Paris Academy in 1807. In addition to Laplace, Monge, and Lacroix, Lagrange was a member of the review committee for this treatise. Did he recognize a close relative of his own theorem on the conservation of the potential energy of an ideal fluid?

Proof of the Malus theorem. We consider two optical media 1 and 2 with the respective indices of refraction n_1 and n_2, separated by a regular surface Λ

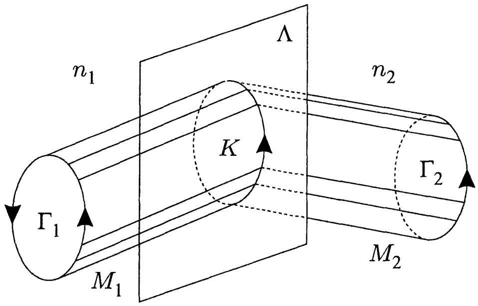

Fig. 6. On the proof of the Malus theorem

(see Fig. 6). Let $u(x) = n^2 v(x)$ be the piecewise smooth vector field defined by $n_1^2 v_1$ and $n_2^2 v_2$ in media 1 and 2, and let Γ_1 be a closed contour in medium 1.

Let M_1 be a tube made by light rays passing through points of the contour Γ_1 that intersects the interface transversally in a closed curve K. Having refracted, these rays form a tube M_2 in medium 2. Let Γ_2 be a closed contour on the tube M_2 homological to the contour K.

First, we prove that

$$\oint_{\Gamma_1} (n_1^2 v_1, dx) = \oint_{\Gamma_2} (n_2^2 v_2, dx) \,. \tag{3.8}$$

If g^t is the phase flow of the field u, then (3.8) implies an analogue of formula (1.10):

$$\int_{g^t(\Gamma)} (u, dx) = \text{const} \,.$$

Because the indices of refraction n_1 and n_2 are constant, Proposition 1 yields

$$u \times \operatorname{rot} u = 0. \tag{3.9}$$

Therefore, the field $\operatorname{rot} u$ is tangent to the surface M_1, and its flow through this surface vanishes. By the Stokes formula, we have

$$\oint_{\Gamma_1} (u, dx) = \oint_K (u, dx) = \oint_K (n_1^2 v_1, dx). \tag{3.10}$$

Similarly,

$$\oint_{\Gamma_2} (u, dx) = \oint_K (u, dx) = \oint_K (n_2^2 v_2, dx). \tag{3.11}$$

The contour K lies on the interface Λ, and by Descartes's law, the vector field $n_1^2 v_1 - n_2^2 v_2$ is orthogonal to Λ. Therefore,

$$\oint_K (n_1^2 v_1 - n_2^2 v_2, dx) = 0,$$

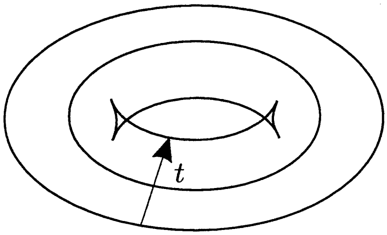

Fig. 7. Appearance of singularities

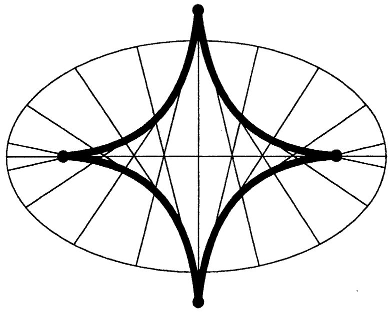

Fig. 8. Caustic of an ellipse

and integrals (3.10) and (3.11) are hence equal to each other. This proves Eq. (3.8).

Now let light rays intersect a regular surface Σ orthogonally. We can assume that the contour Γ_1 lies on Σ. Then the integral in the left-hand side of (3.8) vanishes because the vectors $v_1(x)$ are orthogonal to Γ_1. Therefore, the integral in the right-hand side of (3.8) vanishes for an arbitrary closed contour Γ_2. This implies that the field $v(x)$ is a potential field. $\square$

3.4. Let Σ be a smooth regular surface and v be a vector field of a Hamilton system of rays. We set

$$\Sigma_t = g_v^t(\Sigma), \qquad \Sigma_0 = \Sigma.$$

By the Malus theorem, the surfaces Σ_t are orthogonal to light rays; the eikonal takes constant values on these surfaces.

It turns out that the surfaces Σ_t may have singularities for $t > 0$. This effect is shown in Fig. 7 for the case of an ellipse on the plane; the field v is directed inside the ellipse. The appearance of singularities is connected with the presence of caustics, i.e., enveloping curves for systems of light rays (the Greek word "caustic" means "burning"—concentration of light occurs at the corresponding points). The caustic of the ellipse is shown in Fig. 8; this curve is the astroid. In general, caustics of the surface Σ are also surfaces; singularities of the orthogonal surfaces Σ_t lie exactly on caustics of Σ.

Caustics divide the space into domains, which are filled by light rays with different multiplicities: the same number of rays pass through each point of such a domain. This is why the eikonal has singularities on caustics; it becomes a multivalued function.

A caustic can be described constructively as follows. Let x be a point of the surface Σ. A plane passing through the normal to the surface Σ at the point x interects Σ in a plane curve. Of course, the center of curvature of this curve at the point x lies on the normal. If we rotate the intersecting plane about the normal, then the intersection curve changes. Therefore, the position of

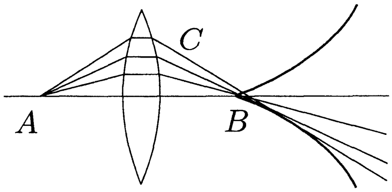

Fig. 9. Caustic on a lens axis

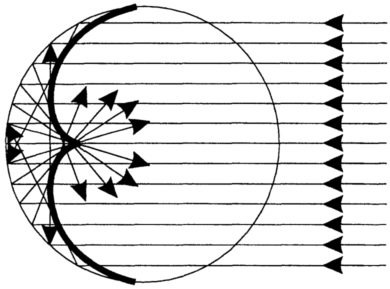

Fig. 10. Caustic in a coffee cup

the center of curvature and the radius of curvature also change. After a half-revolution, the radius of curvature passes its maximum and minimum values. The planes corresponding to these extremals of the curvature intersect the surface Σ at the principal curvature curves, which intersect orthogonally at the point x. The centers of curvature of the principal curves are called the *focal points* of the surface Σ. The infinitesimally close normals to Σ passing through the principal curvature curves intersect each other at these two points. Clearly, the focal points depend on the point $x \in \Sigma$. The set of all focal points coincides with the caustic of Σ. Caustics are also called the *focal surfaces.*

Huygens investigated caustics of plane curves in the 1650s in connection with his theory of evolvents. Some problems for singularities of families of orthogonal plane curves were considered in l'Hospital's manual on the analysis of infinitesimals (1700). In 1852, Cayley investigated the caustic of the three-axis ellipsoid. At the present time, the theory of singularities of Hamilton systems of rays is highly advanced and is a constituent of catastrophe theory (see [7, 9]). As far as the author knows, singularities of Kummer systems of rays have not yet been studied within the framework of catastrophe theory. We note that in practice, caustics usually appear after reflection or refraction of Hamilton systems of rays (see Figs. 9 and 10).

§4. Fermat Principle, Canonical Hamilton Equations, and the Optical–Mechanical Analogy

4.1. In his book *Synthesis for Refraction* (1662), Fermat showed that Descartes's refraction laws can be deduced from a variational principle: light propagates along the path with the minimum duration. In Fig. 11, the refracted ray ABC is longer than the straight segment ADB, but the time of motion for light particles moving along ACB is less than the time of motion for ADB. "Nature acts in the *simplest* and *most available* ways. We assume that this idea must be expressed exactly in these words, not as 'Nature always acts by *shortest* lines.'"

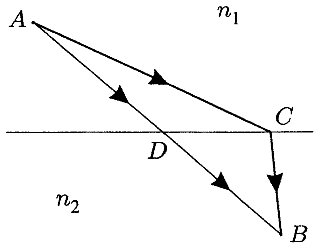

Fig. 11. On the Fermat principle

The reflection law for light rays also conforms with the *Fermat principle.* Therefore, this principle can naturally serve as the base of geometric optics. However, it must be formulated as the stationarity (not minimality) of the light-propagation time because there exists more than one light ray passing through points that lie "after" caustics (see Fig. 9).

Light-propagation time is expressed by the formula

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} L(\dot{x}, x)dt, \qquad L = |\dot{x}| f\left(x, \frac{\dot{x}}{|\dot{x}|}\right), \tag{4.1}$$

where $\dot{x} = v$ is the velocity of light particles and f is refraction index (3.1). The integral

$$\int_{t_1}^{t_2} L(\dot{x}, x)dt \tag{4.2}$$

is the optical length of the path $x : [t_1, t_2] \to E^3$. It is an analogue of the *action functional* (in the sense of Hamilton) in classical mechanics. To emphasize this analogy, we call the integrand in (4.1) the *Lagrange function* or *Lagrangian.* Integral (4.2) is called the *action functional* in the sense of Fermat (the Fermat action). Lagrangian (4.1) possesses the homogeneity property

$$L(\lambda \dot{x}, x) = \lambda L(\dot{x}, x)$$

for all $\lambda > 0$; such Lagrangians are called *parametric Lagrangians*: the value of Fermat action (4.2) depends only on the trajectory of the light particle and the direction of motion and is independent of the parameterization of the trajectory (i.e., of the velocity). If s is a new parameter ($\dot{s} > 0$), then integral (4.2) equals

$$\int_{s_1}^{s_2} L(x', x)\, ds, \qquad (\cdot)' = \frac{d}{ds}(\cdot).$$

Taking this notation into account, we can represent the Fermat principle in the form

$$\delta \int_{s_1}^{s_2} L\, ds = 0, \tag{4.3}$$

where δ is the variation symbol introduced by Lagrange. We now formulate precise definitions. Let $x(s)$, $s_1 \leq s \leq s_2$, be a smooth path. The *variation* of this path is the smooth family of paths

$$x_\alpha(s) = x(s, \alpha), \quad \alpha \in (-\varepsilon, \varepsilon),$$

satisfying the two conditions

1. $x_0(s) = x(s)$ for all $s_1 \leq s \leq s_2$, and
2. $x_\alpha(s_1) = x(s_1)$ and $x_\alpha(s_2) = x(s_2)$ for all α.

The Fermat action along paths $x_\alpha(s)$ is a function of α. The variation δ is the derivative with respect to the parameter α for $\alpha = 0$. Because s_1 and s_2 are independent of α (condition 2), we have

$$\delta \int_{s_1}^{s_2} L\, ds = \int_{s_1}^{s_2} \left(\frac{\partial L}{\partial x'}\, \delta x' + \frac{\partial L}{\partial x}\, \delta x \right) ds. \tag{4.4}$$

Differentiations with respect to t and α commute for smooth functions; therefore, $\delta x' = (\delta x)'$, and we can integrate by parts:

$$\delta \int_{s_1}^{s_2} L\, ds = \frac{\partial L}{\partial x'}\, \delta x \Big|_{s_1}^{s_2} - \int_{s_1}^{s_2} \left[\left(\frac{\partial L}{\partial x'} \right)' - \frac{\partial L}{\partial x} \right] \delta x\, ds.$$

Because $\delta x = 0$ for $s = s_1$ and $s = s_2$, Fermat principle (4.3) becomes

$$\int_{s_1}^{s_2} \left[\left(\frac{\partial L}{\partial x'} \right)' - \frac{\partial L}{\partial x} \right]_{x(s)} \delta x\, ds = 0 \tag{4.5}$$

for all variations $\delta x(s)$ of the path $x(s)$. It is easy to prove (see any textbook in the calculus of variations) that the stationarity criterion for the action functional at the path $x(s)$ is

$$\left(\frac{\partial L}{\partial x'} \right)' - \frac{\partial L}{\partial x} = 0. \tag{4.6}$$

This equation is called the *Euler–Lagrange equation.* In the case where the Lagrangian L is parametric, the Euler–Lagrange equation has the family of solutions $x(\lambda s)$, $\lambda > 0$, in addition to the solution $x(s)$. The light path parameterized by the time t satisfies the equation

$$\left(\frac{\partial L}{\partial \dot{x}} \right)^{\bullet} - \frac{\partial L}{\partial x} = 0. \tag{4.7}$$

In mechanics, this equation is called the *Lagrange equation.* By the definition of the refraction index, the function $x(t)$ satisfies the relation $L = 1$.

Therefore, a smooth path $x(t)$ describes the motion of light particles if the function $x(t)$ satisfies Eq. (4.7) and the relation $L = 1$. We consider the general case where $x(t)$ is a piecewise-smooth curve transversal to interfaces of optical media. Discontinuities of the velocity $\dot{x}(t)$ correspond to instants of refraction or reflection of the light ray. The function $x(t)$ satisfies Lagrange equation (4.7) (and, of course, the relation $L = 1$) between these discontinuities, and at the discontinuity $t = \tau$, the velocities $\dot{x}(\tau - 0)$ and $\dot{x}(\tau + 0)$ satisfy the refraction law or the reflection law. It can be proved that the Fermat action takes a stationary value if and only if the path satisfies the conditions above.

4.2. The Lagrange equations in the explicit form contain second derivatives of x with respect to t and are therefore second-order equations. It is useful to introduce the six-dimensional phase space and pass to first-order equations. The Lagrange equations can be represented in the form of the Hamilton equations by using the *Legendre transformation.* We recall the definition and main properties of this transformation.

We consider the Lagrangian $L(v, x)$ as a function of the velocity $v = \dot{x}$ taking x as the parameter (in optical problems, the function L is formally undefined for $v = 0$). We set

$$y = \frac{\partial L}{\partial v}. \tag{4.8}$$

This is the momentum of the light particle. If the Hess matrix consisting of second partial derivatives

$$\frac{\partial^2 L}{\partial v^2} = \left\| \frac{\partial^2 L}{\partial v_i \partial v_j} \right\| \tag{4.9}$$

is nondegenerate, then by the implicit function theorem, we can express v (locally at least) as a function of y for fixed x. It can be proved that v is the gradient of some function of y (similar to (4.8)). Indeed, we set

$$H(x, y) = y \cdot v - L(v, x)\Big|_{v(y,x)} \tag{4.10}$$

and calculate the gradient of H with respect to y:

$$\frac{\partial H}{\partial y} = v + y\frac{\partial v}{\partial y} - \frac{\partial L}{\partial v}\frac{\partial v}{\partial y} = v + \left(y - \frac{\partial L}{\partial v}\right)\frac{\partial v}{\partial y} = v. \tag{4.11}$$

The correspondence $L \to H$ is called the Legendre transformation; it is involutive, i.e., its square is the identity transformation (this follows from the duality of (4.8) and (4.11) and the symmetry of relation (4.10) under the permutation $H \leftrightarrow L$).

It turns out that

$$\frac{\partial L}{\partial x} + \frac{\partial H}{\partial x} = 0, \tag{4.12}$$

where the momenta y and velocities v are related by (4.8) or (4.11). For the proof, we differentiate (4.10) with respect to x:

$$\frac{\partial H}{\partial x} = y\frac{\partial v}{\partial x} - \frac{\partial L}{\partial x} - \frac{\partial L}{\partial v}\frac{\partial v}{\partial x} = -\frac{\partial L}{\partial x} + \left(y - \frac{\partial L}{\partial v}\right)\frac{\partial v}{\partial x} = -\frac{\partial L}{\partial x}.$$

Theorem 5 (Poisson and Hamilton). *The function $x(t)$ is a solution of Lagrange equation* (4.7) *if and only if the functions*

$$x = x(t) \qquad \text{and} \qquad y = \left.\frac{\partial L}{\partial \dot{x}}\right|_{x(t)}$$

satisfy the equations

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}. \tag{4.13}$$

Differential equations (4.13) are called the *canonical Hamilton equations*, and the function H is called the *Hamiltonian.* The space of conjugate variables x and y is called the *phase space* (due to W. Gibbs).

The first equation in system (4.13) coincides with Eq. (4.11) and is a direct consequence of the Legendre transformation (Poisson in fact obtained this equation). The second equation in (4.13) is implied by Lagrange equation (4.7) after the substitution $\partial L/\partial\dot{x} \to y$; we take relation (4.12) into account. If we use relation (4.11) and the Legendre transformation $H \to L$, then Hamilton equations (4.13) become the Lagrange equation.

4.3. Unfortunately, Theorem 5 is inapplicable in geometric optics because Lagrangian (4.1) is parametric and Hess matrix (4.9) is therefore degenerate. Indeed, by the Euler formula for homogeneous functions, we have

$$\frac{\partial L}{\partial v} \cdot v = L,$$

and the components of the momentum $\partial L/\partial v$ are homogeneous in v with degree 0. Applying the Euler formula to the function $\partial L/\partial v$ at the point $v \neq 0$, we obtain

$$\frac{\partial^2 L}{\partial v^2} v = 0; \tag{4.14}$$

therefore,

$$\det\left|\frac{\partial^2 L}{\partial v^2}\right| \equiv 0.$$

We can overcome this difficulty as follows. We consider the closed set of points $v \in \mathbb{R}^3$ satisfying the equation $L(v,x) = 1$. This surface is called the *indicatrix* at the point $x \in E^3$. The indicatrix can be regarded as the set of velocities of light particles emitted from the point x in all directions. For example, if the optical medium is isotropic with the refraction index $n(x)$,

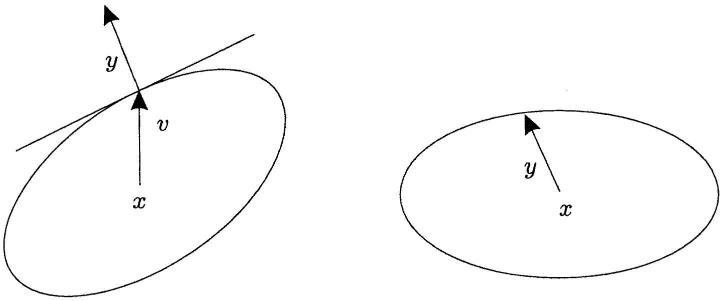

Fig. 12. Indicatrix and figuratrix

then the indicatrix at the point x is the sphere of radius $1/n(x)$. Below, we consider the case where indicatrices are smooth convex surfaces (this case is important in optics).

We introduce the *figuratrix* as the set of momenta $y \in \mathbb{R}^3$ defined analytically by the relations

$$y = \frac{\partial L}{\partial v}, \qquad L(v, x) = 1. \tag{4.15}$$

It can be shown that the figuratrix is also a convex surface.

There exists a unique smooth function $H(x, y)$ positively homogeneous in y (i.e., $H(x, \lambda y) = \lambda H(x, y)$ for all $\lambda > 0$) and such that $H \equiv 1$ for all y belonging to the figuratrix; the function H is linear on each half-line passing through the point $x \in E^3$ and equals 1 at the point of intersection of this half-line and the figuratrix. The transformation

$$v = \frac{\partial H}{\partial y}, \qquad H(x, y) = 1$$

converts the figuratrix into the initial indicatrix. The functions L and H are thus dual to each other, similarly to the duality between the indicatrix and the figuratrix. But this duality is not the Legendre duality. The theory of parametric Hamiltonians is based on the transformations of poles and polars relative the unit sphere (this transformation is well known in geometry).

Theorem 6. *The relations*

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial y} \tag{4.16}$$

hold along light rays.

Proof. We pass to the nonparametric case and apply the Legendre transformation. We introduce the new Lagrangian $L^* = L^2/2$ (it is homogeneous in $\dot{x}$ with degree 2) and show that the Hess matrix $\|\partial^2 L^*/\partial v^2\|$ is positive definite. We fix the point x and consider a regular curve $v = v(s)$, $v' \neq 0$, lying on the indicatrix $L^*(v(s), x) \equiv 1/2$. Differentiating this identity with respect to s, we obtain

$$\frac{\partial L^*}{\partial v} \cdot v' = 0, \qquad \frac{\partial^2 L^*}{\partial v^2} v' \cdot v' = -\frac{\partial L^*}{\partial v} \cdot v''. \tag{4.17}$$

Because $v' \neq 0$ is a tangent vector and the indicatrix is a convex surface, the acceleration v'' is directed inside the indicatrix. Hence,

$$\frac{\partial L^*}{\partial v} \cdot v'' < 0,$$

and by (4.17),

$$\frac{\partial^2 L^*}{\partial v^2} v' \cdot v' > 0. \tag{4.18}$$

Therefore, this quadratic form is positive definite on the tangent plane at each point of the indicatrix.

The Euler formula for a homogeneous function

$$\frac{\partial L^*}{\partial v} \cdot v = 2L^*$$

implies that the derivatives $\partial L^*/\partial v_k$ are first-degree functions homogeneous in v. Applying the Euler formula again, we obtain

$$\frac{\partial^2 L^*}{\partial v^2} v = \frac{\partial L^*}{\partial v},$$

and hence

$$\frac{\partial^2 L^*}{\partial v^2} v \cdot v = \frac{\partial L^*}{\partial v} \cdot v = 2L^* > 0.$$

Therefore, inequality (4.18) is valid for vectors v transversal to the indicatrix. We have proved that the quadratic form in (4.18) is positive definite in the entire $\mathbb{R}^3$ and hence

$$\det \left\| \frac{\partial^2 L^*}{\partial v^2} \right\| > 0.$$

By virtue of the relations

$$\delta \int_{t_1}^{t_2} L^* \, dt = \int_{t_1}^{t_2} L \, \delta L \, dt \qquad \text{and} \qquad L = 1,$$

the function $x(t)$ is an extremal of the variational problem

$$\delta \int_{t_1}^{t_2} L^* \, dt = 0.$$

We introduce the canonical momentum

$$y = \frac{\partial L^*}{\partial \dot{x}} = L \frac{\partial L}{\partial \dot{x}}.$$

On the indicatrix $L = 1$, we have relation (4.15). Because L^* is a second-degree function homogeneous in $\dot{x}$, we see that

$$H^*(x, y) = \frac{\partial L^*}{\partial \dot{x}} \cdot x - L^* = L(\dot{x}, x).$$

We set $H^* = H^2/2$. Obviously, $H \equiv 1$ on the curve $x = x(t)$, $y = y(t)$ describing the motion of light particles. Therefore, H is the function introduced above dual to the parametric Lagrangian.

Applying Theorem 5, we now obtain Eqs. (4.16):

$$\dot{x} = \frac{\partial H^*}{\partial \dot{y}} = H\frac{\partial H}{\partial y} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H^*}{\partial \dot{x}} = -H\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial x}.$$

The theorem is proved. □

Equations (4.16) were obtained by Hamilton in the paper "On systems of rays" submitted to the Irish Academy of Sciences in 1824. Later (1834), Hamilton generalized this result to the dynamics of conservative systems.

4.4. In the particular case of an isotropic medium, $L = |\dot{x}|n(x)$ and $H = |y|/n(x)$.

Theorem 7. *Light rays in an isotropic optical medium $E^3 = \{x\}$ with the refraction index $n(x)$ coincide with the trajectories of a material point in a potential field with the force function $n^2/2$.*

Proof. We write the explicit form of canonical equations (4.16) with the Hamiltonian $H = |y|/n$ at the level $H = 1$:

$$\dot{x} = \frac{y}{|y|n} = \frac{y}{n^2}, \qquad \dot{y} = \frac{|y|}{n^2}\frac{\partial n}{\partial x} = \frac{1}{n}\frac{\partial n}{\partial x}.$$

Introducing the new time τ as $d\tau = n^2 dt$ and letting the prime denote differentiation with respect to τ, we have

$$x' = y, \qquad y' = n\frac{\partial n}{\partial x}$$

and hence

$$x'' = \frac{\partial U}{\partial x}, \qquad U = \frac{n^2}{2}.$$

These equations describe the motion of a material point of unit mass in the potential field with the force function U. The theorem is proved. □

Theorem 7 constitutes the *optical–mechanical analogy*, which was initially established by J. Bernoulli in 1696.

We apply this result to the problem of light rays in an atmosphere where the refraction index weakly depends on the altitude z:

$$n(z) = n_0(1 + \varepsilon z). \tag{4.19}$$

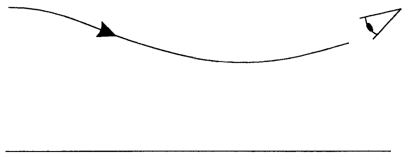

Fig. 13. Mirage

Clearly, the speed of light decreases as the air density increases; therefore, $\varepsilon < 0$. For small ε, we can assume that

$$\frac{n^2(z)}{2} = \frac{n_0^2}{2} - gz, \qquad g = n_0^2\varepsilon.$$

This force function describes the free fall of a massive particle in free space. Therefore, light rays are downward curved parabolas. In particular, the Sun near the horizon seems 1/2 degree higher than its actual position. One more curious phenomenon of the same kind is a mirage: a traveller in the desert seems to see blue water. Monge first explained this phenomenon as follows. Near the heated surface of the Earth, the air is more tenuous, and this leads to the case $\varepsilon > 0$ in formula (4.19). The light rays are upward curved parabolas; the traveller actually sees not a water surface but the blue sky.

Monge first encountered the mirage phenomenon during Napoleon's Egyptian campaign (he participated in it). Monge explained the mirage in a small note published in the *Journal of the Egyptian Institute* founded by Napoleon. As Arago said, "After publication of this note, even simple soldiers ceased to be surprised by a mirage, which had sometimes horrified and perplexed them."

§5. Hamiltonian Form of the Equations of Motion

5.1. First, we recall the main principles of *Lagrangian mechanics*. Positions of a mechanical system are in one-to-one correspondence with points of a *configuration space*, i.e., a smooth manifold M^n. The number $n = \dim M$ is called the *number of degrees of freedom* of the mechanical system. Local coordinates $(x_1, \ldots, x_n) = x$ on M^n are usually called *generalized* or *Lagrangian coordinates*.

As an example, we consider a rigid body in E^3 rotating about a fixed point. This system has three degrees of freedom. Positions of the rigid body can be uniquely assigned to rotations of the space E^3, i.e., orthogonal rotations, which transform fixed orthogonal axes XYZ into a moving frame xyz attached to the body. These rotations are determined by orthogonal 3×3 matrices with the determinant $+1$ (orthogonal matrices with the determinant -1 correspond to transformations that change orientation). The set of all these matrices is the group $SO(3)$ (special orthogonal group of rotations of E^3), which serves

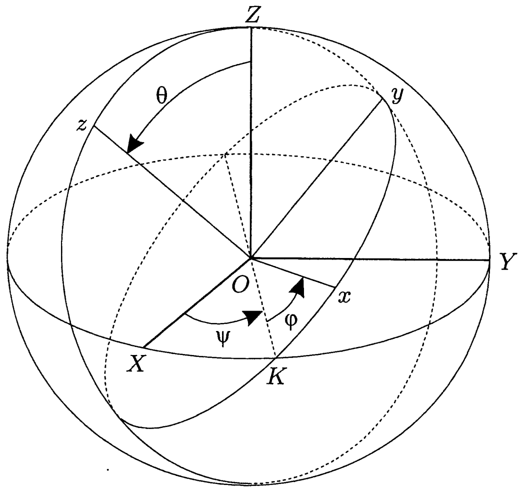

Fig. 14. Euler angles

as the configuration space for the top in the three-dimensional space. It can be proved that the group $SO(3)$ considered as a three-dimensional manifold is isomorphic to the three-dimensional sphere with identified antipodal points. The *Euler angles* θ, φ, ψ (see Fig. 14) are usually taken as local coordinates on $SO(3)$. They are well defined if $\theta \in (0, \pi)$.

Let $t \to x(t)$ be a motion of the system. The set of derivatives of local coordinates with respect to time $(\dot{x}_1, \dots, \dot{x}_n) = \dot{x}$ is called the velocity of the system. The velocities are transformed contravariantly under changes of local coordinates: if $(x'_1, \dots, x'_n) = x'$ are new coordinates, then

$$\dot{x}'_i = \sum_{j=1}^{n} \frac{\partial x'_i}{\partial x_j} \dot{x}_j \tag{5.1}$$

or, briefly,

$$\dot{x}' = J\dot{x},$$

where $J = \|\partial x'_i / \partial x_j\|$ is the Jacobi matrix of the transformation $x \to \dot{x}$. Therefore, velocities are *tangent vectors* to M. In general, tangent vectors are (one time) *contravariant tensors.* Such a tensor is defined at the point $x_0 \in M$ by the set of numbers $v = (v_1, \dots, v_n)$ (components of the tensor) assigned to each coordinate system $(x_1, \dots, x_n)$ in some neighborhood of x_0 and the transformation law that under the change of variables $(x_1, \dots, x_n) \to (x'_1, \dots, x'_n)$, the components of the tensor are transformed as

$$v'_i = \sum_{j=1}^{n} \frac{\partial x'_i}{\partial x_j} v_j.$$

Partial derivatives are calculated only at the point x_0.

The set of all tangent vectors at the point x_0 is an n-dimensional vector space, which is called the *tangent space* and is denoted by $T_{x_0}M$. The union of all tangent spaces

$$\bigcup_{x\in M} T_x M = TM$$

is called the space of the *tangent bundle* of M and plays a significant role in mechanics. This space has the structure of a $2n$-dimensional smooth manifold; sets of coordinates $x_1, \ldots, x_n$ and its derivatives $\dot{x}_1, \ldots, \dot{x}_n$ are local coordinates on TM. A pair $(x, \dot{x})$ is called a *state* of the system, and the space TM itself is called the *state space.*

Inertial properties of the mechanical system are determined by the *kinetic energy*

$$T = \frac{1}{2} \sum_{i,j=1} a_{ij}(x)\dot{x}_i\dot{x}_j. \tag{5.2}$$

This is a smooth function on TM, which is a positive-definite quadratic form on each of the tangent spaces T_xM. From the geometric standpoint, T defines a *Riemannian metric* on M. If the system moves inertially (in the absence of external forces), then its trajectories are geodesic curves of metric (5.2).

A force $F = (F_1, \ldots, F_n)$ affecting the system is a geometric object of another type called a *covector* or (one time) *covariant tensor.* By definition, this is the set of numbers $(u_1, \ldots, u_n) = u$ assigned to each local coordinate system; under a change of coordinates, the components of the covector are transformed as

$$u_i' = \sum_{j=1}^{n} \frac{\partial x_j}{\partial x_i'} u_j \tag{5.3}$$

or, briefly,

$$u' = (J^{\mathrm{T}})^{-1}u.$$

We consider an important example of a covector field. Let $f : M \to \mathbb{R}$ be a smooth function. We set

$$u_i = \frac{\partial f}{\partial x_i}, \quad 1 \le i \le n. \tag{5.4}$$

It can be easily verified that this is a smooth *covector field* on M called the *gradient* of the function f. We emphasize that the gradient of the function is a covector, not a vector. We can assign a tangent vector to the gradient only if a Riemannian metric on M is given.

A covector $u = (u_1, \dots, u_n)$ and a vector $v = (v_1, \dots, v_n)$ can be multiplied by the rule

$$u \cdot v = \sum_{i=1}^{n} u_i v_i. \tag{5.5}$$

By virtue of (5.1) and (5.3), this product is an invariant (i.e., is independent of the choice of local coordinates). For example, if the covector u has form (5.4), then product (5.5) is the derivative of the function f in the direction of the vector field v.

Motions $t \to x(t)$ of the mechanical system with the kinetic energy T under the effect of the force F are solutions of the *Lagrange differential equations*

$$\left(\frac{\partial T}{\partial \dot{x}}\right)^{\bullet} - \frac{\partial T}{\partial x_i} = F_i, \quad 1 \le i \le n. \tag{5.6}$$

The components F_i of the force F, in general, depend on x, $\dot{x}$, and t. It can be proved that the expression in the left-hand side of (5.6) is transformed covariantly under changes of local coordinates. Therefore, expressions in both sides of (5.6) are covectors; this property is usually called the *covariance property* of the Lagrange equations.

A significant role in applications is played by *potential forces*

$$F_i = -\frac{\partial V}{\partial x_i}, \quad 1 \le i \le n,$$

where V is a smooth function on M called the *potential energy.* In this case, Eqs. (5.6) can be rewritten in a more elegant form:

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\bullet} - \frac{\partial L}{\partial x} = 0, \qquad L = T - V. \tag{5.7}$$

The function L is called the *Lagrange function* or *Lagrangian.*

We now see that motions of the mechanical system in a potential field coincide with extremals of the variational problem

$$\delta \int_{t_1}^{t_2} L \, dt = 0, \qquad \delta x(t_1) = \delta x(t_2) = 0.$$

"It is astonishing that in Lagrange's work this statement may be read only between the lines. This explains the strange situation that this relation—mainly through Jacobi's influence—is generally known in Germany, and therefore also in France, as *Hamilton's principle.* In England no one understands this expression; there this equation is known under the correct if intuitive name of principle of stationary action" (F. Klein [37]).

5.2. The invariance property of the Hamilton action with respect to transformation groups of the configuration space is closely connected with *conservation laws*, i.e., integrals of the Lagrange equations. Let $v(x)$ be a vector field on M. We can assign the differential equation

$$\frac{dx}{d\alpha} = v(x), \quad x \in M,$$

to this field; its phase flow is denoted by g_v^α. Let $t \to x(t)$ be a motion of the mechanical system, i.e., a solution of Eq. (5.7). We set

$$x_\alpha(t) = g_v^\alpha(x(t))$$

and calculate the Hamilton action on these curves:

$$I[x_\alpha(t)] = \int_{t_1}^{t_2} L\left(\dot{x}_\alpha(t), x_\alpha(t)\right) dt.$$

Letting δ denote the derivative with respect to α at $\alpha = 0$ and using (4.3), we obtain

$$\delta \int_{t_1}^{t_2} L\, dt = \frac{\partial L}{\partial \dot{x}} \cdot v \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial \dot{x}} \right)^{\cdot} - \frac{\partial L}{\partial x} \right] \cdot v\, dt. \tag{5.8}$$

The second term in the right-hand side vanishes because the function $x_\alpha(t)$ is the solution of the Lagrange equations for $\alpha = 0$. We assume that the Hamilton action is *invariant* with respect to the group g_v^α; then the left-hand side of (5.8) also vanishes, and values of the function

$$f(\dot{x}, x) = \frac{\partial L}{\partial \dot{x}} \cdot v \tag{5.9}$$

are hence constant on each solution of (5.7). We have obtained the famous *Noether theorem* (1918). We note that the derivatives $\partial L/\partial \dot{x}_i$ are transformed covariantly; therefore, $\partial L/\partial \dot{x}$ is a covector called the *momentum* of the mechanical system.

The Noether theorem also holds for nonautonomous systems where the Lagrangian explicitly depends on time. In this case, we introduce the extended configuration space

$$\widetilde{M} = M \times \mathbb{R}_t,$$

where $\mathbb{R}_t$ is the time axis. Curves $t \to x(t)$ can be parameterized by a new parameter τ, $t = t(\tau)$, $t' > 0$ (the prime denotes the derivative with respect to τ). The Hamilton action becomes

$$\int_{t_1}^{t_2} L(\dot{x}, x, t) dt = \int_{\tau_1}^{\tau_2} \widetilde{L}(x', x, t', t) d\tau, \qquad \widetilde{L} = t' L\left(\frac{x'}{t'}, x, t\right).$$

The Lagrangian $\widetilde{L}$ is parametric and looks like Lagrangian (4.1) in geometric optics.

Let $\widetilde{v}$ be a vector field on $\widetilde{M}$ generating the system of equations

$$\frac{dx}{d\alpha} = v(x,t), \qquad \frac{dt}{d\alpha} = u(x,t),$$

where v is a field on M depending on t. If the phase flow of this system leaves the Hamilton action invariant, then Lagrange equations (5.7) have the integral

$$\frac{\partial \widetilde{L}}{\partial x'} \cdot v + \frac{\partial \widetilde{L}}{\partial t'} \cdot u. \tag{5.10}$$

By virtue of the relations

$$\frac{\partial \widetilde{L}}{\partial x'} = \frac{\partial L}{\partial \dot{x}}, \qquad \frac{\partial \widetilde{L}}{\partial t'} = L - \frac{\partial L}{\partial \dot{x}} \cdot \dot{x},$$

integral (5.10) becomes

$$\frac{\partial L}{\partial \dot{x}} \cdot v - \left(\frac{\partial L}{\partial \dot{x}} \cdot \dot{x} - L\right) u. \tag{5.11}$$

We consider the particular case where the Lagrangian L is independent of t. Obviously, we can take $v = 0$ and $u = 1$ as the symmetry field. The Noether integral becomes

$$L - \frac{\partial L}{\partial \dot{x}} \cdot \dot{x}.$$

If $L = T - V$, then this function is the total energy $H = T + V$ with the opposite sign.

We note that the Noether theorem traces to earlier observations of Lagrange and Jacobi on the relation between classical integrals for systems of interacting particles and the invariance properties of the dynamic equations with respect to the Galileo transformation group.

5.3. We now convert n second-order Lagrange equations into $2n$ first-order Hamilton equations. As in Sec. 4, we introduce canonical momenta $(y_1, \dots, y_n) = y$ by setting

$$y_i = \frac{\partial L}{\partial \dot{x}_i}, \quad 1 \le i \le n. \tag{5.12}$$

These covector quantities are elements of the space T_x^*M dual to the tangent space T_xM. For the "natural" Lagrangian $L = T - V$, formula (5.12) becomes

$$y_i = \sum_{j=1}^{n} a_{ij} \dot{x}_j$$

and defines the isomorphism of the vector spaces T_xM and T_x^*M. For which Lagrangians $L(\dot{x}, x, t)$ is mapping (5.12) $T_xM \to T_x^*M$ surjective? The following two conditions are sufficient:

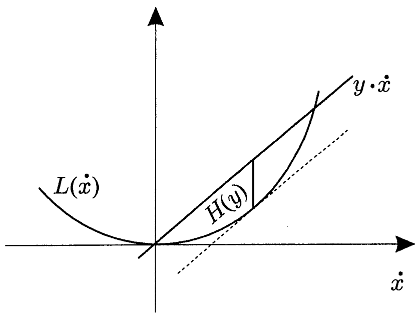

Fig. 15. Legendre transformation

1. the symmetric matrix

$$\left\| \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \right\|$$

   is positive definite;
2. the Lagrangian considered as a function of velocity increases at infinity faster than a linear function:

$$\frac{L(\dot{x})}{|\dot{x}|} \to +\infty \quad \text{as} \quad |\dot{x}| \to \infty.$$

Here, $|\cdot|$ is a norm on $T_x M$. Indeed, in this case, the graph of $L(\dot{x})$ is a convex surface, and the function

$$y \cdot \dot{x} - L(\dot{x})$$

has exactly one critical point for each y (see Fig. 15). This point is just the solution of Eq. (5.12).

We can therefore introduce the Hamilton function (Hamiltonian)

$$H(x, y, t) = y \cdot \dot{x} - L(\dot{x}, x, t)\Big|_{\dot{x} \to y}. \tag{5.13}$$

For fixed t, the Hamiltonian is a smooth function on the $2n$-dimensional manifold that is the union of cotangent spaces

$$T^* M = \bigcup_{x \in M} T_x^* M.$$

Formula (5.13) can be written as

$$H = \frac{\partial L}{\partial \dot{x}} \cdot \dot{x} - L\Big|_{\dot{x} \to y}.$$

If $L = T - V$, then the Hamiltonian coincides with the total mechanical energy: $H = T + V$.

By the Poisson–Hamilton theorem (see Sec. 4), if the function $t \to x(t)$ is a solution of the Lagrange equations, then the functions

$$x(t) \qquad \text{and} \qquad y(t) = \left.\frac{\partial L}{\partial \dot{x}}\right|_{x(t)}$$

satisfy the canonical Hamilton equations

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \qquad \dot{y}_k = -\frac{\partial H}{\partial x_k}, \quad 1 \le k \le n. \tag{5.14}$$

The coordinates x_k and y_k are the *conjugate canonical coordinates*, and the space of the cotangent bundle $P = T^*M$ is the *phase space.*

5.4. In Hamiltonian mechanics, an important role is played by the differential 1-form

$$\varphi = y\,dx - H\,dt = \sum y_i\,dx_i - H\,dt \tag{5.15}$$

called the *energy–momentum form* (the term was suggested by Cartan [20]). For example, Noether integral (5.11) equals the value of the form φ on the symmetry field $\widetilde{v}$ of the space–time:

$$\varphi(\widetilde{v}) = y \cdot v - Hu = \frac{\partial L}{\partial \dot{x}} \cdot v - \left(\frac{\partial L}{\partial \dot{x}} \cdot x - L\right) u.$$

In the general case, where the Hamiltonian depends on time, we consider the *extended phase space* $\widetilde{P} = P \times \mathbb{R}_t$, $\dim \widetilde{P} = 2n + 1$. Energy–momentum form (5.15) is a 1-form on $\widetilde{P}$. We consider its exterior differential

$$\phi = d\varphi = -\left[\sum dx_i \wedge dy_i + \sum \frac{\partial H}{\partial x_i} dx_i \wedge dt + \sum \frac{\partial H}{\partial y_i} dy_i \wedge dt\right]. \tag{5.16}$$

This is an exterior 2-form defined on pairs of vectors tangent to the manifold $\widetilde{P}$. Let

$$\xi = (x_\xi, y_\xi, t_\xi)^{\mathrm{T}} \qquad \text{and} \qquad \eta = (x_\eta, y_\eta, t_\eta)^{\mathrm{T}}$$

be two vectors at the point $z = (x, y, t) \in \widetilde{P}$. Then the value $\phi(\xi, \eta)$ can be represented in the explicit form

$$-A\xi \cdot \eta,$$

where

$$A = \begin{pmatrix} 0 & -E_n & -H_x' \\ E_n & 0 & -H_y' \\ H_x' & H_y' & 0 \end{pmatrix} \tag{5.17}$$

is a skew-symmetric matrix of order $2n + 1$ and E_n is the $n \times n$ unit matrix.

Any 2-form ϕ in an odd-dimensional space is degenerate: for an arbitrary point $z \in \widetilde{P}$, there exists a nonzero vector $\xi \in T_z\widetilde{P}$ such that $\phi(\xi, \eta) = 0$ for all $\eta \in T_z\widetilde{P}$. The vector ξ is called the *vortex vector*. Clearly, vortex vectors are eigenvectors with the eigenvalue zero.

To explain the term "vortex vector," we discuss the following simple example. Let $a = (u, v, w)$ be a smooth vector field in the three-dimensional oriented Euclidean space $E^3 = \{x, y, z\}$. We assign the 1-form

$$\varphi = u\,dx + v\,dy + w\,dz$$

to the field a and consider its exterior differential

$$\begin{aligned}\phi = d\varphi = &\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) dy \wedge dz \\ &+ \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) dz \wedge dx + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \wedge dy. \qquad (5.18)\end{aligned}$$

Let ξ and η be two vectors at the point (x, y, z). By the definition of the exterior product, the value of the form $dx \wedge dy$ at the vectors ξ and η equals the oriented area of the parallelogram spanned by the projections of these vectors on the plane of the variables x and y, i.e., equals the projection of the vector cross product $\xi \times \eta$ on the z axis. Formula (5.18) now becomes

$$\phi(\xi, \eta) = (\xi \times \eta, \operatorname{rot} a) = (\operatorname{rot} a \times \xi, \eta).$$

There exists a linear skew-symmetric operator A such that $(\operatorname{rot} a) \times \xi = A\xi$. Therefore,

$$\phi(\xi, \eta) = (A\xi, \eta). \qquad (5.19)$$

The matrix of this operator (in the Cartesian coordinates x, y, z) is

$$\begin{pmatrix} 0 & \dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} & 0 & \dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z} & \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} & 0 \end{pmatrix}.$$

By virtue of (5.19), the vector $\operatorname{rot} a$ is the vortex vector for the 2-form ϕ. This is the motivation for the terminology in the higher-dimensional case.

The form ϕ is said to be *nonsingular* if the vortex vector $\xi(z)$ is unique (up to a constant nonzero multiplier) for all $z \in \widetilde{P}$. For example, the condition for the nonsingularity of 2-form (5.17) is $\operatorname{rot} a \neq 0$. Because the rank of matrix (5.17) equals $2n$, we see that form (5.16) is nonsingular.

Let $\xi(z)$ be a nonzero smooth vector field on $\widetilde{P}$ consisting of vortex vectors of the form ϕ. Integral curves of this field (obviously, they are independent of the lengths of the vectors ξ) are called *vortex lines*. The form of matrix (5.17) implies that if $\xi = (x_\xi, y_\xi, t_\xi)$ is a vortex vector, then $t_\xi \neq 0$. Vortex lines can be found by integrating the system of differential equations

$$\frac{dz}{ds} = \xi(z), \quad z \in \widetilde{P}.$$

Let $x = x(s)$, $y = y(s)$, $t = t(s)$ be a solution of this system. Because $dt/ds \neq 0$, vortex lines can be parameterized by the time t:

$$x = x(t), \qquad y = y(t). \tag{5.20}$$

Theorem 8. *Functions* (5.20) *satisfy the canonical Hamilton equations with the Hamiltonian H.*

Proof. The vector with the components

$$\frac{\partial H}{\partial y}, \quad -\frac{\partial H}{\partial x}, \quad 1$$

is a vortex vector. □

We emphasize that it is not necessary to take time as a parameter on vortex lines; if

$$\left|\frac{\partial H}{\partial x}\right| + \left|\frac{\partial H}{\partial y}\right| \neq 0,$$

then we choose one of coordinates $x_1, \ldots, x_n$ or $y_1, \ldots, y_n$ as a parameter.

Theorem 8 states the *vortex principle* of Hamiltonian mechanics. This result is contained in Cartan's *Integral Invariants.* Arzhanykh [11] generalized the vortex principle to the cases of systems with nonpotential forces and nonholonomic systems.

§6. Action in the Phase Space and the Poincaré–Cartan Invariant

6.1. Let $\widetilde{P} = P \times \mathbb{R}_t$ be an extended $(2n+1)$-dimensional phase space and $H(x, y, t)$ be a Hamiltonian. We can integrate energy–momentum form (5.15)

$$\varphi = y \cdot dx - H\,dt$$

over curves $\gamma : [0,1] \to \widetilde{P}$:

$$I[\gamma] = \int_\gamma \varphi. \tag{6.1}$$

This integral is called the *action in the extended phase space* or simply the *action.*

Let z be the set of variables x, y, and t. If the curve γ is represented analytically by smooth functions

$$x = x(s), \qquad y = y(s), \qquad t = t(s), \quad 0 \leq s \leq 1,$$

then

$$i[\gamma] = \int_0^1 (y \cdot x' - Ht')\, ds, \tag{6.2}$$

where the prime denotes the derivative with respect to s. If $t' > 0$, $t(0) = t_1$, and $t(1) = t_2$, then we can take t as a parameter, $x = x(t)$, $y = y(t)$. We assume that the coordinates and the momenta are related by

$$\dot{x} = \frac{\partial H}{\partial y}$$

or, equivalently,

$$y = \frac{\partial L}{\partial \dot{x}}.$$

Action (6.2) over this specific curve then coincides with the Hamilton action

$$\int_{t_1}^{t_2} L\Big|_{x(t)}\, dt.$$

We return to the general case and consider the variation of the curve γ, i.e., a smooth one-parameter family of curves γ_α, $-\varepsilon < \alpha < \varepsilon$, $\varepsilon > 0$, where $\gamma_0 = \gamma$. The variation is determined by a smooth function of two variables

$$z = z(s, \alpha), \quad s \in [0,1], \quad \alpha \in (-\varepsilon, \varepsilon).$$

The derivative

$$\frac{\partial z}{\partial \alpha}\bigg|_{\alpha=0}$$

is a smooth vector field u defined on the initial field γ; it is called the *variation field* (the usual notation is δz). We emphasize that we consider variations of a general type and do not assume that the field u vanishes at the endpoints $z_1 = z(0)$ and $z_2 = z(1)$ of the curve γ. As usual, the variation of action (6.1) is the derivative of the function $I[\gamma_\alpha]$ at $\alpha = 0$.

Lemma 1 (on action variation). *We have*

$$\delta I = \varphi(u)\Big|_{z_1}^{z_2} + \int_\gamma i_u \phi, \tag{6.3}$$

where $\phi = d\varphi$ and $i_u\phi = \phi(u, \cdot)$ is the inner product of the field u and the 2-form ϕ.

Proof. The variation field u defined on the curve γ can be extended to a smooth field defined in some neighborhood of γ. The action variation is the derivative along the field u at points of the curve γ:

$$\delta I = \int_\gamma L_u \varphi. \tag{6.4}$$

Using the homotopy formula

$$L_u\varphi = di_u\varphi + i_u d\varphi,$$

we convert (6.4) to the form

$$\delta I = i_u\varphi\Big|_{z_1}^{z_2} + \int_\gamma i_u\phi.$$

The lemma is proved. □

Formula (6.3) was initially obtained by Cartan in *Integral Invariants.* We note that it is universal, i.e., independent of the specific form of the 1-form φ.

Theorem 9. *A smooth curve γ in $\widetilde{P}$ with the endpoints z_1 and z_2 is a vortex line if and only if*

$$\delta I[\gamma] = \varphi(u)\Big|_{z_1}^{z_2} \tag{6.5}$$

for all variation fields u.

Proof. Let γ be a vortex line. Then its tangent vectors annihilate the 2-form ϕ, and, in particular, $i_u\phi = 0$ for tangent vectors. Therefore, by virtue of (6.3),

$$\int_\gamma i_u\phi = 0, \tag{6.6}$$

and we obtain (6.5). Conversely, if (6.6) holds along a curve γ for all vector fields u, then the relation $i_u\phi = 0$ holds for all vectors tangent to γ. Because the vectors u are arbitrary, we see that γ is a vortex line. □

Theorem 9 has the following important consequence.

Corollary 1. *Vortex lines in the extended phase space coincide with extremals of the action in the class of curves with fixed endpoints.*

This is the *action stationarity principle in the phase space* suggested by Helmholtz and Poincaré. There exists an interesting relation between this principle and the Hamilton principle. The action can be represented by the integral

$$\int_{t_1}^{t_2} \mathcal{L}\,dt, \qquad \mathcal{L} = y\cdot\dot{x} - H(x,y,t).$$

The "Lagrangian" $\mathcal{L}$ is degenerate: it is independent of the velocity $\dot{y}$. The Helmholtz–Poincaré principle can be written in the form of the "Hamilton principle:"

$$\delta\int_{t_1}^{t_2} \mathcal{L}\,dt = 0.$$

The Euler–Lagrange equations

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)^{\bullet} = \frac{\partial \mathcal{L}}{\partial x}, \qquad \left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)^{\bullet} = \frac{\partial \mathcal{L}}{\partial y}$$

are equivalent to the canonical Hamilton equations

$$\dot{y} = -\frac{\partial H}{\partial x}, \qquad \dot{x} = \frac{\partial H}{\partial y}.$$

6.2. As a simple consequence of the action stationarity principle in the phase space, we mention the *Whittaker method* for reducing the order of autonomous Hamiltonian systems.

Let $H : P \to \mathbb{R}$ be a Hamiltonian independent of time. Then H is an integral of the canonical equations

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \qquad \dot{y} = -\frac{\partial H}{\partial x_k}, \quad 1 \le k \le n. \tag{6.7}$$

We assume that some partial derivatives of the Hamiltonian H are nonzero at some point of the energy surface

$$\Sigma_h^{2n-1} = \{(x,y) \in P : H(x,y) = h\};$$

for definiteness, let $\partial H/\partial y_n \neq 0$. In this case, the equation

$$H(x_1, \dots, x_n, y_1, \dots, y_n) = h$$

can be solved (at least locally) with respect to y_n:

$$y_n + K(x_1, \dots, x_n, y_1, \dots, y_{n-1}, h) = 0.$$

Because $\dot{x} = \partial H/\partial y_n \neq 0$, phase curves $x = x(t)$, $y = y(t)$ for fixed h can be parameterized by x_n, which is denoted by τ:

$$x_i = x_i(\tau), \qquad y_i = y_i(\tau), \quad 1 \le i \le n-1.$$

Theorem 10 (Whittaker [76]**).** *The functions $x(\tau)$ and $y(\tau)$ satisfy the canonical equations*

$$\frac{dx_i}{d\tau} = \frac{\partial K}{\partial y_i}, \qquad \frac{dy_i}{d\tau} = -\frac{\partial K}{\partial x_i}, \quad 1 \le i \le n-1. \tag{6.8}$$

These equations are called the *Whittaker equations.* In the general case, system (6.8) is nonautonomous. If the coordinate x_n is *cyclic* (i.e., it is not contained in the explicit expression of the Hamiltonian), then the canonical Whittaker equations are autonomous, and the number of their degrees of freedom can be decreased by one. Equations (6.8) can be obtained by direct calculation, but we use the action stationarity principle.

Proof of the Whittaker theorem. Let $x = x(t)$, $y = y(t)$ be a solution of canonical system (6.7). By hypothesis, this curve (denoted by γ) lies on Σ. Therefore, for variations of γ in the class of curves with fixed endpoints lying on Σ, we have

$$\delta I[\gamma] = \delta \int_\gamma (y \cdot \dot{x} - H)\, dt = \delta \int_\gamma y \cdot dx = 0.$$

In the local coordinates $x_1, \dots, x_{n-1}, \dots, x_n, y_1, \dots, y_{n-1}$ on Σ^{2n-1} we have

$$\int_\gamma y \cdot dx = \int_\gamma y_1\, dx_1 + \dots + y_{n-1}\, dx_{n-1} - K\, dx_n,$$

and it hence remains to use the corollary of Theorem 9. □

The Whittaker theorem suggests a method for "autonomizing" Hamilton equations with a Hamiltonian $H(x, y, t)$ depending on time. For this, we add two conjugate canonical variables $x_{n+1} = t$ and y_{n+1} to the phase space P (the dimension of P increases by two) and introduce the new Hamoltonian

$$\mathcal{H} = y_{n+1} + H(x_1, \dots, x_n, y_1, \dots, y_n, x_{n+1}).$$

The Hamilton equations

$$\dot{x}_s = \frac{\partial \mathcal{H}}{\partial y_s}, \qquad \dot{y}_s = -\frac{\partial \mathcal{H}}{\partial x_s}, \quad 1 \le s \le n,$$

coincide with the initial canonical equations (6.7). The first additional equation

$$\dot{x}_{n+1} = \frac{\partial \mathcal{H}}{\partial y_{n+1}} = 1$$

is a trivial identity, and the second additional equation

$$\dot{y}_{n+1} = -\frac{\partial \mathcal{H}}{\partial x_{n+1}}$$

is just the *energy variation theorem* for the initial equations (6.7). Setting $\Sigma^{2n+1} = \{x, y : \mathcal{H} = 0\}$ and using Whittaker's order-reducing method, we obtain the usual canonical equations with the Hamiltonian H.

Remark. The duality of the time t and the energy H taken with the opposite sign can be seen in the explicit expression of the Cartan 1-form

$$\varphi = y \cdot dx - H\, dt.$$

The $2n+2$ variables x, y, t, and h ($h = -H$) are not independent: one of them (namely, h) is a known function of the other variables. The equation $\mathcal{H} = 0$ can be solved with respect to each of these $2n+2$ variables, and the specific choice of such a variable is inessential. For example, in the nonautonomous case, this equation can be solved with respect to t, and h the plays the role of "time."

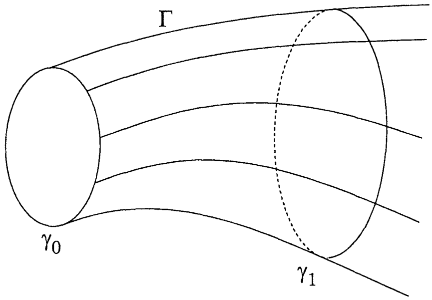

Fig. 16. Vortex tube

6.3. The *Poincaré–Cartan integral invariant*, which is very important in the theory of Hamiltonian systems, can be deduced from Theorem 9. Let γ_0 be a closed path in the extended phase space $\widetilde{P}$; it can be represented by a periodic function

$$z = z_0(\alpha), \quad \alpha \bmod 2\pi.$$

By Sec. 5, for each point of γ_0, there exists a unique vortex line

$$z = z(s, \alpha), \quad s \in \mathbb{R}, \qquad z(0, \alpha) = z_0(\alpha), \tag{6.9}$$

passing through this point. The set of all these vortex lines is a cylindrical surface Γ called the *vortex tube.* The variable s is a parameter on vortex lines; it can be chosen in various ways. Fixing the value of s (for definiteness, setting $s = 1$), we obtain the closed curve γ_1 lying on Γ. Obviously, the curves γ_0 and γ_1 are homotopic on Γ (γ_0 continuously turns into γ_1 as the parameter s changes from 0 to 1).

Theorem 11 (Cartan [20]). *We have*

$$\int_{\gamma_0} \varphi = \int_{\gamma_1} \varphi. \tag{6.10}$$

Proof. The proof is based on Theorem 9. We consider the family γ_α of vortex lines depending on the parameter $\alpha \bmod 2\pi$,

$$z = z(s, \alpha), \quad 0 \leq s \leq 1,$$

and use formula (6.5):

$$\frac{d}{d\alpha} I[\gamma_\alpha] = \varphi\left(\frac{\partial z}{\partial \alpha}\right)\bigg|_{s=1} - \varphi\left(\frac{\partial z}{\partial \alpha}\right)\bigg|_{s=0}.$$

Integrating this equation in α from 0 to 2π, we obtain

$$\int_0^1 \varphi\left(\frac{\partial z}{\partial \alpha}\right)\bigg|_{s=1} d\alpha = \int_0^1 \varphi\left(\frac{\partial z}{\partial \alpha}\right)\bigg|_{s=0} d\alpha,$$

as was required. □

Remark. The Cartan theorem can be proved by applying the Stokes formula to the vortex tube Γ bounded by the honological cycles γ_0 and γ_1 (similar to the proof of the Malus theorem in Sec. 3).

The integral

$$\int_\gamma y \cdot dx - H\, dt \tag{6.11}$$

taken over a closed curve in the extended phase space is called the *Poincaré–Cartan integral invariant.* Cartan called formula (6.10) the generalized principle of conservation of energy–momentum [20]. He also showed that the canonical Hamilton equations with the Hamiltonian H are *unique* differential equations admitting integral invariant (6.11) (see Sec. 11 in [20]).

Theorem 11 has the following important consequence.

Corollary 2 (The Poincaré Theorem [64]**).** *Let γ_0 and γ_1 be sections of the vortex tube Γ by hyperplanes $t =$ const. Then*

$$\int_{\gamma_0} y \cdot dx = \int_{\gamma_1} y \cdot dx.$$

This result can be interpreted as follows. Let γ be a closed contour in the phase space P and $g^t(\gamma)$ be its image under a shift along trajectories of a Hamiltonian system by the time interval t. Then

$$\int_{g^t(\gamma)} y \cdot dx = \text{const}. \tag{6.12}$$

Poincaré himself formulated the theorem thus. This theorem is analogous to the Thomson theorem in hydrodynamics. Integral (6.12) is called the *universal Poincaré integral invariant.* The word "universal" means that integral (6.12) is an invariant for all Hamiltonian systems defined on the same phase space. By the Lee Hwa-Chung theorem [52], any first-order universal integral invariant differs from the Poincaré invariant by only a constant multiplier. Moreover, it was proved that specific Hamiltonian systems with complicated behavior of phase trajectories (for example, the equations in the three-body problem) do not have other integral invariants (see [47]).

§7. Hamilton–Jacobi Method and Huygens Principle

7.1. There exist two main types of mechanical problems: the *Cauchy problem*, where we find the motion of the system from its initial state, and the *boundary value problem*, where we find the motion $t \mapsto x(t)$ satisfying the boundary condition, i.e., taking the values x_0 and x_1 at the respective instants t_0 and t_1. Unlike the Cauchy problem, the boundary value problem is not solvable in the general case. The most effective method for proving its solvability is the variational method: we find the stationary value (usually the minimum) of the Hamilton action in the class of curves with fixed endpoints. For example, in the absence of external forces (in this case, trajectories are geodesic curves of the metric on M determined by the kinetic energy), the boundary value problem has a solution if all motions are unconstrained, i.e., defined for all values of time (the Hopf–Rinow theorem). Another essential distinction between the Cauchy problem and the boundary value problem is that the boundary value problem may have several distinct solutions. The simplest example is flat and high trajectories of projectiles. The Serr theorem yields a more complicated example: any two points of a compact Riemannian manifold can be connected by infinite number of distinct geodesic curves. *Conjugate points* (i.e., points where infinitesimally close trajectories intersect) are obstructions to the uniqueness of the solution of the boundary value problem.

We fix a point $(x_0, t_0) \in M \times \mathbb{R}_t$. We assume that there exists a domain U of the extended configuration space $M \times \mathbb{R}_t$ such that the boundary value problem has a unique solution for the points (x_0, t_0) and (x, t) of U and this solution smoothly depends on x and t. We can calculate the momentum value for each of these solutions and thus lift them into the extended phase space. Calculating the action in the extended phase space on these curves, we obtain a smooth function S of x and t. By the action variation formula (Theorem 9 in Sec. 6), we have

$$dS = \frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial t} dt = y\, dx - H(x, y, t)\, dt.$$

Therefore,

$$y = \frac{\partial S}{\partial x}, \qquad \frac{\partial S}{\partial t} + H = 0.$$

We see that the action S satisfies the equation

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \tag{7.1}$$

Hamilton obtained this equation in 1824 in geometric optics and ten years later generalized it to the mechanics of systems with potential forces. Equation (7.1) is called the *Hamilton partial differential equation* or the *Hamilton–Jacobi equation*, because Jacobi simplified the way to obtain this equation and investigated its most important properties.

If the Hamiltonian H is independent of t, then we can find the solution of Eq. (7.1) in the form

$$S = -ht + W(t),$$

where h = const is the energy constant and the function W satisfies the equation

$$H\left(x, \frac{\partial W}{\partial x}\right) = h. \tag{7.2}$$

For example, we have

$$H = \frac{|y|}{n(x)}$$

in the problem of light propagation in an isotropic medium, where n is the refraction index and the energy constant equals 1. In this case, Eq. (7.2) coincides with eikonal equation (3.7),

$$\sum \left(\frac{\partial W}{\partial x_i}\right)^2 = n^2(x).$$

7.2.

Theorem 12. *Let $S(x,t)$ be a solution of Hamilton–Jacobi equation* (7.1). *Then the n-dimensional surface*

$$\Sigma_t = \left\{x, y : y = \frac{\partial S}{\partial x}\right\} \subset P$$

is an invariant surface of the canonical Hamilton equations with the Hamiltonian H.

The invariance property of the surface Σ_t, which depends on time as on a parameter, can be treated as follows. Let $x(t)$, $y(t)$ be a solution of the Hamilton equations with the initial data

$$x(t_0) = x_0, \qquad y(t_0) = y_0.$$

If the point x_0, y_0 lies on Σ_{t_0}, then the point $x(t)$, $y(t)$ lies on Σ_t for all t.

This property can be formulated more briefly if we consider the extended phase space $\widetilde{P} = P \times \mathbb{R}_t$. The relation $y = \partial S/\partial x$ defines an $(n+1)$-dimensional surface $\widetilde{\Sigma}$ in $\widetilde{P}$. Its invariance means that if a vortex line intersects $\widetilde{\Sigma}$, then it is contained in $\widetilde{\Sigma}$.

In his *New Methods of Celestial Mechanics* (1892), Poincaré mentioned Theorem 12 and gave the general definition of *invariant relations*, which mediate between solutions and integrals (see Sec. 19 in [64]). Poincaré's theorem was rediscovered on repeated occasions by various authors (see, e.g., the treatise of Levi-Civita and Amaldi in Chap. 10 in [53]). Theorem 12 is actually contained in the Monge characteristic theory of first-order partial differential equations. In the Monge theory, differential equations that can explicitly contain the unknown function (unlike the Hamilton–Jacobi equation) are considered; therefore, Theorem 12 has another form in the general case (see [24]).

Proof of Theorem 12. We set

$$f = y - \frac{\partial S}{\partial x}.$$

We must show that $f = 0$ implies $\dot{f} = 0$. Indeed,

$$-\dot{f} = \frac{\partial H}{\partial x}\bigg| + \frac{\partial^2 S}{\partial x^2}\frac{\partial H}{\partial y}\bigg| + \frac{\partial^2 S}{\partial t \partial x}.$$

The vertical line means that we have replaced the momentum with the gradient of S in the expressions for the partial derivatives. Now differentiating (7.1) with respect to x, we obtain $\dot{f} = 0$, as was required. □

In symplectic geometry, the surfaces Σ_t are called the *Lagrange surfaces.* Their characteristic property is that the value of the Poincaré invariant

$$\int_\gamma y \cdot dx$$

equals zero for any closed contour γ homotopic to zero (i.e., this contour can be continuously constricted to a point inside the surface Σ itself). We also call these surfaces *potential surfaces* because of the potentiality property of the momentum field $y(x,t) = \partial S/\partial x$. We note that potential fields and potentials were initially introduced by Lagrange. Looking ahead, we can say that our purpose is to investigate n-dimensional invariant manifolds that are defined by nonpotential momentum fields.

7.3. The Hamilton action S is a function of x and t and also depends on the choice of the point x_0, i.e., on n arbitrary parameters. Because $dt = 0$ for $t = t_0$, the action variation formula yields

$$\frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x_0}dx_0 = y\,dx - H\,dt - y_0\,dx_0.$$

Hence,

$$y = \frac{\partial S}{\partial x}, \qquad -y_0 = \frac{\partial S}{\partial x_0}. \tag{7.3}$$

These $2n$ relations define solutions of the canonical equations as functions of time and the initial data x_0 and y_0.

It is useful to consider n-parameter families of solutions of the Hamilton–Jacobi equation, $S(x,t,c)$, $c = c(x_1,\dots,c_n)$. Such a family is called the *complete integral* of Eq. (7.1) if

$$\det\left\|\frac{\partial^2 S}{\partial x \partial c}\right\| \neq 0. \tag{7.4}$$

By the implicit function theorem, the entire phase space P (or at least a $2n$-dimensional domain in P) is fibered by the family of invariant surfaces

$$\Sigma(c) = \left\{ x, y : y = \frac{\partial S}{\partial x} \right\}.$$

Theorem 13. *If $S(x,t,c)$ is a complete integral of the Hamilton–Jacobi equation, then the general solution of the canonical Hamilton equations can be found from the relations*

$$y = \frac{\partial S}{\partial x}, \qquad -b = \frac{\partial S}{\partial c}, \tag{7.5}$$

where $b = (b_1, \dots, b_n)$ are arbitrary constants.

The minus sign in the second group of Eqs. (7.5) is taken for convenience (cf. (7.3)). We recall that the *general solution* of the system of $2n$ differential Hamilton equations is the family of solutions depending on $2n$ arbitrary constants, which can be expressed via the initial coordinates and momenta. The first equation in (7.5) is an invariant relation (by Theorem 1), and by virtue of this relation, the functions $\partial S/\partial c_1, \dots, \partial S/\partial c_n$ compose the set of independent integrals of the canonical Hamilton equations. Because inequality (7.4) holds, we can find the coordinates x as functions of t and $2n$ arbitrary constants b and c from the second relation in (7.5) by the implicit function theorem. Substituting these relations in the first relation in (7.5), we obtain the momenta as functons of t, b, and c.

Because of Theorem 12, it is sufficient to show that the derivatives $\partial S/\partial x$ are integrals of the Hamilton equations on integral manifolds. Indeed,

$$\left(\frac{\partial S}{\partial c}\right)^{\bullet} = \frac{\partial^2 S}{\partial t \partial c} + \frac{\partial^2 S}{\partial c \partial x}\frac{\partial H}{\partial y}\bigg|_{y=\partial S/\partial x}.$$

Differentiating Eq. 7.1 with respect to c, we see that this sum vanishes.

Theorem 13 was established by Jacobi in 1837. "It should however be observed that the converse of this theorem—namely the theorem that the solution of a partial differential equation such as Hamilton's depends on the solution of a set of ordinary differential equations (the differential equations of the characteristics), which in the case are of the Hamiltonian form, had been discovered by Pfaff and Cauchy (completing the earlier work of Lagrange and Monge) before Hamilton and Jacobi approached the subject from the dynamical side" (Whittaker [76]). The most effective direct method for solving the Hamilton–Jacobi equations is separation of variables: the complete integral is the sum of terms each depending on only one of the variables $x_1, \dots, x_n$, and t.

7.4. Lagrange observed that the *enveloping surface* of the family of solutions of the first-order partial differential equation is also a solution of this equation. This simple result has important applications.

We consider the equation

$$F\left(x_1, \ldots, x_m, z, y_1, \ldots, y_m\right) = 0, \tag{7.6}$$

where z is an unknown function of x and $y_k = \partial z / \partial x_k$. We usually assume that

$$\sum \left(\frac{\partial F}{\partial y_k}\right)^2 \neq 0. \tag{7.7}$$

Of course, the Hamilton–Jacobi equation is of form (7.6), where the role of the independent variables x is played by the generalized coordinates and time. The peculiarity of the Hamilton–Jacobi equation is that it does not explicitly contain the unknown function z.

Each solution $z = z(x)$ of Eq. (7.6) defines a hypersurface (*integral surface*) in the $(m+1)$-dimensional space of variables $\{x_1, \ldots, x_m, z\} = \mathbb{R}^{m+1}$. The vector $y = \{y_1, \ldots, y_m, 1\} \neq 0$ is orthogonal to the integral surface relative to the standard Euclidean metric in $\mathbb{R}^{m+1}$. A hyperplane in $\mathbb{R}^{m+1}$ passing through the point (x, z) orthogonally to the vector y, whose components satisfy Eq. (7.6), is said to be *admissible.* By virtue of condition (7.7), admissible planes passing through the same point of $\mathbb{R}^{m+1}$ form a smooth $(m-1)$-dimensional family. Therefore, solving Eq. (7.6) can be treated as finding hypersurfaces $z = z(x)$ such that all their tangent planes are admissible.

We now consider the family of solutions $z = z(x, c)$ of Eq. (7.6) depending on the parameters $c = (c_1, \ldots, c_r)$. Let $Z(x)$ be the enveloping surface of this family. This means that for each point $(x_0, z_0) \in \mathbb{R}^{m+1}$ satisfying the equation $z_0 = Z(x_0)$, there exists c_0 such that the integral surface $z = z(x, c_0)$ contains the point (x_0, z_0) and the tangent plane of the integral surface at this point coincides with the tangent plane of the surface $z = Z(x)$. The function $x \mapsto Z(x)$ is a solution of Eq. (7.6) because the tangent plane is admissible. This is Lagrange's theorem, which was proved in 1772 in a memoir submitted to the Berlin Academy of Sciences.

In the case where Eq. (7.6) does not contain z (the Hamilton–Jacobi equation is of just this type), Lagrange's theorem can be slightly revised. If $z(x, c)$ is a family of regular solutions of Eq. (7.6), i.e.,

$$\sum \left(\frac{\partial z}{\partial x_k}\right)^2 \neq 0,$$

then

$$\sigma(c) = \{x : z(x, c) = \text{const}\}$$

is a family of regular hypersurfaces in $\mathbb{R}^m = \{x\}$.

Theorem 14. *The enveloping surface of the family of $(m-1)$-dimensional surfaces $\sigma(c)$ (if it exists) is a level surface of a solution of Eq.* (7.6).

We must remember that not all families of multidimensional surfaces (manifolds) have an enveloping surface. In differential geometry, it is proved that a general one-parameter family of curves on the plane and one- and two-parameter families of surfaces in the three-dimensional space have enveloping curves or surfaces. An arbitrary one-parameter family of curves in the three-dimensional space has no enveloping curve or surface in the general case. Thom [73] found the conditions for the existence of an enveloping manifold for general r-parameter p-dimensional manifolds in a q-dimensional space:

$$q - 2p + 1 \leq r \leq q - 1. \tag{7.8}$$

For example, these conditions hold for $p = 1$, $q = 3$, and $r = 2$; therefore, the problem of finding the enveloping surface of a two-parameter family of curves in the three-dimensional space is well posed.

If $z(x, c)$ is a complete integral of the Hamilton–Jacobi equation, then the number of parameters is $r = m - 1$. Because $p = m - 1$ and $q = m \geq 2$, conditions (7.8) are satisfied; hence, general smooth families of hypersurfaces $\sigma(c)$ must have enveloping surfaces.

7.5. Huygens knew Theorem 14 and formulated it for optical problems in his book *The Theory of Light* (1690). This work contains the first rigorous formulation of the wave theory of light proposed by Hooke. Hooke and Newton, who lectured on optics to Cambridge students at the start of his career and advocated the particle theory of light, disputed the reasons for color rings appearing in thin films (these rings are now called Newton's rings, although they were discovered by Hooke and Boyle). As a result, Newton decided to publish nothing in optics while Hooke was alive. In 1704, two years after Hooke's death, Newton published his book *Optics, or a Treatise on the Reflexions, Refractions, Inflexions, and Colours of Light*, in which he collected all his results in the theory of light. To interrelate wave properties of light and the particle theory, Newton assumed that light particles periodically, at stated very small intervals, have "fits" that change the properties of reflection and refraction.

Huygens, who was a Cartesian, denied action at a distance. According to Huygens, light propagation is realized by an immediate mechanical interaction, which causes a propagation of the excitation of an optical medium (light ether) that permeates all bodies. Excitation is the transition from the rest state to oscillation. Each ether particle having been excited causes surrounding particles to oscillate. Therefore, according to Huygens, light is not a stream of moving particles but the propagation process of elastic oscillations of the optical medium. The *Huygens principle* states that if the light excitation has come to some point of the space filled with ether, then this point must be regarded as the center of a new spherical wave. The interaction of secondary waves is such that their enveloping surface is the wavefront. This immediately implies the known laws of reflection and refraction (Fig. 17). Deduction of the main relations in geometric optics (e.g., the canonical Hamilton equations and

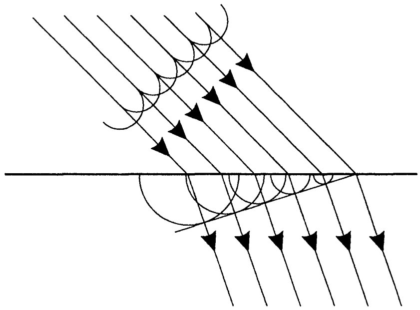

Fig. 17. Light refraction by Huygens

the Hamilton–Jacobi equation) from the Huygens principle and a discussion of the duality between the frameworks of light rays and light waves can be found in Suppl. 1 in [28].

To establish the connection between the Huygens principle and Theorem 14, we consider the eikonal equation. Let $W(x)$ be a solution of this equation. Light rays are orthogonal to the level surfaces

$$\sigma(t) = \{x \in \mathbb{R}^3 : W(x) = t\}$$

and the propagation time of light particles between the surfaces $\sigma(t_1)$ and $\sigma(t_2)$ equals $t_2 - t_1$ (see Sec. 3). We assume that the light excitation is localized on the surface $\sigma_0 = \sigma(0)$ at $t = 0$. Then this excitation arrives at the surface $\sigma(t)$ after the time interval t, i.e., $\sigma(t)$ is the wavefront. Let $c = (c_1, c_2)$ be the coordinates on the surface $\sigma(t)$. We now consider the problem of the propagation of light from points $c \in \sigma(t)$. Let $\sigma_c(\tau)$ be secondary "spherical" wavefronts, i.e., surfaces such that the excitation comes them for time τ. These surfaces are level surfaces of the corresponding solutions of Eq. (3.7). By Theorem 14, the enveloping surface of the family $\sigma_c(\tau)$ is also a level surface of some solution of Eq. (3.7). Because the propagation time of light particles from the initial surface σ_0 to the enveloping surface equals $t + \tau$, it is clear that this enveloping surface coincides with $\sigma(t + \tau)$.

7.6. From the formal standpoint, the Huygens principle implies that secondary wavefronts must have an enveloping surface not only ahead of the primary wavefront but also behind it. Fresnel complemented the Huygens principle with the important assumption (it is connected with the idea of wave superposition) that secondary waves mutually neutralize behind the primary wavefront. He used these ideas to calculate diffraction and interference qualitatively.

After Fresnel, elastic oscillations of the optical medium (ether) became the focus of interest in optics. For example, in 1839, MacCullough obtained equations describing these oscillations and having the form of the famous Maxwell equations. However, these investigations were mostly phenomenological and had no underlying physical ideas.

In the process of developing electromagnetic theory, it became clear that light is electromagnetic waves. We show that the main relations of geometric optics can be obtained from the Maxwell equations. For simplicity, we consider an isotropic nonconducting nonmagnetic medium. Under these assumptions, the Maxwell equations have the form

$$\operatorname{rot} H - \frac{\varepsilon}{c}\frac{\partial E}{\partial t} = 0, \qquad \operatorname{rot} E + \frac{1}{c}\frac{\partial H}{\partial t} = 0, \tag{7.9}$$

$$\operatorname{div} E = 0, \qquad \operatorname{div} H = 0, \tag{7.10}$$

where E and H are electric and magnetic intensities, c is the speed of light, ε is the dielectric inductivity (it depends on the point $x \in \mathbb{R}^3$). We can eliminate H from (7.9) and obtain a second-order equation for the electric field. For this, we apply the operator $\partial/\partial t$ to the first equation and the operator rot to the second equation:

$$\operatorname{rot}\frac{\partial H}{\partial t} - \frac{\varepsilon}{c}\frac{\partial^2 E}{\partial t^2} = 0, \qquad \operatorname{rot}\operatorname{rot} E + \frac{1}{c}\operatorname{rot}\frac{\partial H}{\partial t} = 0.$$

Eliminating $\operatorname{rot}(\partial H/\partial t)$, we obtain

$$\frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2} + \operatorname{rot}\operatorname{rot} E = 0. \tag{7.11}$$

Using the first equation in (7.10), we can show that $\operatorname{rot}\operatorname{rot} E = -\Delta E$, where ΔE denotes the Laplace operator applied to the vector E componentwise. Finally, Eq. (7.11) becomes

$$\frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = \Delta E. \tag{7.12}$$

We consider the case of simple harmonic oscillations, where the components of the field E have the form

$$f(x)e^{i\omega t}, \qquad x = (x_1, x_2, x_3),$$

and $\omega = \text{const}$ is the frequency. Equation (7.12) yields the equation for the amplitude f:

$$\Delta f + k^2 n^2 f = 0, \tag{7.13}$$

where $n = \sqrt{\varepsilon}$ and $k = \omega/c$. Because $1/(2\pi k)$ is the wave length, we see that in the short-wave case, k takes large values. Our purpose is to solve Eq. (7.13) for large k.

We apply the *stationary phase method* proposed by Kelvin and used by Poincaré and Debay [55] in optical problems. We find the solution of Eq. (7.13) in the form

$$u(x)e^{ikS(x)}. \tag{7.14}$$

The amplitude u changes slowly whereas the phase ikS oscillates rapidly. The quasi-classical approximation method in quantum mechanics is also based on the same idea: solutions of the Schrödinger equation are found in form (7.14), and the role of the large parameter k is played by the inverse Planck constant.

Substituting (7.14) in (7.13), we obtain

$$k^2u\left[n^2-\left(\frac{\partial S}{\partial x}\right)^2\right]+o(k)=0.$$

We do not need the explicit expression for $o(k)$ (it is actually linear in k). Dividing the last equation by k^2 and letting k tend to infinity, we obtain the well-known eikonal equation

$$\sum\left(\frac{\partial S}{\partial x_i}\right)^2=n^2. \tag{7.15}$$

Therefore, $n=\sqrt{\varepsilon}$ is the refraction index of the optical medium, $S(x)=$ const is the approximate equation of equiphase surfaces, and normals of these surfaces coincide with the directions of light rays.

The justification of the limit process transforming (7.13) into (7.15) is a delicate problem (see [29, 55] for details).

§8. Hydrodynamics of Hamiltonian Systems

8.1. The Poincaré theorem (Theorem 12 in Sec. 7) gives the invariance criterion for an n-dimensional potential manifold (n is the number of degrees of freedom) admitting a single-valued projection on the configuration space: the potential of the corresponding momentum field satisfies the Hamilton–Jacobi equation. We now specify the conditions for the invariance of n-dimensional potential (vortex) manifolds.

Let Σ_t^n be a manifold in the phase space $P=T^*M$ admitting a single-valued projection on the configuration space M. In the canonical coordinates x, y, this manifold is defined by the equation

$$y=u(x,t), \tag{8.1}$$

where u is a covector field on M (possibly depending on time).

Theorem 15. *The manifold Σ_t is an invariant manifold for the canonical Hamilton equations with the Hamiltonian $H(x,y,t)$ if and only if field* (8.1) *satisfies the equation*

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x}, \tag{8.2}$$

where

$$\operatorname{rot} u = \frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^{\mathrm{T}}$$

is a skew-symmetric $n\times n$ matrix (the curl of the covector field u),

$$v(x,t) = \left.\frac{\partial H}{\partial y}\right|_{y=u(x,t)} \tag{8.3}$$

is a vector field on M, and $h(x,t) = H(x,u(x,t),t)$ is a function on M depending on time as on a parameter.

Proof. Let $f = y - u(x,t)$. We must show that $f = 0$ implies $\dot{f} = 0$, where the dot denotes the derivative by virtue of the canonical equations with the Hamiltonian H. On one hand,

$$\dot{y} = -\left.\frac{\partial H}{\partial x}\right|_{y=u};$$

on the other hand,

$$\dot{y} = \dot{u} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\dot{x} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left.\frac{\partial H}{\partial y}\right|_{y=u}.$$

Therefore,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}v = -\left.\frac{\partial H}{\partial x}\right|_{y=u}. \tag{8.4}$$

By the definition of the function h, we have

$$\frac{\partial h}{\partial x} = \left.\frac{\partial H}{\partial x}\right|_{y=u} + \left.\frac{\partial H}{\partial y}\right|_{y=u}\frac{\partial u}{\partial x} = \left.\frac{\partial H}{\partial x}\right|_{y=u} + \left(\frac{\partial u}{\partial x}\right)^{\mathrm{T}} v.$$

Taking this relation into account, we easily obtain (8.2) from (8.4). □

For the "natural" Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(x,t)y_i y_j + V(x_1,\dots,x_n,t),$$

Eq. (8.2) becomes

$$\frac{\partial u_i}{\partial t} + \sum_{j,k} g_{jk}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)u_k = -\frac{1}{2}\frac{\partial}{\partial x_i}\left(\sum_{j,k} g_{jk}u_j u_k\right) - \frac{\partial V}{\partial x_i} \tag{8.5}$$

for $i = 1, \dots, n$. This is a nonlinear system of n first-order partial differential equations for n unknown functions $u_1, \dots, u_n$. For example, in the case where $n = 3$ and $g_{ij} = \delta_{ij}$, we have $u = v$, and Eqs. (8.5) become the well-known Lamb equations in ideal fluid dynamics, Eq. (1.4). Therefore, Eq. (8.2) is also called the *Lamb equation* in the general case, and the function h is called the *Bernoulli function.* Motions of the system such that their phase trajectories lie on Σ can be found as solutions of the following system of differential equations on M:

$$\dot{x} = v(x,t), \quad x \in M. \tag{8.6}$$

The Poincaré theorem in Sec. 7 is a particular case of Theorem 15. Indeed, let the potential momentum field $u = \partial S/\partial x$ generate an invariant manifold. Then $\operatorname{rot} u = 0$ and (8.2) imply

$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial t} + h\right) = 0;$$

therefore,

$$\frac{\partial S}{\partial t} + h = g,$$

where g is a function of t. Performing the gauge transformation

$$S \to S - \int g(t)\, dt,$$

which does not change the momentum field, we obtain the Hamilton–Jacobi equation for the potential S,

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \tag{8.7}$$

The derivation of Eq. (8.7) from Eq. (8.2) is similar to the derivation of the Lagrange–Cauchy integral from the Lamb equation for potential flows of a barotropic fluid in a potential field (see Sec. 1).

If $\operatorname{rot} u = 0$, then the corresponding invariant manifold Σ was called the *potential* or *Lagrangian* manifold above. Invariant manifolds such that $\operatorname{rot} u \neq 0$ are called vortex manifolds. Our aim is to investigate vortex manifolds.

Equations (8.2) apparently first appeared in the calculus of variations as the *consistency conditions for fields of extremals* (these fields are known to be described by canonical equations). However, the calculus of variations deals with *self-consistent* (potential) fields only. In this case, the field is an n-parameter family of nonintersecting extremals generating an n-dimensional invariant manifold in a $2n$-dimensional phase space (see [14, 28]). The consistency condition for such a field is usually represented in the form of Eq. (8.4), which is an analogue of *Euler equation* (1.2) in hydrodynamics. The Lamb transformation (transition from (8.4) to (8.2)) was used in the theory of Hamiltonian systems to study invariant relations linear in momenta (see [53, 76]).

Arzhanykh [12] generalized the Lamb equation to the case of non-Hamilonian systems (in particular, nonholonomic systems) and used the Hamilton–Jacobi method to solve them exactly. However, Eq. (8.2) had no connections with hydrodynamic ideas until [39].

8.2. We now discuss the problem of the existence of solutions of the Lamb equation. Let M be a compact manifold without boundary and $u_0(x)$ be a smooth covector field defined globally on M.

Theorem 16. *For sufficiently small t, there exists a smooth solution $u(x,t)$ of Eq. (8.2) such that $u(x,0) = u_0(x)$ for all $x \in M$.*

Proof. We consider the family of solutions of the Hamilton equations

$$x = x(t, x_0), \qquad y = y(t, x_0)$$

satisfying the initial conditions

$$x(0, x_0) = x_0, \qquad y(0, x_0) = u_0(x_0).$$

By the implicit function theorem, the mapping $M \to M$ defined as $x = x(t, x_0)$ is a diffeomorphism for sufficiently small t. The desired solution $u(x,t)$ is equal to $y(t, x_0)$, where we must replace x_0 with the solution of the equation $x = x(t, x_0)$ (obviously, this solution is a function of x and t). □

In the analytic case (where the Hamiltonian H is an analytic function on $P \times \mathbb{R}$), Theorem 16 is a simple consequence of the *Cauchy–Kovalevskaya theorem*: because system (8.2) is solved relative to the derivatives $\partial u_1/\partial t, \dots,$ $\partial u_n/\partial t$, its solutions are uniquely determined by the values of the functions $u_1, \dots, u_n$ at $t = 0$ and exist for sufficiently small $|t|$ (we note that they can be found as a convergent ascending power series in t).

For large t, solutions $u(x,t)$ are usually nonsmooth and multivalued (see corresponding examples for potential solutions in Sec. 3). Therefore, the Lamb equation is inapplicable, and we must use the original canonical equations on $P = T^*M$. It is considered that a similar phenomenon is possible in ideal fluid dynamics: the solution of the Euler equation and the continuity equation satisfying smooth initial data becomes nonsmooth and non-well-defined in a finite time. From the physical standpoint, this means that the classical hydrodynamic model becomes inapplicable.

A family of solutions $u(t, x, \alpha)$ (with n parameters $\alpha = (\alpha_1, \dots, \alpha_n)$) of the Lamb equation is called a *complete integral* of this equation if

$$\det \left\| \frac{\partial u}{\partial \alpha} \right\| \neq 0. \tag{8.8}$$

For the potential field $u = \partial S/\partial x$, this condition becomes known condition (7.4),

$$\det \left\| \frac{\partial^2 S}{\partial x \partial \alpha} \right\| \neq 0.$$

Let $x(t, \alpha, \beta)$ be a "total" solution of differential equation (8.6),

$$\dot{x} = v(x, t, \alpha) = \left.\frac{\partial H}{\partial y}\right|_{y=u(x,t,\alpha)}, \tag{8.9}$$

i.e.,

$$\det \left\| \frac{\partial x}{\partial \beta} \right\| \neq 0 \tag{8.10}$$

for $t = 0$ (therefore, this condition holds for small $|t| \neq 0$). We claim that

$$x = x(t, \alpha, \beta), \qquad y = u(x(t, \alpha, \beta), t, \alpha)$$

is the total solution of the canonical Hamilton equations,

$$\det \left\| \frac{\partial (x, y)}{\partial (\alpha, \beta)} \right\| \neq 0.$$

Indeed, the solutions of Eq. (8.9) have the form

$$x = x_0 + tv(x_0, 0, \alpha) + o(t)$$

for small t, where the initial position x_0 depends nonsingularly on n parameters β,

$$\det \left\| \frac{\partial x_0}{\partial \beta} \right\| \neq 0.$$

Then

$$\left\| \frac{\partial (x, y)}{\partial (\alpha, \beta)} \right\| = \begin{pmatrix} 0 & \dfrac{\partial u}{\partial \alpha} \\ \dfrac{\partial x_0}{\partial \beta} & \dfrac{\partial y}{\partial \beta} \end{pmatrix}.$$

By virtue of (8.8) and (8.10), this matrix is nondegenerate.

We assume that the complete integral $u(x, t, \alpha)$ of the Lamb equation is known. An n-dimensional invariant manifold of the Hamilton equations

$$\Sigma(\alpha) = \{x, y : y = u(x, t, \alpha)\}$$

depending on time corresponds to each value of α. By the implicit function theorem, condition (8.8) implies that the entire $2n$-dimensional phase space is fibered into n-dimensional manifolds $\Sigma(\alpha)$. Hence, solving the Hamilton equations in this case can be reduced to solving differential equations (8.8) defined on the n-dimensional configuration space M. By the Jacobi theorem (see Sec. 7), Eqs. (8.9) can be integrated in quadratures for potential fields. This cannot be asserted in the general case without additional assumptions (see Chap. 4).

8.3. Theorem 15 has a series of useful consequences. For example, we consider light propagation in a nonhomogeneous isotropic medium. Light rays are described by the canonical equations with the Hamiltonian $H = |y|/n(x)$, where n is the refraction index. In addition, $H = 1$ on real trajectories. We consider a ray system in $E^3 = \{x\}$; it generates the vector field $v(x)$ of velocities of light particles. The three-dimensional invariant manifold Σ^3 in the six-dimensional phase space $E^3 \times \mathbb{R}^3 = \{x, y\}$ corresponds to this ray system. The corresponding momentum field $u(x)$ can be found from the equations

$$\dot{x} = v = \frac{\partial H}{\partial y} = \frac{y}{|y|n}, \qquad H = 1.$$

This implies $u = vn^2$. Because the field u is stationary and $H \equiv 1$, Lamb equation (8.2) becomes

$$(\operatorname{rot} u)v = 0.$$

In the three-dimensional Euclidean space, this relation is equivalent to

$$v \times \operatorname{rot}(n^2 v) = 0. \tag{8.11}$$

For a homogeneous medium, we have $n = \text{const}$, and Eq. (8.11) implies Proposition 1 in Sec. 3.

8.4. The Lamb equations are useful for solving boundary value problems. In Sec. 7, we discussed the simplest boundary value problem: find the motion $t \to x(t)$, $t_1 \leq t \leq t_2$, satisfying the boundary conditions

$$x(t_1) = a, \qquad x(t_2) = b, \tag{8.12}$$

where a and b are fixed points of the configuration space M. We saw that this problem is closely related to the Hamilton–Jacobi equation.

We now consider a more general problem. Let Σ_1 and Σ_2 be two n-dimensional manifolds in the $2n$-dimensional phase space. Find the solution $x = x(t)$, $y = y(t)$, $t_1 \leq t \leq t_2$, of the Hamilton equations such that the point with the coordinates x and y lies on Σ_1 if $t = t_1$ and on Σ_2 if $t = t_2$. We study the case where the manifolds Σ_1 and Σ_2 admit single-valued projections on M (this assumption does not hold for boundary conditions (8.12)) and can hence be defined by the equations $y = u_1(x)$ and $y = u_2(x)$. The boundary conditions become

$$y(t_1) = u_1(x(t_1)), \qquad y(t_2) = u_2(x(t_2)). \tag{8.13}$$

These relations form an algebraic system of $2n$ equations for finding $2n$ initial values.

The simplest method for solving boundary value problem (8.13) is the *marching method.* We first calculate the solution $u(x,t)$ of the Lamb equation satisfying the initial condition

$$u(x, t_1) = u_1(x).$$

The process of transferring the boundary condition given at $t = t_1$ to all points of the segment $[t_1, t_2]$ is called the *direct run*. In particular, taking $t = t_2$, we have

$$u(x, t_2) = u_2(x);$$

this relation determines the position of the system at $t = t_2$. If the corresponding value $x = x_2$ is unique, then the original boundary value problem has a unique solution, which can be found by integrating system (8.6) consisting of n differential equations with the initial condition $x(t_2) = x_2$. This step in solving the boundary value problem is called the *inverse run*. The momentum values can be calculated from the formula $y = u(x, t)$.

The marching method for solving boundary value problem (8.13) has an advantage over the usual method, where we find the general solution of the Hamilton equations and then select the values of arbitrary constants such that boundary conditions (8.13) hold. This preeminence is especially apparent in numerical computation.

As an example, we consider the linear system

$$P\ddot{x} + Qx = 0, \quad x \in \mathbb{R}^n, \tag{8.14}$$

where P and Q are symmetric $n \times n$ matrices depending on time and P is positive definite for all t. Equation (8.14) is usually called the *Sturm–Liouville equation* and plays an essential role in the theory of the second variation of the action functional; it is the variational Euler–Lagrange equation for the Lagrangian

$$L = \frac{1}{2}(P\dot{x}, \dot{x}) - \frac{1}{2}(Qx, x).$$

Equation (8.14) can be represented in the Hamiltonian form

$$y = P\dot{x}, \qquad \dot{y} + Qx = 0. \tag{8.15}$$

For linear systems, we usually take the linear boundary conditions

$$y = A_1 x + b_1 \quad (t = t_1), \qquad y = A_2 x + b_2 \quad (t = t_2), \tag{8.16}$$

where A_1 and A_2 are square matrices (generally speaking, nonsymmetric) and b_1 and b_2 are constant vectors. If $b_1 = b_2 = 0$, then boundary conditions (8.16) are homogeneous, and we call them *Sturm-type conditions*. In this case, the boundary value problem obviously has the trivial solution $x \equiv 0$. Therefore, we usually consider the problem of the existence nontrivial (nonzero) solutions for a boundary value problem with Sturm-type boundary conditions.

We find the invariant manifold of system (8.15) in the form $y = Ax + b$, where A and b are unknown functions of time. The invariance condition has the form

$$(\dot{A} + AP^{-1}A + Q)x + \dot{b} + AP^{-1}b = 0$$

(the Lamb equation). It is equivalent to two systems of ordinary differential equations

$$\dot{A} + AP^{-1}A + Q = 0, \qquad \dot{b} + AP^{-1}b = 0. \tag{8.17}$$

The first system is the matrix *Riccati equation.* The direct run is solving Eqs. (8.17) with the initial data

$$A(t_1) = A_1, \qquad b(t_1) = b_1.$$

The position of the system at $t = t_2$ can be found from the linear algebraic equation

$$A(t_2)x + b(t_2) = A_2 x + b_2.$$

If the matrix $A(t_2) - A_2$ is nondegenerate, then the solution $x = x_2$ is unique. The inverse run is solving the differential equation

$$\dot{x} = P^{-1}Ax + P^{-1}b$$

with the initial condition $x(t_2) = x_2$.

The invariant manifold $y = Ax + b$ of system (8.15) is a potential manifold only if the matrix A is symmetric. It can be easily shown that if the matrix A satisfying the matrix Riccati equation is symmetric at some instant, then $A^{\mathrm{T}} = A$ for all t. This simple result is a particular case of the general Lagrange theorem about the potentiality of the solutions of the Lamb equation (see Chap. 2). Therefore, if the matrix A_1 is nonsymmetric, then this invariant manifold is a vortex manifold.

8.5. The results in Secs. 8.1 and 8.2 allow a new view of Descartes's vortex theory. According to Descartes, the motion of mechanical systems is described by first-order differential equations in the configuration space (as is the case for point vortices in an ideal fluid; see Sec. 2):

$$\dot{x} = v(x, t), \quad x \in M. \tag{8.18}$$

Actually, the motion of the system is defined not only by the position but also by the velocity at each fixed instant of time. Furthermore, Descartes gave no principles for constructing the field v for different mechanical systems.

A main achievement of Newton was perceiving that the dynamics of real systems are described by second-order differential equations. Of course, Newton had predecessors in this problem: foremost was Galileo, who introduced the notion of acceleration into mechanics and obtained the simplest second-order equations describing the free fall of bodies in free space. To reduce the equations of motion to the investigation of a dynamic system (i.e., to first-order equations), it is necessary to double the dimension of the position space and to introduce the auxiliary phase space. However, we are interested not in the phase trajectories themselves but in their projections on the configuration space.

Solving dynamic problems is possible "inside" the configuration space. For this, it is necessary to solve Lamb equation (8.2), which is a system of partial differential equations on M, and then, using (8.3) to calculate the vector field v from the solution of the Lamb equation, to solve Eq. (8.18). The benchmark data of this approach is also the Hamiltonian of the mechanical system considered. As we saw, using the Lamb equation to solve boundary value problems numerically is much better that the traditional methods based on directly integrating $2n$ differential Hamilton equations. The Lamb equation is especially effective in the cases where we investigate n-parameter families of solutions of Hamiltonian systems (for example, ray systems in optics). By Theorem 16, Lamb equation (8.2) has both potential solutions and vortex solutions. The same motion can be included in n-dimensional families of solutions in different ways. Segregating the class of potential solutions leads to the Hamilton–Jacobi method, which was developed by Huygens for problems in geometric optics.

From the physical standpoint, Eqs. (8.2), (8.3), and (8.18) describe the motion of a *collisionless medium*: particles moving along different trajectories do not interact (in particular, the medium is rarefied insofar as its particles "pierce" each other without collisions). The model of collisionless media with potential fields of initial velocities is used in astrophysics to explain the formation of stellar clusters (see [9] for more details).

8.6. Having in mind the analogy with hydrodynamics, we can ask whether Eq. (8.18) (where the field v is defined by relation (8.3)) has an integral invariant of the form

$$\int_{g^t(D)} \rho(x,t)\, d^n x = \text{const}, \qquad \rho > 0. \tag{8.19}$$

In this case, the density ρ satisfies the differential equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0. \tag{8.20}$$

We can treat invariant (8.19) as the mass conservation law and Eq. (8.20) as the continuity equation.

Proposition 2. *Let $u(t,x,\alpha)$ be the complete integral of the Lamb equation. Then Eq.* (8.18) *has integral invariant* (8.19) *with the density*

$$\rho = \det \left\| \frac{\partial u}{\partial \alpha} \right\|. \tag{8.21}$$

Proof. Relations $y = u(t, x, \alpha)$ can be solved (locally at least) with respect to α:

$$\alpha_1 = f_1(x, y, t), \qquad \dots, \qquad \alpha_n = f_n(x, y, t).$$

It is clear that $f_1, \dots, f_n$ are integrals of the Hamilton equations. We pass from the canonical variables x, y to the coordinates x, α. In the new variables, the Hamilton equations become

$$\dot{\alpha} = 0, \qquad \dot{x} = v(x, t, \alpha). \tag{8.22}$$

By the *Liouville theorem*, the phase flow of the Hamiltonian system preserves the phase volume

$$\operatorname{mes}(D) = \int_D dx_1 \cdots dx_n \, dy_1 \cdots dy_n.$$

In the new variables, this integral equals

$$\operatorname{mes}(D) = \int_D \left| \frac{\partial u}{\partial \alpha} \right| \, d^n x \, d^n \alpha.$$

Therefore, Eqs. (8.22) have the invariant with the density $\rho = |\partial u / \partial \alpha|$. Because $\alpha = \text{const}$, the equation $\dot{x} = v(x, t, \alpha)$ has invariant (8.19) with density (8.21) for fixed α. □

§9. Lamb Equations and the Stability Problem

9.1. In addition to solving boundary value problems, the Lamb equations are useful in stability problems and problems of exact integration of dynamic equations. Setting aside the integration problem to Chap. 4, we now briefly discuss the application of the Lamb equations to solving equilibrium stability problems for mechanical systems.

We study equilibrium conditions for linear systems or, more precisely, gyroscopic stabilization of equilibrium positions. We consider the linear system with n degrees of freedom

$$\ddot{x} + \Gamma \dot{x} + P x = 0, \quad x \in \mathbb{R}^n, \tag{9.1}$$

where Γ is a skew-symmetric matrix and P is a symmetric nondegenerate matrix. We can imagine that a particle moving in $\mathbb{R}^n$ is affected by the *gyroscopic force* $-\Gamma \dot{x}$ and the potential force $-Px$. Gyroscopic forces appear, for example, in the following cases: when passing to a rotating frame of reference (the Coriolis force); when order-reducing systems with symmetries; when investigating the motion of charged particles in magetic fields (the Lorentz force). Gyroscopic forces do not influence the conservation of total mechanical energy,

$$H = \frac{1}{2}(\dot{x}, \dot{x}) + \frac{1}{2}(Px, x). \tag{9.2}$$

Because $\det P \neq 0$ by hypothesis, we find that $x = 0$ is the unique equilibrium point of system (9.1). Because Eqs. (9.1) are linear, the stability of the equilibrium state $x = 0$, $\dot{x} = 0$ is equivalent to the boundedness of all solutions of (9.1). Integral (9.2) immediately implies the following simple, but important, *Lagrange–Kelvin theorem*: if the potential energy $V(x) = (Px, x)/2$ has a minimum at the point $x = 0$, then this equilibrium state remains stable after adding arbitrary gyroscopic forces. The following *Kelvin theorem* is less obvious: if the instability degree is odd (the *instability degree* is the index of the quadratic form V), then gyroscopic stabilization is impossible. A review of results on the problem of gyroscopic stabilization can be found in [18, 35].

9.2. Equations (9.1) can be represented in the form of the Lagrange equations with the Lagrangian

$$L = \frac{1}{2}(\dot{x}, \dot{x}) + \frac{1}{2}(\dot{x}, \Gamma x) - \frac{1}{2}(Px, x).$$

Performing the Legendre transformation, we can obtain the Hamilton equations with a quadratic Hamiltonian and then apply the theory of invariant relations (see Sec. 8). In this case, however, the direct approach is simpler.

We find n-dimensional invariant manifolds Σ defined by the linear relations

$$\dot{x} = Ax, \tag{9.3}$$

where A is an $n \times n$ matrix. The invariance condition for Σ is equivalent to the quadratic matrix equation

$$(A + \Gamma)A + P = 0, \tag{9.4}$$

which replaces the Lamb equation. Setting

$$A = D - \frac{\Gamma}{2},$$

we transform Eq. (9.4) into the form

$$D^2 + \frac{\Gamma D - D\Gamma}{2} + P - \frac{\Gamma^2}{4} = 0. \tag{9.5}$$

In the canonical variables

$$x \qquad \text{and} \qquad y = \frac{\partial L}{\partial \dot{x}} = \dot{x} + \frac{\Gamma x}{2},$$

the invariant plane Σ has the form

$$y = Dx.$$

Therefore, Σ is a potential manifold if and only if the matrix $D = A + \Gamma/2$ is symmetric. Equation (9.3) has the quadratic integral

$$h(x) = H\Big|_{\dot{x}=Ax} = \frac{1}{2}(A^{\mathrm{T}}Ax, x) + \frac{1}{2}(Px, x). \tag{9.6}$$

For invariant potential planes, $h(x) \equiv 0$.

If linear system (9.3) of order n has unbounded solutions, then the equilibrium point $x = 0$ is obviously unstable. Conversely, let the function h have a strict extremum (maximum or minimum) at the point $x = 0$. Then all solutions of system (9.3) are bounded, but this does not imply the stability of the equilibrium position $x = 0$ of system (9.1).

We note that quadratic equation (9.4) has another solution: if A is a solution, then the matrix $A' = A^{\mathrm{T}} - \Gamma$ also satisfies (9.4) (this is an analogue of the Viéte theorem). Therefore, in addition to Σ, system (9.1) has the n-dimensional invariant plane Σ' defined by the equation

$$\dot{x} = A'x = (A^{\mathrm{T}} - \Gamma)x. \tag{9.7}$$

Let h' be the restriction of total energy (9.3) to Σ'. Because $\det P \neq 0$, Eq. (9.4) implies the nondegeneracy of the matrices A and A'.

Proposition 3. *Let* $\det(D - D') \neq 0$. *Then the following assertions hold:*

1. *the invariant manifold* Σ' *is a vortex manifold;*
2. *the phase space* $\mathbb{R}^{2n} = \{x, \dot{x}\}$ *is the direct sum of the n-dimensional invariant planes* Σ *and* Σ';
3. *the equilibrium position of system* (9.1) *is stable if and only if all solutions of linear differential equations* (9.3) *and* (9.7) *are bounded.*

Proof. Because $A = D - \Gamma/2$, we have $A' - D^{\mathrm{T}} - \Gamma/2$. Further, because $\det(D^{\mathrm{T}} - D) \neq 0$, the invariant plane Σ is a vortex manifold. If the n-dimensional planes Σ and Σ' have a common point $x, \dot{x}$ distinct from the equilibrium state, then $Ax = (A^{\mathrm{T}} - \Gamma)x$ for some $x \neq 0$. Therefore, the matrix $A - A^{\mathrm{T}} + \Gamma = D - D^{\mathrm{T}}$ is degenerate, and hence $\mathbb{R}^{2n} = \Sigma \oplus \Sigma'$. Each of linear systems (9.3) and (9.7) have n linearly independent solutions. By virtue of item 2, these $2n$ solutions form a basis in the $2n$-dimensional space of all solutions of system (9.1). $\square$

Remark. The skew-symmetric matrix $D - D^{\mathrm{T}}$ can be nondegenerate only for even n.

Proposition 4 ([43]). *Let the quadratic form h be nondegenerate and its index be odd. Then the equilibrium position $x = 0$ is unstable.*

Proof. We set $h(x) = (Bx, x)/2$, $B = A^{\mathrm{T}}A + P$, and $\det B \neq 0$ (by hypothesis). Because h is an integral of Eq. (9.3), we have $\dot{h} = (x, BAx) = 0$. Therefore, the matrix $C = BA$ is skew-symmetric and nondegenerate, $|C| > 0$, and n is even. Let $f(\lambda) = \det(A - \lambda E)$ be the characteristic polynomial of the matrix A. Because n is even, $f(\lambda) \to +\infty$ as $\lambda \to \infty$. The index of h is odd; therefore, $|B| < 0$. Hence, $f(0) = |A| = |B^{-1}||C| < 0$. By the continuity property, the polynomial f has a real positive root, and the equilibrium position $x = 0$ is therefore unstable. $\square$

Taking (9.5) into account, we can easily show that

$$B = (D^{\mathrm{T}} - D)A. \tag{9.8}$$

Therefore, the quadratic form h is nondegenerate if

$$\det(D^{\mathrm{T}} - D) \neq 0. \tag{9.9}$$

This inequality also guarantees the nondegeneracy of the quadratic form h'.

Proposition 5. *Let condition* (9.9) *hold. Then the index of the quadratic form H is equal to the sum of the indices of the quadratic forms h and h'.*

Proof. Because $|D - D^{\mathrm{T}}| \neq 0$, Proposition 3 implies $\mathbb{R}^{2n} = \Sigma \oplus \Sigma'$. Therefore, any vector $x, \dot{x}$ in $\mathbb{R}^{2n}$ is the sum of the vectors $v, \dot{v}$ and $v', \dot{v}'$ and

$$\dot{v} = Av, \qquad \dot{v}' = A'v'.$$

We use the obvious identity

$$\begin{aligned}\frac{1}{2}(x,x) + \frac{1}{2}(Px,x) = \frac{1}{2}(\dot{v},v) + \frac{1}{2}(Pv,v) + \frac{1}{2}(\dot{v}',\dot{v}')\\ + \frac{1}{2}(Pv',v') + (\dot{v},\dot{v}') + (Pv,v').\end{aligned}$$

By virtue of (9.4), the bilinear form

$$(\dot{v},\dot{v}') + (Pv,v') = ((A+\Gamma)Av,v') + (Pv,v')$$

vanishes. Therefore, the value of the quadratic form H at the point $x, \dot{x}$ is the sum of $h(v)$ and $h'(v')$. In particular, $\operatorname{ind} H = \operatorname{ind} h + \operatorname{ind} h'$. □

Corollary 3. *Let the potential energy V have a maximum and the function h have a strict extremum at the point $x = 0$. Then the equilibrium position $x = 0$ of system* (9.1) *is stable.*

Proof. Because the quadratic form h has a strict extremum, it is nondegenerate. Therefore, by virtue of (9.8), $\det(D^{\mathrm{T}} - D) \neq 0$. In the case considered, $\operatorname{ind} H = n$ and $\operatorname{ind} h$ equals either 0 or n, and $\operatorname{ind} h'$ equals either n or 0 by Proposition 3. We see that the function h' has a strict extremum at the point $x = 0$. The stability of the equilibrium position $x = 0$ is now implied by Proposition 1. □

In some cases, these reasons allow specifying the constructive conditions for gyroscopic stabilization. The stability problem is reduced to the problem of solving quadratic matrix equations.

9.3.

Theorem 17. *If the potential energy has a strict maximum at the point* $x = 0$, $\det \Gamma \neq 0$, *and*

$$\|\Gamma^{-1}\| \cdot \left\|P^{1/2}\right\| < \frac{1}{2}, \tag{9.10}$$

then the equilibrium position $x = 0$ *of system* (9.1) *is stable.*

Here, $\|\cdot\|$ is an arbitrary matrix norm. S. V. Bolotin pointed out this theorem to the author; it is a generalization of an earlier result by A. V. Karapetyan.

Proof. Because the form V has a strict maximum at the point $x = 0$, there exists a positive-definite matrix $P^{1/2}$. We set $A = P^{1/2}X$. The matrix X satisfies the equation

$$XP^{1/2}X + \Gamma X + P^{1/2} = 0$$

or

$$X = \Phi(X) = -\Gamma^{-1}(XP^{1/2}X + P^{1/2}).$$

We show that Φ maps the unit ball $\|X\| \leq 1$ into itself. By virtue of (9.10), we have

$$\begin{aligned}
\|-\Phi(X)\| &\leq \|\Gamma^{-1}\| \cdot \left\|XP^{1/2}X + P^{1/2}\right\| \\
&\leq \|\Gamma^{-1}\| \cdot \left(\left\|P^{1/2}\right\| \cdot \|X\| + \left\|P^{1/2}\right\|\right) \\
&\leq 2\,\|\Gamma^{-1}\| \cdot \left\|P^{1/2}\right\| < 1.
\end{aligned}$$

Therefore, by the Bohl–Brauer theorem, the mapping Φ has a fixed point inside the unit ball $\|X\| \leq 1$.

Setting $x = P^{-1/2}z$, we rewrite integral (9.6) as

$$h = \frac{1}{2}(Xz, z) + \frac{1}{2}(z, z).$$

Because $\|X\| < 1$, the quadratic form h is positive definite, and the equlibrium position $x = 0$ of system (9.1) is hence stable by Corollary 3. □

9.4. It turns out that if condition (9.9) holds, then linear equation (9.3) can be reduced to the canonical form of the Hamilton differential equations. In this case Eq. (9.3) is equivalent to the Lamb equation

$$(D - D^{\mathrm{T}})\dot{x} = -\frac{\partial h}{\partial x}, \tag{9.11}$$

where the quadratic form $h = (Bx, x)/2$ is defined by (9.6). Equation (9.11) has the explicit form

$$\Omega\dot{x} = -Bx, \qquad \Omega = D - D^{\mathrm{T}}. \tag{9.12}$$

Because the skew-symmetric matrix Ω is nondegenerate, there exists a matrix C ($\det C \neq 0$) such that $C^{\mathrm{T}}\Omega C = I$, where

$$I = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}, \qquad m = \frac{n}{2}.$$

Substituting $x = Cz$ in (9.12), we obtain

$$I\dot{z} = -B'z, \qquad B' = C^{\mathrm{T}}BC.$$

This system has the canonical form of the Hamilton equations

$$\dot{z}_i = \frac{\partial h}{\partial z_{m+i}}, \qquad \dot{z}_{m+i} = -\frac{\partial h}{\partial z_i}, \quad 1 \leq i \leq m.$$

The role of the Hamiltonian is played by the function h represented in the new variables z. The variables $z_1, \ldots, z_m$ play the role of canonical coordinates and $z_{m+1}, \ldots, z_{2m}$ play the role of canonical momenta.

In Sec. 1 in Chap. 2, we generalize this result to the case of nonlinear systems.

Chapter 2
General Vortex Theory

§1. Lamb Equations and Hamilton Equations

1.1. The aim in this chapter is to investigate the multidimensional Lamb equation, which interconnects a covector field $u(x,t)$, a vector field $v(x,t)$, and a function $h(x,t)$:

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x}. \tag{1.1}$$

The tensor objects u, v, and h are defined on a connected smooth manifold $M^n = \{x\}$.

The system of differential equations on M^n

$$\dot{x} = v(x,t), \tag{1.2}$$

whose phase flow g_v^t can be considered as a flow of fluid in M^n, corresponds to a vector field v. It turns out that properties of this flow are similar to those of the flow of an ideal fluid in the three-dimensional Euclidean space.

Of course, we can study Eqs. (1.1) and (1.2) independently of their origin in Hamiltonian mechanics. But there is a simpler way, which is better from the methodological standpoint. We show that Eq. (1.1) is the Lamb equation for an n-dimensional invariant surface of some Hamiltonian system. Therefore, we can apply well-known results concerning Hamilton equations in Chap. 1.

Proposition 1. *Let u, v, and h satisfy Eq.* (1.1). *Then in the phase space $P = T^*M$, there exists a Hamiltonian system with an n-dimensional invariant manifold*

$$\Sigma^n = \{x, y : y = u(x,t)\} \tag{1.3}$$

such that the Lamb equation for Σ coincides with Eq. (1.1).

Proof. We set

$$H = \sum (y_i - u_i)\, v_i + h.$$

Because y and u are covectors and v is a vector, this formula well defines a function on T^*M linear in momenta. By virtue of (1.1) and the relations $H|_{y=u} = h$ and

$$\dot{x} = \frac{\partial H}{\partial y} = v(x,t),$$

we see that (1.3) is an invariant manifold of the Hamiltonian system with the Hamiltonian H. □

The following example shows that Eq. (1.1) defines the Hamiltonian ambigously. Let

$$H = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) - \varepsilon_1\varepsilon_2 \sin x_1 \cos x_3 - \varepsilon_2\varepsilon_3 \sin x_2 \cos x_1 - \varepsilon_3\varepsilon_1 \sin x_3 \cos x_2,$$

where $\varepsilon_i = \text{const}$. This reversible Hamiltonian system defined on $P = \mathbb{T}^3 \times \mathbb{R}^3$ admits the invariant relations

$$y_1 = \varepsilon_1 \sin x_3 + \varepsilon_3 \cos x_2,$$
$$y_2 = \varepsilon_2 \sin x_1 + \varepsilon_1 \cos x_3,$$
$$y_3 = \varepsilon_3 \sin x_2 + \varepsilon_2 \cos x_1,$$

which define the flow of an incompressible fluid on the three-dimensional torus $\mathbb{T}^3 = \{x_1, x_2, x_3, \operatorname{mod} 2\pi\}$; the velocity is collinear with its curl (see (1.18) in Chap. 1). For almost all values of ε_1, ε_2, and ε_3, there exist zones of quasi-random particle motion.

1.2.

Theorem 1 (variational principle). *A smooth curve $t \to x(t)$, where $t_1 \le t \le t_2$, is a solution of the system*

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)\,\dot{x} = -\frac{\partial h}{\partial x} \tag{1.4}$$

if and only if this curve yields a stationary value of the functional

$$P = \int_{t_1}^{t_2} (u \cdot \dot{x} - h)\, dt \tag{1.5}$$

in the class of curves with fixed ends.

This assertion is a consequence of the Poincaré–Helmholtz variational principle in the phase space (see Sec. 6 in Chap. 1): the functional P is a restriction of the action functional

$$\int_{t_1}^{t_2} (y \cdot \dot{x} - H)\, dt$$

to the set of curves in the position space M by the invariant relation $y = u(x,t)$. Another proof of this theorem can be obtained by using the coincidence of Eq. (1.4) and Lagrange equations with the Lagrangian $\mathcal{L} = \sum u_i \dot{x}_i - h$.

Birkhoff called the integral of type (1.5) the *Pfaff action* and Eqs. (1.4) the *Pfaff equations.* He considered the case where the skew-symmetric matrix $\operatorname{rot} u$ is nondegenerate (in particular, if n is even). In this case, we can uniquely express $\dot{x}$ as a function of x and t. In his famous *Dynamical Systems* [13], Birkhoff stated the research program for Eqs. (1.4) (in the case where $\det(\operatorname{rot} u) \neq 0$). Results of investigations up to 1983 are collected in Santilli's monograph [70]. Birkhoff himself paid special attention to the case where the field u is explicitly independent of time; he failed to notice that (1.4) becomes a usual Hamiltonian system in this case.

Indeed, by the *Darboux theorem*, there exists a local transformation $x \mapsto z$ such that the matrix

$$J^{\mathrm{T}}(\operatorname{rot} u)J$$

has the form

$$I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where J is the Jacobi matrix $\partial x/\partial z$ and E is the identity matrix of dimension $n/2$. In the new variables $z_1, \dots, z_n$, the equation

$$J^{\mathrm{T}}(\operatorname{rot} u)J\dot{z} = -J^{\mathrm{T}}\frac{\partial h}{\partial x}$$

(i.e., Eq. (1.4)) has the canonical form

$$I\dot{z} = -\frac{\partial h}{\partial z}.$$

The coordinates z_i, $z_{n/2+i}$, $1 \le i \le n/2$, are the conjugate canonical variables.

In what follows, various cases where the curl matrix is degenerate are of special interest.

1.3.

Theorem 2. *System* (1.2) *has the relative integral invariant*

$$\int_{g_v^t(\gamma)} u \cdot dx = \text{const}. \tag{1.6}$$

Here, γ is an arbitrary closed contour on M. Theorem 2 is a consequence of the Poincaré theorem (Sec. 6 in Chap. 1): integral (1.6) is a restriction of the Poincaré invariant to the set of closed curves lying on the invariant surface Σ^n. This simple, but important, result generalizes the *Thomson theorem* on conservation of circulation to the multidimensional case.

Theorem 2 can be proved by direct calculation. For this, we introduce the 1-form

$$\omega = \sum u_i\, dx_i$$

and rewrite Lamb equation (1.1) in the equivalent form

$$\frac{\partial \omega}{\partial t} + i_v d\omega = -dh. \tag{1.7}$$

Using the homotopy formula

$$L_v = i_v d + d i_v,$$

we obtain

$$\frac{\partial \omega}{\partial t} + L_v \omega = dg, \tag{1.8}$$

where $g = i_v\omega - h$. The function g has a simple mechanical sense: this is the Lagrangian in which the velocity $\dot{x}$ is replaced by the vector field v. In the left-hand side of (1.8), we have $\dot{\omega}$, the total derivative with respect to time of the 1-form ω by virtue of system (1.2).

Let γ be a path (1-chain) on M. We set

$$I(t) = \int_{g_v^t(\gamma)} \omega.$$

It is easy to obtain the formula

$$\dot{I} = \int_{g_v^t(\gamma)} \frac{\partial\omega}{\partial t} + L_v\omega \tag{1.9}$$

for the variation of the integral I. If γ is a closed path (i.e., a cycle), then $\dot{I} = 0$ by virtue of (1.8) and the Newton–Leibnitz formula. We note that formula (1.9) is valid for an arbitrary k-form ω and arbitrary k-chain γ.

1.4. Theorem 2 implies the following important consequence.

Theorem 3. *If the field $u(x,t)$ is a potential field for $t = 0$, then it is a potential field for all t.*

This is an analogue of the famous *Lagrange theorem* in ideal fluid dynamics. The proof is very simple. If $u = \partial\varphi/\partial x$ locally for $t = 0$, then integral (1.6) over a sufficiently small closed contour γ equals zero. Therefore, it equals zero for all t. Then the integrand is a closed 1-form, as is well known in analysis.

It can be easily seen that if the field u has a single-valued potential ϕ on the entire M for $t = 0$ (i.e., the form ω is exact), then this property holds for all t.

In the potential case, Lamb equation (1.1) implies the *Lagrange–Cauchy integral.* Indeed, if $u = \partial\varphi/\partial x$, then $\operatorname{rot} u = 0$, and Eq. (1.1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial t} + h\right) = 0.$$

Therefore,

$$\frac{\partial\varphi}{\partial t} + h(x,t) = g(t). \tag{1.10}$$

This is the so-called Lagrange–Cauchy integral. After the substitution

$$\varphi \mapsto \varphi - \int g(t)\,dt,$$

which does not change the field of momenta, the function g in the right-hand side of (1.10) becomes zero.

If Eq. (1.1) is obtained as the Lamb equation for a invariant potential surface of the Hamilton equations, then (1.10) becomes the well-known *Hamilton–Jacobi partial differential equation.*

§2. Reduction to the Autonomous Case

2.1. The general theory of integral invariants was developed by Poincaré; he formulated this theory in the third volume of *New Methods of Celestial Mechanics*. Cartan made a number of important supplements, summarized in his *Integral Invariants* [20]. The main idea in Cartan's book is the "autonomization" of differential equations, i.e., the transition from the position space to the space–time in which the coordinates x and time t are taken as equivalent variables. This consequent relativistic standpoint turned to be very fruitful. We now apply this method to the Lamb equation.

We consider the $(n+1)$-dimensional space–time $\widetilde{M} = M^n \times \mathbb{R}_t$ and let z denote a point in $\widetilde{M}$, i.e., the tuple $(x_1, \dots, x_n, t)$. The space–time has the direct-product structure, but this is insignificant for what follows.

Differential equations (1.2) are replaced by

$$\dot{x} = v(x,t), \qquad \dot{t} = 1,$$

or, in brief,

$$\dot{z} = \tilde{v}(z). \tag{2.1}$$

The field z has the components $v, 1$ in the coordinates $z = \{x, t\}$. It is insignificant for Eq. (2.1) that its solutions are parameterized by the time t. The key role is played by integral curves of the field $\tilde{v}$: they are tangent to a vector of the field $\tilde{v}$ at each of their points. If we parameterize the integral curves on $\widetilde{M}$ by the variable t and then project them on M, we obtain solutions of Eq. (2.1). From this standpoint, it is not the field $\tilde{v}$ itself that is important, but the direction field defined by $\tilde{v}$ (i.e., each vector $\tilde{v}$ can be multiplied by an arbitrary nonzero function of z). In relativistic mechanics, integral curves of the field $\tilde{v}$ are called *world lines.*

Lamb equation (1.1) can be represented in the equivalent form

$$\begin{pmatrix} & & & \frac{\partial u_1}{\partial t} + \frac{\partial h}{\partial x_1} \\ & \operatorname{rot} u & & \vdots \\ & & & \frac{\partial u_n}{\partial t} + \frac{\partial h}{\partial x_n} \\ -\left(\frac{\partial u_1}{\partial t} + \frac{\partial h}{\partial x_1}\right) & \cdots & -\left(\frac{\partial u_n}{\partial t} + \frac{\partial h}{\partial x_n}\right) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ 1 \end{pmatrix} = 0. \tag{2.2}$$

The first n equations (in rows) coincide with the corresponding Lamb equation, and the last equation

$$\sum \left(\frac{\partial u_i}{\partial t} + \frac{\partial h}{\partial x_i}\right) v_i = 0$$

is a simple consequence of (1.1) and the skew-symmetry of the matrix $\operatorname{rot} u$.

The skew-symmetric matrix of order $n+1$ in the left-hand side of (2.2) is the matrix of the curl of the 1-form

$$\varphi = \omega - h\,dt, \qquad \omega = \sum u_i\,dx_i,$$

defined on $\widetilde{M}$. Therefore, Eq. (2.2) can be represented in the invariant form

$$i_{\tilde{v}}\phi = 0, \qquad \phi = d\varphi. \tag{2.3}$$

2.2. According to the terminology introduced in Sec. 5 in Chap. 1, the vector $\tilde{v}$ is the vortex vector for the closed 2-form ϕ. If φ is the stress-energy form in the extended phase space, then Eq. (2.3) expresses the vortex principle of Hamiltonian mechanics (see Sec. 5).

Let n be an even number and ϕ be a nonsingular 2-form. Then the vector $\tilde{v}$ is defined uniquely up to a nonzero multiplier. In the general case, the form ϕ has other linearly independent vortex vectors, for example, the vectors $\tilde{w}$ whose components in the coordinates x, t are $w, 0$ (w is the vortex vector of the 2-form $\Omega = d\omega$ in the n-dimensional space M). The number of independent vortex vectors w is $n - \operatorname{rank}(\operatorname{rot} u)$. These vectors play a crucial role in our theory. Obviously, vectors of the form $\tilde{v} + \tilde{w}$ are also vortex vectors of the form ϕ.

We consider an example from electrodynamics due to Cartan (Sec. 81 in [20]) in which Eq. (2.3) appears. Let E and H be the vectors of the electric and magnetic fields in the free space and c be the speed of light. We consider the tensor of the electromagnetic field $\|F_{ij}\|$, the skew-symmetric matrix

$$\begin{pmatrix} 0 & H_3 & -H_2 & cE_1 \\ -H_3 & 0 & H_1 & cE_2 \\ H_2 & -H_1 & 0 & cE_3 \\ -cE_1 & -cE_2 & -cE_3 & 0 \end{pmatrix}. \tag{2.4}$$

With this matrix, we can associate the exterior 2-form

$$F = \sum F_{ij}\,dx_i \wedge dx_j, \tag{2.5}$$

where x_1, x_2, and x_3 are spatial Cartesian coordinates in the three-dimensional Euclidean space and $x_4 = t$ is time.

The form F is closed, $dF = 0$. This follows from the Maxwell equations

$$\frac{1}{c}\frac{\partial H}{\partial t} + \operatorname{rot} E = 0, \qquad \operatorname{div} H = 0.$$

Therefore, $F = df$, where the 1-form

$$f = \sum f_i\,dx_i$$

is called the 4-potential of the electromagnetic field. The other two Maxwell equations yield the wave equation for the coefficients f_i.

The rank of the skew-symmetric matrix (2.4) can equal four, two, or zero. The determinant of this matrix equals $c^2(E,H)^2$. Therefore, $\operatorname{rank} F = 2$ in the case where the fields E and H are orthogonal and nonzero. This case is the most interesting in the theory of electromagnetic waves.

We consider this case. Let $(E,H) = 0$ and $E^2 + H^2 \neq 0$. Then the closed 2-form F has two vortex vectors whose spatial and temporal components are respectively

$$H, \quad 0 \tag{2.6}$$

and

$$c[E,H], \quad H^2. \tag{2.7}$$

The projections of integral curves of field (2.6) on $\mathbb{R}^3 = \{x_1, x_2, x_3\}$ are magnetic field lines; the spatial components of field (2.7) determine the propagation direction of the electromagnetic wave.

2.3. We can integrate the form φ over oriented curves γ in $\widetilde{M}$ and thus obtain the "action" functional

$$I[\gamma] = \int_\gamma \varphi. \tag{2.8}$$

The lemma on the action variation (see Sec. 6 in Chap. 1) immediately implies the *variational principle*: integral curves on the field $\tilde{v}$ are extremals of functional (2.8) in the class of curves with fixed ends.

This principle is also valid for the vortex fields $\tilde{w}$. Because the t-components of the vectors $\tilde{w}$ are equal to zero, integral curves of the field $\tilde{w}$ (vortex lines) lie on hypersurfaces $t = \text{const}$ and can hence be considered curves in the configuration space M. In this case, the variational principle can be slightly revised: vortex lines are extremals of the functional

$$\int_\gamma \omega,$$

where γ is an oriented curve on M and the time t is considered the parameter.

If the coefficients of the 1-form φ are periodic in t (for example, with the period τ), then we can take a "cylinder," i.e., the direct product of M and the circle $\{t \bmod \tau\}$, as the space–time $\widetilde{M}$. In this case, closed curves could appear among the world lines. It is easy to see that these curves give stationary values to functional (2.8) defined on the space of all closed curves. This simple result can be useful for the proof of the existence of periodic solutions of system (2.1). As an example, we refer to the Conley–Zahnder theorem [23] on the existence of $n+1$ distinct τ-periodic solutions to the Hamilton equations on the $2n$-dimensional torus with a τ-periodic Hamitonian.

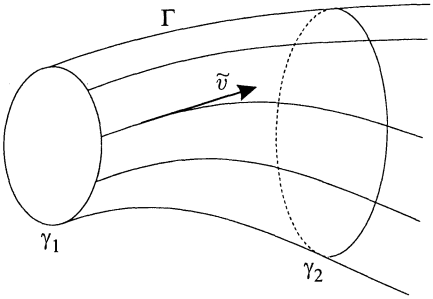

Fig. 18. Tube of world lines

2.4. Let γ_1 be a closed curve in $\widetilde{M}$. A unique world line passes through a given point of γ_1, and the set of all these world lines compose a cylindrical surface Γ (see Fig. 18). Let γ_2 be another closed curve in Γ homological to γ_1. Then

$$\int_{\gamma_1} \varphi = \int_{\gamma_2} \varphi. \tag{2.9}$$

Indeed, by the Stokes theorem, the difference of these integrals is

$$\iint \phi, \qquad \phi = d\varphi,$$

which equals zero because the 2-form ϕ vanishes on each pair of linearly independent vectors tangent to Γ (by (2.3) because $\tilde{v}$ is tangent to Γ by construction).

Equation (2.9) contains the Poincaré–Cartan integral invariant (see Sec. 6 in Chap. 1) as a particular case. To see this, we take the stress-energy 1-form in the extended phase space as the form φ.

If γ_1 and γ_2 are the sections of the world-line tube with the hypersurfaces $t = t_1$ and $t = t_2$, then (2.9) becomes Theorem 2 in Sec. 1.

We now replace world lines with vortex lines, i.e., by integral curves of the vortex field $\tilde{w}$. Because the field $\tilde{w}$ satisfies (2.3), the integral of the 1-form φ over a closed cycle is a *relative invariant* for the system of equations

$$\frac{dz}{d\alpha} = \tilde{w}(z),$$

where α is a real parameter playing the role of time. Because the t-component of the vector $\tilde{w}$ is zero, we obtain an interesting consequence: the system of differential equations on M

$$\frac{dx}{d\alpha} = w(x,t),$$

which determines the vortex lines, and the system $\dot{x} = v(x,t)$ have the integral invariant

$$\int_\gamma \omega$$

for all t. Here γ is an arbitrary closed contour on the configuration space M. This fact generalizes Cartan's result (Sec. 24 in [20]): differential equations of trajectories and differential equations of vortex lines in ideal fluid dynamics have the same linear integral invariant.

2.5. By the Stokes formula

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega, \tag{2.10}$$

the *absolute integral invariant* generated by the $(k+1)$-form $\Omega = d\omega$ corresponds to each relative integral invariant generated by the k-form ω. (This result and general formula (2.10) were obtained by Poincaré.) Therefore, system of differential equations (1.2) has the absolute invariant

$$\iint_\sigma \Omega, \tag{2.11}$$

where σ is an arbitrary two-dimensional surface (with an edge in general).

This result can easily be obtained by direct calculation. We apply exterior differentiation to both sides of (1.8):

$$\frac{\partial \Omega}{\partial t} + L_v \Omega = 0. \tag{2.12}$$

We use the well-known identities

$$\frac{\partial}{\partial t} d = d \frac{\partial}{\partial t}, \qquad dL_v = L_v d, \qquad dd = 0.$$

To conclude the proof of the invariance of integral (2.11), it remains to apply (1.9).

Equation (2.12) can be rewritten in the matrix form

$$\frac{\partial}{\partial t} \operatorname{rot} u + \operatorname{rot}((\operatorname{rot} u)v) = 0. \tag{2.13}$$

More exactly, this is the matrix form of the equation

$$\frac{\partial \Omega}{\partial t} + d\, i_v \Omega = 0. \tag{2.14}$$

The equivalency of Eqs. (2.12) and (2.14) follows because the form Ω is closed, $d\Omega = 0$.

Equation (1.1) in Chap. 1 describing the variation of a solenoidal field "frozen" into a flow can be reduced to form (2.14). For the usual Lamb equation in the three-dimensional Euclidean space, Eq. (2.13) becomes the well-known equation for the curl variation,

$$\frac{\partial}{\partial t}\operatorname{rot} v = \operatorname{rot}(v \times \operatorname{rot} v).$$

2.6. Because system (2.1) has relative integral invariant (2.9), it also has the absolute invariant

$$\int_\sigma \phi, \qquad \phi = d\varphi. \tag{2.15}$$

It is easy to show that the 2-forms Ω and ϕ are related by

$$\phi = \Omega + (i_v\Omega) \wedge dt. \tag{2.16}$$

Indeed, because $\varphi = \omega - h\,dt$, we have

$$\Phi = d\omega - \frac{\partial \omega}{\partial t} \wedge dt - dh \wedge dt.$$

By (1.7), $-dh = \partial\omega/\partial t + i_v\Omega$, which implies (2.16). Formula (2.16) admits a generalization.

Proposition 2. *If system* (1.2) *has the absolute integral invariant with the k-form Ω, then system* (2.1) *has absolute invariant* (2.15) *with the k-form*

$$\phi = \Omega + (-1)^k(i_v\Omega) \wedge dt. \tag{2.17}$$

For $k = 2$, we obtain (2.16). Proposition 2 actually belongs to Cartan (Sec. 30 in [20]); instead of the explicit formula for ϕ, Cartan stated the rule for obtaining it: the differentials dx_i in the explicit expression for Ω should be replaced with the differences $dx_i - v_i\,dt$.

As an example, we consider the motion of a medium in the Euclidean space $E^3 = \{x_1, x_2, x_3\}$; this motion is described by the system of equations

$$\dot{x}_1 = v_1, \qquad \dot{x}_2 = v_2, \qquad \dot{x}_3 = v_3. \tag{2.18}$$

The functions v_i depend on x and t. Let $\rho(x,t)$ be the density of the medium at the point x at the instant t. The mass contained in a three-dimensional volume τ at the instant t equals

$$\int_\tau \rho\,dx_1 \wedge dx_2 \wedge dx_3. \tag{2.19}$$

This integral is, of course, the absolute invariant of system (2.18). Continuity equation (1.3) in Chap. 1 expresses the conservation of integral (2.19).

Applying transformation (2.17) to the 3-form $\Omega = \rho\,d^3x$, we obtain the 3-form

$$\begin{aligned}\phi = \rho(&dx_1 \wedge dx_2 \wedge dx_3 - v_1\,dx_2 \wedge dx_3 \wedge dt \\ &- v_2\,dx_3 \wedge dx_1 \wedge dt - v_3\,dx_1 \wedge dx_2 \wedge dt).\end{aligned}$$

Cartan called this form the *element of matter.* Considering a three-dimensional aggregation of particles of the medium (such that each particle is considered at its individual instant of its motion), we obtain a three-dimensional volume τ^* in the four-dimensional space–time $\{x_1, x_2, x_3, t\}$. The integral of the 3-form ϕ over τ^* is obviously equal to the mass of the particle aggregation considered.

Cartan noted that in the general case, Proposition 2 does not hold for relative invariants. We complement Cartan's observation with the following assertion.

Proposition 3. *If system* (1.2) *has the relative integral invariant with the k-form Ω,*

$$\frac{\partial \Omega}{\partial t} + L_v \Omega = d\Lambda,$$

and the $(k-1)$-form Λ generates the absolute invariant

$$\frac{\partial \Lambda}{\partial t} + L_v \Lambda = 0,$$

then system (2.1) *has the relative invariant with k-form* (2.17) *and*

$$L_v \phi = \tilde{d}\Psi,$$

where

$$\Psi = \Lambda + (-1)^{k-1}(i_v \Lambda) \wedge dt.$$

Here, $\tilde{d}$ denotes the exterior differentiation in the $(n+1)$-dimensional space–time.

2.7. The transition to the space–time allows a simple formulation of the Noether theorem for the Lamb equation. Let $u(z)$ be a vector field on $\widetilde{M}$. The system of differential equations

$$\frac{dz}{d\alpha} = u(z), \quad -\varepsilon < \alpha < \varepsilon, \quad \varepsilon > 0,$$

corresponds to this field; the phase flow g_u^α of the system can be treated as the variation for a sufficiently small α.

Let γ be an arbitrary segment of a world line. We say that action (2.8) is invariant under the one-parameter group g_u^α if

$$\int_{g_u^\alpha(\gamma)} \varphi = \text{const}. \tag{2.20}$$

Theorem 4. *If the action functional is invariant under the transformation group g_u in the space–time, then Eq.* (1.2) *has the first integral $\varphi(u)$.*

Proof. Differentiating (2.20) with respect to α and applying the lemma on the action variation, we obtain

$$0 = \delta I = \varphi(u)\Big|_{z_1}^{z_2} + \int_\gamma i_u\phi, \tag{2.21}$$

where z_1 and z_2 are the endpoints of the segment γ. Because the vector $\tilde{v}$ is tangent to γ and (2.3) holds, the integral in the right-hand side of (2.21) vanishes. Therefore, $\varphi(u)$ takes a constant value on each world line. □

§3. Invariant Volume Forms

3.1. For flows of fluid in $E^3 = \{x\}$, the differential 3-form $\tau = \rho\, d^3x$, where $\rho(x,t)$ is the density of the mass distribution, is the invariant volume form, i.e.,

$$\int_{g_v^t(D)} \tau = \text{const} \tag{3.1}$$

for any measurable three-dimensional domain $D \subset E^3$. Integral relation (3.1) is equivalent to the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0. \tag{3.2}$$

In the case where $\rho = $ const (a homogeneous fluid), this equation takes the simplest form, $\operatorname{div} v = 0$. If the fluid flow is a potential flow (i.e., $v = \partial\varphi/\partial x$), then we obtain the Laplace equation $\Delta\varphi = 0$ for the potential of the velocity field, and φ is hence a harmonic function in E^3. Therefore, in the stationary case, the problem is reduced to the boundary problem for the Laplace equation. All the required information that can give the dynamic Euler equations consists of the Lagrange theorem on the conservation of the flow potential of an ideal fluid.

In this section, we study the problem of the existence of invariant n-forms (volume forms on M) for differential equations (1.2). These forms play the role of densities and correspond to the mass conservation law.

3.2. In what follows, the notion of the *class* of a differential form plays a significant role. Cartan introduced this notion in [20]. We recall that the class of a p-form α at the point $x \in M$ is the codimension of the subspace consisting of vectors $\xi \in T_xM$ such that

$$i_\xi\alpha = i_\xi\, d\alpha = 0.$$

We consider forms of *constant* (i.e., independent of x) *class.*

If the form α is closed, i.e., $d\alpha = 0$, then the class of α is equal to the codimension of the subspace of vectors ξ such that $i_\xi\alpha = 0$. This number is also called the *rank* of the form α. In particular, because any n-form τ on an

n-dimensional manifold is closed, τ is a *volume form* if and only if the class (or rank) of τ equals n. We also note that the class of a closed 2-form is always even.

We need the following two simple assertions (see, e.g., Chap. 6 in [30]):

1. Let n be even, $n = 2s$. A closed 2-form α has the maximal class n if and only if
$$\alpha^s = \underbrace{\alpha \wedge \cdots \wedge \alpha}_{s \text{ times}}$$
is a volume form.
2. Let n be odd, $n = 2s + 1$. Then the class of a 1-form α is maximal (equals n) if and only if $\alpha \wedge (d\alpha)^s$ is a volume form.

In Secs. 1 and 2, we introduced two forms $\omega = \sum u_i\, dx_i$ and $\Omega = d\omega$, which generate the respective relative and absolute integral invariants of system (1.2).

Proposition 4. *Let $n = 2s$ and the class of the 2-form Ω equal n. Then system* (1.2) *has the integral invariant*

$$\int_\gamma \tau, \tag{3.3}$$

where $\tau = \Omega^s$.

Proof. Because the 2-form Ω is closed, τ is a volume form on M^n. Further, by virtue of (2.12),

$$\dot\tau = (\Omega^s)^\bullet = \dot\Omega \wedge \Omega \wedge \cdots \wedge \Omega + \Omega \wedge \dot\Omega \wedge \cdots \wedge \Omega + \cdots = 0.$$

The proposition is proved. □

We note that the famous Liouville theorem on the conservation of the phase volume for Hamiltonian systems is a particular case of Proposition 4.

We now consider the case where $n = 2s + 1$ and the class of the form ω equals n. Then the n-form $\tau = \omega \wedge \Omega^s$ is a volume form. This form is not invariant in the general case; indeed, by (1.8),

$$\begin{aligned}\dot\tau &= \dot\omega \wedge \Omega^s + \omega \wedge \dot\Omega \wedge \Omega \wedge \cdots \wedge \Omega + \cdots \\ &= \dot\omega \wedge \Omega^s = dg \wedge \Omega^s,\end{aligned} \tag{3.4}$$

where $g = i_v\omega - h$ is the "Lagrangian."

Because the form Ω is closed, we have

$$dg \wedge \Omega^s = d(g\Omega^s).$$

Therefore, for a compact manifold M, we obtain

$$\frac{d}{dt}\int_M \tau = \int_M dg \wedge \Omega^s = \int_M d(g\Omega^s) = 0,$$

and the volume τ of the whole M is hence conserved. This remark is informative only in the nonautonomous case.

We consider the important particular case where (1.1) is the Lamb equation for a stationary, invariant n-dimensional surface of a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum g_{ij}(x) y_i y_j.$$

By virtue of the Euler formula for homogeneous functions,

$$i_v\omega = \sum y_i \left.\frac{\partial H}{\partial y_i}\right|_{y=u} = 2h. \tag{3.5}$$

Proposition 5 ([42]). *In the case considered, Eqs.* (1.2) *has the integral invariant* (3.3), *where* $\tau = \omega \wedge \Omega^s$.

Proof. By (3.5), we have $g = 2h - h = h$. In the stationary case, Eq. (1.7) yields $dh = -i_v\Omega$. Therefore,

$$\dot{\tau} = -i_v\Omega \wedge \Omega^s.$$

The relation $\Omega^{s+1} = 0$ implies

$$i_v\Omega^{s+1} = (s+1)i_v\Omega \wedge \Omega^s = 0$$

and hence $\dot{\tau} = 0$. □

3.3. If the classes of the forms ω and Ω are not maximal, then we cannot obtain any volume form, and the probem of the existence of invariant measures for Eq. (1.2) is nontrivial.

In Sec. 8 in Chap. 1, we saw that this problem can be easily solved if we know the complete solution $u(x,t,c)$, $c = (c_1, \dots, c_n)$, of the Lamb equation for the Hamiltonian system with n degrees of freedom. Indeed, for fixed c, Eq. (1.2) has the invariant

$$\int_\gamma \rho\, d^n x,$$

where

$$\rho = \frac{\partial(u_1, \dots, u_n)}{\partial(c_1, \dots, c_n)}. \tag{3.6}$$

The density $\rho(x,t)$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum \frac{\partial(\rho v_i)}{\partial x_i} = 0. \tag{3.7}$$

For potential flows, i.e., $u = \partial\varphi/\partial x$, the complete solution of the Lamb equation becomes the complete integral $\varphi(t,x,c)$ of the Hamilton–Jacobi equation

$$\frac{\partial \varphi}{\partial t} + H\left(x, \frac{\partial \varphi}{\partial x}, t\right) = 0.$$

In this case, formula (3.6) becomes

$$\rho = \det \left\| \frac{\partial^2 \varphi}{\partial x_i \partial c_j} \right\|.$$

Tne nondegeneracy of the matrix of second derivatives for the potential is a feature of the complete integral.

We consider the stationary solution for the Hamiltonian system with the natural Hamiltonian

$$H = \frac{1}{2} \sum a_{ij}(x) y_i y_j + V(x).$$

We have

$$v_i = \sum a_{ij} \frac{\partial \varphi}{\partial x_j},$$

and (3.7) hence takes the explicit form

$$\sum_i \frac{\partial}{\partial x_i} \left(\rho \sum_j a_{ij} \frac{\partial \varphi}{\partial x_j} \right) = 0. \tag{3.8}$$

This is an analogue of the Laplace equation for the velocity potential of a homogeneous fluid. If

$$\rho = c\sqrt{g}, \qquad g = \det \|a_{ij}\|, \qquad c = \text{const},$$

then (3.8) becomes the Laplace–Beltrami equation on M equipped with the Riemannian metric defined by the kinetic energy. In this case, the potential φ is a harmonic function on M.

§4. Vortex Manifolds

4.1. We fix an instant t. We recall that a tangent vector $w \in T_x M$ such that

$$i_w \Omega = 0 \tag{4.1}$$

is called a *vortex vector* at the point $x \in M^n$. In matrix notation, this equation has the form $(\operatorname{rot} u)w = 0$, i.e., the vortex vector is an eigenvector of the skew-symmetric curl matrix belonging to the eigenvalue zero.

By (4.1), all vortex vectors at the point x compose a linear subspace $\Pi_x \subset T_x M$. Let $m = \dim \Pi_x$. It is clear that

$$m = n - \operatorname{rank} \Omega.$$

Because the 2-form Ω is closed, its rank is equal to its class. Obviously, $\operatorname{rank} \Omega$ coincides with the rank of the skew-symmetric matrix $\operatorname{rot} u$ and is even. Particularly, in the case where the dimension of the configuration space is odd, we have $m \geq 1$.

In what follows, we assume that Ω is a form of *constant rank* (or class), i.e., its rank is independent of x in the entire M or in some domain of M. In this case, the family $\{\Pi_x\}$ generates an m-dimensional *distribution* of tangent planes on M^n. A regular m-dimensional manifold $W \subset M$ is called the *integral manifold* of the distribution Π if the m-dimensional plane tangent to W at any point $x \in W$ coincides with Π_x:

$$T_x W = \Pi_x.$$

A distribution Π is said to be *integrable* if for any point $x \in M$, there exists an integral manifold of Π; in other words, the entire M is fibered into m-dimensional integral manifolds of the distribution Π.

Proposition 6. *The distribution* $\{\Pi_x\}$ *of vortex vectors is integrable.*

Proof. Let w_1 and w_2 be two vector fields on M consisting of vortex vectors. If Π is integrable, then fields w_1 and w_2 are tangent to an integral manifold W, and their commutator $[w_1, w_2]$ is therefore also tangent to W, i.e., is a vortex field. This necessary condition for the integrability of the plane distribution Π is also sufficient (see, e.g., [22]). We prove this statement.

Let

$$i_{w_1}\Omega = i_{w_2}\Omega = 0.$$

We use the formula

$$i_{[w_1,w_2]}\Omega = [L_{w_1}, i_{w_2}]\Omega = L_{w_1}(i_{w_2}\Omega) - i_{w_2}(L_{w_1}\Omega) = -i_{w_2}(L_{w_1}\Omega)$$

from Chap. 4 in [30]. On the other hand, for any vortex field w, we have

$$L_w\Omega = d(i_w\Omega) + i_w d\Omega = 0$$

because the 2-form Ω is closed. □

Integral manifolds of the vortex vector distribution are called *vortex manifolds.* They naturally generalize the notion of vortex lines in hydrodynamics. We note that vortex manifolds, generally speaking, differ at distinct times.

Remark. In the proof of the integrability of the vortex vector distribution, we use only the property that the form Ω is closed.

4.2.

Theorem 5. *The flow of the system*

$$\dot{x} = v(x, t) \tag{4.2}$$

transforms vortex manifolds into vortex manifolds.

Proof. We use condition (2.12) for the "freezing" of the form Ω into the flow of system (4.2):

$$\dot{\Omega} = \frac{\partial \Omega}{\partial t} + L_v\Omega = 0. \tag{4.3}$$

This means that the value of the form Ω on any pair of vectors transferred by flow (4.2) is constant. This can be explained as follows. Let $w_1(0)$ and $w_2(0)$ be two vectors tangent to M at the point x_0 for $t = 0$. Let $w_1(t)$ and $w_2(t)$ be their images under the mapping g_v^t (more precisely, under the differential dg^t); they are tangent vectors at the point $x(t) = g^t(x_0)$. Relation (4.3) implies

$$\Omega\left(w_1(t), w_2(t)\right) = \text{const}\,. \tag{4.4}$$

If $w_1(0)$ is a vortex vector, then $\Omega(w_1(0), \cdot\,) = 0$. Therefore, by virtue of (4.4), $\Omega(w_1(t), \cdot\,) = 0$, and $w_1(t)$ is hence a vortex vector for all t.

We find that the differential dg^t of the phase flow transforms the vortex vector distribution at $t = 0$ into the vortex vector distribution at the instant t. Let W_0 be a vortex manifold at the initial instant and $W_t = g^t(W_0)$. Because the vectors $w(0)$ are tangent to the surface W_0, their images $w(t)$ are tangent to W_t. Therefore, W_t is the vortex manifold at the instant t. $\square$

Theorem 5 is a multidimensional analogue of the famous Helmholtz–Thomson theorem on the conservation ("freezing-in") of vortex lines. In the proof of Theorem 5, we use Eq. (4.3) and the property that the form Ω is closed. Therefore, Theorem 5 also includes the theorem on the conservation of magnetic force lines in magetohydrodynamics (see Sec. 1 in Chap. 1).

4.3. We consider the stationary case, where the form ω, field v, and function h are independent of time. In this case, Lamb equation (1.7) becomes

$$i_v\Omega = -dh, \qquad \Omega = d\omega. \tag{4.5}$$

Theorem 6. *In the stationary case, the function h is constant on the integral curves of the field v and on the vortex manifolds.*

Trajectories of the system $\dot{x} = v(x)$ can be called *streamlines.* Hence, Theorem 6 is a multidimensional generalization of the classic *Bernoulli theorem* in ideal fluid dynamics.

Proof. Because $\Omega(v, v) = 0$, (4.5) implies

$$i_v\, dh = \frac{\partial h}{\partial x} \cdot v = 0.$$

Therefore, h is constant on streamlines.

Now let w be an arbitrary vector field consisting of vortex vectors that are tangent to a connected vortex manifold W. Because $i_w\Omega = 0$, (4.5) implies the identity

$$0 = i_w i_v\Omega = -i_w\, dh = -\frac{\partial h}{\partial x} \cdot w.$$

Hence, h is constant on W. $\square$

4.4. The problem of finding vortex manifolds is well simplified if we represent the covector field $u(x,t)$ as the sum

$$\frac{\partial S}{\partial x} + A_1 \frac{\partial B_1}{\partial x} + \cdots + A_k \frac{\partial B_k}{\partial x}, \tag{4.6}$$

where S, A_1, B_1, etc., are some functions of x and t. Because of the relation

$$u = \frac{\partial S'}{\partial x} - \sum B_s \frac{\partial A_s}{\partial x}, \qquad S' = S + \sum A_i B_i,$$

the functions A_s and B_s have the same meaning. In hydrodynamics, the functions S, A_1, B_1, etc., are called the *Clebsch potentials* (see, e.g., Sec. 167 in [51]).

If the potentials $A_1, B_1, \ldots, B_k$ considered as functions of x are independent, then $\operatorname{rank}(\operatorname{rot} u) = 2k$. Because

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \sum \left(\frac{\partial A_s}{\partial x_j} \frac{\partial B_s}{\partial x_i} - \frac{\partial A_s}{\partial x_i} \frac{\partial B_s}{\partial x_j} \right),$$

vortex vectors coincide with vectors tangent to the $(n-2k)$-dimensional surfaces

$$\{x \in M : A_1(x,t) = a_1,\ B_1(x,t) = b_1,\ \ldots,\ B_k(x,t) = b_k\},$$

where $a, b = \text{const}$. Therefore, these surfaces are the required vortex manifolds.

If the 1-form ω has a constant class, then the Clebsch potentials exist (the Darboux theorem). Moreover, the functions $A_1, B_1, \ldots, B_k$ can be taken as the new coordinates, denoted by $x_1, \ldots, x_{2k}$. In an explicit form, we write formulas (4.6) in components as

$$u_1 = \frac{\partial S}{\partial x_1}, \qquad u_2 = \frac{\partial S}{\partial x_2} + x_1, \qquad \ldots,$$
$$u_{2k+1} = \frac{\partial S}{\partial x_{2k+1}}, \qquad \ldots, \qquad u_n = \frac{\partial S}{\partial x_n},$$

and Lamb equation (1.1) as

$$\begin{aligned} \dot{x}_1 &= -\frac{\partial}{\partial x_2}\left(\frac{\partial S}{\partial t} + h\right), & \dot{x}_2 &= \frac{\partial}{\partial x_1}\left(\frac{\partial S}{\partial t} + h\right), & \ldots,\\ \dot{x}_{2k-1} &= -\frac{\partial}{\partial x_{2k}}\left(\frac{\partial S}{\partial t} + h\right), & \dot{x}_{2k} &= \frac{\partial}{\partial x_{2k-1}}\left(\frac{\partial S}{\partial t} + h\right), \end{aligned} \tag{4.7}$$

and

$$\frac{\partial}{\partial x_{2k+1}}\left(\frac{\partial S}{\partial t} + h\right) = \cdots = \frac{\partial}{\partial x_n}\left(\frac{\partial S}{\partial t} + h\right) = 0. \tag{4.8}$$

Relation (4.8) implies that $\partial S/\partial t + h$ is a function depending only on the coordinates $x_1, \ldots, x_{2k}$ and time t. This relation becomes the Hamilton–Jacobi equation for $k = 0$ (i.e., if u is a potential field) and therefore generalizes

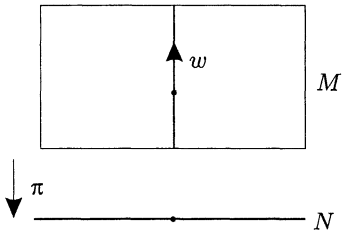

Fig. 19. Quotient system

it. System (4.7) is the closed system of canonical differential equations for the Clebsch potentials with the Hamiltonian $\partial S/\partial t + h$. This generalizes the known results of Clebsch and Stewart [51] on vortex flows of an ideal fluid (for $n = 3$).

Equations (4.8) generalize the second part of the Bernoulli theorem: the function $\partial S/\partial t + h$ is constant on vortex manifolds.

4.5. By Proposition 6, if the 2-form Ω has a constant rank, then the n-dimensional manifold M is fibered into $(n-2k)$-dimensional vortex manifolds W_x $(2k = \operatorname{rank}\Omega)$. We introduce an *equivalence relation* on M by identifying points lying in the same connected component of the vortex manifold. This equivalence relation defines the *quotient space*

$$N = M/W.$$

The set N is locally a smooth two-dimensional manifold, but in the whole, it may have a complicated topological structure. The nature of the difficulty can be seen in the example of a two-dimensional torus fibered by irrational windings. The mapping

$$\pi : M \to N$$

assigning a vortex manifold W_x passing through x to each point $x \in M$ defines a *bundle*: M is a fiber bundle, N is a base of the bundle, and vortex manifolds W_x are fibers.

By the Helmholtz–Thomson theorem (Theorem 5), the phase flow g_v^t of system (1.2) transforms vortex manifolds on M into vortex manifolds. Therefore, the action of g^t on the base N is well defined. Differentiating the mapping

$$z \to g^t(z), \quad z \in N,$$

with respect to t, we obtain the vector field $V(z,t)$ and the system of differential equations on N

$$\dot{z} = V(z,t), \tag{4.9}$$

which is naturally called the *quotient system*.

Clearly, the differential of π transforms the field v into the field V. This assertion is a consequence of the Helmholtz–Thomson theorem. The image of an arbitrary vector field on M under the mapping $d\pi$ is not defined at all, because it depends on the choice of a point on the vortex manifold. Obviously, $d\pi$ annihilates vortex vectors. Therefore, vortex vectors are *vertical.* Moreover, the image of the vector field $v + \sum \lambda_i w_i$ under the mapping $d\pi$ equals V for all λ. To avoid an ambiguity, we do not define a distribution of vertical vectors on M; such vectors are said to be *horizontal.* A distribution of horizontal vectors on M defines a *connection* in the bundle $\pi : M \to N$.

The concept of a connection can be illustrated by the problem of lifting paths in the base N to horizontal paths in M. A path $\Gamma(t) : [a,b] \to M$ is said to be horizontal if the tangent vector $\dot\Gamma$ is horizontal for all $t \in [a,b]$. Let $\gamma : [a,b] \to N$ be an arbitrary smooth path in the base and $\pi(x_0) = \gamma(a)$. It is easy to see that there exists a unique path $\Gamma : [a,b] \to M$ such that $\Gamma(a) = x_0$ and $\pi(\Gamma(t)) = \gamma(t)$ for all t. This construction allows defining the parallel translation of vectors in M. (See monograph [54] for the theory of fibered spaces.)

4.6. It turns out that system (4.9) on the base N takes the form of the Lamb equation with a nondegenerate curl matrix. For the proof, we define vortex manifolds W as common level surfaces for $2k$ independent functions

$$f_1(x,t), \quad \dots, \quad f_{2k}(x,t).$$

We take these functions as the first $2k$ local coordinates and keep the notation $x_1, \dots, x_n$ for the new variables. In the new coordinates, the form of Eqs. (1.1) is preserved, and vortex vectors are linear combinations of the vectors

$$\begin{aligned} w_{2k+1} &= (0,\ \dots,\ 0,\ 1,\ 0,\ \dots,\ 0)^{\mathrm{T}}, \quad \dots, \\ w_n &= (0,\ \dots,\ 0,\ 0,\ 0,\ \dots,\ 1)^{\mathrm{T}}. \end{aligned}$$

Relation (4.1) is equivalent to the series of equalities

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = 0, \quad i = 1, \dots, n, \quad j = 2k+1, \dots, n. \tag{4.10}$$

Let $p \neq q$ be arbitrary subscripts. Then

$$\frac{\partial u_p}{\partial x_j} = \frac{\partial u_j}{\partial x_p} \quad \text{and} \quad \frac{\partial u_q}{\partial x_i} = \frac{\partial u_j}{\partial x_q}.$$

Therefore,

$$\frac{\partial^2 u_p}{\partial x_q \partial x_j} = \frac{\partial^2 u_j}{\partial x_q \partial x_p} \quad \text{and} \quad \frac{\partial^2 u_q}{\partial x_p \partial x_j} = \frac{\partial^2 u_j}{\partial x_p \partial x_q},$$

and we have

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_p}{\partial x_q} - \frac{\partial u_q}{\partial x_p}\right) = 0,$$

that is, the elements of the matrix $\operatorname{rot} u$ are independent of $x_{2k+1}, \ldots, x_n$ in the new variables. Taking (4.10) into account, we find that the closed 2-form Ω has the form

$$\sum_{i,j=1}^{2k} \omega_{ij}\, dx_i \wedge dx_j,$$

where the coefficients ω_{ij} depend only on $x_1, \ldots, x_{2k}$, and t.

By the local Poincaré lemma, we have

$$\Omega = d\left(U_1\, dx_1 + \cdots + U_{2k}\, dx_{2k}\right),$$

where U_s are functions of $x_1, \ldots, x_{2k}$, and t. Because the 1-form

$$\sum_{j=1}^{n} u_j\ dx_j - \sum_{i=1}^{2k} U_i\, dx_i$$

is closed, it is a differential of some function $S(x_1, \ldots, x_n, t)$ by the Poincaré lemma, and hence

$$\begin{aligned} u_i &= U_i + \frac{\partial S}{\partial x_i}, \quad && i = 1, \ldots, 2k, \\ u_j &= \frac{\partial S}{\partial x_j}, && j = 2k+1, \ldots, n. \end{aligned}$$

We find that in the new variables, system (1.1) segregates into two subsystems,

$$\frac{\partial U}{\partial t} + (\operatorname{rot} U)V = -\frac{\partial}{\partial X}\left(h + \frac{\partial S}{\partial t}\right), \tag{4.11}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial S}{\partial t} + h\right) = 0, \qquad j = 2k+1, \ldots, n, \tag{4.12}$$

where X denotes the set of the local coordinates $x_1, \ldots, x_{2k}$ on the base N. According to (4.12), the function $h + \partial S/\partial t$ depends only on $x_1, \ldots, x_{2k}$ and t, and (4.11) is therefore the Lamb equation on N, and the dynamics is governed by the equation $\dot{X} = U(X, t)$. Obviously, the matrix $\operatorname{rot} U$ is nondegenerate. Equation (4.11) can be transformed into the form of the Hamilton differential equations by using the Darboux theorem (similarly to Sec. 4.4).

We note that relations (4.12) are a local generalization of the Bernoulli theorem: the function $h + \partial S/\partial t$ is constant on vortex manifolds for any fixed t. This result can be globalized if we assume that the form $\Omega = d\omega$ is stationary (i.e., is explicitly independent of t). If the configuration space M is simply connected, then there exists a function $S(x, t)$ on M such that $\omega = \tilde{\omega} + dS$, where the 1-form $\tilde{\omega}$ is independent of t. The Lamb equation becomes

$$i_v \Omega = -d\tilde{h}, \qquad \tilde{h} = h + \frac{\partial S}{\partial t}. \tag{4.13}$$

In this case, vortex manifolds are stationary; the function $\tilde{h}$ on them takes values that can depend only on t. Relation (4.13) implies

$$\frac{d\tilde{h}}{dt} = \frac{\partial \tilde{h}}{\partial t}.$$

In particular, if $\tilde{h}$ is explicitly independent of t, then it is constant on streamlines.

§5. Euler Equation

5.1. Let $w(x,t)$ be a smooth vector field on M consisting of vortex vectors. The following assertion holds.

Proposition 7. *The vector field*

$$\frac{\partial w}{\partial t} + [w, v]$$

is also a vortex field.

Corollary 1. *Let $w^{(1)}, \dots, w^{(m)}$ $(m = n - \operatorname{rank}(\operatorname{rot} u))$ be a basis of vortex vector fields. Then*

$$\frac{\partial w^{(i)}}{\partial t} + \left[w^{(i)}, v\right] = \sum_{j=1}^{m} \lambda_{ij} w^{(j)}, \quad i = 1, \dots, m,$$

where λ_{ij} are smooth functions of x and t.

Proof of Proposition 7. Using results in Sec. 2, we reduce Lamb equation (1.1) to the autonomous case. For this, we set

$$\tilde{v} = (v_1, \dots, v_n, 1) \qquad \text{and} \qquad \tilde{w} = (w_1, \dots, w_n, 0),$$

where v_i and w_j are the components of the respective vector fields v and w. By (2.2) and (2.3), $\tilde{v}$ and $\tilde{w}$ are the vortex fields of some closed 2-form ϕ,

$$i_{\tilde{v}}\phi = i_{\tilde{w}}\phi = 0.$$

We use the formula

$$i_{[\tilde{w},\tilde{v}]}\phi = L_{\tilde{v}} i_{\tilde{w}}\phi - i_{\tilde{w}} L_{\tilde{v}}\phi$$

(see Chap. 4 in [30]). Because $i_{\tilde{w}}\phi = 0$ and $L_{\tilde{v}}\phi = i_{\tilde{v}}\, d\phi + d i_{\tilde{v}}\phi = 0$, we find that $[\tilde{w}, \tilde{v}]$ is a vortex vector with the components

$$\frac{\partial w_1}{\partial t} + [w, v]_1, \quad \dots, \quad \frac{\partial w_n}{\partial t} + [w, v]_n, \quad 0. \tag{5.1}$$

If $w^{(1)}, \dots, w^{(m)}$ is a basis of the vortex vectors, then $\tilde{w}^{(1)}, \dots, \tilde{w}^{(m)}, \tilde{v}$ is a basis of the vortex vectors in the extended space. Hence,

$$[\tilde{w}, \tilde{v}] = \sum \lambda_i \tilde{w}^{(i)} + \mu \tilde{v}.$$

Because the t-component of the field $\tilde{v}$ is 1, we have $\mu \equiv 0$ by virtue of (5.1). Therefore, $\partial w/\partial t + [w, v]$ is a vortex vector for all t. □

Remark. Proposition 7 is well known in the three-dimensional case. Poincaré, Zhuravski, and Fridman [27, 62] found the criterion for freezing integral curves of the vector field $w(x,t)$ into the flow generated by the velocity field $v(x,t)$: the vectors

$$\dot{w} - L_w v \qquad \text{and} \qquad w$$

are collinear. Because

$$\dot{w} = \frac{\partial w}{\partial t} + L_v w$$

and

$$L_v w - L_w v = [w, v],$$

we have

$$\frac{\partial w}{\partial t} + [w, v] = \lambda w.$$

5.2. We consider an important particular case, where n is odd and the 2-form Ω is nonsingular for all t. In this case, vortex vectors form a one-dimensional subspace at any point $x \in M$. The following local theorem holds.

Theorem 7 ([39]). *If n is odd and Ω is nonsingular, then there exists a smooth vortex vector field $w(x,t)$ such that*

$$\frac{\partial w}{\partial t} + [w, v] = 0. \tag{5.2}$$

This equation is an analogue of the well-known Euler equation for the momentum variation. Theorem 1 in Chap. 1 is a particular case of Theorem 7. Equation (5.2) is also called the *Euler equation.*

Proof of Theorem 7. By Proposition 7, under our assumption, any nonzero vortex field satisfies the equation

$$\frac{\partial w}{\partial t} + [w, v] = \mu w,$$

where $\mu(x,t)$ is a smooth function. In this equation, we replace w with the vortex field λw and choose the multiplier λ such that λw satisfies (5.2). We thus obtain the equation

$$\frac{\partial \lambda}{\partial t} + L_v \lambda = \mu \lambda \tag{5.3}$$

or, equivalently, $\dot{\lambda} = \mu\lambda$, where the dot denotes the total derivative with respect to time by virtue of system (1.2).

We show that Eq. (5.3) is always locally solvable. We note that (5.3) can be rewritten in the equivalent form

$$L_{\tilde{v}} \lambda = \mu \lambda, \tag{5.4}$$

where $\tilde{v}$ is the natural extension of the field v to the extended space–time $M \times \mathbb{R}_t$. Because $\tilde{v} \neq 0$, the components of $\tilde{v}$ in some new local coordinates $z_1, \ldots, z_{n+1}$ in $M \times \mathbb{R}$ are $1, 0, \ldots, 0$ (by the theorem on straightening vector fields). Therefore, Eq. (5.4) becomes

$$\frac{\partial \lambda}{\partial z_1} = \mu \lambda$$

and can be easily solved:

$$\lambda = \exp\left(\int \mu \, dz_1\right).$$

The theorem is proved. □

5.3. The case where $\dim M = 3$ is of specific interest. Obviously, the 2-form $\Omega = d\omega$ is nonsingular if and only if $\Omega \neq 0$.

We consider a natural mechanical system with the configuration space M; the Hamiltonian of this system is the sum of the kinetic energy T and the potential energy V. In the three-dimensional case, the metric

$$T = \frac{1}{2} \sum g_{ij}(x) \dot{x}_i \dot{x}_j$$

allows calculating the curl of the velocity field v on M; this field may depend on time. We introduce the invariant definition of $\operatorname{rot} v$. For this, we assign a covector field $u = \{u_j\}$ with the components

$$u_j = \sum_i g_{ij} v_i \tag{5.5}$$

to the vector field $v = \{v_i\}$. Let $\omega = \sum u_j \, dx_j$, let $\Omega = d\omega$, let

$$\tau = \sqrt{g}\, d^3 x$$

be a 3-form of oriented volume on M (g is the determinant of the matrix $\|g_{ij}\|$), and let

$$i_{\operatorname{rot} v} \tau = \Omega. \tag{5.6}$$

If the space is Euclidean (i.e., $\|g_{ij}\|$ is the identity matrix), then (5.6) defines the usual curl field. We note that u is the value of the canonical momentum $y = \partial T / \partial \dot{x}$ calculated for the field v. Using (5.6), we can write the components of $\operatorname{rot} v$ explicitly:

$$\frac{1}{\sqrt{g}}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \quad \frac{1}{\sqrt{g}}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \quad \frac{1}{\sqrt{g}}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

In these formulas, the components u_1, u_2, and u_3 are calculated by (5.5). (Another way of deriving the formulas for the curl components can be found, e.g., in the classic book by Veblen [75]; see also [68]).

We now consider another 3-form $\tau' = \rho\tau$ on M, where ρ is a smooth positive function of x and t, and assume that τ' generates the absolute integral invariant

$$\int \tau' \tag{5.7}$$

of system (1.2). This means that the product $\rho\sqrt{g}$ satisfies Liouville equation (3.2). Equation (5.6) implies that the field $\operatorname{rot} v$ is a vortex field.

Theorem 8. *The vortex vector field*

$$w(x,t) = \frac{\operatorname{rot} v}{\rho} \tag{5.8}$$

satisfies Euler equation (5.2).

This theorem is a direct generalization of Theorem 1 in Chap. 1 in compressible fluid dynamics. Here, $M = E^3$, the metric is Euclidean, and invariant (5.7) is the mass of the medium in a moving volume.

Proof of Theorem 8. Relations (5.6) and (5.8) imply

$$i_w\tau' = \Omega.$$

Applying the operator $\partial/\partial t$ to both sides of the last equation, we obtain

$$i_{\partial w/\partial t}\tau' + i_w \frac{\partial \tau'}{\partial t} = \frac{\partial \Omega}{\partial t}. \tag{5.9}$$

Further,

$$L_v i_w \tau' = i_w L_v \tau' + i_{[w,v]}\tau' = L_v \Omega. \tag{5.10}$$

Because the forms Ω and τ' generate the absolute integral invariants, we have

$$\frac{\partial \Omega}{\partial t} + L_v\Omega = 0 \qquad \text{and} \qquad \frac{\partial \tau'}{\partial t} + L_v\tau' = 0. \tag{5.11}$$

Adding (5.9) and (5.10) and taking (5.11) into account, we obtain

$$i_{\partial w/\partial t}\tau' + i_{[w,v]}\tau' = 0.$$

Because the 3-form τ' is nondegenerate, we have

$$\frac{\partial w}{\partial t} + [w,v] = 0.$$

The theorem is proved. □

Remark. In the Euclidean space, the divergence of any curl field is zero. This property also holds in an arbitrary Riemannian space if we define the divergence of the vector field $v = \{v_i\}$ as

$$\frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x_i}\left(\sqrt{g}\, v_i\right).$$

In Riemannian geometry, this quantity is called the *absolute divergence* and the sum

$$\sum \frac{\partial v_i}{\partial x_i} \tag{5.12}$$

is called the *divergence of the vector density* $\{v_i\}$. The latter expression is also invariant (i.e., is independent of the choice of local coordinates) for fields of the form ρv, where ρ is the density of some volume form. The divergence enters Liouville equation (3.2) in exactly such a way. We note that (5.12) is independent of the Riemannian metric.

5.4. Theorem 8 admits a generalization to the case where $n = \dim M$ is an arbitrary odd number. Let $\tau = \sqrt{g}\, d^n x$ be a volume form and the closed 2-form Ω have the constant class $n-1$. We define the vector field $w(x,t)$ by the formula

$$i_w \tau = \Omega^k, \quad n = 2k+1.$$

Because the n-form τ is nondegenerate, the field w is well defined. The field $w = \operatorname{rot} v$ is called the *generalized curl* of the vector field v (see, e.g., [75]). The absolute divergence of this field is obviously equal to zero. Because the class of the 2-form Ω is $2k = n-1$, we find that w is a vortex vector for all values of x and t.

Theorem 9. *If system* (1.2) *has integral invariant* (5.7), *where* $\tau' = \rho\tau$, *then the vector field* w/ρ *satisfies the Euler equation.*

Theorem 9 becomes Theorem 8 for $n = 3$; the assumptions that the 2-form Ω is nonsingular and that the class of this form equals 2 are obviously equivalent.

§6. Vortices in Dissipative Systems

6.1. The energy dissipation in viscous fluid dynamics leads to numerous specific effects, for example, to the diffusion of vortices (see Sec. 2 in Chap. 1). In the dynamics of systems with finitely many degrees of freedom, the fluid friction forces are usually described by the Rayleigh dissipative function [66] if the velocity is sufficiently small.

Let M^n be the configuration space of a mechanical system with n degrees of freedom, $(x_1, \dots, x_n) = x$ be the generalized coordinates, T be the kinetic energy, and V be the potential energy of the system. The equations of motion of the mechanical system with fluid friction have the form

$$\left(\frac{\partial L}{\partial \dot{x}}\right)^{\bullet} - \frac{\partial L}{\partial x} = -\frac{\partial \Phi}{\partial \dot{x}}, \qquad L = T - V, \tag{6.1}$$

where Φ is a positive-definite quadratic form in the generalized velocity $\dot{x}$; this form is called the *Rayleigh function*. Let $H = T + V$ be the total mechanical energy. Equations (6.1) imply that

$$\dot{H} = -2\Phi;$$

this means that the energy in fact dissipates.

We consider the case where $\Phi = -\nu T$, where ν is a known positive function of time (e.g., $\nu = \text{const} > 0$). Applying the Legendre transformation, we pass from Lagrange equations (6.1) to the generalized canonical Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x} - \nu y, \tag{6.2}$$

where $y = (y_1, \ldots, y_n)$, $y = \partial T/\partial \dot{x}$, are canonical momenta and $H(x, y, t)$ is the Hamiltonian, which can be assumed to be an arbitrary known function of x, y, and t.

Our first observation is that system (6.2) with dissipation can be reduced to the form of the usual Hamilton differential equations

$$\dot{X} = \frac{\partial K}{\partial Y}, \qquad \dot{Y} = -\frac{\partial K}{\partial X} \tag{6.3}$$

by using the change of variables

$$X = x, \qquad Y = y\mu(t), \qquad \mu = \exp\left[-\int \nu(t)\,dt\right].$$

The function

$$K = \mu H(X, Y/\mu, t) \tag{6.4}$$

plays the role of the new Hamiltonian. Therefore, system (6.2) can be investigated with the methods of Hamiltonian mechanics. However, we prefer the direct method without using the reduction to system (6.3). We must remember that system (6.2) is autonomous in the stationary case (i.e., where the functions T and V are explicitly independent of t), but this property is lost in passing to (6.3).

6.2. We find n-dimensional invariant surfaces Σ for Eqs. (6.2) in the form

$$y = u(x, t).$$

As above, we set

$$h(x, t) = H(x, u(x, t), t) \qquad \text{and} \qquad \dot{x} = v(x, t) = \left.\frac{\partial H}{\partial y}\right|_{y=u}. \tag{6.5}$$

It can be easily shown that the covector field u, the function h, and the vector field v are related by the equation

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x} - \nu u, \tag{6.6}$$

where

$$\operatorname{rot} u = \left\| \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right\|.$$

Equation (6.6) is called the *generalized Lamb equation.* In hydrodynamics, the additional term $-\nu u$ is the external friction force. For $\nu = 0$, we obtain the usual Lamb equation.

Equation (6.6) can be represented by using exterior differential forms. As usual, we set

$$\omega = \sum u_i \, dx_i, \qquad \Omega = d\omega.$$

Then

$$\frac{\partial \omega}{\partial t} + i_v \Omega = -dh - \nu\omega. \tag{6.7}$$

Applying exterior differentiation to both sides of this equation and using the homotopy formula, we obtain

$$\frac{\partial \Omega}{\partial t} + L_v \Omega = -\nu\Omega. \tag{6.8}$$

Clearly, the left-hand side of this relation is the total derivative of Ω by virtue of system (6.5).

Equation (6.7) can be transformed into

$$\frac{\partial \omega}{\partial t} + L_v \omega = dg - \nu\omega, \tag{6.9}$$

where $g = \omega(v) - h$ is the Lagrangian of the considered system restricted to the invariant surface Σ.

6.3. Let g^t be the flow of system (6.5) and γ be a closed contour on M. The integral

$$I = \int_{g^t(\gamma)} \omega \tag{6.10}$$

is a function of t. We apply the well-known formula

$$\dot{I}(t) = \int_{g^t(\gamma)} \dot{\omega}, \qquad \dot{\omega} = \frac{\partial \omega}{\partial t} + L_v \omega.$$

Because γ is a closed contour, (6.9) implies that

$$\dot{I} = -\nu(t) I$$

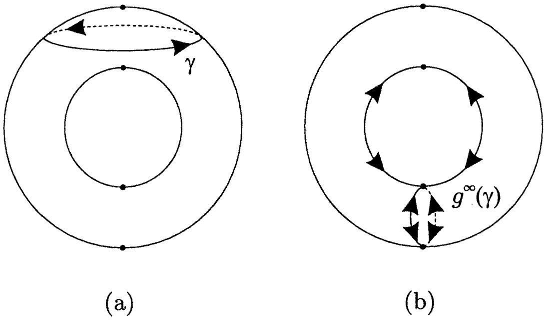

Fig. 20. Evolution of closed paths

and hence

$$I(t) = I(0)\mu(t). \tag{6.11}$$

Therefore, if the integral

$$\int_0^\infty \nu(t)\, dt$$

diverges (e.g., for $\nu = \text{const} \neq 0$), then $I(t)$ monotonically tends to zero as $t \to \infty$. This property can be treated as the *vortex diffusion effect* in finite-dimensional dissipative systems.

This assertion is probably valid for general Rayleigh functions; the following observation supports this. If x_0 is an isolated minimum point of the potential energy V, then x_0 is a point of asymptotically stable equilibrium for system (6.1). In particular, if γ is a closed contour lying in a small neighborhood of x_0, then the contour $g^t(\gamma)$ contracts to x_0 as $t \to +\infty$, and integral (6.10) hence tends to zero.

It is useful to take into account that the evolution of closed paths in dissipative systems can lead to intricate sets. For example, we consider the gradient system

$$\dot{x} = -\frac{\partial \varphi}{\partial x}$$

on the two-dimensional torus realized as a surface of revolution in the three-dimensional space, where φ is a function of the height of the point. Figure 20 shows the evolution of a closed contour lying in a neighborhood of a point of unstable equilibrium: the set $g^t(\gamma)$ tends to a double curve (Fig. 20(b)) as $t \to \infty$.

In viscous fluid dynamics, Eq. (6.8) is replaced by the equation

$$\frac{\partial \Omega}{\partial t} + L_v\Omega = \nu\, \Delta\Omega, \tag{6.12}$$

where Δ is the Laplace operator. Asymptotic properties (as $t \to +\infty$) of solutions to (6.12) on a compact manifold M were studied by Arnol'd. In [6], the author considered effects of the transfer and diffusion of arbitrary k-forms satisfying Eq. (6.12). In our case, the 2-form Ω is closed. In hydrodynamics, the case where M is a three-dimensional torus is of special interest; the investigation of flows with periodic boundary conditions leads to this problem.

Arnol'd showed that if the diffusion coefficient is sufficiently large, then the following assertions hold:

1. in the limit, evolution (6.12) with an arbitrary closed initial condition (i.e., $d\Omega = 0$ for $t = 0$) yields a stationary k-form of the same cohomology class (i.e., with the same values of integrals over k-dimensional cycles);
2. in any cohomology class of closed k-forms, there exists a unique stationary form.

In our case, the 2-form Ω is exact. Therefore, the cohomology class of this form is trivial, and hence $\Omega \to 0$ as $t \to \infty$ (by property 2). In the limit, the integral of the 1-form ω (the fluid circulation) over an arbitrary closed contour γ homotopic to zero vanishes. Evidently,

$$\lim_{t\to+\infty} \oint_{g^t(\gamma)} \omega = 0.$$

6.4. We recall that a covector field u is a potential field ($u = \partial\varphi/\partial x$) if and only if the integral of the 1-form ω over an arbitrary closed contour (contractible to a point) vanishes. Taking this into account, we obtain the *generalized Lagrange theorem* on potential flows from (6.11): if $u(x,t)$ is a potential field for $t = t_0$, then it is a potential field for all t.

Substituting $u = \partial\varphi/\partial x$ in (6.6), we obtain an analogue of the *Lagrange–Cauchy integral*

$$\frac{\partial\varphi}{\partial t} + H\left(x, \frac{\partial\varphi}{\partial x}, t\right) + \nu\varphi = f(t). \tag{6.13}$$

Let $\theta(t)$ be a solution of the ordinary differential equation

$$\dot{\theta} + \nu(t)\theta = f(t).$$

The transformation $\varphi \to \varphi + \theta(t)$ leaves the gradient field invariant and allows setting $f = 0$ in (6.13). For $\nu = 0$, the obtained equation coincides with the Hamilton–Jacobi equation.

In the stationary case (i.e., where φ, ν, and f are explicitly independent of t), Eq. (6.13) was initially obtained by Arzhanykh [12], but he did not relate the substitution $\partial\varphi/\partial x$ to the Lagrange theorem. He proved the following generalization of the Jacobi theorem on the complete integral.

Theorem 10. *Let $\varphi(t, x_1, \ldots, x_n, c_1, \ldots, c_n)$ be a complete integral of Eq. (6.13) for $f = 0$. Then the general solution of Eqs. (6.2) can be found from* *the relations*

$$\frac{\partial\varphi}{\partial c} = -\alpha\mu(t), \qquad \frac{\partial\varphi}{\partial x} = y, \qquad \alpha = (\alpha_1, \dots, \alpha_n) = \text{const}.$$

For $\nu = 0$, we obtain the Jacobi theorem. However, Arzhanykh's theorem can be easily deduced from the Jacobi theorem. For this, we substitute

$$\varphi = \frac{\psi(x,t)}{\mu(t)}$$

in (6.13) for $f = 0$. The function ψ satisfies the equation

$$\frac{\partial\psi}{\partial t} + \mu H\left(x, \frac{1}{\mu}\frac{\partial\psi}{\partial x}, t\right) = 0, \tag{6.14}$$

which explicitly contains derivatives of ψ only. Equation (6.14) is the Hamilton–Jacobi equation for the system with Hamiltonian (6.4).

6.5. Let $w_1(x)$ and $w_2(x)$ be vectors tangent to M at the same point, which are transferred by the flow of system (6.5). Relation (6.8) implies

$$\Omega\left(w_1(t), w_2(t)\right) = \Omega\left(w_1(0), w_2(0)\right)\mu(t).$$

Therefore, if $\Omega\left(w_1(t), w_2(t)\right)$ vanishes for $t = 0$, then this function is identical to zero for all t, and vortex vectors of the closed 2-form Ω are frozen into the flow of system (6.5). This yields the *generalized Helmholtz–Thomson theorem*: the flow of system (6.5) transforms vortex manifolds into vortex manifolds.

6.6. If the 1-form ω has a constant class, then it can be locally transformed into the form

$$\omega = dS + x_1\,dx_2 + \cdots + x_{2k-1}\,dx_{2k},$$

where $2k$ is the rank of the closed 2-form Ω and S is a smooth function of $x_1, \dots, x_n$ and t. The coordinates $x_1, \dots, x_{2k}$ and the function S are the generalized Clebsch potentials (see Sec. 4). In these variables, vortex manifolds are determined by the equations

$$x_1 = a_1, \qquad \dots, \qquad x_{2k} = a_{2k}, \qquad a = \text{const}, \tag{6.15}$$

and generalized Lamb equation (6.6) becomes

$$\begin{aligned} \dot{x}_1 &= -\frac{\partial\varkappa}{\partial x_2} - \nu x_1, \qquad \dot{x}_2 = \frac{\partial\varkappa}{\partial x_1}, \qquad \dots, \\ \dot{x}_{2k-1} &= -\frac{\partial\varkappa}{\partial x_{2k}} - \nu x_{2k-1}, \qquad \dot{x}_{2k} = \frac{\partial\varkappa}{\partial x_{2k-1}}, \end{aligned} \tag{6.16}$$

and

$$\frac{\partial\varkappa}{\partial x_{2k+1}} = \cdots = \frac{\partial\varkappa}{\partial x_n} = 0, \tag{6.17}$$

where $\varkappa = \partial S/\partial t + \nu S + h$.

From (6.15) and (6.17), we obtain the *generalized Bernoulli theorem*: the function $\varkappa$ is constant on vortex manifolds for fixed t. Therefore, Eqs. (6.16) compose a closed system of ordinary differential equations that generalizes the canonical Hamilton equations. Because vortex manifolds are "numbered" by the coordinates $x_1, \ldots, x_{2k}$, we again obtain the Helmholtz–Thomson theorem from this.

Chapter 3
Geodesics on Lie Groups with a Left-Invariant Metric

§1. Euler–Poincaré Equations

1.1. Let $v_1, \dots, v_n$ be tangent vector fields on an n-dimensional configuration space M that are linearly independent at each point of M (or in some part of it). The commutators of these fields can be decomposed in the basis $\{v_k\}$,

$$[v_i, v_j] = \sum c_{ij}^k v_k, \tag{1.1}$$

where the coefficients c_{ij}^k are functions of the point $x \in M$.

The velocity $\dot{x}$ of a system is also a tangent vector, and we can write

$$\dot{x} = \sum \omega_k v_k. \tag{1.2}$$

The coefficients $(\omega_1, \dots, \omega_n) = \omega$ are linear functions on $\dot{x}$; they depend on the choice of the fields v and are called *quasi velocities*. In particular, if we take the vector fields

$$\frac{\partial}{\partial x_1}, \quad \dots, \quad \frac{\partial}{\partial x_n} \tag{1.3}$$

as v_k, then obviously $\omega_k = \dot{x}_k$.

The Lagrangian $L(\dot{x}, x, t)$ of a mechanical system can be represented as a function of ω, x, and t: $L = \mathcal{L}(\omega, x, t)$. In the new variables, the Lagrange equations with the Lagrangian L become

$$\left(\frac{\partial \mathcal{L}}{\partial \omega_k}\right)^{\bullet} = \sum_{i,j=1}^{n} c_{ki}^j \frac{\partial \mathcal{L}}{\partial \omega_j} \omega_i + X_k(\mathcal{L}), \tag{1.4}$$

where X_k is the Lie derivative along the vector field v_k.

Equations (1.4) were first obtained by Poincaré in 1901 [63]. If the operators X_k are chosen as (1.3), then the *Poincaré equations* become the standard Lagrange equations. We should remember that system (1.4) is nonclosed; to close it, we must add relations (1.2).

1.2. We now obtain the Poincaré equations. Let

$$X_i = \sum_s a_{is} \frac{\partial}{\partial x_s}$$

V. V. Kozlov, *Dynamical Systems X*

be the explicit form of the differentiation operator along the field v_i. By the definition of the commutator, we have

$$[X_i, X_j] = X_i X_j - X_j X_i = \sum c_{ij}^k X_k$$

or, equivalently,

$$\sum_l \frac{\partial a_{is}}{\partial x_l} a_{jl} = \sum_l \frac{\partial a_{js}}{\partial x_l} a_{il} + \sum_k c_{ij}^k a_{ks}. \tag{1.5}$$

Considering the component form of (1.2),

$$\dot{x}_s = \sum_k a_{ks}\omega_k, \quad 1 \le s \le n,$$

we obtain

$$\frac{\partial \mathcal{L}}{\partial \omega_k} = \sum \frac{\partial L}{\partial \dot{x}_s} \frac{\partial \dot{x}_s}{\partial \omega_k} = \sum \frac{\partial L}{\partial \dot{x}_s} a_{ks}. \tag{1.6}$$

Using the Lagrange equations, we next have

$$\begin{aligned}\left(\frac{\partial \mathcal{L}}{\partial \omega_k}\right)^{\bullet} &= \sum \left(\frac{\partial L}{\partial \dot{x}_s}\right)^{\bullet} a_{ks} + \sum \frac{\partial L}{\partial \dot{x}_s} \frac{\partial a_{ks}}{\partial x_l} \dot{x}_l \\ &= \sum \frac{\partial L}{\partial x_s} a_{ks} + \sum \frac{\partial L}{\partial \dot{x}_s} \frac{\partial a_{ks}}{\partial x_l} a_{il}\omega_i.\end{aligned}$$

Taking the relations

$$\frac{\partial \mathcal{L}}{\partial x_s} = \frac{\partial L}{\partial x_s} + \sum \frac{\partial L}{\partial \dot{x}_l} \frac{\partial a_{il}}{\partial x_s} \omega_i$$

into account, we convert the obtained relation into the form

$$\left(\frac{\partial \mathcal{L}}{\partial \omega_k}\right)^{\bullet} = \sum \frac{\partial \mathcal{L}}{\partial x_s} a_{ks} + \sum \frac{\partial L}{\partial \dot{x}_s} \left(\frac{\partial a_{ks}}{\partial x_l} a_{il} - \frac{\partial a_{is}}{\partial x_l} a_{kl}\right) \omega_i.$$

Applying (1.5), we now obtain

$$\left(\frac{\partial \mathcal{L}}{\partial \omega_k}\right)^{\bullet} = \sum \frac{\partial L}{\partial \dot{x}_s} a_{js} c_{ki}^j \omega_i + X_k(\mathcal{L}).$$

Using (1.6), we finally obtain Poincaré equations (1.4).

1.3. Assuming that the Lagrangian $\mathcal{L}$ is a function convex in ω and increases at infinity faster than any linear function, we can then perform the Legendre transformation

$$m_k = \frac{\partial \mathcal{L}}{\partial \omega_k}, \qquad \mathcal{H} = (m \cdot \omega - \mathcal{L})_{\omega \to m}.$$

As is known, we then have

$$\omega_k = \frac{\partial \mathcal{H}}{\partial m_k}, \qquad X_k(\mathcal{L}) = -X_k(\mathcal{H}).$$

In the variables x and m, Eqs. (1.4) become

$$\dot{m}_k = \sum_{i,j} c^j_{ki} m_j \frac{\partial H}{\partial m_i} - X_k(\mathcal{H}), \quad 1 \le k \le n. \tag{1.7}$$

These equations are called the *Chetaev equations* [21].

1.4. We now assume that the configuration space M is a Lie group G. We recall that G is simultaneously a group (with the multiplication $\tau, \sigma \mapsto \tau\sigma$) and a smooth manifold and that

1. $\tau \mapsto \tau^{-1}$ is a smooth mapping $G \to G$ and
2. $\tau, \sigma \mapsto \tau\sigma$ is a smooth mapping $G \times G \to G$.

The basic example is $SO(3)$, the rotation group of the three-dimensional Euclidean space. This group consists of orthogonal 3×3 matrices whose determinants are equal to one. A 3×3 matrix is determined by nine arbitrary parameters; six independent orthogonality conditions define a smooth, regular three-dimensional surface in nine-dimensional space, the manifold $SO(3)$. From the topological standpoint, this is a three-dimensional sphere with identified antipodal points. It can be easily verified that the multiplication of matrices is a smooth transformation of this surface. As noted above (see Sec. 5 in Chap. 1), the group $SO(3)$ is the configuration space for the problem of the rotation of a rigid body about a fixed point.

Because G is a group, for arbitrary σ and τ, there exists a unique ρ such that $\rho\sigma = \tau$ ($\rho = \tau\sigma^{-1}$). We let Φ_ρ denote the left-shift operator generated by ρ,

$$\Phi_\rho : \xi \mapsto \rho\xi, \quad \xi \in G.$$

By condition 2, Φ_ρ is a smooth mapping of the group G onto itself. Clearly, $\Phi_\rho^{-1} = \Phi_{\rho^{-1}}$ is also a smooth mapping (condition 1). Therefore, Φ_ρ is a diffeomorphism, and its differential $d\Phi_\rho$ is hence an isomorphism of tangent spaces T_σ and T_τ.

A smooth vector field v on a Lie group G is said to be *left-invariant* if we have

$$d\Phi_\rho v(\sigma) = v(\tau) \tag{1.8}$$

for all τ and σ. We can simplify this condition by taking $\sigma = e$ in (1.7) (where e is the unit of the group G):

$$d\Phi_\tau v(e) = v(\tau) \quad \text{for all } \tau \in G.$$

Therefore, left-invariant vector fields on G form an n-dimensional vector space $\mathfrak{g}$ isomorphic to T_eG. If u and v are smooth vector fields and Φ is a smooth mapping, then

$$d\Phi([u, v]) = [d\Phi(u), d\Phi(v)].$$

Taking relation (1.8) into account, we obtain an important consequence: the commutator of left-invariant vector fields is also a left-invariant vector field. Therefore, the vector space $\mathfrak{g}$ becomes a Lie algebra if we define the multiplication in $\mathfrak{g}$ as the commutation. This operation has the following properties:

1. $[u, v] = -[v, u]$,
2. $[c_1 u_1 + c_2 u_2, v] = c_1 [u_1, v] + c_2 [u_2, v]$ for $c_1, c_2 \in \mathbb{R}$, and
3. $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$ for arbitrary $u, v, w \in \mathfrak{g}$ (the Jacobi identity).

The algebra $\mathfrak{g}$ is called the *Lie algebra* of the Lie group G.

If $v_1, \dots, v_n$ is a set of linearly independent left-invariant vector fields on G, then the coefficients c_{ij}^k in Eq. (1.1) are constant; we call them the *structure constants* of the Lie algebra. Obviously, $c_{ij}^k = -c_{ji}^k$.

Similarly, we can define *right-invariant* vector fields on the Lie group G; these fields are invariant under all right shifts of the group G. It can be easily verified that the vector space of right-invariant fields equipped with the commutation is an algebra isomorphic to the Lie algebra of the group G. (See, e.g., Chevalley's monograph [22] for more details. Applications to differential equations can be found in [61].)

1.5. As an important example, we consider the group $GL(n)$ consisting of all nondegenerate $n \times n$ matrices. Well-known formulas for the product of matrices and the inverse matrix show that $GL(n)$ is a Lie group of dimension n^2. We calculate its Lie algebra.

Let e be the unit matrix. We consider vectors tangent to $GL(n)$ at the point e. Let $t \mapsto x(t)$ be a smooth curve on $GL(n)$ and $x(0) = e$. Then

$$x(t) = e + tA + o(t), \tag{1.9}$$

where A is an $n \times n$ matrix; consequently, $\dot{x}(0) = A$. Therefore, the tangent space T_e coincides with the vector space of all $n \times n$ matrices.

A left shift $x \mapsto zx$ transforms curve (1.9) into the curve $t \mapsto zx(t)$, whose derivative at $t = 0$ is equal to zA. Hence, this shift transforms the vector A from T_e into the vector zA tangent to $GL(n)$ at the point z.

Let $z = \|z_{ij}\|$ and $A = \|a_{ij}\|$. The differentiation operator

$$\sum z_{is} a_{sj} \frac{\partial}{\partial z_{ij}}$$

corresponds to the left-invariant field $z \mapsto zA$. Let $z \mapsto zB$ be another left-invariant vector field. The commutator of these fields at the point $z = e$ is obviously equal to $[A, B] = BA - AB$. Therefore, the Lie algebra $\mathfrak{gl}(n)$ is isomorphic to the space of all real $n \times n$ matrices with the standard commutation. The algebra $\mathfrak{gl}(n)$ plays a significant role in the theory of Lie algebras. By the *Ado theorem*, each Lie algebra is isomorphic to some subalgebra of $\mathfrak{gl}(n)$ with an appropriate n.

Let $z \mapsto Az$ and $z \mapsto Bz$ be two right-invariant vector fields on $GL(n)$. The commutator of these fields is equal to $AB - BA$; it differs from $[A, B]$ only in its sign.

The special orthogonal group $SO(n)$ is very important in applications. This group consists of orthogonal $n \times n$ matrices with the determinant one. Clearly, $SO(n) \subset GL(n)$, $\mathfrak{so}(n) \subset \mathfrak{gl}(n)$, and $\dim SO(n) = n(n-1)/2$. In this case, the matrices $x(t)$ from (1.9) satisfy the condition $xx^{\mathrm{T}} = e$. Therefore,

$$e = (e + tA + o(t))(e + tA^{\mathrm{T}} + o(t)) = e + t(A + A^{\mathrm{T}}) + o(t),$$

whence $A^{\mathrm{T}} + A = 0$. The algebra $\mathfrak{so}(n)$ therefore consists of skew-symmetric $n \times n$ matrices.

1.6. We consider the case where $n = 3$. To every skew-symmetric matrix

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix},$$

there corresponds a vector a from the three-dimensional oriented Euclidean space with the components a_1, a_2, and a_3. It can be easily verified that by this correspondence, the matrix commutator $[A, B]$ becomes the standard vector cross product $a \times b$. This means that the algebra $\mathfrak{so}(3)$ is isomorphic to the space of three-dimensional vectors with the commutator defined as the cross product.

The group $SO(3)$ is a configuration space for the problem of rotation of a rigid body about a fixed point: each state of the body can be obtained from some given state by rotations. Rotation of a rigid body is described by the function $t \mapsto x(t)$, where x is an orthogonal matrix from $SO(3)$. The velocity of rotation $\dot{x}(t)$ is a vector tangent to the group at the point $x(t)$. We can dislocate this vector to the unit of the group (i.e., into the algebra $\mathfrak{so}(3)$) in two natural ways: left shifts and right shifts. As result, we obtain two skew-symmetric matrices, $x^{-1}\dot{x}$ and $\dot{x}x^{-1}$.

Let $R(t)$ be the radius vector of a point of the body in fixed space. Then $R(t) = x(t)R(0)$, and therefore

$$V(t) = \dot{R}(t) = \dot{x}(t)R(0) = \dot{x}x^{-1}R(t).$$

In the three-dimensional oriented space, the skew-symmetric operator $\dot{x}x^{-1}$ is the operator of the cross product $\Omega \times (\cdot)$. Finally, we obtain the *Euler formula* $V = \Omega \times R$. The vector Ω is called the *angular velocity* in fixed space. Therefore, right-invariant vector fields on $SO(3)$ correspond to rotations of a rigid body with a uniform angular velocity about an axis fixed in fixed space.

The orthogonal transformation with the matrix x^{-1} restores the rigid body to the initial state. Hence, $v(t) = x^{-1}(t)V(t)$ is the velocity of the body in a moving frame attached to the body. Therefore, $v = x^{-1}\dot{x}r = \omega \times r$, where ω is the vector of angular velocity and $r = x^{-1}(t)R(t) = R(0)$ is the radius vector of a point of the body in moving space. This immediately implies that left-invariant vector fields on $SO(3)$ correspond to rotations of a rigid body with a uniform angular velocity in moving space.

We choose three pairwise orthogonal axes passing through a fixed point in the rigid body; for example, we can choose the principal axes of inertia of the body. Let v_1, v_2, and v_3 be independent left-invariant fields on $SO(3)$ generated by rotations of the body about these axes with the angular velocity equal to one. By the isomorphism of $\mathfrak{so}(3)$ and the algebra of vectors of the three-dimensional Euclidean space, we obtain the formulas for the commutators

$$[v_1, v_2] = v_3, \qquad [v_2, v_3] = v_1, \qquad [v_3, v_1] = v_2. \tag{1.10}$$

1.7. Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on the group G,

$$\langle \dot{x}, \dot{x} \rangle = \sum a_{ij}(x)\dot{x}_i\dot{x}_j.$$

From the standpoint of dynamics, this metric is defined by the kinetic energy T and specifies the inertial properties of the system, $T = \langle \dot{x}, \dot{x} \rangle / 2$.

The metric $\langle \cdot, \cdot \rangle$ is said to be left-invariant if it preserves its values under left shifts. In other words, the value of the bilinear form $\langle \cdot, \cdot \rangle$ on an arbitrary pair of left-invariant vector fields is constant (i.e., is independent of the point of G).

Let $v_1, \dots, v_n$ be a basis of independent left-invariant fields. The positive-definite Gramm matrix

$$I = \|I_{ij}\|, \qquad I_{ij} = \langle v_i, v_j \rangle = \text{const},$$

is called the inertia tensor of the mechanical system in this basis. By (1.2), we have

$$T = \frac{1}{2}\left\langle \sum \omega_i v_i, \sum \omega_j v_j \right\rangle = \frac{1}{2}\sum I_{ij}\omega_i\omega_j.$$

In the absence of external forces, Poincaré equations (1.4) become

$$\dot{m}_i = \sum c_{ik}^j m_j \omega_k, \qquad m_s = \sum I_{sp}\omega_p. \tag{1.11}$$

These equations are differential equations in the variables ω on the Lie algebra $\mathfrak{g}$ of the group G and in the variables m on the dual vector space $\mathfrak{g}^*$. Equations (1.11) are called the *Euler–Poincaré equations.*

As an example, we consider the case $G = SO(3)$. The left-invariance of the kinetic energy of the rotating top is obvious. Using commutation relations (1.10), we can easily transform Eqs. (1.11) into the form

$$I\dot{\omega} + \omega \times I\omega = 0, \tag{1.12}$$

where ω is the angular velocity and I is the inertia tensor of the body. This is the famous dynamic Euler equation published in 1758.

Equations (1.11) compose a moiety of the equations of motion; we must add geometric equations (1.2) to them. From the standpoint of differential

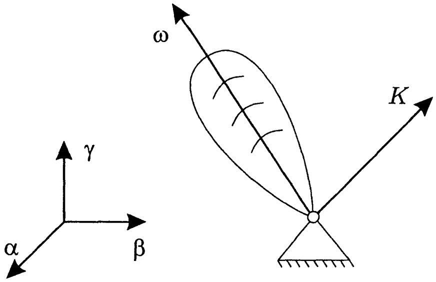

Fig. 21. Rotating top

geometry, these equations describe the geodesics of a left-invariant metric on a Lie group.

§2. Vortex Theory of the Top

2.1. The key problem in applying the general vortex theory developed in Chap. 2 is to find invariant manifolds admitting single-valued projections onto the configuration space. This problem is easily and naturally solved in the case of the Euler top, i.e., for the inertial rotation of a rigid body with a fixed point in the three-dimensional Euclidean space. Many results in this section can be immediately generalized to the more general problem of geodesics in Lie groups with left-invariant metrics.

Let α, β, γ be an orthonormal frame in the fixed space; we consider them as vectors in a moving space attached to a rigid body. Then these vectors are not constant: their evolution is described by the *Poisson equations*

$$\dot{\alpha} + \omega \times \alpha = 0, \qquad \dot{\beta} + \omega \times \beta = 0, \qquad \dot{\gamma} + \omega \times \gamma = 0. \tag{2.1}$$

Equations (2.1) and the dynamic Euler equations form the complete system of equations of motion. The first integrals of this system,

$$(I\omega, \alpha) = c_1, \qquad (I\omega, \beta) = c_2, \qquad (I\omega, \gamma) = c_3, \tag{2.2}$$

mean that the angular momentum vector $K = I\omega$ of the body considered as a vector in the fixed space is constant.

Integrals (2.2) have a simple group theory interpretation. A right-invariant vector field on $SO(3)$ corresponds to a rotation of the top with the uniform angular velocity $\omega = \alpha$. The phase flow of this field consists of left shifts on $SO(3)$. Because the kinetic energy of the top is invariant under left shifts, the Noether theorem (see Sec. 5 in Chap. 1) implies that the equations of motion have the integral

$$\left(\frac{\partial T}{\partial \omega}, \alpha\right) = \text{const}.$$

Obviously, Eqs. (2.2) imply

$$I\omega = c_1\alpha + c_2\beta + c_3\gamma. \tag{2.3}$$

This relation allows representing the angular velocity of the top as a single-valued function on the configuration space. In other words, (2.3) defines a three-dimensional stationary invariant manifold admitting a single-valued projection onto $SO(3)$. In what follows, we study the nontrivial case where $c_1^2 + c_2^2 + c_3^2 \neq 0$.

2.2. For convenience, we consider a moving trihedron formed by the principal axes of inertia of the rigid body relative to the fixed point. The inertia tensor has the diagonal form in these axes, $I = \operatorname{diag}(I_1, I_2, I_3)$. Let ω_1, ω_2, and ω_3 be the projections of the angular velocity vector onto these moving axes. This assumption simplifies the form of the kinetic energy,

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right). \tag{2.4}$$

We now describe invariant manifolds (2.3) in the canonical variables. For this, we introduce the Euler angles θ, φ, and ψ as the generalized coordinates (see Sec. 5 in Chap. 1) uniquely determining the position of the principal axes of inertia of the rigid body relative to the fixed trihedron.

Geometric relations (1.2) become the well-known *kinematic Euler formulas* (1760)

$$\begin{aligned}
\omega_1 &= \dot\psi \sin\theta \sin\varphi + \dot\theta\cos\varphi,\\
\omega_2 &= \dot\psi \sin\theta \cos\varphi - \dot\theta\sin\varphi,\\
\omega_3 &= \dot\psi \cos\theta + \dot\varphi,
\end{aligned} \tag{2.5}$$

and kinetic energy (2.4) becomes a quadratic form in the velocities $\dot\psi$, $\dot\theta$, and $\dot\varphi$. The conjugate momenta are defined as usual,

$$p_\psi = \frac{\partial T}{\partial\dot\psi}, \qquad p_\theta = \frac{\partial T}{\partial\dot\theta}, \qquad p_\varphi = \frac{\partial T}{\partial\dot\varphi}.$$

Using (2.4) and (2.5), we obtain

$$\begin{aligned}
p_\psi &= I_1\omega_1\frac{\partial\omega_1}{\partial\dot\psi} + I_2\omega_2\frac{\partial\omega_2}{\partial\dot\psi} + I_3\omega_3\frac{\partial\omega_3}{\partial\dot\psi}\\
&= I_1\omega_1\sin\theta\sin\varphi + I_2\omega_2\sin\theta\cos\varphi + I_3\omega_3\cos\theta,\\
p_\theta &= I_1\omega_1\cos\varphi - I_2\omega_2\sin\varphi,\\
p_\phi &= I_3\omega_3.
\end{aligned} \tag{2.6}$$

The vectors α, β, and γ can also be expressed in terms of the Euler angles:

$$\alpha = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi \\ -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi \\ \sin\psi\sin\theta \end{bmatrix},$$
$$\beta = \begin{bmatrix} \sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi \\ -\sin\psi\sin\varphi - \cos\theta\cos\psi\cos\varphi \\ \cos\psi\sin\theta \end{bmatrix}, \tag{2.7}$$
$$\gamma = \begin{bmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ \cos\theta \end{bmatrix}$$

(see Whittaker's treatise [76] for the proof).

Considering (2.3) and (2.7), we obtain the final form of the invariant manifolds in the canonical variables from Eqs. (2.6):

$$\begin{aligned} p_\psi &= c_3, \\ p_\theta &= c_1\cos\psi + c_2\sin\psi, \\ p_\phi &= c_1\sin\theta\sin\psi - c_2\sin\theta\cos\psi + c_3\cos\theta. \end{aligned} \tag{2.8}$$

We note that these formulas do not contain the moments of inertia I_1, I_2, and I_3 of the body.

We introduce the fundamental 1-form

$$\omega = p_\psi\, d\psi + p_\theta\, d\theta + p_\varphi\, d\varphi$$

and the 2-form $\Omega = d\omega$.

Proposition 1. *If* $\sum c_k^2 \neq 0$, *then* $\operatorname{rank}(\Omega) = 2$.

Therefore, the three-dimensional invariant manifolds obtained are vortex manifolds.

Proof. Without loss of generality, we can assume that the constant angular momentum vector K is directed along γ: $K = k\gamma$, $k = |K| \neq 0$. Then we can take $c_1 = c_2 = 0$ and $c_3 = k$ in Eqs. (2.8). Hence, $\omega = k(d\psi + \cos\theta\, d\varphi)$ and

$$\Omega = k\sin\theta\, d\varphi \wedge d\theta. \tag{2.9}$$

Because the Euler angles are defined as $0 < \theta < \pi$, we have $\operatorname{rank}(\Omega) = 2$ for these values of θ. Choosing another system of coordinates, we can prove that the 2-form Ω is also nondegenerate at the poles $\theta = 0$ and $\theta = \pi$. □

2.3. If k is fixed, then invariant relations (2.3) (or, equivalently, (2.8)) define the dynamic system

$$\dot{x} = v(x), \qquad x = (\psi, \theta, \varphi) \tag{2.10}$$

on $SO(3)$. Using kinematic Euler formulas (2.5), we can easily rewrite these equations as

$$\begin{aligned}
\dot{\psi} &= k\left(\frac{\sin^2\varphi}{I_1} + \frac{\cos^2\varphi}{I_2}\right), \\
\dot{\theta} &= k\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin\theta\sin\varphi\sin\psi, \\
\dot{\varphi} &= k\cos\theta\left(\frac{1}{I_3} - \frac{\sin^2\varphi}{I_1} - \frac{\cos^2\varphi}{I_2}\right).
\end{aligned} \tag{2.11}$$

Equations (2.11) are well known in the context of exact integration of the Euler problem [76]. Their phase flow defines a stationary "current" on $SO(3)$. We study its properties.

The following assertions hold:

1. The vortex fields w on $SO(3)$ commuting with the velocity field v generate the rotation of a rigid body with the angular velocity $\omega = \mu\gamma$, $\gamma = \text{const}$. In particular, these vortex fields are right-invariant, and all vortex curves are closed. The group $SO(3)$ is fibered by the vortex curves, and the bundle obtained is exactly the *Hopf bundle* well known in topology.
2. These fields are defined by the relations $i_\omega\Omega = 0$, $\omega(w) = \text{const} \neq 0$.
3. The Hamiltonian system on $T^*SO(3)$ with the Hamiltonian

   $$H' = \frac{1}{2}K^2 = \frac{1}{2}\left[(I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2\right]$$

   has the same three-dimensional invariant surfaces (2.8). The corresponding vector fields on $SO(3)$ of form (2.10) are vortex fields commuting with field (2.11).
4. We can use (2.4) and (2.5) to calculate the determinant g of the coefficient matrix g_{ij} for the kinetic energy,

   $$g = I_1I_2I_3\sin^2\theta. \tag{2.12}$$

   Equations (2.10) admit a nontrivial integral invariant

   $$\operatorname{mes}(D) = \int_D \sqrt{g}\,d^3x, \tag{2.13}$$

   which defines a measure on $SO(3)$ invariant under all left and right shifts. We recall that on a compact group, there exists a unique (up to a multiplier) bi-invariant measure called the *Haar measure* [22]. The invariant measure of system (2.10) can also be written as

   $$\int \omega \wedge \Omega. \tag{2.14}$$

5. The metric on $SO(3)$ determined by the kinetic energy of the rigid body allows calculating the curl of the field v. This field is a vortex, and $\operatorname{rot} v$ commutes with v.
6. The Bernoulli integral h is equal to $k^2\left(I^{-1}\gamma,\gamma\right)/2$. The critical points of h are orbits of uniform rotations of the top about principal axes of inertia with a fixed value of K, and the critical values are equal to the energy values at these rotations. If c is a noncritical value of h, then the Bernoulli surface $B_c = \{h(x) = c\}$ is a two-dimensional torus with a conditionally periodic motion.

2.4. The proof of properties 1–6 is based on simple direct calculations. By (2.9), the vector fields

$$\psi' = \mu, \qquad \theta' = 0, \qquad \varphi' = 0 \tag{2.15}$$

are vortex fields. Fields (2.11) and (2.15) commute if $\mu = \text{const}$. Kinematic Euler formulas (2.5) imply that field (2.15) generates the rotation of a rigid body with the angular velocity $\omega = \mu\gamma$, and property 1 is proved.

Because $\mu = \text{const}$, we have $\omega(w) = k\mu = \text{const}$, and we obtain property 2. Property 3 can be proved similarly.

By Proposition 1 in Chap. 1, Eqs. (2.11) have the integral invariant

$$\int v\, d\psi\, d\theta\, d\varphi \tag{2.16}$$

with the density

$$v = \left|\frac{\partial\left(p_\psi, p_\theta, p_\varphi\right)}{\partial\left(c_1, c_2, c_3\right)}\right| = \sin\theta.$$

By (2.12), invariant (2.16) differs from (2.13) by a constant multiplier. The existence of integral invariant (2.14) for system (2.10) is implied by Proposition 2 in Chap. 2. Because $\sin\theta > 0$ for $0 < \theta < \pi$, invariant (2.16) defines a measure on $SO(3)$. The direct proof of the invariance of this measure under left and right shifts by elements of $SO(3)$ can be found, e.g., in [59].

According to the results in Sec. 5 in Chap. 2, the components of the curl of field (2.11) (relative to the intrinsic metric on $SO(3)$ defined by the kinetic energy) are equal to

$$-\frac{k}{\sqrt{I_1 I_2 I_3}}, \quad 0, \quad 0.$$

Therefore, the fields $\operatorname{rot} v$ and v commute, and we have obtained property 5.

The proof of property 6 is similar to the reasoning in Sec. 1 in Chap. 1. If the inertia tensor I is nonspherical, then $h \neq \text{const}$, and the field v is hence not collinear with its curl.

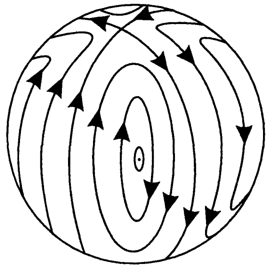

Fig. 22. Dynamic system on the Poisson sphere

2.5. In the case of the Euler top, we can obtain the explicit formulas for the Clebsch potentials. Indeed, we can assume that

$$S = k\psi, \qquad x_1 = k\cos\theta, \qquad x_2 = \varphi. \tag{2.17}$$

On the interval $0 < \theta < \pi$, x_1 is monotonic. In these variables, the fundamental 1-form ω becomes

$$\omega = dS + x_1\, dx_2.$$

Therefore, x_1, x_2, and S are the Clebsch potentials.

Factoring the configuration space, the group $SO(3)$, by closed vortex curves is equivalent to eliminating the precession angle ψ. The right-hand sides of Eqs. (2.11) do not contain ψ, and the equations for θ and φ are therefore the equations on the basis of the fibering of $SO(3)$ by the vortex curves. This basis is diffeomorphic to the two-dimensional sphere, on which θ and φ are the usual spherical coordinates. In the dynamics of a rigid body, this sphere is called the *Poisson sphere.*

According to the general results in Sec. 4 in Chap. 2, the coordinates x_1 and x_2 on the Poisson sphere satisfy the Hamilton differential equations

$$\dot{x}_1 = -\frac{\partial h}{\partial x_2}, \qquad \dot{x}_2 = \frac{\partial h}{\partial x_1},$$

where

$$h = \frac{1}{2}\left(\frac{\sin^2 x_2}{I_1} + \frac{\cos^2 x_2}{I_2}\right)\left(k^2 - x_1^2\right) + \frac{x_1^2}{2I_3}$$

is the Bernoulli function written in the canonical variables x_1 and x_2. The fibering of the Poisson sphere by the level curves of the function h is shown in Fig. 22.

2.6. In conclusion, we briefly discuss the vortex theory of the Euler top with dissipation (Sec. 6 in Chap. 2). In this case, the dynamic Euler equations have the form

$$I\dot{\omega} + \omega \times I\omega = -\nu I\omega,$$

where $\nu(t) > 0$ is the coefficient of viscosity. These equations together with Poisson equations (2.1) yield the relations

$$(I\omega, \alpha) = c_1\mu, \qquad (I\omega, \beta) = c_2\mu, \qquad (I\omega, \gamma) = c_3\mu, \tag{2.18}$$

where $c =$ const and $\dot{\mu} = \nu$. Without loss of generality, we can assume that $c_1 = c_2 = 0$. Taking $c_2\mu = k(t)$, we obtain

$$I\omega = k\gamma$$

from (2.18). This vector-valued equality allows expressing the angular velocity of the body as a single-valued function of its position and time:

$$\dot{x} = v(x,t), \quad x = (\psi, \theta, \varphi) \in SO(3).$$

In the variables ψ, θ, and φ (the Euler angles), this system of equations coincides with Eqs. (2.11) with k a known function of time.

The vector fields

$$\psi' = \lambda, \qquad \theta' = 0, \qquad \varphi' = 0$$

are vortex fields (as in the case $\nu = 0$), and we again obtain the already known bundle of $SO(3)$ by closed vortex lines. Because the right-hand sides of Eqs. (2.11) do not contain the precession angle ψ, the phase flow of Eqs. (2.11) (with a variable coefficient k) transforms vortex lines into vortex lines.

The Clebsch potentials are again of form (2.17). The function S and the Bernoulli function h are now independent of time. Because $\dot{k} = -\nu k$, the Hamiltonian $\mathcal{H}$ in Sec. 6 in Chap. 2 (formula (6.17)) obviously equals h. Because h is independent of ψ, we obtain the generalized Bernoulli theorem: h is constant on vortex lines for fixed t.

§3. Haar Measure

3.1. The main ideas in the vortex theory of the top can be extended to a more general problem of geodesics of left-invariant metrics on Lie groups. Let G be a Lie group with the local coordinates $x = (x_1, \dots, x_n)$, $T(\dot{x}, x)$ be a left-invariant metric, and $\omega_1, \dots, \omega_n$ be independent right-invariant fields on G.

Proposition 2. *The phase flow of a right-invariant or left-invariant field is a respective family of left or right shifts on G.*

This is a well-known fact from Lie group theory (see, e.g., [22]). We can illustrate Proposition 2 with the example of the matrix Lie group $GL(n)$. Because an arbitrary Lie group is locally isomorphic to a subgroup of GL (the *Ado theorem*), Proposition 2 is then implied in the general case.

According to Sec. 1, a right-invariant vector field on $GL(n)$ has the form Az, where A is a constant $n \times n$ matrix and $z \in GL(n)$. This field generates the system of linear differential equations

$$\frac{dz}{d\alpha} = Az. \tag{3.1}$$

As is known,

$$z = e^{A\alpha} z_0$$

is a general solution of Eq. (3.1). The correspondence

$$\alpha \to e^{A\alpha} z_0, \quad z_0 \in GL(n),$$

generates a one-parameter group of left shifts.

Now let $z \to zA$ be a left-invariant field. Then Eq. (3.1) should be replaced with

$$\frac{dz}{d\alpha} = zA.$$

The general solution of this equation

$$z = z_0 e^{A\alpha}$$

generates right shifts on $GL(n)$.

According to Proposition 2, phase flows of right-invariant vector fields $w_1, \ldots, w_n$ are families of left shifts on G. Because the kinetic energy T is left-invariant, the equations of motion have n Noether integrals

$$\frac{\partial T}{\partial \dot{x}} \cdot w_1 = c_1, \quad \ldots, \quad \frac{\partial T}{\partial \dot{x}} \cdot w_n = c_n. \tag{3.2}$$

Because the vector fields $w_1, \ldots, w_n$ are linearly independent at each point of G, linear algebraic system (3.2) can be uniquely solved with respect to the velocity given fixed values of c:

$$\dot{x} = v(x, c), \quad x \in G. \tag{3.3}$$

We recall that on each Lie group, there exists a unique (up to a multiplier) measure invariant under all left shifts (or right shifts). In the case of a unimodular group, this measure (usually called the *Haar measure*) is bi-invariant. The analytic unimodularity criterion is that

$$\sum c_{ik}^k = 0, \quad 1 \le i \le n. \tag{3.4}$$

In particular, all compact groups are unimodular (see [22] for details).

Theorem 1 ([49]). *If c is fixed, then the phase flow of system* (3.3) *preserves the right-invariant measure on G.*

Corollary 1. *If G is a unimodular group, then the phase flow of system* (3.3) *preserves the Haar measure on G.*

Remark. Euler–Poincaré equations (1.11) on the algebra $\mathfrak{g}$ admit an invariant measure with a smooth density if and only if the group G is unimodular (see [41]). In this case, the phase flow of system (1.11) preserves the standard measure $\int d^n\omega$ on $\mathfrak{g}$.

3.2. *Proof of Theorem 1.* We should verify that the integral invariant from Proposition 1 in Chap. 1 (Sec. 3) is a right-invariant measure on G. Let $v_{i1}, \dots, v_{in}$ be components of a right-invariant field v_i. By virtue of (1.2), system (3.3) can be rewritten as

$$\dot{x}_i = \sum_l v_{il}(x)\omega_l(x, c). \tag{3.5}$$

The system of equations (1.2) and (1.11) is Hamiltonian. Therefore, by virtue of the Liouville theorem, the phase flow of this system in the canonical variables

$$x, \quad y = \frac{\partial T}{\partial \dot{x}}$$

preserves the standard measure. The density of this measure in the variables x and ω can be calculated as the Jacobian of the transformation

$$x, y \to x, \omega.$$

We set

$$V = \|v_{ij}\|.$$

Because

$$y_s = \frac{\partial T}{\partial \dot{x}} = \sum \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{x}_s} = \sum m_i \frac{\partial \omega_i}{\partial \dot{x}_s},$$

we have

$$\frac{\partial y}{\partial m} = \frac{\partial \omega}{\partial \dot{x}}.$$

Therefore, the density of the invariant Liouville measure in the variables x and ω equals

$$M = \det \frac{\partial y}{\partial \omega} = \det \left(\frac{\partial y}{\partial m}\frac{\partial m}{\partial \omega} \right) = (\det V)^{-1} \det I, \tag{3.6}$$

where I is the matrix of the inertia operator.

Let

$$w_i = (w_{i1}, \dots, w_{in}), \quad 1 \le i \le n,$$

be right-invariant vector fields on G as in Sec. 3.1. Similarly to (1.2), we set

$$\dot{x}_i = \sum w_{il}s_l, \quad 1 \le i \le n, \tag{3.7}$$

where $s = (s_1, \dots, s_n) \in \mathfrak{g}$ is the velocity of the investigated system. The Noether integrals can be written as

$$f_j = \sum \frac{\partial T}{\partial \dot{x}_l} \frac{\partial \dot{x}_l}{\partial s_j} = \sum m_k \frac{\partial \omega_k}{\partial \dot{x}_l} w_{lj} = c_j, \quad 1 \le i \le n. \tag{3.8}$$

The system of equations (1.2) and (1.11) can be replaced by system (3.5) and the trivial relations

$$\dot{c}_1 = \dots = \dot{c}_n = 0.$$

The density of the Liouville measure in the variables x and c is

$$\rho = M \left(\det \frac{\partial f}{\partial \omega} \right)^{-1}.$$

Obviously, this function gives the density of the integral invariant for system (3.5) with a fixed c.

Setting

$$W = \|w_{ij}\|$$

and applying (3.6) and (3.8), we obtain

$$\rho = (\det V)^{-1} \det \left(\frac{\partial f}{\partial m} \frac{\partial m}{\partial \omega} \right)^{-1} = (\det W)^{-1}. \tag{3.9}$$

The theorem is proved.

3.3. We now consider the system of differential equations on G

$$\dot{x}_i = \sum v_{il} \omega_l, \tag{3.10}$$

where $\omega_1, \dots, \omega_n$ are considered constant parameters. For all ω, the phase flow of system (3.10) is a family of right shifts on G.

Lemma 1. *The function* $\det W^{-1}$ *is the density of the integral invariant of system* (3.10).

Corollary 2. *The density of a right-invariant measure on* G *is* $c \det W^{-1}$, $c = \text{const} \neq 0$.

Indeed, each right shift on G is a shift along trajectories of system (3.10) for appropriate $\omega_1, \dots, \omega_n$.

Proof of Lemma 1. We assume that the position of the system at some state is described by the local coordinates $x_1^0, \dots, x_n^0$ and the right-invariant fields are the vectors $w_1^0, \dots, w_n^0$. We set

$$W^0 = \det \left\| w_{ij}^0 \right\|.$$

Under a right shift on G, the coordinates $x_1^0, \ldots, x_n^0$ become $x_1, \ldots, x_n$ and the tangent vectors $w_1^0, \ldots, w_n^0$ become $w_1, \ldots, w_n$. It is well known that

$$W = JW^0, \qquad dx_1 \cdots dx_n = \det J \, dx_1^0 \cdots dx_n^0, \tag{3.11}$$

where J is the Jacobi matrix of the transformation $x_0 \to x$.

From (3.10), we obtain

$$\det W^{-1} \, dx_1 \cdots dx_n = \det \left(W^0\right)^{-1} \, dx_1^0 \cdots dx_n^0,$$

that is, the phase flow of system (3.10) has the integral invariant with the density $\det W^{-1}$. □

Applying formula (3.9) and Lemma 1 completes the proof of Theorem 1.□

3.4. We can prove the following lemma similarly.

Lemma 2. *The function* $\det V^{-1}$ *is the density of the integral invariant of system* (3.7).

This lemma implies that the density of a left-invariant measure on G is $c \det V^{-1}$, $c = \text{const} \neq 0$. For a unimodular group in particular, we have

$$\frac{\det V}{\det W} = \text{const}\,.$$

It is interesting to prove Lemmas 1 and 2 directly using analytic unimodularity criterion (3.4). More precisely, we prove that for all ω the phase flow of system (3.10) preserves the integral invariant with the density $\det V^{-1}$.

Indeed, considering the matrix

$$K = \frac{\partial \dot{x}}{\partial x} V - \dot{V}, \tag{3.12}$$

where the dot denotes the derivative by virtue of system (3.10), we can easily verify that the jth column of the matrix K equals

$$\sum_{k,j} v_k c_{ij}^k \omega_i$$

and the (k, j)th element of the matrix $V^{-1}K$ equals

$$\sum c_{ij}^k \omega_i.$$

We next premultiply (3.12) by V^{-1} and calculate the traces of the left- and right-hand sides of the result,

$$\operatorname{tr} V^{-1} K = \operatorname{tr} V^{-1} \frac{\partial \dot{x}}{\partial x} V - \operatorname{tr} V^{-1} \dot{V}.$$

By virtue of (3.4), $\operatorname{tr} V^{-1} K = 0$. Further,

$$\operatorname{tr} V^{-1} \partial \dot{x} \frac{\partial x}{V} = \operatorname{tr} \frac{\partial \dot{x}}{\partial x}$$

is the divergence of the right-hand side of system (3.10), and

$$\operatorname{tr} V^{-1}\dot{V} = \frac{d}{dt} \ln \det V.$$

Finally, we obtain the identity

$$\operatorname{tr} \frac{\partial \dot{x}}{\partial x} + \frac{d}{dt} \ln(\det V)^{-1} = 0,$$

which is the Liouville equation for the density of the integral invariant $\det V^{-1}$, as was required.

§4. Poisson Brackets

4.1. The key role in Hamiltonian mechanics is played by the Poisson brackets referred to in Chapter 1 in connection with the dynamics of point vortices. We now recall the definition and main properties of the Poisson brackets and prove several assertions that are used in what follows.

We assign the function

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right),$$

called the *Poisson bracket* of f and g, to each ordered pair of functions f and g on the phase space M^{2n}. The derivative with respect to time of a function $F(x, y, t)$ by virtue of the Hamiltonian system with the Hamiltonian H is obviously equal to

$$\frac{\partial F}{\partial t} + \{F, H\}.$$

Therefore, the canonical equations themselves can be rewritten as

$$\dot{x}_i = \{x_i, H\}, \qquad \dot{y}_i = \{y_i, H\}, \quad 1 \le i \le n.$$

These relations show the specific role of the Poisson bracket in the theory of Hamiltonian systems.

The Poisson bracket has the following properties:

1. $\{f, g\} = -\{g, f\}$ (skew symmetry),
2. $\{c_1 f_1 + c_2 f_2, g\} = c_1 \{f_1, g\} + c_2 \{f_2, g\}$ for all real c_1 and c_2 (bilinearity),
3. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (the Jacobi identity),
4. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ (the Leibnitz rule), and
5. if $\{f, g\} = 0$ for all g, then $f = \text{const}$ (nondegeneracy).

The set of all smooth functions on the phase space C^∞ equipped with the usual addition and multiplication by numbers is a infinite-dimensional vector space. By virtue of properties 1–3, the "inner product" $\{\cdot,\cdot\}$ makes this space a Lie algebra.

The Jacobi identity implies the following simple, but important, theorem.

Theorem 2 (Poisson theorem). *The Poisson bracket of first integrals of the Hamilton equations is also a first integral.*

We note that the Poisson bracket of two integrals can be a constant and even zero. The Poisson theorem implies that all integrals form a Lie subalgebra in the Lie algebra of all functions.

If $\{f, g\} = 0$, we say that the functions f and g commute or are *in involution.* If f and g are also independent of t, we say that f is an integral of the Hamilton equations with the Hamiltonian g; conversely, g is an integral of the Hamilton equations with the Hamiltonian f. This duality explains the term "involution."

4.2. As an important example, we consider Noether integrals (3.2), which in canonical variables have the form

$$F_1 = y \cdot w_1, \quad \dots, \quad F_n = y \cdot w_n. \tag{4.1}$$

Proposition 3. *Linear combinations of the functions $F_1, \dots, F_n$ with the commutator $\{\cdot,\cdot\}$ generate an n-dimensional Lie algebra isomorphic to $\mathfrak{g}$.*

Proof. Because the algebra of right-invariant vector fields on the group G is isomorphic to the Lie algebra $\mathfrak{g}$, we can choose fields $w_1, \dots, w_n$ such that

$$[w_i, w_j] = \sum c_{ij}^k w_k, \tag{4.2}$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$. It is easily to verify the relations

$$\{F_i, F_j\} = y \cdot [w_i, w_j].$$

By virtue of (4.1) and (4.2), we obtain

$$\{F_i, F_j\} = \sum c_{ij}^k F_k. \tag{4.3}$$

The proposition is proved. $\square$

4.3. Let $f_1, \dots, f_n$ be functions on a phase space M^{2n} (possibly depending on t) such that

$$\frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots, y_n)} \neq 0.$$

Then, by the implicit function theorem, the system of equations

$$f_1(x, y, t) = c_1, \quad \dots, \quad f_n(x, y, t) = c_n \tag{4.4}$$

can be solved (at least locally) for the momenta y,

$$y_1 = u_1(x,t,c), \quad \ldots, \quad y_1 = u_1(x,t,c), \tag{4.5}$$

where $c = (c_1, \ldots, c_n)$. We introduce the $n \times n$ matrix of Poisson brackets

$$A = \|\{f_i, f_j\}\| .$$

In the expressions for the Poisson brackets, the momenta should be replaced in accordance with formulas (4.5).

Lemma 3. *We have* $\operatorname{rank}(\operatorname{rot} u) = \operatorname{rank} A$.

Corollary 3. *If the functions $f_1, \ldots, f_n$ are in involution, then the 1-form*

$$\sum u_i(x,t,c)\, dx_i$$

is closed for all t and c.

Proof of Lemma 3. Obviously, the functions

$$F_k(x,t,c) = f_k(x, u(x,t,c), t), \quad 1 \le k \le n, \tag{4.6}$$

are identically equal to c_k. Therefore,

$$\frac{\partial F_k}{\partial x_i} = \frac{\partial f_k}{\partial x_i} + \sum \frac{\partial f_k}{\partial y_j}\frac{\partial u_j}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial F_s}{\partial x_i} = \frac{\partial f_s}{\partial x_i} + \sum \frac{\partial f_s}{\partial y_j}\frac{\partial u_j}{\partial x_i} = 0.$$

Multiplying the first formula by $\partial f_s/\partial y_i$ and the second by $-\partial f_s/\partial y_i$ and summing over i, we obtain the relations

$$\{f_k, f_s\} + \sum_{i,j} \frac{\partial f_k}{\partial y_j}\frac{\partial f_s}{\partial y_j}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = 0, \quad 1 \le k, s \le n. \tag{4.7}$$

We introduce the $n \times n$ matrix

$$B = \left\| \frac{\partial f_i}{\partial y_j} \right\|,$$

where the momenta are also eliminated by (4.5). By assumption, B is a nondegenerate matrix. The set of relations (4.7) can be represented in matrix form as

$$A = -B(\operatorname{rot} u)B^{\mathrm{T}}.$$

As is known, the matrices A and B have the same rank in this case. $\square$

4.4. Let $\Phi(x,y,t)$ be a function on the phase space commuting with the functions $f_1,\dots,f_n$,

$$\{\Phi, f_k\} = 0, \quad 1 \le k \le n. \tag{4.8}$$

We can assign the Hamiltonian system

$$\dot{x} = \frac{\partial \Phi}{\partial y}, \qquad \dot{y} = -\frac{\partial \Phi}{\partial x}$$

to this function. We set

$$w = \left.\frac{\partial \Phi}{\partial y}\right|_{y=u(x,t,c)}.$$

It is clear that w is a vector field on the configuration space.

Lemma 4. *If Φ is a function on $f_1,\dots,f_n$, then w is a vortex field,*

$$(\operatorname{rot} u)w = 0. \tag{4.9}$$

Proof. From (4.4) and (4.5), we obtain the identities

$$y_i \equiv u_i(x,t,f(x,y,t)), \quad 1 \le i \le n.$$

Therefore,

$$\sum_{s=1}^{n} \frac{\partial u_i}{\partial c_s}\frac{\partial f_s}{\partial y_j} = \delta_{ij}, \tag{4.10}$$

where δ_{ij} is the Kronecker symbol. Multiplying (4.7) by the derivatives $\partial u_p/\partial c_k$ and $\partial u_q/\partial c_s$, summing over k and s, and applying (4.10), we obtain

$$\sum_{k,s} \frac{\partial u_p}{\partial c_k}\frac{\partial u_q}{\partial c_s}\{f_k, f_s\} + \frac{\partial u_p}{\partial x_q} - \frac{\partial u_q}{\partial x_p} = 0.$$

To verify (4.9), we multiply the latter relations by

$$w_p = \left.\frac{\partial \Phi}{\partial y_p}\right|_{y=u(x,t,c)}$$

and sum over p from 1 to n. We are interested in

$$\sum_{k,s,p} \frac{\partial u_p}{\partial c_k}\frac{\partial \Phi}{\partial y_p}\frac{\partial u_q}{\partial c_s}\{f_k, f_s\}.$$

By assumption, the function

$$\phi(x,t,c) = \Phi(x,u(x,t,c),t)$$

depends only on $c_1 = f_1, \ldots, c_n = f_n$. Therefore,

$$\sum_p \frac{\partial \Phi}{\partial y_p} \frac{\partial u_p}{\partial c_k} = \frac{\partial \phi}{\partial c_k}.$$

Further, the sum

$$\sum_k \frac{\partial \phi}{\partial c_k} \{f_k, f_s\}$$

is equal to the Poisson bracket $\{\Phi, f_s\}$, which is zero by assumption. □

4.5. We can assign the canonical system of differential equations with the Hamiltonian f,

$$\dot{x} = \frac{\partial f}{\partial y}, \qquad \dot{y} = -\frac{\partial f}{\partial x},$$

to each smooth function f on the phase space. The generating vector field is said to be *Hamiltonian* and is denoted by v_f. Let L_f be the differentiation operator along the field v_f. By the definition of the Poisson bracket,

$$L_f h = \{h, f\}.$$

Lemma 5. *The commutator of the Hamiltonian vector fields v_f and v_g is a Hamiltonian vector field with the Hamiltonian $\{f, g\}$.*

Corollary 4. *If $\{f, g\} = 0$, then $[v_f, v_g] = 0$.*

Proof of Lemma 5. Using the Jacobi identity, we obtain

$$\begin{aligned}(L_g L_f - L_f L_g)\, h &= L_g\{h, f\} - L_f\{h, g\} \\ &= \{\{h, f\}, g\} - \{\{h, g\}, f\} \\ &= -\{\{f, g\}, h\} = L_{\{f,g\}}.\end{aligned}$$

The lemma is proved. □

§5. Casimir Functions and Vortex Manifolds

5.1. We calculate the Poisson bracket of two functions f and h defined on the dual space $\mathfrak{g}^*$; they depend only on the variables $m_1, \ldots, m_n$. To do this, by the common rule, we consider the Hamiltonian system with the Hamiltonian h and calculate the complete derivative of f with respect to time by virtue of this system. In the variables x and m, the Hamilton equations have the form of Chetaev equations (1.7),

$$\dot{m}_k = \sum c_{ik}^j m_j \frac{\partial h}{\partial m_i}, \quad 1 \le k \le n.$$

Because f is independent of x, we do not need the closing equations; therefore,

$$\dot{f} = \{f, h\} = \sum c_{ik}^j m_j \frac{\partial f}{\partial m_k} \frac{\partial h}{\partial m_i}. \tag{5.1}$$

Therefore, the Poisson bracket of functions on $\mathfrak{g}^*$ is also a function on $\mathfrak{g}^*$. Bracket (5.1) is called the *Lie–Poisson bracket*; it was initially introduced by Lie in the theory of transformation groups.

From (5.1), we obtain the important formula

$$\{m_i, m_j\} = \sum c_{ij}^k m_k, \tag{5.2}$$

which is completely analogous to (4.3). It shows that the space of linear functions on $\mathfrak{g}^*$ (canonically isomorphic to the vector space $\mathfrak{g}$) is a Lie algebra relative to the Lie–Poisson bracket; obviously, this algebra is isomorphic to $\mathfrak{g}$.

The Lie–Poisson bracket has properties 1–4. However, it may be degenerate, i.e., there exist nonconstant functions commuting with all functions on $\mathfrak{g}^*$. Such functions are called *Casimir functions.*

We consider an example. In the Euler problem of the free rotation of a top, the Lie–Poisson bracket is defined as

$$\{m_1, m_2\} = m_3, \qquad \{m_2, m_3\} = m_1, \qquad \{m_3, m_1\} = m_2.$$

It is degenerate because the function $k^2 = m_1^2 + m_2^2 + m_3^2$ (the squared angular momentum of the body) commutes with all functions on $(\mathfrak{so}(3))^*$. According to the vortex theory of the Euler top (see Sec. 2), integral curves of the Hamiltonian system with the Hamiltonian k^2 are vortex lines.

5.2. The Casimir functions are closely related to Noether integrals (4.1).

Lemma 6. *For all $i, j = 1, \dots, n$, we have $\{m_i, F_j\} = 0$.*

Proof. We take m_i as the Hamiltonian. Because it is invariant under all left shifts (similarly to the original left-invariant metrics), the equations with the Hamiltonian m_i have n Noether integrals $F_1, \dots, F_n$. □

In the case of the Euler top, for example, Lemma 6 asserts that projections of the angular momentum on the moving and fixed axes commute, which can be easily verified by direct calculation in the canonical variables θ, φ, ψ, p_θ, p_φ, and p_ψ. Indeed, the angular momentum in the direction of the fixed vector γ equals p_ψ (we use the notation in Sec. 2), and its projections on the moving axes equal $I_1\omega_1$, $I_2\omega_2$, and $I_3\omega_3$. Being represented in the canonical variables, they do not contain the precession angle ψ; therefore, the Poisson brackets of these projections with p_ψ are zero.

We pass from the canonical variables x and y in the phase space $T^*G \simeq G \times \mathfrak{g}^*$ to the variables x and m. The Noether integrals then become functions of x and m that are linear in $m_1, \dots, m_n$. Let $x_1, \dots, x_n$ be coordinates in a neighborhood of the unit of G in which $x = 0$.

Lemma 7. *If the right-invariant fields $w_1, \ldots, w_n$ at the unit of G have the forms*

$$(1,0,\ldots,0)^{\mathrm{T}}, \quad \ldots, \quad (0,0,\ldots,1)^{\mathrm{T}}, \tag{5.3}$$

then $F_k = m_k$, $1 \leq k \leq n$, at this point.

Proof. As shown in Sec. 1, left- and right-invariant fields on G can be obtained from a tangent vector at the unit by left and right shifts. Therefore, left-invariant fields at the unit are also of form (5.3), and their component matrix $\|v_{ij}\|$ is the identity matrix. We obtain

$$F_k = \frac{\partial T}{\partial \dot{x}} \cdot w_k = \frac{\partial T}{\partial \omega} \frac{\partial \omega}{\partial \dot{x}} \cdot w_k = \frac{\partial T}{\partial \omega} \cdot \left(\frac{\partial \omega}{\partial \dot{x}}\right)^{\mathrm{T}} w_k = \frac{\partial T}{\partial \omega_k} = m_k$$

from (1.2) and (5.2). □

Lemma 7 has a simple meaning for the Euler top. The unit of $SO(3)$ corresponds to the coincidence of the moving and fixed axes; therefore, at this position, the projections of the angular momentum on the moving and fixed axes are equal.

Proposition 4. *A function of Noether integrals that is independent of the coordinates on a group is a Casimir function, and vice versa.*

Proof. 1. Let a function

$$\Phi\left(F_1, \ldots, F_n\right) \tag{5.4}$$

be independent of the coordinates $x_1, \ldots, x_n$ on the group G. Because $F_k = m_k$ for $x = 0$ (Lemma 2), function (5.4) coincides with the function

$$\Phi\left(m_1, \ldots, m_k\right). \tag{5.5}$$

By Lemma 6, this function (and therefore (5.4)) commutes with all Noether integrals. Because commutation relations (4.3) and (5.2) formally coincide, function (5.5) commutes with each of $m_1, \ldots, m_n$ and is hence a Casimir function.

2. Conversely, let (5.5) be a Casimir function. Then (5.4) commutes with each of the Noether integrals. Therefore, it is independent of a point in G. Because (5.4) and (5.5) coincide for $x = 0$, they are identical. □

5.3. For fixed values of the Noether integrals

$$F_1 = c_1, \quad \ldots, \quad F_n = c_n, \tag{5.6}$$

the canonical momenta can be represented as single-valued functions on G,

$$y_1 = u_1(x,c), \quad \ldots, \quad y_n = u_n(x,c).$$

Let T be a left-invariant metric (kinetic energy) on G represented in the canonical variables. We set

$$\dot{x} = \left.\frac{\partial T}{\partial y}\right|_{y=u(x,c)} = v(x,c), \quad x \in G.$$

Obviously, the field v coincides with field (3.3).

Let $\Phi_1(x,y),\dots,\Phi_k(x,y)$ be Casimir functions. We can assign k vector fields on G,

$$w_1(x,c) = \left.\frac{\partial \Phi_1}{\partial y}\right|_{y=u}, \quad \dots, \quad w_k(x,c) = \left.\frac{\partial \Phi_k}{\partial y}\right|_{y=u},$$

to them. The vector fields v and $w_1,\dots,w_k$ have a clear geometric meaning. Let

$$v_T, \quad v_{\Phi_1}, \quad \dots, \quad v_{\Phi_k} \tag{5.7}$$

be Hamiltonian vector fields on $G \times \mathfrak{g}^*$ generated by the Hamiltonians T, $\Phi_1,\dots,\Phi_k$. Because these functions are left-invariant, the Noether integrals are integrals for each field in (5.7). Therefore, vector fields (5.7) are fields tangent to each of the n-dimensional regular surfaces

$$\Sigma_c = \{F_1 = c_1, \ \dots, \ F_n = c_n\} \subset G \times \mathfrak{g}^*.$$

The images of Hamiltonian fields (5.7) under the natural projection $\Sigma_c \to G$, $(x,y) \mapsto x$, are precisely the vector fields v, $w_1,\dots,w_k$.

Theorem 3 ([45]). *The following assertions hold*:

1. *the fields $w_1,\dots,w_k$ are vortex fields,*
2. *they commute with v and with each other, and*
3. *we have $\omega(w_j) = \mathrm{const}$, where $\omega = \sum u_i\, dx_i$.*

This theorem generalizes assertions 1–3 in the vortex theory of the top (see Sec. 2) to the multidimensional case.

Proof. 1. Assertion 1 is a direct consequence of Lemma 4.

2. Because

$$\{T, \Phi_i\} = 0 \qquad \text{and} \qquad \{\Phi_i, \Phi_j = 0\},$$

Hamiltonian fields (5.7) commute with each other (see Lemma 3). Therefore, projections on G of the restrictions of these fields to the invariant manifolds Σ_c also commute.

3. We note that each of the homogeneous parts of the Maclaurin expansion of a Casimir function by homogeneous forms of the variables $m_1,\dots,m_n$ is itself a Casimir function. We can hence consider only homogeneous polynomials. Furthermore,

$$i_{w_s}\omega = \left(y \cdot \frac{\partial \Phi_s}{\partial y}\right)_{y=u} \tag{5.8}$$

is equal to $p\Phi_s$, where p is the degree of homogeneity of Φ_s (the Euler formula). By Proposition 2, a Casimir function is a polynomial in Noether integrals. Substituting $y = u(x, c)$, we therefore conclude that the right-hand side of (5.8) depends only on $c_1, \dots, c_n$. □

5.4. In connection with Theorem 3, the natural question of whether combinations of the vector fields $w_1, \dots, w_n$ exhaust all vortex fields arises. The answer essentially depends on the constant Noether integrals $c_1, \dots, c_n$. For example, if $c = 0$, then $\omega = 0$, and any nonzero vector field is therefore a vortex field. However, for almost all values of $c_1, \dots, c_n$, there are no other vortex fields. These values are determined by the condition that the rank r of the skew-symmetric matrix of Poisson brackets

$$\|\{F_i, F_j\}\| = \left\| \sum_k c_{ij}^k c_k \right\|$$

is maximum. We note that r is an even number. It can be proved that the number of independent Casimir functions is $k = n - r$. Then, by Lemma 3, all vortex vectors are linear combinations of $w_1, \dots, w_k$.

Because the fields $w_1, \dots, w_k$ are left-invariant, these vectors are linearly independent at all points of G. Because they commute, their linear combinations generate an integrable k-dimensional distribution of tangent vectors on G (the Frobenius theorem). The k-dimensional integral manifolds of this distribution are exactly *vortex manifolds.*

If G is a compact group (as in the case of the Euler top), then the vortex manifolds are closed surfaces. This is an analogue of the closedness property of vortex lines in the case of a top. Because k-dimensional vortex manifolds are compact and have k independent commuting tangent fields, they are k-dimensional tori. The torus containing the unit of the group G is a subgroup of G; it is called the *maximal torus* of G. Maximal tori play a significant role in the classification of compact Lie groups (see, e.g., [1]).

The compactness property of vortex manifolds allows factoring G by vortex manifolds. After the factoring, the system of equations $\dot{x} = v(x, c)$ becomes a Hamiltonian system on the quotient space of even dimension $n - k$. Different aspects of the reduction of systems with symmetries can be found in [10].

5.5. Let $H_1, \dots, H_s$ be functions on the dual space $\mathfrak{g}^*$ that are first integrals of the equations of motion, i.e., the Lie–Poisson bracket of each of these functions with the kinetic energy represented in the variables $m_1, \dots, m_n$ is equal to zero. These integrals can be extended to functions defined on the whole space. We set

$$h_1(x, c) = H_1(x, u(x, c)), \quad \dots, \quad h_s(x, c) = H_s(x, u(x, c)).$$

If $H_1, \dots, H_s$ are not Casimir functions, then for almost all values of $c \in \mathbb{R}^n$, the functions $h_1, \dots, h_s$ are nonconstant on G. In addition, we set

$$v_1 = \left.\frac{\partial H_1}{\partial y}\right|_{y=u}, \quad \dots, \quad v_s = \left.\frac{\partial H_s}{\partial y}\right|_{y=u}.$$

Theorem 4. *The functions $h_1, \dots, h_s$ are Bernoulli functions (they are constant on streamlines and on vortex lines), and the vector fields $v_1, \dots, v_s$ on G commute with the vortex fields $w_1, \dots, w_k$.*

This theorem is implied because the functions $H_1, \dots, H_s$ commute with Noether integrals and Casimir functions. Because we can add kinetic energy to the set of functions $H_1, \dots, H_s$, Theorem 4 is a generalization of assertion 6 in the vortex theory of the top.

We consider the important particular case where

$$s = \frac{n-k}{2} \tag{5.9}$$

and the independent integrals $H_1, \dots, H_s$ are pairwise in involution. Then, obviously,

$$[v_i, v_j] = 0, \quad 1 \le i, j \le s. \tag{5.10}$$

Let

$$B_\alpha = \{x : h_1(x) = \alpha_1, \ \dots, \ h_s(x) = \alpha_s\}$$

be Bernoulli surfaces. For noncritical values of $\alpha \in \mathbb{R}^n$, they are regular manifolds of dimension $s + k$ that, by Theorem 2, admit $s+k$ independent tangent fields

$$v_1, \quad \dots, \quad v_s, \quad w_1, \quad \dots, \quad w_k.$$

By Theorem 4 and (5.10), these fields pairwise commute. Therefore, if G is a compact group, then each connected component of the Bernoulli surface is an $(s+k)$-dimensional torus.

For the Euler top, we have $n = 3$, $k = 1$ (the only Casimir function is the squared angular momentum), and $s = 1$ (if the inertia tensor is not spherical, then the energy integral and the Casimir function are independent). In this case, relation (5.9) holds, and the group $SO(3)$ is fibered to two-dimensional tori, Bernoulli surfaces.

Chapter 4
Vortex Method for Integrating Hamilton Equations

§1. Hamilton–Jacobi Method and the Liouville Theorem on Complete Integrability

1.1. As is known (see Sec. 7 in Chap. 1), the Hamilton–Jacobi method reduces the problem of solving the canonical equations

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \qquad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad 1 \le i \le n, \tag{1.1}$$

where $H = H(x, y, t)$, to investigating the first-order partial differential equation

$$\frac{\partial S}{\partial t} + H\left(x_1, \dots, x_n, \frac{\partial S}{\partial x_1}, \dots, \frac{\partial S}{\partial x_n}, t\right) = 0. \tag{1.2}$$

Let $S(x, t, c)$, $c = (c_1, \dots, c_n)$, be the complete integral of Eq. (1.2), i.e., this function satisfies (1.2) for all c and

$$\det \left\| \frac{\partial^2 S}{\partial x_i \partial c} \right\| \neq 0. \tag{1.3}$$

Then the relations

$$y = \frac{\partial S}{\partial x}, \qquad -a = \frac{\partial S}{\partial c}, \qquad a = (a_1, \dots, a_n), \tag{1.4}$$

hold by the *Jacobi theorem*, and the general solution of system (1.1) can be found from these relations: the coordinates x and the momenta y are functions of t and $2n$ arbitrary constants $a_1, \dots, a_n$ and $c_1, \dots, c_n$.

Condition (1.3) implies that $c_1, \dots, c_n$ as functions of x, y, and t can be found from the first group of equations (at least locally). These functions are independent integrals of Eqs. (1.1) and are pairwise in involution, $\{c_i, c_j\} = 0$ (see Lemma 1 in Chap. 3, Sec. 4).

In 1855, Liouville noted that the last assertion can be converted.

Theorem 1. *If Eqs.* (1.1) *have n independent integrals*

$$F_1(x, y, t), \quad \dots, \quad F_n(x, y, t)$$

that are pairwise in involution, $\{F_i, F_j\} = 0$, then Eqs. (1.1) *are integrable by quadratures.*

Integrable by quadratures means that the complete solution can be found by algebraic operations (including inversion of functions) and calculating integrals of functions of one variable.

1.2. From the methodological standpoint, the proof of the *Liouville theorem* is simpler for the autonomous case, where the functions $H, F_1, \dots, F_n$ are explicitly independent of t and, in particular, H is an integral. Theorem 1 was stated by the French mathematician Boure before Liouville. It is useful to take into account that each Hamiltonian system with the Hamiltonian F_i has the same set of integrals. Such systems are said to be *completely integrable.*

The Hamiltonian vector fields $v_{F_1}, \dots, v_{F_n}$ are tangent to the integral manifolds

$$N_c = \{F_1 = c_1, \ \dots, \ F_n = c_n\}$$

and pairwise commute (Lemma 3 in Chap. 3, Sec. 4). Therefore, each compact component of N_c is an n-dimensional torus with conditionally periodic motions. The geometric version of the Liouville theorem can be found, e.g., in [5].

1.3. We now prove Theorem 1 in the autonomous case. We assume that

$$\frac{\partial(F_1, \dots, F_n)}{\partial(y_1, \dots, y_n)} \neq 0. \tag{1.5}$$

This technical assumption simplifies the proof because it guarantees the independence of the functions $F_1, \dots, F_n$.

By virtue of (1.5), the system of algebraic equations

$$F_1(x, y) = c_1, \quad \dots, \quad F_n(x, y) = c_n$$

can be solved locally with respect to momenta:

$$y_1 = u_1(x, c), \quad \dots, \quad y_n = u_n(x, c).$$

By Corollary 3 (see Sec. 4 in Chap. 3), the 1-form

$$\omega = u_1\, dx_1 + \cdots + u_n\, dx_n \tag{1.6}$$

is the total differential of some function $W(x, c)$. This function can be constructed by calculating integrals of functions of one variable.

We now show that

$$H(x, u(x, c)) = h(c). \tag{1.7}$$

Calculating the derivative of the left-hand side of (1.7) with respect to the coordinate x_i,

$$\frac{\partial H}{\partial x_i} = \sum_j \frac{\partial H}{\partial y_j} \frac{\partial u_j}{\partial x_i} = a_i, \tag{1.8}$$

and taking the relation

$$F_k(x, u(x, c)) \equiv c_k \tag{1.9}$$

into account, we obtain

$$\frac{\partial F_k}{\partial x_i} + \sum_j \frac{\partial F_k}{\partial y_j}\frac{\partial u_j}{\partial x_i} = 0. \tag{1.10}$$

Relations (1.8) and (1.10) imply

$$\sum_i a_i \frac{\partial F_k}{\partial y_i} = \{H, F_k\} + \sum_{i,j} \frac{\partial H}{\partial y_j}\frac{\partial F_k}{\partial y_i}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

Because the functions $F_1, \dots, F_n$ are first integrals and 1-form (1.6) is closed, we see that $a \cdot \xi_k = 0$, where

$$a = (a_1, \dots, a_n)^{\mathrm{T}} \qquad \text{and} \qquad \xi_k = \left(\frac{\partial F_k}{\partial y_1}, \dots, \frac{\partial F_k}{\partial y_n}\right)_{y=u}.$$

By virtue of inequality (1.5), the vectors $\xi_1, \dots, \xi_n$ are linearly independent, and hence $a = 0$. Relation (1.7) is proved.

We set

$$S(x, t, c) = -h(c)t + W(x, c). \tag{1.11}$$

Because the Hamiltonian H is explicitly independent of t and relation (1.7) holds, function (1.11) satisfies Hamilton–Jacobi equation (1.2) for fixed c. It remains to verify inequalities (1.3). Relations (1.9) imply

$$\sum_i \frac{\partial F_k}{\partial y_i}\frac{\partial u_i}{\partial c_j} = \delta_{kj}$$

or, equivalently,

$$\left\|\frac{\partial F}{\partial y}\right\| \left\|\frac{\partial u}{\partial c}\right\| = E.$$

Therefore, the matrix

$$\left\|\frac{\partial u}{\partial c}\right\| = \left\|\frac{\partial^2 W}{\partial x \partial c}\right\| = \left\|\frac{\partial^2 S}{\partial x \partial c}\right\|$$

is nondegenerate.

To complete the proof, it remains to use the Jacobi theorem on the complete integral.

1.4. In the nonautonomous case, we extend the phase space by adding new canonical coordinates $x_{n+1} = t$ and y_{n+1} and take the new Hamiltonian

$$\mathcal{H} = y_{n+1} + H(x_1, \dots, x_n, y_1, \dots, y_n, x_{n+1})$$

(cf. Sec. 6 in Chap. 1). The Hamilton equations

$$\dot{x}_s = \frac{\partial \mathcal{H}}{\partial y_s}, \qquad \dot{y}_s = -\frac{\partial \mathcal{H}}{\partial x_s}, \quad 1 \le s \le n+1, \tag{1.12}$$

are equivalent to the original equations (1.1). The first integrals in Theorem 1 become

$$\mathcal{F}_k = F_k(x_1, \dots, x_n, y_1, \dots, y_n, x_{n+1}), \quad 1 \le k \le n.$$

Obviously,

$$\{\mathcal{F}_k, \mathcal{H}\} = \frac{\partial F_k}{\partial t} + \{F_k, H\} = \dot{F}_k = 0,$$
$$\{\mathcal{F}_i, \mathcal{F}_j\} = \{F_i, F_j\} = 0.$$

Therefore, Hamiltonian system (1.12) with $n+1$ degrees of freedom has $n+1$ independent integrals $\mathcal{H}, \mathcal{F}_1, \dots, \mathcal{F}_n$, which are pairwise in involution. Applying the autonomous version of the Liouville theorem to system (1.12), we complete the proof.

1.5. As we saw in Sec. 1 in Chap. 2, Hamiltonian systems of differential equations have a more general description in the form

$$(\operatorname{rot} u)\dot{x} = -\frac{\partial h}{\partial x}, \tag{1.13}$$

where $(x_1, \dots, x_n) = x$ are local coordinates on a smooth manifold M^n, $\operatorname{rot} u$ is a nondegenerate matrix of the curl of a covector field $u(x)$, and h is a smooth function on M^n. The assumption $\det(\operatorname{rot} u) \ne 0$ implies that n is necessarily even. By the Darboux theorem, system (1.13) can be locally transformed to the canonical form of the Hamilton equations by an appropriate change of variables. Unfortunately, this reduction can be performed explicitly only in particular cases.

The Poisson bracket can be defined not only by using the canonical variables. Let g be a smooth function on M^n. We calculate its derivative by virtue of system (1.13):

$$\dot{g} = \{g, h\} = \sum \gamma_{ij} \frac{\partial g}{\partial x_i} \frac{\partial h}{\partial x_j}, \tag{1.14}$$

where γ_{ij} are elements of the matrix inverse to $\operatorname{rot} u$. Obviously, bracket (1.14) possesses all the properties of the Poisson bracket.

We now assume that system (1.13) has $n/2$ independent integrals

$$f_1 = h, \quad f_2, \quad \dots, \quad f_{n/2} \tag{1.15}$$

such that their Poisson brackets (1.14) vanish. The question arises: Can system (1.13) be integrated by quadratures in this case? The Hamilton–Jacobi method is inapplicable because the Darboux theorem is nonconstructive. However, the Liouville theorem is valid in this case, but its proof is based on another idea.

Let $v_1, \dots, v_{n/2}$ be Hamiltonian fields generated by Hamiltonians (1.15):

$$(\operatorname{rot} u)v_k = -\frac{\partial f_k}{\partial x}, \quad 1 \le k \le n/2.$$

The field v_1 corresponds to the initial vector field (1.13). These fields pairwise commute and are tangent to the n-dimensional integral manifolds

$$I_c = \left\{x \in M : f_1(x) = c_1, \ \ldots, \ f_{n/2} = c_{n/2}\right\}.$$

We have thus obtained the following general situation. There are k vector fields on a k-dimensional manifold I^k, and they are linearly independent at every point of I and pairwise commute. Integral curves of these fields can then be calculated by quadratures. This is a particular case of the integrability of a system of k differential equations having a k-dimensional solvable symmetry group. In our case, phase flows of the vector fields $v_1, \ldots, v_k$ $(k = n/2)$ transform integral curves of each field v_i into integral curves of the same field, and we therefore have a k-dimensional commutative symmetry group (as is known, all Abelian groups are solvable).

The Lie theorem is the "continuous" analogue of the famous Galois theory of the solvability by radicals of an algebraic equation with a solvable root-permutation group. A detailed proof and applications to the Hamilton equations can be found in [46].

We note that compact connected components of nonsingular integral manifolds I_c are $(n/2)$-dimensional tori and there exist angular coordinates on these tori such that the components of the vector fields $v_1, \ldots, v_{n/2}$ are simultaneously constant. Therefore, phase flows of the vector fields v_k are exactly uniform motions (see, e.g., [5]).

§2. Noncommutative Integration of the Hamilton Equations

2.1. Let

$$F_1, \quad \ldots, \quad F_m \tag{2.1}$$

be independent integrals of the Hamiltonian system in the $2n$-dimensional phase space. In the autonomous case, the Hamiltonian can be among them.

The set of functions (2.1) is not necessarily involutive. Let $2r$ be the rank of the skew-symmetric matrix consisting of the Poisson brackets

$$\|\{F_i, F_j\}\|, \quad 1 \le i, j \le m. \tag{2.2}$$

We consider the domain of the phase space in which this rank is constant. We note that the rank of matrix (2.2) takes the maximum value almost everywhere.

The inequality

$$m \le n + r$$

is well known in symplectic geometry. For example, in Cartan's book [20], it is a simple consequence of more considerable results. The assumption

$$m = n + r \tag{2.3}$$

is often called the *condition for noncommutative integrability* of the Hamilton equations. The meaning of this term is explained below. In the particular case where integrals (2.1) commute ($r = 0$), condition (2.3) becomes the well-known Liouville condition for the complete integrability of Hamiltonian systems. Condition (2.3) has a clear sense: if we add some independent functions to set (2.1), then the rank of the matrix of their Poisson brackets necessarily increases.

In the theory of noncommutative integration, closed sets of integrals (2.1) are usually considered, i.e., their Poisson brackets are functions of $F_1, \dots, F_n$. This assumption is natural from the standpoint of the Poisson theorem. It is easy to see that an arbitrary set of first integrals can be extended to a closed set using only differentiation and algebraic operations (see, e.g., [34]). However, the closedness assumption is not always necessary.

2.2. As an example, we refer to *Nekhoroshev's theorem* [60], which was the origin of the theory of noncommutative integration. We assume that the autonomous Hamiltonian system has $n+k$ independent first integrals

$$F_1, \quad \dots, \quad F_{n+k} \tag{2.4}$$

such that the first $n-k$ integrals are in involution with all functions (2.4). We introduce $(n-k)$-dimensional integral manifolds

$$I_c = \{F_1 = c_1, \ \dots, \ F_{n+k} = c_{n+k}\} \tag{2.5}$$

in the $2n$-dimensional phase space. Nekhoroshev proved that if I_c is connected and compact, then it is diffeomorphic to an $(n-k)$-dimensional torus and there exist canonical coordinates J, p, $\varphi \bmod 2\pi$, and q in a neighborhood of this torus such that

$$J_s = J_s(F_1, \dots, F_{n-k}), \quad 1 \le s \le n-k, \tag{2.6}$$

and the conjugate variables p and q depend on all integrals (2.4).

Because the Hamiltonian is among the first $n-k$ integrals, we see that by (2.6),

$$H = H(J_1, \dots, J_{n-k})$$

in the new canonical variables. Therefore, the angular coordinates $\varphi_1, \dots,$ φ_{n-k} vary uniformly in time on the $(n-k)$-dimensional invariant torus:

$$\dot{\varphi}_s = \frac{\partial H}{\partial J_s} = \text{const}, \quad 1 \le s \le n-k.$$

We prove the first part of Nekhoroshev's theorem: compact connected components of integral manifolds (2.5) are tori. For this, we introduce $n-k$ Hamiltonian vector fields $v_1, \dots, v_{n-k}$ generated by the Hamiltonians $F_1, \dots, F_{n-k}$. We see that because these functions, by assumption, are independent and are in involution with all functions (2.4), the fields $v_1, \ldots, v_{n-k}$ are also independent, are tangent to I_c, and pairwise commute. Because $\dim I_c = n - k$, we have obtained what was required (cf. Sec. 1).

In addition, we have proved that the original differential Hamilton equations can be integrated by quadratures. Indeed, we can explicitly construct $(n-k)$-dimensional invariant manifolds and $n-k$ pairwise commuting independent tangent fields. The initial Hamiltonian vector field is among these constructed fields. It remains to use the Lie theorem on the integrability by quadratures of a system possessing a complete Abelian symmetry group.

In Nekhoroshev's theorem, $m = n + k$, and the rank of the matrix of the Poisson brackets of functions (2.4) has the obvious estimate

$$2r \leq n + k - (n - k) = 2k.$$

Hence, the inequality $m \leq n + k$ becomes the equality, and the condition for noncommutative integrability therefore holds.

2.3. The theorem on the toric structure of integral manifolds under general condition (2.3) was proved by Mishchenko and Fomenko in the case where integrals (2.1) generate a finite-dimensional Lie algebra (their Poisson brackets are linear functions of functions (2.1)). Later, Strel'tsov proved this result for closed sets of integrals of the general type. Brailov proved the integrability by quadratures of Hamiltonian systems with closed sets of integrals satisfying condition (2.3). His proof is based on the following idea: if the set of integrals is closed, then one can constructively (i.e., using algebraic operations) find the complete set of commuting tangent fields on integral manifolds.

A review of results in the theory of noncommutative integration of Hamiltonian systems and the connection between this theory and the earlier results of Lie, Cartan, and Dirac can be found in Fomenko's book [26].

2.4. As an example, we consider the Euler top. In this case, $n = 3$ and $m = 4$. The equations of motion have the energy integral H and three Noetherian integrals, projections K_x, K_y, and K_z of the angular momentum to fixed axes. By virtue of the commutation relations

$$\{H, K_x\} = 0, \quad \ldots, \quad \{K_x, K_y\} = K_z, \quad \ldots,$$

the rank of the matrix of the Poisson brackets equals two at typical points; therefore, $r = 1$, and condition (2.3) for noncommutative integrability is fulfilled.

We show how Nekhoroshev's theorem can be used in this problem. We have four (3+1) integrals: H, K^2 (the squared angular momentum), and two projections of the angular momentum K_y and K_z. The functions H and K^2 (in number, $3 - 1$) commute with all integrals.

We also note that H, K^2, and K_y compose a complete set of independent integrals in involution. Therefore, the entire six-dimensional phase space is fibered to three-dimensional invariant tori with conditionally periodic motions. However, because there exists the independent integral K_z, these three-dimensional tori are fibered to two-dimensional tori lying in three-dimensional invariant manifolds, which are determined by the conditions that the projections K_x, K_y, and K_z are constant. The natural projection on the configuration space transforms these two-dimensional tori into the Bernoulli surfaces in the hydrodynamic theory of the Euler top.

2.5. The above considerations fall in the autonomous case. However, condition (2.3) for noncommutative integrability can be easily extended to the nonautonomous case. We assume that integrals (2.1) and the Hamiltonian $H(x, y, t)$ can explicitly depend on time. We extend the phase space by adding the new canonical variables $x_{n+1} = t$ and y_{n+1} and introduce the new Hamiltonian

$$\mathcal{H} = y_{n+1} + H(x, y, x_{n+1}).$$

Functions (2.1) are integrals $\mathcal{F}_s = F_s(x, y, x_{n+1})$ of the extended system:

$$\{\mathcal{F}_s, \mathcal{H}\} = 0, \quad 1 \leq s \leq m. \tag{2.7}$$

In addition, by Sec. 1,

$$\{\mathcal{F}_i, \mathcal{F}_j\} = \{F_i, F_j\}. \tag{2.8}$$

We now have a Hamiltonian system with $n+1$ degrees of freedom, which has $m+1$ independent first integrals

$$\mathcal{H}, \quad \mathcal{F}_1, \quad \ldots, \quad \mathcal{F}_m. \tag{2.9}$$

By virtue of (2.7) and (2.8), the rank of the matrix of their Poisson brackets has not increased and is equal to

$$r = \operatorname{rank} \|\{F_i, F_j\}\|.$$

Therefore, equality (2.3) is also valid for the extended autonomous Hamiltonian system.

These remarks imply the following theorem.

Theorem 2. *If a nonautonomous Hamiltonian system with n degrees of freedom has $n+k$ independent integrals* (2.4) (*generally speaking, depending on time*) *such that the first $n-k$ integrals are in involution with all integrals* (2.4), *then the Hamilton equations are integrable by quadratures.*

§3. Vortex Integration Method

3.1. We first point out some leading considerations. we assume that the Hamiltonian system of differential equations

$$x = \frac{\partial H}{\partial y}, \qquad y = -\frac{\partial H}{\partial x}, \qquad (3.1)$$
$$x = (x_1, \dots, x_n), \qquad y = (y_1, \dots, y_n),$$

has $n+k$ independent first integrals

$$F_1, \quad \dots, \quad F_{n+k} \qquad (3.2)$$

such that the first $n-k$ functions are in involution with all integrals (3.2). In this case, by Sec. 2, condition (2.3) for noncommutative integrability holds. We emphasize that we consider the general nonautonomous case, where integrals (3.2) and the Hamiltonian H can explicitly depend on time.

Similarly to the proof of the Liouville theorem on complete integrability, we consider the algebraic system of n equations

$$F_{1+k}(x, y, t) = c_1, \quad \dots, \quad F_{n+k}(x, y, t) = c_n \qquad (3.3)$$

and additionally assume that

$$\frac{\partial(F_{1+k}, \dots, F_{n+k})}{\partial(y_1, \dots, y_n)} \neq 0. \qquad (3.4)$$

System (3.3) can be solved locally with respect to the canonical momenta:

$$y = u(x, t, c), \qquad c = (c_1, \dots, c_n). \qquad (3.5)$$

Because the n equations in (3.3) define an n-dimensional invariant manifold of system (3.1) for all c, field (3.5) satisfies the Lamb equation

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)v = -\frac{\partial h}{\partial x}, \qquad (3.6)$$
$$v = \left.\frac{\partial H}{\partial y}\right|_{y=u}, \qquad h = H(x, u(x, t, c), t).$$

By virtue of condition (3.4), we have

$$\frac{\partial(u_1, \dots, u_n)}{\partial(c_1, \dots, c_n)} \neq 0, \qquad (3.7)$$

and (3.5) is therefore the *complete integral* of Lamb equation (3.6).

We now ascertain how complete integral (3.5) interrelates with the first integrals $F_1, \dots, F_{n-k}$ from set (3.2), which commute with all other integrals. We set

$$v_i(x, t, c) = \left.\frac{\partial F_i}{\partial y}\right|_{y=u}, \qquad f_i(x, t, c) = F_i(x, u(x, t, c), c).$$

It is clear that v_i is a vector field on the configuration manifold.

Lemma 1. *The function F_i is in involution with the functions $F_{k+1}, \dots,$ F_{k+n} if and only if the relation*

$$(\operatorname{rot} u)v_i = -\frac{\partial f_i}{\partial x} \tag{3.8}$$

holds for all values of the parameters $c_1, \dots, c_n$.

Proof. The relations $\{F_i, F_j\} = 0$, $s \geq k+1$, mean that the functions $F_{k+1}, \dots, F_{k+n}$ are first integrals of the differential Hamilton equations

$$\frac{dx}{d\tau} = \frac{\partial F_i}{\partial y}, \qquad \frac{dy}{d\tau} = -\frac{\partial F_i}{\partial x}, \tag{3.9}$$

where the time t is assumed to be a parameter. Therefore, algebraic equations (3.3) define invariant manifolds of canonical Hamilton equations (3.9) for fixed t. This implies that Lamb equations (3.8) hold (we note that the derivative $\partial u/\partial t$ does not enter these equations). The converse assertion is obvious. $\square$

Therefore, if we have an overabundant set of first integrals (3.2) satisfying the condition in Nekhoroshev's theorem, then we can find the complete integral of Lamb equation (3.6) from Eqs. (3.3). In addition, this complete integral satisfies $n-k$ "truncated" Lamb equations (3.8).

3.2. This reasoning can be inverted.

Theorem 3 ([48]). *Let $u(x, t, c)$ be the complete integral of Lamb equation* (3.6), *and let the following conditions hold:*

a. $\operatorname{rank}(\operatorname{rot} u) = 2k$;

b. *there exist k integrals $F_1(x, y, t), \dots, F_k(x, y, t)$ of Hamilton equations* (3.1) *such that $\{F_i, F_j\} = 0$ for all $1 \leq i, j \leq n$ and the field $u(x, t, c)$ satisfies the equations*

$$\begin{aligned} (\operatorname{rot} u)v_i &= -\frac{\partial f_i}{\partial x}, \quad 1 \leq i \leq k, \\ v_i &= \left.\frac{\partial F_i}{\partial y}\right|_{y=u}, \qquad f_i = F_i(x, u(x, t, c), t) \end{aligned} \tag{3.10}$$

for all c;

c. *$f_1, \dots, f_k$ considered as functions of x are independent.*

Then the original Hamilton equations (3.1) *are noncommutatively integrable.*

We recall that the rank of the matrix $\operatorname{rot} u$ is even because of its skew-symmetry. The method for explicitly integrating Hamilton differential equations using the complete integral of the Lamb equation from Theorem 3 is called the *vortex integration method.*

We consider a particular case where u is a potential solution of the Lamb equation,

$$u = \frac{\partial S(x, t, c)}{\partial x}.$$

Then $\operatorname{rot} u = 0$ and hence $k = 0$. Nondegeneracy condition (3.7) becomes

$$\det \left\| \frac{\partial^2 S}{\partial x \partial c} \right\| \neq 0,$$

and S is the complete integral of the Hamilton–Jacobi equation. Therefore, the classical Hamilton–Jacobi method is a particular case of the vortex integration method.

Proof of Theorem 3. We must specify the set of independent first integrals of Hamilton equations (3.1) satisfying condition (2.3) for noncommutative integrability.

For this, we consider the system of algebraic equations

$$\begin{aligned} &y_1 = u_1(x, t, c), \quad \ldots, \quad y_n = u_n(x, t, c), \\ &c = (c_1, \ldots, c_n). \end{aligned} \tag{3.11}$$

By virtue of condition (3.7), we can find (at least locally) constant parameters $c_1, \ldots, c_n$ as functions of the canonical variables x and y and the time t from this system:

$$c_1 = F_{k+1}(x, y, t), \quad \ldots, \quad c_n = F_{k+n}(x, y, t). \tag{3.12}$$

Because Eqs. (3.11) define the invariant manifolds of Hamilton equations (3.1), functions (3.12) are their first integrals. By virtue of (3.7), the functions $F_{k+1}, \ldots, F_{k+n}$ are independent. Moreover, condition c implies the functional independence of the set of integrals

$$F_1, \quad \ldots, \quad F_k, \quad F_{k+1}, \quad \ldots, \quad F_{k+n}. \tag{3.13}$$

Condition b implies that the first k integrals are in involution with all integrals (3.13) (Lemma 1). Therefore, the rank of the matrix of Poisson brackets for set (3.13) is equal to the rank of the matrix of Poisson brackets for set (3.12). By Lemma 1 in Chap. 3 (Sec. 4), this rank equals $\operatorname{rank}(\operatorname{rot} u) = 2k$.

Finally, for the set of independent first integrals (3.13), we have

$$m = n + k, \qquad r = k,$$

and therefore $m = n + r$. □

Theorem 3 shows that the vortex integration method interrelates with the condition for noncommutative integrability as the Hamilton–Jacobi method interrelates with the Liouville condition for complete integrability of the Hamilton equations.

3.3. As another example, we consider the extreme case where the curl matrix has the maximum rank n. This is the "most vortex" solution of the Lamb equation. In this case, $n = 2k$, and Hamilton equations (3.1) have k involutive integrals satisfying conditions b and c of Theorem 3. It turns out that the Hamilton equations can be integrated by quadratures.

This result, distinguished in [37], has the simplest sense in the autonomous case, where the functions $H = F_1, F_2, \dots, F_{n/2}$ and the field u are explicitly independent of t. Then Lamb equation (3.6),

$$(\operatorname{rot} u)\dot{x} = -\frac{\partial h}{\partial x}, \tag{3.14}$$

is the Hamiltonian system in the $2k$-dimensional phase space $\{x\}$ with the symplectic structure

$$\omega = d(u\,dx) = \sum \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_i \wedge dx_j$$

and the Hamiltonian h (see Sec. 1). By virtue of conditions b and c, the functions $h = f_1, f_2, \dots, f_k$ are independent integrals of Eqs. (3.14). Because $\{F_i, F_j\} = 0$, the functions $f_1, \dots, f_k$ are also in involution relative to the symplectic structure ω. By the Liouville theorem, Eqs. (3.14) are integrable by quadratures. The momenta y can be found from the relations $y = u(x, c)$. The nonautonomous case can be reduced to the autonomous case by extending the phase space (see Sec. 4).

3.4. In [48], I claimed that if the conditions of Theorem 3 hold, then the Hamilton equations are integrable by quadratures. The proof of this assertion contains a gap, which can be closed only if some additional conditions hold. One such case ($n = 2k$) was examined in the preceding subsection.

The problem is closely related to the possibility of explicitly constructing $n-2k$ functions of integrals (3.12) commuting with all functions (3.12). Adding k functions from condition b to them, we obtain $n-k$ functions, which are in involution with $n+k$ integrals (3.13). This problem can be essentially simplified if we assume that set (3.12) is closed, i.e., the Poisson brackets of these functions are expressed through themselves.

The closedness condition for set (3.12) can be formulated as some properties of the curl matrix of covector field (3.11). For this, we introduce n linearly independent covectors

$$a_1 = \frac{\partial u}{\partial c_1}, \quad \dots, \quad a_n = \frac{\partial u}{\partial c_n},$$

which, of course, depend on the point x of the configuration space and on the time t. We can uniquely assign the dual set of linearly independent vectors $b_1, \dots, b_n$ to these covectors,

$$(a_i, b_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

It is easy to verify that

$$b_j = \left.\frac{\partial F_{n+j}}{\partial y}\right|_{y=u}.$$

For brevity, we set $B = \operatorname{rot} u$.

Lemma 2. *The set of functions $F_{k+1}, \ldots, F_{k+n}$ is closed if and only if the functions (Bb_i, b_j) are independent of x and t for all $i, j = 1, \ldots, n$.*

Thereby, verifying the closedness conditions only requires differentiation and algebraic operations (including inversion of functions). The complete integral $u(x, t, c)$ of the Lamb equation is said to be *closed* if the conditions of Lemma 2 hold. In particular, all potential solutions are closed.

Lemma 2 is a simple consequence of relations (4.7) and (4.10) in Chap. 3.

Theorem 4 ([45]). *Let the complete integral $u(x, t, c)$ of the Lamb equation satisfy conditions* a–c *in Theorem 3. Then the Hamilton equations are integrable by quadratures.*

Indeed, by Lemmas 1 and 2, the set of integrals (3.13) of the Hamilton equations is closed relative to the Poisson brackets. Theorem 4 is now implied by the nonautonomous version of Brailov's theorem on the integrability of Hamiltonian systems that have closed sets of integrals satisfying condition (2.3). Because the complete integral of the Hamilton–Jacobi equation generates the closed complete integral of the Lamb equation, the classical Jacobi theorem is a particular case of Theorem 4.

3.5. As an example, we consider Liouville's problem (1858) on the inertial rotation of a nonrigid body: its particles move relative to each other because of internal forces. We choose the inertia axes of the body as the axes of the moving frame. Let ω be the angular velocity of the moving trihedron, K be the angular momentum of the body relative to a fixed point, and $I = \operatorname{diag}(I_1, I_2, I_3)$ be the inertia matrix. The angular momentum and the angular velocity are related by

$$K = I\omega + \lambda, \tag{3.15}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the gyroscopic torque. The additional term in (3.14) is due to the motion of particles of the body.

By the theorem on the variation of the angular momentum, the dynamic Euler equations become

$$\dot{K} = \omega \times K = 0. \tag{3.16}$$

We assume that I_i and λ_i are known functions of time. For example, if the body contains symmetric flywheels rotating freely about their axes, then the principal moments of inertia and gyroscopic torques are constants. Kelvin called such systems *gyrostats*. There are other problems in the dynamics of nonrigid bodies; for example, Seeliger and Chetaev considered a nonrigid body and added the equation for the velocity of "radiant expansion" for closing system (3.15), (3.16). (See Routh's treatise [69] for more details.)

Adding the Poisson equations for unit fixed vectors α, β, and γ,

$$\dot{\alpha} = \alpha \times \omega, \qquad \dot{\beta} = \beta \times \omega, \qquad \dot{\gamma} = \gamma \times \omega, \tag{3.17}$$

to (3.15) and (3.16), we obtain the complete system, which allows determining the orientation of the principal axes of inertia of the body. Equations (3.15) and (3.16) have three momentum integrals

$$(K, \alpha) = c_1, \qquad (K, \beta) = c_2, \qquad (K, \gamma) = c_3, \tag{3.18}$$

which imply the integral $(K, K) = \text{const}$ of Euler equations (3.16).

We find three-dimensional surfaces that have single-valued projections onto the configuration space, the group $SO(3)$. This means that the angular momentum K is a function of α, β, γ, and t. From (3.15) and (3.16), we obtain the partial differential equation

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial \alpha}(\alpha \times \omega) + \frac{\partial K}{\partial \beta}(\beta \times \omega) + \frac{\partial K}{\partial \gamma}(\gamma \times \omega) = K \times \omega, \tag{3.19}$$

where we must replace ω with $I^{-1}(K - \lambda)$. Of course, this is the Lamb equation, but it is represented in noncanonical variables. Passing to Eq. (3.19) is similar to the transition from the Hamilton equations to the Poincaré–Chetaev equations on Lie algebras.

By (3.18), the function

$$K = c_1\alpha + c_2\beta + c_3\gamma \tag{3.20}$$

is a solution of Lamb equation (3.19). In the canonical variables θ, p_θ, ..., this solution is the same as that for the ordinary Euler problem (see formulas (2.8) in Chap. 2). Obviously, complete solution (3.20) is closed, does not contain time, and the curl matrix has the rank two for $|c| \neq 0$.

We assume that Eqs. (3.15) and (3.16) have the integral $F(K, t)$ independent of the momentum integral (K, K). Then Eqs. (3.16) and (3.17) are integrable by quadratures. We note that this fact can be deduced from Theorem 4. Indeed, we have $k = 1$, and the Poincaré–Chetaev equation

$$\dot{K} = K \times \frac{\partial F}{\partial K} \tag{3.21}$$

corresponds to the function F; we must add Poisson equations (3.17) to (3.21). Clearly, surfaces (3.18) are invariant surfaces for Eqs. (3.21) and (3.17). Complete solution (3.20) satisfies "truncated" Lamb equation (3.10) because it is independent of time. Therefore, conditions a and b in the vortex theorem of integration hold; condition c is implied by the independence of the functions F and K^2.

Remark. The result on the integrability by quadratures of Eqs. (3.16) and (3.17) can be obtained using the theory of noncommutative integration of Hamiltonian systems. In this case, $n = 3$, and the functions F, K^2, (K, α), and (K, β) constitute the required set of integrals. The first two functions commute with the others, and the matrix of their Poisson brackets equals two.

It is interesting to write the system of equations on the group $SO(3)$ that generalizes Eqs. (2.11) in Chap. 2 for the Euler top. For definiteness, we consider the case where the angular momentum K is directed along γ: $K = k\gamma$, $k = |K|$. Assuming that the vector γ is vertical, we introduce the Euler angles θ, φ, and ψ, which determine the orientation of the principal inertia axes of the nonrigid body. Using the kinematic Euler formulas, we can obtain the equations of motion on the group $SO(3)$:

$$\begin{aligned}
\dot\theta &= k\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin\theta\sin\varphi\cos\varphi - \frac{\lambda_1\cos\varphi}{I_1} + \frac{\lambda_2\sin\varphi}{I_2}, \\
\dot\varphi &= k\cos\theta\left(\frac{1}{I_3} - \frac{\sin^2\varphi}{I_1} - \frac{\cos^2\varphi}{I_2}\right) + \frac{\lambda_1\sin\varphi\cos\theta}{I_1\sin\theta} \\
&\quad + \frac{\lambda_2\cos\varphi\cos\theta}{I_2\sin\theta} - \frac{\lambda_3}{I_3}, \\
\dot\psi &= k\left(\frac{\sin^2\varphi}{I_1} + \frac{\cos^2\varphi}{I_2}\right) - \frac{\lambda_1\sin\varphi}{I_1\sin\theta} - \frac{\lambda_2\cos\varphi}{I_2\sin\theta}.
\end{aligned} \tag{3.22}$$

These equations have the integral invariant

$$\operatorname{mes}(D) = \iiint\limits_D \sin\theta\, d\theta\, d\varphi\, d\psi,$$

which coincides with the two-sided invariant Haar measure on the group $SO(3)$.

In these variables, the vortex fields have the form

$$\theta' = 0, \qquad \varphi' = 0, \qquad \psi' = \mu,$$

and the vortex lines are defined by the equations $\theta, \varphi = \text{const}$. Because the third equation in system (3.22) does not contain the angle ψ, we obtain the generalized Helmholtz–Thomson theorem: the flow of system (3.22) transforms vortex lines into vortex lines. Factoring the group $SO(3)$ by closed vortex lines yields the first two equations in system (3.22); they compose a closed nonautonomous Hamiltonian system of the Poisson sphere, and the standard area 2-form plays the role of the symplectic structure.

3.6. The vortex method for integrating Hamilton equations includes the problem of explicitly finding the complete integral of the Lamb equations. As is known, integrating the Hamilton–Jacobi equation yields potential solutions. For this, Jacobi developed the method of separation of variables, which was improved by Imshenetsky, Stäckel, and others. Finding nonpotential solutions of the Lamb equations gives many more possibilities; unfortunately, this problem is still little studied.

To show specific difficulties encountered in this direction, we consider the Hamilton equations with the Hamiltonian

$$H = \frac{1}{2}(y, y) + (y, Ax) + V(x), \tag{3.23}$$

where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are conjugate canonical variables, A is a constant matrix, and V is the potential energy. Terms linear in momenta correspond to gyroscopic forces. For definiteness, we consider the case where the matrix A is skew-symmetric, $A^{\mathrm{T}} = -A$. An important example is the system

$$\ddot{x}_1 = \omega\, \dot{x}_2, \qquad \ddot{x}_2 = -\omega\, \dot{x}_1 \tag{3.24}$$

describing the planar motion of a charged particle in a perpendicular magnetic field. These equations are the Lagrange equations with the Lagrangian

$$L = \frac{\dot{x}_1^2 + \dot{x}_2^2}{2} + \frac{\omega}{2}(x_1\dot{x}_2 - x_2\dot{x}_1).$$

The Legendre transformation

$$y_1 = \dot{x}_1 - \frac{\omega x_2}{2}, \qquad y_2 = \dot{x}_2 + \frac{\omega x_1}{2}$$

yields Hamilton equations with the Hamiltonian

$$H = \frac{y_1^2 + y_2^2}{2} + \frac{\omega}{2}(x_2 y_1 - x_1 y_2) + \frac{\omega^2(x_1^2 + x_2^2)}{8}. \tag{3.25}$$

In this example,

$$A = \begin{pmatrix} 0 & \omega/2 \\ -\omega/2 & 0 \end{pmatrix}. \tag{3.26}$$

Obviously, Eqs. (3.24) have a family of two-dimensional invariant manifolds admitting single-valued projections on the configuration plane x_1, x_2:

$$\dot{x}_1 = \omega x_2 + c_1, \qquad \dot{x}_2 = -\omega x_1 + c_2, \qquad c_i = \mathrm{const}\,.$$

It is clear that the particle trajectories are circles (the *Larmor circles*). In canonical variables, these manifolds have the form

$$y_1 = \frac{\omega x_2}{2} + c_1, \qquad y_2 = -\frac{\omega x_1}{2} + c_2. \tag{3.27}$$

If $\omega \neq 0$, then they are vortex manifolds.

For the equations with Hamiltonian (3.23), we find n-dimensional invariant manifolds of the form

$$y = Ax + \frac{\partial S}{\partial x}$$

(cf. (3.27)). Then the Lamb equation becomes

$$\frac{\partial}{\partial x}\frac{\partial S}{\partial t} + 2A\left(2Ax + \frac{\partial S}{\partial x}\right) = -\frac{\partial h}{\partial x}. \tag{3.28}$$

Clearly, the term $4A^2x$ is the gradient of the quadratic form $\Phi = -2(Ax, Ax)$. Equation (3.28) is consistent if the term $2A\,\partial S/\partial x$ is the gradient of some function $R(x)$. The existence criterion for the function R is

$$A\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial x^2}A = 0.$$

We note that for $n = 2$, it is equivalent to the harmonicity of the function S: $\Delta S = 0$. From (3.28), we finally obtain the analogue of the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial x}\right) + 2\left(\frac{\partial S}{\partial x}, Ax\right) + R - \frac{1}{2}(Ax, Ax) + V = 0. \tag{3.29}$$

For $n = 2$, we can take

$$\frac{\partial S}{\partial x_1} = -\frac{\partial R}{\partial x_2}, \qquad \frac{\partial S}{\partial x_2} = \frac{\partial R}{\partial x_1}. \tag{3.30}$$

These equations are the famous Cauchy–Riemann equations; S and R are conjugate harmonic functions. In the stationary case, Eq. (3.29) can be represented as the equation for R:

$$\frac{1}{2}\left[\left(\frac{\partial R}{\partial x_1}\right)^2 + \left(\frac{\partial R}{\partial x_2}\right)^2\right] - \omega\left(\frac{\partial R}{\partial x_1}x_1 + \frac{\partial R}{\partial x_2}x_2\right)$$
$$+\,\omega R - \frac{\omega^2}{8}\left(x_1^2 + x_2^2\right) + V = h, \tag{3.31}$$

where h is the energy constant. Of course, we must remember that it is necessary to add the harmonicity condition for R to Eq. (3.31).

For Hamiltonian (3.25), we have

$$V = \frac{\omega^2\left(x_1^2 + x_2^2\right)}{8}. \tag{3.32}$$

The function $R = c_2x_1 - c_1x_2$, where c_1 and c_2 are arbitrary constants, can serve as the harmonic complete integral, and the energy constant equals

$$h = \frac{c_1^2 + c_2^2}{2}.$$

Invariant manifolds of the Hamiltonian system with Hamiltonian (3.25) can be found in the form

$$y = -Ax + \frac{\partial S}{\partial x}.$$

Then $\operatorname{rot} u = -2A$, and the Lamb equation takes the simpler form

$$\frac{\partial}{\partial x}\frac{\partial S}{\partial t} - 2A\frac{\partial S}{\partial x} = -\frac{\partial h}{\partial x}.$$

Setting

$$2A\frac{\partial S}{\partial x} = \frac{\partial P}{\partial x},$$

we obtain the equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial x}\right) - P - \frac{1}{2}(Ax, Ax) + V = 0,$$

which is similar to Eq. (3.29).

Let $n = 2$, the matrix A have the form (3.26), and the function P satisfy Cauchy–Riemann equations (3.30). We then obtain the equation

$$\frac{1}{2}\left[\left(\frac{\partial P}{\partial x_1}\right)^2 + \left(\frac{\partial P}{\partial x_2}\right)^2\right] - \omega P - \frac{\omega^2}{8}\left(x_1^2 + x_2^2\right) + V = h \tag{3.33}$$

for the stationary case. We must add the Laplace equation $\Delta P = 0$ to this equation.

Although Eq. (3.33) is simpler than Eq. (3.31), it has no potential solutions for potential (3.32).

§4. Complete Integrability of the Quotient System

4.1. As we have seen, if $\operatorname{rank}(\operatorname{rot} u)$ is constant and equals $2k$, then the configuration space M^n is fibered by $(n-2k)$-dimensional vortex manifolds N. Identifying points lying in the same connected component of the vortex manifold, we obtain the $2k$-dimensional quotient space M/N. The generalized Helmholtz–Thomson theorem allows defining a quotient system on M/N, which also has the form of the Lamb equations with a nondegenerate curl matrix (Eqs. (4.11) in Chap. 2).

The main result of the vortex theory of integrating Hamilton equations is the following theorem.

Theorem 5. *Let conditions* a–c *of Theorem 1 in Sec. 3 hold. Then explicit solutions of the quotient system on M/N can be found by quadratures.*

This result has two aspects. First, vortex manifolds of the matrix $\operatorname{rot} u$, where $u(x, t, c)$ is the complete integral of the original Lamb equation, can be found explicitly, and the equations of the quotient system can therefore be represented explicitly. Second, differential equations of the quotient system can be explicitly integrated by quadratures.

4.2. We discuss the second aspect and assume that the quotient system is already transformed into the form of the Lamb equation with a nondegenerate curl:

$$\frac{\partial u}{\partial t} + (\operatorname{rot} u)\dot{x} = -\frac{\partial h}{\partial x}, \qquad x = (x_1, \ldots, x_{2k}). \tag{4.1}$$

As we noted above, such systems are called *Birkhoff systems* (Birkhoff himself called them Pfaff equations). Because the curl is nondegenerate, Eq. (4.1) can be solved with respect to the velocity:

$$\dot{x} = v(x, t). \tag{4.2}$$

We assume that these equations have k independent integrals

$$f_1(x,t), \quad \ldots, \quad f_k(x,t). \tag{4.3}$$

We can uniquely assign a vector field v_s to each of these function by the rule

$$(\operatorname{rot} u)v_s = -\frac{\partial f_s}{\partial x}. \tag{4.4}$$

In the general case, these fields depend on time. The question arises: When do they commute for all fixed t? To answer this question, we introduce the matrix $\Gamma = \|\gamma_{ij}\|$ inverse to the matrix $\operatorname{rot} u$ (as in Sec. 1). The matrix Γ is also skew-symmetric.

Lemma 3. *If*

$$\sum_{i,j} \gamma_{ij} \frac{\partial f_p}{\partial x_i} \frac{\partial f_q}{\partial x_j} = 0, \tag{4.5}$$

then $[v_p, v_q] = 0$.

Proof. Relation (4.4) shows that v_s is the Hamiltonian field (generated by the Hamiltonian f_s) with respect to the symplectic structure defined by the closed nondegenerate 2-form

$$\Omega = d\left(\sum u_i\, dx_i\right).$$

The expression in the left-hand side of (4.5) is the Poisson bracket of the functions f_p and f_q represented in the noncanonical variables $x_1, \ldots, x_{2k}$. We now see that Lemma 3 is a consequence of Corollary 3 in Chap. 3 (Sec. 4). □

In analogy with the Hamiltonian case, we say that the functions f_p and f_q are *in involution*. The system of Birkhoff equations (4.1) is said to be *completely integrable* if integrals (4.3) are pairwise in involution. This terminology is explained by the following proposition.

Proposition 1. *A completely integrable Birkhoff system can be integrated by quadratures.*

This proposition has the following important consequence.

Corollary 1. *Let conditions* a–c *of the theory of vortex integration hold and* $\operatorname{rank}(\operatorname{rot} u) = n$. *Then the Hamilton equations are integrable by quadratures.*

The autonomous version of this assertion was proved in Sec. 3 using the Liouville theorem on complete integrability.

Proof of Proposition 1. We write Eqs. (4.1) and (4.4) in the invariant form:

$$\frac{\partial \omega}{\partial t} + i_v \Omega = -dh, \tag{4.6}$$

$$i_{v_s} \Omega = -df_s. \tag{4.7}$$

Because functions (4.3) are integrals of system (4.2), we have

$$\frac{\partial f_s}{\partial t} + L_v f_s = 0. \tag{4.8}$$

Taking (4.8) into account, we obtain the relation

$$L_v i_{v_s} \Omega = -L_v\, df_s = \frac{\partial}{\partial t} df_s \tag{4.9}$$

from (4.7). On the other hand, Eq. (4.6) implies

$$i_{v_s} \frac{\partial \Omega}{\partial t} + i_{v_s} L_v \Omega = 0. \tag{4.10}$$

Subtracting (4.10) from (4.9) and using (4.7), we obtain

$$i_{\partial v_s/\partial t} \Omega + i_{[v_s, v]} \Omega = 0.$$

Because the 2-form Ω is nondegenerate, we have

$$\frac{\partial v_s}{\partial t} + [v_s, v] = 0. \tag{4.11}$$

We now extend the $2k$-dimensional x-space by adding the time t as a new coordinate. In the $(2k+1)$-dimensional space, we introduce $k+1$ vector fields

$$\widetilde{v}, \quad \widetilde{v}_1, \quad \ldots, \quad \widetilde{v}_k, \tag{4.12}$$

which are obtained from the fields $v, v_1, \ldots, v_k$ by adding the t-components $1, 0, \ldots, 0$ respectively. The fields $v_1, \ldots, v_k$ commute for all t because integrals (4.3) are in involution (see Lemma 3). Therefore,

$$[\widetilde{v}_i, \widetilde{v}_j] = 0, \quad i, j = 1, \ldots, k.$$

It can be easily verified (see Sec. 5 in Chap. 2) that relations (4.11) are equivalent to the commutation relations

$$[\widetilde{v}, \widetilde{v}_s] = 0, \quad s = 1, \ldots, k.$$

We now consider a regular $(k+1)$-dimensional surface

$$I_\alpha = \{x, t : f_1 = \alpha_1, \ \ldots, \ f_k = \alpha_k\} \tag{4.13}$$

in the extended $(2k+1)$-dimensional space. By virtue of (4.8), the field $\widetilde{v}$ is tangent to I_α for all α. The same property for the vector fields $\widetilde{v}_s$ is implied by (4.7). Therefore, $k+1$ vector fields (4.12) are tangent to $(k+1)$-dimensional integral surfaces (4.13), are linearly independent at each point, and pairwise commute. The integrability by quadratures of system (4.1) is now implied by the Lie theorem on the integrability of differential equations with an Abelian symmetry group. □

In Sec. 1 in Chap. 2, we noted that Birkhoff equations (4.1) can be locally reduced to the usual Hamilton equations. However, this reduction is not constructive, and Proposition 1 hence does not follow, formally speaking, from the Liouville theorem on the complete integrability of Hamiltonian systems.

4.3. It remains to show that if conditions a–c hold, then vortex manifolds can be found by quadratures; in this case, the transition to the quotient system can be performed constructively.

We consider the family of common level surfaces for the functions f_s mentioned in condition b:

$$M_\beta^{n-k} = \{x : f_1(x,t) = \beta_1, \ \ldots, \ f_k(x,t) = \beta_k\}.$$

In the nonautonomous case, these surfaces depend on the time t as on a parameter.

We fix the value of t. Let $V_1, \ldots, V_k$ be Hamiltonian vector fields in the $2n$-dimensional space of variables x and y generated by the Hamiltonians $F_1, \ldots, F_k$. These fields pairwise commute, $[V_i, V_j] = 0$, because $\{F_i, F_j\} = 0$. By condition b, the fields V_i are tangent to the n-dimensional surfaces

$$\Sigma_c = \{x, y : y = u(x,t,c)\}$$

(see Lemma 1 in Sec. 3), and their projections $v_1, \ldots, v_k$ on the configuration space are therefore well defined. Because the fields V_i pairwise commute, we obtain $[v_i, v_j] = 0$.

Because $\{F_i, F_j\} = 0$, all functions F_i are the integrals of the vector fields V_j: $V_j(F_i) = 0$. Therefore, $v_j(f_i) = 0$ for all $i, j = 1, \ldots, k$. This means that the fields v_i are tangent to each of the surfaces M_β. On the other hand, there exist independent vortex vector fields $w_1, \ldots, w_{n-2k}$ that are also tangent to M_β. Indeed, by virtue of (3.10), we have

$$\frac{\partial f_i}{\partial x} \cdot w_j = 0.$$

Further, we prove that the vectors

$$v_1, \quad \ldots, \quad v_k, \quad w_1, \quad \ldots, \quad w_{n-2k} \tag{4.14}$$

are linearly independent. We suppose the contrary, i.e., let

$$\sum \lambda_i v_i + \sum \mu_j w_j = 0 \tag{4.15}$$

with some λ and μ, where $\sum|\lambda_i| \neq 0$. Premultiplying (4.15) by $\operatorname{rot} u$ and using (3.10), we obtain

$$\sum \lambda_i (\operatorname{rot} u) v_i = -\sum \lambda_i \frac{\partial f_i}{\partial x} = 0.$$

By condition c, the functions $f_1, \ldots, f_k$ are independent; therefore, all λ_i equal zero. This is a contradiction.

We note that the number of independent tangent fields (4.14) equals the dimension of integral surfaces M_β.

We now find $(n-2k)$-dimensional vortex manifolds, more precisely, intersection surfaces of these manifolds with the $(n-k)$-dimensional surfaces M_β. These intersection surfaces are k-dimensional and are therefore defined by the equations

$$\varphi_1(x) = \gamma_1, \quad \ldots, \quad \varphi_k(x) = \gamma_k, \quad x \in M_\beta,$$

on M_β. By the definition of vortex manifolds, the functions φ_i satisfy the equations

$$w_1(\varphi_i) = \cdots = w_{n-2k}(\varphi_i) = 0, \quad 1 \leq i \leq k. \tag{4.16}$$

To find these functions, we use the additional conditions

$$v_j(\varphi_i) = \delta_{ji}, \quad 1 \leq j \leq k, \tag{4.17}$$

where δ_{ji} is the Kronecker symbol.

First, we must show that systems (4.16) and (4.17) have solutions. Indeed, $[v_i, v_j] = 0$, and the commutators $[v_i, w_j]$ can be linearly expressed through the vortex vectors w. This is implied by Proposition 1 in Chap. 2 (Sec. 5) and the absence in Lamb equation (3.10) of the derivative of the covector field u with respect to the independent variable playing the role of time.

The proof of the existence of solutions to systems (4.16) and (4.17) of partial differential equations is as follows. We introduce new local coordinates y and z on the manifold M_β such that vortex manifolds are defined by the equations $y = \text{const}$. The variables z are coordinates on vortex manifolds. In the new variables, the operators of differentiation along vortex fields become

$$\sum c_j(y, z) \frac{\partial}{\partial z_j}. \tag{4.18}$$

Let

$$\sum a_i \frac{\partial}{\partial y_i} + \sum b_j \frac{\partial}{\partial z_j} \tag{4.19}$$

be the operators corresponding to the commuting fields v. The coefficients a_i do not depend on z because the commutators of operators (4.18) and (4.19) have form (4.18). Operators (4.19) corresponding to distinct fields v_s commute, and the k "truncated" operators

$$\sum a_i(y)\frac{\partial}{\partial y_i} \tag{4.20}$$

also commute therefore. We can now perform the invertible change of variables $y \to \widetilde{y}$ such that operators (4.19) become

$$\frac{\partial}{\partial \widetilde{y}_1}, \quad \dots, \quad \frac{\partial}{\partial \widetilde{y}_k}.$$

In the new variables, we can set

$$\varphi_1 = \widetilde{y}_1, \quad \dots, \quad \varphi_k = \widetilde{y}_k.$$

Because vector fields (4.14) are independent, the partial derivatives of φ_i with respect to local coordinates on M_β can be uniquely determined using only algebraic operations. It is well known that a function can be reconstructed from its partial derivatives using several integrations with respect to one variable. The theorem is completely proved.

§5. Systems with Three Degrees of Freedom

5.1. We consider the simplest vortex solution of the Lamb equation, where

$$\operatorname{rank}(\operatorname{rot} u) = 2 \tag{5.1}$$

and therefore $k = 1$. By condition *b*, we need an integral F of Hamilton equations (3.1) for the integrability of the Hamilton equations in this case.

For autonomous systems, where the Hamiltonian H is independent of time, the problem is simpler: we can take the function $H(x, y)$ as the integral F. Then the field $u(x, c)$ satisfies Eq. (3.10) because this equation coincides with the original autonomous Lamb equation. Condition *c* of the vortex theory of integration becomes

$$d_x H(x, u(x, c)) \neq 0. \tag{5.2}$$

We have obtained the following proposition.

Proposition 2. *If $u(x, c)$ is a complete closed solution of the Lamb equation satisfying conditions* (5.1) *and* (5.2), *then Hamilton equations* (3.1) *are integrable by quadratures.*

5.2. In the case where $n = 3$, the closedness condition for the complete solution in Proposition 2 is not necessary. The proof of this assertion is based on the *Euler–Jacobi theorem* on the integrating factor. This case is of special interest from the standpoint of the analogy between hydrodynamics and the vortex theory of Hamiltonian systems.

Proposition 3 ([45]). *Let $n = 3$ and a complete stationary solution of the Lamb equation satisfying conditions* (5.1) *and* (5.2) *be given. Then the Hamilton equations are integrable by quadratures.*

If $\operatorname{rank}(\operatorname{rot} u)$ is constant, then for $n = 3$, it equals either zero or two. In the first case, we have a potential solution, and the integrability of the Hamilton equations is implied by the Jacobi theorem. If the rank equals two and the function

$$h(x, c) = H(x, u(x, c)) \tag{5.3}$$

is independent of x, then the field

$$v(x, c) = \left.\frac{\partial H}{\partial y}\right|_{y=u} \tag{5.4}$$

is collinear with the curl, and integral curves of the field v can demonstrate chaotic behavior. We recall that for potential flows, the function h depends only on c.

5.3.

Proof of Proposition 3. We consider the dynamic system on M^3 defined by field (5.4):

$$\dot{x} = v(x, c). \tag{5.5}$$

For each $c = (c_1, c_2, c_3)$, it has nonconstant integral (5.3) and the integral invariant with the density

$$\rho(x, c) = \det \left\| \frac{\partial u}{\partial c} \right\| .$$

We prove that under these assumptions, system (5.5) of differential equations on the three-dimensional manifold is integrable by quadratures.

We pass from the coordinates x_1, x_2, x_3 to the new coordinates z_1, z_2, z_3 by setting

$$z_3 = h(x_1, x_2, x_3);$$

then Eqs. (5.5) become

$$\dot{z}_1 = V_1(z_1, z_2, z_3), \qquad \dot{z}_2 = V_2(z_1, z_2, z_3), \qquad \dot{z}_3 = 0.$$

These equations have an integral invariant with the density

$$P(z_1, z_2, z_3) = \rho \det \left\| \frac{\partial x}{\partial z} \right\|.$$

We set $z_3 = \alpha = \text{const}$. Obviously, the function $P(z_1, z_2, \alpha)$ is the density of the integral invariant for the system

$$\dot{z}_1 = V_1(z_1, z_2, \alpha), \qquad \dot{z}_2 = V_2(z_1, z_2, \alpha). \tag{5.6}$$

We see that because

$$\frac{\partial PV_1}{\partial z_1} + \frac{\partial PV_2}{\partial z_2} = 0,$$

the 1-form $P(V_1\, dz_2 - V_2\, dz_1)$ is (locally) the total differential of some function $f(z_1, z_2, \alpha)$:

$$\frac{\partial f}{\partial z_1} = -PV_2, \qquad \frac{\partial f}{\partial z_2} = PV_1.$$

It remains to note that the function f is the nonconstant integral of system (5.6) and can be found by ordinary quadratures. □

5.4. For systems on the plane, the density ρ of the integral invariant was called the *integrating factor* by Euler. Jacobi generalized Euler's observation to the case of a system of n differential equations with $n-2$ independent integrals and an invariant measure. A discussion of the flow structure on integral surfaces of such systems can be found in [38]. The reasoning in the preceding subsection corresponds to the well-known Clebsch theorem in hydrodynamics: if the nonconstant Bernoulli function for a stationary flow is known, then the motion of fluid particles can be found by quadratures. The role of the integral invariant is played by the mass of the matter.

Supplement 1
Vorticity Invariants and Secondary Hydrodynamics

1. We consider the barotropic flow of an ideal fluid in a potential force field. It is described by Lamb equation (1.5) in Chap. 1:

$$\frac{\partial v}{\partial t} = v \times \operatorname{rot} v - \frac{\partial f}{\partial x}. \tag{1}$$

We consider periodic boundary conditions: all characteristics of the flow depend on the coordinates x_1, x_2, x_3 2π-periodically. We can assume that the flow is defined on the three-dimensional Euclidean torus

$$M = \{x_1, x_2, x_3 \bmod 2\pi\}\,.$$

Let $u(x,t)$ be a solenoidal field on M satisfying Eq. (1.1) in Chap. 1:

$$\frac{\partial u}{\partial t} = \operatorname{rot}(u \times v). \tag{2}$$

As noted above, this equation plays an essential role in continuum mechanics. In particular, the field of the curl of the fluid-particle velocity satisfies this equation.

Theorem 1. *The integral*

$$I(t) = \int_M (u, v)\, d^3x \tag{3}$$

is independent of time.

This theorem was proved by Moreau [58] for the case where $u = \operatorname{rot} v$.

Proof. We use the obvious formula

$$\dot{I} = \int_M \left[\left(\frac{\partial u}{\partial t}, v\right) + \left(u, \frac{\partial v}{\partial t}\right)\right] d^3x. \tag{4}$$

Using (1) and (2), we transform the integrand into the form

$$(v \times \operatorname{rot} u, u) + (v, \operatorname{rot}(v \times u)) - \left(\frac{\partial f}{\partial x}, u\right). \tag{5}$$

Taking the well-known relation

$$(\operatorname{rot} a, b) - (\operatorname{rot} b, a) = \operatorname{div}(a \times b)$$

into account, we transform the second term in (5) into the form

$$(v, \operatorname{rot}(v \times u)) = (v \times u, \operatorname{rot} v) + \operatorname{div}((v \times u) \times v).$$

Further, we have

$$\left(\frac{\partial f}{\partial x}, u\right) = \operatorname{div} fu$$

because the field u is solenoidal. We have thus reduced the integrand in (4) to the divergent form:

$$\operatorname{div}((v \times u) \times v - fu).$$

By the Gauss formula, integral (4) vanishes. □

2. Theorem 1 has a series of interesting consequences in the case of an incompressible fluid, where $\operatorname{div} v = 0$. We consider solenoidal fields $w(x,t)$ satisfying the Euler equation

$$\frac{\partial w}{\partial t} = [v, w]. \tag{6}$$

Proposition 1. *If the field w satisfying Eq. (6) is solenoidal for some t, then it is solenoidal for all t.*

Proof. Using the well-known identity in vector analysis

$$\operatorname{rot}(a \times b) = [a, b] + a \operatorname{div} b - b \operatorname{div} a$$

and the incompressibility assumption, we write Eq. (6) as

$$\frac{\partial w}{\partial t} = \operatorname{rot}(v \times w) - v \operatorname{div} w. \tag{7}$$

Applying the operation div to both sides of this relation and taking the identity $\operatorname{div} \operatorname{rot} = 0$ into account, we obtain

$$\frac{\partial}{\partial t} \operatorname{div} w = -L_v(\operatorname{div} w).$$

Therefore,

$$(\operatorname{div} w)^{\bullet} = 0.$$

The proposition is proved. □

The validity of the assumption that the field w is solenoidal follows from Proposition 1. Equation (7) implies that the soleniodal field w satisfies Eq. (2). Therefore, we have the following consequence of Theorem 1.

Corollary. *We have*

$$\int_M (w, v)\, d^3x = \text{const}. \tag{8}$$

We recall that Eq. (2) is the condition for integral curves of the field u to be frozen into the fluid flow and Euler equation (6) is the criterion for the vector field w to be frozen.

3. Integral invariants (8) have an interesting interpretation in the dynamics of an ideal incompressible fluid. We consider the group M of diffeomorphisms leaving the volume element invariant; we let SDiff M denote this infinite-dimensional group. The Lie algebra of the group SDiff M consists of vector fields on M with zero divergence. We define the scalar product of two elements of this algebra (i.e., two soleniodal vector fields v_1 and v_2) as

$$\langle v_1, v_2 \rangle = \int (v_1, v_2)\, d^3x.$$

We now consider the flow of a homogeneous ideal fluid in the domain M; for simplicity, we assume that the density of the fluid equals 1. The continuity equation yields the incompressibility condition $\operatorname{div} v = 0$. Flows of the fluid are described by curves g^t on the group SDiff M: a diffeomorphism $g^t : M \to M$ transforms the initial position of any fluid particle into its position at the time t.

It is easily verified that the kinetic energy of the fluid

$$T = \frac{1}{2}\langle v, v \rangle \tag{9}$$

is the *right-invariant* Riemannian metric on the group SDiff M. In the 1960s, the following important observation was made: flows of an ideal incompressible fluid are geodesic lines of metric (9). This is a consequence of the principle of least action, which can be regarded as the definition of an ideal fluid (see [4, 57, 77]).

As noted in Chap. 3 (Sec. 3), right shifts are included in phase flows of left-invariant fields. Left-invariant fields on the group SDiff M are soleniodal fields on M satisfying Euler equation (6). Therefore, by the Noether theorem, equations of geodesic lines on the group SDiff M have an infinite series of linear integrals (8): $\langle w, v \rangle = \text{const}$.

We note that the Noether integrals in the finite-dimensional case have the same form. Indeed, let

$$T = \frac{1}{2} \sum g_{ij}(x) \dot{x}_i \dot{x}_j$$

be the kinetic energy defining the scalar product

$$\langle \xi, \eta \rangle = \sum g_{ij} \xi_i \eta_j,$$
$$\xi = (\xi_1, \dots, \xi_n), \qquad \eta = (\eta_1, \dots, \eta_n).$$

If $w = (w_1, \dots, w_n)$ is the symmetry field, then the Noether integral has the form

$$\frac{\partial T}{\partial \dot{x}} \cdot w = \sum g_{ij} \dot{x}_i w_j = \langle w, \dot{x} \rangle.$$

The structure of integrals that are linear in velocity for natural mechanical systems and their relation to symmetry groups was studied in the work of

Moris Levy in 1878 (forty years before Emmy Noether's publication). In differential geometry, where the Riemannian metric plays the role of kinetic energy, symmetry fields are usually called *Killing fields*; they were investigated by Killing in 1892.

Because kinetic energy (9) is a nondegenerate quadratic form, the infinite series of integrals (8) allows determining the velocity v of the flow as a function on the group SDiff M. On SDiff M, an infinite-dimensional dynamic system therefore arises, and its phase flow has properties similar to the properties of the stationary flow of a nonviscous fluid. It is interesting to study this system from the hydrodynamic standpoint presented in Chap. 3 (vortex vectors and manifolds, Bernoulli surfaces, invariant measures, etc.): this approach can be called *secondary hydrodynamics.*

4. The Moreau theorem is really a particular case of a general assertion in Sec. 3 in Chap. 2. Let the 1-form ω satisfy the Lamb equation

$$\frac{\partial \omega}{\partial t} + i_v\, d\omega = -dh$$

and $\Omega = d\omega$. If the manifold M is closed and $\dim M = 2s+1$, then

$$\int_M \omega \wedge \Omega^s = \text{const}\,. \tag{10}$$

For flows of an ideal fluid with periodic boundary conditions ($M = T^3$),

$$\omega = v_1\, dx_1 + v_2\, dx_2 + v_3\, dx_3$$

is the circulation 1-form. It can be easily verified that

$$\omega \wedge \Omega = (\operatorname{rot} v, v)\, dx^3.$$

Relation (10) can be generalized if we replace Ω with an arbitrary m-form Φ frozen into the flow:

$$\frac{\partial \Phi}{\partial t} + L_v \Phi = 0.$$

Theorem 2. *Let M be a closed manifold and* $\dim M = ms+1$, *where s is an integer. Then*

$$\int \omega \wedge \Phi^s = \text{const}\,.$$

Proof. Let $\tau = \omega \wedge \Phi^s$. Then

$$\dot{\tau} = \dot{\omega} \wedge \Phi^s + \omega \wedge \dot{\Phi} \wedge \cdots \wedge \Phi + \cdots = \dot{\omega} \wedge \Phi^s = dg \wedge \Phi^s,$$

where $g = \omega(v) - h$ is the Lagrangian. Because the form Φ is closed, we have

$$dg \wedge \Phi^s = d(g\Phi^s),$$

and by the Stokes formula,

$$\frac{d}{dt}\int_M \tau = \int_M dg \wedge \Phi^s = \int_M d(g\Phi^s) = 0.$$

The theorem is proved. $\square$

Supplement 2
Quantum Mechanics and Hydrodynamics

1. As is known in quantum mechanics, the dynamics of a particle of unit mass in a potential field with the potential $V(x)$, $x \in E^3$, are described by the *Schrödinger equation*

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi, \tag{1}$$

where $\psi(x,t)$ is a complex-valued wave function, $\hbar$ is the Planck constant, and $i^2 = -1$. The wave function has the following physical meaning: $|\psi(x,t)|^2$ is the probability density of finding the particle at the point $x \in E^3$ at the time t. Hence, we assume that

$$\int_{E^3} \psi\overline{\psi}\, d^3x = 1. \tag{2}$$

This assumption is reasonable because the integral in the left-hand side of (2) is independent of t for an arbitrary solution of Eq. (1).

Setting

$$\psi = \sqrt{\rho}\, e^{iS/\hbar}, \tag{3}$$

where $\rho(x,t) \geq 0$ and $S(x,t)$ are scalar-valued functions, we obtain the system of equations

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\left(\rho\frac{\partial S}{\partial x}\right) = 0, \tag{4}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + V - \frac{\hbar^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{5}$$

where Δ is the Laplace operator.

As far as the author knows, Eqs. (4) and (5) were first obtained by E. Madelung. They were also studied by Bohm in connection with his "hidden parameters" conjecture, an attempt to avoid the stochastic interpretation of quantum mechanics [33]. As $\hbar \to 0$, Eq. (5) becomes the Hamilton–Jacobi equation describing potential families of trajectories of a classical particle with the Hamiltonian

$$H = \frac{y^2}{2} + V(x,t),$$

and Eq. (4) coincides with the Liouville equation for the density of the integral invariant for the gradient dynamic system

$$\dot{x} = \frac{\partial S}{\partial x}. \tag{6}$$

If we have the complete integral of the Hamilton–Jacobi equation, then Eq. (4) can be solved by quadratures. Substituting the obtained functions S and ρ in (3), we obtain the so-called quasi-classical approximation for the solution of Schrödinger equation (1).

2. Bohm proposed interpreting Eq. (5) in the general case, where $\hbar \neq 0$, as the Hamilton–Jacobi equation for a classical particle in two potential fields with the potentials V and

$$P = -\frac{\hbar^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{7}$$

Bohm called the function P the quantum mechanical potential. Of course, this representation has a real sense only if the function ρ is already known. An enlightening discussion and critical remarks on the Bohm concept can be found in [33].

Actually, Eqs. (4) and (5) have a simple hydrodynamic interpretation. Equation (4) is the continuity equation for velocity field (6), and Eq. (5) is the Lagrange–Cauchy integral for potential flows of an "ideal barotropic" fluid under the influence of a potential mass force with the potential density V. The quantum mechanical potential P plays the role of the pressure function. Unlike the usual hydrodynamic assumptions, P depends not only on the density ρ but also on its derivatives.

Hence, solutions of the Schrödinger equation are in one-to-one correspondence with potential flows of an ideal, generalized barotropic fluid with pressure function (7). This analogy is well known in physics. A discussion and an application to superconductivity can be found in the famous Feynman course (Vol. 3, Chap. 19). The question arises: Do vortex flows of this unreal quantum fluid have a physical meaning?

3. We now turn to the limit transition from quantum mechanics to classical mechanics as $\hbar \to 0$. Equation (5) becomes closed after taking the limit. Let g^t be the flow of dynamic system (6). We assume that the particle is localized at the point $x_0 \in E^3$ at the initial instant, i.e., the probability density ρ is the Dirac function $\delta(x - x_0)$ at $t = 0$. Then it follows from Eq. (4) that the density $\rho(x,t)$ is the δ-function for all t:

$$\rho(x,t) = \delta(x - g^t(x_0)).$$

This can be obtained as follows. Let D be an arbitrary domain in $\mathbb{R}^3$ that does not contain the point x_0. We then have

$$\int_D \rho\, d^3x = 1 \tag{8}$$

for $t = 0$. By (4), ρ is the density of the integral invariant of system (6); therefore, (8) holds for an arbitrary instant t and domain D that does not contain the point $g^t(x_0)$. This reasoning is valid for systems of differential equations of general type (not only gradient systems).

Therefore, if the particle is localized at the point x_0 at $t = 0$, then it is localized at the point $g^t(x_0)$ at the instant t, and its motion law

$$t \to g^t(x_0)$$

satisfies the Newton equation

$$\ddot{x} = -\frac{\partial V}{\partial x}.$$

In the case where $\hbar \neq 0$, the situation is quite different. Equations (4) and (5) are related to each other; therefore, because of the presence of quantum mechanical potential P, the probability density ρ concentrated at the point x_0 at $t = 0$ spreads. This effect, which is closely related to the Heisenberg uncertainty relation, is similar to the diffusion of vortices in a viscous liquid (Sec. 2 in Chap. 1).

We demonstrate the spreading effect for the probability density of a free particle on the line $\mathbb{R} = \{x\}$. The Schrödinger equation becomes

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi. \tag{9}$$

For simplicity, we set $\hbar = 2$. In the momentum representation, solutions of Eq. (9) have the form

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(p)\, e^{i(px - p^2 t)}\, dp, \tag{10}$$

where φ is an arbitrary square-summable function on $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} |\varphi(p)|^2\, dp = 1 \tag{11}$$

(see, e.g., [25]). For example, we take

$$\varphi = \left(\frac{\varepsilon}{\pi}\right)^{1/4} e^{-\varepsilon p^2/2}, \quad \varepsilon > 0.$$

Normalization condition (11) obviously holds. Using the known properties of the Fourier transformation, we obtain the explicit expression for the probability density:

$$\rho(x,t) = \frac{\sqrt{\varepsilon}}{\sqrt{\pi(\varepsilon^2 + 4t^2)}} e^{-\varepsilon x^2/(\varepsilon^2 + 4t^2)}. \tag{12}$$

For $t = 0$, we have

$$\rho = \frac{1}{\sqrt{\pi\varepsilon}} e^{-x^2/\varepsilon}.$$

This function tends to the δ-function as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi\varepsilon}} e^{-x^2/\varepsilon} f(x)\, dx = f(0).$$

On the other hand, from (12), we obtain the asymptotic representation

$$\rho \sim \frac{\sqrt{\varepsilon}}{2\sqrt{\pi} t} e^{-\varepsilon x^2/(4t^2)}$$

for large t. It is clear that for fixed $\varepsilon > 0$, this function tends to zero (uniformly in x) as $t \to \infty$.

Supplement 3
Vortex Theory of Adiabatic Equilibrium Processes

§1. General Vortex Theory

Let M be a smooth manifold, and let $x = (x_1, \dots, x_n)$ be the local coordinates in M. Let ω be a differential 1-form, let v be a vector field in M, and let $h : M \to \mathbb{R}$ be a function. The data ω, v, h may depend on time, t. Vortex theory studies the situation when ω, v, h satisfy the equation

$$\partial\omega/\partial t + i_v d\omega = -dh. \tag{1}$$

We will call equation (1) the generalized Lamb equation. Indeed, it corresponds to the well known equation for the vorticity (the rotor of the velocity) of a barotropic fluid in a potential force field. Equation (1) naturally arises in the study of invariant manifolds in the phase space of Hamiltonian systems that project on the configuration space in a one-to-one fashion. It also arises in geometric optics in the investigation of ray systems simply filling a domain in the three dimensional space. See [1].

Equation (1) is studied in detail in [1]. The crucial object is the set of the vortex fields v that annihilate the closed 2-form $\Omega = d\omega$, i. e., $i_v\Omega = 0$. Fix a time instant t. Then the vortex fields generate an integrable distribution in M. The integrable manifolds of this distribution are the vortex manifolds. Consider the system

$$\dot{x} = v(x, t). \tag{2}$$

By the generalized Helmholtz-Thomson Theorem, the flow of Eq. (2) transforms vortex hypersurfaces into vortex hypersurfaces. The goal of this appendix is to apply these ideas to study the adiabatic processes in equilibrium thermodynamics.

§2. The Heat Flow Form

The traditional exposition of the classical thermodynamics presents difficulties of pedagogical nature. Already F. Klein pointed this out in [2]: "... in order to reach the summit the reader has to overcome the barrier of unfamiliar mathematical concepts. He is expected to follow the trail barely marked by the founding fathers (Carnot, Clausius). However, the general outline of the summit is visible already from the trailhead, if the reader is looking in the right direction."

Suffice it to recall that in the traditional exposition the symbol dQ is used for the differentials of functions and for the infinitesimally small increments of the quantites that are not functions at all. The appropriate framework for the exposition of equilibrium thermodynamics is the calculus of differential forms.

Let $a_1, \ldots, a_n$ be the external parameters of the system, and let τ be the absolute temperature. Of fundamental importance in thermodynamics is the 1-form of the heat flow

$$\omega = dE + \sum A_i da_i. \tag{3}$$

Here E is the internal energy, and A_i is the generalized force corresponding to the Lagrangean coordinate a_i. For instance, if the coordinate in question is the volume, then the generalized force is the pressure. It is a part of the setting in thermodynamics to define the quantities E and A_i as functions of a and τ. The relations $A_i = f_i(a_1, \ldots, a_n, \tau)$ are usually called the equations of state. Besides the volume, one often uses for the generalized coordinates the density, the charge, the magnetization, the electric polarization, the deformation tensor. Then the generalized forces are respectively the chemical potential, the electric potential, the magnetic field, the electric field, the stress tensor.

The differential form

$$\sum A_i da_i \tag{4}$$

is the work of the force A corresponding to the infinitesimal increment da. By the assumptions of thermodynamics, for any fixed τ the form Eq. (4) is closed. If the state space $\{a_1, \ldots, a_n\}$ is $\mathbb{R}^n$, then, by Poincaré's lemma, this form is exact (i.e., it is the differential of a smooth function of a).

We will assume that

$$\sum (\partial A_i / \partial \tau)^2 \neq 0. \tag{5}$$

Therefore the form in Eq. (3) is not closed. Set $\Omega = d\omega$. Then

$$\Omega = \sum \frac{\partial A_i}{\partial \tau} d\tau \wedge da_i. \tag{6}$$

We display below the skew-symmetric $(n+1) \times (n+1)$ matrix corresponding to Ω:

$$\begin{bmatrix} 0 & \frac{\partial A_1}{\partial \tau} & \cdots & \frac{\partial A_n}{\partial \tau} \\ -\frac{\partial A_1}{\partial \tau} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial A_n}{\partial \tau} & 0 & \cdots & 0 \end{bmatrix}. \tag{7}$$

In view of assumption (5), the rank of the differential form Ω is two. Since Ω is closed, its class is equal to its rank.

In fact, in thermodynamics one makes a stronger assumption: the class of ω is two. Denote by i the interior product of a vector and a form. The assumption above means that at every point $(\tau, a_1, \ldots, a_n)$ the codimension of the space of vectors v such that $i_v\omega = i_v\Omega = 0$ is equal to two. It is well known that a 1-form of class two has an integrating factor. More precisely, by Darboux' theorem, locally there are two independent, smooth functions f, g such that $\omega = fdg$.

According to the second law of thermodynamics [3, 4], there is a global representation

$$\omega = \mu dS, \tag{8}$$

where $S = S(\tau, a_1, \dots, a_n)$ is the entropy of the system, and $\mu > 0$. Equilibrium thermodynamics is, in fact, the geometry of differential forms of class 2.

§3. Adiabatic Processes

A thermodynamics process corresponds to an oriented contour $\gamma : \Delta \to M$, where $\Delta \subset \mathbb{R}$ is an interval, and $M = \{\tau, a_1, \dots, a_n\}$ is the state space. The orientation of γ indicates the direction of the process. In the equilibrium thermodynamics a process is interpreted as an infinitely slow succession of equilibria of the system. These processes are called reversible.

The integral of a 1-form over γ is the amount of heat that the system acquires during the process. The integral

$$\int_\gamma \sum A_i da_i$$

is equal to the work of the external forces. A process is adiabatic if there is no inflow or the outflow of heat in the system.

Let $\alpha \in \Delta$ be a smooth parameter. Then τ, a are smooth functions of α. The velocity of the process is given by

$$v_0 = \frac{\partial \tau}{\partial \alpha}, \; v_1 = \frac{\partial a_1}{\partial \alpha}, \; \cdots, \; v_n = \frac{\partial a_n}{\partial \alpha}.$$

Set $v = (v_0, v_1, \dots, v_n)$. Then the process is adiabatic if and only if

$$i_v dS = \frac{\partial S}{\partial \tau} v_0 + \sum \frac{\partial S}{\partial a_i} v_i = 0.$$

Conversely, let $v = v(\tau, a)$ be a vector field in M. Solutions of the corresponding system of diffetential equations

$$\frac{\partial \tau}{\partial \alpha} = v_0(\tau, a), \; \frac{\partial a_i}{\partial \alpha} = v_i(\tau, a), \; i \geq 1 \tag{9}$$

are the quasi-static processes of the thermodynamical system.

If the process is not isothermic ($v_0 \neq 0$), we use the absolute temperature as a parameter on γ. Then $a_i = a_i(\tau)$ and $v_0 = 1$ in Eq. (9). In view of Eq. (8), an adiabatic process satisfies

$$i_v d\omega = i_v(d\tau \wedge dS) = (i_v d\tau) \wedge dS + d\tau \wedge (i_v dS) = dS. \tag{10}$$

This is the generalized Lamb equation, Eq. (1), where the entropy with the minus sign plays the role of the "hamiltonian" h. Therefore, the results of [1] can be applied to Eq. (10).

Vortex vectors $v = (v_0, v_1, \dots, v_n)$ are defined by

$$i_v \Omega = 0, \ \Omega = d\omega.$$

Thus, vortex vectors are the eigenvectors of the skew-symmetric matrix Eq. (7) with the eigenvalue zero. Eqs. (3,7) imply that

$$\frac{\partial A_i}{\partial \tau} = \frac{\partial S}{\partial a_i}, \ i \geq 1. \tag{11}$$

By assumption Eq. (5)

$$v_0 = 0, \ \sum \frac{\partial S}{\partial a_i} v_i = 0.$$

The integral curves of v are called the vortex lines. The latter of the equations above implies the "Bernoulli theorem": The entropy is constant on the vortex lines. In view of the equality $v_0 = 0$, the vortex lines correspond to the isothermic, adiabatic processes.

Recall that the vortex manifolds are the maximal integral manifolds of the integrable distribution of vortex vectors. It is straightforward to see that they are the $(n-1)$-dimensional manifolds

$$\Sigma_S = \{\tau, a : \tau = \text{const}, \ S(\tau, a) = s = \text{const}\}. \tag{12}$$

By the "Helmholtz-Thomson Theorem", the phase flow of the system Eq. (9) (where $v_0 = 1$), transforms vortex manifolds into vortex manifolds.

This observation suggests that the dynamical system Eq. (9) descends to the quotient M/Σ obtained by identifying the points in M that belong to the same submanifold Eq. (12). The points of M/Σ are determined by the parameters τ, s satisfying the equation (compare with Eq. (10))

$$i_v(d\tau \wedge ds) = ds. \tag{13}$$

In the variables τ, s the vector field v has the form

$$v_0 = 1, \ \frac{\partial S}{\partial \tau} = 0.$$

This is the hamiltonian version of the sraightening of a vector field theorem. Indeed, the system Eq. (13) is hamiltonian with respect to the standart symplectic structure (the area form $d\tau \wedge ds$) and the Hamiltonian $-s$. The analog of Eq. (13) in hydrodynamics are the well known Clebsch-Stuart equations [1].

§4. Integral Invariants

Let γ_α be a closed contour in M frozen into the flow of the system Eq. (9) (with $v_0 = 1$). By the generalized Poincaré Theorem [1]

$$\int_{\gamma_\alpha} \omega = \text{const.} \tag{14}$$

We can reformulate this as follows: The work of external forces $A_1, \dots, A_n$ over a closed contour does not depend on the contour.

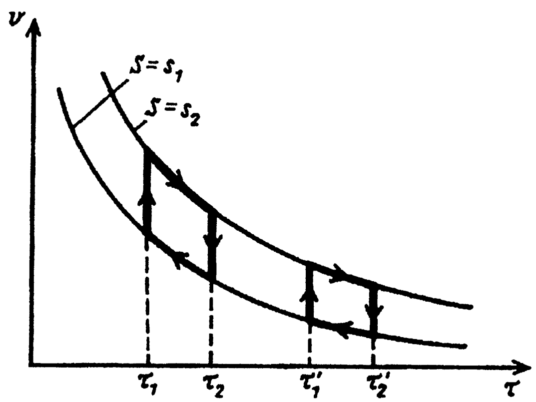

Fig. 1

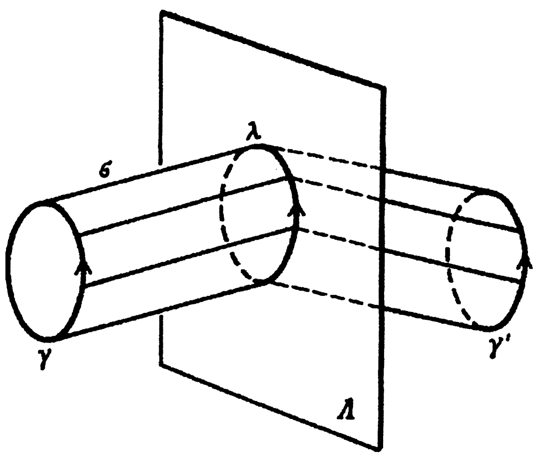

Fig. 2

Example. Let γ and γ' be two Carnot cycles for an ideal gas corresponding to the entropy values s_1 and s_2 respectively. See figure 1. If $\tau_2 - \tau_1 = \tau_2' - \tau_1'$, then γ' is obtained by a translation of γ along the flow of of the system Eq. (9).

Therefore, by eqs. (3,4)

$$\int_\gamma pdv = \int_{\gamma'} pdv. \tag{15}$$

Thus, the work of the heat engine (or the amount of heat obtained by it) on the cycles γ and γ' is the same. On the other hand, the work is equal to $(\tau_2 - \tau_1)(s_2 - s_1)$. This easily follows from the classical equations of Carnot's:

$$\Delta S = q_1/\tau_1 = q_2/\tau_2,$$

where q_1 (resp. q_2) is the amount of heat received by the engine in the isothermic process. Therefore

$$q_2 - q_1 = \frac{\tau_2}{\tau_1} q_1 - q_1 = (\tau_2 - \tau_1)\frac{q_1}{\tau_1} = \Delta\tau\Delta S.$$

Besides, this formula follows immediately from the equation $d\omega = d\tau \wedge dS$ and the Stokes' theorem.

Eq. (15) holds in greater generality. Let γ' be the contour obtained by translating every point of γ along the adiabatic line passing through it by the same amount of temperature, τ. Then Eq. (15) holds for the pair γ, γ'.

Invariance of the integral Eq. (14) is also closely related to the vortex lines. Let $n \geq 2$ and let γ be a closed contour in M. The vortex lines passing through the points of γ form a cylindrical surface σ in M, i. e., a vortex tube. Let now γ' be any cycle in σ, homologous to γ. Then, by a straightforward corollary of Stokes' theorem [1]

$$\int_\gamma \omega = \int_{\gamma'} \omega. \tag{16}$$

This is especially simple to see when $n = 2$. Then $\dim M = 3$ and Ω is nondegenerate: Through every point of M passes a unique vortex line.

Eq. (16) also holds for the phase transitions of the second kind [3, 4]. Then the phase space M is partitioned by a hypersurface Λ into the domains N and N', i.e., $M = N \cup N'$, $N \cap N' = \Lambda$. In each domain a smooth 1-form of the heat flow is given. Let ω and ω' be the respective forms. They extend to smooth forms in the neigborhoods of N and N', so that the extensions agree on Λ, but their differentials $d\omega$ and $d\omega'$ do not. It means that the vortex lines are not differentiable across Λ.

Let σ be a vortex tube in M, and let γ, γ' be homologous cycles in σ contained in N, N' respectively. Then

$$\int_\gamma \omega = \int_{\gamma'} \omega'. \tag{17}$$

Figure 2 illustrates the idea of the proof. Let λ be a cycle in $\sigma \cap \Lambda$, homologous to γ and γ'. By continuity

$$\int_\lambda \omega = \int_{\lambda'} \omega'.$$

Eq. (17) is completely analogous to the Malus theorem in geometric optics [1].

§5. Variational Principle

There is a close connection between the integral invariants and the following variational principle: Solutions of the system Eq. (2) are the extremals of the variational problem

$$\delta\left[\int_{t_1}^{t_2} \omega - h\,dt\right] = 0 \tag{18}$$

in the class of curves with fixed endpoints. This generalizes the variational principle of Helmholtz-Poincaré for the hamiltonian systems.

Let γ be a smooth contour in M parametrized by α, $\alpha_1 \leq \alpha \leq \alpha_2$. Let

$$v = (v_o, v_1, \dots, v_n)$$

be the velocity field Eq. (9). Let I be the functional on the space of curves with fixed endpoints given by

$$I[\gamma] = \int_{\alpha_1}^{\alpha_2} [\omega(v) + S]d\alpha,\ \omega(v) = i_v\omega. \tag{19}$$

Then γ is an extremal of the functional Eq. (19) if and only if
1) The equality $d\tau/d\alpha = 1$ holds (i. e., $v_0 = 1$);
2) The curve γ corresponds to an adiabatic non-isothermic process.
Apparently, variational principles of this type have not been considered in thermodynamics [5, 6].

Eq. (18) implies that the variation of the functional Eq. (19) vanishes on the adiabatic non-isothermic processes. To show the converse, we introduce the Lagrangian

$$\mathcal{L} = \omega(v) + S = \tau\frac{dS}{d\tau} + S.$$

Let $'$ denote the derivative with respect to α. Consider the Euler-Lagrange equations

$$\left(\frac{\partial\mathcal{L}}{\partial\tau'}\right)' = \frac{\partial\mathcal{L}}{\partial\tau},\ \left(\frac{\partial\mathcal{L}}{\partial\alpha'}\right)' = \frac{\partial\mathcal{L}}{\partial\alpha}. \tag{20}$$

The last equation yields

$$\tau' dS/d\alpha = dS/d\alpha.$$

The assumption Eq. (5) and the formula Eq. (11) imply $\tau' = 1$. The first equation of the system Eq. (20) yields

$$\frac{\partial S}{\partial\tau} + \sum\frac{\partial S}{\partial\alpha_i}v_i = 0,$$

which is the adiabatic condition.

Adiabatic isothermic processes correspond to a simpler variational principle. They are stationary for the functional

$$J[\gamma] = \int_\gamma \omega \tag{21}$$

on the class of curves with fixed endpoints. The proof is analogous.

Remark. Functionals Eq. (19) and Eq. (21) on adiabatic processes (non-isothermic and isothermic respectively) are equal to the difference of the values of the function τS at its endpoints. Recall that τS is the difference between the internal energy and the free energy of a thermodynamical system.

§6. Generalizations

The results above hold for a class of processes which is wider than the adiabatic processes. Assume that the field Eq. (9) satisfies

$$L_v S = g(\tau) \tag{22}$$

where g is an arbitrary real function. Taking into account Eq. (22), we obtain

$$i_v \Omega = dS - (L_v S) d\tau = dB, \tag{23}$$

where $B = S - f(\tau)$ and f is an antiderivative of g. Eq. (23) is completely anlogous to Eq. (10), where the function B plays the role of entropy.

References

1. V. V. Kozlov, *General vortex theory*, Izhevsk: Publishing House of Udmurt University 1998. English translation: see present book.
2. F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19 Jahrhundert*, Springer-Verlag 1926. Reprint. Grundlehren der Mathematischen Wissenschaften. Band 24 und 25. Berlin-Heidelberg-New York: Springer-Verlag 1979.
3. M.A. Leontovich, *Introduction into thermodynamics*, Moscow-Leningrad: Gostekhizdat 1952 (in Russian).
4. A. Sommerfeld, *Thermodynamik und Statistik.* (Vorlesungen ber theoretische Physik, Band 5.) Herausgegeben von F. Bopp und J. Meixner. Wiesbaden: Dieterichsche Verlagsbuchhandlung 1952. 2. Aufl., bearbeitet und ergnzt (Vorlesungen ber theoretische Physik. Bd. 5). Leipzig: Akademische Verlagsgesellschaft Geest & Portig K.-G. XIV, 338 S. (1962).
5. I. Gyarmati, *Non-Equilibrium Thermodynamics. Field Theory and Variational Principles.* Springer-Verlag 1970.
6. V.L. Berdichevski, *Variational principles of continuum mechanics*, Moscow: Nauka 1983 (in Russian).

References

1. Adams, J. F. (1969): Lectures on Lie groups. Math. Lect. Note Series. New York–Amsterdam: W. A. Benjamin. Zbl. 206.31604*
2. Aref, H. (1986): The numerical experiment in fluid mechanics. J. Fluid Mech. **173**, 15–41
3. Arnol'd, V. I. (1966): On the topology of three-dimensional steady flows of an ideal fluid. Prikl. Mat. Mekh. **30**, No. 1, 183–185. English transl.: J. Appl. Math. Mech. **30**, 223–226 (1966). Zbl. 156.23002
4. Arnol'd, V. I. (1966): Sur un principe variationnel pour les écoulements stationnaires des liquides parfaits et ses applications aux problèmes de stabilité non linéaires. J. Méc. (Paris) **5**, No. 1, 29–43. Zbl. 161.22903
5. Arnold, V. I. (1979): Mathematical Methods of Classical Mechanics. Moscow: Nauka. English transl.: Grad. Texts in Math. **60**. New York–Heidelberg–Berlin: Springer-Verlag (1978). Zbl. 386.70001
6. Arnol'd, V. I. (1982): Some remarks on the antidynamo theorem. Vestn. MGU. Ser. Mat. Mekh. No. 6, 50–57. English transl.: Mosc. Univ. Math. Bull. **37**, No. 6, 57–66 (1982). Zbl. 532.58034
7. Arnol'd, V. I. (1983): Singularities of systems of rays. Usp. Mat. Nauk. **38**, No. 2, 77–147. English transl.: Russ. Math. Surv. **38**, No. 2, 87–176 (1983). Zbl. 531.58009
8. Arnol'd, V. I. (1990): Huygens and Barrow, Newton and Hooke: Pioneers in Mathematical Analysis and Catastrophe Theory from Evolvents to Quasicrystals. Moscow: Nauka. English transl.: Basel: Birkhäuser Verlag (1990). Zbl. 721.01001
9. Arnold, V. I. (1990): Catastrophe Theory. Moscow: Nauka. English transl.: Berlin: Springer-Verlag (1992). Zbl. 746.58001
10. Arnold, V. I., Kozlov, V. V., Neishtadt, A. I. (1985): Mathematical Aspects of Classical and Celestial Mechanics. Moscow: VINITI. English transl.: Berlin: Springer-Verlag (1997). Zbl. 885.70001
11. Arzhanykh, I. S. (1949): Vortex principle of analytic dynamics. Dokl. Akad. Nauk SSSR **65**, No. 5, 613–616 (Russian)
12. Arzhanykh, I. S. (1965): Field of Momenta. Tashkent: Nauka (Russian)
13. Birkhoff, G. D. (1966): Dynamical Systems. Providence, R. I.: Am. Math. Soc. Zbl. 171.05402
14. Bliss, G. A. (1946): Lectures on the Calculus of Variations. Chicago: Univ. of Chicago Press. Zbl. 063.00459
15. Bolotin, S. V. (1986): A comment on the Routh method and the Hertz hypothesis. Vestn. MGU. Ser. Mat. Mekh. No. 5, 51–53. English transl.: Mosc. Univ. Mech. Bull. **41**, No. 5, 21–25 (1986). Zbl. 624.70016
16. Bolotin, S. V., Kozlov, V. V. (1980): Asymptotic solutions of equations of dynamics. Vestn. MGU. Ser. Mat. Mekh. No. 4, 84–89. English transl.: Mosc. Univ. Mech. Bull. **35**, No. 3–4, 82–88 (1980). Zbl. 457.70022
17. Born, M. (1933): Optik: Ein Lehrbuch der elektromagnetischen Lichttheorie. Berlin: Springer-Verlag
18. Bulatovich, R. M. (1997): On stability of linear potential gyroscopic systems in the cases where potential energy has a maximum. Prikl. Mat. Mekh. **61**, No. 3, 385–389 (Russian)

---

*For the convenience of the reader, references to reviews in Zentralblatt für Mathematik (Zbl.), compiled using the MATH database, have, as far as possible, been included in this bibliography.

19. Campbell, L. J. (1981): Transverse normal models of finite vortex array. Phys. Rev. **24**, No. 2, 514–534
20. Cartan, E. (1922): Leçons sur les invariants intégraux. Paris: Hermann
21. Četayev, N. (1927): Sur les équations de Poincaré. C. R. Acad. Sci. Paris **185**, 1577–1578
22. Chevalley, C. (1946): Theory of Lie Groups. Vol. 1. Princeton Mathematical Series **8**. Princeton, N.J.: Princeton Univ. Press
23. Conley, C. C., Zehnder, E. (1983): The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold. Invent. Math. **73**, 33–49. Zbl. 516.58017
24. Courant, R., Hilbert, D. (1962): Methods of Mathematical Physics. Vol. 2: Partial Differential Equations. New York–London: Interscience. Zbl. 099.29504
25. Faddeev, L. D., Yakubovsky, O. A. (1980): Lectures on Quantum Mechanics. Leningrad: Leningrad Univ. Press (Russian)
26. Fomenko, A. T. (1988): Symplectic Geometry. Moscow: Moscow Univ. Press. English transl.: Adv. Stud. in Contemp. Math. **5**. Amsterdam: Gordon and Breach (1995). Zbl. 873.58031
27. Fridman, A. A. (1934): An Experience on the Hydrodynamics of a Compressible Fluid. Moscow–Leningrad: ONTI (Russian)
28. Gelfand, I. M., Fomin, S. V. (1961): Calculus of Variations. Moscow: Fizmatgiz. English transl.: Englewood Cliffs, N.J.: Prentice-Hall (1965). Zbl. 127.05402
29. Guillemin, V., Sternberg, Sh. (1977): Geometric Asymptotics. Math. Surv. No. 14. Providence, R.I.: Am. Math. Soc. Zbl. 364.53011
30. Godbillon, C. (1985): Geometrie differentielle et mécanique analytique. Collect. Mech. Math. Paris: Hermann. Zbl. 653.53001
31. Hénon, M. (1966): Sur la topologie des lognes de courant dans un cas particularies. C. R. Acad. Sci. Paris **262**, 312–314
32. Hertz, H. (1894): Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt. Leipzig: Gesammelte Werke. Bd. 3. English transl.: The Principles of Mechanics, Presented in a New Form. New York: Dover (1956).
33. Inostr. Lit. (1953): Causality Problems in Quantum Mechanics: Collected Translations. Moscow: Inostr. Lit. (Russian)
34. Izd. Akad. Nauk (1943): Isaac Newton: Collection of Papers for the 300th Birthday. Moscow: Izd. Akad. Nauk (Russian)
35. Karapetyan A. V., Rumyantsev V. V. (1983): Stability of Conservative and Dissipative Systems. Moscow: VINITI (Russian)
36. Kirchhoff, H. (1887): Vorlesungen über mathematische Physik. Mechanik. Leipzig
37. Klein, F. (1979): Development of Mathematics in the 19th Century. Brookline, Mass.: Math. Sci. Press. Zbl. 411.01009
38. Kozlov, V. V. (1980): Methods of Qualitative Analysis in the Dynamics of a Rigid Body. Moscow: Moscow Univ. Press (Russian). Zbl. 557.70009
39. Kozlov, V. V. (1983): The hydrodynamics of Hamiltonian systems. Vestn. MGU. Ser. Mat. Mekh. No. 6, 10–22. English transl.: Mosc. Univ. Mech. Bull. **38**, No. 6, 9–23 (1983). Zbl. 585.76004
40. Kozlov, V. V. (1984): Notes on steady vortex motions of continuous medium. Prikl. Mat. Mekh. **47**, No. 2, 341–342. English transl.: J. Appl. Math. Mech. **47**, 288–289 (1984). Zbl. 564.76116
41. Kozlov, V. V. (1988): Invariant measures of Euler–Poincaré equations on Lie algebras. Funkts. Anal. Prilozhen. **22**, No. 1, 69–70. English transl.: Funct. Anal. Appl. **22**, No. 1, 58–59 (1988). Zbl. 667.58038
42. Kozlov, V. V. (1990): Eddy theory of the top. Vestn. MGU. Ser. Mat. Mekh. No. 4, 56–62. English transl.: Mosc. Univ. Mech. Bull. **45**, No. 4, 26-38 (1990). Zbl. 850.70068

43. Kozlov, V. V. (1992): Linear systems with a quadratic integral. Prikl. Mat. Mekh. **56**, No. 6, 900–906. English transl.: J. Appl. Math. Mech. **56**, No. 6, 803–809 (1992). Zbl. 792.70014
44. Kozlov, V. V. (1993): Dynamical system determined by the Navier–Stokes equations. Rus. J. Math. Phys. **1**, No. 1, 57–69
45. Kozlov, V. V. (1995): Hydrodynamics of noncommutative integration of Hamiltonian systems. In: Dynamical Systems in Classical Mechanics. Am. Math. Soc. Transl. Ser. 2. **168**, 227–238
46. Kozlov, V. V. (1995): Symmetries, Topology, and Resonances in Hamiltonian mechanics. Izhevsk: Izhevsk. Univ. English transl.: Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 31. Berlin: Springer-Verlag Zbl. 843.58068
47. Kozlov, V. V. (1995): Integral invariants of the Hamilton equations. Mat. Zametki **58**, No. 3, 379–393. English transl.: Math. Notes **58**, No. 3, 938–947 (1995). Zbl. 853.58052
48. Kozlov, V. V. (1996): On a generalization of the Hamilton–Jacobi method. Prikl. Mat. Mech. **60**, No. 5, 929–939 (Russian)
49. Kozlov, V. V., Yaroshchuk, V. A. (1993): On the invariant measures of Euler–Poincaré equations on unimodular groups. Vestn. MGU. Ser. Mat. Mekh. No. 2, 91–95. English transl.: Mosc. Univ. Mech. Bull. **48**, No. 2, 45–50 (1993). Zbl. 804.58034
50. Kummer, E. E. (1860): Allgemeine Theorie der geradlinigen Strahlungssysteme. J. Reine Angew. Math. **57**, 189–230
51. Lamb, H. (1993): Hydrodynamics. Cambridge Mathematical Library. Cambridge: Cambridge Univ. Press. Zbl. 828.01012
52. Lee, Hwa-Chung (1947): The universal integral invariants of Hamiltonian systems and application to the theory of canonical transformations. Proc. Roy. Soc. Edinburgh, Sec. A **62**, 237–246. Zbl. 030.05501
53. Levi-Civita, T., Amaldi, U. (1923): Lezioni di meccanica razionale. Vol. 1, 2. Bologna
54. Lichnerowicz, A. (1976): Global Theory of Connections and Holonomy Groups. Leyden, The Netherlands: Noordhoff Intl. Zbl. 337.53031
55. Maslov, V. P. (1988): Asymptotic Methods and Perturbation Theory. Moscow: Nauka (Russian) Zbl. 653.35002
56. Meleshko, V. V., Konstantinov, M. Yu. (1993): Dynamics of vortex structures. Kiev: Naukova Dumka (Russian)
57. Moreau, J.-J. (1959): Une méthode de "cinématique fonctionnelle" en hydrodynamique. C. R. Acad. Sci., Paris **249**, 2156–2158. Zbl. 117.19601
58. Moreau, J.-J. (1961): Constantes d'un flot tourbillonnaire en fluid parfait barotrope. C. R. Acad. Sci., Paris **252**, 2810–2812. Zbl. 151.41703
59. Naimark, M. A. (1958): Linear Representations of the Lorentz Group. Moscow: Nauka (Russian)
60. Nekhoroshev, N. N. (1972): Variables "action–angle" and their generalizations. Tr. Mosk. Mat. Obshch. **26**, 181–198 (Russian)
61. Ovsyannikov, L. V. (1978): Group Analysis of Differential Equations. Moscow: Nauka (Russian)
62. Poincaré, H. (1893): Théorie des tourbillons. Paris: G. de Carre et Naud
63. Poincaré, H. (1901): Sur une forme nouvelle des équations de la mécanique. C. R. Acad. Sci. Paris **132**, 369–371
64. Poincaré, H. (1993): New Methods in Celestial Mechanics. History of Modern Physics and Astronomy. **13**. Bristol: Am. Inst. Phys. Zbl. 776.01009
65. Poston, T., Stewart, I. (1978): Catastrophe Theory and Its Applications. Surv. and Ref. Works in Math. **2**. London–San Francisco–Melbourne: Pitman. Zbl. 382.58006

66. Rayleigh (Strutt, J. W.) (1945): The Theory of Sound. Vols. 1, 2. New York: Dover
67. Rashevsky, P. K. (1947): Geometric Theory of Partial Differential Equations. Moscow–Leningrad: Gostekhizdat (Russian)
68. Rashevsky, P. K. (1967): Riemannian Geometry and Tensor Analysis. Moscow: Nauka (Russian)
69. Routh, E. J. (1955): Treatise on the Dynamics of a System of Rigid Bodies. New York: Dover
70. Santilli, R. M. (1983): Foundations of Theoretical Mechanics 2: Birkhoffian Generalization of Hamiltonian Mechanics. Berlin: Springer-Verlag
71. Sommerfeld, A., Runge, J. (1911): Anwendung der Vectorrechnung auf die Grundlagen der geometrischen Optik. Annalen d. Physik. **35**, 4 Folge, 277–299
72. Souček, J., Souček, V. (1973): The Morse–Sard theorem for real-analytic functions. Theory Nonlin. Operators. Proc. Summer School, Babylon 1971, 121–125. Zbl. 314.57021
73. Thom, R. (1962): Sur la théorie des enveloppes. J. Math. Pur Appl. **41**, No. 2, 177–192
74. Tkachenko, V. M. (1965): On vortex lattices. Zh. Eksp. Teor. Fiz. **49**, No. 6, 1875–1883 (Russian)
75. Veblen, O. (1927): Invariants of Quadratic Differential Forms. Cambridge: Cambridge Univ. Press
76. Whittaker, E. T. (1952): A Treatise on the Analytical Dynamics of Particles and Rigid Bodies with an Introduction to the Problem of Three Bodies. Cambridge: Cambridge Univ. Press
77. Yudovich, V. I. (1961): Planar nonstationary motions of an ideal incompressible fluid. Dokl. Akad. Nauk SSSR **136**, No. 3, 564–567 (Russian)
78. Zhuravlev V. F. (1982): On some properties of gyroscopic systems in connection with the Hertz concept in mechanics. Izv. Akad. Nauk SSSR. Mekh. Tv. Tela **2**, 15–19 (Russian)
79. Ziglin, S. L. (1987): Splitting of the separatrices and the nonexistence of first integrals in systems of differential equations of Hamiltonian type with two degrees of freedom. Izv. Akad. Nauk SSSR, Ser. Mat. **51**, No. 5, 1088–1103. English. transl.: Math. USSR Izv. **31**, No. 2, 407–421 (1988). Zbl. 679.34029
80. Ziglin, S. L. (1990): Nonintegrability of the problem on motion of four point vortices. Dokl. Akad. Nauk SSSR **250**, No. 6, 1296–1300 (Russian)

Index