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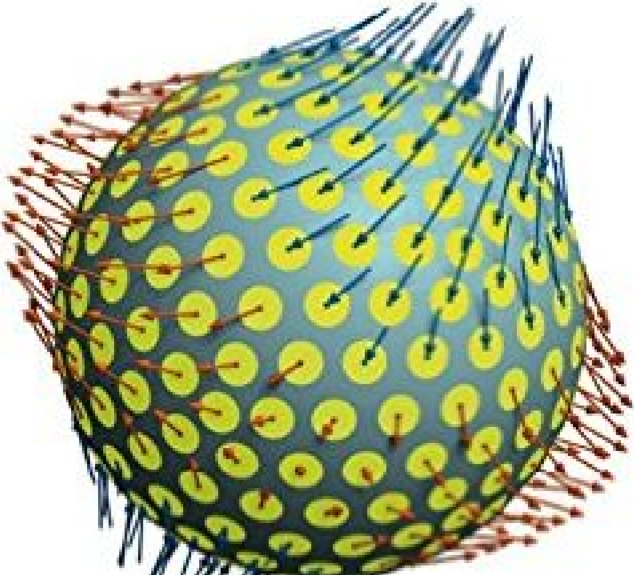
Nail H. Ibragimov

TENSORS AND RIEMANNIAN GEOMETRY

WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS



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With Applications to Differential Equations

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Preface

Tensors are very simple mathematical object from point of view of transformations. Namely, tensor fields in a vector space or on a curved manifold undergo *linear transformations* under changes of the space coordinates. The coefficients of the corresponding linear transformation are expressed in terms of the Jacobian matrix of the changes of coordinates.

Consequently, tensor calculus provides a comprehensive answer to the question on covariant (i.e. independent on a choice of coordinates) representation of equations. Hence, the tensor calculus, Riemannian geometry and theory of relativity are closely connected with transformation groups.

F. Klein (42) has underscored that what is called in physics *the special theory of relativity* is, in fact, a theory of invariants of the Lorentz group. Indeed, central for the Newtonian classical mechanics is the *Galilean relativity principle.* It states that the fundamental equations of dynamics should be invariant under the Galilean transformation which is written, e.g. in the direction of the x-axis, as follows:

$$\overline{x} = x + at$$

and has the generator

$$X = t\frac{\partial}{\partial x} \cdot$$

In the Galilean transformation, the group parameter a is a velocity of a moving frame, and the time t remains invariant. In Einstein's special relativity, the Galilean transformation is replaced by the Lorentz transformation

$$\overline{x} = x\cosh(a/c) + ct\sinh(a/c), \quad \overline{t} = t\cosh(a/c) + (x/c)\sinh(a/c)$$

with the generator

$$X = t\frac{\partial}{\partial x} + \frac{x}{c^2}\frac{\partial}{\partial t},$$

where the constant c is the velocity of light. The Lorentz transformation involves, along with the space variables x, the time variable t as well thus leading to the concept of the four-dimensional space-time known as the Minkowski space. The Galilean relativity is obtained from the special relativity by assuming that $a \ll c$. Indeed, formally letting $(a/c) \to 0$ in the Lorentz transformation, we have:

$$\overline{x} \approx x\Big(1 + \frac{a^2}{2c^2}\Big) + ct \cdot \frac{a}{c} \approx x + at, \quad \overline{t} \approx t\Big(1 + \frac{a^2}{2c^2}\Big) + \frac{x}{c}\frac{a}{c} \approx t.$$

Einstein's theory of general relativity is aimed at replacing Newton's empirical gravitation law by the concept of curvature of space-times. According to this concept, a distribution of matter causes a curvature, and the curvature is perceived as a gravitation.

Furthermore, Riemannian spaces associated with second-order linear differential equations were used by Hadamard (23) in investigation of the Cauchy problem for hyperbolic equations (see also (14)) and by Ovsyannikov (55) in group analysis of hyperbolic and elliptic equations. It is a long tradition, however, to teach partial differential equations without using notation and methods of Riemannian geometry. Accordingly, it is not clarified in most of textbooks why, e.g. the commonly known standard forms for hyperbolic, parabolic and elliptic second-order equations are given in the case of two independent variables, whereas this classification for equations with $n > 2$ variables is given at a fixed point only. Use of Riemannian geometry explains a geometric reason of this difference and shows (28), e.g. that one can obtain a standard form of hyperbolic equations with several independent variables if the associated Riemannian space has a non-trivial conformal group, in particular, the space is conformally flat.

This book is based on my lectures delivered at Novosibirsk and Moscow State Universities in Russia during 1972–1973 and 1988–1990, respectively, Collège de France in 1980, University of the Witwatersrand (Republic of South Africa) during 1995–1997, Blekinge Institute of Technology (Sweden) during 2004–2011 and Ufa State Aviation Technical University (Russia) during 2012–2013.

The necessary information about local and approximate transformation groups as well as symmetries of differential equations can be found in (39).

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Nail H. Ibragimov

Contents

Tensor calculus has been invented by G. Ricci. He called the new branch of mathematics an *absolute differential calculus* and developed it during the ten years of 1887—1896. The tensor calculus provides an elegant language, e.g. for presenting the special and general relativity.

The concept of tensors was motivated by development of *Riemannian geometry* of general manifolds (Riemann, 1854) and by E. B. Christoffel's transformation theory of quadratic differential forms (Christoffel, 1869). Subsequently, the tensor notation has been generally accepted in differential geometry, continuum mechanics and theory of relativity (see (5)).

The tensor calculus and Riemannian spaces furnish a profound mathematical background for theoretical physics and differential equations of mathematical physics.

Chapter 1 contains a collection of selected formulae from the classical vector calculus and an easy to follow introduction to the index notation used in the present book.

Chapter 2 includes a variety of topics on conservation laws from the basic concepts and examples through to modern developments in this field.

Since the present book is designed for graduate courses in differential equations and mathematical modelling, I provide in Chapter 3 a simple introduction to tensors and Riemannian spaces with emphasis on calculations in local coordinates rather than on the global geometric language.

The concepts of isometric, conformal and generalized motions in Riemannian spaces, given in Chapter 4, are useful in various applications in physics and theory of differential equations.

Chapter 1

Preliminaries

1.1 Vectors in linear spaces

This section contains basic notion and useful formulae from vector analysis in n-dimensional linear spaces $\mathbb{R}^n$. Following the convenient traditional notation, vectors of dimensions two ($n = 2$) and three ($n = 3$) are denoted by letters in bold faced type.

1.1.1 Three-dimensional vectors

1.1.1.1 Vector algebra

Let $\boldsymbol{a} \in \mathbb{R}^3$ be a three-dimensional (in particular, two-dimensional) vector. Graphically, $\boldsymbol{a}$ is a directed line segment. Its magnitude is denoted by $|\boldsymbol{a}|$.

The *scalar product* $\boldsymbol{a} \cdot \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar quantity (i.e. a real number) defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta, \tag{1.1.1}$$

where θ $(0 \leq \theta \leq \pi)$ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The scalar product is denoted in the literature by $(\boldsymbol{a}, \boldsymbol{b})$.

The vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ can be represented in the rectangular Cartesian coordinates in the form

$$\boldsymbol{a} = (a^1, a^2, a^3), \quad \boldsymbol{b} = (b^1, b^2, b^3), \tag{1.1.2}$$

which means that

$$\begin{aligned} \boldsymbol{a} &= a^1 \boldsymbol{i} + a^2 \boldsymbol{j} + a^3 \boldsymbol{k}, \\ \boldsymbol{b} &= b^1 \boldsymbol{i} + b^2 \boldsymbol{j} + b^3 \boldsymbol{k}, \end{aligned} \tag{1.1.3}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors along the first, second and third coordinate axes in $\mathbb{R}^3$, respectively. Then the scalar product (1.1.1) is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a^i b^i. \tag{1.1.4}$$

The *vector product* of the vectors (1.1.3) is a vector given by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} \tag{1.1.5}$$

or (see also Eq. (8.2.22))

$$\boldsymbol{a} \times \boldsymbol{b} = (a^2 b^3 - a^3 b^2)\boldsymbol{i} + (a^3 b^1 - a^1 b^3)\boldsymbol{j} + (a^1 b^2 - a^2 b^1)\boldsymbol{k}. \tag{1.1.6}$$

The vector product is also denoted by $[\boldsymbol{a}, \boldsymbol{b}]$ or $[\boldsymbol{a}\,\boldsymbol{b}]$.

Proposition 1.1.1. The vector product defined by Eq. (1.1.5) has the following properties:
(i) the vector product is anticommutative,

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a};$$

(ii) the magnitude of the vector $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the area of the parallelogram with the sides $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e.

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta,$$

where $\theta(0 \leq \theta \leq \pi)$ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$;
(iii) the vector $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to the plane spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ and is such that the triplet $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ forms a right-handed system.

Given three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$, one can produce the *mixed product*

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

called also the *scalar triple product*. Geometrically, the mixed product is equal to the volume of the parallelepiped having $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as its edges, provided that the triplet $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ forms a right-handed system. For the vectors written in the rectangular Cartesian coordinates in the form (1.1.2), the mixed product is given by

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{vmatrix}. \tag{1.1.7}$$

One can consider also the *vector triple product* $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$. It can be computed by the equation

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{1.1.8}$$

1.1.1.2 Differential calculus of vectors

Here we consider vector fields. The calculations are given in the rectangular Cartesian reference frame. In particular, the independent variables are the coordinates x, y, z of the position vector $\boldsymbol{x} = (x, y, z)$.

A *vector field* $\boldsymbol{a} = (a^1, a^2, a^3)$ is a vector function

$$\boldsymbol{a} = \boldsymbol{a}(x, y, z)$$

depending upon the position vector $\boldsymbol{x} = (x, y, z)$.

Hamilton's operator ∇ is a vector given in the rectangular Cartesian coordinates (x, y, z) by

$$\nabla = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z}, \tag{1.1.9}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the x, y and z axes, respectively. In other words, Hamilton's operator is the vector $\nabla = (\nabla_x, \nabla_y, \nabla_z)$ with the components

$$\nabla_x = \frac{\partial}{\partial x}, \quad \nabla_y = \frac{\partial}{\partial y}, \quad \nabla_z = \frac{\partial}{\partial z}.$$

The familiar operations of gradient, divergence and curl are written via Hamilton's operator as follows.

The *gradient* of a scalar field $\phi = \phi(x, y, z)$ is the vector field

$$\operatorname{grad}\phi = \nabla\phi \equiv \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}. \tag{1.1.10}$$

The *divergence* of a vector field $\boldsymbol{a}$ is the scalar product of the vectors ∇ and $\boldsymbol{a} = (a^1, a^2, a^3)$:

$$\operatorname{div}\boldsymbol{a} \stackrel{\text{def}}{=} \nabla\cdot\boldsymbol{a} = \nabla_x a^1 + \nabla_y a^2 + \nabla_z a^3 \equiv \frac{\partial a^1}{\partial x} + \frac{\partial a^2}{\partial y} + \frac{\partial a^3}{\partial z}. \tag{1.1.11}$$

The *curl* or *rotation* of a vector field $\boldsymbol{a} = \boldsymbol{a}(x, y, z)$ is the vector product of the vectors ∇ and $\boldsymbol{a}$:

$$\operatorname{curl}\boldsymbol{a} \equiv \operatorname{rot}\boldsymbol{a} \stackrel{\text{def}}{=} \nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a^1 & a^2 & a^3 \end{vmatrix} \tag{1.1.12}$$

or (see also Eq. (8.2.23))

$$\nabla\times\boldsymbol{a} = \left(\frac{\partial a^3}{\partial y} - \frac{\partial a^2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial a^1}{\partial z} - \frac{\partial a^3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial a^2}{\partial x} - \frac{\partial a^1}{\partial y}\right)\boldsymbol{k}. \tag{1.1.13}$$

Let $\boldsymbol{a}$, $\boldsymbol{b}$ and ϕ, ψ be vector and scalar fields, respectively, and let α, β be arbitrary constants. The operator ∇ together with formulae of *vector algebra* enables one to relate scalar and vector fields through differentiation and to obtain the following equations:

$$\begin{aligned}
&1.\quad \nabla(\alpha\phi + \beta\psi) = \alpha\nabla\phi + \beta\nabla\psi,\\
&2.\quad \nabla\cdot(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha\nabla\cdot\boldsymbol{a} + \beta\nabla\cdot\boldsymbol{b},\\
&3.\quad \nabla\times(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha\nabla\times\boldsymbol{a} + \beta\nabla\times\boldsymbol{b},\\
&4.\quad \nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi,\\
&5.\quad \nabla\cdot(\phi\boldsymbol{a}) = (\nabla\phi)\cdot\boldsymbol{a} + \phi(\nabla\cdot\boldsymbol{a}),\\
&6.\quad \nabla\times(\phi\boldsymbol{a}) = (\nabla\phi)\times\boldsymbol{a} + \phi\nabla\times\boldsymbol{a},\\
&7.\quad \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}),\\
&8.\quad \nabla\cdot(\nabla\phi) \equiv \nabla^2\phi \equiv \Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2},\\
&9.\quad \nabla\cdot(\nabla\times\boldsymbol{a}) = 0,\\
&10.\ \nabla\times(\nabla\phi) = 0,\\
&11.\ \nabla\times(\nabla\times\boldsymbol{a}) = \nabla(\nabla\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}.
\end{aligned} \tag{1.1.14}$$

Note that the 11th equation of (1.1.14) follows from Eq. (1.1.8).

Equations (1.1.14) are often written in the notation (1.1.10)–(1.1.11). For example, the equations 5, 9, 10 and 11 are written:

$$\begin{aligned}
&5'.\quad \operatorname{div}(\phi\,\boldsymbol{a}) = \phi\operatorname{div}\boldsymbol{a} + \boldsymbol{a}\cdot\operatorname{grad}\phi,\\
&9'.\quad \operatorname{div}\operatorname{rot}\boldsymbol{a} = 0,\\
&10'.\ \operatorname{rot}\operatorname{grad}\phi = 0,\\
&11'.\ \operatorname{rot}\operatorname{rot}\boldsymbol{a} = \operatorname{grad}(\operatorname{div}\boldsymbol{a}) - \Delta\,\boldsymbol{a},
\end{aligned}$$

where

$$\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the *Laplacian.*

1.1.1.3 Integral calculus of vectors

Green's theorem: Let V be an arbitrary region in the (x, y) plane with the boundary ∂V. Then

$$\int_{\partial V} Pdx + Qdy = \int_V \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{1.1.15}$$

for any (differentiable) functions $P(x, y)$ and $Q(x, y)$.

Stokes' theorem: Let V be an (orientable) surface in the space (x, y, z) with the boundary ∂V, and let $P(x, y, z), Q(x, y, z)$ and $R(x, y, z)$ be any (differentiable) functions. Then

$$\begin{aligned}\int_{\partial V} Pdx + Qdy + Rdz = \int_V \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\, dz \\ + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\, dx.\end{aligned} \tag{1.1.16}$$

Equation (1.1.16) is written in the vector notation as follows:

$$\int_{\partial V} \boldsymbol{A} \cdot d\boldsymbol{x} = \int_V \operatorname{curl} \boldsymbol{A} \cdot \boldsymbol{dS}.$$

Here $\boldsymbol{A} = (P, Q, R)$, $d\boldsymbol{x} = (dx, dy, dz)$ and $\boldsymbol{dS} = \boldsymbol{\nu} dS$, where $\boldsymbol{\nu}$ is the unit outward normal to the surface V.

The divergence theorem (the Gauss-Ostrogradsky theorem): Let V be a volume in the space (x, y, z) with the closed boundary ∂V and $\boldsymbol{A}$ be any vector field. Then

$$\int_{\partial V} (\boldsymbol{A} \cdot \boldsymbol{\nu})\, dS = \int_V (\nabla \cdot \boldsymbol{A})\, dx\, dy\, dz \equiv \int_V \operatorname{div} \boldsymbol{A}\, dx\, dy\, dz, \tag{1.1.17}$$

where $\boldsymbol{\nu}$ is the unit outward normal to the boundary ∂V of V.

1.1.2 General case

In what follows, we denote by $\mathbb{R}^n$ a real n-dimensional vector (linear) space. Its elements are n-dimensional vectors $x = (x^1, \ldots, x^n)$, where x^i denote the Cartesian coordinates of a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ referred to rectangular axes. Thus, the *scalar product* of vectors $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n) \in \mathbb{R}^n$ are defined by the formula similar to (1.1.4):

$$x \cdot y = \sum_{i=1}^{n} x^i y^i. \tag{1.1.18}$$

The length of a vector $x = (x^1, \ldots, x^n)$ is

$$|x| = \sqrt{x \cdot x} \equiv \sqrt{\sum_{i=1}^{n} (x^i)^2}\,. \tag{1.1.19}$$

The linear vector space $\mathbb{R}^n$ endowed with the scalar product (1.1.18) is called the n-dimensional *Euclidean space.*

The distance $d(x, y) = |x - y|$ between two points $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ is determined by

$$d^2(x, y) = \sum_{i=1}^{n} (y^i - x^i)^2. \tag{1.1.20}$$

A linear transformation

$$\bar{x}^i = a^i_j x^j, \quad \det\|a^i_j\| \neq 0$$

maps the orthogonal Cartesian reference system to what is called an *oblique Cartesian coordinate system.* Let the points $x, y \in \mathbb{R}^n$ have the coordinates x^i, y^i referred to an oblique Cartesian coordinate system. It follows from (1.1.20) that the square of the length of the line segment joining the points $x, y \in \mathbb{R}^n$ is given by

$$d^2(x, y) = \sum_{i,j=1}^{n} g_{ij}(y^i - x^i)(y^j - x^j) \tag{1.1.21}$$

with constant coefficients g_{ij} such that $\det\|g_{ij}\| \neq 0$. The coefficients g_{ij} depend on the coefficients of the linear transformations relating the rectangular and oblique Cartesian coordinate systems.

Let us denote the vector $y - x = (y^1 - x^1, \ldots, y^n - x^n)$ by

$$dx = (dx^1, \ldots, dx^n),$$

and its length by $ds = |dx|$. Then Eqs. (1.1.20) and (1.1.21) are written as

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2 \equiv \sum_{i,j=1}^{n} \delta_{ij} dx^i dx^j \tag{1.1.22}$$

and

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j, \quad g_{ij} = \text{const.}, \tag{1.1.23}$$

respectively. Equation (1.1.22) is written by using the *Kronecker symbols* defined as follows:

$$\delta^{ij} \equiv \delta^i_j \equiv \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.1.24}$$

We will also use in this book (see, e.g. Section 8.2.3) the *permutation symbol* e_{ijk} in three-dimensional spaces. It is skew-symmetric in all three indices and is equal to 1 when the triple ijk is a cyclic permutation of 123. Thus, the permutation symbol is defined by the equations

$$\begin{gathered} \mathrm{e}_{123} = \mathrm{e}_{231} = \mathrm{e}_{312} = 1, \quad \mathrm{e}_{213} = \mathrm{e}_{132} = \mathrm{e}_{321} = -1, \\ \mathrm{e}_{iik} = 0, \quad \mathrm{e}_{iji} = 0, \quad \mathrm{e}_{ijj} = 0, \quad i, j, k = 1, 2, 3. \end{gathered} \tag{1.1.25}$$

1.2 Index notation. Summation convention

It is convenient to lower superscripts and to raise subscripts by means of the Kronecker symbols. Let u^i, v_j, where both the superscript i and the subscript j range from 1 to n, be any two sets of $n > 0$ quantities. One can write

$$u_i = \delta_{ij} u^j, \quad v^i = \delta^{ij} v_j. \tag{1.2.1}$$

Consider the sum

$$\sum_{i=1}^{n} u^i v_i = u^1 v_1 + \cdots + u^n v_n. \tag{1.2.2}$$

To simplify such and more complicated expressions which often occur in what follows, we shall adhere the usual convention to drop the summation symbol when repeated indices appear as superscripts and subscripts. Then Eq. (1.2.2) can be written in the following equivalent compact forms:

$$u^i v_i = u_i v^i = \delta_{ij} u^j v^i = \delta^{ij} u_i v_j. \tag{1.2.3}$$

Similar abbreviation is used in more general situations. For instance,

$$\sum_{i=1}^{n} T_i^i \equiv T_1^1 + \cdots + T_n^n \equiv T_i^i$$

and

$$y_i = \sum_{j=1}^{n} a_{ij} x^j \equiv a_{ij} x^j, \quad i = 1, \ldots, n.$$

The latter equation represents n quantities

$$\begin{aligned} y_1 &= a_{11} x^1 + a_{12} x^2 + \cdots + a_{1n} x^n, \\ y_2 &= a_{21} x^1 + a_{22} x^2 + \cdots + a_{2n} x^n, \\ &\cdots\cdots\cdots\cdots \\ y_n &= a_{n1} x^1 + a_{n2} x^2 + \cdots + a_{nn} x^n. \end{aligned} \tag{1.2.4}$$

Of course, by adopting the summation convention we lose the information that the expression (1.2.3) actually means the sum $u^1 v_1 + \cdots + u^n v_n$ of n terms. However, this convention is widely used because it is advantageous in computations dealing with complicated tensor equations.

Example 1.2.1. The differential of a function $u = u(x^1, \ldots, x^n)$ is written $du = (\partial u / \partial x^i) dx^i$, where i is a superscript in dx^i and a subscript in $\partial u / \partial x^i$.

Exercises

Exercise 1.1. Show that Eqs. (1.1.1) and (1.1.4) give one and the same quantity for the scalar product of two vectors. In other words, verify that $|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \sum_{i=1}^{3} a^i b^i$.

Exercise 1.2. Prove that the vector product (1.1.5) satisfies the property (ii) of Proposition 1.1.1.

Exercise 1.3. Verify the following equations for the mixed product (1.1.7):

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}).$$

Exercise 1.4. Show that $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = 0$ for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Exercise 1.5. Prove Eq. (1.1.8).

Exercise 1.6. Using the notation of Sections 1.1.2 and 1.2, prove that

$$\text{(i) } \frac{\partial x^i}{\partial x^k} = \delta^i_k, \qquad \text{(ii) } \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^j} = \delta^k_j, \quad \text{(iii) } \delta^i_k\, x^k = x^i,$$
$$\text{(iv) } \delta_{ik}\, x^i\, x^k = |x|^2, \quad \text{(v) } \delta^i_i = n.$$

Chapter 2

Conservation laws

2.1 Conservation laws in classical mechanics

2.1.1 Free fall of a body near the earth

The equation describing the free fall of a body with mass m under the earth's gravitation is written

$$m\ddot{x} + mg = 0, \tag{2.1.1}$$

where

$$\dot{x} = \frac{dx}{dt}$$

and g is the acceleration of gravity. The general solution of Eq. (2.1.1) is

$$x(t) = -\frac{g}{2}t^2 + C_1 t + C_2. \tag{2.1.2}$$

By letting $t = 0$ in Eq. (2.1.2) and in the expression

$$v(t) = -gt + C_1 \tag{2.1.3}$$

for the velocity $v = \dot{x}$ one obtains the physical meaning of the integration constants. Namely C_2 is the initial position x_0 of the body and C_1 is its initial velocity v_0 :

$$C_1 = v_0, \quad C_2 = x_0.$$

Thus, the trajectory and the velocity of the body in the free fall can be written in the form

$$x = x_0 + tv_0 - \frac{g}{2}t^2, \quad v = v_0 - gt. \tag{2.1.4}$$

Solving Eqs. (2.1.3) and (2.1.2) with respect to C_1, C_2 one obtains the following conservation laws:

$$\dot{x} + gt = C_1, \quad x - t\dot{x} - \frac{g}{2}t^2 = C_2, \tag{2.1.5}$$

or

$$v(t) + gt = v_0, \quad x(t) - tv(t) - \frac{g}{2}t^2 = x_0.$$

Note that the conservation laws (2.1.5) contain the time t. Eliminating t from two conservation laws (2.1.5) and introducing the new constant

$$C = \frac{1}{2}C_1^2 + gC_2,$$

we obtain the following conservation law that does not involve the time explicitly:

$$\frac{1}{2}v^2 + gx = C. \tag{2.1.6}$$

Equation (2.1.6) expresses the well-known law of conservation of the energy

$$E = \frac{m}{2}v^2 + mgx. \tag{2.1.7}$$

The energy conservation means that

$$\frac{m}{2}v^2 + mgx = \frac{m}{2}v_0^2 + mgx_0.$$

The total energy (2.1.7) is the sum of the kinetic energy

$$T = \frac{m}{2}v^2 \equiv \frac{m}{2}\dot{x}^2$$

and the potential energy

$$U = mgx.$$

Recall that the Lagrangian of a mechanical system is defined as the difference of the kinetic and potential energies. Accordingly, the Lagrangian of free fall of a body with mass m has the form

$$L = \frac{m}{2}\dot{x}^2 - mgx. \tag{2.1.8}$$

Note that the substitution

$$x = y - \frac{g}{2}t^2 \tag{2.1.9}$$

connects Eq. (2.1.1) with the simplest second-order equation

$$\ddot{y} = 0. \tag{2.1.10}$$

It is well known that Eq. (2.1.10) admits the eight-dimensional Lie algebra spanned by operators (see, e.g. (39), Example 3.5)

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t}, \quad X_4 = x\frac{\partial}{\partial t}, \quad X_5 = t\frac{\partial}{\partial x}, \\ &X_6 = x\frac{\partial}{\partial x}, \quad X_7 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}, \quad X_8 = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x}. \end{aligned} \tag{2.1.11}$$

One can verify (solve Exercise 2.1) that the substitution (2.1.9) maps the operators (2.1.11) into the symmetries of Eq. (2.1.1). These symmetries provide the eight-dimensional Lie algebra spanned by the operators

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x},\\
&X_5 = t\frac{\partial}{\partial t} - gt^2\frac{\partial}{\partial x}, \quad X_6 = t^2\frac{\partial}{\partial t} + \left(tx - \frac{gt^3}{2}\right)\frac{\partial}{\partial x},\\
&X_7 = \left(x - \frac{gt^2}{2}\right)\frac{\partial}{\partial t} - 2gtx\frac{\partial}{\partial x},\\
&X_8 = \left(tx + \frac{gt^3}{2}\right)\frac{\partial}{\partial t} + \left(x^2 - \frac{g^2t^4}{4}\right)\frac{\partial}{\partial x}.
\end{aligned} \tag{2.1.12}$$

Let us find the one-parameter group generated by the operator X_6 from (2.1.12). The Lie equations have the form

$$\begin{aligned}
\frac{d\bar{t}}{da} &= \bar{t}^{\,2}, \qquad\qquad \bar{t}\big|_{a=0} = t,\\
\frac{d\bar{t}}{da} &= \bar{t}\bar{x} - \frac{g}{2}\,\bar{t}^{\,3}, \quad \bar{x}\big|_{a=0} = x.
\end{aligned}$$

The solution of the first equation is

$$\bar{t} = \frac{t}{1-at}.$$

Substituting it in the second equation we obtain

$$\frac{d\bar{t}}{da} = \frac{t\bar{x}}{1-at} - \frac{gt^3}{2(1-at)^3}.$$

Upon integrating this non-homogeneous first-order linear equation using the initial condition we have

$$\bar{x} = \frac{x}{1-at} - \frac{gt^3a}{2(1-at)^2}.$$

Thus, the operator X_6 generates the following one-parameter group admitted by Eq. (2.1.1):

$$\bar{t} = \frac{t}{1-at} \qquad \bar{x} = \frac{x}{1-at} - \frac{gt^3a}{2(1-at)^2}. \tag{2.1.13}$$

2.1.2 Fall of a body in a viscous fluid

An approximate mathematical description of the fall of a body under the earth's gravitation in a viscous fluid, e.g. the fall of a meteoroid in the earth's atmosphere, is given by the equation

$$m\ddot{x} + k\dot{x} + mg = 0 \tag{2.1.14}$$

with a positive constant $k \neq 0$. The general solution of Eq. (2.1.14) is

$$x = C_2 - \frac{mg}{k}\,t - C_1 \mathrm{e}^{-\frac{k}{m}t}. \tag{2.1.15}$$

The velocity $v = \dot{x}$ is described by the equation

$$v = -\frac{mg}{k} + C_1 \frac{k}{m}\,\mathrm{e}^{-\frac{k}{m}t}. \tag{2.1.16}$$

Solving the Eqs. (2.1.15) and (2.1.16) with respect to C_1, C_2 one obtains

$$\frac{m}{k}\left(\dot{x} + \frac{mg}{k}\right)\mathrm{e}^{\frac{k}{m}t} = C_1, \quad x + \frac{mg}{k}\,t + \frac{m}{k}\left(\dot{x} + \frac{mg}{k}\right) = C_2,$$

or equivalently

$$(kv + mg)\mathrm{e}^{\frac{k}{m}t} = \widetilde{C}_1, \quad mv + kx + mgt = \widetilde{C}_2. \tag{2.1.17}$$

Eliminating t from two conservation laws (2.1.17) we obtain the following conservation law that does not involve time explicitly:

$$(kv + mg)\,\mathrm{e}^{-\frac{kv}{mg} - \frac{k^2 x}{m^2 g}} = C. \tag{2.1.18}$$

Proposition 2.1.1. The Lagrangian for Eq. (2.1.14) is

$$L = \left(\frac{m}{2}\,\dot{x}^2 + \frac{k}{2}\,x\dot{x} + \frac{k^2}{4m}\,x^2 - mgx\right)\mathrm{e}^{\frac{k}{m}t}. \tag{2.1.19}$$

Proof. The change of the dependent variable

$$x = y\mathrm{e}^{-\frac{k}{2m}t} \tag{2.1.20}$$

maps Eq. (2.1.14) to the form

$$m\ddot{y} - \frac{k^2}{4m}\,y + mg\,\mathrm{e}^{\frac{k}{2m}t} = 0. \tag{2.1.21}$$

It is obvious that the Lagrangian of Eq. (2.1.21) is

$$L = \frac{m}{2}\,\dot{y}^2 + \frac{k^2}{8m}\,y^2 - mgy\,\mathrm{e}^{\frac{k}{2m}t}. \tag{2.1.22}$$

Using the change of variables (2.1.20) we obtain the Lagrangian (2.1.19). □

Upon letting $t = 0$ in Eqs. (2.1.15) and (2.1.16) we have the equations

$$x_0 = C_2 - C_1, \quad v_0 = C_1 \frac{k}{m} - \frac{mg}{k}.$$

Solving these equations for C_1, C_2 and using the notation

$$\gamma = \frac{k}{m}, \tag{2.1.23}$$

we obtain the following expressions for the integration constants via the initial position x_0 and the initial velocity v_0 of the body:

$$C_1 = \frac{v_0}{\gamma} + \frac{g}{\gamma^2}, \quad C_2 = x_0 + \frac{v_0}{\gamma} + \frac{g}{\gamma^2}.$$

Now Eqs. (2.1.15) and (2.1.16) are written as follows:

$$x = x_0 + \frac{v_0}{\gamma} + \frac{g}{\gamma^2} - \frac{gt}{\gamma} - \left(\frac{v_0}{\gamma} + \frac{g}{\gamma^2}\right) \mathrm{e}^{-\gamma t},$$

$$v = -\frac{g}{\gamma} + \left(v_0 + \frac{g}{\gamma}\right) \mathrm{e}^{-\gamma t}.$$

We expand the exponent $\mathrm{e}^{-\gamma t}$ into the Taylor series and obtain:

$$\begin{aligned} x &\approx x_0 + t v_0 - \frac{1}{2} g t^2 + \frac{1}{6} \gamma \left(g t^3 - 3 v_0 t^2\right) + o(\gamma), \\ v &\approx v_0 - g t + \frac{1}{2} \gamma \left(g t^2 - 2 t v_0\right) + o(\gamma). \end{aligned} \tag{2.1.24}$$

When $\gamma = 0$, Equations (2.1.24) coincide with Eqs. (2.1.4) for the free fall.

We rewrite the conservation law (2.1.18) in the notation (2.1.23):

$$(kv + mg)\, \mathrm{e}^{-\gamma \frac{v}{g} - \gamma^2 \frac{x}{g}} = C, \tag{2.1.25}$$

and expand the exponent into the Taylor series to obtain:

$$m(\gamma v + g)\left(1 - \gamma \frac{v}{g} - \gamma^2 \frac{x}{g} + \frac{1}{2}\gamma^2 \frac{v^2}{g^2} + \gamma^3 \frac{xv}{g^2} - \frac{1}{6}\gamma^3 \frac{v^3}{g^3} + \cdots\right) = C.$$

Keeping here the terms up to the order γ^3, we obtain the following approximate conservation law:

$$mg - \gamma^2 \left(\frac{1}{2g} m v^2 + m x\right) + \frac{m}{3g^2}\left(\gamma^3 v^3\right) \approx C. \tag{2.1.26}$$

Since mg is a constant, we can move this term to the right-hand cite, multiply the resulting equation (2.1.26) by the constant factor $-g/\gamma^2$ and arrive at the following form of the approximate conservation law (2.1.26):

$$\frac{1}{2} m v^2 + m g x - \frac{k}{3g} v^3 \approx \text{Const.} \tag{2.1.27}$$

This form of the conservation law is convenient for comparing it with the energy (2.1.7) of the free fall.

Equations (2.1.24) yield that

$$\frac{1}{2} m v^2 + m g x - \frac{k}{3g} v^3 = \frac{1}{2} m v_0^2 + m g x_0 - \frac{k}{3g} v_0^3 + o(\gamma)$$

in accordance with the approximate conservation law (2.1.27).

2.1.3 Discussion of Kepler's laws

In 1609, J. Kepler formulated two of the cardinal principles of modern astronomy known as Kepler's first and second laws. Kepler's first law states that *the orbit of a planet is an ellipse with the sun at one focus.* Kepler's second law tells that *the areas swept out in equal times by the line joining the sun to a planet are equal.* Kepler's third law was published later. It asserts that *the ratio T^2/R^3 of the square of the period T and the cube of the mean distance R from the sun is the same for all planets.* Kepler's laws, based on empirical astronomy, gave an answer to the question of *how* the planets move. They challenged scientists to answer the question of *why* the planets obey these laws. The necessary dynamics had been initiated by Galileo Galilei and developed into modern rational mechanics by Newton in his *Principles.*

According to Newton's gravitation law, the force of attraction between the sun and a planet has the form

$$\boldsymbol{F} = \frac{\mu}{r^3}\boldsymbol{x}, \quad \mu = -GmM,$$

where G is the universal constant of gravitation, m and M are the masses of a planet and the sun, respectively, $\boldsymbol{x} = (x^1, x^2, x^3)$ is the position vector of the planet considered as a particle, $r = |\boldsymbol{x}|$ is the distance of the planet from the sun. Ignoring the motion of the sun under a planet's attraction and using Newton's second law one obtains the system of three second-order ordinary differential equations that can be written in the vector form

$$m\,\frac{d^2\boldsymbol{x}}{dt^2} = \frac{\mu}{r^3}\,\boldsymbol{x}, \quad \mu = \text{const.} \tag{2.1.28}$$

The problem on integration of Eq. (2.1.28) is referred to as *Kepler's problem.*

Newton derived Kepler's laws by solving the differential equation (2.1.28). It can be shown however that the first and the second Kepler's laws are direct consequences of certain conservation laws. Namely, the second Kepler's law can be derived, without integrating the nonlinear equation (2.1.28), from conservation of the *angular momentum*

$$\boldsymbol{M} = m(\boldsymbol{x} \times \boldsymbol{v}), \tag{2.1.29}$$

where the vector $\boldsymbol{v} = \dot{\boldsymbol{x}} \equiv d\boldsymbol{x}/dt$ is the velocity of the planet. The first Kepler's law follows from conservation of the *Laplace vector*

$$\boldsymbol{A} = (\boldsymbol{v} \times \boldsymbol{M}) + \frac{\mu}{r}\,\boldsymbol{x}. \tag{2.1.30}$$

Let us verify that $\boldsymbol{M}$ and $\boldsymbol{A}$ are conserved vectors for Eq. (2.1.28). We have:

$$\frac{d\boldsymbol{M}}{dt} = m(\dot{\boldsymbol{x}} \times \boldsymbol{v}) + m(\boldsymbol{x} \times \dot{\boldsymbol{v}}).$$

Since $\dot{\boldsymbol{x}} = \boldsymbol{v}$, one has $\dot{\boldsymbol{x}} \times \boldsymbol{v} = 0$. Hence,

$$\frac{d\boldsymbol{M}}{dt} = m(\boldsymbol{x} \times \dot{\boldsymbol{v}}). \tag{2.1.31}$$

Invoking that $\dot{\boldsymbol{v}} = \ddot{\boldsymbol{x}}$ and hence $m\ddot{\boldsymbol{x}} = \mu \boldsymbol{x}/r^3$ according to Newton's equation (2.1.28), we reduce Eq. (2.1.31) to the form

$$\frac{d\boldsymbol{M}}{dt} = \frac{\mu}{r^3}\, m(\boldsymbol{x} \times \boldsymbol{x}).$$

Since $\boldsymbol{x} \times \boldsymbol{x} = 0$, we obtain the conservation law

$$\left.\frac{d\boldsymbol{M}}{dt}\right|_{(2.1.28)} = 0, \tag{2.1.32}$$

where the symbol $\big|_{(2.1.28)}$ means evaluated on the solutions of the differential equation (2.1.28). Equation (2.1.32) shows that the vector $\boldsymbol{M}$ is *constant along the trajectory of the planet* under consideration. Therefore $\boldsymbol{M}$ provides a vector valued integral of motion for Eq. (2.1.28).

Consider the Laplace vector $\boldsymbol{A}$ given by Eq. (2.1.30). Differentiating $\boldsymbol{A}$ and taking into account the conservation equation (2.1.32), one obtains:

$$\frac{d\boldsymbol{A}}{dt} = (\dot{\boldsymbol{v}} \times \boldsymbol{M}) + \mu\frac{\boldsymbol{v}}{r} - \mu(\boldsymbol{x}\cdot\boldsymbol{v})\,\frac{\boldsymbol{x}}{r^3},$$

or, substituting the expression (2.1.29) for $\boldsymbol{M}$,

$$\frac{d\boldsymbol{A}}{dt} = m\dot{\boldsymbol{v}} \times (\boldsymbol{x} \times \boldsymbol{v}) + \mu\frac{\boldsymbol{v}}{r} - \mu(\boldsymbol{x}\cdot\boldsymbol{v})\,\frac{\boldsymbol{x}}{r^3}.$$

Using the property (1.1.8),

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c},$$

of the vector triple product one has:

$$\frac{d\boldsymbol{A}}{dt} = m(\boldsymbol{v}\cdot\boldsymbol{v}')\boldsymbol{x} - m(\boldsymbol{x}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} + \mu\frac{\boldsymbol{v}}{r} - \mu(\boldsymbol{x}\cdot\boldsymbol{v})\,\frac{\boldsymbol{x}}{r^3}.$$

Invoking that $m\dot{\boldsymbol{v}} = \mu\boldsymbol{x}/r^3$, one finally arrives at the equation

$$\left.\frac{d\boldsymbol{A}}{dt}\right|_{(2.1.28)} = 0. \tag{2.1.33}$$

Thus, $\boldsymbol{A}$ is constant along the trajectory of the planet.

Let us derive Kepler's second law from the conservation of the angular momentum. Since the vector product $\boldsymbol{x} \times \boldsymbol{v}$ is orthogonal to the position vector $\boldsymbol{x}$ and the origin of the position vector is fixed, it follows from the constancy of the angular momentum $\boldsymbol{M}$ that the position vector and hence the orbit of every planet lies in a fixed plane orthogonal to the constant vector $\boldsymbol{M}$.

Let us choose the rectangular coordinate frame (x, y, z) with the sun at the origin O and the z-axis directed along the vector $\boldsymbol{M}$. Then $z = 0$ on the orbit, i.e. the path of the planet under consideration lies on the (x, y) plane. Consequently, $\boldsymbol{v} = (v^1, v^2, 0)$, where $v^1 = \dot{x},\ v^2 = \dot{y}$. Hence

$$\boldsymbol{M} = m \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ v^1 & v^2 & 0 \end{vmatrix} = m(xv^2 - yv^1)\boldsymbol{k},$$

where $\boldsymbol{k}$ is the unit vector directed along the z axis. Thus, the vector of angular momentum has the form

$$\boldsymbol{M} = (0, 0, M), \tag{2.1.34}$$

where

$$M = m(xv^2 - yv^1). \tag{2.1.35}$$

Then the conservation of the angular momentum is written $M = \text{const}$.

Let us introduce the polar coordinates r, ϕ defined by

$$x = r\cos\phi, \quad y = r\sin\phi.$$

In these coordinates, the velocity $\boldsymbol{v}$ has the components

$$v^1 = \dot{r}\cos\phi - r\dot{\phi}\sin\phi, \quad v^2 = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

and the conservation law (2.1.35) becomes:

$$M = mr^2\dot{\phi} = \text{const}. \tag{2.1.36}$$

The expression $\frac{1}{2}r^2 d\phi$ is the area dS of the infinitesimal sector bounded by by two neighbouring position vectors and an element of the planet's orbit. Hence the angular momentum (2.1.36) can be written as $M = 2m\dot{S}$, where $\dot{S} = dS/dt$ is the sectorial velocity. Therefore, the conservation of angular momentum implies that the sectorial velocity is constant. It follows upon integration that *the position vector $\boldsymbol{x}$ of the planet sweeps out equal areas in equal times.* This is Kepler's second law.

I present now the derivation of Kepler's first law given by Laplace in (47), Book II, Chap. III. Consider the motion of a planet in the rectangular Cartesian coordinates introduced in the above derivation of Kepler's second law. The angular momentum takes in this coordinate system the forms (2.1.34) and (2.1.35). Moreover, the orbit of the planet lies in the (x, y) plane, so that the position vector and the velocity of the planet have the forms $\boldsymbol{x} = (x, y, 0)$ and $\boldsymbol{v} = (v^1, v^2, 0)$, respectively. Therefore,

$$\boldsymbol{v} \times \boldsymbol{M} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v^1 & v^2 & 0 \\ 0 & 0 & M \end{vmatrix} = M(v^2\boldsymbol{i} - v^1\boldsymbol{j}),$$

or

$$\boldsymbol{v} \times \boldsymbol{M} = M(v^2,\ -v^1,\ 0).$$

Therefore the Laplace vector (2.1.30) is written

$$\boldsymbol{A} = M(v^2,\ -v^1,\ 0) + \frac{\mu}{r}(x, y, 0), \tag{2.1.37}$$

where $r = \sqrt{x^2 + y^2}$. According to this formula and the conservation of the Laplace vector, $\boldsymbol{A}$ is a plane constant vector, namely

$$\boldsymbol{A} = (A_1, A_2, 0), \quad A_1,\ A_2 = \text{const.} \tag{2.1.38}$$

Let's calculate the scalar product of the Laplace vector $\boldsymbol{A}$ with the position vector $\boldsymbol{x}$. In virtue of (2.1.37), (2.1.38) and the definition of the scalar product (1.1.4), we have $A_1x + A_2y = M(xv^2 - yv^1) + \mu r$, or using (3.2.3) and (3.2.17):

$$r(A_1 \cos\varphi + A_2 \sin\varphi) = \frac{M^2}{m} + \mu r. \tag{2.1.39}$$

Equation (2.1.39) defines an ellipse. To prove, let's transform Eq. (2.1.39) to the canonical form

$$r = \frac{p}{1 + e\cos\phi}. \tag{2.1.40}$$

This can be done by a proper rotation of the coordinate system. Namely, we set $\varphi = \phi + \theta$, where θ is an unknown constant angle. The left-hand side of Eq. (2.1.39) is written

$$(A_1 \cos\theta + A_2 \sin\theta)\cos\phi + (A_2\cos\theta - A_1\sin\theta)\sin\phi.$$

Let $A_2\cos\theta - A_1\sin\theta = 0$, i.e. $\theta = \arctan(A_2/A_1)$. Then

$$\cos^2\theta = (1 + \tan^2\theta)^{-1} = \frac{A_1^2}{A^2}, \quad \text{where} \quad A^2 = A_1^2 + A_2^2.$$

Hence $\cos\theta = A_1/A,\ \ \sin\theta = A_2/A$. It follows

$$A^1\cos\theta + A^2\sin\theta = A \equiv \sqrt{A_1^2 + A_2^2}.$$

Thus, Eq. (2.1.39) becomes

$$rA\cos\phi = \frac{M^2}{m} + \mu r.$$

This is the canonical equation (2.1.40) of an ellipse with the parameters

$$e = -\frac{A}{\mu}, \quad p = -\frac{M^2}{\mu m},$$

where $M = |\boldsymbol{M}|,\ A = |\boldsymbol{A}|$.

2.2 General discussion of conservation laws

2.2.1 Conservation laws for ODEs

The concept of a conservation law for ODEs (ordinary differential equations) is motivated by the conservation of such quantities as energy, linear and angular momenta, etc. that arise in mechanics, e.g. discussed in Section 2.1. These quantities are conserved in the sense that they are constant on each trajectory of a given dynamical system. Namely, a function $T = T(t, x, v)$ depending on time t, the position coordinates $x = (x^1, \ldots, x^m)$ and the velocity $v = (v^1, \ldots, v^m)$ with $v^\alpha = \dot{x}^\alpha$ is called a *conserved quantity* if it satisfies the equation

$$D_t(T) = 0 \tag{2.2.1}$$

on each trajectory $x = x(t)$ of the dynamical system in question. Here

$$D_t(T) \equiv \frac{\partial T}{\partial t} + \dot{x}^\alpha \frac{\partial T}{\partial x^\alpha} + \dot{v}^\alpha \frac{\partial T}{\partial v^\alpha}$$

is the total derivative with respect to time. If $x = x(t)$ is a given trajectory and $v = \dot{x}(t)$ is the corresponding velocity, and if we denote

$$T(t) = T(t, x(t), \dot{x}(t)),$$

then the conservation equation (2.2.1) is written

$$\frac{dT(t)}{dt} = 0. \tag{2.2.2}$$

The conservation (2.2.2) means that the conserved quantity $T(t, x, v)$ is constant on each trajectory. Therefore T is also called a *constant of motion.*

For example, a free motion of a single particle with the mass m is described by the equation

$$m\ddot{\boldsymbol{x}} = 0. \tag{2.2.3}$$

The energy

$$E = \frac{m}{2}\,|\boldsymbol{v}|^2$$

of the particle is a constant of the free motion. Indeed, its total derivative $D_t(E) = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v}$ vanishes on any trajectory due to Eq. (2.2.3).

In the case of any system of ordinary differential equations, conservation laws are knows as *first integrals.*

The extension of the conservation law (2.2.2) to continuous systems leads to the concepts of conservation laws for PDEs (partial differential equations) with any number $n \geq 1$ of independent variables. This is discussed in the next section.

2.2.2 Conservation laws for PDEs

We will use the following notation ant terminology. Let

$$x = (x^1, \dots, x^n)$$

denote n independent variables and let

$$u = (u^1, \dots, u^m)$$

denote m dependent variables. The set of the first-order partial derivatives

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \quad i = 1, \dots, n;\ \alpha = 1, \dots, m$$

of u with respect to x is denoted by

$$u_{(1)} = \{u_i^\alpha\}.$$

The higher derivatives are denoted likewise, e.g. the set of the second-order partial derivatives is

$$u_{(2)} = \{u_{ij}^\alpha\}.$$

In terms of the total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \tag{2.2.4}$$

we have

$$u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \dots .$$

A locally analytic function $f = f(x, u, u_{(1)}, \dots)$ of a finite number of variables

$$x,\ u,\ u_{(1)},\ u_{(2)},\ u_{(3)}, \dots$$

is called a *differential function*. The highest order of derivatives appearing in f is called the order of the differential function and is denoted by $\mathrm{ord}(f)$, e.g. if $f = f(x, u, u_{(1)}, \dots, u_{(s)})$ then $\mathrm{ord}(f) = s$. The set of all differential functions of finite order is denoted by $\mathcal{A}$. The set $\mathcal{A}$ is a vector space endowed with the usual multiplication of functions.

The following result from the classical variational calculus is useful in dealing with conservation laws for differential equations.

Proposition 2.2.1. A differential function $f(x, u, u_{(1)}, \dots, u_{(s)}) \in \mathcal{A}$ is a divergence,

$$f = D_i(h^i), \quad h^i(x, u, \dots, u_{(s-1)}) \in \mathcal{A}, \tag{2.2.5}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \dots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \dots, m, \tag{2.2.6}$$

where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m \tag{2.2.7}$$

are the variational derivatives with respect to the dependent variables u^α. Here x may denote one or several independent variables.

The statement that (2.2.5) implies (2.2.6) follows from the operator identity (see Exercise 2.7)

$$\frac{\delta}{\delta u^\alpha} D_i = 0.$$

For the proof of the inverse statement that (2.2.6) implies (2.2.5), see (9), Chapter 4, §3.5, (60) and (31), Section 8.4.1.

In the one-dimensional case (one independent variable x and one dependent variable y), the variational derivative (2.2.7) is written

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''} - D_x^3 \frac{\partial}{\partial y'''} + \cdots \tag{2.2.8}$$

and Proposition 2.2.1 is formulated as follows.

Proposition 2.2.2. A differential function $f(x, y, y', \ldots, y^{(s)}) \in \mathcal{A}$ is the total derivative,

$$f = D_x(g), \quad g(x, y, y', \ldots, y^{(s-1)}) \in \mathcal{A}, \tag{2.2.9}$$

if and only if the following equation holds identically in $x, y, y', \ldots$:

$$\frac{\delta f}{\delta y} = 0. \tag{2.2.10}$$

We will consider systems of m differential equations

$$F_\sigma(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \sigma = 1, \ldots, m, \tag{2.2.11}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. The systems (2.2.11) are assumed to be *regularly defined*. This means (see (56), Sections 1.3.3 and 1.5.1) that the Jacobian matrix

$$J = \left\| \frac{\partial F_\sigma}{\partial x}, \frac{\partial F_\sigma}{\partial u}, \frac{\partial F_\sigma}{\partial u_{(1)}}, \ldots, \frac{\partial F_\sigma}{\partial u_{(s)}} \right\|$$

of the functions F_σ with respect to all their arguments has the rank m at all points $(x, u, u_{(1)}, \ldots, u_{(s)})$ satisfying the system (2.2.11). For example, the Korteweg枕e Vries (KdV) equation $u_t - uu_x - u_{xxx} = 0$ is regularly defined, while the same equation written in the form $(u_t - uu_x - u_{xxx})^2 = 0$ is not regularly defined, since the corresponding Jacobian matrix has the zero rank on solutions of the KdV.

Definition 2.2.1. An n-dimensional vector

$$C = (C^1, \ldots, C^n) \tag{2.2.12}$$

satisfying the equation $\operatorname{div} C = 0$ on the solutions of Eqs. (2.2.11), i.e.

$$D_i(C^i)|_{(2.2.11)} = 0 \tag{2.2.13}$$

is called a *conserved vector*, and Eq. (2.2.13) is called a *conservation law* for the system (2.2.11).

Definition 2.2.2. A conserved vector (2.2.12) whose components are differential functions $C^i \in \mathcal{A}$ is called a *local conserved vector*. The corresponding conservation law (2.2.13) also bears the nomenclature *local*.

It is obvious that if the divergence of a vector (2.2.12) vanishes identically, it is a conserved vector for any system of differential equations. This is a *trivial* conserved vector for all differential equations. Another type of *trivial conserved vectors* are provided by those vectors whose components C^i vanish on the solutions of the system (2.2.11). One ignores both types of trivial conserved vectors. In other words, conserved vectors are simplified by considering them up to addition of the trivial conserved vectors.

Remark 2.2.1. Since the conservation equation (2.2.13) is linear with respect to C^i, any linear combination with constant coefficients of a finite number of conserved vectors is again a conserved vector. Moreover, the total differentiations D_k behave like multiplication of conserved vectors by constant factors and hence map a conserved vector (2.2.12) into conserved vectors. Indeed, since the total differentiations commute, Equation. (2.2.13) yields:

$$\big[D_i\left(D_k(C^i)\right)\big]_{(2.2.11)} = D_k\left(D_i(C^i)|_{(2.2.11)}\right) = 0.$$

Therefore, the vector

$$D_k(C) = \big(D_k(C^1), \ldots, D_k(C^n)\big)$$

obtained from the conserved vector (2.2.12) is not considered as a new conserved vector.

Remark 2.2.2. The following less obvious observation is particularly useful in practice. Let

$$C^1\big|_{(2.2.11)} = \widetilde{C}^1 + D_2(H^2) + \cdots + D_n(H^n), \tag{2.2.14}$$

where $H^2, \ldots, H^n \in \mathcal{A}$. Then the conserved vector (2.2.12) can be replaced with the equivalent conserved vector

$$\widetilde{C} = (\widetilde{C}^1, \widetilde{C}^2, \ldots, \widetilde{C}^n) = 0 \tag{2.2.15}$$

with the components

$$\widetilde{C}^1, \quad \widetilde{C}^2 = C^2 + D_1(H^2), \ \ldots, \quad \widetilde{C}^n = C^n + D_1(H^n). \tag{2.2.16}$$

The passage from (2.2.12) to the vector (2.2.15) is based on the commutativity of the total differentiations. Namely, we have

$$D_1 D_2(H^2) = D_2 D_t(H^2), \quad D_1 D_n(H^n) = D_n D_t(H^n),$$

and therefore the conservation equation (2.2.13) for the vector (2.2.12) is equivalent to the conservation equation

$$\big[D_i(\widetilde{C}^i)\big]_{(2.2.11)} = 0$$

for the vector (2.2.15). If $n \geq 3$, the simplification (2.2.16) of the conserved vector can be iterated: if $\widetilde{C}^2$ contains the terms

$$D_3(\widetilde{H}^3) + \cdots + D_n(\widetilde{H}^n)$$

one can subtract them from $\widetilde{C}^2$ and add to $\widetilde{C}^3, \ldots, \widetilde{C}^n$ the corresponding terms

$$D_2(\widetilde{H}^3), \ldots, D_2(\widetilde{H}^n).$$

Note that the conservation law (2.2.13) for Eqs. (2.2.11) can be written in the form

$$D_i(C^i) = \mu^\sigma F_\sigma\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) \tag{2.2.17}$$

with undetermined coefficients $\mu^\sigma = \mu^\sigma(x, u, u_{(1)}, \ldots)$ depending on a finite number of variables $x, u, u_{(1)}, \ldots$. If C^i depend on higher-order derivatives, Equation (2.2.17) is replaced with

$$D_i(C^i) = \mu^\sigma F_\sigma + \mu^{i\sigma} D_i\big(F_\sigma\big) + \mu^{ij\sigma} D_i D_j\big(F_\sigma\big) + \cdots. \tag{2.2.18}$$

Proposition 2.2.3. Let one of the independent variables, e.g. x^1 be time t. Let the components of the conserved vector C decrease rapidly and vanish at the space infinity. Then the conservation equation (2.2.13),

$$\big[D_t(A^1) + D_2(A^2) + \cdots + D_n(A^n)\big]\,\big|_{(2.2.11)} = 0, \tag{2.2.19}$$

implies that the integral

$$T(t) = \int_{\mathbb{R}^{n-1}} C^1(x, u(x), u_{(1)}(x))dx^2 \cdots dx^{n-1} \tag{2.2.20}$$

satisfies the conservation equation similar to Eq. (2.2.2) and hence it is constant along any solution $u = u(x)$ of Eqs. (2.2.11). In other words, the following equation is satisfied:

$$\frac{d}{dt}\int_{\mathbb{R}^{n-1}} C^1\, dx^2 \cdots dx^n = 0. \tag{2.2.21}$$

Accordingly, C^1 is called the *conservation density*, and Eq. (2.2.21) is called the integral form of the conservation law (2.2.13).

Proof. Consider an $(n-1)$-dimensional tube domain Ω in the n-dimensional space $\mathbb{R}^n$ of the variables $x = (t, x^2, \ldots, x^n)$ given by

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{i=2}^{n} (x^i)^2 = r^2, \quad t_1 \le t \le t_2 \right\},$$

where r, t_1 and t_2 are constants such that $r > 0,\ t_1 < t_2$. Let S be the boundary of Ω and let ν be the unit outward normal to the surface S. Applying the n-dimensional version of the divergence theorem (1.1.17) to the domain Ω and using the conservation law (2.2.13) one obtains:

$$\int_S C \cdot \nu \quad d\sigma = \int_\Omega \operatorname{div} C = 0. \tag{2.2.22}$$

Letting $r \to \infty$, we can neglect the integral over the cylindrical surface in the left-hand side of Eq. (2.2.22). In order to obtain integrals over the bases of the cylinder Ω note that at the lower base of the cylinder ($t = t_1$) we have

$$C \cdot \nu = -C^1|_{t=t_1},$$

and at the upper base ($t = t_2$) we have

$$C \cdot \nu = C^1|_{t=t_2}.$$

Therefore, Equation (2.2.22) implies that the function $C^1(x, u(x), u_{(1)}(x), \ldots)$ satisfies the condition

$$\int_{\mathbb{R}^{n-1}} C^1 dx^2 \cdots dx^n \bigg|_{t=t_1} = \int_{\mathbb{R}^{n-1}} C^1 dx^2 \cdots dx^n \bigg|_{t=t_2}$$

for any solution $u(x)$ of the system (2.2.11). Since t_1 and t_2 are arbitrary, the above equation means that the function $T(t)$ given by Eq. (2.2.20) is independent of time for any solution of Eqs. (2.2.11). This completes the proof. $\square$

Often the integral form (2.2.21) of conservation laws is considered to be preferable due to its physical significance. However the *differential form* (2.2.13) carries, in general, more information than the integral form (2.2.21). Using the integral form (2.2.21) one may even lose some nontrivial conservation laws. As an example, consider the two-dimensional Boussinesq equations

$$\begin{aligned} \Delta\psi_t - g\rho_x - f v_z &= \psi_x \Delta\psi_z - \psi_z \Delta\psi_x, \\ v_t + f\psi_z &= \psi_x v_z - \psi_z v_x, \\ \rho_t + \frac{N^2}{g}\psi_x &= \psi_x \rho_z - \psi_z \rho_x \end{aligned} \tag{2.2.23}$$

used in geophysical fluid dynamics for investigating uniformly stratified incompressible fluid flows in the ocean. Here Δ is the two-dimensional Laplacian,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},$$

and ψ is the stream function so that the x, z- components u, w of the velocity (u, v, w) of the fluid are given by

$$u = \psi_z, \quad w = -\psi_x. \tag{2.2.24}$$

Equations (2.2.23) involve the physical constants: g is the gravitational acceleration, f is the Coriolis parameter, and N is responsible for the density stratification of the fluid. Each equation of the system (2.2.23) has the conservation form:

$$\begin{aligned} D_t(\Delta\psi) + D_x(-g\rho + \psi_z \Delta\psi) + D_z(-fv - \psi_x \Delta\psi) &= 0, \\ D_t(v) + D_x(v\psi_z) + D_z(f\psi - v\psi_x) &= 0, \\ D_t(\rho) + D_x\left(\frac{N^2}{g}\psi + \rho\,\psi_z\right) + D_z(-\rho\,\psi_x) &= 0. \end{aligned} \tag{2.2.25}$$

In the integral form (2.2.21) these conservation laws are written

$$\begin{aligned} \frac{d}{dt}\iint \Delta\psi\, dxdz &= 0, \\ \frac{d}{dt}\iint v\, dxdz &= 0, \\ \frac{d}{dt}\iint \rho\, dxdz &= 0. \end{aligned} \tag{2.2.26}$$

We can rewrite the differential conservation equations (2.2.25) in an equivalent form by using the operations (2.2.14)—(2.2.16) of the conserved vectors. Namely, let us apply these operations to the first equation of (2.2.25), i.e. to the conserved vector

$$C^1 = \Delta\psi, \quad C^2 = -g\rho + \psi_z \Delta\psi, \quad C^3 = -fv - \psi_x \Delta\psi. \tag{2.2.27}$$

Noting that

$$C^1 = D_x(\psi_x) + D_z(\psi_z)$$

and using the operations (2.2.14)—(2.2.16) we transform the vector (2.2.27) to the form

$$\widetilde{C}^1 = 0, \quad \widetilde{C}^2 = -g\rho + \psi_{tx} + \psi_z \Delta\psi, \quad \widetilde{C}^3 = -fv + \psi_{tz} - \psi_x \Delta\psi. \tag{2.2.28}$$

The integral conservation equation (2.2.21) for the vector (2.2.28) is trivial, $0 = 0$. Thus, after the transformation of the conserved vector (2.2.27) to the equivalent form (2.2.28) we have lost the first integral conservation law in (2.2.26). But it does not mean that the conserved vector (2.2.28) has no physical significance. Indeed, if to write the differential conservation equation with the vector (2.2.28), we again obtain the first equation of the system (2.2.23):

$$D_x(\widetilde{C}^2) + D_z(\widetilde{C}^3) = \Delta\psi_t - g\rho_x - fv_z - \psi_x\Delta\psi_z + \psi_z\Delta\psi_x.$$

If a conservation law is given in the integral form, the following consequence of Proposition 2.2.1 can be used as a test for conservation density.

Proposition 2.2.4. A function $\tau(t, x^2, \ldots, x^n)$ is a conservation density for Eqs. (2.2.11), i.e. the integral conservation law (2.2.21)

$$\frac{d}{dt}\int_{\mathbb{R}^{n-1}} \tau(t, x^2, \ldots, x^n)\, dx^2 \cdots dx^n = 0$$

is satisfied on solutions of Eqs. (2.2.11), if and only if τ solves the equations

$$\frac{\delta}{\delta u^\alpha}\left[D_t(\tau)|_{(2.2.11)}\right] = 0, \quad \alpha = 1, \ldots, m. \tag{2.2.29}$$

2.3 Conserved vectors defined by symmetries

2.3.1 Infinitesimal symmetries of differential equations

Let us consider a first-order linear differential operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \xi^i, \eta^\alpha \in \mathcal{A}. \tag{2.3.1}$$

We will assume that ξ^i, η^α are arbitrary differential functions and will use the prolonged action of the operator X,

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots + \zeta_{i_1 \cdots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \tag{2.3.2}$$

whose additional coefficients are obtained by means of the usual prolongation formulae (see, e.g. (39), Section 3.3.3)

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\ \zeta_{i_1 i_2}^\alpha &= D_{i_2}(\zeta_{i_1}^\alpha) - u_{j i_1}^\alpha D_{i_2}(\xi^j), \ldots, \\ \zeta_{i_1 \cdots i_s}^\alpha &= D_{i_s}(\zeta_{i_1 \cdots i_{s-1}}^\alpha) - u_{j i_1 \cdots i_{s-1}}^\alpha D_{i_s}(\xi^j). \end{aligned} \tag{2.3.3}$$

Definition 2.3.1. The operator (2.3.1) is called an infinitesimal symmetry of Eqs. (2.2.11) if the following equations are satisfied:

$$XF_\sigma\left(x, u, u_{(1)}, \ldots, u_{(s)}\right)\big|_{(2.2.11)} = 0, \quad \sigma = 1, \ldots, m. \tag{2.3.4}$$

Here the operator X is taken in the prolonged form (2.3.2).

Remark 2.3.1. If $\mathrm{ord}(\xi^i) = 0$ and $\mathrm{ord}(\eta^\alpha) = 0$, i.e. $\xi^i = \xi^i(x, u), \eta^\alpha\eta^\alpha(x, u)$, the operator X is known as a point symmetry. In this case X is the generator of a one-parameter transformation group admitted by Eqs. (2.2.11).

If $\mathrm{ord}(\xi^i) \geq 1$ and/or $\mathrm{ord}(\eta^\alpha) \geq 1$, the infinitesimal symmetry X generates a one-parameter formal group of infinite-order tangent transformations (see (28), Chapter 3) admitted by Eqs. (2.2.11). In this case X is known as an infinitesimal higher-order tangent (or Lie-Bäcklund) symmetry. In particular cases of first-order differential functions ξ^i and η^α the operator X might be a generator of a first-order tangent transformation group.

2.3.2 Euler-Lagrange equations. Noether's theorem

The *variational derivative* (or Euler-Lagrange operator) in the space $\mathcal{A}$ of differential functions is the formal sum

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m, \tag{2.3.5}$$

where the summation is presupposed over the repeated indices $i_1, \ldots, i_s$ running from 1 to n.

Let $L \in \mathcal{A}$ be a differential function of an arbitrary order. The *Euler-Lagrange equations* are defined by

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{2.3.6}$$

The differential function L is called a *Lagrangian.* The following particular cases often occur in applications.

If the Lagrangian L is a first-order differential function, $L = L(x, u, u_{(1)})$, then Eqs. (2.3.6) are written

$$\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u^\alpha_i}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{2.3.7}$$

In the case of a second-order Lagrangian $L = L(x, u, u_{(1)}, u_{(2)})$ the Euler-Lagrange equations (2.3.6) have the form

$$\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u^\alpha_i}\right) + D_i D_k\left(\frac{\partial L}{\partial u^\alpha_{ik}}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{2.3.8}$$

In the classical mechanics, various mechanical systems are characterized by first-order Lagrangians $L = L(t, x, \dot{x})$, where the independent variable is the time t, the dependent variables are the coordinates $x = (x^1, \ldots, x^m)$ of particles of the system, and $\dot{x}$ is the vector with the components

$$\dot{x}^\alpha \equiv \frac{dx^\alpha}{dt}, \quad \alpha = 1, \ldots, m.$$

In this case, the Euler-Lagrange equations (2.3.7) are usually written in the form

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.3.9}$$

where d/dt is understood as the total derivative.

We turn now to Noether's theorem dealing with symmetries and conservation laws for the Euler-Lagrange equations. Recall that the Euler-Lagrange equations (2.3.7) appear as necessary conditions for the functions $u = u(x)$ providing extreme of the following integral (called the variational integral)

$$\int_V L(x, u, u_{(1)}) dx, \tag{2.3.10}$$

where V is an arbitrary n-dimensional volume in the space of the independent variables x. We will deal with groups of transformations

$$\bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \varphi^\alpha(x, u, a), \quad i = 1, \ldots, n; \; \alpha = 1, \ldots, m, \tag{2.3.11}$$

leaving invariant the Euler-Lagrange equations. One of obvious possibilities for the invariance of Eqs. (2.3.7) is the invariance of the variational integral (2.3.10) in the following sense:

$$\int_{\overline{V}} L(\bar{x}, \bar{u}, \bar{u}_{(1)}) d\,\bar{x} = \int_V L(x, u, u_{(1)}) dx, \tag{2.3.12}$$

where $\overline{V}$ is a volume obtained from V by transformation (2.3.11).

Lemma 2.3.1. The invariance condition (2.3.12) is written in terms of the generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{2.3.13}$$

of the group (2.3.11) in the following form:

$$X(L) + LD_i(\xi^i) = 0, \tag{2.3.14}$$

where X is understood as the *prolongation* of the generator (2.3.13) to the derivatives u_i^α involved in the Lagrangian, i.e.

$$X(L) = \xi^i \frac{\partial L}{\partial x^i} + \eta^\alpha \frac{\partial L}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial L}{\partial u_i^\alpha}, \quad \zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j).$$

Proof. Using the rule of change of variables in integrals, we rewrite the left-hand side of Eq. (2.3.12) in the form

$$\int_{\overline{V}} L(\bar{x}, \bar{u}, \bar{u}_{(1)}) d\,\bar{x} = \int_{V} L(\bar{x}, \bar{u}, \bar{u}_{(1)}) J dx, \tag{2.3.15}$$

where $J = \det \| D_j(\bar{x}^i) \|$ is the Jacobian with $\bar{x}^i$ taken from (2.3.11). Here $J > 0$ for sufficiently small a because

$$J|_{a=0} = 1.$$

The rule of differentiation of determinants gives

$$\left.\frac{\partial J}{\partial a}\right|_{a=0} = D_i(\xi^i).$$

Hence

$$J \approx 1 + a D_i(\xi^i).$$

Using (2.3.15) and noting that the volume V is arbitrary, we conclude that the integral equation (2.3.12) is equivalent to the equation

$$L(\bar{x}, \bar{u}, \bar{u}_{(1)}) J = L(x, u, u_{(1)}). \tag{2.3.16}$$

Now we use the infinitesimals

$$\bar{x}^i \approx x^i + a\xi^i, \ \ \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha, \ \ \bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha, \ \ J \approx 1 + a D_i(\xi^i)$$

and obtain

$$L(\bar{x}, \bar{u}, \bar{u}_{(1)}) J \approx L(x, u, u_{(1)}) + a\big[X(L) + L D_i(\xi^i)\big].$$

This equation implies Eq. (2.3.14). □

Noether's theorem (54) can be formulated in the following form convenient for applications ((31), Section 9.7).

Theorem 2.3.1. Let the generator (2.3.13) obey the invariance test (2.3.14). Then the vector $C = (C^1, \ldots, C^n)$ with the components

$$C^i = \xi^i L + W^\alpha \frac{\partial L}{\partial u_i^\alpha}, \quad i = 1, \ldots, n, \tag{2.3.17}$$

is a conserved vector for the Euler-Lagrange equations (2.3.7), i.e.

$$D_i(C^i)\big|_{(2.3.7)} = 0.$$

In Eqs. (2.3.17) and in the expressions of conserved vectors for higher-order Lagrangians given further, I use the following notation:

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha, \quad \alpha = 1, \ldots, m. \tag{2.3.18}$$

Proof. I follow the simple proof given in (31), Section 9.7.2. We first verify that the equation

$$X(L) + LD_i(\xi^i) = D_i\left(\xi^i L + W^\alpha \frac{\partial L}{\partial u_i^\alpha}\right) + W^\alpha\left[\frac{\partial L}{\partial u^\alpha} - D_i\Big(\frac{\partial L}{\partial u_i^\alpha}\Big)\right] \quad (2.3.19)$$

holds for any first-order differential function $L(x, u, u_{(1)})$. Indeed, we write the operator (2.3.13) in the form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial}{\partial u_i^\alpha} + \cdots$$

and arrive at the desired equation (2.3.19) as follows:

$$\begin{aligned} X(L) + LD_i(\xi^i) &= \xi^i D_i(L) + W^\alpha \frac{\partial L}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial L}{\partial u_i^\alpha} + LD_i(\xi^i) \\ &= D_i(\xi^i L) + W^\alpha \frac{\partial L}{\partial u^\alpha} + D_i\Big(W^\alpha \frac{\partial L}{\partial u_i^\alpha}\Big) - W^\alpha D_i\Big(\frac{\partial L}{\partial u_i^\alpha}\Big) \\ &= D_i\Big(\xi^i L + W^\alpha \frac{\partial L}{\partial u_i^\alpha}\Big) + W^\alpha\left[\frac{\partial L}{\partial u^\alpha} - D_i\Big(\frac{\partial L}{\partial u_i^\alpha}\Big)\right]. \end{aligned}$$

Lemma 2.3.1 and Eq. (2.3.19) complete the proof of the theorem. □

In the case of second-order Lagrangians $L(x, u, u_{(1)}, u_{(2)})$ the invariance test for the variational integral has again the same form (2.3.14), but X is understood as its prolongation up to the second-order derivatives $u_{(2)} = \{u_{ij}^\alpha\}$. For higher-order Lagrangians $L(x, u, u_{(1)}, u_{(2)}, \ldots)$ the generator X should be prolonged to all derivatives $u_{(1)}, u_{(2)}, \ldots$ involved in L. With this alteration, Theorem 2.3.1 is modified as follows.

Theorem 2.3.2. Let the generator (2.3.13), admitted by the Euler-Lagrange equations, obey the invariance test (2.3.14). Then in the case of second-order Lagrangians the vector $C = (C^1, \ldots, C^n)$ with the components

$$C^i = \xi^i L + W^\alpha\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right)\right] + D_j\left(W^\alpha\right)\frac{\partial L}{\partial u_{ij}^\alpha} \quad (2.3.20)$$

is a conserved vector for Eqs. (2.3.8). In the case of the Euler-Lagrange equations (2.3.6) with a Lagrangian of an arbitrary order the conserved vector $C = (C^1, \ldots, C^n)$ has the components

$$\begin{aligned} C^i =& \xi^i L + W^\alpha\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) - \cdots\right] \quad (2.3.21) \\ &+ D_j\left(W^\alpha\right)\left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k\left(W^\alpha\right)\left[\frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots\right]. \end{aligned}$$

In (2.3.20) and (2.3.21) the Lagrangian L should be written in the symmetric form with respect to the mixed derivatives $u^\alpha_{ij},\ u^\alpha_{ijk}, \ldots$.

Remark 2.3.2. Proposition 2.2.1 guarantees that addition of a divergence term to the Lagrangian does not change the Euler-Lagrange equations (2.3.6). In other words, one can replace Eq. (2.3.16) by the equation

$$L(\bar{x}, \bar{u}, \bar{u}_{(1)})J = L(x, u, u_{(1)}) + D_i(F^i), \quad F^i \in \mathcal{A},$$

and its equivalent for higher-order Lagrangians. Accordingly, one can deal with generators X that are admitted by the Euler-Lagrange equations and satisfy the divergence condition

$$X(L) + LD_i(\xi^i) = D_i(B^i) \tag{2.3.22}$$

instead of the invariance test (2.3.14). Then the conserved vector is written

$$A^i = C^i - B^i, \quad i = 1, \ldots, n, \tag{2.3.23}$$

where C^i are given by Eqs. (2.3.17), (2.3.20) and (2.3.21) for the first-order, second-order and higher-order Lagrangians, respectively. The invariance test (2.3.14) is a particular case of the divergence condition (2.3.22).

Let us dwell on the case of one independent variable t considered above in discussing the Euler-Lagrange equations (2.3.9),

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = 0, \quad \alpha = 1, \ldots, m,$$

with a first-order Lagrangian

$$L = L(t, x, \dot{x}).$$

Let

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta^\alpha(t, x)\frac{\partial}{\partial x^\alpha} \tag{2.3.24}$$

be a symmetry generator for Eqs. (2.3.9). Then Eqs. (2.3.17) give the following conserved quantity:

$$C = \xi L + (\eta^\alpha - \xi \dot{x}^\alpha)\frac{\partial L}{\partial \dot{x}^\alpha}. \tag{2.3.25}$$

Example 2.3.1. The free motion of a particle with a constant mass m provides a simple example for illustrating Noether's theorem. In this case one deals with the Lagrangian

$$L = \frac{m}{2}|\dot{\boldsymbol{x}}|^2, \tag{2.3.26}$$

where

$$|\dot{\boldsymbol{x}}|^2 = \sum_{\alpha=1}^{3}(\dot{x}^\alpha)^2.$$

The Euler-Lagrange equations (2.3.9) have the form

$$m\ddot{x}^\alpha = 0, \quad \alpha = 1, 2, 3. \tag{2.3.27}$$

Equation (2.3.27) admit the Galilean group with the generators

$$X_0 = \frac{\partial}{\partial t}, \quad X_\alpha = \frac{\partial}{\partial x^\alpha}, \quad X_{\alpha\beta} = x^\beta \frac{\partial}{\partial x^\alpha} - x^\alpha \frac{\partial}{\partial x^\beta}, \quad Y_\alpha = t\frac{\partial}{\partial x^\alpha}, \tag{2.3.28}$$

where $\alpha, \beta = 1, 2, 3$. We will apply Noether's theorem to the generators (2.3.28). It will be convenient to write the conservation quantities using the velocity vector $\boldsymbol{v} = \dot{\boldsymbol{x}}$ with the components v^1, v^2, v^3.

(i) *Time translation.* The generator X_0 of the time translation group has the coordinates

$$\xi = 1, \quad \eta^1 = \eta^2 = \eta^3 = 0.$$

We substitute them in Eq. (2.3.25), denote $E = -C$ and obtain the conservation of the energy

$$E = \frac{m}{2}\,|\boldsymbol{v}|^2. \tag{2.3.29}$$

(ii) *Space translations.* Let us take the translation along the x^1-axis with the generator X_1. This generator has the coordinates

$$\xi = 0, \quad \eta^1 = 1, \quad \eta^2 = \eta^3 = 0.$$

We substitute them in Eq. (2.3.25), denote $C = p^1$ and obtain the conserved quantity

$$p^1 = mv^1.$$

The use of all space translations with the generators X_α furnishes us with the vector valued conservation quantity, namely the linear momentum

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{2.3.30}$$

(iii) *Rotations.* Consider the rotation around the x^3-axis. Its generator X_{12} has the coordinates

$$\xi = 0, \quad \eta^1 = x^2, \quad \eta^2 = -x^1, \quad \eta^3 = 0.$$

Substituting them in Eq. (2.3.25) and denoting $C = M_3$, we obtain the conserved quantity

$$M_3 = m(x^2v^1 - x^1v^2).$$

The use of all rotations with the generators X_{12}, X_{13}, X_{23} furnishes us with the conservation of the angular momentum

$$\boldsymbol{M} = m(\boldsymbol{x} \times \boldsymbol{v}). \tag{2.3.31}$$

(iv) *Galilean transformations.* The Galilean transformations have the generators Y_α. Unlike the translation and rotation generators $X_0, X_\alpha, X_{\alpha\beta}$, the generators

Y_α do not satisfy the invariance condition (2.3.14). Indeed, we take Y_1 in the prolonged form

$$Y_1 = t\frac{\partial}{\partial x^1} + \frac{\partial}{\partial v^1}$$

and obtain

$$Y_1(L) + LD_t(\xi) = mv^1 \equiv D_t(mx^1).$$

Hence, one can use Remark 2.3.2 with

$$B = mx^1.$$

Equation (2.3.25) gives

$$C = mtv^1.$$

Substituting these expressions for C and B in Eq. (2.3.23) we obtain the conserved quantity

$$Q^1 = m(tv^1 - x^1).$$

Using all generators Y_i we arrive at the vector valued conserved quantity

$$\boldsymbol{Q} = m(t\boldsymbol{v} - \boldsymbol{x}). \tag{2.3.32}$$

In the case of a system of particles, conservation of the vector $\boldsymbol{Q}$ is known as the center-of-mass theorem.

Remark 2.3.3. The system of three second-order ordinary differential equations (2.3.27) has six functionally independent firsts integrals. But we have obtained ten conserved quantities (2.3.29)–(2.3.32). Therefore, four of these conserved quantities are functionally dependent on six conserved quantities. Namely, we can express the energy and angular momentum via the conserved vectors (2.3.30) and (2.3.32) as follows:

$$E = \frac{1}{2m}\,|\boldsymbol{p}|^2, \quad \boldsymbol{M} = \frac{1}{m}\,\boldsymbol{p} \times \boldsymbol{Q}. \tag{2.3.33}$$

Example 2.3.2. Consider Kepler's problem described by Eqs. (2.1.28),

$$m\ddot{x}^\alpha = \frac{\mu}{r^3}\,x^\alpha, \quad \alpha = 1,2,3, \tag{2.3.34}$$

with the Lagrangian

$$L = \frac{m}{2}\sum_{\alpha=1}^{3}(\dot{x}^\alpha)^2 - \frac{\mu}{r}. \tag{2.3.35}$$

It is known that Eqs. (2.3.34) admit five linearly independent point symmetries (i.e. operators of the form (2.3.24))

$$X_0 = \frac{\partial}{\partial t}, \quad X_{\alpha\beta} = x^\beta\frac{\partial}{\partial x^\alpha} - x^\alpha\frac{\partial}{\partial x^\beta}, \quad Z = 3t\frac{\partial}{\partial t} + 2x^\alpha\frac{\partial}{\partial x^\alpha} \tag{2.3.36}$$

and the following three infinite-order tangent (Lie-Bäcklund) symmetries ((28), Section 25.1):

$$\hat{X}_\beta = \left(2x^\beta v^\alpha - x^\alpha v^\beta - (\boldsymbol{x}\cdot\boldsymbol{v})\delta^\alpha_\beta\right)\frac{\partial}{\partial x^\alpha}, \quad \beta = 1,2,3. \tag{2.3.37}$$

Application of Noether's theorem to the time translation generator X_0 gives the conservation of the energy

$$E = \frac{m}{2}|\boldsymbol{v}|^2 + \frac{\mu}{r}. \tag{2.3.38}$$

The rotation generators $X_{\alpha\beta}$ provide the angular momentum

$$\boldsymbol{M} = m(\boldsymbol{x}\times\boldsymbol{v}). \tag{2.3.39}$$

Let us turn to the Lie-Bäcklund symmetries (2.3.37). We write the operator $\hat{X}_1$ from (2.3.37) in the prolonged form

$$\begin{aligned}\hat{X}_1 = &-\left(x^2v^2 + x^3v^3\right)\frac{\partial}{\partial x^1} + \left(2x^1v^2 - x^2v^1\right)\frac{\partial}{\partial x^2}\\ &+\left(2x^1v^3 - x^3v^1\right)\frac{\partial}{\partial x^3} + \left[(v^2)^2 + (v^3)^2 + x^2\dot{v}^2 + x^3\dot{v}^3\right]\frac{\partial}{\partial \dot{x}^1}\\ &+\left[v^1v^2 + 2x^1\dot{v}^2 - x^2\dot{v}^1\right]\frac{\partial}{\partial \dot{x}^2} + \left[v^1v^3 + 2x^1\dot{v}^3 - x^3\dot{v}^1\right]\frac{\partial}{\partial \dot{x}^3},\end{aligned} \tag{2.3.40}$$

act on the Lagrangian (2.3.35) and obtain:

$$\begin{aligned}\hat{X}_1(L) = &\,2mx^1\left(v^2\dot{v}^2 + v^3\dot{v}^3\right) - mx^2\left(v^1\dot{v}^2 + v^2\dot{v}^1\right) - mx^3\left(v^1\dot{v}^3 + v^3\dot{v}^1\right)\\ &+\frac{\mu}{r^3}\left[x^1\left(x^2v^2 + x^3v^3\right) - \left((x^2)^2 + (x^3)^2\right)v^1\right].\end{aligned} \tag{2.3.41}$$

Replacing $m\dot{v}^\alpha$ with $\mu\, x^\alpha/r^3$ according to Eqs. (2.3.34), one can verify that Eq. (2.3.41) yields:

$$\hat{X}_1(L) = D_t\left(-2\,\frac{\mu}{r}\,x^1\right). \tag{2.3.42}$$

Hence, the divergence condition (2.3.22) is satisfied with

$$B = -2\,\frac{\mu}{r}\,x^1\,. \tag{2.3.43}$$

Substituting the coordinates

$$\xi = 0, \quad \eta^1 = -x^2v^2 - x^3v^3, \quad \eta^2 = 2x^1v^2 - x^2v^1, \quad \eta^3 = 2x^1v^3 - x^3v^1$$

of the operator $\hat{X}_1$ in Eq. (2.3.25) we obtain

$$C = 2m\left[x^1(v^2)^2 + x^1(v^3)^2 - (x^2v^2 + x^3v^3)v^1\right]. \tag{2.3.44}$$

Now we substitute the expressions (2.3.44) and (2.3.43) of C and B, respectively, in Eqs. (2.3.23), divide the result by 2, and obtain the following conserved quantity:

$$A^1 = m\left[x^1(v^2)^2 + x^1(v^3)^2 - (x^2v^2 + x^3v^3)v^1\right] + \frac{\mu}{r}x^1. \tag{2.3.45}$$

One can verify that Eq. (2.3.45) is identical with the first component of the Laplace vector (2.1.30):

$$\boldsymbol{A} = (\boldsymbol{v} \times \boldsymbol{M}) + \frac{\mu}{r}\boldsymbol{x}.$$

Using all operators (2.3.37) we obtain the Laplace vector (2.1.30).

The generator Z from (2.3.36) does not obey the divergence condition (2.3.22) (see Exercise 2.12). Consequently, it does not furnish a conservation law. However, the scaling transformation group, generated by the operator Z, does not change the ratio t^2/r^3, the constancy of which is a mathematical expression of Kepler's third law (see Section 2.1.3). Hence, Kepler's third law follows directly from the scaling invariance of the equations of motion (2.1.28).

2.3.3 Method of nonlinear self-adjointness

The method of nonlinear self-adjointness (35), (37), (38) significantly extends the possibilities for constructing conservation laws associated with symmetries, because it does not require the existence of a Lagrangian. In particular, it can be applied to any system of ordinary differential equations, to all linear partial differential equations and to systems of nonlinear partial differential equations possessing at least one local conservation law. Furthermore, in contrast to Noether's theorem, it does not require that the number of equations in the system be equal to the number of dependent variables. Thus, we will consider in this section regularly defined systems

$$F_\sigma(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \sigma = 1, \ldots, N, \tag{2.3.46}$$

containing any of number N of differential equations, independently on the number m of dependent variables.

First we extend the notation of Section 2.3.1 by adding N auxiliary dependent variables

$$v = (v^1, \ldots, v^N)$$

to m dependent variables

$$u = (u^1, \ldots, u^m)$$

of Eqs. (2.3.46). We also use the partial derivatives

$$u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \ \ldots, \ u_{(s)} = \{u_{i_1\cdots i_s}^\alpha\},$$

$$v_{(1)} = \{v_i^\sigma\}, \quad v_{(2)} = \{v_{ij}^\sigma\}, \ \ldots, \ v_{(s)} = \{v_{i_1 \cdots i_s}^\sigma\}.$$

Accordingly, we extend the total differentiation (2.2.4) as follows:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + v_i^\sigma \frac{\partial}{\partial v^\sigma} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + v_{ij}^\sigma \frac{\partial}{\partial v_j^\sigma} + \cdots, \tag{2.3.47}$$

so that

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i D_j(u^\alpha), \quad u_{i_1\cdots i_s}^\alpha = D_{i_1} \cdots D_{i_s}(u^\alpha),$$

$$v_i^\sigma = D_i(v^\sigma), \quad v_{ij}^\sigma = D_i D_j(v^\sigma), \quad v_{i_1\cdots i_s}^\sigma = D_{i_1} \cdots D_{i_s}(v^\sigma).$$

Then we associate with the system (2.3.46) the adjoint system

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2.3.48}$$

where the *adjoint operator* F_α^* has the form

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha}. \tag{2.3.49}$$

Here $\mathcal{L}$ is given by

$$\mathcal{L} = \sum_{\sigma=1}^{N} v^\sigma F_\sigma(x, u, u_{(1)}, \ldots, u_{(s)}), \tag{2.3.50}$$

and $\delta/\delta u^\alpha$ is the variational derivative (2.3.5),

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1\cdots i_s}^\alpha}.$$

The differential function $\mathcal{L}$ defined by Eq. (2.3.50) is called the *formal Lagrangian* for Eqs. (2.3.46).

Remark 2.3.4. If the system (2.3.46) is *determined* ($N = m$), then the adjoint system (2.3.48) is also determined. If the system (2.3.46) is *over-determined*, i.e. $N > m$, then the adjoint system (2.3.48) is *sub-definite* since it contains $m < N$ equations for N new dependent variables v. Vice versa, if $N < m$, then the system (2.3.46) is sub-definite and the adjoint system (2.3.48) is over-determined.

Remark 2.3.5. For a linear equation $L[u] = 0$ the adjoint operator defined by (2.3.49) coincides with the classical definition of the adjoint operator $L^*[v]$ determined by the equation

$$vL[u] - uL^*[v] = D_i(p^i).$$

Definition 2.3.2. The system (2.3.46) is said to be nonlinearly self-adjoint if the adjoint system (2.3.48) is satisfied for all solutions of the system (2.3.46) after a substitution

$$v^\sigma = \varphi^\sigma(x, u, u_{(1)}, \ldots), \quad \sigma = 1, \ldots, N, \tag{2.3.51}$$

where not all φ^σ vanish identically.

In this definition, $\varphi^\sigma \in \mathcal{A}$ may be any differential function. If the substitution (2.3.51) does not involve derivatives, i.e.

$$v^\sigma = \varphi^\sigma(x, u), \quad \sigma = 1, \ldots, N, \tag{2.3.52}$$

then Definition 2.3.2 is equivalent to the requirement that the equations

$$F^*_\alpha\big(x, u, \varphi, u_{(1)}, \varphi_{(1)}, \ldots, u_{(s)}, \varphi_{(s)}\big) = \lambda^\sigma_\alpha\, F_\sigma\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) \tag{2.3.53}$$

hold identically in $x, u, u_{(1)}, \ldots, u_{(s)}$ for all $\alpha = 1, \ldots, m$. Here $\lambda^\sigma_\alpha \in \mathcal{A}$ are undeterminate coefficients, φ is the N-dimensional vector

$$\varphi = \big(\varphi^1(x, u), \ldots, \varphi^N(x, u)\big)$$

and $\varphi_{(1)}, \ldots, \varphi_{(s)}$ are the derivatives of φ, e.g.

$$\varphi_{(1)} = \{D_i(\varphi)\}_{i=1}^n \equiv \left\{\frac{\partial \varphi(x, u)}{\partial x^i} + u^\alpha \frac{\partial \varphi(x, u)}{\partial u^\alpha}\right\}_{i=1}^n .$$

If the substitution (2.3.51) is given by a first-order differential substitution

$$v^\sigma = \varphi^\sigma(x, u, u_{(1)}), \quad \sigma = 1, \ldots, N, \tag{2.3.54}$$

then the nonlinear self-adjointness condition (2.3.53) is replaced with

$$F^*_\alpha\big(x, u, \varphi, \ldots, u_{(s)}, \varphi_{(s)}\big) = \lambda^\sigma_\alpha\, F_\sigma + \lambda^{j\sigma}_\alpha\, D_j(F_\sigma). \tag{2.3.55}$$

Example 2.3.3. Let us consider the nonlinear heat equation

$$u_t - k(u)u_{xx} - k'(u)u_x^2 = 0, \quad k'(u) \neq 0.$$

Applying the definition (2.3.49) of the adjoint operator, we obtain the following adjoint equation (2.3.48):

$$v_t + k(u)v_{xx} = 0.$$

We insert in this equation the substitution (2.3.52) written together with the necessary derivatives:

$$v = \varphi(t, x, u), \quad v_t = \varphi_u u_t + \varphi_t, \quad v_x = \varphi_u u_x + \varphi_x,$$
$$v_{xx} = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}$$

and arrive at the following self-adjointness condition (2.3.53):

$$\varphi_u u_t + \varphi_t + k(u)[\varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}] = \lambda[u_t - k(u)u_{xx} - k'(u)u_x^2].$$

The comparison of the coefficients of u_t in both sides of this equation yields $\lambda = \varphi_u$. Then, comparing the terms with u_{xx} we see that $\varphi_u = 0$. Hence $\varphi = \varphi(t, x)$, and the self-adjointness condition has the form

$$\varphi_t + k(u)\varphi_{xx} = 0.$$

Since the latter equation should be satisfied identically in y, x, u, it yields $\varphi_t = 0$, $\varphi_{xx} = 0$, whence $\varphi = C_1 x + C_2$, where $C_1, C_2 = \text{const}$. Thus, the nonlinear heat equation is nonlinearly self-adjoint with the substitution (2.3.52) given by

$$v = C_1\, x + C_2.$$

Example 2.3.4. Let us consider the equation

$$u_{xy} = \sin u\,. \tag{2.3.56}$$

Its adjoint equation is

$$v_{xy} - v\cos u = 0.$$

The calculation shows that Eq. (2.3.56) is not nonlinearly self-adjoint with a substitution of the form (2.3.52), i.e. with

$$v = \varphi(x, y, u).$$

However, one can verify that Eq. (2.3.56) is nonlinearly self-adjoint with the following first-order differential substitution (2.3.54):

$$v = A_1(xu_x - yu_y) + A_2 u_x + A_3 u_y, \tag{2.3.57}$$

where $A_1,\ A_2,\ A_3$ are arbitrary constants. The nonlinear self-adjointness condition (2.3.55) is satisfied in the following form:

$$(v_{xy} - v\cos u)\,|_{(2.3.57)} = (A_1 x + A_2)D_x(u_{xy} - \sin u) + (A_3 - A_1 y)D_y(u_{xy} - \sin u).$$

Example 2.3.5. He have demonstrated recently (19) nonlinear self-adjointness of the Krichever-Novikov equation (44)

$$u_t = u_{xxx} - \frac{3}{2}\,\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x},$$

where $P(u)$ is a cubic polynomial without repeated roots. The substitution (2.3.54) for this equation is given by the *second-order differential function*

$$v = \frac{u_{tx}}{u_x^2} - \frac{u_{xx}}{u_x^3}\,u_t.$$

This substitution is written in (19), Eq. (2.4), by eliminating time derivations.

The following theorem, proved in (35), suggests a simple method for constructing conservation laws associated with symmetries of all nonlinearly self-adjoint equations.

Theorem 2.3.3. Let the system (2.3.46) be nonlinearly self-adjoint and let

$$X = \xi^i(x, u, u_{(1)}, \dots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots)\frac{\partial}{\partial u^\alpha}$$

be any infinitesimal symmetry (Lie point, Lie-Bäcklund or non-local) of the system (2.3.46). Then the vector with the components

$$\begin{aligned} C^i = &\xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots\right] \\ &+ D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right] \end{aligned} \tag{2.3.58}$$

is a conserved vector for the system (2.3.46). Here $\mathcal{L}$ is the formal Lagrangian (2.3.50) and W^α is given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha.$$

In (2.3.58) the formal Lagrangian $\mathcal{L}$ must be written in symmetric form with respect to all the mixed derivatives u_{ij}^α, $u_{ijk}^\alpha, \dots$. The auxiliary dependent variables v^α, together with their derivatives, must be eliminated from the expressions (2.3.58) by means of a substitution (2.3.51).

2.3.4 Short pulse equation

Here we illustrate, following (37), the method of nonlinear self-adjointness by computing a conservation law for the nonlinear equation

$$u_{xt} = u + \frac{1}{2}u^2 u_{xx} + u u_x^2 \tag{2.3.59}$$

known in the literature as the *short pulse equation.* It is used as a mathematical model for describing the propagation of ultrashort optical pulses in nonlinear media, for example, in a silica fibre.

Equation (2.3.59) was derived (up to notation and appropriate scaling the physical variables) in (62) by considering the propagation of linearly polarized light in a one-dimensional medium and assuming that the light propagates in the infrared range. The final step in construction of the model is based on the method of multiple scales.

The formal Lagrangian (2.3.50) for Eq. (2.3.59) has the form

$$\mathcal{L} = v\left(u_{xt} - u - \frac{1}{2}\,u^2 u_{xx} - u u_x^2\right) \tag{2.3.60}$$

and leads to the following adjoint equation (2.3.48):

$$v_{xt} = v + \frac{1}{2}\,u^2 v_{xx}. \tag{2.3.61}$$

The next statement has been proved in (37), Section 10.2.

Proposition 2.3.1. Eq. (2.3.59) is not nonlinearly self-adjoint with a substitution of the form (2.3.52),

$$v = \varphi(t, x, u),$$

but it is nonlinearly self-adjoint with the following first-order differential substitution:

$$v = u_t - \frac{1}{2}\,u^2 u_x. \tag{2.3.62}$$

Let us verify that the differential substitution (2.3.62) satisfies the nonlinear self-adjointness condition (2.3.55) of Eq. (2.3.59). We first obtain

$$v_x = u_{xt} - \frac{1}{2}\,u^2 u_{xx} - u u_x^2,$$

or

$$v_x = F + u, \tag{2.3.63}$$

where

$$F = u_{xt} - u - \frac{1}{2}\,u^2 u_{xx} - u u_x^2.$$

Then we have:

$$v_{xt} = D_t(F) + u_t, \quad v_{xx} = D_x(F) + u_x.$$

Substituting the expressions for v_{xt}, v and v_{xx} in (2.3.61) and invoking the notation for F, we arrive at the condition (2.3.55) in the following form:

$$v_{xt} - v - \frac{1}{2}\,u^2 v_{xx} = \left(D_x - \frac{1}{2}\,u^2\,D_t\right)\left(u_{xt} - u - \frac{1}{2}\,u^2 u_{xx} - u u_x^2\right).$$

Let us construct the conservation laws

$$D_t(C^1) + D_x(C^2) = 0$$

for Eq. (2.3.59). This equation admits the three-dimensional Lie algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u} + x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}.$$

Note that the formal Lagrangian (2.3.60) does not contain derivatives of order higher than two. Consequently, Equations (2.3.58) for computing the conserved vectors are written

$$C^i = W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right)\right] + D_j(W)\frac{\partial\mathcal{L}}{\partial u_{ij}}.$$

In our case we have:

$$C^1 = -WD_x\left(\frac{\partial\mathcal{L}}{\partial u_{tx}}\right) + D_x(W)\frac{\partial\mathcal{L}}{\partial u_{tx}},$$

$$C^2 = W\left[\frac{\partial\mathcal{L}}{\partial u_x} - D_t\left(\frac{\partial\mathcal{L}}{\partial u_{xt}}\right) - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right)\right] + D_t(W)\frac{\partial\mathcal{L}}{\partial u_{xt}} + D_x(W)\frac{\partial\mathcal{L}}{\partial u_{xx}}.$$

Using here the formal Lagrangian (2.3.60) in the symmetric form

$$\mathcal{L} = v\left(\frac{1}{2}\,u_{tx} + \frac{1}{2}\,u_{xt} - u - \frac{1}{2}\,u^2u_{xx} - uu_x^2\right)$$

we obtain

$$C^1 = -\frac{1}{2}Wv_x + \frac{1}{2}\,vD_x(W),$$

$$C^2 = -W\Big(uvu_x + \frac{1}{2}\,v_t - \frac{1}{2}\,u^2v_x\Big) + \frac{1}{2}\,vD_t(W) - \frac{1}{2}\,u^2vD_x(W).$$

We further simplify these expressions by restricting them on the solutions of Eq. (2.3.59) and hence replacing v_x with u due to Eq. (2.3.63). Thus

$$C^1 = -\frac{1}{2}Wu + \frac{1}{2}\,vD_x(W),$$
$$C^2 = -W\Big(uvu_x + \frac{1}{2}\,v_t - \frac{1}{2}\,u^3\Big) + \frac{1}{2}\,vD_t(W) - \frac{1}{2}\,u^2vD_x(W), \tag{2.3.64}$$

where v and v_t should be eliminated via the substitution (2.3.62).

We have to specify the conserved vector by substituting in (2.3.64) the value of W corresponding to a certain symmetry of Eq. (2.3.59). We take the symmetry X_3 given above. The corresponding W is

$$W = u + tu_t - xu_x.$$

Inserting it in Eqs. (2.3.64) and simplifying the resulting vector by means of the changes (2.2.14)—(2.2.16) from Remark 2.2.2, we obtain after routine calculations the conserved vector

$$C^1 = u^2, \quad C^2 = u^2u_xu_t - u_t^2 - \frac{1}{4}u^4 - \frac{1}{4}u^4u_x^2. \tag{2.3.65}$$

The symmetries X_1 and X_2 do not give new conserved vectors (37).

2.3.5 Linear equations

In the case of linear equations, Equations (2.3.58) associate with any symmetry an infinite set of conservation laws. For instance, applying the conservation formula (2.3.58) to the scaling symmetry $X = u\partial/\partial u$ of the heat equation $u_t - \Delta u = 0$ we obtain (see (37), Section 12.1 and the references therein) the conservation law

$$D_t[vu] + \nabla \cdot [u\nabla v - v\nabla u] = 0,$$

where v is an arbitrary solution of the adjoint equation $v_t + \Delta v = 0$ to the heat equation. This conservation law embraces the conservation laws associated with all other symmetries of the heat equation.

Exercises

Exercise 2.1. Derive the Lie algebra with the basis (2.1.12) by applying the substitution (2.1.9) to the symmetries (2.1.11) of Eq. (2.1.10).

Exercise 2.2. Verify that the operators X_5, X_6, X_7 and X_8 from (2.1.12) are admitted by Eq. (2.1.1).

Exercise 2.3. Find the one-parameter groups generated by the operators X_5, X_7 and X_8 from (2.1.12).

Exercise 2.4. Make a qualitative analysis of a fall of a body with a constant mass m in a viscous fluid, similar to that given in Section 2.1.2, considering the following more complicated model instead of Eq. (2.1.14):

$$m\ddot{x} + k\dot{x} - \beta\dot{x}^2 + mg = 0, \quad \text{where } k, \beta = \text{const.} \geq 0. \tag{2.4.1}$$

Exercise 2.5. Obtain the energy (2.1.7) applying Noether's theorem to the Lagrangian (2.1.8) and using the time translation generator $X_1 = \partial/\partial t$.

Exercise 2.6. Prove that

$$\frac{\partial}{\partial u^\alpha} D_i = D_i \frac{\partial}{\partial u^\alpha},$$

$$D_j \frac{\partial}{\partial u_j^\alpha} D_i = D_i \frac{\partial}{\partial u^\alpha} + D_i D_j \frac{\partial}{\partial u_j^\alpha},$$

$$D_j D_k \frac{\partial}{\partial u_{jk}^\alpha} D_i = D_i D_k \frac{\partial}{\partial u_k^\alpha} + D_i D_j D_k \frac{\partial}{\partial u_{jk}^\alpha}.$$

Exercise 2.7. Prove the operator identity (see Proposition 2.2.1)

$$\frac{\delta}{\delta u^\alpha} D_i = 0.$$

Hint: See (31), Section 8.4.1.

Exercise 2.8. Let $f(x, y, y', \dots, y^{(s)}) \in \mathcal{A}$ be a differential function of one independent variable x and one dependent variable y. Prove that if the equation $D_x(f) = 0$ holds identically in all variables $x, y, y', \dots, y^{(s)}$, and $y^{(s+1)}$, then $f = \text{const}$. See (31), Section 8.4.1.

Exercise 2.9. Derive the second equation in (2.3.33).

Exercise 2.10. Derive the energy (2.3.38) and the angular momentum (2.3.38) by applying Noether's theorem to the time translation and the rotation generators X_0 and $X_{\alpha\beta}$, respectively.

Exercise 2.11. Derive Eq. (2.3.42) from (2.3.41).

Exercise 2.12. Show that the operator Z from (2.3.36) and the Lagrangian (2.3.35) do not satisfy the divergence condition (2.3.22).

Chapter 3

Introduction of tensors and Riemannian spaces

3.1 Tensors

3.1.1 Motivation

Definition of covariant and contravariant vectors and tensors can be motivated by the well-known behaviour of partial derivatives and differentials under a change of variables

$$\bar{x}^i = \bar{x}^i(x), \quad i = 1, \ldots, n. \tag{3.1.1}$$

Let $\bar{f}(\bar{x})$ be a function of $\bar{x}$, and let $f(x)$ be the expression of $\bar{f}(\bar{x})$ in the variable x, i.e.

$$f(x) = \bar{f}(\bar{x}(x)).$$

The usual chain rule yields:

$$\frac{\partial f}{\partial x^i} = \frac{\partial \bar{f}}{\partial \bar{x}^k}\frac{\partial \bar{x}^k}{\partial x^i}, \quad i = 1, \ldots, n.$$

Hence the coordinates $f_i = \partial f/\partial x^i$ of the gradient ∇f transform, under the change of variables (3.1.1), are as follows:

$$f_i = \frac{\partial \bar{x}^k}{\partial x^i}\bar{f}_k, \quad i = 1, \ldots, n, \tag{3.1.2}$$

where $f_k = \partial \bar{f}/\partial \bar{x}^k$.

Recall that the differentials dx^i are written in the new variables (3.1.1) as follows:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^k}dx^k, \quad i = 1, \ldots, n. \tag{3.1.3}$$

Taking the inverse transformation (3.1.1), $x^i = x^i(\bar{x})$, one can rewrite Eqs. (3.1.2) and (3.1.3) in the form

$$\bar{f}_i = \frac{\partial x^k}{\partial \bar{x}^i}f_k \tag{3.1.2$'$}$$

and

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^k}d\bar{x}^k, \tag{3.1.3$'$}$$

respectively. Using these equations, one readily obtains the classical theorem on the invariance of the first differential, namely:

$$df = f_i dx^i = \bar{f}_i \, d\bar{x}^i. \tag{3.1.4}$$

3.1.2 Covariant and contravariant vectors

The following definition extends the transformation law (3.1.2) of partial derivatives.

Definition 3.1.1. Any two sets of quantities a_i and $\bar{a}_i$ $(i = 1, \ldots, n)$ related by the equations

$$a_i = \frac{\partial \bar{x}^k}{\partial x^i} \, \bar{a}_k, \quad i = 1, \ldots, n, \tag{3.1.5}$$

define a *covariant vector* with the components a_i and $\bar{a}_i$ in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively.

Thus, *the gradient of any function $f(x)$ is a covariant vector.* Equation (3.1.5) can be rewritten in the form solved with respect to $\bar{a}_j$ (cf. (3.1.2′)):

$$\bar{a}_j = a_i \frac{\partial x^i}{\partial \bar{x}^j}. \tag{3.1.6}$$

Indeed, multiplication of Eq. (3.1.5) by $\partial x^i / \partial \bar{x}^j$ and summation in i yields (see also Exercise 1.6 (ii)):

$$a_i \frac{\partial x^i}{\partial \bar{x}^j} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^j} \bar{a}_k = \delta^k_j \bar{a}_k = \bar{a}_j.$$

Let us extend the transformation law (3.1.3) of differentials to any n quantities $a^i, \quad i = 1, \ldots, n$ (they may be functions of x) and other n quantities $\bar{a}^i$ (they will be regarded as functions of $\bar{x}$) be defined by

$$\bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^k} a^k, \quad i = 1, \ldots, n. \tag{3.1.7}$$

Definition 3.1.2. Two sets of quantities a^i and $\bar{a}^i$ $(i = 1, \ldots, n)$ related by Eq. (3.1.7) define a *contravariant vector* with the components a^i and $\bar{a}^i$ in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively. A contravariant vector is written sometimes $a = (a^1, \ldots, a^n)$.

Lemma 3.1.1. Given an arbitrary contravariant vector a^i and a covariant vector c_i, their *scalar product* (termed also the *inner product*) $a^i c_i$ is invariant under an arbitrary change of variables (3.1.1), i.e.

$$\bar{a}^i \bar{c}_i = a^i c_i. \tag{3.1.8}$$

Proof. Indeed, we have:

$$\bar{a}^i \bar{c}_i = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^j}{\partial \bar{x}^i} a^k c_j = \delta^j_k a^k c_j = a^k c_k. \qquad \square$$

Lemma 3.1.2. Let Eq. (3.1.8) holds for an arbitrary contravariant vector a^i. Then c_i is a covariant vector. Likewise, if Eq. (3.1.8) holds for an arbitrary covariant vector c_i then a^i is a contravariant vector.

Proof. The equation

$$\bar{a}^i \bar{c}_i - a^k c_k = 0 \tag{3.1.9}$$

and Eq. (3.1.7) yield that

$$a^k \Big(\frac{\partial \bar{x}^i}{\partial x^k} \, \bar{c}_i - c_k \Big) = 0.$$

Since a^k are arbitrary quantities, we arrive at Eq. (3.1.5). To prove the second part of the statement, we use Eq. (3.1.5) for c_k,

$$c_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{c}_i,$$

and write Eq. (3.1.9) in the form

$$\Big(\bar{a}^i - \frac{\partial \bar{x}^i}{\partial x^k} \, a^k \Big) \bar{c}_i = 0.$$

It follows that a^i satisfy Eq. (3.1.7) for contravariant vectors. $\square$

3.1.3 Tensor algebra

If $\lambda^i, \mu^i, \xi_i, \eta_i$ are contravariant and covariant vectors then the products

$$a^{ik} = \lambda^i \mu^k,$$

$$a_{ik} = \xi_i \, \eta_k,$$

$$a^i_k = \lambda^i \, \xi_k$$

transform as follows:

$$\bar{a}^{ij} = a^{kl} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l}, \tag{3.1.10}$$

$$\bar{a}_{ij} = a_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j}, \tag{3.1.11}$$

$$\bar{a}^i_j = a^k_l \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} \, . \tag{3.1.12}$$

These transformation laws define contravariant, covariant and mixed tensors of the second order (briefly called 2-tensors). Namely, we use the following definitions.

Definition 3.1.3. Two sets of quantities, a^{ij} and $\bar{a}^{ij}$ related by Eq. (3.1.10) define a *contravariant* 2-*tensor* with the components a^{ij} and $\bar{a}^{ij}$ in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively.

Definition 3.1.4. Two sets of quantities, a_{ij} and $\bar{a}_{ij}$ related by Eq. (3.1.11) define a *covariant* 2-*tensor* with the components a_{ij} and $\bar{a}_{ij}$ in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively.

Definition 3.1.5. Two sets of quantities, a^i_j and $\bar{a}^i_j$ related by Eq. (3.1.12) define a *mixed* 2-*tensor* with the components a^i_j and $\bar{a}^i_j$ in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively.

Tensors of an arbitrary order are defined likewise. In particular, we will need *tensors of order zero* called also *scalars*. They are defined as follows.

Definition 3.1.6. Two quantities, $\varphi(x)$ and $\bar{\varphi}(\bar{x})$ given in the coordinate systems $\{x^i\}$ and $\{\bar{x}^i\}$, respectively, define a *scalar* if

$$\bar{\varphi} = \varphi. \tag{3.1.13}$$

In other words, scalars are *invariants* of the group of arbitrary transformations (3.1.1).

For example, Equation (3.1.4) written in the form $df = \overline{df}$, where

$$df = f_i dx^i, \quad \overline{df} = \bar{f}_i \, d\bar{x}^i,$$

shows that the differential df of a differentiable function $f(x)$ is a scalar. Furthermore, it is manifest from Eq. (3.1.8) that the scalar product of a contravariant and a covariant vector is a scalar (hence, the nomenclature *scalar product*, unlike the accidental *dot product*, has a geometric significance).

One can construct new tensors from given ones by means of operations of addition, multiplication and subtraction.

The operation of *addition* can be applied to tensors having a similar type and leads to a tensor of the same type. For example, the sum of contravariant 2-tensors a^{ij} and b^{ij} is a contravariant 2-tensor

$$c^{ij} = a^{ij} + b^{ij}. \tag{3.1.14}$$

Moreover, we have the same result if we replace the addition by any linear combination, where the coefficients are scalars. For example, if $\varphi(x)$ and $\psi(x)$ are two scalars, then the linear combination

$$p^{ij} = \varphi(x)a^{ij} + \psi(x)b^{ij}$$

of contravariant 2-tensors a^{ij} and b^{ij} is a contravariant 2-tensor.

The operation of *multiplication* can be applied to tensors of any type and leads to a tensor of higher-order. The type of the resulting tensor is the natural combination of the types of its factors. For example, the product of a contravariant vector a^i and a mixed 3-tensor b_l^{jk} is a mixed 4-tensor:

$$c_l^{ijk} = a^i b_l^{jk}. \tag{3.1.15}$$

The operation of *contraction* is often used in tensor calculus. It can be applied to tensors of mixed type and maps tensors into new tensors by lowering their order. For example, the contraction of a mixed 5-tensor a_{rs}^{ijk} in the indices j and s leads to a mixed 3-tensor c_r^{ik} obtained as follows:

$$c_r^{ik} = a_{rj}^{ijk}, \tag{3.1.16}$$

where in the right-hand side the summation in the index j is understood.

Definition 3.1.7. Equations written in terms of tensors, e.g. $T_k^{ij} = 0$, are valid in any system of coordinates and are called *covariant equations.*

3.2 Riemannian spaces

3.2.1 Differential metric form

Recall that the element of length of an n-dimensional Euclidean space, referred to rectangular and oblique Cartesian coordinate systems, is given by Eqs. (1.1.22),

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2$$

and (1.1.23),

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j, \quad g_{ij} = \text{const.},$$

respectively. In curvilinear coordinate systems, g_{ij} may be variable. For example, in spherical coordinates in the space of three dimensions

$$x^1 = r \sin\theta \cos\phi, \quad x^2 = r \sin\theta \sin\phi, \quad x^3 = r \cos\theta, \tag{3.2.1}$$

the infinitesimal length will be given by the following quadratic differential form with variable coefficients (see Example 3.2.1):

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.2.2}$$

G. F. B. Riemann generalized in 1854 the concept of the infinitesimal length in n-dimensional spaces by considering arbitrary quadratic differential forms

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{3.2.3}$$

with coefficients $g_{ij}(x)$ depending on $x = (x^1, \ldots, x^n)$ and symmetric in i, j:

$$g_{ij}(x) = g_{ji}(x), \quad i, j = 1, \ldots, n. \tag{3.2.4}$$

It is assumed that the matrix $\|g_{ij}(x)\|$ in a generic point $x \in V_n$ is non-degenerate, i.e. $\det\|g_{ij}(x)\| \neq 0$. Hence, there exists the inverse matrix $\|g_{ij}(x)\|^{-1}$ with the entries denoted by $g^{ij}(x)$. By definition of an inverse matrix, one has

$$g_{ij}g^{jk} = \delta_i^k. \tag{3.2.5}$$

An n-dimensional space with a metric form (3.2.3) is called a *Riemannian space* of n-dimensions and is denoted by V_n.

It is required that the metric form (3.2.3) be independent on a choice of coordinate systems, i.e. be invariant under an arbitrary change of variables x given by Eq. (3.1.1). Since the differentials dx^i transform as coordinates of a contravariant vector and hence their product $dx^i dx^j$ form a contravariant tensor of the second order, it follows from the invariance of the inner product $ds^2 = g_{ij}(x)dx^i dx^j$ that g_{ij} is a *covariant tensor of the second order.* That is (cf. Eqs. (3.1.11) and (3.1.5)):

$$g_{ij}(x) = \bar{g}_{kl}(\bar{x})\frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^l}{\partial x^j}, \quad i, j = 1, \ldots, n. \tag{3.2.6}$$

It follows that the elements of the inverse matrix

$$\|g^{kl}(x)\| = \|g_{ij}(x)\|^{-1}$$

provide a contravariant symmetric tensor, i.e. they obey Eq. (3.1.10):

$$\bar{g}^{ij}(\bar{x}) = g^{kl}(x)\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^l}, \quad i, j = 1, \ldots, n. \tag{3.2.7}$$

We will often omit the argument in tensors, e.g. $g_{ij}(x)$ will be simply written g_{ij}. The symmetric tensor g_{ij} is called the *metric tensor* of the space V_n with the metric form (3.2.3). The tensor g^{ij} is the contravariant representation of the metric tensor. The transformation law (3.2.6) of the covariant metric tensor can also be written in the following form:

$$\bar{g}_{ij} = g_{kl}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}. \tag{3.2.8}$$

It is convenient to raise the subscripts by means of g^{ij}, e.g.

$$a^i = g^{il}a_l; \quad a^i{}_j = g^{il}a_{lj}; \quad a^i{}_{jk} = g^{il}a_{ljk}; \quad a^{ij}{}_k = g^{il}g^{jm}a_{lmk} \tag{3.2.9}$$

and lower the superscripts by means of g_{ij} :

$$b_i = g_{il} b^l; \quad b_i^{\ j} = g_{il} b^{lj}; \quad b_i^{\ jk} = g_{il} b^{ljk}; \quad b_{ij}^{\ \ k} = g_{il} g_{jm} b^{lmk}. \tag{3.2.10}$$

Let $x_0 \in V_n$ be a generic point, i.e. $\det\|g_{ij}(x_0)\| \neq 0$. Then the metric form (3.2.3) can be reduced, at the point x_0, to the canonical form

$$ds^2 = (dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^n)^2. \tag{3.2.11}$$

We say that V_n is a space of signature $(+\cdots+-\cdots-)$ where the signs "+" appear p times. The theory of relativity deals with four-dimensional Riemannian spaces V_4 of signature $(+---)$. The signature does not depend on the choice of x_0.

Example 3.2.1. Consider the three-dimensional Euclidean space. In the rectangular Cartesian coordinates the metric has the form $ds^2 = dx^2 + dy^2 + dz^2$ and the metric tensor is written $g_{ij} = \delta_{ij}$. Let us find the metric form

$$ds^2 = \overline{g}_{ij} d\bar{x}^i d\bar{x}^j$$

and the metric tensor $\overline{g}_{ij}$ in the spherical coordinates

$$\bar{x}^1 = r, \quad \bar{x}^2 = \theta, \quad \bar{x}^3 = \phi \tag{3.2.12}$$

connected with the Cartesian coordinates by the equations

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta. \tag{3.2.13}$$

Using Eqs. (3.2.13) we obtain the differentials

$$\begin{aligned} dx &= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi, \\ dy &= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi, \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta, \end{aligned}$$

substitute them in $dx^2 + dy^2 + dz^2$ and arrive at the following representation of the metric form in the spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2.$$

Using the notation (3.2.12), we have the metric tensor

$$\|\overline{g}_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\,\sin\theta)^2 \end{Vmatrix}.$$

Example 3.2.2. The four-dimensional Riemannian space with the metric form

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{3.2.14}$$

where the constant c is the velocity of light in the empty space, is known as the *Minkowski space*, or the *Minkowski space-time* (52). It belongs to the physically significant class of four-dimensional Riemannian spaces having the signature $(+ - --)$.

3.2.2 Geodesics. The Christoffel symbols

The length s of a curve in V_n:

$$x^i = x^i(\sigma), \quad \sigma_1 \le \sigma \le \sigma_2,$$

joining the points $x_1, x_2 \in V_n$ so that

$$x^i(\sigma_1) = x_1^i, \quad x^i(\sigma_2) = x_2^i,$$

is given by the integral

$$s = \int_{\sigma_1}^{\sigma_2} L d\sigma, \tag{3.2.15}$$

where

$$L = \sqrt{g_{jk}(x)\dot{x}^j\dot{x}^k}, \quad \dot{x}^j = \frac{dx^j(\sigma)}{d\sigma}. \tag{3.2.16}$$

If the curve has an extremal length, i.e. it provides an extremum to the integral (3.2.15), it is called a *geodesic* joining the points x_1 and x_2 in the space V_n. The condition to be a geodesic is thus given by the Euler-Lagrange equations (2.3.6) with the Lagrangian (3.2.16):

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, n. \tag{3.2.17}$$

Theorem 3.2.1. If instead of a general parameter σ we use the arc length s of the curve measured from a fixed point x_1, then Eqs. (3.2.17) of geodesics are written

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i = 1, \ldots, n, \tag{3.2.18}$$

where the coefficients Γ^i_{jk} are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \tag{3.2.19}$$

and are known as the *Christoffel symbols*.

Proof. Equation (3.2.16) yields

$$\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\dot{x}^j\dot{x}^k}{\sqrt{g_{jk}\dot{x}^j\dot{x}^k}}, \qquad \frac{\partial L}{\partial \dot{x}^i} = \frac{g_{ij}\dot{x}^j}{\sqrt{g_{jk}\dot{x}^j\dot{x}^k}},$$

whence, using the equation

$$\sqrt{g_{jk}\dot{x}^j\dot{x}^k} = \frac{ds}{d\sigma},$$

we have

$$\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{\dot{x}^j\dot{x}^k}{\frac{ds}{d\sigma}}, \qquad \frac{\partial L}{\partial \dot{x}^i} = \frac{g_{ij}\dot{x}^j}{\frac{ds}{d\sigma}}.$$

Finally, letting $\sigma = s$ we obtain

$$\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{dx^j}{ds}\frac{dx^k}{ds}, \qquad \frac{\partial L}{\partial \dot{x}^i} = g_{ij}\frac{dx^j}{ds}.$$

Differentiating the second equation, we have:

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial g_{ij}}{\partial x^k}\frac{dx^k}{ds}\frac{dx^j}{ds} + g_{ij}\frac{d^2x^j}{ds^2}.$$

Hence, the Euler-Lagrange equations (3.2.17) are written as follows:

$$g_{ij}\frac{d^2x^j}{ds^2} + \frac{\partial g_{ij}}{\partial x^k}\frac{dx^k}{ds}\frac{dx^j}{ds} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Rewriting these equations in the symmetric form with respect to j and k :

$$g_{ij}\frac{d^2x^j}{ds^2} + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

multiplying by g^{il} and summing for i, we obtain

$$\frac{d^2x^l}{ds^2} + \frac{1}{2}g^{il}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Invoking Eqs. (3.2.19), we arrive at Eqs. (3.2.18), thus proving the theorem. □

Example 3.2.3. In the Euclidean space referred to a rectangular Cartesian coordinate frame, the metric tensor is $g_{ij} = \delta_{ij}$. Consequently, $\Gamma^i_{jk} = 0$ and Eqs. (3.2.18) have the form

$$\frac{d^2x^i}{ds^2} = 0.$$

Hence the geodesics are straight lines:

$$x^i = x^i_0 + \lambda^i s, \quad i = 1, \ldots, n.$$

Remark 3.2.1. The Christoffel symbols (3.2.19) do not behave as a tensor under changes of coordinates, see (17), Section 7.

3.2.3 Covariant differentiation. The Riemann tensor

With the aid of Christoffel symbols one defines the covariant differentiation of tensors on the Riemannian space V_n. Covariant differentiation takes tensors again into tensors. Covariant derivatives, e.g. of a *scalar* a, and of covariant and contravariant vectors a_i and a^i are defined by the following formulae:

$$a_{,k} = \frac{\partial a}{\partial x^k}, \tag{3.2.20}$$

$$a_{i,k} = \frac{\partial a_i}{\partial x^k} - a_j \Gamma^j_{ik}, \tag{3.2.21}$$

$$a^i_{,k} = \frac{\partial a^i}{\partial x^k} + a^j \Gamma^i_{jk}. \tag{3.2.22}$$

The covariant differentiation is often indicated by a subscript preceded by a comma. For repeated covariant differentiation I will use only one comma, e.g. $a_{i,jk}$ for the second covariant derivative of a covariant vector a_i.

The covariant differentiation of higher-order covariant, contravariant and mixed tensors is obtained merely by iterating the formulae (3.2.21) and (3.2.22). For example, in the case of second-order tensors we have

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{il}\Gamma^l_{jk} - a_{lj}\Gamma^l_{ik}, \tag{3.2.23}$$

$$a^{ij}_{,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{il}\Gamma^j_{lk} + a^{lj}\Gamma^i_{lk}, \tag{3.2.24}$$

$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^l_j \Gamma^i_{lk} - a^i_l \Gamma^l_{jk}. \tag{3.2.25}$$

Note, that for scalars the double covariant derivative does not depend on the order of differentiation,

$$a_{,jk} = a_{,kj}.$$

However, this similarity with the usual differentiation is violated, in general, when dealing with vectors and tensors of higher order. Namely, one can prove the following:

$$a_{i,jk} = a_{i,kj} + a_l R^l_{ijk},$$
$$a^i_{,jk} = a^i_{,kj} - a^l R^i_{ljk},$$

and so on. Here

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk} \tag{3.2.26}$$

is a tensor of the fourth order called the *Riemann tensor* (or the *Riemann-Christoffel tensor*) of the Riemannian space V_n. It is obvious from Eq. (3.2.26) that the Riemann tensor is skew-symmetric in jk :

$$R^l_{ijk} = -R^l_{ikj}. \tag{3.2.27}$$

According to the above formulae for the double differentiation, the successive covariant differentiations of tensors in V_n permute if and only if the Riemann tensor of the space V_n vanishes identically:

$$R^l_{ijk} = 0, \quad i, j, k, l = 1, \ldots, n. \tag{3.2.28}$$

3.2.4 Flat spaces

The following statement has a great geometrical importance. For its proof, see e.g. (17), Section 10.

Theorem 3.2.2. The metric form (3.2.3),

$$ds^2 = g_{ij}(x)dx^i dx^j,$$

of a Riemannian space V_n can be reduced by an appropriate change of variables (3.1.1) to a form (3.2.3) with constant coefficients,

$$ds^2 = \bar{g}_{ij}\bar{d}x^i\bar{d}x^j, \quad \bar{g}_{ij} = \text{const.},$$

and hence to the canonical form (3.2.11), in a neighborhood of a regular point x_0 (not only at the point x_0) if and only if the Riemann tensor R^l_{ijk} of V_n vanishes (Eqs. (3.2.28)).

Hint of the proof. Due to the transformation law (3.2.8) of the metric tensor, or equivalently (3.2.8), one has to demonstrate that the compatibility condition of the over-determined system of differential equations

$$\bar{g}^{ij} = g^{kl}(x)\frac{\partial \bar{x}^i(x)}{\partial x^k}\frac{\partial \bar{x}^j(x)}{\partial x^l}, \quad \bar{g}^{ij} = \text{const.},$$

for unknown functions $\bar{x}^i(x)$ is given by R^l_{ijk}. □

In view of Theorem 3.2.2, the Riemann tensor R^l_{ijk} is also called the *curvature tensor* of the space V_n.

Definition 3.2.1. A Riemannian space whose curvature tensor vanishes is called a *flat space* and is denoted by S_n.

Remark 3.2.2. Since the Christoffel symbols (3.2.19) do not transform as a tensor under changes of coordinates (Remark 3.2.1), the equations $\Gamma^i_{jk} = 0$ are not covariant (see Definition 3.1.7), i.e. not invariant under changes of variables x^i. Furthermore, it follows from Theorem 3.2.2 that one can nullify the Christoffel symbols by an appropriate change of variables if and only if the Riemann tensor R^l_{ijk} vanishes. Accordingly, the nonlinear equations (3.2.18) for geodesics can be linearized by a change of variables only in flat spaces.

Contracting the indices l and k in the Riemann-Christoffel tensor (3.2.26), one obtains the *Ricci tensor*:

$$R_{ij} \stackrel{\text{def}}{=} R^k_{ijk} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{mk}. \tag{3.2.29}$$

Finally, multiplication of the Ricci tensor by g^{ij} followed by contraction yields the *scalar curvature* of the space V_n:

$$R = g^{ij} R_{ij}. \tag{3.2.30}$$

3.3 Application to ODEs

S. Lie presented in (48), §1, the complete description of all linearizable second-order ordinary differential equations

$$y'' = f(x, y, y'), \tag{3.3.1}$$

i.e. those equations that can be obtained from linear equations

$$y'' + a(x)y' + b(x)y = 0 \tag{3.3.2}$$

by a change of variables. Recall that any linear equation (3.3.2) of the second order can be reduced to the simplest form

$$\frac{d^2u}{dt^2} = 0 \tag{3.3.3}$$

by an appropriate regular change of variables

$$t = \phi(x, y), \quad u = \psi(x, y), \tag{3.3.4}$$

namely, by the change of variables (see e.g. (35), Section 3.3.2)

$$t = \int \frac{\mathrm{e}^{-\int a(x)dx}}{z^2(x)}\, dx, \quad u = \frac{y}{z(x)}, \tag{3.3.5}$$

where $z(x) \neq 0$ is any solution of Eq. (3.3.2).

As a first step, Lie showed that the linearizable second-order equations are at most cubic in the first derivative, i.e. belong to the family of equations of the form

$$y'' + F_3(x, y)y'^{\,3} + F_2(x, y)y'^{\,2} + F_1(x, y)y' + F(x, y) = 0. \tag{3.3.6}$$

However, not every equation of the form (3.3.6) with arbitrary coefficients $F_3(x, y)$, $\ldots, F(x, y)$ is linearizable. Therefore Lie investigated in (48) (an outline of Lie's approach is given in (40)) the conditions that guarantee the possibility of linearization of Eq. (3.3.6). Using a projective geometric reasoning, he found the following over-determined system of four auxiliary equations for two dependent variables w, z :

$$\begin{aligned}
\frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x},\\
\frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3,\\
\frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2,\\
\frac{\partial z}{\partial y} &= -\,zw + FF_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y}.
\end{aligned} \tag{3.3.7}$$

In terms of this system, Lie demonstrated the linearization test: Equation (3.3.6) is linearizable if and only if the over-determined system (3.3.6) is integrable. We gave in (40) an alternative proof of Lie's linearization test based on techniques of Riemannian geometry. I present here an outline of our approach.

Recall that Eq. (3.3.3) describes the straight lines on the (x, y) plane. Hence, to prove the linearization test, we have to find all equations (3.3.6) whose integral curves

$$x = x(t), \quad y = y(t), \tag{3.3.8}$$

can be straighten out by a change of variables. Considering the first equation (3.3.8) as a change of the independent variable and using the notation

$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt},$$

we have:

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}, \quad y'' = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{1}{\dot{x}} = \frac{1}{\dot{x}^3}\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right).$$

Then the left-hand side of Eq. (3.3.6) becomes

$$\begin{aligned}
y'' + F_3\, y'^{\,3} + F_2\, y'^{\,2} + {}& F_1\, y' + F\\
&= \frac{1}{\dot{x}^3}\left(\dot{x}\ddot{y} - \dot{y}\ddot{x} + F_3\,\dot{y}^3 + F_2\,\dot{x}\dot{y}^2 + F_1\,\dot{x}^2\dot{y} + F\,\dot{x}^3\right),
\end{aligned}$$

and hence Eq. (3.3.6) can be written in the form

$$\dot{x}\left[\ddot{y} + \alpha\dot{y}^2 + \gamma\dot{x}\dot{y} + F\dot{x}^2\right] - \dot{y}\left[\ddot{x} - (F_3\,\dot{y}^2 + \beta\dot{x}\dot{y} + \delta\dot{x}^2)\right] = 0, \tag{3.3.9}$$

where

$$\alpha + \beta = F_2, \quad \gamma + \delta = F_1. \tag{3.3.10}$$

Taking projections of Eq. (3.3.9) in the (x, y) plane, we split it into the system

$$\begin{aligned} &\ddot{x} - F_3\,\dot{y}^2 - \beta\dot{x}\dot{y} - \delta\dot{x}^2 = 0, \\ &\ddot{y} + \alpha\dot{y}^2 + \gamma\dot{x}\dot{y} + F\dot{x}^2 = 0. \end{aligned} \tag{3.3.11}$$

Thus, we have transformed Eq. (3.3.6) cubic in y' into the system of equations (3.3.11) quadratic in $\dot{x}, \dot{y}$.

Equations (3.3.11) have the form of Eqs. (3.2.18) of geodesics,

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\,\dot{x}^k = 0, \quad i = 1, 2, \tag{3.3.12}$$

where $x^1 = x, x^2 = y$, and the Christoffel symbols Γ^i_{jk} are given by

$$\begin{aligned} &\Gamma^1_{11} = -\delta, \quad \Gamma^1_{12} = \Gamma^1_{21} = -\tfrac{1}{2}\beta, \quad \Gamma^1_{22} = -F_3, \\ &\Gamma^2_{11} = F, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \tfrac{1}{2}\,\gamma, \qquad \Gamma^2_{22} = \alpha. \end{aligned} \tag{3.3.13}$$

As mentioned in Remark 3.2.2, one can nullify the Christoffel symbols by an appropriate change of variables, and hence straighten out the geodesics if and only if the Riemann tensor R^l_{ijk} vanishes. Therefore we compute the Riemann tensor by applying Eqs. (3.2.26) to the Christoffel symbols (3.3.13). Due to Eq. (3.2.27), $R^l_{ijk} = -R^l_{ikj}$, we have to calculate only

$$R^l_{112} = 0, \quad R^l_{212} = 0, \quad l = 1, 2.$$

The calculation gives:

$$\begin{aligned} R^1_{112} &= -\frac{1}{2}\,\beta_x + \delta_y - \frac{1}{4}\,\beta\,\gamma + FF_3, \\ R^1_{212} &= \frac{1}{2}\,\beta_y - (F_3)_x - \frac{1}{4}\,\beta^2 + \delta F_3 + \frac{1}{2}\,\gamma F_3 - \frac{1}{2}\,\alpha\,\beta, \\ R^2_{112} &= \frac{1}{2}\,\gamma_x - F_y + \frac{1}{4}\,\gamma^2 - \alpha\,F + \frac{1}{2}\,\gamma\,\delta - \frac{1}{2}\,\beta\,F, \\ R^2_{212} &= \alpha_x - \frac{1}{2}\,\gamma_y - FF_3 + \frac{1}{4}\,\beta\,\gamma. \end{aligned} \tag{3.3.14}$$

Now we substitute the expressions (3.3.14) in the equations $R^l_{ijk} = 0$, let

$$\alpha = F_2 - \beta, \quad \delta = F_1 - \gamma$$

according to Eqs. (3.3.10), and obtain the following equations:

$$\frac{1}{2}\beta_x = \frac{1}{4}\beta\gamma - FF_3 - \frac{1}{3}(F_1)_y + \frac{2}{3}(F_2)_x,$$

$$\frac{1}{2}\beta_y = -\frac{1}{4}\beta^2 + \frac{1}{2}\beta F_2 + \frac{1}{2}\gamma F_3 + (F_3)_x - F_1F_3,$$

$$\frac{1}{2}\gamma_x = \frac{1}{4}\gamma^2 - \frac{1}{2}\beta F - \frac{1}{2}\gamma F_1 + F_y + FF_2,$$

$$\frac{1}{2}\gamma_y = -\frac{1}{4}\beta\gamma + FF_3 - \frac{1}{3}(F_2)_x + \frac{2}{3}(F_1)_y\,.$$

These equations coincide with Lie's four auxiliary equations (3.3.7) after introducing the notation

$$w = \frac{\beta}{2}, \quad z = \frac{\gamma}{2}.$$

One can verify that the compatibility conditions

$$w_{xy} = w_{yx}, \quad z_{xy} = z_{yx}$$

of the over-determined system (3.3.7) are given by the equations

$$\begin{aligned} 3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} &= (3F_1F_3 - F_2^2)_x - 3(FF_3)_y - 3F_3F_y + F_2(F_1)_y,\\ 3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} &= 3(FF_3)_x + (F_1^2 - 3FF_2)_y + 3F(F_3)_x - F_1(F_2)_x. \end{aligned} \tag{3.3.15}$$

Equations (3.3.15) provide a convenient form of Lie's linearization test.

Exercises

Exercise 3.1. Prove the invariance of the first differential, i.e. derive Eq. (3.1.4), $f_i dx^i = \bar{f}_i\, d\bar{x}^i$.

Exercise 3.2. Solve Eqs. (3.1.7), $\bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^k}\, a^k \quad (i = 1, \ldots, n)$, with respect to a^k.

Answer.

$$a^k = \frac{\partial x^k}{\partial \bar{x}^i}\,\bar{a}^i, \quad i = 1, \ldots, n.$$

Exercise 3.3. Give the definition of 3-tensors.

Exercise 3.4. Let the quantities $g_{ik}(x)$ be symmetric, i.e. $g_{ik} = g_{ki}$. Prove that $ds^2 = g_{ik}(x)dx^i dx^k$ is a scalar if and only if g_{ik} is a covariant 2-tensor.

Exercise 3.5. Let g_{ij} and g^{ij} be the covariant and contravariant representations, respectively, of the metric tensor of an n-dimensional space V_n. Verify that $g_{ij}g^{ij} = n$.

Exercise 3.6. Show, using Eq. (3.2.25) that $\delta^i_{j,k} = 0$.

Exercise 3.7. Show, using (3.2.23) and (3.2.24), that the metric tensor behaves as a constant with respect to the covariant differentiation, namely:

$$g_{ij,k} = 0, \quad g^{ij}_{\ \ ,k} = 0.$$

Exercise 3.8. Find the Christoffel symbols

$$\overline{\Gamma}^i_{jk}(\bar{x}) = \frac{1}{2}\overline{g}^{il}\left(\frac{\partial \overline{g}_{lj}}{\partial \bar{x}^k} + \frac{\partial \overline{g}_{lk}}{\partial \bar{x}^j} - \frac{\partial \overline{g}_{jk}}{\partial \bar{x}^l}\right)$$

in the Euclidean space $\mathbb{R}^3$ referred to the spherical coordinates (3.2.12) by using Eqs. (3.2.13).

Exercise 3.9. Use the result of Exercise 3.8 and find the solution of the equations of geodesics

$$\frac{d^2\bar{x}^i}{ds^2} + \overline{\Gamma}^i_{jk}(\bar{x})\frac{d\bar{x}^j}{ds}\frac{d\bar{x}^k}{ds} = 0, \quad i = 1, 2, 3,$$

in the Euclidean space $\mathbb{R}^3$ referred to the spherical coordinates.

Exercise 3.10. Find the geodesics in the Minkowski space with the metric form $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

Exercise 3.11. Using the definition (3.2.21) of the covariant derivative of covariant vectors, show that

$$a_{i,k} - a_{k,i} = \frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i}.$$

Exercise 3.12. Obtain metric tensor $\overline{g}_{ij}$ in the spherical coordinates (3.2.12), given in Example 3.2.1, using the transformation law (3.2.7) or (3.2.8) of contravariant or covariant tensors, respectively.

Exercise 3.13. Show that the change of variables (3.3.5) reduces the linear ODE (3.3.2) to the simples form (3.3.3).

Exercise 3.14. Obtain the components (3.3.14) of the Riemann tensor from the Christoffel symbols (3.3.13).

Exercise 3.15. Prove Theorem 3.2.2 following the hint provided after formulation of the theorem.

Exercise 3.16. Obtain Eqs. (3.3.15) from the compatibility conditions $w_{xy} = w_{yx}$, $z_{xy} = z_{yx}$ of the over-determined system (3.3.7).

Chapter 4

Motions in Riemannian spaces

4.1 Introduction

Consider a Riemannian space V_n with a metric tensor g_{ij}. Let G be a group of transformations

$$\bar{x}^i = f^i(x, a), \quad i = 1, \ldots, n, \tag{4.1.1}$$

of points $x = (x^1, \ldots, x^n) \in V_n$. We impose the condition

$$f^i(x, 0) = x^i, \quad i = 1, \ldots, n,$$

and write the infinitesimal transformation in the first-order of precision in the parameter a :

$$\bar{x}^i \approx x^i + a\xi^i(x),$$

where

$$\xi^i(x) = \frac{\partial f^i(x, a)}{\partial a}\bigg|_{a=0}, \quad i = 1, \ldots, n. \tag{4.1.2}$$

It is convenient to represent the infinitesimal transformation by the first-order linear differential operator

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{4.1.3}$$

known as the *generator of the group* (4.1.1). Furthermore, we extend the action of the transformations (4.1.1) to the metric tensor in accordance with the transformation law (3.2.6):

$$g_{ij} = \overline{g}_{kl}\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}, \quad i, j = 1, \ldots, n. \tag{4.1.4}$$

In what follows, we will deal with multi-parameter groups G.

We say that G is the group of *isometric motions* if the transformations (4.1.1) and (4.1.4) do not change the *shape and size* of any geometric figures in V_n. If the transformations keep unaltered the shape of figures but not their size, then G is called the group of *conformal motions* in V_n. In general, an arbitrary multi-parameter group G does not leave invariant neither size nor the shape of figures. If, nevertheless the transformations (4.1.1) and (4.1.4) leave unaltered some of features of geometric configurations we call G a group of *generalized motions* in V_n.

4.2 Isometric motions

4.2.1 Definition

Isometric motions are widely used in elementary geometry in two- and three-dimensional Euclidean spaces (a particular case of Riemannian spaces).

Definition 4.2.1. The group of transformations (4.1.1) is called a *group of isometric motions* in the space V_n if all components of the metric tensor $g_{ij}(x)$ are invariant under the transformations (4.1.1) and (4.1.4), i.e.

$$\overline{g}_{ij} = g_{ij}. \tag{4.2.1}$$

According to this definition and Eq. (4.1.4), the isometric motions are described by the following over-determined system of nonlinear partial differential equations of the first order:

$$g_{ij}(x) = g_{kl}(f)\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}. \tag{4.2.2}$$

Here $g_{ij}(x),\ i,j = 1,\ldots,n$, are given functions and $f = (f^1,\ldots,f^n)$ is an unknown vector function to be determined from Eqs. (4.2.2).

One can say geometrically that isometric motions do not change the shape and the size of figures.

4.2.2 Killing equations

Let us determine the infinitesimal isometric motions. If one differentiates (4.2.2) with respect to a at $a = 0$ and uses (4.1.2), one obtains the following over-determined system of linear partial differential equations of the first order for the unknown functions ξ^i:

$$\xi^k\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = 0, \quad i,j = 1,\ldots,n. \tag{4.2.3}$$

The equations (4.2.3) were obtained by W. Killing in 1892 and are known as the *Killing equations.* A solution $\xi = (\xi^1,\ldots,\xi^n)$ of the Killing equations is termed also a *Killing vector.*

The following identity holds:

$$\xi^k\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = \xi_{i,j} + \xi_{j,i}, \tag{4.2.4}$$

where $\xi_i = g_{ik}\xi^k$ is the covariant representation of the the vector ξ^i, and $\xi_{i,j}$ denotes the covariant derivative. The tensor at the right-hand side of the equation

(4.2.4) is called the *Lie derivative* and is denoted by $\mathcal{L}g_{_{ij}}$. Thus,

$$\mathcal{L}g_{_{ij}} = \xi_{i,j} + \xi_{j,i}. \tag{4.2.5}$$

Now the Killing equations (4.2.3) are written in the covariant form

$$\xi_{i,j} + \xi_{j,i} = 0 \tag{4.2.6}$$

or

$$\mathcal{L}g_{_{ij}} = 0. \tag{4.2.7}$$

4.2.3 Isometric motions on the plane

Let us find the group of isometric motions on the (x, y) plane. In the Cartesian coordinate system $x^1 = x,\ x^2 = y$ we have $g_{ij} = \delta_{ij}$. The condition (4.2.2) for isometric motions has the form

$$\delta_{ij} = \delta_{kl}\frac{\partial f^k}{\partial x^i}\frac{\partial f^l}{\partial x^j}. \tag{4.2.8}$$

The Killing equations (4.2.3) are written

$$\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i} = 0 \tag{4.2.9}$$

and provide the system of three equations for two functions $\xi^1(x, y), \xi^2(x, y)$:

$$\frac{\partial \xi^1}{\partial x} = 0, \quad \frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^2}{\partial x} = 0, \quad \frac{\partial \xi^2}{\partial y} = 0.$$

The first and the third equations yield that $\xi^1 = \xi^1(y)$ and $\xi^2 = \xi^2(x)$, respectively. Therefore the second equation is written

$$\frac{d\xi^1(y)}{dy} = -\frac{d\xi^2(x)}{dx} = C, \quad C = \text{const.},$$

whence

$$\xi^1 = C_1 + C\,y, \quad \xi^2(x) = C_2 - C\,x.$$

We arrive at the following generators:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Hence, the continuous group of isometric motions is a three-parameter group composed of translations of x and y and rotations in the (x, y) plane. Furthermore, it is clear from Eqs. (4.2.8) that the reflection $\overline{x} = -x, \overline{y} = y$ is an independent discrete isometric transformation. It is also clear that all other reflections compositions of the above transformations. Thus, the general group of isometric motions comprises the following four different transformations:

$$
\begin{aligned}
&\text{(i)}\ \ \overline{x} = x + a_1, \quad \overline{y} = y;\\
&\text{(ii)}\ \ \overline{x} = x, \quad \overline{y} = y + a_2;\\
&\text{(iii)}\ \overline{x} = x\cos a_3 + y \sin a_3,\\
&\qquad \overline{y} = y \cos a_3 - x \sin a_3;\\
&\text{(iv)}\ \overline{x} = -x, \quad \overline{y} = y.
\end{aligned}
\tag{4.2.10}
$$

4.2.4 Maximal group of isometric motions

We will use here the following definition (see also Definition 9.1.1 in Section 9.1.3).

Definition 4.2.2. Riemannian spaces V_n of *constant curvature* are defined by the equation

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}), \quad K = \text{const.},$$

where R^m_{jkl} is the covariant Riemann tensor:

$$R_{ijkl} = g_{im}R^m_{jkl}.$$

Riemann proved (1854) that the spaces V_n of constant curvature are characterized by the condition that their metric can be written, in appropriate coordinates, in the form

$$ds^2 = \frac{1}{\theta^2}\sum_i \epsilon_i (dx^i)^2, \quad \theta = 1 + \frac{K}{4}\sum_i \epsilon_i (x^i)^2, \tag{4.2.11}$$

where each $\epsilon_i = \pm 1$, in agreement with the signature of V_n, and $K = \text{const}$.

Theorem 4.1. The order of the group of isometries in any Riemannian space V_n of the dimension $n \geq 3$ does not exceed

$$\frac{n(n+1)}{2} \tag{4.2.12}$$

and this maximal value is attained only for the spaces of constant curvature.

The group of isometric motions in the space of constant curvature with the metric (4.2.11) has the following $n(n+1)/2$ generators:

$$
\begin{aligned}
X_i &= \left(\frac{K}{2}x^i x^j + (2-\theta)\epsilon_i \delta^{ij}\right)\frac{\partial}{\partial x^j},\\
X_{ij} &= \epsilon_j x^j \frac{\partial}{\partial x^i} - \epsilon_i x^i \frac{\partial}{\partial x^j} \quad (i, j = 1, \ldots, n).
\end{aligned}
\tag{4.2.13}
$$

Here, the indices i, j in the expressions $\epsilon_i x^i, \epsilon_j x^j$ and $\epsilon_i \delta^{ij}$ are fixed (no summation), and δ^{ij} is the Kronecker symbol.

4.3 Conformal motions

4.3.1 Definition

Definition 4.3.1. Transformations in the space V_n define a *group of conformal transformations* (alias *conformal motions*), of briefly a *conformal group*, if they change the fundamental metric form ds^2 according to the rule

$$d\bar{s}^2 = \Phi(x,a) ds^2, \quad \Phi(x,a) \neq 0.$$

This equation is equivalently written in terms of the metric tensor as follows:

$$g_{ij} = \Phi(x,a)\bar{g}_{ij}. \tag{4.3.1}$$

The geometric significance of conformal motions is that they keep unaltered the shape of figures, but may change their size. Note that isometric motions provide the particular case of conformal motions.

4.3.2 Generalized Killing equations

Multiplying Eqs. (4.3.1) by

$$\frac{\partial f^i}{\partial x^k}\frac{\partial f^j}{\partial x^l}$$

then summing with respect to i, j and using (4.1.4) we obtain

$$g_{ij}(f(x,a))\frac{\partial f^i(x,a)}{\partial x^k}\frac{\partial f^j(x,a)}{\partial x^l} = \Phi(x,a) g_{kl}(x).$$

We now differentiate the result with respect to the group parameter a at $a = 0$ and, changing the indices, obtain the following *generalized Killing equations* (see Eqs. (6.1.19)):

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = \phi(x) g_{ij},$$

where

$$\phi(x) = \left.\frac{\partial \Phi(x,a)}{\partial a}\right|_{a=0}.$$

The covariant form of the generalized Killing equations is (cf. Eqs. (4.2.6))

$$\xi_{i,j} + \xi_{j,i} = \phi\, g_{ij}, \quad i, j = 1, \ldots, n. \tag{4.3.2}$$

4.3.3 Conformally flat spaces

Definition 4.3.2. A Riemannian space $\widetilde{V}_n$ with the metric form $d\widetilde{s}^{\,2}$ is said to be conformal to a space V_n with the metric form ds^2 if $d\widetilde{s}^{\,2} = h(x)ds^2$.

We shall assume that the function $h(x)$ figuring in this definition is positive, and write the conformal correspondence between $\widetilde{V}_n$ and V_n as the following relation between their metric tensors:

$$\widetilde{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x) \tag{4.3.3}$$

Remark 4.3.1. A group which is conformal in V_n is conformal also in any space conformal to V_n but, generally speaking, with a different function $\phi(x)$ appearing in the generalized Killing equations (4.3.2).

The conformal correspondence (4.3.3) implies the following relations:

$$\begin{aligned} \Gamma^k_{ij} &= \widetilde{\Gamma}^k_{ij} - \delta^k_i \sigma_{,j} - \delta^k_j \sigma_{,i} + \widetilde{g}_{ij} \widetilde{g}^{\,kl} \sigma_{,l}\,; \\ R &= e^{2\sigma}(\widetilde{R} - 2(n-1)\widetilde{\Delta}_2\sigma + (n-1)(n-2)\widetilde{\Delta}_1\sigma); \\ C^l_{ijk} &= \widetilde{C}^l_{ijk}, \end{aligned} \tag{4.3.4}$$

where C^l_{ijk} is defined by (when $n > 3$)

$$\begin{aligned} C^l_{ijk} = R^l_{ijk} &+ \frac{1}{n-2}(\delta^l_j R_{ik} - \delta^l_k R_{ij} + g_{ik} R^l_j - g_{ij} R^l_k) \\ &+ \frac{R}{(n-1)(n-2)}(\delta^l_k g_{ij} - \delta^l_j g_{ik}). \end{aligned} \tag{4.3.5}$$

and is known as the *Weil tensor.*

Remark 4.3.2. Riemannian spaces that are conformal to a flat space are called *conformally flat spaces.* Any two-dimensional Riemannian space V_2 is conformally flat space since an arbitrary quadratic form ds^2 in two variables can be reduced to the diagonal form

$$ds^2 = \lambda(x^1, x^2)\,[(dx^1)^2 \pm (dx^2)^2].$$

In higher dimensions, the conformally flat spaces are described by the following theorem (see (17)).

Theorem 4.3.1. The space V_n with $n > 3$ is conformally flat if and only if

$$C^l_{ijk} = 0 \quad (i, j, k, l = 1, \dots, n).$$

In the case $n = 3$ the Weil tensor vanishes identically, and the criterion for V_3 to be conformal to a flat space is written

$$R_{ijk} = 0,$$

where R_{ijk} is the following tensor:

$$R_{ijk} = \frac{2}{n-2}(L_{ik,j} - L_{ij,k}), \quad L_{ik} = -R_{ik} + \frac{R}{2(n-1)} g_{ik}\,.$$

Theorem 4.3.2. The order of the group of conformal transformations in any Riemannian space V_n of the dimension $n \geq 3$ does not exceed

$$\frac{(n+1)(n+2)}{2} \tag{4.3.6}$$

and this maximal value is attained only for the conformally flat spaces.

For convenience, let us write the generators of the conformal group in an arbitrary conformally flat space. To this end, it suffices to consider only flat spaces with positive definite metrics. Indeed, for the case of arbitrary signature it is enough to formally replace the corresponding real variables x^i by $\sqrt{-1}\,x^i$. Thus, let S_n be a flat space. In the Cartesian coordinate system, where $g_{ij} = \delta_{ij}$, the generalized Killing equations (4.3.2) become

$$\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i} = \phi \delta_{ij}. \tag{4.3.7}$$

Calculating their general solution, one obtains the following generators of the group of conformal transformations in S_n :

$$\begin{aligned} &X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} \quad (i<j), \\ &Z = x^i \frac{\partial}{\partial x^i}, \quad Y_i = \left(2x^i x^j - |x|^2 \delta^{ij}\right) \frac{\partial}{\partial x^j}, \quad (i,j = 1,\ldots,n). \end{aligned} \tag{4.3.8}$$

4.4 Generalized motions

The theory of generalized motions was developed in 1969 and used it in general relativity for constructing so-called *partially invariant solutions* of the Einstein equations (see (28), Chapter 1, and the references therein).

4.4.1 Generalized motions, their invariants and defect

The change of the length ds under the transformations (4.1.1),

$$\overline{x}^{\,i} = f^i(x,a),$$

is measured by the difference between

$$d\bar{s}^2 = g_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j$$

and

$$ds^2 = \bar{g}_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j$$

where (see (4.1.4))

$$g_{ij} = \overline{g}_{kl}\frac{\partial f^k(x)}{\partial x^i}\frac{\partial f^l(x)}{\partial x^j}.$$

Definition 4.4.1. A function

$$J(\|g_{ij}\|) = J(g_{11}, g_{12}, \ldots, g_{nn})$$

is called an *invariant* of the group G of transformations (4.1.1) in the Riemannian space V_n with the metric tensor g_{ij} if $J(\|g_{ij}\|)$ is invariant under the extended transformations (4.1.1) and (4.1.4):

$$J(\|g_{ij}\|) = J(\|\overline{g}_{ij}\|). \tag{4.4.1}$$

Equation (4.4.1) can be written in the form

$$J(\|g_{ij}(\bar{x})\|) = J(\|\overline{g}_{ij}(\bar{x})\|).$$

Since $\overline{g}_{ij}(\bar{x}) = g_{ij}(x)$, the above equation means that the value of the function J at the point $\bar{x} \in V_n$ which is the image of $x \in V_n$ under the transformation (4.1.1) is the same as the value of J at the original point x. It is evident from the definition that any group of transformations in V_n has at most $n(n+1)/2$ invariants and this maximal value is attained only for isometric motions.

Example 4.4.1. Thus, the group of isometric motions has

$$\frac{n(n+1)}{2}$$

independent invariants, namely all components of the metric tensor:

$$J_{ij} = g_{ij}, \quad i \leq j;\ i, j = 1, \ldots, n.$$

It means geometrically that all lengths and angles are invariant.

Example 4.4.2. The group of conformal motions has for

$$\frac{n(n+1)}{2} - 1$$

independent invariants the ratios, e.g. (assuming that $g_{11} \neq 0$)

$$J_{ik} = \frac{g_{ik}}{g_{11}}, \quad i \leq k;\ i = 1, \ldots, n;\ k = 2, \ldots, n.$$

It means geometrically that all angles are invariant.

An arbitrary multi-parameter group G of transformations in V_n does not have, in general, any invariants except a trivial invariant $J = \text{const}$. Therefore, we single out the cases when it does by the following definition.

Definition 4.4.2. Let G be a group of transformations in a Riemannian space V_n. We say that G is a group of *generalized motions* if it has at least one non-trivial invariant $J(\|g_{ij}(\bar{x})\|)$.

The following theorem provides the necessary and sufficient condition for G to be a group of *generalized motions* and gives the number of invariants.

Theorem 4.4.1. Let G_r be an r-parameter group of transformations (4.1.1) with basic generators

$$X_\nu = \xi^i_{(\nu)}(x)\frac{\partial}{\partial x^i}, \quad \nu = 1, 2, \ldots, r. \tag{4.4.2}$$

The group G_r has precisely

$$\frac{n(n+1)}{2} - \delta(V_n, G_r) \tag{4.4.3}$$

functionally independent invariants in V_n. Here

$$\delta(V_n, G_r) = \text{rank}\|\mathcal{L}_{(\nu)}g_{ij}\|, \tag{4.4.4}$$

where

$$\mathcal{L}_{(\nu)}g_{ij} = \xi_{(\nu)i,j} + \xi_{(\nu)j,i} = \xi^k_{(\nu)}\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k_{(\nu)}}{\partial x^j} + g_{jk}(x)\frac{\partial \xi^k_{(\nu)}(x)}{\partial x^i}$$

is the Lie derivative (cf. (4.2.4)), and $\|\mathcal{L}_{(\nu)}g_{ij}\| \equiv \|\xi_{(\nu)i,j} + \xi_{(\nu)j,i}\|$ is the matrix whose rows are indexed by ν and columns by the double index ij :

$$\|\mathcal{L}_{(\nu)}g_{ij}\| = \begin{Vmatrix} \mathcal{L}_{(1)}g_{11} & \mathcal{L}_{(1)}g_{12} & \cdots & \mathcal{L}_{(1)}g_{nn} \\ \mathcal{L}_{(2)}g_{11} & \mathcal{L}_{(2)}g_{12} & \cdots & \mathcal{L}_{(2)}g_{nn} \\ \vdots & \vdots & & \vdots \\ \mathcal{L}_{(r)}g_{11} & \mathcal{L}_{(r)}g_{12} & \cdots & \mathcal{L}_{(r)}g_{nn} \end{Vmatrix}.$$

Thus, G_r is a group of generalized motions in V_n if

$$\text{rank}\|\mathcal{L}_{(\nu)}g_{ij}\| < \frac{n(n+1)}{2}.$$

Definition 4.4.3. The number $\delta(V_n, G_r)$ defined by the equation (4.4.4) is called the *defect of the group G_r of generalized motions in V_n*.

4.4.2 Invariant family of spaces

Let G be a group of generalized motions in V_n. We introduce the *invariant family of spaces*, denoted by $\boldsymbol{G(V_n)}$, by the following definition.

Definition 4.4.4. The *invariant family of spaces* $\boldsymbol{G(V_n)}$ is the set of n-dimensional Riemannian spaces determined by the following properties:
(i) $V_n \in \boldsymbol{G(V_n)}$,
(ii) $\boldsymbol{G(V_n^*)} \subset \boldsymbol{G(V_n)}$ for every $V_n^* \in \boldsymbol{G(V_n)}$,
(iii) $\boldsymbol{G(V_n)}$ is the minimal set obeying (i—ii).

The set $\boldsymbol{G(V_n)}$ depends on $\delta = \delta(V_n, G)$ arbitrary functions of x.

Example 4.4.3. For the group of isometric motions we have $\mathcal{L}_{(\nu)} g_{ij} = 0$, and hence $\delta(V_n, G) = 0$. Consequently, the invariant family comprises only one space, namely $\boldsymbol{G(V_n)} = V_n$.

Example 4.4.4. For the group of conformal motions the generalized Killing equations

$$\mathcal{L}_{(\nu)} g_{ij} = \phi_{(\nu)}(x) g_{ij}$$

yield $\delta(V_n, G) = 1$. Consequently, the invariant family involves one arbitrary function, $\sigma(x)$, and comprises all spaces conformal to the given space V_n, i.e. $\boldsymbol{G(V_n)} = \{V_n^*\}$, where

$$V_n^* : \quad g^*_{ij}(x) = \sigma(x) g_{ij}(x).$$

Example 4.4.5. Non-conformal motions with the defect $\delta = 1$ in the Euclidean space $\mathbb{R}^4 : g_{ij} = \delta_{ij}$. The following groups have the defect $\delta = 1 :$

$$G_4 : \ X_1 = h(x^4)\left(\frac{\partial}{\partial x^l} - x^l \frac{\partial}{\partial x^4}\right), \quad h(x^4) \text{ is any fixed function,}$$

$$X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = \frac{\partial}{\partial x^3}, \quad X_{23} = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}.$$

$$G_5 : \ X_1 = \exp(x^4)\left(\frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^4}\right), \quad X_i = \frac{\partial}{\partial x^i} \quad (i = 2, 3, 4),$$

$$X_{23} = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}.$$

$$G_\infty : \ X_k = \frac{\partial}{\partial x^k}, \ X_{kl} = x^l \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^l}, \quad X_f = f(x^4) \frac{\partial}{\partial x^4}.$$

In G_∞, we have $k, l = 1, 2, 3$ and $f(x^4)$ is an arbitrary (not fixed) function.

All three groups have the following 9 invariants:

$$J_{ik} = g_{ik}, \quad i = 1, \ldots, 4; \ k = 1, 2, 3.$$

Accordingly, the invariant family $\boldsymbol{G}(\mathbb{R}^4)$ for all three groups is determined by the metric tensor

$$\|g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma(x) \end{Vmatrix}.$$

Remark 4.4.1. The condition (ii) of Definition 4.4.4 implies that $\delta(V_n^*, G) \leq \delta(V_n, G)$ for any $V_n^* \in \boldsymbol{G}(\boldsymbol{V_n})$. Of particular interest are the following two extremal situations.

(1°) There exists $V_n^* \in \boldsymbol{G}(\boldsymbol{V_n})$ with $\delta(V_n^*, G) = 0$. This means that the group G of motions in V_n is a group of isometric motions in some member V_n^* of the invariant family $\boldsymbol{G}(\boldsymbol{V_n})$.

(2°) We have $\delta(V_n^*, G) = \delta(V_n, G)$ for all $V_n^* \in \boldsymbol{G}(\boldsymbol{V_n})$, and hence $G(V_n^*) = \boldsymbol{G}(\boldsymbol{V_n})$ for all $V_n^* \in \boldsymbol{G}(\boldsymbol{V_n})$.

Remark 4.4.2. Let G be the group of conformal motions in V_n. We will call G a trivial conformal group in V_n if the condition (1°) of Remark 4.4.1 holds, and a nontrivial conformal group otherwise. In the latter case V_n is called a *space with nontrivial conformal group.* See also Section 6.2.1.

Exercises

Exercise 4.1. Find the group of isometric motions in the three-dimensional Euclidean space $\mathbb{R}^3$ in the Cartesian coordinates, i.e. $g_{ij} = \delta_{ij}$.

Exercise 4.2. Find the expression of the infinitesimal isometric motions in the Euclidean space $\mathbb{R}^3$ referred to the spherical coordinates (3.2.12). Verify, in these coordinates, that the Killing equations are satisfied.

Exercise 4.3. Find the group of isometric motions in the Minkowski space with the metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

The second part of the book provides a flexible text for students and their teachers, spanning a variety of topics on applications of the Riemannian geometry in the theory of partial differential equations with emphasis on hyperbolic equations.

Chapter 5

Riemannian spaces associated with linear PDEs

5.1 Covariant form of second-order equations

Consider second-order linear differential equations

$$g^{ij}(x)u_{ij} + b^k(x)u_k + c(x)u = 0 \tag{5.1.1}$$

with non-degenerate principal part, i.e.

$$\det\|g^{ij}(x)\| \neq 0.$$

We denote by $g_{ij}(x)$ the entries of the inverse matrix

$$\|g_{ij}(x)\| = \|g^{ij}(x)\|^{-1}$$

and consider the Riemannian space V_n with the metric

$$ds^2 = g_{ij}(x)dx^i dx^j. \tag{5.1.2}$$

The space V_n is called the *Riemannian space associated with* Eq. (5.1.1).

Since Eq. (5.1.1) can be multiplied by any non-vanishing function, e.g. by $\mathrm{e}^{2\sigma(x)}$, we actually associate to Eq. (5.1.1) the family of conformal Riemannian spaces $\widetilde{V}_n$ with the metric tensors (cf. (4.3.3)):

$$\widetilde{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)}g_{ij}(x). \tag{5.1.3}$$

Equation (5.1.1),

$$g^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + b^k(x)\frac{\partial u}{\partial x^k} + c(x)u = 0$$

can be written in the *covariant form* in the associated Riemannian space V_n. Namely, according to Eqs. (3.2.20) and (3.2.21), the second covariant derivative is given by

$$u_{,ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}. \tag{5.1.4}$$

Therefore, we rewrite Eq. (5.1.1) in the form

$$g^{ij}(x)\left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right) + \left(b^k + g^{ij}\Gamma^k_{ij}\right)\frac{\partial u}{\partial x^k} + c(x)u = 0$$

and arrive at the following covariant form of Eq. (5.1.1):

$$g^{ij}u_{,ij} + a^k u_{,k} + cu = 0. \tag{5.1.5}$$

Here $u_{,k} = \partial u/\partial x^k$ is a covariant vector, and the coefficients

$$a^k = b^k + g^{ij}\Gamma^k_{ij}, \tag{5.1.6}$$

unlike b^k, behave as a contravariant vector in V_n (Exercise 5.1).

In particular, if $c = 0$ and $a^k = 0, \ k = 1, \ldots, n$, Eq. (5.1.5) is written in the form

$$\Delta_2 u \equiv g^{ij}u_{,ij} = 0. \tag{5.1.7}$$

In Riemannian geometry, the covariant expressions

$$\Delta_1 u = g^{ij}\, u_{,i}u_{,j} \equiv g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} \tag{5.1.8}$$

and

$$\Delta_2 u = g^{ij}\, u_{,ij} \equiv g^{ij}\left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}\right) \tag{5.1.9}$$

are called *Beltrami's differential parameters* of the first and second order, respectively.

The covariant form (5.1.5) is convenient, e.g. for writing differential equations in different coordinates.

Example 5.1.1. Let us find the expression of the Laplace equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{5.1.10}$$

in the spherical coordinates $x^1 = r, \ x^2 = \theta, \ x^3 = \phi$, where

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

Since the Laplace equation (5.1.10) is written in the Cartesian coordinates in $\mathbb{R}^3$, where $g^{ij} = \delta^{ij}, \ g_{ij} = \delta_{ij}$, all Christoffel symbols vanish, and the Laplacian Δu in Eq. (5.1.10) is identical with Beltrami's parameter $\Delta_2 u$ given in (5.1.7) when $g^{ij} = \delta^{ij}$:

$$\delta^{ij}u_{,ij} = 0.$$

Hence, $\Delta u = \Delta_2 u$ in any coordinate system. Therefore we take the Laplace equation in the covariant form

$$\Delta_2 u \equiv g^{ij}u_{,ij} = 0. \tag{5.1.11}$$

and consider it in the spherical coordinates. As shown in Example 3.2.1, the metric tensor is written in the spherical coordinates as

$$\|g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\,\sin\theta)^2 \end{Vmatrix}, \tag{5.1.12}$$

and hence $\|g^{ij}\| = \|g_{ij}\|^{-1}$ has the form

$$\|g^{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\,\sin\theta)^{-2} \end{Vmatrix}. \tag{5.1.13}$$

For these metric tensors, the Christoffel symbols (3.2.19) are written as follows:

$$\begin{aligned} \Gamma^1_{jk} &= -\frac{1}{2}\frac{\partial g_{jk}}{\partial r}, \\ \Gamma^2_{jk} &= \frac{1}{2r^2}\left(\frac{\partial g_{2j}}{\partial x^k} + \frac{\partial g_{2k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial \theta}\right), \\ \Gamma^3_{jk} &= \frac{1}{2(r\sin\theta)^2}\left(\frac{\partial g_{3j}}{\partial x^k} + \frac{\partial g_{3k}}{\partial x^j}\right). \end{aligned} \tag{5.1.14}$$

However, we do not need all of them. Indeed, in our case Beltrami's differential parameter (5.1.9) has the form

$$\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{(r\sin\theta)^2}\frac{\partial^2 u}{\partial \phi^2} - \left(\Gamma^k_{11} + \frac{1}{r^2}\Gamma^k_{22} + \frac{1}{(r\sin\theta)^2}\Gamma^k_{33}\right)\frac{\partial u}{\partial x^k}.$$

One can readily obtain from (5.1.14) that $\Gamma^k_{11} = 0$, and that the only non-vanishing components of Γ^k_{22} and Γ^k_{33} are

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2\theta, \quad \Gamma^2_{33} = -\sin\theta\,\cos\theta. \tag{5.1.15}$$

Substituting (5.1.15) in the above expression for $\Delta_2 u$, we arrive at the following form of the Laplace equation in the spherical coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{(r\sin\theta)^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial \theta} = 0. \tag{5.1.16}$$

5.2 Conformally invariant equations

It has been shown by L. Ovsyannikov (55), Chapter 6, that the group admitted by Eq. (5.1.5) is a subgroup of the group of conformal transformations in the associated Riemannian space V_n. It also follows from his calculations that the following equation admits the whole conformal group:

$$g^{ij}\,u_{,ij} + \frac{n-2}{4(n-1)}Ru = 0, \tag{5.2.1}$$

where R is the scalar curvature of the space V_n.

Definition 5.2.1. Equation (5.1.5) is said to be conformally invariant if it admits the whole group of conformal transformations in the associated Riemannian space V_n.

Conformally invariant equations will be discussed in more detail in Chapter 6. In particular, it will be shown that Eq. (5.2.1) is only in the conformally invariant equation in the spaces V_4 of normal hyperbolic type (i.e. of signature $(- - - +)$) with nontrivial conformal group.

Exercises

Exercise 5.1. Show that the coefficients (5.1.6),

$$a^k = b^k + g^{ij}\Gamma^k_{ij}$$

in the covariant equation (5.1.5) behave as a contravariant vector.

Exercise 5.2. Rewrite directly the Laplace equation (5.1.11) in the spherical coordinates to obtain Eq. (5.1.16).

Exercise 5.3. Any linear equation in two independent variables can be written in a standard form in a certain domain even if the equation has variable coefficients. Hence, one defines the type (hyperbolic, elliptic and parabolic) of these equations in certain domains. In the case of $n \geq 3$ independent variables, unlike the case of two variables, the type of linear equations with variable coefficients are defined only at a fixed point. Explain the geometric reason in terms of the associated Riemannian spaces.

Chapter 6

Geometry of linear hyperbolic equations

6.1 Generalities

6.1.1 Covariant form of determining equations

We will write the linear second-order partial differential equations with n independent variables $x = (x^1, \ldots, x^n)$ in the covariant form (5.1.5),

$$F[u] \equiv g^{ij}(x)u_{,ij} + a^i(x)u_{,i} + c(x)u = 0, \tag{6.1.1}$$

where $u_{,i}$ and $u_{,ij}$ are the covariant derivatives in the associated Riemannian space V_n with the metric

$$ds^2 = g_{ij}(x)dx^i dx^j. \tag{6.1.2}$$

Recall that $g_{ij}(x)$ and $g^{ij}(x)$ are the entries of the mutually inverse matrices,

$$\|g_{ij}(x)\| = \|g^{ij}(x)\|^{-1}.$$

They provide two representations, *covariant* and *contravariant,* of the metric tensor of the space V_n.

Any equation (6.1.1), due to its linearity and homogeneity, has the following obvious symmetries:

$$X_* = \tau(x)\frac{\partial}{\partial u}, \quad X_\diamond = u\frac{\partial}{\partial u}. \tag{6.1.3}$$

We will ignore them and consider only the non-trivial symmetries

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \sigma(x)u\frac{\partial}{\partial u}, \tag{6.1.4}$$

where $\sigma(x)$ is determined up to adding any constant because one can subtract from X the operator $X_\diamond$ multiplied by any constant factor.

It is shown in (55) that the condition that the operator (6.1.4) is admitted by Eq. (6.1.1) can be written in the form

$$X(F[u]) = f(x,u)F[u]$$

with an undetermined coefficient $f(x,u)$, where the action of the operator X is extended to the first and second derivatives by the usual prolongation procedure. Furthermore, it is demonstrated that the system of determining equations obtained from the above condition can be written in terms of the invariants

$$K_{ij} = a_{i,j} - a_{j,i} \quad (i,j = 1,\dots,n), \tag{6.1.5}$$

$$H = -2c + a^i_{,i} + \frac{1}{2}a^i a_i + \frac{n-2}{2(n-1)}R. \tag{6.1.6}$$

Namely, the determining equations are reduced in (55), Chapter 6, to the following *covariant equations:*

$$\xi_{i,j} + \xi_{j,i} = \mu(x) g_{ij}, \tag{6.1.7}$$

$$(K_{il}\xi^l)_{,j} - (K_{jl}\xi^l)_{,i} = 0, \tag{6.1.8}$$

$$\xi^k H_{,k} + \mu H = 0, \tag{6.1.9}$$

$$\frac{\partial}{\partial x^i}\Big(\sigma + \frac{n-2}{2}\mu + a_k\xi^k\Big) + \xi^k K_{ik} = 0. \tag{6.1.10}$$

Here R is the scalar curvature of V_n, indices i,j run over the values $1,\dots,n$. The covariant vectors $\xi_i,\ a_i$ are determined by the equations

$$\xi_i = g_{ij}\,\xi^j, \quad a_i = g_{ij}\,a^j. \tag{6.1.11}$$

Equations (6.1.7) are the generalized Killing equations (4.3.2) for the conformal group in V_n. Equations (6.1.8) and (6.1.9) impose additional constraints on the operator (6.1.4). Hence, the *group admitted by* Eq. (6.1.1) *is a subgroup of the conformal group* or at most the conformal group itself. Once $\xi^i(x)$ and $\mu(x)$ are determined from Eqs. (6.1.7)–(6.1.9), one can find $\sigma(x)$ by solving Eqs. (6.1.10). Solvability of the over-determined system of n equations (6.1.10) for the unknown function σ is guaranteed by Eqs. (6.1.8).

Definition 6.1.1. Equations (6.1.1) admitting the whole conformal group of the space V_n is called a *conformally invariant equation* in V_n.

6.1.2 Equivalence transformations

The linearity and homogeneity of Eq. (6.1.1) do not alter under the *equivalence transformations* defined as follows.

Definition 6.1.2. Two equations of the form (6.1.1) are said to be *equivalent* if they are obtained from each other by the following transformations called *equivalence transformations:*

(a) Arbitrary change of coordinates

$$x'^i = x'^i(x), \quad i = 1,\dots,n, \tag{6.1.12}$$

(b) Linear transformation of the dependent variable,

(c) Multiplication of Eq. (6.1.1) by a non-vanishing function.

It is convenient to replace (b) and (c) by their combinations

$$\overline{F}[u] = \mathrm{e}^{-\nu(x)}\, F[\mathrm{e}^{\nu(x)}\, u] \tag{6.1.13}$$

and

$$\widetilde{F}[u] = \mathrm{e}^{-2\lambda(x)}\, F[u]. \tag{6.1.14}$$

The qualification of the quantities (6.1.5) and (6.1.6) as *invariants* reflects the following properties of K_{ij} and H proved in (55), §29.

Theorem 6.1.1. The quantities H and K_{ij} are a scalar and a covariant 2-tensor, respectively. It means that they behave under the change of coordinates (6.1.12) as follows (cf. (3.1.11)):

$$H' = H, \quad K'_{ij} = K_{ml}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}. \tag{6.1.15}$$

Furthermore, they are invariant under the transformation (6.1.13):

$$\overline{H} = H, \quad \overline{K}_{ij} = K_{ij}, \tag{6.1.16}$$

and undergo the following transformations under the transformation (6.1.14):

$$\widetilde{H} = \mathrm{e}^{-2\lambda(x)}\, H, \quad \widetilde{K}_{ij} = K_{ij}. \tag{6.1.17}$$

6.1.3 Existence of conformally invariant equations

Theorem 6.1.2. A conformally invariant equation exists in any space V_n.

Proof. Let us take Eq. (6.1.1) with $a^i = 0$ $(i = 1, \ldots, n)$. Then Eqs. (6.1.5), (6.1.6) yield

$$K_{ij} = 0, \quad H = -2c + \frac{n-2}{2(n-1)}R.$$

Now we can set $H = 0$ by taking

$$c = \frac{n-2}{4(n-1)}R.$$

The corresponding equation (6.1.1) has the form (5.2.1):

$$\Diamond[u] = 0, \quad \text{where} \ \ \Diamond[u] = g^{ij}\, u_{,ij} + \frac{n-2}{4(n-1)}\, Ru. \tag{6.1.18}$$

Let G be the group of conformal motions in V_n. The generators

$$X = \xi^i(x)\frac{\partial}{\partial x^i} + \sigma(x)u\frac{\partial}{\partial u} \tag{6.1.4}$$

of this group satisfy the generalized Killing equations (6.1.7):

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} = \mu(x) g_{ij}\,. \tag{6.1.19}$$

Furthermore, Equations (6.1.8) and (6.1.9) are satisfied because $K_{ij} = 0,\ H = 0$. Finally, Equation (6.1.10) serves for determining the coefficient $\sigma(x)$ in the operator (6.1.4). Namely, invoking that this coefficient should be determined up to adding a constant, we have from Eq. (6.1.10):

$$\sigma = \frac{2-n}{2}\,\mu. \tag{6.1.20}$$

We conclude that Eq. (6.1.18) is conformally invariant. Its conformal symmetries (6.1.4) are determined by Eqs. (6.1.19) and (6.1.20). □

The conformally invariant operator $\diamondsuit[u]$ defined by Eq. (6.1.18) has remarkable properties. Two of them are given by the following statements.

Theorem 6.1.3. Equation (6.1.1) is equivalent to Eq. (6.1.18) if and only if its invariants (6.1.5) and (6.1.6) vanish (compare with Theorem 5.3.1 in (35)):

$$K_{ij} = 0, \quad H = 0. \tag{6.1.21}$$

The proof of this theorem is essentially based on the following statement.

Lemma 6.1.1. Let Eq. (6.1.1),

$$g^{ij}(x) u,_{ij} + a^i(x) u,_i + c(x) u = 0, \tag{6.1.22}$$

satisfy the condition that its invariants (6.1.5) vanish, $K_{ij} = 0$. Then Eq. (6.1.1) can be reduced by an appropriate equivalence transformation (6.1.13) to an equation with the coefficients $a^i(x) = 0,\ i = 1, \ldots, n$.

Proof. Eq. (6.1.5) for K_{ij} can be written in the form (see Exercise 3.11)

$$K_{ij} = \frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}.$$

Then the equation $K_{ij} = 0$ becomes identical with the well-known necessary and sufficient condition for a_i to have the form

$$a_i(x) = \frac{\partial \phi(x)}{\partial x^i}.$$

On the other hand, the reckoning shows that the equivalence transformation (6.1.13) maps Eq. (6.1.5) to an equation with

$$\bar{a}_i(x) = a_i(x) + 2\,\frac{\partial \nu(x)}{\partial x^i}.$$

Therefore, using the transformation (6.1.13), $\overline{F}[u] = \mathrm{e}^{-\nu(x)}\, F[\mathrm{e}^{\nu(x)}\, u]$, with $\nu(x) = -\phi(x)/2$ we obtain $\bar{a}_i(x) = 0$. Invoking the connection (6.1.11) between the covariant and contravariant components of vectors, we conclude that $\bar{a}^i(x) = 0$ $(i = 1, \ldots, n)$, thus completing the proof. $\square$

Theorem 6.1.4. The operators $\diamondsuit[u]$ and $\widetilde{\diamondsuit}[u]$ in conformal spaces V_n and $\widetilde{V}_n$ with metric tensors $g_{ij}(x)$ and $\tilde{g}_{ij}(x) = e^{2\theta(x)} g_{ij}(x)$, respectively, are connected by the equation

$$\widetilde{\diamondsuit}[u] = e^{-\frac{n+2}{2}\,\theta(x)} \diamondsuit[u e^{\frac{n-2}{2}\,\theta(x)}]. \tag{6.1.23}$$

6.2 Spaces with nontrivial conformal group

6.2.1 Definition of nontrivial conformal group

Definition 6.2.1. A group G of conformal motions in V_n is called *trivial* if G is the group of isometric motions in V_n or in some space V_n^* conformal to V_n. If the conformal group is nontrivial, we say that V_n is a space with nontrivial conformal group. In other words, V_n is a space with nontrivial conformal group if V_n and all conformal spaces V_n^* possess the conformal group that is wider than the group of isometric motions.

Flat spaces, and hence all conformally flat spaces have nontrivial conformal groups. The following theorem ((68), (65), the proof is given also in (28), Section 8.4) states that in the case of definite metric, the conformally flat spaces are the only Riemannian spaces with nontrivial conformal group.

Theorem 6.2.1. A space V_n of dimension $n \geq 3$ with a definite metric has a nontrivial conformal group if and only if V_n is conformally flat.

6.2.2 Classification of four-dimensional spaces

Our emphasis in this chapter is on hyperbolic equations with $n = 4$ independent variables. Consequently, we will deal with four-dimensional spaces of normal hyperbolic type, i.e. with Riemannian spaces V_4 whose metric (6.1.2) has the signature $(- - -+)$.

A classification of spaces V_4 of normal hyperbolic type with nontrivial conformal group due to R.F. Biljalov is discussed in (58), Chapter VII. This result can be formulated in the following convenient form.

Lemma 6.2.1. A four-dimensional space V_4 of normal hyperbolic type is a space with nontrivial conformal group if and only if its contravariant metric tensor g^{ij} can be reduced, by passing to a conformal space and choosing an appropriate coordinate system

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = t,$$

to the form given by the entries of the following matrix:

$$\|g^{ij}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -f(x-t) & -\varphi(x-t) & 0 \\ 0 & -\varphi(x-t) & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f - \varphi^2 > 0. \tag{6.2.1}$$

Conversely, any metric of the form (6.2.1) defines a space with nontrivial conformal group. The order of the conformal group in these spaces can equal to 6, 7 or 15 (if V_4 is conformally flat).

In the case of arbitrary functions $f(x-t)$ and $\varphi(x-t)$ satisfying only the hyperbolicity condition $f - \varphi^2 > 0$, the space V_4 with the metric tensor (6.2.1) has the 6-parameter conformal group with the generators

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \\ &X_4 = (t+x)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\ &X_5 = y\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) - F(x-t)\frac{\partial}{\partial y} - \Phi(x-t)\,\frac{\partial}{\partial z}, \\ &X_6 = z\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) - \Phi(x-t)\frac{\partial}{\partial y} + (t-x)\frac{\partial}{\partial z}, \end{aligned} \tag{6.2.2}$$

where

$$F(\sigma) = \int f(\sigma)\,d\sigma, \quad \Phi(\sigma) = \int \varphi(\sigma)\,d\sigma. \tag{6.2.3}$$

In particular cases, e.g. when

$$\varphi = 0, \quad f(x-t) = \mathrm{e}^{x-t},$$

the space has a 7-dimensional conformal group.

The inverse matrix to the matrix (6.2.1) has the form

$$\|g_{ij}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\Delta} & -\dfrac{\varphi}{\Delta} & 0 \\ 0 & -\dfrac{\varphi}{\Delta} & \dfrac{f}{\Delta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Delta = \varphi^2 - f. \tag{6.2.4}$$

Substituting (6.2.1) and (6.2.4) in (3.2.19), one obtains the following non-vanishing Christoffel symbols (and those given by the symmetry $\Gamma^k_{ji} = \Gamma^k_{ij}$):

$$\begin{aligned}
&\Gamma^1_{22} = \frac{1}{2}\left(\frac{1}{\Delta}\right)', \quad \Gamma^1_{23} = -\frac{1}{2}\left(\frac{\varphi}{\Delta}\right)', \quad \Gamma^1_{33} = \frac{1}{2}\left(\frac{\varphi^2}{\Delta}\right)', \\
&\Gamma^2_{12} = -\Gamma^2_{24} = \frac{1}{2}\left[\Delta\left(\frac{1}{\Delta}\right)' + \frac{\varphi\varphi'}{\Delta}\right], \\
&\Gamma^2_{13} = -\Gamma^2_{34} = -\varphi\Gamma^2_{12} - \frac{\varphi'}{2}, \\
&\Gamma^3_{12} = -\Gamma^3_{24} = \frac{\varphi'}{2\Delta}, \quad \Gamma^3_{13} = -\Gamma^3_{34} = -\frac{\varphi\varphi'}{2\Delta}, \\
&\Gamma^4_{ij} = \Gamma^1_{ij} \quad (i, j = 2, 3),
\end{aligned} \tag{6.2.5}$$

where the prime indicates the differentiation, e.g. $\varphi' = d\varphi(\sigma)/d\sigma$.

The following statement ((46), see also (28), Section 8.5) isolates among the spaces V_4 with the metric tensor (6.2.1) those which are conformally-flat, i.e. conformal to the Minkowski space.

Theorem 6.2.2. The space V_4 with the metric tensor (6.2.1) is conformally-flat only in the following three cases:

$$\begin{aligned}
&\text{(i)} f = \frac{\beta\varphi}{\sqrt{\beta^2 - 1}} + \frac{4\alpha^2 - \beta^2}{4(\beta^2 - 1)}, \quad \varphi = \frac{\alpha \tan[a(x - t) + b]}{\sqrt{1 + \beta^2 \tan^2[a(x - t) + b]}} + \frac{\beta}{2\sqrt{\beta^2 - 1}}. \\
&\text{(ii)} f = \alpha^2, \qquad \varphi = \frac{\alpha[a(x - t) + b]}{\sqrt{1 + [a(x - t) + b]^2}}. \\
&\text{(iii)} f = [a(x - t) + b]^{-2} + \alpha^2, \quad \varphi = \alpha.
\end{aligned} \tag{6.2.6}$$

Here a, b, α, and β are any constants (they are not the same in the cases (i), (ii), (iii)) satisfying the hyperbolicity condition $f - \varphi^2 > 0$.

The proof requires the solution of the system of second-order ordinary differential equations for the functions $f(x-t)$ and $\varphi(x-t)$. These differential equations are obtained by nullifying the Weil tensor (4.3.5) of the space V_4 the metric tensor (6.2.1), in accordance with Theorem 4.3.1.

The result is particularly simple in the case $\varphi = 0$. Then, according to the first line in Eqs. (6.2.6), the metric tensor (6.2.1) is conformally equivalent to the Minkowski metric only if

$$f = [a(x-t) + b]^{-2}$$

with constants a, b not vanishing simultaneously.

6.2.3 Uniqueness theorem

Theorem 6.2.3. Any conformally invariant equation (6.1.1) in a space V_4 of normal hyperbolic type with nontrivial conformal group is equivalent to Eq. (6.1.18):

$$g^{ij}(x)u_{,ij} + \frac{1}{6}Ru = 0. \tag{6.2.7}$$

Proof. It is sufficient to provide the proof for the spaces with the metric tensor (6.2.1). We will use the first four operators from (6.2.2). Upon introducing the coordinates

$$x'^1 = x + t, \quad x'^2 = y, \quad x'^3 = z, \quad x'^4 = x - t,$$

these operators are written

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x'^1}, \quad X_2 = \frac{\partial}{\partial x'^2}, \quad X_3 = \frac{\partial}{\partial x'^3}, \\ X_4 &= 2x'^1\frac{\partial}{\partial x'^1} + x'^2\frac{\partial}{\partial x'^2} + x'^3\frac{\partial}{\partial x'^3}. \end{aligned}$$

Substituting the coordinates of the operators X_1, X_2, X_3 in Eqs. (6.1.8) written in the form (see Exercise 6.6)

$$\xi^l\frac{\partial K_{ij}}{\partial x'^l} + K_{lj}\frac{\partial \xi^l}{\partial x'^i} + K_{il}\frac{\partial \xi^l}{\partial x'^j} = 0, \tag{6.2.8}$$

we obtain

$$\frac{\partial K_{ij}}{\partial x'^\alpha} = 0, \quad \alpha = 1, 2, 3.$$

Hence,

$$K_{ij} = K_{ij}(x'^4).$$

Now we apply Eqs. 6.2.8 to X_4 and obtain

$$K_{ij} = 0.$$

Let us solve Eq. (6.1.9). Considering the generalized Killing equations (6.1.19) one can verify that $\mu = 0$ for the operators X_α ($\alpha = 1, 2, 3$) and $\mu \neq 0$ for the operator X_4. Accordingly, using Eq. (6.1.9) for the operators X_1, X_2, X_3 one obtains

$$H = H(x'^4).$$

Now we substitute the coordinates of X_4 in Eq. (6.1.9) and have

$$H = 0.$$

Thus, Equations (6.1.21) are satisfied and Theorem 6.1.3 completes the proof.

□

6.2.4 On spaces with trivial conformal group

The space V_4 with the metric form

$$ds^2 = dt^2 - (1+t)dx^2 - dy^2 - dz^2, \quad t \geq 0, \tag{6.2.9}$$

provides and example of a space of normal hyperbolic type with trivial conformal group. Indeed, treating (x, y, z, t) as (x^1, x^2, x^3, x^4), we write the generalized Killing equations (6.1.19) in the form

$$\begin{gathered}
\frac{\partial \xi^1}{\partial x} + \frac{1}{2(1+t)}\xi^4 = \frac{\partial \xi^2}{\partial y} = \frac{\partial \xi^3}{\partial z} = \frac{\partial \xi^4}{\partial t} = \frac{\mu}{2},\\
(1+t)\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^2}{\partial x} = (1+t)\frac{\partial \xi^1}{\partial z} + \frac{\partial \xi^3}{\partial x} = (1+t)\frac{\partial \xi^1}{\partial t} - \frac{\partial \xi^4}{\partial x} = 0,\\
\frac{\partial \xi^2}{\partial z} + \frac{\partial \xi^3}{\partial y} = \frac{\partial \xi^2}{\partial t} - \frac{\partial \xi^4}{\partial y} = \frac{\partial \xi^3}{\partial t} - \frac{\partial \xi^4}{\partial z} = 0.
\end{gathered}$$

Solving these equations we obtain the five-parameter group of conformal transformations with the following generators:

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\\
&X_5 = \frac{1}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + (1+t)\frac{\partial}{\partial t}.
\end{aligned} \tag{6.2.10}$$

As mentioned in Lemma 6.2.1, the order of the conformal group in any space with nontrivial conformal group equals 6, 7 or 15. Therefore we conclude that the metric (6.2.9) defines a space with trivial conformal group. It means that there exists a conformal space $\widetilde{V}_4$ where the operators (6.2.9) generate the group of isometric motions, i.e. satisfy the Killing equations. The reckoning shows (see (28), Section 8.1) that the $\widetilde{V}_4$ has the metric

$$d\tilde{s}^2 = (1+t)^{-2}\big[dt^2 - (1+t)dx^2 - dy^2 - dz^2\big]. \tag{6.2.11}$$

Remark 6.2.1. In spaces with trivial conformal group there exist infinite number conformally invariant equations which are not equivalent to each other. Indeed, if V_4 is a space with trivial conformal group, there exists a conformal space $\widetilde{V}_4$ such that the conformal group G in V_4 becomes the group of isometric motions in $\widetilde{V}_4$ so that $\mu = 0$. Now we consider the equations (6.1.1) in $\widetilde{V}_4$ with the coefficients

$$a^i = 0, \quad i = 1, \cdots, 4,$$

and

$$H = \text{const.}$$

It is manifest from the determining equations (6.1.7)—(6.1.9) that these equations admit the group G, and hence, they are conformally invariant in V_4. Not all of these equations equivalent, e.g. the equations with $H = 0$ and with $H = 1$ because during equivalence transformations the quantity H can acquire at most a nonzero factor.

6.3 Standard form of second-order equations

It is a long tradition to teach partial differential equations without using concepts of Riemannian geometry. Accordingly, it is not clarified why, e.g. the commonly known standard forms for hyperbolic, parabolic and elliptic second-order equations are given exclusively in the case of two independent variables. The second-order linear equations with $n > 2$ variables are classified into the hyperbolic, parabolic and elliptic types at a fixed point only. Use of Riemannian geometry explains a geometric reason for this difference. Namely, the possibility of reducing elliptic and hyperbolic equations in two independent variables to the well-known standard form

$$u_{xx} \pm u_{yy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0 \tag{6.3.1}$$

is based on Remark 4.3.2 according to which all two-dimensional Riemannian spaces V_2 are conformally flat, i.e. their metric can be written in the diagonal form $ds^2 = \lambda(x, y)\,[(dx)^2 \pm (dy)^2]$ in appropriate variables x, y. The associated differential equation will be written in these variables precisely in the standard form (6.3.1).

It is obvious that the reduction to the standard form is also possible in conformally flat spaces of any dimension.

Moreover, the geometric vantage point shows that one can obtain a standard form of hyperbolic equations (6.1.1) with n independent variables not only in the case when the associated Riemannian space V_n is conformally flat, but also for all spaces V_n with nontrivial conformal group.

We will restrict our consideration to the physically significant case $n = 4$.

6.3.1 Curved wave operator in V_4 with nontrivial conformal group

Consider the conformally invariant equation (6.2.7) in *curved space-times*, i.e. in Riemannian spaces of normal hyperbolic time, in general, with non-vanishing curvature tensor R^l_{ijk}. It is natural to call the left-hand side of Eq. (6.2.7) the *curved wave operator* [1] and denote it by

$$\diamondsuit[u] = g^{ij}(x)u_{,ij} + \frac{1}{6}Ru. \tag{6.3.2}$$

The aim of this section is to reduce the wave equation

$$\diamondsuit[u] = 0 \tag{6.3.3}$$

in the spaces V_4 with nontrivial conformal group to the simplest form, which will be considered as a standard form.

Theorem 6.3.1. The wave equation (6.3.3) in the spaces V_4 with nontrivial conformal group can be reduced by equivalence transformations to the following *standard form:*

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0, \tag{6.3.4}$$

where

$$f(x-t) - \varphi^2(x-t) > 0.$$

Proof. It is sufficient to provide the proof for the spaces with the metric tensor (6.2.1). The scalar curvature (3.2.30) in this metric is written:

$$R = g^{ij}R_{ij} = R_{44} - R_{11} - fR_{22} - 2\varphi R_{23} - R_{33}.$$

Using the definition (3.2.29) of the Ricci tensor,

$$R_{ij} = \frac{\partial\Gamma^k_{ik}}{\partial x^j} - \frac{\partial\Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{mk},$$

and the following relations obtained from Eqs. (6.2.5):

$$\Gamma^i_{4j} - \Gamma^i_{1j}, \quad \frac{\partial\Gamma^k_{ij}}{\partial x^4} = -\frac{\partial\Gamma^k_{ij}}{\partial x^1} \quad (i,j,k = 1,\dots,4),$$

we find the required components of the Ricci tensor:

$$R_{11} = R_{44}, \quad R_{22} = R_{23} = R_{33} = 0.$$

1 This justifies the use of the symbol $\diamondsuit[u]$ in Eqs. (6.3.2) and (6.1.18).

Therefore $R = 0$, and hence the curved wave operator (6.3.2) in the metric (6.2.1) is written

$$\diamondsuit[u] = g^{ij}u_{,ij} \equiv g^{ij}u_{ij} - g^{ij}\Gamma^k_{ij}\,u_k. \tag{6.3.5}$$

Let us find the quantities

$$\Gamma^k = g^{ij}\Gamma^k_{ij}.$$

Equations (6.2.5) yield:

$$\Gamma^1 = \Gamma^4 = [\ln|\Delta|^{-1/2}]\,', \quad \Gamma^2 = \Gamma^3 = 0.$$

I provide here the calculation of Γ^1. Other components of Γ^k are computed likewise. We have:

$$\Gamma^1 = -\frac{1}{2}f\left(\frac{1}{\Delta}\right)' + \varphi\left(\frac{\varphi}{\Delta}\right)^1 - \frac{1}{2}\left(\frac{\varphi^2}{\Delta}\right)' = \frac{1}{2}(\varphi^2 - f)\left(\frac{1}{\Delta}\right)' = -\frac{1}{2}\frac{\Delta'}{\Delta}.$$

Hence,

$$\Gamma^1 = (\ln|\Delta|^{-1/2})'.$$

The calculation shows (see Proof of Lemma 6.1.1) that the equivalence transformation (6.1.13)

$$\overline{F}[u] = \mathrm{e}^{-\nu}F[u\mathrm{e}^{\nu}], \quad \text{where} \quad \nu = \ln|\Delta|^{1/4},$$

maps the operator (6.3.5) into the curved wave operator with $\bar{a}^i = \Gamma^i$:

$$\diamondsuit[u] = g^{ij}u_{,ij} + \bar{a}^i u_{,i} = g^{ij}u_{ij}.$$

Substituting here the tensor g^{ij} given by (6.2.1), we verify that the wave equation (6.3.3) is written in the form (6.3.4). □

6.3.2 Standard form of hyperbolic equations with nontrivial conformal group

The calculations of the previous section lead to the following result.

Theorem 6.3.2. The linear second-order hyperbolic equations (5.1.1):

$$g^{ij}(x)u_{ij} + b^k(x)u_k + c(x)u = 0$$

in the spaces V_4 with nontrivial conformal group can be reduced by equivalence transformations to the following *standard form:*

$$u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} + b^i u_i + cu = 0, \tag{6.3.6}$$

where

$$f(x-t) - \varphi^2(x-t) > 0$$

and b^i, c are arbitrary functions of x, y, z, t.

Exercises

Exercise 6.1. Verify that the operators (6.2.2) solve the generalized Killing equations (6.1.19).and single out the generators of the group of isometric motions.

Exercise 6.2. Single out from the operators (6.2.2) the generators of the group of isometric motions.

Exercise 6.3. Calculate the Christoffel symbols (6.2.5).

Exercise 6.4. Find the generators of the conformal group in the space with the metric tensor (6.2.1) when

$$\varphi = 0, \quad f(x-t) = \mathrm{e}^{x-t}.$$

Hint. There are seven linearly independent generators.

Exercise 6.5. Find the generators of the conformal group in the space with the metric tensor (6.2.1) when

$$\varphi = 0, \quad f(x-t) = \mathrm{e}^{-(x-t)^2}.$$

Hint. There are six linearly independent generators.

Exercise 6.6. Show that Eqs. (6.1.8) can be written in the form (6.2.8).

Hint. First verify that the following equations hold:

$$\frac{\partial K_{ij}}{\partial x^l} + \frac{\partial K_{jl}}{\partial x^i} + \frac{\partial K_{li}}{\partial x^j} = 0.$$

Chapter 7

Solution of the initial value problem

7.1 The Cauchy problem

We will discuss here the Cauchy problem for the wave equation (6.3.3) in the spaces V_4 with nontrivial conformal group. The curved wave operator will be written in the form

$$\diamondsuit[u] = u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz}. \tag{7.1.1}$$

We will derive the solution of the general Cauchy problem

$$\diamondsuit[u] = 0, \quad u|_{t=0} = g(x,y,z), \quad u_t|_{t=0} = h(x,y,z) \tag{7.1.2}$$

with smooth initial data $g(x,y,z)$ and $h(x,y,z)$.

7.1.1 Reduction to a particular Cauchy problem

Lemma 7.1.1. The solution of the general Cauchy problem (7.1.2) can be reduced to the solution of the particular Cauchy problem

$$\diamondsuit[u] = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = h(x,y,z). \tag{7.1.3}$$

Proof. The operator (7.1.1) commutes with $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, i.e.

$$\diamondsuit\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\diamondsuit. \tag{7.1.4}$$

Therefore, if v and w solve the particular Cauchy problem (7.1.3) with the data $v_t|_{t=0} = g(x,y,z)$, and $w_t|_{t=0} = g_x(x,y,z) - h(x,y,z)$, respectively:

$$\begin{aligned} \diamondsuit[v] = 0, &\quad v|_{t=0} = 0, \quad v_t|_{t=0} = g, \\ \diamondsuit[w] = 0, &\quad w|_{t=0} = 0, \quad w_t|_{t=0} = g_x - h, \end{aligned}$$

then the function

$$u = v_t + v_x - w \tag{7.1.5}$$

provides the solution to the Cauchy problem (7.1.2). $\square$

7.1.2 Fourier transform and solution of the particular Cauchy problem

The Fourier transformation

$$\hat{u}(x,\lambda,\mu,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-i(\lambda y+\mu z)} u(x,y,z,t)dydz,$$

maps the Cauchy problem (7.1.3) to the Cauchy problem

$$\hat{L}[\hat{u}] = 0,\ \hat{u}|_{t=0} = 0, \quad \hat{u}_t|_{t=0} = \hat{h}(x,\lambda,\mu) \tag{7.1.6}$$

for the operator

$$\hat{L}[\hat{u}] = \hat{u}_{tt} - \hat{u}_{xx} + (f\lambda^2 + 2\varphi\lambda\mu + \mu^2)\hat{u} \tag{7.1.7}$$

with two independent variables t, x and the parameters λ, μ.

We rewrite the operator (7.1.7) in the characteristic variables

$$\alpha = x - t, \quad \beta = x + t$$

and rewrite the equation $\hat{L}[\hat{u}] = 0$ in the form

$$\hat{u}_{\alpha\beta} - \frac{1}{4}(f\lambda^2 + 2\varphi\lambda\mu + \mu^2)\hat{u} = 0. \tag{7.1.8}$$

One can readily verify that in the new independent variables

$$\bar{t} = \frac{1}{2}(x+t), \quad \bar{x} = -\frac{1}{2}[\lambda^2 F(x-t) + 2\lambda\mu\Phi(x-t) + \mu^2(x-t)].$$

Equation (7.1.8) is written as the telegraph equation

$$\hat{u}_{\bar{x}\bar{t}} + \hat{u} = 0$$

with the known Riemann's function

$$R(\bar{\xi},\bar{\tau};\bar{x},\bar{t}) = J_0\Big(\sqrt{4(\bar{t}-\bar{\tau})(\bar{x}-\bar{\xi})}\Big),$$

where J_o is the Bessel function. Returning to the old variables x and t, we obtain the following Riemann's function for the equation $\hat{L}[\hat{u}] = 0$:

$$R(\xi,\tau;x,t) = J_0\Big(\sqrt{(x-\xi+t-\tau)[\lambda^2(F_0-F)+2\lambda\mu(\Phi_0-\Phi)+\mu^2(\xi-x-\tau+t)]}\Big),$$

where

$$F_0 = F(\xi-\tau), \quad F = F(x-t),$$
$$\Phi_0 = \Phi(\xi-\tau), \quad \Phi = \Phi(x-t).$$

Having Riemann's function one can solve the Cauchy problem (7.1.6) using the formula
$$\hat{u}(x,\lambda,\mu,t) = \frac{1}{2}\int_{x-t}^{x+t} \hat{h}(\xi,\lambda,\mu)R(\xi,0;x,t)d\xi.$$
Substituting here the Fourier transform $\hat{h}(\xi,\lambda,\mu)$ of $h(\xi,y,z)$ we have:
$$\hat{u}(x,\lambda,\mu,t) = \frac{1}{4\pi}\int_{x-t}^{x+t} R(\xi,0;x,t)\left[\int_{\mathbb{R}^2} e^{-i(\lambda\eta+\mu\zeta)}h(\xi,\eta,\zeta)d\eta d\zeta\right]d\xi. \quad (7.1.9)$$

The inverse Fourier transform of the function (7.1.9) provides the following solution to the particular Cauchy problem (7.1.3):
$$u(x,y,z,t) = \frac{1}{4\pi}\int_{x-t}^{x+t} d\xi \int_{\mathbb{R}^2} I\cdot h(\xi,\eta,\zeta)d\eta d\zeta. \quad (7.1.10)$$
Here
$$I = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i[\lambda(\eta-y)+\mu(\zeta-z)]}J_0(k\sqrt{Q(\lambda,\mu)})d\lambda d\mu,$$
where $k=\sqrt{x+t-\xi}$ and $Q(\lambda,\mu)$ is the quadratic form
$$Q(\lambda,\mu) = \alpha^2\lambda^2 + 2b\lambda\mu + c^2\mu^2 \quad (7.1.11)$$
with the coefficients
$$a^2 = F(\xi) - F(x-t),\ b = \Phi(\xi) - \Phi(x-t),\ c^2 = \xi - (x-t).$$

7.1.3 Simplification of the solution

Let us rewrite the integral I in a more convenient form. The condition
$$-\Delta(\sigma) = f(\sigma) - \varphi^2(\sigma) > 0$$
guarantees that the quadratic form
$$q(\sigma;\lambda,\mu) = f(\sigma)\lambda^2 + 2\varphi(\sigma)\lambda\mu + \mu^2$$
having the discriminant $-\Delta(\sigma)$ is positive definite. Hence, the quadratic form (7.1.11) is also positive definite since it has the form
$$Q(\lambda,\mu) = \int_{x-t}^{\xi} q(\sigma;\lambda,\mu)d\sigma.$$
It follows that the discriminant $a^2c^2-b^2$ of the quadratic form $Q(\lambda,\mu)$ is positive. Taking this into account, we make the change of variables $\lambda,\mu;\eta,\zeta$:
$$\lambda = \frac{1}{a}\left(\bar{\lambda} - \frac{b\,\bar{\mu}}{\sqrt{a^2c^2-b^2}}\right),\ \mu = \frac{a\,\bar{\mu}}{\sqrt{a^2c^2-b^2}}$$

and

$$\bar{\eta} - \bar{y} = \frac{1}{a}(\eta - y), \quad \bar{\zeta} - \bar{z} = \frac{1}{\sqrt{a^2c^2 - b^2}}\left[a(\zeta - z) - \frac{b}{a}(\eta - y)\right]. \qquad (7.1.12)$$

In these variables we have:

$$Q = \bar{\lambda}^2 + \bar{\mu}^2,$$

$$\lambda(\eta - y) + \mu(\zeta - z) = \bar{\lambda}(\bar{\eta} - \bar{y}) + \bar{\mu}(\bar{\zeta} - \bar{z}),$$

$$d\eta\, d\zeta = \sqrt{a^2c^2 - b^2}\, d\bar{\eta}\, d\bar{\zeta}, \quad d\lambda\, d\mu = \frac{1}{\sqrt{a^2c^2 - b^2}}\, d\bar{\lambda}\, d\bar{\mu}.$$

Therefore the integral I becomes

$$I = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-i[\bar{\lambda}(\bar{\eta}-\bar{y})+\bar{\mu}(\bar{\zeta}-\bar{z})]} J_0\left(k\sqrt{\bar{\lambda}^2 + \bar{\mu}^2}\right) \frac{d\bar{\lambda}\, d\bar{\mu}}{\sqrt{a^2c^2 - b^2}}.$$

After standard calculation by using the well-known formula for the Fourier transform of spherically symmetric functions and properties of the Bessel function J_0 one finally obtains:

$$I = \frac{1}{\sqrt{a^2c^2 - b^2}} \int_0^\infty J_0(kr)J_0(\rho r) r dr = \frac{\delta(k - \rho)}{\rho\sqrt{a^2c^2 - b^2}},$$

where δ is Dirac's delta-function and

$$\rho = \sqrt{(\bar{\eta} - \bar{y})^2 + (\bar{\zeta} - \bar{z})^2}.$$

Now we can evaluate the inner integral in (7.1.10). We will use the polar coordinates ρ, θ on the plane of variables $\bar{\eta}, \bar{\zeta}$:

$$\bar{\eta} - \bar{y} = \rho\cos\theta, \ \bar{\zeta} - \bar{z} = \rho\sin\theta.$$

Substituting here the expressions (7.1.12) for $\bar{\eta} - \bar{y}$ and $\bar{\zeta} - \bar{z}$ we can obtain the functions $\eta(\rho, \theta)$ and $\zeta(\rho, \theta)$. Now we can calculate the mentioned integral. We have:

$$\int_{\mathbb{R}^2} I \cdot h(\xi, \eta, \zeta) d\eta\, d\zeta = \int_0^{2\pi} d\theta \int_0^\infty I \cdot h(\xi, \eta(\rho, \theta), \zeta(\rho, \theta))\sqrt{a^2c^2 - b^2}\, \rho d\rho$$

$$= \int_0^{2\pi} h(\xi, \eta(k, \theta), \zeta(k, \theta)) d\theta.$$

Determining the functions $\eta(k, \theta)$, $\zeta(k, \theta)$ from Eqs. (7.1.12) one obtains

$$\int_{\mathbb{R}^2} I \cdot h(\xi, \eta, \zeta) d\eta\, d\zeta = \int_0^{2\pi} h\left(\xi, y + ak\cos\theta, z + \frac{b}{a}\, k\cos\theta + \frac{\sqrt{a^2c^2 - b^2}}{a}\, k\sin\theta\right) d\theta.$$

Using this expression and introducing the quantities

$$\begin{aligned} A &= \sqrt{(x+t-\xi)[F(\xi)-F(x-t)]}, \\ B &= \frac{x+t-\xi}{A}[\Phi(\xi)-\Phi(x-t)], \\ C &= \sqrt{t^2-(x-\xi)^2-B^2}. \end{aligned} \tag{7.1.13}$$

one can write the solution (7.1.10) of the Cauchy problem (7.1.3) in the form

$$u(x,y,z,t) = T[h], \tag{7.1.14}$$

where

$$T[h] = \frac{1}{4\pi}\int_{x-t}^{x+t} d\xi \int_0^{2\pi} h(\xi, y + A\cos\theta, z + B\cos\theta + C\sin\theta)d\theta. \tag{7.1.15}$$

7.1.4 Verification of the solution

Let us verify (see (28), Section 12.3) that the function $u(x,y,z,t)$ defined by Eqs. (7.1.14) and (7.1.15) solves the Cauchy problem (7.1.3). It is obvious that the initial conditions are satisfied. Therefore it remains to check that the function (7.1.14) annuls the operator (7.1.1), i.e. solves the equation

$$\diamondsuit[u] = u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0. \tag{7.1.16}$$

We introduce the variables

$$\alpha = x - t, \quad \beta = x + t$$

and rewrite the operator $\diamondsuit[u]$ in the form

$$\diamondsuit[u] = 4u_{\alpha\beta} + f(\alpha)u_{yy} + 2\varphi(\alpha)u_{yz} + u_{zz}. \tag{7.1.17}$$

In these variables, the solution formula (7.1.14) becomes

$$u(\alpha,\beta,y,z) = -\frac{1}{4\pi}\int_{\alpha}^{\beta} d\xi \int_0^{2\pi} h(\xi, y + A\cos\theta, z + B\cos\theta + C\sin\theta)d\theta, \tag{7.1.18}$$

where the quantities (7.1.13) are written as follows:

$$\begin{aligned} A &= \sqrt{(\beta-\xi)[F(\xi)-F(\alpha)]}, \\ B &= \frac{\beta-\xi}{A}[\Phi(\xi)-\Phi(\alpha)], \\ C &= \sqrt{(\alpha+\xi)(\beta-\xi)-B^2}. \end{aligned} \tag{7.1.19}$$

One can verify that the functions (7.1.19) satisfy the relations

$$
\begin{aligned}
&A_\alpha A_\beta = -\frac{1}{4} f(\alpha), \quad AB_\beta = BA_\beta, \quad AC_\beta = CA_\beta, \\
&A_\alpha B_\beta + A_\beta B_\alpha = -\frac{1}{2}\varphi(\alpha), \quad B_\alpha B_\beta + C_\alpha C_\beta = -\frac{1}{4}, \\
&AA_{\alpha\beta} = A_\alpha B_\beta, \quad BB_{\alpha\beta} = B_\alpha B_\beta, \quad CC_{\alpha\beta} = C_\alpha C_\beta.
\end{aligned}
\tag{7.1.20}
$$

Differentiating (7.1.18) we obtain

$$
\begin{aligned}
u_{\alpha\beta} = &\frac{1}{4\pi}\int_\alpha^\beta \left\{ P + \int_0^{2\pi} \left[h_{yy} A_\alpha A_\beta \cos^2\theta \right.\right. \\
&+ h_{yz}((A_\alpha C_\beta + A_\beta C_\alpha)\cos\theta\sin\theta + (A_\alpha B_\beta + A_\beta B_\alpha)\cos^2\theta) \\
&\left.\left. + h_{zz}(B_\alpha B_\beta \cos^2\theta + (B_\alpha C_\beta + B_\beta C_\alpha)\cos\theta\sin\theta + C_\alpha C_\beta \sin^2\theta) \right] d\theta \right\} d\xi,
\end{aligned}
\tag{7.1.21}
$$

where

$$
P = \int_0^{2\pi} [h_y A_{\alpha\beta} cos\theta + h_z(B_{\alpha\beta}\cos\theta + C_{\alpha\beta}\sin\theta)]d\theta.
$$

We rewrite the expression for P using integration by parts, e.g.

$$
\begin{aligned}
\int_0^{2\pi} h_y \cos\theta d\theta &= \int_0^{2\pi} h_y d(\sin\theta) = -\int_0^{2\pi} \sin\theta d(h_y) \\
&= \int_0^{2\pi} [h_{yy} A \sin^2\theta + h_{yz}(B\sin^2\theta - C\sin\theta\cos\theta)]d\theta,
\end{aligned}
$$

and obtain

$$
\begin{aligned}
P = \int_0^{2\pi} &\{h_{yy} AA_{\alpha\beta}\sin^2\theta + h_{yz}[(AB_{\alpha\beta} + BA_{\alpha\beta})\sin^2\theta - (AC_{\alpha\beta} + CA_{\alpha\beta})\cos\theta \\
&\sin\theta] + h_{zz}[BB_{\alpha\beta}\sin^2\theta - (BC_{\alpha\beta} + CB_{\alpha\beta})\cos\theta\sin\theta + CC_{\alpha\beta}\cos^2\theta]\}d\theta.
\end{aligned}
$$

We substitute this expression for P in (7.1.21), use the relations (7.1.20) and obtain

$$
4u_{\alpha\beta} = \frac{1}{4\pi}\int_\alpha^\beta d\xi \int_0^{2\pi} -[f(\alpha)h_{yy} + 2\varphi(\alpha)h_{yx} + h_{zz}]d\theta. \tag{7.1.22}
$$

The computation of the derivatives of $u(\alpha, \beta, y, z)$ with respect to y and z is straightforward. The result of this computation combined with (7.1.22) yields the desired equation $\diamondsuit[u] = 0$.

7.1.5 Comparison with Poisson's formula

It is well known that in the case of the classical wave operator

$$\Box u = u_{tt} - u_{xx} - u_{yy} - u_{zz}, \tag{7.1.23}$$

the solution to the Cauchy problem

$$\Box u = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = h(x, y, z)$$

is given by Poisson's formula

$$u(x, y, z, t) = \frac{1}{4\pi t} \int_{S_t} h dS, \tag{7.1.24}$$

where the integral is taken over the two-dimensional sphere S_t having the radius t and the center at the point (x, y, z). Let us introduce the polar coordinates on the (y, z) plane. Then for the points $(\xi, \eta, \zeta) \in S_t$ we have

$$\xi = \xi, \quad \eta = y + \rho \cos\theta, \quad \zeta = z + \rho \sin\theta,$$

where the coordinates (ξ, ρ, θ) satisfy the conditions

$$(\xi - x)^2 + \rho^2 = t^2; \quad x - t \le \xi \le x + t, \quad 0 \le \theta \le 2\pi.$$

Now we have $dS = td\xi \, d\theta$, and Poisson's formula (7.1.24) becomes

$$u(x, y, z, t) = \frac{1}{4\pi} \int_{x-t}^{x+t} d\xi \int_0^{2\pi} h\Big(\xi, y + \sqrt{t^2 - (\xi - x)^2} \cos\theta, z + \sqrt{t^2 - (\xi - x)^2} \sin\theta\Big) d\theta. \tag{7.1.25}$$

Let us compare (7.1.25) with (7.1.14)-(7.1.15). The curved wave operator (7.1.1) coincides with the operator (7.1.23) when

$$f = 1, \quad \varphi = 0.$$

In this case

$$F(\sigma) = \sigma, \quad \Phi = \text{const}.$$

Therefore Eqs. (7.1.13) yield:

$$A = \sqrt{t^2 - (\xi - x)^2}, \quad B = 0, \quad C = \sqrt{t^2 - (\xi - x)^2}\,.$$

Now it is manifest that when $f = 1, \varphi = 0$, the solution (7.1.14) and (7.1.15) coincides with Poisson's formula (7.1.25).

7.1.6 Solution of the general Cauchy problem

The formulae (7.1.5) and (7.1.14) provide the following representation of the solution to the general Cauchy problem (7.1.2):

$$u = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) T[g] - T\left[\frac{\partial g}{\partial x} - h\right], \tag{7.1.26}$$

where the operator T is defined by Eq. (7.1.15).

7.2 Geodesics in spaces with nontrivial conformal group

In this section we will solve the equations for geodesics in the spaces V_4 with the tensor g^{ij} of the type (6.2.1). We will also calculate the square of geodesic distance $\Gamma(x_0, x)$ between points $x_0, x \in V_4$. It is useful in discussing the Huygens principle in spaces with nontrivial conformal group.

7.2.1 Outline of the approach

The general approach for computing the geodesic distance in a Riemannian space V_n is as follows. Let us assume that the geodesic line passing through the fixed point x_0 and an arbitrary point $x \in V_n$ in a vicinity of the point x_0 is parameterized by means of the arc s length counted from the point x_0. Then the coordinates x^i of the point x are functions $x^i = x^i(s)$ satisfying the system of differential equations (3.2.18),

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \quad (i = 1, \ldots, n), \tag{7.2.1}$$

and the initial conditions

$$x^i|_{s=0} = x_0^i, \quad \left.\frac{dx^i}{ds}\right|_{s=0} = \alpha^i \quad (i = 1, \ldots, n). \tag{7.2.2}$$

According to the definition (3.2.3) of the metric form in V_n,

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{3.2.3}$$

the constant vector $\alpha = (\alpha^1, \ldots, \alpha^n)$ satisfies the condition

$$g_{ij}(x_0)\alpha^i\alpha^j = 1. \tag{7.2.3}$$

Indeed, one has along the geodesic curve:

$$dx^i = \frac{dx^i(s)}{ds} ds.$$

Substituting this in (3.2.3), dividing the resulting equation by ds^2, then letting $s = 0$ and invoking the conditions (7.2.2), one arrives at Eq. (7.2.3).

Let

$$x^i = x^i(s; x_0, \alpha) \quad (i = 1, \ldots, n) \tag{7.2.4}$$

be the solution of the problem (7.2.1), (7.2.2). Solving Eqs. (7.2.4) with respect to α^i one obtains

$$\alpha^i = \psi^i(s; x_0, x) \quad (i = 1, \ldots, n), \tag{7.2.5}$$

when $|\alpha|$ is sufficiently small. Substituting the expressions (7.2.5) for α^i in Eq. (7.2.3) and solving the resulting equation

$$g_{ij}(x_0)\psi^i(s; x_0, x)\psi^j(s; x_0, x) = 1 \tag{7.2.6}$$

with respect to s, one obtains the square of the geodesic distance $s(x_0, x)$ between the points x_0 and x, i.e. the function

$$\Gamma(x_0, x) = [s(x_0, x)]^2.$$

7.2.2 Equations of geodesics in spaces with nontrivial conformal group

Let return to the spaces V_4 with the metric tensor (6.2.1). We will use the notation (6.2.3) for the primitives of f and φ the following abbreviations:

$$\begin{aligned}
&x_0 = (\xi, \eta, \zeta, \tau), \quad x = (x, y, z, t), \quad \alpha = (\alpha, \beta, \gamma, \delta),\\
&f_0 = f(\xi - \tau), \quad f = f(x - t), \quad \varphi_0 = \varphi(\xi - \tau), \quad \varphi = \varphi(x - t),\\
&F_0 = F(\xi - \tau), \quad F = F(x - t), \quad \Phi_0 = \Phi(\xi - \tau), \quad \Phi = \Phi(x - t),\\
&\Delta = \det \|g^{ij}\| = \varphi^2 - f, \quad \Delta_0 = \varphi_0^2 - f_0.
\end{aligned} \tag{7.2.7}$$

Substituting the expressions (6.2.5) of the Christoffel symbols in Eqs. (7.2.1), we obtain the following equations for the geodesics:

$$\begin{aligned}
\frac{d^2x}{ds^2} + \frac{1}{2}\left(\frac{1}{\Delta}\right)'\left(\frac{dy}{ds} - \varphi\frac{dz}{ds}\right)^2 + \frac{\varphi'}{\Delta}\left(\frac{dy}{ds} - \varphi\frac{dz}{ds}\right)\frac{dz}{ds} &= 0,\\
\frac{d}{ds}\left[\frac{1}{\Delta}\left(\frac{dy}{ds} - \varphi\frac{dz}{ds}\right)\right] &= 0,\\
\frac{d^2z}{ds^2} + \frac{1}{\Delta}\left(\frac{dy}{ds} - \varphi\frac{dz}{ds}\right)\frac{d\varphi}{ds} &= 0,\\
\frac{d^2(x-t)}{ds^2} &= 0,
\end{aligned} \tag{7.2.8}$$

where

$$\frac{d\varphi}{ds} = \varphi' \frac{d(x-t)}{ds}, \quad \text{etc.}$$

7.2.3 Solution of equations for geodesics

Let us solve the system (7.2.8). Using the initial conditions (7.2.2) and the notation (7.2.7) one obtains from the fourth and second equations (7.2.8):

$$x - t = \xi - \tau + (\alpha - \delta)s, \tag{7.2.9}$$

$$\varphi\frac{dz}{ds} - \frac{dy}{ds} = \frac{\gamma\varphi_0 - \beta}{\Delta_0}\Delta. \tag{7.2.10}$$

Therefore the third equation (7.2.8) becomes

$$\frac{d^2 z}{ds^2} = \frac{\gamma\varphi_0 - \beta}{\Delta_0}\frac{d\varphi}{ds}.$$

Now the above equations yield:

$$\frac{dz}{ds} = \frac{1}{\Delta_0}\left[(\gamma\varphi_0 - \beta)\varphi + \beta\varphi_0 - \gamma f_0\right] \tag{7.2.11}$$

and

$$\frac{dy}{ds} = \frac{1}{\Delta_0}[(\gamma\varphi_0 - \beta)f + (\beta\varphi_0 - \gamma f_0)]. \tag{7.2.12}$$

Solving Eqs. (7.2.11)—(7.2.12) by using Eq. (7.2.9) one obtains

$$y = \eta + a(F - F_0) + b(\Phi - \Phi_0), \tag{7.2.13}$$

$$z = \zeta + a(\Phi - \Phi_0) + b(\alpha - \delta)s, \tag{7.2.14}$$

where

$$a = \frac{\gamma\varphi_0 - \beta}{(\alpha - \delta)\Delta_0}, \quad b = \frac{\beta\varphi_0 - \gamma f_0}{(\alpha - \delta)\Delta_0}. \tag{7.2.15}$$

By virtue of Eqs. (7.2.10) and (7.2.11) the first equation of the system (7.2.8) takes the form

$$\frac{d^2 x}{ds^2} + \frac{\gamma\varphi_0 - \beta}{\Delta_0^2}\left[\frac{1}{2}(\gamma\varphi_0 - \beta)f' + (\beta\varphi_0 - \gamma f_0)\varphi'\right] = 0$$

and yields

$$x = \xi + \left(\alpha + \beta a - \frac{\alpha - \delta}{2}a^2 f_0\right)s - \frac{1}{2}a^2(F - F_0) - ab(\Phi - \Phi_0). \tag{7.2.16}$$

Collecting the above formulas, we conclude that the solution to the system (7.2.8) with the initial conditions (7.2.2) is given by Eqs. (7.2.9), (7.2.13), (7.2.14) and (7.2.16). In other words, the geodesics in the curved spaces V_4 with the metric tensor (6.2.1) have the following form:

$$\begin{aligned}
x &= \xi + \Big(\alpha + \beta a - \frac{\alpha-\delta}{2}a^2 f_0\Big)s - \frac{1}{2}a^2(F - F_0) - ab(\Phi - \Phi_0),\\
y &= \eta + a(F - F_0) + b(\Phi - \Phi_0),\\
z &= \zeta + a(\Phi - \Phi_0) + b(\alpha - \delta)s,\\
t &= \tau + \Big(\delta + \beta a - \frac{\alpha-\delta}{2}a^2 f_0\Big)s - \frac{1}{2}a^2(F - F_0) - ab(\Phi - \Phi_0).
\end{aligned}$$

One can readily rewrite the solution in the form (7.2.4).

7.2.4 Computation of the geodesic distance

Substituting the elements g_{ij} of the matrix (6.2.4) in Eq. (7.2.3) and using the notation (7.2.15) one obtains

$$\begin{aligned}
g_{ij}(x_0)\alpha^i\alpha^j &= \delta^2 - \alpha^2 + \frac{1}{\Delta_0}(\beta^2 - 2\beta\gamma\varphi_0 + \gamma^2 f_0)\\
&= -(\alpha - \delta)(\alpha + \delta + a\beta + b\gamma).
\end{aligned}$$

Hence, Equation (7.2.3) becomes

$$-(\alpha - \delta)(\alpha + \delta + \beta a + \gamma b) = 1. \tag{7.2.17}$$

Equations (7.2.15) provide

$$\beta = (\alpha - \delta)(a f_0 + b\varphi_0), \quad \gamma = (\alpha - \delta)(a\varphi_0 + b). \tag{7.2.18}$$

Furthermore, Equation (7.2.13) and (7.2.14) yield

$$\begin{aligned}
a &= \frac{1}{\Theta}[(y - \eta)(\alpha - \delta)s - (z - \zeta)(\Phi - \Phi_0)],\\
b &= \frac{1}{\Theta}[(z - \zeta)(F - F_0) - (y - \eta)(\Phi - \Phi_0)],
\end{aligned} \tag{7.2.19}$$

where

$$\Theta = (F - F_0)(\alpha - \delta)s - (\Phi - \Phi_0)^2. \tag{7.2.20}$$

When s is small, Equation (7.2.20) can be written in the form

$$\Theta = -\Delta_0(\alpha - \delta)^2 s^2 + o(s^2). \tag{7.2.21}$$

On the other hand, Equation (7.2.17) guarantees that $\alpha - \delta \neq 0$. We also know that $\Delta_0 \neq 0$ because the matrix $\|g^{ij}\|$ is non-degenerate. Therefore Eq. (7.2.21) shows that $\Theta \neq 0$ when $s \neq 0$ is sufficiently small.

Equations (7.2.18) provide that

$$\beta a + \gamma b = (\alpha - \delta)(a^2 f_0 + 2ab\varphi_0 + b^2).$$

Furthermore, we obtain from Eqs. (7.2.16) and (7.2.18):

$$2\alpha = 2\frac{x-\xi}{s} - (\alpha - \delta)(a^2 f_0 + 2ab\varphi_0) + \frac{1}{s}[a^2(F - F_0) + 2ab(\Phi - \Phi_0)].$$

Therefore writing $\alpha + \delta = 2\alpha - (\alpha - \delta)$, we have

$$\alpha + \delta + a\beta + b\gamma = \frac{1}{s}[2(x-\xi) - (\alpha - \delta)s + a^2(F - F_0) + 2ab(\Phi - \Phi_0) + b^2(\alpha - \delta)s].$$

Substituting here the values of $(\alpha - \delta)s$ obtained from Eq. (7.2.9) and the values of a and b given by Eqs. (7.2.19) we obtain:

$$\begin{aligned}\alpha + \delta + \beta a + \gamma b = \frac{1}{s}\Big\{ & x - \xi + t - \tau + \frac{1}{A}\big[(y-\eta)^2(x - \xi - t + \tau) \\ & -2(y-\eta)(z-\zeta)(\Phi - \Phi_0) + (z-\zeta)^2(F - F_0)\big]\Big\}.\end{aligned}$$

Now we substitute this expression for $\alpha + \delta + \beta a + \gamma b$ and the value for $\alpha - \delta$ obtained from Eq. (7.2.9) in Eq. (7.2.17), multiply the result by s^2 and arrive at the following formula of the square of the geodesic distance:

$$\begin{aligned}\Gamma(x_o, x) = {} & (t-\tau)^2 - (x-\xi)^2 \\ & - \frac{x - \xi - t + \tau}{(x - \xi - t + \tau)(F - F_0) - (\Phi - \Phi_0)^2}[(x - \xi - t + \tau)(y-\eta)^2 \\ & - 2(\Phi - \Phi_0)(y-\eta)(z-\zeta) + (F - F_0)(z-\zeta)^2].\end{aligned} \tag{7.2.22}$$

If $f = 1, \varphi = 0$, Equation (7.2.22) gives the square of the geodesic distance

$$\Gamma_*(x_0, x) = (t-\tau)^2 - (x-\xi)^2 - (y-\eta)^2 - (z-\zeta)^2 \tag{7.2.23}$$

between the points $x_o = (\xi, \eta, \zeta, \tau)$ and $x = (x, y, z, t)$ measured in the Minkowski metric (3.2.14) with $c = 1$,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{7.2.24}$$

7.3 The Huygens principle

Originally this principle was introduced by Christiaan Huygens (8) formulated as a geometric method for constructing wave fronts. It has served to clarify the main features of light propagation. The mathematical aspects of Huygens' principle were discussed in terms of the wave equation by Kirchhoff (1882), Beltrami (1889) and Volterra (1894). For a long time, several distinct meanings were attributed to this principle. We owe the current understanding of the mathematical nature of what is called the Huygens principle to Hadamard (see (23) and references therein). He gave the definitive classification of different meanings of the principle. One of them (known as Hadamard's "minor premise") is the statement on existence of a sharp rear front of waves governed by the hyperbolic equation (6.1.1),

$$F[u] \equiv g^{ij}(x)u,_{ij} + a^i(x)u,_i + c(x)u = 0. \tag{6.1.1}$$

Recall that the Cauchy problem for Eq. (6.1.1) is a problem of determining the solution which assumes given values of u and its normal derivatives on a space-like hypersurface, called the *initial manifold.* The given initial values are termed the *Cauchy data.* In general, the solution $u(x)$ at a point $x = (x^1, \ldots, x^n)$ depends on the Cauchy data in the interior of the intersection of the initial manifold and the retrograde characteristic conoid with the apex x.

Existence of a sharp rear front of waves means that the solution $u(x)$ depends only on the data taken on the intersection of the initial manifold and the retrograde characteristic conoid. In this case the waves propagate without diffusion. In our discussion, the Huygens principle is identified with Hadamard's "minor premise".

The Huygens principle holds, in particular, for the classical wave equations with an arbitrary even number $n \geq 4$ of independent variables as well as for equations that are equivalent to these equations. The case $n = 4$ is of our interest because of its physical significance. In this case, examples of equations that are not equivalent to the classical wave equation but satisfy the Huygens principle have been constructed in (20) and independently in (41). Namely, it has been shown by different methods the the Huygens principle is satisfied for Eq. (7.1.16),

$$u_{tt} - u_{xx} - f(x - t)u_{yy} - 2\varphi(x - t)u_{yz} - u_{zz} = 0.$$

7.3.1 Huygens' principle for classical wave equation

For the classical wave operator

$$\square[u] = u_{tt} - u_{xx} - u_{yy} - u_{zz}, \tag{7.3.1}$$

the Huygens principle states that the wave, carrying a disturbance initially limited in space and time, does not leave any trace behind.

In terms of the wave equation

$$\square[u] = 0, \tag{7.3.2}$$

the Huygens principle means that the solution of an arbitrary Cauchy problem at any point $x = (x, y, z, t)$ is determined by the Cauchy data taken on the intersection of the initial manifold and the characteristic cone

$$\Gamma_*(x_0, x) = 0 \tag{7.3.3}$$

with apex at the point x. Indeed, if we take the initial manifold in the form $\tau = 0$, then Eq. (7.2.23) shows that the intersection of the characteristic cone (7.3.3) with $\tau = 0$ is given by the equation

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = t^2$$

describing the sphere S_t of the radius t with the center at the point (x, y, z). Therefore the Huygens principle follows from Poisson's formula (7.1.24).

Remark 7.3.1. For equations

$$\square[u] + a^i(x, y, z, t)u_i + c(x, y, z, t)u = 0$$

that are not equivalent to the wave equation (7.3.2) the solution of the Cauchy problem is determined by the Cauchy data taken inside the sphere S_t, i.e. in the region

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \leq t^2.$$

Therefore these equations do not satisfy the Huygens principle.

7.3.2 Huygens' principle for the curved wave operator in V_4 with nontrivial conformal group

For the curved wave operator (7.1.1) the Huygens principle is formulated in the same way as for the operator (7.3.1), the only difference being that the characteristic cone (7.3.3) is replaced by the characteristic conoid

$$\Gamma(x_0, x) = 0, \tag{7.3.4}$$

where $\Gamma(x_0, x)$ is the square of the geodesic distance given by Eq. (7.2.22).

Validity of the Huygens principle for the conformally invariant equations in spaces V_4 with nontrivial conformal group follows from the representation (7.1.26) of the solution of the Cauchy problem (7.1.2) for the curved wave operator (7.1.1).

The following theorem solves the problem on the Huygens principle for spaces with nontrivial conformal group (see (28) and the references therein).

Theorem 7.3.1. In the spaces V_4 of normal hyperbolic type with nontrivial conformal group the Huygens principle holds for Eq. (6.1.1), $F[u] = 0$, if and only if the operator $F[u]$ is equivalent to the curved wave operator $\diamondsuit[u]$ given by Eq. (7.1.1).

7.3.3 On spaces with trivial conformal group

We do not know any example of a space V_4 of normal hyperbolic type with trivial conformal group containing a Huygens type equation (see e.g. (21) and (27), §19). Moreover, it has been demonstrated in (51) that if a space V_4 with trivial conformal group has the vanishing Ricci tensor (3.2.29), $R_{ij} = 0$, then it does not contain a Huygens type equation. One of physically important consequences of this general result is that the Huygens principle is not satisfied in the Schwarzschild space.

Exercises

Exercise 7.1. Find the square of the geodesic distance

$$\Gamma(x_0, x) = [s(x_0, x)]^2$$

in the Minkowski space with the metric (3.2.14),

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

Exercise 7.2. Prove the operator identity (7.1.4).

Exercise 7.3. Find the square of the geodesic distance $\Gamma(x_0, x)$ in in the space with the metric tensor (6.2.1) when

$$\varphi = 0, \;\; f(x - t) = \mathrm{e}^{x-t}.$$

The present time of past things is our memory;
the present time of present things is our sight;
the present time of future things is our expectation.
In what space therefore do we measure the time?

St. Augustine (63)

Henceforth space by itself and time by itself
are doomed to fade away into mere shadows,
and only a kind of union of the two
will preserve an independent reality.

H. Minkowski (52)

Special Relativity is a mainstay of modern theoretical and mathematical physics. It provides a theoretical background for developing such topics as *General Relativity and Cosmology, Electrodynamics, Quantum Theory*, etc. Nowadays, we possess numerous comprehensive and well-written expositions of the theory of relativity by celebrated authors that contain a physical motivation and applications (see, e.g. (45)).

Chapter 8 is a brief introduction to the mathematical background of the special relativity. Derivation of the conservation laws in relativistic mechanics is based on the Lie group approach rather than on physical considerations. Two fundamental relativistic (i.e. Lorentz invariant) equations of theoretical physics, the Maxwell and Dirac equations, are discussed from the group point of view. A possibility offered by groups of generalized motions for constructing particular solutions to the Einstein equations in the theory of general relativity is also discussed in Chapter 8.

Chapter 9 contains a detailed discussion of the relativistic dynamics in the de Sitter space in terms of the approximate groups. The presentation is based on (32).

Chapter 8
Brief introduction to relativity

8.1 Special relativity

8.1.1 Space-time intervals

An event occurring in a material particle is described by the place where it occurred and the time when it occurred. Thus, events are represented by points in a four-dimensional space of three space coordinates and the time. Points in this four-dimensional space are called *world points.* A totality of world points corresponding to each particle, describe a line called a *world line.* This line determines the positions of the particle in all moments of time. The world line of a particle in uniform motion with a constant velocity is a straight line.

In what follows, world points will be defined by three space coordinates (x, y, z) referred to the rectangular Cartesian frame and by time by t.

Inertial system of reference is a system of space coordinates and time such that the velocity of a freely moving body referred to this coordinates is constant.

Principle of relativity asserts that *the laws of nature are identical in all inertial systems of reference.* That is the equations describing the laws of natural phenomena are invariant under the invertible transformations of space-time coordinates (x, y, z, t) from one inertial system to another. The commonly known example is the *Galilean principle of relativity* in classical mechanics.

Fundamental assumption hinted by experiments on the light propagation is that the *velocity of propagation of interactions is a universal constant,* namely, the velocity c of light in empty space. Its numerical value is

$$c = 2.99793 \times 10^{10} \quad \text{cm/sec.}$$

Special relativity formulated by A. Einstein (15) in 1905 is the combination of the principle of relativity with the assumption on the finiteness of velocity of propagation of interactions.

Let a signal propagate, in an inertial system of reference, with the light velocity c. We suppose that one sends the signal at time t_1 from a point (x_1, y_1, z_1). Let the signal arrive at point (x_2, y_2, z_2) at time t_2. Since the signal propagates with velocity c, the distance

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

covered in time $t_2 - t_1$ is equal to $c(t_2 - t_1)$. Hence the coordinates of the world points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are related by

$$c(t_2 - t_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Let's write this relation in the symmetric form

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0. \tag{8.1.1}$$

The quantity

$$s_{12} = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}$$

is called the *interval* between the world points (or events) (x_1, y_1, z_1, t_1) and $(x_2, y_2,\ z_2, t_2)$. The infinitesimal interval ds is defined then by (3.2.14):

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

The assumption that light velocity c is universally constant implies that the differential form (3.2.14) is invariant with respect to any change of coordinates of world points. Hence (3.2.14) defines the metric of a four-dimensional Riemannian space of signature $(+ - --)$. This is a *flat space*, namely the Minkowski space considered in Example 3.2.2.

8.1.2 The Lorentz group

The group of isometric motions in the Minkowski space with the metric (3.2.14) is the non-homogeneous Lorentz group with ten generators

$$\begin{aligned} &X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{0i} = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}\,x^i\frac{\partial}{\partial t}, \quad (i = 1, 2, 3),\\ &X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad (i < j;\ i = 1, 2;\ j = 1, 2, 3). \end{aligned} \tag{8.1.2}$$

8.1.3 Relativistic principle of least action

One can develop a theory of relativity in any four-dimensional Riemannian space V_4 with a metric form (3.2.3) of the space-time signature $(+ - --)$. We discuss here the relativistic mechanics of particles, more specifically, the motion of a free material particle. In this case, the motion of a particle with mass m is determined by the following

PRINCIPLE OF LEAST ACTION: *A particle moves so that its world line* $x^i = x^i(s),\ i = 1, \ldots, 4,$ *is a geodesic.*

This means that the motion of a particle in the space with the metric (3.2.3)

$$ds^2 = g_{ij}(x)dx^i dx^j,$$

is determined by the Lagrangian

$$L = -mc\sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}. \tag{8.1.3}$$

Here $x = (x^1, \ldots, x^4)$, and $\dot{x} = dx/ds$ is the derivative of the four-vector x with respect to the arc length s measured from a fixed point x_0. Thus, the equations of free motion are the Euler-Lagrange equations (2.3.6) with the Lagrangian (8.1.3). These equations have the form (3.2.18),

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i = 1, \ldots, 4,$$

where the coefficients Γ^i_{jk} are the *Christoffel symbols* given by (3.2.19):

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right).$$

A subsequent development of the relativistic mechanics is based, in accordance with Klein [1910], on the group of isometric motions. The generators

$$X = \eta^i(x)\frac{\partial}{\partial x^i} \tag{8.1.4}$$

of the isometric motions in the space V_4 with the metric form (3.2.3) are determined by the *Killing equations* (4.2.3):

$$\eta^k\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \eta^k}{\partial x^j} + g_{jk}\frac{\partial \eta^k}{\partial x^i} = 0 \quad (i \le j), \quad i, j = 1, \ldots, 4. \tag{8.1.5}$$

8.1.4 Relativistic Lagrangian

According to Section 8.1.1, the *Special Relativity* is based on the Minkowski space-time metric (3.2.14).

Further, the vector of space variables is denoted by

$$\boldsymbol{x} = (x^1, x^2, x^3),$$

and the physical velocity

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$$

is a three-dimensional vector

$$\boldsymbol{v} = (v^1, v^2, v^3).$$

The scalar and vector products are denoted by $(\boldsymbol{x} \cdot \boldsymbol{v})$ and $\boldsymbol{x} \times \boldsymbol{v}$, respectively. Equation (3.2.14) written in the form

$$ds = c\sqrt{1-\beta^2}\,dt, \quad \beta^2 = |\boldsymbol{v}|^2/c^2, \tag{8.1.6}$$

yields:

$$\frac{d\boldsymbol{x}}{ds} = \frac{\boldsymbol{v}}{c\sqrt{1-\beta^2}}, \quad \frac{dt}{ds} = \frac{1}{c\sqrt{1-\beta^2}}.$$

The action integral is given by (see Section 8.1.3)

$$S = \alpha \int_a^b ds,$$

where the integral is taken along the world line of the particle between two arbitrary world points a and b and α is a normalizing factor. We represent the action as an integral with respect to the time by substituting the expression (8.1.6) for ds:

$$S = \alpha c \int_{t_1}^{t_2} \sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}\,dt. \tag{8.1.7}$$

Thus, we arrive at the action integral with the Lagrangian

$$L = \alpha c \sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}. \tag{8.1.8}$$

The factor α is to be found from the requirement that the function (8.1.8) must go over into the Lagrangian (see Eq. (2.3.26))

$$L = \frac{m}{2}|\boldsymbol{v}|^2 \tag{8.1.9}$$

in the classical limit

$$\frac{|\boldsymbol{v}|^2}{c^2} \to 0.$$

We expand the function (8.1.8) in powers of $|\boldsymbol{v}|^2/c^2$ and neglect terms of higher order to obtain

$$L = \alpha c \sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}} \approx \alpha c - \frac{\alpha |\boldsymbol{v}|^2}{2c}.$$

Omitting the constant term αc and comparing with the classical Lagrangian (8.1.9) one arrives at

$$\alpha = -mc.$$

Thus, the relativistic Lagrangian for a free particle with the mass m is

$$L = -mc^2\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}, \tag{8.1.10}$$

where

$$|\boldsymbol{v}|^2 = \sum_{i=1}^{3}(v^i)^2.$$

8.1.5 Conservation laws in relativistic mechanics

We write the generators (8.1.2) of the Lorentz group in the form (2.3.24),

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta^i(t,x)\frac{\partial}{\partial x^i}. \tag{8.1.11}$$

The conserved quantity (2.3.25) is written as

$$T = (\eta^i - \xi v^i)\frac{\partial L}{\partial v^i} + \xi L, \quad i = 1,2,3. \tag{8.1.12}$$

Let's apply the formula (8.1.12) to the time translation with the generator X_0. Here

$$\xi = 1, \quad \eta^1 = \eta^2 = \eta^3 = 0.$$

The formula (8.1.12) yields (by setting $E = -T$) the relativistic energy:

$$E = \frac{mc^2}{\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}}. \tag{8.1.13}$$

Similarly, by using the operators X_i and X_{ij}, one easily obtains the *relativistic momentum*

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}} \tag{8.1.14}$$

and the *relativistic angular momentum*

$$\boldsymbol{M} = \boldsymbol{x} \times \boldsymbol{p}, \tag{8.1.15}$$

respectively.

The generators X_{0i} of the Lorentz transformations give rise to the vector

$$\boldsymbol{Q} = \frac{m(\boldsymbol{x} - t\boldsymbol{v})}{\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}}. \tag{8.1.16}$$

The conservation of the vector (8.1.16), in the case of N-body problem, provides the relativistic center-of-mass theorem.

8.2 The Maxwell equations

8.2.1 Introduction

We will consider the Maxwell equations in vacuum

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{8.2.1}$$

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0, \tag{8.2.2}$$

where $\boldsymbol{E} = (E^1, E^2, E^3)$ is the electric field and $\boldsymbol{H} = (H^1, H^2, H^3)$ is the magnetic field. The independent variables are the time t and the vector $\boldsymbol{x}$ of spatial coordinates. In rectangular Cartesian coordinates $\boldsymbol{x} = (x, y, z)$ the system of equations (8.2.1) and (8.2.2) is written

$$\begin{aligned} &E_y^3 - E_z^2 + H_t^1 = 0, \qquad H_y^3 - H_z^2 - E_t^1 = 0,\\ &E_z^1 - E_x^3 + H_t^2 = 0, \qquad H_z^1 - H_x^3 - E_t^2 = 0,\\ &E_x^2 - E_y^1 + H_t^3 = 0, \qquad H_x^2 - H_y^1 - E_t^3 = 0, \end{aligned} \tag{8.2.1$'$}$$

$$E_x^1 + E_y^2 + E_z^3 = 0, \qquad H_x^1 + H_y^2 + H_z^3 = 0. \tag{8.2.2$'$}$$

The system (8.2.1′) and (8.2.2′) is over-determined because it contains eight equation for six components of the three-dimensional vectors $\boldsymbol{E}, \boldsymbol{H}$. Since the Euler-Lagrange equations (2.3.6) provide a *determined system*, the over-determined systems of the Maxwell equations (8.2.1) and (8.2.2) cannot have a usual Lagrangian written in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$. On the other hand, the evolutionary part (8.2.1) of the Maxwell equations is a determined system and may have a Lagrangian. We will see further that it does indeed. However, Equations (8.2.1) taken without Eqs. (8.2.2) do not admit either the proper Lorentz or the conformal transformations of the Minkowski space. From symmetry point of view, the additional equations (8.2.2) guarantee the Lorentz and the conformal invariance, but they destroy the Lagrangian.

In this section we illustrate, following (34), how to attain a harmony between these two contradictory properties. We will derive the conservation laws via Noether's theorem for all symmetries of the system (8.2.1) and (8.2.2).

8.2.2 Symmetries of Maxwell's equations

Equations (8.2.1) and (8.2.2) are invariant under the translations of time t and the position vector $\boldsymbol{x}$ as well as the simultaneous rotations of the vectors $\boldsymbol{x}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ due to the vector formulation of Maxwell's equations. The generators of these transformation provide the following seven infinitesimal symmetries:

$$
\begin{aligned}
&X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z},\\
&X_{12} = y\frac{\partial}{\partial x} - x\tfrac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + H^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^2},\\
&X_{13} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^3},\\
&X_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^2} - E^2\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial H^3}.
\end{aligned}
\tag{8.2.3}
$$

It was discovered by Lorentz (50) that that the Maxwell equations (8.2.1) and (8.2.2) are invariant under the hyperbolic rotations known today as the Lorentz transformations. They provide the infinitesimal symmetries

$$
\begin{aligned}
&X_{01} = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + E^2\frac{\partial}{\partial H^3} + H^3\frac{\partial}{\partial E^2} - E^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial E^3},\\
&X_{02} = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} + E^3\frac{\partial}{\partial H^1} + H^1\frac{\partial}{\partial E^3} - E^1\frac{\partial}{\partial H^3} - H^3\frac{\partial}{\partial E^1},\\
&X_{03} = t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} + E^1\frac{\partial}{\partial H^2} + H^2\frac{\partial}{\partial E^1} - E^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial E^2}.
\end{aligned}
\tag{8.2.4}
$$

It was shown later by Bateman (3), (4) and Cunningham (10) that Eqs.

(8.2.1) and (8.2.2) admit the conformal transformations with the generators

$$
\begin{aligned}
Y_1 &= \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t} \\
&\quad -\left(4xE^1 + 2yE^2 + 2zE^3\right)\frac{\partial}{\partial E^1} - \left(4xH^1 + 2yH^2 + 2zH^3\right)\frac{\partial}{\partial H^1} \\
&\quad -\left(4xE^2 - 2yE^1 - 2tH^3\right)\frac{\partial}{\partial E^2} - \left(4xH^2 - 2yH^1 + 2tE^3\right)\frac{\partial}{\partial H^2} \\
&\quad -\left(4xE^3 - 2zE^1 + 2tH^2\right)\frac{\partial}{\partial E^3} - \left(4xH^3 - 2zH^1 - 2tE^2\right)\frac{\partial}{\partial H^3}, \\
Y_2 &= 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2 - z^2 + t^2\right)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + 2yt\frac{\partial}{\partial t} \\
&\quad -\left(4yE^1 - 2xE^2 - 2tH^3\right)\frac{\partial}{\partial E^1} - \left(4yH^1 - 2xH^2 + 2tE^3\right)\frac{\partial}{\partial H^1} \\
&\quad -\left(4yE^2 + 2xE^1 + 2zE^3\right)\frac{\partial}{\partial E^2} - \left(4yH^2 + 2xH^1 + 2zH^3\right)\frac{\partial}{\partial H^2} \\
&\quad -\left(4yE^3 - 2zE^2 + 2tH^1\right)\frac{\partial}{\partial E^3} - \left(4yH^3 - 2zH^2 - 2tE^1\right)\frac{\partial}{\partial H^3}, \\
Y_3 &= 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + \left(z^2 - x^2 - y^2 + t^2\right)\frac{\partial}{\partial z} + 2zt\frac{\partial}{\partial t} \\
&\quad -\left(4zE^1 - 2xE^3 + 2tH^2\right)\frac{\partial}{\partial E^1} - \left(4zH^1 - 2xH^3 - 2tE^2\right)\frac{\partial}{\partial H^1} \\
&\quad -\left(4zE^2 - 2yE^3 - 2tH^1\right)\frac{\partial}{\partial E^2} - \left(4zH^2 - 2yH^3 + 2tE^1\right)\frac{\partial}{\partial H^2} \\
&\quad -\left(4zE^3 + 2yE^2 + 2xE^1\right)\frac{\partial}{\partial E^3} - \left(4zH^3 + 2yH^2 + 2xH^1\right)\frac{\partial}{\partial H^3}, \\
Y_4 &= 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} + 2tz\frac{\partial}{\partial z} + \left(x^2 + y^2 + z^2 + t^2\right)\frac{\partial}{\partial t} \\
&\quad -\left(4tE^1 + 2yH^3 - 2zH^2\right)\frac{\partial}{\partial E^1} - \left(4tH^1 - 2yE^3 + 2zE^2\right)\frac{\partial}{\partial H^1} \\
&\quad -\left(4tE^2 + 2zH^1 - 2xH^3\right)\frac{\partial}{\partial E^2} - \left(4tH^2 - 2zE^1 + 2xE^3\right)\frac{\partial}{\partial H^2} \\
&\quad -\left(4tE^3 - 2yH^1 + 2xH^2\right)\frac{\partial}{\partial E^3} - \left(4tH^3 + 2yE^1 - 2xE^2\right)\frac{\partial}{\partial H^3}.
\end{aligned}
\tag{8.2.5}
$$

Furthermore, Eqs. (8.2.1) and (8.2.2) admit the dilation generators

$$
Z_1 = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}\cdot\frac{\partial}{\partial \boldsymbol{H}} \equiv \sum_{k=1}^{3} E^k\frac{\partial}{\partial E^k} + \sum_{k=1}^{3} H^k\frac{\partial}{\partial H^k}, \tag{8.2.6}
$$

$$
Z_2 = t\frac{\partial}{\partial t} + \boldsymbol{x}\frac{\partial}{\partial \boldsymbol{x}} \equiv t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \tag{8.2.7}
$$

due to homogeneity, and the superposition generators

$$S = \boldsymbol{E}_*(\boldsymbol{x},t)\cdot\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}_*(\boldsymbol{x},t)\cdot\frac{\partial}{\partial \boldsymbol{H}} \tag{8.2.8}$$

due to linearity, where $\boldsymbol{E}_*(\boldsymbol{x},t),\ \boldsymbol{H} = \boldsymbol{H}_*(\boldsymbol{x},t)$ solve Eqs. (8.2.1) and (8.2.2).

Finally, Equations (8.2.1) and (8.2.2) admit the rotations in the space of the electric and magnetic fields (duality rotation):

$$\overline{\boldsymbol{E}} = \boldsymbol{E}\cos\alpha - \boldsymbol{H}\sin\alpha, \quad \overline{\boldsymbol{H}} = \boldsymbol{H}\cos\alpha + \boldsymbol{E}\sin\alpha$$

with the generator (*duality generator*)

$$Z_0 = \boldsymbol{E}\cdot\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\cdot\frac{\partial}{\partial \boldsymbol{E}} \equiv \sum_{k=1}^{3}\left(E^k\frac{\partial}{\partial H^k} - H^k\frac{\partial}{\partial E^k}\right). \tag{8.2.9}$$

Solution of the determining equations for symmetries led to the following result (25) on the Lie point symmetries of the Maxwell equations.

Theorem 8.2.1. The operators (8.2.3)—(8.2.9) span the Lie algebra of the maximal point transformation group admitted by Eqs. (8.2.1) and (8.2.2). This algebra comprises the 17-dimensional subalgebra spanned by the operators (8.2.3)—(8.2.7) and (8.2.9), and the infinite-dimensional ideal consisting of the superposition generators (8.2.8).

8.2.3 General discussion of conservation laws

Conservation laws for the Maxwell equations are written in the differential form (2.2.13) as follows:

$$\big[D_t(\tau) + \operatorname{div}\boldsymbol{\chi}\big]_{(8.2.1)-(8.2.2)} = 0, \tag{8.2.10}$$

where τ is the *density* of the conservation law and the vector $\boldsymbol{\chi} = (\chi^1,\chi^2,\chi^3)$ is termed the *flux*. The divergence $\operatorname{div}\boldsymbol{\chi}$ of the flux vector is given by

$$\operatorname{div}\boldsymbol{\chi} \equiv \nabla\cdot\boldsymbol{\chi} = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3). \tag{8.2.11}$$

The symbol $\big|_{(8.2.1)-(8.2.1)}$ means that Eq. (8.2.10) is satisfied on the solutions of the Maxwell equations (8.2.1) and (8.2.2). The density τ and flux $\boldsymbol{\chi}$ may depend, in general, on the variables $t, \boldsymbol{x}$ as well as on the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{H}$ and their derivatives.

If the time derivatives of $\boldsymbol{E}, \boldsymbol{H}$ have been eliminated by using Eqs. (8.2.1), then τ and χ^1,χ^2,χ^3 do not involve the time derivatives $\boldsymbol{E}_t, \boldsymbol{H}_t$, and Eq. (8.2.10) takes the form

$$\big(D_t(\tau)\big|_{(8.2.1)} + \operatorname{div}\boldsymbol{\chi}\big)\big|_{(8.2.2)} = 0, \tag{8.2.12}$$

where $D_t(\tau)\big|_{(8.2.1)}$ is obtained from $D_t(\tau)$ by substituting $\boldsymbol{E}_t, \boldsymbol{H}_t$ given by Eqs. (8.2.1), whereas the sign $\big|_{(8.2.2)}$ indicates that the terms proportional to div$\boldsymbol{E}$ and div$\boldsymbol{H}$ have been eliminated by using Eqs. (8.2.2).

The integral form (2.2.21) of the conservation equation (8.2.10) is

$$\frac{d}{dt}\int \tau\, dxdydz = 0, \tag{8.2.13}$$

where d/dt is identified with the total derivative D_t.

We will begin the construction of conservation laws by considering the evolutionary part (8.2.1) of the Maxwell equations. In this case, Proposition 2.2.1 leads to the following statement.

Theorem 8.2.2. A function $\tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{H}_x, \boldsymbol{E}_y, \boldsymbol{H}_y, \boldsymbol{E}_z, \boldsymbol{H}_z)$ is a conservation density for Eqs. (8.2.1) if and only if it satisfies the equations

$$\frac{\delta}{\delta \boldsymbol{E}}\left[D_t(\tau)\big|_{(8.2.1)}\right] = 0, \quad \frac{\delta}{\delta \boldsymbol{H}}\left[D_t(\tau)\big|_{(8.2.1)}\right] = 0. \tag{8.2.14}$$

Proof. The statement of this theorem is a particular case of the general test (2.2.29) for conservation densities. But I will give an independent proof. We will use Proposition 2.2.1 in the following form:

$$\frac{\delta}{\delta \boldsymbol{E}}(\operatorname{div}\boldsymbol{\chi}) = 0, \quad \frac{\delta}{\delta \boldsymbol{H}}(\operatorname{div}\boldsymbol{\chi}) = 0. \tag{8.2.15}$$

Here the vector valued variational derivatives are given by

$$\frac{\delta}{\delta \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}} - D_j\frac{\partial}{\partial \boldsymbol{E}_j} + \cdots, \quad \frac{\delta}{\delta \boldsymbol{H}} = \frac{\partial}{\partial \boldsymbol{H}} - D_j\frac{\partial}{\partial \boldsymbol{H}_j} + \cdots, \tag{8.2.16}$$

where the index j runs over 1, 2, 3, and the following notation is assumed:

$$\begin{gathered} D_1 = D_x, \quad D_2 = D_y, \quad D_3 = D_z, \\ \boldsymbol{E}_1 = \boldsymbol{E}_x, \quad \boldsymbol{E}_2 = \boldsymbol{E}_y, \quad \boldsymbol{E}_3 = \boldsymbol{E}_z, \\ \boldsymbol{H}_1 = \boldsymbol{H}_x, \quad \boldsymbol{H}_2 = \boldsymbol{H}_y, \quad \boldsymbol{H}_3 = \boldsymbol{H}_z. \end{gathered} \tag{8.2.17}$$

Since we consider Eqs. (8.2.1) without Eqs. (8.2.2), the conservation equation (8.2.12) is written

$$D_t(\tau)\big|_{(8.2.1)} + \operatorname{div}\boldsymbol{\chi} = 0.$$

Taking from the latter equation the variational derivatives (8.2.16) and invoking Eqs. (8.2.15) we arrive at Eqs. (8.2.14). This completes the proof. □

It is manifest that any conservation density for Eqs. (8.2.1) is a conservation density for the whole system (8.2.1) and (8.2.2) of the Maxwell equations as well, but not vice versa. The conditions for conservation densities for the system (8.2.1) and (8.2.2) are weaker than Eqs. (8.2.14). These conditions are formulated in the following theorem proved in (34) (see also (36)).

Theorem 8.2.3. A function $\tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{H}_x, \ldots)$ is a conservation density for the Maxwell equations (8.2.1) and (8.2.2) if and only if the following equations are satisfied:

$$\begin{aligned} &\frac{\delta}{\delta \boldsymbol{E}}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta \boldsymbol{E}}\right|_{(8.2.2)}\right] = 0, \quad \frac{\delta}{\delta \boldsymbol{H}}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta \boldsymbol{E}}\right|_{(8.2.2)}\right] = 0,\\ &\frac{\delta}{\delta \boldsymbol{E}}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta \boldsymbol{H}}\right|_{(8.2.2)}\right] = 0, \quad \frac{\delta}{\delta \boldsymbol{H}}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta \boldsymbol{H}}\right|_{(8.2.2)}\right] = 0. \end{aligned} \tag{8.2.18}$$

The coordinate form of these equations is

$$\begin{aligned} &\frac{\delta}{\delta E^i}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta E^k}\right|_{(8.2.2)}\right] = 0, \quad \frac{\delta}{\delta H^i}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta E^k}\right|_{(8.2.2)}\right] = 0,\\ &\frac{\delta}{\delta E^i}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta H^k}\right|_{(8.2.2)}\right] = 0, \quad \frac{\delta}{\delta H^i}\left[\left.\frac{\delta D_t(\tau)\big|_{(8.2.1)}}{\delta H^k}\right|_{(8.2.2)}\right] = 0, \end{aligned} \tag{8.2.19}$$

where $i, k = 1, 2, 3$.

Remark 8.2.1. The variational derivatives (8.2.16) are written in coordinates as follows:

$$\frac{\delta}{\delta E^i} = \frac{\partial}{\partial E^i} - D_j \frac{\partial}{\partial E^i_j} + \cdots, \quad \frac{\delta}{\delta H^i} = \frac{\partial}{\partial H^i} - D_j \frac{\partial}{\partial H^i_j} + \cdots, \tag{8.2.20}$$

where the indices i, j run over 1, 2, 3. Accordingly, two vector equations (8.2.15) are written as six scalar equations

$$\frac{\delta}{\delta E^i}\left[D_t(\tau)\big|_{(8.2.1)}\right] = 0, \quad \frac{\delta}{\delta H^i}\left[D_t(\tau)\big|_{(8.2.1)}\right] = 0, \quad i = 1, 2, 3. \tag{8.2.21}$$

Remark 8.2.2. It is convenient to use the permutation symbol (1.1.25) when dealing with vector products of three-dimensional vectors. The vector product (1.1.5) of $\boldsymbol{a} = (a^1, a^2, a^3)$ and $\boldsymbol{b} = (b^1, b^2, b^3)$ can be defined by

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \mathrm{e}_{ijk} a^j b^k, \quad i = 1, 2, 3. \tag{8.2.22}$$

Accordingly, the definition (1.1.12) of $\nabla \times \boldsymbol{a}$ can be written in the form

$$(\nabla \times \boldsymbol{a})_i = \sum_{j,k=1}^{3} \mathrm{e}_{ijk} \nabla_j a^k = \sum_{j,k=1}^{e} \mathrm{e}_{ijk} a^k_j, \quad i = 1, 2, 3. \tag{8.2.23}$$

Example 8.2.1. The function $\tau = \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)$ is a conservation density for Eqs. (8.2.1). Indeed, we have

$$D_t(\tau)\big|_{(8.2.1)} = \big(\boldsymbol{E} \cdot \boldsymbol{E}_t + \boldsymbol{H} \cdot \boldsymbol{H}_t\big)\big|_{(8.2.1)} = \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) - \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}).$$

Therefore Eq. (8.2.23) yields

$$D_t(\tau)\big|_{(8.2.1)} = E^i \sum_{j,k=1}^{3} \mathrm{e}_{ijk} H^k_j - H^k \sum_{j,i=1}^{3} \mathrm{e}_{kji} E^i_j. \tag{8.2.24}$$

Now one can easily verify that Eqs. (8.2.21) for conservation densities are satisfied. For example,

$$\begin{aligned}\frac{\delta}{\delta E^i}\left[D_t(\tau)\big|_{(8.2.1)}\right] &= \sum_{j,k=1}^{3} \mathrm{e}_{ijk} H^k_j + \sum_{j,i=1}^{3} \mathrm{e}_{kji} D_j(H^k)\\ &= \sum_{j,k=1}^{3} \mathrm{e}_{ijk} H^k_j + \sum_{j,i=1}^{3} \mathrm{e}_{kji} H^k_j\\ &= \sum_{j,k=1}^{3} \mathrm{e}_{ijk} H^k_j - \sum_{j,i=1}^{3} \mathrm{e}_{ijk} H^k_j = 0.\end{aligned}$$

The verification of the second equation (8.2.21) is similar (Exercise 8.4).

Example 8.2.2. The *Poynting vector* $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ is not a conservation density for the evolutionary part (8.2.1) of Maxwell's equation, but is a conservation density for the system (8.2.1) and (8.2.2).

Indeed, it is shown in (34), Example 1.2, that Eqs. (8.2.14) are not satisfied. For instance, the time derivative of its first component of the Poynting vector, $\sigma_1 = E^2 H^3 - E^3 H^2$, is equal to

$$D_t(\sigma_1)\big|_{(8.2.1)} = E^2(E^1_y - E^2_x) + H^3(H^1_z - H^3_x) + E^3(E^1_z - E^3_x) + H^2(H^1_y - H^2_x).$$

Therefore, e.g.

$$\frac{\delta D_t(\sigma_1)\big|_{(8.2.1)}}{\delta E^1} = -E^2_y - E^3_z \neq 0.$$

On the other hand it is demonstrated in (34), Example 1.3, that Eqs. (8.2.18) are satisfied, and hence $\boldsymbol{\sigma}$ is a conservation density for Eqs. (8.2.1) and (8.2.2).

8.2.4 Evolutionary part of Maxwell's equations

Here we deal with the evolutionary part (8.2.1),

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0.$$

of Maxwell's equations. It is known that if the equations

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0$$

are satisfied at the initial time $t = 0$, then Eqs. (8.2.1) guarantee that they hold at any time t. Hence, Equations (8.2.2) can be regarded as initial conditions.

The Lagrangian for the system (8.2.1) can be obtained from the formal Lagrangian

$$\mathcal{L} = \boldsymbol{V} \cdot \left(\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{W} \cdot \left(\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{8.2.25}$$

The adjoint system (2.3.48) with this formal Lagrangian $\mathcal{L}$ is written

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{E}} \equiv \operatorname{curl} \boldsymbol{V} + \frac{\partial \boldsymbol{W}}{\partial t} = 0, \quad \frac{\delta \mathcal{L}}{\delta \boldsymbol{H}} \equiv \operatorname{curl} \boldsymbol{W} - \frac{\partial \boldsymbol{V}}{\partial t} = 0$$

and coincides with Eqs. (8.2.1) if $\boldsymbol{V} = \boldsymbol{E}, \boldsymbol{W} = \boldsymbol{H}$. Hence, the the system (8.2.1) is self-adjoint. Consequently, the system of equations (8.2.1) has the following Lagrangian ((33), Section 3):

$$L = \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{8.2.26}$$

Its coordinate form is

$$\begin{aligned} L = {} & E^1 \left(E^3_y - E^2_z + H^1_t\right) + E^2 \left(E^1_z - E^3_x + H^2_t\right) + E^3 \left(E^2_x - E^1_y + H^3_t\right) \\ & + H^1 \left(H^3_y - H^2_z - E^1_t\right) + H^2 \left(H^1_z - H^3_x - E^2_t\right) + H^3 \left(H^2_x - H^1_y - E^3_t\right). \end{aligned}$$

Remark 8.2.3. The system of the Maxwell equations with electric charges and currents is written

$$\nabla \times \boldsymbol{E} + \tfrac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{j} = 0, \tag{8.2.27}$$

$$\nabla \cdot \boldsymbol{E} = \rho, \quad \nabla \cdot \boldsymbol{H} = 0, \tag{8.2.28}$$

In this case the Lagrangian for the evolutionary part (8.2.27) has the form (see (1), (61) and (64))

$$L = \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} - 2\boldsymbol{j}\right).$$

By separating Eqs. (8.2.1) from Eqs. (8.2.2) we loose certain symmetries of the Maxwell equations. Namely, the symmetries of Eqs. (8.2.1) are given by the following theorem proved in (34).

Theorem 8.2.4. The maximal Lie algebra admitted by the evolutionary equations (8.2.1) of Maxwell's system comprises the 10-dimensional subalgebra spanned by the operators (8.2.3), (8.2.6), (8.2.7), (8.2.9), and the infinite dimensional ideal (8.2.8).

The conservation laws associated with the symmetries of Eqs. (8.2.1) can be computed by Noether's theorem. The following notation is used here. The independent variables x^i $(i = 0, 1, 2, 3)$ and the dependent variables u^α $(\alpha = 1, \ldots, 6)$ are

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

and

$$u^1 = E^1, \quad u^2 = E^2, \quad u^3 = E^3, \quad u^4 = H^1, \quad u^5 = H^2, \quad u^6 = H^3,$$

respectively. The symmetry generators are written in the form

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}.$$

Since the Lagrangian (8.2.26) vanishes on the solutions of Eqs. (8.2.1), one can use the invariance condition (2.3.14) and Eq. (2.3.14) for computing the conserved vectors in the reduced forms

$$X(L) = 0 \tag{8.2.29}$$

and

$$C^i = \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial L}{\partial u_i^\alpha}, \tag{8.2.30}$$

respectively. The divergence condition (2.3.22) and Eq. (2.3.23) for computing the conserved vectors are simplified likewise:

$$X(L) = D_i(B^i) \tag{8.2.31}$$

and

$$C^i = \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial L}{\partial u_i^\alpha} - B^i, \tag{8.2.32}$$

respectively. We will denote

$$C^0 = \tau, \quad C^1 = \chi^1, \quad C^2 = \chi^2, \quad C^3 = \chi^3 \tag{8.2.33}$$

and write the conservation laws in the form (8.2.12):

$$D_t(\tau) + \operatorname{div} \boldsymbol{\chi} = 0.$$

8.2.4.1 Time translation

Let us apply the formula (8.2.30) to the time translation generator

$$X_0 = \frac{\partial}{\partial t}.$$

In this case

$$\xi^0 = 1, \quad \xi^1 = \xi^2 = \xi^3 = 0, \quad \eta^\alpha = 0, \quad \eta^\alpha - \xi^j u_j^\alpha = -u_t^\alpha.$$

Denoting by π_0 and $\boldsymbol{\chi}_0$ the density and the flux of the conserved vector provided by X_0 and using the coordinate form of the Lagrangian (8.2.26), we obtain:

$$\pi_0 = -u_t^\alpha \frac{\partial \mathcal{L}}{\partial u_t^\alpha} = -E_t^k \frac{\partial L}{\partial E_t^k} - H_t^k \frac{\partial L}{\partial H_t^k} = \sum_{k=1}^{3} \left(H^k E_t^k - E^k H_t^k\right),$$

or

$$\pi_0 = \boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t. \tag{8.2.34}$$

The flux components are calculated likewise. For instance,

$$\chi_0^1 = -u_t^\alpha \frac{\partial \mathcal{L}}{\partial u_x^\alpha} = E^2 E_t^3 - E^3 E_t^2 + H^2 H_t^3 - H^3 H_t^2\,.$$

Computing the other components of the flux we obtain

$$\boldsymbol{\chi}_0 = \left(\boldsymbol{E} \times \boldsymbol{E}_t\right) + \left(\boldsymbol{H} \times \boldsymbol{H}_t\right). \tag{8.2.35}$$

Thus, the time translation leads to the conservation equation

$$D_t\left(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\right) + \nabla \cdot \left(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\right) = 0. \tag{8.2.36}$$

The equation should be satisfied on the solutions of Eqs. (8.2.1). The integral form (8.2.13) of the conservation law (8.2.36) is

$$\frac{d}{dt} \int \left(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\right) dx dy dz = 0. \tag{8.2.37}$$

Remark 8.2.4. Upon eliminating in (8.2.34) the time-derivatives by using Eqs. (8.2.1) the differential conservation law (8.2.36) is written in the form

$$D_t\left[\boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) + \boldsymbol{H} \cdot (\nabla \times \boldsymbol{H})\right] + \operatorname{div}\left[\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\right] = 0.$$

It was discovered in this form by Lipkin (49) (see also (64)).

8.2.4.2 Spatial translations

Applying the above procedure to the generators of spatial translations X_1, X_2, X_3 from (8.2.3), one obtains the conservation densities

$$\begin{aligned} \pi_1 &= \boldsymbol{H} \cdot \boldsymbol{E}_x - \boldsymbol{E} \cdot \boldsymbol{H}_x, \\ \pi_2 &= \boldsymbol{H} \cdot \boldsymbol{E}_y - \boldsymbol{E} \cdot \boldsymbol{H}_y, \\ \pi_3 &= \boldsymbol{H} \cdot \boldsymbol{E}_z - \boldsymbol{E} \cdot \boldsymbol{H}_z \end{aligned} \tag{8.2.38}$$

and the fluxes

$$\begin{aligned}\boldsymbol{\chi}_1 &= (\boldsymbol{E}\times\boldsymbol{E}_x) + (\boldsymbol{H}\times\boldsymbol{H}_x),\\ \boldsymbol{\chi}_2 &= (\boldsymbol{E}\times\boldsymbol{E}_y) + (\boldsymbol{H}\times\boldsymbol{H}_y),\\ \boldsymbol{\chi}_3 &= (\boldsymbol{E}\times\boldsymbol{E}_z) + (\boldsymbol{H}\times\boldsymbol{H}_z).\end{aligned} \tag{8.2.39}$$

The corresponding conservation equations are written

$$D_t(\pi_k) + \nabla\cdot(\boldsymbol{E}\times\boldsymbol{E}_k + \boldsymbol{H}\times\boldsymbol{H}_k) = 0, \quad k = 1, 2, 3, \tag{8.2.40}$$

where

$$\boldsymbol{E}_k = \frac{\partial \boldsymbol{E}}{\partial x^k}, \quad \boldsymbol{H}_k = \frac{\partial \boldsymbol{H}}{\partial x^k}.$$

The integral conservation laws are written in the vector form

$$\frac{d}{dt}\int \boldsymbol{\pi}\, dxdydz = 0, \tag{8.2.41}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (8.2.38).

8.2.4.3 Rotations

The invariance under the rotations with the generators X_{12}, X_{13}, X_{23} from (8.2.3) provides the vector valued integral conservation law

$$\frac{d}{dt}\int \left[2(\boldsymbol{E}\times\boldsymbol{H}) + (\boldsymbol{x}\times\boldsymbol{\pi})\right] dxdydz = 0. \tag{8.2.42}$$

The corresponding flux can be found in (34).

8.2.4.4 Duality rotations

The duality generator Z_0 given by Eq. (8.2.9) gives the conservation density

$$\tau = \frac{1}{2}(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2)$$

and the flux $\boldsymbol{\chi}$ is the Poynting vector:

$$\boldsymbol{\chi} = (\boldsymbol{E}\times\boldsymbol{H}).$$

Thus duality rotations lead to the well known law of conservation of energy

$$D_t\left(\frac{|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2}{2}\right)\Bigg|_{(8.2.1)} + \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = 0.$$

It is written in the integral form as follows (see Example 8.2.1):

$$\frac{d}{dt}\int \frac{1}{2}(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2) dxdydz = 0. \tag{8.2.43}$$

8.2.4.5 Dilations

The invariance test (8.2.29) is satisfied for the dilation generators Z_1 and Z_2 given by (8.2.6) and (8.2.7), respectively. Accordingly, Equation (8.2.30) for constructing conserved vectors is applicable. However, the calculation shows that the generator Z_1 of dilations of the dependent variables gives $\tau = 0, \boldsymbol{\chi} = 0$, and hence it does not provide a nontrivial conservation law.

Let us consider the dilations of the independent variables with the generator Z_2. For this operator, Equation (8.2.30) for $\tau = C^0$ is written

$$\tau = -\big(t\boldsymbol{E}_t + x\boldsymbol{E}_x + y\boldsymbol{E}_y + z\boldsymbol{E}_z\big)\frac{\partial L}{\partial \boldsymbol{E}_t} - \big(t\boldsymbol{H}_t + x\boldsymbol{H}_x + y\boldsymbol{H}_y + z\boldsymbol{H}_z\big)\frac{\partial L}{\partial \boldsymbol{H}_t}.$$

Substituting the expression (8.2.26) for the Lagrangian one obtains

$$\tau = t\pi_0 + \boldsymbol{x} \cdot \boldsymbol{\pi}, \tag{8.2.44}$$

where π_0 is given by (8.2.34) and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (8.2.38). The similar calculations give the flux

$$\boldsymbol{\chi} = t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3, \tag{8.2.45}$$

where the vector $\boldsymbol{\chi}_0$ is given by Eq. (8.2.35), and the vectors $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \boldsymbol{\chi}_3$ are given by Eqs. (8.2.39). Thus, the dilations of the independent variables give the conservation law

$$D_t\big(t\pi_0 + \boldsymbol{x} \cdot \boldsymbol{\pi}\big) + \nabla \cdot \big(t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3\big) = 0. \tag{8.2.46}$$

Its integral representation is

$$\frac{d}{dt}\int \big(t\pi_0 + \boldsymbol{x} \cdot \boldsymbol{\pi}\big)\, dxdydz = 0. \tag{8.2.47}$$

8.2.4.6 Conservation laws provided by superposition

For the superposition generator (8.2.8),

$$S = \boldsymbol{E}_*(\boldsymbol{x}, t) \cdot \frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}_*(\boldsymbol{x}, t) \cdot \frac{\partial}{\partial \boldsymbol{H}},$$

Equation (8.2.30) for calculating the conservation density is written

$$\tau = \boldsymbol{E}_*\frac{\partial \mathcal{L}}{\partial \boldsymbol{E}_t} + \boldsymbol{H}_*\frac{\partial \mathcal{L}}{\partial \boldsymbol{H}_t}$$

and yields:

$$\tau = \boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H}. \tag{8.2.48}$$

The flux components are computed likewise, e.g.

$$\chi^1 = E_*^k \frac{\partial \mathcal{L}}{\partial E_x^k} + H_*^k \frac{\partial \mathcal{L}}{\partial H_x^k} = E_*^2 E^3 - E_*^3 E^2 + H_*^2 H^3 - H_*^3 H^2.$$

Computing the other flux components we obtain the following flux:

$$\boldsymbol{\chi} = \big(\boldsymbol{E}_* \times \boldsymbol{E}\big) + \big(\boldsymbol{H}_* \times \boldsymbol{H}\big). \tag{8.2.49}$$

Thus, the linear superposition principle provides the infinite set of conservation laws

$$D_t\big(\boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H}\big) + \nabla \cdot \big(\boldsymbol{E}_* \times \boldsymbol{E} + \boldsymbol{H}_* \times \boldsymbol{H}\big) = 0. \tag{8.2.50}$$

Their integral form is [2]

$$\frac{d}{dt}\int (\boldsymbol{H}_* \cdot \boldsymbol{E} - \boldsymbol{E}_* \cdot \boldsymbol{H})\, dxdydz = 0. \tag{8.2.51}$$

Here the couple $\boldsymbol{E}_*, \boldsymbol{H}_*$ is any solution of the evolutionary part of Maxwell's equations, in particular, it may be a solution of the system (8.2.1) and (8.2.2).

Example 8.2.3. Taking an arbitrary constant solution

$$\boldsymbol{E}_* = \text{const.}, \quad \boldsymbol{H}_* = \text{const.}$$

we obtain from (8.2.51) the conservation equations

$$\frac{d}{dt}\int \boldsymbol{E}\, dxdydz = 0, \quad \frac{d}{dt}\int \boldsymbol{H}\, dxdydz = 0 \tag{8.2.52}$$

corresponding to the particular cases of the superposition symmetry (8.2.8),

$$S_1 = \frac{\partial}{\partial \boldsymbol{E}}, \quad S_2 = \frac{\partial}{\partial \boldsymbol{H}}.$$

Example 8.2.4. Substituting in (8.2.51) the traveling wave solution

$$\boldsymbol{E}_* = \begin{pmatrix} 0 \\ f(x-t) \\ g(x-t) \end{pmatrix}, \quad \boldsymbol{H}_* = \begin{pmatrix} 0 \\ -g(x-t) \\ f(x-t) \end{pmatrix}$$

of Eqs. (8.2.1) and (8.2.2) we obtain the conservation law

$$\frac{d}{dt}\int \big\{f(x-t)\,[E^3 - H^2] - g(x-t)\,[E^2 + H^3]\big\}\, dxdydz = 0 \tag{8.2.53}$$

with two arbitrary functions, $f(x-t)$ and $g(x-t)$.

Remark 8.2.5. If $(\boldsymbol{E}, \boldsymbol{H})$ is identical with the solution $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, then the conservation law (8.2.50) is trivial because its density and the flux vanish. This is in accordance with the fact that when $\boldsymbol{E}_* = \boldsymbol{E},\ \boldsymbol{H}_* = \boldsymbol{H}$ the superposition generator S coincides with the dilation generator Z_1 which does not provide a nontrivial conservation law.

2 The conservation law (8.2.51) should not be confused with the *Lorentz reciprocity theorem* expressed by the equation $\nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{H}_* - \boldsymbol{E}_* \times \boldsymbol{H}\big) = 0$, see (24), Sections 3—8.

8.2.5 Conservation laws of Eqs. (8.2.1) and (8.2.2)

Bessel-Hagen (6) applied Noether's theorem to the the 15-parameter conformal group and, using the variational formulation of Maxwell's equations in terms of the 4-potential for the electromagnetic field, derived 15 conservation laws. In this way, he obtained, along with the well-known theorems on conservation of energy, momentum, angular momentum and the relativistic center-of-mass theorem, five new conservation laws. He wrote about the latter: "The future will show if they have any physical significance". To the best of my knowledge, a physical interpretation and utilization of Bessel-Hagen's new conservation laws is still an open question. Meanwhile, Lipkin (49) discovered ten new conservation laws involving first derivatives of the electric and magnetic vectors. It was shown later (64) that Lipkin's conservation laws were associated with translations and Lorentz transformations. For a review of further investigations in this direction, see (18) and (30).

We will present here the conservation laws obtained applying Noether's theorem to the Lagrangian of the evolutionary part of Maxwell's equations. In passing from Eqs. (8.2.1) to the whole system (8.2.1) and (8.2.2) of Maxwell's equations, the conservation laws associated with the translations, the duality rotations, the dilations and the linear superposition do not change. However, the rotations, the Lorentz transformations and the conformal transformations lead to new conservation laws for the system (8.2.1) and (8.2.2).

8.2.5.1 Splitting of conservation law (8.2.42)

According to Example (8.2.2), the Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ is not a conservation density for the evolutionary part (8.2.1) of Maxwell's equation, but is a conservation density for the system (8.2.1) and (8.2.2). Consequently, the conservation law (8.2.42),

$$\frac{d}{dt}\int \left[2\big(\boldsymbol{E} \times \boldsymbol{H}\big) + \big(\boldsymbol{x} \times \boldsymbol{\pi}\big)\right] dxdydz = 0 \tag{8.2.42}$$

for the evolutionary part of Maxwell's equation gives rise to two different conservation laws, namely the conservation of the linear momentum

$$\frac{d}{dt}\int \big(\boldsymbol{E} \times \boldsymbol{H}\big)\, dxdydz = 0 \tag{8.2.54}$$

and the independent conservation law

$$\frac{d}{dt}\int \big(\boldsymbol{x} \times \boldsymbol{\pi}\big)\, dxdydz = 0, \tag{8.2.55}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (8.2.38).

Thus, the rotation generators X_{12}, X_{13}, X_{23} from (8.2.3) provide *two* vector valued conservation laws (8.2.54), (8.2.55) for Eqs. (8.2.1) and (8.2.2)

8.2.5.2 Conservation laws due to Lorentz symmetry

By the action of the generators (8.2.4) of the Lorentz transformations on the Lagrangian (8.2.26) we obtain, e.g.

$$X_{0i}(L)\big|_{(8.2.1)} = -E^i(\nabla \cdot \boldsymbol{H}) + H^i(\nabla \cdot \boldsymbol{E}),$$

where X_{0i} is prolonged to the derivatives involved in L. Therefore the operators X_{0i} satisfy the invariance test (8.2.29) on the solutions of the Maxwell equations (8.2.1) and (8.2.2):

$$X_{0i}(L)\big|_{(8.2.1)-(8.2.2)} = 0.$$

Consequently, the conservation laws associated with the Lorentz transformations can be computed by means of Eqs. (8.2.30). This procedure shows that the Lorentz invariance provides the following vector valued conservation law for Eqs. (8.2.1) and (8.2.2):

$$\frac{d}{dt}\int (\pi_0 \boldsymbol{x} - t\boldsymbol{\pi})\, dxdydz = 0, \tag{8.2.56}$$

where π_0 is given by Eq. (8.2.34), and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (8.2.38).

8.2.5.3 Conservation laws due to conformal symmetry

The generators (8.2.5) of the conformal transformation satisfy the invariance test (8.2.29). Hence the conservation laws associated with the Lorentz transformations can be constructed by means of Eqs. (8.2.30). The calculation shows that the generators Y_1, Y_2, Y_3 from (8.2.5) lead to the vector valued conservation density

$$\boldsymbol{\tau} = -4\left[\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H})\right] + 2(\boldsymbol{x} \cdot \boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}.$$

Using the test for conservation densities, given by Theorem 8.2.3 in Section 8.2.3, one can verify that the terms of $\boldsymbol{\tau}$ that are linear in the independent variables and those quadratic in these variables provide two independent conservation densities. Therefore we conclude that the invariance of Maxwell's equations (8.2.1) and (8.2.2) with respect to the conformal transformations generated by Y_1, Y_2, Y_3 provides two vector valued conservation laws. Namely, the conservation of the angular momentum

$$\frac{d}{dt}\int \big[\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H})\big] dxdydz = 0, \tag{8.2.57}$$

and the following new vector valued conservation law:

$$\frac{d}{dt}\int \big[2(\boldsymbol{x} \cdot \boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}\big] dxdydz = 0, \tag{8.2.58}$$

where $\boldsymbol{x} = (x, y, z)$, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$.

The operator Y_4 from (8.2.5) leads to the conservation law

$$\frac{d}{dt}\int \big[2t(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (t^2 + x^2 + y^2 + z^2)\pi_0\big]dxdydz = 0. \tag{8.2.59}$$

8.2.5.4 Summary of conservation laws

The Noether theorem can be applied to over-determined systems of differential equations, provided that the system in question contains a sub-system having a Lagrangian. In the case of the Maxwell equations the appropriate sub-system is provided by the evolution equations (8.2.1) The results on conservation laws obtained in the previous sections are summarized in the following theorem. For the sake of brevity, the conservation equations are written in the integral form.

Theorem 8.2.5. The symmetries of Eqs. (8.2.1) and (8.2.2) and the Lagrangian (8.2.26) of the evolutionary part (8.2.1) of Maxwell's equations provide the following conservation laws for the Maxwell equations (8.2.1) and (8.2.2).

Classical conservation laws:

$$\frac{d}{dt}\int \big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)dxdydz = 0 \quad \text{(energy conservation)}, \tag{8.2.60}$$

$$\frac{d}{dt}\int \big(\boldsymbol{E}\times\boldsymbol{H}\big)\,dxdydz = 0 \qquad \text{(linear momentum)}, \tag{8.2.61}$$

$$\frac{d}{dt}\int \big[\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{H})\big]dxdydz = 0 \quad \text{(angular momentum)}. \tag{8.2.62}$$

Non-classical conservation laws:

$$\frac{d}{dt}\int \pi_0\,dxdydz = 0, \tag{8.2.63}$$

$$\frac{d}{dt}\int \boldsymbol{\pi}\,dxdydz = 0, \tag{8.2.64}$$

$$\frac{d}{dt}\int \big(\boldsymbol{x}\times\boldsymbol{\pi}\big)\,dxdydz = 0, \tag{8.2.65}$$

$$\frac{d}{dt}\int \big(t\pi_0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big)\,dxdydz = 0, \tag{8.2.66}$$

$$\frac{d}{dt}\int (\pi_0\boldsymbol{x} - t\boldsymbol{\pi})\,dxdydz = 0, \tag{8.2.67}$$

$$\frac{d}{dt}\int \big[2(\boldsymbol{x}\cdot\boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}\big]dxdydz = 0, \tag{8.2.68}$$

$$\frac{d}{dt}\int \big[2t(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (t^2 + x^2 + y^2 + z^2)\pi_0\big]dxdydz = 0, \tag{8.2.69}$$

$$\frac{d}{dt}\int \big(\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}\big)\,dxdydz = 0. \tag{8.2.70}$$

Here π_0 is given by Eq. (8.2.34) and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (8.2.38). The infinite set of conservation laws (8.2.70) involves an arbitrary solution $(\boldsymbol{E}_*, \boldsymbol{H}_*)$ of the Maxwell equations (8.2.1) and (8.2.2).

8.2.5.5 Symmetries associated with conservation laws

The correspondence between the conservation laws summarized above and the symmetries of the Maxwell equations is given in the following table.

Conservation law	Symmetry
Energy conservation (8.2.60)	Duality symmetry Z_0 (8.2.9)
Linear momentum (8.2.61)	Rotation symmetries X_{12}, X_{13}, X_{23} (8.2.3)
Angular momentum (8.2.62)	Conformal symmetries Y_1, Y_2, Y_3 from (8.2.5)
Conservation law (8.2.63)	Time translation X_0 from (8.2.3)
Conservation law (8.2.64)	Spatial translations X_1, X_2, X_3 from (8.2.3)
Conservation law (8.2.65)	Rotation symmetries X_{12}, X_{13}, X_{23} (8.2.3)
Conservation law (8.2.66)	Space-time dilation Z_2 (8.2.7)
Conservation law (8.2.67)	Lorentz symmetries X_{01}, X_{02}, X_{03} (8.2.4)
Conservation law (8.2.68)	Conformal symmetries Y_1, Y_2, Y_3 from (8.2.5)
Conservation law (8.2.69)	Conformal symmetry Y_4 from (8.2.5)
Conservation law (8.2.70)	Superposition symmetry S (8.2.8)

8.3 The Dirac equation

One of fundamental equations in quantum mechanics is the Dirac equation

$$m\psi + \gamma^k \frac{\partial \psi}{\partial x^k} = 0, \quad m = \text{const.} \tag{8.3.1}$$

The dependent variable $\psi = (\psi^1, \psi^2, \psi^3, \psi^4)$ is a four-dimensional column vector with complex valued components. The independent variables compose the four-dimensional vector $x = (x^1, x^2, x^3, x^4)$, where x^1, x^2, x^3 are the real valued spatial variables and x^4 is the complex variable defined by $x^4 = ict$ with t being time and c the light velocity. Furthermore, γ^k are the following 4×4 complex matrices called the Dirac matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

8.3.1 Lagrangian obtained from the formal Lagrangian

Four complex equations (8.3.1) can be written as eight real valued equations by considering the Dirac equation together with the conjugate equation

$$m\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k = 0. \tag{8.3.2}$$

Here $\widetilde{\psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3, \widetilde{\psi}_4)$ is the four-dimensional row vector defined by

$$\widetilde{\psi} = \psi^{*T}\gamma^4, \tag{8.3.3}$$

where the indices $*$ and T indicate the transition to the complex conjugate and the transposition, respectively. The components of the row vector $\widetilde{\psi}$ will be denoted by subscripts

Taking the formal Lagrangian (2.3.50) for the system (8.3.1) and (8.3.2) in the form

$$\mathcal{L} = u\left(m\psi + \gamma^k \frac{\partial \psi}{\partial x^k}\right) + \left(m\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k\right) v, \tag{8.3.4}$$

where u is a four-dimensional column vector, v is a four-dimensional row vector, we obtain the following adjoint system (2.3.48):

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \psi} &\equiv mu - \frac{\partial u}{\partial x^k}\gamma^k = 0, \\ \frac{\delta \mathcal{L}}{\delta \widetilde{\psi}} &\equiv mv + \gamma^k \frac{\partial v}{\partial x^k} = 0. \end{aligned} \tag{8.3.5}$$

The adjoint system (8.3.5) coincides with Eqs. (8.3.1) and (8.3.2) upon setting

$$v = \psi, \quad u = \widetilde{\psi}. \tag{8.3.6}$$

Hence, the system (8.3.1) and (8.3.2) is self-adjoint. Therefore we follow the procedure described in (33), Theorem 2.4. Namely, we substitute in the formal Lagrangian (8.3.4) the values of v and u given by Eqs. (8.3.6), divide by two and obtain the following Lagrangian [3] for the system (8.3.1) and (8.3.2):

$$L = \frac{1}{2}\left[\widetilde{\psi}\left(m\psi + \gamma^k \frac{\partial \psi}{\partial x^k}\right) + \left(m\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k\right)\psi\right]. \tag{8.3.7}$$

3 It is well known in the literature, see e.g. (7).

One can readily verify that the Euler-Lagrange equations (2.3.6) for this L give the system (8.3.1) and (8.3.2):

$$\frac{\delta L}{\delta \psi} = m\widetilde{\psi} - \frac{\partial \widetilde{\psi}}{\partial x^k}\gamma^k, \quad \frac{\delta \mathcal{L}}{\delta \widetilde{\psi}} = m\psi + \gamma^k \frac{\partial \psi}{\partial x^k}.$$

8.3.2 Symmetries

The material of this and the next section is taken from (27), §29.

We note first of all that the Dirac equations (8.3.1) and (8.3.2) are linear and homogeneous. Accordingly, they admit the infinite group G_+ with the superposition generator

$$X_\varphi = \varphi^k(x)\frac{\partial}{\partial \psi^k} + \widetilde{\varphi}_k(x)\frac{\partial}{\partial \widetilde{\psi}_k} \tag{8.3.8}$$

and the following generator of multiplication by a real valued parameter:

$$X_d = \psi^k \frac{\partial}{\partial \psi^k} + \widetilde{\psi}_k \frac{\partial}{\partial \widetilde{\psi}_k} \tag{8.3.9}$$

Here the column vector $\varphi(x)$ with the components $\varphi^1(x), \ldots, \varphi^4(x)$ is any solution of Eq. (8.3.1). The row vector $\widetilde{\varphi}(x) = (\widetilde{\varphi}_1(x), \widetilde{\varphi}_2(x), \widetilde{\varphi}_3(x), \widetilde{\varphi}_4(x))$ is obtained from $\varphi(x)$ by the formula (8.3.3) and solves the conjugate equation (8.3.2). The group G^+ is a normal subgroup of the full group admitted by Eqs. (8.3.1) and (8.3.2). Hereafter, by the admitted group we shall mean the quotient group G/G^+ by the normal subgroup G_+.

The Dirac equation (8.3.1) is a relativistic equation, i.e. it is Lorentz invariant. Namely, Eqs. (8.3.1) and (8.3.2) admit the 10-parameter group with the generators

$$X = X^0 + (S\psi)^k \frac{\partial}{\partial \psi^k} + (\widetilde{\psi}\widetilde{S})_k \frac{\partial}{\partial \widetilde{\psi}_k}, \tag{8.3.10}$$

where the operator

$$X^0 = \xi^k(x)\frac{\partial}{\partial x^k} \tag{8.3.11}$$

runs over the set of the generators (see (4.3.8))

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} \quad (i < j), \quad (i, j = 1, \ldots, 4), \tag{8.3.12}$$

of the Lorentz group, S and $\widetilde{S}$ are the following 4×4-matrices:

$$S = \frac{1}{8}\sum_{k,l=4}^{4} \frac{\partial \xi^k}{\partial x^l}\left(\gamma^k\gamma^l - \gamma^l\gamma^k - 3\delta^{kl}\right), \quad \widetilde{S} = \gamma^4 S^{*T}\gamma^4. \tag{8.3.13}$$

The Lorentz group is not the largest symmetry group of the Dirac equations. First we note that these equations are invariant not only under the real valued dilations generated by the operator (8.3.9), but under the multiplication by any complex number as well. Therefore Eqs. (8.3.1) and (8.3.2) admit the phase transformations

$$\overline{\psi} = \psi \mathrm{e}^{ia}, \quad \overline{\widetilde{\psi}} = \widetilde{\psi}\, \mathrm{e}^{-ia}. \tag{8.3.14}$$

The second equation in (8.3.14) is derived from the first one merely by definition (8.3.3):

$$\overline{\widetilde{\psi}} = \overline{\psi}^{\,*T} \gamma^4. \tag{8.3.15}$$

Furthermore, it is possible that Eqs. (8.3.1) and (8.3.2) admit transformations mixing the variables ψ and $\widetilde{\psi}$. Indeed, is has been demonstrated in (26) that there are precisely two one-parameter groups of this type, namely:

$$\overline{\psi} = \psi \cosh a + \gamma^4 \gamma^2 \widetilde{\psi}^T \sinh a, \tag{8.3.16}$$

and

$$\overline{\psi} = \psi \cosh a + i\gamma^4 \gamma^2 \widetilde{\psi}^T \sinh\, a. \tag{8.3.17}$$

The transformations (8.3.16) and (8.3.17) should be completed by adding the transformations of the variable $\widetilde{\psi}$ obtained via Eq. (8.3.15).

The result is formulated as follows.

Theorem 8.3.1. The maximal quotient group G/G^+ of point transformations admitted by Eqs. (8.3.1) and (8.3.2) is the 13-parameter group consisting of the 10-parameter Lorentz group with the generators (8.3.10)—(8.3.13) and three one-parameter groups (8.3.14),(8.3.16), (8.3.17).

Consider now the case $m = 0$. The conformal invariance of Eq. (8.3.1) with $m = 0$ was established by Dirac (13). Then Pauli (57) discovered three more symmetries. Namely, he showed that Eqs. (8.3.1) and (8.3.2) with $m = 0$ admit the following three one-parameter groups:

$$\overline{\psi} = \psi \cos a + \gamma^3 \gamma^1 \widetilde{\psi}^T \sin a, \tag{8.3.18}$$

$$\overline{\psi} = \psi \cos a + i\gamma^3 \gamma^1 \widetilde{\psi}^T \sin a, \tag{8.3.19}$$

$$\overline{\psi} = \psi \cosh a - \gamma^5 \psi \sinh a, \tag{8.3.20}$$

The group composed by (8.3.18)—(8.3.20) together with the transformation

$$\overline{\psi} = \psi \cos a + i\gamma^5 \psi \sin a \tag{8.3.21}$$

is known as the 4-parameter Pauli group. The matrix γ^5 used in Eqs. (8.3.20) and (8.3.21) is defined by

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4. \tag{8.3.22}$$

The transformations (8.3.18)—(8.3.21) should be completed by adding the transformations of the variable $\widetilde{\psi}$ obtained via Eq. (8.3.15).

Combining the above results, we obtain the following theorem.

Theorem 8.3.2. The maximal quotient group G/G^+ of point transformations admitted by Eqs. (8.3.1) and (8.3.2) with $m = 0$ is the 22-parameter group. It contains seven one-parameter groups (8.3.14), (8.3.16), (8.3.17), (8.3.18)—(8.3.21) and the 15-parameter conformal group of the Minkowski space with the generators (8.3.10), where $S, \widetilde{S}$ are given by Eqs. (8.3.13), X^0 has the form (8.3.10) and runs over the operators (see (4.3.8))

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} \quad (i < j),\\ Z &= x^i \frac{\partial}{\partial x^i}, \quad Y_i = \left(2x^i x^j - |x|^2 \delta^{ij}\right) \frac{\partial}{\partial x^j}, \quad (i, j = 1, \dots, 4). \end{aligned} \tag{8.3.23}$$

8.3.3 Conservation laws

Theorem 2.3.1 from Section 2.3.2 allows to construct the conservation laws associated with the symmetries presented in Section 8.3.2. Calculation provides the following results (see (27)).

Let $m \neq 0$. In this case Theorem 2.3.1 is applicable to the superposition generator (8.3.8) and to all 13 symmetries given in Theorem 8.3.1, i.e. to the generators (8.3.10)—(8.3.12) of the Lorentz group and to the transformations (8.3.14),(8.3.16), (8.3.17). Equations (2.3.17) of Theorem 2.3.1 associate with these symmetries the conservation laws

$$D_k(A^k_\varphi) = 0, \quad D_k(P^k_i) = 0, \quad D_k(M^k_{ij}) = 0, \quad D_k(C^k_\alpha) = 0 \ (\alpha = 1, 2, 3)$$

with the following conserved vectors.

The superposition generator (8.3.8) provides the conserved vector with the components

$$A^k_\varphi = \widetilde{\psi}\gamma^k \varphi(x) - \widetilde{\varphi}(x)\gamma^k \psi, \quad k = 1, \dots, 4. \tag{8.3.24}$$

Taking all possible solutions $\varphi(x)$ of the Dirac equation (8.3.1), one obtains an infinite set of conserved vectors.

The Lorentz group generators $X_i,\ X_{ij}$ lead to the conserved vectors

$$P^k_i = \frac{1}{2}\left(\frac{\partial \widetilde{\psi}}{\partial x^i}\gamma^k \psi - \widetilde{\psi}\gamma^k \frac{\partial \psi}{\partial x^i}\right), \tag{8.3.25}$$

$$M^k_{ij} = \frac{1}{4}\delta_{is}\delta_{jl}\left[\widetilde{\psi}(\gamma^k\gamma^s\gamma^l + \gamma^s\gamma^l\gamma^k)\psi\right] + \delta_{jl}x^l P^k_i - \delta_{is}x^s P^k_j \quad (j < l), \tag{8.3.26}$$

where $i, j, k = 1, \dots, 4$. The conserved quantities (8.3.25) and (8.3.26) are known in Quantum mechanics as the *energy-momentum tensor* and the *angular momentum tensor*, respectively (7).

The phase transformation (8.3.14) is responsible for conservation of the *electric current*

$$C_1^k = \widetilde{\psi}\gamma^k\psi. \tag{8.3.27}$$

Finally, the transformations (8.3.16), (8.3.17) provide two additional conserved vectors. They are linear combinations of the conserved vectors

$$C_2^k = \psi^T\gamma^4\gamma^2\gamma^k\psi, \tag{8.3.28}$$
$$C_3^k = \widetilde{\psi}\gamma^k\gamma^4\gamma^2\widetilde{\psi}^T. \tag{8.3.29}$$

The linear combination of the conserved vectors (8.3.24)—(8.3.29) contains all conserved vectors associated with the point symmetries of the Dirac equations (8.3.1) and (8.3.2).

Let $m = 0$. Then, using the additional symmetries given in Theorem 8.3.2, one obtains in addition to (8.3.24)—(8.3.29) the following conserved vectors:

$$A^k = x^i P_i^k, \tag{8.3.30}$$
$$B_i^k = 2x^j M_{ji}^k + |x|^2 P_i^k, \tag{8.3.31}$$
$$C_4^k = \widetilde{\psi}\gamma^k\gamma^5\psi. \tag{8.3.32}$$

In Equations (8.3.30) and (8.3.31) P_i^k and M_{ji}^k are the energy-momentum tensor and the angular momentum tensor, respectively.

8.4 General relativity

8.4.1 The Einstein equations

Central for the general relativity are Einstein's field equations

$$R_{ik} - \frac{1}{2}Rg_{ik} = T_{ik}, \quad (i, k = 1, \dots, 4). \tag{8.4.1}$$

Here T_{ik} is the *energy-momentum tensor* defined by a distribution of masses of gas or other particles, electromagnetic fields, etc. It satisfies the condition

$$T^{ik}_{\ ,k} = 0, \tag{8.4.2}$$

where

$$T^{ik} = g^{im}g^{kl}T_{ml}. \tag{8.4.3}$$

The left-hand side

$$R_{ik} - \frac{1}{2} R g_{ik}$$

of Eq. (8.4.1) is a tensor known as the *Einstein tensor.*

A remarkable property of the Einstein equations (8.4.1) is their general covariance, i.e., the independence from the choice of coordinate systems.

In the empty space, i.e. when

$$T_{ik} = 0,$$

the Einstein equations (8.4.1) take the form (see Exercise 8.13)

$$R_{ik} = 0, \quad (i, k = 1, \dots, 4). \tag{8.4.4}$$

8.4.2 The Schwarzschild space

One of widely known solutions of the Einstein equations (8.4.4) is provided by the Schwarzschild space with the metric

$$ds^2 = \left(1 - \frac{1}{r}\right) dt^2 - \left(1 - \frac{1}{r}\right)^{-1} dr^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{8.4.5}$$

It is obtained by substituting the general central-symmetric metric

$$ds^2 = A(t, r)dt^2 + B(t, r)dr^2 + C(t, r)(\sin^2\theta d\varphi^2 + d\theta^2) + D(t, r)drdt \tag{8.4.6}$$

in Einstein's empty space field equations (8.4.4). Here t is time, and r, θ, and φ are the spherical coordinates (cf. (3.2.1) and (3.2.2)). Central symmetry means that the space V_4 with the metric (8.4.6) admits the three-parameter group of space rotations as the group of isometric motions.

8.4.3 Discussion of Mercury's parallax

In the concluding "Common scholium" of his "Principia" (53), Newton wrote: "The gravity to the Sun ... is reduced with the distance to the Sun. This reduction is proportional to the square of the distance, which follows from the fact that planets' aphelions are at rest, and even up to the farthest comets' aphelions provided that these aphelions are at rest. But so far I have not been able to derive the reason for these gravity properties from the phenomena, and I do not devise any hypotheses... It is enough to know that gravity really exists, acts according to the laws stated and is sufficient to explain all motions of celestial bodies."

About 150 years later it was however found that the planets' aphelions (or perihelions) are not at rest but are slowly moving. For example, the observed parallax of Mercury is about $43''$ per century. It is very small. However, small effects are sometimes of fundamental significance for the theory, all the more so if we deal with description of real phenomena. Specifically, the anomaly in the motion of Mercury has not been given a satisfactory explanation on the basis of Newton's gravitation law, despite the efforts of greatest scientists. The explanation given by Einstein in 1915 was the first experimental justification of the general relativity theory. All these facts are well known, and a wonderful critical survey can be found in (67), Chap. 8, Sec. 6. Here we consider a point that has been mentioned in (29).

Einstein's explanation (16) of the parallax of Mercury's perihelion is based on the assumption that the space near the Sun is not plane but has the Schwarzschild metric (8.4.5). But in passing from the plane Minkowski space to the curved Schwarzschild space the Huygens principle on the existence of a sharp rear front of waves is violated. This is because the Schwarzschild space has a trivial conformal group (see Section 7.3.3).

Thus, in connection with the theoretical explanation of the observed anomaly in the planets' motion a new problem arises. It can be stated as the following alternative.

1. The explanation by passing to the Schwarzschild metric is adequate to the phenomenon in the approximation required. Then the Huygens principle is not valid, and hence light signals undergo distortions. And we should estimate the distortion level from the view point of possible observation.

2. The Huygens principle holds in the real world. Then we should explain the parallax of Mercury's perihelion without any contradiction with this principle. This task requires thorough physical analysis of the equations of motion for a particle in curved space-times with nontrivial conformal group. The problem is facilitated by the fact that such spaces are described completely by the metrics of the form (see Chapter 6)

$$ds^2 = e^{\mu(x)}[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - 2f(x^1 - x^0)dx^2dx^3 - g(x^1 - x^0)(dx^3)^2]$$

depending on two functions f and g of one variable $x^1 - x^0$ and one function $\mu(x)$ depending on all variables $x = (x^0, x^1, x^2, x^3)$. The de Sitter space considered in Chapter 9 is a simple particular case of these spaces.

8.4.4 Solutions based on generalized motions

Schwarzschild's solution opened a way for obtaining different types of solutions to the Einstein equations by seeking spaces V_4 possessing given groups as groups of isometric motions (see, e.g. (58) and the references therein). There are numerous physically interesting solutions obtained in this way. They are invariant solutions in terms of group analysis.

Introduction of generalized motions opened new possibilities and showed that there are the following 20 different types of exact solutions of type $[\delta, \rho]$:

$$\begin{aligned}
\delta = 0: &\ \rho = 3, 2, 1, 0\\
\delta = 1: &\ \rho = 3, 2, 1, 0\\
\delta = 2: &\ \rho = 3, 2, 1, 0\\
\delta = 3: &\ \rho = 3, 2, 1, 0\\
\delta = 4: &\ \rho = 3, 2, 1, 0\\
\delta = 5: &\ \rho = 3, 2, 1, 0
\end{aligned}$$

Here ρ is the *rank* of the solution and δ its *defect.* If a solution has type $[\delta, \rho]$, then δ components of metric tensor g_{ij} depend on 4 variables x, and $4 - \delta$ of g_{ij} depend only on ρ invariant variables $\lambda = \lambda(x)$. The classical approach based on groups of isometric motions, furnish only four types of solution with $\delta = 0$. All of them are are invariant solutions. Invariant solutions of differential equations were introduced by S. Lie. The solutions with defect $\delta \neq 0$ belong to the type of so-called *partially invariant solutions* introduced by L.V. Ovsyannikov (55), (56).

Example 8.4.1. This example illustrates the algorithm for constructing partially invariant solutions with defect $\delta \neq 0$. Let us take the group G_5 with the generators

$$X_i = \frac{\partial}{\partial x^i} \quad (i = 1, \ldots, 4), \quad X_5 = x^1 \frac{\partial}{\partial x^2}$$

and look for a solution of type $[1, 0]$. The requirement $\delta = 1$ yields (see (28), p. 108) that the metric has the form

$$ds^2 = -[f(x)dx^1 - dx^2]^2 + 2dx^1 dx^3 + C(dx^3)^2(dx^4)^2, \quad C = \text{const.},$$

where $f(x) = f(x^1, \ldots, x^4)$ is an arbitrary function. The non-vanishing Christoffel symbols are

$$\begin{array}{llll}
\Gamma^1_{11} = f f_3, & \Gamma^1_{12} = -\dfrac{1}{2} f_3, & \Gamma^2_{11} = f^2 f_3 - f f_2 - f_1, & \\
\Gamma^2_{12} = -\dfrac{1}{2} f f_3, & \Gamma^2_{13} = -\dfrac{1}{2} f_3, & \Gamma^2_{14} = -\dfrac{1}{2} f_4, & \Gamma^3_{11} = f^2 f_2, \\
\Gamma^3_{12} = -f f_2, & \Gamma^3_{13} = -\dfrac{1}{2} f f_3, & \Gamma^3_{22} = f_2, & \Gamma^3_{23} = \dfrac{1}{2} f_3, \\
\Gamma^3_{14} = -\dfrac{1}{2} f f_4, & \Gamma^3_{24} = \frac{1}{2} f_4, & \Gamma^4_{11} = -f f_4, & \Gamma^4_{12} = \frac{1}{2} f_4,
\end{array}$$

where $f_i = \partial f / \partial x^i$. Inserting them in the definition of the Ricci tensor R_{ij} and solving the Einstein equations (8.4.4), we obtain

$$C = 0, \quad f = h(x^1) x^2 + x^4 \sqrt{-2(h'(x^1) + h(x^1)^2)}, \tag{8.4.7}$$

where $h(x^1)$ is any function satisfying the condition

$$h'(x^1) + h(x^1)^2 < 0.$$

The resulting metric

$$ds^2 = -(f dx^1 - dx^2)^2 + 2 dx^1 dx^3 - (dx^4)^2 \tag{8.4.8}$$

with a function $f(x)$ of the form (8.4.7) defines a space V_4 which satisfies Einstein's equations and admits G_5 as a group of motions with defect $\delta = 1$. If $h(x^1)$ is an arbitrary function, the curvature tensor does not vanish, e.g.

$$R^4_{141} = \frac{3}{2}\left[h'(x^1) + a^2\right] \neq 0.$$

Exercises

Exercise 8.1. Verify that the Maxwell equations (8.2.1) and (8.2.2) admit the duality rotations

$$\overline{\boldsymbol{E}} = \boldsymbol{E} \cos\alpha - \boldsymbol{H} \sin\alpha, \quad \overline{\boldsymbol{H}} = \boldsymbol{H} \cos\alpha + \boldsymbol{E} \sin\alpha$$

with the generator (8.2.9).

Exercise 8.2. Show that the components of the vector product defined by Eq. (8.2.2) coincide with the components of vector given by Eq. (1.1.6):

$$(\boldsymbol{a} \times \boldsymbol{b}) = \left(a^2 b^3 - a^3 b^2,\ a^3 b^1 - a^1 b^3,\ a^1 b^2 - a^2 b^1\right).$$

Exercise 8.3. Show that Eq. (8.2.23) gives the same expression for $\nabla \times \boldsymbol{a}$ as Eq. (1.1.13):

$$\nabla \times \boldsymbol{a} = \left(a_y^3 - a_z^2,\ a_z^1 - a_x^3,\ a_x^2 - a_y^1\right).$$

Exercise 8.4. Verify the validity of the second equation (8.2.21) for $D_t(\tau)\big|_{(8.2.1)}$ given by Eq. (8.2.24) in Example 8.2.1.

Exercise 8.5. Verify the validity of Eqs. (8.2.21) by using the expanded form of Eq. (8.2.24),

$$\begin{aligned} D_t(\tau)\big|_{(8.2.1)} = {} & E^1(H_y^3 - H_z^2) + E^2(H_z^1 - H_x^3) + E^3(H_x^2 - H_y^1) \\ & - H^1(E_y^3 - E_z^2) - H^2(E_z^1 - E_x^3) - H^3(E_x^2 - E_y^1). \end{aligned}$$

Exercise 8.6. Verify by straightforward calculations that the conservation equation (8.2.36)

$$D_t\big(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\big) + \nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\big) = 0$$

is satisfied on the solutions of Eqs. (8.2.1). See (34) or (36), Section 3.

Exercise 8.7. Write the Dirac equations (8.3.10), (8.3.1) and (8.3.2) in the expanded form.

Exercise 8.8. Write the operator (8.3.10),

$$X = X^0 + (S\psi)^k \frac{\partial}{\partial \psi^k} + (\widetilde{\psi}\widetilde{S})_k \frac{\partial}{\partial \widetilde{\psi}_k},$$

in the expanded form when X^0 is the rotation generator

$$X^0 = X_{12} \equiv x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}.$$

Exercise 8.9. Verify that the operator X obtained in Exercise 8.8 is admitted by the Dirac equations (8.3.1) and (8.3.2).

Exercise 8.10. Show that the matrix $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ (see Eq. (8.3.22)) satisfies the equation $\gamma^5\gamma^5 = I$, where I is the unit matrix.

Exercise 8.11. Using Exercise 8.10, show that the transformation (8.3.20),

$$\overline{\psi} = \psi \cosh a - \gamma^5 \psi \sinh a,$$

can be written in the form

$$\overline{\psi} = \psi \mathrm{e}^{-a\gamma^5}.$$

Exercise 8.12. Using Exercise 8.10, show that the transformation (8.3.21),

$$\overline{\psi} = \psi \cos a + i\gamma^5 \psi \sin a,$$

can be written in the form

$$\overline{\psi} = \psi \mathrm{e}^{ia\gamma^5}.$$

Exercise 8.13. Obtain the expression (8.3.25) for the energy-momentum tensor P_i^k by applying the conservation formula (2.3.17) to the translation generators X_i from (8.3.12).

Exercise 8.14. Obtain the expression (8.3.26) for the angular momentum tensor M_{ij}^k by applying the conservation formula (2.3.17) to the generators X_{ij} from (8.3.12).

Exercise 8.15. Let V_4 be any Riemannian space with the vanishing Einstein tensor, i.e.

$$R_{ik} - \frac{1}{2} R g_{ik} = 0.$$

Show that the scalar curvature R of V_4 vanishes.

Hint. Use multiplication of the given equation by g^{ij} followed by contraction and see Exercise 3.5.

Chapter 9

Relativity in de Sitter space

In this chapter we discuss the relativistic mechanics in the de Sitter space by using an approximate representation of the de Sitter group. Namely, the de Sitter group is considered as a perturbation of the Poincaré group by a small curvature. In order to illustrate a utility of the approximate approach, a derivation of exact transformations of the de Sitter group is presented, then an independent and simple computation of the first-order approximate representation of this group is suggested. Modified relativistic conservation laws for the free motion of a particle and unusual properties of neutrinos in the de Sitter space are discussed. The material of this chapter is taken from (32).

9.1 The de Sitter space

Two astronomers who live in the de Sitter world and have different de Sitter clocks might have an interesting conversation concerning the real or imaginary nature of some world events.

F. Klein (43)

9.1.1 Introduction

In 1917, de Sitter (11) suggested a solution for Einstein's equations of general relativity and discussed in several papers its potential value for astronomy. This solution is a four-dimensional space-time of constant Riemannian curvature and is called the de Sitter space.

Felix Klein (43) gave a remarkably complete projective-geometric analysis of the de Sitter metric in the spirit of his earlier work (42) where he mentioned (see (42), p. 287) that "what modern physicists call the theory of relativity is the theory of invariants of the four-dimensional space-time x, y, z, t (the Minkowski space) with respect to a certain group, namely the Lorentz group" and suggested to identify the notions "theory of relativity" and "theory of invariants of a group of transformations" thus obtaining numerous types of theories of relativity. The *de Sitter universe* (the space-time of a constant curvature) admitting, like the Minkowski space, a ten-parameter group of isometric motions, *the de Sitter group*, served Klein as a perfect illustration. Klein's idea consisted in using the de Sitter group for developing the relativity theory comprising, along with the light velocity, another empirical constant, namely the curvature of the universe. The latter would encapsulate all three possible types of spaces with constant curvature: elliptic, hyperbolic (the Lobachevsky space), and parabolic (the Minkowski space as a limiting case of zero curvature).

The following opinions of the world authorities in this field give a comprehensive idea of utility and complexity in developing relativistic mechanics in the de Sitter universe.

> Dirac (12):
>
> *"The equations of atomic physics are usually formulated in terms of space-time of the space of the special theory of relativity. ... Nearly all of the more general spaces have only trivial groups of operations which carry the spaces over into themselves, so they spoil the connexion between physics and group theory. There is an exception, however, namely the de Sitter space (without no local gravitational fields). This space is associated with a very interesting group, and so the study of the equations of atomic physics in this space is of special interest, from mathematical point of view."*
>
> Synge ((66), Chapter VII):
>
> *"The success of the special theory of relativity in dealing with those phenomena which do not involve gravitation suggests that, if we are to work instead with a de Sitter universe, the curvature K must be very small in comparison with significant physical quantities of like dimensions (K has the dimension of $\sec^{-2}$). Without good reason one does not feel inclined to complicate the simplicity of the Minkowski space-time by introducing curvature. Nevertheless the de Sitter universe is interesting in itself. It opens up new vistas, introducing us to the idea that space may be finite, and this seems to satisfy some mental need in us, for infinity is one of those things which we find difficulty in comprehending."*
>
> Gürsey (22):
>
> *"Since the de Sitter group gives such a close approximation to the empirical Poincaré group and, in addition, is based on the structure of the observed universe in accordance with Mach's principle, there seems now sufficient motivation for the detailed study of this group.*

What can we expect from such study? The curvature of our universe being as small as it is, can we hope to obtain any results not already given by the Lorentz group? A tentative answer is that we should be optimistic for two reasons. Firstly, the translation group no longer being valid in the de Sitter space, we shall lose the corresponding laws of energy and momentum conservation. The deviations from these laws with the usual definitions of energy and momentum based on the Poincaré group should manifest themselves in the cosmological scale. These laws, however, will be replaced by the laws of the conservation of observable corresponding to the new displacement operators which in the de Sitter group have taken the place of translations. It is in the light of these new definitions that such vague motions as the creation of matter in the expanding universe should be discussed. Secondly, we should see if new results are implied for elementary particles. Because the structure of the de Sitter group is so widely different from the Poincaré group its representations will have a totally different character, so that the concept of particle, that we have come to associate with the representations of a kinematical group will need revision. Another point worth investigating is the possibility of having new symmetry principles connected with the global properties of the de Sitter group."

One of obstructing factors is that in the de Sitter group usual shifts of space-time coordinates are substituted by transformations of a rather complex form (see Section 1.3). Therefore invariants and invariant equations also become much more complicated as compared to Lorentz-invariant equations. In order to simplify the theory of dynamics in the de Sitter space by preserving its qualitative characteristics, an approximate group approach is employed here. The approach is based on recently developed theory of approximate groups with a small parameter (2). In this connection the de Sitter group is considered as a perturbation of the Poincaré group by a small constant curvature K. The *approximate representation* of the de Sitter group obtained in this way can be dealt with as easily as the Poincaré group. To elicit possible new effects specific for dynamics in the de Sitter universe it is sufficient to calculate first-order perturbations with respect to the curvature of the universe because, according to cosmological data, the curvature is a small constant $K \sim 10^{-54}\ \text{cm}^{-2}$.

9.1.2 Reminder of the notation

See Chapter. 3. Let V_n be an n-dimensional Riemannian space with the metric

$$ds^2 = g_{ij}(x)dx^i dx^j, \tag{9.1.1}$$

where $x = (x^1, \dots, x^n)$. We use the convention on summation in repeated indices. It is assumed that the matrix $\|g_{ij}(x)\|$ is symmetric, $g_{ij}(x) = g_{ji}(x)$, and is non-degenerate, i.e. $\det\|g_{ij}(x)\| \neq 0$ in a generic point $x \in V_n$. Hence, there exists the inverse matrix $\|g_{ij}(x)\|^{-1}$ with the entries denoted by $g^{ij}(x)$. By definition of an inverse matrix, one has $g_{ij}g^{jk} = \delta_i^k$.

The equations of geodesics in V_n are written (see Section 3.2.2)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i = 1, \dots, n, \tag{9.1.2}$$

where the coefficients Γ^i_{jk}, known as the *Christoffel symbols,* are given by

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{9.1.3}$$

It is manifest from (9.1.3) that $\Gamma^i_{jk} = \Gamma^i_{kj}$.

With the aid of Christoffel symbols one defines the covariant differentiation of tensors on the Riemannian space V_n. Covariant differentiation takes tensors again into tensors. Covariant derivatives, e.g. of a *scalar* a, and of covariant and contravariant vectors a_i and a^i are defined as follows:

$$a_{,k} = \frac{\partial a}{\partial x^k}.$$
$$a_{i,k} = \frac{\partial a_i}{\partial x^k} - a_j\Gamma^j_{ik}.$$
$$a^i_{,k} = \frac{\partial a^i}{\partial x^k} + a^j\Gamma^i_{jk}.$$

The covariant differentiation will be indicated by a subscript preceded by a comma. For repeated covariant differentiation we use only one comma, e.g. $a_{i,jk}$ denotes the second covariant derivative of a covariant vector a_i.

The covariant differentiation of higher-order covariant, contravariant and mixed tensors is obtained merely by iterating the above formulae. For example, in the case of second-order tensors we have

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{il}\Gamma^l_{jk} - a_{lj}\Gamma^l_{ik},$$
$$a^{ij}_{\ \ ,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{il}\Gamma^j_{lk} + a^{lj}\Gamma^i_{lk},$$
$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^l_j\Gamma^i_{lk} - a^i_l\Gamma^l_{jk}.$$

For scalars the double covariant derivative does not depend on the order of differentiation, $a_{,jk} = a_{,kj}$. Indeed,

$$a_{,jk} = \frac{\partial^2 a}{\partial x^j\partial x^k} - \Gamma^i_{jk}\frac{\partial a}{\partial x^i}, \quad a_{,kj} = \frac{\partial^2 a}{\partial x^k\partial x^j} - \Gamma^i_{kj}\frac{\partial a}{\partial x^i}, \tag{9.1.4}$$

and the equation $\Gamma^i_{jk} = \Gamma^i_{kj}$ yields that $a_{,jk} = a_{,kj}$. However, this similarity with the usual differentiation is violated, in general, when dealing with vectors and tensors of higher order. Namely, one can prove that

$$\begin{aligned} a_{i,jk} &= a_{i,kj} + a_l R^l_{ijk}, \\ a^i_{,jk} &= a^i_{,kj} - a^l R^i_{ljk}, \end{aligned} \tag{9.1.5}$$

etc. Here

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk} \tag{9.1.6}$$

is a mixed tensor of the fourth order called the *Riemann tensor.* It is also used in the form of the covariant tensor of the fourth order defined by

$$R_{hikl} = g_{hl} R^l_{ijk}. \tag{9.1.7}$$

According to Eqs. (9.1.5), the successive covariant differentiations of tensors in V_n permute only if the Riemann tensor vanishes identically:

$$R^l_{ijk} = 0, \qquad i, j, k, l = 1, \ldots, n. \tag{9.1.8}$$

The Riemannian spaces satisfying the condition (9.1.8) are said to be *flat.* The flat spaces are characterized by the following statement:

The metric form (9.1.1) of a Riemannian space V_n can be reduced by an appropriate change of variables x to the form

$$ds^2 = (dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^n)^2$$

in a neighborhood of a regular point x_0 (not only at the point x_0) if and only if the Riemann tensor R^l_{ijk} of V_n vanishes.

Contracting the indices l and k in the Riemann tensor (9.1.6), one obtains the following tensor of the second order called the *Ricci tensor*:

$$R_{ij} \stackrel{\text{def}}{=} R^k_{ijk} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{mk}. \tag{9.1.9}$$

Finally, multiplication of the Ricci tensor by g^{ij} followed by contraction yields the *scalar curvature* of the space V_n:

$$R = g^{ij} R_{ij}. \tag{9.1.10}$$

9.1.3 Spaces of constant Riemannian curvature

Recall the notions of *curvature* and *spaces of constant curvature* introduced by Riemann (59) (for a detailed discussion, see (17), Sections 25—27).

A pair of contravariant vector-fields λ^i and μ^i in a Riemannian space V_n is called an *orientation* in V_n. Riemann introduced a so-called *geodesic surface* S at a point $x \in V_n$ as a locus of geodesics through x in the directions

$$a^i(x) = \tau\lambda^i(x) + \sigma\mu^i(x)$$

with parameters τ and σ. Then the *curvature* K of V_n at $x \in V_n$ is defined as the Gaussian curvature of S. The *Riemannian curvature* K can be expressed via Riemann's tensor $R_{ijkl} = g_{im}R^m_{jkl}$ as follows:

$$K = \frac{R_{ijkl}\lambda^i\mu^j\lambda^k\mu^l}{(g_{ik}g_{jl} - g_{il}g_{jk})\lambda^i\mu^j\lambda^k\mu^l}.$$

Definition 9.1.1. A Riemannian space V_n with the metric (9.1.1) is called a space of a *constant Riemannian curvature* if K is constant, i.e.

$$R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}), \quad K = \text{const.}$$

The following form of the line element (I keep Riemann's notation) in the spaces of constant curvature:

$$ds = \frac{1}{1 + \frac{\alpha}{4}\sum x^2}\sqrt{\sum dx^2}, \quad \alpha = \text{const.}, \tag{9.1.11}$$

was obtained by Riemann ((59), Section II.4), whose name it bears.

In the case of an arbitrary signature, the Riemannian spaces V_n of constant curvature are characterized by the condition that their metric (9.1.1) can be written in appropriate coordinates in the form (see (17), Section 27):

$$ds^2 = \frac{1}{\theta^2}\sum_{i=1}^{n} \epsilon_i (dx^i)^2, \quad \theta = 1 + \frac{K}{4}\sum_{i=1}^{n} \epsilon_i (x^i)^2, \tag{9.1.12}$$

where each $\epsilon_i = \pm 1$, in agreement with the signature of V_n, and $K = \text{const.}$ Equation (9.1.12) is also referred to as Riemann's form of the metric of the spaces of constant curvature.

9.1.4 Killing vectors in spaces of constant curvature

The generators

$$X = \xi^i(x)\frac{\partial}{\partial x^i} \tag{9.1.13}$$

of the *isometric motions* in the space V_n with the metric (9.1.1) are determined by the *Killing equations*

$$\xi^l \frac{\partial g_{ij}}{\partial x^l} + g_{il}\frac{\partial \xi^l}{\partial x^j} + g_{jl}\frac{\partial \xi^l}{\partial x^i} = 0, \quad i, j = 1, \ldots, n. \tag{9.1.14}$$

The solutions $\xi = (\xi^1, \dots, \xi^n)$ of Eqs. (9.1.13) are often referred to as the *Killing vectors*. Let us write Eqs. (9.1.14) for the metric (9.1.12). We have:

$$g_{ij} = \frac{1}{\theta^2}\,\epsilon_i\delta_{ij}, \qquad \frac{\partial g_{ij}}{\partial x^l} = -\frac{K}{\theta^3}\,\epsilon_l x^l \epsilon_i \delta_{ij}, \tag{9.1.15}$$

where δ_{ij} is the Kronecker symbol. The indices i and l in the expressions $\epsilon_i\delta_{ij}$ and $\epsilon_l x^l$, respectively, are fixed (no summation in these indices). Upon substituting the expressions (9.1.15), Equations (9.1.14) become

$$\epsilon_i\delta_{il}\frac{\partial \xi^l}{\partial x^j} + \epsilon_j\delta_{jl}\frac{\partial \xi^l}{\partial x^i} - \frac{K}{\theta}\,\epsilon_i\delta_{ij}\sum_{l=1}^{n}\epsilon_l x^l \xi^l = 0,$$

or (no summation in i and j)

$$\epsilon_i\frac{\partial \xi^i}{\partial x^j} + \epsilon_j\frac{\partial \xi^j}{\partial x^i} - \frac{K}{\theta}\,\epsilon_i\delta_{ij}\sum_{l=1}^{n}\epsilon_l x^l \xi^l = 0, \quad i, j = 1, \dots, n. \tag{9.1.16}$$

The spaces V_n of constant Riemannian curvature are distinguished by the remarkable property that they are the only spaces possessing the largest group of isometric motions, i.e. the maximal number $\frac{n(n+1)}{2}$ of the Killing vectors (see, e.g. (17), Section 71). Namely, integration of Eqs. (9.1.16) yields the $\frac{n(n+1)}{2}$ -dimensional Lie algebra spanned by the operators

$$X_i = \left[\frac{K}{2}x^i x^j + (2-\theta)\epsilon_i\delta^{ij}\right]\frac{\partial}{\partial x^j}, \tag{9.1.17}$$

$$X_{ij} = \epsilon_j x^j\frac{\partial}{\partial x^i} - \epsilon_i x^i\frac{\partial}{\partial x^j} \quad (i < j),$$

where $i, j = 1, \dots, n$, and δ^{ij} is the Kronecker symbol. In the expressions $\epsilon_i x^i$ and $\epsilon_i\delta^{ij}$ the index i is fixed (no summation). Likewise, there is no summation in the index j in the expression $\epsilon_j x^j$.

Remark 9.1.1. The direct solution of the Killing equations (9.1.16) requires tedious calculations. Therefore, I give in Section 9.3.2 a simple method based on our theory of continuous approximate transformation groups.

9.1.5 Spaces with positive definite metric

The spaces V_n of constant Riemannian curvature can be represented as hypersurfaces in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let us dwell on four-dimensional Riemannian spaces V_4.

The surface of the four-dimensional sphere with the radius ρ :

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 + \zeta_5^2 = \rho^2 \tag{9.1.18}$$

in the five-dimensional Euclidean space $\mathbb{R}^5$ represents the Riemannian space V_4 with the positive definite metric having the constant curvature

$$K = \rho^{-2}. \tag{9.1.19}$$

Introducing the coordinates x^μ on the sphere by means of the stereographic projection:

$$x^\mu = \frac{2\zeta_\mu}{1 + \zeta_5 \rho^{-1}}, \quad \mu = 1, \dots, 4, \tag{9.1.20}$$

one can rewrite the metric of the space V_4 in Riemann's form (9.1.12):

$$ds^2 = \frac{1}{\theta^2} \sum_{\mu=1}^{4} (dx^\mu)^2, \tag{9.1.21}$$

where

$$\theta = 1 + \frac{K}{4}\sigma^2, \quad \sigma^2 = \sum_{\mu=1}^{4} (x^\mu)^2. \tag{9.1.22}$$

Remark 9.1.2. The inverse transformation to (9.1.20) has the form

$$\zeta_\mu = \frac{x^\mu}{\theta}, \quad \frac{\zeta_5}{\rho} = \frac{1}{\theta}\left(1 - \frac{K}{4}\sigma^2\right). \tag{9.1.23}$$

It is obtained by adding

$$\frac{K}{4}\sigma^2 = \frac{1 - \zeta_5\rho^{-1}}{1 + \zeta_5\rho^{-1}}$$

to Eq. (9.1.20).

Let us consider the generators

$$X = \xi^\mu(x) \frac{\partial}{\partial x^\mu}$$

of the group of isometric motions in the space V_4 of constant curvature with the positive definite metric (9.1.21). In this case the Killing equations

$$\xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + g_{\mu\alpha} \frac{\partial \xi^\alpha}{\partial x^\nu} + g_{\nu\alpha} \frac{\partial \xi^\alpha}{\partial x^\mu} = 0 \tag{9.1.24}$$

have the form (9.1.16) with all $\epsilon_i = +1$, i.e.

$$\frac{\partial \xi^\mu}{\partial x^\nu} + \frac{\partial \xi^\nu}{\partial x^\mu} - \frac{K}{\theta}(x \cdot \xi)\delta_{\mu\nu} = 0, \quad \mu, \nu = 1, \dots, 4, \tag{9.1.25}$$

where $x \cdot \xi = \sum\limits_{\alpha=1}^{4} x^\alpha \xi^\alpha$ is the scalar product of the vectors x and ξ.

According to (9.1.17), Eqs. (9.1.25) have 10 linearly independent solutions that provide the following 10 generators:

$$X_1 = \left[1 + \frac{K}{4}\big(2(x^1)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^1} + \frac{K}{2}x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right),$$

$$X_2 = \left[1 + \frac{K}{4}\big(2(x^2)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^2} + \frac{K}{2}x^2\left(x^1\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right),$$

$$X_3 = \left[1 + \frac{K}{4}\big(2(x^3)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^3} + \frac{K}{2}x^3\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^4}\right),$$

$$X_4 = \left[1 + \frac{K}{4}\big(2(x^4)^2 - \sigma^2\big)\right]\frac{\partial}{\partial x^4} + \frac{K}{2}x^4\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}\right),$$

$$X_{\mu\nu} = x^\nu\frac{\partial}{\partial x^\mu} - x^\mu\frac{\partial}{\partial x^\nu} \quad (\mu < \nu, \quad \mu = 1,2,3;\ \nu = 1,2,3,4), \tag{9.1.26}$$

where $\sigma^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$ defined in (9.1.22).

Let us verify that, e.g. the coordinates

$$\xi^1 = 1 + \frac{K}{4}\big[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\big],$$

$$\xi^2 = \frac{K}{2}x^1x^2, \quad \xi^3 = \frac{K}{2}x^1x^3, \quad \xi^4 = \frac{K}{2}x^1x^4$$

of the operator X_1 satisfy the Killing equations. We write

$$\xi^\mu = \Big(1 - \frac{K}{4}\sigma^2\Big)\delta_{1\mu} + \frac{K}{2}x^1x^\mu, \quad \mu = 1,\dots,4,$$

and obtain

$$\sum_{\mu=1}^{4} x^\mu\xi^\mu = \Big(1 + \frac{K}{4}\sigma^2\Big)x^1, \quad \frac{\partial\xi^\mu}{\partial x^\nu} = \frac{K}{2}(x^1\delta_{\mu\nu} + x^\mu\delta_{1\nu} - x^\nu\delta_{1\mu}).$$

Whence,

$$x\cdot\xi = \theta x^1, \quad \frac{\partial\xi^\mu}{\partial x^\nu} + \frac{\partial\xi^\nu}{\partial x^\mu} = Kx^1\delta_{\mu\nu},$$

and hence the Killing equations (9.1.25) are obviously satisfied.

The six operators $X_{\mu\nu}$ from (9.1.26) correspond to the rotational symmetry of the metric (9.1.21) inherited from the rotational symmetry of the sphere (9.1.18) written in the variables (9.1.20). The other four operators in (9.1.26) can also be obtained as a result of the rotational symmetry of the sphere (9.1.18). To this end one can utilize the substitution (9.1.20) and rewrite the rotation generators on the plane surfaces $(\zeta_1, \zeta_5), \dots, (\zeta_4, \zeta_5)$ in the variables x. Unlike this external geometrical construction, the Killing equations give an internal definition of the group of motions independent of embedding V_4 in the five-dimensional Euclidean space.

Remark 9.1.3. The structure constants c_{mn}^{p} of the Lie algebra L_{10} spanned by the operators (9.1.26) are determined by the commutators

$$(X_\mu, X_\nu) = KX_{\mu\nu}, \quad (X_{\mu\nu}, X_\mu) = X_\nu,$$
$$(X_{\mu\nu}, X_\alpha) = 0 \quad (\alpha \neq \mu, \alpha \neq \nu),$$
$$(X_{\mu\nu}, X_{\alpha\beta}) = \delta_{\mu\alpha}X_{\nu\beta} + \delta_{\nu\beta}X_{\mu\alpha} - \delta_{\mu\beta}X_{\nu\alpha} - \delta_{\nu\alpha}X_{\mu\beta}.$$

It follows that the determinant of the matrix $A_{mn} = c_{mp}^{q}c_{nq}^{p}$ is equal to $-6K^4$, and hence does not vanish if $K \neq 0$. Thus, by E. Cartan's theorem, the group of isometric motions of the metric (9.1.21) with $K \neq 0$ is semi-simple. This fact is essential in investigation of representations of the de Sitter group (see (22) and the references therein).

Remark 9.1.4. It is manifest from Riemann's form (9.1.12) that the spaces of constant Riemannian curvature are conformally flat. In particular, the space V_4 with the metric (9.1.21) is conformal to the four-dimensional Euclidean space and has the metric tensor

$$g_{\mu\nu} = \frac{1}{\theta^2}\delta_{\mu\nu}, \quad g^{\mu\nu} = \theta^2\delta^{\mu\nu}, \quad \mu, \nu = 1, \ldots, 4. \tag{9.1.27}$$

Its Christoffel symbols $\Gamma_{\mu\nu}^{\alpha}$ and the scalar curvature R are given by

$$\Gamma_{\mu\nu}^{\alpha} = \frac{K}{2\theta}(\delta_{\mu\nu}x^\alpha - \delta_\mu^\alpha x^\nu - \delta_\nu^\alpha x^\mu), \quad R = -12K. \tag{9.1.28}$$

In Eqs. (9.1.27) and (9.1.28), $\delta_{\mu\nu}, \delta^{\mu\nu}$ and δ_μ^ν denote the Kronecker symbol.

9.1.6 Geometric realization of the de Sitter metric

In order to arrive at the de Sitter space V_4 we set

$$x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = ct\sqrt{-1}, \tag{9.1.29}$$

where c is the velocity of light in vacuum. Then, according to Eqs. (9.1.20), the variable ζ_4 is imaginary whereas $\zeta_1, \zeta_2, \zeta_3$ are real variables. As for the fifth coordinate, ζ_5, it is chosen from the condition that the ratio ζ_5/ρ is a real quantity. Hence, either both variables ζ_5 and ρ are real, or both of them are purely imaginary. If both ζ_5 and ρ are real, Equation (9.1.18) yields that the de Sitter metric is represented geometrically by the surface (see (12))

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 - \zeta_4^2 + \zeta_5^2 = \rho^2. \tag{9.1.30}$$

If both ζ_5 and ρ are imaginary, we have the surface

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 - \zeta_4^2 - \zeta_5^2 = -\rho^2. \tag{9.1.31}$$

Inserting (9.1.29) in (9.1.21) and denoting $-ds^2$ by ds^2 in accordance with the usual physical convention to interpret ds as the interval between *world events*, we obtain the following *de Sitter metric*:

$$ds^2 = \frac{1}{\theta^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{9.1.32}$$

where

$$\theta = 1 - \frac{K}{4}(c^2t^2 - x^2 - y^2 - z^2). \tag{9.1.33}$$

Equations (9.1.19), (9.1.30) and (9.1.31) show that the de Sitter space can have positive or negative curvature depending on the choice of the surface (9.1.30) or (9.1.31), respectively. The geometry of these surfaces is discussed by F. Klein (43).

9.2 The de Sitter group

The generators of the de Sitter group are given in Section 9.2.1 (see Eqs. (9.2.2)). Computation of the group transformations $x'^\mu = f^\mu(x,a)$ by solving the Lie equations

$$\frac{dx'^\mu}{da} = \xi^\mu(x'), \quad x'^\mu|_{a=0} = x^\mu, \quad \mu = 1, \ldots, 4, \tag{9.2.1}$$

for the generators (9.2.2) given further requires complicated calculations, in accordance with Synge's opinion cited in the preamble to this chapter. The result of such calculations for the operator X_1 from (9.2.2) will be provided in Section 9.2.4. First we will check how these transformations may look like by discussing conformal transformations in $\mathbb{R}^3$.

9.2.1 Generators of the de Sitter group

Rewriting the operators (9.1.26) in the variables x, y, z, t defined by Eqs. (9.1.29) we obtain the generators

$$\begin{aligned}
X_1 &= \left[1 + \frac{K}{4}(x^2 - y^2 - z^2 + c^2t^2)\right]\frac{\partial}{\partial x} + \frac{K}{2}x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right),\\
X_2 &= \left[1 + \frac{K}{4}(y^2 - x^2 - z^2 + c^2t^2)\right]\frac{\partial}{\partial y} + \frac{K}{2}y\left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right),\\
X_3 &= \left[1 + \frac{K}{4}(z^2 - x^2 - y^2 + c^2t^2)\right]\frac{\partial}{\partial z} + \frac{K}{2}z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right)
\end{aligned} \tag{9.2.2}$$

$$X_4 = \frac{1}{c^2}\left[1 - \frac{K}{4}(c^2t^2 + x^2 + y^2 + z^2)\right]\frac{\partial}{\partial t} - \frac{K}{2}t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad Y_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t}, \qquad i, j = 1, 2, 3,$$

of the *de Sitter group*, i.e. of the group of isometric motions of the metric (9.1.32). The de Sitter group is a generalization of the *Poincaré group* (non-homogeneous *Lorentz group*). Namely, the operators X_{ij} and Y_i generate the usual homogeneous Lorentz group (the rotations and Lorentz transformations). The operators X_1, X_2, X_3 and X_4 in (9.2.2) can be considered as the generators of the "generalized space-translations" and the "generalized time-translations", respectively.

9.2.2 Conformal transformations in $\mathbb{R}^3$

Let us consider a relatively simple problem on determining the one-parameter group of conformal transformations in the Euclidean space $\mathbb{R}^3$ generated by the operator

$$X = (y^2 + z^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z}. \tag{9.2.3}$$

In this case the Lie equations (9.2.1) are written

$$\begin{aligned}
\frac{dx'}{da} &= y'^2 + z'^2 - x'^2, \quad & x'\big|_{a=0} &= x,\\
\frac{dy'}{da} &= -2x'y', & y'\big|_{a=0} &= y,\\
\frac{dz'}{da} &= -2x'z', & z'\big|_{a=0} &= z.
\end{aligned} \tag{9.2.4}$$

To simplify the integration of Eqs. (9.2.4), one can introduce canonical variables u, v, w, where u, v are two functionally independent invariants, i.e. two different solutions of the equation

$$X(f) \equiv (y^2 + z^2 - x^2)\frac{\partial f}{\partial x} - 2xy\frac{\partial f}{\partial y} - 2xz\frac{\partial f}{\partial z} = 0, \tag{9.2.5}$$

and w is a solution of the equation

$$X(w) = 1. \tag{9.2.6}$$

Then rewriting the operator X in the new variables by the formula

$$X = X(u)\frac{\partial}{\partial u} + X(v)\frac{\partial}{\partial v} + X(w)\frac{\partial}{\partial w} \tag{9.2.7}$$

and invoking that $X(u) = X(v) = 0,\ X(w) = 1$, one obtains

$$X = \frac{\partial}{\partial w}.$$

Hence, the transformation of our group has the form

$$u' = u, \qquad v' = v, \qquad w' = w + a. \tag{9.2.8}$$

The solution of Eqs. (9.2.4) is obtained by substituting in (9.2.8) the expressions for u, v, w via x, y, z, and the similar expressions for u', v', w' via x', y', z'. Thus, the problem has been reduced to determination of u and v from Eq. (9.2.5), and w from Eq. (9.2.6). Let us carry out the calculations.

Calculation of invariants u and v requires the determination of two independent first integrals of the characteristic system for Eq. (9.2.5):

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}.$$

The second equation of this system, $dy/y = dz/z$, yields the invariant

$$u = \frac{z}{y}.$$

Substituting $z = uy$ into the first equation of the characteristic system one obtains

$$\frac{dx}{x^2 - (1 + u^2)y^2} = \frac{dy}{2xy}$$

or, setting $p = x^2$,

$$\frac{dp}{dy} = \frac{p}{y} - (1 + u^2)y.$$

Taking into account that u is constant on the solutions of the characteristic system, we readily integrate this first-order linear equation and obtain

$$p = vy - (1 + u^2)y^2, \quad v = \text{const}. \tag{9.2.9}$$

Expressing the constant of integration v from (9.2.9) and substituting $p = x^2$, $u = z/y$, we obtain the second invariant

$$v = \frac{r^2}{y}, \quad r^2 = x^2 + y^2 + z^2.$$

In order to solve Eq. (9.2.6) we rewrite the operator (9.2.3) in the variables u, v, y using the transformation formula similar to (9.2.7):

$$X = X(u)\frac{\partial}{\partial u} + X(v)\frac{\partial}{\partial v} + X(y)\frac{\partial}{\partial y}.$$

Since $X(u) = 0, X(v) = 0,\ X(y) = -2xy$, and Eq. (9.2.9) upon substitution $p = x^2$ yields $x = \sqrt{vy - (1+u^2)y^2}$, the operator (9.2.3) becomes

$$X = -2y\sqrt{vy - (1+u^2)y^2}\,\frac{\partial}{\partial y}.$$

Therefore Eq. (9.2.6) takes the form

$$\frac{dw}{dy} = -\frac{1}{2y\sqrt{vy - (1+u^2)y^2}},$$

whence, ignoring the constant of integration, we obtain

$$w = \sqrt{\frac{1}{vy} - \frac{1+u^2}{v^2}} = \frac{x}{r^2}.$$

Now we substitute the resulting expressions

$$u = \frac{z}{y}, \quad v = \frac{r^2}{y}, \quad w = \frac{x}{r^2}$$

and the similar expressions for the primed variables in Eqs. (9.2.8) and obtain

$$\frac{z'}{y'} = \frac{z}{y}, \quad \frac{r'^2}{y'} = \frac{r^2}{y}, \quad \frac{x'}{r'^2} = \frac{x}{r^2} + a.$$

Solution of these equations with respect to x', y', z', furnishes the following transformations of the conformal group with the generator (9.2.3):

$$\begin{aligned} x' &= \frac{x + ar^2}{1 + 2ax + a^2r^2}, \\ y' &= \frac{y}{1 + 2ax + a^2r^2}, \\ z' &= \frac{z}{1 + 2ax + a^2r^2}. \end{aligned} \tag{9.2.10}$$

9.2.3 Inversion

An alternative approach to calculation of the conformal transformations (9.2.10) is based on Liouville's theorem (J. Liouville, 1847) stating that any conformal mapping in the three-dimensional Euclidean space is a composition of translations, rotations, dilations and the *inversion*

$$x_1 = \frac{x}{r^2}, \quad y_1 = \frac{y}{r^2}, \quad z_1 = \frac{z}{r^2}. \tag{9.2.11}$$

The inversion (9.2.11) is the reflection with respect to the unit sphere. It preserves the angles, i.e. represents a particular case of conformal transformations.

It carries the point (x, y, z) with the radius r to the point (x_1, y_1, z_1) with the radius $r_1 = 1/r$ (therefore it is also called a *transformation of reciprocal radii*) and hence, the inverse transformation

$$x = \frac{x_1}{r_1^2}, \quad y = \frac{y_1}{r_1^2}, \quad z = \frac{z_1}{r_1^2} \tag{9.2.11$'$}$$

coincides with (9.2.11). If the inversion (9.2.11) is denoted by S, then the latter property means that $S^{-1} = S$. According to Liouville's theorem, the one-parameter transformation group (9.2.10) can be obtained merely as a conjugate group from a group of translations H along the x-axis. Indeed, if one introduces the new variables (9.2.11′) in the generator of translations $X_1 = \partial/\partial x_1$ according to the formula (cf. (9.2.7))

$$X = SX_1 \equiv X_1(x)\frac{\partial}{\partial x} + X_1(y)\frac{\partial}{\partial y} + X_1(z)\frac{\partial}{\partial z}, \tag{9.2.12}$$

then by virtue of

$$X_1(x) = \frac{y_1^2 + z_1^2 - x_1^2}{r_1^4} = y^2 + z^2 - x^2,$$
$$X_1(y) = -2\frac{x_1 y_1}{r_1^4} = -2xy, \quad X_1(z) = -2xz,$$

one obtains the operator (9.2.3). Therefore, the group G with the generator (9.2.3) is obtained from the translation group

$$H: \; x_2 = x_1 + a, \quad y_2 = y_1, \quad z_2 = z_1 \tag{9.2.13}$$

by the conjugation

$$G = SHS. \tag{9.2.14}$$

Indeed, mapping the point (x, y, z) to the point (x_1, y_1, z_1) by means of the inversion (9.2.11), we write the transformation (9.2.13) in the original variables

$$x_2 = \frac{x + ar^2}{r^2}, \quad y_2 = \frac{y}{r^2} \quad z_2 = \frac{z}{r^2}. \tag{9.2.15}$$

Now according to (9.2.14) let us make one more inversion

$$x' = \frac{x_2}{r_2^2}, \quad y' = \frac{y_2}{r_2^2} \quad z' = \frac{z_2}{r_2^2}.$$

Substituting here the expressions (9.2.15) and the expression

$$r_2^2 = \frac{(x + ar^2)^2 + y^2 + z^2}{r^4} = \frac{1 + 2ax + a^2 r^2}{r^2},$$

we arrive precisely to the transformations (9.2.10) of the group G.

Note that the inversion, like any conformal transformation, is admitted by the Laplace equation. Specifically, the Laplace equation

$$\Delta u \equiv u_{xx} + u_{yy} + u_{zz} = 0$$

is invariant under the transformation (9.2.11) of the independent variables supplemented by the following transformation of the dependent variable u :

$$u_1 = ru. \tag{9.2.16}$$

Thus, Equations (9.2.11) and (9.2.16) define a symmetry transformation

$$x_1 = \frac{x}{r^2}, \quad y_1 = \frac{y}{r^2}, \quad z_1 = \frac{z}{r^2}, \quad u_1 = ru, \tag{9.2.17}$$

of the Laplace equation known as the *Kelvin transformation.*

9.2.4 Generalized translation in direction of x-axis

Integration of the Lie equations for the generator X_1 from (9.3.25) yields the following one-parameter "generalized translation" group in the direction of the x-axis:

$$\begin{aligned}
\overline{x} &= \frac{2\sqrt{\varepsilon}\,x\,\cos(2a\sqrt{\varepsilon}) + (1-\varepsilon\sigma^2)\sin(2a\sqrt{\varepsilon})}{\sqrt{\varepsilon}(1+\varepsilon\sigma^2) + \sqrt{\varepsilon}(1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2\varepsilon\,x\,\sin(2a\sqrt{\varepsilon})},\\
\overline{y} &= \frac{2y}{1+\varepsilon\sigma^2 + (1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},\\
\overline{z} &= \frac{2z}{1+\varepsilon\sigma^2 + (1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},\\
\overline{t} &= \frac{2t}{1+\varepsilon\sigma^2 + (1-\varepsilon\sigma^2)\cos(2a\sqrt{\varepsilon}) - 2x\sqrt{\varepsilon}\sin(2a\sqrt{\varepsilon})},
\end{aligned} \tag{9.2.18}$$

where $\sigma^2 = x^2 + y^2 + z^2 - c^2t^2$ and $\varepsilon = K/4$.

9.3 Approximate de Sitter group

In this section, the de Sitter group is considered as a perturbation of the Poincaré group by a small constant curvature K. The *approximate representation* of the de Sitter group obtained here can be dealt with as easily as the Poincaré group. A brief introduction to the theory of approximate transformation groups given in Section 9.3.1 is sufficient for our purposes. For a motivation and a detailed presentation of the theory of approximate transformation groups, see (2).

9.3.1 Approximate groups

Let $f(x,\varepsilon)$ and $g(x,\varepsilon)$ be analytical functions, where $x = (x^1,\dots,x^n)$ and ε is a small parameter. We say that the functions f and g are approximately equal and write $f \approx g$ if $f - g = o(\varepsilon)$. The same notation is used for functions depending additionally on a parameter a (group parameter).

Consider a vector-function $f(x,a,\varepsilon)$ with components

$$f^1(x,a,\varepsilon),\dots,f^n(x,a,\varepsilon)$$

satisfying the "initial conditions" $f^i(x,0,\varepsilon) \approx x^i$ $(i = 1,\dots,n)$. We will write these conditions in the vector form

$$f(x,0,\varepsilon) \approx x.$$

Furthermore, we assume that the function f is defined and smooth in a neighborhood of $a = 0$ and that, in this neighborhood, $a = 0$ is the only solution of the equation

$$f(x,a,\varepsilon) \approx x.$$

Definition 9.3.1. An *approximate transformation*

$$x' \approx f(x,a,\varepsilon) \tag{9.3.1}$$

in $\mathbb{R}^n$ is the set of all invertible transformations

$$x' = g(x,a,\varepsilon)$$

with $g(x,a,\varepsilon) \approx f(x,a,\varepsilon)$.

Definition 9.3.2. We say that (9.3.1) is a one-parameter *approximate transformation group* with the group parameter a if

$$f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon). \tag{9.3.2}$$

Let us write the transformations (9.3.1) with the precision $o(\varepsilon)$ in the form

$$x' \approx f_0(x,a) + \varepsilon f_1(x,a), \tag{9.3.3}$$

and denote

$$\xi_0(x) = \left.\frac{\partial f_0(x,a)}{\partial a}\right|_{a=0}, \quad \xi_1(x) = \left.\frac{\partial f_1(x,a)}{\partial a}\right|_{a=0}. \tag{9.3.4}$$

Theorem 9.3.1. If (9.3.3) obeys the approximate group property (9.3.2) then the following *approximate Lie equation* holds:

$$\frac{d(f_0 + \varepsilon f_1)}{da} = \xi_0(f_0 + \varepsilon f_1) + \varepsilon \xi_1(f_0 + \varepsilon f_1) + o(\varepsilon). \tag{9.3.5}$$

Conversely, given any smooth vector-function $\xi(x,\varepsilon) \approx \xi_0(x) + \varepsilon\xi_1(x) \not\approx 0$, the solution $x' \approx f(x,a,\varepsilon)$ of the *approximate Cauchy problem*

$$\frac{dx'}{da} \approx \xi(x',\varepsilon), \tag{9.3.6}$$

$$x'\big|_{a=0} \approx x \tag{9.3.7}$$

determines a one-parameter approximate transformation group with the parameter a.

The proof of this theorem is similar to the proof of Lie's theorem for exact groups (2). However, we have to specify the notion of the approximate Cauchy problem. The approximate differential equation (9.3.6) is considered here as a family of differential equations

$$\frac{dz'}{da} = \tilde{\xi}(x',\varepsilon) \tag{9.3.8}$$

with the right-hand sides $\tilde{\xi}(x,\varepsilon) \approx \xi(x,\varepsilon)$. Likewise, the approximate initial condition (9.3.7) is treated as the set of equations

$$x'\big|_{a=0} = \alpha(x,\varepsilon), \quad \alpha(x,\varepsilon) \approx x. \tag{9.3.9}$$

Definition 9.3.3. We define the solution of the approximate Cauchy problem (9.3.6), (9.3.7) as a solution to any problem (9.3.8), (9.3.9) considered with the precision $o(\varepsilon)$.

Proposition 9.3.1. The solution of the approximate Cauchy problem given by Definition 9.3.3 is unique.

Proof. Indeed, according to the theorem on continuous dependence on parameters of the solution to Cauchy's problem, solutions for all problems of the form (9.3.8), (9.3.9) coincide with the precision $o(\varepsilon)$.

Definition 9.3.3 shows that in order to find solution of the approximate Lie equation (9.3.5) with the initial condition (9.3.7) it suffices to solve the following exact Cauchy problem:

$$\begin{aligned} &\frac{df_0}{da} = \xi_0(f_0), \quad f_0\big|_{a=0} = x, \\ &\frac{df_1}{da} = \sum_{k=1}^{n} \frac{\partial\xi_0(f_0)}{\partial x^k} f_1^k + \xi_1(f_0), \quad f_1\big|_{a=0} = 0. \end{aligned} \tag{9.3.10}$$

Equations (9.3.10) are obtained from (9.3.5) by separating principal terms with respect to ε. □

Example 9.3.1. To illustrate the method, let us find the approximate transformation group

$$x' = x_0' + \varepsilon x_1', \quad y' = y_0' + \varepsilon y_1'$$

on the (x, y) plane defined by the generator

$$X = (1 + \varepsilon x^2)\frac{\partial}{\partial x} + 2\varepsilon xy\frac{\partial}{\partial y}.$$

The above operator X is a one-dimensional analogue of the operator X_1 from (9.2.2). We have:

$$\xi_0 = (1, 0), \quad \xi_1 = (x^2, 2xy),$$

and the Cauchy problem (9.3.10) is written in the form

$$\frac{dx_0'}{da} = 1, \quad \frac{dy_0'}{da} = 0, \quad x_0'\big|_{a=0} = x, \quad y_0'\big|_{a=0} = y; \tag{9.3.11}$$

$$\frac{dx_1'}{da} = (x_0')^2, \quad \frac{dy_1'}{da} = 2x_0'y_0', \quad x_1'\big|_{a=0} = y_1'\big|_{a=0} = 0. \tag{9.3.12}$$

Equations (9.3.11) yield $x_0' = x + a$, $y_0' = y$, and hence Eqs. (9.3.12) take the form

$$\frac{dx_1'}{da} = (x + a)^2, \quad \frac{dy_1'}{da} = 2y(x + a), \quad x_1'|_{a=0} = y_1'|_{a=0} = 0,$$

whence

$$x_1' = x^2a + xa^2 + \frac{1}{3}a^3, \quad y_1' = 2xya + ya^2.$$

Thus, the approximate transformation group is given by

$$x' = x + a + (x^2a + xa^2 + \frac{1}{3}a^3)\varepsilon, \quad y' = y + (2xya + ya^2)\varepsilon. \tag{9.3.13}$$

It is instructive to compare the approximate transformations with the corresponding exact transformation group. The latter is obtained by solving the (exact) Lie equations

$$\frac{dx'}{da} = 1 + \varepsilon x'^2, \quad \frac{dy'}{da} = 2\varepsilon x'y', \quad x'\big|_{a=0} = x, \quad y'\big|_{a=0} = y.$$

These equations yield:

$$x' = \frac{\sin(\delta a) + \delta x\cos(\delta a)}{\delta\big[\cos(\delta a) - \delta x\sin(\delta a)\big]}, \quad y' = \frac{y}{\big[\cos(\delta a) - \delta x\sin(\delta a)\big]^2}, \tag{9.3.14}$$

where $\delta = \sqrt{\varepsilon}$. The approximate transformation (9.3.13) can be obtained from (9.3.14) by singling out the principal linear terms with respect to ε.

9.3.2 Simple method of solution of Killing's equations

Now we will carry out the program of an approximate representation of the de Sitter group. Let us first apply the approximate approach to differential equations outlined in Section 9.3.1 to the Killing equations (9.1.25) for the metric (9.1.21). Setting

$$\xi = \xi_0 + K\xi_1 + o(K), \tag{9.3.15}$$

we obtains from (9.1.25) the *approximate Killing equations*

$$\frac{\partial \xi_0^\mu}{\partial x^\nu} + \frac{\partial \xi_0^\nu}{\partial x^\mu} + K\left[\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} - (x\cdot\xi_0)\delta_{\mu\nu}\right] \approx 0, \tag{9.3.16}$$

whence (cf. Eqs. (9.3.10))

$$\frac{\partial \xi_0^\mu}{\partial x^\nu} + \frac{\partial \xi_0^\nu}{\partial x^\mu} = 0, \tag{9.3.17}$$

and

$$\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} - (x\cdot\xi_0)\delta_{\mu\nu} = 0. \tag{9.3.18}$$

Equations (9.3.17) define the group of isometric motions in the Euclidean space, namely rotations and translations, so that

$$\xi_0^\mu = a_\nu^\mu x^\nu + b^\mu \tag{9.3.19}$$

with constant coefficients a_ν^μ and b^μ. The coefficients a_ν^μ are skew-symmetric, i.e. $a_\nu^\mu = -a_\mu^\nu$, therefore we have

$$x\cdot\xi_0 = b\cdot x \equiv \sum_{\mu=1}^{4} b^\mu x^\mu. \tag{9.3.20}$$

Due to the obvious symmetry of Eqs. (9.3.18) it is sufficient to find their particular solution letting $b = (1,0,0,0)$. Then Eq. (9.3.20) yields $x\cdot\xi_0 = x^1$ and Eqs. (9.3.18) take the form

$$\frac{\partial \xi_1^\mu}{\partial x^\nu} + \frac{\partial \xi_1^\nu}{\partial x^\mu} = x^1\delta_{\mu\nu}. \tag{9.3.21}$$

Solving Eqs. (9.3.21) with $\mu = \nu$ one obtains

$$\begin{aligned} \xi_1^1 &= \frac{1}{4}(x^1)^2 + \varphi^1(x^2,x^3,x^4), \quad & \xi_1^2 &= \frac{1}{2}x^1x^2 + \varphi^2(x^1,x^3,x^4),\\ \xi_1^3 &= \frac{1}{2}x^1x^3 + \varphi^3(x^1,x^2,x^4), \quad & \xi_1^4 &= \frac{1}{2}x^1x^4 + \varphi^4(x^1,x^2,x^3). \end{aligned} \tag{9.3.22}$$

Using Eqs. (9.3.22), we arrive at the following Eqs. (9.3.21) with $\mu = 1:$

$$\frac{\partial \varphi^1}{\partial x^\nu} + \frac{\partial \varphi^\nu}{\partial x^1} + \frac{1}{2}x^\nu = 0, \quad \nu = 2,3,4,$$

whence

$$\frac{\partial^2 \varphi^1}{(\partial x^\nu)^2} + \frac{1}{2} = 0, \quad \nu = 2,3,4.$$

A particular solution to the latter equations is given by

$$\varphi^1 = -\frac{1}{4}\big[(x^2)^2 + (x^3)^2 + (x^4)^2\big]. \tag{9.3.23}$$

One can easily verify that substitution of (9.3.23) and $\varphi^2 = \varphi^3 = \varphi^4 = 0$ in (9.3.22) yields a solution to Eqs. (9.3.21). Finally, combining Eqs. (9.3.22), (9.3.23), (9.3.15) and Eqs. (9.3.19) with $a^\mu_\nu = 0, b = (1,0,0,0)$, one obtains the following solution of the approximate Killing equations (9.3.16):

$$\xi^1 = 1 + \frac{K}{4}[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2],\ \xi^\nu = \frac{K}{2}x^1x^\nu, \quad \nu = 2,3,4. \tag{9.3.24}$$

The vector (9.3.24) gives the operator X_1 from (9.1.26) and hence, according to Section 9.1.5, it is an exact solution of the Killing equations (9.1.25). The fact that the approximate solution coincides with the exact solution of the Killing equations is a lucky accident and has no significance in what follows. Of importance for us is the simplification achieved by the approximate approach to solving the Killing equations. All other operators (9.1.26) can be obtained by renumbering the coordinates in (9.3.24). The transition from (9.1.26) to the generators of the de Sitter group has been given in Section 9.2.1.

9.3.3 Approximate representation of de Sitter group

To illustrate the method, I will find the approximate representation of the one-parameter group for one of the generators (9.2.2). Let us take $\varepsilon = K/4$ as a small parameter and write the coordinates of the first operator (9.2.2),

$$X_1 = [1 + \varepsilon(x^2 - y^2 - z^2 + c^2t^2)]\frac{\partial}{\partial x} + 2\varepsilon x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right), \tag{9.3.25}$$

in the form $\xi = \xi_0 + \varepsilon\xi_1$. Then

$$\xi_0 = (1,0,0,0), \quad \xi_1 = (x^2 - y^2 - z^2 + c^2t^2, 2xy, 2xz, 2xt).$$

In order to find the corresponding group of approximate transformations

$$x' = x'_0 + \varepsilon x'_1, \quad y' = y'_0 + \varepsilon y'_1, \quad z' = z'_0 + \varepsilon z'_1, \quad t' = t'_0 + \varepsilon t'_1,$$

we have to solve the Cauchy problem (9.3.10) comprising, in our case, the equations

$$\begin{aligned} &\frac{dx'_0}{da} = 1, \quad &&\frac{dy'_0}{da} = 0, \quad &&\frac{dz'_0}{da} = 0, \quad &&\frac{dt'_0}{da} = 0, \\ &x'_0|_{a=0} = x, \quad &&y'_0|_{a=0} = y, \quad &&z'_0|_{a=0} = z, \quad &&t'_0|_{a=0} = t, \end{aligned} \tag{9.3.26}$$

and

$$\begin{aligned} &\frac{dx_1'}{da} = x_0'^2 - y_0'^2 - z_0'^2 + c^2 t_0'^2, \\ &\frac{dy_1'}{da} = 2x_0' y_0', \quad \frac{dz_1'}{da} = 2x_0' z_0', \quad \frac{dt_1'}{da} = 2x_0' t_0', \\ &x_1'|_{a=0} = y_1'|_{a=0} = z_1'|_{a=0} = t_1'|_{a=0} = 0. \end{aligned} \tag{9.3.27}$$

In order to solve the system (9.3.26), (9.3.27), we proceed as in Example 9.3.1. Namely, we solve Eqs. (9.3.26) and obtain

$$x_0' = x + a, \quad y_0' = y, \quad z_0' = z, \quad t_0' = t. \tag{9.3.28}$$

Then we substitute the expressions (9.3.28) in the differential equations from (9.3.27) and arrive at the following equations:

$$\begin{aligned} &\frac{dx_1'}{da} = x^2 - y^2 - z^2 + c^2 t^2 + 2xa + a^2, \\ &\frac{dy_1'}{da} = 2xy + 2ya, \quad \frac{dz_1'}{da} = 2xz + 2za, \quad \frac{dt_1'}{da} = 2xt + 2ta. \end{aligned}$$

Their integration by using the initial conditions yields

$$\begin{aligned} x_1' &= (x^2 - y^2 - z^2 + c^2 t^2)a + xa^2 + \frac{1}{3}a^3, \\ y_1' &= 2xya + ya^2, \\ z_1' &= 2xza + za^2, \\ t_1' &= 2xta + ta^2. \end{aligned} \tag{9.3.29}$$

Combining the equations (9.3.28), (9.3.29) and substituting $\varepsilon = K/4$, we obtain the one-parameter approximate transformation group

$$\begin{aligned} x' &= x + a + \frac{K}{4}[(x^2 - y^2 - z^2 + c^2 t^2)a + xa^2 + \frac{1}{3}a^3], \\ y' &= y + \frac{K}{4}(2xya + ya^2), \\ z' &= z + \frac{K}{4}(2xza + za^2), \\ t' &= t + \frac{K}{4}(2xta + ta^2) \end{aligned} \tag{9.3.30}$$

with the first generator (9.2.2):

$$X_1 = \left(1 + \frac{K}{4}(x^2 - y^2 - z^2 + c^2 t^2)\right)\frac{\partial}{\partial x} + \frac{K}{2}x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right).$$

The transformations (9.3.30) can be called a *generalized translation* in the de Sitter universe. In the case $K = 0$ they coincide with the usual translation $x' = x + a$ along the x-axis, whereas for a small curvature $K \neq 0$ they have little deviation from the translation. Compare also with the exact one-parameter group transformation (9.2.18) obtained by integration of the exact Lie equations for the generator (9.3.25).

Exercise 9.3.1. Find the approximate group with the generator X_4 of the generalized time-translations from (9.2.2).

9.4 Motion of a particle in de Sitter space

9.4.1 Introduction

It is accepted in theoretical physics that the free motion of a particle with mass m in a curved space-time, i.e. in a Riemannian space V_4 of the normal hyperbolic type with the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{9.4.1}$$

moves along geodesic curves in V_4. More specifically, it means that the free motion of the particle is determined by the Lagrangian

$$\mathcal{L} = -mc\sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}. \tag{9.4.2}$$

Here the constant c is the light velocity in vacuum and the dot in $\dot{x}^\mu$ indicates the differentiation with respect to the parameter σ of a curve

$$x^\mu = x^\mu(\sigma), \quad (\mu = 1, \ldots, 4)$$

along which the particle is moving. According to Section 9.1.2, the Euler-Lagrange equations with the Lagrangian (9.4.2) are the equations of geodesic curves in V_4.

Taking for the parameter σ the arc length s measured from a fixed point x_0, we conclude that in the problem of the free motion of a particle we have one independent variable s and four dependent variables x^μ satisfying the the second-order ordinary differential equations (9.1.2):

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \quad \lambda = 1, \ldots, 4, \tag{9.4.3}$$

where the coefficients $\Gamma^\lambda_{\mu\nu}$ are the *Christoffel symbols* given by (9.1.3):

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right), \quad \lambda, \mu, \nu = 1, 2, 3, 4. \tag{9.4.4}$$

The infinitesimal symmetries of these equations will be written in the form

$$X = \tau(s, x)\frac{\partial}{\partial s} + \xi^\mu(s, x)\frac{\partial}{\partial x^\mu}. \tag{9.4.5}$$

In this notation, Equations (2.3.17) for conserved vectors become one-dimensional and provide the following equation for calculating *conserved quantities*:

$$A = (\xi^\mu - \tau \dot{x}^\mu)\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \tau \mathcal{L}. \tag{9.4.6}$$

On the other hand, in our case the variational integral (2.3.10) is the integral of the element of the length of the arc ds in the space V_4. Accordingly, it is invariant with respect to the group of isometric motions in the space V_4. Therefore, the admitted operators (9.4.5) have $\tau = 0$ and reduce to the form

$$X = \xi^\mu(x)\frac{\partial}{\partial x^\mu}, \tag{9.4.7}$$

where the functions $\xi^\mu(x)$ satisfy the Killing equations (9.1.24).

Let us substitute the coordinates of the operator (9.4.7) in Eq. (9.4.6). We have

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -\frac{mc}{\sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}}\, g_{\mu\nu}(x)\dot{x}^\nu.$$

Since the parameter of the curves is the arc length s, Equation (9.4.1) yields $g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu = 1$, and hence

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -mc\, g_{\mu\nu}(x)\dot{x}^\nu.$$

Substituting these expressions in Eq. (9.4.6) and invoking that $\tau = 0$, we obtain the following equation for calculating conserved quantities:

$$A = -mc\, g_{\mu\nu}\, \xi^\mu\, \dot{x}^\nu. \tag{9.4.8}$$

In what follows, we will use Eq. (9.4.8) in the following form:

$$T = mc\, g_{\mu\nu}\, \xi^\mu\, \frac{dx^\nu}{ds}. \tag{9.4.9}$$

9.4.2 Conservation laws in Minkowski space

The relativistic conservation laws are obtained by applying the above procedure to the Minkowski space with the metric (3.2.14),

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.$$

The group of isometric motions in the Minkowski space is the Lorentz group with the generators (cf. (9.2.2))

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad Y_i = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t} \tag{9.4.10}$$

where $i, j = 1, 2, 3$.

Let us apply the conservation formula (9.4.9) to the generators (9.4.10). We will write Eq. (9.4.9) in the form

$$T = mc \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, \xi^{\mu}\, \frac{dx^{\nu}}{ds}, \tag{9.4.11}$$

where

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{9.4.12}$$

We will denote the spatial vector by $\boldsymbol{x} = (x^1, x^2, x^3)$. Accordingly, the physical velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$ is a three-dimensional vector $\boldsymbol{v} = (v^1, v^2, v^3)$. We also use the usual symbols $(\boldsymbol{x} \cdot \boldsymbol{v})$ and $\boldsymbol{x} \times \boldsymbol{v}$ for the scalar and vector products, respectively.

Writing Eq. (3.2.14) in the form

$$ds = c\sqrt{1-\beta^2}\, dt, \quad \beta^2 = |\boldsymbol{v}|^2/c^2, \tag{9.4.13}$$

we obtain from (9.4.2) the following *relativistic Lagrangian*, specifically, the Lagrangian for the free motion of a particle in the special theory of relativity:

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2}. \tag{9.4.14}$$

Furthermore, we have:

$$\frac{dx^0}{ds} = \frac{1}{c\sqrt{1-\beta^2}}, \quad \frac{dx^i}{ds} = \frac{v^i}{c\sqrt{1-\beta^2}}, \quad i = 1, 2, 3. \tag{9.4.15}$$

Now the conserved quantity (9.4.11) is written:

$$T = \frac{m}{\sqrt{1-\beta^2}}\left[c^2\xi^0 - (\boldsymbol{\xi} \cdot \boldsymbol{v})\right], \tag{9.4.16}$$

where

$$(\boldsymbol{\xi} \cdot \boldsymbol{v}) = \sum_{i=1}^{3} \xi^i v^i.$$

Let us take the generator X_0 from (9.4.10). Substituting its coordinates $\xi^0 = 1, \xi^i = 0$ in (9.4.16) we obtain Einstein's *relativistic energy:*

$$E = \frac{mc^2}{\sqrt{1-\beta^2}}\,. \tag{9.4.17}$$

Likewise, substituting in (9.4.16) the coordinates of the operators X_i and X_{ij}, we arrive at the *relativistic momentum*

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\beta^2}} \tag{9.4.18}$$

and the *relativistic angular momentum*

$$\mathbf{M} = \boldsymbol{x} \times \boldsymbol{p}, \tag{9.4.19}$$

respectively.

The generators Y_i of the Lorentz transformations give rise to the vector

$$\mathbf{Q} = \frac{m(\boldsymbol{x} - t\boldsymbol{v})}{\sqrt{1-\beta^2}}\,. \tag{9.4.20}$$

In the case of N-body problem, conservation of the vector (9.4.20) furnishes the relativistic center-of-mass theorem.

9.4.3 Conservation laws in de Sitter space

In the notation (9.4.12), the de Sitter space metric (9.1.32) is written

$$ds^2 = \frac{1}{\theta^2}\big(c^2dt^2 - |d\boldsymbol{x}|^2\big), \quad \theta = 1 - \frac{K}{4}\big(c^2t^2 - |\boldsymbol{x}|^2\big), \tag{9.4.21}$$

and Equations. (9.4.13), (9.4.14), (9.4.15) and (9.4.16) are replaced by

$$ds = \frac{c}{\theta}\sqrt{1-\beta^2}\,dt, \quad \beta^2 = |\boldsymbol{v}|^2/c^2, \tag{9.4.22}$$

$$\mathcal{L} = -mc^2\theta^{-1}\sqrt{1-\beta^2}, \tag{9.4.23}$$

$$\frac{dx^0}{ds} = \frac{\theta}{c\sqrt{1-\beta^2}}, \quad \frac{dx^i}{ds} = \frac{\theta v^i}{c\sqrt{1-\beta^2}}, \quad i = 1,2,3, \tag{9.4.24}$$

and

$$T = \frac{m}{\theta\sqrt{1-\beta^2}}[c^2\xi^0 - (\boldsymbol{\xi}\cdot\boldsymbol{v})], \tag{9.4.25}$$

respectively.

Let us apply (9.4.25) to the generalized time-translation generator X_4 from (9.2.2) written in the form

$$X_0 = \left[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2)\right]\frac{\partial}{\partial t} - \frac{K}{2}c^2t\,x^i\frac{\partial}{\partial x^i}. \tag{9.4.26}$$

Substituting the coordinates

$$\xi^0 = 1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2), \quad \xi^i = -\frac{K}{2}c^2tx^i, \quad i = 1,2,3,$$

of the operator (9.4.26) in (9.4.25), we obtain the conserved quantity

$$T_0 = \frac{mc^2}{\theta\sqrt{1-\beta^2}}\left[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2) + \frac{K}{2}t(\boldsymbol{x}\cdot\boldsymbol{v})\right].$$

One can readily verify that the following equation holds:

$$\frac{1}{\theta}\left[1-\frac{K}{4}(c^2t^2+|\boldsymbol{x}|^2)\right]$$

Therefore, T_0 the yields the *energy of a free particle in the de Sitter space:*

$$\mathcal{E}=\frac{mc^2}{\sqrt{1-\beta^2}}\left[1+\frac{K}{2\theta}(t\boldsymbol{v}-\boldsymbol{x})\cdot\boldsymbol{x}\right]. \tag{9.4.27}$$

If $K=0$, we have $\mathcal{E}=E$, where E is the relativistic energy (9.4.17). If $K\neq 0$, (9.4.27) yields in the linear approximation with respect to K :

$$\mathcal{E}\approx E\left[1-\frac{K}{2}(\boldsymbol{x}-t\boldsymbol{v})\cdot\boldsymbol{x}\right]$$

Similar calculations for the operators X_i from (9.2.2) provide the momentum

$$\boldsymbol{\mathcal{P}}=\frac{m}{\theta\sqrt{1-\beta^2}}\left[(2-\theta)\boldsymbol{v}-\frac{K}{2}(c^2t-\boldsymbol{x}\cdot\boldsymbol{v})\boldsymbol{x}\right]. \tag{9.4.28}$$

In the linear approximation with respect to K, it is written:

$$\boldsymbol{\mathcal{P}}\approx\boldsymbol{p}-\frac{K}{2}E\left[t(\boldsymbol{x}-t\boldsymbol{v})+\frac{1}{c^2}(\boldsymbol{x}\times\boldsymbol{v})\times\boldsymbol{x}\right]$$

where E and $\boldsymbol{p}$ are the relativistic energy (9.4.17) and momentum (9.4.18), respectively. It is manifest that $\boldsymbol{\mathcal{P}}=\boldsymbol{p}$ if $K=0$.

By using the infinitesimal rotations X_{ij} and the generators Y_i of the Lorentz transformations from (9.2.2), one obtains the angular momentum

$$\boldsymbol{\mathcal{M}}=\frac{1}{2-\theta}(\boldsymbol{x}\times\boldsymbol{\mathcal{P}})$$

and the vector

$$\boldsymbol{\mathcal{Q}}=\frac{m(\boldsymbol{x}-t\boldsymbol{v})}{\theta\sqrt{1-\beta^2}},$$

respectively.

9.4.4 Kepler's problem in de Sitter space

Let us extend the Lagrangian (9.4.23) of the free motion of a particle to Kepler's problem in the de Sitter space. We will require that the generalized Lagrangian possesses the basic properties of the Lagrangian in the classical Kepler problem, namely, its invariance with respect to the rotations and time translations. Furthermore, we require that when $K = 0$ and $\beta^2 \to 0$, the generalized Lagrangian assumes the classical value

$$L = \frac{1}{2}m|\boldsymbol{v}|^2 + \frac{\alpha}{|\boldsymbol{x}|}, \quad \alpha = \text{const.} \tag{9.4.29}$$

Therefore, starting from (9.4.23), we seek for the Lagrangian of Kepler's problem in the de Sitter space in the form

$$\mathcal{L} = -mc^2\theta^{-1}\sqrt{1-\beta^2} + \frac{\alpha}{|\boldsymbol{x}|}\theta^s(1-\beta^2)^l \tag{9.4.30}$$

with undetermined constants s and l. The action integral

$$\int \mathcal{L}dt \tag{9.4.31}$$

with the Lagrangian (9.4.30) is invariant with respect to rotations. Hence, the invariance test (see (28))

$$\widetilde{X}_0(\mathcal{L}) + D_t(\xi^0)\mathcal{L} = 0 \tag{9.4.32}$$

of the action integral (9.4.31) with respect to the generalized time-translations with the generator (9.4.26),

$$X_0 = \left[1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2)\right]\frac{\partial}{\partial t} - \frac{K}{2}c^2t\,x^i\frac{\partial}{\partial x^i},$$

will be the only additional condition for determining the unknown constants s and l in (9.4.30). Here $\widetilde{X}_0$ is the prolongation of the operator X_0 to $\boldsymbol{v}$ and ξ^0 is its coordinate at $\frac{\partial}{\partial t}$:

$$\xi^0 = 1 - \frac{K}{4}(c^2t^2 + |\boldsymbol{x}|^2).$$

The computation shows that

$$\widetilde{X}_0(\mathcal{L}) + D_t(\xi^0)\mathcal{L} = \frac{\alpha K}{2|\boldsymbol{x}|}\theta^s(1-\beta^2)^l\left[(1-2l)\frac{\boldsymbol{x}\cdot\boldsymbol{v}}{c^2} + st\right]. \tag{9.4.33}$$

Therefore, Eq. (9.4.32) yields that $s = 0$ and $l = \frac{1}{2}$. This proves the following.

Theorem 9.4.1. The Lagrangian (9.4.30) is invariant under the group of generalized time-translations if and only if it has the form

$$\mathcal{L} = \left(-mc^2\theta^{-1} + \frac{\alpha}{|\boldsymbol{x}|}\right)\sqrt{1-\beta^2}. \tag{9.4.34}$$

Remark 9.4.1. A similar theorem on the uniqueness of an invariant Lagrangian is not valid in the Minkowsky space. Indeed, as follows from (9.4.33), for $K = 0$ Eq. (9.4.32) is satisfied for any Lagrangian (9.4.30). Thus, Theorem 9.4.1 manifests a significance of the curvature K.

Let us check that the Lagrangian (9.4.34) takes the classical value (9.4.29) when $K = 0$ and $\beta^2 \to 0$. Taking in (9.4.34) the approximation

$$\sqrt{1-\beta^2} \approx 1 - \beta^2 = 1 - \frac{1}{2}\frac{|\boldsymbol{v}|^2}{c^2}$$

and letting $\beta^2 \to 0$, we obtain

$$\mathcal{L} = -mc^2 + \frac{1}{2}\, m|\boldsymbol{v}|^2 + \frac{\alpha}{|\boldsymbol{x}|}.$$

Ignoring the constant term $-mc^2$, we arrive at (9.4.29).

9.5 Curved wave operator

Recall that the *curved wave operator* in a space-time V_4 with the metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu,$$

where $x = (x^1, \dots, x^4)$, is defined by Eq. (6.3.2) and has the form

$$\diamondsuit[u] = g^{\mu\nu} u_{,\mu\nu} + \frac{1}{6}\, Ru, \tag{9.5.1}$$

where R is the scalar curvature of V_4 and $u_{,\mu\nu}$ is the second-order covariant derivative. Upon substituting the expression (9.1.4) for the covariant differentiation, Equation (9.5.1) is written

$$\diamondsuit[u] = g^{\mu\nu}\left(\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu}\frac{\partial a}{\partial x^\lambda}\right) + \frac{1}{6}Ru. \tag{9.5.2}$$

The characteristic property of the corresponding wave equation

$$\diamondsuit[u] = 0, \tag{9.5.3}$$

is that it is the only conformally invariant equation in V_4 provided that V_4 has a nontrivial conformal group. Moreover, in V_4 with a nontrivial conformal group, (9.5.3) is the only equation obeying the Huygens principle.

The operator (9.5.2) in the de Sitter space with the metric (9.1.32),

$$ds^2 = \frac{1}{\theta^2}(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{9.5.4}$$

can be written explicitly by using the expressions (9.1.28) and the change of variables (9.1.29). However, it is simpler to use the connection (6.1.23) between the operators (9.5.1) in mutually conformal spaces.

Namely, let V_4 have a metric tensor $g_{\mu\nu}$ and let $\widetilde{V}_4$ be a conformal space with the metric tensor

$$\widetilde{g}_{\mu\nu}(x) = H^2(x) g_{\mu\nu}(x), \quad \mu, \nu = 1, \dots, 4.$$

Then the curved wave operators $\diamondsuit$ and $\widetilde{\diamondsuit}$ in V_4 and $\widetilde{V}_4$, respectively, are related by the equation

$$\widetilde{\diamondsuit}[u] = H^{-3} \diamondsuit [Hu]. \tag{9.5.5}$$

In the case of the metric (9.5.4) we have $H = \theta^{-1}$. Substituting in Eq. (9.5.5) the operator (9.5.1) in the de Sitter space instead of $\widetilde{\diamondsuit}[u]$ and the usual wave operator in the Minkowski space instead of $\diamondsuit$, we obtain the following expression for the curved wave operator in the de Sitter space:

$$\diamondsuit[u] = \theta^3 \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \left(\frac{u}{\theta} \right). \tag{9.5.6}$$

Hence, we arrive at the following statement.

Theorem 9.5.1. The solution to the wave equation

$$\diamondsuit u = 0 \tag{9.5.7}$$

in the de Sitter space is given by

$$u = \theta\, v, \tag{9.5.8}$$

where v is the solution of the usual wave equation,

$$v_{tt} - c^2 (u_{xx} + u_{yy} + u_{zz}) = 0, \tag{9.5.9}$$

and

$$\theta = 1 - \frac{K}{4} (c^2 t^2 - x^2 - y^2 - z^2). \tag{9.5.10}$$

Consequently, one can obtain the solution, e.g. of the Cauchy problem

$$u\big|_{t=0} = u_0(x, y, z), \quad u_t\big|_{t=0} = u_1(x, y, z)$$

for Eq. (9.5.7) by substituting in (9.5.8) the solution v of the following Cauchy problem (see the expression for θ in Eqs. (9.4.21)) for Eq. (9.5.9):

$$v\big|_{t=0} = \left(1 + \frac{K}{4} |\boldsymbol{x}|^2\right)^{-1} u_0(x, y, z), \quad v_t\big|_{t=0} = \left(1 + \frac{K}{4} |\boldsymbol{x}|^2\right)^{-1} u_1(x, y, z).$$

9.6 Neutrinos in de Sitter space

In the present section, a specific symmetry of the Dirac equations for particles with zero mass (neutrinos) in the de Sitter space-time is considered in the framework of the theory of approximate groups. A theoretical conclusion of this section is that, in the de Sitter universe with a constant curvature $K \neq 0$, a massless neutrino splits into two "massive" neutrinos. The question remains open of whether this conclusion has a real physical significance and of how one can detect the splitting of neutrinos caused by curvature.

9.6.1 Two approximate representations of Dirac's equations in de Sitter space

Dirac's equations in the de Sitter space (12) for particles with zero mass (neutrinos) can be written in the linear approximation with respect to the curvature K as follows:

$$\gamma^\mu \frac{\partial \phi}{\partial x^\mu} - \frac{3}{4} K (x \cdot \gamma) \phi = 0. \tag{9.6.1}$$

Here γ^μ are the usual Dirac matrices in the Minkowski space, ϕ is a four-dimensional complex vector, and

$$(x \cdot \gamma) = \sum_{\mu=1}^{4} x^\mu \gamma^\mu.$$

The following propositions can be proved by direct calculations.

Proposition 9.6.1. The substitution

$$\psi = \phi - \frac{3}{8} K |x|^2 \phi \tag{9.6.2}$$

reduces Eq. (9.6.1), in the first order of precision with respect to K, to the Dirac equation in the Minkowski space,

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = 0. \tag{9.6.3}$$

Note that Eq. (9.6.3), due to homogeneity, is invariant under the following transformation of the variables x:

$$y^\mu = i x^\mu, \quad i = \sqrt{-1}. \tag{9.6.4}$$

Therefore, by setting

$$\phi(x) = \varphi(y), \tag{9.6.5}$$

we can rewrite Eq. (9.6.2) in the form

$$\psi = \varphi + \frac{3}{8}K|y|^2\varphi. \tag{9.6.6}$$

The invariance of Eq. (9.6.3) with respect to the transformation (9.6.4) means that

$$\gamma^\mu \frac{\partial \psi}{\partial y^\mu} = 0. \tag{9.6.7}$$

Substitution of (9.6.6) into Eq. (9.6.7) yields:

$$\gamma^\mu \frac{\partial \varphi}{\partial y^\mu} + \frac{3}{4}K(y\cdot\gamma)\varphi = 0. \tag{9.6.8}$$

Equation (9.6.8) coincides with the Dirac equation (9.6.1) in the de Sitter space-time with the curvature $(-K)$.

Proposition 9.6.2. The combined system of equations (9.6.1), (9.6.8) inherits all symmetries of the usual Dirac equation (9.6.3). Namely, the system (9.6.1), (9.6.8) is invariant under the approximate representation of the de Sitter group and under the transformation (9.6.4) - (9.6.5). Moreover, it is conformally invariant in the first order of accuracy with respect to K.

9.6.2 Splitting of neutrinos by curvature

The above calculations show that the Dirac equations in the de Sitter universe have the following peculiarities due to curvature.

9.6.2.1 Effective mass

Equation (9.6.1) can be regarded as a Dirac equation

$$\gamma^\mu \frac{\partial \phi}{\partial x^\mu} + m\phi = 0 \tag{9.6.9}$$

with the variable "effective" mass

$$m = -\frac{3}{4}K(x\cdot\gamma). \tag{9.6.10}$$

Then, in the framework of the usual relativistic theory, neutrinos will have *small but nonzero* mass. It follows from Proposition 9.6.2 that the "massive" neutrinos move with velocity of light and obey the Huygens principle on existence of a sharp rear front.

9.6.2.2 Splitting of neutrinos

Since Eqs. (9.6.3) and (9.6.7) coincide, there is no preference between two transformations (9.6.2) and (9.6.6), and hence between Eqs. (9.6.1) and (9.6.8). Consequently, a "massless" neutrino given by Eq. (9.6.3) splits into two "massive" neutrinos. These "massive" neutrinos have "effective" masses

$$m_1 = -\frac{3}{4}K(x \cdot \gamma)$$

and

$$m_2 = \frac{3}{4}K(x \cdot \gamma)$$

and are described by Eqs. (9.6.1) and (9.6.8), respectively. These two particles are distinctly different if and only if $K \neq 0$. One of them, namely given by Eq. (9.6.1), can be regarded as a *proper neutrino* and the other given by Eq. (9.6.8) as an *antineutrino.*

9.6.2.3 Neutrino as a compound particle

I suggest the following interpretation of the above mathematical observations. In the de Sitter universe with a curvature $K \neq 0$, a neutrino is a compound particle, namely *neutrino–antineutrino* with the total mass

$$m = m_1 + m_2 = 0. \tag{9.6.11}$$

It is natural to assume that only the first component of the compound is observable and is perceived as a massive neutrino. The counterpart to the neutrino provides the validity of the zero-mass-neutrino model and has the real nature in the *anti-universe* with the curvature $(-K)$. A physical relevance of this model can be manifested, however, only by experimental observations.

Exercises

Exercise 9.1. Deduce the operators (9.1.26) by both suggested approaches, namely by the change of variables (9.1.20) in the rotation generators of the sphere (9.1.18), and by solving the Killing equations (9.1.25).

Exercise 9.2. Find the Christoffel symbols for the de Sitter metric (9.1.32),

$$ds^2 = \frac{1}{\theta^2}(c^2dt^2 - dx^2 - dy^2 - dz^2),$$

in the first order of precision with respect to the curvature K.

Exercise 9.3. Find the geodesics in the de Sitter metric (9.1.32) in the first order of precision with respect to the curvature K.

Exercise 9.4. Is the Huygens principle satisfied for the curved wave operator (9.5.6) in the de Sitter space? Justify your answer.

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