Graduate Texts in Mathematics 27

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General Topology

Springer

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This book was originally published in 1955 by Van Nostrand in the "University Series in Higher Mathematics" (Editorial Board: M.A. Stone, L. Nirenberg, S.S. Chern).

Mathematics Subject Classification (1991): 54-01

Library of Congress Cataloging in Publication Data

Kelley, John L.
General topology.
(Graduate texts in mathematics; 27)
Reprint of the 1955 ed. published by Van Nostrand, New York in The University series in higher mathematics, M. A. Stone, L. Nirenberg, and S. S. Chern, eds.
Bibliography: p. 282
Includes index.
1. Topology. I. Title. II. Series.
QA611.K4 1975 514'.3 75-14364

Printed on acid-free paper.

Printed and bound by R. R. Donnelley & Sons, Harrisonburg, VA.
Printed in the United States of America.

9 8 7 6

ISBN 0-387-90125-6 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-90125-6 Springer-Verlag Berlin Heidelberg New York SPIN 10634178

PREFACE

This book is a systematic exposition of the part of general topology which has proven useful in several branches of mathematics. It is especially intended as background for modern analysis, and I have, with difficulty, been prevented by my friends from labeling it: What Every Young Analyst Should Know.

The book, which is based on various lectures given at the University of Chicago in 1946–47, the University of California in 1948–49, and at Tulane University in 1950–51, is intended to be both a reference and a text. These objectives are somewhat inconsistent. In particular, as a reference work it offers a reasonably complete coverage of the area, and this has resulted in a more extended treatment than would normally be given in a course. There are many details which are arranged primarily for reference work; for example, I have taken some pains to include all of the most commonly used terminology, and these terms are listed in the index. On the other hand, because it is a text the exposition in the earlier chapters proceeds at a rather pedestrian pace. For the same reason there is a preliminary chapter, not a part of the systematic exposition, which covers those topics requisite to the main body of work that I have found to be new to many students. The more serious results of this chapter are theorems on set theory, of which a systematic exposition is given in the appendix. This appendix is entirely independent of the remainder of the book, but with this exception each part of the book presupposes all earlier developments.

There are a few novelties in the presentation. Occasionally the title of a section is preceded by an asterisk; this indicates that the section constitutes a digression. Other topics, many of equal or greater interest, have been treated in the problems. These problems are supposed to be an integral part of the discussion. A few of them are exercises which are intended simply to aid in understanding the concepts employed. Others are counter examples, marking out the boundaries of possible theorems. Some are small theories which are of interest in themselves, and still others are introductions to applications of general topology in various fields. These last always include references so that the interested reader (that elusive creature) may continue his reading. The bibliography includes most of the recent contributions which are pertinent, a few outstanding earlier contributions, and a few "cross-field" references.

I employ two special conventions. In some cases where mathematical content requires "if and only if" and euphony demands something less I use Halmos' "iff." The end of each proof is signalized by ▌. This notation is also due to Halmos.

J. L. K.

Berkeley, California
February 1, 1955

ACKNOWLEDGMENTS

It is a pleasure to acknowledge my indebtedness to several colleagues.

The theorems surrounding the concept of even continuity in chapter 7 are the joint work of A. P. Morse and myself and are published here with his permission. Many of the pleasanter features of the appended development of set theory are taken from the unpublished system of Morse, and I am grateful for his permission to use these; he is not responsible for inaccuracies in my writing. I am also indebted to Alfred Tarski for several conversations on set theory and logic.

I owe thanks to several colleagues who have read part or all of the manuscript and made valuable criticisms. I am particularly obliged to Isaku Namioka, who has corrected a grievous number of errors and obscurities in the text and has suggested many improvements. Hugo Ribeiro and Paul R. Halmos have also helped a great deal with their advice.

Finally, I tender my very warm thanks to Tulane University and to the Office of Naval Research for support during the preparation of this manuscript. This book was written at Tulane University during the years 1950–52; it was revised in 1953, during tenure of a National Science Foundation Fellowship and a sabbatical leave from the University of California.

J. L. K.

April 21, 1961

A number of corrections have been made in this printing of the text. I am indebted to many colleagues, and especially to Krehe Ritter, for bringing errors to my attention.

J. L. K.

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Chapter 0

PRELIMINARIES

The only prerequisites for understanding this book are a knowledge of a few of the properties of the real numbers and a reasonable endowment of that invaluable quality, mathematical maturity. All of the definitions and basic theorems which are assumed later are collected in this first chapter. The treatment is reasonably self-contained, but, especially in the discussion of the number system, a good many details are omitted. The most profound results of the chapter are theorems of set theory, of which a systematic treatment is given in the appendix. Because the chapter is intended primarily for reference it is suggested that the reader review the first two sections and then turn to chapter one, using the remainder of the chapter if need arises. Many of the definitions are repeated when they first occur in the course of the work.

SETS

We shall be concerned with sets and with members of sets. "Set," "class," "family," "collection," and "aggregate" are synonymous,* and the symbol ε denotes membership. Thus $x \varepsilon A$ if and only if x is a member (an element, a point) of A. Two sets are identical iff they have the same members, and equality is

* This statement is not strictly accurate. There are technical reasons, expounded in the appendix, for distinguishing between two different sorts of aggregates. The term "set" will be reserved for classes which are themselves members of classes. This distinction is of no great importance here; with a single non-trivial exception, each class which occurs in the discussion (prior to the appendix) is also a set.

always used to mean identity. Consequently, $A = B$ if and only if, for each x, $x \in A$ when and only when $x \in B$.

Sets will be formed by means of braces, so that $\{x: \cdots$ *(proposition about* $x) \cdots\}$ is the set of all points x such that the proposition about x is correct. Schematically, $y \in \{x: \cdots$ *(proposition about* $x) \cdots\}$ if and only if the corresponding proposition about y is correct. For example, if A is a set, then $y \in \{x: x \in A\}$ iff $y \in A$. Because sets having the same members are identical, $A = \{x: x \in A\}$, a pleasant if not astonishing fact. It is to be understood that in this scheme for constructing sets "x" is a dummy variable, in the sense that we may replace it by any other variable that does not occur in the proposition. Thus $\{x: x \in A\} = \{y: y \in A\}$, but $\{x: x \in A\} \neq \{A: A \in A\}$.

There is a very useful rule about the construction of sets in this fashion. If sets are constructed from two different propositions by the use of the convention above, and if the two propositions are logically equivalent, then the constructed sets are identical. The rule may be justified by showing that the constructed sets have the same members. For example, if A and B are sets, then $\{x: x \in A \textit{ or } x \in B\} = \{x: x \in B \textit{ or } x \in A\}$, because y belongs to the first iff $y \in A$ or $y \in B$, and this is the case iff $y \in B$ or $y \in A$, which is correct iff y is a member of the second set. All of the theorems of the next section are proved in precisely this way.

SUBSETS AND COMPLEMENTS; UNION AND INTERSECTION

If A and B are sets (or families, or collections), then A is a **subset** (**subfamily, subcollection**) of B if and only if each member of A is a member of B. In this case we also say that A is **contained** in B and that B **contains** A, and we write the following: $A \subset B$ and $B \supset A$. Thus $A \subset B$ iff for each x it is true that $x \in B$ whenever $x \in A$. The set A is a **proper subset** of B (A is properly contained in B and B properly contains A) iff $A \subset B$ and $A \neq B$. If A is a subset of B and B is a subset of C, then clearly A is a subset of C. If $A \subset B$ and $B \subset A$, then $A = B$, for in this case each member of A is a member of B and conversely.

The **union (sum, logical sum, join)** of the sets A and B, written $A \cup B$, is the set of all points which belong either to A or to B; that is, $A \cup B = \{x: x \varepsilon A \textit{ or } x \varepsilon B\}$. It is understood that "or" is used here (and always) in the non-exclusive sense, and that points which belong to both A and B also belong to $A \cup B$. The **intersection (product, meet)** of sets A and B, written $A \cap B$, is the set of all points which belong to both A and B; that is, $A \cap B = \{x: x \varepsilon A \textit{ and } x \varepsilon B\}$. The **void set (empty set)** is denoted 0 and is defined to be $\{x: x \neq x\}$. (Any proposition which is always false could be used here instead of $x \neq x$.) The void set is a subset of every set A because each member of 0 (there are none) belongs to A. The inclusions, $0 \subset A \cap B \subset A \subset A \cup B$, are valid for every pair of sets A and B. Two sets A and B are **disjoint**, or **non-intersecting**, iff $A \cap B = 0$; that is, no member of A is also a member of B. The sets A and B **intersect** iff there is a point which belongs to both, so that $A \cap B \neq 0$. If $\mathfrak{A}$ is a family of sets (the members of $\mathfrak{A}$ are sets), then $\mathfrak{A}$ is a **disjoint family** iff no two members of $\mathfrak{A}$ intersect.

The **absolute complement** of a set A, written $\sim A$, is $\{x: x \notin A\}$. The **relative complement** of A with respect to a set X is $X \cap \sim A$, or simply $X \sim A$. This set is also called the **difference** of X and A. For each set A it is true that $\sim\sim A = A$; the corresponding statement for relative complements is slightly more complicated and is given as part of 0.2.

One must distinguish very carefully between "member" and "subset." The set whose only member is x is called **singleton** x and is denoted $\{x\}$. Observe that $\{0\}$ is not void, since $0 \varepsilon \{0\}$, and hence $0 \neq \{0\}$. In general, $x \varepsilon A$ if and only if $\{x\} \subset A$.

The two following theorems, of which we prove only a part, state some of the most commonly used relationships between the various definitions given above. These are basic facts and will frequently be used without explicit reference.

1 Theorem *Let A and B be subsets of a set X. Then $A \subset B$ if and only if any one of the following conditions holds:*

$$A \cap B = A, \quad B = A \cup B, \quad X \sim B \subset X \sim A,$$

$$A \cap X \sim B = 0, \quad \textit{or} \quad (X \sim A) \cup B = X.$$

2 THEOREM *Let A, B, C, and X be sets. Then:*

(a) $X \sim (X \sim A) = A \cap X$.
(b) (*Commutative laws*) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.
(c) (*Associative laws*) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$.
(d) (*Distributive laws*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(e) (*De Morgan formulae*) $X \sim (A \cup B) = (X \sim A) \cap (X \sim B)$ *and* $X \sim (A \cap B) = (X \sim A) \cup (X \sim B)$.

PROOF Proof of (a): A point x is a member of $X \sim (X \sim A)$ iff $x \in X$ and $x \notin X \sim A$. Since $x \notin X \sim A$ iff $x \notin X$ or $x \in A$, it follows that $x \in X \sim (X \sim A)$ iff $x \in X$ and either $x \notin X$ or $x \in A$. The first of these alternatives is impossible, so that $x \in X \sim (X \sim A)$ iff $x \in X$ and $x \in A$; that is, iff $x \in X \cap A$. Hence $X \sim (X \sim A) = A \cap X$. Proof of first part of (d): A point x is a member of $A \cap (B \cup C)$ iff $x \in A$ and either $x \in B$ or $x \in C$. This is the case iff either x belongs to both A and B or x belongs to both A and C. Hence $x \in A \cap (B \cup C)$ iff $x \in (A \cap B) \cup (A \cap C)$, and equality is proved. ∎

If $A_1, A_2, \cdots, A_n$ are sets, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is the union of the sets and $A_1 \cap A_2 \cap \cdots \cap A_n$ is their intersection. It does not matter how the terms are grouped in computing the union or intersection because of the associative laws. We shall also have to consider the union of the members of non-finite families of sets and it is extremely convenient to have a notation for this union. Consider the following situation: for each member a of a set A, which we call an index set, we suppose that a set X_a is given. Then the union of all the X_a, denoted $\bigcup \{X_a: a \in A\}$, is defined to be the set of all points x such that $x \in X_a$ for some a in A. In a similar way the intersection of all X_a for a in A, denoted $\bigcap \{X_a: a \in A\}$, is defined to be $\{x: x \in X_a \textit{ for each } a \textit{ in } A\}$. A very important special case arises when the index set is itself a family $\mathcal{A}$ of sets and X_A is the set A for each A in $\mathcal{A}$. Then the foregoing definitions become: $\bigcup \{A: A \in \mathcal{A}\} = \{x: x \in A \textit{ for some } A \textit{ in } \mathcal{A}\}$ and $\bigcap \{A: A \in \mathcal{A}\} = \{x: x \in A \textit{ for each } A \textit{ in } \mathcal{A}\}$.

There are many theorems of an algebraic character on the union and intersection of the members of families of sets, but we shall need only the following, the proof of which is omitted.

3 THEOREM *Let A be an index set, and for each a in A let X_a be a subset of a fixed set Y. Then:*

(a) *If B is a subset of A, then* $\bigcup\{X_b: b \in B\} \subset \bigcup\{X_a: a \in A\}$ *and* $\bigcap\{X_b: b \in B\} \supset \bigcap\{X_a: a \in A\}$.
(b) *(De Morgan formulae)* $Y \sim \bigcup\{X_a: a \in A\} = \bigcap\{Y \sim X_a: a \in A\}$ *and* $Y \sim \bigcap\{X_a: a \in A\} = \bigcup\{Y \sim X_a: a \in A\}$.

The De Morgan formulae are usually stated in the abbreviated form: the complement of the union is the intersection of the complements, and the complement of an intersection is the union of the complements.

It should be emphasized that a reasonable facility with this sort of set theoretic computation is essential. The appendix contains a long list of theorems which are recommended as exercises for the beginning student. (See the section on elementary algebra of classes.)

4 *Notes* In most of the early work on set theory the union of two sets A and B was denoted by $A + B$ and the intersection by AB, in analogy with the usual operations on the real numbers. Some of the same algebraic laws do hold; however, there is compelling reason for not following this usage. Frequently set theoretic calculations are made in a group, a field, or a linear space. If A and B are subsets of an (additively written) group, then $\{c: c = a + b$ *for some a in A and some b in B*$\}$ is a natural candidate for the label "$A + B$," and it is natural to denote $\{x: -x \in A\}$ by $-A$. Since the sets just described are used systematically in calculations where union, intersection, and complement also appear, the choice of notation made here seems the most reasonable.

The notation used here for construction of sets is the one most widely used today, but "$\underset{x}{E}$" for "the set of all x such that" is also used. The critical feature of a notation of this sort is the following: one must be sure just which is the dummy variable. An example will clarify this contention. The set of all squares of positive numbers might be denoted quite naturally by $\{x^2: x > 0\}$, and, proceeding, $\{x^2 + a^2: x < 1 + 2a\}$ also has a natural meaning. Unfortunately, the latter has three possible natural meanings, namely: $\{z: \textit{for some } x \textit{ and some } a,\ z = x^2 + a^2 \textit{ and } x < 1 + 2a\}$, $\{z: \textit{for some } x,\ z = x^2 + a^2 \textit{ and } x < 1 + 2a\}$, and $\{z: \textit{for some } a,\ z = x^2 + a^2 \textit{ and } x < 1 + 2a\}$. These sets are quite different, for the first depends on neither x nor a, the second is dependent on a, and the third depends on x. In slightly more technical terms one says that "x" and "a" are both dummies in the first, "x" is a dummy in the second, and "a" in the third. To avoid ambiguity, in each use of the brace notation the first space after the brace and preceding the colon is always occupied by the dummy variable.

Finally, it is interesting to consider one other notational feature. In reading such expressions as "$A \cap (B \cup C)$" the parentheses are essential. However, this could have been avoided by a slightly different choice of notation. If we had used "$\cup AB$" instead of "$A \cup B$," and similarly for intersection, then all parentheses could be omitted. (This general method of avoiding parentheses is well known in mathematical logic.) In the modified notation the first distributive law and the associative law for unions would then be stated: $\cap A \cup BC = \cup \cap AB \cap AC$ and $\cup A \cup BC = \cup \cup ABC$. The shorthand notation also reads well; for example, $\cup AB$ is the union of A and B.

RELATIONS

The notion of set has been taken as basic in this treatment, and we are therefore faced with the task of defining other necessary concepts in terms of sets. In particular, the notions of ordering and function must be defined. It turns out that these may be treated as relations, and that relations can be defined rather naturally as sets having a certain special structure. This section is therefore devoted to a brief statement of the definitions and elementary theorems of the algebra of relations.

Suppose that we are given a relation (in the intuitive sense) between certain pairs of objects. The basic idea is that the relation may be represented as the set of all pairs of mutually related objects. For example, the set of all pairs consisting of a number and its cube might be called the cubing relation. Of course, in order to use this method of realization it is necessary that we have available the notion of ordered pair. This notion can be defined in terms of sets.* The basic facts which we need here are: each ordered pair has a first coordinate and a second coordinate, and two ordered pairs are equal (identical) if and only if they have the same first coordinate and the same second coordinate. The ordered pair with first coordinate x and second coordinate y is denoted (x,y). Thus $(x,y) = (u,v)$ if and only if $x = u$ and $y = v$.

It is convenient to extend the device for the formation of sets so that $\{(x,y): \cdots\}$ is the set of all pairs (x,y) such that $\cdots$. This convention is not strictly necessary, for the same set is obtained by the specification: $\{z:$ *for some* x *and some* y, $z = (x,y)$ *and* $\cdots\}$.

A **relation** is a set of ordered pairs; that is, a relation is a set, each member of which is an ordered pair. If R is a relation we write xRy and $(x,y) \in R$ interchangeably, and we say that x is **R-related** to y if and only if xRy. The **domain** of a relation R is the set of all first coordinates of members of R, and its **range** is the set of all second coordinates. Formally, *domain* $R = \{x:$ *for some* y, $(x,y) \in R\}$ and *range* $R = \{y:$ *for some* x, $(x,y) \in R\}$. One of the simplest relations is the set of all pairs (x,y) such that x is a member of some fixed set A and y is a member of some fixed set B. This relation is the **cartesian product** of A and B and is denoted by $A \times B$. Thus $A \times B = \{(x,y)$: $x \in A$ *and* $y \in B\}$. If B is non-void the domain of $A \times B$ is A. It is evident that every relation is a subset of the cartesian product of its domain and range.

The **inverse** of a relation R, denoted by R^{-1}, is obtained by reversing each of the pairs belonging to R. Thus $R^{-1} = \{(x,y)$: $(y,x) \in R\}$ and xRy if and only if $yR^{-1}x$. For example, $(A \times B)^{-1} = B \times A$ for all sets A and B. The domain of the inverse of a relation R is always the range of R, and the range of R^{-1} is the domain of R. If R and S are relations their **composition**, $R \circ S$ (sometimes written RS), is defined to be the set of all pairs (x,z)

* An honest treatment of the problem is given in the appendix, where N. Wiener's definition of ordered pair is used. The ingenious notion of representing relations in this fashion is due to C. S. Peirce. A very readable account of the elementary relation algebra will be found in A. Tarski [1].

such that for some y it is true that $(x,y) \in S$ and $(y,z) \in R$. Composition is generally not commutative. For example, if $R = \{(1,2)\}$ and $S = \{(0,1)\}$, then $R \circ S = \{(0,2)\}$ and $S \circ R$ is void. The **identity** relation on a set X (the **identity on X**), denoted Δ or $\Delta(X)$, is the set of all pairs of the form (x,x) for x in X. The name is derived from the fact that $\Delta \circ R = R \circ \Delta = R$ whenever R is a relation whose range and domain are subsets of X. The identity relation is also called the **diagonal,** a name suggestive of its geometric position in $X \times X$.

If R is a relation and A is a set, then $R[A]$, the set of all R-relatives of points of A, is defined to be $\{y: xRy \textit{ for some } x \textit{ in } A\}$. If A is the domain of R, then $R[A]$ is the range of R, and for arbitrary A the set $R[A]$ is contained in the range of R. If R and S are relations and $R \subset S$, then clearly $R[A] \subset S[A]$ for every A.

There is an extensive calculus of relations, of which the following theorem is a fragment.

5 THEOREM *Let R, S, and T be relations and let A and B be sets. Then:*

(a) $(R^{-1})^{-1} = R$ *and* $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

(b) $R \circ (S \circ T) = (R \circ S) \circ T$ *and* $(R \circ S)[A] = R[S[A]]$.

(c) $R[A \cup B] = R[A] \cup R[B]$ *and* $R[A \cap B] \subset R[A] \cap R[B]$.

More generally, if there is given a set X_a for each member a of a non-void index set A then:

(d) $R[\bigcup\{X_a: a \in A\}] = \bigcup\{R[X_a]: a \in A\}$ *and* $R[\bigcap\{X_a: a \in A\}] \subset \bigcap\{R[X_a]: a \in A\}$.

PROOF As an example we prove the equality: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. A pair (z,x) is a member of $(R \circ S)^{-1}$ iff $(x,z) \in R \circ S$, and this is the case iff for some y it is true that $(x,y) \in S$ and $(y,z) \in R$. Consequently $(z,x) \in (R \circ S)^{-1}$ iff $(z,y) \in R^{-1}$ and $(y,z) \in S^{-1}$ for some y. This is precisely the condition that (z,x) belong to $S^{-1} \circ R^{-1}$. ∎

There are several special sorts of relations which occur so frequently in mathematics that they have acquired names. Aside from orderings and functions, which will be considered in detail in the following sections, the types listed below are probably the most useful. Throughout the following it will be convenient to suppose that R is a relation and that X is the set of all points which belong to either the domain or the range of R; that is, X = (domain R) $\cup$ (range R). The relation R is **reflexive** if and only if each point of X is R-related to itself. This is entirely equivalent to requiring that the identity Δ (or $\Delta(X)$) be a subset of R. The relation R is **symmetric**, provided that xRy whenever yRx. Algebraically, this requirement may be phrased: $R = R^{-1}$. At the other extreme, the relation R is **anti-symmetric** iff it is never the case that both xRy and yRx. In other words, R is anti-symmetric iff $R \cap R^{-1}$ is void. The relation R is **transitive** iff whenever xRy and yRz then xRz. In terms of the composition of relations, the relation R is transitive if and only if $R \circ R \subset R$. It follows that, if R is transitive, then $R^{-1} \circ R^{-1} = (R \circ R)^{-1} \subset R^{-1}$, and hence the inverse of a transitive relation is transitive. If R is both transitive and reflexive, then $R \circ R \supset R \circ \Delta$ and hence $R \circ R = R$; in the usual terminology, such a relation is idempotent under composition.

An **equivalence** relation is a reflexive, symmetric, and transitive relation. Equivalence relations have a very simple structure, which we now proceed to describe. Suppose that R is an equivalence relation and that X is the domain of R. A subset A of X is an **equivalence class** (an R-equivalence class) if and only if there is a member x of A such that A is identical with the set of all y such that xRy. In other words, A is an equivalence class iff there is x in A such that $A = R[\{x\}]$. The fundamental result on equivalence relations states that the family $\mathfrak{A}$ of all equivalence classes is disjoint, and that a point x is R-related to a point y if and only if both x and y belong to the same equivalence class. The set of all pairs (x,y) with x and y in a class A is simply $A \times A$, which leads to the following concise formulation of the theorem.

6 THEOREM *A relation R is an equivalence relation if and only if there is a disjoint family $\mathfrak{A}$ such that $R = \bigcup\{A \times A: A \in \mathfrak{A}\}$.*

PROOF If R is an equivalence relation, then R is transitive: if yRx and zRy, then zRx. In other words, if xRy, then $R[\{y\}] \subset R[\{x\}]$. But R is symmetric (xRy whenever yRx), from which it follows that, if xRy, then $R[\{x\}] = R[\{y\}]$. If z belongs to both $R[\{x\}]$ and $R[\{y\}]$, then $R[\{x\}] = R[\{z\}] = R[\{y\}]$, and consequently two equivalence classes either coincide or are disjoint. If y and z belong to the equivalence class $R[\{x\}]$, then, since $R[\{y\}] = R[\{x\}]$, it follows that yRz or, in other words, $R[\{x\}] \times R[\{x\}] \subset R$. Hence the union of $A \times A$ for all equivalence classes A is a subset of R, and since R is reflexive, if xRy, then $(x,y) \in R[\{x\}] \times R[\{x\}]$. Hence $R = \bigcup\{A \times A\colon A \in \mathfrak{A}\}$. The straightforward proof of the converse is omitted. ∎

We are frequently interested in the behavior of a relation for points belonging to a subset of its domain, and frequently the relation possesses properties for these points which it fails to have for all points. Given a set X and a relation R one may construct a new relation $R \cap (X \times X)$ whose domain is a subset of X. For convenience we will say that a relation R has a **property on** X, or that R **restricted to** X has the property iff $R \cap (X \times X)$ has the property. For example, R is transitive on X iff $R \cap (X \times X)$ is a transitive relation. This amounts to asserting that the defining property holds for points of X; in this case, whenever x, y, and z are points of X such that xRy and yRz, then xRz.

FUNCTIONS

The notion of function must now be defined in terms of the concepts already introduced. This offers no difficulty if we consider the following facts. Whatever a function is, its graph has an obvious definition as a set of ordered pairs. Moreover, there is no information about the function which cannot be derived from its graph. In brief, there is no reason why we should attempt to distinguish between a function and its graph.

A **function** is a relation such that no two distinct members have the same first coordinate. Thus f is a function iff the members of f are ordered pairs, and whenever (x,y) and (x,z) are members of f, then $y = z$. We do not distinguish between a function and its graph. The terms **correspondence**, **transformation**, **map**, **operator**, and **function** are synonymous. If f is a function and x is a point of its domain (the set of all first coordinates of members of f), then $f(x)$, or f_x is the second coordinate of the unique member of f whose first coordinate is x. The point $f(x)$

is the **value** of f at x, or the **image** of x under f, and we say that f **assigns** the value $f(x)$ to x, or **carries** x into $f(x)$. A function f is **on** X iff X is its domain and it is **onto** Y iff Y is its range (the set of second coordinates of members of f, sometimes called the set of values). If the range of f is a subset of Y, then f is **to** Y, or **into** Y. In general a function is many to one, in the sense that there are many pairs with the same second coordinate or, equivalently, many points at which the function has the same value. A function f is **one to one** iff distinct points have distinct images; that is, if the inverse relation, f^{-1}, is also a function.

A function is a set, and consequently two functions, f and g, are identical iff they have the same members. It is clear that this is the case iff the domain of f is identical with the domain of g and $f(x) = g(x)$ for each x in this domain. Consequently, we may define a function by specifying its domain and the value of the function at each member of the domain. If f is a function on X to Y and A is a subset of X, then $f \cap (A \times Y)$ is also a function. It is called the **restriction** of f to A, denoted $f|A$, its domain is A, and $(f|A)(x) = f(x)$ for x in A. A function g is the restriction of f to some subset iff the domain of g is a subset of the domain of f, and $g(x) = f(x)$ for x in the domain of g; that is, iff $g \subset f$. The function f is called an **extension** of g iff $g \subset f$. Thus f is an extension of g iff g is the restriction of f to some subset of the domain of f.

If A is a set and f is a function, then, following the definition given for arbitrary relations, $f[A] = \{y: \textit{for some } x \textit{ in } A, (x,y) \in f\}$; equivalently, $f[A]$ is $\{y: \textit{for some } x \textit{ in } A, y = f(x)\}$. The set $f[A]$ is called the image of A under f. If A and B are sets, then, by theorem 0.5, $f[A \cup B] = f[A] \cup f[B]$ and $f[A \cap B] \subset f[A] \cap f[B]$, and similar formulae hold for arbitrary unions and intersections. It is not true in general that $f[A \cap B] = f[A] \cap f[B]$, for disjoint sets may have intersecting images. If f is a function, then the set $f^{-1}[A]$ is called the **inverse (inverse image, counter image)** of A under f. The inverse satisfies the following algebraic rules.

7 THEOREM *If f is a function and A and B are sets then*

(a) $f^{-1}[A \sim B] = f^{-1}[A] \sim f^{-1}[B]$,

(b) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$, *and*
(c) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$.

More generally, if there is given a set X_c for each member c of a non-void index set C then

(d) $f^{-1}[\bigcup\{X_c: c \in C\}] = \bigcup\{f^{-1}[X_c]: c \in C\}$, *and*
(e) $f^{-1}[\bigcap\{X_c: c \in C\}] = \bigcap\{f^{-1}[X_c]: c \in C\}$.

PROOF Only part (e) will be proved. A point x is a member of $f^{-1}[\bigcap\{X_c: c \in C\}]$ if and only if $f(x)$ belongs to this intersection, which is the case iff $f(x) \in X_c$ for each c in C. But the latter condition is equivalent to $x \in f^{-1}[X_c]$ for each c in C; that is, $x \in \bigcap\{f^{-1}[X_c]: c \in C\}$. ∎

The foregoing theorem is often summarized as: the inverse of a function preserves relative complements, unions, and intersections. It should be noted that the validity of these formulae does not depend upon the sets A and B being subsets of the range of the function. Of course, $f^{-1}[A]$ is identical with the inverse image of the intersection of A with the range of f. However, it is convenient not to restrict the notation here (and the corresponding notation for images under f) to subsets of the range (respectively, the domain).

The composition of two functions is again a function by a straightforward argument. If f is a function, then $f^{-1} \circ f$ is an equivalence relation, for $(x,y) \in f^{-1} \circ f$ if and only if $f(x) = f(y)$. The composition $f \circ f^{-1}$ is a function; it is the identity on the range of f.

8 *Notes* There are other notations for the value of a function f at a point x. Besides $f(x)$ and f_x, all of the following are in use: (f,x), (x,f), fx, xf, and $\cdot fx$. The first two of these are extremely convenient in dealing with certain dualities, where one is considering a family F of functions, each on a fixed domain X, and it is desirable to treat F and X in a symmetric fashion. The notations "fx" and "xf" are obvious abbreviations of the notation we have adopted; whether the "f" is written to the left or to the right of "x" is clearly a matter of taste. These two share a disadvantage which is possessed by the "$f(x)$" notation. In certain rather complicated situations the notation is ambiguous, un-

less parentheses are interlarded liberally. The last notation (used by A. P. Morse) is free from this difficulty. It is unambiguous and does not require parentheses. (See the comments on union and intersection in 0.4.)

There is a need for a bound variable notation for a function. For example, the function whose domain is the set of all real numbers and which has the value x^2 at the point x should have a briefer description. A possible way out of this particular situation is to agree that x is the identity function on the set of real numbers, in which case x^2 might reasonably be the squaring function. The classical device is to use x^2 both for the function and for its value at the number x. A less confusing approach is to designate the squaring function by $x \to x^2$. This sort of notation is suggestive and is now coming into common use. It is not universal and, for example, the statement $(x \to x^2)(t) = t^2$ would require explanation. Finally it should be remarked that, although the arrow notation will undoubtedly be adopted as standard, the λ-convention of A. Church has technical advantages. (The square function might be written as $\lambda x\colon x^2$.) No parentheses are necessary to prevent ambiguity.

ORDERINGS

An ordering **(partial ordering, quasi-ordering)** is a transitive relation. A relation $<$ **orders (partially orders)** a set X iff it is transitive on X. If $<$ is an ordering and $x < y$, then it is customary to say that x **precedes** y or x is **less than** y (relative to the order $<$) and that y **follows** x and y is **greater than** x. If A is contained in a set X which is ordered by $<$, then an element x of X is an **upper bound** of A iff for each y in A either $y < x$ or $y = x$. Similarly an element x is a **lower bound** of A if x is less than or equal to each member of A. Of course, a set may have many different upper bounds. An element x is a **least upper bound** or **supremum** of A if and only if it is an upper bound and is less than or equal to every other upper bound. (In other words, a supremum is an upper bound which is a lower bound for the set of all upper bounds.) In the same way, a **greatest lower bound** or **infimum** is an element which is a lower bound and is greater than or equal to every other lower bound. A set X is **order-complete** (relative to the ordering $<$) if and only if each non-void subset of X which has an upper bound has a supremum. It is a little surprising that this condition on upper bounds is entirely equivalent to the corresponding statement for lower bounds. That is:

9 THEOREM *A set X is order-complete relative to an ordering if and only if each non-void subset which has a lower bound has an infimum.*

PROOF Suppose that X is order-complete and that A is a non-void subset which has a lower bound. Let B be the set of all lower bounds for A. Then B is non-void and surely every member of the non-void set A is an upper bound for B. Hence B has a least upper bound, say, b. Then b is less than or equal to each upper bound of B, and in particular b is less than or equal to each member of A, and hence b is a lower bound of A. On the other hand, b is itself an upper bound of B; that is, b is greater than or equal to each lower bound of A. Hence b is a greatest lower bound of A. The converse proposition may be proved by the same sort of argument, or, directly, one may apply the result just proved to the relation inverse to $<$. ∎

It should be remarked that the definition of ordering is not very restrictive. For example, $X \times X$ is an ordering of X, but a rather uninteresting one. Relative to this ordering each member of X is an upper bound, and in fact a supremum, of every subset. The more interesting orderings satisfy the further condition: if x is less than or equal to y and y is also less than or equal to x, then $y = x$. In this case there is at most one supremum for a set, and at most one infimum.

A **linear ordering** (**total**, **complete**, or **simple** ordering) is an ordering such that:

(a) *If $x < y$ and $y < x$, then $x = y$, and*
(b) *$x < y$ or $y < x$ whenever x and y are distinct members of the union of the domain and the range of $<$.*

It should be noticed that a linear ordering is not necessarily reflexive. However, agreeing that $x \leq y$ iff $x < y$ or $x = y$, the relation $\leq$ is always a reflexive linear ordering if $<$ is a linear ordering. Following the usual convention, a relation is said **to linearly order** a set X iff the relation restricted to X is a linear ordering. A set with a relation which linearly orders it is called a **chain.** Clearly suprema and infima are unique in chains. The remaining theorems in this section will concern chains, although it will be evident that many of the considerations apply to less restricted orderings.

A function f on a set X to a set Y is **order preserving (monotone, isotone)** relative to an order $<$ for X and an order $\prec$ for Y iff $f(u) \prec f(v)$ or $f(u) = f(v)$ whenever u and v are points of X such that $u \leqq v$. If the ordering $\prec$ of Y is simply $Y \times Y$, or if the ordering $<$ of X is the void relation, then f is necessarily order preserving. Consequently one cannot expect that the inverse of a one-to-one order preserving function will always be order preserving. However, if X and Y are chains and f is one to one and isotone, then necessarily f^{-1} is isotone, for if $f(u) \prec f(v)$ and $f(u) \neq f(v)$, then it is impossible that $v < u$ because of the order-preserving property.

Order-complete chains have a very special property. Suppose that X and Y are chains, that X_0 is a subset of X, and that f is an order-preserving function on X_0 to Y. The problem is: Does there exist an isotone extension of f whose domain is X? Unless some restriction is made on f the answer is "no," for, if X is the set of all positive real numbers, X_0 is the subset consisting of all numbers which are less than one, $Y = X_0$ and f is the identity map, then it is easy to see that there is no isotone extension. (Assuming an extension f^-, what is $f^-(1)$?) But this example also indicates the nature of the difficulty, for X_0 is a subset of X which has an upper bound and $f[X_0]$ has no upper bound. If an isotone extension f^- exists, then the image under f^- of an upper bound for a set A is surely an upper bound for $f[A]$. A similar statement holds for lower bounds, and it follows that, if a subset A of X_0 is **order-bounded** in X (that is, it has both an upper and lower bound in X), then the image $f[A]$ is order-bounded in Y. The following theorem asserts that this condition is also sufficient for the existence of an isotone extension.

10 THEOREM *Let f be an isotone function on a subset X_0 of a chain X to an order-complete chain Y. Then f has an isotone ex-*

tension whose domain is X if and only if f carries order-bounded sets into order-bounded sets. (More precisely stated, the condition is that, if A is a subset of X_0 which is order-bounded in X, then $f[A]$ is order-bounded in Y.)

PROOF It has already been observed that the condition is necessary for the existence of an isotone extension, and it remains to prove the sufficiency. We must construct an isotone extension of a given function f. First we note that if A is a subset of X_0 which has a lower bound in X, then $f[A]$ has a lower bound, for, choosing a point x in A, the set $\{y: y \in A \text{ and } y \leqq x\}$ is order-bounded, hence its image under f is order-bounded, and a lower bound for this image is also a lower bound for $f[A]$. A similar statement applies to upper bounds. For each x in X let L_x be the set of all members of X_0 which are less than or equal to x; that is, $L_x = \{y: y \leqq x \text{ and } y \in X_0\}$. If L_x is void, then x is a lower bound for X_0, hence $f[X_0]$ has an infimum v, and we define $f^-(x)$ to be v. If L_x is not void, then, since x is an upper bound for L_x, the set $f[L_x]$ has an upper bound and hence a supremum, and we define $f^-(x) = \sup f[L_x]$. The straightforward proof that f^- is an isotone extension of f is omitted. ∎

In certain cases the isotone extension of a function is unique. One such case will occur in treating the decimal expansion of a real number. Without attempting to get the best result of the sort, we give a simple sufficient condition for uniqueness which will apply.

11 THEOREM *Let f and g be isotone functions on a chain X to a chain Y, let X_0 be a subset of X on which f and g agree, and let Y_0 be $f[X_0]$. A sufficient condition that $f = g$ is that Y_0 intersect every set of the form $\{y: u < y < v, u \neq y \text{ and } y \neq v\}$, where u and v are members of Y such that $u < v$.*

PROOF If $f \neq g$, then $f(x) \neq g(x)$ for some x in X, and we may suppose that $f(x) < g(x)$. Each point of X_0 which is less than or equal to x maps under f into a point less than or equal to $f(x)$, because f is isotone, and each point which is greater than or equal to x maps under g into a point greater than or equal to $g(x)$, because g is isotone. It follows that no point of X_0 maps into the set $\{y: f(x) < y < g(x), f(x) \neq y \textit{ and } y \neq g(x)\}$, and the theorem is proved. ▮

12 *Notes* There is a natural way to embed a chain in an order-complete chain which is an abstraction of Dedekind's construction of the real numbers from the set of rational numbers. The process can also be applied to less restricted orderings, as shown by H. M. MacNeille (see Birkhoff [1; 58]). The pattern is very suggestive of the compactification procedure for topological spaces (chapter 5).

ALGEBRAIC CONCEPTS

In this section a few definitions from elementary algebra are given. For the most part these notions are used in the problems. The terminology is standard, and it seems worth while to summarize the few notions which are required.

A **group** is a pair, $(G,\cdot)$ such that G is a non-void set and $\cdot$, called the group operation, is a function on $G \times G$ to G such that: (a) the operation is associative, that is, $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ for all elements x, y and z of G; (b) there is a neutral element, or identity, e, such that $e\cdot x = x\cdot e = x$ for each x in G; and (c) for each x in G there is an inverse element x^{-1} such that $x\cdot x^{-1} = x^{-1}\cdot x = e$. If the group operation is denoted $+$, then the element inverse to x is usually written $-x$. Following the usual custom, the value of the function $\cdot$ at (x,y) is written $x\cdot y$ instead of the usual functional notation $\cdot(x,y)$, and if no confusion seems likely, the symbol $\cdot$ may be omitted entirely and the group operation indicated by juxtaposition. We shall sometimes say (imprecisely) that G is a group. If A and B are subsets of G, then $A\cdot B$, or simply AB, is the set of all elements of the form $x\cdot y$ for some x in A and some y in B. The set $\{x\}\cdot A$ is also denoted by $x\cdot A$ or simply xA, and similarly for operation on the right. The group is **abelian**, or **commutative**, iff $x\cdot y = y\cdot x$ for all members x and y of G. A group H is a subgroup of G iff $H \subset G$ and the group operation of H is that of G, restricted to $H \times H$. A subgroup H is **normal (distinguished, invariant)** iff $x\cdot H = H\cdot x$ for each x in G. If H is a subgroup of G a **left coset** of H is a subset which is of the form $x\cdot H$ for some x in G. The family of all left cosets is denoted by G/H. If H is normal and A and B belong to G/H, then $A \cdot B$ is also a member, and, with this definition of group operation, G/H is a group, called the **quotient** or **factor** group. A function f on a group G to a group H is a **homomorphism**, or **representation**, iff $f(x \cdot y) = f(x) \cdot f(y)$ for all members x and y of G. The **kernel** of f is the set $f^{-1}[e]$; it is always an invariant subgroup. If H is an invariant subgroup of G, then the function whose value at x is $x \cdot H$ is a homomorphism, usually called the **projection**, or **quotient map**, of G onto G/H.

A **ring** is a triple $(R,+,\cdot)$ such that $(R,+)$ is an abelian group and $\cdot$ is a function on $R \times R$ to R such that: the operation is associative, and the distributive laws $u \cdot (x + y) = u \cdot x + u \cdot y$ and $(u + v) \cdot x = u \cdot x + v \cdot x$ hold for all members x, y, u, and v of R. A **subring** is a subset which, under the ring operations restricted, is a ring, and a ring **homomorphism** or **representation** is a function f on a ring to another ring such that $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all members x and y of the domain. An additive subgroup I of a ring R is a **left ideal** iff $xI \subset I$ for each x in R, and is a **two-sided ideal** iff $xI \subset I$ and $Ix \subset I$ for each x in R. If I is a two-sided ideal, R/I is, with the proper addition and multiplication, a ring, and the projection of R onto R/I is a ring homomorphism. A **field** is a ring $(F,+,\cdot)$ such that F has at least two members, and $(F \sim \{0\},\cdot)$, where 0 is the element neutral with respect to $+$, is a commutative group. The operation $+$ is the **addition** operation, $\cdot$ is the **multiplication**, and the element neutral with respect to multiplication is the **unit**, 1. It is customary, when no confusion results, to replace $\cdot$ by juxtaposition, and, ignoring the operations, to say that "F is a field." A **linear space**, or **vector space**, over a field F (the **scalar** field of the space) is a quadruple $(X,\oplus,\cdot,F)$, such that $(X,\oplus)$ is an abelian group and $\cdot$ is a function on $F \times X$ to X such that for all members x and y of X, and all members a and b of F, $a \cdot (b \cdot x) = (a \cdot b) \cdot x$, $(a + b) \cdot x = a \cdot x \oplus b \cdot x$, $a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y$, and $1 \cdot x = x$. A **real linear space** is a linear space over the field of real numbers. The notion of linear space can also be formulated in a slightly different fashion. The family of all homomorphisms of an abelian group into itself becomes, with addition defined pointwise and with composition of functions as multiplication, a ring, called the **endomorphism ring** of the group. A linear space over a field F is a quadruple $(X, \oplus, \cdot, F)$ such that $(X, \oplus)$ is an abelian group and $\cdot$ is a ring homomorphism of F into the endomorphism ring of $(X, \oplus)$ which carries the unit, 1, into the identity homomorphism.

A linear space $(Y, \oplus, \odot, F)$ is a **subspace** of a linear space $(X, +, \cdot, F)$ iff $Y \subset X$ and the operations $+$ and $\cdot$ agree with $\oplus$ and $\odot$ where the latter are defined. The family X/Y of cosets of X modulo a subspace Y may be made into a linear space if addition and scalar multiplication are defined in the obvious way. The projection f of X onto X/Y then has the property that $f(a \cdot x + b \cdot y) = a \cdot f(x) + b \cdot f(y)$ for all members a and b of F and all x and y in X. Such a function is called a **linear function.** If f is a linear function the set $f^{-1}[0]$ is called the **null space** of f; the null space of a linear function is a linear subspace of the domain (provided the operations of addition and scalar multiplication are properly defined).

Suppose f is a linear function on X to Y and g is a linear map of X onto Z such that the null space of f contains the null space of g. Then there is a unique linear function h on Z to Y such that $f = h \circ g$ (explicitly, $h(z)$ is the unique member of $f \circ g^{-1}[z]$). (The function h is said to be **induced** by f and g.) A particular consequence of this fact is that each linear function may be written as a projection into a quotient space followed by a one-to-one linear function.

THE REAL NUMBERS

This section is devoted to the proof of a few of the most important results concerning the real numbers.

An **ordered field** is a field F and a subset P, called the set of **positive elements,** such that

(a) *if x and y are members of P, then $x + y$ and xy are also members; and*

(b) *if x is a member of F, then precisely one of the following statements is true:* $x \in P$, $-x \in P$, or $x = 0$.

One easily verifies that $<$ is a linear ordering of F, where, by definition, $x < y$ iff $y - x \in P$. The usual simple propositions about adding and multiplying inequalities hold. The members x of F such that $-x \in P$ are **negative.**

It will be assumed that the real numbers are an ordered field which is order-complete, in the sense that every non-void subset which has an upper bound has a least upper bound, or supremum. By 0.9 this last requirement is entirely equivalent to the statement that each non-void subset which has a lower bound has a greatest lower bound, or infimum.

We first prove a few propositions about integers. An **inductive set** is a set A of real numbers such that $0 \in A$, and whenever $x \in A$, then $x + 1 \in A$. A real number x is a **non-negative integer** iff x belongs to every inductive set. In other words, the set ω of non-negative integers is defined to be the intersection of the members of the family of all inductive sets. Each member of ω is actually non-negative because the set of all non-negative numbers is inductive. It is evident that ω is itself an inductive set and is a subset of every other inductive set. It follows that **(principle of mathematical induction)** each inductive subset of ω is identical with ω. A proof which relies on this principle is a **proof by induction.** We prove the following little theorem as an example: if p and q are non-negative integers and $p < q$, then $q - p \in \omega$. First observe that the set consisting of 0 and all numbers of the form $p + 1$ with p in ω is inductive, and hence each non-zero member of ω is of the form $p + 1$. Next, let A be the set of all non-negative integers p such that $q - p \in \omega$ for each larger member q of ω. Surely $0 \in A$, and let us suppose that p is a member of A and that q is an arbitrary member of ω which is larger than $p + 1$. Then $p < q - 1$ and therefore $q - 1 - p \in \omega$, because $p \in A$ and $q - 1 \in \omega$. Consequently $p + 1 \in A$, hence A is an inductive set, and therefore $A = \omega$. It is equally simple to show that the sum of two members of ω is a member of ω, and it follows that the set $\{x: x \in \omega \text{ or } -x \in \omega\}$ is a group. It is the **group of integers.**

There is another form of the principle of mathematical induction which is frequently convenient, namely: *each non-void subset A of ω has a smallest member.* To prove this proposition consider the set B of all members of ω which are lower bounds for A; that is, $B = \{p: p \in \omega \text{ and } p \leqq q \text{ for all } q \text{ in } A\}$. The set B

is not inductive, for, if $q \in A$, then $q + 1 \notin B$. Since $0 \in B$ it follows that there is a member p of B such that $p + 1 \notin B$. If $p \in A$, then clearly p is the smallest member of A; otherwise there is a member q of A such that $p < q < p + 1$. But then $q - p$ is a non-zero member of ω and hence $q - p - 1$ is a negative member of ω, which is impossible.

It is possible to **define a function by induction** in the following sense. For each non-negative integer p let $\omega_p = \{q\colon q \in \omega \textit{ and } q \leq p\}$. Suppose that we seek a function on ω, that the functional value a at 0 is given, and for each function g on a set ω_p there is given $F(g)$, the value of the desired function at $p + 1$. Thus the value desired at $p + 1$ may depend on all of the values for smaller integers. In these circumstances it is true that there is a unique function f on ω such that $f(0) = a$ and $f(p + 1) = F(f \mid \omega_p)$ for each p in ω. (The function $f \mid \omega_p$ is the function f restricted to the set ω_p.) This proposition is frequently considered to be obvious, but the proof is not entirely trivial.

13 THEOREM *Suppose a is given and $F(g)$ is given whenever g is a function whose domain is of the form ω_p for some p in ω. Then there is a unique function f such that $f(0) = a$ and $f(p + 1) = F(f \mid \omega_p)$ for each p in ω.*

PROOF Let $\mathfrak{F}$ be the family of all functions g such that the domain of g is a set ω_p for some p in ω, $g(0) = a$, and for each member q of ω such that $q \leq p - 1$, $g(q + 1) = F(g \mid \omega_q)$. (Intuitively, the members of $\mathfrak{F}$ are initial segments of the desired function.) The family $\mathfrak{F}$ has the very important property: if g and h are members of $\mathfrak{F}$, then either $g \subset h$ or $h \subset g$. To prove this it is necessary to show that $g(q) = h(q)$ for each q belonging to the domain of both. Suppose this is false, and let q be the smallest integer such that $g(q) \neq h(q)$. Then $q \neq 0$, because $g(0) = h(0) = a$, and hence $g(q) = F(g \mid \omega_{q-1})$ which, since g and h agree for values smaller than q, is $F(h \mid \omega_{q-1}) = h(q)$, and this is a contradiction. Now let $f = \bigcup\{g\colon g \in \mathfrak{F}\}$. Then the members of f are surely ordered pairs, and if $(x,y) \in g \in \mathfrak{F}$ and $(x,z) \in h \in \mathfrak{F}$, then (x,y) and (x,z) both belong to g or both to h, and hence $y = z$. Consequently f is a function, and it must be shown that it is the required function. First, because $\{(0,a)\} \in \mathfrak{F}$, $f(0) = a$.

Next, if $q + 1$ belongs to the domain of f, then for some g in $\mathfrak{F}$, $q + 1$ is a member of the domain of g, and hence $f(q + 1) = g(q + 1) = F(g \mid \omega_q) = F(f \mid \omega_q)$. Finally, to show that the domain of f is ω, suppose that q is the first member of ω which is not in the domain of f. Then $q - 1$ is the last member of the domain of f, and $f \cup \{(q,F(f))\}$ is a member of $\mathfrak{F}$. Hence q belongs to the domain of f, which is a contradiction. ∎

The foregoing theorem can be used systematically in showing the elementary properties of the real numbers. For example, if b is a positive number and p an integer, b^p is defined as follows. In the foregoing theorem, let $a = 1$ and for each function g with domain ω_p let $F(g) = bg(p)$. Then $f(0) = 1$ and $f(p + 1) = bf(p)$ for each p in ω, if f is the function whose existence is guaranteed by the theorem. Letting $b^p = f(p)$, it follows that $b^0 = 1$, and $b^{p+1} = bb^p$, from which one can show by induction that $b^{p+q} = b^p b^q$ for all members p and q of ω. If b^{-p} is defined to be $1/b^p$ for each non-negative integer p, then the usual elementary proof shows that $b^{p+q} = b^p b^q$ for all integers p and q.

So far in this discussion of the real numbers we have not used the fact that the field of real numbers is order-complete. We now prove a simple, but noteworthy, consequence of order completeness. First, the set ω of non-negative integers does not have an upper bound, for, if x were a least upper bound of ω, then $x - 1$ would not be an upper bound, and hence $x - 1 < p$ for some p in ω. But then $x < p + 1$ and this contradicts the fact that x was supposed to be an upper bound. Consequently, if x is a positive real number and y is a real number, then $px > y$ for some positive integer p because there is a member p of ω which is larger than y/x. An ordered field for which this proposition is true is said to have an **Archimedean order.**

We will need the fact that each non-negative real number has a b-adic expansion, where b is an arbitrary integer greater than one. Roughly speaking, we want to write a number x as the sum of multiples of powers of b, the multiples (digits) being non-negative integers less than b. Of course, the b-adic expansion of a number may fail to be unique—in the decimal expansion, .9999··· (all nines) and 1.000··· (all zeros) are to be expansions of the same real number. The expansion itself is a function which assigns to each integer an integer between 0 and $b - 1$, such that (since we want only a finite number of non-zero integers before the decimal point) there is a first non-zero digit. Formally, a is a b-**adic expansion** iff a is a function on the integers to ω_{b-1} ($= \{q: q \in \omega \text{ and } q \leq b - 1\}$), such that there is a smallest integer p for which a_p ($= a(p)$) is not zero. A b-adic expansion a is **rational** iff there is a last non-zero digit (that is, for some integer p, $a_q = 0$ whenever $q > p$). For each rational b-adic expansion a there is a simple way of assigning a corresponding real number $r(a)$. Except for a finite number of integers p the number $a_p b^{-p}$ is zero, and the sum of $a_p b^{-p}$ for p in this finite set is the real number $r(a)$. We write $r(a) = \Sigma\{a_p b^{-p}: p \text{ an integer}\}$. A real number which is of this form is a b-**adic rational.** These numbers are precisely those of the form, qb^{-p}, for integers p and q. Let E be the set of all b-adic expansions. Then E is linearly ordered by dictionary order; in detail, a b-adic expansion a precedes a b-adic expansion c in **dictionary order (lexicographic order)** iff for the smallest integer p such that $a_p \neq c_p$ it is true that $a_p < c_p$. It is very easy to see that, like a dictionary, E is actually linearly ordered by $<$. The correspondence r is order preserving, and this is the key to the following proposition.

14 THEOREM *Let E be the set of b-adic expansions, let R be the set of rational expansions, and for a in R let* $r(a) = \Sigma\{a_p b^{-p}$: p an integer$\}$. *Then there is a unique isotone extension $\bar{r}$ of r whose domain is E, and $\bar{r}$ maps $E \sim R$ onto the positive real numbers in a one-to-one fashion.*

PROOF According to theorem 0.10 there will be an isotone extension $\bar{r}$ of r iff r carries each subset of R which is order-bounded in E into an order-bounded subset of the real numbers. But for each a in E there is evidently b in R such that $b > a$, and it follows that, if a subset A of R has a for an upper bound, then $r(b)$ is an upper bound for $f[A]$. A similar argument applies to lower bounds, and we conclude that r carries order-bounded sets into order-bounded sets and consequently has an isotone extension $\bar{r}$ whose domain is E.

To show the extension is unique it is sufficient, by 0.11, to prove that, for non-negative real numbers x and y, if $x < y$, then there is a in R such that $x < r(a) < y$. Because $b^p > p$ for each non-negative integer p (a fact which is easily proved by induction), and because the set of non-negative integers is not bounded, there is an integer p such that $b^p > 1/(y - x)$. Then $b^{-p} < (y - x)$. There is an integer q such that $qb^{-p} \geqq y$ because the ordering is Archimedean, and since there is a smallest such integer q, it may be supposed that $(q - 1)b^{-p} < y$. It follows that $(q - 1)b^{-p} > x$ because b^{-p} is less than $(y - x)$ and this proves that there is a b-adic rational, $(q - 1)b^{-p}$, which is the image of a member of R and lies between x and y. Consequently the correspondence $\bar{r}$ is unique.

Next, we show that the correspondence $\bar{r}$ is one to one on $E \sim R$. It is straightforward to see that $\bar{r}$ is one to one on R, and this fact is assumed in the following. Suppose that $a \in E$, $c \in E \sim R$, and $a < c$. Then for the first value of p such that a_p and c_p are different, $a_p < c_p$. The expansion d, such that for $q < p$, $d_q = a_q$, for $q > p$, $d_q = 0$, and $d_p = a_p + 1$, is a member of R which is greater than a, and since c does not have a last non-zero digit, $a < d < c$. Repeating, there is a member e of R such that $a < d < e < c$. Then, since on R the function $\bar{r}$ is one to one, $\bar{r}(a) \leqq \bar{r}(d) < \bar{r}(e) \leqq \bar{r}(c)$, and $\bar{r}$ is therefore one to one on $E \sim R$.

Finally, it must be shown that the image of $E \sim R$ under $\bar{r}$ is the set of all positive numbers. First notice that for every pair of members c and d of R for which $c < d$ there is a in $E \sim R$ such that $c < a < d$, and consequently for positive real numbers x and y with $x < y$ there is a in $E \sim R$ such that $x < \bar{r}(a) < y$. If now x is a positive real number which is not the image of a member of $E \sim R$, let $F = \{a\colon a \in E \sim R \text{ and } \bar{r}(a) < x\}$. If the set F has a supremum c then, if $\bar{r}(c) < x$ no point of $E \sim R$ maps into the interval $(\bar{r}(c),x)$, and if $\bar{r}(c) > x$, then ($\bar{r}$ is order preserving) no point of $E \sim R$ maps into the interval $(x,\bar{r}(c))$. In either event a contradiction results, and the theorem will follow if it is shown that each non-empty subset of $E \sim R$ which has an upper bound has a supremum: that is, $E \sim R$ is order-complete. Suppose then that F is a non-void subset of $E \sim R$ which has an upper bound. Then there is a smallest integer p

such that $a_p \neq 0$ for some a in F. Define c_q to be zero for $q < p$, let F_p be the set of all members a of F with non-zero p-th digit a_p, and let $c_p = \max \{a_p: a \in F_p\}$. Continue inductively, letting F_{p+1} be the set of all members a of F_p such that $a_q = c_q$ for $q = p$, and let $c_{p+1} = \max \{a_{p+1}: a \in F_{p+1}\}$. No one of the sets F_p can be void and without difficulty one sees that the expansion c obtained by this construction is an upper bound of F, and in fact a supremum, and that $c \in E \sim R$. ▌

The foregoing theorem will be used for b equal to two, three, and ten. The b-adic expansions are then called **dyadic, triadic,** and **decimal,** respectively.

COUNTABLE SETS

A set is finite iff it can be put into one-to-one correspondence with a set of the form $\{p: p \in \omega \text{ and } p < q\}$, for some q in ω. A set A is **countably infinite** iff it can be put into one-to-one correspondence with the set ω of non-negative integers; that is, iff A is the range of some one-to-one function on ω. A set is **countable** iff it is either finite or countably infinite.

15 THEOREM *A subset of a countable set is countable.*

PROOF Suppose A is countable, f is one to one on ω with range A, and that $B \subset A$. Then f, restricted to $f^{-1}[B]$, is a one-to-one function on a subset of ω with range B, and if it can be shown that $f^{-1}[B]$ is countable, then a one-to-one function onto B can be constructed by composition. The proof therefore reduces to showing that an arbitrary subset C of ω is countable. Let $g(0)$ be the first member of C, and proceeding inductively, for p in ω, let $g(p)$ be the first member of C different from $g(0), g(1), \cdots, g(p-1)$. If this choice is impossible for some p then g is a function on $\{q: q \in \omega \textit{ and } q < p\}$ with range C, and C is finite. Otherwise (using 0.13 on the construction of functions by induction) there is a function g on ω such that, for each p in ω, $g(p)$ is the first member of C different from $g(0), g(1), \cdots, g(p-1)$. Clearly g is one to one. It is easily verified by induction that $g(p) \geq p$ for all p, and hence it follows from the choice of $g(p+1)$ that each member p of C is one of the numbers $g(q)$ for $q \leq p$. Therefore the range of g is C. ▌

16 THEOREM *If the domain of a function is countable, then the range is also countable.*

PROOF It is sufficient to show that, if A is a subset of ω and f is a function on A onto B, then B is countable. Let C be the set of all members x of A such that, if $y \in A$ and $y < x$, then $f(x) \neq f(y)$; that is, C consists of the smallest member of each of the sets $f^{-1}[y]$ for y in B. Then $f \mid C$ maps C onto B in a one-to-one fashion, and since C is countable by 0.15, so is B. ▌

17 THEOREM *If $\mathcal{A}$ is a countable family of countable sets, then $\bigcup\{A: A \in \mathcal{A}\}$ is countable.*

PROOF Because $\mathcal{A}$ is countable there is a function F whose domain is a subset of ω and whose range is $\mathcal{A}$. Since $F(p)$ is countable for each p in ω, it is possible to find a function G_p on a subset of $\{p\} \times \omega$ whose range is $F(p)$. Consequently there is a function (the union of the functions G_p) on a subset of $\omega \times \omega$ whose range is $\bigcup\{A: A \in \mathcal{A}\}$, and the problem reduces to showing that $\omega \times \omega$ is countable. The key to this proof is the observation that, if we think of $\omega \times \omega$ as lying in the upper right-hand part of the plane, the diagonals which cross from upper left to lower right contain only a finite number of members of $\omega \times \omega$. Explicitly, for n in ω, let $B_n = \{(p,q): (p,q) \in \omega \times \omega \text{ and } p + q = n\}$. Then B_n contains precisely $n + 1$ points, and the union $\bigcup\{B_n: n \in \omega\}$ is $\omega \times \omega$. A function on ω with range $\omega \times \omega$ may be constructed by choosing first the members of B_0, next those of B_1, and so on. The explicit definition of such a function is left to the reader. ▌

The **characteristic function** of a subset A of a set X is the function f such that $f(x) = 0$ for x in $X \sim A$ and $f(x) = 1$ for x in A. A function f on a set X which assumes no value other than zero and one is called a characteristic function; it is clearly *the* characteristic function of $f^{-1}[1]$. The function which is zero everywhere is the characteristic function of the void set, and the function which is identically one on X is the characteristic function of X. Two sets have the same characteristic functions iff they are identical, and hence there is a one-to-one correspondence between the family of all characteristic functions on a set X and the family of all subsets of X.

If ω is the set of non-negative integers, the family of all characteristic functions on ω may be put into one-to-one correspondence with the set F of all dyadic expansions a such that $a_p = 0$ for $p < 0$. The family of all finite subsets of ω corresponds in a one-to-one way to the subfamily G of F consisting of rational dyadic expansions. We now use the classical Cantor process to prove that F is uncountable.

18 THEOREM *The family of all finite subsets of a countably infinite set is countable, but the family of all subsets is not.*

PROOF In view of the remarks preceding the statement of the theorem it is sufficient to show that the set F of all dyadic expansions a with $a_p = 0$ for p negative is uncountable, and that the subset G of F consisting of rational expansions is countable. Suppose that f is a one-to-one function on ω with range F. Let a be the member of F such that $a_p = 1 - f(p)_p$ for each non-negative integer p. That is, the p-th digit of a is one minus the p-th digit of $f(p)$. Then $a \in F$ and clearly, for each p in ω, $a \neq f(p)$ because a and $f(p)$ differ in the p-th digit. It follows that a does not belong to the range of f, and this is a contradiction. Hence F is uncountable.

It remains to be proved that G is countable. For p in ω let $G_p = \{a\colon a \in G \text{ and } a_q = 0 \text{ for } q > p\}$. Then G_0 contains just two elements, and since there are precisely twice as many members in G_{p+1} as in G_p, it follows that G_p is always finite. Hence $G = \bigcup\{G_p\colon p \in \omega\}$ is countable. ∎

The natural correspondence between F and a subset of the real numbers is, according to 0.14, one to one on $F \sim G$. Since G is countable, $F \sim G$ must be uncountable. Hence

19 COROLLARY *The set of all real numbers is uncountable.*

CARDINAL NUMBERS

Many of the theorems on countability are special cases of more general theorems on cardinal numbers. The set ω of non-negative integers played a special role in the above and, in a more general way, this role may be occupied by sets (of which ω is one) called cardinal numbers. Let us agree that two sets,

A and B, are **equipollent** iff there is a one-to-one function on A with range B. It turns out that for every set A there is a unique cardinal number C such that A and C are equipollent. If C and D are distinct cardinal numbers, then C and D are not equipollent but one of the cardinal numbers, say C, and a proper subset of the other are equipollent. In this case C is said to be the **smaller cardinal number** and we write $C < D$. With this definition of order the family of all cardinal numbers is linearly ordered, and even more, every non-void subfamily has a least member. (These facts are proved in the appendix.)

Accepting the facts in the previous paragraph for the moment it follows that, if A and B are sets, then there is a one-to-one function on A to a subset of B, or the reverse, because there are cardinal numbers C and D such that A and C, and B and D, respectively, are equipollent. Suppose now that there is a one-to-one function on A to a subset of B and also a one-to-one function on B to a subset of A. Then C and a subset of D are equipollent, and D and a subset of C are equipollent, from which it follows, since the ordering of the cardinal numbers is linear, that $C = D$. Hence A and B are equipollent. This is the classical Schroeder-Bernstein theorem. We give a direct proof of this theorem which is independent of the general theory of cardinal numbers because the proof gives non-trivial additional information.

20 THEOREM *If there is a one-to-one function on a set A to a subset of a set B and there is also a one-to-one function on B to a subset of A, then A and B are equipollent.*

PROOF Suppose that f is a one-to-one map of A into B and g is one to one on B to A. It may be supposed that A and B are disjoint. The proof of the theorem is accomplished by decomposing A and B into classes which are most easily described in terms of parthenogenesis. A point x (of either A or B) is an ancestor of a point y iff y can be obtained from x by successive application of f and g (or g and f). Now decompose A into three sets: let A_E consist of all points of A which have an even number of ancestors, let A_O consist of points which have an odd number of ancestors, and let A_I consist of points with infinitely many ancestors. Decompose B similarly and observe: f maps A_E onto B_O and A_I onto B_I, and g^{-1} maps A_O onto B_E. Hence the function which agrees with f on $A_E \cup A_I$ and agrees with g^{-1} on A_O is a one-to-one map of A onto B. ▌

21 *Notes* The foregoing proof does not use the axiom of choice, which is interesting but not very important. It is important to notice that the function desired was constructed from the two given functions by a countable process. Explicitly, if f is a one-to-one function on A to B and g is one to one on B to A, if $E_0 = A \sim g[B]$, $E_{n+1} = g \circ f[E_n]$ for each n, and if $E = \bigcup \{E_n : n \in \omega\}$, then the function h which is equal to f on E and equal to g^{-1} on $A \sim E$ is a one-to-one map of A onto B. (More precisely, $h = (f \mid E) \cup (g^{-1} \mid A \sim E)$.) The importance of this result lies in the fact that, if f and g have certain pleasant properties (such as being Borel functions), then h retains these properties.

The intuitively elegant form of the proof of theorem 0.20 is due to G. Birkhoff and S. MacLane.

ORDINAL NUMBERS

Except for examples, the ordinal numbers will not be needed in the course of this work. However, several of the most interesting counter examples are based on extremely elementary properties of the ordinals and it seems proper to state here the few facts which are necessary for these. (The ordinal numbers are constructed and these and other properties proved in the appendix.)

22 SUMMARY *There is an uncountable set Ω', which is linearly ordered by a relation $<$ in such a way that:*

(a) *Every non-void subset of Ω' has a smallest element.*
(b) *There is a greatest element Ω of Ω'.*
(c) *If $x \in \Omega'$ and $x \neq \Omega$, then the set of all members of Ω' which precede x is countable.*

The set Ω' is the set of all ordinals which are less than or equal to Ω, **the first uncountable ordinal.** A linearly ordered set such that every non-void subset has a least element is **well ordered.**

In particular, each non-void subset of a well-ordered set has an infimum. Since every subset of Ω' has an upper bound, namely, Ω, it follows by 0.9 that every non-void subset of Ω' has a supremum. One of the curious facts about Ω' is the following.

23 THEOREM *If A is a countable subset of Ω' and $\Omega \notin A$, then the supremum of A is less than Ω.*

PROOF Assume that A is a countable subset of Ω' and that $\Omega \notin A$. For each member a of A the set $\{x: x \leqq a\}$ is countable and hence the union of all such sets is countable. This union is $\{x: x \leqq a \textit{ for some } a \textit{ in } A\}$ and the supremum b of the union is therefore an upper bound for A. The point b has only a countable number of predecessors relative to the ordering, and hence $b \neq \Omega$. It follows that the supremum of A is less than Ω. ∎

One member of Ω' deserves special notice. The first member of Ω' which does not have a finite number of predecessors is the **first non-finite ordinal** and is denoted ω. The symbol ω has already been used to denote the set of non-negative integers. In the construction of the ordinal numbers it turns out that the first non-finite ordinal is, in fact, the set ω of non-negative integers!

CARTESIAN PRODUCTS

If A and B are sets the cartesian product $A \times B$ has been defined as the set of all ordered pairs (x,y) such that $x \in A$ and $y \in B$. It is useful to extend the definition of cartesian product to families of sets, just as the notion of union and intersection was extended to arbitrary families of sets. Suppose that for each member a of an index set A there is given a set X_a. The **Cartesian product** of the sets X_a, written $\times\{X_a: a \in A\}$, is defined to be the set of all functions x on A such that $x(a) \in X_a$ for each a in A. It is customary to use subscript notation rather than the usual functional notation, so that $\times\{X_a: a \in A\} = \{x: x \textit{ is a function on } A \textit{ and } x_a \in X_a \textit{ for } a \textit{ in } A\}$. The definition is initially a little surprising but it is actually a precise statement of the intuitive concept: a point x of the product consists of a point (namely, x_a) selected from each of the sets X_a. The set X_a is the **a-th coordinate set,** and the point x_a is the **a-th coordi-**

nate of the point x of the product. The function P_a which carries each point x of the product onto its a-th coordinate x_a is the **projection** into the a-th coordinate set. That is, $P_a(x) = x_a$. The map P_a is also called the **evaluation at** a.

There is an important special case of a cartesian product. Suppose that the coordinate set X_a is a fixed set Y for each a in the index set A. Then the cartesian product $\times\{X_a: a \in A\} = \times\{Y: a \in A\} = \{x: x \text{ is a function on } A \text{ to } Y\}$. Thus $\times\{Y: a \in A\}$ is precisely the set of all functions on A to Y, sometimes written Y^A. A familiar instance is real **Euclidean** n-**space**. This is the set of all real-valued functions on a set consisting of the integers $0, 1, \cdots, n-1$, and the i-th coordinate of a member x is x_i.

There is another interesting special case. Suppose the index set is itself a family $\mathcal{A}$ of sets, and that for each A in $\mathcal{A}$ the A-th coordinate set is A. In this case the cartesian product $\times\{A: A \in \mathcal{A}\}$ is the family of all functions x on $\mathcal{A}$ such that $x_A \in A$ for each A in $\mathcal{A}$. These functions, members of the cartesian product, are sometimes called **choice functions** for $\mathcal{A}$, since intuitively the function x "chooses" a member x_A from each set A. If the empty set is a member of $\mathcal{A}$, then there is clearly no choice function for $\mathcal{A}$; that is, the cartesian product is void. If the members of $\mathcal{A}$ are not empty it is still not entirely obvious that the cartesian product is non-void, and, in fact, the question of the existence of a choice function for such a family turns out to be quite delicate. The next section is devoted to several propositions, each equivalent to a positive answer to the question. We shall assume as an axiom the most convenient one of these propositions. (A different choice is made in the appendix; together with the next section, this shows the equivalence of the various statements.) With unusual self-restraint we refrain from discussing the philosophical implications.

HAUSDORFF MAXIMAL PRINCIPLE

If $\mathcal{A}$ is a family of sets (or a collection of families of sets) a member A is the **largest member** of $\mathcal{A}$ if it contains every other member; that is, if A is larger than every other member of $\mathcal{A}$.

Similarly, A is the **smallest member** of the family iff A is contained in each member. It is frequently of importance to know that a family has a largest member or a smallest member. Clearly the largest and smallest members are unique when they exist. However, even in cases where the family $\mathcal{A}$ has no largest member, there may be a member such that no other member properly contains A, although there are members which neither contain nor are contained in A. Such a member is called a maximal member of the family. Formally, A is a **maximal** member of $\mathcal{A}$ iff no member of $\mathcal{A}$ properly contains A. Similarly A is a **minimal** member of $\mathcal{A}$ iff no member of $\mathcal{A}$ is properly contained in A. It is very easy to make examples of families which have no maximal member, or families in which each member is both maximal and minimal (for example a disjoint family). In general, some special hypothesis must be added to ensure the existence of maximal members.

A family $\mathfrak{N}$ of sets is a **nest** (sometimes called a **tower** or a **chain**) iff, whenever A and B are members of the family, then either $A \subset B$ or $B \subset A$. This is precisely the same thing as saying that the family $\mathfrak{N}$ is linearly ordered by inclusion, or, in our terminology, that $\mathfrak{N}$ with the inclusion relation is a chain. If $\mathfrak{N} \subset \mathcal{A}$ and $\mathfrak{N}$ is a nest, then $\mathfrak{N}$ is a **nest in** $\mathcal{A}$. We know that a family of sets may fail to have a maximal element. Let us consider the collection of all nests in a fixed family $\mathcal{A}$ and ask if among these there is a maximal nest. That is, for each family $\mathcal{A}$, is there a nest $\mathfrak{M}$ in $\mathcal{A}$ which is properly contained in no nest in $\mathcal{A}$? We assume the following statement as an axiom.

24 HAUSDORFF MAXIMAL PRINCIPLE *If $\mathcal{A}$ is a family of sets and $\mathfrak{N}$ is a nest in $\mathcal{A}$, then there is a maximal nest $\mathfrak{M}$ in $\mathcal{A}$ which contains $\mathfrak{N}$.*

The next theorem lists a number of important consequences of the Hausdorff maximal principle. Before stating the results we review some of the terminology which is commonly used in this connection. A family $\mathcal{A}$ of sets is of **finite character** iff each finite subset of a member of $\mathcal{A}$ is a member of $\mathcal{A}$, and each set A, every finite subset of which belongs to $\mathcal{A}$, itself belongs to $\mathcal{A}$. If $<$ is an ordering of a set A, then a subset B which is linearly ordered by $<$ is called a chain in A. A **maximal element** of the ordered set A is an element x such that x follows each comparable element of A; that is, if $y \in A$, then either y precedes x or x does not precede y. A relation $<$ is a well ordering of a set A iff $<$ is a linear ordering of A such that each non-void subset has a first member (a member which is less than or equal to every other member). If there exists a well ordering of A, then we say that A can be well ordered.

25 THEOREM

(a) MAXIMAL PRINCIPLE *There is a maximal member of a family* $\mathcal{A}$ *of sets, provided that for each nest in* $\mathcal{A}$ *there is a member of* $\mathcal{A}$ *which contains every member of the nest.*

(b) MINIMAL PRINCIPLE *There is a minimal member of a family* $\mathcal{A}$, *provided that for each nest in* $\mathcal{A}$ *there is a member of* $\mathcal{A}$ *which is contained in every member of the nest.*

(c) TUKEY LEMMA *There is a maximal member of each non-void family of finite character.*

(d) KURATOWSKI LEMMA *Each chain in a (partially) ordered set is contained in a maximal chain.*

(e) ZORN LEMMA *If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.*

(f) AXIOM OF CHOICE *If* X_a *is a non-void set for each member* a *of an index set* A, *then there is a function* c *on* A *such that* $c(a) \in X_a$ *for each* a *in* A.

(g) ZERMELO POSTULATE *If* $\mathcal{A}$ *is a disjoint family of non-void sets, then there is a set* C *such that* $A \cap C$ *consists of a single point for every* A *in* $\mathcal{A}$.

(h) WELL-ORDERING PRINCIPLE *Each set can be well ordered.*

PROOF We sketch the proof of each of these propositions, leaving a good many of the details to the reader.

Proof of (a): Choose a maximal nest $\mathfrak{M}$ in $\mathcal{A}$ and let A be a member of $\mathcal{A}$ containing $\bigcup\{M: M \in \mathfrak{M}\}$. Then A is a maximal member of $\mathcal{A}$, for if A is properly contained in a member B of $\mathcal{A}$, then $\mathfrak{M} \cup \{B\}$ is a nest in $\mathcal{A}$ which properly contains $\mathfrak{M}$, which is a contradiction.

Proof of (b): A proof very much like the one above is clearly possible. However, one may use (a) instead, by letting $X =$

$\bigcup\{A: A \in \mathcal{A}\}$, letting $\mathcal{C}$ be the family of complements relative to X of members of $\mathcal{A}$, observing that because of the De Morgan formulae $\mathcal{C}$ satisfies the hypothesis of (a), hence has a maximal member M, and that $X \sim M$ is surely a minimal member of $\mathcal{A}$.

Proof of (c): The proof is based on the maximal principle (a). Let $\mathcal{A}$ be a family which is of finite character, let $\mathfrak{N}$ be a nest in $\mathcal{A}$, and let $A = \bigcup\{N: N \in \mathfrak{N}\}$. Each finite subset F of A is necessarily a subset of some member of $\mathfrak{N}$, for we may choose a finite subfamily of the nest $\mathfrak{N}$ whose union contains F, and this finite subfamily has a largest member which then contains F. Consequently $A \in \mathcal{A}$. Then $\mathcal{A}$ satisfies the hypothesis of (a) and therefore has a maximal member.

Proof of (d): Suppose B is a chain in the partially ordered set A. Let $\mathcal{A}$ be the family of all chains in A which contain B. If $\mathfrak{N}$ is a nest in $\mathcal{A}$, then it can be directly verified that $\bigcup\{N: N \in \mathfrak{N}\}$ is again a member, so that $\mathcal{A}$ satisfies the hypothesis of (a) and consequently has a maximal member.

Proof of (e): Choose an upper bound for a maximal chain.

Proof of (f): Recall that a function is a set of ordered pairs such that no two members have the same first coordinate. Let $\mathfrak{F}$ be the family of all functions f such that the domain of f is a subset of A and $f(a) \in X_a$ for each a in the domain of f. (The members of $\mathfrak{F}$ are "fragments" of the function we seek.) The following argument shows that $\mathfrak{F}$ is a family of finite character. If f is a member of $\mathfrak{F}$, then every subset of f, and in particular every finite subset, is also a member of $\mathfrak{F}$. On the other hand, if f is a set, each finite subset of which belongs to $\mathfrak{F}$, then the members of f are ordered pairs, no two different pairs have the same first coordinate, and consequently f is a function. Moreover, if a is a member of the domain of f, then $\{(a,f(a))\} \in \mathfrak{F}$ and hence $f(a) \in X_a$, and it follows that $f \in \mathfrak{F}$. Because $\mathfrak{F}$ is a family of finite character there is a maximal member c of $\mathfrak{F}$, and it is only necessary to show that the domain of c is A. If a is a member of A which is not a member of the domain of c, then, since X_a is non-void, there is a member y of X_a and $c \cup \{(a,y)\}$ is itself a function and is a member of $\mathfrak{F}$, which contradicts the fact that c is maximal.

Proof of (g): Apply the axiom of choice to the index set $\mathcal{A}$ with $X_A = A$ for each A in $\mathcal{A}$.

Proof of (h): Suppose that X is the (non-void) set which is to be well ordered. Let $\mathcal{A}$ be the family of all non-void subsets of X, and let c be a choice function for $\mathcal{A}$; that is, c is a function on $\mathcal{A}$ such that $c(A) \in A$ for each A in $\mathcal{A}$. The idea of the proof is to construct an ordering $\leqq$ such that for each "initial segment" A the first point which follows A in the ordering is $c(X \sim A)$. Explicitly, define a set A to be a *segment* relative to an order $<$ iff each point which precedes a member of A is itself a member of A. In particular the void set is a segment. Let $\mathcal{C}$ be the class of all reflexive linear orderings $\leqq$ which satisfy the conditions: the domain D of $\leqq$ is a subset of X and for each segment A other than D the first point of $D \sim A$ is $c(X \sim A)$. It is almost evident that each member of $\mathcal{C}$ is a well ordering, for if B is a non-void subset of the domain of a member $\leqq$ and $A = \{y$: *$y \leqq x$ and $y \neq x$ for each x in B*$\}$, then $c(X \sim A)$ is the first member of B. Suppose that $\leqq$ and $\leq$ are members of $\mathcal{C}$, that D is the domain of $\leqq$, and that E is the domain of $\leq$. Let A be the set of all points x such that the sets $\{y: y \leqq x\}$ and $\{y: y \leq x\}$ are identical and such that on these sets the two orderings agree. Then A is a segment relative to both $\leqq$ and $\leq$. If A is not identical with either D or E, then $c(X \sim A)$ is the first point of each of these sets which does not belong to A; but then $c(X \sim A) \in A$ in view of the definition of A. It follows that $A = D$ or $A = E$. Thus any two members of $\mathcal{C}$ are related as follows: the domain of one member is a segment relative to the other, and the two orderings agree on this segment. Using this fact it is not hard to see that the union $\prec$ of the members of $\mathcal{C}$ is itself a member of $\mathcal{C}$; it is the largest member of $\mathcal{C}$. If F is the domain of $\prec$, then $F = X$, for otherwise the point $c(X \sim F)$ may be adjoined at the end of the ordering (more precisely, $\prec \cup (F \times \{c(X) \sim F)\}$ is a member of $\mathcal{C}$ which properly contains $\prec$). The theorem follows. ∎

26 *Notes* Each of the propositions listed above is actually equivalent to the Hausdorff maximal principle, and any one of these might reasonably be assumed as an axiom. In the ap-

pendix the maximal principle is derived from the axiom of choice.

The derivation of the well ordering principle from the choice axiom which is given above is essentially that of Zermelo [1]. A proof which uses 0.25(e) is also quite feasible. It may be noted that the union of a nest of well orderings is generally not a well ordering, so that a direct application of the maximal principle to the family of well orderings is impossible.

It should be remarked that the labelling of the various propositions in 0.25 is somewhat arbitrary. The Hausdorff maximal principle was used independently by C. Kuratowski, R. L. Moore, and M. Zorn in forms approximating those above.

Finally it may be noted that, although the formulation of Tukey's lemma which is given is more or less standard, it does not imply (directly) the most commonly cited applications (for example, each group contains a maximal abelian subgroup). There is a more general form which states (very roughly): if a family $\mathcal{a}$ of sets is defined by a (possibly infinite) number of conditions such that each condition involves only finitely many points, then $\mathcal{a}$ has a maximal member.

Chapter 1

TOPOLOGICAL SPACES

TOPOLOGIES AND NEIGHBORHOODS

A **topology** is a family $\mathfrak{J}$ of sets which satisfies the two conditions: the intersection of any two members of $\mathfrak{J}$ is a member of $\mathfrak{J}$, and the union of the members of each subfamily of $\mathfrak{J}$ is a member of $\mathfrak{J}$. The set $X = \bigcup\{U: U \in \mathfrak{J}\}$ is necessarily a member of $\mathfrak{J}$ because $\mathfrak{J}$ is a subfamily of itself, and every member of $\mathfrak{J}$ is a subset of X. The set X is called the **space** of the topology $\mathfrak{J}$ and $\mathfrak{J}$ is a **topology for** X. The pair $(X,\mathfrak{J})$ is a **topological space.** When no confusion seems possible we may forget to mention the topology and write "X is a topological space." We shall be explicit in cases where precision is necessary (for example if we are considering two different topologies for the same set X).

The members of the topology $\mathfrak{J}$ are called **open** relative to $\mathfrak{J}$, or $\mathfrak{J}$-open, or if only one topology is under consideration, simply open sets. The space X of the topology is always open, and the void set is always open because it is the union of the members of the void family. These may be the only open sets, for the family whose only members are X and the void set is a topology for X. This is not a very interesting topology, but it occurs frequently enough to deserve a name; it is called the **indiscrete** (or **trivial**) topology for X, and $(X,\mathfrak{J})$ is then an **indiscrete topological space.** At the other extreme is the family of all subsets of X, which is the **discrete** topology for X (then $(X,\mathfrak{J})$ is a **discrete topological space**). If $\mathfrak{J}$ is the discrete topology, then every subset of the space is open.

The discrete and the indiscrete topology for a set X are respectively the largest and the smallest topology for X. That is, every topology for X is contained in the discrete topology and contains the indiscrete topology. If $\mathfrak{J}$ and $\mathfrak{U}$ are topologies for X, then, following the convention for arbitrary families of sets, $\mathfrak{J}$ is smaller than $\mathfrak{U}$ and $\mathfrak{U}$ is larger than $\mathfrak{J}$ iff $\mathfrak{J} \subset \mathfrak{U}$. In other words, $\mathfrak{J}$ is smaller than $\mathfrak{U}$ iff each $\mathfrak{J}$-open set is $\mathfrak{U}$-open. In this case it is also said that $\mathfrak{J}$ is **coarser** than $\mathfrak{U}$ and $\mathfrak{U}$ is **finer** than $\mathfrak{J}$. (Unfortunately, this situation is described in the literature by both of the statements: $\mathfrak{J}$ is **stronger** than $\mathfrak{U}$ and $\mathfrak{J}$ is **weaker** than $\mathfrak{U}$.) If $\mathfrak{J}$ and $\mathfrak{U}$ are arbitrary topologies for X it may happen that $\mathfrak{J}$ is neither larger nor smaller than $\mathfrak{U}$; in this case, following the usage for partial orderings, it is said that $\mathfrak{J}$ and $\mathfrak{U}$ are not **comparable.**

The set of real numbers, with an appropriate topology, is a very interesting topological space. This is scarcely surprising since the notion of a topological space is an abstraction of some interesting properties of the real numbers. The **usual topology** for the real numbers is the family of all those sets which contain an open interval about each of their points. That is, a subset A of the set of real numbers is open iff for each member x of A there are numbers a and b such that $a < x < b$ and the **open interval** $\{y: a < y < b\}$ is a subset of A. Of course, we must verify that this family of sets is indeed a topology, but this offers no difficulty. It is worth noticing that, conveniently, an open interval is an open set.

A set U in a topological space $(X,\mathfrak{J})$ is a **neighborhood** ($\mathfrak{J}$-neighborhood) of a point x iff U contains an open set to which x belongs. A neighborhood of a point need not be an open set, but every open set is a neighborhood of each of its points. Each neighborhood of a point contains an open neighborhood of the point. If $\mathfrak{J}$ is the indiscrete topology the only neighborhood of a point x is the space X itself. If $\mathfrak{J}$ is the discrete topology, then every set to which a point belongs is a neighborhood of it. If X is the set of real numbers and $\mathfrak{J}$ is the usual topology, then a neighborhood of a point is a set containing an open interval to which the point belongs.

1 THEOREM *A set is open if and only if it contains a neighborhood of each of its points.*

PROOF The union U of all open subsets of a set A is surely an open subset of A. If A contains a neighborhood of each of its points, then each member x of A belongs to some open subset of A and hence $x \in U$. In this case $A = U$ and therefore A is open. On the other hand, if A is open it contains a neighborhood (namely, A) of each of its points. ∎

The foregoing theorem evidently implies that a set is open iff it is a neighborhood of each of its points.

The **neighborhood system** of a point is the family of all neighborhoods of the point.

2 THEOREM *If $\mathfrak{U}$ is the neighborhood system of a point, then finite intersections of members of $\mathfrak{U}$ belong to $\mathfrak{U}$, and each set which contains a member of $\mathfrak{U}$ belongs to $\mathfrak{U}$.*

PROOF If U and V are neighborhoods of a point x, there are open neighborhoods U_0 and V_0 contained in U and V respectively. Then $U \cap V$ contains the open neighborhood $U_0 \cap V_0$ and is hence a neighborhood of x. Thus the intersection of two (and hence of any finite number of) members of $\mathfrak{U}$ is a member. If a set U contains a neighborhood of a point x it contains an open neighborhood of x and is consequently itself a neighborhood. ∎

3 *Notes* Fréchet [1] first considered abstract spaces. The concept of a topological space developed during the following years, accompanied by a good deal of experimentation with definitions and fundamental processes. Much of the development of the theory may be found in Hausdorff's classic work [1] and, a little later, in the early volumes of *Fundamenta Mathematicae*. There are actually two fundamental concepts which have grown out of these researches: that of a topological space and that of a uniform space (chapter 7). The latter notion, which has been formalized relatively recently (A. Weil [1]), owes much to the study of topological groups.

Standard references on general topology include:

Alexandroff and Hopf [1] (the first two chapters), Bourbaki [1], Fréchet [2], Kuratowski [1], Lefschetz [1] (the first chapter), R. L. Moore [1], Newman [1], Sierpinski [1], Tukey [1], Vaidyanathaswamy [1], and G. T. Whyburn [1].

CLOSED SETS

A subset A of a topological space $(X,\mathfrak{J})$ is **closed** iff its relative complement $X \sim A$ is open. The complement of the complement of the set A is again A, and hence a set is open iff its complement is closed. If $\mathfrak{J}$ is the indiscrete topology the complement of X and the complement of the void set are the only closed sets; that is, only the void set and X are closed. It is always true that the space and the void set are closed as well as open, and it may happen, as we have just seen, that these are the only closed sets. If $\mathfrak{J}$ is the discrete topology, then every subset is closed and open. If X is the set of real numbers and $\mathfrak{J}$ the usual topology, then the situation is quite different. A **closed interval** (that is, a set of the form $\{x: a \leqq x \leqq b\}$) is fortunately closed. An open interval is not closed and a **half-open interval** (that is, a set of the form $\{x: a < x \leqq b\}$ or $\{x: a \leqq x < b\}$ where $a < b$) is neither open nor closed. Indeed—(problem 1.J)—the only sets which are both open and closed are the space and the void set.

According to the De Morgan formulae, 0.3, the union (intersection) of the complements of the members of a family of sets is the complement of the intersection (respectively union). Consequently, the union of a finite number of closed sets is necessarily closed and the intersection of the members of an arbitrary family of closed sets is closed. These properties characterize the family of closed sets, as the following theorem indicates. The simple proof is omitted.

4 THEOREM *Let $\mathfrak{F}$ be a family of sets such that the union of a finite subfamily is a member, the intersection of an arbitrary nonvoid subfamily is a member, and $X = \bigcup\{F: F \in \mathfrak{F}\}$ is a member. Then $\mathfrak{F}$ is precisely the family of closed sets in X relative to the topology consisting of all complements of members of $\mathfrak{F}$.*

ACCUMULATION POINTS

The topology of a topological space can be described in terms of neighborhoods of points and consequently it must be possible to formulate a description of closed sets in terms of neighborhoods. This formulation leads to a new classification of points in the following way. A set A is closed iff $X \sim A$ is open, and hence iff each point of $X \sim A$ has a neighborhood which is contained in $X \sim A$, or equivalently, is disjoint from A. Consequently, A is closed iff for each x, if every neighborhood of x intersects A, then $x \in A$. This suggests the following definition.

A point x is an **accumulation point** (sometimes called **cluster** point or **limit** point) of a subset A of a topological space $(X,\mathfrak{J})$ iff every neighborhood of x contains points of A other than x. Then it is true that each neighborhood of a point x intersects A if and only if x is either a point of A or an accumulation point of A. The following theorem is then clear.

5 THEOREM *A subset of a topological space is closed if and only if it contains the set of its accumulation points.*

If x is an accumulation point of A it is sometimes said, in a pleasantly suggestive phrase, that there are points of A arbitrarily near x. If we pursue this imagery it appears that an indiscrete topological space is really quite crowded, for each point x is an accumulation point of every set other than the void set and the set $\{x\}$. On the other hand, in a discrete topological space, no point is an accumulation point of a set. If X is the set of real numbers with the usual topology a variety of situations can arise. If A is the open interval $(0,1)$, then every point of the closed interval $[0,1]$ is an accumulation point of A. If A is the set of all non-negative rationals with squares less than 2, then the closed interval $[0,\sqrt{2}]$ is the set of accumulation points. If A is the set of all reciprocals of integers, then 0 is the only accumulation point of A, and the set of integers has no accumulation points.

6 THEOREM *The union of a set and the set of its accumulation points is closed.*

PROOF If x is neither a point nor accumulation point of A, then there is an open neighborhood U of x which does not intersect A. Since U is a neighborhood of each of its points, no one of these is an accumulation point of A. Hence the union of the set A and the set of its accumulation points is the complement of an open set. ∎

The set of all accumulation points of a set A is sometimes called the **derived** set of A.

CLOSURE

The **closure** ($\mathfrak{J}$-closure) of a subset A of a topological space $(X,\mathfrak{J})$ is the intersection of the members of the family of all closed sets containing A. The closure of A is denoted by A^-, or by $\bar{A}$. The set A^- is always closed because it is the intersection of closed sets, and evidently A^- is contained in each closed set which contains A. Consequently A^- is the smallest closed set containing A and it follows that A is closed if and only if $A = A^-$. The next theorem describes the closure of a set in terms of its accumulation points.

7 THEOREM *The closure of any set is the union of the set and the set of its accumulation points.*

PROOF Every accumulation point of a set A is an accumulation point of each set containing A, and is therefore a member of each closed set containing A. Hence A^- contains A and all accumulation points of A. On the other hand, according to the preceding theorem, the set consisting of A and its accumulation points is closed and it therefore contains A^-. ▌

The function which assigns to each subset A of a topological space the value A^- might be called the closure function, or closure operator, relative to the topology. This operator determines the topology completely, for a set A is closed iff $A = A^-$. In other words, the closed sets are simply the sets which are fixed under the closure operator. It is instructive to enquire: Under what circumstances is an operator which is defined for all subsets of a fixed set X the closure operator relative to some topology for X? It turns out that four very simple properties serve to describe closure. First, because the void set is closed, the closure of the void set is void; and, second, each set is contained in its closure. Next, because the closure of each set is closed, the closure of the closure of a set is identical with the closure of the set (in the usual algebraic terminology, the closure operator is idempotent). Finally, the closure of the union of two sets is the union of the closures, for $(A \cup B)^-$ is always a closed set containing A and B, and therefore contains A^- and B^- and hence $A^- \cup B^-$; on the other hand, $A^- \cup B^-$ is a closed set containing $A \cup B$ and hence also $(A \cup B)^-$.

A **closure operator** on X is an operator which assigns to each subset A of X a subset A^c of X such that the following four statements, the **Kuratowski closure axioms**, are true.

(a) *If* 0 *is the void set*, $0^c = 0$.
(b) *For each* A, $A \subset A^c$.
(c) *For each* A, $A^{cc} = A^c$.
(d) *For each* A *and* B, $(A \cup B)^c = A^c \cup B^c$.

The following theorem of Kuratowski shows that these four statements are actually characteristic of closure. The topology defined below is the topology **associated** with a closure operator.

8 THEOREM *Let* c *be a closure operator on* X, *let* $\mathfrak{F}$ *be the family of all subsets* A *of* X *for which* $A^c = A$, *and let* $\mathfrak{J}$ *be the family of complements of members of* $\mathfrak{F}$. *Then* $\mathfrak{J}$ *is a topology for* X, *and* A^c *is the* $\mathfrak{J}$*-closure of* A *for each subset* A *of* X.

PROOF Axiom (a) shows that the void set belongs to $\mathfrak{F}$, and (d) shows that the union of two members of $\mathfrak{F}$ is a member of $\mathfrak{F}$. Consequently the union of any finite subfamily (void or not) of $\mathfrak{F}$ is a member of $\mathfrak{F}$. Because of (b), $X \subset X^c$, so that $X = X^c$, and the union of the members of $\mathfrak{F}$ is then X. In view of theorem 1.4, it will follow that $\mathfrak{J}$ is a topology for X if it is shown that the intersection of the members of any non-void subfamily of $\mathfrak{F}$ is a member of $\mathfrak{F}$. To this end, first observe that, if $B \subset A$, then $B^c \subset A^c$, because $A^c = [(A \sim B) \cup B]^c = (A \sim B)^c \cup B^c$. Now suppose that $\mathfrak{A}$ is a non-void subfamily of $\mathfrak{F}$ and that $B = \bigcap \{A: A \in \mathfrak{A}\}$. The set B is contained in each member of $\mathfrak{A}$, and therefore $B^c \subset \bigcap \{A^c: A \in \mathfrak{A}\} = \bigcap \{A: A \in \mathfrak{A}\} = B$. Since $B \subset B^c$, it follows that $B = B^c$ and $B \in \mathfrak{F}$. This shows that $\mathfrak{J}$ is a topology, and it remains to show that A^c is A^-, the $\mathfrak{J}$-closure of A. By definition, A^- is the intersection of all the $\mathfrak{J}$-closed sets, that is, the members of $\mathfrak{F}$, which contain A. By axiom (c), $A^c \in \mathfrak{F}$, and hence $A^- \subset A^c$; since $A^- \in \mathfrak{F}$ and $A^- \supset A$ it follows that $A^- \supset A^c$ and hence $A^- = A^c$. ∎

INTERIOR AND BOUNDARY

There is another operator defined on the family of all subsets of a topological space, which is very intimately related to the closure operator. A point x of a subset A of a topological space is an **interior point** of A iff A is a neighborhood of x, and the set of all interior points of A is the **interior** of A, denoted A^0. (In the usual terminology, the relation "is an interior point of" is the inverse of the relation "is a neighborhood of.") It is convenient to exhibit the connection between this notion and the earlier concepts before considering examples.

9 THEOREM *Let A be a subset of a topological space X. Then the interior A^0 of A is open and is the largest open subset of A. A set A is open if and only if $A = A^0$. The set of all points of A which are not points of accumulation of $X \sim A$ is precisely A^0. The closure of $X \sim A$ is $X \sim A^0$.*

PROOF If a point x belongs to the interior of a set A, then x is a member of some open subset U of A. Every member of U is also a member of A^0, and consequently A^0 contains a neighborhood of each of its points and is therefore open. If V is an open subset of A and $y \in V$, then A is a neighborhood of y and so $y \in A^0$. Hence A^0 contains each open subset of A and it is therefore the largest open subset of A. If A is open, then A is surely identical with the largest open subset of A; hence A is open iff $A = A^0$. If x is a point of A which is not an accumulation point of $X \sim A$, then there is a neighborhood U of x which does not intersect $X \sim A$ and is therefore a subset of A. Then A is a neighborhood of x and $x \in A^0$. On the other hand, A^0 is a neighborhood of each of its points and A^0 does not intersect $X \sim A$, so that no point of A^0 is an accumulation point of $X \sim A$. Finally, since A^0 consists of the points of A which are not accumulation points of $X \sim A$, the complement, $X \sim A^0$, is precisely the set of all points which are either points of $X \sim A$ or accumulation points of $X \sim A$; that is, the complement is the closure $(X \sim A)^-$. ∎

The last statement of the foregoing theorem deserves a little further consideration. For convenience, let us denote the relative complement $X \sim A$ by A'. Then A'', the complement of the complement of A, is again A (we sometimes say $'$ is an operator of period two). The preceding result can then be stated as $A^{0\prime} = A'^{-}$, and, it follows, taking complements, that $A^0 = A'^{-\prime}$. Thus the interior of A is the complement of the closure of the complement of A. If A is replaced by its complement it follows that $A^- = A'^{0\prime}$, so that the closure of a set is the complement of the interior of the complement.*

If X is an indiscrete space the interior of every set except X itself is void. If X is a discrete space, then each set is open and closed and consequently identical with its interior and with its closure. If X is the set of real numbers with the usual topology, then the interior of the set of all integers is void; the interior of a closed interval is the open interval with the same endpoints. The interior of the set of rational numbers is void, and the closure of the interior of this set is consequently void. The closure of the set of rational numbers is the set X of all numbers, and the interior of this set is X again. Thus the interior of the closure of a set may be quite different from the closure of the interior; that is, the interior operator and the closure operator do not generally commute.

There is one other operator which occurs frequently enough to justify its definition. The **boundary** of a subset A of a topological space X is the set of all points x which are interior to neither A nor $X \sim A$. Equivalently, x is a point of the boundary iff each neighborhood of x intersects both A and $X \sim A$. It is clear that the boundary of A is identical with the boundary of $X \sim A$. If X is indiscrete and A is neither X nor void, then the boundary of A is X, while if X is discrete the boundary of every subset is void. The boundary of an interval of real numbers, in the usual topology for the reals, is the set whose only members are the endpoints of the interval, regardless of whether the interval is open, closed, or half-open. The boundary of the

* An amusing and instructive problem suggests itself. From a given subset A of a topological space, how many different sets can be constructed by successive applications, in any order, of closure, complementation and interior? From the remarks in the above paragraph and the fact that $A^{--} = A^-$, this reduces to: how many distinct sets may be formed from a single set A, by alternate applications of complementation and the closure operator? The surprising answer is given in problem 1.E.

set of rationals, or the set of irrationals, is the set of all real numbers.

It is not difficult to discover the relations between boundary, closure, and interior. The following theorem, whose proof we omit, summarizes the facts.

10 THEOREM *Let A be a subset of a topological space X and let $b(A)$ be the boundary of A. Then $b(A) = A^- \cap (X \sim A)^- = A^- \sim A^0$, $X \sim b(A) = A^0 \cup (X \sim A)^0$, $A^- = A \cup b(A)$ and $A^0 = A \sim b(A)$.*

A set is closed if and only if it contains its boundary and is open if and only if it is disjoint from its boundary.

BASES AND SUBBASES

In defining the usual topology for the set of real numbers we began with the family $\mathfrak{B}$ of open intervals, and from this family constructed the topology $\mathfrak{J}$. The same method is useful in other situations and we now examine the construction in detail. A family $\mathfrak{B}$ of sets is a **base for a topology** $\mathfrak{J}$ iff $\mathfrak{B}$ is a subfamily of $\mathfrak{J}$ and for each point x of the space, and each neighborhood U of x, there is a member V of $\mathfrak{B}$ such that $x \in V \subset U$. Thus the family of open intervals is a base for the usual topology of the real numbers, in view of the definition of the usual topology and the fact that open intervals are open relative to this topology.

There is a simple characterization of bases which is frequently used as a definition: A subfamily $\mathfrak{B}$ of a topology $\mathfrak{J}$ is a base for $\mathfrak{J}$ iff each member of $\mathfrak{J}$ is the union of members of $\mathfrak{B}$. To prove this fact, suppose that $\mathfrak{B}$ is a base for the topology $\mathfrak{J}$ and that $U \in \mathfrak{J}$. Let V be the union of all members of $\mathfrak{B}$ which are subsets of U and suppose that $x \in U$. Then there is W in $\mathfrak{B}$ such that $x \in W \subset U$, and consequently $x \in V$. Hence $U \subset V$ and since V is surely a subset of U, $V = U$. To show the converse, suppose $\mathfrak{B} \subset \mathfrak{J}$ and each member of $\mathfrak{J}$ is the union of members of $\mathfrak{B}$. If $U \in \mathfrak{J}$, then U is the union of the members of a subfamily of $\mathfrak{B}$, and for each x in U there is V in $\mathfrak{B}$ such that $x \in V \subset U$. Consequently $\mathfrak{B}$ is a base for $\mathfrak{J}$.

Although this is a very convenient method for the construction of topologies, a little caution is necessary because not every family of sets is the base for a topology. For example, let X consist of the integers 0, 1, and 2, let A consist of 0 and 1, and let B consist of 1 and 2. If $\mathcal{S}$ is the family whose members are X, A, B and the void set, then $\mathcal{S}$ cannot be the base for a topology because: by direct computation, the union of members of $\mathcal{S}$ is always a member, so that if $\mathcal{S}$ were the base of a topology that topology would have to be $\mathcal{S}$ itself, but $\mathcal{S}$ is not a topology because $A \cap B \notin \mathcal{S}$. The reason for this situation is made clear by the following theorem.

11 THEOREM *A family $\mathcal{B}$ of sets is a base for some topology for the set $X = \bigcup\{B: B \in \mathcal{B}\}$ if and only if for every two members U and V of $\mathcal{B}$ and each point x in $U \cap V$ there is W in $\mathcal{B}$ such that $x \in W$ and $W \subset U \cap V$.*

PROOF If $\mathcal{B}$ is a base for some topology, U and V are members of $\mathcal{B}$ and $x \in U \cap V$ then, since $U \cap V$ is open, there is a member of $\mathcal{B}$ to which x belongs and which is a subset of $U \cap V$. To show the converse, let $\mathcal{B}$ be a family with the specified property and let $\mathfrak{I}$ be the family of all unions of members of $\mathcal{B}$. A union of members of $\mathfrak{I}$ is itself a union of members of $\mathcal{B}$ and is therefore a member of $\mathfrak{I}$, and it is only necessary to show that the intersection of two members U and V of $\mathfrak{I}$ is a member of $\mathfrak{I}$. If $x \in U \cap V$, then we may choose U' and V' in $\mathcal{B}$ such that $x \in U' \subset U$ and $x \in V' \subset V$, and then a member W of $\mathcal{B}$ such that $x \in W \subset U' \cap V' \subset U \cap V$. Consequently $U \cap V$ is the union of members of $\mathcal{B}$, and $\mathfrak{I}$ is a topology. ∎

We have just seen that an arbitrary family $\mathcal{S}$ of sets may fail to be the base for any topology. With admirable persistence we vary the question and enquire whether there is a unique topology which is, in some sense, generated by $\mathcal{S}$. Such a topology should be a topology for the set X which is the union of the members of $\mathcal{S}$, and each member of $\mathcal{S}$ should be open relative to the topology; that is, $\mathcal{S}$ should be a subfamily of the topology. This raises the question: Is there a smallest topology for X which contains $\mathcal{S}$? The following simple result will enable us to exhibit this smallest topology.

12 THEOREM *If $\mathcal{S}$ is any non-void family of sets the family of all*

finite intersections of members of $\mathcal{S}$ *is the base for a topology for the set* $X = \bigcup\{S: S \in \mathcal{S}\}$.

PROOF If $\mathcal{S}$ is a family of sets let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{S}$. Then the intersection of two members of $\mathcal{B}$ is again a member of $\mathcal{B}$ and, applying the preceding theorem, $\mathcal{B}$ is the base for a topology. ▮

A family $\mathcal{S}$ of sets is a **subbase for a topology** $\mathfrak{J}$ iff the family of finite intersections of members of $\mathcal{S}$ is a base for $\mathfrak{J}$ (equivalently, iff each member of $\mathfrak{J}$ is the union of finite intersections of members of $\mathcal{S}$). In view of the preceding theorem every non-empty family $\mathcal{S}$ is the subbase for some topology, and this topology is, of course, uniquely determined by $\mathcal{S}$. It is the smallest topology containing $\mathcal{S}$ (that is, it is a topology containing $\mathcal{S}$ and is a subfamily of every topology containing $\mathcal{S}$).

There will generally be many different bases and subbases for a topology and the most appropriate choice may depend on the problem under consideration. One rather natural subbase for the usual topology for the real numbers is the family of half-infinite open intervals; that is, the family of sets of the form $\{x: x > a\}$ or $\{x: x < a\}$. Each open interval is the intersection of two such sets, and this family is consequently a subbase. The family of all sets of the same form with a rational is a less obvious and more interesting subbase. (See problem 1.J.)

A space whose topology has a countable base has many pleasant properties. Such spaces are said to satisfy the **second axiom of countability.** (The terms **separable** and **perfectly separable** are also used in this connection, but we shall use neither.)

13 THEOREM *If A is an uncountable subset of a space whose topology has a countable base, then some point of A is an accumulation point of A.*

PROOF Suppose that no point of A is an accumulation point and that $\mathcal{B}$ is a countable base. For each x in A there is an open set containing no point of A other than x, and since $\mathcal{B}$ is a base we may choose B_x in $\mathcal{B}$ such that $B_x \cap A = \{x\}$. There is then a one-to-one correspondence between the points of A and the members of a subfamily of $\mathcal{B}$, and A is therefore countable. ▮

A sharper form of this theorem is stated in problem 1.H.

A set A is **dense** in a topological space X iff the closure of A is X. A topological space X is **separable** iff there is a countable subset which is dense in X. A separable space may fail to satisfy the second axiom of countability. For example, let X be an uncountable set with the topology consisting of the void set and the complements of finite sets. Then every non-finite set is dense because it intersects every non-void open set. On the other hand, suppose that there is a countable base $\mathcal{B}$ and let x be a fixed point of X. The intersection of the family of all open sets to which x belongs must be $\{x\}$, because the complement of every other point is open. It follows that the intersection of those members of the base $\mathcal{B}$ to which x belongs is $\{x\}$. But the complement of this countable intersection is the union of a countable number of finite sets, hence countable, and this is a contradiction. (Less trivial examples occur later.) There is no difficulty in showing that a space with a countable base is separable.

14 Theorem *A space whose topology has a countable base is separable.*

PROOF Choose a point out of each member of the base, thus obtaining a countable set A. The complement of the closure of A is an open set which, being disjoint from A, contains no non-void member of the base and is hence void. ∎

A family $\mathcal{A}$ is a **cover** of a set B iff B is a subset of the union $\bigcup\{A: A \in \mathcal{A}\}$; that is, iff each member of B belongs to some member of $\mathcal{A}$. The family is an **open cover** of B iff each member of $\mathcal{A}$ is an open set. A **subcover** of $\mathcal{A}$ is a subfamily which is also a cover.

15 Theorem (Lindelöf) *There is a countable subcover of each open cover of a subset of a space whose topology has a countable base.*

PROOF Suppose A is a set, $\mathcal{A}$ is an open cover of A, and $\mathcal{B}$ is a countable base for the topology. Because each member of $\mathcal{A}$ is the union of members of $\mathcal{B}$ there is a subfamily $\mathcal{C}$ of $\mathcal{B}$ which also covers A, such that each member of $\mathcal{C}$ is a subset of some member of $\mathcal{A}$. For each member of $\mathcal{C}$ we may select a containing member of $\mathfrak{A}$ and so obtain a countable subfamily $\mathfrak{D}$ of $\mathfrak{A}$. Then $\mathfrak{D}$ is also a cover of A because $\mathfrak{C}$ covers A. Hence $\mathfrak{A}$ has a countable subcover. ∎

A topological space is a **Lindelöf space** iff each open cover of the space has a countable subcover.

Since the second axiom of countability has been mentioned, it seems only proper that the first be stated. This axiom concerns a localized form of the notion of a base. A **base for the neighborhood system** of a point x, or a **local base** at x, is a family of neighborhoods of x such that every neighborhood of x contains a member of the family. For example, the family of open neighborhoods of a point is always a base for the neighborhood system. A topological space satisfies the **first axiom of countability** if the neighborhood system of every point has a countable base. It is clear that each topological space which satisfies the second axiom of countability also satisfies the first; on the other hand, any uncountable discrete topological space satisfies the first axiom (there is a base for the neighborhood system of each point x which consists of the single neighborhood $\{x\}$) but not the second (the cover whose members are $\{x\}$ for all x in X has no countable subcover). The second axiom of countability is therefore definitely more restrictive than the first.

It is worth noticing that, if $U_1, U_2, \cdots, U_n, \cdots$ is a countable local base at x, then a new local base $V_1, V_2, \cdots, V_n, \cdots$ can be found such that $V_n \supset V_{n+1}$ for each n. The construction is simple: let $V_n = \bigcap\{U_k: k \leqq n\}$.

A **subbase for the neighborhood system** of a point x, or a **local subbase** at x, is a family of sets such that the family of all finite intersections of members is a local base. If $U_1, U_2, \cdots, U_n, \cdots$ is a countable local subbase, then $V_1, V_2, \cdots, V_n, \cdots$, where $V_n = \bigcap\{U_k: k \leqq n\}$ is a countable local base. Hence the existence of a countable local subbase at each point implies the first axiom of countability.

RELATIVIZATION; SEPARATION

If $(X,\mathfrak{J})$ is a topological space and Y is a subset of X we may construct a topology $\mathfrak{U}$ for Y which is called the **relative to-**

pology, or the **relativization of $\mathfrak{J}$ to Y.** The relative topology $\mathfrak{U}$ is defined to be the family of all intersections of members of $\mathfrak{J}$ with Y; that is, U belongs to the relative topology $\mathfrak{U}$ iff $U = V \cap Y$ for some $\mathfrak{J}$-open set V. It is not difficult to see that $\mathfrak{U}$ is actually a topology. Each member U of the relative topology $\mathfrak{U}$ is said to be **open in Y,** and its relative complement $Y \sim U$ is **closed in Y.** The $\mathfrak{U}$-closure of a subset of Y is its **closure in Y.** Each subset Y of X is both open and closed in itself, although Y may be neither open nor closed in X. The topological space $(Y,\mathfrak{U})$ is called a **subspace** of the space $(X,\mathfrak{J})$. More formally, an arbitrary topological space $(Y,\mathfrak{U})$ is a subspace of another space $(X,\mathfrak{J})$ iff $Y \subset X$ and $\mathfrak{U}$ is the relativization of $\mathfrak{J}$.

It is worth noticing that, if $(Y,\mathfrak{U})$ is a subspace of $(X,\mathfrak{J})$ and $(Z,\mathfrak{V})$ is a subspace of $(Y,\mathfrak{U})$, then $(Z,\mathfrak{V})$ is a subspace of $(X,\mathfrak{J})$. This transitivity relation will often be used without explicit mention.

Suppose that $(Y,\mathfrak{U})$ is a subspace of $(X,\mathfrak{J})$ and that A is a subset of Y. Then A may be either $\mathfrak{J}$-closed or $\mathfrak{U}$-closed, a point y may be either a $\mathfrak{U}$ or a $\mathfrak{J}$-accumulation point of A, and A has both a $\mathfrak{J}$ and a $\mathfrak{U}$-closure. The relations between these various notions are important.

16 THEOREM *Let $(X,\mathfrak{J})$ be a topological space, let $(Y,\mathfrak{U})$ be a subspace, and let A be a subset of Y. Then:*

(a) *The set A is $\mathfrak{U}$-closed if and only if it is the intersection of Y and a $\mathfrak{J}$-closed set.*
(b) *A point y of Y is a $\mathfrak{U}$-accumulation point of A if and only if it is a $\mathfrak{J}$-accumulation point.*
(c) *The $\mathfrak{U}$-closure of A is the intersection of Y and the $\mathfrak{J}$-closure of A.*

PROOF The set A is closed in Y iff its relative complement $Y \sim A$ is of the form $V \cap Y$ for some $\mathfrak{J}$-open set V, but this is true iff $A = (X \sim V) \cap Y$ for some V in $\mathfrak{J}$. This proves (a), and (b) follows directly from the definition of the relative topology and the definition of accumulation point. The $\mathfrak{U}$-closure of A is the union of A and the set of its $\mathfrak{U}$-accumulation points, and hence by (b) it is the intersection of Y and the $\mathfrak{J}$-closure of A. ∎

If $(Y,\mathfrak{U})$ is a subspace of $(X,\mathfrak{J})$ and Y is open in X, then each set open in Y is also open in X because it is the intersection of an open set and Y. A similar statement, with "closed" replacing "open" everywhere, is also true. However, knowing that a set is open or closed in a subspace generally tells very little about the situation of the set in X. If X is the union of two sets Y and Z and if A is a subset of X such that $A \cap Y$ is open in Y and $A \cap Z$ is open in Z, then one might hope that A is open in X. But this is not always true, for if Y is an arbitrary subset of X and $Z = X \sim Y$, then $Y \cap Y$ and $Y \cap Z$ are open in Y and Z respectively. There is one important case, in which this result does hold. Two subsets A and B are **separated** in a topological space X iff $A^- \cap B$ and $A \cap B^-$ are both void. This definition of separation involves the closure operation in X. However, the apparent dependence on the space X is illusory, for A and B are separated in X if and only if neither A nor B contains a point or an accumulation point of the other. This condition may be restated in terms of the relative topology for $A \cup B$, in view of part (b) of the foregoing theorem, as: both A and B are closed in $A \cup B$ (or equivalently A (or B) is both open and closed in $A \cup B$) and A and B are disjoint. As an example, notice that the open intervals (0,1) and (1,2) are separated subsets of the real numbers with the usual topology and that there is a point, 1, belonging to the closure of both. However, (0,1) is not separated from the closed interval [1,2] because 1, which is a member of [1,2], is an accumulation point of (0,1).

Three theorems on separation will be needed in the sequel.

17 THEOREM *If Y and Z are subsets of a topological space X and both Y and Z are closed or both are open, then $Y \sim Z$ is separated from $Z \sim Y$.*

PROOF Suppose that Y and Z are closed subsets of X. Then Y and Z are closed in $Y \cup Z$ and therefore $Y \sim Z = ((Y \cup Z) \sim Z)$ and $Z \sim Y$ are open in $Y \cup Z$. It follows that both $Y \sim Z$ and $Z \sim Y$ are open in $(Y \sim Z) \cup (Z \sim Y)$, and since they are complements relative to this set both are closed in $(Y \sim Z) \cup$

$(Z \sim Y)$. Consequently $Y \sim Z$ and $Z \sim Y$ are separated. A dual argument applies to the case where both Y and Z are open in X. ▮

18 THEOREM *Let X be a topological space which is the union of subsets Y and Z such that $Y \sim Z$ and $Z \sim Y$ are separated. Then the closure of a subset A of X is the union of the closure in Y of $A \cap Y$ and the closure in Z of $A \cap Z$.*

PROOF The closure of a union of two sets is the union of the closures and hence $A^- = (A \cap Y)^- \cup (A \cap Z \sim Y)^-$. Consequently $A^- \cap Y = [(A \cap Y)^- \cap Y] \cup [(A \cap Z \sim Y)^- \cap Y]$. The set $(Z \sim Y)^-$ is disjoint from $Y \sim Z$, hence $(Z \sim Y)^- \subset Z$, and it follows that $(A \cap Z \sim Y)^-$ is a subset of $(A \cap Z)^- \cap Z$. Similarly $A^- \cap Z$ is the union of $(A \cap Z)^- \cap Z$ and a subset of $(A \cap Y)^- \cap Y$. Consequently $A^- = (A^- \cap Y) \cup (A^- \cap Z) = [(A \cap Y)^- \cap Y] \cup [(A \cap Z)^- \cap Z]$, and the theorem is proved. ▮

19 COROLLARY *Let X be a topological space which is the union of subsets Y and Z such that $Y \sim Z$ and $Z \sim Y$ are separated. Then a subset A of X is closed (open) if $A \cap Y$ is closed (open) in Y and $A \cap Z$ is closed (open) in Z.*

PROOF If $A \cap Y$ and $A \cap Z$ are closed in Y and Z respectively, then, by the preceding theorem, A is necessarily identical with its closure and is therefore closed. If $A \cap Y$ and $A \cap Z$ are open in Y and Z respectively, then $Y \cap X \sim A$ and $Z \cap X \sim A$ are closed in Y and in Z and hence $X \sim A$ is closed and A is open. ▮

CONNECTED SETS

A topological space $(X,\mathfrak{J})$ is **connected** iff X is not the union of two non-void separated subsets. A subset Y of X is connected iff the topological space Y with the relative topology is connected. Equivalently, Y is connected iff Y is not the union of two non-void separated subsets. Another equivalence follows from the discussion of separation: A set Y is connected iff the only subsets of Y which are both open and closed in Y are Y and the void set. From this form it follows at once that any indiscrete space is connected. A discrete space containing more than one point is not connected. The real numbers, with the usual topology, are connected (problem 1.J), but the rationals, with the usual topology of the reals relativized, are not connected. (For any irrational a the sets $\{x: x < a\}$ and $\{x: x > a\}$ are separated.)

20 THEOREM *The closure of a connected set is connected.*

PROOF Suppose that Y is a connected subset of a topological space and that $Y^- = A \cup B$, where A and B are both open and closed in Y^-. Then each of $A \cap Y$ and $B \cap Y$ is open and closed in Y, and since Y is connected, one of these two sets must be void. Suppose that $B \cap Y$ is void. Then Y is a subset of A and consequently Y^- is a subset of A because A is closed in Y^-. Hence B is void, and it follows that Y^- is connected. ∎

There is another version of this theorem which is apparently stronger, which states that, if Y is a connected subset of X and if Z is a set such that $Y \subset Z \subset Y^-$, then Z is connected. However, the stronger form is an immediate consequence of applying the foregoing theorem to Z with the relative topology.

21 THEOREM *Let $\mathfrak{A}$ be a family of connected subsets of a topological space. If no two members of $\mathfrak{A}$ are separated, then $\bigcup\{A: A \in \mathfrak{A}\}$ is connected.*

PROOF Let C be the union of the members of $\mathfrak{A}$ and suppose that D is both open and closed in C. Then for each member A of $\mathfrak{A}$, $A \cap D$ is open and closed in A, and since A is connected either $A \subset D$ or $A \subset C \sim D$. Now if A and B are members of $\mathfrak{A}$ it is impossible that $A \subset D$ and $B \subset C \sim D$, for in this case A and B, being respectively subsets of the separated sets D and $C \sim D$, would be separated. Consequently either every member of $\mathfrak{A}$ is a subset of $C \sim D$ and D is void, or every member of $\mathfrak{A}$ is a subset of D and $C \sim D$ is void. ∎

A **component** of a topological space is a maximal connected subset; that is, a connected subset which is properly contained in no other connected subset. A component of a subset A is a component of A with the relative topology; that is, a maximal connected subset of A. If a space is connected, then it is its only component. If a space is discrete, then each component consists of a single point. Of course, there are many spaces which are not discrete which have components consisting of a single point—for example, the space of rational numbers, with the (relativized) usual topology.

22 Theorem *Each connected subset of a topological space is contained in a component, and each component is closed. If A and B are distinct components of a space, then A and B are separated.*

proof Let A be a non-void connected subset of a topological space and let C be the union of all connected sets containing A. In view of the preceding theorem, C is surely connected, and if D is a connected set and contains C, then, since $D \subset C$, it follows that $C = D$. Hence C is a component. (If A is void, and the space is not, a set consisting of a single point is contained in a component, and hence so is A.) Each component C is connected and hence, by 1.20, the closure C^- is connected. Therefore C is identical with C^- and C is closed. If A and B are distinct components and are not separated, then their union is connected, by 1.21, which is a contradiction. ∎

It is well to end our remarks on components with a word of caution. If two points, x and y, belong to the same component of a topological space, then they always lie in the same half of a separation of the space. That is, if the space is the union of separated sets A and B, then both x and y belong to A or both x and y belong to B. The converse of this proposition is false. It may happen that two points always lie in the same half of a separation but nevertheless lie in different components. (See problem 1.P.)

PROBLEMS

A LARGEST AND SMALLEST TOPOLOGIES

(a) The intersection of any collection of topologies for X is a topology for X.

(b) The union of two topologies for X may not be a topology for X (unless X consists of at most two points).

(c) For any collection of topologies for X there is a unique largest topology which is smaller than each member of the collection, and a unique smallest topology which is larger than each member of the collection.

B TOPOLOGIES FROM NEIGHBORHOOD SYSTEMS

(a) Let $(X,\mathfrak{I})$ be a topological space and for each x in X let $\mathfrak{U}_x$ be the family of all neighborhoods of x. Then:

(i) If $U \in \mathfrak{U}_x$, then $x \in U$.
(ii) If U and V are members of $\mathfrak{U}_x$, then $U \cap V \in \mathfrak{U}_x$.
(iii) If $U \in \mathfrak{U}_x$ and $U \subset V$, then $V \in \mathfrak{U}_x$.
(iv) If $U \in \mathfrak{U}_x$, then there is a member V of $\mathfrak{U}_x$ such that $V \subset U$ and $V \in \mathfrak{U}_y$ for each y in V (that is, V is a neighborhood of each of its points).

(b) If $\mathfrak{U}$ is a function which assigns to each x in X a non-void family $\mathfrak{U}_x$ satisfying (i), (ii), and (iii), then the family $\mathfrak{I}$ of all sets U, such that $U \in \mathfrak{U}_x$ whenever $x \in U$, is a topology for X. If (iv) is also satisfied, then $\mathfrak{U}_x$ is precisely the neighborhood system of x relative to the topology $\mathfrak{I}$.

Note Various methods of describing a topological space have been investigated intensively. Kuratowski's three closure axioms may be replaced by a single condition, as shown by Monteiro [1] and by Iseki [1]. It is also possible to use the notion of separation as a primitive (Wallace [1], Krishna Murti [1] and Szymanski [1]); the notion of derived set may also be used as primitive (for information and references see Monteiro [2] and Ribeiro [3]). The relation between various operations has been studied by Stopher [1].

C TOPOLOGIES FROM INTERIOR OPERATORS

If ^{i} is an operator which carries subsets of X into subsets of X, and $\mathfrak{I}$ is the family of all subsets such that $A^i = A$, under what conditions will $\mathfrak{I}$ be a topology for X and ^{i} the interior operator relative to this topology?

D ACCUMULATION POINTS IN T_1-SPACES

A topological space is a *T_1-space* iff each set which consists of a single point is closed. (We sometimes say, inaccurately, that "points are closed.")

(a) For any set X there is a unique smallest topology $\mathfrak{I}$ such that $(X,\mathfrak{I})$ is a T_1-space.

(b) If X is infinite and $\mathfrak{I}$ is the smallest topology such that $(X,\mathfrak{I})$ is a T_1-space, then $(X,\mathfrak{I})$ is connected.

(c) If $(X,\mathfrak{I})$ is a T_1-space, then the set of accumulation points of each subset is closed. A sharper result (C. T. Yang): A necessary and sufficient condition that the set of accumulation points of each subset be closed is that the set of accumulation points of $\{x\}$ be closed for each x in X.

Note There is a sequence of successively stronger requirements which may be put upon the topology of a space. A topological space is a T_0-*space* iff for each pair x and y of distinct points, there is a neighborhood of one point to which the other does not belong. In slightly different terminology, the space is a T_0-space iff for distinct points x and y either $x \notin \{y\}^-$ or $y \notin \{x\}^-$. We will define T_2 and T_3-spaces later. The terminology is due to Alexandroff and Hopf [1].

E KURATOWSKI CLOSURE AND COMPLEMENT PROBLEM

If A is a subset of a topological space, then at most 14 sets can be constructed from A by complementation and closure. There is a subset of the real numbers (with the usual topology) from which 14 different sets can be so constructed. (First notice that if A is the closure of an open set, then A is the closure of the interior of A; that is, for such sets $A = A'^{-\prime-}$ where $'$ denotes complementation.)

F EXERCISE ON SPACES WITH A COUNTABLE BASE

If the topology of a space has a countable base, then each base contains a countable subfamily which is also a base.

G EXERCISE ON DENSE SETS

If A is dense in a topological space and U is open, then $U \subset (A \cap U)^-$.

H ACCUMULATION POINTS

Let X be a space, each subspace of which is Lindelöf, let A be an uncountable subset, and let B be the subset consisting of all points x of A such that each neighborhood of x contains uncountably many points of A. Then $A \sim B$ is countable, and consequently each neighborhood of a point of B contains uncountably many points of B.

Note The accumulation points of a set A may be classified according to the least cardinal number of the intersection of A and a neighborhood of the point. If there is also a cardinal number restriction on a base for the topology then several inequalities result. Theorems 1.13, 1.14, and 1.15 all have generalizations applying to spaces with a base of a given cardinal.

I THE ORDER TOPOLOGY

Let X be a set, linearly ordered by a relation $<$ which is anti-symmetric (it is false that $x < x$). The *order topology* (the $<$ order topology) has a subbase consisting of all sets of the form: $\{x: x < a\}$ or $\{x: a < x\}$ for some a in X.

(a) The order topology for X is the smallest topology in which order is continuous, in the following sense: if a and b are members of X and $a < b$, then there are neighborhoods U of a and V of b such that, whenever $x \in U$ and $y \in V$, then $x < y$.

(b) Let Y be a subset of a set X which is linearly ordered by $<$. Then Y is linearly ordered by $<$, but the $<$ order topology for Y may not be the relativized $<$ order topology for X.

(c) If X, with the order topology, is connected, then X is order-complete (that is, each non-void set with an upper bound has a supremum).

(d) If there are points a and b in X such that $a < b$ and there is no point c such that $a < c < b$, then X is not connected. Such an ordering is said to have a *gap*. Show that X is connected relative to the order topology iff X is order-complete and there are no gaps.

J PROPERTIES OF THE REAL NUMBERS

Let R be the set of real numbers with the usual topology.

(a) An additive subgroup of the reals which contains more than one member is either dense in R or has a smallest positive element. In particular, the set of rational numbers is dense in R.

(b) The usual topology for the reals is identical with the order topology. The usual topology has a countable base.

(c) A closed subgroup of R is either countable or identical with R. A connected subgroup is either $\{0\}$ or R and an open subgroup is necessarily identical with R.

(d) (A. P. Morse) A *proper* interval is a half-open, open, or closed interval which contains more than one point. If $\mathcal{A}$ is an arbitrary family of proper intervals, then there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcup\{B: B \in \mathcal{B}\} = \bigcup\{A: A \in \mathcal{A}\}$. (Observe that a disjoint family of proper intervals is countable, and show that all but a countable number of points of $\bigcup\{A: A \in \mathcal{A}\}$ are interior points of members of $\mathcal{A}$.)

(e) The family $\mathcal{S}$ of all proper intervals is a subbase for the discrete topology $\mathfrak{J}$ for R. The space $(R,\mathfrak{J})$ is not a Lindelöf space, although every cover by members of $\mathcal{S}$ has a countable subcover. (Contrast with the Alexander theorem 5.6.)

Note Further properties of the real numbers are stated in the next problem.

K HALF-OPEN INTERVAL SPACE

Let X be the set of real numbers and let $\mathfrak{J}$ be the topology for X which has for a base the family $\mathfrak{B}$ of all half-open intervals $[a,b) = \{x: a \leq x < b\}$ where a and b are real numbers. A $\mathfrak{J}$-accumulation point of a set is called an *accumulation point from the right*, and accumulation points from the left are similarly defined.

(a) Members of the base $\mathfrak{B}$ are both open and closed. The space $(X,\mathfrak{J})$ is not connected.

(b) The space $(X,\mathfrak{J})$ is separable but $\mathfrak{J}$ has no countable base. (For every x in X each base must contain a set whose infimum is x.)

(c) Each subspace of $(X,\mathfrak{J})$ is a Lindelöf space. (See 1.J(d).)

(d) If A is a set of real numbers then the set of all points of A which are not accumulation points from the right is countable. More generally, the set of points of A which are not accumulation points from both the right and the left is countable. (See 1.H.)

(e) Every subspace of $(X,\mathfrak{J})$ is separable.

L HALF-OPEN RECTANGLE SPACE

Let Y be $X \times X$, where X is the space of the preceding problem, and let $\mathfrak{U}$ be the topology which has as a base the family of all $A \times B$, where A and B are members of the topology $\mathfrak{J}$ of the preceding example.

(a) The space $(Y,\mathfrak{U})$ is separable.

(b) The space $(Y,\mathfrak{U})$ contains a subspace which is not separable. (For example, $\{(x,y): x + y = 1\}$.)

(c) The space $(Y,\mathfrak{U})$ is not a Lindelöf space. (If each open cover of Y has a countable subcover, then every closed subspace has the same property. Consider $\{(x,y): x + y = 1\}$.)

Note The spaces described in 1.K and 1.L are among the stock counter-examples of general topology. We enumerate other pathological features in 4.I. P. R. Halmos first observed that the product (in a sense to be made specific in chapter 3) of Lindelöf spaces may fail to be a Lindelöf space.

M EXAMPLE (THE ORDINALS) ON 1ST AND 2ND COUNTABILITY

Let Ω' be the set of all ordinals less than or equal to the first uncountable ordinal Ω, let X be $\Omega' \sim \{\Omega\}$, and let ω be the set of all non-negative integers, each with the order topology.

(a) ω is discrete and satisfies the 2nd axiom of countability.

(b) X satisfies the first but not the second axiom of countability.

(c) Ω' satisfies neither axiom of countability; if U is a separable subspace of Ω', then U is itself countable.

N COUNTABLE CHAIN CONDITION

A topological space satisfies the *countable chain condition* iff each disjoint family of open sets is countable. A separable space satisfies the countable chain condition, but not conversely. (Consider an uncountable set with the topology consisting of the void set and the complements of countable sets.) There are more complicated examples (see the Helly space of 5.M) which satisfy the first countability axiom and are separable, but fail to satisfy the second axiom of countability.

O THE EUCLIDEAN PLANE

The Euclidean plane is the set of all pairs of real numbers and the *usual topology* for the plane has a base which consists of all cartesian products $A \times B$ where A and B are open intervals with rational endpoints. This base is countable and the plane is consequently separable.

(a) The usual topology of the plane has a base which consists of all open discs, $\{(x,y): (x - a)^2 + (y - b)^2 < r^2\}$, where a, b, and r are rational numbers.

(b) Let X be the set of all points in the plane with at least one irrational coordinate, and let X have the relative topology. Then X is connected.

P EXAMPLE ON COMPONENTS

Let X be the following subset of the Euclidean plane, with the usual topology relativized. For each positive integer n let $A_n = \{1/n\} \times [0,1]$, where $[0,1]$ is the closed interval, and let X be the union of the sets A_n, with $(0,0)$ and $(0,1)$ adjoined. Then $\{(0,0)\}$ and $\{(0,1)\}$ are components of X, but each open and closed subset of X contains neither or both of the points.

Q THEOREM ON SEPARATED SETS

If X is a connected topological space, Y is a connected subset and $X \sim Y = A \cup B$, where A and B are separated, then $A \cup Y$ is connected.

R FINITE CHAIN THEOREM FOR CONNECTED SETS

Let $\mathcal{A}$ be a family of connected subsets of a topological space satisfying the condition: if A and B belong to $\mathcal{A}$, then there is a finite sequence $A_0, A_1, \cdots A_n$, of members of $\mathfrak{A}$ such that $A_0 = A$, $A_n = B$, and, for each i, the sets A_i and A_{i+1} are not separated. Then $\bigcup\{A: A \in \mathfrak{A}\}$ is connected. From this fact deduce 1.21.

S LOCALLY CONNECTED SPACES

A topological space is *locally connected* iff for each point x and each neighborhood U of x the component of U to which x belongs is a neighborhood of x.

(a) Each component of an open subset of a locally connected space is open.

(b) A topological space is locally connected iff the family of open connected subsets is a base for the topology.

(c) If points x and y of a locally connected space X belong to different components, then there are separated subsets A and B of X such that $x \in A$, $y \in B$, and $X = A \cup B$.

Note For many other properties of locally connected spaces and for generalizations, see G. T. Whyburn [1] and R. L. Wilder [1].

T THE BROUWER REDUCTION THEOREM

The usual statement of the theorem is as follows. Let X be a topological space satisfying the second axiom of countability. A property P of subsets of X is called *inductive* iff whenever each member of a countable nest of closed sets has P, then the intersection has P. A set A is *irreducible* with respect to P iff no proper closed subset of A has P. Then: If a closed subset A of X possesses an inductive property P, there is an irreducible closed subset of A which possesses P.

The theorem can be stated more formally in terms of a family of sets (the family of all sets possessing P).

(a) State and prove the theorem in this form. Assume that the topological space is such that every subspace is a Lindelöf space.

(b) If $(X,\mathfrak{J})$ is an arbitrary topological space can any result of this general sort be affirmed? (See 0.25.)

Chapter 2

MOORE-SMITH CONVERGENCE

INTRODUCTION

This chapter is devoted to the study of Moore-Smith convergence. It will turn out that the topology of a space can be described completely in terms of convergence, and the major part of the chapter is devoted to this description. We shall also characterize those notions of convergence which can be described as convergence relative to some topology. This project is similar in purpose to the theory of Kuratowski closure operators; it yields a useful and intuitively natural way of specifying certain topologies. However, the importance of convergence theory extends beyond this particular application, for the fundamental constructions of analysis are limit processes. We are interested in developing a theory which will apply to convergence of sequences, of double sequences, to summation of sequences, to differentiation and integration. The theory which we develop here is by no means the only possible theory, but it is unquestionably the most natural.

Sequential convergence furnishes the pattern on which the theory is developed, and we therefore list a few definitions and theorems on sequences to indicate this pattern. These will be particular cases of the theorems proved later.

A sequence is a function on the set ω of non-negative integers. A sequence of real numbers is a sequence whose range is a subset of the set of real numbers. The value of a sequence S at n is denoted, interchangeably, by S_n or $S(n)$. A sequence S is in a set A iff $S_n \varepsilon A$ for each non-negative integer n, and S is eventually in A iff there is an integer m such that $S_n \varepsilon A$ whenever $n \geqq m$. A sequence of real numbers converges to a number s relative to the usual topology iff it is eventually in each neighborhood of s. Using these definitions it turns out that, if A is a set of real numbers, then a point s belongs to the closure of A iff there is a sequence in A which converges to s, and s is an accumulation point of A iff there is a sequence in $A \sim \{s\}$ which converges to s.

We shall want to construct subsequences of a sequence. A sequence S may converge to no point and yet, by a proper construction, a sequence may be obtained from it which converges. We wish to select an integer N_i, for each i in ω, such that S_{N_i} converges. Restated, we want to find a sequence N of integers so that the composition $S \circ N(i) = S_{N_i} = S(N(i))$ converges. If no other requirement is made this is easy enough; if $N_i = 0$ for each i, then $S \circ N$ converges to S_0 since $S \circ N(i) = S_0$ for each i. Of course, an additional condition must be imposed so that the behavior of a subsequence is related to the behavior of the sequence for large integers. The usual condition is that N be strictly monotonically increasing; that is, if $i > j$, then $N_i > N_j$. This condition is unnecessarily stringent, and we impose instead the requirement that, as i becomes large, N_i also becomes large. Formally, then, T is a subsequence of a sequence S iff there is a sequence N of non-negative integers such that $T = S \circ N$ (equivalently, $T_i = S_{N_i}$ for each i) and for each integer m there is an integer n such that $N_i \geqq m$ whenever $i \geqq n$.

The set of points to which the subsequences of a given sequence converge satisfy a condition obtained by weakening the requirement of convergence. A sequence S is frequently in a set A iff for each non-negative integer m there is an integer n such that $n \geqq m$ and $S_n \varepsilon A$. This is precisely the same thing as saying that S is not eventually in the complement of A; intuitively, a sequence is frequently in A if it keeps returning to A. A point s is a cluster point of a sequence S iff S is frequently in each neighborhood of s. Then, if a sequence of real numbers is eventually in a set so is every subsequence, and consequently if a sequence converges to a point so does every subsequence. Each cluster point of a sequence is a limit point of a subsequence, and conversely.

The definitions and statements above are phrased so as to be applicable to any topological space, but unfortunately the theorems, in this generality, are false. (See the problems at the end of this chapter.) This unhappy situation is remedied by noticing that very few of the properties of the integers are used in proving theorems on sequences of real numbers. It is almost evident (although we have not given the proofs) that we need only certain properties of the ordering. Strictly speaking, convergence of sequences involves not only the function S on the non-negative integers ω, but also the ordering, $\geqq$, of ω. For convenience, in the work on convergence, we modify slightly the definition of sequence and agree that a sequence is an ordered pair $(S,\geqq)$ where S is a function on the integers, and we discuss convergence of the pair $(S,\geqq)$. (It will turn out that convergence of the pair $(S,\leqq)$ is also meaningful, but quite different.) Mention of the order will be omitted if no confusion is likely, and convergence of a sequence S will always mean convergence of the pair $(S,\geqq)$.

It is also convenient to have a bound variable (dummy variable) notation for sequences, and accordingly, if S is a function on the non-negative integers ω, $\{S_n, n \in \omega, \geqq\}$ is defined to be the pair $(S,\geqq)$. If A is a subset of ω, then convergence of $\{S_n, n \in A, \geqq\}$ will also be meaningful and will be related to the convergence of $(S,\geqq)$.

After this lengthy introduction the notion of convergence is almost self-evident, lacking a single fact. Which properties of the order $\geqq$ are used? These properties are listed below, and by using them the usual arguments of sequential convergence, with small modifications, are valid.

1 *Notes* E. H. Moore's study of unordered summability of sequences [1] led to the theory of convergence (Moore and Smith [1]). The generalization of the notion of subsequence which we will use is also due to Moore [2]. Garrett Birkhoff [3] applied Moore-Smith convergence to general topology; the form in which we give the theory is approximately that of J. W. Tukey [1]. See McShane [1] for an extremely readable expository account.

The problems at the end of the chapter contain a brief discussion of another theory of convergence and appropriate references.

DIRECTED SETS AND NETS

A binary relation $\geqq$ **directs** a set D if D is non-void and

(a) if m, n and p are members of D such that $m \geqq n$ and $n \geqq p$, then $m \geqq p$;
(b) if $m \in D$, then $m \geqq m$; and
(c) if m and n are members of D, then there is p in D such that $p \geqq m$ and $p \geqq n$.

We say that m follows n in the order $\geqq$ and that n precedes m iff $m \geqq n$. In the usual language of relations (see chapter 0) the condition (a) states that $\geqq$ is transitive on D, or partially orders D, and (b) states that $\geqq$ is reflexive on D. The condition (c) is special in character.

There are several natural examples of sets directed by relations. The real numbers as well as the set ω of non-negative integers are directed by $\geqq$. Observe that 0 is a member of ω which follows every other member in the order $\leqq$. It is also noteworthy that the family of all neighborhoods of a point in a topological space is directed by $\subset$ (the intersection of two neighborhoods is a neighborhood which follows both in the ordering $\subset$). The family of all finite subsets of a set is, on the other hand, directed by $\supset$. Any set is directed by agreeing that $x \geqq y$ for all members x and y, so that each element follows both itself and every other element.

A **directed set** is a pair $(D,\geqq)$ such that $\geqq$ directs D. (This is sometimes called a **directed system**.) A **net** is a pair $(S,\geqq)$ such that S is a function and $\geqq$ directs the domain of S. (A net is sometimes called a directed set.) If S is a function whose domain contains D and D is directed by $\geqq$, then $\{S_n, n \in D,\geqq\}$ is the net $(S|D,\geqq)$ where $S|D$ is S restricted to D. A net $\{S_n, n \in D,\geqq\}$ is **in** a set A iff $S_n \in A$ for all n; it is **eventually in** A iff there is an element m of D such that, if $n \in D$ and $n \geqq m$, then $S_n \in A$. The net is **frequently in** A iff for each m in D there is n in D such that $n \geqq m$ and $S_n \in A$. If $\{S_n, n \in D,\geqq\}$

is frequently in A, then the set E of all members n of D such that $S_n \in A$ has the property: for each $m \in D$ there is $p \in E$ such that $p \geqq m$. Such subsets of D are called **cofinal**. Each cofinal subset E of D is also directed by $\geqq$ because for elements m and n of E there is p in D such that $p \geqq m$ and $p \geqq n$, and there is then an element q of E which follows p. We have the following obvious equivalence: a net $\{S_n, n \in D, \geqq\}$ is frequently in a set A iff a cofinal subset of D maps into the set A, and this is the case iff the net is not eventually in the complement of A.

A net $(S,\geqq)$ in a topological space $(X,\mathfrak{I})$ **converges** to s relative to $\mathfrak{I}$ iff it is eventually in each $\mathfrak{I}$-neighborhood of s. The notion of convergence depends on the function S, the topology $\mathfrak{I}$, and the ordering $\geqq$. However, in cases where no confusion is likely to result we may omit all mention of $\mathfrak{I}$ or of $\geqq$ or of both and simply say "the net S (or the net $\{S_n, n \in D\}$) converges to s." If X is a discrete space (every subset is open), then a net S converges to a point s iff S is eventually in $\{s\}$: that is, from some point on S is constantly equal to s. On the other hand, if X is indiscrete (the only open sets are X and the void set), then every net in X converges to every point of X. Consequently a net may converge to several different points.

It is easy to describe the accumulation points of a set, the closure of a set, and in fact the topology of a space in terms of convergence. The arguments are slight variants of those usually given for sequences of real numbers.

2 THEOREM *Let X be a topological space. Then:*

(a) *A point s is an accumulation point of a subset A of X if and only if there is a net in $A \sim \{s\}$ which converges to s.*
(b) *A point s belongs to the closure of a subset A of X if and only if there is a net in A converging to s.*
(c) *A subset A of X is closed if and only if no net in A converges to a point of $X \sim A$.*

PROOF If s is an accumulation point of A, then for each neighborhood U of s there is a point S_U of A which belongs to $U \sim \{s\}$. The family $\mathfrak{U}$ of all neighborhoods of s is directed by $\subset$, and if U and V are neighborhoods of s such that $V \subset U$, then $S_V \in V$

$\subset U$. The net $\{S_U, U \in \mathfrak{U}, \subset\}$, therefore converges to s. On the other hand, if a net in $A \sim \{s\}$ converges to s, then this net has values in every neighborhood of s and $A \sim \{s\}$ surely intersects each neighborhood of s. This establishes the statement (a). To prove (b), recall that the closure of a set A consists of A together with all the accumulation points of A. For each accumulation point s of A there is, by the preceding, a net in A converging to s; for each point s of A any net whose value at every element of its domain is s converges to s. Therefore each point of the closure of A has a net in A converging to it. Conversely, if there is a net in A converging to s, then every neighborhood of s intersects A and s belongs to the closure of A. Proposition (c) is now obvious. ∎

We have noticed that, in general, a net in a topological space may converge to several different points. There are spaces in which convergence is unique in the sense that, if a net S converges to a point s and also to a point t, then $s = t$. A topological space is a **Hausdorff space** (T_2-space, or **separated** space) iff whenever x and y are distinct points of the space there exists disjoint neighborhoods of x and y.

3 THEOREM *A topological space is a Hausdorff space if and only if each net in the space converges to at most one point.*

PROOF If X is a Hausdorff space and s and t are distinct points of X, then there are disjoint neighborhoods U and V of s and t respectively. Since a net cannot be eventually in each of two disjoint sets it is clear that no net in X converges to both s and t. To establish the converse assume that X is not a Hausdorff space and that s and t are distinct points such that every neighborhood of s intersects every neighborhood of t. Let $\mathfrak{U}_s$ be the family of neighborhoods of s and $\mathfrak{U}_t$ the family of neighborhoods of t; then both $\mathfrak{U}_s$ and $\mathfrak{U}_t$ are directed by $\subset$. We order the cartesian product $\mathfrak{U}_s \times \mathfrak{U}_t$ by agreeing that $(T,U) \geqq (V,W)$ iff $T \subset V$ and $U \subset W$. Clearly the cartesian product is directed by $\geqq$. For each (T,U) in $\mathfrak{U}_s \times \mathfrak{U}_t$ the intersection $T \cap U$ is non-void, and hence we may select a point $S_{(T,U)}$ from $T \cap U$. If $(V,W) \geqq (T,U)$, then $S_{(V,W)} \in V \cap W \subset T \cap U$ and consequently the net $\{S_{(T,U)}, (T,U) \in \mathfrak{U}_s \times \mathfrak{U}_t, \geqq\}$ converges to both s and t. ∎

If $(X,\mathfrak{I})$ is a Hausdorff space and a net $\{S_n, n \in D, \geq\}$ in X converges to s we write $\mathfrak{I}$-lim $\{S_n, n \in D, \geq\} = s$. When no confusion seems possible this will be abbreviated: lim $\{S_n: n \in D\} = s$ or $\lim_n S_n = s$. The use of "limit" should be restricted to nets in a Hausdorff space so that the usual rule concerning substitution of equals for equals may remain valid. If lim $\{S_n: n \in D\} = s$ and lim $\{S_n: n \in D\} = t$, then $s = t$, since we always use equality in the sense of identity. As a matter of fact we shall occasionally use the notation $\lim_n S_n = s$ to mean S converges to s in cases where the space is not Hausdorff.

The device used in the preceding proof is often useful. If $(D,\geq)$ and $(E,\succ)$ are directed sets, then the cartesian product $D \times E$ is directed by $\gg$, where $(d,e) \gg (f,g)$ iff $d \geq f$ and $e \succ g$. The directed set $(D \times E,\gg)$ is the **product directed set.** We also want to define the product of a family of directed sets. Suppose for each a in a set A we are given a directed set $(D_a,>_a)$. The cartesian product $\times\{D_a: a \in A\}$ is the set of all functions d on A such that d_a $(= d(a))$ is a member of D_a for each a in A. The product directed set is $(\times\{D_a: a \in A\},\geq)$ where, if d and e are members of the product $d \geq e$ iff $d_a >_a e_a$ for each a in A. The **product order** is $\geq$. Of course, it must be verified that the product directed set is, in fact, directed. If d and e are members of the cartesian product $\times\{D_a: a \in A\}$, then for each a there is a member f_a of D_a which follows both d_a and e_a in the order $>_a$, and consequently the function f whose value at a is f_a follows both d and e in the order $\geq$. An important special case of the product directed set is that in which all coordinate sets D_a are identical and all relations $>_a$ are identical. In this case $\times\{D: a \in A\}$ is simply the set D^A of all functions on A to D, which is directed by the convention that d follows e iff $d(a)$ follows $e(a)$ for each member a of A. This is, for example, precisely the usual ordering of the set of all real valued functions on the set of real numbers.

The next result on limits is related to the closure axiom: $A^{--} = A^-$. It is important because it replaces an iterated limit by a single limit. The situation is as follows: Consider the class of all functions S such that $S(m,n)$ is defined whenever m belongs to a directed set D and n belongs to a directed set E_m. We want to find a net R with values in this domain such that $S \circ R$ converges to $\lim_m \lim_n S(m,n)$ whenever S is a function to a topological space and this iterated limit exists. It is interesting to notice that the solution of this problem requires Moore-Smith convergence, for, considering double sequences, no sequence whose range is a subset of $\omega \times \omega$ can have this property. The construction which yields a solution to the problem is a variant of the diagonal process. Let F be the product directed set $D \times \times\{E_n: n \in D\}$, and for each point (m,f) of F let $R(m,f) = (m,f(m))$. Then R is the required net.

4 THEOREM ON ITERATED LIMITS *Let D be a directed set, let E_m be a directed set for each m in D, let F be the product $D \times \times\{E_m: m \in D\}$, and for (m,f) in F let $R(m,f) = (m,f(m))$. If $S(m,n)$ is a member of a topological space for each m in D and each n in E_m, then $S \circ R$ converges to* $\lim_m \lim_n S(m,n)$ *whenever this iterated limit exists.*

PROOF Suppose $\lim_m \lim_n S(m,n) = s$ and that U is an open neighborhood * of s. We must find a member (m,f) of F such that, if $(p,g) \geqq (m,f)$, then $S \circ R(p,g) \in U$. Choose m in D so that $\lim_n S(p,n) \in U$ for each p following m and then, for each such p, choose a member $f(p)$ of E_p such that $S(p,n) \in U$ for all n following $f(p)$ in E_p. If p is a member of D which does not follow m let $f(p)$ be an arbitrary member of E_p. If $(p,g) \geqq (m,f)$, then $p \geqq m$, hence $\lim_n S(p,n) \in U$, and (since $g(p) \geqq f(p)$) $S \circ R\,(p,g) = S(p,g(p)) \in U$. ▮

SUBNETS AND CLUSTER POINTS

Following the pattern discussed in the introduction to the chapter we now define the generalization of subsequence and prove the hoped-for theorems.

* The existence of an open neighborhood of s is essential to the proof. The iterated limit theorem, the fact that the family of open neighborhoods of a point is a local base, and the closure axiom "$A^{--} = A^-$" are intimately related. Convergence has been studied in spaces with a structure less restrictive than a topology. See Ribeiro [1].

A net $\{T_m, m \in E\}$ is a **subnet** of a net $\{S_n, n \in D\}$ iff there is a function N on E with values in D such that

(a) $T = S \circ N$, or equivalently, $T_i = S_{N_i}$ for each i in E; and
(b) for each m in D there is n in E with the property that, if $p \geqq n$, then $N_p \geqq m$.

Since there seems to be no possibility of confusion we omit specific mention of the orderings involved. The second condition states, intuitively, "as p becomes large so does N_p." From this condition it is immediately clear that, if S is eventually in a set A, then the subnet $S \circ N$ of S is also eventually in A. This is a very important fact and the definition of subnet is designed to obtain precisely this result. Notice that each cofinal subset E of D is directed by the same ordering, and that $\{S_n, n \in E\}$ is a subnet of S. (Let N be the identity function on E, and the second condition of the definition becomes the requirement that E be cofinal.) This is a standard way of constructing subnets, and it is unfortunate that this simple variety of subnet is not adequate for all purposes. (2.E.)

There is a special sort of subnet which is adequate for almost all purposes. Suppose N is a function on the directed set E to the directed set D such that N is isotone ($N_i \geqq N_j$ if $i \geqq j$) and the range of N is cofinal in D. Then clearly $S \circ N$ is a subnet of S for each net S. The subnet constructed in the proof of the following lemma is of this sort (as remarked by K. T. Smith).

5 LEMMA *Let S be a net and $\mathfrak{A}$ a family of sets such that S is frequently in each member of $\mathfrak{A}$, and such that the intersection of two members of $\mathfrak{A}$ contains a member of $\mathfrak{A}$. Then there is a subnet of S which is eventually in each member of $\mathfrak{A}$.*

PROOF The intersection of any two members of $\mathfrak{A}$ contains a member of $\mathfrak{A}$ and therefore $\mathfrak{A}$ is directed by $\subset$. Let $\{S_n, n \in D\}$ be a net which is frequently in each member of $\mathfrak{A}$ and let E be the set of all pairs (m,A) such that $m \in D$, $A \in \mathfrak{A}$, and $S_m \in A$. Then E is directed by the product ordering for $D \times \mathfrak{A}$, for if (m,A) and (n,B) are members of E there is C in $\mathfrak{A}$ such that $C \subset A \cap B$ and p in D such that p follows both m and n and

$S_p \in C$; then $(p,C) \in E$ and (p,C) follows both (m,A) and (n,B). For (m,A) in E let $N(m,A) = m$. Then N is clearly isotone, and the range of N is cofinal in D ($\{S_n, n \in D\}$ is frequently in each member of $\mathfrak{A}$). Consequently $S \circ N$ is a subnet of S. Finally, if A is a member of $\mathfrak{A}$, if m is an arbitrary member of D such that $S_m \in A$, and if (n,B) is a member of E which follows (m,A), then $S \circ N(n,B) = S_n \in B \subset A$; it follows that $S \circ N$ is eventually in A. ▮

We now apply this lemma to convergence in a topological space. A point s of the space is a **cluster point** of a net S iff S is frequently in every neighborhood of s. A net may have one, many, or no cluster points. For example, if ω is the set of non-negative integers, then $\{n, n \in \omega\}$ is a net which has no cluster point relative to the usual topology for the real numbers. The other sort of extreme occurs if S is a sequence whose range is the set of all rational numbers (such a sequence exists because the set of rationals is countable). It is easy to see that this sequence is frequently in each open interval, and consequently every real number is a cluster point. If a net converges to a point, then this point is surely a cluster point, but it is possible that a net may have a single cluster point and fail to converge to this point. For example, consider the sequence $-1, 1, -1, 2, -1, 3, -1 \cdots$, constructed by alternating -1 and the sequence of positive integers. Then -1 is the unique cluster point of the sequence, but the sequence fails to converge to -1.

6 THEOREM *A point s in a topological space is a cluster point of a net S if and only if some subnet of S converges to s.*

PROOF Let s be a cluster point of S and let $\mathfrak{U}$ be the family of all neighborhoods of s. Then the intersection of two members of $\mathfrak{U}$ is again a member of $\mathfrak{U}$, and S is frequently in each member of $\mathfrak{U}$. Consequently the preceding lemma applies and there is a subnet of S which is eventually in each member of $\mathfrak{U}$, that is, converges to s. If s is not a cluster point of S, then there is a neighborhood U of s such that S is not frequently in U, and therefore S is eventually in the complement of U. Then each subnet of S is eventually in the complement of U and hence cannot converge to s. ▮

The following is a characterization of cluster points in terms of closure.

7 THEOREM *Let $\{S_n, n \in D\}$ be a net in a topological space and for each n in D let A_n be the set of all points S_m for $m > n$. Then s is a cluster point of $\{S_n, n \in D\}$ if and only if s belongs to the closure of A_n for each n in D.*

PROOF If s is a cluster point of $\{S_n, n \in D\}$, then for each n, A_n intersects each neighborhood of s because $\{S_n, n \in D\}$ is frequently in each neighborhood. Therefore s is in the closure of each A_n. If s is not a cluster point of $\{S_n, n \in D\}$ there is a neighborhood U of s such that $\{S_n, n \in D\}$ is not frequently in U. Hence for some n in D, if $m \geq n$, then $S_m \notin U$, so that U and A_n are disjoint. Consequently s is not in the closure of A_n. ∎

SEQUENCES AND SUBSEQUENCES

It is of some interest to know when a topology can be described in terms of sequences alone, not only because it is a convenience to have a fixed domain for all nets, but also because there are properties of sequences which fail to generalize. The most important class of topological spaces for which sequential convergence is adequate are those satisfying the first countability axiom: the neighborhood system of each point has a countable base. That is, for each point x of the space X there is a countable family of neighborhoods of x such that every neighborhood of x contains some member of the family. In this case we may replace "net" by "sequence" in almost all of the preceding theorems.

It should be noticed that a sequence may have subnets which are not subsequences.

8 THEOREM *Let X be a topological space satisfying the first axiom of countability. Then:*

(a) *A point s is an accumulation point of a set A if and only if there is a sequence in $A \sim \{s\}$ which converges to s.*
(b) *A set A is open if and only if each sequence which converges to a point of A is eventually in A.*

(c) *If s is a cluster point of a sequence S there is a subsequence of S converging to s.*

PROOF Suppose that s is an accumulation point of a subset A of X, and that $U_0, U_1, \cdots U_n \cdots$ is a sequence which is a base for the neighborhood system of s. Let $V_n = \bigcap\{U_i: i = 0, 1, \cdots, n\}$. Then the sequence $V_0, V_1, \cdots, V_n \cdots$ is also a base for the neighborhood system of s and, moreover, $V_{n+1} \subset V_n$ for each n. For each n select a point S_n from $V_n \cap (A \sim \{s\})$, thus obtaining a sequence $\{S_n, n \in \omega\}$ which evidently converges to s. This establishes half of (a), and the converse is obvious. If A is a subset of X which is not open, then there is a sequence in $X \sim A$ which converges to a point of A. Such a sequence surely fails to be eventually in A, and part (b) of the theorem follows. Finally, suppose that s is a cluster point of a sequence S and that $V_0, V_1 \cdots$ is a sequence which is a base for the neighborhood system of s such that $V_{n+1} \subset V_n$ for each n. For every non-negative integer i, choose N_i such that $N_i \geq i$ and S_{N_i} belongs to V_i. Then surely $\{S_{N_i}, i \in \omega\}$ is a subsequence of S which converges to s. ▌

*CONVERGENCE CLASSES

It is sometimes convenient to define a topology by specifying what nets converge to which points. For example, if $\mathfrak{F}$ is a family of functions each on a fixed set X to a topological space Y it is natural to specify that a net $\{f_n, n \in D\}$ converges to a function g iff $\{f_n(x), n \in D\}$ converges to $g(x)$ for each x in X. (This sort of convergence is discussed in some detail in chapter 3.) Having made such a specification the question naturally arises: Is there a topology for $\mathfrak{F}$ such that this convergence is convergence relative to the topology? An affirmative answer would enable us to use the machinery developed for topological spaces to investigate the structure of $\mathfrak{F}$.

The problem may be formally phrased as follows. If $\mathcal{C}$ is a class consisting of pairs (S,s), where S is a net in X and s a point, when is there a topology $\mathfrak{I}$ for X such that $(S,s) \in \mathcal{C}$ iff S converges to s relative to the topology $\mathfrak{I}$? From the preceding discussion of convergence we know several properties which $\mathcal{C}$ must possess if such a topology exists. We shall say that $\mathcal{C}$ is a **convergence class** for X iff it satisfies the conditions listed below.* For convenience, we say that S converges ($\mathcal{C}$) to s or that $\lim_n S_n \equiv s$ ($\mathcal{C}$) iff $(S,s) \in \mathcal{C}$.

(a) *If S is a net such that $S_n = s$ for each n, then S converges ($\mathcal{C}$) to s.*
(b) *If S converges ($\mathcal{C}$) to s, then so does each subnet of S.*
(c) *If S does not converge ($\mathcal{C}$) to s, then there is a subnet of S, no subnet of which converges ($\mathcal{C}$) to s.*
(d) *(Theorem 2.4 on iterated limits) Let D be a directed set, let E_m be a directed set for each m in D, let F be the product $D \times \times\{E_m\colon m \in D\}$ and for (m,f) in F let $R(m,f) = (m,f(m))$. If* $\lim_m \lim_n S(m,n) \equiv s$ ($\mathcal{C}$), *then $S \circ R$ converges ($\mathcal{C}$) to s.*

It has previously been shown that convergence in a topological space satisfies (a), (b), and (d). Statement (c) is easily established, in this case, by the argument: If a net $\{S_n, n \in D\}$ fails to converge to a point s, then it is frequently in the complement of a neighborhood of s, and hence for a cofinal subset E of D, $\{S_n, n \in E\}$ is in the complement. But clearly $\{S_n, n \in E\}$ is a subnet, no subnet of which converges to s.

We now show that every convergence class is actually derived from a topology.

9 THEOREM *Let $\mathcal{C}$ be a convergence class for a set X, and for each subset A of X let A^c be the set of all points s such that, for some net S in A, S convergences ($\mathcal{C}$) to s. Then ^{c} is a closure operator, and $(S,s) \in \mathcal{C}$ if and only if S converges to s relative to the topology associated with ^{c}.*

PROOF It is first shown that ^{c} is a closure operator. (See 1.8.) Since a net is a function on a directed set, and the set is non-void by definition, $(0)^c$ is void. In view of condition (a) on constant nets, for each member s of a set A there is a net S which converges ($\mathcal{C}$) to s, and hence $A \subset A^c$. If $s \in A^c$, then because of the definition of the operator ^{c} $s \in (A \cup B)^c$ and consequently

* The first three of these, with "net" replaced by "sequence," are Kuratowski's modification of the Fréchet axioms for limit space. See Kuratowski [1].

$A^c \subset (A \cup B)^c$ for each set B. Therefore $A^c \cup B^c \subset (A \cup B)^c$. To show the opposite inclusion suppose that $\{S_n, n \in D\}$ is a net in $A \cup B$, and suppose that $\{S_n, n \in D\}$ converges $(\mathcal{C})$ to s. If $D_A = \{n\colon n \in D \text{ and } S_n \in A\}$, and $D_B = \{n\colon n \in D \text{ and } S_n \in B\}$, then $D_A \cup D_B = D$. Hence either D_A or D_B is cofinal in D, and consequently either $\{S_n, n \in D_A\}$ or $\{S_n, n \in D_B\}$ is a subnet of $\{S_n, n \in D\}$ which also converges $(\mathcal{C})$ to s by virtue of condition (b). Hence $s \in A^c \cup B^c$ and we have shown that $A^c \cup B^c = (A \cup B)^c$. It must now be shown that $A^{cc} = A^c$, and condition (d) is precisely what is needed. If $\{T_m, m \in D\}$ is a net in A^c which converges $(\mathcal{C})$ to t, then for each m in D there are a directed set E_m and a net $\{S(m,n), n \in E_m\}$ which converge $(\mathcal{C})$ to T_m. Condition (d) then exhibits a net which converges $(\mathcal{C})$ to t and consequently $t \in A^c$. Hence $A^{cc} = A^c$.

The more delicate part of the proof, that of showing that convergence $(\mathcal{C})$ is identical with convergence relative to the topology $\mathfrak{I}$ associated with the operator ^{c}, remains. First, suppose $\{S_n, n \in D\}$ converges $(\mathcal{C})$ to s and S does not converge to s relative to $\mathfrak{I}$. Then there is an open neighborhood U of s such that $\{S_n, n \in D\}$ is not eventually in U. Hence there is a cofinal subset E of D such that $S_n \in X \sim U$ for n in E. Since $\{S_n, n \in E\}$ is a subnet of $\{S_n, n \in D\}$ this subnet in $X \sim U$ converges $(\mathcal{C})$ to s by condition (b). Hence $X \sim U \neq (X \sim U)^c$, and U is not open relative to $\mathfrak{I}$, which is a contradiction.

Finally, suppose that a net P converges to a point r relative to the topology $\mathfrak{I}$ and fails to converge $(\mathcal{C})$. Then by (c) there is a subnet $\{T_m, m \in D\}$, no subnet of which converges $(\mathcal{C})$ to r, and a contradiction results if we construct such a subnet. For each m in D let $B_m = \{n\colon n \in D \text{ such that } n \geq m\}$ and let A_m be the set of all T_n for n in B_m. Because $\{T_m, m \in D\}$ converges relative to $\mathfrak{I}$ to r, r must lie in the closure of each A_m. Consequently, for each m in D there are a directed set E_m and a net $\{U(m,n), n \in E_m\}$ in A_m, such that the composition $\{T \circ U(m,n), n \in E_m\}$ converges $(\mathcal{C})$ to r. Condition (d) on convergence classes now applies. If $R(m,f) = (m,f(m))$ for each (m,f) in $D \times \times\{E_m, m \in D\}$, then $T \circ U \circ R$ converges $(\mathcal{C})$ to r. Moreover, if $p \geq m$, then $U \circ R(p,f) = U(p,f(p)) \in B_m$; that is, $U \circ R(p,f) \geq m$. It follows that $T \circ U \circ R$ is a subnet of T, and the theorem follows. ∎

The preceding theorem sets up a one-to-one correspondence between the topologies for a set X and the convergence classes on it. This correspondence is order inverting in the following sense. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are convergence classes and $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are the associated topologies, then $\mathcal{C}_1 \subset \mathcal{C}_2$ if and only if $\mathfrak{I}_2 \subset \mathfrak{I}_1$. (This fact is immediately evident from the definition of convergence.) We also notice that the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ is a convergence class in view of the four characteristic properties of such classes. It is easy to see that the topology associated with $\mathcal{C}_1 \cap \mathcal{C}_2$ is the smallest topology which is larger than each of $\mathfrak{I}_1$ and $\mathfrak{I}_2$, and dually, the convergence class of $\mathfrak{I}_1 \cap \mathfrak{I}_2$ is the smallest convergence class which is larger than each of $\mathcal{C}_1$ and $\mathcal{C}_2$.

PROBLEMS

A EXERCISE ON SEQUENCES

Let X be a countable set with a topology consisting of the void set together with all sets whose complements are finite. What sequences converge to what points?

B EXAMPLE: SEQUENCES ARE INADEQUATE

Let Ω' be the set of ordinal numbers which are less than or equal to the first uncountable ordinal Ω, and let the topology be the order topology. Then Ω is an accumulation point of $\Omega' \sim \{\Omega\}$, but no sequence in $\Omega' \sim \{\Omega\}$ converges to Ω.

C EXERCISE ON HAUSDORFF SPACES: DOOR SPACES

A topological space is a *door space* iff every subset is either open or closed. A Hausdorff door space has at most one accumulation point, and if x is a point which is not an accumulation point, then $\{x\}$ is open. (If U is an arbitrary neighborhood of an accumulation point y, then $U \sim \{y\}$ is open.)

D EXERCISE ON SUBSEQUENCES

Let N be a sequence of non-negative integers such that no integer occurs more than a finite number of times; that is, for each m, the set $\{i: N_i = m\}$ is finite. Then if $\{S_n, n \in \omega\}$ is any sequence, $\{S_{N_i}, i \in \omega\}$ is a subsequence. If $\{S_n, n \in \omega\}$ is a sequence in a topological space, and N is an arbitrary sequence of non-negative integers, then $\{S_{N_i}, i \in \omega\}$ is either a subsequence of $\{S_n, n \in \omega\}$ or else has a cluster point.

E EXAMPLE: COFINAL SUBSETS ARE INADEQUATE

Let X be the set of all pairs of non-negative integers with the topology described as follows: For each point (m,n) other than $(0,0)$ the set $\{(m,n)\}$ is open. A set U is a neighborhood of $(0,0)$ iff for all except a finite number of integers m the set $\{n: (m,n) \notin U\}$ is finite. (Visualizing X in the Euclidean plane, a neighborhood of $(0,0)$ contains all but a finite number of the members of all but a finite number of columns.)

(a) The space X is a Hausdorff space.

(b) Each point of X is the intersection of a countable family of closed neighborhoods.

(c) The space is a Lindelöf space; that is, each open cover has a countable subcover.

(d) No sequence in $X \sim \{(0,0)\}$ converges to $(0,0)$. (If a sequence S in $X \sim \{(0,0)\}$ converges to $(0,0)$, then it is eventually in the complement of each column, and the sequence has only a finite number of values in each column.)

(e) There is a sequence S in $X \sim \{(0,0)\}$ with $(0,0)$ as a cluster point, and S restricted to any cofinal subset of the integers fails to converge.

Note This example is due to Arens [1].

F MONOTONE NETS

Let X be an order-complete chain; that is, X is linearly ordered by a relation $>$, such that each non-void subset of X which has an upper bound has a supremum. Let X have the order topology (1.I). A net $(S,\succ)$ in X is monotone increasing (decreasing) iff whenever $m \succ n$, then $S_m \geqq S_n (S_n \geqq S_m)$.

(a) Each monotone increasing net in X whose range is bounded (there is x in X such that $x \geqq S_n$ for all n) converges to the supremum of its range.

(b) If X is the set of all real numbers with the usual order or if X is the set of all ordinal numbers less than the first uncountable ordinal, then each monotone increasing (decreasing) net whose range has an upper (lower) bound converges to the supremum (infimum) of its range.

G INTEGRATION THEORY, JUNIOR GRADE

Let f be a real-valued function whose domain includes a set A, let $\mathfrak{A}$ be the family of all finite subsets of A, and for each F in $\mathfrak{A}$ let $S_F = \sum\{f(a): a \in F\}$. Then $\mathfrak{A}$ is directed by $\supset$ and $\{S_F, F \in \mathfrak{A}, \supset\}$

is a net. If this net converges f is *summable* over A and the number to which the net converges is the *unordered sum* of f over A, denoted $\sum\{f(a)\colon a \in A\}$ or simply $\sum_A f$.

(a) If f is non-negative (non-positive), then f is summable iff there is an upper bound (lower bound) for the sums over finite subsets of A. (Use the preceding problem on monotone nets.)

(b) Let $A_+ = \{a\colon f(a) \geq 0\}$ and $A_- = \{a\colon f(a) < 0\}$. Then f is summable over A iff it is summable over both A_+ and A_-. If f is summable over A, then $\sum_A f = \sum_{A_+} f + \sum_{A_-} f$.

(c) A function f is summable over A iff $|f|$ is summable over A, where $|f|(a) = |f(a)|$.

(d) If f is summable on a set A, then f is zero outside some countable subset of A. (If f is different from zero at every point of some uncountable subset, then, for some positive integer n, $\{a\colon f(a) \geqq 1/n\}$ is uncountable.)

(e) If f and g are summable over A and r and s are real numbers, then $rf + sg$ is summable over A and $\sum_A(rf + sg) = r\sum_A f + s\sum_A g$.

(f) If f is summable over A, and B and C are disjoint subsets of A, then f is summable over each of B and C and $\sum_{B \cup C} f = \sum_B f + \sum_C f$.

(g) If x is a sequence of real numbers, then the *ordered sum* ("sum of the series") is the limit of the sequence S_n where $S_n = \sum\{x_i\colon i = 0, 1, \cdots, n\}$. In other words, the ordered sum is the limit $\{S_F, F \in \mathcal{B}\}$, where $\mathcal{B}$ is the family of all sets which are of the form $\{m\colon m \leqq n\}$ for some n. This is a subnet of the net defining the unordered sum. The sequence x is *absolutely summable* iff the sequence $|x|$, where $|x|_n = |x_n|$, has an ordered sum. The unordered sum of x over the integers exists iff the sequence is absolutely summable, and in this case, the unordered and ordered sums are equal.

(h) (*Fubinito*) Let f be a real-valued function on a cartesian product $A \times B$. Then:

(i) If f is summable over $A \times B$, then $\sum_{A\times B} f = \sum\{\sum\{f(a,b)\colon b \in B\}\colon a \in A\}$. (The latter is one of the two *iterated* sums.)

(ii) If, for each member a of A, $f(a,b)$ is either non-negative for all b or non-positive for all b, if $F(a) = \sum\{f(a,b)\colon b \in B\}$, and if F is summable over A, then f is summable over $A \times B$.

(iii) In general, both iterated sums may exist and f may fail to be summable. In fact, if both A and B are countably infinite and F and G are arbitrary real functions on A and on B respectively, then there is f on $A \times B$ such that $\sum\{f(a,b);\ a \in A\} = G(b)$ and $\sum\{f(a,b)\colon b \in B\} = F(a)$ for all b in B and all a in A.

Notes The results stated in this problem are those which are needed to develop measure theory using unordered summation instead of absolutely convergent series. All the results except (d), (g) and (h,ii) can be established in a much more general situation; in chapter 7 the problem will be reexamined using the notion of completeness. The order-theoretic treatment above gives some insight into more complicated examples of integration.

Historically, unordered summation was the forerunner of Moore-Smith convergence. (Moore [1].)

H INTEGRATION THEORY, UTILITY GRADE

Let f be a bounded real-valued function on the closed interval of real numbers $[a,b]$. A *subdivision* S of $[a,b]$ is a finite family of closed intervals, covering $[a,b]$, such that any two intervals have at most one point in common. The length of an interval I is denoted $|I|$, and for a subdivision S the *mesh*, $\|S\|$, is the maximum of $|I|$ for I in S. We direct the family of subdivisions in two different ways:

(i) $S \geqq S'$ iff S is a *refinement* of S', in the sense that each member of S is a subset of a member of S'; and
(ii) $S \gg S'$ iff $\|S\| \leqq \|S'\|$.

Let $M_f(I)$ be the supremum of f on I, and let $m_f(I)$ be the infimum. The *upper* and *lower Darboux* sums corresponding to the subdivision S are defined to be $D_f(S) = \sum\{|I|M_f(I) : I \in S\}$ and $d_f = \sum\{|I|m_f(I) : I \in S\}$ respectively. The Riemann sums are more complicated. A *choice function* for a subdivision S is a function c on S such that $c(I) \in I$ for each I in S. The set of all pairs (S,c), such that S is a subdivision and c is a choice function for S, is ordered in two ways: $(S,c) \succ (S',c')$ iff $S \geqq S'$ and $(S,c) \succ\succ (S',c')$ iff $S \gg S'$. For a pair (S,c) the *Riemann sum* is $R_f(S,c) = \sum\{|I|f(c(I)) : I \in S\}$.

The basic computation is made in terms of the ordering by refinement.

(a) The nets $(D_f,\geqq)$ and $(d_f,\geqq)$ are monotonically decreasing and increasing respectively, and hence converge.

(b) $d_f(S) \leqq R_f(S,c) \leqq D_f(S)$ for all subdivisions S and all choice functions c.

(c) For each positive number e there is a $\succ$-cofinal subset of the set of pairs (S,c) such that $R_f(S,c) + e \geqq D_f(S)$. (There is also a dual proposition.)

(d) The net $(R_f,\succ)$ converges iff $\lim (D_f,\geqq) = \lim (d_f,\geqq)$. If $(R_f,\succ)$ converges, then $\lim (R_f,\succ) = \lim (D_f,\geqq) = \lim (d_f,\geqq)$.

(e) The net $(R_f,\succ)$ is a subnet of $(R_f,\succ\succ)$.

(f) The net $(R_f,\succ\succ)$ converges iff $\lim (D_f,\geqq) = \lim (d_f,\geqq)$. If $(R_f,\succ\succ)$ converges $\lim (R_f,\succ\succ) = \lim (R_f,\succ)$.

Notes The Riemann integral of f is usually defined to be the limit of $(R_f,\succ\succ)$. The advantage of considering refinement as well as mesh is, here, essentially a matter of technique. If instead of considering finite subdivisions and length of intervals we consider countable subdivisions and let $|I|$ be the Lebesgue measure of I, the net $(R_f,\succ)$ converges to the usual Lebesgue integral of f, while $(R_f,\succ\succ)$ may not. Further, a definition of the refinement type may be used to integrate certain functions whose values lie in a vector space. (See Hille [1], chapter 3.) An integral of the Darboux type requires that the range of the function to be integrated be partially ordered. The Daniell integral and various generalizations (Bourbaki [2], McShane [2] and [3], and M. H. Stone [1]) are essentially of this sort. There is another standard way of introducing an integral, via a completion process with respect to a metric, which has many advantages (Halmos [1]).

I MAXIMAL IDEALS IN LATTICES

A *lattice* is a non-void set X with a reflexive partial ordering $\geqq$ such that for every pair x and y of members of X there is a (unique) smallest element $x \vee y$ which is greater than each of x and y and a (unique) largest element $x \wedge y$ which is smaller than each. The elements $x \vee y$ and $x \wedge y$ are respectively the *join* and the *meet* of x and y. The lattice is *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, and z in X. A subset A of X is an *ideal* (a *dual ideal*) iff whenever $y \geqq x$ and $y \in A$, then $x \in A$, and if y and z belong to A so does $y \vee z$ (respectively, whenever $x \geqq y$ and $y \in A$, then $x \in A$, and if $y \in A$ and $z \in A$, then $y \wedge z \in A$).

Let A and B be disjoint subsets of a distributive lattice X such that A is an ideal and B is a dual ideal. Then there are disjoint sets A' and B' such that A' is an ideal containing A, B' is a dual ideal containing B, and $A' \cup B' = X$.

The proof of this proposition is broken down into a sequence of lemmas.

(a) The family of all ideals which contain A and are disjoint from B contains a maximal member A'. (See 0.25.) Similarly there is a dual ideal B' which contains B, is disjoint from A', and is maximal with respect to these properties.

(b) The smallest ideal which contains A' and a member c of X is $\{x : x \leqq c \textit{ or } x \leqq c \vee y \textit{ for some } y \textit{ in } A'\}$. Since A' is maximal, if c

does not belong to either A' or B, then $c \vee x \in B$ for some x in A'. (If $z \geqq x \in B$, then $z \in B$.)

(c) If c belongs to neither A' nor B', then there is x in A' and y in B' such that $c \vee x \in B'$ and $c \wedge y \in A'$. Then $(c \vee x) \wedge y = (c \wedge y) \vee (x \wedge y)$ belongs to both A' and B'.

Notes This theorem is due to M. H. Stone [2]; it is the best form of one of the basic facts about ordered sets. It is used in the next two problems and it is the fact underlying the most important results on compactness (chapter 5). An application of some form of the maximal principle seems to be essential to its proof. It has been stated in the literature that this theorem (or, more precisely, a corollary to the theorem which occurs in problem 2. K) implies the axiom of choice, but I do not know whether this is the case. Finally, the definition of distributivity which is given above is unduly restrictive. Either of the two equalities implies the other (Birkhoff [1]).

J UNIVERSAL NETS

A net in a set X is said to be *universal* iff for each subset A of X the net is eventually in A or eventually in $X \sim A$.

(a) If a universal net is frequently in a set it is eventually in the set. Hence a universal net in a topological space converges to each of its cluster points.

(b) If a net is universal, then each subnet is also universal. If S is a universal net in X and f is a function on X to Y, then $f \circ S$ is a universal net in Y.

(c) *Lemma* If S is a net in X, then there is a family $\mathcal{C}$ of subsets of X such that: S is frequently in each member of $\mathcal{C}$, the intersection of two members of $\mathcal{C}$ belongs to $\mathcal{C}$, and for each subset A of X either A or $X \sim A$ belongs to $\mathcal{C}$. (Either show that there is a family $\mathcal{C}$ maximal with respect to the first two listed properties and demonstrate that it possesses the third, or apply 2.I, letting $\mathcal{A}$ be the family of all sets A such that S is eventually in $X \sim A$, $\mathcal{B}$ the family of all B such that S is eventually in B, and let the ordering be $\subset$.)

(d) There is a universal subnet of each net in X. (Use the preceding result and 2.5.)

K BOOLEAN RINGS: THERE ARE ENOUGH HOMOMORPHISMS

A *Boolean ring* is a ring $(R,+,\cdot)$ such that $r \cdot r = r$ and $r + r = 0$ for each r in R. The field of integers modulo 2 is denoted I_2.

(a) A Boolean ring is commutative. (Observe that $(r + s)\cdot(r + s) = r + s$.)

(b) If $(R,+,\cdot)$ is a Boolean ring, then multiplication of members of R by members of I_2 can be defined so that R is an algebra over I_2.

(c) The *symmetric difference* $A \Delta B$ of two sets A and B is defined to be $(A \cup B) \sim (A \cap B)$. If $\mathcal{A}$ is the family of all subsets of a set X, then $(\mathcal{A},\Delta,\cap)$ is a Boolean ring with unit.

(d) Let X be a set and let $I_2{}^X$ be the family of all functions on X to I_2. Define addition and multiplication of functions pointwise (that is, $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$). Then $(I_2{}^X,+,\cdot)$ is a Boolean ring with unit and is isomorphic to $(\mathcal{A},\Delta,\cap)$ where $\mathcal{A}$ is the family of all subsets of X.

(e) The *natural ordering* of a Boolean ring is defined by agreeing that $r \geqq s$ iff $r \cdot s = s$. The relation $\geqq$ partially orders R in such a way that the least element which follows both r and s is $r \vee s = r + s + r \cdot s$ and the greatest element which precedes both r and s is $r \wedge s = r \cdot s$. Each of $\vee$ and $\wedge$ are associative operations and the following distributive laws hold: $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$ and $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$.

(f) Recall that S is an ideal in a Boolean ring $(R,+,\cdot)$ iff S is an additive subgroup and $r \cdot s \in S$ whenever $r \in R$ and $s \in S$; the ideal S is maximal iff $R \neq S$ and no ideal other than R properly contains S. There is a one-to-one correspondence between maximal ideals in R and homomorphisms into I_2 which are not identically zero. (The kernel of such a homomorphism is a maximal ideal.)

(g) A necessary and sufficient condition that S be an ideal in a Boolean ring is that $r \vee s \in S$ whenever r and s are members of S and $t \in S$ whenever t precedes a member of S in the natural order (that is, $t \leqq$ some member of S). A subset T of R is called a *dual ideal* iff $r \wedge s \in T$ whenever r and s are members of T and $t \in T$ whenever t follows a member of T. If $r \in R$, then $\{s: r \geqq s\}$ is an ideal and $\{s: s \geqq r\}$ is a dual ideal. If S is an ideal, T is a disjoint dual ideal, and $S \cup T = R$, then the function which is zero on S and one on T is a homomorphism of R into I_2. (In a Boolean ring of sets an ideal is frequently called an $\cap$-ideal and a dual ideal a $\cup$-ideal.)

(h) *Theorem* If S is an ideal in a Boolean ring and T is a dual ideal which is disjoint from S, then there is a homomorphism of the ring into I_2 which is zero on S and one on T. In particular, if r is a non-zero member of the ring there is a homomorphism h of the ring such that $h(r) = 1$. (In other words, there are enough homomorphisms to distinguish members of the ring. A proof of this theorem may be based on 2.I.)

(i) If X is a topological space and $\mathcal{B}$ is the family of all subsets of X which are both open and closed, then $(\mathcal{B},\Delta,\cap)$ is a Boolean algebra.

(j) Not all Boolean algebras are isomorphic to an algebra of all subsets of a set. (Show by example that there are countable Boolean algebras.)

Note This investigation is completed in 5.S.

L FILTERS

A theory of convergence has been built on the concept of filter. A *filter* $\mathfrak{F}$ in a set X is a family of non-void subsets of X such that

(i) the intersection of two members of $\mathfrak{F}$ always belongs to $\mathfrak{F}$; and
(ii) if $A \in \mathfrak{F}$ and $A \subset B \subset X$, then $B \in \mathfrak{F}$.

In the terminology of the previous problem a filter is a proper dual ideal in the Boolean ring of all subsets of X. A filter $\mathfrak{F}$ *converges* to a point x in a topological space X iff each neighborhood of x is a member of $\mathfrak{F}$ (that is, the neighborhood system of x is a subfamily of $\mathfrak{F}$).

(a) A subset U is open iff U belongs to every filter which converges to a point of U.

(b) A point x is an accumulation point of a set A iff $A \sim \{x\}$ belongs to some filter which converges to x.

(c) Let ϕ_x be the collection of all filters which converge to a point x. Then $\bigcap \{\mathfrak{F}: \mathfrak{F} \in \phi_x\}$ is the neighborhood system of x.

(d) If $\mathfrak{F}$ is a filter converging to x and $\mathfrak{G}$ is a filter which contains $\mathfrak{F}$, then $\mathfrak{G}$ converges to x.

(e) A filter in X is an *ultrafilter* iff it is properly contained in no filter in X. If $\mathfrak{F}$ is an ultrafilter in X and the union of two sets is a member of $\mathfrak{F}$, then one of the two sets belongs to $\mathfrak{F}$. In particular, if A is a subset of X, then either A or $X \sim A$ belongs to $\mathfrak{F}$. (Problem 2.I again.)

(f) One might suspect that filters and nets lead to essentially equivalent theories. Grounds for this suspicion may be found in the following facts:

(i) If $\{x_n, n \in D\}$ is a net in X, then the family $\mathfrak{F}$ of all sets A such that $\{x_n, n \in D\}$ is eventually in A is a filter in X.
(ii) Let $\mathfrak{F}$ be a filter in X and let D be the set of all pairs (x,F) such that $x \in F$ and $F \in \mathfrak{F}$. Direct D by agreeing that $(y,G) \geq (x,F)$ iff $G \subset F$, and let $f(x,F) = x$. Then $\mathfrak{F}$ is precisely the family of all sets A such that the net $\{f(x,F), (x,F) \in D\}$ is eventually in A.

Notes The definition of filter is due to H. Cartan; his treatment of convergence is given in full in Bourbaki [1]. Proposition (c) is a remark of W. H. Gottschalk; (f) is part of the folklore of the subject.

Chapter 3

PRODUCT AND QUOTIENT SPACES

It is the purpose of this chapter to investigate two methods of constructing new topological spaces from old. One of these involves assigning a standard sort of topology to the cartesian product of spaces, thus building a new space from those originally given. For example, the Euclidean plane is the product space of the real numbers (with the usual topology) with itself, and Euclidean n-space is the product of the real numbers n times. In chapter 4 arbitrary cartesian products of the real numbers will serve as standard spaces with which one may compare other topological spaces.

The second method of constructing a new space from a given one depends on dividing the given space X into equivalence classes, each of which is a point of the newly constructed space. Roughly speaking, we "identify" the points of certain subsets of X, so obtaining a new set of points, which is then assigned the "quotient" topology. For example, the equivalence classes of real numbers modulo the integers are assigned a topology so that the resulting space is a "copy" of the unit circle in the plane.

Both of these methods of constructing spaces are motivated by making certain functions continuous. We therefore begin by defining continuity and proving a few simple propositions about it.

CONTINUOUS FUNCTIONS

For convenience we review some of the terminology and a few elementary propositions about functions (chapter 0). The words "function," "map," "mapping," "correspondence," "operator," and "transformation" are synonymous. A function f is said to be on X iff its domain is X. It is to Y, or into Y, iff its range is a subset of Y and it is onto Y if its range is Y. The value of f at a point x is $f(x)$ and $f(x)$ is also called the image under f of x. If B is a subset of Y, then the inverse under f of B, $f^{-1}[B]$, is $\{x: f(x) \in B\}$. The inverse under f of the intersection (union) of the members of a family of subsets of Y is the intersection (union) of the inverses of the members; that is, if Z_c is a subset of Y for each member c of a set C, then $f^{-1}[\bigcap\{Z_c: c \in C\}] = \bigcap\{f^{-1}[Z_c]: c \in C\}$, and similarly for unions. If $y \in Y$, then $f^{-1}[\{y\}]$, the inverse of the set whose only member is y, is abbreviated $f^{-1}[y]$. The image $f[A]$ of a subset A of X is the set of all points y such that $y = f(x)$ for some x in A. The image of the union of a family of subsets of X is the union of the images, but, in general, the image of the intersection is not the intersection of the images. A function f is one to one iff no two distinct points have the same image, and in this case f^{-1} is the function inverse to f. (Notice that the notation is arranged so that, roughly speaking, square brackets occur in the designations of subsets of the range and domain of a function, and parentheses in the designation of members. For example, if f is one to one onto Y and $y \in Y$, then $f^{-1}(y)$ is the unique point x of X such that $f(x) = y$, and $f^{-1}[y] = \{x\}$.)

A map f of a topological space $(X,\mathfrak{J})$ into a topological space $(Y,\mathfrak{U})$ is **continuous** iff the inverse of each open set is open. More precisely, f is continuous with respect to $\mathfrak{J}$ and $\mathfrak{U}$, or $\mathfrak{J}$-$\mathfrak{U}$ continuous, iff $f^{-1}[U] \in \mathfrak{J}$ for each U in $\mathfrak{U}$. The concept depends on the topology of both the range and the domain space, but we follow the usual practice of suppressing all mention of the topologies when confusion is unlikely. There are one or two propositions about continuity which are quite important, although almost self-evident. First, if f is a continuous function on X to Y and g is a continuous function on Y to Z, then the composition $g \circ f$ is a continuous function on X to Z, for $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$ for each subset V of Z, and using first the continuity of g, then that of f, it follows that if V is open so is $(g \circ f)^{-1}[V]$. If f is a continuous function on X to Y, and A is a subset of X, then the restriction of f to A, $f|A$, is also continuous with respect to the relative topology for A, for if U is open in Y, then $(f|A)^{-1}[U] = A \cap f^{-1}[U]$, which is open in A. A function f such that $f|A$ is continuous is **continuous on** A. It may happen that f is continuous on A but fails to be continuous on X.

The following is a list of conditions, each equivalent to continuity; it is useful because it is frequently necessary to prove functions continuous.

1 THEOREM *If X and Y are topological space and f is a function on X to Y, then the following statements are equivalent.*

(a) *The function f is continuous.*
(b) *The inverse of each closed set is closed.*
(c) *The inverse of each member of a subbase for the topology for Y is open.*
(d) *For each x in X the inverse of every neighborhood of $f(x)$ is a neighborhood of x.*
(e) *For each x in X and each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f[V] \subset U$.*
(f) *For each net S (or $\{S_n, n \in D\}$) in X which converges to a point s, the composition $f \circ S$ ($\{f(S_n), n \in D\}$) converges to $f(s)$.*
(g) *For each subset A of X the image of the closure is a subset of the closure of the image; that is, $f[A^-] \subset f[A]^-$.*
(h) *For each subsct B of Y, $f^{-1}[B]^- \subset f^{-1}[B^-]$.*

PROOF (a) $\leftrightarrow$ (b): This is a simple consequence of the fact that the inverse of a function preserves relative complements; that is, $f^{-1}[Y \sim B] = X \sim f^{-1}[B]$ for every subset B of Y.

(a) $\leftrightarrow$ (c): If f is continuous then the inverse of a member of a subbase is open because each subbase member is open. Conversely, since each open set V in Y is the union of finite intersections of subbase members, $f^{-1}[V]$ is the union of finite intersections of the inverses of subbase members; if these are open, then the inverse of each open set is open.

(a) $\rightarrow$ (d): If f is continuous, $x \in X$, and V is a neighborhood of $f(x)$, then V contains an open neighborhood W of $f(x)$ and

$f^{-1}[W]$ is an open neighborhood of x which is a subset of $f^{-1}[V]$; consequently $f^{-1}[V]$ is a neighborhood of x.

(d) $\rightarrow$ (e): Assuming (d), if U is a neighborhood of $f(x)$, then $f^{-1}[U]$ is a neighborhood of x such that $f[f^{-1}[U]] \subset U$.

(e) $\rightarrow$ (f): Assuming (e), let S be a net in X which converges to a point s. Then if U is a neighborhood of $f(s)$ there is a neighborhood V of s such that $f[V] \subset U$, and since S is eventually in V, $f \circ S$ is eventually in U.

(f) $\rightarrow$ (g): Assuming (f), let A be a subset of X and s a point of the closure A. Then there is a net S in A which converges to s, and $f \circ S$ converges to $f(s)$, which is therefore a member of $f[A]^-$. Hence $f[A^-] \subset f[A]^-$.

(g) $\rightarrow$ (h): Assuming (g), if $A = f^{-1}[B]$, then $f[A^-] \subset f[A]^- \subset B^-$ and hence $A^- \subset f^{-1}[B^-]$. That is, $f^{-1}[B]^- \subset f^{-1}[B^-]$.

(h) $\rightarrow$ (b): Assuming (h), if B is a closed subset of Y, then $f^{-1}[B]^- \subset f^{-1}[B^-] = f^{-1}[B]$ and $f^{-1}[B]$ is therefore closed. ∎

There is also a localized form of continuity which is useful.* A function f on a topological space X to a topological space Y is **continuous at a point** x iff the inverse under f of each neighborhood of $f(x)$ is a neighborhood of x. It is easy to give characterizations of the form of 3.1(e) and 3.1(f) for continuity at a point. Evidently f is continuous iff it is continuous at each point of its domain.

A **homeomorphism**, or **topological transformation**, is a continuous one-to-one map of a topological space X onto a topological space Y such that f^{-1} is also continuous. If there exists a homeomorphism of one space onto another, the two spaces are said to be **homeomorphic** and each is a **homeomorph** of the other. The identity map of a topological space onto itself is always a homeomorphism, and the inverse of a homeomorphism is again a homeomorphism. It is also evident that the composition of two homeomorphisms is a homeomorphism. Consequently the collection of topological spaces can be divided into equivalence classes such that each topological space is homeomorphic to every member of its equivalence class and to these spaces only. Two topological spaces are **topologically equivalent** iff they are homeomorphic.

* If f is defined on a subset A of a topological space, then continuity at points of the closure A^- may also be defined (see 3.D); several useful propositions result.

Two discrete spaces, X and Y, are homeomorphic iff there is a one-to-one function on X onto Y, that is, iff X and Y have the same cardinal number. This is true because every function on a discrete space is continuous, regardless of the topology of the range space. It is also true that two indiscrete spaces (the only open sets are the space and the void set) are homeomorphic iff there is a one-to-one map of one onto the other, because each function into an indiscrete space is continuous regardless of the topology of the domain space. In general, it may be quite difficult to discover whether two topological spaces are homeomorphic. The set of all real numbers, with the usual topology, is homeomorphic to the open interval $(0,1)$, with the relative topology, for the function whose value at a member x of $(0,1)$ is $(2x - 1)/x(x - 1)$ is easily proved to be a homeomorphism. However, the interval $(0,1)$ is not homeomorphic to $(0,1) \cup (1,2)$, for if f were a homeomorphism (or, in fact, just a continuous function) on $(0,1)$ with range $(0,1) \cup (1,2)$, then $f^{-1}[(0,1)]$ would be a proper open and closed subset of $(0,1)$, and $(0,1)$ is connected. This little demonstration was achieved by noticing that one of the spaces is connected, the other is not, and the homeomorph of a connected space is again connected. A property which when possessed by a topological space is also possessed by each homeomorph is a **topological invariant**. The proof that two spaces are not homeomorphic usually depends on exhibiting a topological invariant which is possessed by one but not by the other. A property which is defined in terms of the members of the space and the topology turns out, automatically, to be a topological invariant. Besides connectedness, the property of having a countable base for the topology, having a countable base for the neighborhood system of each point, being a T_1 space or being a Hausdorff space, are all topological invariants. Formally, topology is the study of topological invariants.*

PRODUCT SPACES

There is a standard way of topologizing the cartesian product of a collection of topological spaces. The construction is ex-

* A *topologist* is a man who doesn't know the difference between a doughnut and a coffee cup.

tremely important and consequently we examine minutely the properties of the topology introduced. Let X and Y be topological spaces and let $\mathfrak{B}$ be the family of all cartesian products $U \times V$ where U is an open subset of X and V is an open subset of Y. The intersection of two members of $\mathfrak{B}$ is a member of $\mathfrak{B}$, because $(U \times V) \cap (R \times S) = (U \cap R) \times (V \cap S)$, and consequently $\mathfrak{B}$ is the base for a topology for $X \times Y$ by theorem 1.11. This topology is called the **product topology** for $X \times Y$. A subset W of $X \times Y$ is open relative to the product topology if and only if for each member (x,y) of W there are open neighborhoods U of x and V of y such that $U \times V \subset W$. The spaces X and Y are **coordinate spaces,** and the functions P_0 and P_1 which carry a point (x,y) of $X \times Y$ into x and into y respectively are the **projections** into the coordinate spaces. These projections are continuous functions, for if U is open in X, $P_0^{-1}[U]$ is $U \times Y$, which is open. Continuity of the projections actually serves to characterize the product topology in the following sense. Suppose $\mathfrak{I}$ is a topology for $X \times Y$ such that each of the projections is continuous. Then if U is open in X and V is open in Y the set $U \times V$ is open relative to $\mathfrak{I}$, for $U \times V = P_0^{-1}[U] \cap P_1^{-1}[V]$ and this set is open relative to $\mathfrak{I}$ because the projections are continuous. Consequently $\mathfrak{I}$ is larger than the product topology and the product topology is therefore the smallest topology for which the projections into coordinate spaces are continuous.

There is no difficulty in extending this definition of product topology to cartesian product of any finite number of coordinate spaces. If each of $X_0, X_1, \cdots X_{n-1}$ is a topological space, then a base for the product topology for $X_0 \times X_1 \times \cdots \times X_{n-1}$ is the family of all products $U_0 \times U_1 \times \cdots U_{n-1}$ where each U_i is open in X_i. In particular, if each X_i is the set of real numbers with the usual topology, then the product space is **Euclidean n-space** E_n. The members of E_n are real-valued functions on the set $0, 1, \cdots n - 1$, the value of the function x at the integer i being x_i $(= x(i))$.

The product topology for the cartesian product of an arbitrary family of topological spaces will now be defined. Suppose we are given a set X_a for each member a of an index set A. The cartesian product $\times\{X_a: a \in A\}$ is defined to be the set of all functions x on A such that $x_a \in X_a$ for each a in A. The set X_a is the a-th coordinate set and the projection P_a of the product into the a-th coordinate set is defined by $P_a(x) = x_a$. Suppose that a topology $\mathfrak{J}_a$ is given for each coordinate set. The construction * of the product topology is motivated by the requirement that each projection P_a is to be continuous. In order to attain continuity of the projections it is necessary and sufficient that each set of the form $P_a^{-1}[U]$ be open, where U is an open subset of X_a. The family of all sets of this form is a subbase for a topology; it is clearly the smallest topology such that projections are continuous. The **product topology** is this topology. The members of the defining subbase are of the form $\{x: x_a \in U\}$ where U is open in X_a; they are, intuitively, cylinders over open sets in the coordinate spaces. It is sometimes said that elements of the subbase consist of sets obtained by "restricting the a-th coordinate to lie in an open subset of the a-th coordinate space." A base for the product topology is the family of all finite intersections of subbase elements. A member U of this base is of the form $\bigcap\{P_a^{-1}[U_a]: a \in F\} = \{x: x_a \in U_a \textit{ for each } a \textit{ in } F\}$ where F is a finite subset of A and U_a is open in X_a for each a in F. It is to be emphasized that these are *finite* intersections. It is not true that $\times\{U_a: a \in A\}$ is always open relative to the product topology if U_a is open in X_a for each a. The **product space** is the cartesian product with the product topology.

The projections of a product space into the coordinate spaces have another very useful property. A function f on a topological space X to another space Y is **open (interior)** iff the image of each open set is open; that is, if U is open in X, then $f[U]$ is open in Y.

2 THEOREM *The projection of a product space into each of its coordinate spaces is open.*

PROOF Let P_c be the projection of $\times\{X_a: a \in A\}$ into X_c. In order to show that P_c is open it is sufficient to show that the image of a neighborhood of a point x in the product is a neighborhood of $P_c(x)$, and it may be assumed that the neighborhood

* This description of the product topology is due to N. Bourbaki.

in the product space is a member of the defining base for the product topology. Suppose that $x \in V = \{y: y_a \in U_a \text{ for } a \text{ in } F\}$, where F is a finite subset of A and U_a is open in X_a for each a in F. We construct a copy of X_c which contains the point x. For $z \in X_c$ let $f(z)_c = z$, and for $a \neq c$ let $f(z)_a = x_a$. Then $P_c \circ f(z) = z$. If $c \notin F$, then clearly $f[X_c] \subset V$ and $P_c[V] = X_c$ which is open. If $c \in F$, then $f(z) \in V$ iff $z \in U_c$ and $P_c[V] = U_c$. The theorem follows. (As a matter of fact, the function f defined in this proof is a homeomorphism, a fact that is occasionally useful.) ∎

It might be conjectured that the projection of a closed set in a product space is closed. This, however, is easily seen to be false, for in the Euclidean plane the set $\{(x,y): xy = 1\}$ has a non-closed projection on each coordinate space.

There is an extremely useful characterization of continuity of a function whose range is a subset of a product space.

3 THEOREM *A function f on a topological space to a product $\times\{X_a: a \in A\}$ is continuous if and only if the composition $P_a \circ f$ is continuous for each projection P_a.*

PROOF If f is continuous, then $P_a \circ f$ is always continuous because P_a is continuous. If $P_a \circ f$ is continuous for each a, then for each open subset U of X_a the set $(P_a \circ f)^{-1}[U] = f^{-1}[P_a^{-1}[U]]$ is open. It follows that the inverse under f of each member of the defining subbase for the product topology is open, and hence (3.1c) f is continuous. ∎

Convergence in a product space can be described very simply in terms of the projections.

4 THEOREM *A net S in a product space converges to a point s if and only if its projection in each coordinate space converges to the projection of s.*

PROOF Since the projection into each coordinate space is continuous, if $\{S_n, n \in D\}$ is a net in the cartesian product $\times\{X_a: a \in A\}$ which converges to a point s, then the net $\{P_a(S_n), n \in D\}$ surely converges to $P_a(s)$. To show the converse, let $\{S_n, n \in D\}$ be a net such that $\{P_a(S_n), n \in D\}$ converges to s_a for each a in A. Then for each open neighborhood U_a of s_a, $\{P_a(S_n), n \in D\}$

is eventually in U_a, consequently $\{S_n, n \in D\}$ is eventually in $P_a^{-1}[U_a]$, and hence $\{S_n, n \in D\}$ must eventually be in each finite intersection of sets of the form $P_a^{-1}[U_a]$. Since the family of such finite intersections is a base for the neighborhood system of s in the product topology, $\{S_n, n \in D\}$ converges to s. ∎

Convergence in the product topology is called **coordinatewise convergence**, or **pointwise convergence**. The latter term is used most frequently in the case in which all coordinate spaces are identical. In this important special case the cartesian product $\times\{X: a \in A\}$ is simply the set of all functions on A to X, usually denoted X^A. A net $\{F_n, n \in D\}$ in X^A converges to f in the topology of pointwise convergence iff the net $\{F_n(a), n \in D\}$ converges to $f(a)$ for each a in A. This fact makes the terminology, "pointwise convergence," seem reasonable. The product topology is also called the **simple** topology in this case.

It is natural to ask whether the product of topological spaces inherits properties which are possessed by the coordinate spaces. For example, we might ask, in case each coordinate space is a Hausdorff space or satisfies the first or second axiom of countability, whether the product space also has these properties. The following theorems answer these questions.

5 THEOREM *The product of Hausdorff spaces is a Hausdorff space.*

PROOF If x and y are distinct members of the product $\times\{X_a: a \in A\}$, then $x_a \neq y_a$ for some a in A. If each coordinate space is Hausdorff, then there are disjoint open neighborhoods U and V of x_a and y_a respectively and $P_a^{-1}[U]$ and $P_a^{-1}[V]$ are disjoint neighborhoods of x and y in the product. ∎

Recall that an indiscrete topological space is one in which the only open sets are the void set and the space.

6 THEOREM *Let X_a be a topological space satisfying the first axiom of countability for each member a of an index set A. Then the product $\times\{X_a: a \in A\}$ satisfies the first axiom of countability if and only if all but a countable number of the spaces X_a are indiscrete.*

PROOF Suppose that B is a countable subset of A, that X_a is indiscrete for a in $A \sim B$, and that x is a point in the product space. For each a in A choose a countable base $\mathfrak{U}_a$ for the neighborhood system of x_a in X_a. Then $\mathfrak{U}_a = \{X_a\}$ if a is in $A \sim B$. Consider the family of all finite intersections of sets of the form $P_a^{-1}[U]$ for a in A and U in $\mathfrak{U}_a$. This is a countable family because $P_a^{-1}[U] = \times\{X_b\colon b \in A\}$ if $a \in A \sim B$. But the family of these finite intersections is a base for the neighborhood system of x and consequently the product space satisfies the first axiom of countability.

To prove the converse suppose that B is an uncountable subset of A such that for each a in B there is a neighborhood of x_a in X_a which is a proper subset of X_a, and suppose that there is a countable base $\mathfrak{U}$ for the neighborhood system of x. Each member U of $\mathfrak{U}$ contains a member of the defining base for the product topology, and consequently, except for a finite number of members a of A, $P_a[U] = X_a$. Since B is uncountable, there is a member a of B such that $P_a[U] = X_a$ for every U in $\mathfrak{U}$. But there is an open neighborhood V of x_a which is a proper subset of X_a, and clearly no member of $\mathfrak{U}$ is a subset of $P_a^{-1}[V]$ since each member of $\mathfrak{U}$ projects onto X_a. This is a contradiction. ∎

It is also true that the coordinate spaces inherit certain properties of a product space. If a product space is Hausdorff, so is each coordinate space, and if the product space has a countable local base at each point, then so does each coordinate space. These propositions are easy to establish, and the proofs are omitted.

7 *Notes* Tychonoff defined the product topology and proved the most important properties in two classic papers (Tychonoff [1] and [2]). His results are now among the standard tools of general topology. (See also chapter 5.) Prior to Tychonoff's work a great deal of investigation had been done on the convergence of sequences of functions relative to the topology of pointwise convergence. Many difficulties occur in this work because the topology cannot be completely described by sequential convergence, at least in the most interesting cases. (See problem 3.W.)

QUOTIENT SPACES

We begin by reviewing briefly the considerations which led to the definition of the product topology. If f is a function on a set X with values in a topological space Y, then it is always possible to assign a topology to X such that f is a continuous function. One obvious and uninteresting topology which has this property is the discrete topology; a more interesting topology is the family $\mathfrak{I}$ of all sets of the form $f^{-1}[U]$ for U open in Y. This family is evidently a topology because the inverse of a function preserves unions. Each topology, relative to which f is continuous, contains $\mathfrak{I}$ and consequently $\mathfrak{I}$ is the smallest topology for which f is continuous. If we are given a family of functions, a function f_a for each member a of an index set A, then the topology, a subbase for which is the family of all sets of the form $f_a^{-1}[U]$ for a in A and U open in the range of f_a, has precisely the same properties. This is the method by which the product topology was defined.

It is the purpose of this section to investigate the reciprocal situation. If f is a function on a topological space X with range Y, how may Y be topologized so that f is continuous? If a subset U of Y is open in a topology relative to which f is continuous, then $f^{-1}[U]$ is open in X. On the other hand, the family $\mathfrak{U}$ of all subsets U such that $f^{-1}[U]$ is open in X is a topology for Y because the inverse of an intersection (or union) of members of the family is the intersection (union) of the inverses. The topology $\mathfrak{U}$ is therefore the largest topology for Y such that the function f is continuous; it is called the **quotient topology** for Y (the quotient topology relative to f and the topology of X). A subset B of Y is closed relative to the quotient topology iff $f^{-1}[Y \sim B] = X \sim f^{-1}[B]$ is open in X. Hence B is closed iff $f^{-1}[B]$ is closed.

Without some severe limitation on f very little can be said about the quotient topology. Consequently we consider only functions belonging to one of two dual categories. Recall that f, a function on a topological space with values in another space, is open iff the image of each open set is open. A function f is said to be **closed** iff the image of each closed set is closed. It has already been observed that projection of the Euclidean plane onto its first coordinate space is an open map which is not closed, and subspaces of the plane give examples of maps which are closed but not open, and maps which are neither open nor closed. The subspace $X = \{(x,y): x = 0 \textit{ or } y = 0\}$, consisting of the two axes, is mapped onto the reals by the projection $P(x,y) = x$. The image of a small neighborhood of (0,1) is mapped into the single point 0. Consequently P is not an open map on X, but it is easy to verify that it is closed. If (0,0) is removed, leaving $X \sim \{(0,0)\}$, then P on this subspace is neither open nor closed (the image of the closed set $\{(x,y): y = 0 \textit{ and } x \neq 0\}$ is not closed).

It is apparent from the definition that the notion of an open or a closed map depends on the topology of the range space. However, if a map f is continuous and either open or closed, then the topology of the range is entirely determined by the map f and the topology of the domain.

8 THEOREM *If f is a continuous map of the topological space $(X,\mathfrak{I})$ onto the space $(Y,\mathfrak{U})$ such that f is either open or closed, then $\mathfrak{U}$ is the quotient topology.*

PROOF If f is an open map and U is a subset of Y such that $f^{-1}[U]$ is open, then $U = f[f^{-1}[U]]$ is open relative to $\mathfrak{U}$. Consequently, if f is open, each set open relative to the quotient topology is open relative to $\mathfrak{U}$, and the quotient topology is smaller than $\mathfrak{U}$. If f is continuous as well as open, then since the quotient topology is the largest for which f is continuous, $\mathfrak{U}$ is the quotient topology. To prove the theorem for a closed function f it is only necessary to replace "open" by "closed" in each of the preceding statements. ∎

If f is a function on a topological space to a product space, then f is continuous iff the composition of f with each projection is continuous. There is an analogue of this proposition for quotient spaces.

9 THEOREM *Let f be a continuous map of a space X onto a space Y and let Y have the quotient topology. Then a function g on Y to a topological space Z is continuous if and only if the composition $g \circ f$ is continuous.*

PROOF If U is open in Z and $g \circ f$ is continuous, then $(g \circ f)^{-1}[U] = f^{-1}[g^{-1}[U]]$, which is open in X, and $g^{-1}[U]$ is therefore open in Y by the definition of the quotient topology. The converse is clear. ▌

It is almost evident that the quotient topology and the properties of open or closed maps have little to do with the range space. In fact, if f is a continuous map of a topological space X onto a space Y with the quotient topology, then a topological copy of Y may be reconstructed from X, its topology, and the family of all sets of the form $f^{-1}[y]$ for y in Y. The construction goes as follows. Let $\mathfrak{D}$ be the family of all subsets of X of the form $f^{-1}[y]$ for y in Y, and let P be the function on X to $\mathfrak{D}$ whose value at x is $f^{-1}[f(x)]$. For each member y of Y let $g(y) = f^{-1}[y]$, so that g is a one-to-one map of Y onto $\mathfrak{D}$. Then $g \circ f = P$, and $f = g^{-1} \circ P$. If $\mathfrak{D}$ is assigned the quotient topology (relative to P) the preceding theorem 3.9 shows the continuity of g (since $g \circ f = P$) and the continuity of g^{-1} (since $g^{-1} \circ P = f$). Consequently g is a homeomorphism.

The preceding remarks show that the range space is essentially extraneous to the discussion, and the remaining theorems of the section will be formulated so as to display this fact. As a preliminary we consider briefly the families of subsets of a fixed set X. A **decomposition (partition)** of X is a disjoint family $\mathfrak{D}$ of subsets of X whose union is X. The **projection (quotient map)** of X onto the decomposition $\mathfrak{D}$ is the function P whose value at x is the unique member of $\mathfrak{D}$ to which x belongs. There is an equivalent way of describing a decomposition. Given $\mathfrak{D}$, define a relation R on X by agreeing that a point x is R related to a point y iff x and y belong to the same member of the decomposition. Formally, the relation R **of the decomposition** $\mathfrak{D}$ is the subset of $X \times X$ consisting of all pairs (x,y) such that x and y belong to the same member of $\mathfrak{D}$, or, briefly, $R = \bigcup\{D \times D: D \in \mathfrak{D}\}$. If P is the projection of X onto $\mathfrak{D}$, then $R = \{(x,y): P(x) = P(y)\}$. The relation R is an equivalence relation: that is, it is reflexive, symmetric, and transitive (see chapter 0). Reciprocally, each equivalence relation on X defines a family of subsets (the equivalence classes) which is a decomposition of X. If R is an equivalence relation on X, then X/R is defined to be the family of equivalence classes. If A is a subset of X, then $R[A]$ is the set of all points which are R relatives of points of A; that is, $R[A] = \{y\colon (x,y) \in R \textit{ for some } x \textit{ in } A\}$. Equivalently, $R[A] = \bigcup\{D\colon D \in X/R \textit{ and } D \cap A \textit{ is non-void}\}$. If x is a point of X, then we abbreviate $R[\{x\}]$ as $R[x]$. The set $R[x]$ is the equivalence class to which x belongs, and if P is the projection of X onto the decomposition, then $P(x) = R[x]$.

We assume for the rest of the section that X is a fixed topological space, R is an equivalence relation on X, and that P is the projection of X onto the family X/R of equivalence classes. The **quotient space** is the family X/R with the quotient topology (relative to P). If $\mathfrak{a} \subset X/R$, then $P^{-1}[\mathfrak{a}] = \bigcup\{A\colon A \in \mathfrak{a}\}$ and hence $\mathfrak{a}$ is open (closed) relative to the quotient topology iff $\bigcup\{A\colon A \in \mathfrak{a}\}$ is open (respectively closed) in X.

10 Theorem *Let P be the projection of the topological space X onto the quotient space X/R. Then the following statements are equivalent.*

(a) *P is an open mapping.*
(b) *If A is an open subset of X, then $R[A]$ is open.*
(c) *If A is a closed subset of X, then the union of all members of X/R which are subsets of A is closed.*

If "open" and "closed" are interchanged in (a), (b), *and* (c) *the resulting statements are equivalent.*

PROOF It is first shown that (a) is equivalent to (b). For each subset A of X, the set $R[A] = P^{-1}[P[A]]$. If P is open and A is open, then, since P is continuous, $P^{-1}[P[A]]$ is open. If $P^{-1}[P[A]]$ is open for each open set A, then, since by the definition of the quotient topology $P[A]$ is open, P is an open mapping. To prove (b) equivalent to (c) notice that the union of all members of X/R which are subsets of A is $X \sim R[X \sim A]$, and this set is closed for each closed A iff $R[X \sim A]$ is open whenever $X \sim A$ is open. A proof of the dual proposition is obtained by interchanging "open" and "closed" throughout. ▌

If X is a Hausdorff space or satisfies one of the axioms of countability it is natural to ask whether the quotient space X/R necessarily inherits these properties. Without some drastic restriction the answer is "no." For example, if X is the set of real numbers with the usual topology and R is the set of pairs (x,y) such that $x - y$ is rational, then the quotient space X/R is indiscrete, and the projection P of X onto X/R is open. Consequently an open map may carry a Hausdorff space into a non-Hausdorff space. An example of a closed map which carries a Hausdorff space onto a non-Hausdorff space or carries a space satisfying the first axiom of countability onto a space which fails to satisfy the axiom, is slightly more complex but not difficult. (3.R, 4.G.) There is an additional hypothesis which is sometimes useful. It is sometimes assumed that R, which is a set of ordered pairs, is closed in the product space $X \times X$. This condition may be restated: if x and y are members of X which are not R related, then there is a neighborhood W of (x,y) in the product space $X \times X$ which is disjoint from R. Such a neighborhood W contains a neighborhood of the form $U \times V$, where U and V are neighborhoods of x and y respectively, and $U \times V$ is disjoint from R iff there is no point of U which is R related to a point of V. That is, R is closed in $X \times X$ iff, whenever x and y are points of X which are not R related, then there are neighborhoods U and V of x and y respectively such that no point of U is R related to a point of V. Equivalently, no member of X/R intersects both U and V.

11 THEOREM *If the quotient space X/R is Hausdorff, then R is closed in the product space $X \times X$.*

If the projection P of a space X onto the quotient space X/R is open, and R is closed in $X \times X$, then X/R is a Hausdorff space.

PROOF If X/R is a Hausdorff space and $(x,y) \notin R$, then $P(x) \neq P(y)$ and there are disjoint open neighborhoods U of $P(x)$ and V of $P(y)$. The sets $P^{-1}[U]$ and $P^{-1}[V]$ are open, and since their images under P are disjoint, no point of $P^{-1}[U]$ is R related to a point of $P^{-1}[V]$. Therefore $P^{-1}[U] \times P^{-1}[V]$ is a neighborhood of (x,y) which is disjoint from R, and R is closed. The first statement of the theorem is proved. Suppose now that P is open, R is closed in $X \times X$, and $P(x)$ and $P(y)$ are distinct members of X/R. Then x is not R related to y and, since R is closed, there are open neighborhoods U and V of x and y respectively such that no point of U is R related to a point of V. Hence the images of U and V are disjoint, and since P is open they are open neighborhoods of $P(x)$ and $P(y)$ respectively. ∎

Closed maps have been studied rather extensively under a different name. A decomposition $\mathfrak{D}$ of a topological space X is **upper semi-continuous** iff for each D in $\mathfrak{D}$ and each open set U containing D there is an open set V such that $D \subset V \subset U$ and V is the union of members of $\mathfrak{D}$. (See problem 3.F for the origin of the term "upper semi-continuous.")

12 THEOREM *A decomposition $\mathfrak{D}$ of a topological space X is upper semi-continuous if and only if the projection P of X onto $\mathfrak{D}$ is closed.*

PROOF According to theorem 3.10, P is a closed map iff for each open subset U of X it is true that the union V of all members of $\mathfrak{D}$ which are subsets of U is an open set. If P is closed, $D \in \mathfrak{D}$ and $D \subset U$, then V is the required open set and hence $\mathfrak{D}$ is upper semi-continuous. To prove the converse suppose that $\mathfrak{D}$ is upper semi-continuous and that U is an open subset of X. Let V be the union of all members of $\mathfrak{D}$ which are subsets of U. If $x \in V$, then $x \in D \subset U$ for some D in $\mathfrak{D}$. Hence by upper semi-continuity there is an open set W, the union of members of $\mathfrak{D}$, such that $D \subset W \subset U$. Then W is a subset of V and consequently V is a neighborhood of x. The set V is open because it is a neighborhood of each of its points, and it follows from 3.10 that P is a closed map. ∎

If A and B are disjoint closed subsets of X one may define the decomposition $\mathfrak{D}$ of X whose members are A, B, and all sets $\{x\}$ for x in $X \sim (A \cup B)$. The quotient space of this decomposition is sometimes called "the space obtained by identifying all points of A and identifying all points of B." It is very easy to verify that $\mathfrak{D}$ is upper semi-continuous, and if X is Hausdorff the relation $R = \bigcup\{D \times D: D \in \mathfrak{D}\}$ is closed in $X \times X$. One might suppose that with this simple construction the quotient space might inherit pleasant properties of the space X. Unfortunately, even in this case X may be Hausdorff or satisfy the first or second countability axiom and the corresponding proposition for the quotient space be false.

13 *Note* The notion of upper semi-continuous collection was introduced by R. L. Moore in the late twenties; open mappings were first studied intensively by Aronszajn a little later (Aronszajn [2]). Many of the results of the preceding section will be found in Whyburn [2].

PROBLEMS

A CONNECTED SPACES

The image under a continuous map of a connected space is connected.

B THEOREM ON CONTINUITY

Let A and B be subsets of a topological space X such that $X = A \cup B$, and $A \sim B$ and $B \sim A$ are separated. If f is a function on X which is continuous on A and continuous on B, then f is continuous on X. (See 1.19.)

C EXERCISE ON CONTINUOUS FUNCTIONS

If f and g are continuous functions on a topological space X with values in a Hausdorff space Y, then the set of all points x in X such that $f(x) = g(x)$ is closed. Consequently, if f and g agree on a dense subset of X ($f(x) = g(x)$ for x belonging to a dense subset of X), then $f = g$.

D CONTINUITY AT A POINT; CONTINUOUS EXTENSION

Let f be defined on a subset X_0 of a topological space X with values in a Hausdorff space Y; then f *is continuous at* x iff x belongs to the closure of X_0 and for some member y of the range the inverse of each neighborhood of y is the intersection of X_0 and a neighborhood of x.

(a) A function f is continuous at x iff $x \in \overline{X_0}$ and whenever S and T are nets in X_0 converging to x then $f \circ S$ and $f \circ T$ converge to the same point of Y.

(b) Let C be the set of points at which f is continuous and let f' be the function on C whose value at a point x is the member y of the range space which is given by the definition of continuity at a point (more precisely, the graph of f' is the intersection of $C \times Y$ with the closure of the graph of f). The function f' has the property: If U is open in X, then $f'[U] \subset f[U]^-$. The function f' is continuous, provided Y has the property: The family of closed neighborhoods of each point of Y is a base for the neighborhood system of the point. (Such topological spaces are called *regular*. The requirement that Y be regular is essential here, as shown by Bourbaki and Dieudonné [1].)

E EXERCISE ON REAL-VALUED CONTINUOUS FUNCTIONS

Let f and g be real-valued functions on a topological space, let f and g be continuous with respect to the usual topology for the real numbers, and let a be a fixed real number.

(a) The function af, whose value at x is $af(x)$, is continuous. (Show that the function which carries the real number r into ar is continuous, and use the fact that the composition of continuous functions is continuous.)

(b) The function $|f|$, whose value at x is $|f(x)|$, is continuous.

(c) If $F(x) = (f(x), g(x))$, then F is continuous relative to the usual topology for the Euclidean plane. (Verify that F followed by projection into a coordinate space is continuous.)

(d) The functions $f + g$, $f - g$, and $f \cdot g$ are continuous, and if g is never zero, then f/g is continuous. (First show that $+$, $-$, and $\cdot$ are continuous functions on the Euclidean plane to the space of real numbers. (See also 3.S.))

(e) The functions $\max [f,g] = [(f + g) + |f - g|]/2$ and $\min [f,g] = [(f + g) - |f - g|]/2$ are continuous.

F UPPER SEMI-CONTINUOUS FUNCTIONS

A real-valued function f on a topological space X is *upper semi-continuous* iff the set $\{x: f(x) \geqq a\}$ is closed for each real number a. The *upper* topology $\mathfrak{U}$ for the set R of real numbers consists of the void set, R, and all sets of the form $\{t: t < a\}$ for a in R. If $\{S_n, n \in D\}$ is a net of real numbers, then $\lim\sup \{S_n: n \in D\}$ is defined to be $\lim \{\sup \{S_m: m \in D \text{ and } m \geqq n\}: n \in D\}$ where this limit is taken relative to the usual topology for the real numbers.

(a) A net $\{S_n, n \in D\}$ of real numbers converges to s relative to $\mathfrak{U}$ iff $\lim\sup \{S_n: n \in D\} \geqq s$.

(b) If f is a real-valued function on X, then f is upper semi-continuous iff f is continuous relative to the upper topology $\mathfrak{U}$, and this is the case iff $\lim\sup \{f(x_n): n \in D\} \leqq f(x)$ whenever $\{x_n, n \in D\}$ is a net in X converging to a point x.

(c) If f and g are upper semi-continuous and t is a non-negative real number, then $f + g$ and tf are upper semi-continuous.

(d) If F is a family of upper semi-continuous functions such that $i(x) = \inf \{f(x): f \in F\}$ exists for each x in X, then i is upper semi-continuous. (Observe that $\{x: i(x) \geqq a\} = \bigcap \{\{x: f(x) \geqq a\}: f \in F\}$.)

(e) If f is a bounded real-valued function on X, then there is a smallest upper semi-continuous function f^- such that $f^- \geqq f$. If $\mathfrak{V}$ is the family of neighborhoods of a point x and $S_V = \sup \{f(y): y \in V\}$, then $f^-(x) = \lim \{S_V, V \in \mathfrak{V}, \subset\}$.

(f) A real-valued function g is called *lower semi-continuous* iff $-g$ is upper semi-continuous. If f is a bounded real-valued function, let $f_- = -(-f)^-$ and let the *oscillation* of f, Q_f, be defined by $Q_f(x) = f^-(x) - f_-(x)$ for x in X. Then Q_f is upper semi-continuous, and f is continuous iff $Q_f(x) = 0$ for all x in X.

(g) Let f be a non-negative real valued function on X, let R have the usual topology, and let $G = \{(x,t): 0 \leqq t \leqq f(x)\}$ have the relativized product topology of $X \times R$. Let $\mathfrak{D}$ be the decomposition of G into "vertical slices"; that is, sets of the form $(\{x\} \times R) \cap G$. If the decomposition $\mathfrak{D}$ is upper semi-continuous, then f is upper semi-continuous. (The converse is also true but the simplest proof requires theorem 5.12.)

G EXERCISE ON TOPOLOGICAL EQUIVALENCE

(a) Any two open intervals of real numbers, with the relativized usual topology for the reals, are homeomorphic.

(b) Any two closed intervals are homeomorphic, and any two half-open intervals are homeomorphic.

(c) No open interval is homeomorphic to a closed or half-open interval, and no closed interval is homeomorphic to a half-open interval.

(d) The subspace $\{(x,y): x^2 + y^2 = 1\}$ of the Euclidean plane is not homeomorphic to a subspace of the space of real numbers.

(Certain of the foregoing spaces have one or more points x such that the complement of $\{x\}$ is connected.)

H HOMEOMORPHISMS AND ONE-TO-ONE CONTINUOUS MAPS

Given two topological spaces X and Y, a one-to-one continuous map of Y onto X and a one-to-one continuous map of X onto Y: then X and Y are not necessarily homeomorphic. (Let the space X consist of a countable number of disjoint half-open intervals and a countable number of *isolated* points (points x such that $\{x\}$ is open). Let Y consist of a countable number of open intervals and a countable number of isolated points. Observe that a countable number of half-open intervals can be mapped in a one-to-one continuous way onto an open interval. I believe this example is due to R. H. Fox.)

I CONTINUITY IN EACH OF TWO VARIABLES

Let X and Y be topological spaces, $X \times Y$ the product space, and let f be a function on $X \times Y$ to another topological space. Then $f(x,y)$

is **continuous in** x iff for each y the function $f(\ ,y)$ whose value at x is $f(x,y)$, is continuous. Similarly, $f(x,y)$ is continuous in y iff for each $x \in X$, the function $f(x,\ )$ such that $f(x,\ )(y) = f(x,y)$, is continuous. If f is continuous on the product space, then $f(x,y)$ is continuous in x and in y, but the converse is false. (The classical example is the real-valued function f on the Euclidean plane such that $f(x,y) = xy/(x^2 + y^2)$ and $f(0,0) = 0$.)

J EXERCISE ON EUCLIDEAN n-SPACE

A subset A of Euclidean n-space E_n is *convex* iff for every pair x and y of points of A and every real number t, such that $0 \leqq t \leqq 1$, the point $tx + (1 - t)y$ is a member of A. (We define $(tx + (1 - t)y)_i = tx_i + (1 - t)y_i$.) Then any two non-void open convex subsets of E_n are homeomorphic. What of closed convex subsets?

K EXERCISE ON CLOSURE, INTERIOR AND BOUNDARY IN PRODUCTS

Let X and Y be topological spaces and let $X \times Y$ be the product space. For each set C let C^b be the boundary of C. Then, if A and B are subsets of X and Y respectively,

(a) $(A \times B)^- = A^- \times B^-$,

(b) $(A \times B)^0 = A^0 \times B^0$, and

(c) $(A \times B)^b = (A \times B)^- \sim (A \times B)^0 = ((A^b \cup A^0) \times (B^b \cup B^0)) \sim (A^0 \times B^0) = (A^b \times B^b) \cup (A^b \times B^0) \cup (A^0 \times B^b) = (A^b \times B^-) \cup (A^- \times B^b)$.

L EXERCISE ON PRODUCT SPACES

Suppose that, for each member a of an index set A, X_a is a topological space. Let B and C be disjoint subsets of A such that $A = B \cup C$. Then the product space $\times\{X_b: b \in B\} \times \times\{X_c: c \in C\}$ is homeomorphic to the product space $\times\{X_a: a \in A\}$. For each fixed topological space X the product X^A is homeomorphic to $X^B \times X^C$ and $(X^B)^C$ is homeomorphic to $X^{B \times C}$, all spaces being given the product topology.

M PRODUCT OF SPACES WITH COUNTABLE BASES

The product topology has a countable base iff the topology of each coordinate space has a countable base and all but a countable number of the coordinate spaces are indiscrete.

N EXAMPLE ON PRODUCTS AND SEPARABILITY

Let Q be the closed unit interval and let X be the product space Q^Q. Let A be the subset of X consisting of characteristic functions of points;

more precisely, $x \in A$ iff for some q in Q, $x(q) = 1$ and x is zero on $Q \sim \{q\}$.

(a) The space X is separable. (The set of all x in X with finite range (sometimes called step functions) are dense in X. There is also a countable subset of this set which is dense in X.)

(b) The set A, with the relative topology, is discrete and not separable.

(c) There is a single accumulation point x of A in X, and if U is a neighborhood of x, then $A \sim U$ is finite.

O PRODUCT OF CONNECTED SPACES

The product of an arbitrary family of connected topological spaces is connected. (Fix a point x in the product, and let A be the set of all points y such that there is a connected subset to which both x and y belong. Show that A is dense.)

P EXERCISE ON T_1-SPACES

The product of T_1-spaces is a T_1-space. If $\mathfrak{D}$ is a decomposition of a topological space, then the quotient space is T_1 iff the members of $\mathfrak{D}$ are closed.

Q EXERCISE ON QUOTIENT SPACES

The projection of a topological space X into the quotient space X/R is a closed map iff, for each subset A of X, $R[A]^- \subset R[A^-]$. The projection is open iff $R[A^0] \subset R[A]^0$ for each subset A. ($^-$ and ^{0} are the closure and interior respectively.)

R EXAMPLE ON QUOTIENT SPACES AND DIAGONAL SEQUENCES

Let X be the Euclidean plane with the usual topology, let A be the set of all points (x,y) with $y = 0$, and let the decomposition $\mathfrak{D}$ consist of A and all sets $\{(x,y)\}$ with $(x,y) \notin A$. Then $\mathfrak{D}$, with the quotient topology, has the following properties.

(a) The projection of X onto the quotient space is closed.

(b) There is a countable number of neighborhoods of A whose intersection is $\{A\}$.

(c) For each non-negative integer m the sequence $\{(m,1/(n+1)), n \in \omega\}$ converges, in the quotient space, to A. If $\{N_n, n \in \omega\}$ is a subsequence of the sequence of non-negative integers, then the sequence $\{(n,1/(N_n+1)), n \in \omega\}$ does not converge to A. (The latter might be called a diagonal of the original family of sequences.)

(d) The quotient space does not satisfy the first axiom of countability.

Note This example is due to R. S. Novosad.

S TOPOLOGICAL GROUPS

A triple $(G,\cdot,\mathfrak{I})$ is a *topological group* iff $(G,\cdot)$ is a group, $(G,\mathfrak{I})$ is a topological space, and the function whose value at a member (x,y) of $G \times G$ is $x \cdot y^{-1}$ is continuous relative to the product topology for $G \times G$. When confusion is unlikely all mention of the group operation $\cdot$ and the topology $\mathfrak{I}$ is suppressed, and we say "G is a topological group." If X and Y are subsets of G, then $X \cdot Y$ is the set of all points z of G such that $z = x \cdot y$ for some x in X and some y in Y. If x is a point of G we abbreviate $\{x\} \cdot Y$ and $Y \cdot \{x\}$ to $x \cdot Y$ and $Y \cdot x$ respectively, and Y^{-1} is defined to be $\{x: x^{-1} \varepsilon Y\}$.

(a) If X, Y, and Z are subsets of G, then $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ and $(X \cdot Y)^{-1} = Y^{-1} \cdot X^{-1}$.

(b) Let $(G,\cdot)$ be a group and $\mathfrak{I}$ a topology for G. Then $(G,\cdot,\mathfrak{I})$ is a topological group iff for each x and y in G and each neighborhood W of $x \cdot y^{-1}$ there are neighborhoods U of x and V of y such that $U \cdot V^{-1} \subset W$. Equivalently, $(G,\cdot,\mathfrak{I})$ is a topological group iff i and m are continuous, where $i(x) = x^{-1}$ and $m(x,y) = x \cdot y$.

(c) If G is a topological group, then i, where $i(x) = x^{-1}$, is a homeomorphism of G onto G. For each a in G both L_a and R_a (called the *left* and *right translations* by a) are homeomorphisms, where $L_a(x) = a \cdot x$ and $R_a(x) = x \cdot a$.

It is very important to notice that the topology of a topological group is determined by the neighborhood system of a member of the group. This fact (precisely stated below) permits the "localization" of many notions.

(d) If G is a topological group and $\mathfrak{U}$ is the neighborhood system of the identity, then a subset A of G is open iff $x^{-1} \cdot A \,\varepsilon\, \mathfrak{U}$ for each x in A or equivalently if $A \cdot x^{-1} \,\varepsilon\, \mathfrak{U}$ for each x in A. The closure of the subset A is $\bigcap \{U \cdot A: U \,\varepsilon\, \mathfrak{U}\} = \bigcap \{A \cdot U: U \,\varepsilon\, \mathfrak{U}\}$. (Notice that $x \,\varepsilon\, U \cdot A$ iff $(U^{-1} \cdot x) \cap A$ is not void.)

(e) The family $\mathfrak{U}$ of neighborhoods of the identity e of a topological group has the properties:

(i) if U and V belong to $\mathfrak{U}$, then $U \cap V \,\varepsilon\, \mathfrak{U}$;
(ii) if $U \,\varepsilon\, \mathfrak{U}$ and $U \subset V$, then $V \,\varepsilon\, \mathfrak{U}$;
(iii) if $U \,\varepsilon\, \mathfrak{U}$, then for some $V \,\varepsilon\, \mathfrak{U}$, $V \cdot V^{-1} \subset U$; and
(iv) for each U in $\mathfrak{U}$ and each x in G, $x \cdot U \cdot x^{-1} \,\varepsilon\, \mathfrak{U}$.

On the other hand, given a group G and a non-void family $\mathfrak{U}$ of non-void subsets satisfying these four propositions there is a unique topology $\mathfrak{J}$ for G such that $(G,\cdot,\mathfrak{J})$ is a topological group and $\mathfrak{U}$ is the neighborhood system of the identity element.

(f) Every group, with the discrete topology or with the indiscrete topology, is a topological group If G is the set of real numbers, then $(G,+,\mathfrak{J})$, where $\mathfrak{J}$ is the usual topology, is a topological group and $(G \sim \{0\},\cdot,\mathfrak{J})$ is also a topological group. If G is the set of all integers, p is a prime and $\mathfrak{U}$ is the family of all subsets U of G such that for some positive integer k every integral multiple of p^k belongs to U, then $\mathfrak{U}$ is the neighborhood system of 0 relative to a topology $\mathfrak{J}$ such that $(G,+,\mathfrak{J})$ is a topological group.

(g) A topological group is a Hausdorff space whenever it is T_0-space. (That is, if x and y are distinct elements there is either a neighborhood of x to which y does not belong or the reverse. Observe that if $x \notin U\cdot y$, then $x\cdot y^{-1} \notin U$, and if $V^{-1}\cdot V \subset U$, then $V\cdot x \cap V\cdot y$ is void.)

(h) If U is open and X is an arbitrary subset of a topological group, then $U\cdot X$ and $X\cdot U$ are open. However, both X and Y may be closed subsets and $X\cdot Y$ may fail to be closed. (Consider the Euclidean plane with the usual addition with $X = Y = \{(x,y): y = 1/x^2\}$.)

(i) A cartesian product $\times\{G_a: a \in A\}$ of groups is a group under the operation: $(x\cdot y)_a = x_a\cdot y_a$ for each a in A. The product, with the product topology, is a topological group and the projection into each coordinate space is a continuous open homomorphism.*

Note Bourbaki [1], Pontrjagin [1], and Weil [2] are standard references on topological groups; see also Chevalley [1].

T SUBGROUPS OF A TOPOLOGICAL GROUP

(a) A subgroup of a topological group is, with the relative topology, a topological group.

(b) The closure of a subgroup is a subgroup and the closure of an invariant subgroup is invariant (invariant = normal = distinguished).

(c) Every subgroup with non-void interior is open and closed. A subgroup H is either closed or $H^- \sim H$ is dense in H^-.

(d) The smallest subgroup which contains a fixed open subset of a topological group is both open and closed.

(e) The component of the identity in a topological group is an invariant subgroup.

* Some authors use the term "representation" to mean continuous homomorphism, and the term "homomorphism" to mean a continuous homomorphism which is an open map onto its range.

(f) A discrete (with the relative topology) normal subgroup of a connected topological group is a subset of the center. (For a fixed member h of the subgroup H consider the map of G into H which carries x into $x^{-1} \cdot h \cdot x$.)

U FACTOR GROUPS AND HOMOMORPHISMS

Let G be a topological group, H a subgroup, G/H the family of left cosets (sets of the form $x \cdot H$ for some x in G). Then G/H with the quotient topology is a *homogeneous space.* If H is an invariant subgroup, then G/H is a group, called the factor group or quotient group.

(a) The projection of a topological group G onto the homogeneous quotient space G/H is open and continuous. (Show that the union of all left cosets which intersect an open set U is $U \cdot H$ and apply 3.10.)

(b) If H is an invariant subgroup, then G/H, with the quotient topology, is a topological group and the projection is a continuous, open homomorphism.

(c) The map of the homogeneous space which carries an element A into $a \cdot A$, where a is a fixed member of G, is a homeomorphism.

(d) If f is a homomorphism of a topological group G into another group H, then f is continuous iff the inverse of a neighborhood of the identity element of H is a neighborhood of the identity of G.

(e) If f is a continuous homomorphism of the topological group G into a topological group J, then the map of G onto $f[G]$, where $f[G]$ has the quotient topology, is a continuous open homomorphism, and the identity map of $f[G]$, with the quotient topology, into J is continuous. Hence each continuous homomorphism may be "factored" into a continuous open homomorphism followed by a continuous one-to-one homomorphism. If f is a continuous open homomorphism of G onto J, then J is topologically isomorphic to G/K where K is the kernel of f.

(f) If $J \subset H \subset G$ and J and H are invariant subgroups of G, then H/J is a subgroup of G/J, the quotient topology for H/J is the relative quotient topology for G/J, and the map of G/J into G/H which carries A into $A \cdot H$ is continuous and open, and hence $(G/J)/(H/J)$ is topologically isomorphic to G/H.

V BOX SPACES

A base for the *box* topology for the cartesian product $\times\{X_a : a \in A\}$ is the family of all sets $\times\{U_a : a \in A\}$ where U_a is open in X_a for each a in A. Hence the cartesian product of open sets is open relative to the box topology.

(a) Projection into each coordinate space is, relative to the box topology, continuous and open.

(b) Let Y be the cartesian product of the real numbers an infinite number of times; that is, $Y = R^A$, where R is the set of real numbers and A is an infinite set. With the box topology Y does not satisfy the first countability axiom, and the component of Y to which a point y belongs is the set of all points x such that $\{a: x_a \neq y_a\}$ is finite. (Let x and y be points of Y whose coordinates differ for an infinite set $a_0, a_1, \cdots, a_p \cdots$ of members of A. Let Z be the set of all z in Y such that for some k, $p| z(a_p) - x(a_p) |/| x(a_p) - y(a_p) | < k$ for all p. Then Z is open and closed, $x \in Z$ and $y \notin Z$.)

(c) Prove the results of (b) for the product of an infinite number of connected, Hausdorff topological groups, each of which contains at least two points. Show first that the product of topological groups is, with the box topology, a topological group.

W FUNCTIONALS ON REAL LINEAR SPACES

Let $(X,+,\cdot)$ be a real linear space. A real-valued linear function on X is called a *linear functional.* The set Z of all linear functionals on X is, with the natural definition of addition and scalar multiplication, a real linear space. It is clear that Z is a subset of the product $R^X = \times\{R: x \in X\}$, where R is the set of real numbers. The relativized product topology for Z is called the *weak** or w^*-topology (the *simple* topology). (The space Z is a subgroup of R^X, which is a topological group according to 3.S(i); however, the following results do not require the propositions on topological groups.)

The following propositions characterize w^*-dense subspaces of Z and w^*-continuous linear functionals.

(a) If $f, g_1, \cdots, g_n$ are members of Z and $f(x) = 0$ whenever $g_i(x) = 0$ for each i, then there are real numbers $a_1, \cdots, a_n$ such that $f = \sum\{a_i g_i: i = 1, \cdots, n\}$. (Consider the map G of X into E^n defined by $(G(x))_i = g_i(x)$. Show that there is an induced map F (see chapter 0) such that $f = F \circ G$.)

(b) *Density lemma* Let Y be a linear subspace of Z such that for each non-zero member x of X there is g in Y such that $g(x) \neq 0$. Then Y is w^*-dense in Z. (To show that $f \in Y^-$ it is necessary to prove that for each finite subset $x_1, \cdots, x_n$ of X there is a member of Y which approximates f at each of $x_1, \cdots, x_n$. Show there is g in Y such that $g(x_i) = f(x_i)$ for each i, $i = 1, \cdots, n$.)

(c) *Evaluation theorem* A linear functional F on Z is w^*-continuous iff it is an evaluation; that is, iff for some x in X it is true that

$F(g) = g(x)$ for all g in Z. (If F is w^*-continuous, then for some $x_1, \cdots, x_n$ in X and some positive real numbers $r_1, \cdots, r_n$ it is true that $|F(g)| < 1$ whenever $|g(x_i)| < r_i$ for each i. Show that, if $g(x_i) = 0$ for each i, then $F(g) = 0$.)

Notes The concept of the product topology grew out of the study of sequential convergence relative to the w^*-topology. The latter has been studied extensively (see, for example, Banach [1]). There were several awkward situations which arose in this study, which have been somewhat clarified by further topological developments. One might define the sequential closure of a set to be the union of the set and all limit points of sequences in the set, and agree that a set is sequentially closed iff it is identical with its sequential closure. Then it is not hard to see that a set may be sequentially closed relative to the w^*-topology but may fail to be w^*-closed. This is not a serious criticism if sequential convergence is the object under study. However, the really damaging fact is that the sequential closure of a set may fail to be sequentially closed; that is, sequential closure is not a Kuratowski closure operator. Because of this the machinery of general topology does not apply to the sequential closure operator, and *ad hoc* arguments are necessary for each conclusion. See Banach [1; 208 ff] for further discussion and examples.

X REAL LINEAR TOPOLOGICAL SPACES

A *real linear topological space* (r.l.t.s) is a quadruple $(X,+,\cdot,\mathfrak{J})$ such that $(X,+,\cdot)$ is a real linear space, $(X,+,\mathfrak{J})$ is a topological group, and the scalar multiplication, $\cdot$, is a continuous function on $X \times$ (*real numbers*) to X. Recall that a subset K of a real linear space is convex iff, whenever $0 \leqq t \leqq 1$ and x and y are members of K, then $t\cdot x + (1 - t)\cdot y \in K$.

(a) The function which, for a fixed real number a, $a \neq 0$, carries each member x of a real linear topological space into $a\cdot x$ is a homeomorphism.

(b) The cartesian product of real linear topological spaces is, with addition and scalar multiplication defined coordinate-wise, and with the product topology, a r.l.t.s.

(c) If Y is a linear subspace of the r.l.t.s. X, then Y, with the relative topology, is a r.l.t.s., and X/Y, with the quotient topology, is a r.l.t.s.

(d) Let K be a convex subset of a r.l.t.s. X and f a linear functional on X. Then f is continuous on K iff, for each real number t, the set $f^{-1}[t] \cap K$ is closed in K. (If $\{x_n, n \in D\}$ is a net in K, converging to a member x of K such that $\{f(x_n), n \in D\}$ fails to converge to $f(x)$, choose for n in a cofinal subset of D a point y_n on the segment from x_n to x such that $f(y_n)$ is a constant different from $f(x)$.)

(e) If f is a real-valued linear function (that is, a linear functional) on a r.l.t.s. X, then f is continuous iff $\{x: f(x) = 0\}$ is closed.

Notes The concept of a linear topological space is relatively recent (Kolmogoroff [1] and v. Neumann [1]); it is a notion which grew out of the study of the weak and weak* topologies for a Banach space and its adjoint. Much of the elementary theory of linear topological spaces is a direct application of the theory of topological groups; the results which distinguish the theory from that of topological groups all depend on convexity arguments. (This is a perfectly normal state of affairs; the chief use of the scalar multiplication, which is the only distinguishing feature, is in convexity arguments.) The few results on r.l.t. spaces which occur in the problems of this book do not constitute an adequate introduction to the theory because we do not list the propositions on convexity which are essential to a serious study. The following are suggested as reading references: Bourbaki [3], Nachbin [1], and Nakano [1]. The first of these contains a study of linear topological spaces over a topologized (not necessarily commutative) field.

Chapter 4

EMBEDDING AND METRIZATION

The development of general topology has followed an evolutionary pattern which occurs frequently in mathematics. One begins by observing similarities and recurring arguments in several situations which superficially seem to bear little resemblance to each other. We then attempt to isolate the concepts and methods which are common to the various examples, and if the analysis has been sufficiently penetrating we may find a theory containing many or all of our examples, which in itself seems worthy of study. It is in precisely this way, after much experimentation, that the notion of a topological space was developed. It is a natural product of a continuing consolidation, abstraction, and extension process. Each such abstraction, if it is to contain the examples from which it was derived in more than a formal way, must be tested to find whether we have really found the central ideas involved. This testing is usually done by comparing the abstractly constructed object with the objects from which it derived. In this case we want to find whether a topological space, at least under some reasonable restrictions, must necessarily be one of the particular concrete spaces from which the notion is derived. The "standard" examples with which we compare spaces are cartesian products of unit intervals and metric spaces. In this chapter the elementary properties of metric and pseudo-metric spaces are developed, and necessary and sufficient conditions are given under which a space is a copy of a metric space or of a subspace of the cartesian product of intervals.

A word of caution: the notion of a topological space by no means includes all of the properties which metric spaces possess. In chapter 6 a different and more penetrating abstraction of the concept of a metric space is made.

EXISTENCE OF CONTINUOUS FUNCTIONS

In this section we prove four lemmas, all part of a program to construct real-valued continuous functions on topological spaces. For the moment we are concerned with a rather special sort of topological space. A space is **normal** * iff for each disjoint pair of closed sets, A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A **T_4-space** is a normal space which is T_1 ($\{x\}$ is closed for each x). If it is agreed that a set U is a **neighborhood of a set** A iff A is a subset of the interior U^0 of U, then the definition of normality can be stated: a space is normal iff disjoint closed sets have disjoint neighborhoods. There is another restatement of the condition which is also suggestive. A family of neighborhoods of a set is a **base for the neighborhood system** of the set iff every neighborhood of the set contains a member of the family. If W is a neighborhood of a closed subset A of a normal space X, then there are disjoint open sets U and V such that $A \subset U$ and $X \sim W^0 \subset V$, and hence the arbitrary neighborhood W of A contains the closed neighborhood U^-. Consequently the family of closed neighborhoods of a closed set A is a base for the neighborhood system of A if the space is normal. The converse is also true, for if A and B are disjoint closed sets and W is a closed neighborhood of A which is contained in $X \sim B$, then W^0 and $X \sim W$ are disjoint open neighborhoods of A and B respectively.

Every discrete space and every indiscrete space is normal and consequently a normal space need not be Hausdorff and may fail to satisfy the first or second axiom of countability. However, a T_4-space (T_1 and normal) is surely a Hausdorff space. A closed

* This nomenclature is an excellent example of the time-honored custom of referring to a problem we cannot handle as abnormal, irregular, improper, degenerate, inadmissible, and otherwise undesirable. A brief discussion of the abnormalities of the class of normal spaces occurs in the problems at the end of the chapter.

subset of a normal space is, with the relative topology, normal. However, subspaces, products, and quotient spaces of normal spaces may not be normal. (See 4.E, 4.F.)

There is a condition which for T_1-spaces is intermediate to Hausdorff and normal, and under certain circumstances implies normality. A topological space is **regular** iff for each point x and each neighborhood U of x there is a closed neighborhood V of x such that $V \subset U$; that is, the family of closed neighborhoods of each point is a base for the neighborhood system of the point. An equivalent statement: for each point x and each closed set A, if $x \notin A$, then there are disjoint open sets U and V such that $x \in U$ and $A \subset V$. A regular space which is also T_1 is called a **T_3-space.** Recall that a Lindelöf space is a topological space such that each open cover has a countable subcover.

1 LEMMA (TYCHONOFF) *Each regular Lindelöf space is normal.*

PROOF Suppose A and B are closed disjoint subsets of X. Because X is regular, for each point of A there is a neighborhood whose closure fails to intersect B and consequently the family $\mathfrak{U}$ of all open sets whose closures do not intersect B is a cover of A. Similarly, the family $\mathfrak{V}$ of all open sets whose closures do not intersect A is a cover of B, and $\mathfrak{U} \cup \mathfrak{V} \cup \{X \sim (A \cup B)\}$ is a cover of X. There is then a sequence $\{U_n, n \in \omega\}$ of members of $\mathfrak{U}$ which covers A, and a sequence $\{V_n, n \in \omega\}$ of members of $\mathfrak{V}$ which covers B. Let $U_n' = U_n \sim \bigcup\{V_p^-: p \leqq n\}$ and let $V_n' = V_n \sim \bigcup\{U_p^-: p \leqq n\}$. Since $U_n' \cap V_m$ is void for $m \leqq n$, it follows that $U_n' \cap V_m'$ is void for $m \leqq n$. Applying the same argument with the roles of U and V interchanged, $U_n' \cap V_m'$ is void for all m and n and consequently $\bigcup\{U_n': n \in \omega\}$ is disjoint from $\bigcup\{V_n': n \in \omega\}$. Finally, $V_p^- \cap A$ and $U_p^- \cap B$ are void for all p and hence the open disjoint sets $\bigcup\{U_n': n \in \omega\}$ and $\bigcup\{V_n': n \in \omega\}$ contain A and B respectively. ∎

In particular, a regular topological space satisfying the second axiom of countability is always normal.

We now begin the construction of continuous real-valued functions. If A and B are disjoint closed sets, we want to construct a continuous real-valued function which is zero on A and one on B, with all values in the closed interval [0,1]. Instead of constructing the function f directly we construct sets which correspond (approximately) to sets of the form $\{x: f(x) < t\}$. The two following lemmas show the relation between a family of subsets and a real-valued function.

2 Lemma *Suppose that for each member t of a dense subset D of the positive reals F_t is a subset of a set X such that*

(a) *if $t < s$, then $F_t \subset F_s$; and*
(b) $\bigcup\{F_t: t \in D\} = X$.

For x in X let $f(x) = \inf\{t: x \in F_t\}$. Then $\{x: f(x) < s\} = \bigcup\{F_t: t \in D$ and $t < s\}$ and $\{x: f(x) \leq s\} = \bigcap\{F_t: t \in D$ and $t > s\}$ for each real number s.

PROOF The calculation is direct. The set $\{x: f(x) < s\} = \{x: \inf\{t: x \in F_t\} < s\}$, and since the infimum is less than s iff some member of $\{t: x \in F_t\}$ is less than s, the set $\{x: f(x) < s\}$ is the set of all x such that for some t, $t < s$ and $x \in F_t$; that is, $\bigcup\{F_t: t \in D$ *and* $t < s\}$. This establishes the first equality. To prove the second notice that $\inf\{t: x \in F_t\} \leq s$ if for each u greater than s there is $t < u$ such that $x \in F_t$. Conversely, if for each t in D such that $t > s$ it is true that $x \in F_t$, then $\inf\{t: x \in F_t\} \leq s$ because D is dense in the set of positive numbers. Consequently the set of all x such that $f(x) = \inf\{t: x \in F_t\} \leq s$ is $\{x:$ *if* $t \in D$ *and* $t > s$, *then* $x \in F_t\} = \bigcap\{F_t: t \in D$ *and* $t > s\}$. ∎

3 Lemma *Suppose that for each member t of a dense subset D of the positive reals F_t is an open subset of a topological space X such that*

(a) *if $t < s$, then the closure of F_t is a subset of F_s; and*
(b) $\bigcup\{F_t: t \in D\} = X$.

Then the function f such that $f(x) = \inf\{t: x \in F_t\}$ is continuous.

PROOF According to 3.1 a function is continuous if the inverse of each member of some subbase for the topology of the range space is open, and the family of all sets of the form $\{t: t < s\}$ or $\{t: t > s\}$, for real numbers s, is a subbase for the usual topology for the set of real numbers. Consequently, to show f continuous it is sufficient to show that $\{x: f(x) < s\}$ is open and $\{x: f(x) \leq s\}$

is closed for each real s. In view of the previous lemma the first of these, $\{x: f(x) < s\}$, is the union of open sets F_t and is therefore open. With reference again to the previous lemma, $\{x: f(x) \leqq s\} = \bigcap \{F_t: t \in D \text{ and } t > s\}$, and the proof will be complete if we show this set is identical with $\bigcap \{F_t^-: t \in D \text{ and } t > s\}$. Since $F_t \subset F_t^-$ for each t, surely $\bigcap \{F_t: t \in D \text{ and } t > s\} \subset \bigcap \{F_t^-: t \in D \text{ and } t > s\}$. On the other hand, for each t in D with $t > s$ there is r in D such that $s < r < t$, and hence such that $F_r^- \subset F_t$. The reverse inclusion follows. ∎

The principal result of the section is now easily proved.

4 Lemma (Urysohn) *If A and B are disjoint closed subsets of a normal space X, then there is a continuous function f on X to the interval* $[0,1]$ *such that f is zero on A and one on B.*

PROOF Let D be the set of positive dyadic rational numbers (that is, the set of all numbers of the form $p2^{-q}$, where p and q are positive integers). For t in D and $t > 1$ let $F(t) = X$, let $F(1) = X \sim B$ and let $F(0)$ be an open set containing A such that $F(0)^-$ is disjoint from B. For t in D and $0 < t < 1$ write t in the form $t = (2m + 1)2^{-n}$ and choose, inductively on n, $F(t)$ to be an open set containing $F(2m2^{-n})^-$ and such that $F(t)^- \subset F((2m + 2)2^{-n})$. This choice is possible because X is normal. Let $f(x) = \inf \{t: x \in F(t)\}$. The previous lemma shows that f is continuous. The function f is zero on A because $A \subset F(t)$ for each t in D, and f is one on B because $F(t) \subset X \sim B$ for $t \leqq 1$ and $F(t) = X$ for $t > 1$. ∎

EMBEDDING IN CUBES

The cartesian product of closed unit intervals, with the product topology, is called a cube. A cube is then the set Q^A of all functions on a set A to the closed unit interval Q, with the topology of pointwise, or coordinate-wise, convergence. The cube is used as a standard sort of space, and we want to describe those topological spaces which are homeomorphic to subspaces of cubes. The device used to accomplish this description is simple but noteworthy; it will be used again in other connections.

Suppose that F is a family of functions such that each member

f of F is on a topological space X to a space Y_f (the range may be different for different members of the family). There is then a natural mapping of X into the product $\times\{Y_f: f \in F\}$ which is defined by mapping a point x of X into the member of the product whose f-th coordinate is $f(x)$. Formally, the **evaluation** map e is defined by: $e(x)_f = f(x)$. It turns out that e is continuous if the members of F are continuous and e is a homeomorphism if, in addition, F contains "enough functions." A family F of functions on X **distinguishes points** iff for each pair of distinct points x and y there is f in F such that $f(x) \neq f(y)$. The family **distinguishes points and closed sets** iff for each closed subset A of X and each member x of $X \sim A$ there is f in F such that $f(x)$ does not belong to the closure of $f[A]$.

5 EMBEDDING LEMMA *Let F be a family of continuous functions, each member f being on a topological space X to a topological space Y_f. Then:*

(a) *The evaluation map e is a continuous function on X to the product space $\times\{Y_f: f \in F\}$.*
(b) *The function e is an open map of X onto $e[X]$ if F distinguishes points and closed sets.*
(c) *The function e is one to one if and only if F distinguishes points.*

PROOF The map e followed by projection P_f into the f-th coordinate space is continuous because $P_f \circ e(x) = f(x)$. Consequently, by theorem 3.3, e is continuous. To prove statement (b) it is sufficient to show that the image under e of an open neighborhood U of a point x contains the intersection of $e[X]$ and a neighborhood of $e(x)$ in the product. Choose a member f of F such that $f(x)$ does not belong to the closure of $f[X \sim U]$. The set of all y in the product such that $y_f \notin f[X \sim U]^-$ is open, and evidently its intersection with $e[X]$ is a subset of $e[U]$. Hence e is an open map of X onto $e[X]$. Statement (c) is clear. ∎

The preceding lemma reduces the problem of embedding a space topologically in a cube to the problem of finding a "rich" set of continuous real-valued functions on the space. There are topological spaces on which each continuous real-valued function is constant. For example, any indiscrete space has this property. There are less trivial examples; there are regular Hausdorff spaces on which every real continuous function is constant.* A topological space X is called **completely regular** iff for each member x of X and each neighborhood U of x there is a continuous function f on X to the closed unit interval such that $f(x) = 0$ and f is identically one on $X \sim U$. It is clear that the family of all continuous functions on a completely regular space to the unit interval $[0,1]$ distinguishes points and closed sets, in the sense of the preceding lemma. (The converse statement is also true, but will not be needed here.) If a completely regular space is T_1 ($\{x\}$ is closed for each x), then the family of continuous functions on the space to $[0,1]$ also distinguishes points. A completely regular T_1-space is called a **Tychonoff** space. If X is a Tychonoff space and F is the family of all continuous functions on X to $[0,1]$, then the embedding lemma 4.5 shows that the evaluation map of X into the cube Q^F is a homeomorphism. Thus each Tychonoff space is homeomorphic to a subspace of a cube. This fact is actually characteristic of Tychonoff spaces, as will presently be demonstrated.

Each normal T_1-space is a Tychonoff space in view of Urysohn's lemma 4.4. Each completely regular space is regular, for if U is a neighborhood of x and f is a continuous function which is zero at x and one on $X \sim U$, then $V = \{y: f(y) < \frac{1}{2}\}$ is an open set whose closure is contained in $\{y: f(y) \leq \frac{1}{2}\}$, which is a subset of U. For T_1-spaces there is a hierarchy of so-called separation axioms: Hausdorff, regular, completely regular, and normal. Except for normality, these properties are hereditary, in the sense that each subspace of a space X enjoys the property if X does. The product of spaces of each of these types is again of the same type, excepting, again, normality. The proofs of these facts are left as problems (4.H) except for the following, which is needed now.

6 THEOREM *The product of Tychonoff spaces is a Tychonoff space.*

PROOF For convenience, let us agree that a continuous function f on a topological space X to the closed unit interval is *for* a

* See Hewitt [1] and Novak [1]. For other facts on separation axioms see van Est and Freudenthal [1].

pair (x,U) iff x is a point, U is a neighborhood of x, $f(x) = 0$, and f is identically one on $X \sim U$. If $f_1, \cdots, f_n$ are functions for $(x,U_1), \cdots, (x,U_n)$, where n is a positive integer, and if $g(x) = \sup \{f_i(x) : i = 1, \cdots, n\}$, then g is a function for $(x, \bigcap \{U_i: i = 1, \cdots, n\})$. Consequently the space is completely regular if for each x and each neighborhood U of x belonging to some subbase for the topology there is a function for (x,U). If X is the product $\times \{X_a : a \in A\}$ of Tychonoff spaces and $x \in X$ let U_a be a neighborhood of x_a in X_a. If f is a function for (x_a,U_a), then $f \circ P_a$, where P_a is the projection into the a-th coordinate space, is a function for $(x,P_a^{-1}[U_a])$. The family of sets of the form $P_a^{-1}[U_a]$ is a subbase for the product topology and hence the product space is completely regular. Since the product of T_1-spaces is a T_1-space the theorem follows. ∎

7 EMBEDDING THEOREM *In order that a topological space be a Tychonoff space it is necessary and sufficient that it be homeomorphic to a subspace of a cube.*

PROOF The closed unit interval is a Tychonoff space, and hence a cube, being a product of unit intervals, is a Tychonoff space by 4.6. Each subspace of a cube is therefore a Tychonoff space. It has already been observed that if X is a Tychonoff space and F the set of all continuous functions on X to the closed unit interval Q, then (by the embedding lemma 4.5) the evaluation map is a homeomorphism of X into the cube Q^F. ∎

METRIC AND PSEUDO-METRIC SPACES

There are many topological spaces in which the topology is derived from a notion of distance. A **metric** for a set X is a function d on the cartesian product $X \times X$ to the non-negative reals such that for all points x, y, and z of X,

(a) $d(x,y) = d(y,x)$,
(b) (*triangle inequality*) $d(x,y) + d(y,z) \geq d(x,z)$,
(c) $d(x,y) = 0$ if $x = y$, and
(d) if $d(x,y) = 0$, then $x = y$.

The last of these conditions is inessential for many purposes. A function d which satisfies only (a) and (b) and (c) is called a **pseudo-metric** (sometimes an **écart**, although "écart" is also used in a slightly different sense). All of the definitions of this section will be made for pseudo-metrics, it being understood that the same definitions are to hold with "pseudo-metric" replaced by "metric."

A **pseudo-metric** space is a pair (X,d) such that d is a pseudo-metric for X. For members x and y of X the number $d(x,y)$ is the **distance** (if confusion seems possible the d-distance) from x to y. If r is a positive number the set $\{y: d(x,y) < r\}$ is the **open sphere** of d-radius r about x, or briefly the open r-sphere about x, and $\{y: d(x,y) \leq r\}$ is the **closed r-sphere** about x. The intersection of two open spheres may not be a sphere. However, if $d(x,y) < r$ and $d(x,z) < s$, then each point w such that $d(w,x) < \min [r - d(x,y), s - d(x,z)]$ is a member of both the open r-sphere about y and the open s-sphere about z because of the triangle inequality. Consequently the intersection of two open spheres contains an open sphere about each of its points, and hence the family of all open spheres is the base for a topology for X (see 1.11). This topology is the **pseudo-metric topology** for X. Observe that each closed sphere is closed relative to the pseudo-metric topology.

Let X be a set and define $d(x,y)$ to be zero if $x = y$ and one otherwise. Then d is a metric for X and the open 1-sphere about each point x is $\{x\}$; hence $\{x\}$ is open relative to the metric topology and the space is discrete. The closed 1-sphere about each point is X and it follows that the closure of an open r-sphere may be different from the closed r-sphere. If d is defined to be zero for all pairs (x,y) in $X \times X$, then d is not a metric, but is a pseudo-metric. Then the open r-sphere about each point is the entire space, and the pseudo-metric topology for X is the indiscrete topology. If X is the set of all real numbers and $d(x,y) = |x - y|$ then d is a metric for X; it is called the **usual metric** for the real numbers. The usual metric topology is fortunately the usual topology for the reals.

The distance from a point x to a non-void subset A of a pseudo-metric space is defined to be $D(A,x) = \inf \{d(x,y): y \in A\}$.

8 THEOREM *If A is a fixed subset of a pseudo-metric space, then the distance from a point x to A is a continuous function of x relative to the pseudo-metric topology.*

PROOF Since $d(x,z) \leqq d(x,y) + d(y,z)$ it follows, taking lower bounds for z in A, that $D(A,x) \leqq d(x,y) + D(A,y)$. The same inequality holds with x and y interchanged and hence $|D(A,x) - D(A,y)| \leqq d(x,y)$. Consequently, if y is in the open r-sphere about x, then $|D(A,x) - D(A,y)| < r$ and continuity follows. ∎

9 THEOREM *The closure of a set A in a pseudo-metric space is the set of all points which are zero distance from A.*

PROOF Since $D(A,x)$ is continuous in x the set $\{x: D(A,x) = 0\}$ is closed and contains A and hence contains the closure A^- of A. If $y \notin A^-$, then there is a neighborhood of y, which may be taken to be an open r-sphere, which does not intersect A. Consequently $D(A,y) \geqq r$ and hence $\{x: D(A,x) = 0\} \subset A^-$. Therefore $A^- = \{x: D(A,x) = 0\}$. ∎

10 THEOREM *Each pseudo-metric space is normal.*

PROOF Let A and B be disjoint closed subsets of a pseudo-metric space X, and let $D(A,x)$ and $D(B,x)$ be the distance from x to A and B respectively. Let $U = \{x: D(A,x) - D(B,x) < 0\}$ and let $V = \{x: D(A,x) - D(B,x) > 0\}$. The function $D(A,x) - D(B,x)$ is continuous in x and therefore U and V are open. Clearly U is disjoint from V, and using 4.9 it follows that $A \subset U$ and $B \subset V$. ∎

11 THEOREM *Every pseudo-metric space satisfies the first axiom of countability. The second is satisfied if and only if the space is separable.*

PROOF A set is open relative to the pseudo-metric topology iff it contains an open sphere about each of its points. Therefore the family of open spheres about a point x is a base for the neighborhood system of x. Since each open sphere about x contains a sphere with rational radius there is a countable base for the neighborhood system and the space satisfies the first axiom of countability. Each space which satisfies the second axiom of countability is separable, so it remains to prove that a separable pseudo-metric space has a countable base for its topology. Let Y be a countable dense subset and let $\mathcal{U}$ be the family of all open spheres with rational radii about members of Y. Surely $\mathcal{U}$ is countable. If U is a neighborhood of a point x there is, for some positive r, an open r-sphere about x which is contained in U. Let s be a positive rational number less than r, let y be a point of Y such that $d(x,y) < s/3$, and let V be the open $2s/3$ sphere about y. Then $x \in V \subset U$ and hence $\mathcal{U}$ is a base for the topology. ∎

12 THEOREM *A net* $\{S_n, n \in D\}$ *in a pseudo-metric space* (X,d) *converges to a point* s *if and only if* $\{d(S_n,s), n \in D\}$ *converges to zero.*

PROOF A net $\{S_n, n \in D\}$ converges to s iff the net is eventually in each open r-sphere about s, but this is true iff $\{d(S_n,s), n \in D\}$ is eventually in each open r-sphere about 0 in the space of real numbers with the usual metric. ∎

The **diameter** of a subset A of a pseudo-metric space (X,d) is $\sup \{d(x,y): x \in A \text{ and } y \in A\}$. If this supremum does not exist the diameter is said to be infinite. It is interesting to notice that the property of having a finite diameter is not a topological invariant.

13 THEOREM *Let* (X,d) *be a pseudo-metric space, and let* $e(x,y) = \min\,[1,d(x,y)]$. *Then* (X,e) *is a pseudo-metric space whose topology is identical with that of* (X,d).

Consequently each pseudo-metric space is homeomorphic to a pseudo-metric space of diameter at most one.

PROOF To prove that e is a pseudo-metric it is sufficient to show that if a, b, and c are non-negative numbers such that $a + b \geqq c$, then $\min\,[1,a] + \min\,[1,b] \geqq \min\,[1,c]$, for the latter inequality becomes the triangle inequality for e if we set $a = d(x,y)$, $b = d(y,z)$ and $c = d(x,z)$. If either $\min\,[1, a]$ or $\min\,[1,b]$ is one the inequality is surely correct since $\min\,[1,c] \leqq 1$. If neither of these is one the inequality $a + b \geqq c \geqq \min\,[1,c]$ completes the proof. Consequently e is a pseudo-metric for X. The family of all open r-spheres, for r less than one, is a base for the pseudo-metric topology. Since this family is the same whether d or e is used as pseudo-metric, the two pseudo-metric topologies are identical. Clearly the e-diameter of X is at most one. ∎

The product of uncountably many topological spaces does not generally satisfy the first axiom of countability (see 3.6) and consequently one cannot expect to find a pseudo-metric for the product of arbitrarily many pseudo-metric spaces such that the pseudo-metric topology is the product topology. For countable products the situation is pleasant. Because of the previous theorem we restrict our attention to spaces of diameter at most one.

14 THEOREM *Let* $\{(X_n,d_n),\ n \in \omega\}$ *be a sequence of pseudo-metric spaces, each of diameter at most one, and define* d *by:* $d(x,y) = \sum\{2^{-n}d_n(x_n,y_n)\colon n \in \omega\}$. *Then* d *is a pseudo-metric for the cartesian product, and the pseudo-metric topology is the product topology.*

PROOF The simple proof that d is a pseudo-metric is omitted. (Problem 2.G on summability contains the necessary machinery.) To show the two topologies identical, first observe that, if V is a 2^{-p}-sphere about a point x of the product and $U = \{y\colon d_n(x_n,y_n) < 2^{-p-n-2}$ *for* $n \leqq p + 2\}$, then $U \subset V$, for if $y \in U$, then $d(x,y) < \sum\{2^{-p-n-2}\colon n = 0, \cdots, p+2\} + \sum\{2^{-n}\colon n = p+3, \cdots\} < 2^{-p-1} + 2^{-p-1} = 2^{-p}$. But U is a neighborhood of x in the product topology and it follows that each set which is open relative to the pseudo-metric topology is open relative to the product topology. To show the converse consider a member U of the defining subbase of the product topology. Then U is of the form $\{x\colon x_n \in W\}$ where W is open in X_n. For x in U there is an open r-sphere about x_n which is a subset of W, and since $d(x,y) \geqq 2^{-n}d_n(x_n,y_n)$ the open $r2^{-n}$-sphere about x is a subset of U. Therefore each member of the defining subbase, and consequently each member of the product topology, is open relative to the pseudo-metric topology. ∎

If (X,d) and (Y,e) are pseudo-metric spaces and f is a map of X onto Y, then f is an **isometry** (a d-e isometry) iff $d(x,y) = e(f(x),f(y))$ for all points x and y of X. Every isometry is a continuous open map (relative to the two pseudo-metric topologies) because the image of each open r-sphere about x is an open r-sphere about $f(x)$. The composition of two isometrics is again an isometry and if an isometry is one to one the inverse is also an isometry. On a metric space an isometry is necessarily one to one and an isometry of a metric space onto a metric space is a homeomorphism. The collection of all metric spaces is divided into equivalence classes of mutually isometric spaces. Each property which, when possessed by a metric space, is also possessed by each isometric metric space, is a **metric invariant.** A metric invariant is not necessarily a topological invariant (for example, consider the property of being of infinite diameter).

Each pseudo-metric space differs but little, in one sense, from a metric space. In making this statement precise it is convenient to agree that the **distance between two subsets,** A and B, of a pseudo-metric space is $D(A,B) = \text{dist}\,(A,B) = \inf\,\{d(x,y)\colon x \in A \text{ and } y \in B\}$. It is generally not true that D is a pseudo-metric, for the space X is zero distance from every non-void subset and the triangle inequality fails. However, D is actually a metric for the members of the decomposition which we want to consider. For a pseudo-metric space (X,d) let $\mathfrak{D}$ be the family of all sets of the form $\{x\}^-$. Because of 4.9, $\{x\}^-$ is precisely the set of all points y such that $d(x,y) = 0$, and the decomposition $\mathfrak{D}$ is the quotient X/R where R is the relation $\{(x,y)\colon d(x,y) = 0\}$.

15 THEOREM *Let (X,d) be a pseudo-metric space, let $\mathfrak{D}$ be the family of all sets $\{x\}^-$ for x in X, and for members A and B of $\mathfrak{D}$ let $D(A,B) = \text{dist}\,(A,B)$. Then $(\mathfrak{D},D)$ is a metric space whose topology is the quotient topology for $\mathfrak{D}$, and the projection of X onto $\mathfrak{D}$ is an isometry.*

PROOF A point u is a member of $\{x\}^-$ iff $d(u,x) = 0$, and this is true iff $x \in \{u\}^-$. If $u \in \{x\}^-$ and $v \in \{y\}^-$, then $d(u,v) \leq d(u,x) + d(x,y) + d(y,v) = d(x,y)$. Consequently, since in this case it is also true that $x \in \{u\}^-$ and $y \in \{v\}^-$, $d(u,v) = d(x,y)$. It follows that for members A and B of $\mathfrak{D}$, $D(A,B)$ is identical with $d(x,y)$ for every x in A and every y in B. Therefore $(\mathfrak{D},D)$ is a metric space and the projection of X onto $\mathfrak{D}$ is an isometry. If U is an open set in X and $x \in U$, then, for some $r > 0$, U contains an open r-sphere about x, and hence contains $\{x\}^-$. The projection of X onto $\mathfrak{D}$ is therefore an open map relative to the quotient topology for $\mathfrak{D}$, by 3.10. The projection is also open relative to the metric topology derived from D and hence, by 3.8, these two topologies are identical. ▌

METRIZATION

Given a topological space $(X,\mathfrak{I})$, it is natural to ask whether there is a metric for X such that $\mathfrak{I}$ is the metric topology. Such a metric **metrizes** the topological space and the space is said to be **metrizable.** Similarly, a topological space is **pseudo-metrizable** iff there is a pseudo-metric such that the topology is the pseudo-metric topology. A pseudo-metric is a metric if and only if the topology is T_1 (that is, iff $\{x\}$ is closed for each point x) and it follows that a space is metrizable if and only if it is T_1 and pseudo-metrizable. The theorems of this section are stated for metrizable spaces; the corresponding theorems for pseudo-metrizable spaces will be self-evident.

The two principal theorems of the section give necessary and sufficient conditions that a topological space be, respectively, metrizable and separable, and metrizable. The first of these is the classical metrization theorem of Urysohn; all of the pieces of its proof are already available and it is simply a matter of fitting the facts together. The second theorem has been proved only recently (its history is given in the notes at the end of the section). It turns out that a mild variant of Urysohn's procedure proves the sufficiency of the conditions imposed, but the necessity requires a new sort of construction. A further study of the concepts introduced here is made in the last section of chapter 5. Finally, the entire problem of metrization is approached from a different point of view in chapter 7; however, the results obtained there do not include the theorems of this section.

The pattern for a proof of metrizability is very simple. According to 4.14 the product of countably many pseudo-metric spaces is pseudo-metrizable. According to the embedding lemma 4.5, if F is a family of continuous functions on a T_1-space X, where a member f of F maps X into a space Y_f, then the evaluation map of X into $\times\{Y_f: f \in F\}$ is a homeomorphism whenever

F distinguishes points and closed sets (that is, if A is a closed subset of X and x is a member of $X \sim A$, then $f(x) \notin f[A]^-$ for some member f of F). The problem of metrizing a T_1-space X then reduces to that of finding a countable family of continuous functions, each on X to some pseudo-metrizable space, such that F distinguishes points and closed sets. (A pseudo-metrizable T_1-space is necessarily metrizable.)

For convenience, let Q^ω denote the product of the closed unit interval with itself countably many times; that is Q^ω is the set of all functions on the non-negative integers to the closed unit interval Q, with the product topology.

16 METRIZATION THEOREM (URYSOHN) *A regular T_1-space whose topology has a countable base is homeomorphic to a subspace of the cube Q^ω and is hence metrizable.*

PROOF In view of the remarks preceding the theorem it is sufficient to show that there is a countable family of continuous functions on X to Q which distinguishes points and closed sets. Let $\mathcal{B}$ be a countable base for the topology of X and let $\mathcal{A}$ be the set of all pairs (U,V) such that U and V belong to $\mathcal{B}$ and $U^- \subset V$. Surely $\mathcal{A}$ is countable. For each pair (U,V) in $\mathcal{A}$ choose a continuous function f on X to Q such that f is zero on U and one on $X \sim V$ (such a function exists because of the Tychonoff lemma 4.1 and the Urysohn lemma 4.4) and let F be the family of functions so obtained. Then F is countable and it remains to be proved that F distinguishes points and closed sets. If B is closed and $x \in X \sim B$ choose a member V of $\mathcal{B}$ such that $x \in V \subset X \sim B$ and choose U in $\mathcal{B}$ such that $x \in U^- \subset V$. Then $(U,V) \in \mathcal{A}$, and if f is the corresponding member of F, then $f(x) = 0 \notin \{1\} = f[B]^-$. ∎

It is easy to describe the class of topological spaces to which the foregoing metrization theorem applies.

17 THEOREM *If X is a T_1-space, then the following are equivalent:*

(a) *X is regular and there is a countable base for its topology.*
(b) *X is homeomorphic to a subspace of the cube Q^ω.*
(c) *X is metrizable and separable.*

PROOF The previous theorem shows that (a) $\rightarrow$ (b). The cube Q^ω is metrizable, by 4.14, and satisfies the second axiom of countability (3.M). Hence each subspace is metrizable and satisfies the second axiom of countability and is therefore separable. Hence (b) $\rightarrow$ (c). (Caution: it is not true that a subspace of a separable space is necessarily separable.) Finally (c) $\rightarrow$ (a), for if X is metrizable and separable, then it is surely regular and by 4.11 it satisfies the second axiom of countability. ▌

The metrization theorem for spaces which are not necessarily separable depends heavily on the ideas which we have already exploited. A brief discussion of methodology will indicate where the procedure used so far can be improved. The construction of a metric for X is accomplished by finding a family of mappings of X into pseudo-metrizable spaces. But observe: so far the only space which has been used as the range space is the unit interval Q. Stated in slightly different form, if f is a function on X to Q, then one may construct a pseudo-metric for X by letting $d(x,y) = |f(x) - f(y)|$. The Urysohn metrization is accomplished by using a countable number of pseudo-metrics of this sort, and the problem is to generalize this construction. If F is a family of functions on X to Q, then a possible candidate for a pseudo-metric is the sum: $\sum\{|f(x) - f(y)|: f \in F\}$. This sum must be continuous in x and y in order that the identity map of X into the pseudo-metric space (X,d) be continuous, and a condition much weaker than finiteness of the family F will ensure continuity. It is sufficient, to obtain continuity, that for each point x of X there be a neighborhood U of x such that all but a finite number of the members of F vanish on U; in other words, a sort of local finiteness suffices. This notion of local finiteness is the key to the problem.

A family $\mathcal{A}$ of subsets of a topological space is **locally finite** iff each point of the space has a neighborhood which intersects only finitely many members of $\mathcal{A}$. It follows immediately from the definition that a point is an accumulation point of the union $\bigcup\{A: A \in \mathcal{A}\}$ iff it is an accumulation point of some member of $\mathcal{A}$, and hence the closure of the union is the union of the closures; that is, $[\bigcup\{A: A \in \mathcal{A}\}]^- = \bigcup\{A^-: A \in \mathcal{A}\}$. It is also evident that the family of all closures of members of $\mathcal{A}$ is locally finite.

A family $\mathfrak{A}$ is **discrete** if each point of the space has a neighborhood which intersects at most one member of $\mathfrak{A}$. A discrete family is locally finite, and if $\mathfrak{A}$ is discrete, then the family of closures of members of $\mathfrak{A}$ is also discrete. Finally, a family $\mathfrak{A}$ is **σ-locally finite (σ-discrete)** if and only if it is the union of a countable number of locally finite (respectively, discrete) subfamilies.

The following metrization theorem can now be stated. Its proof is contained in the sequence of lemmas which follows the statement.

18 METRIZATION THEOREM *The following three conditions on a topological space are equivalent.*

(a) *The space is metrizable.*
(b) *The space is T_1 and regular, and the topology has a σ-locally finite base.*
(c) *The space is T_1 and regular, and the topology has a σ-discrete base.*

It is clear that condition (c) implies (b) and it will be proved that (b) implies (a), and finally that (a) implies (c). The first step in the proof is a variant of Tychonoff's lemma, 4.1.

19 LEMMA *A regular space whose topology has a σ-locally finite base is normal.*

PROOF If A and B are disjoint closed subsets of the space X, then there are open covers $\mathfrak{U}$ and $\mathfrak{V}$ of A and B respectively such that the closure of each member of $\mathfrak{U}$ is disjoint from B, the closure of each member of $\mathfrak{V}$ is disjoint from A, and both $\mathfrak{U}$ and $\mathfrak{V}$ are subfamilies of a σ-locally finite base $\mathfrak{B}$. It follows that $\mathfrak{U} = \bigcup\{\mathfrak{U}_n: n \in \omega\}$ and $\mathfrak{V} = \bigcup\{\mathfrak{V}_n: n \in \omega\}$ where $\mathfrak{U}_n$ and $\mathfrak{V}_n$ are locally finite families. Let $U_n = \bigcup\{W: W \in \mathfrak{U}_n\}$ and let $V_n = \bigcup\{W: W \in \mathfrak{V}_n\}$. Then $U_n^- = \bigcup\{W^-: W \in \mathfrak{U}_n\}$, and hence U_n^- is disjoint from B and similarly V_n^- is disjoint from A. This is precisely the situation which occurs in the proof of 4.1, and as there, the proof is completed by letting $U_n' = U_n \sim \bigcup\{V_k^-: k \leq n\}$, $V_n' = V_n \sim \bigcup\{U_k^-: k \leq n\}$. The union of the sets U_n' and the union of the sets V_n' are the required disjoint neighborhoods of A and B respectively. ∎

The following lemma now completes the proof that the conditions listed in 4.18 are sufficient for metrizability.

20 LEMMA *A regular T_1-space whose topology has a σ-locally finite base is metrizable.*

PROOF It will be shown that there is a countable family D of pseudo-metrics on the space X such that each member of D is continuous on $X \times X$ and such that for each closed subset A of X and each point x of $X \sim A$ there is a member d of D such that the d-distance from x to A is positive. This will prove metrizability, for the map of X into each of the pseudo-metric spaces (X,d) will then be continuous, and 4.5 and 4.14 will apply just as for the Urysohn theorem. The problem is then to construct the family D. Let $\mathcal{B}$ be a σ-locally finite base for the topology of X, and suppose that $\mathcal{B} = \bigcup\{\mathcal{B}_n: n \in \omega\}$ where each $\mathcal{B}_n$ is locally finite. For every ordered pair of integers m and n and for each member U of $\mathcal{B}_m$, let U' be the union of all members of $\mathcal{B}_n$ whose closures are contained in U. Because $\mathcal{B}_n$ is locally finite the closure of U' is a subset of U, and there is a continuous function f_U on X to the unit interval which is one on U' and zero on $X \sim U$ by 4.19 and 4.4. Let $d(x,y) = \sum\{|f_U(x) - f_U(y)|: U \in \mathcal{B}_m\}$. The continuity of d on $X \times X$ is a straightforward consequence of the local finiteness of $\mathcal{B}_m$. Finally, let D be the family of pseudo-metrics so obtained; since one pseudo-metric was constructed for each ordered pair of integers, D is countable. If A is a closed subset of X and $x \in X \sim A$, then for some m and some U in $\mathcal{B}_m$ it is true that $x \in U \subset X \sim A$, and for some n and some V in $\mathcal{B}_n$ it is true that $x \in V$ and $V^- \subset U$. For the pseudo-metric d constructed for this pair it is clear that the d-distance from x to A is at least one. ∎

The most interesting part of the proof of the metrization theorem remains. It must be proved that each metric space has a σ-discrete base. A stronger result than this is true and, since the more potent theorem will be needed later, we introduce a new concept. A cover $\mathcal{B}$ of a set X is a **refinement** of a cover $\mathcal{A}$ iff each member of $\mathcal{B}$ is a subset of a member of $\mathcal{A}$. For example, in a metric space, the family of all open spheres of radius one half is a refinement of the family of all open spheres of radius one. The following theorem states that any open cover of a pseudo-metric space has an open refinement which is σ-discrete. This will imply that each pseudo-metric topology has a σ-discrete base, for one may select a σ-discrete refinement $\mathcal{B}_n$ of the cover consisting of all open spheres of radius $1/n$, and the union of the families $\mathcal{B}_n$ is then a σ-discrete base. This fact completes the proof of the metrization theorem 4.18.

21 THEOREM *Each open cover of a pseudo-metrizable space has an open σ-discrete refinement.*

PROOF Let $\mathcal{U}$ be an open cover of the pseudo-metric space (X,d). The first step in the proof is the decomposition of each member U of $\mathcal{U}$ into "concentric disks." For each positive integer n and each member U of $\mathcal{U}$ let U_n be the set of all members x of U such that dist $[x,X \sim U] \geq 2^{-n}$. Because of the triangle inequality it is clear that dist $[U_n,X \sim U_{n+1}] \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}$. Chóose a relation $<$ which well orders the family $\mathcal{U}$ (see 0.25h) and for each positive integer n and each member U of $\mathcal{U}$ let $U_n^* = U_n \sim \bigcup \{V_{n+1}: V \in \mathcal{U} \textit{ and } V < U\}$. For each U and V in $\mathcal{U}$ and each positive integer n it is true that $U_n^* \subset X \sim V_{n+1}$, or $V_n^* \subset X \sim U_{n+1}$, depending on whether U follows or precedes V in the ordering. In either case dist $[U_n^*,V_n^*] \geq 2^{-n-1}$. It follows that if $U_n^\sim$ is defined to be the set of all points x such that the distance from x to U_n^* is less than 2^{-n-3}, then dist $[U_n^\sim,V_n^\sim] \geq 2^{-n-2}$ and hence for each fixed n the family of all sets of the form $U_n^\sim$ is discrete. Let $\mathcal{V}$ be the family of $U_n^\sim$ for all n and all U in $\mathcal{U}$. Then $\mathcal{V}$ is an open cover of X, for if U is the first member of $\mathcal{U}$ to which x belongs, then surely $x \in U_n^\sim$ for some n. Evidently $U_n^\sim \subset U$, and consequently $\mathcal{V}$ is a σ-discrete open refinement of $\mathcal{U}$. ∎

22 *Notes* There are really two metrization problems. The topological problem has just been treated and the problem of uniform metrization will be considered in chapter 7 (statement and history are given there). Curiously enough a satisfactory solution of the latter was found much earlier than a solution of the former. Urysohn's theorem, although treating only a special case, was certainly the most satisfactory theorem of the topo-

logical sort until very recently. The key to the present reasonably satisfactory situation was furnished by two papers. Dieudonné [1] initiated a study of spaces with the property that each open cover has an open locally finite refinement (paracompact spaces; see chapter 5). A. H. Stone [1] showed that each metrizable space is paracompact (a special case of this theorem was earlier demonstrated by C. H. Dowker [1]). The σ-locally finite characterization was then discovered independently, by Nagata [1] and by Smirnov [1]. The σ-discrete characterization is due to Bing [1]. The proof of necessity (4.21) of the metrizability conditions is actually an initial fragment of Stone's proof of paracompactness.

Smirnov [2] has also showed that paracompactness together with local metrizability implies metrizability.

Finally a brief statement of the role of pseudo-metrizable spaces might be made. Most of the spaces which occur naturally in analysis are pseudo-metric rather than metric, and even in the metrization problem a construction via pseudo-metrics was convenient. Of course, one may always replace a pseudo-metric space by a related metric space (theorem 4.15), but the process of taking quotient spaces becomes a bit tedious and for most purposes the requirement $d(x,y) = 0$ iff $x = y$ is completely irrelevant. One is tempted to work exclusively with pseudo-metrics, but this has disadvantages, for example, when one seeks to construct a topological map. A possible way out is to redefine "topological map" to mean a relation which induces a one-to-one intersection and union-preserving map on the topologies.

PROBLEMS

A REGULAR SPACES

(a) Let X be a regular space and let $\mathfrak{D}$ be the family of all subsets of the form $\{x\}^-$ for x in X. Then $\mathfrak{D}$ is a decomposition of X, the projection of X onto the quotient space $\mathfrak{D}$ is both open and closed, and the quotient space is regular Hausdorff. (If A is a subset of X which is either open or closed, then $\{x\}^- \subset A$ whenever $x \in A$.)

(b) The product of regular spaces is again regular.

B CONTINUITY OF FUNCTIONS ON A METRIC SPACE

A function f on a pseudo-metric space (X,d) to a pseudo-metric space (Y,e) is continuous iff for each x in X and each $\epsilon > 0$ there is $\delta > 0$ such that $e(f(x),f(y)) < \epsilon$ if $d(x,y) < \delta$.

C PROBLEM ON METRICS

Let f be a continuous real-valued function defined on the set of all non-negative real numbers, such that $f(x) = 0$ iff $x = 0$, f is non-decreasing, and $f(x + y) \leqq f(x) + f(y)$ for all non-negative numbers x and y. (A function satisfying this last condition is called *subadditive*.) If (X,d) is a metric space and $e(x,y) = f(d(x,y))$, then (X,e) is a metric space, and the metric topology of the space (X,e) is identical with that of (X,d). (A particular case of this result which occurs frequently in the literature: $f(x) = x/(1 + x)$.)

D HAUSDORFF METRIC FOR SUBSETS

Let (X,d) be a metric space of finite diameter, and let $\mathcal{A}$ be the family of all closed subsets. For $r > 0$ and A in $\mathcal{A}$ let $V_r(A) = \{x: \text{dist}(x,A) < r\}$, and define, for members A and B of $\mathcal{A}$, $d'(A,B) = \inf\{r: A \subset V_r(B) \text{ and } B \subset V_r(A)\}$. d' is the Hausdorff metric; it is not the same as the distance between sets used in the text.

(a) $(\mathcal{A},d')$ is a metric space, and the map which carries x in X into $\{x\}$ in $\mathcal{A}$ is an isometry of X onto a subspace of $\mathcal{A}$.

(b) The topology of the Hausdorff metric for $\mathcal{A}$ is not determined by the metric topology for X. For example, let X be the set of all positive real numbers, let $d(x,y) = |x/(1 + x) - y/(1 + y)|$, and let $e(x,y) = \min[1,|x - y|]$. Then the metric topologies of (X,d) and (X,e) are identical, but those of $(\mathcal{A},d')$ and $(\mathcal{A},e')$ are different. (In $(\mathcal{A},d')$ the set of all positive integers is an accumulation point of the family of all its finite subsets.)

Note For information and references on this topic see Michael [2].

E EXAMPLE (THE ORDINALS) ON THE PRODUCT OF NORMAL SPACES

The product of normal spaces is not generally normal.* Let Ω_0 be the set of all ordinal numbers less than the first uncountable ordinal Ω and let Ω' be $\Omega_0 \cup \{\Omega\}$, each with the order topology.

(a) *Interlacing lemma* Let $\{x_n, n \in \omega\}$ and $\{y_n, n \in \omega\}$ be two se-

* It is possible to do part of this problem a little more efficiently using methods from the following chapter. However, the facts given here will be useful later. I believe the example is due to J. Dieudonné and A. P. Morse, independently.

quences in Ω_0 such that $x_n \leqq y_n \leqq x_{n+1}$ for each n. Then both sequences converge, and to the same point of Ω_0.

(b) If A and B are closed disjoint subsets of Ω_0, then Ω is not an accumulation point of both A and B.

(c) Both Ω_0 and Ω' are normal. (If A and B are closed disjoint subsets and the first point of $A \cup B$ belongs to A, find a finite sequence $a_0, b_0, a_1, \cdots a_n$ (or b_n) such that $a_i \in A$, $b_i \in B$, no point of A is between a_i and b_i, and no point of B is between b_i and a_{i+1}, for each i. The intervals $(a_i,b_i]$ are both open and closed.)

(d) If f is a function on Ω_0 to Ω_0 such that $f(x) \geqq x$ for each x, then for some x in Ω_0, the point (x,x) is an accumulation point of the graph of f. (Define a sequence, inductively, such that $x_{n+1} = f(x_n)$, observe that $x_n \leqq f(x_n) \leqq x_{n+1}$, and use the interlacing lemma.)

(e) The product $\Omega_0 \times \Omega'$ is not normal. (Let A be the set of all points (x,x), and let $B = \Omega_0 \times \{\Omega\}$. If U is a neighborhood of A let $f(x)$ be the smallest ordinal larger than x such that $(x,f(x)) \notin U$. Then (d) applies.)

F EXAMPLE (THE TYCHONOFF PLANK) ON SUBSPACES OF NORMAL SPACES

A subspace of a normal space may fail to be normal. Let Ω' be the set of ordinal numbers not greater than the first uncountable ordinal Ω, and let ω' be the set of ordinals not greater than the first infinite ordinal, ω, each with the order topology. The product $\Omega' \times \omega'$ is called the *Tychonoff plank*. It is not difficult to prove directly that the plank is normal; however, this fact is an immediate consequence of a theorem of the next chapter. Let X be $(\Omega' \times \omega') \sim \{(\Omega,\omega)\}$, so that X is the plank with a corner point removed. Let A be the set of all points of X with first coordinate Ω and let B be the set of all points with second coordinate ω. Then there are no disjoint neighborhoods of A and B. (If U is a neighborhood of A, then for x in ω let $f(x)$ be the first ordinal such that if $y > f(x)$, then $(y,x) \in U$. The supremum of the values of f is less than Ω.)

G EXAMPLE: PRODUCTS OF QUOTIENTS AND NON-REGULAR HAUSDORFF SPACES

Let X be a regular Hausdorff space which is not normal, and let A and B be disjoint closed sets such that each neighborhood of A intersects each neighborhood of B. Let Δ be the set of all (x,x) for x in X (Δ is the identity relation on X).

(a) Let $R = \Delta \cup (A \times A)$. Then R is closed in $X \times X$ and the quotient space X/R is a Hausdorff space which is not regular. (The members of the quotient space are A, and $\{x\}$ for x in $X \sim A$.)

(b) Let $S = \Delta \cup (A \times A) \cup (B \times B)$. Then S is closed in $X \times X$, but X/S is not a Hausdorff space. (The members of X/S are A, B, and $\{x\}$ for each x in $X \sim (A \cup B)$.)

(c) There is a natural map of $X \times X$ onto $(X/S) \times (X/S)$ which carries (x,y) into $(S[x],S[y])$. It is natural to ask whether this map is open, provided X/S is given the quotient topology and $(X/S) \times (X/S)$ and $X \times X$ are given the product topologies. (This is equivalent to asking whether the product of quotients is topologically equivalent to the quotient of the product.) If S is the relation defined in (b), then the map is not open. (Consider the neighborhood $X \times X \sim (A \times A \cup B \times B \cup \Delta)$ of $A \times B$.)

H HEREDITARY, PRODUCTIVE, AND DIVISIBLE PROPERTIES

A property P of a space is *hereditary* iff each subspace of a space with P also has P; it is *productive* iff the product of spaces enjoying P has P; and it is *divisible* iff the quotient space of each space with P has P. Consider the properties: T_1, H = Hausdorff, R = regular, CR = completely regular, T = Tychonoff, N = normal, C = connected, S = separable, C_{I} = first axiom of countability, C_{II} = second countability axiom, M = metrizable, and L = Lindelöf. The following table is filled out + or −, depending on whether the property at the head of the column is or is not of the sort listed on the left. Show by example (most of the necessary examples have already been mentioned in the problems) or proof that the listing is correct.

	T_1	H	R	CR	T	N	C	S	C_{I}	C_{II}	M	L
Hereditary	+	+	+	+	+	−	−	−	+	+	+	−
Productive	+	+	+	+	+	−	+	−	−	−	−	−
Divisible	−	−	−	−	−	−	+	+	−	−	−	+

Quite different results are obtained if one varies the problem by considering only closed subspaces, or only open maps.

I HALF-OPEN INTERVAL SPACE

Let X be the set of all real numbers with the half-open interval topology (a base is the family of all half-open intervals $[a,b)$; see 1.K and 1.L). Then:

(a) X is regular.

(b) X is normal. (Recall that every open cover of X has a countable subcover.)

(c) The product space $X \times X$ is not normal. (Let $Y = \{(x,y): x + y = 1\}$, let A be the set of all members of Y with first coordinate irrational, and let $B = Y \sim A$. Assume that U and V are disjoint neighborhoods of A and B, and for x in A let $f(x) = \sup \{e: [x,e) \times [1 - x,e) \subset U\}$. Then f is a function on the set of all irrational numbers and f is never zero. The contradiction depends on the fact that for some positive integer n there is a rational number which is an accumulation point of $\{x: f(x) \geqq 1/n\}$. This fact is an immediate consequence of the theorem that the space of real numbers (with the usual topology) is of the second category (see chapter 7), but a direct proof seems awkward.)

Note This example is due to Sorgenfrey [1].

J THE SET OF ZEROS OF A REAL CONTINUOUS FUNCTION

A subset of a topological space is called a G_δ iff it is the intersection of the members of a countable family of open sets.

(a) If f is a continuous real valued function on X, then $f^{-1}[0]$ is a G_δ. (The set $\{0\}$ is a G_δ in the space of all real numbers.)

(b) If A is a closed G_δ in a normal topological space X, then there exists a continuous real-valued function f such that $A = f^{-1}[0]$.

K PERFECTLY NORMAL SPACES

A topological space is called *perfectly normal* iff it is normal and each closed subset is a G_δ.

(a) Each pseudo-metrizable space is perfectly normal.

(b) The product of an uncountable number of unit intervals is not perfectly normal. (A G_δ in such a space cannot consist of a single point.)

L CHARACTERIZATION OF COMPLETELY REGULAR SPACES

A topological space is completely regular iff it is homeomorphic to a subspace of a product of pseudo-metric spaces.

M UPPER SEMI-CONTINUOUS DECOMPOSITION OF A NORMAL SPACE

The image of a normal topological space under a closed continuous map is a normal space.

Chapter 5

COMPACT SPACES

The notion of a compact topological space is (like every concept studied in this book) an abstraction of certain important properties of the set of real numbers. The classic theorem of Heine-Borel-Lebesgue asserts that every open cover of a closed and bounded subset of the space of real numbers has a finite subcover. This theorem has extraordinarily profound consequences, and, like most good theorems, its conclusion has become a definition. A topological space is **compact (bicompact)** if and only if each open cover has a finite subcover.* A subset A of a topological space is compact iff it is, with the relative topology, compact; equivalently A is compact iff every cover of A by sets which are open in X has a finite subcover.

EQUIVALENCES

This section is devoted to characterizations of compactness in terms of closed sets, convergence, bases, and subbases.

A family $\mathcal{A}$ of sets has the **finite intersection property** iff the intersection of the members of each finite subfamily of $\mathcal{A}$ is non-void. The De Morgan formulae (0.2) on complements furnish the connection between this notion and the concept of compactness.

* The term "compact" has also been used to denote "sequentially compact" and "countably compact" (in the terminology of the problems at the end of this chapter). N. Bourbaki and his colleagues reserve the term "compact" for compact Hausdorff spaces.

1 THEOREM *A topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-void intersection.*

PROOF If $\mathcal{A}$ is a family of subsets of a topological space X, then, according to the De Morgan formulae, $X \sim \bigcup \{A: A \in \mathcal{A}\} = \bigcap \{X \sim A: A \in \mathcal{A}\}$ and hence $\mathcal{A}$ is a cover of X iff the intersection of the complements of the members of $\mathcal{A}$ is void. The space X is compact iff each family of open sets, such that no finite subfamily covers X, fails to be a cover, and this is true iff each family of closed sets which possesses the finite intersection property has a non-void intersection. ∎

2 THEOREM *A topological space X is compact if and only if each net in X has a cluster point.*

Consequently, X is compact if and only if each net in X has a subnet which converges to some point of X.

PROOF Let $\{S_n, n \in D\}$ be a net in the compact topological space X and for each n in D let A_n be the set of all points S_m for $m \geqq n$. Then the family of all sets A_n has the finite intersection property because D is directed by $\geqq$, and consequently the family of all closures A_n^- also has the finite intersection property. Since X is compact there is a point s which belongs to each A_n^-, and according to theorem 2.7 such a point s is a cluster point of the net $\{S_n, n \in D\}$. To prove the converse proposition let X be a topological space in which every net has a cluster point and let $\mathcal{A}$ be a family of closed subsets of X such that $\mathcal{A}$ has the finite intersection property. Define $\mathcal{B}$ to be the family of all finite intersections of members of $\mathcal{A}$; then $\mathcal{B}$ has the finite intersection property and since $\mathcal{A} \subset \mathcal{B}$, it is sufficient to show $\bigcap \{B: B \in \mathcal{B}\}$ non-void. The intersection of two members of $\mathcal{B}$ is a member of $\mathcal{B}$ and therefore $\mathcal{B}$ is directed by $\subset$. If we choose a member S_B from each B in $\mathcal{B}$, then $\{S_B, B \in \mathcal{B}\}$ is a net in X and consequently has a cluster point s. If B and C are members of $\mathcal{B}$ such that $C \subset B$, then $S_C \in C \subset B$; therefore the net $\{S_B, B \in \mathcal{B}\}$ is eventually in the closed set B and hence the cluster point s belongs to B. Therefore s belongs to each member of $\mathcal{B}$ and the intersection of the members of $\mathcal{B}$ is non-void. Finally, the second statement of the theorem follows from the fact (2.6) that a point is a cluster point of a net iff some subnet converges to it. ∎

Under certain circumstances compactness can be characterized in terms of the existence of accumulation points of subsets. The following sequence of lemmas and the subsequent theorem indicate the situation. The problems at the end of the chapter show that the limitations imposed are necessary. It is convenient to use a variant of the notion of accumulation point in stating the results. A point x is an **ω-accumulation point** of a set A iff each neighborhood of x contains infinitely many points of A. Each ω-accumulation point of a set is also an accumulation point, and if the space is T_1 the converse holds.

3 LEMMA *Every sequence in a topological space has a cluster point if and only if every infinite set has an ω-accumulation point.*

PROOF Suppose that every sequence has a cluster point and that A is an infinite subset. Then there is a sequence of distinct points (a one-to-one sequence) in A, and each cluster point of such a sequence is clearly an ω-accumulation point of A. Conversely, if every infinite subset of a topological space has an accumulation point and $\{S_n, n \in \omega\}$ is a sequence in the space, then one of two situations must occur. Either the range of the sequence is infinite, in which case each ω-accumulation point of this infinite set is a cluster point of the sequence, or else the range of the sequence is finite. In the latter case, for some point x of the space, $S_n = x$ for infinitely many non-negative integers n, and x is a cluster point of the sequence. ∎

4 LEMMA *If X is a Lindelöf space and every sequence in X has a cluster point, then X is compact.*

PROOF It must be shown that each open cover of X has a finite subcover. Because of the hypothesis it may be assumed that the open cover consists of sets $A_0, A_1, \cdots, A_n \cdots$, for n in ω. Proceeding inductively, let $B_0 = A_0$ and for each p in ω let B_p be the first of the sequence of A's which is not covered by $B_0 \cup B_1 \cup \cdots \cup B_{p-1}$. If this choice is impossible at any stage, then the sets already selected are the required finite subcover. Otherwise it is possible to select a point b_p in B_p for each p in ω such that

$b_p \notin B_i$ for $i < p$. Let x be a cluster point of this sequence. Then $x \in B_p$ for some p, and since x is a cluster point, $b_q \in B_p$ for some $q > p$. But this is a contradiction. ▌

The following theorem summarizes information on sequences and subsequences, accumulation points and compactness.

5 THEOREM *If X is a topological space, then the conditions below are related as follows. For all spaces* (a) *is equivalent to* (b) *and* (d) *implies* (a). *If X satisfies the first axiom of countability, then* (a), (b), *and* (c) *are equivalent. If X satisfies the second axiom of countability, then all four conditions are equivalent. If X is pseudo-metric, then each of the four conditions implies that X satisfies the second countability axiom and all four are equivalent.*

(a) *Every infinite subset of X has an ω-accumulation point.*
(b) *Every sequence in X has a cluster point.*
(c) *For each sequence in X there is a subsequence converging to a point of X.*
(d) *The space X is compact.*

PROOF Lemma 5.3 states that (a) is equivalent to (b) and since a sequence is a net, 5.2 shows that (d) always implies (b). If X satisfies the first axiom of countability then (b) and (c) are equivalent by 2.8. If X satisfies the second axiom of countability, then every open cover has a countable subcover, lemma 5.4 applies, and hence all four statements are equivalent. If X is pseudo-metric, then X satisfies the first axiom of countability, the first three conditions are equivalent, each is implied by compactness, and the theorem will be proved if it is shown that a pseudo-metric space such that each infinite subset has an accumulation point is separable and hence satisfies the second axiom of countability. Suppose that X is such a pseudo-metric space. For r positive consider the family of all sets A such that the distance between any two distinct points of A is at least r. It is easily seen that this family has a maximal member A_r by 0.25. The set A_r must be finite, for the $r/2$ sphere about each point of X contains at most one member of A_r and therefore A_r has no accumulation point. Moreover, the r-sphere about each point x of X must intersect A_r because A_r is maximal and otherwise x could be adjoined to A_r. Finally the union A of sets A_r, for r

the reciprocal of a positive integer, is surely countable and A is clearly dense in X. ∎

If $\mathcal{B}$ is a base for the topology of a compact space X and $\mathcal{A}$ is a cover of X by members of $\mathcal{B}$, then there is a finite subcover of $\mathcal{A}$. Conversely, suppose that $\mathcal{B}$ is a base for the topology and that every cover by members of $\mathcal{B}$ has a finite subcover. If $\mathcal{C}$ is an arbitrary open cover of X define $\mathcal{A}$ to be the family of all members of $\mathcal{B}$ which are subsets of some member of $\mathcal{C}$. Because $\mathcal{B}$ is a base, the family $\mathcal{A}$ is a cover of X, and consequently there is a finite subcover $\mathcal{A}'$ of $\mathcal{A}$. For each member of $\mathcal{A}'$ we may select a member of $\mathcal{C}$ which contains it, and the result is a finite subcover of $\mathcal{C}$. This shows that, if "a base for a topology is compact," then the space is compact. This is a useful but not a very profound result. The corresponding theorem on subbases is both profound and useful.

6 THEOREM (ALEXANDER) *If $\mathcal{S}$ is a subbase for the topology of a space X such that every cover of X by members of $\mathcal{S}$ has a finite subcover, then X is compact.*

PROOF For brevity let us agree that a family of subsets of X is inadequate iff it fails to cover X, and is finitely inadequate iff no finite subfamily covers X. Then the definition of compactness of X can be stated: each finitely inadequate family of open sets is inadequate. Observe that the class of finitely inadequate families of open sets is of finite character and therefore each finitely inadequate family is contained in a maximal family by Tukey's lemma 0.25(c). Such a maximal finitely inadequate family $\mathcal{A}$ has a special property which is established as follows.* If $C \notin \mathcal{A}$ and C is open, then by maximality there is a finite subfamily $A_1, \cdots A_m$ of $\mathcal{A}$ such that $C \cup A_1 \cup \cdots A_m = X$. Hence no open set containing C belongs to $\mathcal{A}$. If D is another open set and $D \notin \mathcal{A}$, then there is $B_1, \cdots, B_n$ in $\mathcal{A}$ such that $D \cup B_1 \cup \cdots \cup B_n = X$ and $(C \cap D) \cup A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_n = X$ by a simple set theoretic calculation. It follows that $C \cap D \notin \mathcal{A}$. Consequently, if no member of a finite family of open sets belongs to $\mathcal{A}$, then no open set containing the intersection belongs to $\mathcal{A}$; restated, if a member of $\mathcal{A}$ contains a finite intersection $C_1 \cap C_2 \cdots \cap C_p$ of open sets, then some $C_i \in \mathcal{A}$.

* Problem 2.I is precisely the result needed here.

The proof of the theorem is now straightforward. Suppose that $\mathcal{S}$ is a subbase such that each open cover by subbase elements has a finite subcover (that is, each finitely inadequate subfamily is inadequate) and suppose that $\mathcal{B}$ is a finitely inadequate family of open subsets of X. Then there is a maximal family $\mathcal{A}$ of this sort containing $\mathcal{B}$ and it is sufficient to show that $\mathcal{A}$ is inadequate. The family $\mathcal{S} \cap \mathcal{A}$ of all members of $\mathcal{A}$ which belong to $\mathcal{S}$ is finitely inadequate and hence $\mathcal{S} \cap \mathcal{A}$ does not cover X. Consequently the theorem will be proved if it is shown that each point in $\bigcup\{A: A \in \mathcal{A}\}$ belongs to $\bigcup\{A: A \in \mathcal{S} \cap \mathcal{A}\}$. Because $\mathcal{S}$ is a subbase each point x of a member A of $\mathcal{A}$ belongs to some finite intersection of members of $\mathcal{S}$ which is contained in A. The paragraph above shows that some one of this finite family belongs to $\mathcal{A}$, hence $\bigcup\{A: A \in \mathcal{A}\} = \bigcup\{A: A \in \mathcal{S} \cap \mathcal{A}\}$, and the theorem is proved. ▌

COMPACTNESS AND SEPARATION PROPERTIES

In this section the consequences of compactness in conjunction with the so-called separation axioms will be examined. In each case the theorem proved is the assumed separation axiom (Hausdorff, regular, completely regular) with the word "point" replaced by "compact set." A simple but important corollary on continuous mappings of compact spaces into Hausdorff spaces is derived, and finally we prove a separation theorem of A. D. Wallace which includes most of the earlier theorems.

It is always true that a closed subset A of a compact space X is compact, for each net in A has a subnet which converges to a point which belongs to A because A is closed. (A proof based directly on the definition of compactness is almost as simple.) The converse theorem is false, for if A is a proper non-void subset of an indiscrete space X (only X and the void set are open), then A is surely compact but not closed. This cannot happen if X is a Hausdorff space.

7 THEOREM *If A is a compact subset of a Hausdorff space X and x is a point of $X \sim A$, then there are disjoint neighborhoods of x and of A.*

Consequently each compact subset of a Hausdorff space is closed.

PROOF Since X is Hausdorff there is a neighborhood U of each point y of A such that x does not belong to the closure U^-. Because A is compact there is a finite family $U_0, U_1, \cdots, U_n$ of open sets covering A such that $x \notin U_i^-$ for $i = 0, 1, \cdots, n$. If $V = \bigcup\{U_i: i = 0, 1, \cdots, n\}$, then $A \subset V$ and $x \notin V^-$. Consequently $X \sim V^-$ and V are disjoint neighborhoods of x and A. ▮

8 THEOREM *Let f be a continuous function carrying the compact topological space X onto the topological space Y. Then Y is compact, and if Y is Hausdorff and f is one to one then f is a homeomorphism.*

PROOF If $\mathfrak{A}$ is an open cover of Y, then the family of all sets of the form $f^{-1}[A]$, for A in $\mathfrak{A}$, is an open cover of X which has a finite subcover. The family of images of members of the subcover is a finite subfamily of $\mathfrak{A}$ which covers Y and consequently Y is compact. Suppose that Y is Hausdorff and f is one to one. If A is a closed subset of X, then A is compact and hence its image $f[A]$ is compact and therefore closed. Then $(f^{-1})^{-1}[A]$ is closed for each closed set A and f^{-1} is continuous. ▮

9 THEOREM *If A and B are disjoint compact subsets of a Hausdorff space X, then there are disjoint neighborhoods of A and B.*
Consequently each compact Hausdorff space is normal.

PROOF For each x in A there is by 5.7 a neighborhood of x and a neighborhood of B which are disjoint. Consequently there is a neighborhood U of x whose closure is disjoint from B, and since A is compact there is a finite family $U_0, U_1, \cdots, U_n$ such that U_i^- is disjoint from B for $i = 0, 1, \cdots, n$ and $A \subset V = \bigcup\{U_i: i = 0, 1, \cdots, n\}$. Then V is a neighborhood of A and $X \sim V^-$ is a neighborhood of B which is disjoint from V. ▮

10 THEOREM *If X is a regular topological space, A a compact subset, and U a neighborhood of A, then there is a closed neighborhood V of A such that $V \subset U$.*
Consequently each compact regular space is normal.

PROOF Because X is regular, for each x in A there is an open neighborhood W of x such that $W^- \subset U$, and by compactness there is a finite open cover $W_0, W_1, \cdots, W_n$ of A such that $W_i^- \subset U$ for each i. Then $V = \bigcup \{W_i^-: i = 0, 1, \cdots, n\}$ is the required neighborhood of A. ▌

11 THEOREM *If X is a completely regular space, A is a compact subset and U is a neighborhood of A, then there is a continuous function f on X to the closed interval* [0,1] *such that f is one on A and zero on $X \sim U$.*

PROOF For each x in A there is a continuous function g which is one at x and zero on $X \sim U$. The set $\{y: g(y) > \frac{1}{2}\}$ is open in X and hence if h is defined by $h(y) = \min [2g(y),1]$, then h is continuous, has values in [0,1], is zero on $X \sim U$, and is one on a neighborhood of x. Because A is compact there is a finite family $h_0, h_1, \cdots h_n$ of continuous functions on X to [0,1] such that $A \subset \bigcup \{h_i^{-1}[1]: i = 0, 1, \cdots, n\}$ and each h_i is zero on $X \sim U$. The function f whose value at x is $\max \{h_i(x): i = 0, 1, \cdots, n\}$ is the required function. ▌

Each of the last two theorems has a formulation which is superficially different; the statement "A is compact and U a neighborhood of A" can be replaced by "if A is compact and B is a disjoint closed set," and the conclusion changed in the obvious way.

Most of the results of this section are easy consequences of the following theorem.

12 THEOREM (WALLACE) *If X and Y are topological spaces, A and B are compact subsets of X and Y respectively, and W is a neighborhood of $A \times B$ in the product space $X \times Y$, then there are neighborhoods U of A and V of B such that $U \times V \subset W$.*

PROOF For each member (x,y) of $A \times B$ there are open neighborhoods R of x and S of y such that $R \times S \subset W$. Since B is compact, for a fixed x in A there are neighborhoods R_i of x and corresponding open sets S_i, for $i = 0, 1, \cdots n$, such that $B \subset Q = \bigcup \{S_i: i = 0, 1, \cdots, n\}$. If $P = \bigcap \{R_i: i = 0, 1, \cdots n\}$, then P is a neighborhood of x and Q is a neighborhood of B such that $P \times Q \subset W$. Since A is compact there are open sets P_i

in X and Q_i in Y, for $i = 0, 1, \cdots m$, such that each Q_i is a neighborhood of B, $P_i \times Q_i \subset W$, and $A \subset \bigcup \{P_i: i = 0, 1, \cdots m\} = U$. Then U and $V = \bigcap \{Q_i: i = 0, 1, \cdots, m\}$ are neighborhoods of A and B respectively, $U \times V$ is a subset of W, and the theorem follows. ∎

PRODUCTS OF COMPACT SPACES

The classical theorem of Tychonoff on the product of compact spaces is unquestionably the most useful theorem on compactness. It is probably the most important single theorem of general topology. This section is devoted to the Tychonoff theorem and a few of its consequences.

13 THEOREM (TYCHONOFF) *The cartesian product of a collection of compact topological spaces is compact relative to the product topology.*

PROOF Let $Q = \times\{X_a: a \in A\}$ where each X_a is a compact topological space and Q has the product topology. Let $\mathcal{S}$ be the subbase for the product topology consisting of all sets of the form $P_a^{-1}[U]$ where P_a is the projection into the a-th coordinate space and U is open in X_a. In view of theorem 5.6 the space Q will be compact if each subfamily $\mathcal{A}$ of $\mathcal{S}$, such that no finite subfamily of $\mathcal{A}$ covers Q, fails to cover Q. For each index a let $\mathcal{B}_a$ be the family of all open sets U in X_a such that $P_a^{-1}[U] \in \mathcal{A}$. Then no finite subfamily of $\mathcal{B}_a$ covers X_a and hence by compactness there is a point x_a such that $x_a \in X_a \sim U$ for each U in $\mathcal{B}_a$. The point x whose a-th coordinate is x_a then belongs to no member of $\mathcal{A}$ and consequently $\mathcal{A}$ is not a cover. ∎

We give an alternate proof of Tychonoff's theorem which does not depend on the Alexander theorem 5.6.

ALTERNATE PROOF (BOURBAKI) It will be proved that if $\mathcal{B}$ is a family of subsets of the product and $\mathcal{B}$ has the finite intersection property, then $\bigcap \{B^-: B \in \mathcal{B}\}$ is not void. The class of all families which possess the finite intersection property is of finite character and consequently we may assume that $\mathcal{B}$ is maximal with respect to this property by Tukey's lemma 0.25(c). Because $\mathcal{B}$

is maximal each set which contains a member of $\mathfrak{B}$ belongs to $\mathfrak{B}$ and the intersection of two members of $\mathfrak{B}$ belongs to $\mathfrak{B}$. Moreover, if C intersects each member of $\mathfrak{B}$, then $C \in \mathfrak{B}$ by maximality.* Finally, the family of projections of members of $\mathfrak{B}$ into a coordinate space X_a has the finite intersection property and it is therefore possible to choose a point x_a in $\bigcap\{P_a[B]^-: B \in \mathfrak{B}\}$. The point x whose a-th coordinate is x_a then has the property: each neighborhood U of x_a intersects $P_a[B]$ for every B in $\mathfrak{B}$, or equivalently $P_a^{-1}[U] \in \mathfrak{B}$, for each neighborhood U of x_a in X_a. Therefore finite intersections of sets of this form belong to $\mathfrak{B}$. Then each neighborhood of x which belongs to the defining base for the product topology belongs to $\mathfrak{B}$ and hence intersects each member of $\mathfrak{B}$. Therefore x belongs to B^- for each B in $\mathfrak{B}$, and the theorem is proved. ∎

Several important applications of Tychonoff's theorem occur in the chapter on function spaces; for the moment we consider a very simple consequence. A subset of a pseudo-metric space is **bounded** iff it is of finite diameter. Thus a subset of the space of real numbers is bounded iff it has both an upper and lower bound. The following is the classical theorem of Heine-Borel-Lebesgue.

14 THEOREM *A subset of Euclidean n-space is compact if and only if it is closed and bounded.*

PROOF Let A be a compact subset of E_n. Then A is closed because E_n is a Hausdorff space. Because of compactness A can be covered by a finite family of open spheres of radius one, and because each of these is bounded A is bounded. To prove the converse suppose that A is a closed and bounded subset of E_n. Let B_i be the image of A under the projection into the i-th coordinate space, and notice that each B_i is bounded because the projection decreases distances. Then $A \subset \times\{B_i: i = 0, 1, \cdots, n-1\}$, and this set is a subset of a product of closed bounded intervals of real numbers. Since A is a closed subset of the product, and the product of compact spaces is compact, the proof reduces to showing that a closed interval $[a,b]$ is compact relative to the usual topology. Let $\mathfrak{C}$ be an open cover of $[a,b]$ and

* We are evidently reproving part of proposition 2.I.

let c be the supremum of all members x of $[a,b]$ such that some finite subfamily of $\mathcal{C}$ covers $[a,x]$. (The set is not void because a is a member.) Choose U in $\mathcal{C}$ such that $c \in U$, and choose a member d of the open interval (a,c) such that $[d,c] \subset U$. There is a finite subfamily of $\mathcal{C}$ which covers $[a,d]$, and this family with U adjoined covers $[a,c]$. Unless $c = b$ the same finite subfamily covers an interval to the right of c, which contradicts the choice of c. The theorem follows. ∎

The closed unit interval is compact and consequently each cube (the product of closed unit intervals) is compact. The following characterization of Tychonoff spaces (completely regular T_1-spaces) is then almost self-evident.

15 THEOREM *A topological space is a Tychonoff space if and only if it is homeomorphic to a subspace of a compact Hausdorff space.*

PROOF By 4.6, each Tychonoff space is homeomorphic to a subset of a cube, which is a compact Hausdorff space. Conversely, each compact Hausdorff space is normal and consequently (Urysohn's lemma 4.4) is a Tychonoff space, and each subspace is therefore a Tychonoff space. ∎

The product of more than a finite number of non-compact spaces fails to be compact in a rather spectacular way. A set in a topological space is **nowhere-dense** in the space iff its closure has a void interior.

16 THEOREM *If an infinite number of the coordinate spaces are non-compact, then each compact subset of the product is nowhere dense.*

PROOF Suppose that $\times\{X_a: a \in A\}$ has a compact subset B with an interior point x. Then B contains a neighborhood U of x which is a member of the defining base and is therefore of the form $\bigcap\{P_a^{-1}[V_a]: a \in F\}$, where F is a finite subset of A and V_a is open in X_a. If b is a member of $A \sim F$, then $P_b[B] = X_b$ and X_b is therefore compact because it is the continuous image of a compact space. Hence all but a finite number of the coordinate spaces are compact. ∎

LOCALLY COMPACT SPACES

A topological space is **locally compact** iff each point has at least one compact neighborhood. A compact space is automatically locally compact, every discrete space is locally compact, and each closed subspace of a locally compact space is itself locally compact (the intersection of a closed set and a compact set is a closed subset of the latter, and hence compact). Many of the pleasant properties of compact spaces are shared by locally compact spaces. The following proposition is a convenient tool for the study of such spaces.

17 THEOREM *If X is a locally compact topological space which is either Hausdorff or regular, then the family of closed compact neighborhoods of each point is a base for its neighborhood system.*

PROOF Let x be a point of X, C a compact neighborhood of x, and U an arbitrary neighborhood of x. If X is regular, then there is a closed neighborhood V of x which is a subset of the intersection of U and the interior of C, and evidently V is closed and compact. If X is Hausdorff and W is the interior of $U \cap C$, then, since W^- is a compact Hausdorff space, W contains a closed compact set V which is a neighborhood of x in W^- by 5.9; but V is also a neighborhood of x in W (that is, with respect to the relativized topology for W) and is therefore a neighborhood of x in X. ∎

In particular it follows that every locally compact Hausdorff space is regular; actually a stronger statement is true.

18 THEOREM *If U is a neighborhood of a closed compact subset A of a regular locally compact topological space X, then there is a closed compact neighborhood V of A such that $A \subset V \subset U$.*

Moreover, there is a continuous function f on X to the closed unit interval such that f is zero on A and one on $X \sim V$.

PROOF For each point x of A there is a neighborhood W which is a closed compact subset of U. Since A is compact it may be covered by a finite family of such neighborhoods and their union is a closed compact neighborhood V of A. Then V with the relative topology is a regular compact space which is therefore normal (5.10). Hence there is a continuous function g on V to the closed unit interval such that g is zero on A and one on $V \sim V^0$ (V^0 is the interior of V). Let f equal g on V and one on $X \sim V$. Then f is continuous because V^0 and $X \sim V$ are separated and f is continuous on V and $X \sim V^0$. (Problem 3.B.) ▌

It follows that each locally compact, regular, topological space is completely regular and each locally compact Hausdorff space is a Tychonoff space.

It is not true that the continuous image of a locally compact space is locally compact, for every discrete space is locally compact and each topological space is the continuous one-to-one image of a discrete space (using the same set, the discrete topology, and the identity function). If a function is both open and continuous, then the image of a compact neighborhood of a point is a compact neighborhood of the image point, and consequently the image of a locally compact space is locally compact. This simple fact and an earlier result give a precise description of those product spaces which are locally compact.

19 THEOREM *If a product is locally compact, then each coordinate space is locally compact and all except a finite number of coordinate spaces are compact.*

PROOF If a product is locally compact, then each coordinate space is locally compact because projection into a coordinate space is open. If infinitely many coordinate spaces are non-compact, then each compact subset of the product is nowhere dense, according to 5.16, and no point has a compact neighborhood. ▌

QUOTIENT SPACES

In this section the investigation of quotient spaces which was begun in chapter 3 is continued. We are interested in the consequences of compactness and the single theorem of the section summarizes some of the pleasant properties which result from the additional assumption. It has already been observed that the continuous image of a compact space is compact, but without additional hypotheses the image space may still be quite unattractive. For example, if X is the closed unit interval with the usual topology and $\mathfrak{D}$ is the decomposition consisting of all subsets of the form $\{x: x - a \text{ is rational}\}$, then the quotient space is compact and the projection onto the quotient space is open, but the quotient topology is indiscrete (only the space and the void set open). It turns out that, if the members of $\mathfrak{D}$ are compact and the decomposition is upper semi-continuous, then the quotient space inherits many of the properties of X.

20 THEOREM *Let X be a topological space, let $\mathfrak{D}$ be an upper semi-continuous decomposition of X whose members are compact, and let $\mathfrak{D}$ have the quotient topology. Then $\mathfrak{D}$ is, respectively, Hausdorff, regular, locally compact, or satisfies the second axiom of countability, provided X has the corresponding property.*

PROOF For convenience let us agree that a subset of X is admissible iff it is the union of members of $\mathfrak{D}$. In view of the definition of upper semi-continuity each neighborhood in X of a member A of $\mathfrak{D}$ contains an admissible neighborhood, and hence the image under projection of a neighborhood of A in X is a neighborhood of A in $\mathfrak{D}$. Moreover, projection carries closed sets into closed sets (3.12). Suppose that X is a Hausdorff space and that A and B are distinct members of $\mathfrak{D}$. Then by 5.9 there are disjoint neighborhoods (in X) of A and B, these contain disjoint admissible neighborhoods, and the projections of the latter are the required disjoint neighborhoods of A and B in $\mathfrak{D}$. If X is regular, $A \in \mathfrak{D}$, and $\mathfrak{U}$ is a neighborhood of A in $\mathfrak{D}$, then the union U of the members of $\mathfrak{U}$ is a neighborhood of A in X. In view of 5.10 there is a closed neighborhood of A contained in U, and the image under projection of this neighborhood is the required neighborhood of A in $\mathfrak{D}$. If X is locally compact, then evidently there is a compact neighborhood of each member of $\mathfrak{D}$, and the image under projection is a compact neighborhood in $\mathfrak{D}$.

Finally, suppose there is a countable base $\mathfrak{B}$ for the topology of X. The family $\mathfrak{U}$ of unions of finite subfamilies of $\mathfrak{B}$ is countable. For each member U of $\mathfrak{U}$ let U' be the union of all members of $\mathfrak{D}$ which are subsets of U, and let $\mathfrak{J}$ be the family of all sets U' for U in $\mathfrak{U}$. Then the images of the members of $\mathfrak{J}$ are open and it will be shown that the collection of images is a base for the quotient topology. This will follow if for each A in $\mathfrak{D}$ and each neighborhood V of A there is U in $\mathfrak{I}$ such that $A \subset U \subset V$. But A may be covered by a finite number of the members of $\mathfrak{B}$ such that the union W of these members, which is a member of $\mathfrak{U}$, is contained in V. If $U = W'$, then $U \in \mathfrak{I}$ and $A \subset U \subset V$, and the theorem follows. ∎

There is an interesting corollary to this theorem. If X is separable metric and the members of an upper semi-continuous decomposition are compact, then the quotient space is Hausdorff, normal, and satisfies the second axiom of countability, and is consequently metrizable.

COMPACTIFICATION

In studying a non-compact topological space X it is often convenient to construct a space which contains X as a subspace and is itself compact. For example, it is frequently useful to adjoin two points, $+\infty$ and $-\infty$, to the space of real numbers. The resulting space is sometimes called the *extended* real numbers; it is linearly ordered by agreeing that $+\infty$ is the largest member and $-\infty$ is the smallest. With this ordering (an extension of the usual ordering) it turns out that every non-void subset of the extended real numbers has both an infimum and a supremum and the space is compact relative to its order topology (5.C). The extended reals are a *compactification* of the space of real numbers, in a sense which will presently be made precise. Of course, this device is primarily a convenience. It does not add to our knowledge of the real numbers. However, it does permit the use of the standard compactness arguments and it simplifies many proofs materially.

The simplest sort of compactification of a topological space is made by adjoining a single point. This procedure is familiar in analysis, for in function theory the complex sphere is constructed by adjoining a single point, ∞, to the Euclidean plane and specifying that the neighborhoods of ∞ are the complements of bounded subsets of the plane. This construction can be duplicated for an arbitrary topological space; the clue to the topology to be introduced in the enlarged space is the fact that the complement of an open neighborhood of ∞ in the complex sphere is compact.

The **one point compactification** * of a topological space X is the set $X^* = X \cup \{\infty\}$ with the topology whose members are the open subsets of X and all subsets U of X^* such that $X^* \sim U$ is a closed compact subset of X. Of course, it must be verified that this specification gives a topology for X^*. This verification is made in the proof of the following proposition.

21 THEOREM (ALEXANDROFF) *The one point compactification X^* of a topological space X is compact and X is a subspace. The space X^* is Hausdorff if and only if X is locally compact and Hausdorff.*

PROOF A set U is open in X^* iff (a) $U \cap X$ is open in X and (b) whenever $\infty \in U$, then $X \sim U$ is compact. Consequently finite intersections and arbitrary unions of sets open in X^* intersect X in open sets. If ∞ is a member of the intersection of two open subsets of X^*, then the complement of the intersection is the union of two closed compact subsets of X and is therefore closed and compact. If ∞ belongs to the union of the members of a family of open subsets of X^*, then ∞ belongs to some member U of the family, and the complement of the union is a closed subset of the compact set $X \sim U$ and is therefore closed and compact. Consequently X^* is a topological space and X is a subspace. If $\mathfrak{U}$ is an open cover of X^*, then ∞ is a member of some U in $\mathfrak{U}$ and $X \sim U$ is compact, and hence there is a finite subcover of $\mathfrak{U}$. Therefore X^* is compact. If X^* is a Hausdorff space, then its open subspace X is a locally compact Hausdorff space. Finally it must be shown that X^* is a Hausdorff space if X is a locally compact Hausdorff space. It is only necessary to show that, if $x \in X$, then there are disjoint neighborhoods of x and ∞. But since X is locally compact and Hausdorff there is a closed compact neighborhood U of x in X and $X^* \sim U$ is the required neighborhood of ∞. ▌

If X is a compact topological space, then ∞ is an isolated point of the one point compactification (that is, $\{\infty\}$ is both open and closed). Conversely, if ∞ is an isolated point of X^*, then X is closed in X^* and is therefore compact.

The one point compactification is of a very special sort, and

* This definition is actually incomplete until ∞ is defined. Any element which is not a member of X, for example X, will do.

we wish to consider other methods of embedding a topological space in a compact space. It is convenient to allow a topological embedding rather than insist that the original be actually a subspace of the constructed compact space. With this in mind, a **compactification** of a topological space X is defined to be a pair (f,Y), where Y is a compact topological space and f is a homeomorphism of X onto a dense subspace of Y. (To be consistent, the one point compactification of X should be the pair (i,X^*), where i is the identity function.) A compactification (f,Y) is called Hausdorff iff Y is a Hausdorff space. A relation is defined on the collection of all compactifications of a space X by agreeing that $(f,Y) \geqq (g,Z)$ iff there is a continuous map h of Y onto Z such that $h \circ f = g$. Equivalently $(f,Y) \geqq (g,Z)$ iff the function $g \circ f^{-1}$ on $f[X]$ to Z has a continuous extension h which carries Y onto Z. If the function h can be taken to be a homeomorphism, then (f,Y) and (g,Z) are said to be **topologically equivalent.** In this case both of the relations $(f,Y) \geqq (g,Z)$ and $(g,Z) \geqq (f,Y)$ hold, for h^{-1} is a continuous map of Z onto Y such that $f = h^{-1} \circ g$.

22 THEOREM *The collection of all compactifications of a topological space is partially ordered by* $\geqq$. *If* (f,Y) *and* (g,Z) *are Hausdorff compactifications of a space and* $(f,Y) \geqq (g,Z) \geqq (f,Y)$, *then* (f,Y) *and* (g,Z) *are topologically equivalent.*

PROOF If $(f,Y) \geqq (g,Z) \geqq (h,U)$, where these are compactifications of a space X, then there are continuous functions j on Y to Z and k on Z to U such that $g = j \circ f$ and $h = k \circ g$ and hence $h = k \circ j \circ f$ and $(f,Y) \geqq (h,U)$. Consequently $\geqq$ partially orders the collection of all compactifications of X. If (f,Y) and (g,Z) are Hausdorff compactifications each of which follows the other relative to the ordering $\geqq$, then both $f \circ g^{-1}$ and $g \circ f^{-1}$ have continuous extensions j and k to all of Z and Y respectively. Since $k \circ j$ is the identity map on the dense subset $g[X]$ of Z and Z is Hausdorff, $k \circ j$ is the identity map of Z onto itself and similarly $j \circ k$ is the identity map of Y onto Y. Consequently (f,Y) and (g,Z) are topologically equivalent. ∎

The smallest compactification of a compact Hausdorff space X is X itself (more precisely (i,X) where i is the identity map on

X). One would expect that the one point compactification of a non-compact space would be the smallest relative to the ordering $\geqq$. If we restrict our attention to Hausdorff compactifications this is actually the case (a corollary to 5.G), although it is easy to see that there is generally no compactification which is smaller than every other. On the other hand, if X is a space which has a Hausdorff compactification (by 5.15 such a space is a Tychonoff space), then there is a largest compactification which we now construct.

For each topological space X let $F(X)$ be the family of all continuous functions on X to the closed unit interval Q. The cube $Q^{F(X)}$ (the product of the unit interval Q taken $F(X)$ times) is compact by the Tychonoff theorem. The evaluation map e carries a member x of X into the member $e(x)$ of $Q^{F(X)}$ whose f-th coordinate is $f(x)$ for each f in $F(X)$. Evaluation is a continuous map of X into the cube $Q^{F(X)}$, and if X is a Tychonoff space, then e is a homeomorphism of X onto a subspace of $Q^{F(X)}$. (The embedding lemma 4.5 states these facts explicitly.) The Stone-Čech compactification is the pair $(e, \beta(X))$ where $\beta(X)$ is the closure of $e[X]$ in the cube $Q^{F(X)}$. We take time out for a lemma before showing the crucial property of this compactification.

23 LEMMA *If f is a function on a set A to a set B and f^* is the map of Q^B into Q^A defined by $f^*(y) = y \circ f$ for all y in Q^B, then f^* is continuous.*

PROOF A map into a product space is continuous iff the map followed by each projection is continuous (3.3). If a is a member of A, then $P_a \circ f^*(y) = P_a(y \circ f) = y(f(a))$. But $y(f(a))$ is simply the projection of y into the $f(a)$-coordinate space of Q^B and this is a continuous map. ▮

The construction outlined in this lemma is worthy of notice, for it is used systematically in dealing with function spaces. Observe that the function f^* induced by f goes in the direction opposite to that of f, in the sense that f carries A into B while f^* carries Q^B into Q^A.

With the aid of this lemma the principal theorem on the Stone-Čech compactification becomes a routine though mildly intricate calculation.

24 THEOREM (STONE-ČECH) *If X is a Tychonoff space and f is a continuous function on X to a compact Hausdorff space Y, then there is a continuous extension of f which carries the compactification $\beta(X)$ into Y. (More precisely, if $(e,\beta(X))$ is the Stone-Čech compactification, then $f \circ e^{-1}$ can be extended to a continuous function on $\beta(X)$ to Y.)*

PROOF Given f define f^* on $F(Y)$ to $F(X)$ by letting $f^*(a) = a \circ f$ for each a in $F(Y)$. Continuing, define f^{**} on $Q^{F(X)}$ to $Q^{F(Y)}$ by letting $f^{**}(q) = q \circ f^*$ for each q in $Q^{F(X)}$. Let e be the evaluation map of X into $Q^{F(X)}$ and let g be the evaluation map of Y into $Q^{F(Y)}$. The following diagram shows the situation.

$$\begin{tikzcd} \beta(X) \subset Q^{F(X)} \arrow[r, "f^{**}"] & Q^{F(Y)} \supset \beta(Y) \\ X \arrow[u, "e"] \arrow[r, "f"] & Y \arrow[u, "g"'] \end{tikzcd}$$

The map e is a homeomorphism, and the map g is a homeomorphism of Y onto $\beta(Y)$ because Y is compact Hausdorff. The map f^{**} is continuous by lemma 5.23 and, if it is shown that $f^{**} \circ e = g \circ f$, then it will follow that $g^{-1} \circ f^{**}$ is the required continuous extension of $f \circ e^{-1}$. If x is a member of X and h a member of $F(Y)$, then $(f^{**} \circ e)(x)(h) = (e(x) \circ f^*)(h) = e(x)(h \circ f) = h \circ f(x) = g(f(x))(h) = (g \circ f)(x)(h)$ because of the definitions of f^{**}, f^*, e, and g respectively. The theorem follows. ∎

The extension property of the foregoing theorem shows that the Stone-Čech compactification $(e,\beta(X))$ follows every other Hausdorff compactification in the ordering $\geqq$ and is therefore the largest such compactification. If (f,Y) has this extension property, then $(f,Y) \geqq (e,\beta(X))$ and consequently is topologically equivalent to $(e,\beta(X))$ by 5.22. Hence the compactification $(e,\beta(X))$ is characterized (to a topological equivalence) by the extension property of theorem 5.24.

25 *Note* The results above (M. H. Stone [6] and Čech, [1]) furnish a maximal compactification. Many other smaller compactifications have been constructed for various purposes. There is a very large literature on the subject and it is only possible to

cite a few sample contributions. For a recent contribution to one of the oldest compactification theories (Carathéodory's prime end theory) see Ursell and Young [1]. Freudenthal [1] examines a compactification which is maximal in a class much more restricted than that majorized by $\beta(X)$. A general discussion of compactification is given by Myškis ([1], [2], and [3]). He distinguishes between "external" descriptions of a compactification (such as that of $\beta(X)$, and the almost periodic compactification of a group as sketched in 7.T) and "internal" descriptions (for example the Alexandroff one point compactification and the Wallman (5.R)). The relation between internal and external description of a compactification is frequently the key to the usefulness of the notion. Certain parts of the internal structure of $\beta(X)$ have been discussed (see Nagata [2], Smirnov [3], and Wallace [2]). The compactification $\beta(X)$ is also related to the notion of absolute closure; see, for example, M. H. Stone [6], A. D. Alexandroff [1], Katětov [1], and Ramanathan [1].

LEBESGUE'S COVERING LEMMA

There is an extremely useful lemma of Lebesgue which states that, if $\mathfrak{U}$ is an open cover of a closed interval of real numbers, then there is a positive numbei r such that, if $|x - y| < r$, then both x and y belong to some member of the cover. In a certain sense each open cover covers the points of the interval "uniformly." In this section we prove this lemma and a topological variant which will apply to arbitrary compact spaces. The latter result may be considered to be an introduction to the ideas of the next section on paracompactness.

26 THEOREM *If $\mathfrak{U}$ is an open cover of a compact subset A of a pseudo-metric space (X,d), then there is a positive number r such that the open r-sphere about each point of A is contained in some member of $\mathfrak{U}$.*

PROOF Let $U_1, \cdots, U_n$ be a finite subcover of the open cover $\mathfrak{U}$ of A, let $f_i(x) = \operatorname{dist}[x, X \sim U_i]$, and let $f(x) = \max[f_i(x): i = 1, \cdots, n]$. Then each f_i is continuous and consequently f is continuous. Each point of A belongs to some U_i and hence

$f(x) \geqq f_i(x) > 0$ for each x in A. The set $f[A]$ is a compact subset of the positive real numbers and consequently there is a positive real number r such that $f(x) > r$ for all x in A. Hence for each x in A there is i such that $f_i(x) > r$ and it follows that the open r-sphere about x is contained in U_i. ▮

There is a useful corollary of the foregoing theorem. If A is a compact subset of a pseudo-metric space and U is a neighborhood of A, then there is a positive number r such that U contains the open r-sphere about every point of A; that is, the distance from A to $X \sim U$ is positive.

Theorem 5.26 may be rephrased in a suggestive way. If V is the set of all pairs of points of X such that $d(x,y) < r$, then $V[x] = \{y: (x,y) \in V\}$ is simply the open r-sphere about x. The set V is an open subset of $X \times X$ and contains the diagonal Δ (the set of all pairs (x,x) for x in X). The foregoing theorem then implies the following topological result: If $\mathfrak{U}$ is an open cover of a compact pseudo-metric space, then there is a neighborhood V of the diagonal in $X \times X$ such that for each point x the set $V[x]$ is contained in some member of $\mathfrak{U}$. This variant of the Lebesgue lemma turns out to be correct for arbitrary compact regular spaces.

A cover $\mathfrak{U}$ of a topological space is called an **even** cover iff there is a neighborhood V of the diagonal in $X \times X$ such that for each x the set $V[x]$ is contained in some member of $\mathfrak{U}$. In other words, the family of all sets of the form $V[x]$ refines $\mathfrak{U}$. Recall that a cover $\mathfrak{A}$ is a refinement of $\mathfrak{U}$ iff each member of $\mathfrak{A}$ is a subset of some member of $\mathfrak{U}$, and that a family $\mathfrak{B}$ of sets is locally finite iff there is a neighborhood of each point of the space which intersects only finitely many members of $\mathfrak{B}$. A family of sets is **closed** iff each member is closed.

27 THEOREM *If an open cover of a space has a closed locally finite refinement then it is an even cover.*

Consequently each open cover of a compact regular space is even.

PROOF Let $\mathfrak{U}$ be an open cover of a topological space X and let $\mathfrak{A}$ be a closed locally finite refinement. For each A in $\mathfrak{A}$ choose a member U_A of $\mathfrak{U}$ such that $A \subset U_A$, and let $V_A = (U_A \times U_A) \cup ((X \sim A) \times (X \sim A))$. Evidently V_A is an open neighborhood of the diagonal in $X \times X$, and, if $x \in A$, then $V_A[x] = U_A$. Letting $V = \bigcap \{V_A : A \in \mathcal{A}\}$, it follows that for each point x the set $V[x] \subset V_A[x] = U_A$ and consequently the family of sets of the form $V[x]$ is a refinement of $\mathcal{U}$. It remains to be proved that V is a neighborhood of the diagonal. For each point (x,x) of the diagonal choose a neighborhood W of x such that W intersects only finitely many members of $\mathcal{A}$. If $W \cap A$ is void, then $W \subset X \sim A$ and $W \times W \subset V_A$. It follows that V contains the intersection of $W \times W$ with a finite number of the sets V_A and is therefore a neighborhood of (x,x).

Finally, if X is compact and regular, then each open cover $\mathcal{U}$ has a closed finite refinement (cover X by open subsets whose closures refine $\mathcal{U}$) and hence each open cover is even. ▌

* *PARACOMPACTNESS*

A topological space is paracompact iff it is regular * and each open cover has an open locally finite refinement. The purpose of this section is to prove the equivalence of paracompactness and a number of other conditions. The methods used are closely related to those of chapter 6.

Recall that a family $\mathcal{A}$ of subsets of a topological space is discrete iff there is a neighborhood of each point of the space which intersects at most one member of the family. The family $\mathcal{A}$ is σ-discrete (σ-locally finite) iff it is the union of countably many discrete (respectively locally finite) subfamilies. The principal theorem of the section can now be stated; its proof is given in the sequence of lemmas which follows the statement.

28 THEOREM *If X is a regular topological space, then the following statements are equivalent.*

(a) *The space X is paracompact.*
(b) *Each open cover of X has a locally finite refinement.*
(c) *Each open cover of X has a closed locally finite refinement.*
(d) *Each open cover of X is even.*
(e) *Each open cover of X has an open σ-discrete refinement.*
(f) *Each open cover of X has an open σ-locally finite refinement.*

* The usual definition of paracompact specifies "Hausdorff" instead of "regular." It is not hard to show that a Hausdorff space is regular if each open cover has an open locally finite refinement.

The pattern of proof is (a) $\to$ (b) $\to$ (c) $\to$ (d) $\to$ (e) $\to$ (f) $\to$ (b) $\to$ (a). The first of these implications is clear, and the following lemma demonstrates the second.

29 LEMMA *If X is regular and each open cover has a locally finite refinement, then each open cover has a closed locally finite refinement.*

PROOF If $\mathfrak{U}$ is an open cover of X, then there is an open cover $\mathfrak{V}$ such that the family of closures of members of $\mathfrak{V}$ refines $\mathfrak{U}$, because X is regular. (For each x, if $x \in U$ there is an open neighborhood V of x such that $V^- \subset U$.) Let $\mathfrak{A}$ be a locally finite refinement of $\mathfrak{V}$. Then the family $\mathfrak{B}$ of all closures of members of $\mathfrak{A}$ is locally finite, and each member of $\mathfrak{B}$ is a subset of V^- for some V in $\mathfrak{V}$. Hence $\mathfrak{B}$ is the required closed locally finite refinement of $\mathfrak{U}$. ∎

For any topological space an open cover which has a closed locally finite refinement is even, according to 5.27. Hence statement (c) of the theorem implies (d). Before proving the next implication we prove two lemmas which are of some interest in themselves. For convenience we review some of the facts which will be needed (see the section on relations in chapter 0). If U is a subset of $X \times X$ and $x \in X$, then $U[x]$ is the set of all points y such that $(x,y) \in U$. If A is a subset of X, then $U[A] = \{y: (x,y) \in U \textit{ for some } x \textit{ in } A\}$; clearly $U[A]$ is the union of the sets $U[x]$ for x in A. The set $\{(x,y): (y,x) \in U\}$ is denoted by U^{-1}, and U is called symmetric if $U = U^{-1}$. The set $U \cap U^{-1}$ is always symmetric. If U and V are subsets of $X \times X$, then $U \circ V$ is the set of all pairs (x,z) such that for some y in X it is true that $(x,y) \in V$ and $(y,z) \in U$. In other words $(x,z) \in U \circ V$ iff $(x,z) \in V^{-1}[y] \times U[y]$ for some y, and consequently $U \circ V$ is the union of the sets $V^{-1}[y] \times U[y]$ for y in X. In particular if V is symmetric, then $V \circ V = \bigcup \{V[y] \times V[y]: y \in X\}$. Finally, for each subset A of X it is true that $(U \circ V)[A] = U[V[A]]$.

30 LEMMA *Let X be a topological space such that each open cover is even. If U is a neighborhood of the diagonal in $X \times X$ then there is a symmetric neighborhood V of the diagonal such that $V \circ V \subset U$.*

PROOF For each point x of X there is a neighborhood $W(x)$ such that $W(x) \times W(x) \subset U$, because U is a neighborhood of the diagonal. The family $\mathcal{W}$ of all sets of the form $W(x)$ is an open cover of X and there is therefore a neighborhood R of the diagonal such that the family of all sets $R[x]$ refines $\mathcal{W}$, and hence $R[x] \times R[x] \subset U$ for each x. Finally, let $V = R \cap R^{-1}$. Then V is a symmetric neighborhood of the diagonal and $V[x] \times V[x] \subset U$ for all x. Since $V \circ V$ is the union of the sets $V[x] \times V[x]$ it follows that $V \circ V \subset U$. ▌

The preceding lemma has the following intuitive content. Let us say two points x and y are at most U-distance apart if $(x,y) \in U$. Then there is V such that, if x and y, and y and z, are at most V-distance apart, then x and z are at most U-distance apart.

The following lemma shows that paracompact spaces satisfy a very strong normality condition.

31 LEMMA *Let X be a topological space such that each open cover is even and let $\mathcal{A}$ be a locally finite (or a discrete) family of subsets of X. Then there is a neighborhood V of the diagonal in $X \times X$ such that the family of all sets $V[A]$ for A in $\mathcal{A}$ is locally finite (respectively discrete).*

PROOF If $\mathcal{A}$ is a locally finite family of subsets there is an open cover $\mathcal{U}$ of X such that each member of $\mathcal{U}$ intersects only finitely many of the members of the family $\mathcal{A}$. Let U be a neighborhood of the diagonal such that the sets $U[x]$ refine $\mathcal{U}$. By the preceding lemma there is a neighborhood V of the diagonal such that $V \circ V \subset U$, and it may be supposed that $V = V^{-1}$. If $V \circ V[x] \cap A$ is void, then $V[x]$ must be disjoint from $V[A]$ because: if $y \in V[x] \cap V[A]$, then $(y,x) \in V^{-1} = V$, $(z,y) \in V$ for some z in A, and hence $(z,x) \in V \circ V$. Then $z \in V \circ V[x]$ and this is a contradiction. Consequently if $V[x]$ intersects $V[A]$, then $V \circ V[x]$ intersects A, and it follows that the family of all sets $V[A]$ for A in $\mathcal{A}$ is locally finite. If "finitely many" is replaced by "at most one," then a proof of the corresponding proposition for discrete families is obtained. ▌

If V is an open subset of $X \times X$, then $V[x]$ is open for every point x of X, because $V[x]$ is the inverse image of V under the continuous map which carries each point y of X into (x,y). If

A is a subset of X, then $V[A]$ is open because it is the union of the sets $V[x]$ for x in A. Consequently the preceding lemma permits us to enlarge each member of a locally finite or discrete family to an open set and still preserve the character of the family. In particular, if each open cover $\mathfrak{U}$ of a regular space has a locally finite refinement $\mathfrak{A}$, then the lemma applies (we have shown that (b) $\to$ (c) $\to$ (d) in 5.28) and there is an open neighborhood V of the diagonal such that the family of all sets $V[A]$ for A in $\mathfrak{A}$ is locally finite. The latter family may fail to be a refinement of $\mathfrak{U}$, but this is easily remedied by choosing U_A in $\mathfrak{U}$ such that $A \subset U_A$ and then letting $W_A = U_A \cap V[A]$. The family which is constructed in this fashion is clearly an open locally finite refinement of $\mathfrak{U}$ and it follows that the space is paracompact; that is, (b) $\to$ (a) in 5.28.

There is an obvious corollary to 5.31. A family consisting of two closed disjoint subsets is evidently discrete and hence:

32 Corollary *A paracompact space is normal.*

The proof of 5.28 will be complete if we establish two facts: If X is regular and each open cover is even, then each open cover has an open σ-discrete refinement, and if each open cover of X has an open σ-locally finite refinement, then each open cover has a locally finite refinement. (Evidently (e) $\to$ (f) in 5.28.)

33 Lemma *If X is a space such that each open cover is even, then every open cover of X has an open σ-discrete refinement.*

PROOF The proof, like that of 4.21, is an application of A. H. Stone's trick. (This lemma can be deduced from 4.21 and the results of chapter 6.) Because of lemma 5.31 it is sufficient to find a σ-discrete refinement of an open cover $\mathfrak{U}$, since such a σ-discrete refinement can then be "expanded" to an open σ-discrete refinement. Let V be an open neighborhood of the diagonal such that the family of all sets $V[x]$ for x in X refines $\mathfrak{U}$. Let $V_0 = V$ and select, inductively, V_n to be an open symmetric neighborhood of the diagonal such that $V_n \circ V_n \subset V_{n-1}$ for each positive integer n. Let $U_1 = V_1$ and, inductively, let $U_{n+1} = V_{n+1} \circ U_n$. It is easy to see that $U_n \subset V_0$ for each n, and it follows that for each n the family of all $U_n[x]$ for x in X refines

$\mathfrak{U}$. Choose a relation $<$ which well-orders X (see 0.25) and for each n and each x let $U_n^*(x) = U_n[x] \sim \bigcup\{U_{n+1}[y]: y < x\}$. For each fixed n the family $\mathfrak{U}_n$ of all sets $U_n^*(x)$ is discrete, as may be demonstrated as follows. Clearly $U_n^*(x)$ is disjoint from $V_{n+1}[U_n^*(y)]$ if $x \neq y$ because of the construction. If for some z in X the neighborhood $V_{n+1}[z]$ intersects $U_n^*(y)$, then $z \in V_{n+1}[U_n^*(y)]$ and $V_{n+1}[U_n^*(y)]$ is a neighborhood of z which intersects no set $U_n^*(x)$ for $x \neq y$. It follows that the family $\mathfrak{U}_n$ is discrete and it remains to be proved that each point of X belongs to some member of some $\mathfrak{U}_n$. For x in X choose y to be the first point of X such that x belongs to $U_n[y]$ for some n. Then surely $x \in U_n^*(y)$ for some n. ∎

34 LEMMA *If each open cover of a space has an open σ-locally finite refinement, then each open cover has a locally finite refinement.*

PROOF Let $\mathfrak{U}$ be an open cover and let $\mathfrak{V}$ be an open σ-locally finite refinement. Suppose that $\mathfrak{V} = \bigcup\{\mathfrak{V}_n: n \in \omega\}$ where each $\mathfrak{V}_n$ is an open locally finite family. For each n and each member V of $\mathfrak{V}_n$ let $V^* = V \sim \bigcup\{U: U \in \mathfrak{V}_k \textit{ for some } k < n\}$, and let $\mathfrak{W}$ be the family of all sets of the form V^*. Then $\mathfrak{W}$ is a cover of X and a refinement of $\mathfrak{U}$. Finally, for x in X let n be the first integer such that x belongs to some member V of $\mathfrak{V}_n$. Then V is a neighborhood of x which is disjoint from every member of $\mathfrak{W}$ save those which were constructed from the families $\mathfrak{V}_k$ for $k \leqq n$. It follows that $\mathfrak{W}$ is locally finite. ∎

Theorem 4.21 states that each open cover of a pseudo-metrizable space has an open σ-discrete refinement. This fact and theorem 5.28 of this section then give the corollary:

35 COROLLARY *Each pseudo-metrizable space is paracompact.*

In conclusion it should be remarked that subspaces, quotients, and products of paracompact spaces are usually not paracompact. Moreover, a space may be locally metrizable, locally compact, Hausdorff, normal, and satisfy the first axiom of countability and still fail to be paracompact. The requisite examples are given in the problems at the end of this chapter.

36 *Notes* There is another characterization of paracompactness which might be added to the list given in 5.28. A regular space

is paracompact iff it is fully normal (see problem 5.V). This characterization is due to A. H. Stone [1]. The equivalences (b), (c), (e), and (f) of theorem 5.28 are due to E. Michael [1]. As far as I know, equivalence (d) was first noticed by J. S. Griffin and myself.

The σ-discrete characterization of paracompactness might well be taken as a definition of countable dimension (see Hurewicz and Wallman [1; 32] and Eilenberg [1]). There is an F_σ-theorem (Michael *loc. cit.*) which is also suggestive of dimension theory.

PROBLEMS

A EXERCISE ON REAL FUNCTIONS ON A COMPACT SPACE

(a) If A is a non-void compact subset of the space of real numbers, then both the supremum and the infimum of A belong to A.

(b) Each continuous real valued function f on a compact space X assumes a maximum and a minimum value. That is, there are points x and y of the space such that $f(x)$ and $f(y)$ are respectively the supremum and infimum of f on X.

(c) Let f be a continuous real valued function f on a compact space X. If f is always positive, then f is bounded away from zero, in the sense that there is $e > 0$ such that $f(x) > e$ for x in X.

B COMPACT SUBSETS

(a) The intersection of two compact subsets of a topological space may fail to be compact. The intersection of the members of an arbitrary family of closed and compact subsets is closed and compact. (Clearly two compact subsets with non-compact intersection must be subsets of a space which is not Hausdorff. Let X be the product of the space of real numbers and an indiscrete space which has two members.)

(b) The closure of a compact subset of a topological space may fail to be compact. However, the closure of a compact subset of a regular space is compact.

(c) If A and B are disjoint closed subsets of a pseudo-metric space and A is compact, then there is a member x of A such that dist (A,B) = dist $(x,B) > 0$. (The function dist (x,B) is continuous in x and is positive for x in A.)

(d) If A and B are disjoint closed and compact subsets of a pseudo-metric space, then there are members x of A and y of B such that $d(x,y) = \text{dist}(A,B)$.

C COMPACTNESS RELATIVE TO THE ORDER TOPOLOGY

Let X be a set which is linearly ordered by a relation $<$ and let X have the order topology (see 1.I). Then every closed, order-bounded subset of X is compact iff X is order-complete relative to $<$. (The family of all subsets of X of the form $\{x: a < x\}$ or $\{x: x < a\}$ is a subbase for the order topology for X and Alexander's subbase theorem 5.6 applies. A proof which is independent of 5.6 can be made via the argument which was used in 5.14.)

D ISOMETRIES OF COMPACT METRIC SPACES

Let X and Y be metric spaces, let X be compact, let f be an isometry of X onto a subspace of Y, and let g be an isometry of Y onto a subspace of X. Then f maps X onto Y. (If h is an isometry of X onto a proper subset of itself and $x \in X \sim h[X]$ let $a = \text{dist}(x,h[X])$. Define a sequence inductively by letting $x_0 = x$ and $x_{n+1} = h(x_n)$ and prove that, if $m \neq n$, then $d(x_m,x_n) \geqq a$.)

E COUNTABLY COMPACT AND SEQUENTIALLY COMPACT SPACES

A topological space is *countably compact* iff every countable open cover has a finite subcover. A space is *sequentially compact* iff every sequence has a convergent subsequence.

(a) A space is countably compact iff each sequence has a cluster point.

(b) A T_1-space is countably compact iff every infinite set has an accumulation point. (See 5.3.)

(c) A T_1-space is countably compact iff every infinite open cover has a proper subcover. (If A is an infinite set with no accumulation point, then each subset of A is closed. One may construct an open cover $\mathfrak{U}$ by choosing an open neighborhood of each point of A which contains no other point of A and then adjoining, if necessary, $X \sim A$. Then $\mathfrak{U}$ has no proper subcover. On the other hand, if $\mathfrak{V}$ is an open cover with no proper subcover then each member V of $\mathfrak{V}$ contains a point belonging to no other member of $\mathfrak{V}$.)

(d) A space satisfying the first countability axiom is countably compact iff it is sequentially compact (5.5).

(e) With the order topology, the set Ω_0 of all ordinal numbers less than the first uncountable ordinal Ω is locally compact, Hausdorff, satisfies the first axiom of countability, and is sequentially compact, but is not compact.

Note Proposition (c) is due to Arens and Dugundji [1].

F COMPACTNESS; THE INTERSECTION OF COMPACT CONNECTED SETS

(a) Let $\mathfrak{A}$ be a family of closed compact sets such that $\bigcap\{A: A \in \mathfrak{A}\}$ is a subset of an open set U. Then there is a finite subfamily $\mathfrak{F}$ of $\mathfrak{A}$ such that $\bigcap\{A: A \in \mathfrak{F}\} \subset U$.

(b) If $\mathfrak{A}$ is a family of compact subsets of a Hausdorff space X such that finite intersections of members of $\mathfrak{A}$ are connected, then $\bigcap\{A: A \in \mathfrak{A}\}$ is connected.

G PROBLEM ON LOCAL COMPACTNESS

If X is a Hausdorff space and Y is a dense locally compact subspace, then Y is open.

H NEST CHARACTERIZATION OF COMPACTNESS

A topological space X is compact iff each nest of closed non-void sets has a non-void intersection. (Recall that a nest is a family of sets which is linearly ordered by inclusion. If each nest of closed non-void sets has a non-void intersection and $\mathfrak{A}$ is a family of closed sets with the finite intersection property, let $\mathfrak{B}$ be a maximal family of closed sets which contains $\mathfrak{A}$ and has the finite intersection property, and let $\mathfrak{N}$ be a maximal nest in $\mathfrak{B}$. Examination of the properties of $\mathfrak{B}$ and of $\mathfrak{N}$ leads to a proof. An entirely different proof can be based on well ordering, using part of the procedure outlined in the next problem.)

I COMPLETE ACCUMULATION POINTS

A point x is a *complete accumulation point* of a subset A of a topological space iff for each neighborhood U of x the sets A and $A \cap U$ have the same cardinal number. A topological space is compact iff each infinite subset has a complete accumulation point. (If X is not compact choose an open cover $\mathfrak{A}$ with no finite subcover such that the cardinal number c of $\mathfrak{A}$ is as small as possible. Let C be a well-ordered set of cardinal c such that the set of predecessors of each member has a cardinal less than c. (It is shown in the appendix that c is such a set.) Let f be a one-to-one map of C onto $\mathfrak{A}$. Then for each member b of C the union $\bigcup\{f(a): a < b\}$ does not cover X and, in fact, the complement of this union must have cardinal number at least as great as c

It is therefore possible to choose x_b from the complement such that $x_a \neq x_b$ for $a < b$. Consider the set of all x_b.)

J EXAMPLE: UNIT SQUARE WITH DICTIONARY ORDER

Let X be the cartesian product of the closed unit interval Q with itself ordered by dictionary (lexicographic) order. (That is, $(a,b) < (c,d)$ iff $a < c$ or $a = c$ and $b < d$.) With the order topology X is compact, connected, and Hausdorff. It satisfies the first countability axiom but is not separable and is hence not metrizable.

K EXAMPLE (THE ORDINALS) ON NORMALITY AND PRODUCTS

The product of a locally compact, normal Hausdorff space and a compact Hausdorff space may fail to be normal. (The difficult part has already been established in 4.E and it is only necessary to show that Ω' and Ω_0 are compact and locally compact Hausdorff respectively. Ω' is the space of ordinals less than or equal to Ω and Ω_0 is the set of ordinals less than Ω, each with the order topology.)

L THE TRANSFINITE LINE

Let A be a well-ordered set, let the half-open interval $[0,1)$ have the usual order, let $A \times [0,1)$ have the dictionary (lexicographic) order, and let $A \times [0,1)$ have the order topology. Discuss the properties of this space.

M EXAMPLE: THE HELLY SPACE

The *Helly space* is the family H of all non-decreasing functions on the closed unit interval Q with values in Q. It is a subset of the product space Q^Q, and its topology is the relative product topology. The space H has the following properties:

(a) H is compact Hausdorff. (It is a closed subspace of Q^Q.)

(b) H satisfies the first axiom of countability and is hence sequentially compact. (The set of points of discontinuity of each member of H is countable. This fact, and the fact that Q is separable, must be used in constructing a countable base for the neighborhood system of a point h of H.)

(c) H is separable. (A countable dense set can be constructed using the rationals.)

(d) H is not metric. (For t in Q let $f_t(x)$ be 0 for $x < t$, 1 for $x > t$, and let $f_t(t) = \frac{1}{2}$. The family A of all functions of the form f_t is uncountable and no member of A is an accumulation point of A. But each subspace of a compact metric space is separable.)

N EXAMPLES ON CLOSED MAPS AND LOCAL COMPACTNESS

(a) Let X be the space of real numbers with the usual topology, let I be the set of integers, and let $\mathfrak{D}$ be the decomposition whose members are I and all sets $\{x\}$ for x in $X \sim I$. Then the projection of X onto the quotient space is closed and continuous, but the quotient space is not locally compact nor does it satisfy the first axiom of countability.

(b) Let Ω_0 be the set of all ordinal numbers less than Ω, with the order topology, let A be a closed uncountable set whose complement is also uncountable, and let $\mathfrak{D}$ be the decomposition whose members are A and all sets $\{x\}$ for x in $\Omega_0 \sim A$. Then the projection of Ω_0 onto the quotient space is continuous and closed and the quotient space is compact, but it fails to satisfy the first axiom of countability. (Use the interlacing lemma 4.E.)

O CANTOR SPACES

The *Cantor discontinuum* (*middle third* set) is the set of all members of the closed unit interval which have a triadic expansion in which the digit one does not occur. (It will be convenient throughout this problem to use only irrational triadic expansions, that is, expansions which are not identically zero from some point on. Each real number has a unique irrational expansion, as noted in 0.14.) The discontinuum is called the middle third set because: The (open) middle third of the interval [0,1] is precisely the set of numbers whose triadic expansions have ones in the first place after the "decimal" point. The middle third of each of the remaining intervals consists of points whose expansions have ones in the second but not the first place. Continuing, it is clear that the discontinuum can be obtained by successive deletion of middle thirds.

A product space 2^A (that is, all functions on a set A to the discrete space whose only members are 0 and 1, with the product topology) is called a *Cantor space*.

(a) The Cantor discontinuum is homeomorphic to 2^ω. For x in 2^ω let $f(x)$ be the member of [0,1] whose triadic expansion has the digit $2x(p)$ in the p-th place.)

(b) Each point of the discontinuum is an accumulation point and the complement of the discontinuum is an open dense subset of the real numbers.

(c) If A is a closed non-void subset of 2^ω, then there is a continuous function r on 2^ω to A such that $r(x) = x$ for x in A. (It is a little easier to see the proof if one looks at the Cantor discontinuum, which is the homeomorphic image of 2^ω.)

(d) Each compact Hausdorff space is the continuous image of a closed subset of some Cantor space. (Let F be the family of all functions f on 2 such that $f(0)$ and $f(1)$ are closed subsets of the compact Hausdorff space X and $f(0) \cup f(1) = X$. If x is a member of 2^F and $f \in F$, then $f(x_f)$ is a closed subset of X. The intersection $\bigcap \{f(x_f): f \in F\}$ is void or consists of a single point, and in the latter case this point is defined to be $\phi(x)$. One can prove that the domain of ϕ is a closed subset of 2^F; if U is a subset of X, then $\phi^{-1}[U] = \{x: x$ *is a member of domain* ϕ *and* $\bigcap \{f(x_f): f \in F\} \subset U\}$.)

(e) Each compact metric space X is the continuous image of 2^ω. (Instead of the family F of the previous proof one may construct a smaller family which will play the same role. If $U_0, \cdots, U_n, \cdots$ is a base for the topology of X let $f_n(0) = U_n^-$ and $f_n(1) = X \sim U_n$.)

(f) Each Cantor space 2^A satisfies the countable chain condition; that is, each disjoint family of open sets is countable. (If $\mathfrak{U}$ is a disjoint family of open subsets of 2^A, then one may suppose that the members of $\mathfrak{U}$ belong to the defining base for the product topology; each member is, in a natural sense, the intersection of a finite number of half-spaces. For some integer n there is then an infinite (in fact, uncountable) disjoint family, each member of which is the intersection of precisely n half-spaces. A simple argument on disjointness completes the proof.

There is a shorter, more sophisticated proof. A Cantor space with coordinatewise addition, modulo 2, is a compact topological group and hence there is a Haar measure (see Halmos [1; 254]). Since this measure is finite and is positive for open sets the countable chain condition is clear.)

(g) Not every compact Hausdorff space is the continuous image of the Cantor set. (The one point compactification of an uncountable discrete space does not satisfy the countable chain condition.)

Notes Proposition (b) is due to Cantor, (e) to P. Alexandroff and Urysohn, and (f) and (g) to J. W. Tukey. Proposition (g) is also a corollary of some results of Szpilrajn [1].

P CHARACTERIZATION OF THE STONE-ČECH COMPACTIFICATION

Let (f,Y) be a Hausdorff compactification of the topological space X such that for each bounded continuous real-valued function g on X the function $g \circ f^{-1}$ has a continuous extension. Then (f,Y) is topologically equivalent to the Stone-Čech compactification $(e,\beta(X))$. (Consider the definition of $\beta(X)$.)

Q EXAMPLE (THE ORDINALS) ON COMPACTIFICATION

Let Ω' be the set of all ordinal numbers less than or equal to Ω, and let $\Omega_0 = \Omega' \sim \{\Omega\}$. Assign each the order topology. Then the Stone-Čech compactification $\beta(\Omega_0)$ is homeomorphic to Ω'. (This will follow from the preceding problem if it is shown that every bounded real-valued continuous function f on Ω_0 is eventually constant,* in the sense that for some x in Ω_0, if $y > x$, then $f(y) = f(x)$. If f is a bounded continuous real-valued function and r and s are real numbers such that $r > s$, then the interlacing lemma 4.E shows that one of the sets $\{x: f(x) \geqq r\}$ and $\{x: f(x) \leqq s\}$ is countable. Using this fact it is not hard to see that f is eventually constant. The hypothesis that f be bounded is actually not essential.)

Note This result is due to Tong [1].

R THE WALLMAN COMPACTIFICATION

Let X be a T_1-space, let $\mathfrak{F}$ be the family of all closed subsets of X, and let $w(X)$ be the collection of all subfamilies $\mathfrak{A}$ of $\mathfrak{F}$ which possess the finite intersection property and are maximal in $\mathfrak{F}$ relative to this property.

(a) If $\mathfrak{A} \in w(X)$, then the intersection of two members of $\mathfrak{A}$ is a member of $\mathfrak{A}$; dually, if A and B are members of $\mathfrak{F} \sim \mathfrak{A}$, then $A \cup B$ is a member of $\mathfrak{F} \sim \mathfrak{A}$. (See 2.I.)

(b) For each point x of X let $\phi(x) = \{A\colon A \in \mathfrak{F} \textit{ and } x \in A\}$. Then ϕ is a one-to-one map of X into $w(X)$.

(c) For each open subset U of X let $U^* = \{\mathfrak{A}\colon \mathfrak{A} \in w(X) \textit{ and } A \subset U \textit{ for some } A \textit{ in } \mathfrak{A}\}$. Then $w(X) \sim U^* = \{\mathfrak{A}\colon X \sim U \in \mathfrak{A}\}$. If U and V are open subsets of X, then $(U \cap V)^* = U^* \cap V^*$ and $(U \cup V)^* = U^* \cup V^*$.

(d) Let $w(X)$ have the topology with a base the family of all sets of the form U^* for U open in X. Then $w(X)$ is compact, the map ϕ is continuous, and $\phi(X)$ is dense in $w(X)$. (Show compactness via the finite intersection property argument for complements of members of the base.)

(e) If X is normal, then $w(X)$ is Hausdorff.

(f) If f is a bounded continuous real-valued function on X, then $f \circ \phi^{-1}$ may be extended continuously to all of $w(X)$. (If a continuous extension is impossible, then by a little argument it can be shown that there are closed disjoint subsets R and S of the reals such that $f^{-1}[R]$

* This curious property of Ω_0 has been used by E. Hewitt [1] in constructing a regular Hausdorff space X such that every continuous real-valued function on X is constant.

and $f^{-1}[S]$ are disjoint but the closures of the images under ϕ of these sets intersect. But if A and B are closed disjoint subsets of X, then $\{\mathfrak{A}: A \in \mathfrak{A}\}$ and $\{\mathfrak{A}: B \in \mathfrak{A}\}$ are disjoint and closed in $w(X)$.)

(g) If $w(X)$ is Hausdorff, then the Wallman compactification is topologically equivalent to the Stone-Čech compactification. (See 5.P.)

Notes The principal virtue of the Wallman compactification (Wallman [1]) lies in the fact that the correspondence carrying U into U^* preserves finite intersections and unions. Moreover, the topology for X is carried onto a base for the topology for $w(X)$ by the correspondence, and from this fact it follows that the dimension of X (in the covering sense) and the dimension of $w(X)$ are identical, and X and $w(X)$ have isomorphic Čech homology groups. See Samuel [1] for a related construction.

S BOOLEAN RINGS: STONE REPRESENTATION THEOREM

Let $(R,+,\cdot)$ be a Boolean ring (see 2.K), let S' be the set of all ring homomorphisms of R into I_2 (= the integers mod 2), and let $S = S' \sim \{0\}$, where 0 is the homomorphism which is identically zero. Then S' is a subset of the product I_2^R. The *Stone space* of the ring R is S with the relative product topology (I_2 is assigned the discrete topology).

A *Boolean space* is a Hausdorff space such that the family of all sets which are both compact and open is a base for the topology. A Boolean space is automatically locally compact. The *characteristic ring* of a Boolean space is the ring of all continuous functions f into I_2 such that $f^{-1}[1]$ is compact (that is, all functions to I_2 which vanish outside a compact set; sometimes called functions with a compact support).

(a) The Stone space of a Boolean ring R is a Boolean space and is compact whenever R has a unit. (In this case $S = \{h\colon h \in S'$ *and* $h(1) = 1\}$.)

(b) *Stone-Weierstrass mod* 2 Let $\mathfrak{F}$ be the characteristic ring of a Boolean space X and let $\mathfrak{G}$ be a subring of $\mathfrak{F}$ which has the two point property (that is, for distinct points x and y of X and for a and b in I_2 there is g in $\mathfrak{G}$ such that $g(x) = a$ and $g(y) = b$). Then $\mathfrak{F} = \mathfrak{G}$.

(If X is compact, then $\mathfrak{G}$ has the two point property whenever $1 \in \mathfrak{G}$ and $\mathfrak{G}$ distinguishes points, in the sense that for distinct points x and y of X there is g in $\mathfrak{G}$ such that $g(x) \neq g(y)$. A routine but instructive compactness argument serves to establish (b). One might begin by showing that for a compact subset Y of X and a point x of $X \sim Y$ there is g in $\mathfrak{G}$ such that $g(x) = 0$ and g on Y is one.)

(c) *Representation theorem* Each Boolean ring is isomorphic (under the evaluation map) to the characteristic ring of its Stone space. (For

r in R the evaluation at r, $e(r)$, is the function on S whose value at a member s of S is $s(r)$. This theorem depends on the existence of enough homomorphisms 2.K and the foregoing proposition (b).)

(d) If X is a Boolean space, $\mathfrak{F}$ its characteristic ring, and $\mathfrak{I}$ a maximal proper ideal in $\mathfrak{F}$, then $\mathfrak{I} = \{f\colon f(x) = 0\}$ for some x in X. (Show first that unless there is a point at which all members of $\mathfrak{I}$ vanish, then $\mathfrak{I} = \mathfrak{F}$.)

(e) *Dual representation theorem* If X is a Boolean space, then X is homeomorphic (under the evaluation map) to the Stone space of its characteristic ring. (A maximal ideal is the set of zeros of a unique homomorphism into I_2 and every such set of zeros is a maximal ideal. The preceding proposition (d) shows essentially that the evaluation map carries X onto the Stone space.)

Notes The results above are due to M. H. Stone [3].

There is an interesting variation of the process of representing a Boolean space. If X is a Boolean space let $\mathfrak{F}$ be the ring of all continuous functions on X to I_2. (The requirement that $f^{-1}[1]$ be compact is omitted.) The evaluation map of X into the Stone space S of $\mathfrak{F}$ turns out to be a homeomorphism again, but S is compact and it is, in fact, homeomorphic to the Stone-Čech compactification $\beta(X)$. We omit the proof of this fact as well as the characterizations of ideals and subrings of a Boolean ring in terms of the Stone space.

Finally, this problem is so arranged that the pattern can be transferred to the algebra of all continuous real-valued functions f on a locally compact Hausdorff space X such that, for $e > 0$, $\{x\colon |f(x)| \geq e\}$ is compact. The most difficult step in reproducing the pattern is the Stone-Weierstrass theorem, 7.R, of which (b) above is a miniature. It also turns out that, if X is a Tychonoff space, then the space of all real homomorphisms of the algebra of bounded continuous functions on X is homeomorphic to $\beta(X)$, very much like the situation sketched in the previous paragraph.

T COMPACT CONNECTED SPACES (THE CHAIN ARGUMENT)

Let (X,d) be a compact pseudo-metric space. For each positive number e, define an *e-chain* from a point x of X to a point y to be a finite sequence of points, the first of which is x, the last y, such that the distance between successive points is less than e. For each subset A of X, $C_e(A)$ is defined to be the set of all points which can be joined to points of A by an e-chain and $C(A)$ is defined to be $\bigcap\{C_e(A)\colon e > 0\}$. An equivalent definition: Let $V_0(A) = A$, $V_1(A) = \{x\colon \text{dist}\,(x,A) < e\}$

and inductively $V_{n+1}(A) = V_1(V_n(A))$. Set $C_e(A) = \bigcup \{V_n(A): n \in \omega\}$.

(a) For each $e > 0$ and each set A the set $C_e(A)$ is open and closed.

(b) If A is a connected subset of X, then $C(A)$ is connected. Hence $C(\{x\})$ is the component C_x of X about x for each point x. (If $C(A)$ is the union of disjoint closed sets B and D let $f = [\text{dist}\,(B,D)]/3$ and show by 5.G that $C_e(A) \subset \{x: \text{dist}\,(x,B \cup D) < f\}$ for some positive e.)

(c) If A is a subset of X, then $C(A) = \bigcup \{C_x: x \in A^-\}$. (If $x \notin C(A)$, then $x \notin C_e(A)$ for some positive e.)

(d) The decomposition of X into components is upper semi-continuous.

(e) If X is connected and U is an open neighborhood of a point x, then the closure of some component of U intersects $X \sim U$. (If not, there is a compact neighborhood V of the closure of the component which is contained in U. The component about x of V is contained in the interior V^0 of V and using (c) one can show that there are open and closed subsets of V containing $V \sim V^0$ and x respectively.)

(f) No closed connected subset of X which contains more than one point is the union of a countable disjoint family of closed subsets. (Proposition (e) plays a critical role in this proof. If the set $\bigcup \{A_n: n \in \omega\}$ is closed and connected and the sets A_n are closed and disjoint it is possible to find a closed connected set which is disjoint from A_1 and intersects more than one of the sets A_n.)

(g) Let X be the subset $\{(x,y): x^2y^2 = 1\}$ of the Euclidean plane with the usual metric. Then X is locally compact and any two points can be joined by an e-chain for each $e > 0$, but X is not connected.

Notes The results of this problem generalize very naturally to compact Hausdorff (or compact regular) spaces. The even covering theorem 5.27 gives the necessary mechanism.

Lest proposition (e) make one over-optimistic on the properties of connected sets, the classic example of Knaster and Kuratowski [1] should be mentioned. There is a connected subspace X of the Euclidean plane and a point x of X such that $X \sim \{x\}$ contains no connected set.

U FULLY NORMAL SPACES

If $\mathfrak{U}$ is a family of subsets of a set X and x is a point of X, then the *star* at x of $\mathfrak{U}$ is the union of the members of $\mathfrak{U}$ to which x belongs. A cover $\mathfrak{V}$ is a *star-refinement* of $\mathfrak{U}$ iff the family of stars of $\mathfrak{V}$ at points of X is a refinement of $\mathfrak{U}$. A topological space is *fully normal* iff each open cover has an open star-refinement. Then: A regular topological space is fully normal iff it is paracompact. (If X is paracompact the even covering property together with 5.30 yields an easy proof of full normality. On the other hand, if X is fully normal, $\mathfrak{U}$ is an open cover and $\mathfrak{V}$ is an open star-refinement of $\mathfrak{U}$, then $\bigcup\{V \times V: V \in \mathfrak{V}\}$ is a neighborhood of the diagonal.)

Note The definition of full normality is due to J. W. Tukey [1], who proved many useful properties. The equivalence with paracompactness was proved by A. H. Stone [1].

V POINT FINITE COVERS AND METACOMPACT SPACES

A family of subsets of X is *point finite* iff no point of X belongs to more than a finite number of members of the family. A topological space is *metacompact* iff each open cover has a point finite refinement.

(a) Let $\mathfrak{U}$ be a point finite open cover of a normal space X. Then it is possible to select an open set $G(U)$ for each U in $\mathfrak{U}$ in such a way that $G(U)^- \subset U$ and the family of all sets $G(U)$ is a cover of X. (Choose a maximal member of the class of all functions F satisfying the conditions: the domain of F is a subfamily of $\mathfrak{U}$, $F(U)$ is an open set whose closure is contained in U for each U in the domain of F and $\bigcup\{F(U): U \in \text{domain } F\} \cup \bigcup\{V: V \in \mathfrak{U} \text{ and } V \notin \text{domain } F\} = X$. Point finiteness of $\mathfrak{U}$ implies the existence of a maximal F.)

(b) A point finite cover of a set has a minimal subcover (that is, a subcover no proper subfamily of which is a cover).

(c) A metacompact T_1-space is countably compact (see 5.E) iff it is compact.

Note Propositions (b) and (c) are taken directly from Arens and Dugundji [1].

W PARTITION OF UNITY

A *partition of unity* on a topological space X is a family F of continuous functions on X to the set of non-negative real numbers such that $\sum\{f(x): f \in F\} = 1$ for each x in X, and all but a finite number of members of F vanish on some neighborhood of each point of X. A partition F of unity is *subordinate* to a cover $\mathfrak{U}$ of X iff each member of F vanishes outside some member of $\mathfrak{U}$. Then: For each locally finite open cover $\mathfrak{U}$ of a normal space there is a partition of unity which is subordinate to $\mathfrak{U}$. A slightly stronger result may be proved: If $\mathfrak{U}$ is a locally finite open cover of a normal space, then it is possible to select a non-negative continuous function f_U for each U in $\mathfrak{U}$ such that f_U is 0 outside U and is everywhere less than or equal to one, and $\sum\{f_U(x): U \in \mathfrak{U}\} = 1$ for all x. (See 5.V(a) above.)

Note As far as I know, this result (approximately) is due independently to Hurewicz, Bochner, and Dieudonné.

X THE BETWEEN THEOREM FOR SEMI-CONTINUOUS FUNCTIONS

Let g and h be, respectively, lower and upper semi-continuous real-valued functions on a paracompact space X, and suppose that $h(x) < g(x)$ for all x in X. Then there is a continuous real-valued function p on X such that $h(x) < p(x) < g(x)$ for each x. (Let $\mathfrak{U}$ be the family of all open subsets U of X such that the supremum of h on U is less than the infimum of g on U, and let F be a partition of unity which is subordinate to $\mathfrak{U}$. For each f in F choose k_f such that, if $f(x) \neq 0$, then $h(x) < k_f < g(x)$, and let $p(x) = \sum\{k_f f(x): f \in F\}$. The value of p at a point x is then an average of numbers, all of which lie between $h(x)$ and $g(x)$.)

Notes The result above can be improved by first finding a countable refinement for the family $\mathfrak{U}$. The proposition then holds for countably paracompact spaces (that is, spaces such that each countable open cover has a locally finite refinement). The converse of the sharpened form of the theorem is true. Dowker [2] has proved the equivalence of: (1) X is countably paracompact and normal, (2) the product of X and the closed unit interval is normal, and (3) the proposition above. Dowker also shows that a perfectly normal space (normal and each closed subset is a G_δ) is countably paracompact. It is not known whether a normal Hausdorff space must be countably paracompact.

Y PARACOMPACT SPACES

(a) Each regular Lindelöf space is paracompact.

(b) A topological space is defined to be *σ-compact* iff it is the union of a countable family of compact subsets. Each σ-compact space is a Lindelöf space.

(c) If a regular space is the union of the members of an open discrete family of Lindelöf subspaces, then it is paracompact. Consequently each locally compact group is paracompact. (Consider the family of cosets modulo the smallest subgroup containing a fixed compact neighborhood of the identity.)

(d) The half-open interval space of problems 1.K and 4.I is regular and Lindelöf and hence paracompact. The cartesian product of this space with itself is not normal and is therefore not paracompact.

(e) With the order topology the set of all ordinals which are less than the first uncountable ordinal is not paracompact. (Consider the cover consisting of all sets of the form $\{x: x < a\}$. The supremum of each member of an arbitrary refinement of this cover is less than Ω.)

Notes Proposition (a) above is due to Morita [1]. For further information on paracompactness (an F_σ-theorem, products, etc.) see Michael [1]. Bing [1] has studied a normality condition which is intermediate to normality and paracompactness. In this connection it might be emphasized that lemma 5.31 states a noteworthy normality property of paracompact spaces.

Chapter 6

UNIFORM SPACES

There are several properties of metric spaces which are not topological but are closely connected with topological properties. We give examples of the sort of connections contemplated, postponing the definitions and proofs. The property of being a Cauchy sequence is not a topological invariant, for the map f such that $f(x) = 1/x$ is a homeomorphism of the space of positive real numbers onto itself which carries the Cauchy sequence $\{1/(n+1): n \in \omega\}$ into the non-Cauchy sequence $\{n+1, n \in \omega\}$. However, it is possible to derive topological results from statements about Cauchy sequences; for example, a subset A of the space of all real numbers is closed if and only if each Cauchy sequence in A converges to some point of A. The reverse sort of implication may also occur; thus, each continuous function on a compact metric space is uniformly continuous. In this case we deduce from a topological premise (that the space is compact) a non-topological conclusion (that a function is uniformly continuous). This chapter is devoted to a study of quasi-topological results of this sort.

The mathematical construct employed in studying uniformity properties is called a uniform space. A brief discussion will indicate how this notion, which is due to A. Weil [1], applies.

A sequence $\{x_n, n \in \omega\}$ in a pseudo-metric space (X,d) is called a Cauchy sequence iff $d(x_m,x_n)$ converges to zero as m and n become large. This notion is not meaningful in an arbitrary topological space; in order to define a Cauchy sequence it is necessary to know, in some sense, for what pairs the distance $d(x,y)$ is small. This statement may be made precise in the following way. If $V_{d,r} = \{(x,y): d(x,y) < r\}$, then $\{x_n, n \in \omega\}$ is a Cauchy sequence iff for each positive r it is true that (x_m,x_n) is a member of $V_{d,r}$ for m and n large. The notion of uniform continuity can also be formulated in terms of the family of all sets of the form $V_{d,r}$. This suggests consideration of a set X and a special family of subsets of $X \times X$.

If X is a topological group, then a sequence $\{x_n, n \in \omega\}$ may be called a Cauchy sequence iff $x_m x_n^{-1}$ is near the identity e of the group when m and n are large. Again, the information needed to make this definition is information about pairs of points. We need to know which pairs of points (x,y) are such that xy^{-1} is near the identity e. For each neighborhood U of e let $V_U = \{(x,y): xy^{-1} \in U\}$. Then clearly the family of all sets of the form V_U determines which sequences are Cauchy.

A uniform space is defined to be a set X together with a family of subsets of $X \times X$ which satisfies certain natural conditions. This follows the pattern suggested by both of the preceding examples. However, it should be emphasized that this is by no means the only framework in which uniformity can be studied. It is possible to study a set X together with a distinguished family of pseudo-metrics for X, or to distinguish a collection of covers of X which are to be uniform covers (roughly in the sense of the Lebesgue covering lemma 5.26). One may also consider "metrics" with values in a structure less restricted than that of the real numbers. All of these notions are essentially equivalent, as indicated in the problems at the end of the chapter.

Finally, it must be said that there are uniformity properties of metric spaces which apparently do not generalize to less restricted situations. The last section is devoted to a study of some of these.

UNIFORMITIES AND THE UNIFORM TOPOLOGY

We will be concerned with subsets of a cartesian product $X \times X$ of a set with itself. These subsets are relations in the sense of chapter 0, and for convenience we review some of the earlier definitions and results about them. A relation is a set of ordered pairs, and if U is a relation the inverse relation U^{-1} is the set of all pairs (x,y) such that $(y,x) \in U$. The operation of taking inverses is involutory in the sense that $(U^{-1})^{-1}$ is always U. If $U = U^{-1}$, then U is called symmetric. If U and V are relations, then the composition $U \circ V$ is the set of all pairs (x,z) such that for some y it is true that $(x,y) \in V$ and $(y,z) \in U$. Composition is associative, that is, $U \circ (V \circ W) = (U \circ V) \circ W$, and it is always true that $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. The set of all pairs (x,x) for x in X is called the identity relation, or the diagonal, and is denoted by $\Delta(X)$ or simply Δ. For each subset A of X the set $U[A]$ is defined to be $\{y: (x,y) \in U$ *for some* x *in* $A\}$, and if x is a point of X, then $U[x]$ is $U[\{x\}]$. For each U and V and each A it is true that $(U \circ V)[A] = U[V[A]]$. Finally a simple lemma will be needed.

1 LEMMA *If* V *is symmetric, then* $V \circ U \circ V = \bigcup \{V[x] \times V[y]: (x,y) \in U\}$.

PROOF By definition $V \circ U \circ V$ is the set of all pairs (u,v) such that $(u,x) \in V$, $(x,y) \in U$ and $(y,v) \in V$ for some x and some y. Since V is symmetric this is the set of all (u,v) such that $u \in V[x]$ and $v \in V[y]$ for some (x,y) in U. But $u \in V[x]$ and $v \in V[y]$ iff $(u,v) \in V[x] \times V[y]$, and hence $V \circ U \circ V = \{(u,v): (u,v) \in V[x] \times V[y]$ *for some* (x,y) *in* $U\} = \bigcup \{V[x] \times V[y]: (x,y) \in U\}$. ∎

A **uniformity** for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ such that

(a) each member of $\mathcal{U}$ contains the diagonal Δ;
(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
(c) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some V in $\mathcal{U}$;
(d) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and
(e) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ is a **uniform space**.

The metric antecedents of the conditions above are not hard to discern. The first is derived from the condition that $d(x,x) = 0$ and the second derives from the symmetry condition $d(x,y) = d(y,x)$. The third is a vestigal form of the triangle inequality

—it says roughly that for r-spheres there are $(r/2)$-spheres. The fourth and fifth resemble axioms for the neighborhood system of a point and they will be used to derive the corresponding properties for a neighborhood system relative to a topology which will presently be defined.

There may be many different uniformities for a set X. The largest of these is the family of all those subsets of $X \times X$ which contain Δ and the smallest is the family whose only member is $X \times X$. If X is the set of real numbers the **usual uniformity** for X is the family $\mathfrak{U}$ of all subsets U of $X \times X$ such that $\{(x,y): |x - y| < r\} \subset U$ for some positive number r. Each member of $\mathfrak{U}$ is a neighborhood of the diagonal Δ (the line with equation $y = x$), but it is to be emphasized that not every neighborhood of the diagonal is a member of $\mathfrak{U}$. For example, the set $\{(x,y): |x - y| < 1/(1 + |y|)\}$ is a neighborhood of Δ but not a member of $\mathfrak{U}$.

It is not generally true that the union or the intersection of two uniformities for X is a uniformity. However, the union of a collection of uniformities generates a uniformity in a rather natural sense. A subfamily $\mathfrak{B}$ of a uniformity $\mathfrak{U}$ is a **base** for $\mathfrak{U}$ iff each member of $\mathfrak{U}$ contains a member of $\mathfrak{B}$. If $\mathfrak{B}$ is a base for $\mathfrak{U}$, then $\mathfrak{B}$ determines $\mathfrak{U}$ entirely, for a subset U of $X \times X$ belongs to $\mathfrak{U}$ iff U contains a member of $\mathfrak{B}$. A subfamily $\mathfrak{S}$ is a **subbase** for $\mathfrak{U}$ iff the family of finite intersections of members of $\mathfrak{S}$ is a base for $\mathfrak{U}$. These definitions are entirely analogous to the definitions of base and subbase for a topology.

2 THEOREM *A non-void family $\mathfrak{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if*

(a) *each member of $\mathfrak{B}$ contains the diagonal Δ;*
(b) *if $U \in \mathfrak{B}$, then U^{-1} contains a member of $\mathfrak{B}$;*
(c) *if $U \in \mathfrak{B}$, then $V \circ V \subset U$ for some V in $\mathfrak{B}$; and*
(d) *the intersection of two members of $\mathfrak{B}$ contains a member.*

The straightforward proof of this proposition is omitted.

The property of being a subbase for some uniformity is less easy to characterize. However, the following simple result is adequate for our needs.

3 THEOREM *A family* $\mathcal{S}$ *of subsets of* $X \times X$ *is a subbase for some uniformity for* X *if*

(a) *each member of* $\mathcal{S}$ *contains the diagonal* Δ,
(b) *for each* U *in* $\mathcal{S}$ *the set* U^{-1} *contains a member of* $\mathcal{S}$, *and*
(c) *for each* U *in* $\mathcal{S}$ *there is* V *in* $\mathcal{S}$ *such that* $V \circ V \subset U$.

In particular, the union of any collection of uniformities for X *is the subbase for a uniformity for* X.

PROOF It must be shown that the family $\mathcal{B}$ of finite intersections of members of $\mathcal{S}$ satisfies the conditions of 6.2. This follows easily from the observation: If $U_1, \cdots U_n$ and $V_1, \cdots, V_n$ are subsets of $X \times X$, if $U = \bigcap \{U_i : i = 1, \cdots, n\}$ and $V = \bigcap \{V_i : i = 1, \cdots, n\}$, then $V \subset U^{-1}$ (or $V \circ V \subset U$) whenever $V_i \subset U_i^{-1}$ (respectively, $V_i \circ V_i \subset U_i$) for each i. ∎

If $(X,\mathcal{U})$ is a uniform space the **topology** $\mathfrak{J}$ **of the uniformity** $\mathcal{U}$, or the **uniform topology**, is the family of all subsets T of X such that for each x in T there is U in $\mathcal{U}$ such that $U[x] \subset T$. (This is precisely the generalization of the metric topology, which is the family of all sets which contain a sphere about each point.) It must be verified that $\mathfrak{J}$ is indeed a topology, but this offers no difficulty: In view of the definition, the union of members of $\mathfrak{J}$ is surely a member of $\mathfrak{J}$. If T and S are members of $\mathfrak{J}$ and $x \in T \cap S$, then there are U and V in $\mathcal{U}$ such that $U[x] \subset T$ and $V[x] \subset S$, and hence $(U \cap V)[x] \subset T \cap S$; consequently $T \cap S \in \mathfrak{J}$ and $\mathfrak{J}$ is a topology.

The relation between a uniformity and the uniform topology will now be examined.

4 THEOREM *The interior of a subset* A *of* X *relative to the uniform topology is the set of all points* x *such that* $U[x] \subset A$ *for some* U *in* $\mathcal{U}$.

PROOF It must be shown that the set $B = \{x: U[x] \subset A$ *for some* U *in* $\mathcal{U}\}$ is open relative to the uniform topology, for B surely contains every open subset of A and, if B is open, then it must necessarily be the interior of A. If $x \in B$, then there is a member U of $\mathcal{U}$ such that $U[x] \subset A$ and there is V in $\mathcal{U}$ such that $V \circ V \subset U$. If $y \in V[x]$, then $V[y] \subset V \circ V[x] \subset U[x] \subset A$, and hence $y \in B$. Hence $V[x] \subset B$ and B is open. ∎

It follows immediately that $U[x]$ is a neighborhood of x for each U in the uniformity $\mathfrak{U}$, and consequently the family of all sets $U[x]$ for U in $\mathfrak{U}$ is a base for the neighborhood system of x (the family is actually identical with the neighborhood system but this is of no great importance). The following proposition is then clear.

5 THEOREM *If* $\mathfrak{B}$ *is a base (or subbase) for the uniformity* $\mathfrak{U}$, *then for each* x *the family of sets* $U[x]$ *for* U *in* $\mathfrak{B}$ *is a base (subbase respectively) for the neighborhood system of* x.

The uniform topology for X may be used to construct a product topology for $X \times X$. As might be expected, members of the uniformity have a special structure relative to this topology.

6 THEOREM *If* U *is a member of the uniformity* $\mathfrak{U}$, *then the interior of* U *is also a member; consequently the family of all open symmetric members of* $\mathfrak{U}$ *is a base for* $\mathfrak{U}$.

PROOF The interior of a subset M of $X \times X$ is the set of all (x,y) such that, for some U and some V in $\mathfrak{U}$, $U[x] \times V[y] \subset M$. Since $U \cap V \in \mathfrak{U}$ the interior of M is $\{(x,y)\colon V[x] \times V[y] \subset M$ *for some* V *in* $\mathfrak{U}\}$. If $U \in \mathfrak{U}$ there is a symmetric member V of $\mathfrak{U}$ such that $V \circ V \circ V \subset U$ and, according to lemma 6.1, $V \circ V \circ V = \bigcup \{V[x] \times V[y]\colon (x,y) \in V\}$. Hence every point of V is an interior point of U and, since the interior of U contains V, it is a member of $\mathfrak{U}$. ∎

In view of the foregoing theorem every member of a uniformity is a neighborhood of the diagonal. It is to be emphasized that the converse of this proposition is false. There may be many very different uniformities for X, all having the same topology and hence the same family of neighborhoods of the diagonal.

7 THEOREM *The closure, relative to the uniform topology, of a subset* A *of* X *is* $\bigcap \{U[A]\colon U \in \mathfrak{U}\}$. *The closure of a subset* M *of* $X \times X$ *is* $\bigcap \{U \circ M \circ U\colon U \in \mathfrak{U}\}$.

PROOF A point x belongs to the closure of a subset A of X iff $U[x]$ intersects A for each U in $\mathfrak{U}$. But $U[x]$ intersects A iff $x \in U^{-1}[A]$, and since each member of $\mathfrak{U}$ contains a symmetric member, $x \in A^-$ iff $x \in U[A]$ for each U in $\mathfrak{U}$. The first statement is then proved. Similarly, if U is a symmetric member of $\mathfrak{U}$, then $U[x] \times U[y]$ intersects a subset M of $X \times X$ iff $(x,y) \in U[u] \times U[v]$ for some (u,v) in M, that is, iff $(x,y) \in \bigcup\{U[u] \times U[v]: (u,v) \in M\}$. Since by lemma 6.1 this last set is $U \circ M \circ U$ it follows that $(x,y) \in M^-$ iff $(x,y) \in \bigcap\{U \circ M \circ U: U \in \mathfrak{U}\}$. ∎

8 THEOREM *The family of closed symmetric members of a uniformity $\mathfrak{U}$ is a base for $\mathfrak{U}$.*

PROOF If $U \in \mathfrak{U}$ and V is a member of $\mathfrak{U}$ such that $V \circ V \circ V \subset U$, then $V \circ V \circ V$ contains the closure of V in view of the preceding theorem; hence U contains a closed member W of $\mathfrak{U}$ and $W \cap W^{-1}$ is a closed symmetric member. ∎

It will be shown presently that a uniform space (more precisely a space with a uniform topology) is always completely regular. At the moment it is easy to see that such a space is regular, for each neighborhood of a point x contains a neighborhood $V[x]$ such that V is a closed member of $\mathfrak{U}$, and $V[x]$ is consequently closed. Therefore a space with a uniform topology is a Hausdorff space iff each set consisting of a single point is closed. Since the closure of the set $\{x\}$ is $\bigcap\{U[x]: U \in \mathfrak{U}\}$, the space is Hausdorff iff $\bigcap\{U: U \in \mathfrak{U}\}$ is the diagonal Δ. In this case $(X,\mathfrak{U})$ is said to be **Hausdorff** or **separated.**

UNIFORM CONTINUITY; PRODUCT UNIFORMITIES

If f is a function on a uniform space $(X,\mathfrak{U})$ with values in a uniform space $(Y,\mathfrak{V})$, then f is **uniformly continuous** relative to $\mathfrak{U}$ and $\mathfrak{V}$ iff for each V in $\mathfrak{V}$ the set $\{(x,y): (f(x), f(y)) \in V\}$ is a member of $\mathfrak{U}$. This condition may be rephrased in several ways. For each function f on X to Y let f_2 be the induced function on $X \times X$ to $Y \times Y$ which is defined by $f_2(x,y) = (f(x),f(y))$. Then f is uniformly continuous iff for each V in $\mathfrak{V}$ there is U in $\mathfrak{U}$ such that $f_2[U] \subset V$. We also have: if $\mathfrak{S}$ is a subbase for $\mathfrak{V}$, then f is uniformly continuous iff $f_2^{-1}[V] \in \mathfrak{U}$ for each V in $\mathfrak{S}$, because f_2^{-1} preserves unions and intersections. If Y is the set of real numbers and $\mathfrak{V}$ is the usual uniformity, then it follows that f is uniformly continuous iff for each positive number r there is U in $\mathfrak{U}$ such that $|f(x) - f(y)| < r$ whenever $(x,y) \in U$. If X is also the space of real numbers with the usual uniformity, then f is uniformly continuous iff for each positive number r there is a positive number s such that $|f(x) - f(y)| < r$ whenever $|x - y| < s$.

It is evident that, if f is on X to Y and g is a function on Y, then $(g \circ f)_2 = g_2 \circ f_2$, and from this it follows that the composition of two uniformly continuous functions is again uniformly continuous. If f is one-to-one map of X onto Y and both f and f^{-1} are uniformly continuous, then f is a **uniform isomorphism,** and the spaces X and Y (more precisely $(X,\mathfrak{U})$ and $(Y,\mathfrak{V})$) are said to be **uniformly equivalent.** The composition of two uniform isomorphisms, the inverse of a uniform isomorphism, and the identity map of a space onto itself are all uniform isomorphisms, and consequently the collection of all uniform spaces is divided into equivalence classes, consisting of uniformly equivalent spaces. A property which when possessed by one uniform space is also possessed by every uniformly isomorphic space is a **uniform invariant.** With a few exceptions the properties studied in this chapter are uniform invariants.

As might be expected, uniform continuity implies continuity relative to the uniform topology.

9 THEOREM *Each uniformly continuous function is continuous relative to the uniform topology, and hence each uniform isomorphism is a homeomorphism.*

PROOF Let f be a uniformly continuous function on $(X,\mathfrak{U})$ to $(Y,\mathfrak{V})$ and let U be a neighborhood of $f(x)$. Then there is V in $\mathfrak{V}$ such that $V[f(x)] \subset U$, and $f^{-1}[V[f(x)]] = \{y : f(y) \in V[f(x)]\} = \{y : (f(x), f(y)) \in V\} = f_2^{-1}[V][x]$, and this is a neighborhood of x. Hence $f^{-1}[U]$ is a neighborhood of x and continuity is proved. ∎

If f is a function on a set X to a uniform space $(Y,\mathfrak{V})$, then it is not generally true that the family of all sets $f_2^{-1}[V]$ for V in $\mathfrak{V}$ is a uniformity for X. The difficulty is that there may be a subset of $X \times X$ which contains some set $f_2^{-1}[V]$, but is not the inverse of any subset of $Y \times Y$. However, this difficulty is not profound; the family of all $f_2^{-1}[V]$ is the base for a uniformity $\mathfrak{U}$ for X, as we now verify. It is clear that f_2^{-1} preserves inclusions, intersections, and inverses (that is, $f_2^{-1}[V^{-1}] = [f_2^{-1}[V]]^{-1}$), and consequently it is only necessary to show that for each member U of $\mathcal{V}$ there is V in $\mathcal{V}$ such that $f_2^{-1}[V] \circ f_2^{-1}[V] \subset f_2^{-1}[U]$. But if $V \circ V \subset U$ and (x,y) and (y,z) belong to $f_2^{-1}[V]$, then both $(f(x),f(y))$ and $(f(y),f(z))$ belong to V, and hence $(f(x),f(z)) \in V \circ V$. It follows that the family of inverses of members of $\mathcal{V}$ is indeed a base for a uniformity $\mathcal{U}$ for X. It is clear that f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$, and in fact $\mathcal{U}$ is smaller than every other uniformity for which f is uniformly continuous.

If $(X,\mathcal{U})$ is a uniform space and Y is a subset of X, then in view of the preceding discussion there is a smallest uniformity $\mathcal{V}$ such that the identity map of Y into X is uniformly continuous. It is clear that the members of $\mathcal{V}$ are simply the intersections of the members of $\mathcal{U}$ with $Y \times Y$ (sometimes called the trace of $\mathcal{U}$ on $Y \times Y$). The uniformity $\mathcal{V}$ is called the **relativization** of $\mathcal{U}$ to Y, or the **relative uniformity** for Y, and $(Y, \mathcal{V})$ is called a uniform subspace of the space $(X,\mathcal{U})$. We omit the simple verification of the fact that the topology of the relative uniformity $\mathcal{V}$ is the relativized topology of $\mathcal{U}$.

We have seen that there is always a unique smallest uniformity which makes a map of a set X into a uniform space uniformly continuous. This proposition may be extended to a family F of functions such that each member f of F maps X into a uniform space $(Y_f,\mathcal{U}_f)$. The family of all sets of the form $f_2^{-1}[U] = \{(x,y): (f(x),f(y)) \in U\}$, for f in F and U in $\mathcal{U}_f$, is a subbase for a uniformity $\mathcal{U}$ for X, and $\mathcal{U}$ is the smallest uniformity such that each map f is uniformly continuous. (Theorem 6.3 shows that the family of sets of the form $f_2^{-1}[U]$ is a subbase for a uniformity, and evidently $\mathcal{U}$ makes each f uniformly continuous and is smaller than every other uniformity with this property.) It is in precisely this way that the product uniformity is defined. If $(X_a,\mathcal{U}_a)$ is a uniform space for each member a of an index set A, then the **product uniformity** for $\times\{X_a: a \in A\}$ is the smallest uniformity such that projection into each coordinate space is uniformly continuous. The family of all sets of the form $\{(x,y): (x_a,y_a) \in U\}$, for a in A and U in $\mathcal{U}_a$, is a subbase for the product uniformity. If x is a member of the product, then a subbase for the neighborhood system of x (relative to the uniform topology) may be constructed from the subbase for the product uniformity. Hence the family of all sets of the form $\{y: (x_a,y_a) \in U\}$ is a subbase for the neighborhood system of x. It follows that a base for the neighborhood system of x relative to the topology of the product uniformity is the family of finite intersections of sets of the form $\{y: y_a \in U[x_a]\}$ for a in A and U in $\mathfrak{U}_a$. But the same family is also a base for the neighborhood system of x relative to the product topology, and consequently the product topology is the topology of the product uniformity. This statement is the first half of the following theorem.

10 THEOREM *The topology of the product uniformity is the product topology.*

A function f on a uniform space to a product of uniform spaces is uniformly continuous if and only if the composition of f with each projection into a coordinate space is uniformly continuous.

PROOF If f is uniformly continuous with values in the product $\times\{X_a: a \in A\}$, then each projection P_a is uniformly continuous and the composition $P_a \circ f$ is uniformly continuous. If $P_a \circ f$ is uniformly continuous for each a in A and U is a member of the uniformity of X_a, then $\{(u,v): (P_a \circ f(u), P_a \circ f(v)) \in U\}$ is a member of the uniformity $\mathfrak{V}$ of the domain of f. But this set can be written in the form $f_2^{-1}[\{(x,y): (x_a,y_a) \in U\}]$. Hence the inverse under f_2 of each member of a subbase for the product uniformity belongs to $\mathfrak{V}$ and f is therefore uniformly continuous. ∎

The next proposition begins the development of the relation between uniformities and pseudo-metrics for X.

11 THEOREM *Let $(X,\mathfrak{U})$ be a uniform space and let d be a pseudo-metric for X. Then d is uniformly continuous on $X \times X$ relative to the product uniformity if and only if the set $\{(x,y): d(x,y) < r\}$ is a member of $\mathfrak{U}$ for each positive number r.*

PROOF Let $V_{d,r} = \{(x,y): d(x,y) < r\}$. It must be shown that $V_{d,r} \in \mathfrak{U}$ for each positive r iff d is uniformly continuous with respect to the product uniformity for $X \times X$. If U is a member of $\mathfrak{U}$, then the sets $\{((x,y),(u,v)): (x,u) \in U\}$ and $\{((x,y),(u,v)): (y,v) \in U\}$ belong to the product uniformity, and it is easy to see that the family of all sets of the form $\{((x,y),(u,v)): (x,u) \in U$ *and*

$(y,v) \in U\}$ is a base for the product uniformity. Hence if d is uniformly continuous, then for each positive r there is U in $\mathfrak{U}$ such that, if (x,u) and (y,v) belong to U, then $|d(x,y) - d(u,v)| < r$. In particular, letting $(u,v) = (y,y)$, it follows that, if $(x,y) \in U$, then $d(x,y) < r$. Then $U \subset V_{d,r}$ and consequently $V_{d,r} \in \mathfrak{U}$. To prove the converse observe that, if both (x,u) and (y,v) belong to $V_{d,r}$, then $|d(x,y) - d(u,v)| < 2r$ because $d(x,y) \leqq d(x,u) + d(u,v) + d(y,v)$ and $d(u,v) \leqq d(x,u) + d(x,y) + d(y,v)$. It follows that, if $V_{d,r} \in \mathfrak{U}$ for each positive r, then d is uniformly continuous. ∎

METRIZATION

The purpose of this section is to compare uniform spaces and pseudo-metrizable spaces. The comparison is an example of the standard procedure for testing the effectiveness of a generalization. The generalization is compared with the mathematical object which it purports to generalize in order to discover the extent to which the basic concepts have been isolated. In this case (as in many other instances) the comparison yields a representation of the generalized object in terms of its progenitor. A uniformity will be assigned to each family of pseudo-metrics for a set X, and the principal result of the section states that every uniformity is derived in this fashion from the family of its uniformly continuous pseudo-metrics. It will also be shown that a uniformity can be derived from a single pseudo-metric if and only if the uniformity has a countable base.

Each pseudo-metric d for a set X generates a uniformity in the following way. For each positive number r let $V_{d,r} = \{(x,y): d(x,y) < r\}$. Clearly $(V_{d,r})^{-1} = V_{d,r}$, $V_{d,r} \cap V_{d,s} = V_{d,t}$ where $t = \min\,[r,s]$, and $V_{d,r} \circ V_{d,r} \subset V_{d,2r}$. It follows that the family of all sets of the form $V_{d,r}$ is a base for a uniformity for X. This uniformity is called the **pseudo-metric uniformity**, or the **uniformity generated by** d. A uniform space $(X,\mathfrak{U})$ is said to be **pseudo-metrizable** (or **metrizable**) if and only if there is a pseudo-metric (metric, respectively) d such that $\mathfrak{U}$ is the uniformity generated by d. The uniformity generated by a pseudo-metric d can be described in another way. According to 6.11 a pseudo-metric d is uniformly continuous relative to a uniformity $\mathfrak{V}$ (more precisely, relative to the product uniformity constructed from $\mathcal{V}$) if and only if $V_{d,r} \in \mathcal{V}$ for each positive r. The uniformity $\mathcal{U}$ derived from d can then be characterized as the smallest uniformity which makes d uniformly continuous on $X \times X$. It should be noticed that the pseudo-metric topology is identical with the uniform topology of $\mathcal{U}$, because $V_{d,r}[x]$ is the open r-sphere about x and the family of sets of this form is a base for the neighborhood system of x relative to both topologies.

The crucial step in the metrization theorem for uniform spaces is provided by the following lemma.

12 METRIZATION LEMMA *Let* $\{U_n, n \in \omega\}$ *be a sequence of subsets of* $X \times X$ *such that* $U_0 = X \times X$, *each* U_n *contains the diagonal, and* $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ *for each* n. *Then there is a non-negative real-valued function* d *on* $X \times X$ *such that*

(a) $d(x,y) + d(y,z) \geq d(x,z)$ *for all* x, y, *and* z; *and*
(b) $U_n \subset \{(x,y): d(x,y) < 2^{-n}\} \subset U_{n-1}$ *for each positive integer* n.

If each U_n *is symmetric, then there is a pseudo-metric* d *satisfying condition* (b).

PROOF Define a real-valued function f on $X \times X$ by letting $f(x,y) = 2^{-n}$ iff $(x,y) \in U_{n-1} \sim U_n$ and $f(x,y) = 0$ iff (x,y) belongs to each U_n. The desired function d is constructed from its "first approximation" f by a chaining argument. For each x and each y in X let $d(x,y)$ be the infimum of $\sum\{f(x_i,x_{i+1}): i = 0, \cdots, n\}$ over all finite sequences $x_0, x_1, \cdots, x_{n+1}$ such that $x = x_0$ and $y = x_{n+1}$. It is evident that d satisfies the triangle inequality and since $d(x,y) \leq f(x,y)$ it follows that $U_n \subset \{(x,y): d(x,y) < 2^{-n}\}$. If each U_n is symmetric, then $f(x,y) = f(y,x)$ for each pair (x,y) and consequently d is a pseudo-metric in this case. The proof is completed by showing that $f(x_0,x_{n+1}) \leq 2\sum\{f(x_i,x_{i+1}): i = 0, \cdots, n\}$, from which it will follow that, if $d(x,y) < 2^{-n}$, then $f(x,y) < 2^{-n+1}$, hence $(x,y) \in U_{n-1}$, and $\{(x,y): d(x,y) < 2^{-n}\} \subset U_{n-1}$. The proof is by induction on n, and the inequality is clearly valid for $n = 0$. For convenience, call the number $\sum\{f(x_i,x_{i+1}): i = r, \cdots, s\}$ the length of the chain from r to $s + 1$, and let a be the length of the chain from

0 to $n + 1$. Let k be the largest integer such that the chain from 0 to k is of length at most $a/2$, and notice that the chain from $k + 1$ to $n + 1$ has length at most $a/2$. By the induction hypothesis, each of $f(x_0,x_k)$ and $f(x_{k+1},x_{n+1})$ is at most $2(a/2) = a$, and surely $f(x_k,x_{k+1})$ is at most a. If m is the smallest integer such that $2^{-m} \leqq a$, then (x_0,x_k), (x_k,x_{k+1}) and (x_{k+1},x_{n+1}) all belong to U_m and therefore $(x_0,x_{n+1}) \in U_{m-1}$. Hence $f(x_0,x_{n+1}) \leqq 2^{-m+1} \leqq 2a$ and the lemma is proved. ∎

If a uniformity $\mathfrak{U}$ for X has a countable base V_0, V_1, $\cdots$, V_n $\cdots$, then it is possible to construct by induction a family U_0, U_1, $\cdots$, U_n $\cdots$ such that each U_n is symmetric, $U_n \circ U_n \circ U_n \subset U_{n-1}$ and $U_n \subset V_n$ for each positive integer n. The family of sets U_n is then a base for $\mathfrak{U}$, and upon applying the metrization lemma it follows that the uniform space $(X,\mathfrak{U})$ is pseudo-metrizable. Hence:

13 Metrization theorem *A uniform space is pseudo-metrizable if and only if its uniformity has a countable base.*

This theorem clearly implies that a uniform space is metrizable iff it is Hausdorff and its uniformity has a countable base.

14 *Notes* To the best of my knowledge this theorem first appears in Alexandroff and Urysohn [2]. These authors were seeking a solution to the topological metrization problem (see 4.18), and the result they state is (approximately): a topological Hausdorff space $(X,\mathfrak{J})$ is metrizable iff there is a uniformity with a countable base such that $\mathfrak{J}$ is the uniform topology. This is a rather unsatisfactory solution to the topological metrization problem but (with a slightly strengthened conclusion) is precisely the metrization theorem for uniform spaces. Chittenden [1] first proved a "uniform" form of 6.13 and his proof was later drastically simplified by A. H. Frink [1] and by Aronszajn [1]. The preceding proof is Bourbaki's arrangement of Frink's. The first appearance of 6.13 in the form just given occurs in André Weil's classic monograph [1] in which he introduces the notion of uniform space. ∎

A uniformity for a set X may be derived from a family P of pseudo-metrics in the following fashion. Letting $V_{p,r} = \{(x,y): p(x,y) < r\}$, the family of all sets of the form $V_{p,r}$ for p in P and r positive is the subbase for a uniformity $\mathcal{U}$ for X. This uniformity $\mathcal{U}$ is defined to be the **uniformity generated by *P***. The uniformity may be described in several instructive ways. According to 6.11 a pseudo-metric p is uniformly continuous on $X \times X$ relative to the product uniformity derived from $\mathcal{V}$ iff $V_{p,r} \in \mathcal{V}$ for each positive r. Consequently the uniformity generated by P is the smallest uniformity which makes each member p of P uniformly continuous on $X \times X$. Another description: For a fixed member p of P the family of all sets $V_{p,r}$ for r positive is a base for the uniformity of the pseudo-metric space (X,p). If $\mathcal{V}$ is a uniformity for X, then the identity map of $(X,\mathcal{V})$ into (X,p) is uniformly continuous iff $V_{p,r} \in \mathcal{V}$ for each positive r. It follows that the uniformity $\mathcal{U}$ is the smallest such that for each p in P the identity map of X into (X,p) is uniformly continuous. This fact yields yet another description. Let Z be the product $\times\{X: p \in P\}$ (that is, the product of X with itself as many times as there are members of P) and let f be the map of X into Z defined by $f(x)_p = x$ for each x in X and each p in P. Let the p-th coordinate space of this product be assigned the uniformity of the pseudo-metric p, and let Z have the product uniformity. The projection of Z into the p-th coordinate space is the identity map of X onto the pseudo-metric space (X,p), and it therefore follows from 6.10 that the uniformity generated by P is the smallest having the property that the map of X into Z is uniformly continuous. But f is one to one and is consequently a uniform isomorphism of X onto a subspace of the product of pseudo-metric spaces.

It is clearly of some importance to know which uniformities are generated by families of pseudo-metrics—this might be called the generalized metrization problem for uniform spaces. The solution to the problem is a direct application of the preceding results. Let $(X,\mathcal{U})$ be a uniform space and let P be the family of all pseudo-metrics for X which are uniformly continuous on $X \times X$. The uniformity generated by P is smaller than $\mathcal{U}$ in view of 6.11. But the metrization lemma 6.12 shows that for each member U of $\mathcal{U}$ there is a member p of P such that $\{(x,y):$

$p(x,y) < 1/4\}$ is contained in U, and hence $\mathcal{U}$ is smaller than the uniformity generated by P. Thus:

15 THEOREM *Each uniformity for X is generated by the family of all pseudo-metrics which are uniformly continuous on $X \times X$.*

There is an interesting corollary to the foregoing theorem. It has already been observed that, if a uniformity $\mathcal{U}$ for X is generated by a family P of pseudo-metrics, then the space is uniformly isomorphic to a subspace of a product of pseudo-metric spaces, and it is possible to sharpen this result if $(X,\mathcal{U})$ is Hausdorff. The uniformity $\mathcal{U}$ is the smallest which makes the identity map of X into the pseudo-metric space (X,p) uniformly continuous for each p in P. The space (X,p) is isometric under a map h_p to a metric space (X_p,p^*), by theorem 4.15, and it follows that $\mathcal{U}$ is the smallest uniformity making each of the maps h_p uniformly continuous. If a map h of X into $\times\{X_p: p \in P\}$ is defined by letting $h(x)_p = h_p(x)$, then by 6.10 the uniformity $\mathcal{U}$ is the smallest such that h is uniformly continuous. If $(X,\mathcal{U})$ is Hausdorff, then h must be one to one, and in this case h is a uniform isomorphism. The preceding theorem then implies the following result (Weil [1]).

16 THEOREM *Each uniform space is uniformly isomorphic to a subspace of the product of pseudo-metric spaces and each uniform Hausdorff space is uniformly isomorphic to a subspace of the product of metric spaces.*

The preceding theorem yields a characterization of those topologies which can be the uniform topology for some uniformity, for a topological space is completely regular if and only if it is homeomorphic to a subspace of a product of pseudo-metrizable spaces (4.L).

17 COROLLARY *A topology $\mathfrak{J}$ for a set X is the uniform topology for some uniformity for X if and only if the topological space $(X,\mathfrak{J})$ is completely regular.*

The remainder of this section is devoted to a clarification of the relationship between uniformities and pseudo-metrics. A family P of pseudo-metrics for a set X is said to be a gage iff there is a uniformity $\mathcal{U}$ for X such that P is the family of all pseudo-metrics which are uniformly continuous on $X \times X$ relative to the product uniformity derived from $\mathcal{U}$. The family P is called the gage of the uniformity $\mathcal{U}$ and $\mathcal{U}$ is the uniformity of P ($\mathcal{U}$ is generated by P according to 6.15). Every family of pseudo-metrics generates a uniformity; it will also be said to generate the gage of this uniformity. A direct description of the gage generated by a family P of pseudo-metrics is possible. The family of all sets of the form $V_{p,r}$ for p in P and r positive is a subbase for the uniformity of the gage, and hence a pseudo-metric q is uniformly continuous on the product iff for each positive number s the set $V_{q,s}$ contains some finite intersection of sets $V_{p,r}$ for p in P. This remark establishes the following proposition.

18 Theorem *Let P be a family of pseudo-metrics for a set X and let Q be the gage generated by P. Then a pseudo-metric q belongs to Q if and only if for each positive number s there is a positive number r and a finite subfamily $p_1, \cdots, p_n$ of P such that $\bigcap \{V_{p_i,r}: i = 1, \cdots, n\} \subset V_{q,s}$.*

Each concept which is based on the notion of a uniformity can be described in terms of a gage because each uniformity is completely determined by its gage. The following theorem is a dictionary of such descriptions. Recall that p-dist $(x,A) = \inf\{p(x,y): y \in A\}$ is the p-distance from a point x to a set A.

19 Theorem *Let $(X,\mathcal{U})$ be a uniform space and let P be the gage of $\mathcal{U}$. Then:*

(a) *The family of all sets $V_{p,r}$ for p in P and r positive is a base for the uniformity $\mathcal{U}$.*

(b) *The closure relative to the uniform topology of a subset A of X is the set of all x such that p-dist $(x,A) = 0$ for each p in P.*

(c) *The interior of a set A is the set of all points such that for some p in P and some positive number r the sphere $V_{p,r}[x] \subset A$.*

(d) *Suppose P' is a subfamily of P which generates P. A net $\{S_n, n \in D\}$ in X converges to a point s if and only if $\{p(S_n,s), n \in D\}$ converges to zero for each p in P'.*

(e) *A function f on X to a uniform space $(Y,\mathcal{V})$ is uniformly continuous if and only if for each member q of the gage Q of $\mathcal{V}$ it is true that $q \circ f_2 \in P$. (Recall $f_2(x,y) = (f(x),f(y))$.)*

Equivalently, f is uniformly continuous if and only if for each q in Q and each positive number s there is p in P and r positive such that, if $p(x,y) < r$, then $q(f(x),f(y)) < s$.

(f) *If $(X_a,\mathcal{U}_a)$ is a uniform space for each member a of an index set A and P_a is the gage of $\mathcal{U}_a$ then the gage of the product uniformity for $\times\{X_a: a \in A\}$ is generated by all pseudo-metrics of the form $q(x,y) = p_a(x_a,y_a)$ for a in A and p_a in P_a.*

The proof is omitted. It is a straightforward application of earlier results.

COMPLETENESS

This section is devoted to a number of elementary theorems based on the concept of a Cauchy net. A uniform space will be called complete iff each Cauchy net in the space converges to some point. The two most useful results of the section state that the product of complete spaces is complete, and that a uniformly continuous function f to a complete Hausdorff space has a uniformly continuous extension whose domain is the closure of the domain of f.

It will be supposed throughout that X is a set, $\mathcal{U}$ is a uniformity for X, and P is the gage of $\mathcal{U}$ (that is, P is the family of all pseudo-metrics for X which are uniformly continuous on $X \times X$). The definitions will be given in terms of both $\mathcal{U}$ and P, and the proofs use the formulation which is most convenient for the problem under consideration. The set $\{(x,y): p(x,y) < r\}$ will be denoted by $V_{p,r}$.

A net $\{S_n, n \in D\}$ in the uniform space $(X,\mathcal{U})$ is a **Cauchy net** iff for each member U of $\mathcal{U}$ there is N in D such that $(S_m,S_n) \in U$ whenever both m and n follow N in the ordering of D. This definition may be rephrased in terms of a net in $X \times X$. In this form it is stated: the net $\{S_n, n \in D\}$ is a Cauchy net iff the net $\{(S_m,S_n), (m,n) \in D \times D\}$ is eventually in each member of $\mathcal{U}$. (It is understood that $D \times D$ is given the product ordering.)

The family of all sets of the form $V_{p,r}$ for p in the gage P and r positive is a base for the uniformity $\mathcal{U}$, and it follows that $\{S_n, n \in D\}$ is a Cauchy net iff $\{(S_m,S_n), (m,n) \in D \times D\}$ is eventually in each set of the form $V_{p,r}$. In other words, $\{S_n, n \in D\}$ is a Cauchy net if and only if $\{p(S_m,S_n), (m,n) \in D \times D\}$ converges to zero for each pseudo-metric p belonging to the gage P.

There is a simple lemma about Cauchy nets which is used often enough to deserve a formal statement.

20 LEMMA *A net $\{S_n, n \in D\}$ in a uniform space $(X,\mathcal{U})$ is a Cauchy net if and only if either of the following statements is true.*

(a) *The net $\{(S_m,S_n), (m,n) \in D \times D\}$ is eventually in each member of some subbase for the uniformity $\mathcal{U}$.*

(b) *The net $\{p(S_m,S_n), (m,n) \in D \times D\}$ converges to zero for each p in some family of pseudo-metrics which generates the gage P.*

PROOF If a family Q of pseudo-metrics generates P, then the family of all $V_{p,r}$ for p in Q and r positive is a subbase for the uniformity, so that the proof of (b) reduces to that of (a). To prove (a) notice that, if a net (for example $\{(S_m,S_n), (m,n) \in D \times D\}$) is eventually in each of a finite number of sets, it is then eventually in their intersection. ∎

The following proposition relates Cauchy nets to convergence relative to the uniform topology.

21 THEOREM *Each net which converges to a point relative to the uniform topology is a Cauchy net. A Cauchy net converges to each of its cluster points.*

PROOF If $\{S_n, n \in D\}$ converges to a point s, then $\{d(S_n,s), n \in D\}$ converges to zero for each member d of the gage P. Since $d(S_m,S_n) \leq d(S_m,s) + d(S_n,s)$, it follows that $\{d(S_m,S_n), (m,n) \in D \times D\}$ converges to zero and the net is therefore a Cauchy net. Suppose that $\{S_n, n \in D\}$ is a Cauchy net and s is a cluster point. Then for d in P and r positive there is N in D such that, if $m \geq N$ and $n \geq N$, then $d(S_m,S_n) < r/2$. Since s is a cluster point, there is p in D such that $d(S_p,s) \leq r/2$ and $p \geq N$. Then $d(S_n,s) \leq d(S_n,S_p) + d(S_p,s) < r$ if $n \geq N$, and it follows that the net converges to s. ∎

A uniform space is complete iff every Cauchy net in the space converges to a point of the space. Evidently each closed subspace of a complete space $(X,\mathfrak{U})$ is complete. If $(X,\mathfrak{U})$ is Hausdorff and $(Y,\mathfrak{V})$ is a complete subspace, then Y is closed in X, for a net in Y which converges to a point x of X is necessarily a Cauchy net, and x is the unique limit point. This obvious result is one of the most useful facts about completeness.

22 THEOREM *A closed subspace of a complete space is complete, and a complete subspace of a Hausdorff uniform space is closed.*

Before proceeding it may be worth while to mention several examples of complete spaces. If the uniformity $\mathfrak{U}$ is the largest possible uniformity for X (that is, consists of all subsets of $X \times X$ which contain the diagonal), then $(X,\mathfrak{U})$ is complete. The smallest uniformity for X also yields a complete space. If a uniform space $(X,\mathfrak{U})$ is compact relative to the uniform topology, then it is complete, for every net has a cluster point and consequently by theorem 6.21 each Cauchy net converges to some point. The space of real numbers is complete relative to the usual uniformity. This may be seen by verifying that each Cauchy net is eventually in some bounded subset A of the space of real numbers and is therefore eventually in the compact set A^-.

There is a characterization of completeness which is suggestive of compactness. Recall that a family of sets has the finite intersection property iff no finite intersection of members of the family is void, and a topological space is compact iff the intersection of the members of each family of closed sets with the finite intersection property is non-void. To describe completeness another qualification is put on the family. A family $\mathfrak{A}$ of subsets of a uniform space $(X,\mathfrak{U})$ contains small sets iff for each U in $\mathfrak{U}$ there is a member A of $\mathfrak{A}$ such that A is a subset of $U[x]$ for some point x. Another formulation is: for each U in $\mathfrak{U}$ there is A in $\mathfrak{A}$ such that $A \times A \subset U$. In terms of the gage P of the uniform space, a family $\mathfrak{A}$ contains small sets iff for each positive r and each d in P there is A in $\mathfrak{A}$ such that the d-diameter of A is less than r. We omit the proof that these three statements are equivalent.

23 THEOREM * *A uniform space is complete if and only if each family of closed sets which has the finite intersection property and contains small sets has a non-void intersection.*

PROOF Let $(X,\mathfrak{U})$ be a complete uniform space and $\mathfrak{A}$ a family of closed sets which has the finite intersection property and contains small sets. If $\mathfrak{F}$ is the family of all finite intersections of members of $\mathfrak{A}$, then $\mathfrak{F}$ is directed by $\subset$, and for each F in $\mathfrak{F}$ we may choose a point x_F in F. The net $\{x_F, F \in \mathfrak{F}\}$ is a Cauchy net because, if A and B follow a member F of $\mathfrak{F}$ in the ordering $\subset$ (that is, $A \subset F$ and $B \subset F$), then x_A and x_B belong to F, and $\mathfrak{F}$ contains small sets. Consequently, $\{x_F: F \in \mathfrak{F}\}$ converges to a point and since the net is eventually in each member of $\mathfrak{F}$ the point must belong to every member of $\mathfrak{F}$. Hence the intersection $\bigcap\{A: A \in \mathfrak{A}\}$ is non-void. To prove the converse let $\{x_n, n \in D\}$ be a Cauchy net, and for each n in D let A_n be the set of all points x_m for $m \geq n$. Then the family $\mathfrak{A}$ of all sets of the form A_n has the finite intersection property, and since the net is Cauchy the family $\mathfrak{A}$ contains small sets. There is hence a point y which belongs to the intersections of the closures, $\bigcap\{A_n^-: n \in D\}$, and, according to 2.7, the point y is a cluster point of the net $\{x_n, n \in D\}$. Since $\{x_n, n \in D\}$ is a Cauchy net it converges to y. ∎

One might suspect that a uniform space satisfying the first axiom of countability would be complete if every Cauchy sequence in the space converged to a point of the space. Unfortunately this suspicion is unfounded, but the following feeble result is correct.

24 THEOREM *A pseudo-metrizable uniform space is complete if and only if every Cauchy sequence in the space converges to a point.*

PROOF If a uniform space is complete, then each Cauchy net in X, and in particular each Cauchy sequence in X, converges to a point. On the other hand, suppose that (X,d) is a pseudo-metric space such that every Cauchy sequence converges to a point, and that $\mathfrak{A}$ is a family of closed subsets of X which has the finite intersection property and contains small sets. For

* A filter is a Cauchy filter if it contains small sets. Then the theorem can be stated: a space is complete iff each Cauchy filter converges to some point.

each non-negative integer n select a member A_n of $\mathcal{A}$ which is of diameter less than 2^{-n} and select a point x_n belonging to A_n. If m and n are large, then $d(x_m,x_n)$ is small because x_m and x_n belong to A_m and A_n respectively, these two sets intersect, and each has small diameter. Hence $\{x_n, n \in \omega\}$ is a Cauchy sequence and therefore converges to a point y of X. If B is an arbitrary member of $\mathcal{A}$, then dist $(x_n,B) < 2^{-n}$ because B intersects A_n, and it follows that y belongs to the closure of B. Since $\mathcal{A}$ is a family of closed sets y belongs to every member of $\mathcal{A}$. ▌

The usual method of proving completeness consists in showing the space in question is uniformly isomorphic to a closed subspace of a product of complete spaces and then appealing to the following theorem. The proof of this theorem requires the fact that the image of a Cauchy net under a uniformly continuous map is a Cauchy net—a fact which is evident from the definition.

25 THEOREM *The product of uniform spaces is complete if and only if each coordinate space is complete.*

A net in the product is a Cauchy net if and only if its projection into each coordinate space is a Cauchy net.

PROOF Suppose that $(Y_a,\mathcal{U}_a)$ is a complete uniform space for each member a of an index set A. For each a the projection of a Cauchy net into Y_a is a Cauchy net and hence converges to a point, say, y_a. Then the net in the product converges to the point y with a-th coordinate y_a and consequently the product is complete. The simple proof of the converse is omitted.

If $\{x_n, n \in D\}$ is a net in the product which projects into a Cauchy net in each coordinate space, then for each member U of $\mathcal{U}_a$ the net $\{(x_m,x_n), (m,n) \in (D \times D)\}$ is eventually in the inverse under projection of U. That is, $\{(x_m,x_n), (m,n) \in (D \times D)\}$ is eventually in $\{(x,z)\colon (x_a,z_a) \in U\}$. Since the family of sets of this form is a subbase for the product uniformity it follows (6.20) that $\{x_n, n \in D\}$ is a Cauchy net. ▌

A function f is **uniformly continuous on a subset** A of a uniform space $(X,\mathcal{U})$ iff its restriction to A, $f \mid A$, is uniformly continuous with respect to the relativized uniformity. If the range space is complete and Hausdorff * and f is uniformly continuous

* This requirement is not necessary for the existence of an extension, but is necessary for the uniqueness.

on its domain A, then there is a unique uniformly continuous extension whose domain is the closure of A.

26 THEOREM *Let f be a function whose domain is a subset A of a uniform space $(X,\mathfrak{U})$ and whose values lie in a complete Hausdorff uniform space $(Y,\mathfrak{V})$. If f is uniformly continuous on A, then there is a unique uniformly continuous extension f^- of f whose domain is the closure of A.*

PROOF The function f is a subset of $X \times Y$ (we do not distinguish between a function and its graph) and the desired extension is the closure f^- of f in $X \times Y$. (A pair (x,y) belongs to f^- iff there is a net in A converging to x such that the image net converges to y.) The domain of f^- is evidently the closure of A. We will show that, if W is a member of $\mathfrak{V}$, then there is U in $\mathfrak{U}$ such that, if (x,y) and (u,v) are members of f^- and $x \in U[u]$, then $y \in W[v]$. Since Y is Hausdorff this will show that f^- is a function and that f^- is uniformly continuous. Choose a member V of $\mathfrak{V}$ which is closed and symmetric and such that $V \circ V \subset W$ and choose a member U of $\mathfrak{U}$ which is open and symmetric and such that $f[U[x]] \subset V[f(x)]$ for each x in A; suppose (x,y) and (u,v) belong to f^- and $x \in U[u]$. Then the intersection of $U[x]$ and $U[u]$ is open and there is consequently z in A such that both x and u belong to $U[z]$. Both y and v belong to the closure of $f[U[z]]$, by the definition of f^-, and hence both y and v belong to $V[f(z)]$. Hence $(y,v) \in V \circ V \subset W$ and $y \in W[v]$. ▌

COMPLETION

It is the purpose of this section to show that each uniform space is uniformly isomorphic to a dense subspace of a complete uniform space. It is therefore possible to adjoin "ideal elements" to a uniform space in such a way as to obtain a complete uniform space. The procedure is suggestive of the compactification process of chapter 5, but there is one significant difference: the completion of a uniform space is (essentially) unique.

For a metric space X it is possible to find a complete metric space X^* such that X is *isometric* to a dense subspace of X^* (not just uniformly isomorphic). We base the general construction of a completion on this preliminary result.

27 THEOREM *Each metric (or pseudo-metric) space can be mapped by a one-to-one isometry onto a dense subset of a complete metric (respectively pseudo-metric) space.*

PROOF It is only necessary to prove the theorem for a pseudo-metric space (X,d), since the corresponding result for metric spaces then follows from 4.15. Let X^* be the class of all Cauchy sequences in X, and for members S and T of X^* let $d^*(S,T)$ be the limit of $d(S_m,T_m)$ as m becomes large (formally, the limit of $\{d(S_m,T_m),\ m \in \omega\}$). It is easy to verify that d^* is a pseudo-metric for X^*. Let F be the map which carries each point x of X into the sequence which is constantly equal to x; that is, $F(x)_n = x$ for all n. Evidently F is a one-to-one isometry and it remains to prove that $F[X]$ is dense in X^* and X^* is complete. The first of these statements is almost self-evident; if $S \in X^*$ and n is large, then $F(S_n)$ is near S. To show X^* complete, first observe that it is sufficient to show that each Cauchy sequence in $F[X]$ converges to a point of X^* because $F[X]$ is dense in X^*. Finally, each Cauchy sequence in $F[X]$ is of the form $F \circ S = \{F(S_n),\ n \in \omega\}$, where S is a Cauchy sequence in X, and $F \circ S$ converges in X^* to the member S of X^*. ∎

Each uniform space is uniformly isomorphic to a subspace of a product of pseudo-metric spaces, and each Hausdorff uniform space is uniformly isomorphic to a product of metric spaces, by 6.16. The preceding theorem implies that a metric or pseudo-metric space is uniformly isomorphic to a subspace of a complete space of the same sort. It follows without difficulty that:

28 THEOREM *Each uniform space is uniformly isomorphic to a dense subspace of a complete uniform space. Each Hausdorff uniform space is uniformly isomorphic to a dense subspace of a complete Hausdorff uniform space.*

A completion of a uniform space $(X,\mathfrak{U})$ is a pair, $(f,(X^*,\mathfrak{U}^*))$ where $(X^*,\mathfrak{U}^*)$ is a complete uniform space and f is a uniform isomorphism of X into a dense subspace of X^*. The completion is Hausdorff iff $(X^*,\mathfrak{U}^*)$ is a Hausdorff uniform space. The foregoing theorem can then be stated: *Each (Hausdorff) uniform space has a (Hausdorff) completion.*

There is a uniqueness property for Hausdorff completions. If f and g are uniform isomorphisms of X onto dense subspaces of complete Hausdorff uniform spaces X^* and X^{**}, then both $g \circ f^{-1}$ and $f \circ g^{-1}$ have uniformly continuous extensions to all of X^* and X^{**} respectively, by 6.26. It follows that the extension of $g \circ f^{-1}$ is a uniform isomorphism of X^* onto X^{**}. Stated roughly: *the Hausdorff completion of a Hausdorff uniform space is unique to a uniform isomorphism.*

COMPACT SPACES

Each completely regular topology $\mathfrak{I}$ for a set X is the uniform topology for some uniformity $\mathfrak{U}$, but the uniformity is usually not unique. If $(X,\mathfrak{I})$ is compact and regular, then it turns out that there is precisely one uniformity whose topology is $\mathfrak{I}$. In this case the topology determines the uniformity, topological invariants are uniform invariants, and the theory takes a particularly simple form. This section is devoted to a proof of the uniqueness theorem just quoted and to two other propositions. As before, we use either the uniformity of a space or the corresponding gage of uniformly continuous pseudo-metrics as convenience dictates.

29 THEOREM *If $(X,\mathfrak{U})$ is a compact uniform space, then every neighborhood of the diagonal Δ in $X \times X$ is a member of $\mathfrak{U}$ and every pseudo-metric which is continuous on $X \times X$ is a member of the gage of $\mathfrak{U}$.*

PROOF Let $\mathfrak{B}$ be the family of closed members of $\mathfrak{U}$ and let V be an arbitrary open neighborhood of Δ. If $(x,y) \in \bigcap \{U: U \in \mathfrak{B}\}$, then, since $\mathfrak{B}$ is a base for $\mathfrak{U}$, y belongs to every neighborhood of x and hence (x,y) belongs to every neighborhood of Δ. It follows that $\bigcap \{U: U \in \mathfrak{B}\}$ is a subset of V. Since each member U of $\mathfrak{B}$ is compact and V is open the intersection of some finite subfamily of $\mathfrak{B}$ is also a subset of V and hence $V \in \mathfrak{U}$.

If a pseudo-metric d for X is continuous on $X \times X$, then for each positive r the set $\{(x,y): d(x,y) < r\}$ is a neighborhood of the diagonal. Hence d is uniformly continuous and therefore belongs to the gage of $\mathfrak{U}$. ∎

Each compact regular topological space is completely regular and its topology is therefore the uniform topology for some uniformity. This uniformity has just been identified.

30 Corollary *If* $(X,\mathfrak{J})$ *is a compact regular topological space, then the family of all neighborhoods of the diagonal* Δ *is a uniformity for* X *and* $\mathfrak{J}$ *is the uniform topology.*

There is another corollary.

31 Theorem *Each continuous function on a compact uniform space to a uniform space is uniformly continuous.*

PROOF If f is a continuous function on X to Y, then f_2, where $f_2(x,y) = (f(x),f(x))$, is a continuous function on $X \times X$ to $Y \times Y$. Consequently if d belongs to the gage of Y the composition $d \circ f_2$ is continuous on $X \times X$. It follows from theorem 6.29 that $d \circ f_2$ belongs to the gage of X, and hence the function f is uniformly continuous. ∎

Each compact uniform space $(X,\mathfrak{U})$ can be written as the union of a finite number of small sets, in the sense that for each pseudo-metric d belonging to the gage of $\mathfrak{U}$ and each positive r there is a finite cover of X by sets of d-diameter less than r. This is a direct consequence of compactness, since X can be covered by a finite number of $r/3$ spheres about points and each of these is of diameter less than r. A uniform space $(X,\mathfrak{U})$ is **totally bounded** (or **precompact**) iff X is the union of a finite number of sets of d-diameter less than r for each pseudo-metric d of the gage of $\mathfrak{U}$ and each positive r. In terms of $\mathfrak{U}$ this can be stated: for each U in $\mathfrak{U}$ the set X is the union of a finite number of sets B such that $B \times B \subset U$, or, equivalently, for each U in $\mathfrak{U}$ there is a finite subset F of X such that $U[F] = X$. A subset Y of a uniform space is called totally bounded iff Y, with the relativized uniformity, is totally bounded.

There is a simple but very useful relation between compactness and total boundedness.

32 Theorem *A uniform space* $(X,\mathfrak{U})$ *is totally bounded if and only if each net in* X *has a Cauchy subnet.*

Consequently a uniform space is compact if and only if it is totally bounded and complete.

PROOF Suppose S is a net in a totally bounded uniform space $(X,\mathfrak{U})$. The existence of a Cauchy subnet is an obvious consequence of problem 2.J, but we sketch the proof without using the earlier result. Let $\mathfrak{A}$ be the family of all subsets A of X such that S is frequently in A. Then $\{X\} \subset \mathfrak{A}$ and by the maximal principle 0.25 there is a maximal subfamily $\mathfrak{B}$ of $\mathfrak{A}$ which contains $\{X\}$ and has the finite intersection property. Because of maximality it is true that, if a finite union $B_1 \cup \cdots \cup B_n$ of members of $\mathfrak{A}$ belongs to $\mathfrak{B}$, then $B_i \in \mathfrak{B}$ for some i (see 2.I for details). Since X is totally bounded it may be covered by a finite number of small sets, and it follows that $\mathfrak{B}$ contains small sets. Finally, it follows from 2.5 that there is a subnet of S which is eventually in each member of $\mathfrak{B}$, and evidently this subnet is Cauchy.

If $(X,\mathfrak{U})$ is not totally bounded, then for some U in $\mathfrak{U}$ and for every finite subset F of X it is true that $U[F] \neq X$. It follows that one may find by induction a sequence $\{x_n, n \in \omega\}$ such that $x_n \notin U[x_p]$ if $p < n$. Clearly the sequence $\{x_n, n \in \omega\}$ has no Cauchy subnet.

Finally, if $(X,\mathfrak{U})$ is complete and totally bounded, then each net has a subnet which converges to a point of X and hence the space is compact. It has already been observed that a compact space is complete. ▮

There is one other very useful lemma concerning compact spaces. The proposition is an extension of the Lebesgue covering lemma 5.26. A cover of a subset A of a uniform space $(X,\mathfrak{U})$ is a **uniform cover** iff there is a member U of $\mathfrak{U}$ such that the set $U[x]$ is a subset of some member of the cover for every x in A (that is, the family of $U[x]$ for x in A refines the cover). In terms of the gage of the uniformity $\mathfrak{U}$, a cover of A is uniform iff there is a member d of the gage and a positive number r such that the open sphere of d-radius r about each point of A is contained in some member of the cover.

33 THEOREM *Each open cover of a compact subset of a uniform space is a uniform cover.*

In particular, each neighborhood of a compact subset A contains a neighborhood of the form $U[A]$ where U is a member of the uniformity.

PROOF Let $\mathfrak{A}$ be an open cover of the compact subset A of the uniform space $(X,\mathfrak{U})$. Then for each x in A there is U in $\mathfrak{U}$ such that $U[x]$ is a subset of some member of $\mathfrak{A}$, and hence there is V in $\mathfrak{U}$ such that $V \circ V[x]$ is a subset of some member of $\mathfrak{A}$. Choose a finite number of members $x_1, \cdots, x_n$ of A and $V_1, \cdots, V_n$ of $\mathfrak{U}$ such that the sets $V_i[x_i]$ cover A and for each i it is true that $V_i \circ V_i[x_i]$ is a subset of some member of $\mathfrak{A}$. Finally, let $W = \bigcap \{V_i : i = 1, \cdots, n\}$. Then for each point y of A for some i the point y belongs to $V_i[x_i]$ and hence $W[y] \subset W \circ V_i[x_i] \subset V_i \circ V_i[x_i]$. Consequently $W[y]$ is a subset of some member of $\mathfrak{A}$. ∎

FOR METRIC SPACES ONLY

This section is devoted to two propositions concerning complete metric spaces. The results are among the most useful consequences of completeness, and it is unfortunate that no generalization to complete uniform spaces seems possible. The first proposition is the classic theorem of Baire on category; this theorem and one or two related results occupy most of the section. The last theorem of the section states that the image under a continuous uniformly open map of a complete metric space is again complete, provided the range space is Hausdorff. The proof relies on a lemma which we state in considerably more general form than is necessary for this proposition. The lemma (essentially a formalization of an argument of Banach) also yields directly the closed graph and open mapping theorems of normed linear space theory. (See problem 6.**R**.)

34 THEOREM (BAIRE) *Let X be either a complete pseudo-metric space or a locally compact regular space. Then the intersection of a countable family of open dense subsets of X is itself dense in X.*

PROOF We prove the theorem for locally compact regular spaces, adding in parentheses the modifications necessary to establish it for a complete pseudo-metric space. Suppose that $\{G_n, n \in \omega\}$ is a sequence of dense open subsets of X and that U is an arbitrary open non-void subset of X. It must be shown that $U \cap \bigcap \{G_n : n \in \omega\}$ is non-void. To this end choose inductively an open set V_0 such that V_0^- is a compact subset of $U \cap G_0$ (such that V_0^- is a subset of $U \cap G_0$ and has diameter less than one), and then for each positive integer n choose V_n such that V_n^- is a subset of $V_{n-1} \cap G_n$ (and the diameter of V_n is less than $1/n$). This choice is possible because G_n is dense and open. The family of all sets V_n^- for non-negative integers n has the finite intersection property, consists of closed sets and V_0^- is compact (the family contains small sets). Hence $\bigcap\{V_n^-: n \in \omega\}$ is non-void, and since $V_{n+1}^- \subset U \cap G_n$ it follows that $U \cap \bigcap\{G_n: n \in \omega\}$ is non-void. ∎

It should be remarked that the Baire theorem is a hybrid in that a topological conclusion (the intersection of a countable number of dense open sets is dense) is deduced from a non-topological premise (that the space is complete pseudo-metric). There is a purely topological statement which is equivalent. If $(X,\mathfrak{I})$ is a topological space such that for some pseudo-metric d for X the space (X,d) is complete and $\mathfrak{I}$ is the pseudo-metric topology, then the same conclusion holds. (Topological spaces for which there exists such a complete metric have been characterized in a different way, as noted in 6.K.)

A terminology has been devised which is very convenient in discussing questions related to the Baire theorem. A subset A of a topological space is **nowhere dense** in X iff the interior of the closure of A is void; otherwise stated, A is nowhere dense in X iff the open set $X \sim A^-$ is dense in X. It is evident that the finite union of nowhere dense sets is nowhere dense. A subset A of X is **meager** in X or of the **first category** in X iff A is the union of a countable family of nowhere dense sets. The Baire theorem can then be stated: the complement of a meager subset of a complete metric space is dense. (The complement of a meager set is sometimes called **co-meager** or **residual** in X.)

A set A is **non-meager** or of the **second category** in X iff it is not meager in X. The following result is a sort of a localization theorem. From the fact that a set A is non-meager we deduce the existence of points x such that A intersects each neighborhood of x in a non-meager set. It is sometimes said that A is of the second category at such points.

35 THEOREM *Let A be a subset of a topological space X and let $M(A)$ be the union of all open sets V such that $V \cap A$ is meager in X. Then $A \cap M(A)^-$ is meager in X.*

PROOF Let $\mathfrak{U}$ be a disjoint family of open sets which is maximal with respect to the property: if $U \in \mathfrak{U}$, then $U \cap A$ is meager. Such a family $\mathfrak{U}$ exists because of the maximal principle 0.25. Let $W = \bigcup\{U: U \in \mathfrak{U}\}$. The proof reduces to showing that $W \cap A$ is meager, for if this is known then $A \cap W^-$ is meager because $W^- \sim W$ is nowhere dense, and from the maximality of $\mathfrak{U}$ it follows that W^- contains every open set V such that $V \cap A$ is meager. To show that $W \cap A$ is meager, for each U in $\mathfrak{U}$ write $U \cap A$ in the form $\bigcup\{U_n: n \in \omega\}$ where U_n is nowhere dense. Then, because the family $\mathfrak{U}$ is disjoint, the set $\bigcup\{U_n: U \in \mathfrak{U}\}$ is nowhere dense for each non-negative integer n. Hence $W \cap A$ is meager. ∎

An important consequence of the preceding theorem is that if a subset A of a topological space is non-meager then there is a non-void open set V such that the intersection of A with every neighborhood of each point of V^- is non-meager.

The concluding theorem of this chapter shows that completeness is preserved by certain mappings. A map of a uniform space $(X,\mathfrak{U})$ into a uniform space $(Y,\mathfrak{V})$ is **uniformly open** iff for each U in $\mathfrak{U}$ there is V in $\mathfrak{V}$ such that $f[U[x]] \supset V[f(x)]$ for each x in X. It is not true that uniformly open maps preserve completeness for arbitrary uniform spaces; Köthe [1] has given an example of a complete linear topological space and a closed subspace such that the quotient space is not complete. The theorem, like the Baire theorem, is peculiar to pseudo-metric spaces.

The proof of the theorem which is given here depends on a lemma which has other profound consequences (see 6.**R**). The lemma concerns a relation R between points of a pseudo-metric space (X,d) and a uniform space $(Y,\mathfrak{V})$; that is, R is a subset of $X \times Y$. Let $U_r = \{(x,y): d(x,y) < r\}$, so that $U_r[x]$ is simply the r-sphere about x.

36 LEMMA *Let R be a closed subset of the product of a complete pseudo-metric space (X,d) with the uniform space $(Y,\mathfrak{V})$ and suppose that for each positive r there is V in $\mathfrak{V}$ such that $R[U_r[x]]^-$ contains $V[y]$ for each (x,y) in R. Then for each r and each positive e it is true that $R[U_{r+e}[x]] \supset R[U_r[x]]^- \supset V[y]$.*

PROOF The critical fact needed for the proof is: if A is a subset of X and $v \in R[A]^-$, then there is a set B of arbitrarily small diameter such that $v \in R[B]^-$ and $A \cap B$ is not void. This is true because: if r is arbitrary, if V is a symmetric member of $\mathcal{V}$ such that $R[U_r[x]]^- \supset V[y]$ for each member (x,y) of R, if v' is a point of $R[A]$ such that $v' \in V[v]$, and if u is a point of A such that $(u,v') \in R$, then $v \in V[v'] \subset R[U_r[u]]^-$, and the diameter of $U_r[u]$ is at most $2r$.

The lemma is now established as follows. Suppose that $v \in R[U_r[x]]^-$. It will be shown that $v \in R[U_{r+e}[x]]$, which will complete the proof. Let $A_0 = U_r[x]$, and select inductively, for each positive integer n, a subset A_n of X such that $v \in R[A_n]^-$, $A_n \cap A_{n-1}$ is not void, and the diameter of A_n is less than $e2^{-n}$. Since X is complete there is evidently a point u such that each neighborhood W of u contains some A_n (hence $v \in R[W]^-$). Clearly $d(x,u) < r + e$. For each neighborhood W of u and each neighborhood Z of v it is true that $R[W]$ intersects Z, and hence there is (u',v') in R with u' in W and v' in Z; that is, $R \cap (W \times Z)$ is non-void. Since R is closed $(u,v) \in R$ and the proof is complete. ▮

Suppose now that f is uniformly open and continuous, that X is complete and pseudo-metrizable, that Y is Hausdorff, and that Y^* is a Hausdorff completion of Y. Then (the graph of) f is a subset of $X \times Y^*$ which is closed because f is continuous, and satisfies the condition of the preceding lemma because the map of X into Y is uniformly open. Then the lemma implies that f is a uniformly open map of X into Y^*. Finally, since $f[X]$ contains $V[f[X]]$ for some V in $\mathcal{V}$, it must be true that $f[X]$ is closed (and open) in Y^*; hence $f[X]$ is complete.

37 COROLLARY *Let f be a continuous uniformly open map of a complete pseudo-metrizable space into a Hausdorff uniform space. Then the range of the map f is complete.*

PROBLEMS

A EXERCISE ON CLOSED RELATIONS

Let X and Y be topological spaces and let R be a closed subset of $X \times Y$. If A is a compact subset of X, then $R[A]$ is a closed subset of

Y. (If $y \notin R[A]$, then $A \times \{y\}$ is contained in the open set $(X \times Y) \sim R$, and theorem 5.12 may be applied.)

B EXERCISE ON THE PRODUCT OF TWO UNIFORM SPACES

Let $(X,\mathfrak{U})$ and $(Y,\mathfrak{V})$ be uniform spaces and for each U in $\mathfrak{U}$ and each V in $\mathfrak{V}$ let $W(U,V) = \{((x,y),(u,v)): (x,u) \in U \text{ and } (y,v) \in V\}$.

(a) The family of sets of the form $W(U,V)$ is a base for the product uniformity for $X \times Y$.

(b) If R is a subset of $X \times Y$, then $W(U,V)[R] = V \circ R \circ U^{-1} = \bigcup \{U[x] \times V[y]: (x,y) \in R\}$.

(c) The closure of a subset R of $X \times Y$ is $\bigcap \{V \circ R \circ U^{-1}: U \in \mathfrak{U} \text{ and } V \in \mathfrak{V}\}$.

C A DISCRETE NON-METRIZABLE UNIFORM SPACE

It should be observed that a uniform space $(X,\mathfrak{U})$ may fail to be metrizable even though the topology of $\mathfrak{U}$ is metrizable. Let Ω_0 be the set of all ordinals which are less than the first uncountable ordinal Ω, and for each member a of Ω_0 let $U_a = \{(x,y): x = y \text{ or } x \geqq a \text{ and } y \geqq a\}$. Then the family of all sets of the form U_a is a base for a uniformity $\mathfrak{U}$ for Ω_0 (observe that $U_a = U_a \circ U_a = U_a^{-1}$). The topology of this uniformity is the discrete topology and hence metrizable, but the uniform space $(\Omega_0,\mathfrak{U})$ is not metrizable.

D EXERCISE: UNIFORM SPACES WITH A NESTED BASE

Let $(X,\mathfrak{U})$ be a Hausdorff uniform space and suppose that a base $\mathfrak{B}$ for $\mathfrak{U}$ is linearly ordered by inclusion. Then either $(X,\mathfrak{U})$ is metrizable or the intersection of every countable family of open subsets of X is open.

E EXAMPLE: A VERY INCOMPLETE SPACE (THE ORDINALS)

Let Ω_0 be the set of all ordinals less than the first uncountable ordinal Ω, and let $\mathfrak{I}$ be the order topology for Ω_0. Then there is a unique uniformity for Ω_0 whose topology is $\mathfrak{I}$ and Ω_0 is not complete relative to this uniformity. (Using the methods of problem 4.E show that, if U is an open subset of $\Omega_0 \times \Omega_0$ which contains the diagonal, then for some x it is true that $(y,z) \in U$ whenever $y > x$ and $z > x$. Then show that a uniformity whose topology is $\mathfrak{I}$ must be identical with the relativized uniformity of the compact space $\Omega' = \{x: x \leqq \Omega\}$.)

Note This property of Ω_0 was observed by Dieudonné [5]. Doss [1] has characterized topological spaces which, like Ω_0, have a unique uniformity.

F THE SUBBASE THEOREM FOR TOTAL BOUNDEDNESS

The uniform space analogue of Alexander's theorem 5.6 on compact subbases is: Let $(X,\mathfrak{U})$ be a uniform space such that for each member U of some subbase for $\mathfrak{U}$ there is a finite cover $A_1, \cdots, A_n$ of X such that $A_i \times A_i \subset U$ for each i. Then the space $(X,\mathfrak{U})$ is totally bounded.

Consequently the product of uniform spaces is totally bounded if and only if each coordinate space is totally bounded.

The Tychonoff product theorem 5.13 for completely regular spaces may be derived from the preceding proposition and 6.32.

G SOME EXTREMAL UNIFORMITIES

(a) If $(X,\mathfrak{J})$ is a Tychonoff space, then the uniformity of the Stone-Čech compactification of X, relativized to X, is the smallest uniformity such that each bounded real-valued continuous function is uniformly continuous.

(b) If $(X,\mathfrak{J})$ is a completely regular space, then there is a largest uniformity $\mathfrak{V}$ for X whose topology is $\mathfrak{J}$. This uniformity may be described alternately as the smallest which makes uniformly continuous each continuous map into a metric space, or each continuous map into a uniform space. Explicitly, V is a member of $\mathfrak{V}$ iff V is a neighborhood of the diagonal in $X \times X$ and there is a sequence $\{V_n, n \in \omega\}$ of symmetric neighborhoods of the diagonal such that $V_0 \subset V$ and $V_{n+1} \circ V_{n+1} \subset V_n$ for each n in ω.

Note These two constructions are examples of a method which has been used before. If F is an arbitrary family of functions on X, each member f mapping X into a uniform space Y_f, then there is a smallest uniformity which makes each f uniformly continuous (or equivalently, makes the natural map into $\times\{Y_f : f \in F\}$ uniformly continuous).

For further information on some extremal uniformities see Shirota [1].

H UNIFORM NEIGHBORHOOD SYSTEMS

A *uniform neighborhood system* for a set X is a correspondence V and an ordering $\geq$ such that the following conditions are satisfied:

(i) $V_a(x)$ is a subset of X to which x belongs, for each member a of an index set A and each point x of X;
(ii) the relation $\geq$ directs the index set A;
(iii) if $a \geq b$, then $V_a(x) \subset V_b(x)$ for all x;
(iv) for each member a of A there is b in A such that $y \in V_a(x)$ whenever $x \in V_b(y)$; and

(v) for each member a of A there is b in A such that $z \in V_a(x)$ whenever $y \in V_b(x)$ and $z \in V_b(y)$.

(a) If $(V,\geqq)$ is a uniform neighborhood system for X, then the family of all sets of the form $\{(x,y): y \in V_a(x)\}$, for a an arbitrary member of A is the base of a uniformity $\mathfrak{U}$ for X. This uniformity is called the *uniformity of the system*. This uniformity has the property that: for each a in A, for some U in $\mathfrak{U}$, $U[x] \subset V_a(x)$ for all x, and for each U in $\mathfrak{U}$ for some a in A, $V_a(x) \subset U[x]$ for all x.

(b) Let $\mathfrak{U}$ be a uniformity for X, and let $V_U(x) = U[x]$ for each member U of $\mathfrak{U}$ and each member x of X. Then $\mathfrak{U}$ is directed by $\subset$ and $(V,\subset)$ is a uniform neighborhood system for X whose uniformity is $\mathfrak{U}$.

(c) Let P be the gage of a uniformity $\mathfrak{U}$ for X, let A be the cartesian product of P and the set of positive real numbers, and direct A by agreeing that $(p,r) \geqq (q,s)$ iff $r \leqq s$ and $p(x,y) \geqq q(x,y)$ for all x and y in X. If $V_{p,r}(x) = \{y: p(x,y) < r\}$, then $(V,\geqq)$ is a uniform neighborhood system for X whose uniformity is $\mathfrak{U}$.

Note It is evident from the foregoing that "indexed" neighborhoods may be used to discuss uniformity and that the theory so obtained is identical with that of uniform spaces. These facts are due to Weil [1].

I ÉCARTS AND METRICS

An *écart* for a set X is a non-negative real-valued function e on $X \times X$ such that

(i) $e(x,y) = 0$ iff $x = y$ and
(ii) for each positive number s there is a positive number r such that $e(x,z) < s$ whenever $e(x,y)$ and $e(y,z)$ are both less than r.

If e is an écart for X then there is a non-negative function p on $X \times X$ such that

(i) $p(x,y) = 0$ iff $x = y$;
(ii) $p(x,y) + p(y,z) \geqq p(x,z)$ for all x, y, and z in X; and
(iii) for each positive s there is a positive number r such that $p(x,y) < s$ whenever $e(x,y) < r$ and, similarly, $e(x,y) < s$ whenever $p(x,y) < r$.

If $e(x,y) = e(y,x)$ for all x and y then p may be taken to be a metric.

Note This is essentially Chittenden's metrization theorem (see 6.14). The "metrization" of a topological space by a function d satisfying all of the requirements for a metric except "$d(x,y) = d(y,x)$" has been investigated by Ribeiro [2] and by Balanzat [1].

The term "écart" has been used by some authors to mean a distance function taking values in a structure less restricted than that of the real numbers (for example, a partially ordered set). For treatments of uniformity based on ideas of this sort see Appert [1], Colmez [1], Cohen and Goffman [1], Gomes [1], Kalisch [1], and Lasalle [1].

J UNIFORM COVERING SYSTEMS

Let Φ be a collection of covers of a set X such that:

(i) if $\mathfrak{A}$ and $\mathfrak{B}$ are members of Φ, then there is a member of Φ which is a refinement of both $\mathfrak{A}$ and $\mathfrak{B}$;
(ii) if $\mathfrak{A} \in \Phi$, then there is a member of Φ which is a star refinement of $\mathfrak{A}$; and
(iii) if $\mathfrak{A}$ is a cover of X and some refinement of $\mathfrak{A}$ belongs to Φ, then $\mathfrak{A}$ belongs to Φ.

Let $\mathfrak{U}$ be the uniformity for X such that the family of all sets of the form $\bigcup\{A \times A: A \in \mathfrak{A}\}$ for $\mathfrak{A}$ in Φ is a base for $\mathfrak{U}$. Then Φ is precisely the family of all covers of X which are uniform relative to $\mathfrak{U}$.

Note Description of a uniformity by means of covers has been used very effectively by J. W. Tukey [1]; a very early use of this general sort was made by Alexandroff and Urysohn [2].

K TOPOLOGICALLY COMPLETE SPACES: METRIZABLE SPACES

A topological space $(X,\mathfrak{J})$ is called *metrically topologically complete* iff there is a metric d for X such that (X,d) is complete and $\mathfrak{J}$ is the metric topology. A topological space $(X,\mathfrak{J})$ is an *absolute* G_δ iff it is metrizable and is a G_δ (a countable intersection of open sets) in every metric space in which it is topologically embedded. Then: A topological space is metrically topologically complete if and only if it is an absolute G_δ. The proof depends on a sequence of lemmas.

(a) Let (X,d) be a complete metric space, let U be an open subset of X, for x in U let $f(x) = 1/\text{dist}\,(x,X \sim U)$, and let $d^*(x,y) = d(x,y) + |f(x) - f(y)|$. Then d^* is a metric, U is a complete relative to d^*, and the d and d^* topologies for U are identical.

(b) A G_δ in a complete metric space is homeomorphic to a complete metric space. (If $U = \bigcap\{U_n: n \in \omega\}$ consider the map of U into the product of the complete metric spaces (U_n,d_n^*), where d_n^* is constructed from d and U_n as in (a).)

(c) If there is a homeomorphism of a dense subset Y of a Hausdorff space X onto a complete metric space Z, then Y is a G_δ in X. (For each integer n let U_n be the set of all points x of X such that the image of some neighborhood of x is of diameter less than $1/n$. Then the homeomorphism f can be extended continuously to a continuous map f^- of $\bigcap \{U_n: n \in \omega\}$ into Z and $f^{-1} \circ f^-$ must be the identity.)

Note These are classical results; (b) is due to Alexandroff [1] and to Hausdorff [2] and (c) is due to Sierpinski [2].

L TOPOLOGICALLY COMPLETE SPACES: UNIFORMIZABLE SPACES

A topological space $(X,\mathfrak{J})$ is said to be *topologically complete* iff there is a uniformity $\mathfrak{U}$ for X such that $(X,\mathfrak{U})$ is complete and $\mathfrak{J}$ is the uniform topology.

(a) If $\mathfrak{U}$ and $\mathfrak{V}$ are uniformities for X such that $\mathfrak{U} \subset \mathfrak{V}$, if $(X,\mathfrak{U})$ is complete, and if the topology of $\mathfrak{U}$ is identical with that of $\mathfrak{V}$, then $(X,\mathfrak{V})$ is complete. Hence a completely regular space is topologically complete iff it is complete relative to the largest uniformity whose topology is $\mathfrak{J}$.

(b) Let $(X,\mathfrak{U})$ be a complete uniform space, let F be an F_σ (a countable union of closed sets) and let $x \in X \sim F$. Then there is a continuous real-valued function on X which is positive on F and 0 at x. Consequently there is an open set V and a uniformity $\mathfrak{V}$ for V such that V contains F, $x \notin V$, $(V,\mathfrak{V})$ is complete, and the topology of $\mathfrak{V}$ is identical with the relativized topology of $\mathfrak{U}$. (Recall the device used in 6.K(a).)

(c) If $(X,\mathfrak{U})$ is a complete uniform space and Y is a subset of X which is the intersection of the members of a family of F_σ's, then Y, with the relativized uniform topology, is topologically complete. (See 6.K.)

(d) Each paracompact space X is topologically complete. (Consider the uniformity consisting of all neighborhoods of the diagonal. A Cauchy net which converges to no point of X must, for each point x, be eventually in the complement of some neighborhood of x, and the application of the even covering property of paracompact spaces leads to a contradiction.)

Note The problem of topological completeness has been studied by Dieudonné [6]; in particular he has shown that each metrizable space is topologically complete (this is a consequence of either (c) or (d) above). Shirota [2] has proved several interesting and profound theorems on topological completeness, in a direction connected with work of Hewitt [2]. See also Umegaki[1].

I conjecture that a completely regular space X is paracompact iff

(i) the family of all neighborhoods of the diagonal is a uniformity, and
(ii) X is topologically complete.

Neither (i) or (ii) is in itself sufficient to imply paracompactness. A non-paracompact space satisfying (i) is exhibited in 6.E. The condition (i) implies normality (if A and B are disjoint closed sets choose a symmetric U such that $U \circ U \subset (X \sim A) \times (X \sim A) \cup (X \sim B) \times (X \sim B)$ and consider $U[A]$ and $U[B]$; a stronger normality condition may be obtained by a similar argument, as shown by H. J. Cohen [1]). However, the product of uncountably many copies of the space of real numbers is complete and not normal (A. H. Stone [1]).

The F_σ condition encountered in (c) above is suggestive of the work of Smirnov [3] on normality.

M THE DISCRETE SUBSPACE ARGUMENT; COUNTABLE COMPACTNESS

(a) If a subset A of a uniform space $(X,\mathfrak{U})$ is not totally bounded, then there is a member U of $\mathfrak{U}$ and an infinite subset B of A such that $U[x]$ is disjoint from $U[y]$ for every pair of distinct points of B; equivalently, there is a pseudo-metric d in the gage of $\mathfrak{U}$ such that $d(x,y) \geqq 1$ for distinct points x and y of B. (A set such as B might be called uniformly discrete.)

(b) A subset A of a topological space $(X,\mathfrak{T})$ is called *relatively countably compact* iff each sequence in A has a cluster point in X. Each relatively countably compact subset of a completely regular space $(X,\mathfrak{T})$ is totally bounded relative to the largest uniformity whose topology is $\mathfrak{T}$. If $(X,\mathfrak{T})$ is topologically complete a subset is relatively countably compact iff its closure is compact, and a closed subset is compact iff it is countably compact.

N INVARIANT METRICS

A pseudo-metric p for a set X is said to be *invariant* under the members of a family F of one-to-one maps of X onto itself, or simply *F-invariant*, iff $p(x,y) = p(f(x),f(y))$ for all x and y in X and all f in F.

A member U of a uniformity $\mathfrak{U}$ for X is called *F-invariant*, provided $(x,y) \in U$ iff $(f(x),f(y)) \in U$ for all f in F. Then: The family of F-invariant pseudo-metrics which are uniformly continuous on $X \times X$ generates the uniformity $\mathfrak{U}$ if and only if the family of F-invariant members of $\mathfrak{U}$ is a base. (See 6.12.)

Note This is a straightforward generalization of the metrization theorem for topological groups which is stated in the next problem.

O TOPOLOGICAL GROUPS: UNIFORMITIES AND METRIZATION

Let $(G,\mathfrak{I})$ be a topological group, and for each neighborhood U of the identity let $U_L = \{(x,y): x^{-1}y \in U\}$ and let $U_R = \{(x,y): xy^{-1} \in U\}$. Consider the following uniformities for G: the *left uniformity* $\mathcal{L}$ having as a base the family of all sets U_L with U a neighborhood of the identity, the *right uniformity* $\mathcal{R}$ with all U_R as a base, and the *two-sided uniformity* $\mathcal{U}$ having $\mathcal{L} \cup \mathcal{R}$ as a subbase.

(a) The topology $\mathfrak{I}$ is the topology of each of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{U}$.

(b) The uniformity $\mathcal{L}$ (respectively $\mathcal{R}$) is generated by the family of all left-invariant (right-invariant) pseudo-metrics which are continuous on $G \times G$. (See 6.N.)

(c) Let I be the family of all neighborhoods of the identity e which are invariant under inner automorphisms. Then I is a base for the neighborhood system of e iff the family of all pseudo-metrics which are both left and right invariant and are continuous on $G \times G$ generates a uniformity whose topology is $\mathfrak{I}$. (If U is an invariant neighborhood of e, then $U_L = U_R$, and this set is invariant under both left and right translation. If p is left and right invariant, then $p(e,y) = p(x^{-1}ex,x^{-1}yx)$.)

(d) Let G be the set of all real-valued functions of the form $g(x) = ax + b$ where $a \neq 0$. Then G is a group under composition and may be topologized by agreeing that g is near the identity iff a is near 1 and $|b|$ is near zero. For this group $\mathcal{L} \neq \mathcal{R}$ and there is no two-sided invariant metric. (The fact that $\mathcal{L} \neq \mathcal{R}$ follows directly from inspection of the defining bases. To see that no invariant metric exists show that, for each g, if $a \neq 1$, then there is f in G such that the constant coefficient of $f^{-1} \circ g \circ f$ is arbitrarily large.)

Note The existence of left-, right- or two-sided invariant metrics for G follows from the foregoing under the additional hypothesis that there is a countable base for the neighborhood system of e. The existence of left-invariant metrics is due to Birkhoff [2] and to Kakutani [1]. The two-sided invariant theorem is due to Klee [1].

It should be remarked that the requirement that a topological group be metrizable with a two-sided invariant metric is very stringent. In particular, a locally compact group of this sort has a Haar measure which is invariant under both right and left translation.

P ALMOST OPEN SUBSETS OF A TOPOLOGICAL GROUP

A subset A of a topological space X is *almost open* in X, or satisfies the *condition of Baire*, iff there is a meager set B such that the symmetric difference $(A \sim B) \cup (B \sim A)$ is open.

(a) A subset A is almost open in X iff there are meager sets B and C such that $(A \sim B) \cup C$ is open. Countable unions and complements of almost open sets are almost open. Every Borel set is almost open. (The family of *Borel* sets is the smallest family $\mathfrak{B}$ such that $\mathfrak{B}$ contains all open sets, and countable unions and complements of members of $\mathfrak{B}$ belong to $\mathfrak{B}$.)

(b) *Banach-Kuratowski-Pettis Theorem* If A contains a non-meager almost open subset of a topological group X, then AA^{-1} is a neighborhood of the identity element. (If A is non-meager so is X, and because X is a topological group each non-void open subset is also non-meager. For each almost open subset B of X let B^* be the union of all open sets U such that $U \cap (X \sim B)$ is meager. Then $(xB)^* = xB^*$ and $(B \cap C)^* = B^* \cap C^*$ if C is also almost open. Hence $xA^* \cap A^* = (xA \cap A)^*$ and if $xA^* \cap A^*$ is non-void, then $xA \cap A$ is non-void. Then $A^*(A^*)^{-1} = \{x\colon xA^* \cap A^* \textit{ is non-void}\} \subset \{x\colon xA \cap A \textit{ is non-void}\} = AA^{-1}$.)

(c) An almost open subgroup of a non-meager topological group X is either meager in X or open and closed in X.

(d) The requirement "almost open" cannot be omitted from theorem (c). There is a subgroup Y of the group X of real numbers such that the quotient X/Y is countably infinite, and since for each member Z of X/Y there is a homeomorphism of X onto itself carrying Y onto Z it follows that Y is not meager in X. (Let B be a Hamel base for X relative to the rational numbers, let C be a countably infinite subset of B, and let Y be the set of all finite linear rational combinations of members of $B \sim C$.)

Note For history and references on theorem (b) see Pettis [1]. The construction in (d) is not peculiar to the real numbers; a related phenomenon occurs in the much more general situation. The basic idea is due to Hausdorff; the sharpest known results in this direction are found in Pettis [2], where history and further references are also given.

Q COMPLETION OF TOPOLOGICAL GROUPS

Let $(G,\cdot,\mathfrak{J})$ be a topological group, let $\mathfrak{L}$ be its left uniformity, $\mathfrak{R}$ its right uniformity, and $\mathfrak{U}$ its two-sided uniformity ($\mathfrak{U}$ is the smallest uniformity which is larger than each of $\mathfrak{L}$ and $\mathfrak{R}$). It has been noted that $\mathfrak{J}$ is the topology of each of $\mathfrak{L}$, $\mathfrak{R}$, and $\mathfrak{U}$.

(a) $(G,\mathfrak{L})$ is complete iff $(G,\mathfrak{R})$ is complete. A net is Cauchy relative to $\mathfrak{U}$ iff it is Cauchy relative to each of $\mathfrak{L}$ and $\mathfrak{R}$. If $(G,\mathfrak{L})$ is complete so is $(G,\mathfrak{U})$. The uniform space $(G,\mathfrak{L})$ is complete, provided $(G,\mathfrak{U})$ is complete, and the group has the property: if $\{x_n, n \in D\}$ is a Cauchy net relative to $\mathcal{L}$, then $\{(x_n)^{-1}, n \in D\}$ is also a Cauchy net relative to $\mathcal{L}$. (Equivalently, $\mathcal{L}$ and $\mathcal{R}$ have the same Cauchy nets.) Left translation by a fixed member of the group is $\mathcal{L}$-uniformly continuous, right translation is $\mathcal{R}$-uniformly continuous, and inversion (x into x^{-1}) is $\mathcal{U}$-uniformly continuous. Multiplication ((x,y) into xy) is usually not uniformly continuous.

(b) *Theorem* Let $(G,\cdot,\mathfrak{J})$ be a Hausdorff topological group, let $(H,\mathcal{V})$ be a Hausdorff completion of the uniform space $(G,\mathcal{U})$, and let $\mathcal{S}$ be the topology of $\mathcal{V}$. Then the group operation $\cdot$ can be extended in a unique way such that $(H,\cdot,\mathcal{S})$ becomes a topological group and $\mathcal{V}$ becomes its two-sided uniformity.

(c) The preceding theorem yields a topological group completion relative to the right uniformity, provided $\mathcal{L}$ and $\mathcal{R}$ have the same Cauchy nets. But in view of (a) this condition is necessary for the existence of "right completion." The condition is not always satisfied. For example, let G be the group of all homeomorphisms of the closed unit interval $[0,1]$ onto itself with composition for group operation and with the topology of the (right invariant) metric: $d(f,g) = \sup \{|f(x) - g(x)|: x \in [0,1]\}$. There is a sequence $\{f_n, n \in \omega\}$ in G which converges uniformly to a function which is not one to one, and the sequence $\{(f_n)^{-1}, n \in \omega\}$ is therefore not Cauchy relative to the left uniformity. The group G is already complete relative to the two-sided uniformity $\mathcal{U}$, for $\mathcal{U}$ is the uniformity of the metric: $d(x,y) + d(x^{-1},y^{-1})$.

(d) *Theorem* Let $(G,\cdot,\mathfrak{J})$ be a metrizable topological group, let d be a right invariant metric metrizing G, and let $d^*(x,y) = d(x,y) + d(x^{-1},y^{-1})$. Then the two-sided uniformity $\mathcal{U}$ is the uniformity of the metric d^*. The uniform space $(G,\mathcal{U})$ is complete iff G is complete relative to some metric whose topology is $\mathfrak{J}$. (Equivalently, iff G is a G_δ in each metrizable space in which it is topologically embedded.) If $\mathcal{L}$ and $\mathcal{R}$ have the same Cauchy sequences and G is complete relative to some metric whose topology is $\mathfrak{J}$, then G is complete relative to every right invariant metric whose topology is $\mathfrak{J}$. (See 6.K and 6.P.)

Note There are two important special cases in which "right-handed completion" may be accomplished. If there is a totally bounded neighborhood of the identity of the group, or if inversion (the map carrying x into x^{-1}) is uniformly continuous on some neighborhood of the identity, then each right Cauchy net is also a left Cauchy net and the two-sided completion yields also a right completion. These results may be proved directly without great difficulty; they are given in Bourbaki [1] and Weil [2]. The example of (c) is due to Dieudonné 3], and the result (d) is due to Klee [1].

The result of part (d)—the deduction of completeness from metric topological completeness—cannot be extended to non-metrizable groups. (See 7.M.)

R CONTINUITY AND OPENNESS OF HOMOMORPHISMS: THE CLOSED GRAPH THEOREM

Throughout this problem G and H will be Hausdorff topological groups, $\mathfrak{U}$ will be the family of all neighborhoods of the identity in G, and $\mathfrak{V}$ will be the corresponding family in H.

(a) *Closed graph theorem* Let G be a topological group, let H be a metrizable topological group which is complete relative to its right uniformity, and let f be a homomorphism of G into H such that

(i) the graph of f is a closed subset of $G \times H$, and
(ii) the closure of $f^{-1}[V]$ belongs to $\mathfrak{U}$ whenever $V \in \mathfrak{V}$.

Then f is continuous.

Dually, a homomorphism g of H into G is open if

(i)* the graph of g is a closed subset of $H \times G$, and
(ii)* the closure of $g[V]$ belongs to $\mathfrak{U}$ whenever $V \in \mathfrak{V}$.

(The proof is made by applying lemma 6.36 to the relations f^{-1} and g respectively. Use a right invariant metric for H. H is complete relative to each right invariant metric which metrizes H.)

(b) If in the preceding theorem it is assumed that H is a Lindelöf space (each open cover has a countable subcover) and G is non-meager, then condition (ii) is automatically satisfied; if further $g[H] = G$, then (ii)* is also automatically satisfied. If G and H are linear topological space, f and g are linear functions, $g[H] = G$, and G is non-meager, then (ii) and (ii)* are automatically satisfied. (If $V \in \mathfrak{V}$, then $f[G]^- \subset Vf[G]$, and if H is Lindelöf, then $f[G]$ is covered by a countable number of translates of V by members of $f[G]$. The closures of inverses under f of these translates are mutually homeomorphic and must have non-void interiors if G is not meager. Hence $f^{-1}[V]^-$ contains an open set and $(f^{-1}[V^{-1}V])^- \supset (f^{-1}[V^{-1}]f^{-1}[V])^- \supset f^{-1}[V^{-1}]^- f^{-1}[V]^- = (f^{-1}[V]^-)^{-1}(f^{-1}[V]^-)$. It follows that $f^{-1}[V]^- \in \mathfrak{U}$ for each V in $\mathfrak{V}$ and a similar argument applies to g. In the linear topological space case it is possible to use scalar multiples instead of translates of members of $\mathfrak{V}$.)

(c) If H is a locally compact topological group, then the closed graph theorem is valid; that is, (i) and (ii) of (a) imply continuity, and dually.

(This is a simpler result than that above. It depends on the lemma 6.A.)

Note The closed graph theorem for complete normed linear spaces is due to Banach [1;41]. Every known form of the theorem requires drastic countability or compactness assumptions on H. A counter example to a number of attractive conjectures may be constructed as follows. Let G be an arbitrary infinite dimensional complete normed linear space, and let H be G with the topology such that a base for the neighborhoods of 0 is the family of all convex sets which contain a line segment in every direction. The identity map g of H onto G is continuous and satisfies (i)* and (ii)* above (see 6.Ua). The space H has many pleasant properties: for example, it is complete, and the uniform boundedness theorem (6.Ub) holds for it. Nevertheless g is evidently not open.

S SUMMABILITY

Let f be a function whose domain includes a set A and whose values line in a complete abelian Hausdorff topological group G. Let $\mathfrak{A}$ be the family of finite subsets of A, and for F in $\mathfrak{A}$ let S_F be the sum of $f(a)$ for a in F. The family $\mathfrak{A}$ is directed by $\supset$, and $\{S_F, F \in \mathfrak{A}, \supset\}$ is a net in G. If this net converges to a member s of G, then f is said to be *summable over* A, s is defined to be the sum of f over A, and we write $s = \sum\{f(a): a \in A\} = \sum_A f$.

(a) *Cauchy criterion for summability* The function f is summable over A iff for each neighborhood U of 0 in G there is a finite subset B of A such that for every finite subset C of $A \sim B$ it is true that $\sum_C f \in U$. Hence a function summable over A is summable over each subset of A.

(b) If f and g are summable over A, then $f + g$ (where $(f + g)(x) = f(x) + g(x)$) is summable over A and $\sum_A (f + g) = \sum_A f + \sum_A g$.

(c) If f is defined and summable over A and $\mathfrak{B}$ is a disjoint family of subsets of A which cover A, then $\sum_A f = \sum\{\sum\{f(b): b \in B\}: B \in \mathfrak{B}\}$. However, from the existence of the iterated sum it is not possible to deduce summability over A. (See 2.G for a special case in which the existence of the iterated sum implies summability over A.)

T UNIFORMLY LOCALLY COMPACT SPACES

A uniform space $(X,\mathfrak{U})$ is *uniformly locally compact* iff there is a member U of $\mathfrak{U}$ such that $U[x]$ is compact for each x in X. In particular, each locally compact topological group is uniformly locally compact relative to its left and its right uniformity.

(a) Let $(X,\mathfrak{U})$ be a uniform space, let U be a member of $\mathfrak{U}$, let $U_0 = U$ and $U_n = U \circ U_{n-1}$ for each positive integer n. Then for each subset A of X the set $\bigcup\{U_n[A]: n \in \omega\}$ is both open and closed.

(b) If U is a closed neighborhood of the diagonal in $X \times X$, A is a compact subset of X, and $U \circ U[x]$ is compact for each x in A, then $U[A]$ is compact. ($U[A]$ is closed by 6.A.)

(c) A connected uniformly locally compact space $(X,\mathfrak{U})$ is σ-compact (that is, X is the union of a countable family of compact subsets).

(d) Each uniformly locally compact space is the union of a disjoint open family of σ-compact subspaces. Hence each such space is paracompact.

(e) Let $(X,\mathfrak{J})$ be a topological space. Then there is a uniformity $\mathfrak{U}$ whose topology is $\mathfrak{J}$ such that $(X,\mathfrak{U})$ is uniformly locally compact iff $(X,\mathfrak{J})$ is locally compact and paracompact. (See 5.28.)

Note Part (a) is essentially the chain argument of 5.T. It may be noted that the propositions on components and connected sets of 5.T cannot be extended to uniformly locally compact spaces.

U THE UNIFORM BOUNDEDNESS THEOREM

(a) Let X be a real linear topological space which is not meager in itself and let K be a closed convex subset of X such that $K = -K$ and K contains a line segment in each direction (that is, for each x in X there is a positive real number t such that $sx \in K$ if $0 \leqq s \leqq t$). Then K is a neighborhood of 0. (Show that K is not meager in X. Then by 6.P, $K - K$ is a neighborhood of 0 and convexity implies that $2K$ is a neighborhood of 0.)

(b) *Theorem* Let F be a family of continuous linear functions on a non-meager linear topological space X to a normed linear space Y and suppose that $\sup \{\| f(x) \|: f \in F\}$ is finite for each point x of X. Then for some neighborhood U of 0 in X it is true that $\sup \{\| f(x) \|: x \in U$ and $f \in F\}$ is finite. (Use the foregoing proposition to show that, if S is the unit sphere about 0 in Y, then $\bigcap \{f^{-1}[S]: f \in F\}$ is a neighborhood of 0 in X.)

Note Part (b) is the classic Banach-Steinhaus theorem. (Banach [1;80].) The formulation is clearly capable of some generalization; the basic idea of such generalization is that of proposition (a). In the terminology of the next chapter the conclusion of (b) can be stated: F is equicontinuous at 0.

V BOOLEAN σ-RINGS

A Boolean ring $(B,+,\cdot)$ is a *σ-ring* iff each countable subset has a least upper bound relative to the natural ordering of B (see 2.K). Natural examples of Boolean σ-rings are:

(i) The ring $(\mathcal{L},\Delta,\cap)$ where $\mathcal{L}$ is the family of all Lebesgue measurable subsets of $[0,1]$, or the ring $\mathcal{L}$ modulo the family of $\mathfrak{N}$ of all sets of measure zero is a σ-ring. (Here Δ is symmetric difference. The family $\mathfrak{N}$ is actually a σ-ideal, in the obvious sense.)

(ii) The ring $(\mathcal{A}/\mathfrak{M},\Delta,\cap)$, where $\mathcal{A}$ is the family of all Borel subsets of $[0,1]$ and $\mathfrak{M}$ is the subfamily consisting of meager Borel sets.

It is the purpose of this problem to exhibit a representation theorem of the type (ii) for an arbitrary Boolean σ-ring. Throughout $\mathcal{B}$ will be the family of all compact open subsets of a locally compact Boolean space X. There is no loss in generality in restricting attention to rings of the type $(\mathcal{B},\Delta,\cap)$. (See the Stone representation theorem, 5.S.)

(a) If $(\mathcal{B},\Delta,\cap)$ is a Boolean σ-ring, then the closure of the union of a countable subfamily of $\mathcal{B}$ is a member of $\mathcal{B}$ (that is, the closure of the union of a countable family of compact open subsets of X is compact and open).

(b) Let $\mathcal{A}$ be the smallest family of subsets of X such that $\mathcal{B} \subset \mathcal{A}$ and countable unions and symmetric differences of members of $\mathcal{A}$ belong to $\mathcal{A}$. Let $\mathfrak{M}$ be the family of all meager subsets of X. Then for each member A of $\mathcal{A}$ there is a unique member B of $\mathcal{B}$ such that $A \Delta B \in \mathfrak{M}$. (See 6.P(a).)

(c) *Theorem* The σ-ring $\mathcal{A}$ is (additively) the direct sum of $\mathcal{B}$ and the σ-ideal $\mathcal{A} \cap \mathfrak{M}$. Hence $\mathcal{B}$ is isomorphic to the Boolean σ-ring $\mathcal{A}$ modulo the σ-ideal $\mathcal{A} \cap \mathfrak{M}$.

Note The results of this problem are due to Loomis [1]. A space which has the property that the closure of an open set is open (such as the Stone space of a Boolean σ-ring which satisfies a countable chain condition) is sometimes called *extremally disconnected.* The space of real-valued bounded Borel functions on a compact space of this sort decomposes, in a way analogous to proposition (c), into continuous functions and functions vanishing outside a meager set. For this and other results see M. H. Stone [4] and also Dixmier [1].

Chapter 7

FUNCTION SPACES

This chapter is devoted to function spaces. That is, the elements of the spaces are functions on a fixed set X to a fixed topological or uniform space Y. Almost all of the development concerns spaces of functions which are continuous relative to a topology for X. Briefly, the purpose of the study is to define topologies and uniformities for sets of continuous functions, and to prove compactness, completeness, and continuity properties for the resulting spaces.

Most of the results of the chapter have their origins in the early theory of real variables. However, the theorems on joint continuity and the compact open topology are relatively recent; they are due primarily to Fox [1]. Further information on function spaces may be found in Arens [2], Bourbaki [1], Myers [2], and Tukey [1].

POINTWISE CONVERGENCE

One topology for a function space has already been investigated rather extensively. If F is a family of functions, each on a set X to a topological space Y, then F is contained in the product $Y^X = \times\{Y: x \in X\}$. The topology $\mathcal{P}$ of pointwise convergence (coordinatewise convergence, simple convergence) or simply the **pointwise topology** for F is the relativized product topology. A net $\{f_n, n \in D\}$ converges to g iff $\{f_n(x), n \in D\}$ converges to $g(x)$ for each x in X (see 3.4). A subbase for $\mathcal{P}$ is the family of all subsets of the form $\{f: f(x) \in U\}$, where x is a point of X and U is open in Y. For each point x of X there is a function e_x on F, which is called the evaluation at x (or the projection into the x-th coordinate space) which is defined by $e_x(f) = f(x)$ for all f in F. Evaluation at x is continuous and open relative to $\mathcal{P}$ (theorem 3.2), and $\mathcal{P}$ is the smallest topology for F such that each evaluation is continuous. A function g on a topological space to F is continuous relative to $\mathcal{P}$ iff $e_x \circ g$ is continuous for each point x of X (theorem 3.3). It is clear that the pointwise topology depends only on the family of functions and the topology of Y. A topology for X, if such is given, does not enter into the definitions or the theorems. If Y is Hausdorff or regular, then the space F inherits the same property (3.5 and 4.A), but in general Y may be locally compact or satisfy the first or second axiom of countability and F may fail to have these properties (3.6 and 5.19).

A characterization of those function spaces which are compact relative to the topology is an immediate consequence of the Tychonoff theorem, 5.13, on the product of compact spaces. Before stating the result let us agree, for convenience, that a family F of functions on a set X to a topological space Y is **pointwise closed** iff F is a closed subset of the product space Y^X. If A is a subset of X, then $F[A]$ is defined to be the set of all points $f(x)$ for x in A and f in F. If $x \in X$, then $F[\{x\}]$ is abbreviated to $F[x]$. If e_x is the evaluation at x, then clearly $e_x[F] = F[x]$.

1 THEOREM *In order that a family F of functions on a set X to a topological space Y be compact relative to the topology of pointwise convergence it is sufficient that*

(a) *F be pointwise closed in Y^X, and*
(b) *for each point x of X the set $F[x]$ has a compact closure.*

If Y is a Hausdorff space the conditions (a) *and* (b) *are also necessary.*

PROOF The family F is not only a subfamily of Y^X but is also contained in $\times\{F[x]^-: x \in X\}$. If condition (b) is satisfied, then this product is a compact subset of Y^X by the Tychonoff product theorem, and if F is pointwise closed, then F is compact. The sufficiency of (a) and (b) is then proved. If Y is a Hausdorff space and F is compact relative to the pointwise topology, then F is closed by 5.7. The set $F[x]$ is compact and closed because the evaluation at each point x is a continuous map of F into the Hausdorff space Y. ∎

The preceding theorem is more important than casual consideration of the topology of pointwise convergence might indicate. The pointwise topology is in many ways unnatural. For example, let X be a set and for each finite subset A of X let C_A be the characteristic function of A (that is, $C_A(x) = 1$ if $x \in A$ and $C_A(x) = 0$ if $x \notin A$). The family $\mathcal{A}$ of all finite subsets of X is directed by $\supset$, and consequently $\{C_A, A \in \mathcal{A}\}$ is a net of functions on X to the closed unit interval. This net converges to the function e which is identically one, because $\{x\} \in \mathcal{A}$ for each point x, and if $A \supset \{x\}$, then $C_A(x) = 1$. Now a topology such that the characteristic function of a finite set is "near" the unit function is obviously unsuitable for many purposes. The more interesting topologies are those for which convergence is more restricted, that is, the larger topologies. But observe: if $(F,\mathfrak{J})$ is compact and $\mathfrak{J}$ is larger than the topology $\mathcal{P}$ of pointwise convergence, then the identity map i of $(F,\mathfrak{J})$ onto $(F,\mathcal{P})$ is continuous, and if $(F,\mathcal{P})$ is a Hausdorff space, then i must be a homeomorphism. Consequently if $(F,\mathfrak{J})$ is compact, $(F,\mathcal{P})$ is Hausdorff, and $\mathfrak{J}$ is larger than the pointwise topology, then $\mathfrak{J}$ is identical with the topology of pointwise convergence. This simple remark indicates the standard method of proving a function space F compact relative to a topology $\mathfrak{J}$. One first shows that F is compact relative to the topology of pointwise convergence and then proves that $\mathcal{P}$-convergence of a net in F implies $\mathfrak{J}$-convergence. If Y is Hausdorff there can be no loss in restricting attention to these two propositions, for if either fails F is not compact relative to $\mathfrak{J}$.

It is sometimes convenient to consider pointwise convergence for points in a subset of the domain space. Suppose F is a family of functions, each on a set X to a topological space Y, and suppose that A is a subset of X. There is a natural map R of F into the product space Y^A, obtained by mapping each member f of F into its restriction to A: that is, $R(f) = f \mid A$ for each f in

F. The smallest topology $\mathcal{P}_A$ for F such that R is continuous evidently consists of the inverses under R of the open subsets Y^A. This topology is that of **pointwise convergence on *A*.** A subbase for $\mathcal{P}_A$ is the family of sets of the form $\{f: f(x) \in U\}$ for x in A and U open in Y, and a net $\{f_n, n \in D\}$ in F converges to g relative to $\mathcal{P}_A$ iff $\{f_n(x), n \in D\}$ converges to $g(x)$ for each x in A. The map R will be one to one iff, whenever f and g are distinct members of F, then for some point x of A it is true that $f(x) \neq g(x)$. A subset A of X for which this is the case is said **to distinguish members** of the family F.

2 Theorem *Let F be a family of functions, each on a set X to a Hausdorff space Y, and let A be a subset of X. The family F with the topology $\mathcal{P}_A$ of pointwise convergence on A is a Hausdorff space if and only if A distinguishes members of F. If F is compact relative to the topology of pointwise convergence on X and if A distinguishes members of F, then $\mathcal{P}$ and $\mathcal{P}_A$ are identical.*

PROOF The product space Y^A is a Hausdorff space and, in view of the definition of $\mathcal{P}_A$, F with this topology will be Hausdorff iff the restriction map R is one to one. This is the case iff A distinguishes members of F. The identity map i of $(F,\mathcal{P})$ onto $(F,\mathcal{P}_A)$ is always continuous since $\mathcal{P}_A \subset \mathcal{P}$. If $(F,\mathcal{P})$ is compact and $(F,\mathcal{P}_A)$ is Hausdorff, then i is a homeomorphism and $\mathcal{P} = \mathcal{P}_A$. ▌

If the range space is a uniform space, then the topology of pointwise convergence is the topology of a uniformity.

If F is a family of functions on a set X to a uniform space $(Y,\mathcal{V})$, then F is a subset of the product $\times\{Y: x \in X\}$ and the relativized product uniformity is called the **uniformity of pointwise convergence** (or of simple convergence). This is sometimes abbreviated as the $\mathcal{P}$ uniformity. Its properties have already been studied (for example, 6.25).

If A is a subset of X, then the uniformity of pointwise convergence on A, or simply the $\mathcal{P}_A$ uniformity, is defined to be the smallest uniformity which makes the restriction map R of F into the family of all functions on A to Y uniformly continuous. The following simple facts about this uniformity are listed without proof.

3 THEOREM *Let F be a family of functions on a set X to a uniform space $(Y,\mathcal{V})$ and let A be a subset of X. Then the uniformity of pointwise convergence on A has the properties:*

(a) *The family of all sets of the form $\{(f,g): (f(x),g(x)) \in V\}$ for V in $\mathcal{V}$ and x in A is a subbase for the $\mathcal{P}_A$ uniformity.*
(b) *The topology of the $\mathcal{P}_A$ uniformity is the topology of pointwise convergence on A.*
(c) *A net $\{f_n, n \in D\}$ is a Cauchy net if and only if $\{f_n(x), n \in D\}$ is a Cauchy net for each x in A.*
(d) *If $(Y,\mathcal{V})$ is complete and $R[F]$ is closed in Y^A relative to pointwise convergence on A, then F is complete relative to the $\mathcal{P}_A$ uniformity.*

COMPACT OPEN TOPOLOGY AND JOINT CONTINUITY

Given a topology for a family F of functions on a topological space X to a topological space Y one might reasonably ask whether $f(x)$ is continuous simultaneously in f and in x. Stated somewhat more formally, the question is: for which topologies for F is the map $F \times X$ which carries (f,x) onto $f(x)$ continuous, if $F \times X$ is given the product topology? This section is devoted to a brief examination of this question. It turns out that there is a particular function space topology which is related to this problem, and we begin by defining this topology and establishing some elementary properties. The section is devoted entirely to topological questions; connections with a uniformity for function spaces will be established later. Throughout the section F will be a family of functions, each on a topological space X to a topological space Y.

For convenience, for each subset K of X and each subset U of Y, define $W(K,U)$ to be the set of all members of F which carry K into U; that is, $W(K,U) = \{f: f[K] \subset U\}$. The family of all sets of the form $W(K,U)$, for K a compact subset of X and U open in Y, is a subbase for the **compact open** topology for F. The family of finite intersections of sets of the form $W(K,U)$ is then a base for the compact open topology; each member of this base is the form $\bigcap \{W(K_i,U_i): i = 0, 1, \cdots, n\}$, where each K_i

is a compact subset of X and each U_i is an open subset of Y. The fact that each set consisting of a single point is compact makes comparison with the pointwise topology simple.

4 THEOREM *The compact open topology* $\mathcal{C}$ *contains the topology* $\mathcal{P}$ *of pointwise convergence. The space* $(F,\mathcal{C})$ *is a Hausdorff space if the range space* Y *is Hausdorff, and is regular if* Y *is regular and the members of* F *are continuous.*

PROOF For each x in X and each open subset U of Y the set $W(\{x\},U) = \{f: f(x) \in U\}$ belongs to $\mathcal{C}$ because $\{x\}$ is compact. Hence $\mathcal{P} \subset \mathcal{C}$, for the family of all sets of this form is a subbase for the pointwise topology $\mathcal{P}$. If Y is a Hausdorff space, then $(F,\mathcal{P})$ is also a Hausdorff space, by 3.5, and if U and V are disjoint $\mathcal{P}$-neighborhoods of members of F they are also $\mathcal{C}$-neighborhoods. Therefore $(F,\mathcal{C})$ is Hausdorff.

Finally, assume that Y is regular; it must be shown that each neighborhood of each member f of F contains a closed neighborhood. It is sufficient to prove that each neighborhood of f which belongs to a subbase for $\mathcal{C}$ contains a closed neighborhood, for each neighborhood of f contains a finite intersection of neighborhoods belonging to the subbase. Suppose that $f \in W(K,U)$ where K is compact and U is an open subset of Y. Then $f[K]$ is compact, and since Y is regular there is by 5.10 a closed neighborhood V of $f[K]$ such that $V \subset U$. Surely $f \in W(K,V) \subset W(K,U)$ and evidently $W(K,V)$ is a neighborhood of f. It remains to show that $W(K,V)$ is closed. But $W(K,V)$ is the intersection of the sets $W(\{x\},V)$ for x in K, and each of the sets $W(\{x\},V)$ is $\mathcal{P}$-closed and hence $\mathcal{C}$-closed. ∎

There is no hope of showing that, if Y is normal or satisfies the first or second axiom of countability, then $(F,\mathcal{C})$ has these properties, for if X is discrete the only compact sets are finite and hence $\mathcal{C}$ is identical with the topology of pointwise convergence. The product of normal spaces or spaces satisfying one of the countability axioms may fail to have the corresponding property and hence F with the topology $\mathcal{C}$ also may fail to have the property.

Let P be the map of $F \times X$ into Y which carries (f,x) into $f(x)$. Each topology for F gives rise to a product topology for

$F \times X$, and one may ask whether P is continuous relative to this product topology. A topology for F is said to be **jointly continuous** iff the map P of $F \times X$ into Y is continuous. It is very easy to see that the topology of pointwise convergence is usually not jointly continuous. The discrete topology is jointly continuous, for if U is an open subset of Y, then $P^{-1}[U] = \{(f,x): f(x) \in U\} = \bigcup \{\{f\} \times f^{-1}[U]: f \in F\}$, which is the union of open sets (assuming that F is a family of continuous functions). If a topology for F is jointly continuous, then each larger topology is also jointly continuous. Consequently the natural problem is to find the smallest jointly continuous topology, if such exists. It turns out that there is generally no such smallest topology; however, a slight relaxation of the conditions for joint continuity yields a precise description of the compact open topology. A topology for a family F of functions is **jointly continuous on a set** A iff the map P is continuous on $F \times A$, where $P(f,x) = f(x)$. (Caution: This does not mean that P is continuous at the points of $F \times A$; the condition is that the restriction $P \mid (F \times A)$ be continuous.) A topology for F is **jointly continuous on compacta** iff it is jointly continuous on each compact subset of the domain space. Each member f of such a family is necessarily continuous on each compact set K (that is, $f \mid K$ is continuous).

5 THEOREM *Each topology which is jointly continuous on compacta is larger than the compact open topology $\mathcal{C}$. If X is regular or Hausdorff and each member of F is continuous on every compact subset of X, then $\mathcal{C}$ is jointly continuous on compacta.*

PROOF Suppose a topology $\mathfrak{J}$ for F is jointly continuous on compacta, U is an open subset of Y, K is a compact subset of X, and P is the map such that $P(f,x) = f(x)$. It must be shown that $W(K,U)$ is $\mathfrak{J}$-open, where $W(K,U) = \{f: f[K] \subset U\}$. The set $V = (F \times K) \cap P^{-1}[U]$ is open in $F \times K$ because $\mathfrak{J}$ is jointly continuous on compacta. If $f \in W(K,U)$, then $\{f\} \times K \subset V$, and since $\{f\} \times K$ is compact there is a $\mathfrak{J}$-neighborhood N of f such that $N \times K \subset P^{-1}[U]$ by theorem 5.12. In other words, each member of the $\mathfrak{J}$-neighborhood N of f is a member of the compact open neighborhood $W(K,U)$. It follows that $W(K,U)$ is $\mathfrak{J}$-open and the first statement of the theorem is proved. To prove the second assertion, suppose K is a compact subset of X, $x \in K$, U is open in Y, and $(f,x) \in P^{-1}[U]$. Then, since f is continuous on K, there is a compact set M which is a neighborhood of x in K such that $f[M] \subset U$ (recall that X is either Hausdorff or regular). Then $W(M,U) \times M$ is a neighborhood of (f,x) in $F \times K$ and is contained in $P^{-1}[U]$. Joint continuity on K follows. ▌

It may be noticed that, if X is locally compact, then a topology is jointly continuous on compacta iff it is jointly continuous. Hence, if X is a locally compact regular space, then the compact open topology for a family of continuous functions is the smallest jointly continuous topology.

If a topology $\mathfrak{J}$ for a family F is jointly continuous on compacta, then $\mathfrak{J} \supset \mathcal{C} \supset \mathcal{P}$, where $\mathcal{C}$ is the compact open topology and $\mathcal{P}$ is the pointwise. If $(F,\mathfrak{J})$ is compact and the range space is Hausdorff, then $(F,\mathcal{P})$ is Hausdorff and consequently $\mathfrak{J} = \mathcal{C} = \mathcal{P}$. This fact shows the necessity of one of the conditions given for $\mathcal{C}$-compactness in the next theorem. The result is given in a rather curious form in order to be directly applicable to the later problem.

6 THEOREM *Let X be a topological space which is either regular or Hausdorff, let Y be a Hausdorff space, let C be the family of all unctions on X to Y which are continuous on each compact subset of X, and let $\mathcal{C}$ and $\mathcal{P}$ be respectively the compact open and the pointwise topologies. Then a subfamily F of C is $\mathcal{C}$-compact if and only if*

(a) *F is $\mathcal{C}$-closed in C,*
(b) *$F[x]$ has a compact closure for each member x of X, and*
(c) *the topology $\mathcal{P}$ for the $\mathcal{P}$-closure of F in Y^X is jointly continuous on compacta.*

PROOF Suppose F is $\mathcal{C}$-compact. The space $(C,\mathcal{C})$ is Hausdorff because Y is Hausdorff and hence F is $\mathcal{C}$-closed in C. Evaluation at a point x is $\mathcal{P}$-continuous, hence $\mathcal{C}$-continuous, and the image $F[x]$ of F is therefore compact. The topologies $\mathcal{C}$ and $\mathcal{P}$ for F coincide because F is $\mathcal{C}$-compact and $\mathcal{P}$-Hausdorff, hence F is

$\mathcal{P}$-closed in Y^X, and by 7.5 the topology $\mathcal{C}$ (and hence $\mathcal{P}$) for F is jointly continuous on compacta. This completes the proof that conditions (a), (b), and (c) are necessary.

Assuming conditions (a), (b), and (c), let F^- be the $\mathcal{P}$-closure of F in Y^X. Condition (b) states that $F[x]^-$ is compact for each x, and since F^- is a closed subset of the $\mathcal{P}$-compact set $\times\{F[x]^-: x \in X\}$ it follows that F^- is $\mathcal{P}$-compact. By (c) the topology $\mathcal{P}$ for F^- is jointly continuous on compacta. Consequently each member of F^- is continuous on each compactum and $F^- \subset C$. Theorem 5.5 implies that the topology $\mathcal{P}$ for F^- is larger than $\mathcal{C}$, and hence these two topologies for F^- coincide. By (a) the family F is $\mathcal{C}$-closed in C and is hence $\mathcal{C}$ (and $\mathcal{P}$) closed in the subset F^- of C; in fact, $F^- = F$, and F is $\mathcal{C}$-compact. ▌

7 *Notes* The family C of all functions which are continuous on every compact subset coincides with the family of all continuous functions if the space is either locally compact or satisfies the first axiom of countability (see theorem 7.13 and the discussion preceding it). It is usually the family of all continuous functions which is of interest; however, the mathematical structure (and not my whim) is responsible for the appearance of the class C. The class also shows up a little later in a discussion of completeness.

The relation between the compact open topology and joint continuity was first studied by Fox [1], who showed that the compact open topology for a family of continuous functions is smaller than each jointly continuous topology and is itself jointly continuous if the domain space is locally compact. For proof of the fact that there is generally no smallest jointly continuous topology see Arens [2].

UNIFORM CONVERGENCE

This section is devoted to the study of a uniformity for a family F of functions on a set X to a uniform space $(Y,\mathcal{V})$. The uniformity is independent of any topology which may be assigned to the set X, but one of the principal results is that the family of functions continuous relative to a topology for X is closed in the space of all functions on X to Y. That is, the uniform limit of continuous functions is continuous.

The uniformity of uniform convergence is the largest which will be considered and the uniformity of pointwise convergence is the smallest. Both of these may be considered as special instances of uniform convergence on the members of a family $\mathfrak{a}$ of sets. This concept is investigated briefly; a uniformity is constructed for each family $\mathfrak{a}$ of subsets of X, and the elementary properties are derived.

Let F be a family of functions on a set X to a uniform space $(Y,\mathfrak{V})$. For each member V of $\mathfrak{V}$ let $W(V)$ be the set * of all pairs (f,g) such that $(f(x),g(x)) \in V$ for each x in X. Then $W(V)[f]$ is the set of all g such that $g(x) \in V[f(x)]$ for every x in X. It is easy to see that $W(V^{-1}) = (W(V))^{-1}$, $W(U \cap V) = W(U) \cap W(V)$, and $W(U \circ V) \supset W(U) \circ W(V)$ for all members U and V of $\mathfrak{V}$. Consequently the family of all sets $W(V)$ for V in $\mathfrak{V}$ is a base for a uniformity $\mathfrak{U}$ for F by theorem 6.2. The family $\mathfrak{U}$ is the **uniformity of uniform convergence,** or simply the **u.c. uniformity.** The topology of $\mathfrak{U}$ is the **topology of uniform convergence,** or the **u.c. topology.**

It is clear that $\mathfrak{U}$ is larger than the uniformity of pointwise convergence, for if y is an arbitrary member of X and $V \in \mathfrak{V}$, then $\{(f,g)\colon (f(x),g(x)) \in V \textit{ for all } x \textit{ in } X\} \subset \{(f,g)\colon (f(y),g(y)) \in V\}$, and hence each member of the defining base for $\mathfrak{U}$ is a subset of a member of the defining subbase for the pointwise uniformity. It follows that the u.c. topology is larger than the pointwise. It is also easy to see directly that uniform convergence implies pointwise convergence, for a net $\{f_n, n \in D\}$ in F converges to g relative to the u.c. topology iff the net is eventually in $W(V)[g]$ for each V in $\mathfrak{V}$, and this is true iff there is some m in D such that, when $n \geq m$, then $f_n(x) \in V[g(x)]$ for all x in X. The following theorem lists other elementary properties of the uniformity $\mathfrak{U}$.

8 THEOREM *Let F be the family of all functions on a set X to a uniform space $(Y,\mathfrak{V})$ and let $\mathfrak{U}$ be the uniformity of uniform con-*

* The set $W(V)$ may be described very simply in terms of the usual notation for relations: $W(V) = \{(f,g)\colon g \circ f^{-1} \subset V\}$. This statement is clear since $g \circ f^{-1}$ is precisely the set of all pairs $(f(x),g(x))$ with x in X. It is also clear that $W(V) = \{(f,g)\colon g \subset V \circ f\}$ and $W(V)[f] = \{g\colon g \subset V \circ f\} = \{g\colon g(x) \in V[f(x)] \textit{ for each } x \textit{ in } X\}$.

vergence. Then:

(a) *The uniformity* $\mathcal{U}$ *is generated by the family of all pseudo-metrics of the form* $d^*(f,g) = \sup \{d(f(x),g(x)) \colon x \in X\}$, *where* d *is a bounded member of the gage of* $(Y,\mathcal{V})$.
(b) *A net* $\{f_n, n \in D\}$ *in* F *converges uniformly to* g *if and only if it is a Cauchy net relative to* $\mathcal{U}$ *and* $\{f_n(x), n \in D\}$ *converges to* $g(x)$ *for each* x *in* X.
(c) *If* $(Y,\mathcal{V})$ *is complete so is the uniform space* $(F,\mathcal{U})$.

PROOF To prove part (a) observe that the family of all sets of the form $\{(y,z) \colon d(y,z) \leq r\}$, for r positive and for d a bounded member of the gage of $\mathcal{V}$, is a base for $\mathcal{V}$. This is true because for each pseudo-metric e the pseudo-metric $d = \min [1,e]$ is bounded and has the same uniformity. But $\{(f,g) \colon d^*(f,g) \leq r\} = \{(f,g) \colon d(f(x),g(x)) \leq r \text{ for each } x \text{ in } X\} = W(\{(y,z) \colon d(y,z) \leq r\})$, where W is the correspondence used above in defining the u.c. uniformity. It follows that d^* belongs to the gage of $\mathcal{U}$ and that pseudo-metrics of this form generate the gage.

One half of the proposition (b) is obvious, and it is only necessary to show that, if a Cauchy net $\{f_n, n \in D\}$ converges pointwise to g, then it converges uniformly to g. Let V be an arbitrary closed symmetric member of $\mathcal{V}$, and choose m in D such that, if $n \geq m$ and $p \geq m$, then $f_p(x) \in V[f_n(x)]$ for each x in X. Such a choice is possible because the net is assumed to be Cauchy relative to $\mathcal{U}$. Since $V[f_n(x)]$ is closed and $f_p(x)$ converges to $g(x)$ it follows that $g(x) \in V[f_n(x)]$ and hence $f_n(x) \in V[g(x)]$ for each $n \geq m$ and every x in X, and (b) is established. Proposition (c) is an immediate consequence of (b) and of the fact that the product of complete spaces is complete. ∎

The following theorem states the principal properties of $\mathcal{U}$ for a family of continuous functions.

9 THEOREM *Let* F *be the family of all continuous functions on a topological space* X *to a uniform space* $(Y,\mathcal{V})$, *and let* $\mathcal{U}$ *be the uniformity of uniform convergence. Then:*

(a) *The family* F *is closed in the space of all functions on* X *to* Y, *and consequently* $(F,\mathcal{U})$ *is complete if* $(Y,\mathcal{V})$ *is complete.*
(b) *The topology of uniform convergence is jointly continuous.*

PROOF Proposition (a) is proved by showing that the set of non-continuous functions is an open subset of the space G of all functions on X to Y. If f is not continuous at a point x of X there is a member V of $\mathcal{V}$ such that $f^{-1}[V[f(x)]]$ is not a neighborhood of x. Choose a symmetric member W of $\mathcal{V}$ such that $W \circ W \circ W \subset V$. It will be proved that if g is a function such that $(g(y),f(y)) \in W$ for each y, then $g^{-1}[W[g(x)]]$ is not a neighborhood of x and hence g is not continuous. It will follow that $G \sim F$ is open relative to the topology of uniform convergence. If $(g(y),f(y)) \in W$ for each y, then $g \subset W \circ f$ and $g^{-1} \subset f^{-1} \circ W^{-1} = f^{-1} \circ W$ and hence $g^{-1} \circ W \circ g \subset f^{-1} \circ W \circ W \circ W \circ f \subset f^{-1} \circ V \circ f$. Therefore $g^{-1}[W[g(x)]]$ is a subset of $f^{-1}[V[f(x)]]$ and is not a neighborhood of x.

The proof of (b) remains. To show continuity of the map of $F \times X$ into Y at a point (f,x) it is only necessary to verify that for V in $\mathcal{V}$, if $y \in f^{-1}[V[f(x)]]$ and $g(z) \in V[f(z)]$ for all z, then $g(y) \in V[f(y)] \subset V \circ V[f(x)]$. ∎

A number of useful uniformities are constructed by considering uniform convergence on each of a family $\mathcal{A}$ of subsets of the domain space. Explicitly, if F is a family of functions on a set X to a uniform space $(Y,\mathcal{V})$ and $\mathcal{A}$ is a family of subsets of X, then the uniformity of **uniform convergence on members of** $\mathcal{A}$, abbreviated $\mathcal{U} \mid \mathcal{A}$, has for a subbase the family of all sets of the form $\{(f,g)\colon (f(x),g(x)) \in V \textit{ for all } x \textit{ in } A\}$, for V in $\mathcal{V}$ and A in $\mathcal{A}$. This uniformity may be described in another way. For each A in $\mathcal{A}$ let R_A be the map which carries f into the restriction of f to A; that is, $R_A(f) = f|A$. Then R_A carries F into a family of functions on A to Y, this family may be assigned the uniformity of uniform convergence, and the uniformity $\mathcal{U} \mid \mathcal{A}$ may be described as the smallest which makes each R_A uniformly continuous.

The preceding propositions on uniform convergence imply corresponding results about the $\mathcal{U} \mid \mathcal{A}$ uniformity. The simple proofs are omitted.

10 THEOREM *Let X be a topological space, let $(Y,\mathcal{V})$ be a uniform space, let $\mathcal{A}$ be a family of subsets of X which covers X, let G be the family of all functions on X to Y, and let F be the family of all*

functions which are continuous on each member of $\mathcal{A}$. *Then:*

(a) *The uniformity* $\mathcal{U} \mid \mathcal{A}$ *of uniform convergence on members of* $\mathcal{A}$ *is larger than the uniformity of pointwise convergence and smaller than that of uniform convergence on* X.

(b) *A net* $\{f_n, n \in D\}$ *converges to* g *relative to the topology of* $\mathcal{U} \mid \mathcal{A}$ *if and only if it is a Cauchy net (relative to* $\mathcal{U} \mid \mathcal{A}$*) and converges to* g *pointwise.*

(c) *If* $(Y,\mathcal{V})$ *is complete, then* G *is complete relative to* $\mathcal{U} \mid \mathcal{A}$.

(d) *The family* F *is closed in* G *relative to the topology of* $\mathcal{U} \mid \mathcal{A}$, *and hence if* $(Y,\mathcal{V})$ *is complete so is* $(F,\mathcal{U} \mid \mathcal{A})$.

(e) *The topology of* $\mathcal{U} \mid \mathcal{A}$ *for* F *is jointly continuous on each member of* $\mathcal{A}$.

It should be emphasized that the family of continuous functions may fail to be complete relative to $\mathcal{U} \mid \mathcal{A}$. If $\mathcal{A}$ is the family of all sets $\{x\}$ for x in X, then $\mathcal{U} \mid \mathcal{A}$ is simply the uniformity of pointwise convergence, and the family of continuous functions is generally not complete relative to this uniformity. If $\mathcal{A}$ is such that continuity on each member of $\mathcal{A}$ implies continuity on X, then proposition (d) above shows $\mathcal{U} \mid \mathcal{A}$ completeness of the family of continuous functions on X to a complete space. In particular, this is the case if there is a neighborhood of each point of X which belongs to $\mathcal{A}$.

UNIFORM CONVERGENCE ON COMPACTA

In this section two distinct lines of investigation will be combined. Suppose that F is a family of continuous functions on a topological space X to a uniform space $(Y,\mathcal{V})$. The uniformity of **uniform convergence on compacta** is the uniformity $\mathcal{U} \mid \mathcal{C}$, where $\mathcal{C}$ is the family of all compact subsets of X. The topology of $\mathcal{U} \mid \mathcal{C}$ is sometimes called the **topology of compact convergence.** It will be proved that this topology is identical with the compact open topology which is constructed from the topology of X and the topology of the uniformity $\mathcal{V}$. Thus the uniformity $\mathcal{U} \mid \mathcal{C}$ depends on the uniformity $\mathcal{V}$ for Y, but the topology of $\mathcal{U} \mid \mathcal{C}$ depends only on the topology of $\mathcal{V}$. The uniformity $\mathcal{U} \mid \mathcal{C}$ is particularly useful in case the space X has a "rich" supply of compact sets, and the section concludes with a brief examination of spaces satisfying a "richness" condition.

11 THEOREM *Let F be a family of continuous functions on a topological space X to a uniform space $(Y,\mathcal{V})$. Then the topology of uniform convergence on compacta is the compact open topology.*

PROOF Let K be a compact subset of X, U an open subset of Y, let $f \in F$, and suppose that $f[K] \subset U$. Then $f[K]$ is compact and by 6.33 there is V in $\mathcal{V}$ such that $V[f[K]] \subset U$. It is then clear that, if g is a function such that $g(x) \in V[f(x)]$ for each x in K, then $g[K] \subset U$ also. Consequently each set of the form $\{f: f[K] \subset U\}$ is open relative to the topology of $\mathcal{U} \mid \mathcal{C}$, and the compact open topology is therefore smaller than that of $\mathcal{U} \mid \mathcal{C}$.

To prove the converse it must be shown that for each compact subset K of X, each V in $\mathcal{V}$, and each continuous f there are compact subsets $K_1, \cdots K_n$ of X and open subsets $U_1, \cdots U_n$ of Y such that $f[K_i] \subset U_i$, and if $g[K_i] \subset U_i$ for each i then $g(x) \in V[f(x)]$ for each x in K. Choose a closed symmetric member W of $\mathcal{V}$ such that $W \circ W \circ W \subset V$, choose $x_1, \cdots x_n$ in K such that the sets $W[f(x_i)]$ cover $f[K]$, let $K_i = K \cap f^{-1}[W[f(x_i)]]$, and let U_i be the interior of $W \circ W[f(x_i)]$. If $g[K_i] \subset U_i$ for each i, then: for each x in K there is i such that $x \in K_i$, hence $g(x) \in W \circ W[f(x_i)]$, and since $f(x) \in W[f(x_i)]$ it follows that $(g(x), f(x)) \in W \circ W \circ W \subset V$. ∎

If the uniform space $(Y,\mathcal{V})$ is complete and $\mathcal{A}$ is a family of subsets of the topological space X then the family of all functions on X to Y which are continuous on each member of $\mathcal{A}$ is $\mathcal{U} \mid \mathcal{A}$-complete, according to 7.10. In order that the family of all continuous functions be complete it is then sufficient that $\mathcal{A}$ satisfy the condition: a function is continuous whenever it is continuous on each member of $\mathcal{A}$. If f is a function on X to Y and B is a subset of Y, then this condition would be implied by: if $A \cap f^{-1}[B]$ is closed for each member A of $\mathcal{A}$, then $f^{-1}[B]$ is closed. In particular, the space of all continuous functions on X to Y is complete relative to uniform convergence on compacta if X satisfies the condition: if a subset A of X intersects each closed compact set in a closed set, then A is closed. Such a topological space is called a **k**-space. It is clear that the family $\mathcal{C}$ of closed compact sets determines the topology of a k-space entirely, for A is closed iff $A \cap C \varepsilon \mathcal{C}$ for each C in $\mathcal{C}$. By complementation it follows that a subset U of a k-space is open iff $U \cap C$ is open in C for each closed compact set C.

The following is evident in view of the definition of k-space and the remarks preceding.

12 THEOREM *The family of all continuous functions on a k-space to a complete uniform space is complete relative to uniform convergence on compacta.*

The two most important examples of k-spaces are given in the following.

13 THEOREM *If X is a Hausdorff space which is either locally compact or satisfies the first axiom of countability, then X is a k-space.*

PROOF In each case the proof proceeds by assuming that B is a non-closed subset of X and showing that for some closed compact set C the intersection $B \cap C$ is not closed. Suppose x is an accumulation point of B which does not belong to B. If X is locally compact there is a compact neighborhood U of x and the intersection $B \cap U$ is not closed because x is an accumulation point but not a member. If X satisfies the first axiom of countability, then there is a sequence $\{y_n, n \varepsilon \omega\}$ in $B \sim \{x\}$ which converges to x, and the set which is the union of $\{x\}$ and the set of all points y_n is clearly compact, but its intersection with B is not closed. ∎

COMPACTNESS AND EQUICONTINUITY

This is the first of two sections devoted to the problem of finding conditions for compactness of a family of functions relative to the compact open topology. The conclusion desired is topological, and the sharpest results are obtained from purely topological premises. However, the arguments are simpler for uniformities and the discussion of this section concerns maps into a uniform space. The last section of the chapter treats the purely topological problem.

Let F be a family of maps of a topological space X into a uniform space $(Y,\mathfrak{V})$. The family F is **equicontinuous at a point** x if and only if for each member V of $\mathfrak{V}$ there is a neighborhood U of x such that $f[U] \subset V[f(x)]$ for every member f of F. Equivalently, F is equicontinuous at x iff $\bigcap \{f^{-1}[V[f(x)]]: f \in F\}$ is a neighborhood of x for each V in $\mathfrak{V}$. Roughly speaking, F is equicontinuous at x iff there is a neighborhood of x whose image under every member of F is small.

14 THEOREM *If F is equicontinuous at x, then the closure of F relative to the topology $\mathcal{P}$ of pointwise convergence is also equicontinuous at x.*

PROOF If V is a closed member of the uniformity of Y, then the class of all functions f which satisfy the condition $f[U] \subset V[f(x)]$ is evidently closed relative to the topology $\mathcal{P}$ of pointwise convergence because it is identical with $\bigcap \{\{f: (f(y),f(x)) \in V\}: y \in U\}$. It follows that the pointwise closure of F is equicontinuous. ∎

A family F of functions is **equicontinuous** iff it is equicontinuous at every point. In view of the preceding theorem the closure of an equicontinuous family relative to the topology of pointwise convergence is also equicontinuous; in particular the members of the closure are continuous functions. The topology of pointwise convergence has other noteworthy properties for equicontinuous families.

15 THEOREM *If F is an equicontinuous family, then the topology of pointwise convergence is jointly continuous and hence coincides with the topology of uniform convergence on compacta.*

PROOF To prove that the map of $F \times X$ into Y is continuous at (f,x) let V be a member of the uniformity of Y and let U be a neighborhood of x such that $g[U] \subset V[g(x)]$ for all g in F. If g is a member of the $\mathcal{P}$-neighborhood $\{h: h(x) \in V[f(x)]\}$ of f and $y \in U$, then $g(y) \in V[g(x)]$ and $g(x) \in V[f(x)]$. Consequently $g(y) \in V \circ V[f(x)]$, and joint continuity follows. Each jointly continuous topology is larger than the compact open by 7.5, and the compact open topology coincides with that of uniform convergence on compacta by 7.11. ∎

The preceding theorem implies that an equicontinuous family of functions is compact relative to the topology of uniform convergence on compacta if it is compact relative to the pointwise topology $\mathcal{P}$, and the Tychonoff product theorem gives sufficient conditions for $\mathcal{P}$-compactness. In this way equicontinuity together with certain other conditions implies compactness of a family of functions. An implication in the reverse direction, from compactness to equicontinuity, is shown in the following theorem.

16 THEOREM *If a family F of functions on a topological space X to a uniform space $(Y,\mathcal{V})$ is compact relative to a jointly continuous topology, then F is equicontinuous.*

PROOF Suppose that x is a fixed point of X and V is a symmetric member of $\mathcal{V}$. The theorem will follow if it is shown that there is a neighborhood U of x such that $g[U] \subset V \circ V[g(x)]$ for each g in F. Because the topology for F is jointly continuous there is for each member f of F a neighborhood G of f and a neighborhood W of x such that $G \times W$ maps into $V[f(x)]$. If $g \in G$ and $w \in W$, then $g(x)$ and $g(w)$ both belong to $V[f(x)]$ and hence $g(w) \in V \circ V[g(x)]$. That is, $g[W] \subset V \circ V[g(x)]$ for each g in G. Because F is compact there is a finite family $G_1, \cdots, G_n$ covering F and corresponding neighborhoods $W_1, \cdots, W_n$ of x such that $g[W_i] \subset V \circ V[g(x)]$ for each g in G_i. If we let U be the intersection of the neighborhoods W_i of x, it is clear that $g[U] \subset V \circ V[g(x)]$ for every g in F. ∎

The Ascoli theorem for locally compact spaces is an immediate consequence of the preceding results. It is obtained from 7.6 by replacing the condition "the pointwise topology $\mathcal{P}$ for the $\mathcal{P}$-closure of F is jointly continuous on compacta" by "the family F is equicontinuous." The latter condition implies the former (7.14 and 7.15) and compactness implies equicontinuity by 7.16. (A proof which does not depend on 7.6 is also simple to construct.)

17 ASCOLI THEOREM *Let C be the family of all continuous functions on a regular locally compact topological space to a Hausdorff uniform space, and let C have the topology of uniform convergence*

on compacta. Then a subfamily F of C is compact if and only if

(a) *F is closed in C,*
(b) *$F[x]$ has a compact closure for each member x of X, and*
(c) *the family F is equicontinuous.*

A form of the Ascoli theorem is true for families of functions on a k-space (a space such that a set is closed whenever its intersection with every closed compact set is closed). A variant of the notion of equicontinuity is required. A family F of functions is **equicontinuous on a set** A iff the family of all restrictions of members of F to A is an equicontinuous family. A family which is equicontinuous at every point of A is equicontinuous on A, but the converse proposition is false. However, a family which is equicontinuous on A is equicontinuous at each point of the interior of A.

The proof of the following theorem is omitted. It is a straightforward application of 7.6, the results of this section and the fact that a function on a k-space is continuous if it is continuous on each compact set.*

18 Ascoli theorem *Let C be the family of all continuous functions on a k-space X which is either Hausdorff or regular to a Hausdorff uniform space Y, and let C have the topology of uniform convergence on compacta. Then a subfamily F of C is compact if and only if*

(a) *F is closed in C,*
(b) *the closure of $F[x]$ is compact for each x in X, and*
(c) *F is equicontinuous on every compact subset of X.*

• *EVEN CONTINUITY*

This section is devoted to the proof of an Ascoli theorem for topological spaces. The pattern of attack is much the same as the foregoing except that a topological concept replaces the (uni-

* It is evident that the condition "X is a k-space" may be omitted from the hypothesis of the theorem if the family C of continuous functions is replaced by the family of all functions which are continuous on each compact set. However, the same result may be obtained by applying the given theorem to X with the topology $\mathfrak{I}$ such that a set A is $\mathfrak{I}$-closed iff $A \cap B$ is closed for every closed compact set B.

form) concept of equicontinuity. The connections between the two concepts are discussed briefly at the end of the section.

Let F be a family of functions, each on a topological space X to a topological space Y. The concept of even continuity can be described intuitively by the statement: for each x in X, y in Y, and f in F, if $f(x)$ is near y, then f maps points near x into points near y. Explicitly, the family F is **evenly continuous** iff for each x in X, each y in Y, and each neighborhood U of y there is a neighborhood V of x and a neighborhood W of y such that $f[V] \subset U$ whenever $f(x) \in W$. The close connection between this definition and joint continuity may be emphasized by the restatement: F is evenly continuous iff for each x in X and y in Y and for each neighborhood U of y there are neighborhoods V of x and W of y such that $\{f: f \in F \textit{ and } f(x) \in W\} \times V$ is carried into U by the natural map. The crucial property of evenly continuous families is easily demonstrated.

19 THEOREM *Let F be an evenly continuous family of functions on a topological space X to a regular space Y and let $\mathcal{P}$ be the topology of pointwise convergence. Then the $\mathcal{P}$-closure F^- of F is evenly continuous and $\mathcal{P}$ is jointly continuous on F^-.*

PROOF The latter statement of the theorem is evident from the second formulation of the definition of even continuity, since $\{f: f \in F \textit{ and } f(x) \in W\}$ is $\mathcal{P}$-open whenever W is open in Y. To show that the $\mathcal{P}$-closure of F is evenly continuous suppose $x \in X$, $y \in Y$ and U is a neighborhood of y. Because Y is regular it may be supposed that U is closed. Let V be a neighborhood of x and W an open neighborhood of y such that, if $f \in F$ and $f(x) \in W$, then $f[V] \subset U$, and suppose that $\{g_n, n \in D\}$ is a net in F which converges pointwise to g and $g(x) \in W$. Then $\{g_n(x), n \in D\}$ is eventually in W; hence for each z in V it is true that $\{g_n(z), n \in D\}$ is eventually in U and therefore $g(z) \in U$. This shows that $g[V] \subset U$. ∎

Sufficient conditions for compactness of an evenly continuous family of functions are more or less self-evident in view of the preceding result and 7.6. The following proposition shows the necessity of the conditions given in the Ascoli theorem.

20 THEOREM *If a family F of continuous functions on a topological space X to a regular Hausdorff space Y is compact relative to a jointly continuous topology, then F is evenly continuous.*

PROOF The identity map of the compact space F into F with the topology of pointwise convergence is continuous, and since the latter topology is Hausdorff, the two topologies coincide. The pointwise topology for F is therefore jointly continuous. Suppose that $x \in X$, $y \in Y$, and U is an open neighborhood of y. Let W be a closed neighborhood of y such that $W \subset U$, and observe that the set K of all members f of F such that $f(x) \in W$ is pointwise closed and hence compact. If P is the function such that $P(f,x) = f(x)$, then the compact set $K \times \{x\}$ is contained in $P^{-1}[U]$, and since P is continuous there is a neighborhood V of x such that $K \times V \subset P^{-1}[U]$ by 5.12. That is, if $v \in V$ and $f(x) \in W$, then $f(v) \in U$. ∎

21 ASCOLI THEOREM *Let C be the family of all continuous functions on a regular locally compact space X to a regular Hausdorff space Y, and let C have the compact open topology. Then a subset F of C is compact if and only if*

(a) *F is closed in C,*
(b) *the closure of $F[x]$ is compact for each x in X, and*
(c) *F is evenly continuous.*

PROOF If F is compact relative to the compact open topology conditions (a), (b), and (c) follow from 7.6 and 7.20. If F satisfies (a), (b), and (c), then the pointwise closure of F is an evenly continuous family on which the pointwise topology is jointly continuous, by 7.19. Compactness follows from 7.6. ∎

The foregoing theorem can be extended to k-spaces in the same fashion that 7.17 was extended. A family F of functions is **evenly continuous on a set** A iff the family of all restrictions of members of F to A is evenly continuous. With this definition the Ascoli theorem (21) can be proved for k-spaces X if condition (c) is replaced by "F is evenly continuous on each compact subset of X." The straightforward proof of this fact is omitted.

The section is concluded with two propositions which clarify the relation between even continuity and equicontinuity.

22 THEOREM *An equicontinuous family of functions on a topological space to a uniform space is evenly continuous.*

PROOF Suppose that F is an equicontinuous family of functions on X to Y, that $x \in X$ and $y \in Y$, and that U is a neighborhood of y. Then one may assume that U is the sphere of d-radius r about y, where d is a pseudo-metric belonging to the gage of Y and $r > 0$. Since F is equicontinuous at x there is a neighborhood V of x such that, if $z \in V$, then $d(f(x),f(z)) < r/2$ for all f in F. Consequently, if $z \in V$ and $f(x)$ belongs to the sphere of d-radius $r/2$ about y, then $f(z) \in U$. ▌

In a certain sense equicontinuity is the result of "uniformizing" even continuity with respect to the range space, and, as might be expected, equicontinuity may be deduced from even continuity in the presence of a suitable compactness condition.

23 THEOREM * *If F is an evenly continuous family of functions on a topological space X to a uniform space Y, and x is a point of X such that $F[x]$ has a compact closure, then F is equicontinuous at x.*

PROOF Suppose d is a member of the gage of Y and $r > 0$. For each y in $F[x]^-$ there are neighborhoods W of y and V of x such that, if $f(x) \in W$, then $f[V]$ is contained in the sphere of d-radius $r/2$ about y. Because $F[x]^-$ is compact, there is a finite number of neighborhoods W_i of points y_i of $F[x]^-$ and corresponding neighborhoods V_i of x, for $i = 1, \cdots, n$, such that the family of all W_i covers $F[x]^-$, and such that, if $f(x) \in W_i$, then $f[V_i]$ is a subset of the sphere of d-radius $r/2$ about y_i. Consequently, if $T = \bigcap \{V_i: i = 0, 1, \cdots, n\}$ and $f \in F$, then $f(x)$ belongs to W_i for some i, and since $f[T]$ is a subset of some sphere of d-radius $r/2$, $d(f(x),f(y)) < r$ for each y in T. Hence F is equicontinuous. ▌

Notes The results of this section are due to A. P. Morse and myself. Another form of the Ascoli theorem for topological spaces has been obtained by Gale [1].

* This theorem is false if the condition "$F[x]$ has a compact closure" is replaced by "$F[x]$ is totally bounded".

PROBLEMS

A EXERCISE ON THE TOPOLOGY OF POINTWISE CONVERGENCE

The set of all continuous real-valued functions on a Tychonoff space X is dense, relative to the topology of pointwise convergence, in the set of all real-valued functions on X.

B EXERCISE ON CONVERGENCE OF FUNCTIONS

Let f be a continuous real-valued function on the closed unit interval $[0,1]$ such that $f(0) = f(1) = 0$ and f is not identically zero. Let $g_n(x) = f(x^n)$ for each non-negative integer n. Then $\{g_n, n \in \omega\}$ converges pointwise (but not uniformly) to the function h which is identically zero. The union of $\{h\}$ and the set of all g_n is compact relative to the pointwise topology but is not compact relative to the topology of uniform convergence.

C POINTWISE CONVERGENCE ON A DENSE SUBSET

Let F be an equicontinuous family of functions on a topological space X to a uniform space and let A be a dense subset of X. Then the uniformity of pointwise convergence on X is identical with the uniformity of pointwise convergence on A.

D THE DIAGONAL PROCESS AND SEQUENTIAL COMPACTNESS

Prior to the proof of the Tychonoff product theorem the diagonal process, as outlined below, was the standard method of proving compactness of a family of functions. Recall that a topological space is called sequentially compact if each sequence in the space has a subsequence which converges to a point of the space.

(a) The product of a countable number of sequentially compact topological spaces is sequentially compact. (Suppose $\{Y_m, m \in \omega\}$ is a sequence of sequentially compact spaces and $\{f_n, n \in \omega\}$ is a sequence in the product $\times\{Y_m: m \in \omega\}$. Choose an infinite subset A_0 of ω such that $\{f_n(0), n \in A_0\}$ converges to a point of Y_0, and continue inductively, choosing an infinite subset A_{k+1} of A_k such that $\{f_n(k+1), n \in A_{k+1}\}$ converges to a point of Y_{k+1}. If N_k is the k-th member of A_k, then $\{f_{N_k}, k \in \omega\}$ is the required subsequence.)

(b) Let Y be a sequentially compact uniform space, let X be a separable topological space, and let F be an equicontinuous family of functions on X to Y which is closed in Y^X relative to the topology of pointwise convergence. Then F is sequentially compact relative to the pointwise topology (or the compact open topology). (Use 7.C and observe that each Cauchy sequence in Y has a limit point.)

Note Some very beautiful results on countable compactness of function spaces have been obtained recently by Grothendieck [1]. His results apply directly to some interesting linear topological space problems.

E DINI'S THEOREM

If a monotonically increasing net $\{f_n, n \in D\}$ of continuous real-valued functions on a topological space X converges pointwise to a continuous function f, then the net converges to f uniformly on compacta. (This is a straightforward compactness argument. If C is a compact subset of X let $A_n = \{(x,y): x \in C \text{ and } f_n(x) \leq y \leq f(x)\}$ and observe that the intersection of the sets A_n for n in D is simply the graph of $f \mid C$.)

F CONTINUITY OF AN INDUCED MAP

Let X and Y be sets, let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of X and of Y respectively, let F be the family of all functions on X to a uniform space $(Z,\mathcal{U})$, and let G be the family of all functions on Y to $(Z,\mathcal{U})$. If T is a map of X into Y the *induced map* T^* of G into F is defined by $T^*(g) = g \circ T$ for g in G. If for each member A of $\mathcal{A}$ the set $T[A]$ is contained in some member of $\mathcal{B}$, then T^* is uniformly continuous relative to the uniformities $\mathcal{U} \mid \mathcal{A}$ for F and $\mathcal{U} \mid \mathcal{B}$ for G (uniform convergence on members of $\mathcal{A}$ and of $\mathcal{B}$ respectively). In particular T^* is always uniformly continuous relative to the uniformity of uniform convergence and is continuous relative to that of pointwise convergence if $\mathcal{B}$ covers Y. If X and Y are topological spaces and T is continuous, then T^* is uniformly continuous relative to uniform convergence on compacta.

Note The continuity of certain other naturally induced maps has been studied by Arens and Dugundji [2].

G UNIFORM EQUICONTINUITY

A family F of functions on a uniform space $(X,\mathcal{U})$ to a uniform space $(Y,\mathcal{V})$ is *uniformly equicontinuous* iff for each member V of $\mathcal{V}$ there is U in $\mathcal{U}$ such that $(f(x),f(y)) \in V$ whenever $f \in F$ and $(x,y) \in U$.

(a) A family F is uniformly equicontinuous iff it is uniformly jointly continuous, in the sense that the natural map of $F \times X$ into Y is uniformly continuous when the uniformity of F is that of uniform convergence and $F \times X$ has the product uniformity.

(b) The pointwise closure of a uniformly equicontinuous family is uniformly equicontinuous.

(c) If X is compact and F is equicontinuous, then F is uniformly equicontinuous.

Note The proofs of the foregoing propositions require no new methods. A more detailed treatment of the subject is given in Arens [2] and in Bourbaki [1].

H EXERCISE ON THE UNIFORMITY $\mathfrak{U} \mid \mathfrak{A}$

Let X be a set, let $\mathfrak{A}$ be a cover of X which is directed by $\supset$ (that is, for A and B in $\mathfrak{A}$ there is C in $\mathfrak{A}$ such that $C \supset A \cup B$), let $(Y,\mathfrak{V})$ be a uniform space, and let F be the family of functions on X to Y with the uniformity $\mathfrak{U} \mid \mathfrak{A}$ of uniform convergence on members of $\mathfrak{A}$. Finally, suppose that S is a net in F and that for each member A of $\mathfrak{A}$ there is given a subnet $\{S \circ T_A(m), m \in E_A\}$ of S which converges to a member s of F uniformly on A. Give an explicit formula for a subnet of S which converges to s relative to the topology of $\mathfrak{U} \mid \mathfrak{A}$.

I CONTINUITY OF EVALUATION

If F is a family of functions on a set X to a set Y, then X is mapped by evaluation into a family G of functions on F to Y; explicitly, the evaluation $E(x)$ at a point x of X is defined by $E(x)(f) = f(x)$ for all f in F. Let $(X,\mathfrak{U})$ and $(Y,\mathfrak{V})$ be uniform spaces and let G have the uniformity of uniform convergence on members of a family $\mathfrak{A}$ of subsets of F. Then the evaluation map E of X into G is continuous if each member of $\mathfrak{A}$ is equicontinuous, and evaluation is uniformly continuous if each member of $\mathfrak{A}$ is uniformly equicontinuous.

J SUBSPACES, PRODUCTS, AND QUOTIENTS OF k-SPACES

(a) There are Tychonoff spaces which are not k-spaces, and since every Tychonoff space can be embedded in a compact Hausdorff space it follows that not every subspace of a k-space is a k-space. (See the example 2.E.)

(b) The product of uncountably many copies of the real line is not a k-space. (Let A be the subset of the product consisting of all members x such that for some non-negative integer n each coordinate of x is equal to n except for a set of at most n indices, and on this set x is zero. Then A is not closed, but $A \cap C$ is compact for each compact set C.)

(c) Let X be a k-space, let R be an equivalence relation on X, and let X/R have the quotient topology. If X/R is a Hausdorff space, then it is a k-space.

K THE k-EXTENSION OF A TOPOLOGY

Let $(X,\mathfrak{J})$ be a Hausdorff space. The *k-extension of* $\mathfrak{J}$ is defined to be the family $\mathfrak{J}_k$ of all subsets U of X such that $U \cap C$ is open in C for every compact set C (equivalently, A is $\mathfrak{J}_k$-closed iff $A \cap C$ is $\mathfrak{J}$-compact for every $\mathfrak{J}$-compact set C).

(a) If C is a $\mathfrak{J}$-compact subset of X, then the relativization of $\mathfrak{J}$ to C is identical with that of $\mathfrak{J}_k$. Consequently a set is $\mathfrak{J}$-compact iff it is $\mathfrak{J}_k$-compact.

(b) The space $(X,\mathfrak{J}_k)$ is a k-space.

(c) A function on X is $\mathfrak{J}_k$-continuous iff it is $\mathfrak{J}$-continuous on every compact subset of X.

(d) The topology $\mathfrak{J}_k$ is the largest topology which agrees with $\mathfrak{J}$ on compact sets (in the sense that the relativization to a compact set is identical with the relativization of $\mathfrak{J}$).

L CHARACTERIZATION OF EVEN CONTINUITY

A family F of functions on a topological space X to a topological space Y is evenly continuous if and only if for each net $\{(f_n,x_n), n \in D\}$ in $F \times X$ such that $\{x_n, n \in D\}$ converges to x and $\{f_n(x), n \in D\}$ converges to y it is true that $\{f_n(x_n), n \in D\}$ converges to y.

M CONTINUOUS CONVERGENCE

Let F be a family of continuous functions, each on a space X to a space Y. A net $\{f_n, n \in D\}$ *converges continuously* to a member f of F iff it is true that $\{f_n(x_n), n \in D\}$ converges to $f(x)$ whenever $\{x_n, n \in D\}$ is a net in X converging to a point x.

(a) A topology $\mathfrak{J}$ for F is jointly continuous iff a net in F converges continuously to a member f whenever it $\mathfrak{J}$-converges to f.

(b) If a sequence in F converges to f relative to the compact open topology, then it converges to f continuously.

(c) Suppose that X satisfies the first axiom of countability and that F, with the compact open topology $\mathfrak{C}$, also satisfies this axiom. Then $\mathfrak{C}$ is jointly continuous and a sequence in F $\mathfrak{C}$-converges to a member f iff it converges continuously to f.

N THE ADJOINT OF A NORMED LINEAR SPACE

Let X be a real normed linear space and let X^*, its adjoint, be the space of all continuous real-valued linear functions on X. The *norm topology* for X^* is defined by: $\|f\| = \sup \{|f(x)|: \|x\| \leq 1\}$. The topology of pointwise convergence for X^* is called the w^*-topology.

A subset F of X^* is called *w^*-bounded* iff for each member x of X the set of all $f(x)$ with f in F is bounded.

(a) The space X^* is not complete relative to the w^*-topology unless every linear function on X is continuous. (See 3.W. Assume that there are enough continuous linear functionals on X to distinguish points—this fact is a consequence of the Hahn-Banach theorem, Banach [1;27].)

(b) *Theorem* (*Alaoglu*) The unit sphere in X^* is compact relative to the w^*-topology. Hence each norm bounded w^*-closed subset of X^* is w^*-compact. (The unit sphere is a closed subset of the product $\times\{[-\|x\|, \|x\|]: x \in X\}$.)

(c) The space X^* with the w^*-topology is paracompact and hence topologically complete. (See 5.Y and 6.L.)

(d) If a subset F of X^* is equicontinuous, then its w^*-closure is equicontinuous. If F is equicontinuous, then the w^*-closure of F is w^*-compact. If the w^*-closure of F is w^*-compact, then F is w^*-bounded. (Observe that F is equicontinuous iff it is norm bounded.)

(e) If X is non-meager, and in particular if X is complete, then each w^*-bounded subset F of X^* is equicontinuous. (Apply 6.U(b), or apply 6.U(a) to the set $\{x: |f(x)| \leq 1 \textit{ for each } f \textit{ in } F\}$.)

(f) The hypothesis "X is non-meager" cannot be omitted from (e). (Consider the space X of all real sequences which are zero save on a finite set, with the norm $\|x\| = \sum\{|x_n|: n \in \omega\}$. If $f_n(x) = nx_n$, then the sequence $\{f_n, n \in \omega\}$ converges to zero relative to the w^*-topology.)

Note The principal results of this problem are more or less classical and certain of them may clearly be extended to less restricted situations. However, the equivalences resulting from (d) and (e) do not hold for an arbitrary complete linear topological space. In connection with (f) it is interesting to note that a w^*-compact *convex* subset of the adjoint of a normed linear space X is always equicontinuous; the proof of this fact is not entirely trivial.

O TIETZE EXTENSION THEOREM *

(a) Let X be a normal topological space, let A be a closed subset, and let f be a continuous function on A to the closed interval $[-1,1]$. Then f has a continuous extension g which carries X into $[-1,1]$. (Let $C = \{x: f(x) \leq -1/3\}$ and let $D = \{x: f(x) \geq 1/3\}$. By Urysohn's lemma there is f_1 on X to $[-1/3, 1/3]$ such that f_1 is $-1/3$ on C and $1/3$ on

* This theorem occurs here because the proof requires the fact that the uniform limit of continuous functions is continuous. In all honesty I should admit that there are three problems in earlier chapters where the same fact is used.

D. Evidently $|f(x) - f_1(x)| \leq 2/3$ for all x in A. The same sort of argument may be applied to the function $f - f_1$.)

Note Dugundji [1], Dowker [3], and Hanner [1] have proved interesting extensions of Tietze's theorem.

P DENSITY LEMMA FOR LINEAR SUBSPACES OF $C(X)$

Let X be a topological space, let $C(X)$ be the space of all bounded continuous real-valued functions on X, and let $C(X)$ have the topology of uniform convergence (equivalently, norm $C(X)$ by $\|f\| = \sup\{|f(x)|: x \in X\}$). A subset L of $C(X)$ is said to have the *two-set property* iff for closed disjoint subsets A and B of X and for each closed interval $[a,b]$ there is a member f of L such that f maps X into $[a,b]$, f is a on A, and f is b on B. Each linear subspace of $C(X)$ which has the two-set property is dense in $C(X)$. (If g is an arbitrary member of $C(X)$ and dist $(g,L) > 0$ choose h in L such that dist (g,L) is approximately $\|g - h\|$. If $k = g - h$, then dist (k,L) = dist (g,L) which is approximately $\|k\|$. Show that there is a member f of L such that $\|k - f\| \leq 2\|k\|/3$.)

Q THE SQUARE ROOT LEMMA FOR BANACH ALGEBRAS *

A real (or complex) *Banach algebra* is an algebra A over the real (complex) numbers together with a norm such that A is a complete normed linear space and multiplication satisfies the condition: $\|xy\| \leq \|x\| \|y\|$. (In terms of the usual operator norm the algebra A can be described as a Banach space with an associative multiplication such that multiplication on the left by a fixed element x is a linear operator of norm at most $\|x\|$.) Throughout the following, A is a fixed (real or complex) Banach algebra.

A function f on D to a normed linear space is *absolutely summable* iff $\sum\{\|f(n)\|: n \in D\}$ exists.

(a) Each function in A which is absolutely summable is summable. If

$\{x_n, n \in \omega\}$ and $\{y_m, m \in \omega\}$ are absolutely summable, then
$\{x_n y_m, (m,n) \in \omega \times \omega\}$ is absolutely summable, and
$\sum\{x_n: n \in \omega\}\sum\{y_m: m \in \omega\} = \sum\{x_n y_m: (m,n) \in \omega \times \omega\}$.

(The usefulness of this result lies in the fact that the last sum may be computed by grouping the summands in a more or less arbitrary fashion. See 6.S.)

* This proposition is given here essentially as a preliminary to the Stone-Weierstrass theorem. However, the lemma is of some importance in a more general situation and is consequently stated for an arbitrary Banach algebra.

(b) Let a_n be the n-th binomial coefficient in the expansion of $(1 - t)^{1/2}$ about $t = 0$. Then $a_0 = 1$, a_n is negative for n positive, $\sum\{a_n : n \in \omega\} = 0$, and $\sum\{a_n a_{p-n} : n \in \omega \text{ and } n \leqq p\}$ is 1, -1 and 0 for $p = 0$, $p = 1$, and $p > 1$, respectively. (Alternatively, one may define the coefficients a_n recursively so that the last stated relation is satisfied. After verifying that $a_n < 0$ for n positive observe that the partial sums $\sum\{a_n t^n : n < p\}$ are monotonically decreasing in n and bounded below by $(1 - t)^{1/2}$ for $0 \leqq t < 1$—hence also for $t = 1$.)

(c) If the algebra has a unit u and if $\|x - u\| \leqq 1$, then there is an element y in the algebra such that $x = y^2$. Explicitly, y may be taken to be $\sum\{a_n(u - x)^n : n \in \omega\}$, where a_n is defined as in (b). (Here it is assumed that $x^0 = u$. The element y may also be written in the form: $y = \sum\{a_n[(u - x)^n - u] : n \geqq 1\}$. In this form it is clear that y is the limit of polynomials in x and that these polynomials may be taken to be without constant coefficients.)

Note It is evident that a great deal more information can be obtained by means of the methods sketched above (for example, if $\|x\| < 1$, then $\sum\{x^n : n \in \omega\}$ is the multiplicative inverse of $u - x$). For a systematic treatment of Banach algebras see Loomis [2] and Hille [1].

R THE STONE-WEIERSTRASS THEOREM

(a) Let X be a compact topological space, let $C(X)$ be the algebra of all continuous real-valued functions on X, and let $C(X)$ have the norm: $\|f\| = \sup\{|f(x)| : x \in X\}$. Then a subalgebra R of $C(X)$ is dense in $C(X)$ if it has the *two-point property:* for distinct points x and y of X and for each pair a and b of real numbers there is f in R such that $f(x) = a$ and $f(y) = b$.

In particular R is dense if the constant functions belong to R and R distinguishes points (in the sense that, if $x \neq y$, then $f(x) \neq f(y)$ for some f in R).

The proof is accomplished by a sequence of lemmas.

(i) If $f \in R$, then $|f|$ belongs to the closure R^- of R, where $|f|(x) = |f(x)|$. (Take the square root of f^2 using 7.P.)

(ii) If f and g belong to a subalgebra, then $\max[f,g]$ and $\min[f,g]$ belong to its closure. (Here $\max[f,g](x) = \max[f(x),g(x)]$. Observe that $\max[a,b] = [(a + b) + |a - b|]/2$ and $\min[a,b] = [(a + b) - |a - b|]/2$.)

(iii) If the subalgebra R has the two-point property, $f \in C(X)$, $x \in X$, and $e > 0$, then there is g in R^- such that $g(x) = f(x)$ and $g(y) < f(y) + e$ for all y in X. (Using compactness of X, take the minimum of a suitably chosen finite family of functions.)

The theorem now follows from (iii) by taking the maximum of a properly chosen finite family of functions.

(b) If X is a topological space and the family $C(X)$ of all continuous real-valued functions on X is given the topology of uniform convergence on compacta, then each subalgebra of $C(X)$ which has the two-point property is dense in $C(X)$.

Note This is unquestionably the most useful known result on $C(X)$. The corresponding theorem for complex-valued functions is false (consider, for example, the functions which are continuous on the unit disk in the plane and are analytic in its interior). See M. H. Stone [5] for a more detailed discussion.

S STRUCTURE OF $C(X)$

Throughout this problem X, Y, and Z will be compact Hausdorff spaces and $C(X)$, $C(Y)$, and $C(Z)$ will be the algebras of all continuous real-valued functions on X, Y, and Z, respectively. A *real homomorphism* of an elgebra is a homomorphism into the real numbers.

(a) For each continuous function F on X to Y let F^* be the induced map of $C(Y)$ into $C(X)$ defined by $F^*(h) = h \circ F$ for all h in $C(Y)$. Then

(i) F^* is a homomorphism of $C(Y)$ into $C(X)$;
(ii) F maps X onto Y iff F^* is an isomorphism of $C(Y)$ onto a subalgebra of $C(X)$ which contains the unit;
(iii) F is one to one iff F^* maps $C(Y)$ onto $C(X)$;
(iv) if G is a continuous map of Y into Z, then $(G \circ F)^* = F^* \circ G^*$; and
(v) if F is a topological map of X onto Y, then $(F^{-1})^* = (F^*)^{-1}$.

(b) The topology of $C(X)$ is entirely determined by the algebraic operations. In detail: $f \geqq g$ iff $f - g$ is the square of an element of $C(X)$, and $\|f\| = \inf \{k: -ku \leqq f \leqq ku\}$ where u is the function which is identically one. If ϕ is a real homomorphism of $C(X)$, then $|\phi(f)| \leqq \|f\|$ and, unless ϕ is identically zero, $\phi(u) = 1$.

(c) Let S be the set of all real homomorphisms ϕ of $C(X)$ such that $\phi(u) = 1$, let S have the topology of pointwise convergence, and let E be the evaluation map of X into S (that is, $E(x)(f) = f(x)$). Then E is a topological map of X onto S. (Show that S is compact, use the Stone-Weierstrass theorem to show that the evaluation map D of $C(X)$ into $C(S)$ is an isomorphism of $C(X)$ onto $C(S)$, verify that $E^* = D^{-1}$, and use (a).)

(d) The space X is metrizable if and only if $C(X)$ is separable. (This result is not needed for the rest of the problem; it is given simply as an exercise in the use of (c).)

(e) If H is a homomorphism of $C(Y)$ into $C(X)$ which carries the unit of $C(Y)$ into the unit of $C(X)$, then there is a unique continuous map F of X into Y such that $H = F^*$. (The homomorphism H induces a map of the real homomorphisms on $C(X)$ into real homomorphisms on $C(Y)$.)

(f) Let R be a closed subalgebra of $C(X)$ such that $u \in R$, let F be the map of X into $\times\{f[X]: f \in R\}$ which is defined by $F(x)_f = f(x)$, and let Y be the range of F. Then R is the range of the induced isomorphism F^* of $C(Y)$ into $C(X)$.

(g) Let I be a closed ideal in $C(X)$ and let $Z = \{x: f(x) = 0$ *for all* f *in* $I\}$. Then I is the set of all members of $C(X)$ which vanish identically on Z. (If Z is empty, then there is a member of I which vanishes at no point and therefore has an inverse. Consider the subalgebra $C + I$, where C is the set of constant functions. Because Z is non-void the set $C + I$ is closed, and (f) may be applied.)

Notes Quite a bit is known about the structure of $C(X)$. Further information and references are given in a review of the subject by S. B. Myers [1]. See also Hewitt [2].

The line of attack outlined in the preceding problem is not the only one possible—the fundamental facts (the Stone-Weierstrass theorem, the Tychonoff product theorem, and the Tietze theorem) may be used in various ways to yield the desired results. However, the pattern used above is, in part, an example of a general method. To each member of a certain collection of objects (in this case compact Hausdorff spaces X) there is associated another object (in this case the Banach algebras $C(X)$). Moreover, to each of a specified class of maps of the original objects (continuous maps in the case at hand) there is assigned an induced map satisfying certain conditions (for example (iv) and (v) of (a)). In this case the induced maps "go in the direction opposite" that of the inducing maps—such a correspondence is called *contravariant*. The assignment of the Stone-Čech compactification of a Tychonoff space, together with the obvious induced maps, furnishes an example where the induced map is in the same direction as the original—a *covariant* correspondence.

This general method of investigation has been used most successfully by Eilenberg and Steenrod in their axiomatic treatment of homology theory [1]. The method itself was first studied by Eilenberg and MacLane. The study of objects and maps might be called the galactic theory, continuing the analogy whereby the study of a topological space is called global.

T COMPACTIFICATION OF GROUPS; ALMOST PERIODIC FUNCTIONS

It is natural to attempt to map a topological group into a dense subgroup of a compact topological group in somewhat the same way that a Tychonoff space is embedded in its Stone-Čech compactification. A topological embedding is usually impossible—a complete group is closed in each Hausdorff group in which it is topologically and isomorphically embedded. However, a number of interesting results can be obtained; the propositions that follow are intended to be an introduction to these. The development is motivated by the observation: If ϕ is a continuous homomorphism of a topological group G into a compact group H and if g is a continuous real-valued function on H, then $g \circ \phi$ has the property that the set of all left translates is totally bounded (relative to the uniformity of uniform convergence).

Throughout it is assumed that G is a fixed topological group. For each bounded real-valued function f on the group G and each x in G the *left translate* of f by x, $L_x(f)$, is defined by: $L_x(f)(y) = f(x^{-1}y)$. The space of bounded real functions is metrized by $d(f,g) = \sup \{|f(x) - g(x)|: x \in X\}$ and the *left orbit* X_f of a function f is defined to be the closure, relative to the metric topology, of the set of all left translates of f. A function f is called *left almost periodic* iff X_f is compact.

Let A be the set of all continuous left almost periodic functions on G. Then for each x in G the left translation L_x maps A into A. Topologize the space of all maps of A into A by pointwise convergence, and let $\alpha[G]$ be the closure relative to this topology of the set of left translations.

(a) *Lemma* Let (X,d) be a compact metric space and let K be the group (under composition) of all isometries of K into itself. Then the topology (for K) of uniform convergence on X is the topology of the metric: $d^*(R,S) = \sup \{d(R(x),S(x)): x \in X\}$ and this is identical with the topology of pointwise convergence on X. The group K with this topology is a compact topological group.

(b) $\alpha[G]$ is compact. (Observe that $\alpha[G] \subset \times\{X_f: f \in A\}$.)

(c) Each member of $\alpha[G]$ is an isometry which carries each left orbit X_f onto itself. The natural map of $\alpha[G]$ into the product space $\times\{K_f: f \in A\}$, where K_f is the group of isometries of X_f, is a topological isomorphism. Hence $\alpha[G]$ is a topological group.

(d) If A is given the topology of pointwise convergence on G and $\alpha[G]$ (a subset of A^A) has the resulting product topology, then the two topologies for $\alpha[G]$ coincide. Hence $R_n \to R$ in $\alpha[G]$ iff $R_n(f)(x) \to R(f)(x)$ for all f in A and all x in G.

(e) The map L of G into $\alpha[G]$ which carries a member x of G into L_x is a continuous homomorphism. The smallest topology for G which makes L continuous is identical with the smallest topology which makes each member f of A continuous. ($\alpha[G]$ may also be described as the completion, relative to the smallest uniformity which makes each f in A uniformly continuous, of G modulo the subgroup of members of G which are not distinguished from the identity by members of A.)

(f) If g is a continuous real function on $\alpha[G]$, then $g \circ L \in A$. If $f \in A$ and g is the function on $\alpha[G]$ defined by $g(R) = R^{-1}(f)(e)$, then $f = g \circ L$ and g is continuous. The family of continuous real functions on $\alpha[G]$ is isometric (and isomorphic) to A.

(g) If ϕ is a continuous homomorphism of G into a compact topological group H, then there is a continuous homomorphism θ of $\alpha[G]$ into H such that $\phi = \theta \circ L$. (More generally, for H arbitrary ϕ induces a natural homomorphism θ of $\alpha[G]$ into $\alpha[H]$ such that $\theta \circ L = L \circ \phi$. See the definition of α.)

There are several obvious corollaries to the preceding development; for example, a function is left periodic iff it is right periodic, and the class A is a Banach algebra which is isomorphic to the algebra of all continuous functions on the compact group $\alpha[G]$.

(h) The term "almost periodic" is derived from an alternate description of the class A. A member x of G is called a *left e-period* of a real function f iff $|f(x^{-1}y) - f(y)| < e$ for all y in G. Let A_e be the set of all left e-periods of a continuous function f. Then the following are equivalent:

(i) There is a homomorphism ϕ of G into a compact group H and a continuous real-valued function h on H such that $g = h \circ \phi$.
(ii) The set of left translates of f is totally bounded relative to the uniformity of uniform convergence.
(iii) For each positive number e there is a finite subset B of G such that $G = BA_e$.

(The connection between (ii) and (iii) is clarified by observing that $|L_x(f)(z) - L_y(f)(z)| < e$ for all z iff $y^{-1}x$ is a left e-period.)

Notes The results above are due primarily to Weil [2]. The equivalence of parts (ii) and (iii) of (h) is a classical theorem of Bochner. Loomis [2] investigates almost periodic functions by showing first that the set of all left almost periodic functions on a group satisfies the conditions which characterize a Banach algebra of functions, and then defining $\alpha[G]$ to be the set of real homomorphisms of this Banach algebra.

Proposition (a) suggests the general problem of topologizing a homeomorphism group in such a fashion as to obtain a topological group. For results in this direction and for references see Arens [3] and Dieudonné [4].

Appendix

ELEMENTARY SET THEORY

This appendix is devoted to elementary set theory. The ordinal and cardinal numbers are constructed and the most commonly used theorems are proved. The non-negative integers are defined and Peano's postulates are proved as theorems.

A working knowledge of elementary logic is assumed, but acquaintance with formal logic is not essential. However, an understanding of the nature of a mathematical system (in the technical sense) helps to clarify and motivate some of the discussion. Tarski's excellent exposition [1] describes such systems very lucidly and is particularly recommended for general background.

This presentation of set theory is arranged so that it may be translated without difficulty into a completely formal language.* In order to facilitate either formal or informal treatment the introductory material is split into two sections, the second of which is essentially a precise restatement of part of the first. It may be omitted without loss of continuity.

The system of axioms adopted is a variant of systems of Skolem and of A. P. Morse and owes much to the Hilbert-Bernays-von Neumann system as formulated by Gödel. The formulation used here is designed to give quickly and naturally a foundation for mathematics which is free from the more obvious paradoxes.

* That is, it is possible to write the theorems in terms of logical constants, logical variables, and the constants of the system, and the proofs may be derived from the axioms by means of rules of inference. Of course, a foundation in formal logic is necessary for this sort of development. I have used (essentially) Quine's meta-axioms for logic [1] in making this kind of presentation for a course.

For this reason a finite axiom system is abandoned and the development is based on eight axioms and one axiom scheme * (that is, all statements of a certain prescribed form are accepted as axioms).

It has been convenient to state as theorems many propositions which are essentially preliminary to the desired results. This clutters up the list of theorems, but it permits omission of many proofs and abbreviation of others. Most of the devices used are more or less evident from the statements of the definitions and theorems.

THE CLASSIFICATION AXIOM SCHEME

Equality is always used in the sense of logical identity; '$1 + 1 = 2$' is to mean that '$1 + 1$' and '2' are names of the same object. Besides the usual axioms for equality an unrestricted substitution rule is assumed: in particular the result of changing a theorem by replacing an object by its equal is again a theorem.

There are two primitive (undefined) constants besides '$=$' and the other logical constants. The first of those is 'ε,' which is read 'is a member of' or 'belongs to.' The second constant is denoted, rather strangely, '$\{\cdot\cdot : \cdots\}$' and is read 'the class of all $\cdot\cdot$ such that $\cdots$.' It is the *classifier*. A remark on the use of the term 'class' may clarify matters. The term does not appear in any axiom, definition, or theorem, but the primary interpretation † of these statements is as assertions about classes (aggregates, collections). Consequently the term 'class' is used in the discussion to suggest this interpretation.

Lower case Latin letters are (logical) variables. The difference between a constant and variable lies entirely in the substitution rules. For example, the result of replacing a variable in a theorem by another variable which does not occur in the theorem is

* Actually, an axiom scheme for definition is also assumed without explicit statement. That is, statements of a certain form, which in particular involve one new constant and are either an equivalence or an identity, are accepted as definitions and are treated in precisely the same fashion as theorems. The axiom scheme of definition is in the fortunate position of being justifiable in the sense that, if the definitions conform with the prescribed rules, then no new contradictions and no real enrichment of the theory results. These results are due to S. Lésniewski.

† Presumably other interpretations are also possible.

again a theorem, but there is no such substitution rule for constants.

I Axiom of extent * *For each* x *and each* y *it is true that* $x = y$ *if and only if for each* z, $z \varepsilon x$ *when and only when* $z \varepsilon y$.

Thus two classes are identical iff every member of each is a member of the other. We shall frequently omit 'for each x' or 'for each y' in the statement of a theorem or definition. If a variable, for example 'x,' occurs and is not preceded by 'for each x' or 'for some x' it is understood that 'for each x' is to be prefixed to the theorem or definition in question.

The first definition assigns a special name to those classes which are themselves members of classes. The reason for this dichotomy among classes is discussed a little later.

1 Definition x *is a set iff for some* y, $x \varepsilon y$.

The next task is to describe the use of the classifier. The first blank in the classifier constant is to be occupied by a variable and the second by a formula, for example $\{x: x \varepsilon y\}$. We accept as an axiom the statement: $u \varepsilon \{x: x \varepsilon y\}$ iff u is a set and $u \varepsilon y$. More generally, each statement of the following form is supposed to be an axiom: $u \varepsilon \{x: \cdots x \cdots\}$ iff u is a set and $\cdots u \cdots$. Here '$\cdots x \cdots$' is supposed to be a formula and '$\cdots u \cdots$' is supposed to be the formula which is obtained from it by replacing every occurrence of 'x' by 'u.' Thus $u \varepsilon \{x: x \varepsilon y$ *and* $z \varepsilon x\}$ iff u is a set and $u \varepsilon y$ and $z \varepsilon u$.

This axiom scheme is precisely the usual intuitive construction of classes except for the requirement 'u is a set.' This requirement is very evidently unnatural and is intuitively quite undesirable. However, without it a contradiction may be constructed simply on the basis of the axiom of extent. (See theorem 39 and the discussion preceding it.) This complication, which necessitates a good bit of technical work on the existence of sets, is simply the price paid to avoid obvious inconsistencies. Less obvious inconsistencies may very possibly remain.

* One is tempted to make this the definition of equality, thus dispensing with one axiom and with all logical presuppositions about equality. This is perfectly feasible. However, there would be no unlimited substitution rule for equality and one would have to assume as an axiom: If $x \varepsilon z$ and $y = x$, then $y \varepsilon z$

THE CLASSIFICATION AXIOM SCHEME (Continued)

A precise statement of the classification axiom scheme requires a description of formulae. It is agreed that: *

(a) The result of replacing 'α' and 'β' by variables is, for each of the following, a formula.

$$\alpha = \beta \qquad \alpha \in \beta$$

(b) The result of replacing 'α' and 'β' by variables and 'A' and 'B' by formulae is, for each of the following, a formula

if A, then B $\quad$ A iff B $\quad$ it is false that A
A and B $\quad$ A or B
for every α, A $\quad$ for some α, A
$\beta \in \{\alpha: A\}$ $\quad$ $\{\alpha: A\} \in \beta$ $\quad$ $\{\alpha: A\} \in \{\beta: B\}$

Formulae are constructed recursively, beginning with the primitive formulae of (a) and proceeding via the constructions permitted by (b).

II Classification axiom-scheme *An axiom results if in the following 'α' and 'β' are replaced by variables, 'A' by a formula $\mathcal{A}$ and 'B' by the formula obtained from $\mathcal{A}$ by replacing each occurrence of the variable which replaced α by the variable which replaced β:*
For each β, $\beta \in \{\alpha: A\}$ if and only if β is a set and B.

ELEMENTARY ALGEBRA OF CLASSES

The axioms already stated permit the deduction of a number of theorems directly from logical results. The deduction is straightforward and only an occasional proof is given.

2 Definition $x \cup y = \{z: z \in x \text{ or } z \in y\}$.

3 Definition $x \cap y = \{z: z \in x \text{ and } z \in y\}$.

* This circuitous sort of language is unfortunately necessary. Using the convention o quotation marks for names, for example 'Boston' is the name of Boston, if $\mathcal{A}$ is a formula and $\mathcal{B}$ is a formula, then '$\mathcal{A} \to \mathcal{B}$' is not a formula. For example, if $\mathcal{A}$ is '$x = y$' and $\mathcal{B}$ is '$y = z$', then ''$x = y$' $\to$ '$y = z$'' is not a formula. Formulae (for example '$x = y$') contain no quotation marks. Instead of '$\mathcal{A} \to \mathcal{B}$' we want to discuss the result of replacing 'α' by $\mathcal{A}$ and 'β' by $\mathcal{B}$ in '$\alpha \to \beta$.' This sort of circumlocution can be avoided by using Quine's corner convention

The class $x \cup y$ is the *union* of x and y, and $x \cap y$ is the *intersection* of x and y.

4 THEOREM *$z \varepsilon x \cup y$ if and only if $z \varepsilon x$ or $z \varepsilon y$, and $z \varepsilon x \cap y$ if and only if $z \varepsilon x$ and $z \varepsilon y$.*

PROOF From the classification axiom $z \varepsilon x \cup y$ iff $z \varepsilon x$ or $z \varepsilon y$ and z is a set. But in view of the definition 1 of set, $z \varepsilon x$ or $z \varepsilon y$ and z is a set iff $z \varepsilon x$ or $z \varepsilon y$. A similar argument proves the corresponding result for intersection. ▌

5 THEOREM *$x \cup x = x$ and $x \cap x = x$.*

6 THEOREM *$x \cup y = y \cup x$ and $x \cap y = y \cap x$.*

7 THEOREM* *$(x \cup y) \cup z = x \cup (y \cup z)$ and $(x \cap y) \cap z = x \cap (y \cap z)$.*

These theorems state that union and intersection are, in the usual sense, commutative and associative operations. The distributive laws follow.

8 THEOREM *$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ and $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.*

9 DEFINITION *$x \notin y$ if and only if it is false that $x \varepsilon y$.*

10 DEFINITION $\sim x = \{y : y \notin x\}$.

The class $\sim x$ is the *complement* of x.

11 THEOREM $\sim(\sim x) = x$.

12 THEOREM (DE MORGAN) *$\sim(x \cup y) = (\sim x) \cap (\sim y)$ and $\sim(x \cap y) = (\sim x) \cup (\sim y)$.*

PROOF Only the first of the two statements will be proved. For each z, $z \varepsilon \sim(x \cup y)$ iff z is a set and it is false that $z \varepsilon x \cup y$, because of the classification axiom and the definition 10 of complement. Using theorem 4, $z \varepsilon x \cup y$ iff $z \varepsilon x$ or $z \varepsilon y$. Consequently, $z \varepsilon \sim(x \cup y)$ iff z is a set and $z \notin x$ and $z \notin y$; that is, iff $z \varepsilon \sim x$ and $z \varepsilon \sim y$. Using 4 again, $z \varepsilon \sim(x \cup y)$ iff $z \varepsilon (\sim x)$

* There would be no necessity for parentheses if the constant '$\cup$' occurred first in the definition; that is, '$\cup xy$' instead of '$x \cup y$.' In this case the first part of the theorem would read: $\cup \cup xyz = \cup x \cup yz$.

$\cap (\sim y)$. Hence $\sim(x \cup y) = (\sim x) \cap (\sim y)$ because of the axiom of extent. ∎

13 DEFINITION $x \sim y = x \cap (\sim y)$.

The class $x \sim y$ is the *difference* of x and y, or the *complement* of y relative to x.

14 THEOREM $x \cap (y \sim z) = (x \cap y) \sim z$.

The proposition '$x \cup (y \sim z) = (x \cup y) \sim z$' is unlikely, although at this stage it is impossible to exhibit a counter example. To be a little more precise, the negation of the proposition cannot be proved on the basis of the axioms so far assumed; it is possible to make a model for this initial part of the system such that $x \notin y$ for each x and each y (there are no sets). The negation of the proposition can be proved on the basis of axioms which will presently be assumed.

15 DEFINITION $0 = \{x: x \neq x\}$.

The class 0 is the *void class*, or *zero*.

16 THEOREM $x \notin 0$.

17 THEOREM $0 \cup x = x$ *and* $0 \cap x = 0$.

18 DEFINITION $\mathcal{U} = \{x: x = x\}$.

The class $\mathcal{U}$ is the *universe*.

19 THEOREM $x \in \mathcal{U}$ *if and only if* x *is a set.*

20 THEOREM $x \cup \mathcal{U} = \mathcal{U}$ *and* $x \cap \mathcal{U} = x$.

21 THEOREM $\sim 0 = \mathcal{U}$ *and* $\sim \mathcal{U} = 0$.

22 DEFINITION* $\bigcap x = \{z$: for each y, if $y \in x$, then $z \in y\}$.

23 DEFINITION $\bigcup x = \{z$: for some y, $z \in y$ and $y \in x\}$.

The class $\bigcap x$ is the *intersection* of the members of x. Note that the members of $\bigcap x$ are members of members of x and may or may not belong to x. The class $\bigcup x$ is the *union* of the members-

* A bound variable notation for the intersection of the members of a family is not needed in this appendix, and consequently a notation is adopted which is simpler than that used in the rest of the book.

bers of x. Observe that a set z belongs to $\bigcap x$ (or to $\bigcup x$) iff z belongs to every (respectively, to some) member of x.

24 Theorem $\bigcap 0 = \mathcal{U}$ *and* $\bigcup 0 = 0$.

PROOF $z \in \bigcap 0$ iff z is a set and z belongs to each member of 0. Since (theorem 16) there is no member of 0, $z \in \bigcap 0$ iff z is a set, and by 19 and the axiom of extent $\bigcap 0 = \mathcal{U}$. The second assertion is also easy to prove. ▌

25 Definition $x \subset y$ *iff for each* z, *if* $z \in x$, *then* $z \in y$.

A class x is a *subclass* of y, or is *contained in* y, or y *contains* x iff $x \subset y$. It is absolutely essential that '$\subset$' not be confused with 'ε.' For example, $0 \subset 0$ but it is false that $0 \in 0$.

26 Theorem $0 \subset x$ *and* $x \subset \mathcal{U}$.

27 Theorem $x = y$ *iff* $x \subset y$ *and* $y \subset x$.

28 Theorem *If* $x \subset y$ *and* $y \subset z$, *then* $x \subset z$.

29 Theorem $x \subset y$ *iff* $x \cup y = y$.

30 Theorem $x \subset y$ *iff* $x \cap y = x$.

31 Theorem *If* $x \subset y$, *then* $\bigcup x \subset \bigcup y$ *and* $\bigcap y \subset \bigcap x$.

32 Theorem *If* $x \in y$, *then* $x \subset \bigcup y$ *and* $\bigcap y \subset x$.

The preceding definitions and theorems are used very frequently—often without explicit reference.

EXISTENCE OF SETS

This section is concerned with the existence of sets and with the initial steps in the construction of functions and other relations from the primitives of set theory.

III Axiom of subsets *If* x *is a set there is a set* y *such that for each* z, *if* $z \subset x$, *then* $z \in y$.

33 Theorem *If* x *is a set and* $z \subset x$, *then* z *is a set.*

PROOF According to the axiom of subsets, if x is a set there is y such that, if $z \subset x$, then $z \in y$, and hence by the definition 1, z is a set. (Observe that this proof does not use the full strength of the axiom of subsets since the argument does not require that y be a set.) ▮

34 THEOREM $0 = \bigcap \mathcal{U}$ *and* $\mathcal{U} = \bigcup \mathcal{U}$.

PROOF If $x \in \bigcap \mathcal{U}$, then x is a set and since $0 \subset x$ it follows from 33 that 0 is a set. Then $0 \in \mathcal{U}$ and each member of $\bigcap \mathcal{U}$ belongs to 0. It follows that $\bigcap \mathcal{U}$ has no members. Clearly (that is, theorem 26) $\bigcup \mathcal{U} \subset \mathcal{U}$. If $x \in \mathcal{U}$, then x is a set and by the axiom of subsets there is a set y such that, if $z \subset x$, then $z \in y$. In particular $x \in y$, and since $y \in \mathcal{U}$ it follows that $x \in \bigcup \mathcal{U}$. Consequently $\mathcal{U} \subset \bigcup \mathcal{U}$ and equality follows. ▮

35 THEOREM *If* $x \neq 0$, *then* $\bigcap x$ *is a set.*

PROOF If $x \neq 0$, then for some y, $y \in x$. But y is a set and since $\bigcap x \subset y$ by 32 it follows from 33 that $\bigcap x$ is a set. ▮

36 DEFINITION $2^x = \{y : y \subset x\}$.

37 THEOREM $\mathcal{U} = 2^{\mathcal{U}}$.

PROOF Every member of $2^{\mathcal{U}}$ is a set and consequently belongs to $\mathcal{U}$. Each member of $\mathcal{U}$ is a set and is contained in $\mathcal{U}$ (theorem 26) and hence belongs to $2^{\mathcal{U}}$. ▮

38 THEOREM *If* x *is a set, then* 2^x *is a set, and for each* y, $y \subset x$ *iff* $y \in 2^x$.

It is interesting to notice that the existence of sets is not provable on the basis of the axioms so far enunciated, but it is possible to prove that there is a class which is not a set. Letting R be $\{x : x \notin x\}$, by the classifier axiom $R \in R$ iff $R \notin R$ and R is a set. It follows that R is not a set. Observe that, if the classifier axiom did not contain the "is a set" qualification, then an outright contradiction, $R \in R$ iff $R \notin R$, would result. This is the Russell paradox. A consequence of this argument is that $\mathcal{U}$ is not a set, because $R \subset \mathcal{U}$ and 33 applies. (The regularity axiom will imply that $R = \mathcal{U}$; this axiom also yields a different proof that $\mathcal{U}$ is not a set.)

39 THEOREM $\mathcal{U}$ *is not a set.*

40 DEFINITION $\{x\} = \{z\colon \text{if } x \in \mathcal{U}, \text{ then } z = x\}$.

Singleton x is $\{x\}$.

This definition is an example of a technical device which is very convenient. If x is a set, then $\{x\}$ is a class whose only member is x. However, if x is not a set, then $\{x\} = \mathcal{U}$ (these statements are theorems 41 and 43). Actually, the primary interest is in the case where x is a set, and for this case the same result is given by the more natural definition $\{z\colon z = x\}$. However, it simplifies statements of results greatly if computations are arranged so that $\mathcal{U}$ is the result of applying the computation outside its pertinent domain.

41 THEOREM *If x is a set, then, for each y, $y \in \{x\}$ iff $y = x$.*

42 THEOREM *If x is a set, then $\{x\}$ is a set.*

PROOF If x is a set $\{x\} \subset 2^x$ and 2^x is a set. ▮

43 THEOREM $\{x\} = \mathcal{U}$ *if and only if x is not a set.*

PROOF If x is a set, then $\{x\}$ is a set and consequently is not equal to $\mathcal{U}$. If x is not a set, then $x \notin \mathcal{U}$ and $\{x\} = \mathcal{U}$ by the definition. ▮

44 THEOREM *If x is a set, then $\bigcap\{x\} = x$ and $\bigcup\{x\} = x$; if x is not a set, then $\bigcap\{x\} = 0$ and $\bigcup\{x\} = \mathcal{U}$.*

PROOF Use 34 and 41. ▮

IV **Axiom of union** *If x is a set and y is a set so is $x \cup y$.*

45 DEFINITION $\{xy\} = \{x\} \cup \{y\}$.

The class $\{xy\}$ is an *unordered pair.*

46 THEOREM *If x is a set and y is a set, then $\{xy\}$ is a set and $z \in \{xy\}$ iff $z = x$ or $z = y$; $\{xy\} = \mathcal{U}$ if and only if x is not a set or y is not a set.*

47 THEOREM *If x and y are sets, then $\bigcap\{xy\} = x \cap y$ and $\bigcup\{xy\} = x \cup y$; if either x or y is not a set, then $\bigcap\{xy\} = 0$ and $\bigcup\{xy\} = \mathcal{U}$.*

ORDERED PAIRS: RELATIONS

This section is devoted to the properties of ordered pairs and relations. The crucial property for ordered pairs is theorem 55: if x and y are sets, then $(x,y) = (u,v)$ iff $x = u$ and $y = v$.

48 DEFINITION $(x,y) = \{\{x\}\{xy\}\}$.

The class (x,y) is an *ordered pair.*

49 THEOREM *(x,y) is a set if and only if x is a set and y is a set; if (x,y) is not a set, then $(x,y) = \mathcal{U}$.*

50 THEOREM *If x and y are sets, then $\bigcup(x,y) = \{xy\}$, $\bigcap(x,y) = \{x\}$, $\bigcup\bigcap(x,y) = x$, $\bigcap\bigcap(x,y) = x$, $\bigcup\bigcup(x,y) = x \cup y$ and $\bigcap\bigcup(x,y) = x \cap y$.*

If either x or y is not a set, then $\bigcup\bigcap(x,y) = 0$, $\bigcap\bigcap(x,y) = \mathcal{U}$, $\bigcup\bigcup(x,y) = \mathcal{U}$, and $\bigcap\bigcup(x,y) = 0$.

51 DEFINITION 1^{st} coord $z = \bigcap\bigcap z$.

52 DEFINITION 2^{nd} coord $z = (\bigcap\bigcup z) \cup ((\bigcup\bigcup z) \sim \bigcup\bigcap z)$.

These definitions will be used, with one exception, only in the case where z is an ordered pair. The *first coordinate* of z is 1^{st} coord z and the *second coordinate* of z is 2^{nd} coord z.

53 THEOREM 2^{nd} coord $\mathcal{U} = \mathcal{U}$.

54 THEOREM *If x and y are sets* 1^{st} coord $(x,y) = x$ *and* 2^{nd} coord $(x,y) = y$. *If either of x and y is not a set, then* 1^{st} coord $(x,y) = \mathcal{U}$ *and* 2^{nd} coord $(x,y) = \mathcal{U}$.

PROOF If x and y are sets, then the equality for 1^{st} *coord* is immediate from 50 and 51. The equality for 2^{nd} *coord* reduces to showing that $y = (x \cap y) \cup ((x \cup y) \sim x)$, by 50 and 52. It is straightforward to see that $(x \cup y) \sim x = y \sim x$ and by the distributive law $(y \cap x) \cup (y \cap \sim x)$ is $y \cap (x \cup \sim x) = y \cap \mathcal{U} = y$. If either x or y is not a set, then, using 50 it is easy to compute 1^{st} *coord* (x,y) and 2^{nd} *coord* (x,y). ▌

55 THEOREM *If x and y are sets and $(x,y) = (u,v)$, then $x = u$ and $y = v$.*

56 DEFINITION *r is a relation if and only if for each member z of r there is x and y such that $z = (x,y)$.*

A *relation* is a class whose members are ordered pairs.

57 DEFINITION $r \circ s = \{u$: for some x, some y and some z, $u = (x,z)$, $(x,y) \in s$ and $(y,z) \in r\}$.

The class $r \circ s$ is the *composition* of r and s.

To avoid excessive notation we agree that $\{(x,z): \cdots\}$ is to be identical with $\{u$: *for some x, some z, $u = (x,z)$ and* $\cdots\}$. Thus $r \circ s = \{(x,z)$: *for some y, $(x,y) \in s$ and $(y,z) \in r$*$\}$.

58 THEOREM $(r \circ s) \circ t = r \circ (s \circ t)$.

59 THEOREM $r \circ (s \cup t) = (r \circ s) \cup (r \circ t)$ *and* $r \circ (s \cap t) \subset (r \circ s) \cap (r \circ t)$.

60 DEFINITION $r^{-1} = \{(x,y): (y,x) \in r\}$.

If r is a relation r^{-1} is the *relation inverse to r.*

61 THEOREM $(r^{-1})^{-1} = r$.

62 THEOREM $(r \circ s)^{-1} = s^{-1} \circ r^{-1}$.

FUNCTIONS

Intuitively, a function is to be identical with the class of ordered pairs which is its graph. All functions are single-valued, and consequently two distinct ordered pairs belonging to a function must have different first coordinates.

63 DEFINITION *f is a function if and only if f is a relation and for each x, each y, each z, if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$.*

64 THEOREM *If f is a function and g is a function so is $f \circ g$.*

65 DEFINITION domain $f = \{x$: for some y, $(x,y) \in f\}$.

66 DEFINITION range $f = \{y$: for some x, $(x,y) \in f\}$.

67 THEOREM domain $\mathcal{U} = \mathcal{U}$ *and* range $\mathcal{U} = \mathcal{U}$.

PROOF If $x \in \mathcal{U}$, then $(x,0)$ and $(0,x)$ belong to $\mathcal{U}$ and hence x belongs to *domain* $\mathcal{U}$ and *range* $\mathcal{U}$. ▌

68 DEFINITION $f(x) = \bigcap\{y: (x,y) \in f\}$.

Hence $z \in f(x)$ if z belongs to the second coordinate of each member of f whose first coordinate is x.

The class $f(x)$ is the *value* of f at x or the *image* of x under f. It is to be noticed that if x is a subset of *domain* f, then $f(x)$ is not $\{y$: *for some* z, $z \in x$ *and* $y = f(z)\}$.

69 THEOREM *If* $x \notin$ domain f, *then* $f(x) = \mathcal{U}$; *if* $x \in$ domain f, *then* $f(x) \in \mathcal{U}$.

PROOF If $x \notin$ *domain* f, then $\{y: (x,y) \in f\} = 0$, and $f(x) = \mathcal{U}$ (theorem 24). If $x \in$ *domain* f, then $\{y: (x,y) \in f\} \neq 0$ and (theorem 35) $f(x)$ is a set. ▮

The foregoing theorem does not require that f be a function.

70 THEOREM *If f is a function, then* $f = \{(x,y): y = f(x)\}$.

71 THEOREM * *If f and g are functions, then $f = g$ if and only if $f(x) = g(x)$ for each x.*

The two following axioms † further delineate the class of all sets.

V Axiom of substitution *If f is a function and* domain f *is a set, then* range f *is a set.*

VI Axiom of amalgamation *If x is a set so is* $\bigcup x$.

72 DEFINITION $x \times y = \{(u,v): u \in x$ *and* $v \in y\}$.

The class $x \times y$ is the *cartesian product* of x and y.

73 THEOREM *If u and y are sets so is* $\{u\} \times y$.

PROOF Clearly one can construct a function (namely, $\{(w,z)$: $w \in y$ *and* $z = (u,w)\}$) whose domain is y and whose range is $\{u\} \times y$. Then apply the axiom of substitution. ▮

* This theorem would not be true if $f(x)$ had been defined to be the union of the second coordinates of the members of f with first coordinate x. For then, if $y \in \mathcal{U}$ and $y \notin$ *domain* f, then $f(y) = 0$, and, if $g = f \cup \{(y,0)\}$, then $g(x) = f(x)$ for each x and f is not equal to g.

† These two axioms may be replaced by the single axiom: if f is a function and domain f is a set, then $\bigcup$ range f is a set. (In the bound variable notation used earlier in the book this can be stated very naturally: if d is a set and $x(a)$ is a set for each a in d, then $\bigcup\{x(a): a \in d\}$ is a set.) To obtain V and VI from the above one may proceed roughly as follows: For V, given f make a new function whose members are of the form $(x,\{f(x)\})$. For VI, given x consider the function whose members are of the form (u,u) with u in x.

74 THEOREM *If x and y are sets so is $x \times y$.*

PROOF Let f be the function such that *domain* $f = x$ and $f(u) = \{u\} \times y$ for u in x. (There is a unique function of this sort; namely, $f = \{(u,z)\colon u \in x \text{ and } z = \{u\} \times y\}$.) Because of the axiom of substitution, *range* f is a set. By a direct computation range $f = \{z\colon \text{for some } u, u \in x \text{ and } z = \{u\} \times y\}$. Consequently $\bigcup$ *range* f, which by the axiom of amalgamation is a set, is $x \times y$. ▌

75 THEOREM *If f is a function and* domain f *is a set, then f is a set.*

PROOF For $f \subset (\text{domain } f) \times (\text{range } f)$. ▌

76 DEFINITION $y^x = \{f\colon f \text{ is a function, domain } f = x \text{ and range } f \subset y\}$.

77 THEOREM *If x and y are sets so is y^x.*

PROOF If $f \in y^x$, then $f \subset x \times y$, which is a set, and hence $f \in 2^{x \times y}$ (theorem 38) and $2^{x \times y}$ is a set. Since $y^x \subset 2^{x \times y}$ it follows from the axiom of subsets that y^x is a set. ▌

For convenience, three more definitions are made.

78 DEFINITION *f is on x if and only if f is a function and $x =$* domain f.

79 DEFINITION *f is to y if and only if f is a function and* range $f \subset y$.

80 DEFINITION *f is onto y if and only if f is a function and* range $f = y$.

WELL ORDERING

Many of the results of this section are not needed in the development of the integers, ordinals, and cardinals which follows. They are included here because they are interesting in themselves and because the methods are simplified forms of the constructions used later.

Since the basic constructive results have now been proved we are able to assume a somewhat less pedestrian pace.

81 DEFINITION *$x\,r\,y$ if and only if $(x,y) \in r$.*

If $x\,r\,y$, then x is *r-related* to y or x *r-precedes* y.

82 DEFINITION *r connects x if and only if when u and v belong to x either $u\,r\,v$ or $v\,r\,u$ or $v = u$.*

83 DEFINITION *r is transitive in x if and only if, when u, v, and w are members of x and $u\,r\,v$ and $v\,r\,w$, then $u\,r\,w$.*

If r is transitive in x, then r *orders* x. The terminology 'u r-precedes v' is especially suggestive if u and v belong to x and r orders x.

84 DEFINITION *r is asymmetric in x if and only if, when u and v are members of x and $u\,r\,v$, then it is not true that $v\,r\,u$.*

Restated, if $u \in x$ and $v \in x$ and u r-precedes v, then v does not r-precede u.

85 DEFINITION *$x \neq y$ if and only if it is false that $x = y$.*

86 DEFINITION *z is an r-first member of x if and only if $z \in x$ and if $y \in x$, then it is false that $y\,r\,z$.*

87 DEFINITION *r well-orders x if and only if r connects x and if $y \subset x$ and $y \neq 0$, then there is an r-first member of y.*

88 THEOREM *If r well-orders x, then r is transitive in x and r is asymmetric in x.*

PROOF If $u \in x$, $v \in x$, $u\,r\,v$, and $v\,r\,u$, then $\{uv\} \subset x$ and consequently there is an r-first member z of $\{uv\}$. Either $z = u$ or $z = v$, and hence it is either false that $v\,r\,u$ or that $u\,r\,v$. This contradiction shows that r is asymmetric in x. If r fails to be transitive in x, then for some members u, v, and w of x it is true that $u\,r\,v$, $v\,r\,w$, and $w\,r\,u$, since r connects x. But then $\{u\} \cup \{v\} \cup \{w\}$ fails to have an r-first member. ▌

89 DEFINITION *y is an r-section of x if and only if $y \subset x$, r well-orders x, and for each u and v such that $u \in x$, $v \in y$, and $u\,r\,v$ it is true that $u \in y$.*

That is, a subset y of x is an r-section of x iff r well-orders x and no member of $x \sim y$ r-precedes a member of y.

90 THEOREM *If $n \neq 0$ and each member of n is an r-section of x, then $\bigcup n$ and $\bigcap n$ are r-sections of x.*

91 THEOREM *If y is an r-section of x and $y \neq x$, then $y =$ $\{u\colon u \in x \text{ and } u\, r\, v\}$ for some v in x.*

PROOF If y is an r-section of x and $y \neq x$, then there is an r-first member v of $x \sim y$. If $u \in x$ and $u\, r\, v$, then, since v is the r-first member of $x \sim y$, $u \notin x \sim y$ and hence $u \in y$. Therefore $\{u\colon u \in x \text{ and } u\, r\, v\} \subset y$. On the other hand, if $u \in y$, then since $v \notin y$ and y is an r-section, it is false that $v\, r\, u$ and hence it is true that $u\, r\, v$. The required equality follows. ▌

92 THEOREM *If x and y are r-sections of z, then $x \subset y$ or $y \subset x$.*

93 DEFINITION * *f is r-s order preserving if and only if f is a function, r well-orders* domain f, *s well-orders* range f, *and $f(u)\, s\, f(v)$ whenever u and v are members of* domain f *such that $u\, r\, v$.*

94 THEOREM *If x is an r-section of y and f is an r-r order-preserving function on x to y, then for each u in x it is false that $f(u)\, r\, u$.*

PROOF It must be shown that $\{u\colon u \in x \text{ and } f(u)\, r\, u\}$ is void. If not there is an r-first member v of this class. Then $f(v)\, r\, v$, and if $u\, r\, v$, then $u\, r\, f(u)$ or $u = f(u)$. Since $f(v)\, r\, v$, then $f(v)\, r\, f(f(v))$ or $f(v) = f(f(v))$, but since f is r-r order preserving $f(f(v))\, r\, f(v)$ and this is a contradiction. ▌

Thus an r-r order-preserving function on an r-section cannot map a member of its domain into an r-predecessor.

A proof such as that of theorem 94 which depends on considering the r-first element for which the theorem fails is a *proof by induction.*

95 DEFINITION *f is a* 1-1 *function if and only if both f and f^{-1} are functions.*

This is the equivalent to the statement that f is a function and if x and y are distinct members of *domain f*, then $f(x) \neq f(y)$.

96 THEOREM *If f is r-s order preserving, then f is a* 1-1 *function and f^{-1} is s-r order preserving.*

* In this appendix there is no need to consider order-preserving functions (as in chapter 0) whose domain and range are not well-ordered. For the sake of simplicity the earlier terminology is modified.

PROOF If $f(u) = f(v)$, then it is impossible that $u\, r\, v$ or $v\, r\, u$, for in this case $f(u)\, s\, f(v)$ or $f(v)\, s\, f(u)$. Hence $u = v$ and f is 1-1. If $f(u)\, s\, f(v)$, then $u \neq v$, and if $v\, r\, u$, then $f(v)\, s\, f(u)$, which is a contradiction. Therefore f^{-1} is s-r order preserving. ∎

97 THEOREM *If f and g are r-s order preserving,* domain f *and* domain g *are r-sections of x and* range f *and* range g *are s-sections of y, then $f \subset g$ or $g \subset f$.*

PROOF By theorem 92 either *domain* $f \subset$ *domain* g or *domain* $g \subset$ *domain* f, and the theorem will follow if it is proved that $f(u) = g(u)$ for all u belonging to the domain of both f and g. If the class $\{z: z \in (\text{domain } f) \cap (\text{domain } g) \text{ and } g(z) \neq f(z)\}$ is not empty there is an r-first member u. Then $f(u) \neq g(u)$ and it may be supposed that $f(u)\, s\, g(u)$. Since *range* g is an s-section, $g(v) = f(u)$ for some v in x and $v\, r\, u$ because g^{-1} is order preserving. But u is the r-first point at which the functions differ, and therefore $f(v) = g(v) = f(u)$ which is a contradiction. ∎

98 DEFINITION *f is r-s order preserving in x and y if and only if r well-orders x, s well-orders y, f is r-s order preserving,* domain f *is an r-section of x, and* range f *is an s-section of y.*

According to theorem 97, if f and g are both r-s order preserving in x and y, then $f \subset g$ or $g \subset f$.

99 THEOREM *If r well-orders x and s well-orders y, then there is a function f which is r-s order preserving in x and y such that either* domain $f = x$ *or* range $f = y$.

PROOF Let $f = \{(u,v): u \in x, \text{ and for some function } g \text{ which is } r\text{-}s \text{ order preserving in } x \text{ and } y,\ u \in \text{domain } g \text{ and } (u,v) \in g\}$. Because of the preceding theorem, f is a function, and it is easy to see that its domain is an r-section of x and its range is an s-section of y. Hence f is r-s order preserving in x and y and it remains to show that either *domain* $f = x$ or *range* $f = y$. If not, there is an r-first member u of $x \sim$ (*domain* f) and an s-first member v of $y \sim$ (*range* f), and the function $f \cup \{(u,v)\}$ is easily seen to be r-s order preserving in x and y. Then $(u,v) \in f$ by definition of f and hence $u \in$ *domain* f. This is a contradiction. ∎

In one case it is possible to state which of the alternatives in the conclusion of the preceding theorem occurs, for if x is a set and y is not, then it is impossible that range $f = y$ because of the axiom of substitution.

100 THEOREM *If r well-orders x, s well-orders y, x is a set, and y is not a set, then there is a unique r-s order-preserving function in x and y whose domain is x.*

ORDINALS

In this section the ordinal numbers are defined and the fundamental properties established. Another axiom is assumed before beginning the discussion of ordinals.

It is a priori possible that there are classes x and y such that x is the only member of y and y is the only member of x. More generally, it is possible that there is a class z whose members exist by taking in each other's laundry, in the sense that every member of z consists of members of z. The following axiom explicitly denies this possibility by requiring that each non-void class z have at least one member whose elements do not belong to z.

VII Axiom of regularity *If $x \neq 0$ there is a member y of x such that $x \cap y = 0$.*

101 THEOREM $x \notin x$.

PROOF If $x \in x$, then x is a non-void set and is the only member of $\{x\}$. By the axiom of regularity there is y in $\{x\}$ such that $y \cap \{x\} = 0$, and necessarily $y = x$. But then $y \in y \cap \{x\}$, which is a contradiction. ▌

102 THEOREM *It is false that $x \in y$ and $y \in x$.*

PROOF If $x \in y$ and $y \in x$, then both x and y are sets and are the only members of $\{z: z = x \text{ or } z = y\}$. Applying the axiom of regularity to the latter class leads to a contradiction, just as in the proof of the preceding theorem. ▌

Of course, this theorem may be generalized to more than two sets. The axiom of regularity actually implies another strong result, intuitively stated as follows: it is impossible that there be a sequence such that $x_{n+1} \varepsilon x_n$ for each n. A precise statement of the result must be deferred.

103 Definition $E = \{(x,y) : x \varepsilon y\}$.

The class E is the *ε-relation*. Notice that if $x \varepsilon y$ and y is not a set, then $(x,y) = \mathcal{U}$, by theorem 49, and $(x,y) \notin E$.

104 Theorem *E is not a set.*

Proof If $E \varepsilon \mathcal{U}$, then $\{E\} \varepsilon \mathcal{U}$ and $(E, \{E\}) \varepsilon E$. Recall that $(x,y) = \{\{x\}\{xy\}\}$ and, if (x,y) is a set, $z \varepsilon (x,y)$ iff $z = \{x\}$ or $z = \{xy\}$. Consequently $E \varepsilon \{E\} \varepsilon \{\{E\}\{E\{E\}\}\} \varepsilon E$. But if $a \varepsilon b \varepsilon c \varepsilon a$, then, upon application of the axiom of regularity to $\{x : x = a \text{ or } x = b \text{ or } x = c\}$, a contradiction results. ▌

An informal discussion of the structure of the first few ordinals may be conceptually enlightening.* The first ordinal will be 0, the next $1 = 0 \cup \{0\}$, the next $2 = 1 \cup \{1\}$, and the next $3 = 2 \cup \{2\}$. Observe 0 is the only member of 1, that 0 and 1 are the only members of 2, and 0, 1, and 2 are the only members of 3. Each ordinal preceding 3 is not only a member but also a subset of 3. Ordinals are defined so that this very special sort of structure results.

105 Definition† *x is full iff each member of x is a subset of x.*

In other words, x is full iff each member of a member of x is a member of x.

The following definition is due to R. M. Robinson.

106 Definition *x is an ordinal if and only if E connects x and x is full.*

That is, given two members of x, one is a member of the other, and each member of a member of x belongs to x.

107 Theorem *If x is an ordinal E well-orders x.*

* The discussion is not precisely accurate, in that it has not been proved that 0 is a set; in fact, with the axioms at our disposal this is not provable. The existence of sets (and hence the fact that 0 is a set) results from the axiom of infinity, which is stated at the beginning of the next section.

† The term 'complete' is usually used instead of 'full,' but 'complete' has been used earlier in a different sense.

PROOF If u and v are members of x and $u\,E\,v$, then (theorem 102) it is false that $v\,E\,u$ and hence E is asymmetric in x. If y is a non-void subset of x, then by the axiom of regularity there is u in y such that $u \cap y = 0$. Then no member of y belongs to u and u is the E-first member of y. ▌

108 THEOREM *If x is an ordinal, $y \subset x$, $y \neq x$, and y is full, then $y \,\varepsilon\, x$.*

PROOF If $u\,E\,v$ and $v\,E\,y$, then $u\,E\,y$ because y is full. Hence y is an E-section of x. Consequently there is a member v of x such that $y = \{u\colon u \,\varepsilon\, x \text{ and } u\,E\,v\}$ by theorem 91. Since every member of v is a member of x, $y = \{u\colon u \,\varepsilon\, v\}$ and $y = v$. ▌

109 THEOREM *If x is an ordinal and y is an ordinal, then $x \subset y$ or $y \subset x$.*

PROOF The class $x \cap y$ is full and by the preceding theorem either $x \cap y = x$ or $x \cap y \,\varepsilon\, x$. In the first case $x \subset y$. If $x \cap y \,\varepsilon\, x$, then $x \cap y \notin y$ since in this case $x \cap y \,\varepsilon\, x \cap y$. Since $x \cap y \notin y$ the preceding theorem implies that $x \cap y = y$ and therefore $y \subset x$. ▌

110 THEOREM *If x is an ordinal and y is an ordinal, then $x \,\varepsilon\, y$ or $y \,\varepsilon\, x$ or $x = y$.*

111 THEOREM *If x is an ordinal and $y \,\varepsilon\, x$, then y is an ordinal.*

PROOF It is clear that E connects y because x is full and E connects x. The relation E is transitive on y because E well-orders x and $y \subset x$. Consequently if $u\,E\,v$ and $v\,E\,y$, then $u\,E\,y$ and hence y is full. ▌

112 DEFINITION $R = \{x\colon x \text{ is an ordinal}\}$.

113 THEOREM * *R is an ordinal and R is not a set.*

PROOF The last two theorems show that E connects R and that R is full; hence R is an ordinal. If R were a set, then $R \,\varepsilon\, R$ and this is impossible. ▌

In view of theorem 110, R is the only ordinal which is not a set.

* This theorem is essentially the statement of the Burali-Forti paradox—historically the first of the paradoxes of intuitive set theory.

114 THEOREM *Each E-section of R is an ordinal.*

PROOF If an E-section x of R is not equal to R, then by 91 there is a member v of R such that $x = \{u: u \varepsilon R \text{ and } u \varepsilon v\}$. Since each member of v is an ordinal, $x = \{u: u \varepsilon v\} = v$. ▮

115 DEFINITION *x is an ordinal number if and only if $x \varepsilon R$.*

116 DEFINITION *$x < y$ if and only if $x \varepsilon y$.*

117 DEFINITION *$x \leqq y$ if and only if $x \varepsilon y$ or $x = y$.*

118 THEOREM *If x and y are ordinals, then $x \leqq y$ if and only if $x \subset y$.*

119 THEOREM *If x is an ordinal, then $x = \{y: y \varepsilon R \text{ and } y < x\}$.*

120 THEOREM *If $x \subset R$, then $\bigcup x$ is an ordinal.*

PROOF That E connects $\bigcup x$ follows from theorems 110 and 111, and that $\bigcup x$ is full follows from the fact that members of x are full. ▮

It is not hard to see that if x is a subset of R, then the ordinal $\bigcup x$ is the first ordinal which is greater than or equal to each member of x, and that $\bigcup x$ is a set iff x is a set. These results will not be needed, however.

121 THEOREM *If $x \subset R$ and $x \neq 0$, then $\bigcap x \varepsilon x$.*

Indeed, in this case $\bigcap x$ is the E-first member of x.

122 DEFINITION $x + 1 = x \cup \{x\}$.

123 THEOREM *If $x \varepsilon R$, then $x + 1$ is the E-first member of $\{y: y \varepsilon R \text{ and } x < y\}$.*

PROOF It is easy to verify that E connects $x + 1$ and that $x + 1$ is full and is hence an ordinal. If there is u such that $x < u$ and $u < x + 1$, then since x is a set and $u \varepsilon x \cup \{x\}$ either $u \varepsilon x$ and $x \varepsilon u$ or $u = x$ and $x \varepsilon u$. Both of these conclusions are impossible (theorems 101 and 102) and the desired conclusion is established. ▮

124 THEOREM *If $x \varepsilon R$, then $\bigcup(x + 1) = x$.*

125 DEFINITION $f | x = f \cap (x \times \mathcal{U})$.

This definition will be used only in case f is a relation. In this case $f \mid x$ is a relation and is called the *restriction* of f to x.

126 THEOREM *If f is a function, $f \mid x$ is a function whose domain is $x \cap$ (domain f) and $(f \mid x)(y) = f(y)$ for each y in domain $f \mid x$.*

The final theorem on ordinals asserts that (intuitively) it is possible to define a function on an ordinal so that its value at any member of its domain is given by applying a predetermined rule to the earlier values of the function. A little more precisely, given g it is possible to find a unique function f on an ordinal such that $f(x) = g(f \mid x)$ for each ordinal number x. The value $f(x)$ is then completely determined by g and the values of f at ordinal numbers preceding x. Application of this theorem is *defining a function by transfinite induction.* The proof is similar to that of theorem 99 and the same sort of preliminary lemma is needed.

127 THEOREM *Let f be a function such that* domain f *is an ordinal and $f(u) = g(f \mid u)$ for u in* domain f. *If h is also a function such that* domain h *is an ordinal and $h(u) = g(h \mid u)$ for u in* domain h, *then $h \subset f$ or $f \subset h$.*

PROOF Since both *domain f* and *domain h* are ordinals it may be assumed that *domain $f \subset$ domain h* (either this or the reverse inclusion follows from 109) and it remains to be proved that $f(u) = h(u)$ for u in *domain f*. Assuming the contrary, let u be the E-first member of *domain f* such that $f(u) \neq h(u)$. Then $f(v) = h(v)$ for each ordinal v preceding u and hence $f \mid u = h \mid u$. Then $f(u) = g(f \mid u) = h(u)$ and this is a contradiction. ∎

128 THEOREM *For each g there is a unique function f such that* domain f *is an ordinal and $f(x) = g(f \mid x)$ for each ordinal number x.*

PROOF Let $f = \{(u,v)$: *$u \in R$ and there is a function h such that* domain *h is an ordinal, $h(z) = g(h \mid z)$ for z in* domain *h and $(u,v) \in h\}$*. From the preceding theorem it follows that f is a function, and it is evident that the domain of f is an E-section of R and is hence an ordinal. Moreover, if h is a function on an ordinal such that $h(z) = g(h|z)$ for z in *domain* h, then $h \subset f$, and if $z \,\varepsilon$ *domain* h, then $f(z) = g(f|z)$.

Finally, suppose $x \,\varepsilon\, R \sim$ (*domain* f). Then $f(x) = \mathcal{U}$ by theorem 69 and since *domain* f is a set f is a set (theorem 75). If $g(f \mid x) = g(f) = \mathcal{U}$, then the equality $f(x) = g(f \mid x)$ follows. Otherwise $g(f)$ is a set (theorem 69 again). In this case if y is the E-first member of $R \sim$ (*domain* f) and $h = f \cup \{(y,g(f))\}$, then the domain of h is an ordinal and $h(z) = g(h \mid z)$ for z in *domain* h. Hence $h \subset f$ and $y \,\varepsilon$ *domain* f which is a contradiction. Consequently, $g(f) = \mathcal{U}$ and the theorem is proved. ∎

The mechanics of this theorem deserves a little comment. If *domain* f is not R, then $g(f) = \mathcal{U}$ and $f(x) = \mathcal{U}$ for each ordinal number x such that *domain* $f \leqq x$. If $g(0) = \mathcal{U}$, then $f = 0$.

INTEGERS *

In this section the integers are defined and Peano's postulates are derived as theorems. The real numbers may be constructed from the integers (see Landau [1]) by use of these postulates and the two facts: the class of integers is a set (theorem 138), and it is possible to define a function on the integers by induction (theorem 0.13; this fact may also be derived as a corollary to 128). Another axiom is needed.

VIII Axiom of infinity *For some* y, y *is a set*, $0 \,\varepsilon\, y$ *and* $x \cup \{x\} \,\varepsilon\, y$ *whenever* $x \,\varepsilon\, y$.

In particular 0 is a set because 0 is contained in a set.

129 DEFINITION *x is an integer if and only if x is an ordinal and E^{-1} well-orders x.*

130 DEFINITION *x is an E-last member of y if and only if x is an E^{-1}-first member of y.*

131 DEFINITION $\omega = \{x\colon x \text{ is an integer}\}$.

132 THEOREM *A member of an integer is an integer.*

PROOF A member of an integer x is an ordinal and is a subset of x and x is well-ordered by E^{-1}. ∎

* Non-negative integers.

133 THEOREM *If* $y \in R$ *and* x *is an E-last member of* y, *then* $y = x + 1$.

PROOF By theorem 123, $x + 1$ is the E-first member of $\{z: z \in R \text{ and } x < z\}$. Then $x + 1 \leqq y$ because $y \in R$ and $x < y$. Since x is the E-last member of y and $x < x + 1$, it is false that $x + 1 < y$. ▮

134 THEOREM *If* $x \in \omega$, *then* $x + 1 \in \omega$.

135 THEOREM $0 \in \omega$ *and if* $x \in \omega$, *then* $0 \neq x + 1$.

That is, 0 is the successor of no integer.

136 THEOREM *If* x *and* y *are members of* ω *and* $x + 1 = y + 1$, *then* $x = y$.

PROOF By theorem 124, if $x \in R$, then $\bigcup(x + 1) = x$. ▮

The following theorem is the *principle of mathematical induction.*

137 THEOREM *If* $x \subset \omega$, $0 \in x$ *and* $u + 1 \in x$ *whenever* $u \in x$, *then* $x = \omega$.

PROOF If $x \neq \omega$ let y be the E-first member of $\omega \sim x$, and notice that $y \neq 0$. Since $y \subset y + 1$ and $y + 1$ is an integer there is an E-last member u of y, and clearly $u \in x$. Then $y = u + 1$ by theorem 133 and hence $y \in x$. This is a contradiction. ▮

Theorems 134, 135, 136, and 137 are Peano's axioms for integers. The next theorem implies that ω is a set.

138 THEOREM $\omega \in R$.

PROOF By the axiom of infinity there is a set y such that $0 \in y$ and, if $x \in y$, then $x + 1 \in y$. By mathematical induction (that is, the previous theorem) $\omega \cap y = \omega$, and hence ω is a set because $\omega \subset y$. Since ω consists of ordinal numbers, E connects ω and ω is full because each member of an integer is an integer. ▮

THE CHOICE AXIOM

We now state the last axiom and derive two powerful consequences.

139 DEFINITION *c is a choice function if and only if c is a function and $c(x) \varepsilon x$ for each member x of* domain c.

Intuitively, a choice function is a simultaneous selection of a member from each set belonging to domain c.

The following is a strong form of Zermelo's postulate or the axiom of choice.

IX Axiom of choice *There is a choice function c whose domain is $\mathcal{U} \sim \{0\}$.*

The function c selects a member from every non-void set.

140 THEOREM *If x is a set there is a* 1-1 *function whose range is x and whose domain is an ordinal number.*

PROOF The plan of proof is to construct, by transfinite induction, a function satisfying the requirements of the theorem. Let g be the function such that $g(h) = c(x \sim range\ h)$ for each set h, where c is a choice function satisfying the axiom of choice. Applying theorem 128 there is a function f such that *domain* f is an ordinal and $f(u) = g(f \mid u)$ for each ordinal number u. Then $f(u) = c(x \sim range\ (f \mid u))$, and if $u \varepsilon$ *domain* f, then $f(u) \varepsilon x \sim range\ (f \mid u)$. Now f is 1-1, for $f(v) = f(u)$ and $u < v$, then $f(v) \varepsilon\, range\ (f \mid v)$, which contradicts the fact that $f(v) \varepsilon x \sim range\ (f \mid v)$. Since f is 1-1 it is impossible that *domain* $f = R$, for in this case f^{-1} is a function whose domain is a subclass of x and is hence a set, then *range* f^{-1} is a set because of the axiom of substitution and R is not a set. Consequently *domain* $f \varepsilon R$. Because *domain* $f \not\varepsilon$ *domain* f, f (*domain* f) $= \mathcal{U}$ and therefore $c\ (x \sim range\ f) = \mathcal{U}$. Since the domain of c is $\mathcal{U} \sim \{0\}$, $x \sim range\ f = 0$. It follows immediately that f is a function satisfying the requirements of the theorem. ▮

141 DEFINITION *n is a nest if and only if, whenever x and y are members of n, then $x \subset y$ or $y \subset x$.*

The next result is a lemma which is needed for the proof of theorem 143.

142 THEOREM *If n is a nest and each member of n is a nest, then $\bigcup n$ is a nest.*

PROOF If $x \varepsilon m$, $m \varepsilon n$, $y \varepsilon p$, and $p \varepsilon n$, then either $m \subset p$ or $p \subset m$ because n is a nest. Suppose $m \subset p$. Then $x \varepsilon p$ and $y \varepsilon p$ and since p is a nest, $x \subset y$ or $y \subset x$. ▌

The following theorem is the *Hausdorff maximal principle.* It asserts the existence of a maximal nest in any set. The proof is closely related to that of 140.

143 THEOREM *If x is a set there is a nest n such that $n \subset x$ and if m is a nest, $m \subset x$, and $n \subset m$, then $m = n$.*

PROOF The proof is by transfinite induction; intuitively we select a nest and then a larger nest, and "keep going," knowing that, because R is not a set, the set of all nests which are contained in x will be exhausted before the class R of ordinals. For each h let $g(h) = c(\{m\colon m \text{ is a nest, } m \subset x \text{ and for } p \text{ in range } h,\ p \subset m \text{ and } p \neq m\})$, where c is a choice function satisfying the axiom of choice. (Intuitively select $g(h)$ to be a nest in x containing properly each previously selected nest.) By theorem 128 there is a function f such that *domain* f is an ordinal and $f(u) = g(f \mid u)$ for each ordinal number u. From the definition of g it follows that, if $u \varepsilon$ *domain* f, then $f(u) \subset x$ and $f(u)$ is a nest, and if u and v are members of *domain* f and $u < v$, then $f(u) \subset f(v)$ and $f(u) \neq f(v)$. Consequently f is 1-1, f^{-1} is a function and, since x is a set, *domain* $f \varepsilon R$. Since $f(\textit{domain } f) = \mathfrak{U}$, $g(f) = \mathfrak{U}$; consequently there is no nest m which is contained in x and properly contains each member of *range* f. Finally, $\bigcup(\textit{range } f)$ is a nest which contains every member of *range* f, and consequently there is no nest m which is contained in x and properly contains $\bigcup(\textit{range } f)$. ▌

CARDINAL NUMBERS

In this section cardinal numbers are defined and the most commonly used properties are proved. The proofs lean heavily on the earlier results.

144 DEFINITION $x \approx y$ *if and only if there is a* 1-1 *function* f *with* domain $f = x$ *and* range $f = y$.

If $x \approx y$, then x is *equivalent* to y, or x and y are *equipollent.*

145 Theorem $x \approx x$.

146 Theorem *If $x \approx y$, then $y \approx x$.*

147 Theorem *If $x \approx y$ and $y \approx z$, then $x \approx z$.*

148 Definition *x is a cardinal number if and only if x is an ordinal number and, if $y \in R$ and $y < x$, then it is false that $x \approx y$.*

That is, a cardinal number is an ordinal number which is not equivalent to any smaller ordinal.

149 Definition $C = \{x: x \text{ is a cardinal number}\}$.

150 Theorem *E well-orders C.*

151 Definition $P = \{(x,y): x \approx y \text{ and } y \in C\}$.

The class P consists of all pairs (x,y) such that x is a set and y is a cardinal number equivalent to x. For each set x the cardinal number $P(x)$ is the *power* of x or the *cardinal* of x.

The basic facts needed for the following sequence of results have already been demonstrated.

152 Theorem *P is a function,* domain $P = \mathcal{U}$ *and* range $P = C$.

PROOF Theorem 140 is the essential step. ▌

153 Theorem *If x is a set, then $P(x) \approx x$.*

154 Theorem *If x and y are sets, then $x \approx y$ if and only if $P(x) = P(y)$.*

155 Theorem $P(P(x)) = P(x)$.

PROOF If x is not a set, then $P(x) = \mathcal{U}$ by theorem 69 and $P(\mathcal{U}) = \mathcal{U}$. ▌

156 Theorem *$x \in C$ if and only if x is a set and $P(x) = x$.*

157 Theorem *If $y \in R$ and $x \subset y$, then $P(x) \leqq y$.*

PROOF By theorem 99 there is a 1-1 function f which is E-E order preserving in x and R, such that *domain* $f = x$ or *range* $f = R$. Since x is a set and R is not, *domain* $f = x$. By theorem 94, $f(u) \leqq u$ for u in x and consequently x is equivalent to an ordinal less than or equal to y. ▌

158 Theorem *If y is a set and $x \subset y$, then $P(x) \leqq P(y)$.*

The following is the Schroeder-Bernstein theorem. It can be proved directly without the axiom of choice (theorem 0.20).

159 Theorem *If x and y are sets, $u \subset x$, $v \subset y$, $x \approx v$, and $y \approx u$, then $x \approx y$.*

PROOF Using 158, $P(x) = P(v) \leqq P(y) = P(u) \leqq P(x)$. ▌

160 Theorem *If f is a function and f is a set, then $P(\text{range } f) \leqq P(\text{domain } f)$.*

PROOF If f is on x onto y and c is a choice function satisfying the choice axiom there is a function g such that *domain* $g = y$ and $g(v) = c(\{u: v = f(u)\})$ for v in y. Consequently y is equivalent to a subset of x. ▌

The following classic theorem is due to Cantor.

161 Theorem *If x is a set, then $P(x) < P(2^x)$.*

PROOF The function, whose domain is x and whose value at a member u of x is $\{u\}$, is 1-1 and hence x is equivalent to a subset of 2^x and $P(x) \leqq P(2^x)$. If $P(x) = P(2^x)$ there is a 1-1 function f whose domain is x and range is 2^x. Then there is a member u of x such that $f(u) = \{v: v \in x \text{ and } v \notin f(v)\}$. But then $u \in f(u)$ iff $u \notin f(u)$, which is a contradiction. ▌

The foregoing is structurally similar to that of the Russell paradox.

162 Theorem *C is not a set.*

PROOF If C is a set, then $\bigcup C$ is a set, $P(2^{\bigcup C}) \in C$ and hence $P(2^{\bigcup C}) \subset \bigcup C$. Therefore $P(2^{\bigcup C}) \leqq P(\bigcup C)$, which is a contradiction. ▌

After some preliminaries we divide the cardinals into two classes, the finite cardinals and the infinite cardinals, and prove a few special properties for each class.

163 Theorem *If $x \in \omega$, $y \in \omega$ and $x + 1 \approx y + 1$, then $x \approx y$.*

PROOF If f is a 1-1 function on $x + 1$ onto $y + 1$ there is a 1-1 function g on $x + 1$ onto $y + 1$ such that $g(x) = y$; for example, let g be $(f \sim \{(x,f(x))\} \cup \{(f^{-1}(y),y)\}) \cup \{(f^{-1}(y),f(x))\} \cup \{(x,y)\}$. Then $g \mid x$ is a 1-1 function on x onto y. ▮

164 THEOREM $\omega \subset C$.

PROOF The proof is by induction. Apply the preceding theorem to the first integer which is equivalent to a smaller integer to obtain a contradiction, thus proving that each integer is a cardinal number. ▮

165 THEOREM $\omega \varepsilon C$.

PROOF If $\omega \approx x$ and $x \varepsilon \omega$, then $x \subset x + 1 \subset \omega$, and hence $P(x + 1) = P(x)$. This contradicts the preceding theorem, which states that each integer is a cardinal number. ▮

166 DEFINITION *x is finite if and only if $P(x) \varepsilon \omega$.*

167 THEOREM *x is finite if and only if there is r such that r well-orders x and r^{-1} well-orders x.*

PROOF If $P(x) \varepsilon \omega$, then E and E^{-1} well-order $P(x)$, and since $x \approx P(x)$ there is no difficulty finding r such that both r and r^{-1} well-order x. Conversely, if r and r^{-1} well-order x, then by 99 there is a 1-1 function f which is r-E order preserving in x and R such that either *domain* $f = x$ or *range* $f = R$. If $\omega \subset$ *range* f, then r^{-1} does not well-order x because ω has no E last element. Consequently *range* $f \varepsilon \omega$, *domain* $f = x$, and the theorem follows. ▮

Each of the following sequence of theorems on finite sets can be proved by induction on the power of a set or by constructing a well ordering and applying 167. Examples of both sorts of proof will be given.

168 THEOREM *If x and y are finite so is $x \cup y$.*

PROOF If both r and r^{-1} well-order x and both s and s^{-1} well-order y, then, using r for points of x, s for points of $y \sim x$, and letting each member of $y \sim x$ follow every point of x, one can construct an ordering of the required type for $x \cup y$. ▮

169 THEOREM *If x is finite and each member of x is finite, then $\bigcup x$ is finite.*

PROOF One may proceed by induction on $P(x)$. Explicitly, consider the set s of all integers u such that, if $P(x) = u$ and each member of x is finite, then $\bigcup x$ is finite. Clearly 0 belongs to the set s. If $u \in s$, $P(x) = u + 1$, and each member of x is finite, then one may split x into two subsets, one of which has power u and one of which is a singleton. The induction hypothesis and the preceding theorem then show that $\bigcup x$ is finite. Hence $s = \omega$. ▮

170 THEOREM *If x and y are finite so is $x \times y$.*

PROOF The class $x \times y$ is the union of the members of a finite class, the members being of the form $\{v\} \times y$ for v in x. ▮

171 THEOREM *If x is finite so is 2^x.*

PROOF If y is an integer, then the subsets of $y + 1$ can be divided into two classes: those which are subsets of y, and those which are the union of a subset of y and $\{y\}$. This gives the necessary basis for an inductive proof of the theorem. ▮

172 THEOREM *If x is finite, $y \subset x$ and $P(y) = P(x)$, then $x = y$.*

PROOF It is sufficient to consider the case where x is an integer. Suppose $y \subset x$, $y \neq x$, $P(y) = x$, and $x \in \omega$. Then $x \neq 0$ and hence $x = u + 1$ for some integer u. Because $y \neq x$ there is a subset of u which is equivalent to y and hence $P(y) \leqq u$. But $P(y) = x = u + 1$, and this contradicts the fact that each integer is a cardinal number. ▮

The property of theorem 172, that a finite set is equivalent to no proper subset, actually characterizes finite sets.

173 THEOREM *If x is a set and x is not finite, then there is a subset y of x such that $y \neq x$ and $x \approx y$.*

PROOF Since x is a set and is not finite, $\omega \subset P(x)$. There is a function f on $P(x)$ such that $f(u) = u + 1$ for u in ω, and for $f(u) = u$ for u in $P(x) \sim \omega$. The function f is 1-1 and *range* $f = P(x) \sim \{0\}$. Since $P(x) \approx x$ the theorem follows. ▮

174 THEOREM *If $x \in R \sim \omega$, then $P(x + 1) = P(x)$.*

PROOF Clearly $P(x) \leqq P(x + 1)$. Since x is not finite there is a subset u of x such that $u \neq x$ and $u \approx x$. Consequently there is a 1-1 function f on $x + 1$ such that $f(y) \varepsilon u$ for y in x and $f(x) \varepsilon x \sim u$. Hence $P(x + 1) \leqq P(x)$. ∎

The principal remaining theorem depends on an order which will be assigned to the cartesian product $R \times R$. An intuitive description of this order may be instructive. It is to be a well ordering, and on $\omega \times \omega$ it is to have the property that the class of all predecessors of a member (x,y) of $\omega \times \omega$ is finite (a generalization of this fact is the key to the usefulness of the order). Picture $\omega \times \omega$ as a subset of the Euclidean plane and divide $\omega \times \omega$ into classes, putting in the same class pairs (x,y) and (u,v) such that the maximum of x and y is identical with the maximum of u and v. Each class then consists of two sides of a square, and the ordering is arranged so that points on smaller squares precede points on large squares. For points on the sides of the same square the ordering proceeds along the upper edge and to the right, up to but not including the corner point, and then along the right-hand edge upward, ending with the corner point.

If x and y are ordinals, the larger of them is $x \cup y$. This motivates the following definition.

175 DEFINITION max $[x,y] = x \cup y$.

176 DEFINITION $\ll = \{z$: *for some* (u,v) *in* $R \times R$ *and some* (x,y) *in* $R \times R$, $z = ((u,v),(x,y))$, *and* max $[u,v] <$ max $[x,y]$, *or* max $[u,v] =$ max $[x,y]$ *and* $u < x$, *or* max $[u,v] =$ max $[x,y]$ *and* $u = x$ *and* $v < y\}$.

177 THEOREM $\ll$ *well-orders* $R \times R$.

The proof is a straightforward but tedious application of the definition and the fact that $<$ well-orders R.

178 THEOREM *If* $(u,v) \ll (x,y)$, *then* $(u,v) \varepsilon (\max [x,y] + 1) \times (\max [x,y] + 1)$.

PROOF Surely max $[u,v] \leqq$ max $[x,y]$, and hence max $[u,v] \subset$ max $[x,y]$. Since the ordinals u and v are subsets of max $[x,y]$ they are members of max $[x,y] + 1$. ∎

179 THEOREM *If* $x \varepsilon C \sim \omega$, *then* $P(x \times x) = x$.

PROOF We proceed by induction, supposing x to be the first member of $C \sim \omega$ for which the theorem fails. There is by 99 a function f which is $\ll$-E order preserving in $x \times x$ and R, such that either *domain* $f = x \times x$ or *range* $f = R$. Since $x \times x$ is a set and R is not, *domain* $f = x \times x$. We show that, if $(u,v) \in x \times x$, then $f((u,v)) < x$, and the theorem follows. By the preceding theorem the class of all predecessors of (u,v) is a subset of $(\max[u,v] + 1) \times (\max[u,v] + 1)$. If $x = \omega$, then both u and v are finite because $\max[u,v] < x$; by 170, $(\max[u,v] + 1) \times (\max[u,v] + 1)$ is finite, hence $f((u,v))$ has only a finite number of predecessors and $f((u,v)) < x$. If $x \neq \omega$, and $\max[u,v]$ is not finite, then by 174, $P(\max[u,v] + 1) = P(\max[u,v]) < x$ and hence $P(f((u,v))) < x$ and $f((u,v)) < x$. ▌

180 THEOREM *If x and y are members of C, one of which fails to belong to ω, then* $P(x \times y) = \max[P(x),P(y)]$.

The members of $C \sim \omega$ are called *infinite*, or *transfinite*, cardinal numbers.

There are many important and useful theorems on cardinal numbers which have not been given in the preceding list; see, for example, Fraenkel [1] for further information and references. This discussion will be concluded with a brief statement on one of the classic unsolved problems of set theory.

181 THEOREM *There is a unique $<$-$<$ order-preserving function with domain R and range $C \sim \omega$.*

PROOF There is, by 99, a unique $<$-$<$ order-preserving function f in R and $C \sim \omega$ such that either *domain* $f = R$ or *range* $f = C \sim \omega$. Since every E-section of R and of $C \sim \omega$ is a set and neither R nor $C \sim \omega$ is a set, it is impossible that *domain* $f \neq R$ or *range* $f \neq C \sim \omega$. ▌

The unique $<$-$<$ order-preserving function whose existence is guaranteed by the previous theorem is usually denoted by $\aleph$. Thus $\aleph(0)$ (or $\aleph_0$) is ω. The next cardinal $\aleph_1$ is also denoted by Ω; it is the first uncountable ordinal. Since $P(2^{\aleph_0}) > \aleph_0$ it follows that $P(2^{\aleph_0}) \geq \aleph_1$. The equality of these two cardinals is an extremely attractive conjecture. It is called the *hypothesis of the continuum*. The *generalized hypothesis of the continuum is* the statement: if x is an ordinal number, then $P(2^{\aleph_x}) = \aleph_{x+1}$. Neither hypothesis has been proved or disproved. However, Gödel [1] has proved the beautiful metamathematical theorem: If, on the basis of the hypothesis of the continuum, a contradiction is constructed, then a contradiction may be found without assuming the hypothesis of the continuum. The same situation prevails with respect to the generalized hypothesis of the continuum and the axiom of choice.

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ISBN 0-387-90125-6
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9 780387 901251

ISBN 0-387-**90125**-6
ISBN 3-540-**90125**-6

目　　录

第零章　预备知识

理解本书的唯一前提，只需要知道实数的少数性质和具有适当程度的数学修养．以后要用到的所有定义和基本定理都汇集于这个头一章里．这里的论述在一定程度上是自成系统的．但是，特别是在数系的讨论中，有不少细节被略去了．本章最深刻的一些结果是集论的定理，而它的系统论述在附录中给出．由于这一章本来是打算当作参考资料的，因此建议读者先复习一下头两节，然后开始学第一章，当感到需要的时候，可再利用本章的其余部分．许多定义当它们第一次在书中出现时，我们予以重述．

集

我们将要论及集和集的元．“集”、“类”、“族”均为同义语[1]，而符号 $\in$ 表示元的从属关系．所以，当且仅当 x 是 A 的一个元(一个元素，或一个点)时，方可以写 $x \in A$．两个集是恒等的，当且仅当它们有相同的元，并且相等通常总是意指恒等，因此，$A = B$ 的充要条件是：对于每一个 x，$x \in A$ 当且仅当 $x \in B$．

集将要借助于括号来构成，因此 $\{x:\cdots$(关于 x 的命题)$\cdots\}$ 是使得关于 x 的命题是正确的所有点 x 的集．也就是说，$y \in \{x:\cdots$(关于 x 的命题)$\cdots\}$ 当且仅当关于 y 的相应命题是正确的．例如，假定 A 是一个集，则 $y \in \{x: x \in A\}$ 当且仅当 $y \in A$．因为具有相同元的两个集是恒等的，所以 $A = \{x: x \in A\}$，这即使不是一件惊奇的事实，也是一件使人愉快的事实．在构造集的这

1) 这种说法不是绝对准确的，在附录内将要说明，由于技术上的原因，将把类分成不同的两种．我们把“集”这一术语保留给它们本身是类的元的那种类．在此集与类的区别不是十分重要的；除了唯一的一个并非无足轻重的例外，即每一个类当它在讨论中出现时（在附录以前），也是一个集．

一方案中，“x”是一个哑变数，其含意是我们可以用不曾出现在这个命题中的任何其它变数来代替它．于是$\{x: x\in A\}=\{y: y\in A\}$，但是 $\{x:x\in A\}\neq\{A:A\in A\}$．

在这种形式下，关于集的构造有一个很有用的法则．如果两个集是由两个不同的命题利用上面规定的方式构成的，同时假定这两个命题逻辑上是等价的，则构成的集是相等的．这个法则可以通过证明构造成的集具有相同的元来说明它是合理的．例如，假定A与B是两个集，则 $\{x:x\in A$ 或 $x\in B\}=\{x:x\in B$或 $x\in A\}$，因为y属于第一个集当且仅当 $y\in A$ 或 $y\in B$，而这种情况成立当且仅当 $y\in B$ 或 $y\in A$．这一结论成立当且仅当y为第二个集的一个元．下一节的所有定理正是用这种方法加以证明的．

子集与余集；并与交

如果A与B是两个集（或族），则A是B的一个**子集**(**子族**)当且仅当A的每个元是B的一个元．在这种情况下，我们可以说A**被包含**在B中或者B**包含**A，并写成下面的形式：$A\subset B$或者$B\supset A$．于是 $A\subset B$ 当且仅当对于每一x只要 $x\in A$，则有 $x\in B$．集A是B的一个**真子集**（A真正地被包含在B中或B真正地包含A）当且仅当 $A\subset B$ 同时 $A\neq B$．如果A是B的一个子集，同时B又是C的一个子集，那末显然A是C的一个子集．如果 $A\subset B$ 同时 $B\subset A$，则 $A=B$，在这种情况下A的每一个元也是B的一个元，反之亦然．

集A与B的**并**(**和、逻辑和**)记作 $A\cup B$，它是至少属于A或B之一的所有点的集；也就是说 $A\cup B=\{x:x\in A$ 或 $x\in B\}$．在此采用“或”字并没有两者不可兼的意思．也就是说既属于A又属于B的点也属于 $A\cup B$．集A与B的**交**，记作 $A\cap B$，它是同时属于A与B之所有点的集；也就是说，$A\cap B=\{x:x\in A$ 同时 $x\in B\}$．**空集**用 0 来表示[1]，并定义为$\{x:x\neq x\}$（任何一个伪命题可以用在

1) 空集往往用符号$\varnothing$来表示，以便同数 0 相区别．——译者注

此处来代替 $x \neq x$)．空集是任一集 A 的一个子集，因为 0（没有一个元)的每个元属于 A．对于每一对集 A 与 B，包含关系 $0 \subset A \cap B \subset A \subset A \cup B$ 皆成立．两个集 A 与 B 叫做**不相交的**当且仅当 $A \cap B = 0$；也就是说，A 的任何元都不是 B 的元．两个集 A 与 B 叫做**相交的**当且仅当存在一个点同时属于这两个集，因此 $A \cap B \neq 0$．如果 $\mathscr{A}$ 是一个集族（$\mathscr{A}$ 的元均为集），那末 $\mathscr{A}$ 叫做一个**不相交族**当且仅当 $\mathscr{A}$ 的任意两个元都不相交．

一个集 A 的**绝对余集**记作 $\sim A$，它是 $\{x: x \notin A\}$．A 关于一个集 X 的**相对余集**是 $X \cap \sim A$，或者简单地记作 $X \sim A$．这样的集又称为 X 与 A 之**差**．对于每一个集 A 皆有 $\sim \sim A = A$ 成立；关于相对余集的相应说法较复杂，所以把它作为定理 2 的一部分来给出．

必须很仔细地区分“元”与“子集”．仅有一个元 x 的集称为**单点集**，并用 $\{x\}$ 来表示．注意 $\{0\}$ 不是空集，因为 $0 \in \{0\}$，所以 $0 \neq \{0\}$．在一般情况下 $x \in A$ 当且仅当 $\{x\} \subset A$．

下面两个定理表述出上面给出的各种定义之间最常用到的一些关系．这些关系都是一些基本的事实，今后用到时常常不再明确地指出来．在此我们只证这两个定理的一部分．

1 定理． 设 A 与 B 为一集 X 的两个子集，则 $A \subset B$ 当且仅当下列条件之一成立．

$$A \cap B = A;\ B = A \cup B;\ X \sim B \subset X \sim A;$$
$$A \cap X \sim B = 0;\ \text{或}\ (X \sim A) \cup B = X.$$

2 定理． 设 A，B，C 与 X 均为集，则

(a) $X \sim (X \sim A) = A \cap X$.

(b)（交换律）$A \cup B = B \cup A$ 且 $A \cap B = B \cap A$.

(c)（结合律）$A \cup (B \cup C) = (A \cup B) \cup C$ 且
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

(d)（分配律）$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 且
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(e)（De Morgan 公式）$X \sim (A \cup B) = (X \sim A) \cap (X \sim B)$ 且 $X \sim (A \cap B) = (X \sim A) \cup (X \sim B)$.

证明． (a) 的证明．一个点 x 是 $X \sim (X \sim A)$ 的一个元当且仅当 $x \in X$ 同时 $x \notin X \sim A$．由于 $x \notin X \sim A$ 当且仅当 $x \notin X$ 或 $x \in A$，从而推出 $x \in X \sim (X \sim A)$ 当且仅当 $x \in X$ 并且不是 $x \notin X$ 便是 $x \in A$．但这两种情况的第一种是不可能的，所以 $x \in X \sim (X \sim A)$ 当且仅当 $x \in X$ 同时 $x \in A$；也就是当且仅当 $x \in X \cap A$．

(d) 的第一部分之证明．一个点 x 是 $A \cap (B \cup C)$ 的一个元当且仅当 $x \in A$ 同时不是 $x \in B$ 便是 $x \in C$．这种情况又当且仅当 x 不是同时属于 A 与 B 便是同时属于 A 与 C．故 $x \in A \cap (B \cup C)$ 当且仅当 $x \in (A \cap B) \cup (A \cap C)$，从而等式得证． ▌

如果 $A_1, A_2, \cdots, A_n$ 均为集，则 $A_1 \cup A_2 \cup \cdots \cup A_n$ 是这些集的并，而 $A_1 \cap A_2 \cap \cdots \cap A_n$ 是它们的交．由于结合律成立，在计算并与交时，把各集不论怎样结合起来都是无妨的．我们还要考虑非有限集族的元的并，有了这种并的记号是极其方便的．考虑下面的情况：对于一个我们称为指标集的集 A 的每一元 a，假定给定一个集 X_a，于是所有 X_a 的并用 $\bigcup\{X_a : a \in A\}$ 来表示，而它被定义为对于 A 中某一 a 使得 $x \in X_a$ 之所有点 x 的集．类似的方法，对于在 A 中的 a 所有 X_a 的交用 $\bigcap\{X_a : a \in A\}$ 来表示，而它被定义为 $\{x$：对于 A 中的每一 a，$x \in X_a\}$．一个很重要的特殊情况如下：指标集本身是一个集族 $\mathscr{A}$ 并且对于 $\mathscr{A}$ 中每个 A，X_A 就是集 A，这时上面的定义变成：$\bigcup\{A : A \in \mathscr{A}\} = \{x$：对于 $\mathscr{A}$ 中的某一 A，$x \in A\}$ 同时 $\bigcap\{A : A \in \mathscr{A}\} = \{x$：对于在 $\mathscr{A}$ 中的每一个 A，$x \in A\}$．

关于集族的元的并与交有许多具有代数特征的定理，但我们只需要下面的一个，而它的证明被略去了．

3 定理． 设 A 为一指标集，并且对于在 A 中的每一 a 令 X_a 为一个固定集 Y 的一个子集，则

(a) 如果 B 是 A 的一个子集，则 $\bigcup\{X_b : b \in B\} \subset \bigcup\{X_a : a \in A\}$ 且 $\bigcap\{X_b : b \in B\} \supset \bigcap\{X_a : a \in A\}$．

(b) (De Morgan 公式) $Y \sim \bigcup\{X_a : a \in A\} = \bigcap\{Y \sim X_a : a \in A\}$

且 $Y \sim \bigcap\{X_a: a \in A\} = \bigcup\{Y \sim X_a: a \in A\}$.

De Morgan 公式通常以扼要的形式叙述为：并的余集等于余集的交，并且交的余集等于余集的并。

应当强调指出：适当地熟练这类集论的运算是很重要的。附录中包含有一长串定理，我们建议初学者把它作为练习（参看关于类的初等代数那一节）。

4 注记. 在大多数集论的早期著作中，与对实数的通常运算相类似，两个集 A 与 B 的并曾记作 $A + B$，且交记作 AB. 一些相同的代数定律也成立；然而由于迫不得已的原因，下面不采用这种习惯的用法。通常集论运算是在一个群、一个域、或者一个线性空间内取的。如果 A 与 B 是一个（记作加法的）群的两个子集，则集 $\{c: c = a + b$，对于 A 内的某个 a 与 B 内的某个 $b\}$ 自然选用记号"$A + B$"，同时很自然的用 $-A$ 来表示集 $\{x: -x \in A\}$. 由于在系统地应用刚才定义的集进行运算时，集的并、交以及余集也要出现，可见本书采用的记号似乎是最合理的。本书关于集的构造所用的记号是现今使用最广的一种，但是"使得……所有 x 的集"也用记号 $\underset{x}{E}$ 表示。这类记号有下面的弱点：那就是必须确定哪一个是哑变数。现通过下面的例子来说明这种论点。所有正数平方的集可以很自然的用 $\{x^2: x > 0\}$ 表示，继之 $\{x^2 + a^2: x < 1 + 2a\}$ 也有很自然的含意。不幸的是后者可能有三种自然的含意，即：$\{z$: 对于某个 x 与某个 a，$z = x^2 + a^2$ 且 $x < 1 + 2a\}$，$\{z$: 对于某个 x，$z = x^2 + a^2$ 且 $x < 1 + 2a\}$，以及 $\{z$: 对于某个 a，$z = x^2 + a^2$ 且 $x < 1 + 2a\}$. 这些集是完全不同的，因为第一个集既不依赖 x 也不依赖 a，第二个集依赖 a，而第三个集依赖 x. 用稍许专门一点的术语来讲，"x"与"a"在第一个集里都是哑的，"x"在第二个集里是哑的，而"a"在第三个集里是哑的。为了避免含混，每当应用括号记号时，第一个括号之后和冒号之前恒用哑变数来占据。

最后，考虑另一记法的特点是有意义的。在读象"$A \cap (B \cup C)$"这种表达式时括号是最要紧的。然而若选用一种与此稍许不同

的记法就可避免这一点．如果我们用“$\cup AB$”代替了“$A\cup B$”，同时对于交也用类似的方法，于是所有的括号能够被略去．（这种避免括号的一般方法，在数理逻辑里是有名的．）在这种修改的记号里第一分配律和对并的结合律可表述为：$\cap A\cup BC=\cup\cap AB\cap AC$ 同时 $\cup A\cup BC=\cup\cup ABC$．这种速写记法读起来也方便；例如，$\cup AB$ 读为 A 与 B 之并．

关　　系

集的概念在此论述中被取作基础，所以我们面临的任务是用集的术语去定义其它必需的概念，尤其是必需定义序与函数的概念．但这些概念均可当作关系来处理，而关系又能够很自然的被当作具有某种特殊结构的集来定义．因此本节提供关系代数的定义和初等定理的一个简要陈述．

假定我们已给某确定的对象的序偶之间一种关系．（在直观的意义下．）其基本的想法是：关系可表示为所有相互有关的对象的序偶之集．例如，一个数和它的立方构成之所有序偶的集可称为立方关系．自然为了使用这种实现方法，需要有方便的序偶概念，而这个概念能够用集的术语来定义[1]．在此我们需要的基本事实是：每一序偶有一个第一坐标与一个第二坐标，并且两个序偶相等（恒等）当且仅当它们有相同的第一坐标与相同的第二坐标．具有第一坐标 x 与第二坐标 y 的序偶用 (x,y) 来表示．于是当且仅当 $x=u$ 且 $y=v$ 时 $(x,y)=(u,v)$．

为了方便起见，我们推广构造集的方法，使记号 $\{(x,y):\cdots\}$ 表示合于……的所有序偶 (x,y) 之集．然而，严格地说，这种规定并不是必要的．因为同一个集可用较详细的说明：$\{z$：对于某个 x 与某个 y，$z=(x,y)$ 且……$\}$而得到．

1）这个问题的确切说法将在附录中给出．那里采用了 N. Wiener 的序偶定义．用这种方式巧妙地表示关系之思想是属于 C. S. Peirce 的，基本关系代数的清晰论述能在 A. Tarski[1] 中找到．

一个**关系**是一个序偶的集；即一个关系是一个集，而它的每个元是一个序偶. 如果 R 是一个关系，我们用 $\boldsymbol{xRy}$ 简记 $(x, y)\in R$. 同时当且仅当 xRy 时，我们称 x **R-相关于** y. 一个关系 R 的**定义域**是 R 中成员的所有第一个坐标之集，而它的**值域**是所有第二个坐标之集. 用式子的写法 R 的**定义域** $=\{x$: 对于某个 y, $(x, y)\in R\}$ 且 R 的**值域** $=\{y$: 对于某个 x, $(x, y)\in R\}$. 最简单的关系之一是 x 为某一指定集 A 的元而 y 为某一指定集 B 的元所构成之所有序偶 (x, y) 的集. 这个关系是 A 与 B 的**笛卡儿** (Cartesian) **乘积**，并用 $A\times B$ 来表示. 于是 $A\times B=\{(x, y): x\in A$ 且 $y\in B\}$. 如果 B 非空，则 $A\times B$ 的定义域为 A. 因此每个关系显然是它的定义域与值域的笛卡儿积的一个子集.

一个关系 R 之**逆**是用对调属于 R 的每个序偶而得到的，并以 R^{-1} 来表示. 于是 $R^{-1}=\{(x, y): (y, x)\in R\}$ 并且当且仅当 $yR^{-1}x$ 时 xRy. 例如，对所有的集 A 与 B, $(A\times B)^{-1}=B\times A$. 一个关系 R 之逆的定义域恒为 R 的值域，并且 R^{-1} 的值域为 R 的定义域. 如果 R 与 S 是两个关系，它们的**合成** $R\circ S$(有时写成 RS) 定义为：对于某个 y, 使 $(x, y)\in S$ 且 $(y, z)\in R$ 的所有序偶 (x, z) 之集. 合成一般是不可交换的. 例如，如果 $R=\{(1, 2)\}$ 且 $S=\{(0, 1)\}$, 则 $R\circ S=\{(0, 2)\}$, 但 $S\circ R$ 却是空集. 在集 X 上的**恒等关系**(**在 X 上恒同**)是指对于在 X 中的 x, 所有形为 (x, x) 的序偶所成之集，而它用 Δ 或 $\Delta(X)$ 来表示. 这个名字是由每当 R 是一个值域及定义域均为 X 之子集的关系时，恒有 $\Delta\circ R=R\circ\Delta=R$ 成立而产生的. 恒等关系又被称为**对角线**，此名暗示了它在 $X\times X$ 中的几何位置.

如果 R 是一个关系，而 A 是一个集，则所有与 A 中的点 R-相关的点所构成的集 $R[A]$ 被定义为$\{y$: 对于 A 中的某个 x, $xRy\}$. 如果 A 是 R 的定义域，则 $R[A]$ 是 R 的值域，并且对于任意的 A, 集 $R[A]$ 被包含在 R 的值域中. 如果 R 与 S 是两个关系且 $R\subset S$, 则对于每个 A, 显然 $R[A]\subset S[A]$.

对于关系有大量的运算，下面的定理就是其中的一部分.

5 定理. 设 R, S 与 T 皆为关系，又设 A 与 B 是两个集，则

(a) $(R^{-1})^{-1}=R$ 且 $(R\circ S)^{-1}=S^{-1}\circ R^{-1}$.

(b) $R\circ(S\circ T)=(R\circ S)\circ T$ 且 $(R\circ S)[A]=R[S[A]]$.

(c) $R[A\cup B]=R[A]\cup R[B]$ 且

$R[A\cap B]\subset R[A]\cap R[B]$.

更一般地，如果对于一个非空指标集 A 的每个元 a，给定一个集 X_a，则

(d) $R[\bigcup\{X_a: a\in A\}]=\bigcup\{R[X_a]: a\in A\}$ 且

$R[\bigcap\{X_a: a\in A\}]\subset\bigcap\{R[X_a]: a\in A\}$.

证明．作为一个例子，我们证明等式：$(R\circ S)^{-1}=S^{-1}\circ R^{-1}$. 一个序偶 (z, x) 是 $(R\circ S)^{-1}$ 的一个元当且仅当 $(x, z)\in R\circ S$，而这种情况就是当且仅当对于某个 y, $(x, y)\in S$ 并且 $(y, z)\in R$. 所以 $(z, x)\in(R\circ S)^{-1}$ 当且仅当对于某个 y, $(z, y)\in R^{-1}$ 且 $(y, x)\in S^{-1}$. 可这正好是 (z, x) 属于 $S^{-1}\circ R^{-1}$ 的条件．|

有几种特殊类型的关系，由于它们经常在数学中出现，所以给它们起了名字．这里暂且不谈序与函数，因为它们在下一节里将被详加讨论．下面所列举的类型也许是最有用的．

下面始终假定 R 是一个关系，X 是所有属于 R 的定义域或值域的点构成的集；即 $X=$ (R 的定义域) $\cup$ (R 的值域).

关系 R 叫做**自反的**当且仅当 X 的每个点 R-相关于它自己时，而这完全等价于要求恒等关系 Δ (或 $\Delta(X)$) 是 R 的一个子集．

倘若只要 xRy 便有 yRx，则称关系 R 是**对称的**．用代数式子表示，即为：$R=R^{-1}$. 在另一极端，称关系 R 叫做**反对称的**当且仅当不出现 xRy 与 yRx 同时成立的情况．换言之，称 R 为反对称的当且仅当 $R\cap R^{-1}$ 为空集．

称关系 R 为**传递的**当且仅当若 xRy 且 yRz 则 xRz. 用关系合成的术语来讲，关系 R 是传递的当且仅当 $R\circ R\subset R$. 于是推出如果 R 是传递的，则 $R^{-1}\circ R^{-1}=(R\circ R)^{-1}\subset R^{-1}$. 所以传递关系之逆仍是传递的．如果 R 既是传递的又是自反的，则 $R\circ R\supset R\circ\Delta$，故 $R\circ R=R$；用习惯上的术语来讲，这样的一个关系在合成之

下是幂等的.

一个**等价**关系是一个既自反又对称和传递的关系. 等价关系具有很简单之结构. 现在我们来描述它. 假定R是一个等价关系且X是R的定义域,X的一个子集A是一个等价类(一个R-等价类)当且仅当在A中存在一个元x使得A与合于xRy之所有y的集恒等. 换句话说,A为一个等价类当且仅当在A中存在x使得$A=R[\{x\}]$. 关于等价关系的基本结论证明了所有等价类的族$\mathscr{A}$是互不相交的,并且点$x R$-相关于点y当且仅当x与y都属于同一等价类. 用类A中的x与y所构成的所有序偶之集是简单的$A\times A$. 由此导出下面定理之简明叙述.

6 定理. 一个关系R是一个等价关系当且仅当存在一个互不相交族$\mathscr{A}$使得 $R=\bigcup\{A\times A: A\in\mathscr{A}\}$.

证明. 如果R是一个等价关系,则R是传递的: 若yRx且zRy, 则zRx. 换句话说, 如果xRy, 则$R[\{y\}]\subset R[\{x\}]$. 但由于R是对称的. (只要yRx便有xRy). 于是推出如果xRy, 则$R[\{x\}]=R[\{y\}]$. 假定z同时属于$R[\{x\}]$与$R[\{y\}]$, 则$R[\{x\}]=R[\{z\}]=R[\{y\}]$, 所以两个等价类不是重合便是互不相交. 如果$y$与$z$属于等价类$R[\{x\}]$, 则由$R[\{y\}]=R[\{x\}]$推得$yRz$, 换句话说,也就是$R[\{x\}]\times R[\{x\}]\subset R$. 故对所有等价类$A$, $A\times A$的并是R的一个子集, 又由于R是自反的, 所以如果xRy, 则$(x, y)\in R[\{x\}]\times R[\{x\}]$. 故$R=\bigcup\{A\times A: A\in\mathscr{A}\}$. 反过来的简单证明在此省略. |

我们的兴趣经常在于了解: 一个关系在属于它的定义域的一个子集的那些点上的性质,并且对于那些点关系所具有的性质,经常对所有的点却不成立. 已给一个集X和一个关系R, 可以构造一个新的关系$R\cap(X\times X)$, 它的定义域是X的一个子集. 为了方便起见,我们说关系R**在X上具有性质**,或关系R**限制在X上**具有性质当且仅当$R\cap(X\times X)$具有此性质. 例如,R在X上传递当且仅当$R\cap(X\times X)$是一个传递关系. 这等于说定义的性质对于在X中的点成立; 在这种情况下只要x、y及z都是X中使得

xRy 且 yRz 的点，则 xRz.

函　　数

现在函数的概念必须用已经引进的概念来定义. 如果我们考虑到下面的事实，这种企图并不困难. 不管一个函数是什么，而它的图象作为序偶的集确有一个显然的定义. 此外，没有关于函数的信息不能由它的图象导出. 简而言之，没有什么原因要我们全力去找一个函数与它的图象之间的区别.

一个**函数**是使得没有两个不同元具有相同之第一坐标的一种关系. 于是称 f 是一个函数当且仅当 f 的元都是序偶，并且只要 (x, y) 与 (x, z) 为 f 的元，则 $y = z$. 一个函数与它的图象之间我们不加以区分.

对应、**变换**、**映射**、**算子**以及**函数**，这些术语均为同义语. 如果 f 是一个函数且 x 是它的定义域（f 的所有元中第一个坐标之集）内的一个点，则 $f(x)$ 或 f_x 是 f 的第一个坐标为 x 的唯一的元之第二个坐标. 点 $f(x)$ 是 f 在 x 的**值**，或者是在 f 的映射下 x 的**象**，并且称 f 对于 x **指定值** $f(x)$，或者把 x **变成** $f(x)$. 称一个函数 f **在 X 上**当且仅当 X 是它的定义域.（f 的元中第二个坐标之集有时称为值集.）如果 f 的值域是 Y 的一个子集，则 f 是**到 Y**，或**到 Y 内**的. 一般来讲，在下述意义下，一个函数是多对一的. 即可能有许多序偶具有同一第二个坐标，这也就是说，函数在许多点上取同一值. 称一个函数是**一对一**的当且仅当不同的点有不同的象；也就是说假定逆关系 f^{-1} 仍是一个函数.

一个函数是一个集，因此两个函数 f 与 g 恒等当且仅当它们有相同的元，显然这种情况也就是当且仅当 f 的定义域与 g 的定义域是相同的，并且对于在此定义域中的每个 x，$f(x) = g(x)$. 所以我们可以用指定函数的定义域和函数在此定义域每个元上的值来确定一个函数. 如果 f 是在 X 上到 Y 的一个函数，并且 A 是 X 的一个子集，则 $f \cap (A \times Y)$ 也是一个函数. 它称为 f 在 A 上的

限制，并用 $f|_A$ 来表示，$f|_A$ 的定义域为 A，同时对于在 A 中的 x，$(f|_A)(x) = f(x)$．一个函数 g 是 f 在某个子集上的限制当且仅当 g 的定义域是 f 定义域的一个子集，并且对于在 g 定义域中的 x，$g(x) = f(x)$；即当且仅当 $g \subset f$．函数 f 被称为 g 的一个**扩张**当且仅当 $g \subset f$．于是 f 是 g 的一个扩张当且仅当 g 是 f 在 f 的定义域的某个子集上的限制．

如果 A 是一个集且 f 是一个函数，则依照对任意关系给出的定义，有 $f[A] = \{y$：对在 A 中的某个 x，$(x, y) \in f\}$；相当于说 $f[A]$ 等于 $\{y$：对在 A 中的某个 x，$y = f(x)\}$，集 $f[A]$ 称为在 f 的映射下 A 的象．如果 A 与 B 是两个集，则由定理 5，$f[A \cup B] = f[A] \cup f[B]$ 且 $f[A \cap B] \subset f[A] \cap f[B]$，类似的公式对任意交与任意并也成立．一般来说，$f[A \cap B] = f[A] \cap f[B]$ 不成立．因为互不相交的集可能有相交的象．如果 f 是一个函数，则集 $f^{-1}[A]$ 称为在 f 的映射下 A 的**逆（逆象、反象）**．逆满足下面的代数规则．

7 定理． 如果 f 是一个函数，而 A 与 B 是两个集，则

(a) $f^{-1}[A \sim B] = f^{-1}[A] \sim f^{-1}[B]$；

(b) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$；

(c) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$．

更一般地，如果对于非空指标集 C 的每一个元 c，有一个集 X_c 存在，则

(d) $f^{-1}[\bigcup\{X_c : c \in C\}] = \bigcup\{f^{-1}[X_c] : c \in C\}$；

(e) $f^{-1}[\bigcap\{X_c : c \in C\}] = \bigcap\{f^{-1}[X_c] : c \in C\}$．

证明．将仅证明 (e)．

点 x 为 $f^{-1}[\bigcap\{X_c : c \in C\}]$ 之元的充要条件是 $f(x)$ 属于此交，而这种情况当且仅当对于在 C 中的 c，$f(x) \in X_c$．但后面的条件等价于对每个 C 中的 c，$x \in f^{-1}[X_c]$；也就是说，$x \in \bigcap\{f^{-1}[X_c] : c \in C\}$．|

上面的定理经常扼要地说成：一个函数的逆保持相对余集、并以及交．应当注意这些公式的正确性，并不依赖于集 A 与 B 为

f 值域的子集．当然 $f^{-1}[A]$ 同 A 与 f 值域交的逆象是相同的，然而这里将记号(相应的对于 f 映射下象的记号)限制在值域(分别地，定义域)的子集上是不方便的．

两个函数的合成经简单的论证可知它仍是一个函数．如果 f 是一个函数，则 $f^{-1}\circ f$ 是一个等价关系，因为当且仅当 $f(x)=f(y)$ 时，$(x, y)\in f^{-1}\circ f$．合成 $f\circ f^{-1}$ 是一个函数；它是在 f 的值域上的一个恒等关系．

8 注记．关于函数 f 在 x 点的值还有另外一些记号．除了 $f(x)$ 与 f_x 之外，下面的记号：(f, x)、(x, f)、fx、xf 以及 $\cdot fx$ 均有人采用．头两个记号在处理具有某些对偶性时是极其方便的．在那里考虑一个函数族 F，每个的定义域都在指定的 X 上，并且以对称的形式来看待 F 与 X 有预期的好处．fx 与 xf 这两个记号显然是我们已采用记号的缩写；至于说"f"是写在"x"的左边还是右边，显然是一个爱好的问题．这两个记号也有记号"$f(x)$"所具有的不便之处．在某些相当复杂的情况下，除非加上大量括号，记号的含意是不够明确的．最后的记号(已为 A. P. Morse 所采用)免除了这种困难，它含义明确同时不需要括号(参看关于并与交的注记 4)．

对于一个函数，约束变量的记号是需要的，例如，定义域是所有实数的集，而在 x 点取值为 x^2 的函数应该有一个较简单的记述方法．可以从这种特殊的情况出发，把 x 视为实数集上的恒等函数，那末在此情况下 x^2 便有理由为一平方函数．这种经典的手法是把函数与它在 x 的值都用 x^2 来表示．一种消除混淆的建议是用 $x\to x^2$ 来记平方函数，现在建议的这种记号正在逐渐地被广泛采用．自然它也不是万能的．比如记述 $(x\to x^2)(t)=t^2$ 就需要加以解释．最后应该注意到虽然箭头这种记号无疑将要作为标准形式而加以使用．但是 A. Church 的 λ-规定仍具有技术上的方便．(平方函数可写成 $\lambda x: x^2$．) 为了消除混淆，不加括号是必要的．

序

一个**序**(**半序**、**拟序**)是一个传递关系．一个关系$<$**序化**(**半序化**)一个集X当且仅当它在X上是传递的．如果$<$是一个序并且$x<y$，则通常称x**在**y**之前**或者x**小于**y(关于序$<$)并称y**在**x**之后**或者y**大于**x．如果A包含在一个被$<$序化的集X中，则X中的一个元素x叫A的一个**上界**当且仅当对于在A中的每个y不是$y<x$便是$y=x$．类似地，如果x小于或等于A中的每个元，则元素x称为A的一个**下界**．自然一个集可能有许多不同的上界．一个元素x称为A的**最小上界**或者**上确界**当且仅当它是一个上界并且它小于或等于所有其它上界．(换言之，上确界是一个上界，同时又是所有上界集的下界．)以同样的方法，**最大下界**或者**下确界**是一个元素，此元素是一个下界并且大于或等于所有其它下界．称一个集X是**有序完备**的(关于序$<$)当且仅当X的每个有上界的非空子集具有上确界．这个关于上界的条件完全等价于对于下界的相应论述稍稍有点使人感到奇怪．也就是说：

9 定理．一个集X关于一个序是有序-完备的，当且仅当它的每个有下界之非空子集具有下确界．

证明．假定X是有序-完备的，而且A是它的一个有下界的非空子集．令B为A的所有下界的集，于是B非空，并且确保了非空集A的每个元是B的上界．故B有一个最小上界，设为b，从而b小于或等于B的每个上界，特别是b小于等于A的每个元，所以b是A的下界．另一方面，b自身是B的一个上界；也就是说b大于等于A的每个下界．故b是A的最大下界．

逆命题也可以用类似的论述来证明，或者可以直接把刚证明的结果应用到$<$的逆关系上．|

应该注意到序的定义的条件是很弱的．例如$X\times X$是X的一个序，但它没什么意思．关于这个序X的每个元是所有子集的上界，事实上是一个上确界．比较有意义的序需要满足附加的条

件：如果 x 小于等于 y，同时 y 又小于等于 x，则 $x = y$. 在这种情况下，对于一个集至多存在一个上确界与一个下确界。

线性序（**完全**、或者**简单序**）是一个序，它使得：

(a) 如果 $x < y$ 且 $y < x$，则 $x = y$.

(b) 只要 x 与 y 都是 $<$ 的定义域和值域的并的不同元，则不是 $x < y$ 便是 $y < x$.

应注意到一个线性序勿需是自反的. 但是我们约定 $x \leq y$ 当且仅当 $x < y$ 或者 $x = y$. 如果 $<$ 是一个线性序，则关系 $\leq$ 恒为一个自反的线性序. 下面称一个关系**线性序化**一个集 X 当且仅当这个关系在 X 上的限制是一个线性序. 一个集具有一个将它线性序化的关系称为一个**链**. 显然，上确界与下确界在链中都是唯一的. 尽管有许多讨论应用到较少限制的序上显然成立，但本节剩下的定理仍侧重于链.

在一个集 X 上到一个集 Y 的函数 f 称为关于 X 内的序 $<$ 与 Y 内的序 $\prec$ 是**保序的**（**单调的**）当且仅当只要 u 与 v 均为在 X 中使得 $u \leq v$ 的点，便有 $f(u) \prec f(v)$ 或 $f(u) = f(v)$. 如果 Y 的序 $\prec$ 是简单的 $Y \times Y$，或者如果 X 的序是空关系，则 f 必然是保序的. 于是不能期望一一对应的保序函数之逆恒为保序的. 然而如果 X 与 Y 是两个链，并且 f 是一对一的同时单调的，则 f^{-1} 必然是保序的. 因为如果 $f(u) \prec f(v)$ 且 $f(u) \neq f(v)$，但由保序的性质 $v < u$ 是不可能的.

有序完备链具有一个很特别的性质. 假定 X 与 Y 是两个链，X_0 是 X 的一个子集，并且 f 是在 X_0 上到 Y 的一个保序函数. 问：是否存在一个 f 的保序扩张，它的定义域为 X? 除非对 f 加上某些限制，不然回答是"否定的". 因为如果 X 是所有正实数的集，X_0 由所有小于 1 的数所组成的子集，$Y = X_0$ 且 f 是恒等映射，于是容易看出不存在保序扩张.（假定有一个保序扩张 $\bar{f}$，那末 $\bar{f}(1)$ 等于什么呢?）但是此例表明了困难的实质，因为 X_0 是 X 的一个子集，它有上界，而 $f[X_0]$ 却没有上界. 如果有一个保序扩张 $\bar{f}$ 存在，则对于集 A 的一个上界在 $\bar{f}$ 映射下的象必为 $f[A]$ 的一个上

界. 类似地论述对于下界也成立，同时推出如果 X_0 的一个子集 A 在 X 中是**序-有界的**(也就是说它在 X 中既有一个上界又有一个下界)，则象 $f[A]$ 在 Y 中是序-有界的. 下面的定理断言此条件关于一个保序扩张的存在性也是充分的.

10 定理. 设 f 为在链 X 的一个子集 X_0 上到一个有序完备链 Y 的一个保序函数，则 f 具有一个定义域为 X 的保序扩张当且仅当 f 把序-有界集变成序-有界集. (更确切地说，这个条件是如果 X_0 的一个子集 A 在 X 中是序-有界的，则 $f[A]$ 在 Y 中是序-有界的.)

证明. 已经看到这个条件对于一个保序扩张的存在性是必要的，剩下的只要证明充分性. 我们必须造一个已知函数 f 的保序扩张. 首先我们注意到如果 A 是在 X 中有下界的 X_0 的一个子集，则 $f[A]$ 有下界. 因为在 A 中取一点 x，集 $\{y:y\in A$ 且 $y\leq x\}$ 是序-有界的，故在 f 的映射下它的象是序-有界的，同时此象的下界也是 $f[A]$ 的下界. 类似地论述可应用到上界. 对于 X 中的每个点 x，令 L_x 是 X_0 中小于等于 x 的所有元之集；也就是说 $L_x=\{y:y\leq x$ 且 $y\in X_0\}$. 如果 L_x 是空集，则 x 是 X_0 的一个下界. 故 $f[X_0]$ 有一个下确界 v，此时我们定义 $\bar{f}(x)$ 等于 v. 如果 L_x 不空，则由于 x 是 L_x 的一个上界，集 $f[L_x]$ 有一个上界，所以有一个上确界，此时我们定义 $\bar{f}(x)=\sup f[L_x]$. 而 $\bar{f}$ 是 f 的保序扩张是很容易直接验证的，在此从略. ∎

在某些情况下一个函数的保序扩张是唯一的. 譬如这种情况将在讨论实数的十进展开式里出现. 在此我们并没有打算获得关于这方面的最佳结果，而仅仅给出了将要用到的关于唯一性的一个简单之充分条件.

11 定理. 设 f 与 g 是两个在链 X 上到链 Y 的保序函数设 X_0 是 X 的一个子集，而在 X_0 上 f 与 g 相同，又设 Y_0 等于 $f[X_0]$. 于是 $f=g$ 的一个充分条件是 Y_0 与每个形为：$\{y:u<y<v, u\neq y$ 且 $y\neq v\}$ 之集相交. 这里 u 与 v 是 Y 中使得 $u<v$ 的元.

证明. 如果 $f\neq g$，则对于 X 中的某个 x，$f(x)\neq g(x)$，我们

不妨假定$f(x) < g(x)$. 在X_0中小于等于x的每个点在f的映射下变成小于等于$f(x)$的点,此因f是保序的. 又大于等于x的每个点在g的映射下变成大于等于$g(x)$的点,此因g是保序的. 这样便推出了X_0中的点不可能映入集$\{y: f(x) < y < g(x), f(x) \neq y$ 且 $y \neq g(x)\}$内,从而定理得证. ∎

12 注记. 在一个有序完备链中嵌入一个链,有一种很自然的方法,而它是 Dedekind 的由有理数去构造实数的方法之抽象. 这种方法也能应用到较少限制的序上. 正如 H. M. MacNeille 所指出的那样(参看 Birkhoff [1;58]). 这种想法对拓扑空间紧扩张方法(第五章)有巨大启发.

代 数 概 念

在本节里选取了初等代数中已有的少数定义. 而这些概念的绝大多数都是用于问题中. 在需要扼要讲述一些概念时,这些标准术语看起来是很有价值的.

一个群是一个序偶$(G, \cdot)$,它使得G为一个非空集,而被称为群运算的·是在$G \times G$上到G的一个函数,它使得:

(a) 运算是结合的,即对于G中的所有元x, y与z, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(b) 存在一个中性元,或者恒等元e,它使得对于G中的每个x有 $e \cdot x = x \cdot e = x$;

(c) 对于G中的每个x存在一个逆元素x^{-1},它使得$x \cdot x^{-1} = x^{-1} \cdot x = e$.

如果群运算用+来表示,x的逆元通常记作$-x$. 下面通常习惯把函数·在(x, y)的值记作$x \cdot y$以代替常用的函数符号·(x, y).同时一旦在看起来不会引起混淆时,记号·完全可以省去,而群运算就用并列来表示,有时我们(不严格地)说G是一个群. 如果A与B是G的两个子集,则$A \cdot B$或者简单地记成AB是对于在A中的某个x与B中的某个y,所有形为$x \cdot y$的元之集. 集

$\{x\}\cdot A$ 也用 $x\cdot A$ 来表示，或者简单地记成 xA. 关于运算在右边情况类似. 一个群 G 称为 **Abel 群**或者**交换群**当且仅当对于 G 中的所有元素 x 与 y, $x\cdot y=y\cdot x$. 群 H 称为 G 的一个子群当且仅当 $H\subset G$ 且 H 的群运算是 G 的群运算在 $H\times H$ 上的限制. 一个子群 H 称为**正规群**（**不变群**）当且仅当对于 G 中的每个 x, $x\cdot H=H\cdot x$. 如果 H 是 G 的一个子群，H 的**左陪集**是对于 G 中的某个 x 形为 $x\cdot H$ 的一个子集. 所有左陪集的族用 G/H 来表示. 如果 H 是正规的且 A 与 B 属于 G/H，则 $A\cdot B$ 也是一个元，再加之，由这种群运算的定义得知 G/H 是一个群，我们称它为**商群**或者**因子群**. 一个在群 G 上到群 H 的函数 f 是**同态**或者**表示**当且仅当对于 G 中的所有元 x 与 y, $f(x\cdot y)=f(x)\cdot f(y)$. 函数 f 的**核**是集 $f^{-1}[\{e\}]$; 它永远是一个不变子群. 如果 H 是 G 的一个不变子群，则在 x 的值为 $x\cdot H$ 的函数是同态，通常称为 G 到 G/H 上的**射影**、或者**商映射**.

环是一个三元组 $(R,+,\cdot)$，它使得 $(R,+)$ 是一个 Abe 群，同时 $\cdot$ 是一个在 $R\times R$ 上到 R 的函数，此函数使得：这个运算是结合的，并且对于 R 中的所有元 x、y、u 与 v 分配律 $(u+v)\cdot(x+y)=u\cdot x+u\cdot y+v\cdot x+v\cdot y$ 成立[1]. 一个**子环**是一个子集，它在环运算的限制下是一个环，而且一个环**同态**或者**表示**是在一个环上到另一环的函数使得对于定义域中的所有元 x 与 y, $f(x+y)=f(x)+f(y)$ 和 $f(x\cdot y)=f(x)\cdot f(y)$ 成立. 环 R 的一个加法子群 I 称为一个**左理想**当且仅当对于 R 中的每个 x,

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1) D. W. Jonah 于1959年指出（见 *Amer. Math. Monthly*, **66** (1959) p. 38）此处给环下的定义与通常关于环的定义是不等价的. 本书的条件严格弱于通常环的定义的条件. 所谓通常的环的定义是：一个环是一个三元组 $(R,+,\cdot)$ 使得 $(R,+)$ 是一个交换群且 $\cdot$ 是 $R\times R$ 到 R 内的一个函数，使得对于任意的 $a,b,c\in R$, 满足

$$(a\cdot b)\cdot c=a\cdot(b\cdot c);\qquad a\cdot(b+c)=a\cdot b+a\cdot c;$$
$$(b+c)\cdot a=b\cdot a+c\cdot a.$$

我们还可以证明：$(R,+,\cdot)$ 是一个环（通常的）当且仅当 $(R,+,\cdot)$ 是按 Kelley 意义下的环且 $0\cdot 0=0$，这里 0 表加群 $(R,+)$ 的恒等元，这个证明留给读者. ——校者注

$xI \subset I$. I 称为一个**双边理想**当且仅当对于 R 中的每个 x，$xI \subset I$ 同时 $Ix \subset I$. 如果 I 是双边理想，R/I 按照正常加法与乘法构成一个环，并且 R 到 R/I 的射影是环同态. 一个**域**是一个环$(F, +, \cdot)$，它使得 F 至少有两个元且 $(F \sim \{0\}, \cdot)$ 是一个交换群. 这里 0 是关于 $+$ 的中性元. 在此运算 $+$ 是**加法**运算，$\cdot$ 是**乘法**，同时关于乘法的中性元是单位元. 当结果不至于产生混淆时，乘法在习惯上用并列来代替 $\cdot$. 暂且不谈这些运算，我们称"F 是一域". 在域 F 上（空间的**纯量域**）的**线性空间**或**向量空间**是一个四元组 $(X, \oplus, \cdot, F)$，它使得 $(X, \oplus)$ 是一个 Abel 群，并且 $\cdot$ 是在 $F \times X$ 上到 X 的函数，它使得对于 X 中的所有元 x 与 y 以及 F 中的所有元 a 与 b，$a \cdot (b \cdot x) = (a \cdot b) \cdot x$，$(a+b) \cdot x = a \cdot x \oplus b \cdot x$，$a \cdot (x \oplus y) = a \cdot x \oplus b \cdot y$，且 $1 \cdot x = x$. 一个**实线性空间**是在实数域上的一个线性空间. 线性空间的概念尚能叙述成稍许不同的形式. 一个 Abel 群到它自身的一切同态组成的族，以点式相加的加法和以函数的合成为乘法，构成一个环，叫这个群的**自同态环**. 在域 F 上的线性空间是一个四元组 $(X, \oplus, \cdot, F)$，它使得 $(X, \oplus)$ 是一个 Abel 群并且 $\cdot$ 是 F 映到 $(X, \oplus)$ 的自同态环内的一个环同态，并把单位元映为恒等同态[1].

线性空间 $(Y, \oplus, \odot, F)$ 是线性空间 $(X, +, \cdot, F)$ 的一个子空间当且仅当 $Y \subset X$，同时运算 $+$ 与 $\cdot$ 以及 $\oplus$ 与 $\odot$，在后者定义的集上相一致. 如果加法与纯量乘法按惯用的方法定义，那末 X 模子空间 Y 的陪集族 X/Y 能构成线性空间. 于是 X 到 X/Y 上的射影 f 对于 F 的所有元 a 与 b 和 X 中的所有 x 与 y 具有性质：$f(ax + by) = a \cdot f(x) + b \cdot f(y)$. 这样的函数称为**线性函数**. 如果 f

1) 这一句中说运算 $\cdot$ 是一环同态不妥. 这句话改成下列的容易直接证明的命题较为确切.

命题. 设 F 是一个域，1 表其乘法单位元（可逆元），R 是交换群 $(X, \oplus)$ 的自同态环. 设两个函数 $\varphi: F \to R$ 和 $\odot$: $F \times X \to X$ 间有关系：

$$\varphi(a)(x) = a \odot x, \quad a \in F, x \in X,$$

则 $(X, \oplus, \odot)$ 是域 F 上的一个线性空间当且仅当 φ 是一个环同态且 $\varphi(1)$ 是 R 内的恒同(同态). ——校者注

是线性函数，集 $f^{-1}[0]$ 称为 f 的**零空间**；线性函数的零空间是定义域的线性子空间（假定加法与纯量乘法的运算确实已下了定义）.

假定 f 是在 X 上到 Y 的线性函数，而 g 是 X 到 Z 上的线性映射，并且它使得 f 的零空间包含 g 的零空间．于是存在唯一的在 Z 上到 Y 的线性函数 h 使得 $f = h \cdot g$($h(z)$很明显是 $f \cdot g^{-1}[z]$ 的唯一元)．（函数 h 称为由 f 与 g **导出的**.）这个事实的一个特殊推论是：每个线性函数可以写成到一个商空间内的一个射影随后作用于一个一对一的线性函数.

实　　数

本节的主要精力放在与实数有关的少数最重要结果的证明上.

一个**有序域**是一个域 F 和一个被我们称为**正元素**的子集 P，使得：

(a) 如果 x 与 y 均为 P 的元，则 $x+y$ 与 $x \cdot y$ 仍为 P 的元；

(b) 如果 x 是 F 的一个元，则下述论断恰好有一个成立：$x \in P$，$-x \in P$，或者 $x = 0$.

定义 $x < y$ 当且仅当 $y - x \in P$．容易验证 $<$ 是 F 的一个线性序．通常关于对不等式相加与相乘的简单命题是成立的．F 中使得 $-x \in P$ 的那些元 x 称为**负的**.

实数被假定成为一个有序域，而它是有序完备的即每个有上界的非空子集有最小上界或者上确界．由定理 9 最后这个要求完全等价于：每个有下界的非空子集有最大下界或者下确界.

我们首先证明一些关于**整数**的命题．一个**归纳集**是一个实数集 A，而它使得 $0 \in A$，同时只要 $x \in A$，则 $x+1 \in A$．称一个实数 x 是**非负整数**当且仅当 x 属于每一个归纳集．换句话说，非负整数集 ω 被定义为所有归纳集族的元之交．ω 的每个元实际上是非负的，因为所有非负数的集是归纳的．显然 ω 自身是一个归纳

集，并且是其它归纳集的子集．于是推出(**数学归纳原理**) ω 的每个归纳子集恒等于 ω 凡依据这个原理的证明均称为**归纳法证明**．我们证明下面这个小定理作为一个例子：如果 p 与 q 均为非负整数且 $p<q$，则 $q-p\in\omega$．首先注意由 0 和所有形为 $p+1$ 的数(此处 p 属于 ω)所组成的集是归纳的．所以 ω 的每个非 0 元形为 $p+1$．其次令 A 为所有非负整数 p 使得对于 ω 中的每个较大元 q，$q-p\in\omega$ 之集．故必然有 $0\in A$．如果我们假定 p 是 A 的一个元且 q 是 ω 中任意一个大于 $p+1$ 的元，于是 $p<q-1$，所以 $q-1-p\in\omega$，这是由于 $p\in A$ 和 $q-1\in\omega$ 的缘故．从而 $p+1\in A$，故 A 是一个归纳集，所以 $A=\omega$．同样简单地可以证明 ω 的两个元之和仍为 ω 的一个元，于是推得集 $\{x:x\in\omega$ 或 $-x\in\omega\}$ 是一个群，它是**整数群**．

数学归纳原理的另一种形式常常是很方便的，即：ω 的每个非空子集 A 有一个最小元．欲证此命题我们考虑 ω 中 A 之下界的所有元之集 B；即 $B=\{p:p\in\omega$ 且对于 A 中的一切 q，$p\leqq q\}$．集 B 不是归纳的，因为如果 $q\in A$，则 $q+1\notin B$．由于 $0\in B$ 推得 B 中存在一个元 p 使得 $p+1\notin B$．如果 $p\in A$，则显然 p 是 A 的最小元；否则 A 中存在一个元 q 使得 $p<q<p+1$．于是 $q-p$ 是 ω 的非 0 元，而 $q-p-1$ 是 ω 的负元，这是不可能的．

在下面的意义下，**用归纳法定义一个函数**是可能的．对于每个非负整数 p，令 $\omega_p=\{q:q\in\omega$ 且 $q\leqq p\}$．假定我们欲求 ω 上的一个函数，已知它在 0 之函数值为 a，并且对于在集 ω_p 上的每个函数 g 给定 $F(g)$，而它等于欲求函数在 $p+1$ 的值．于是在 $p+1$ 上欲求的函数值可能依赖所有小于 $p+1$ 的整数值．在这种情况下，ω 上存在唯一的函数 f 使得 $f(0)=a$，并且对于 ω 中的每个 p，$f(p+1)=F(f|\omega_p)$ 成立．(函数 $f|\omega_p$ 是函数 f 在集 ω_p 上的限制．)这个命题经常被认为是显然的．可是它的证明并不十分简单．

13 定理．假设已知 a，并且对于在 ω 中的某个 p 只要 g 是以 ω_p 为定义域的函数时 $F(g)$ 已知，则存在唯一的函数 f 使得 $f(0)=$

a，并且对于 ω 中的每个 p，$f(p+1)=F(f|\omega_p)$.

证明．令 $\mathscr{F}$ 为所有函数 g 的族，此处 g 的定义域是 ω_p，$p\in\omega$，$g(0)=a$ 且对于 ω 中每个合于 $q\leqq p-1$ 的元 q，$g(q+1)=F(g|\omega_q)$.（直观上，$\mathscr{F}$ 的元是欲求函数开始的一段.）族 $\mathscr{F}$ 有很重要的性质：如果 g 与 h 都是 $\mathscr{F}$ 的元，则不是 $g\subset h$ 便是 $h\subset g$. 欲证这一点必须证明对于同时属于这两个函数定义域的每个 q，$g(q)=h(q)$. 如若不然，令 q 为使得 $g(q)\neq h(q)$ 的最小整数，因为 $g(0)=h(0)=a$，所以 $q\neq 0$，故 $g(q)=F(g|\omega_{q-1})$. 由于对于所有小于 q 的值 g 与 h 都相同，且 $F(h|\omega_{q-1})=h(q)$. 于是得一矛盾.

现令 $f=\bigcup\{g:g\in\mathscr{F}\}$，则 f 的元无疑是序偶. 如果 $(x,y)\in g\in\mathscr{F}$ 且 $(x,z)\in h\in\mathscr{F}$，则 (x,y) 与 (x,z) 同时属于 g 或者同时属于 h，故 $y=z$，从而 f 是一个函数. 但是还必须证明它是欲求的函数. 首先因为 $\{(0,a)\}\in\mathscr{F}$，$f(0)=a$. 其次如果 $q+1$ 属于 f 的定义域，那末对于 $\mathscr{F}$ 中某个 g，$q+1$ 是 g 的定义域中的一个元，故 $f(q+1)=g(q+1)=F(g|\omega_q)=F(f|\omega_q)$. 最后来证明 f 的定义域为 ω，假定 q 是 ω 中不属 f 定义域之首元，则 $q-1$ 是 f 定义域之末元，并且 $f\cup\{(q,F(f))\}$ 是 $\mathscr{F}$ 的元. 所以 q 属于 f 的定义域. 于是得一矛盾. ∎

上面的定理能够系统地用于证明实数的基本性质. 例如，假设 b 是一个正数且 p 是一个整数，b^p 定义如下：在上面的定理中，令 $a=1$ 且对于每个以 ω_p 为定义域的函数 g，令 $F(g)=bg(p)$，于是 $f(0)=1$ 且对于 ω 中的每个 p，$f(p+1)=bf(p)$. 假定 f 是这样的函数，它的存在性已为上面的定理所保证. 现令 $b^p=f(p)$，于是推得 $b^0=1$ 和 $b^{p+1}=bb^p$，但据此能够用归纳法证明对于 ω 中的所有元 p 与 q，$b^{p+q}=b^p\cdot b^q$. 如果对于每个非负整数 p，b^{-p} 定义为 $1/b^p$，那末常用的初等方法可以证明对一切整数 p 与 q，$b^{p+q}=b^p\cdot b^q$.

至此在实数的这种讨论中，我们尚未用到实数域是有序完备的这个事实. 现在我们证明一个简单的、但又是非常值得注意的

关于有序完备性的推论．首先非负整数集 ω 没有上界．因为如果 x 是 ω 的最小上界，那末 $x-1$ 将不是上界．于是对于 ω 中的某个 p 有 $x-1<p$．但这样一来 $x<p+1$ 便与 x 曾被假定为上界的事实相矛盾．结果，如果 x 是一个正实数且 y 是一个实数，则对于某个正整数 p，$px>y$．此因 ω 中存在一个大于 y/x 的元 p．对此命题成立的有序域称为具有 **Archimede 序**．

我们尚需用到每个非负实数有一个 b-进展式这件事．在此 b 是任意一个大于 1 的整数．粗略地说，我们是想把数 x 写成 b 的幂再乘上一个倍数的这种项的和，而此倍数(数值)是一个小于 b 的非负整数．自然一个数的 b-进展式可能不唯一．譬如，在十进展开式中.9999…(所有为 9)与 1.0000…(所有为 0) 均为同一个实数的展式．实际上，这种展式的本身是一个函数，它把每一个整数映成 0 与 $b-1$ 之间的一个整数，并使得有第一个非零数字存在(由于小数点以前只需要有限个非零整数)．形式地说，称 a 是一个 **b-进展式**当且仅当 a 是一个在整数集上到 $\omega_{b-1}(=\{q:q\in\omega$ 且 $q\leq b-1\})$ 的函数，并使得有一个最小整数 p 存在，对于它 $a_p(=a(p))$ 不为零．一个 b-进展式称为**有理的**当且仅当存在一个最后的非零数字(也就是说，对于某一整数 p，只要 $q>p$，则 $a_q=0$)．对于每个有理 b-进展式有一种简单方法，使它对应于一个实数 $r(a)$．除了对于有限个整数 p 外，数 a_pb^{-p} 为零，对在此有限集中的 p，a_pb^{-p} 之和等于实数 $r(a)$．我们记 $r(a)=\sum\{a_pb^{-p}:p$ 为整数$\}$．具有这种形式的实数是 **b-进有理的**，而这些数对于整数 p 与 q 恰好都是 qb^{-p} 这种形式．令 E 为所有 b-进展式之集，则 E 按字典序是线性有序的；详细的来说，按**字典序**一个 b-进展式 a 在一个 b-进展式 c 之前是指对于使得 $a_p\neq c_p$ 的最小整数 p，$a_p<c_p$ 成立．很容易看出，好象一个字典，实际上 E 按 $<$ 是线性有序的．对应的 r 是保序的，而这也正是下面命题的关键．

14 定理． 设 E 为 b-进展式之集．设 R 为有理展式之集．并且对于在 R 中的 a，令 $r(a)=\sum\{a_pb^{-p}:p$ 为整数$\}$，则 r 存在唯一的保序扩张 $\bar{r}$，它的定义域为 E，并且 $\bar{r}$ 以一一对应的方式把 E

$\sim R$映射到正实数集上.

证明. 按照定理 10, r 有一个保序扩张 $\bar{r}$ 当且仅当 r 把 R 的在 E 中序-有界集的每个子集变成实数的一个序-有界子集. 但是对于在 E 中的每个 a, 在 R 中显然存在 b 使得 $b > a$, 从而推出: 如果 R 的一个子集 A 以 a 为上界, 则 $r(b)$ 为 $r[A]$ 的上界. 类似地论述可以应用到下界,于是 r 把序-有界集变成序-有界集. 因此它有一个保序扩张 $\bar{r}$, 其定义域为 E.

欲证这种扩张是唯一的. 由定理11只要证明对于非负实数 x 与 y, 如果 $x<y$,则在 R 中存在 a 使得 $x<r(a)<y$ 就足够了. 因为对于每个非负整数 p, $b^p > p$ (这一点很容易用归纳法证明). 又因为非负整数集是无界的, 所以存在一个整数 p 使得 $b^p > 1/(y-x)$. 于是 $b^{-p} < (y-x)$. 因为此序是 Archimede 序,所以存在一个整数 q 使得 $qb^{-p} \geqq y$. 又由于这样的整数 q 存在一个最小的,所以可以假定 $(q-1)b^{-p} < y$. 由此推出$(q-1)b^{-p} > x$,此因 b^{-p} 小于$(y-x)$,这证明了存在一个 b-进有理数$(q-1)b^{-p}$, 它是 R 中的一个元之象, 并居于 x 与 y 之间. 从而对应 $\bar{r}$ 是唯一的.

其次我们证明此对应 $\bar{r}$ 在 $E\sim R$ 上是一对一的. 很容易看出 $\bar{r}$ 在 R 上是一对一的. 所以下面把这个事实作为假定. 设 $a\in E$, $c\in E\sim R$ 且 $a<c$, 则对于第一个使得 a_p 与 c_p 不相等的 p 值有 $a_p < c_p$. 展式 d, 使得对于 $q<p$, $d_q = a_q$, 对于 $q>p$, $d_q = 0$ 同时 $d_p = a_p + 1$, 是 R 中一个大于 a 的元. 又由于 c 没有一个最后的非零数字, 所以 $a<d<c$. 根据同样的道理在 R 中存在一个元 e 使得 $a<d<e<c$. 于是由 R 上的函数 $\bar{r}$ 是一对一的, 推得 $\bar{r}(a)\leqq\bar{r}(d)<\bar{r}(e)\leqq\bar{r}(c)$. 所以 $\bar{r}$ 在 $E\sim R$ 上是一对一的.

最后还必须证明在 $\bar{r}$ 的映射下 $E\sim R$ 的像是所有正数之集. 首先注意对于 R 的合于 $c<d$ 之元偶 c 与 d, 在 $E\sim R$ 中存在 a 使得 $c<a<d$. 从而对合于 $x<y$ 之正实数 x 与 y, 在 $E\sim R$ 中存在 a 使得 $x<\bar{r}(a)<y$. 现在假设 x 是一个正实数,而它不是

$E\sim R$内元的像．令 $F=\{a:a\in E\sim R$ 且 $\bar{r}(a)<x\}$．如果集F有上确界 c，假定 $\bar{r}(c)<x$，则没有$E\sim R$中点映入区间（$\bar{r}(c)$, x），又若$\bar{r}(c)>x$，则（$\bar{r}$ 保序）没有$E\sim R$的点映入区间（x，$\bar{r}(c)$）．但是无论哪一种情况均得一矛盾之结果．所以说如果证明了在$E\sim R$中的每个有上界的非空子集必有上确界：即$E\sim R$是有序完备的，则本定理得证．于是假定F是$E\sim R$中具有上界的一个非空子集，则存在一个最小的整数p使得对于在F中的a，$a_p\neq 0$．对于$q<p$定义c_q为零．令F_p为F中具有非零第p个数值 a_p 的所有元a之集．同时令 $c_p=\max\{a_p:a\in F_p\}$．用归纳法继续令F_{p+1}为 F_p中对于$q=p$使得$a_q=c_q$之所有元a的集．同时令 $c_{p+1}=\max\{a_{p+1}:a\in F_{p+1}\}$．所有集 F_p 中不可能有任何一个是空集．并且不难看出用这种构造法所得到的展式c是F的一个上界．实际上是一个上确界，并且 $c\in E\sim R$．|

在上面的定理用于b等于 2、3 和 10 时，对应的b–进展式分别称为**二进**，**三进**和**十进**的．

可　数　集

一个集称为有限的当且仅当它能与某个形如$\{p:p\in\omega$ 且 $p\leq q\}$ 的集一一对应，此处q是ω的某个元．集A是**可数无穷的**是指它能与非负整数集ω构成一一对应；也就是说A是在ω上的某个一对一函数的值域．一个集是**可数的**是指它不是有限便是可数无穷的．

15 定理． 一个可数集的子集是可数的．

证明．假设A是可数的，f是在ω上以A为值域的一对一函数且 $B\subset A$．于是f在 $f^{-1}[B]$ 上的限制是在ω的子集上以B为值域的一对一函数，如果能够证明 $f^{-1}[B]$ 是可数的，则到B上的一对一函数能够用合成法造出．所以本证明化为证明ω的任意子集C是可数的．令$g(0)$为C的首元，并且对ω中的p继续归纳地令$g(p)$为C中不同于 $g(0), g(1), \cdots, g(p-1)$ 的首元．如果

这种选法对某个 p 是不可能的，则 g 为在 $\{q: q\in\omega$ 且 $q<p\}$ 上以 C 为值域的一个函数，并且 C 是有限的. 否则（利用定理 13 关于用归纳法造函数的办法）在 ω 上存在一个函数 g 使得对于 ω 中的每个 p，$g(p)$ 是 C 中不同于 $g(0)$、$g(1)$、$\cdots g(p-1)$ 的首元. 显然 g 为一对一的. 用归纳法容易验证对于所有的 p，$g(p)\geqq p$. 故由 $g(p+1)$ 的选法推得 C 的每个元 p 等于数 $g(q)$ 之一，这里 $q\leqq p$. 因此 g 的值域为 C. ▎

16 定理. 如果一个函数的定义域是可数的，则其值域也是可数的.

证明. 只要证明如果 A 是 ω 的子集且 f 是 A 上到 B 上的函数，则 B 是可数的就够了. 令 C 为 A 中使得如果 $y\in A$ 和 $y<x$，则 $f(x)\neq f(y)$ 的所有元 x 的集；也就是说 C 是每个集 $f^{-1}[y]$ 之最小元所组成. 于是 $f|C$ 在一对一的形式下映 C 到 B 上. 因此由定理 15 知 C 为可数的，所以 B 也可数. ▎

17 定理. 如果 $\mathscr{A}$ 是可数集的可数族，则 $\bigcup\{A: A\in\mathscr{A}\}$ 是可数的.

证明. 因为 $\mathscr{A}$ 是可数的，存在一个函数 F 它的定义域是 ω 的一个子集，而它的值域是 $\mathscr{A}$. 由于对 ω 中每一 p，$F(p)$ 是可数的，所以在 $\{p\}\times\omega$ 的子集上有一个函数 G_p，它的值域为 $F(p)$. 从而在 $\omega\times\omega$ 的子集上存在一个函数（函数 G_p 的并），它的值域为 $\bigcup\{A: A\in\mathscr{A}\}$. 于是问题化为证明 $\omega\times\omega$ 是可数的. 这个证明的关键是要注意：如果我们认为 $\omega\times\omega$ 位于平面的右上部，则由左上到右下穿过的对角线仅仅包含 $\omega\times\omega$ 的有限个元. 为明显计，对于在 ω 中的 n，令 $B_n=\{(p,q):(p,q)\in\omega\times\omega$ 且 $p+q=n\}$，则 B_n 恰含 $n+1$ 个点，并且并 $\bigcup\{B_n; n\in\omega\}$ 等于 $\omega\times\omega$. 在 ω 上以 $\omega\times\omega$ 为值域的函数可以通过首先选 B_0 的元，然后再选 B_1 的元，以此类推来构造. 这样一个函数的明显定义留给读者补充. ▎

集 X 的一个子集 A 的**特征函数**是这样的一个函数 f: 对于 $X\sim A$ 中的 x，$f(x)=0$；而对于 A 中的 x，$f(x)=1$. 在集 X 上一个函数 f 假定只取值零与 1 称为特征函数；它显然是 $f^{-1}[1]$ 的特征

函数．处处为零的函数是空集的特征函数，而在 X 上恒等于 1 的函数是 X 的特征函数．两个集有相同的特征函数当且仅当它们相等．所以在 X 上的所有特征函数的族与 X 的所有子集之族之间存在一一对应．

如果 ω 是非负整数集，所有在 ω 上的特征函数族能与当 $p>0$ 时 $a_p=0$ 的所有这种二进展式 a 之集 F 构成一一对应．ω 的所有有限子集的族与由有理二进展式所组成的 F 之子族 G 一一对应．现在我们用经典的 Cantor 方法来证明 F 是不可数的．

18 定理．一个可数无限集的所有有限子集的族是可数的，但是所有子集的族却不然．

证明．由于定理前面注记只需证明对负的 p 具有 $a_p=0$ 的所有二进展式 a 之集 F 是不可数的，同时由有理展式组成的 F 之子集 G 是可数的就足够了．假设 f 是在 ω 上以 F 为值域的一对一函数．令 a 是 F 中使得对每个非负整数 p，$a_p=1-f(p)_p$ 之元，即 a 的第 p 个下标等于 1 减去 $f(p)$ 的第 p 个下标．于是 $a\in F$ 且对于在 ω 中的每个 p 显然有 $a\neq f(p)$，因为 a 与 $f(p)$ 的第 p 个下标不同．从而推出 a 不属于 f 的值域．于是得一矛盾．故 F 是不可数的．

剩下的是要证明 G 为可数的．对于在 ω 中的 p 令 $G_p=\{a: a\in G$ 且对于 $q>p,\ a_q=0\}$．于是 G_0 正好包含两个元素，并且 G_{p+1} 的元的个数恰好为 G_p 的两倍．所以推出 G_p 恒有限．故 $G=\bigcup\{G_p: p\in\omega\}$ 是可数的．|

依照定理 14 在 F 和实数的子集之间的自然对应，在 $F\sim G$ 上是一对一的．由于 G 是可数的，所以 $F\sim G$ 必然是不可数的．故有

19 系．所有实数的集是不可数的．

基　　数

关于可数性的许多定理都是关于基数的较一般的定理的特殊

情形．非负整数集 ω 在上面起着特殊的作用，而在更一般的方法里，这一作用可以用被称为基数的集（ω 是其中之一）所代替．我们称两个集 A 与 B **等势**当且仅当存在一个在 A 上且以 B 为值域的一对一函数．由此推出对于每个集 A 存在唯一的基数 C 使得 A 与 C 等势．如果 C 与 D 是两个不相同的基数，则 C 与 D 不等势，但基数之一，譬如说 C，与另一个集的真子集等势．在这种情况下，C 被称为**较小的基数**，同时写成 $C < D$．依照这种序的定义所有基数的族是线性序化的．甚至于，每一非空子族具有最小元(这些事实留在附录中给予证明)．

当知道了前面一段的这些事实后，推得如果 A 与 B 是两个集，则存在一个 A 上到 B 的一个子集的一对一函数，或者反之，这是因为存在两个基数 C 与 D 使得 A 与 C 以及 B 与 D 分别等势．现在假定存在一个 A 上到 B 的一个子集的一对一函数，同时又存在 B 上到 A 的一个子集的一对一函数．则 C 与 D 的一个子集等势，同时 D 与 C 的一个子集等势．又由于基数的序是线性的，从而推得 $C=D$．故 A 与 B 等势．这就是经典的 Schroeder-Bernstein 定理．我们给出这个与基数一般理论无关的定理的直接证明，原因在于这个证明给予了非平凡的附加信息．

20 定理．*如果存在集 A 上到 B 的一个子集的一个一对一函数，同时存在 B 上到 A 的一个子集的一个一对一函数，则 A 与 B 等势．*

证明．假定 f 是 A 到 B 内的一个一对一映射，并且 g 是 B 上到 A 内的一个一对一映射．可以假设 A 与 B 互不相交．这个定理的证明是用把 A 与 B 分解的方法完成的，而它最容易用单性生殖的术语来描述．点 x（属于 A 或者 B）为点 y 的祖先当且仅当 y 能够由 x 依次作用 f 与 g（或者 g 与 f）而得到．现把 A 分成三个集：令 A_E 表 A 中有偶数个祖先的点组成之集，令 A_O 表有奇数个祖先的点组成之集，并令 A_I 表有无限多个祖先的点组成的集．类似地分解 B，同时注意：f 映 A_E 到 B_O 上，A_I 到 B_I 上，又 g^{-1} 映 A_O 到 B_E 上．故在 $A_E \cup A_I$ 上与 f 相一致并在 A_O 上与 g^{-1} 相一致的

函数就是 A 到 B 上的一个一对一映射. ∎

21 注记. 上面的证明并没有用到选择公理，这一点虽然有它的价值，但不很重要. 而其重要的在于要注意到欲求函数是由两个已知函数通过可数过程构造出来. 显然如果 f 是一个在 A 上到 B 的一对一函数，并且 g 是一个在 B 上到 A 的一对一函数，如果 $E_0 = A \sim g[B]$，对于每个 n，$E_{n+1} = g\circ f[E_n]$，同时如果 $E = \bigcup\{E_n: n \in \omega\}$，则在 E 上等于 f 且在 $A \sim E$ 上等于 g^{-1} 的函数 h 是 A 到 B 上的一对一映象. (更确切地说，$h = (f|E) \cup (g^{-1}|A \sim E$.) 这个结果的重要性是基于这样的事实：如果 f 与 g 具有某些有趣的性质(譬如是 Bore 函数)，则 h 保持这些性质.

定理 20 的这一直观而又简练的证明形式是属于 G. Birkhoff 和 S. MacLane 的.

序　　数

除了例子之外，序数在本书中是不需要的. 然而几个最有意义的反例全都建筑在序数的基本性质上，所以看起来在此适当地叙述一点与此有关的事实是需要的. (序数的构造以及这些还有其他性质的证明均留在附录中给出.)

22 综述. 存在一个不可数集 Ω'，它被关系 $<$ 线性序化. 并且合于:

(a) Ω' 的每个非空子集有一个最小元.

(b) Ω' 有一个最大元 Ω.

(c) 如果 $x \in \Omega'$ 且 $x \neq \Omega$，则 Ω' 中的在 x 前面的所有元之集是可数的.

集 Ω' 是所有小于等于 Ω 的序数集，而 Ω 是**第一不可数序数**. 使得每个非空子集有一个最小元的线性有序集是**良序的**. 特别是良序集的每个非空子集具有下确界. 由于 Ω' 的每个子集有上界，即 Ω，于是利用定理 9 推得 Ω' 的每个非空子集有上确界. 关于 Ω' 的很多稀奇事实之一是下面的

23 定理．如果A是Ω'的一个可数子集且$\Omega \notin A$，则A的上确界小于Ω．

证明．假定A是Ω'的一个可数子集且$\Omega \notin A$．对于A中的每个元a集$\{x: x \leqq a\}$是可数的．故所有这种集的并是可数的．这个并等于$\{x:$对于A内的某个$a, x \leqq a\}$．因此这个并的上确界b是A的一个上界．点b相对于这个序仅有可数个在前面的元，故$b \neq \Omega$．于是推得A的上确界小于Ω．|

Ω'内有一个元应给予特别的注意．在Ω'内其前面不止有限多个序数的序数中的最小者称为**第一非有限序数**并记为ω．符号ω已被用来表示非负整数集．由序数的构造可知，第一非有限序数实际上就是非负整数集ω！

笛卡儿乘积

如果A与B是两个集，笛卡儿乘积$A \times B$被定义为所有使得$x \in A$且$y \in B$的序偶(x, y)之集．把笛卡儿乘积的定义推广到集族上，正如把并与交的概念推广到任意的集族上一样是有用的．假定对一个指标集A的每个元a，给定一个集X_a．**集X_a的笛卡儿乘积**，记作$\times\{X_a: a \in A\}$，被定义为在A上且对A中的每个a使得$x(a) \in X_a$的所有函数x的集．习惯上宁可用下标的记号而不愿用函数的记号，所以$\times\{X_a: a \in A\} = \{x: x$是$A$上的一个函数且对于在$A$中的$a, x_a \in X_a\}$．这个定义在一开始可能稍微感到有点奇怪，但实际上它是直观概念的精确叙述：乘积的点x由选自每个集X_a的点（即x_a）所组成．集X_a是**第a个坐标集**，并且点x_a是乘积的点x**的第a个坐标**．函数P_a把乘积的每个点x映射到第a个坐标x_a上，是到第a个坐标集内的**射影**；这也就是说$P_a(x) = x_a$．映射P_a又称为**在a的计值**．

笛卡儿乘积有一个重要的特殊情况．假定对于在指标集A中的每个a坐标集X_a为一固定的集Y，则笛卡儿乘积$\times\{X_a: a \in A\} = \times\{Y: a \in A\} = \{x: x$是在$A$上到$Y$的函数$\}$．于是$\times\{Y: a \in A\}$的

确是在 A 上到 Y 的所有函数的集．有时写成 Y^A．一个熟悉的例子是实数 **n 维欧氏空间**．这是一个在由整数 $0, 1, \cdots, n-1$ 组成的集上的所有实值函数之集，并且元 x 的第 i 个坐标为 x_i．

还有另一个有意义的特殊情况．假设指标集是一个集族 $\mathscr{A}$ 自身，并且对于 $\mathscr{A}$ 中的每个 A，第 A 个坐标集为 A．在这种情况下，笛卡儿乘积 $\times\{A: A \in \mathscr{A}\}$ 是在 $\mathscr{A}$ 上所有函数 x 的族，它使得对于 $\mathscr{A}$ 中的每个 A，$x_A \in A$．作为笛卡儿乘积的元，这些函数有时称为对于 $\mathscr{A}$ 的选择函数．因为直观上函数 x 从每个集 A 中"选择"一个元 x_A．如果空集是 $\mathscr{A}$ 的一个元，则对于 $\mathscr{A}$ 显然不存在选择函数；也就是说这种笛卡儿乘积是空的．如果 $\mathscr{A}$ 的元均不空，但笛卡儿乘积非空却不是十分明显的．事实上，对于这样一个族选择函数的存在性问题推证起来尚颇需一些技巧．下面一节中所研究的几个命题，其每一个都等价于这个问题的正面回答．同时我们把这些命题中最方便的一个作为一个公理来假定．（不同的选择在附录中给出；这与下面一节合起来说明了这些不同的说法是等价的．）此外我们还以难得的自我克制，而不去讨论哲学上的牵连．

Hausdorff 极大原理

设 $\mathscr{A}$ 为一个集族（或者集族的集体）．一个元 A 称为 $\mathscr{A}$ 的**最大元**，如果 A 包含所有其它的元；也就是说如果 A 大于 $\mathscr{A}$ 中所有其它的元．类似地 A 是族中的**最小元**，当且仅当 A 被包含在每个元中．欲知一个族有无最大元或者最小元通常是很重要的．显然当它们存在时，最大与最小元都是唯一的．然而，即使在族 $\mathscr{A}$ 没有最大元的情况下，虽然有一些元既不包含 A 也不被 A 所包含，但可能没有其它元真正包含 A．这样的一个元称为这个族中的极大元．形式地说，A 为 $\mathscr{A}$ 的一个**极大元**当且仅当 $\mathscr{A}$ 中没有元真正包含 A．类似地 A 为 $\mathscr{A}$ 的一个**极小元**当且仅当 $\mathscr{A}$ 中没有元真正被 A 所包含．造一些族，它没有极大元或者它的每个元同时

为极大和极小的例子是很容易的(例如一个互不相交族)。在一般情况下，为了保证极大元的存在必须加上某些特殊的假定.

一个集族$\mathfrak{N}$称为**一个套**(有时称为**塔**或者**链**)当且仅当只要A与B是族中的两个元，则不是$A\subset B$便是$B\subset A$. 而这同说一个族$\mathfrak{N}$被包含关系线性序化，或者用我们的术语来讲族$\mathfrak{N}$按包含关系构成一个链完全是一回事. 如果$\mathfrak{N}\subset \mathscr{A}$且$\mathfrak{N}$是一个套，则称$\mathfrak{N}$是**$\mathscr{A}$中的一个套**.我们知道一个集族可能没有极大元素.现在让我们考虑在一个指定的族$\mathscr{A}$中所有套的集，并问它们之中是否有一个极大套存在；也就是说对于每一族$\mathscr{A}$，在$\mathscr{A}$中是否存在一个套$\mathfrak{M}$在$\mathscr{A}$中没有套真正地包含它？我们假定把下面的论述当作一个公理.

24 Hausdorff 极大原理. 如果$\mathscr{A}$是一个集族且$\mathfrak{N}$为$\mathscr{A}$中的一个套，则在$\mathscr{A}$中存在一个极大套$\mathfrak{M}$包含$\mathfrak{N}$.

下面定理列举了豪斯道夫极大原理的几个重要推论. 在论述这些结果之前，让我们回顾一下通常与此有关的一些术语. 一个集族$\mathscr{A}$称为是**有限特征的**当且仅当$\mathscr{A}$中元的每个有限子集是$\mathscr{A}$的元，同时每个集A，若它的一切有限子集属于$\mathscr{A}$，它本身属于$\mathscr{A}$. 如果$<$是集A的一个序，则被$<$线性序化的子集B称为在A中的一个链. 有序集A的一个**极大元**是属于A中每个可比较元素之后的一个元素x；也就是说如果$y\in A$，则或是y在x之前或者x不在y之前. 关系$<$是集A的一个良序是指$<$为A的一个线性序，并使得A的每个非空子集有首元(此元小于等于每个其他的元). 如果A存在一个良序，则我们称A能良序化.

25 定理.

(a) **极大原理.** 倘若对于集族$\mathscr{A}$中的每个套，$\mathscr{A}$中有一个元包含此套所有元，则集族$\mathscr{A}$存在一个极大元.

(b) **极小原理.** 倘若对于在集族$\mathscr{A}$中的每个套，$\mathscr{A}$中有一个元包含在此套的所有元中，则族$\mathscr{A}$存在一个极小元.

(c) **Tukey 引理.** 每个具有有限特征的集族存在一个极大元.

(d) **Kuratowski 引理.** 有序(半序)集中的每个链被包含在一个极大链中.

(e) **Zorn 引理.** 如果在半序集中的每个链有上界，则此集存在一个极大元素.

(f) **选择公理.** 如果对于指标集 A 的每个元 a，X_a 是非空集，则在 A 上存在一个函数 c 使得对于 A 中的每个 a，$c(a)\in X_a$.

(g) **Zermelo 公设** 如果 $\mathscr{A}$ 是一个互不相交的非空集的族，则存在一个集 C 使得对于 $\mathscr{A}$ 中每个 A，$A\cap C$ 由单点组成.

(h) **良序原理.** 每个集都能良序化.

证明. 我们粗略的说一下每个命题的证明，但很多细节留给读者.

(a) 的证明. 在 $\mathscr{A}$ 中选一个极大套 $\mathfrak{M}$，并令 A 为 $\mathscr{A}$ 中一个包含 $\bigcup\{M: M\in\mathfrak{M}\}$ 的元，则 A 是 $\mathscr{A}$ 的一个极大元. 因为如果 A 真正被包含在 $\mathscr{A}$ 的一个元 B 内，则 $\mathfrak{M}\cup\{B\}$ 是 $\mathscr{A}$ 中的一个套，它真正包含 $\mathfrak{M}$. 从而得到一个矛盾.

(b) 的证明. 看起来上面那种证法是显然能行得通的. 不管怎么样，(b) 的证明可借助令 $X=\bigcup\{A: A\in\mathscr{A}\}$，令 $\mathscr{C}$ 为 $\mathscr{A}$ 的元相对于 X 之余族的方法而用 (a) 代替. 据 De Morgan 公式 $\mathscr{C}$ 满足 (a) 的假设，故存在一个极大元 M，于是 $X\sim M$ 一定是 $\mathscr{A}$ 的一个极小元.

(c) 的证明. 这个证明是基于极大原理 (a). 令 $\mathscr{A}$ 为一个具有有限特征的族，令 $\mathfrak{N}$ 为 $\mathscr{A}$ 中的一个套，并令 $A=\bigcup\{N: N\in\mathfrak{N}\}$. A 的每个有限子集 F 必为 $\mathfrak{N}$ 的某个元之子集，因为我们可以选取套 $\mathfrak{N}$ 的一个有限子族，使它的并包含 F，并且此有限子族有一个最大元包含 F. 从而 $A\in\mathscr{A}$. 故 $\mathscr{A}$ 满足 (a) 的假设，所以存在一个极大元.

(d) 的证明. 假定 B 为半序集 A 中的一个链. 令 $\mathscr{A}$ 为 A 中包含 B 之所有链的族. 如果 $\mathfrak{N}$ 是 $\mathscr{A}$ 中的一个套，则能直接验证 $\bigcup\{N: N\in\mathfrak{N}\}$ 仍是一个元，于是 $\mathscr{A}$ 满足 (a) 的假设，从而有一个

极大元存在.

(e) 的证明. 对于一个极大链选一个上界.

(f) 的证明. 回顾一个函数是没有两个元有相同第一个坐标的序偶集. 设 $\mathscr{F}$ 为所有这样的函数 f 的族: f 的定义域是 A 的一个子集且对于在 f 定义域中的每个 a, $f(a)\in X_a$. ($\mathscr{F}$ 的元均为我们欲求函数的"片段".)下面我们证明 $\mathscr{F}$ 为具有有限特征的族. 如果 f 为 $\mathscr{F}$ 的一个元,则 f 的每个子集,特别是有限子集仍为 $\mathscr{F}$ 的一个元. 另一方面,如果 f 是一个集,它的每个有限子集属于 $\mathscr{F}$,则 f 的元均为序偶,没有两个不同的序偶有相同的第一个坐标,从而 f 是一个函数. 加之,如果 a 为 f 定义域中的一个元,则 $\{a, f(a)\}\in\mathscr{F}$, 故 $f(a)\in X_a$, 从而推得 $f\in\mathscr{F}$.

因为 $\mathscr{F}$ 是一个具有有限特征的族. 所以 $\mathscr{F}$ 有一个极大元 c, 现只需证明 c 的定义域为 A. 假定 a 为 A 的一个元而非 c 的定义域中的元, 于是由 X_a 非空推得在 X_a 中存在一个元 y 且 $c\cup\{(a,y)\}$ 自身是一个函数. 同时它又是 $\mathscr{F}$ 的一个元. 但这与 c 为极大元相矛盾.

(g) 的证明. 把选择公理应用到对于在 $\mathscr{A}$ 中的每个 A, $X_a - A$ 的指标集 $\mathscr{A}$ 上.

(h) 的证明. 设 X 是一个欲良序化的(不空)集,令 $\mathscr{A}$ 为 X 的所有非空子集的族,同时设 c 是对于 $\mathscr{A}$ 的一个选择函数;也就是说 c 是一个在 $\mathscr{A}$ 上使得对于 $\mathscr{A}$ 中的每个 A, $c(A)\in A$ 的函数. 这个证明的想法是构造一个序 $\leqq$ 使得对于每个"初始段" A, 在此序中 A 后面的第一个点是 $c(X\sim A)$. 明确地说, 定义一个集 A 为相对于序 $<$ 的一个段当且仅当在 A 的一个元之前的每个点是 A 自身的一个元,特别地,空集是一个段. 令 $\mathscr{C}$ 为满足下面条件的所有自反线性序 $\leqq$ 的类: $\leqq$ 的定义域 D 为 X 的一个子集且对于每个不同于 D 的段 A, $D\sim A$ 的第一个点是 $c(X\sim A)$. $\mathscr{C}$ 的每个元是一个良序几乎是显然的, 因为如果 B 是元 $\leqq$ 定义域的一个非空子集且 $A=\{y$: 对于 B 中的每个 x, $y\leqq x$ 且 $y\neq x\}$,则 $c(X\sim A)$ 为 B 的首元. 假定 $\leqq$ 与 $\leqslant$ 都是 $\mathscr{C}$ 的元, D 为 $\leqq$ 的定义域,而 E 为 $\leqslant$

的定义域．令 A 为使得集 $\{y:y\leqq x\}$ 与 $\{y:y\preccurlyeq x\}$ 恒等且在这些集上两个序一致所有点 x 的集．于是 A 同时相对于 $\leqq$ 与 $\preccurlyeq$ 是一个段．如果 A 既不与 D 又不与 E 相等，则 $c(X\sim A)$ 属于此二集且不属于 A 的第一个点；但由 A 的定义知 $c(X\sim A)\in A$．从而推出 $A=D$ 或者 $A=E$．于是 $\mathscr{C}$ 的任意两个元均有下面的关系：任何一个元的定义域是相对于另一个元的一个段，并且在这个段上两个序是一致的．利用这个事实不难看到 $\mathscr{C}$ 的元之并 $\prec$ 自身是 $\mathscr{C}$ 的一个元；它是 $\mathscr{C}$ 的最大元．如果 F 是 $\prec$ 的定义域，则 $F=X$，因为倘若不然，点 $c(X\sim F)$ 可能接在这个序的最后（更确切地说，$\prec\cup(F\times\{c(X\sim F)\})$ 是 $\prec$ 的一个元，它真包含 $\prec$）．于是定理得证．|

26 注记． 上面列举的每个命题实际上都等价于 Hausdorff 极大原理．并且它们之中的任何一个均有理由被当成一个公理来假定．在附录中极大原理由选择公理导出．

上面给出的从选择公理推导良序原理的方法本质上是依据 Zermelo [1]．然而应用定理 25 (e) 来证明它也是完全有可能的．注意一个良序的套的并一般不再是一个良序，所以想把极大原理直接应用到良序的族上是不可能的．

应当注意在定理 25 中不同命题的符号有点随意性，Hausdorff 极大原理曾被 C. Kuratowski，R. L. Moore 和 M. Zorn 以与上面相近的形式彼此独立使用过．

最后还需注意：虽然已给的 Tukey 引理的形式多少有些典型，但它不能直接地推出多数通常列举的一些应用（例如，每个群包含一个极大的 Abel 子群）．它的更一般形式叙述为（很粗略地）：如果一个集族 $\mathscr{A}$ 被一些（可能无穷多个）条件所定义，而每一个条件仅仅涉及有限多个点，则 $\mathscr{A}$ 有一个极大元．

第一章 拓扑空间

拓扑和邻域

拓扑是指这样的集族 $\mathscr{T}$，它满足两个条件：$\mathscr{T}$ 的任意两个元的交和$\mathscr{T}$的每一个子族的元的并仍为$\mathscr{T}$的元．集 $X = \bigcup\{U: U \in \mathscr{T}\}$ 必须是 $\mathscr{T}$ 的元，因为 $\mathscr{T}$ 是它自己的一个子族；另外 $\mathscr{T}$ 的每一个元都是X的子集．我们称集X为拓扑 $\mathscr{T}$ 的**空间**，而 $\mathscr{T}$ 叫做是**关于X的一个拓扑**．又 $(X, \mathscr{T})$ 称为**拓扑空间**．当不会引起误解时，我们可以简写成“X 为拓扑空间”．但在必须确切说明的情况下，我们还要明显地指出(譬如对同一个集X考虑两种不同的拓扑)．

拓扑 $\mathscr{T}$ 的元叫做关于 $\mathscr{T}$ 的**开集**或 $\mathscr{T}$-开集，如果在讨论中只有一种拓扑，那末就简称为开集．拓扑的空间X恒为开集；空集也恒为开集，因为它是空族的元的并．这些可能是仅有的开集，因为只含X和空集为其元的族是关于X的一个拓扑．虽然这不是一种很有趣的拓扑，但它也时常出现，值得给予一个名称；我们称它为关于X的**平庸**(或平凡)**拓扑**，于是$(X, \mathscr{T})$为**平庸拓扑空间**．另一种极端情形是X的所有子集的族，它叫做关于X的**离散拓扑**(此时 $(X, \mathscr{T})$ 为**离散拓扑空间**)．若$\mathscr{T}$为离散拓扑，则空间的每一个子集为开集．

显然关于集X的离散和平庸拓扑分别为关于X的最大和最小拓扑，即关于X的每一个拓扑均包含在离散拓扑内并且包含平庸拓扑．设$\mathscr{T}$ 和 $\mathscr{U}$ 为关于X的拓扑，则根据对任意的集族所用的说法，当$\mathscr{T} \subset \mathscr{U}$ 时称$\mathscr{T}$ **小于** $\mathscr{U}$ 或 $\mathscr{U}$ **大于** $\mathscr{T}$．换言之，$\mathscr{T}$ 小于 $\mathscr{U}$ 当且仅当每一个 $\mathscr{T}$-开集皆为 $\mathscr{U}$-开集．在这种情况下也称 $\mathscr{T}$ 粗于 $\mathscr{U}$ 或 $\mathscr{U}$ 细于 $\mathscr{T}$ (不幸的是，这种情况在文献中用两

种不同的语句来描述：$\mathscr{T}$ 强于 $\mathscr{U}$ 和 $\mathscr{T}$ 弱于 $\mathscr{U}$）. 若 $\mathscr{T}$ 和 $\mathscr{U}$ 为任意两个关于 X 的拓扑，则可能出现 $\mathscr{T}$ 既不大于又不小于 $\mathscr{U}$ 的情形；这时根据对半序所用的说法，称 $\mathscr{T}$ 和 $\mathscr{U}$ 为**不可比较的**.

带有适当拓扑的实数集是一个很有趣的拓扑空间. 这是不足为奇的，因为拓扑空间的概念就是实数的某些有趣性质的抽象. 关于实数的**通常拓扑**是指所有这样的集组成的族，对其内的每一点，它包含含有该点的一个开区间. 即实数集的子集 A 为开集当且仅当对 A 的每一个元 x 存在数 a 和 b 使得 $a<x<b$ 并且**开区间** $\{y: a<y<b\}$ 是 A 的子集. 自然，我们必须证明该集族的确是一个拓扑，但这一点不难实现. 还值得指出的是此时开区间恒为开集.

拓扑空间 $(X, \mathscr{T})$ 内的集 U 叫做点 x 的**邻域**（$\mathscr{T}$-邻域）当且仅当 U 包含含有 x 的一个开集. 虽然一个点的邻域不必一定为开集，但每一个开集一定是其内每一点的邻域. 又一个点的每一个邻域均包含该点的一个开邻域. 若 $\mathscr{T}$ 为平庸拓扑，则点 x 仅有的邻域就是空间 X 它自己. 若 $\mathscr{T}$ 为离散拓扑，则每一个含有点 x 的集皆为 x 的邻域. 若 X 为实数集，$\mathscr{T}$ 为通常拓扑，则点 x 的邻域是包含含有 x 的一个开区间的集.

1 定理. 一个集为开的当且仅当它包含其内每一点的一个邻域.

证明. 显然集 A 的一切开子集的并 U 仍为 A 的开子集. 于是若 A 包含其内每一点的一个邻域，则 A 的每一个元 x 属于 A 的某个开子集，即 $x\in U$. 从而 $A=U$，亦即 A 为开集.

另一方面若 A 为开集，则它包含其内每一点的一个邻域（即 A）.|

这个定理显然蕴含：一个集为开的当且仅当它是它的每一点的邻域.

一个点的邻域系是指该点的所有邻域的族.

2 定理. 若 $\mathscr{U}$ 为一个点的邻域系，则 $\mathscr{U}$ 的元的有限交属于 $\mathscr{U}$，并且每一个包含 $\mathscr{U}$ 的一个元的集也属于 $\mathscr{U}$.

证明. 若U和V为点x的邻域,则有开邻域U_0和V_0分别包含在U和V内,故$U \cap V$包含开邻域$U_0 \cap V_0$,从而它是x的一个邻域. 于是$\mathscr{U}$的两个(因而有限多个)元的交为$\mathscr{U}$的元.

另外若集U包含点x的一个邻域,则它包含x的一个开邻域,从而它自己也是x的一个邻域. ▌

3 注记. Fréchet[1] 首先考虑了抽象空间. 在随后的年代里,随着对于概念和基本方法的大量探讨,拓扑空间的概念发展起来了. 该理论的发展情况差不多都可以在 Hausdorff 的经典著作[1]和稍后的 *Fundamenta Mathematicae* 的最初几卷中找到. 从这些探索中,实际上两个基本概念就已经出现了:拓扑空间和一致空间(第六章[1]). 而较为新近才出现的后一概念(A. Weil[1]),主要是由于拓扑群的研究.

关于一般拓扑的标准参考书包括:

Alexandroff 和 Hopf [1](前两章),Bourbaki[1],Fréchet[2],Kuratowski [1],Lefschetz [1](第一章),R. L. Moore [1],Newman [1],Sierpinski [1],Tukey [1],Vaidyanathaswamy [1]和 G. T. Whyburn [1].

闭　集

拓扑空间$(X, \mathscr{T})$的子集A叫做**闭集**,当且仅当它的余集$X \sim A$为开集.因为集A的余集的余集仍为A,故集为开集当且仅当它的余集为闭集. 若$\mathscr{T}$为平庸拓扑,则X的余集和空集的余集为仅有的闭集,即只有空集和X为闭集.空间X和空集既是闭集,又是开集总是正确的,但可能出现,正如我们已经看到的,这两个集是仅有的闭集. 若$\mathscr{T}$为离散拓扑,则每一个子集都是闭集和开集. 若X为实数集,$\mathscr{T}$为通常拓扑,则情况完全不同. 虽然**闭区间**(即形如$\{x: a \leqslant x \leqslant b\}$的集必为闭集,但开区间就不是闭集并且**半开区**

1) 此处原书误为第七章,以下关于原书的印刷错误都由译者加以订正,而不再一一指明. ——译者注

间（即形如 $\{x:a<x\leqslant b\}$ 或 $\{x:a\leqslant x<b\}$ 的集，其中 $a<b$）既非开集，又非闭集．问题 1.J 还将说明这时整个空间和空集确实就是仅有的既开又闭的集．

根据 De Morgan 公式（预备知识定理 3），集族的元的余集的并（交）为交（并）的余集．因此，有限多个闭集的并为闭集，并且任意一族闭集的交也为闭集．下面的定理将表明这些性质恰好就是所有闭集的族的特征，但我们略去它的简单的证明．

4 定理． 设 $\mathscr{F}$ 为一个集族，它具有性质：有限子族的并和任意非空子族的交仍为其元，并且 $X=\bigcup\{F:F\in\mathscr{F}\}$ 亦为其元，则 $\mathscr{F}$ 恰为 X 关于由所有 $\mathscr{F}$ 的元的余集所组成的拓扑的所有闭集的族．

聚　　点

我们知道拓扑空间的拓扑能够借助于点的邻域来描述，因此通过邻域也一定能够描述闭集，这种描述就引出了点的按下述方法的一种新的分类．集 A 为闭的当且仅当 $X\sim A$ 为开的，从而当且仅当 $X\sim A$ 的每一个点有一个邻域被包含在 $X\sim A$ 内，或等价地，与 A 不相交．因此 A 为闭集当且仅当对每一个 x，若 x 的每一个邻域与 A 相交，则 $x\in A$．从而导出下面的定义．

我们称点 x 为拓扑空间 $(X,\mathscr{T})$ 的子集 A 的**聚点**（有时叫做**凝聚点或极限点**），当且仅当 x 的每一个邻域包含异于 x 的 A 中的点．于是，点 x 的每一个邻域与 A 相交当且仅当 x 或者是 A 的点，或者是 A 的聚点．由此易知下列定理为真．

5 定理． 拓扑空间的子集为闭集当且仅当它包含它的所有聚点．

如果 x 为 A 的聚点，那末我们有时用这样一种很有启发性的说法来描述它，即“存在 A 的点任意接近 x”．当我们采用这种观点时，就会发现平庸拓扑空间的确是“完全挤满的”，因为其中每一个点 x 都是每一个异于空集和集 $\{x\}$ 的集的聚点．另一方面，在

离散拓扑空间中任意点都不是一个集的聚点．若 X 为具有通常拓扑的实数集，则可能出现多种情况．若 A 为开区间 $(0,1)$，则闭区间 $[0,1]$ 的每一点均为 A 的聚点．若 A 为所有平方小于 2 的非负有理数的集，则闭区间 $[0,\sqrt{2}]$ 为 A 的所有聚点的集．若 A 为所有整数的倒数的集，则 0 为 A 的仅有的聚点，另外所有整数的集没有聚点．

6 定理．任意集与它的所有聚点的集的并恒为闭集．

证明．若 x 既非 A 的点，又非 A 的聚点，则有 x 的开邻域 U，它与 A 不相交．又从 U 是其内每一点的邻域可推出 U 的每一点都不是 A 的聚点，故集 A 和它的聚点的集的并为一个开集的余集．|

集 A 的一切聚点的集有时叫做 A 的**导集**．

闭　　包

拓扑空间 $(X,\mathscr{T})$ 的子集 A 的**闭包**（$\mathscr{T}$-**闭包**）是指所有包含 A 的闭集的族的元的交．A 的闭包记为 A^- 或 $\overline{A}$．集 A^- 恒为闭集，因为它是闭集的交；又易见 A^- 包含在每一个包含 A 的闭集内．因此 A^- 为包含 A 的最小闭集，从而 A 为闭集当且仅当 $A=A^-$．下面的定理将利用集的聚点来描述它的闭包．

7 定理．任何集的闭包是该集和它的所有聚点的集的并．

证明．　因为集 A 的每一个聚点也是每一个包含 A 的集的聚点，亦即为每一个包含 A 的闭集的元，故 A^- 包含 A 和 A 的所有聚点．

另一方面，根据上一定理，由 A 和它的所有聚点所组成的集为闭集，因而它包含 A^-．|

我们把在拓扑空间的每一个子集 A 处取值 A^- 的函数叫做关于拓扑的闭包函数或闭包算子．这个算子完全决定了拓扑，因为集 A 为闭集当且仅当 $A=A^-$．换言之，闭集仅仅是关于闭包算子不变的集．现在来考虑如下的富有启发性的问题：在什么情况下对确定的集 X 的一切子集所定义的算子就是关于 X 的某个拓扑

的闭包算子？为此先考察描述闭包的四个很简单的性质．首先，因为空集为闭集，故空集的闭包也为空集．其次，任何集都包含在它的闭包内．第三，因为每一个集的闭包皆为闭集，故任何集的闭包的闭包与该集的闭包相同（在通常的代数术语下，这就是说，闭包算子是幂等的）．最后，两个集的并的闭包恰为闭包的并，因为从 $(A\cup B)^-$ 为包含 A 和 B 的闭集可推出它包含 A^- 和 B^-，故它也包含 $A^-\cup B^-$；另一方面 $A^-\cup B^-$ 为包含 $A\cup B$ 的闭集，从而也包含 $(A\cup B)^-$．

所谓 X 上的**闭包算子**就是这样的一个算子，它变 X 的每一个子集 A 为 X 的子集 A^c 并且满足如下的四个条件，即 **Kuratowski 的闭包公理**：

(a) 若 0 为空集，则 $0^c=0$；

(b) 对每一个 A，$A\subset A^c$；

(c) 对每一个 A，$A^{cc}=A^c$；

(d) 对每一个 A 和 B，$(A\cup B)^c=A^c\cup B^c$．

下面的 Kuratowski 定理证明了这四个条件实际上也就是闭包的特征，而以下所定义的拓扑就叫做与闭包算子**相关联的**拓扑．

8 定理．设 c 为 X 上的闭包算子，又设 $\mathscr{F}$ 为所有使得 $A^c=A$ 的 X 的子集 A 组成的族，再设 $\mathscr{T}$ 为所有 $\mathscr{F}$ 的元的余集的族，则 $\mathscr{T}$ 为关于 X 的拓扑并且对 X 的每一个子集 A，A^c 为 A 的 $\mathscr{T}$-闭包．

证明．公理 (a) 说明空集属于 $\mathscr{F}$，而公理 (d) 又说明了 $\mathscr{F}$ 的两个元的并仍为 $\mathscr{F}$ 的元，因此，$\mathscr{F}$ 的任何有限子族（空或非空）的并为 $\mathscr{F}$ 的元．由于 (b) $X\subset X^c$，所以 $X=X^c$，且所有 $\mathscr{F}$ 的元的并恰为 X．于是根据定理 1.4，欲证 $\mathscr{T}$ 为关于 X 的拓扑，只须证 $\mathscr{F}$ 的任何非空子族的元的交也为 $\mathscr{F}$ 的元．为此，首先注意若 $B\subset A$，则 $B^c\subset A^c$，这是因为 $A^c=[(A\sim B)\cup B]^c=(A\sim B)^c\cup B^c$．今设 $\mathscr{A}$ 为 $\mathscr{F}$ 的非空子族，$B=\bigcap\{A:A\in\mathscr{A}\}$，则因 B 包含在 $\mathscr{A}$ 的每一个元内，故 $B^c\subset\bigcap\{A^c:A\in\mathscr{A}\}=\bigcap\{A:A\in\mathscr{A}\}=B$，又 $B\subset B^c$，从而 $B=B^c$，即 $B\in\mathscr{F}$．这就证明了

$\mathscr{T}$ 是一个拓扑.

剩下要证的是 A^c 为 A 的 $\mathscr{T}$-闭包 A^-. 根据定义 A^-为所有包含 A 的 $\mathscr{T}$-闭集(即$\mathscr{F}$的元)的交,再根据公理 (c) $A^c \in \mathscr{F}$,故 $A^- \subset A^c$. 又从 $A^- \in \mathscr{F}$ 和 $A^- \supset A$ 可推出 $A^- \supset A^c$, 总之 $A^- = A^c$.|

内 部 和 边 界

现在考虑另一个定义在拓扑空间的所有子集的族上的算子,它与闭包算子有着很密切的联系. 我们称拓扑空间的子集 A 的点 x 为 A 的**内点**,当且仅当 A 是 x 的一个邻域,又 A 的所有内点的集叫做 A 的**内部**,记作 A^0. (按通常的术语关系"…是…的一个内点"是关系"…是…的一个邻域"的逆关系.)为方便起见,在考察具体例子之前先列举这个概念和以前的各个概念之间的联系.

9 定理. 设 A 为拓扑空间 X 的子集,则 A 的内部 A^0 为开集并且它是 A 的最大开子集. 又 A 为开集当且仅当 $A=A^0$. A 中所有不是 $X \sim A$ 的聚点的点所组成的集恰为 A^0. $X \sim A$ 的闭包就是 $X \sim A^0$.

证明. 若点 x 属于集 A 的内部, 则 x 为 A 的某个开子集 U 的元,而 U 的每一个元也是 A^0 的元,故 A^0 包含其内每一点的一个邻域,即为开集. 若 V 为 A 的开子集, $y \in V$, 则 A 为 y 的邻域, 故 $y \in A^0$, 因而 A^0 包含 A 的每一个开子集, 即它是 A 的最大开子集.

若 A 为开集,则 A 自然与它的最大开子集相同,故 A 为开集当且仅当 $A = A^0$.

若 x 为 A 中的点,而不是 $X \sim A$ 的聚点,则有 x 的邻域 U,它与 $X \sim A$ 不相交,从而是 A 的子集,故 A 是 x 的邻域,亦即 $x \in A^0$. 另一方面, A^0 是其内每一点的邻域并且 A^0 与 $X \sim A$ 不相交, 于是没有 A^0 中的点,它是 $X \sim A$ 的聚点.

最后,因为 A^0 是由 A 中不是 $X \sim A$ 的聚点的点所组成,故它

的余集 $X\sim A^c$ 恰好就是所有这样的点的集，它或者是 $X\sim A$ 的点，或者是 $X\sim A$ 的聚点，这表明该余集即为闭包 $(X\sim A)^-$. ∎

上一定理的最后一个结论值得作一些更进一步的讨论．为方便起见，我们把余集 $X\sim A$ 记为 A'. 此时 A''，即 A 的余集的余集仍为 A(我们有时称$'$为周期等于 2 的算子)．因此，前面的这个结果又可以改写成 $A^{0'}=A'^-$. 再取余集就得到 $A^0=A'^{-'}$，即 A 的内部为 A 的余集的闭包的余集．若以 A' 代替 A，则又有 $A^-=A'^{0'}$，故集的闭包为余集的内部的余集[1].

若 X 为平庸空间，则除 X 自己外每一个集的内部为空集．若 X 为离散空间，则每一个集既开又闭，故与它的内部、也与它的闭包相同．若 X 为具有通常拓扑的实数集，则整数集的内部为空集，而闭区间的内部为具有相同端点的开区间．有理数集的内部为空集，从而它的内部的闭包仍为空集．另一方面，有理数集的闭包为实数集 X，于是它的闭包的内部亦为 X. 因此，集的闭包的内部可以完全不同于内部的闭包，即内部算子和闭包算子一般是不可交换的．

还有一个另外的算子，它时常出现，以至于需要我们专门去阐述它的定义．所谓拓扑空间 X 的子集 A 的**边界**就是指所有这样的点的集，它既不属于 A 的内部，又不属于 $X\sim A$ 的内部．等价地，x 为边界的点当且仅当 x 的每一个邻域与 A 和 $X\sim A$ 都相交．显然，A 的边界与 $X\sim A$ 的边界相同．若 X 平庸，并且 A 既不是 X，又不是空集，则 A 的边界为 X；又当 X 离散时每一个子集的边界为空集．另外在通常拓扑下任何实数区间的边界只含有该区间的端点，并且它与区间是否为开、闭或半开无关；而有理数集或无理数集的边界就是所有实数的集．

我们容易看出边界、闭包和内部之间的关系．而下面的略去证

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1) 有一个有趣而又有启发性的问题，也就是通过任意次序连续运用闭包，余集和内部算子，从拓扑空间的一个给定的子集 A 能够作出多少个不同的集？根据上一段的讨论以及 $A^{--}=A^-$，问题又可以化为：通过交替应用余集和闭包算子，从 A 能够作出多少个不同的集？它的意想不到的解答将在问题 1.E 中给出．

明的定理就概括了这些事实.

10 定理. 设 A 为拓扑空间 X 的子集,又设 $b(A)$ 为 A 的边界,则 $b(A)=A^-\cap(X\sim A)^-=A^-\sim A^0$, $X\sim b(A)=A^0\cup(X\sim A)^0$, $A^-=A\cup b(A)$ 并且 $A^0=A\sim b(A)$.

一个集为闭集当且仅当它包含它的边界,并且为开集当且仅当它与它的边界不相交.

基 和 子 基

在定义实数集的通常拓扑时我们是先从所有开区间的族 $\mathscr{B}$ 出发,然后再由它作出拓扑 $\mathscr{T}$. 同样的方法也可以应用于其它的情形,现在我们来详细的考察这种作法. 我们称集族 $\mathscr{B}$ 为**拓扑 $\mathscr{T}$ 的基**当且仅当 $\mathscr{B}$ 是 $\mathscr{T}$ 的子族并且对空间的每一点 x 和 x 的每一个邻域 U 有 $\mathscr{B}$ 的元 V 使得 $x\in V\subset U$. 于是,所有开区间的族为实数集的通常拓扑的基,这是根据通常拓扑的定义以及开区间恒为该拓扑的开集的事实.

基有一个简单的特征,它常常被当作定义:拓扑 $\mathscr{T}$ 的子族 $\mathscr{B}$ 为 $\mathscr{T}$ 的基当且仅当 $\mathscr{T}$ 的每一个元为 $\mathscr{B}$ 的元的并. 现在来证明这个事实. 假设 $\mathscr{B}$ 为拓扑 $\mathscr{T}$ 的基, $U\in\mathscr{T}$, 命 V 为 $\mathscr{B}$ 中所有为 U 的子集的元的并,并设 $x\in U$, 则有 $\mathscr{B}$ 中的 W 使得 $x\in W\subset U$, 即 $x\in V$, 于是 $U\subset V$, 又从 V 的作法可知它是 U 的子集,故有 $V=U$. 今证其逆,假设 $\mathscr{B}\subset\mathscr{T}$ 并且 $\mathscr{T}$ 的每一个元为 $\mathscr{B}$ 的元的并. 于是若 $U\in\mathscr{T}$, 则它是 $\mathscr{B}$ 的一个子族的元的并,从而对 U 中的每一个 x 有 $\mathscr{B}$ 中的 V 使得 $x\in V\subset U$, 即 $\mathscr{B}$ 为 $\mathscr{T}$ 的基.

虽然这是构造拓扑的一种很方便的方法,但谨慎一点是必要的,因为并非每一个集族都是某个拓扑的基. 例如,设 X 由整数 0, 1 和 2 组成,又设 A 与 B 分别由 0 和 1 与 1 和 2 组成,若命 $\mathscr{S}$ 为这样的族,它的元是 X, A, B 和空集,则 $\mathscr{S}$ 不是某个拓扑的基:因为通过直接计算可知 $\mathscr{S}$ 的元的并仍为它的元,故若 $\mathscr{S}$ 为某个拓扑的基,则该拓扑就必定是 $\mathscr{S}$ 它自己;但 $\mathscr{S}$ 不是一个拓扑,

因为 $A\cap B\notin\mathscr{S}$.下面的定理将清楚地说明出现这种现象的原因.

11 定理 集族 $\mathscr{B}$ 为关于集 $X=\bigcup\{B:B\in\mathscr{B}\}$ 的某个拓扑的基当且仅当对 $\mathscr{B}$ 的每两个元 U 和 V 与 $U\cap V$ 中的每一点 x 有 $\mathscr{B}$ 中的元 W 使得 $x\in W$ 并且 $W\subset U\cap V$.

证明. 若 $\mathscr{B}$ 为某个拓扑的基, U 和 V 为 $\mathscr{B}$ 的元并且 $x\in U\cap V$, 则因 $U\cap V$ 为开集, 故有 $\mathscr{B}$ 的元使得 x 属于它并且它还是 $U\cap V$ 的子集.

今证其逆,设 $\mathscr{B}$ 为具有所述性质的族, $\mathscr{T}$ 为所有 $\mathscr{B}$ 的元的并的族. $\mathscr{T}$ 的元的并本身也是 $\mathscr{B}$ 的元的并,故它亦为 $\mathscr{T}$ 的元,因此只须证 $\mathscr{T}$ 的两个元 U 和 V 的交仍为 $\mathscr{T}$ 的元. 若 $x\in U\cap V$,则可选出 $\mathscr{B}$ 中的 U' 和 V' 使得 $x\in U'\subset U$ 并且 $x\in V'\subset V$,此时又有 $\mathscr{B}$ 的元 W 使得 $x\in W\subset U'\cap V'\subset U\cap V$, 从而 $U\cap V$ 为 $\mathscr{B}$ 的元的并. 即 $\mathscr{T}$ 是一个拓扑. |

我们已经看到任意的集族 $\mathscr{S}$ 可以不是某个拓扑的基. 因此,我们把问题加以改变,来考虑是否有一个唯一的拓扑,它在某种意义下由 $\mathscr{S}$ 所生成. 自然如此的拓扑应当是关于集 X 的一个拓扑,其中 X 为 $\mathscr{S}$ 的元的并;并且 $\mathscr{S}$ 的每一个元应当是关于该拓扑的开集, 即 $\mathscr{S}$ 为这个拓扑的子族. 这样就又把问题转化为: 是否有关于 X 的一个最小拓扑它包含 $\mathscr{S}$?下面的简单结果就给出了这个最小拓扑.

12 定理. 若 $\mathscr{S}$ 为任意非空的集族,则 $\mathscr{S}$ 的元的所有有限交的族是关于集 $X=\bigcup\{S:S\in\mathscr{S}\}$ 的某个拓扑的基.

证明. 若 $\mathscr{S}$ 为一个集族, $\mathscr{B}$ 为 $\mathscr{S}$ 的元的所有有限交的族, 则 $\mathscr{B}$ 的两个元的交仍为 $\mathscr{B}$ 的元,再应用上一定理便知 $\mathscr{B}$ 是某个拓扑的基. |

我们称集族 $\mathscr{S}$ 为**拓扑 $\mathscr{T}$ 的子基**当且仅当 $\mathscr{S}$ 的元的所有有限交的族为 $\mathscr{T}$ 的基(即当且仅当 $\mathscr{T}$ 的每一个元是 $\mathscr{S}$ 的元的有限交的并). 根据上一定理, 每一个非空的族 $\mathscr{S}$ 是某个拓扑的子基,并且该拓扑自然唯一的由 $\mathscr{S}$ 所决定,同时它还是包含 $\mathscr{S}$ 的最小拓扑(即它是包含 $\mathscr{S}$ 的拓扑并且还是每一个包含 $\mathscr{S}$ 的拓扑的

子族).

一般地说，一个拓扑是有许多不同的基和子基，但我们可根据所讨论的问题从中选出最适合的一种．对于实数集的通常拓扑，最自然的子基是：所有半无限开区间的族，即所有形如 $\{x:x>a\}$ 或 $\{x:x<a\}$ 的集组成的族．因为每一个开区间是两个如此的集的交，故该族是一个子基．又带有有理数 a 的所有同上形式的集组成的族是一个并不太明显，但却更为有趣的子基(见问题 1.J).

拓扑具有可数基的空间有许多很好的性质．这样的空间叫做满足**第二可数性公理**(对此，也可使用**可分**和**完备可分**等术语，但我们都不准备采用).

13 定理．若 A 为具有可数基的拓扑空间的不可数子集，则有 A 的某个点为 A 的聚点.

证明．假设 A 的任何点均不为 A 的聚点，又设 $\mathscr{B}$ 为可数基，则对 A 中的每一个 x 有一个不包含异于 x 的 A 的点的开集，故有 $\mathscr{B}$ 中的 B_x 使得 $B_x\cap A=\{x\}$，于是 A 的点和 $\mathscr{B}$ 的某个子族的元之间就建立了一个一一对应的关系，从而 A 为可数集.|

该定理的一个更深刻的形式将陈述于问题 1.H.

我们称集 A 在拓扑空间 X 中**稠密**当且仅当 A 的闭包为 X．称拓扑空间 X 为**可分**当且仅当存在在 X 中稠密的可数子集．可分空间可以不满足第二可数性公理．例如，设 X 为一个不可数集，它的拓扑由有限集的余集和 X 本身组成，则每一个非有限的子集皆稠密，因为它与每一个开集都相交．另一方面，假设 X 有可数基 $\mathscr{B}$，命 x 为 X 的确定的点，则所有使得 x 属于它的开集的族的交为 $\{x\}$，因为每一个另外的点的余集必为开集；由此可见，所有使得 x 属于它的 $\mathscr{B}$ 的元的交亦为 $\{x\}$，但该可数多个 $\mathscr{B}$ 的元的交的余集为可数多个有限集的并，从而为可数集，于是得到矛盾．(较不平凡的例子以后将会遇到.)至于具有可数基的空间必为可分的事实则是不难证明的.

14 定理．具有可数基的拓扑空间必为可分.

证明．从基的每一个元中选取一个点，于是得到一个可数集 A．因为 A 的闭包的余集为开集并且与 A 不相交，故它不包含基的非空的元，从而必为空集．|

我们称族 $\mathscr{A}$ 为集 B 的**覆盖**当且仅当 B 是并集 $\bigcup\{A: A\in\mathscr{A}\}$ 的子集，即当且仅当 B 的每一个元属于 $\mathscr{A}$ 的某个元．称 $\mathscr{A}$ 为 B 的**开覆盖**当且仅当覆盖 $\mathscr{A}$ 的每一个元是开集．而覆盖 $\mathscr{A}$ 的仍为覆盖的子族叫做 $\mathscr{A}$ 的**子覆盖**．

15 定理 (Lindelöf)．具有可数基的拓扑空间的子集的每一个开覆盖恒有可数子覆盖．

证明．假设 A 为一个子集，$\mathscr{A}$ 为 A 的开覆盖，而 $\mathscr{B}$ 是拓扑的可数基，则因 $\mathscr{A}$ 的每一个元是 $\mathscr{B}$ 的元的并，故有 $\mathscr{B}$ 的子族 $\mathscr{C}$，它也覆盖 A 并且使得 $\mathscr{C}$ 的每一个元是 $\mathscr{A}$ 的某个元的子集．对 $\mathscr{C}$ 的每一个元，我们选定一个 $\mathscr{A}$ 的元包含它，于是得到 $\mathscr{A}$ 的一个可数子族 $\mathscr{D}$．这时 $\mathscr{D}$ 仍为 A 的覆盖，因为 $\mathscr{C}$ 覆盖 A；从而 $\mathscr{A}$ 有可数子覆盖．|

拓扑空间称为 **Lindelöf 空间**当且仅当空间的每一个开覆盖恒有可数子覆盖．

第二可数性公理前面已经陈述，现在再来陈述第一可数性公理．这个公理与基的概念的局部形式有关．所谓**点 x 的邻域系的基**，或 **x 处的局部基**就是指这样的邻域族，它使得 x 的每一个邻域包含该族的某个元．例如每一点的所有开邻域的族恒为该点的邻域系的基．我们称拓扑空间满足**第一可数性公理**，假如每一点的邻域系都有可数基．显然，每一个满足第二可数性公理的拓扑空间也满足第一可数性公理．另一方面，任何不可数的离散拓扑空间满足第一可数性公理（每一点 x 的邻域系有一个基，它仅由一个邻域 $\{x\}$ 组成），但不满足第二可数性公理（由所有 $\{x\}, x\in X$ 组成的覆盖没有可数子覆盖）．因此，第二可数性公理要比第一可数性公理有更多的限制．

值得注意，如果 $U_1, U_2, \cdots, U_n, \cdots$ 为 x 处的一个可数局部基，那末一定可找到一个新的局部基 $V_1, V_2, \cdots, V_n, \cdots$ 使得对

每一个 n 有 $V_n \supset V_{n+1}$. 作法很简单，只须令 $V_n = \bigcap\{U_k : k \leqslant n\}$.

点 x 的邻域系的子基，或 **x 处的局部子基**是指这样的集族，使得它的元的所有有限交的族为一个局部基. 若 U_1, U_2, $\cdots$, U_n, $\cdots$为一个可数局部子基，则 V_1, V_2, $\cdots$, V_n, $\cdots$为一个可数局部基，其中 $V_n = \bigcap\{U_k : k \leqslant n\}$. 因此，从每一点处的可数局部子基的存在即可推出第一可数性公理.

相对化；分离性

若$(X, \mathscr{T})$ 为拓扑空间，Y 为 X 的子集，则我们可作出 Y 的一个拓扑 $\mathscr{U}$，它叫做 $\mathscr{T}$ 对于 Y 的**相对拓扑**或**相对化**. 相对拓扑 $\mathscr{U}$ 定义为 $\mathscr{T}$ 的所有元与 Y 的交的族，即 U 属于相对拓扑 $\mathscr{U}$ 当且仅当对于某个 $\mathscr{T}$-开集 V，$U = V \cap Y$. 不难看出，$\mathscr{U}$ 的确是一个拓扑. 相对拓扑 $\mathscr{U}$ 的每一个元 U 叫做 **Y 内开的**集，而把它的相对余集 $Y \sim U$ 叫做**在 Y 内闭的集**，又 Y 的子集的 $\mathscr{U}$-闭包是指它**在 Y 内的闭包**. X 的每一个子集 Y 在它自己内既开又闭，虽然 Y 在 X 内可以既不开又不闭. 我们称拓扑空间$(Y, \mathscr{U})$为空间$(X, \mathscr{T})$的**子空间**. 更正式地说，我们称拓扑空间 $(Y, \mathscr{U})$ 为另一个空间$(X, \mathscr{T})$的子空间当且仅当 $Y \subset X$ 并且 $\mathscr{U}$ 是 $\mathscr{T}$ 的相对化.

值得注意，若 $(Y, \mathscr{U})$ 为 $(X, \mathscr{T})$ 的子空间，而 $(Z, \mathscr{V})$ 为 $(Y, \mathscr{U})$ 的子空间则 $(Z, \mathscr{V})$ 为 $(X, \mathscr{T})$ 的子空间. 这个传递关系常常在一些没有明显指出的情况下被利用.

假设 $(Y, \mathscr{U})$ 为 $(X, \mathscr{T})$ 的子空间，又设 A 为 Y 的子集，则 A 可以是 $\mathscr{T}$-闭集，也可以是 $\mathscr{U}$-闭集，点 y 可以是 A 的 $\mathscr{U}$-聚点，也可以是 A 的 $\mathscr{T}$-聚点并且 A 有一个 $\mathscr{T}$-闭包和一个 $\mathscr{U}$-闭包. 显然这些不同概念之间的关系是重要的.

16 定理. 设 $(X, \mathscr{T})$ 为拓扑空间，$(Y, \mathscr{U})$ 为其子空间，又设 A 为 Y 的子集，则

(a) 集 A 为 $\mathscr{U}$-闭集当且仅当它是 Y 和一个 $\mathscr{T}$-闭集的交.

(b) Y 的点 y 为 A 的 $\mathscr{U}$-聚点当且仅当它为 $\mathscr{T}$-聚点.

(c) A 的 $\mathscr{U}$-闭包是 Y 和 A 的 $\mathscr{T}$-闭包的交.

证明. 因为集 A 在 Y 内闭当且仅当 $Y \sim A$ 可表成 $V \cap Y$ 的形式,其中 V 为某个 $\mathscr{T}$-开集,而这又当且仅当对某个 $\mathscr{T}$ 中的 V 有 $A = (X \sim V) \cap Y$,故 (a) 获证.

而 (b) 直接由相对拓扑和聚点的定义即可推出.

至于 (c),则因 A 的 $\mathscr{U}$-闭包为 A 和它的所有 $\mathscr{U}$-聚点的集的并,故再由 (b) 便知它是 Y 与 A 的 $\mathscr{T}$-闭包的交.|

若 $(Y, \mathscr{U})$ 为 $(X, \mathscr{T})$ 的子空间并且 Y 是 X 的开集,则每一个在 Y 内开的集也是 X 的开集,因为它为某个开集和 Y 的交. 显然,处处以"闭"来代替"开"所得的相似结论也成立. 然而,一般的说从一个集在某个子空间内开或者闭,并不能得出该集在 X 内的情况. 假设 X 是两个集 Y 和 Z 的并,而 A 为 X 的子集,它使得 $A \cap Y$ 在 Y 内开,并且 $A \cap Z$ 在 Z 内开,则我们自然期望此时 A 为 X 的开集,可是这并不成立,因为如果 Y 为 X 的任意子集并且 $Z = X \sim Y$,那末 $Y \cap Y$ 和 $Y \cap Z$ 分别在 Y 和 Z 内开. 但有一种重要的特殊情形,对于这种情形该结论成立. 我们称两个集 A 和 B 在拓扑空间 X 内**分离**当且仅当 $A^- \cap B$ 和 $A \cap B^-$ 均为空集. 在这个分离性的定义中,虽然包含有在 X 内的闭包算子,然而这并不表明它对空间 X 的依赖性,因为 A 和 B 在 X 内分离当且仅当 A 和 B 彼此都不含有另一个集的点和聚点. 根据上一定理的 (b),这个条件还可以通过关于 $A \cup B$ 的相对拓扑来描述:即 A 和 B 在 $A \cup B$ 内闭(这等价于,A(或 B) 在 $A \cup B$ 内既开又闭) 并且 A 和 B 互不相交. 作为例子,开区间$(0,1)$ 和 $(1,2)$是带有通常拓扑的实数集的分离子集;但 $(0,1)$ 与闭区间 $[1,2]$ 并不分离,因为 1 是 $[1,2]$ 的元,同时又是 $(0,1)$ 的聚点.

关于分离性有三个以后需要的定理.

17 定理. 若 Y 和 Z 为拓扑空间 X 的子集,并且 Y 和 Z 同为闭集或同为开集,则 $Y \sim Z$ 与 $Z \sim Y$ 分离.

证明. 假设 Y 和 Z 是 X 的闭子集,则 Y 和 Z 在 $Y \cup Z$ 内闭,故 $Y \sim Z = ((Y \cup Z) \sim Z)$ 和 $Z \sim Y$ 在 $Y \cup Z$ 内开,即 $Y \sim Z$ 和

$Z \sim Y$ 在 $(Y \sim Z) \cup (Z \sim Y)$ 内开，再注意 $Y \sim Z$ 和 $Z \sim Y$ 关于该集互为余集便知它们在 $(Y \sim Z) \cup (Z \sim Y)$ 内闭，从而 $Y \sim Z$ 和 $Z \sim Y$ 分离.

对偶的讨论可以应用于 Y 和 Z 是 X 的开子集的情形. ∎

18 定理. 设 X 为拓扑空间，它是子集 Y 和 Z 的并并且 $Y \sim Z$ 和 $Z \sim Y$ 分离，则 X 的子集 A 的闭包为 $A \cap Y$ 在 Y 内的闭包和 $A \cap Z$ 在 Z 内的闭包的并.

证明. 因为两个集的并的闭包为闭包的并，故 $A^- = (A \cap Y)^- \cup (A \cap Z \sim Y)^-$，于是 $A^- \cap Y = [(A \cap Y)^- \cap Y] \cup [(A \cap Z \sim Y)^- \cap Y]$. 又从 $(Z \sim Y)^-$ 与 $Y \sim Z$ 不相交可推出 $(Z \sim Y)^- \subset Z$，即 $(A \cap Z \sim Y)^-$ 为 $(A \cap Z)^- \cap Z$ 的一个子集. 相似地，$A^- \cap Z$ 为 $(A \cap Z)^- \cap Z$ 和 $(A \cap Y)^- \cap Y$ 的一个子集的并. 从而 $A^- = (A^- \cap Y) \cup (A^- \cap Z) = [(A \cap Y)^- \cap Y] \cup [(A \cap Z)^- \cap Z]$. ∎

19 系. 设 X 为拓扑空间，它是子集 Y 和 Z 的并并且 $Y \sim Z$ 和 $Z \sim Y$ 分离，则 X 的子集 A 为闭(开)集，只要 $A \cap Y$ 在 Y 内闭(开)并且 $A \cap Z$ 在 Z 内闭(开).

证明. 若 $A \cap Y$ 和 $A \cap Z$ 分别在 Y 和 Z 内闭，则由上一定理可知 A 必须与它的闭包相同，即为闭集.

若 $A \cap Y$ 和 $A \cap Z$ 分别在 Y 和 Z 内开，则 $Y \cap X \sim A$ 和 $Z \cap X \sim A$ 在 Y 和 Z 内闭，故 $X \sim A$ 为闭集，即 A 为开集. ∎

连　通　集

我们称拓扑空间$(X, \mathscr{T})$为**连通**当且仅当 X 不是两个非空分离子集的并. 称 X 的子集 Y 为连通当且仅当带有相对拓扑的拓扑空间 Y 为连通. 等价地，Y 为连通当且仅当 Y 不是两个非空分离子集的并. 从分离性的讨论还可以得到另一种等价形式：Y 为连通当且仅当在 Y 内既开又闭的 Y 的子集只有 Y 和空集. 从这个形式我们立即可推出任何平庸空间为连通，而包含多于一个点的离散空间为非连通. 又带有通常拓扑的实数集为连通（问题 1.J）. 但

有理数集关于实数集的通常拓扑的相对拓扑非连通。(因为对任何无理数 a，集 $\{x:x<a\}$ 和 $\{x:x>a\}$ 分离.)

20 定理. 连通集的闭包仍为连通集.

证明. 假设 Y 为某个拓扑空间的连通子集并且 $Y^- = A\cup B$，其中 A 和 B 在 Y^- 内既开又闭，则 $A\cap Y$ 和 $B\cap Y$ 在 Y 内既开又闭，故从 Y 为连通可推出这两个集中必有一个为空集. 假设 $B\cap Y$ 为空集，则 Y 为 A 的子集，因为 A 在 Y^- 内闭，从而 Y^- 为 A 的子集，于是 B 为空集，即 Y^- 连通.|

这个定理有一个表面上显得更强的形式，即若 Y 为 X 的连通子集，又 Z 满足 $Y\subset Z\subset Y^-$，则 Z 为连通. 事实上它就是上一定理应用于带有相对拓扑的 Z 的一个明显的推论.

21 定理. 设 $\mathscr{A}$ 为某个拓扑空间的连通子集的族，若 $\mathscr{A}$ 中没有两个元为分离，则 $\bigcup\{A:A\in\mathscr{A}\}$ 为连通.

证明. 设 C 为 $\mathscr{A}$ 的所有元的并，又设 D 在 C 内既开又闭，则对 $\mathscr{A}$ 的每一个元 A，$A\cap D$ 在 A 内既开又闭，又因 A 为连通，故 $A\subset D$ 或 $A\subset C\sim D$. 另外，若 A 和 B 为 $\mathscr{A}$ 的元，则不可能有 $A\subset D$ 和 $B\subset C\sim D$，因为在这种情况下 A 和 B 分别为分离的集 D 和 $C\sim D$ 的子集，从而也是分离的. 因此，或者 $\mathscr{A}$ 的每一个元为 $C\sim D$ 的子集并且 D 为空集，或者 $\mathscr{A}$ 的每一个元为 D 的子集并且 $C\sim D$ 为空集.|

拓扑空间的**连通区**指的是极大的连通子集，即为一个连通子集并且它不真包含在另外的连通子集内. 又子集 A 的连通区是指带有相对拓扑的 A 的连通区，即为 A 的一个极大的连通子集. 若空间为连通，则它是它自己仅有的连通区. 若空间离散，则每一个连通区仅由一个点组成. 自然，还有许多不是离散的空间，它的连通区也仅由一个点组成，譬如带有通常拓扑的相对拓扑的有理数空间就是一个例子.

22 定理. 拓扑空间的每一个连通子集包含在某个连通区内并且每一个连通区皆为闭集. 若 A 和 B 为空间的不同连通区，则 A 和 B 分离.

证明．设 A 为拓扑空间的非空连通子集，C 为包含 A 的所有连通子集的并，则由上一定理可知 C 也是连通子集．若 D 为包含 C 的一个极大连通子集，则因由 C 的作法有 $D\subset C$，故 $C=D$，即 C 为一个连通区．（若 A 为空集并且空间为非空，则因由一个点组成的集包含在某个连通区内，故 A 更是如此．）

因为每一个连通区 C 均为连通，故由定理 1.20，闭包 C^- 也为连通，从而 C 与 C^- 相同，即 C 为闭集．

若 A 和 B 为不同的连通区并且不分离，则由定理 1.21，它的并也为连通，于是发生矛盾．|

用一段足以引起注意的话来结束我们对连通区的讨论将是有益的，若 x 和 y 两点属于拓扑空间的同一个连通区，则它恒位于该空间的任何分离的同一侧；即若空间为分离的集 A 和 B 的并，则 x 和 y 同属于 A，或同属于 B．不过其逆不真，即可能出现：虽然两点恒位于分离的同一侧，但它却属于不同的连通区．（见问题 1.P.）

问　　题

A　最大和最小拓扑

(a) X 的任何一族拓扑的交是 X 的拓扑．

(b) X 的两个拓扑的并不必是 X 的拓扑（除非 X 至多由两个点组成）．

(c) 对 X 的任何一族拓扑，在小于这个族中每个元的拓扑中有唯一一个最大的拓扑，并且在大于这个族中每个元的拓扑中有唯一一个最小的拓扑．

B　从邻域系导出拓扑

(a) 设 $(X,\mathscr{T})$ 为拓扑空间，又对 X 中的每一个 x，命 $\mathscr{U}_x$ 为 x 的所有邻域的族，则有：

(i) 若 $U\in\mathscr{U}_x$，则 $x\in U$；

(ii) 若 U 和 V 为 $\mathscr{U}_x$ 的元，则 $U\cap V\in\mathscr{U}_x$；

(iii) 若 $U\in\mathscr{U}_x$ 并且 $U\subset V$，则 $V\in\mathscr{U}_x$；

(iv) 若 $U\in\mathscr{U}_x$，则有 $\mathscr{U}_x$ 的某个元 V 使得 $V\subset U$ 并且对每一个 V 中的 y 有 $V\in\mathscr{U}_y$（即 V 是它自己的每一个点的邻域）．

(b) 若 $\mathscr{U}$ 为一个函数，它将 X 中的每一个 x 映为一个满足 (i)，(ii)

和 (iii) 的族 $\mathscr{U}_x$，则所有使得当 $x\in U$ 时有 $U\in\mathscr{U}_x$ 的集 U 的族 $\mathscr{T}$ 为 X 的一个拓扑．如果 (iv) 也满足，那末 $\mathscr{U}_x$ 恰好就是 x 关于拓扑 $\mathscr{T}$ 的邻域系．

注　关于描述拓扑空间的各种方法已经有充分的研究．Kuratowski 的三个闭包公理可以用一个单一的条件来代替，这是由 Monteiro[1] 和 Iseki[1] 证明的．此外可以利用分离概念来作为原始概念(Wallace[1]，Krishna Murti[1] 和 Szymanski [1])，又导集概念也可以用来作为原始概念 (情况和文献见 Monteiro[2]和 Ribeiro[3])．至于各种运算之间的关系已由 Stopher[1] 所研究．

C　从内部算子导出拓扑

设 i 为一个算子，它变 X 的子集为 X 的子集，又设 $\mathscr{T}$ 为所有使得 $A^i=A$ 的子集的族，问在何种条件下 $\mathscr{T}$ 是 X 的拓扑并且 ^{i} 是关于该拓扑的内部算子？

D　T_1-空间内的聚点

拓扑空间叫做 **T_1-空间**，当且仅当每一个仅由一个点组成的集为闭集．(有时我们不很精确的把它说成"点是闭的"．)

(a) 对任何集 X 有唯一一个最小拓扑 $\mathscr{T}$ 使得 $(X,\mathscr{T})$ 是 T_1-空间．

(b) 若 X 是无限集，$\mathscr{T}$ 是使得 $(X,\mathscr{T})$ 为 T_1-空间的最小拓扑，则 $(X,\mathscr{T})$ 为连通．

(c) 若 $(X,\mathscr{T})$ 是 T_1-空间，则每一个子集的所有聚点的集为闭集．一个更为深刻的结果 (C. T. Yang) 是：每一个子集的所有聚点的集为闭集的充要条件为对 X 中的每一个 x，$\{x\}$ 的所有聚点的集为闭集．

注　有一系列依次增强的要求可以加在空间的拓扑上．我们称拓扑空间为 **T_0-空间**当且仅当对不同的点 x 与 y，其中的一个点有邻域使得另一个点不属于它．一种稍许不同的说法是：空间为 T_0-空间当且仅当对不同的点 x 与 y 或者 $x\notin\{y\}^-$，或者 $y\notin\{x\}^-$．

以后我们还要定义 T_2 和 T_3-空间．这一术语是属于 Alexandroff 和 Hopf[1] 的．

E　Kuratowski 的闭包和余集问题

若 A 是拓扑空间的一个子集，则利用闭包和余集算子从 A 至多可作出 14 个集．另外，存在实数(关于通常拓扑)的子集，从它的确可作出 14 个不同的集．(首先注意若 A 是开集的闭包，则 A 是 A 的内部的闭包；这就是说，对如

此的集，$A = A'^{-'-}$，其中 $'$ 表示余集算子.）

F 关于具有可数基的空间的习题

若空间的拓扑具有可数基，则每一个基含有一个可数子族，它也是一个基.

G 关于稠密集的习题

若 A 在拓扑空间中稠密，U 是开集，则 $U \subset (A \cap U)^-$.

H 聚点

设 X 为一个空间，它的每一个子空间是 Lindelöf 空间，又设 A 为一个不可数子集，B 是由 A 中所有这样的点 x 所组成的子集，它使得 x 的每一个邻域包含 A 中的不可数多个点，则 $A \sim B$ 为可数集，从而 B 的每一个点的邻域包含有 B 的不可数多个点.

注 集 A 的聚点可以按照 A 和该点的邻域的交的最小基数加以分类. 若对拓扑的基也加以基数的限制，则可得到几个不等式. 定理 1.13，1.14 和 1.15 都能推广到带有给定基数的基的空间.

I 序拓扑

设 X 为关于反对称（不可能有 $x<x$）关系 $<$ 的线性有序集，则**序拓扑**（$<$ 序拓扑）是指由所有形如：$\{x: x<a\}$ 或 $\{x: a<x\}$ 对某个 $a \in X$，的集所组成的子基所确定的拓扑.

(a) X 的序拓扑是使得序在下列意义下为连续的最小拓扑：若 a 和 b 是 X 的元并且 $a<b$，则有 a 的邻域 U 和 b 的邻域 V 使当 $x \in U, y \in V$ 时有 $x<y$.

(b) 设 Y 是关于 $<$ 为线性有序的集 X 的子集，则 Y 关于 $<$ 也为线性有序集，但 Y 的 $<$ 序拓扑可以不是 X 的 $<$ 序拓扑的相对拓扑.

(c) 若 X 关于序拓扑为连通，则 X 是序完备的（即每一个有上界的非空集必有上确界）.

(d) 若有 X 中的 a 和 b 使得 $a<b$，并且不存在 c 使得 $a<c<b$，则 X 为非连通. 如此的序叫做**有裂缝的**. 证明 X 关于序拓扑为连通当且仅当 X 是序完备的和无裂缝的.

J 实数的性质

设 R 是带有通常拓扑的实数集。

(a) 包含多于一个元的实数的加法子群或者在 R 中稠密，或者有一个最小的正元素。特别地，有理数集在 R 中稠密。

(b) 实数的通常拓扑与序拓扑一致。又通常拓扑具有可数基。

(c) R 的闭子群或者为可数集，或者与 R 相同。连通子群或者为 $\{0\}$，或者为 R，且开子群必与 R 相同。

(d) (A. P. Morse) **真区间**是指包含多于一个点的半开，开，或闭的区间。若 $\mathscr{A}$ 是真区间的任意的族，则有 $\mathscr{A}$ 的可数子族 $\mathscr{B}$ 使得 $\bigcup\{B: B\in\mathscr{B}\} = \bigcup\{A: A\in\mathscr{A}\}$。(注意互不相交的真区间的族必为可数，并且证明除 $\bigcup\{A: A\in\mathscr{A}\}$ 的可数多个点外都是 $\mathscr{A}$ 的元的内点.)

(e) 所有真区间的族 $\mathscr{S}$ 是 R 的离散拓扑 $\mathscr{T}$ 的子基。空间 $(R, \mathscr{T})$ 不是 Lindelöf 空间，虽然由 $\mathscr{S}$ 的元组成的每一个覆盖有可数子覆盖。(与 Alexander 的定理 5.6 对比.)

注 实数的更进一步的性质将陈述于下一问题中。

K 半开区间空间

设 X 为实数集，又设 $\mathscr{T}$ 为 X 的拓扑，它具有所有半开区间 $[a,b) = \{x: a\leq x<b\}$ 的族 $\mathscr{B}$ 的基，其中 a 和 b 为实数。将集的 $\mathscr{T}$-聚点叫做**右方聚点**，相似的可以定义左方聚点。

(a) 基 $\mathscr{B}$ 的元是既开又闭的。又空间 $(X, \mathscr{T})$ 为非连通。

(b) 空间 $(X, \mathscr{T})$ 为可分，但不具有可数基。(对 X 中的每一个 x 而言，每一个基必定包含一个以 x 为下确界的集.)

(c) $(X, \mathscr{T})$ 的每一个子空间是 Lindelöf 空间。(见问题 1.J(d).)

(d) 若 A 是一个实数的集，则 A 的所有非右方聚点的点的集为可数集。更一般地，A 的所有非右方与非左方聚点的点的集为可数集。(见问题1.H.)

(e) $(X, \mathscr{T})$ 的每一个子空间为可分。

L 半开矩形空间

设 Y 为 $X\times X$，其中 X 为上一问题中的空间，又设 $\mathscr{U}$ 是具有所有形如 $A\times B$ 的集为基的拓扑，其中 A 和 B 为上一问题中的拓扑 $\mathscr{T}$ 的元。

(a) 空间 $(Y, \mathscr{U})$ 为可分。

(b) 空间 $(Y, \mathscr{U})$ 包含有不可分的子空间。(例如 $\{(x,y): x+y=1\}$.)

(c) 空间 $(Y, \mathscr{U})$ 不是 Lindelöf 空间。(若 Y 的每一个开覆盖有可数

子覆盖，则每一个闭子空间具有相同的性质。再考察 $\{(x,y):x+y=1\}$.)

注 问题 1.K 和 1.L 所描述的空间是一般拓扑中常见的反例。我们在问题 4.I 中将再列举其它的病态性质。P. R. Halmos 首先注意到 Lindelöf 空间的乘积（在第三章的特定意义下）可以不再是 Lindelöf 空间。

M 关于第一和第二可数性公理的例子（序数）

设 Ω' 为所有小于或等于第一个不可数序数 Ω 的序数的集。设 X 为 $\Omega'\sim\{\Omega\}$，又设 ω 为所有非负整数的集，并且每一个都带有序拓扑。

(a) ω 是离散的并且满足第二可数性公理。

(b) X 满足第一但不满足第二可数性公理。

(c) Ω' 不满足两个可数性公理；若 U 是 Ω' 的可分子空间，则 U 自己是可数集。

N 可数链条件

拓扑空间叫做满足**可数链条件**当且仅当每一个互不相交的开集的族为可数。可分空间必满足可数链条件，但其逆不真。（考虑一个不可数集，它的拓扑由空集和可数集的余集组成。）此外，还存在有更复杂的例子，它满足第一可数性公理并且可分，但不满足第二可数性公理（见问题 5.M 的 Helly 空间）。

O 欧几里德平面

欧几里德平面是指所有实数偶的集，并且它带有以所有笛卡儿乘积 $A\times B$ 为基的拓扑，即所谓平面上的**通常拓扑**，其中 A 和 B 是具有有理端点的开区间。这个基为可数，因而平面为可分。

(a) 平面的通常拓扑具有由所有开圆 $\{(x,y):(x-a)^2+(y-b)^2<r^2\}$ 所组成的基，其中 a, b 和 r 为有理数。

(b) 设 X 为平面内所有至少有一个坐标为无理数的点的集，X 具有相对拓扑，则 X 为连通。

P 关于连通区的例子

设 X 为欧几里德平面的如下的子集，其拓扑为通常拓扑的相对拓扑。对每一个正整数 n，命 $A_n=\{1/n\}\times[0,1]$，其中 $[0,1]$ 为闭区间，再命 X 为所有集 A_n 的并再添加 $(0,0)$ 和 $(0,1)$ 两个点，则 $\{(0,0)\}$ 和 $\{(0,1)\}$ 为 X

的连通区，但对 X 的每一个既开又闭的子集，或者这两点都不属于它，或者这两点皆属于它。

Q　关于分离集的定理

若 X 为连通拓扑空间，Y 为连通子集，并且 $X \sim Y = A \cup B$，其中 A 和 B 是分离的，则 $A \cup Y$ 为连通。

R　关于连通集的有限链定理

设 $\mathscr{A}$ 为拓扑空间的连通子集的族并且满足条件：若 A 和 B 属于 $\mathscr{A}$，则有 $\mathscr{A}$ 的元的有限序列 $A_0, A_1, \cdots, A_n$ 使得 $A_0 = A$，$A_n = B$ 并且对每一个 i，集 A_i 和 A_{i+1} 都不分离，则 $\bigcup\{A: A \in \mathscr{A}\}$ 为连通。另外，从这一事实又可导出定理 1.21。

S　局部连通空间

拓扑空间叫做**局部连通**当且仅当对每一个 x 和 x 的每一个邻域 U，U 中含 x 的连通区为 x 的一个邻域。

(a) 局部连通空间的开子集的每一个连通区仍为开集。

(b) 拓扑空间为局部连通当且仅当所有连通开子集的族为拓扑的基。

(c) 若局部连通空间 X 的点 x 和 y 属于不同的连通区，则存在 X 的分离子集 A 和 B 使得 $x \in A$，$y \in B$ 并且 $X = A \cup B$。

注　关于局部连通空间的许多其它性质和推广，见 G. T. Whyburn[1]和 R.L. Wilder [1]。

T　Brouwer 收缩定理

该定理通常陈述如下。设 X 为满足第二可数性公理的拓扑空间。X 的子集的性质 P 叫做**诱导的**当且仅当可数的闭集套的每一个元具有性质 P 时，它的交也具有性质 P。又集合 A 叫做关于 P **不可约**，当且仅当 A 没有具有性质 P 的真闭子集，则：当 X 的闭子集 A 具有性质 P 时存在 A 的一个不可约闭子集也具有性质 P。

借助于一个集族（所有具有性质 P 的集的族）该定理可以得到更形式地陈述。

(a) 在这种形式下陈述并且证明该定理，假定拓扑空间具有性质：它的每一个子空间均为 Lindelöf 空间。

(b) 若 $(X, \mathscr{T})$ 为任意的拓扑空间，则是否能够证实任何这种一般类型的结果？（见预备知识定理 25。）

第二章 Moore-Smith 收敛

引　　论

这一章主要讨论 Moore-Smith 收敛．我们将证明空间的拓扑完全可以通过收敛来描述，本章的较大篇幅也集中在这个方面．我们也将对那些能够描述成关于某一个拓扑的收敛性的收敛概念进行刻划．这个方案在目的上是与 Kuratowski 的闭包算子理论相似的，它成为确定某些拓扑的一种有用而又直观的自然方法．然而，收敛理论的重要性超过了这一特殊的应用，因为分析的基本结构就是极限过程．我们有兴趣的是展开这样的一种理论，它可以应用于序列和重序列的收敛，并且也可以应用于序列的求和以及微分和积分．应当指出，这里我们所展开的理论决不是一种唯一可能的理论，但无疑它是最自然的．

序列的收敛性给出了我们所要展开的理论的雏型，因此我们先提出关于序列的一些定义和定理来说明这个雏型，这实际上是以后所证明的定理的特殊情形．

序列是指在非负整数集 ω 上的函数．而实数序列为值域是实数集的子集的序列．又序列 S 在 n 处的值记为 S_n 或 $S(n)$．我们称 S 在集 A 内当且仅当对每一个非负整数 n 有 $S_n \in A$，又称 S 基本上在 A 内当且仅当存在整数 m 使得当 $n \geqslant m$ 时有 $S_n \in A$．而实数序列关于通常拓扑收敛于数 s 当且仅当它基本上在 s 的每一个邻域内．利用这些定义便知，若 A 为一个集，则点 s 属于 A 的闭包当且仅当存在 A 中的序列收敛于 s，并且 s 为 A 的聚点当且仅当存在 $A \sim \{s\}$ 中的序列收敛于 s．

我们还需要去构造一个序列的子序列．一个序列 S 可能不收敛于任何点，但还可以用适当方法由它构造出另一个序列，后者是

收敛的．即对 ω 中每个 i 我们希望选取整数 N_i 使得 S_{N_i} 收敛．也就是说，希望找出整数序列 N 使得合成 $S\circ N(i)=S_{N_i}=S(N(i))$ 收敛．如果没有其它的要求，那末这是容易作出的，譬如，若对每一个 i，命 $N_i=0$，则 $S\circ N$ 就收敛于 S_0，因为对每一个 i 有 $S\circ N(i)=S_0$． 自然我们还必须加上另外的条件使得子序列的性态与序列在大整数处的值的性态有关．通常的条件是 N 严格地单调增加，即当 $i>j$ 时有 $N_i>N_j$．这个条件过于苛刻，我们再把它换成当 i 变大时 N_i 也变大． 更精确的说，T 为序列 S 的子序列当且仅当存在非负整数序列 N 使得 $T=S\circ N$（这等价于对每一个 i 有 $T_i=S_{N_i}$）并且对每一个整数 m 存在整数 n 使当 $i\geqslant n$ 时 $N_i\geqslant m$．

一个给定序列的一切子序列所收敛的点的集满足一种通过减弱收敛的要求而得到的条件．我们称序列 S 常常在集 A 内当且仅当对每一个非负整数 m 存在整数 n 使得 $n\geqslant m$ 并且 $S_n\in A$．这相当于 S 不是基本上在 A 的余集内；直观的说，一个序列常常在 A 内也就是它总能够保持回到 A．又称点 s 为序列 S 的聚点当且仅当 S 常常在 s 的每一个邻域内．显然，若一个实数序列基本上在某个集内，则它的每一个子序列也基本上在该集内，从而若一个序列收敛于某个点，则它的每一个子序列也收敛于该点．另外，序列的每一个聚点均为某个子序列的极限点，并且其逆亦真．

对以上定义与断言加以修改要使它们能适用于任何拓扑空间，然而不幸的是，在这种一般情况下有关的定理却并不成立(附于章后的问题)．注意到在证明关于实数序列的定理时只利用了整数的很少的性质，这种不愉快的情形就能够得到补救． 几乎明显的是(虽然我们不给出证明)我们只须要序的某些性质．严格地说，序列的收敛不仅包含非负整数 ω 上的函数 S，而且包含 ω 的序 $\geqslant$．因此为方便起见，在关于收敛的讨论中我们稍微修改一下序列的定义，认为序列是一个序偶 $(S,\geqslant)$，其中 S 是整数上的函数，而我们讨论的则是 $(S,\geqslant)$ 的收敛（$(S,\leqslant)$ 的收敛也是有意义的，但完全不同)．又当不会引起误解时，序的叙述可以略去，这

时序列 S 的收敛就表示 $(S, \geqslant)$ 的收敛.

关于序列概念，如果再带上一个变化范围，那末就更为方便，此因当 S 为非负整数 ω 上的函数时就把 $\{S_n, n \in \omega, \geqslant\}$ 定义为 $(S, \geqslant)$. 又若 A 为 ω 的子集，则 $\{S_n, n \in A, \geqslant\}$ 的收敛也是有意义的并且它与 $(S, \geqslant)$ 的收敛有关.

在这么一个长篇引言之后，收敛概念几乎自明，但还缺少一个这样的事实. 也就是究竟要用到序 $\geqslant$ 的哪些性质? 这些性质下面我们将加以指出，并且利用它，关于序列收敛的通常的讨论在稍微修改的情况下即可成立.

1 注记. E. H. Moore 的序列的无序和的研究[1] 引出了收敛的理论 (Moore 和 Smith [1]). 我们所用的子序列概念的推广也属于 Moore[2]. Garrett Birkhoff[3] 把 Moore-Smith 收敛应用于一般拓扑；而我们所给出的理论的形式则接近于 J. W. Tukey [1]. 关于一种非常易读的解说见 Mcshane [1].

在本章最后的问题中包含有另外的收敛理论的一个简要的讨论以及适当的文献.

有 向 集 和 网

我们称关系 $\geqslant$ 使集 D 为**有向**，假如 D 为非空并且满足:

(a) 若 m, n 和 p 为 D 的元并且使得 $m \geqslant n$, $n \geqslant p$, 则 $m \geqslant p$;

(b) 若 $m \in D$, 则 $m \geqslant m$;

(c) 若 m 和 n 为 D 的元，则有 D 中的 p 使得 $p \geqslant m$ 并且 $p \geqslant n$.

又称在序 $\geqslant$ 中 m 在 n 之后或 n 在 m 之前当且仅当 $m \geqslant n$. 若利用关系的通常术语(见预备知识)，则条件 (a) 表明 $\geqslant$ 在 D 上为传递，或使 D 为半序集，而条件 (b) 表明 $\geqslant$ 在 D 上为反身. 至于条件 (c) 则是此处所特有的.

下面给出几个通过关系使集为有向的自然的例子. 首先实数

集和非负整数集ω一样关于$\geqslant$为有向；而零是ω的元，同时它在序$\leqslant$中在每一个其它的元之后．特别值得注意的是拓扑空间内每一点的所有邻域的族关于$\subset$为有向(两个邻域的交仍为一个邻域，并且它在序$\subset$中同时在这两个邻域之后)；另外，一个集的一切有限子集的族关于$\supset$为有向．最后，任何集关于这样的$\geqslant$为有向，即$x\geqslant y$对所有元x和y成立，此时每一个元同时在它自己和每一个其它的元之后．

有向集是指$(D,\geqslant)$并且$\geqslant$使D为有向（有时这叫做**有向系**）．又**网**指的是$(S,\geqslant)$，其中S为一个函数并且$\geqslant$使S的定义域为有向(有时这也叫做有向集)．若S为一个函数，它的定义域包含D并且D关于$\geqslant$为有向集，则$\{S_n, n\in D, \geqslant\}$表示网$(S|D,\geqslant)$，其中$S|D$是$S$在$D$上的限制．我们称网$\{S_n, n\in D, \geqslant\}$**在集$A$内**当且仅当对一切$n$有$S_n\in A$，又称它**最终地在$A$内**当且仅当存在$D$的元$m$使得当$n\in D$并且$n\geqslant m$时有$S_n\in A$．我们再称它**常常在$A$内**当且仅当对每一个$D$中的$m$存在$D$中的$n$使得$n\geqslant m$并且$S_n\in A$．若$\{S_n, n\in D, \geqslant\}$常常在$A$内，则所有使得$S_n\in A$的$D$的元$n$的集$E$具有性质：对每一个$m\in D$存在$p\in E$使得$p\geqslant m$．$D$的具有如此性质的子集叫做**共尾子集**．显然$D$的每一个共尾子集$E$也关于$\geqslant$为有向集，因为对$E$的元$m$和$n$有$D$中的$p$使得$p\geqslant m$，$p\geqslant n$，而此时又有$E$的元$q$它在$p$之后．另外我们有如下的明显的等价性：网$\{S_n, n\in D, \geqslant\}$常常在集$A$内当且仅当存在$D$的共尾子集映到$A$内，而这又当且仅当该网不是基本上在$A$的余集内．

我们称网$(S,\geqslant)$在拓扑空间$(X,\mathscr{T})$内关于$\mathscr{T}$**收敛于**s当且仅当它基本上在s的每一个$\mathscr{T}$-邻域内．虽然收敛的概念依赖于函数S，拓扑$\mathscr{T}$和序$\geqslant$，但在不会引起误解的情况下我们叙述时将略去$\mathscr{T}$或$\geqslant$或它们两者，简称为网S(或网$\{S_n, n\in D\}$)收敛于s．若X为离散空间（每一个子集皆为开集），则网S收敛于点s当且仅当S基本上在$\{s\}$内，即S从某个点起都恒等于s．另一方面，若X为平庸空间（X和空集为仅有的开集），则X中的每一

个网都收敛于X的每一个点. 因此，一个网可以同时收敛于几个不同的点.

借助于收敛的概念我们容易描述集的聚点，集的闭包，因而事实上也就描述了空间的拓扑. 而它的论证则是通常对实数序列的处理的一种简单的变形.

2 定理. 设X为拓扑空间，则:

(a) 点s为X的子集A的聚点当且仅当存在$A\sim\{s\}$中的网收敛于s.

(b) 点s属于X的子集A的闭包当且仅当存在A中的网收敛于s.

(c) X的子集A为闭集当且仅当不存在A中的网收敛于$X\sim A$的点.

证明. 若s为A的聚点，则对s的每一个邻域U存在A的点S_U，它属于$U\sim\{s\}$. 因为s的所有邻域的族$\mathscr{U}$关于$\subset$为有向集，并且若U和V为s的邻域，同时满足$V\subset U$，则$S_V\in V\subset U$，故网$\{S_U, U\in\mathscr{U}, \subset\}$收敛于$s$. 另一方面，若有$A\sim\{s\}$中的网收敛于$s$，则该网在$s$的每一个邻域内有值，即$A\sim\{s\}$确与$s$的每一个邻域相交. 这就证明了命题(a).

今证(b)，首先回顾一下：A的闭包是由A和A的一切聚点所组成. 因为根据上一段的证明对A的每一个聚点s有A中的网收敛于s，又对A的每一点s，显然任何在其定义域的每一个元处取值s的网都收敛于s，故对A的闭包的每一点有A中的网收敛于它. 反之，若有A中的网收敛于s，则s的每一个邻域与A相交，即s属于A的闭包.

命题(c)现在则是明显的. ▎

我们已经看到：在一般的拓扑空间内，一个网可以收敛于几个不同的点. 但也有这样的空间，在其内收敛按下列意义唯一：若网S收敛于点s并且也收敛于点t，则$s=t$. 我们称拓扑空间为 **Hausdorff 空间**（T_2-空间或**分离空间**）当且仅当x和y为空间的不同的点时存在x和y的互不相交的邻域.

3 定理． 拓扑空间为 Hausdorff 空间当且仅当空间中的每一个网都至多收敛于一个点．

证明．若 X 为 Hausdorff 空间，s 和 t 为 X 的不同的点，则有 s 和 t 的互不相交的邻域 U 和 V，又任意的网都不可能同时基本上在两个互不相交的集内，故显然不存在 X 中的网同时收敛于 s 和 t．

今证其逆，假设 X 不是 Hausdorff 空间并且 s 和 t 是不同的点，它使得 s 的每一个邻域与 t 的每一个邻域相交．又设 $\mathscr{U}_s$ 为 s 的邻域系，$\mathscr{U}_t$ 为 t 的邻域系，则 $\mathscr{U}_s$ 和 $\mathscr{U}_t$ 关于 $\subset$ 为有向集．若我们在笛卡儿乘积 $\mathscr{U}_s\times\mathscr{U}_t$ 内，规定 $(T,U)\geqslant(V,W)$ 当且仅当 $T\subset V$ 及 $U\subset W$，则易见 $\mathscr{U}_s\times\mathscr{U}_t$ 关于 $\geqslant$ 为有向集．因为对 $\mathscr{U}_s\times\mathscr{U}_t$ 中的每一个 (T,U)，交 $T\cap U$ 为非空，故从 $T\cap U$ 可选出一个点 $S_{(T,U)}$．又由于当 $(V,W)\geqslant(T,U)$ 时 $S_{(V,W)}\in V\cap W\subset T\cap U$，所以网 $\{S_{(T,U)},(T,U)\in\mathscr{U}_s\times\mathscr{U}_t,\geqslant\}$ 同时收敛于 s 和 t．|

若 $(X,\mathscr{T})$ 为 Hausdorff 空间并且网 $\{S_n,n\in D,\geqslant\}$ 在 X 中收敛于 s，则记作 $\mathscr{T}\text{-}\lim\{S_n,n\in D,\geqslant\}=s$，当不会引起误解时，又简记为 $\lim\{S_n:n\in D\}=s$ 或 $\lim\limits_n S_n=s$．一般的说，"极限"的使用只限于对 Hausdorff 空间中的网，这样使得等量替换规则仍然成立．若 $\lim\{S_n:n\in D\}=s$ 且 $\lim\{S_n:n\in D\}=t$，因为我们总是在恒同意义下使用等式，就得到 $s=t$．其实在非 Hausdorff 空间的情况下，我们也偶尔会应用记号 $\lim\limits_n S_n=s$ 来表示 S 收敛于 s．

在上一定理的证明中所采用的方法是很有用的．即若 $(D,\geqslant)$ 和 $(E,\succ)$ 为有向集，则笛卡儿乘积 $D\times E$ 关于 $\gg$ 为有向集，其中 $(d,e)\gg(f,g)$ 当且仅当 $d\geqslant f$ 及 $e\succ g$．通常我们把有向集 $(D\times E,\gg)$ 叫做**乘积有向集**．更一般地，我们还需要定义一族有向集的乘积．假设对集 A 中的每一个 a 给定了一个有向集 $(D_a,\succ_a)$，则笛卡儿乘积 $\times\{D_a:a\in A\}$ 是所有 A 上的这样的函数 d 的集，它使得对 A 中的每一个 a，$d_a(=d(a))$ 必为 D_a 的元．而乘积

有向集是指 $\{\times\{D_a;a\in A\},\geqslant\}$，其中对乘积的元 d 和 e，$d\geqslant e$ 当且仅当对 A 中的每一个 a 有 $d_a>_a e_a$，又**乘积序**指的是 $\geqslant$. 自然，必须证明乘积有向集确实是一个有向集. 事实上，若 d 和 e 为笛卡儿乘积 $\times\{D_a;a\in A\}$ 的元，则对每一个 a 有 D_a 的元 f_a，它在序 $>_a$ 中同时在 d_a 和 e_a 之后，因此在 a 处取值 f_a 的函数 f 在序 $\geqslant$ 中同时在 d 和 e 之后. 乘积有向集有一种重要的特殊情形，即所有的坐标集 D_a 相同并且所有的关系 $>_a$ 也相同. 在这种情况下 $\times\{D;a\in A\}$ 就简单地成为所有从 A 到 D 的函数的集 D^A，而 d 在 e 之后当且仅当对 A 中的每一个 a，$d(a)$ 在 $e(a)$ 之后. 譬如，实数集上的一切实值函数的集的通常的序就是一个这样的例子.

关于极限的下一结果是与闭包公理：$A^{--}=A^-$ 有关的. 这很重要，因为它表明可以通过单重极限来代替累次极限. 详细情况如下：考察所有这样的函数 S 的类，它使得当 m 属于有向集 D 并且 n 属于有向集 E_m 时 $S(m,n)$ 有定义. 我们来寻找取值于 S 的定义域的网 R 使得当 S 为到某拓扑空间的函数并且 $\lim\limits_m\lim\limits_n S(m,n)$ 存在时 $S\circ R$ 收敛于该累次极限. 有趣的是，这个问题的解决就需要 Moore-Smith 收敛，因为当考虑重序列时就没有值域为 $\omega\times\omega$ 的子集的序列能够具有上述性质. 而解决的方法，实际上则是对角线方法的一种变形. 设 F 为乘积有向集 $D\times\times\{E_n;n\in D\}$ 并且对 F 中的每一个点 (m,f) 命 $R(m,f)=(m,f(m))$，则 R 就是所需要的网.

4 关于累次极限的定理. 设 D 为有向集，又设对 D 中的每一个 m，E_m 为有向集，再设 F 为乘积 $D\times\times\{E_m;m\in D\}$ 并且对 F 中的 (m,f) 命 $R(m,f)=(m,f(m))$. 若对 D 中的每一个 m 和 E_m 中的每一个 n，$S(m,n)$ 为某个拓扑空间的元，则当累次极限存在时 $S\circ R$ 就收敛于 $\lim\limits_m\lim\limits_n S(m,n)$.

证明. 假设 $\lim\limits_m\lim\limits_n S(m,n)=s$ 并且 U 是 s 的开邻域[1]，则

1) s 的开邻域的存在是证明的本质. 累次极限定理，点的开邻域系是局部基的事实和闭包公理 "$A^{--}=A^-$" 有着密切的联系. 至于在具有比拓扑限制更少的构造的空间内的收敛也已经被研究，见 Ribeiro[1].

我们必须找出 F 的元 (m, f) 使得当 $(p, g) \geqslant (m, f)$ 时有 $S \circ R(p, g) \in U$. 先选 D 中的 m 使得 $\lim_n S(p, n) \in U$ 对每一个在 m 之后的 p 成立，再对每一个如此的 p 选 E_p 的元 $f(p)$ 使对一切在 $f(p)$ 之后的 n 有 $S(p, n) \in U$. 又当 p 为 D 的元，而不在 m 之后时，命 $f(p)$ 为 E_p 的任意元. 于是，若 $(p, g) \geqslant (m, f)$，则 $p \geqslant m$，故 $\lim_n S(p, n) \in U$，又 $g(p) \geqslant f(p)$，从而 $S \circ R(p, g) = S(p, g(p)) \in U$.|

子网和聚点

现在接着在本章的引论中所讨论的雏型的基础上，我们来定义子序列的推广并且证明所期望的定理.

我们称网 $\{T_m, m \in D\}$ 为网 $\{S_n, n \in E\}$ 的**子网**当且仅当存在定义在 D 上并且取值于 E 内的函数 N 使得

(a) $T = S \circ N$，这等价于，对 D 中的每一个 i 有 $T_i = S_{N_i}$;

(b) 对 E 中的每一个 m 有 D 中的 n 使得若 $P \geqslant n$，则 $N_P \geqslant m$.

因为似乎没有产生误解的可能，故我们在子网的记号中略去了序$\geqslant$. 第二个条件，直观的说，就是"当 p 变大时 N_p 也变大". 从这个条件显然立刻可推出若 S 基本上在集 A 内，则 S 的子网 $S \circ N$ 也基本上在 A 内. 这是一个很重要的事实，并且这里对子网所采取的定义正是为了得到这个结果. 注意 D 的每一个共尾子集 E 关于同一个序为有向集，因而 $\{S_n, n \in E\}$ 为 S 的子网(设 N 为 E 上的恒等函数，则定义的第二个条件就变成要求 E 为共尾子集). 这是构造子网的一种标准的方法，然而，不幸的是这种简单种类的子网并不能满足所有的目的 (问题 2.E.).

有一种特殊类型的子网，它几乎满足所有的目的. 假设 N 是从有向集 E 到有向集 D 的函数并且保序 (即当 $i \geqslant j$ 时有 $N_i \geqslant N_j$)，同时它的值域为 D 的共尾子集，则易见对每一个网 S，$S \circ N$ 为 S 的子网. 下列引理证明中所作出的子网就属于这种类型 (这由 K. T. Smith 指出).

5 引理. 设 S 为网，又设 $\mathscr{A}$ 为集族，它使得 S 常常在 $\mathscr{A}$ 的每一个元内并且还使得 $\mathscr{A}$ 的两个元的交包含 $\mathscr{A}$ 的一个元，则有 S 的一个子网，它基本上在 $\mathscr{A}$ 的每一个元内.

证明. 因为 $\mathscr{A}$ 的任何两个元的交包含 $\mathscr{A}$ 的一个元，故 $\mathscr{A}$ 关于 $\subset$ 为有向集. 设 $\{S_n, n\in D\}$ 为这样的网，它常常在 $\mathscr{A}$ 的每一个元内，又设 E 为所有这样的 (m, A) 的集，它使得 $m\in D$, $A\in\mathscr{A}$ 并且 $S_m\in A$，则 E 关于 $D\times\mathscr{A}$ 的乘积序为有向集，因为若 (m, A) 和 (n, B) 为 E 的元，则有 $\mathscr{A}$ 中的 C 使得 $C\subset A\cap B$ 和 D 中的 p 使得 p 同时在 m 与 n 之后并且 $S_p\in C$，即 $(p, C)\in E$ 并且 (p, C) 同时在 (m, A) 与 (n, B) 之后. 对 E 中的 (m, A)，命 $N(m, A)=m$，则 N 显然保序并且 N 的值域为 D 的共尾子集（$\{S_n, n\in D\}$ 常常在 $\mathscr{A}$ 的每一个元内）. 因此 $S\circ N$ 为 S 的子网. 最后，设 A 为 $\mathscr{A}$ 的元，m 为 D 的元，它使得 $S_m\in A$，于是，若 (n, B) 为 E 的元，它在 (m, A) 之后，则 $S\circ N(n, B)=S_n\in B\subset A$，故 $S\circ N$ 基本上在 A 内. |

现在我们把这个引理应用于拓扑空间内的收敛. 我们称空间的点 s 为网 S 的**聚点**当且仅当 S 常常在 s 的每一个邻域内. 一个网可以有一个，多个或没有聚点. 例如，若 ω 为非负整数集，则 $\{n, n\in\omega\}$ 是一个网，它关于实数的通常拓扑没有聚点. 还有另一种极端情形，若 S 是一个序列，它的值域是有理数集（如此的序列是存在的，因为有理数集为可数集），则易见该序列常常在每一个开区间内，因而每一个实数都是它的聚点. 若一个网收敛于某个点，则该点必为聚点，但可能有这样的网，它有唯一的聚点，然而却并不收敛于该点. 例如，考虑由交替取 -1 和正整数序列所作出的序列 $-1, 1, -1, 2, -1, 3, -1, \cdots$，这时 -1 为该序列的唯一的聚点，但该序列不收敛于 -1.

6 定理. 拓扑空间中的点 s 为网 S 的聚点当且仅当有 S 的某个子网收敛于 s.

证明. 设 s 为 S 的聚点，又设 $\mathscr{U}$ 为所有 s 的邻域的族，则因 $\mathscr{U}$ 的两个元的交仍为 $\mathscr{U}$ 的一个元并且 S 常常在 s 的每一个邻

域内，故应用上一引理有 S 的一个子网，它基本上在 $\mathscr{U}$ 的每一个元内，即收敛于 s.

若 s 不是 S 的聚点，则有 s 的一个邻域 U 使得 S 不常常在 U 内，故 S 基本上在 U 的余集内，于是 S 的每一个子网基本上在 U 的余集内，从而不收敛于 s.|

下面再利用闭包来给出聚点的一种刻划.

7 定理. 设 $\{S_n, n\in D\}$ 为拓扑空间中的网并且对 D 中的每一个 n，命 A_n 为所有使得 $m>n$ 的点 S_m 的集，则 s 为 $\{S_n, n\in D\}$ 的聚点当且仅当对 D 中的每一个 n，s 属于 A_n 的闭包.

证明. 若 s 为 $\{S_n, n\in D\}$ 的聚点，则对每一个 n，A_n 与 s 的每一个邻域相交，因为 $\{S_n, n\in D\}$ 常常在每一个邻域内. 故 s 在每一个 A_n 的闭包内.

若 s 不是 $\{S_n, n\in D\}$ 的聚点，则有 s 的一个邻域 U 使得 $\{S_n, n\in D\}$ 不常常在 U 内，故对某个 D 中的 n 当 $m\geqslant n$ 时有 $S_m\notin U$，即 U 与 A_n 不相交. 因而 s 不在 A_n 的闭包内.|

序列和子序列

我们有兴趣知道在什么情况下拓扑可以只通过序列来描述，这不仅因为此时有一个对一切网都确定的定义域，比较方便，而且因为有些序列的性质不能推广. 这种拓扑空间中最重要的一类就是满足第一可数性公理的空间：即每一点的邻域系均有可数基，亦即对空间 X 的每一点 x 有 x 的邻域系的一个可数子族使得 x 的每一个邻域都包含该族的某个元. 在这种情况下，几乎上面的所有定理都可以用序列来代替网.

应当注意：序列可以有不是子序列的子网.

8 定理. 设 X 为满足第一可数性公理的拓扑空间，则：

(a) 点 s 为集 A 的聚点当且仅当有 $A\sim\{s\}$ 中的序列收敛于 s.

(b) 集 A 为开集当且仅当每一个收敛于 A 中的点的序列都基

本上在 A 内.

(c) 若 s 为序列 S 的聚点,则有 S 的子序列收敛于 s.

证明. 假设 s 为 X 的子集 A 的聚点, 并且序列 U_0, U_1, $\cdots$, U_n, $\cdots$ 是 s 的邻域系的基. 命 $V_n = \bigcap\{U_i : i = 0, 1, \cdots, n\}$, 则序列 V_0, V_1, $\cdots$, V_n, $\cdots$ 也是 s 的邻域系的基, 而且对每一个 n 有 $V_{n+1} \subset V_n$. 对每一个 n, 选 $V_n \cap (A - \{s\})$ 中的一个点 S_n, 于是得到一个序列 $\{S_n, n \in \omega\}$, 显然它收敛于 s. 这样就得到了命题 (a) 的一半,至于其逆是明显的.

若 A 为 X 的子集,它不是开集,则有 $X \sim A$ 的一个序列,它收敛于 A 中的点,如此的序列一定不能基本上在 A 内,从而又得到了定理中的命题 (b).

最后,假设 s 为序列 S 的聚点,并且序列 V_0, V_1, $\cdots$ 是使得对每一个 n 有 $V_{n+1} \subset V_n$ 的 s 的邻域系的基. 对每一个非负整数 i, 选 N_i 使得 $N_i \geqslant i$ 并且 S_{N_i} 属于 V_i. 这时 $\{S_{N_i}, i \in \omega\}$ 就是 S 的一个收敛于 s 的子序列. ▌

*收　敛　类

我们知道有时通过规定什么样的网才收敛于空间的什么点来定义一个拓扑是方便的. 例如若 $\mathscr{F}$ 为从一个确定的集 X 到拓扑空间 Y 的一个函数族, 则自然规定网 $\{f_n, n \in D\}$ 收敛于函数 g 当且仅当对 X 中的每一个 x, $\{f_n(x), n \in D\}$ 收敛于 $g(x)$ (这种类型的收敛在第三章中将要更详细的讨论),于是自然地产生这样的问题: 是否有 $\mathscr{F}$ 的一个拓扑使得这种收敛就是关于该拓扑的收敛? 它的肯定的回答就使得我们能够利用对拓扑空间所发展了的工具来研究 $\mathscr{F}$ 的构造.

这个问题可以正式叙述如下: 若 $\mathscr{C}$ 是一个由 (S, s) 组成的类, 其中 S 为 X 中的网, s 为点, 则何时有 X 的一个拓扑 $\mathscr{T}$ 使得 $(S,s) \in \mathscr{C}$ 当且仅当 S 关于拓扑 $\mathscr{T}$ 收敛于 s? 从前面对收敛的讨论中,我们已经知道当如此的拓扑存在时 $\mathscr{C}$ 所必须具有的几个性

质．我们称 $\mathscr{C}$ 为关于 X 的**收敛类**当且仅当它满足下面所指出的四个条件[1]．为方便起见，又称 $S(\mathscr{C})$ 收敛于 s 或 $\lim_n S_n = s(\mathscr{C})$ 当且仅当 $(S, s) \in \mathscr{C}$.

(a) 若 S 是这样的网它使得对每一个 n 有 $S_n = s$，则 $S(\mathscr{C})$ 收敛于 s.

(b) 若 $S(\mathscr{C})$ 收敛于 s，则 S 的每一个子网也 $(\mathscr{C})$ 收敛于 s.

(c) 若 S 不 $(\mathscr{C})$ 收敛于 s，则有 S 的一个子网使得该子网没有收敛于 s 的子网.

(d) (关于累次极限的定理 2.4)设 D 为有向集，又设对 D 中的每一个 m，E_m 为有向集，再设 F 为乘积 $D \times \times \{E_m: m \in D\}$ 并且对 F 中的 (m, f)，命 $R(m, f) = (m, f(m))$．若 $\lim_m \lim_n S(m, n) = s$ $(\mathscr{C})$，则 $S \circ R(\mathscr{C})$ 收敛于 s.

前面已经证明在拓扑空间内的收敛一定满足(a)，(b)和(d)．至于命题 (c) 通过下面的论证也容易得到：若网 $\{S_n, n \in D\}$ 不收敛于点 s，则它常常在 s 的某个邻域的余集内，故有 D 的共尾子集 E 使得 $\{S_n, n \in E\}$ 在该余集内，又易见 $\{S_n, n \in E\}$ 是一个子网并且它没有收敛于 s 的子网.

现在我们来证明每一个收敛类实际上是由某个拓扑所导出.

9 定理． 设 $\mathscr{C}$ 为关于集 X 的一个收敛类，又对 X 的每一个子集 A，命 A^c 为所有这样的点 s 的集，它使得对 A 中的某个网 S 有 $S(\mathscr{C})$ 收敛于 s，则 ^{c} 为一个闭包算子，并且 $(S, s) \in \mathscr{C}$ 当且仅当 S 关于与 ^{c} 相伴随的拓扑收敛于 s.

证明．首先证明 ^{c} 为一个闭包算子（参看定理 1.8）．因为网是定义在有向集上的函数，并且由定义可知该有向集为非空，故 $(0)^c$ 为空集．根据关于常数网的条件 (a)，对集 A 的每一个元 s 有一个网 S 收敛于 s，因而 $A \subset A^c$．若 $s \in A^c$，则因由算子 ^{c} 的定义有 $S \in (A \cup B)^c$，故 $A^c \subset (A \cup B)^c$ 对每一个集 B 成立，即 $A^c \cup B^c \subset (A \cup B)^c$．今证反过来的包含关系也成立，假设 $\{S_n, n \in D\}$

1) 其中的前三个条件当以“序列”代替网时也就是 Kuratowski 对极限空间的 Fréchet 公理的修正形式．见 Kuratowski [1]．

为 $A\cup B$ 中的网并且它 ($\mathscr{C}$) 收敛于 s，命 $D_A=\{n:n\in D$ 且 $S_n\in A\}$，$D_B=\{n:n\in D$ 且 $S_n\in B\}$，则 $D_A\cup D_B=D$，故 D_A 或 D_B 为 D 的共尾子集，从而 $\{S_n,n\in D_A\}$ 或 $\{S_n,n\in D_B\}$ 为 $\{S_n,n\in D\}$ 的子网并且由条件 (b) 它也 ($\mathscr{C}$) 收敛于 s，于是 $s\in A^c\cup B^c$，这就证明了 $A^c\cup B^c=(A\cup B)^c$. 现在还必须证明 $A^{cc}=A^c$，同时说明条件 (d) 确实是需要的. 若 $\{T_m,m\in D\}$ 为 A^c 中的网并且它 ($\mathscr{C}$) 收敛于 t，则对 D 中的每一个 m 存在有向集 E_m 和网 $\{S(m,n),n\in E_m\}$，使得它 ($\mathscr{C}$) 收敛于 T_m，然后根据条件 (d) 就有一个网 ($\mathscr{C}$) 收敛于 t，即 $t\in A^c$，因而 $A^{cc}=A^c$.

定理证明更巧妙的部份是：证明 ($\mathscr{C}$) 收敛和关于与算子 ^{c} 相伴随的拓扑 $\mathscr{T}$ 的收敛一致. 先假设 $\{S_n,n\in D\}$ ($\mathscr{C}$) 收敛于 s 并且关于 $\mathscr{T}$ 不收敛于 s，则有 s 的开邻域 U 使得 $\{S_n,n\in D\}$ 基本上不在 U 内，故存在 D 的共尾子集 E 使得对 E 中的所有 n 有 $S_n\in X\sim U$，从而 $\{S_n,n\in E\}$ 为 $\{S_n,n\in D\}$ 的子网并且由条件 (b) 该子网在 $X\sim U$ 内 ($\mathscr{C}$) 收敛于 s，于是 $X\sim U\neq(X\sim U)^c$，即 U 关于 $\mathscr{T}$ 不是开集，矛盾.

最后，假设网 P 关于拓扑 $\mathscr{T}$ 收敛于 r 并且不 ($\mathscr{C}$) 收敛于 s，则由 (c) 有子网 $\{T_m,m\in D\}$ 使得它没有 ($\mathscr{C}$) 收敛于 r 的子网，如果我们能够再作出一个如此的子网，那末就得到矛盾. 对 D 中的每一个 m，命 $B_m=\{n:n\in D$ 且 $n\geqslant m\}$，又命 A_m 为所有使得 $n\in B_m$ 的 T_n 的集，则因 $\{T_m,m\in D\}$ 关于 $\mathscr{T}$ 收敛于 r，故 r 必须位于每一个 A_m 的闭包内，由此可见对 D 中的每一个 m 有有向集 E_m 和 B_m 中的网 $\{U(m,n),n\in E_m\}$ 使得合成 $\{T\circ U(m,n),n\in E_m\}$ ($\mathscr{C}$) 收敛于 r. 于是再利用关于收敛类的条件 (d) 便知若对 $D\times\mathsf{X}\{E_m,m\in D\}$ 中的每一个 (m,f)，命 $R(m,f)=(m,f(m))$，则 $T\circ U\circ R$ ($\mathscr{C}$) 收敛于 r，而且当 $p\geqslant m$ 时有 $U\circ R(p,f)=U(p,f(p))\in B_m$，即 $U\circ R(p,f)\geqslant m$，这表明 $T\circ U\circ R$ 为 T 的子网，于是定理获证. |

上面定理建立了集 X 的拓扑与在其上的收敛类之间的一一对应. 并且这个对应在下列意义下是反序的：若 $\mathscr{C}_1$ 和 $\mathscr{C}_2$ 为收敛

类，$\mathscr{T}_1$ 和 $\mathscr{T}_2$ 为相伴随的拓扑，则 $\mathscr{C}_1 \subset \mathscr{C}_2$ 当且仅当 $\mathscr{T}_2 \subset \mathscr{T}_1$（这个事实由收敛的定义立即推出）. 另外，交 $\mathscr{C}_1 \cap \mathscr{C}_2$ 仍为收敛类，这只须根据如此的类的四个特征性质即得，又易见与 $\mathscr{C}_1 \cap \mathscr{C}_2$ 相伴随的拓扑就是大于 $\mathscr{T}_1$ 和 $\mathscr{T}_2$ 的最小拓扑，对偶的有 $\mathscr{T}_1 \cap \mathscr{T}_2$ 的收敛类是大于 $\mathscr{C}_1$ 和 $\mathscr{C}_2$ 的最小收敛类.

问　　题

A　关于序列的习题

设 X 为可数集，它的拓扑由空集和所有余集为有限的集所组成. 问什么样的序列收敛于什么样的点？

B　例子：序列是不充分的

设 Ω' 为小于或等于第一个不可数序数 Ω 的序数的集，又设拓扑为序拓扑，则 Ω 为 $\Omega' \sim \{\Omega\}$ 的聚点，但没有 $\Omega' \sim \{\Omega\}$ 中的序列收敛于 Ω.

C　关于 Hausdorff 空间的习题：门空间

拓扑空间叫做**门空间**当且仅当每一个子集非开即闭. Hausdorff 门空间至多有一个聚点，并且若 x 为非聚点的点，则 $\{x\}$ 为开集.（若 U 是聚点 y 的任意邻域，则 $U \sim \{y\}$ 为开集.）

D　关于子序列的习题

设 N 为非负整数的序列，它使得任意整数都只能出现有限多次；即对每一个 m，集 $\{i: N_i = m\}$ 为有限集，则当 $\{S_n, n \in \omega\}$ 为任意序列时 $\{S_{N_i}, i \in \omega\}$ 为子序列. 若 $\{S_n, n \in \omega\}$ 为拓扑空间中的序列，而 N 为非负整数的任意序列，则 $\{S_{N_i}, i \in \omega\}$ 或者是 $\{S_n, n \in \omega\}$ 的子序列，或者是它有聚点.

E　例子：共尾子集是不充分的

设 X 为所有非负整数偶的集，它的拓扑描述如下：对每一个异于 $(0, 0)$ 的点 (m, n)，集 $\{(m, n)\}$ 为开集，又集 U 为 $(0, 0)$ 的邻域，假如对除有限多个外的所有整数 m，集 $\{n: (m, n) \notin U\}$ 为有限集.（在欧几里德平面内实现 X，这时 $(0, 0)$ 的邻域包含除有限多个列外的所有列的除有限多个外的所有

的元.)

(a) 空间 X 为 Hausdorff 空间.

(b) X 的每一个点是可数多个闭邻域的族的交.

(c) 空间 X 为 Lindelöf 空间;即每一个开覆盖有可数子覆盖.

(d) 不存在 $X\sim\{(0,0)\}$ 中的序列收敛于 $(0,0)$. (若 $X\sim\{(0,0)\}$ 中的序列 S 收敛于 $(0,0)$,则它基本上在每一列的余集内,并且该序列在每一列内只有有限多个值.)

(e) 存在 $X\sim\{(0,0)\}$ 中的序列 S, 它以 $(0,0)$ 为聚点,并且它在任意整数的共尾子集上的限制都不收敛.

注 该例子属于 Arens[1].

F 单调网

设 X 为有序完备的链;即 X 关于关系 $>$ 为线性有序集并且使得 X 的每一个有上界的非空子集有上确界.又设 X 的拓扑为序拓扑(问题 1.I). X 中的网 $(S,\succ)$ 叫做单调上升(下降)当且仅当对 $m\succ n$ 有 $S_m\geq S_n$ $(S_n\geq S_m)$ 成立.

(a) X 中每一个值域有界(存在 $x\in X$ 使得 $x\geq S_n$ 对一切 n 成立)的单调上升的网收敛于它的值域的上确界.

(b) 若 X 为带有通常序的实数集,或 X 为所有小于第一个不可数序数的序数的集,则每一个值域有上(下)界的单调上升(下降)的网收敛于它的值域的上确界(下确界).

G 积分理论,初级形式

设 f 为实值函数,它的定义域包含集 A, 又设 $\mathscr{A}$ 为所有 A 的有限子集组成的族,并且对 $\mathscr{A}$ 中的每一个 F, 命 $S_F=\Sigma\{f(a):a\in F\}$, 则 $\mathscr{A}$ 关于 $\supset$ 为有向集并且 $\{S_F, F\in\mathscr{A},\supset\}$ 为网. 若该网收敛,则称 f 在 A 上**可和**,并且它所收敛的值叫做 f 在 A 上的**无序和**,记为 $\Sigma\{f(a):a\in A\}$, 或简记为 $\Sigma_A f$.

(a) 若 f 非负(非正),则 f 可和当且仅当它在 A 的所有有限子集上的和有上界(下界). (利用关于单调网的上一问题.)

(b) 设 $A_+=\{a:f(a)\geq 0\}$ 并且 $A_-=\{a:f(a)<0\}$,则 f 在 A 上可和当且仅当 f 在 A_+ 和 A_- 上皆可和. 若 f 在 A 上可和, 则 $\Sigma_A f=\Sigma_{A_+}f+\Sigma_{A_-}f$.

(c) 函数 f 在 A 上可和当且仅当 $|f|$ 在 A 上可和,其中 $|f|(a)=|f(a)|$.

(d) 若 f 在集 A 上可和, 则 f 在 A 的某个可数子集外为零. (若 f 在某个不可数子集的每一点处异于零,则对某个正整数 n, $\{a:f(a)\geq 1/n\}$ 为不可

数集.)

(e) 若 f 和 g 在 A 上可和并且 r 和 s 为实数，则 $rf+sg$ 在 A 上可和并且 $\Sigma_A(rf+sg)=r\Sigma_A f+s\Sigma_A g$.

(f) 若 f 在 A 上可和，并且 B 和 C 为 A 的互不相交的子集，则 f 在 B 和 C 的每一个上可和，并且 $\Sigma_{B\cup C}f=\Sigma_B f+\Sigma_C f$.

(g) 若 x 为实数序列，则其序和("级数的和")是指序列 S_n 的极限，其中 $S_n=\Sigma\{x_i:i=0,1,\cdots,n\}$. 换言之，序和为 $\{S_F, F\in\mathscr{B}\}$ 的极限，其中 $\mathscr{B}$ 为所有形如 $\{m:m\leqslant n\}$ 的集组成的族. 显然，该网是定义无序和的网的子网. 又序列 x 叫做绝对可和当且仅当序列 $|x|$ 存在序和，其中 $|x|_n=|x_n|$. 于是 x 在整数上存在无序和当且仅当该序列为绝对可和，并且在这种情况下无序和与序和相等.

(h) (Fubinito) 设 f 为笛卡儿乘积 $A\times B$ 上的实值函数，则:

(i) 若 f 在 $A\times B$ 上可和，则 $\Sigma_{A\times B}f=\Sigma\{\Sigma\{f(a,b):b\in B\}:a\in A\}$. (后者是两个累次和中的一个.)

(ii) 若对 A 的每一个元 a, $f(a,b)$ 或者对所有 b 为非负，或者对所有 b 为非正，又若 $F(a)=\Sigma\{f(a,b):b\in B\}$ 并且 F 在 A 上可和，则 f 在 $A\times B$ 上可和.

(iii) 一般的说，可能两个累次和同时存在，而 f 为不可和. 事实上，若 A 和 B 均为可数无限集并且 F 和 G 分别为 A 和 B 上的任意实函数，则有 $A\times B$ 上的 f 使得 $\Sigma\{f(a,b):a\in A\}=G(b)$ 和 $\Sigma\{f(a,b):b\in B\}=F(a)$ 分别对 B 中的一切 b 和 A 中的一切 a 成立.

注 在该问题中所陈述的结果是利用无序和代替绝对收敛级数来发展测度论所需要的. 所有这些结果除 (d), (g) 和 (h, ii) 外可以在许多更一般的情况下得到证明;第七章我们还要利用完备性概念再来考察这个问题. 以上关于序的理论的处理将有助于更复杂的积分例子的考察.

另外在历史上，无序和是 Moore-Smith 收敛的先驱. (Moore [1].)

I 积分理论，实用形式

设 f 为实数的闭区间 $[a,b]$ 上的有界实值函数. 规定 $[a,b]$ 的子分划 S 为有限多个闭区间的族，它覆盖 $[a,b]$ 并且使得任何两个区间至多有一个公共点. 将区间 I 的长度记为 $|I|$, 又对子分划 S 定义网孔 $\|S\|$ 为 S 中的 I 的 $|I|$ 的最大值. 另外我们用两种不同方法使所有子分划的族为有向:

(i) $S\geqslant S'$ 当且仅当 S 是 S' 的加细，亦即 S 的每个元都是 S' 某个元的子

集；

(ii) $S \gg S'$ 当且仅当 $\|S\| \leqslant \|S'\|$.

命 $M_f(I)$ 为 f 在 I 上的上确界，$m_f(I)$ 为下确界。又定义相应于子分划 S 的上与下 Darboux 和为 $D_f(S) = \Sigma\{|I| M_f(I): I \in S\}$ 与 $d_f(S) = \Sigma\{|I| m_f(I): I \in S\}$. Riemann 和则更为复杂。规定子分划 S 的选择函数为 S 上的这样的函数 c，它使得对 S 中的每一个 I 有 $c(I) \in I$. 现在再用两种方法使所有 S 为子分划并且 c 为 S 的选择函数的序偶 (S, c) 的集为有序：$(S,c) \succ (S',c')$ 当且仅当 $S \geqslant S'$ 和 $(S, c) \succ\succ (S, c')$ 当且仅当 $S \gg S'$. 最后定义相应于序偶 (S, c) 的 Riemann 和为 $R_f(S, c) = \Sigma\{|I| f(c(I)): I \in S\}$.

借助于加细的序即可作出有关的基本运算。

(a) 网 $(D_f, \geqslant)$ 和 $(d_f, \geqslant)$ 分别为单调下降和单调上升，并且因而收敛。

(b) $d_f(S) \leqslant R_f(S, c) \leqslant D_f(S)$ 对一切子分划 S 和一切选择函数 c 成立。

(c) 对每一个正数 e 存在所有序偶 (S, c) 的集的 $\succ$-共尾子集使得 $R_f(S,c) + e \geqslant D_f(S)$. (还有一个对偶的命题.)

(d) 网 $(R_f, \succ)$ 收敛当且仅当 $\lim (D_f, \geqslant) = \lim(d_f, \geqslant)$. 若 $(R_f, \succ)$ 收敛，则 $\lim(R_f, \succ) = \lim (D_f, \geqslant) = \lim (d_f, \geqslant)$.

(e) 网 $(R_f, \succ)$ 为网 $(R_f, \succ\succ)$ 的子网。

(f) 网 $(R_f, \succ\succ)$ 收敛当且仅当 $\lim (D_f, \geqslant) = \lim (d_f, \geqslant)$. 若 $(R_f, \succ\succ)$ 收敛，则 $\lim (R_f, \succ\succ) = \lim (R_f, \succ)$.

注　f 的 Riemann 积分通常是定义为 $(R_f, \succ\succ)$ 的极限。考虑加细的序和网孔的序都一样方便，只是一个技巧的问题。如果我们代替有限的子分划和区间的长度，考虑可数的子分划和 I 的 Lebesgue 测度 $|I|$，那末网 $(R_f, \succ)$ 就收敛到 f 的通常的 Lebesgue 积分，但 $(R_f, \succ\succ)$ 可以并不如此。此外，加细型的定义还可以应用于取值矢量空间的某种函数的积分。(见 Hille [1]第三章)。又 Darbour 型积分要求被积函数的值域为半序集。而 Daniell 积分和各种推广 (Bourbaki [2], McShane [2] 和 [3] 以及 M. H. Stone [1]) 本质上也都是这种类型。还有一种引入积分的标准方法，它是通过关于某种度量的完备化过程的途径，显然这也带来了许多方便 (Halmos [1]).

I　格内的极大理想

格是指这样的非空集 X，它带有一个反身的半序 $\geqslant$ 使得对 X 的每一对元 x 与 y 有(唯一的)最小元 $x \vee y$ 它大于 x 与 y 和 (唯一的) 最大元 $x \wedge y$ 它小

于 x 与 y. 而元 $x\vee y$ 和 $x\wedge y$ 就叫做 x 与 y 的并和交. 格称为分布格当且仅当 $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ 与 $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$ 对 X 中的一切 x, y 和 z 成立. 又 X 的子集 A 称为理想(对偶理想)当且仅当从 $y\geqslant x$ 和 $y\in A$ 可推出 $x\in A$ 并且从 $y\in A$ 和 $z\in A$ 可推出 $y\vee z\in A$ (从 $x\geqslant y$ 和 $y\in A$ 可推出 $x\in A$ 并且从 $y\in A$ 和 $z\in A$ 可推出 $y\wedge z\in A$).

设 A 和 B 为分布格 X 的互不相交子集并且 A 为理想, B 为对偶理想, 则有互不相交的集 A' 和 B' 使得 A' 为包含 A 的理想, B' 为包含 B 的对偶理想并且 $A'\cup B'=X$.

该命题的证明分解为一系列引理.

(a) 所有包含 A 并且与 B 不相交的理想的族有一个极大元 A' (见预备知识定理 25). 相似地, 有一个对偶理想 B', 它包含 B, 与 A' 不相交, 并且关于这些性质为极大.

(b) 包含 A' 和 X 的元 c 的最小理想为 $\{x: x\leqslant c$ 或 $x\leqslant c\vee y$ 对 A' 中的某个 y 成立$\}$. 因 A' 为极大, 故若 c 不属于 A' 或 B, 则 $c\vee x\in B$ 对 A' 中的某个 x 成立. (若 $z\geqslant x\in B$, 则 $z\in B$.)

(c) 若 c 不属于 A' 和 B', 则有 $x\in A'$ 和 $y\in B'$ 使得 $c\vee x\in B'$ 并且 $c\wedge y\in A'$. 此时 $(c\vee x)\wedge y=(c\wedge y)\vee(x\wedge y)$ 属于 A' 和 B'.

注 该定理属于 M. H. Stone[2]; 它是关于有序集的一个基本事实的最佳形式. 它将应用于以下的两个问题, 并且它还是关于紧性(第五章)的最重要结果的基础. 而极大原理的某种形式的应用似乎就是其证明的本质所在. 文献中已经陈述过从该定理(或更精确地说, 问题 2.K 中的定理的一个系)可推出选择公理, 但我不知道是否确实如此[1]. 最后, 上面所给出的分配性的定义限制是过多了. 其实从这两个等式中的一个即可推出另一个(Birkhoff[1]).

J 万有网

集 X 中的网叫做万有网当且仅当对 X 的每一个子集 A 该网或者基本上在 A 内, 或者基本上在 $X\sim A$ 内.

(a) 若万有网常常在某个集内, 则它就基本上在该集内. 因此, 拓扑空

1) 这个问题的答案是肯定的. 这是由于选择公理等价于这样一个定理: 若 B 是一个 Boole 代数且 $S\subset B$ 合于 $0\notin S$, 则 B 有一个关于与 S 不交的极大的理想. (见 H. Rubin 与 J. Rubin 所著《Equivalents of the Axiom of Choice》的 p.42 上的 AL_5.) 这一定理 (AL_5) 又可被证明与定理 2.11 的 (b) 等价, 而后者的证明又是基于问题 2.I 的定理. ——校者注

间中的万有网恒收敛于它的每一个聚点．

(b) 若一个网为万有网，则它的每一个子网亦为万有网．若 S 为 X 中的万有网并且 f 是从 X 到 Y 的函数，则 $f\circ S$ 为 Y 中的万有网．

(c) 引理．若 S 为 X 中的网，则有 X 的一族子集 $\mathscr{C}$ 使得：S 常常在 $\mathscr{C}$ 的每一个元内，$\mathscr{C}$ 的两个元的交属于 $\mathscr{C}$ 并且对 X 的每一个子集 A，或者 A，或者 $X\sim A$ 属于 $\mathscr{C}$．（或者先证存在一族 $\mathscr{C}$ 关于所指出的前两个性质为极大，再证它具有第三个性质；或者应用问题 2.I，命 $\mathscr{A}$ 为所有使得 S 基本上在 $X\sim A$ 内的集 A 的族，$\mathscr{B}$ 为所有使得 S 基本上在 B 内的 B 的族，而序则取为 $\subset$．）

(d) X 中的每一个网都存在万有子网．（利用上一结果和定理 2.5．）

K　Boole 环：存在足够多的同态

Boole 环是指这样的环 $(R,+,\cdot)$，它使得 $r\cdot r=r$ 和 $r+r=0$ 对每一个 $r\in R$ 成立．又将模 2 的整数域记为 I_2．

(a) Boole 环恒为交换环．（注意 $(r+s)\cdot(r+s)=r+s$．）

(b) 若 $(R,+,\cdot)$ 为 Boole 环，则可定义 R 的元与 I_2 的元的乘法，因而 R 为 I_2 上的一个代数．

(c) 定义两个集 A 和 B 的对称差 $A\triangle B$ 为 $(A\cup B)\sim(A\cap B)$．于是若 $\mathscr{A}$ 为集 X 的所有子集的族，则 $(\mathscr{A},\triangle,\cap)$ 为具有单位元的 Boole 环．

(d) 设 X 为一个集又设 I_2^X 为所有从 X 到 I_2 的函数的族．若定义函数的加法和乘法为点式加法和点式乘法（即 $(f+g)(x)=f(x)+g(x)$，$(f\cdot g)(x)=f(x)\cdot g(x)$），则 $(I_2^X,+,\cdot)$ 为具有单位元的 Boole 环并且同构于 $(\mathscr{A},\triangle,\cap)$，其中 $\mathscr{A}$ 为 X 的所有子集的族．

(e) 若定义 Boole 环的自然序为：$r\geqslant s$ 当且仅当 $r\cdot s=s$，则在这个半序下，R 中在 r 与 s 之后的最小元是 $r\vee s=r+s-r\cdot s$，而在 r 与 s 之前的最大元为 $r\wedge s=r\cdot s$．运算 $\vee$ 和 $\wedge$ 都满足结合律且使得下列分配律成立：$r\wedge(s\vee t)=(r\wedge s)\vee(r\wedge t)$，$r\vee(s\wedge t)=(r\vee s)\wedge(r\vee t)$．

(f) 回顾一下，S 称为 Boole 环 $(R,+,\cdot)$ 的理想当且仅当 S 是一个加法子群并且 $r\cdot s\in S$ 当 $r\in R$，$s\in S$ 时；又理想 S 叫做极大理想当且仅当 $R\neq S$ 并且不存在异于 R 的真包含 S 的理想．存在 R 的极大理想与从 R 到 I_2 内不恒等于零的同态之间的一一对应．（如此的同态的核是一个极大理想．）

(g) S 为 Boole 环的理想的充要条件为当 r 和 s 为 S 的元时有 $r\vee s\in S$ 并且当 t 关于自然序在 S 的某个元之前（即 $t\leqslant S$ 的某个元）时有 $t\in S$．回顾一下，R 的子集 T 叫做对偶理想当且仅当 r 和 s 为 T 的元时有 $r\wedge s\in T$ 并且当

t 在 T 的某个元之后时有 $t\in T$. 若 $r\in R$, 则 $\{s:r\geqslant s\}$ 是一个理想并且 $\{s:s\geqslant r\}$ 是一个对偶理想。又若 S 为理想，T 为不相交的对偶理想并且 $S\cup T=R$, 则在 S 上为零并且在 T 上为 1 的函数是从 R 到 I_2 内的一个同态。（在集的 Boole 环内通常把理想叫做 $\cap$-理想，而把对偶理想叫做 $\cup$-理想.）

(h) 定理. 若 S 为 Boole 环的理想，T 为与 S 不相交的对偶理想，则有从该环到 I_2 内的同态，它在 S 上为零并且在 T 上为 1. 特别，若 r 为环的非零元，则有环的同态 h 使得 $h(r)=1$.（换言之，存在有足够多的同态来区别环的成员。该定理的一种证明是基于问题 2.I 的.）

(i) 若 X 为拓扑空间，$\mathscr{B}$ 为 X 的所有既开又闭的子集的族，则$(\mathscr{B},\triangle,\cap)$ 为 Boole 代数。

(j) 并非一切 Boole 代数都必同构于某个集的所有子集的代数。（证明是通过作出可数的 Boole 代数的例子.）

注 这项研究在问题 5.S 中将得到完善。

L 滤子

关于收敛理论也还可以建立在滤子概念的基础上. 集 X 中的滤子 $\mathscr{F}$ 是指 X 的一族非空子集并且满足

(i) $\mathscr{F}$ 的两个元的交仍然属于 $\mathscr{F}$;

(ii) 若 $A\in\mathscr{F}$ 并且 $A\subset B\subset X$, 则 $B\in\mathscr{F}$.

根据上一问题的术语，所谓滤子就是 X 的所有子集的 Boole 环的一个真对偶理想。又拓扑空间 X 中的滤子 $\mathscr{F}$ 收敛于点 x 当且仅当 x 的每一个邻域均为 $\mathscr{F}$ 的元(即 x 的邻域系为 $\mathscr{F}$ 的一个子族).

(a) 子集 U 为开集当且仅当 U 属于每一个收敛于 U 内的点的滤子.

(b) 点 x 为集 A 的聚点当且仅当 $A\sim\{x\}$ 属于某个收敛于 x 的滤子.

(c) 设 Φ_x 为所有收敛于 x 的滤子的全体，则$\bigcap\{\mathscr{F}:\mathscr{F}\in\Phi_x\}$ 即为 x 的邻域系.

(d) 若 $\mathscr{F}$ 为收敛于 x 的滤子，而 $\mathscr{G}$ 为包含 $\mathscr{F}$ 的滤子，则 $\mathscr{G}$ 也收敛于 x.

(e) X 中的滤子叫做超滤子当且仅当它不能真包含在 X 中的任一滤子内. 若 $\mathscr{F}$ 为 X 中的超滤子并且某两个集的并为 $\mathscr{F}$ 的元，则这两个集之一必属于 $\mathscr{F}$. 特别，若 A 为 X 的子集，则或者 A, 或者 $X\sim A$ 属于 $\mathscr{F}$.（再一次应用问题 2.I.）

(f) 我们可以猜想滤子和网实际上引导出本质上等价的理论. 猜想的基础是如下的事实:

(i) 若 $\{x_n, n\in D\}$ 为 X 中的网，则所有使得 $\{x_n, n\in D\}$ 基本上在其内的集 A 的全体组成的族 $\mathscr{F}$ 为 X 中的滤子。

(ii) 设 $\mathscr{F}$ 为 X 中的滤子，又设 D 为所有使得 $x\in F$ 并且 $F\in\mathscr{F}$ 的 (x, F) 的集，并且规定当 $G\subset F$ 时，$(y,G)\geqslant(x,F)$ 使 D 为有向，再命 $f(x,F)=x$，则 $\mathscr{F}$ 恰为所有使得网 $\{f(x,F),(x,F)\in D\}$ 基本上在 A 内的集 A 的族。

注 滤子的定义是属于 H. Cartan 的；他对收敛的处理在 Bourbaki [1] 中已详细地给出。命题 (c) 是 W. H. Gottschalk 的一个注记；而命题 (f) 则是有关这个论题的"口头文献"的一部分。

第三章　乘积空间和商空间

本章的目的是研究从给定的空间来构造新的拓扑空间的两种方法．其中的第一种方法是对空间的笛卡儿乘积指定一种标准拓扑，于是从给定的一些空间就作出了一个新的空间．例如，欧几里德平面是实数（具有通常拓扑）和它自己的乘积空间，而欧几里德 n-空间是实数的 n 次乘积．在第四章中实数的任意多次的笛卡儿乘积还提供了与其它拓扑空间相比较的一类标准空间．

从一个给定的空间构造新的空间的第二种方法是依赖于把所给定的空间 X 分成等价类．并且此时将每一个等价类看成新构造的空间的一个点．粗糙地说，即我们“叠合”X 的某些子集的点，得到一个新的点集，然后再指定“商”拓扑．例如，实数的所有 mod 整数的等价类对这种拓扑所得到的空间就是平面的单位圆的一个“模型”．

这两种构造空间的方法都是为了使得某些函数连续而引出的．因此，我们开始先定义连续性并且证明关于它的一些简单命题．

连　续　函　数

为方便起见，先回顾一下关于函数的若干术语和一些初等命题（预备知识）．我们称函数（“函数”与“映像”，“映射”，“对应”，“算子”以及“变换”是同义的）f 在 X 上当且仅当它的定义域为 X．又称 f 为到 Y 内当且仅当它的值域为 Y 的子集，另外称它为到 Y 上，假如它的值域为 Y．f 在点 x 处的值记为 $f(x)$，而 $f(x)$ 就叫做 x 关于 f 的像．若 B 为 Y 的子集，则 B 关于 f 的逆 $f^{-1}[B]$ 指的是 $\{x: f(x)\in B\}$．显然，Y 的子集族的元的交（并）关于 f 的逆为这些元

的逆的交（并）；即若对集C的每一个元 c，Z_c 为Y的子集，则 $f^{-1}[\bigcap\{Z_c: c\in C\}]=\bigcap\{f^{-1}[Z_c]: c\in C\}$ 并且对并有相似的结果成立．若 $y\in Y$，则仅含一个元 y 的集的逆 $f^{-1}[\{y\}]$简记为$f^{-1}[y]$．X 的子集A的像 $f[A]$ 是指所有使得对A中的某个 x 有 $y=f(x)$ 的点 y 的集．易见，X的子集族的并的像为像的并，然而，一般地说交的像就不是像的交．最后，我们称 f 为一对一当且仅当不存在具有相同的像的两个不同的点，并且这时 f^{-1} 就是 f 的逆函数．（注意，此处的记号是这样使用的，粗糙地说，方括号表示作用到函数的值域或定义域的子集上，而圆括号表示作用到元上．例如若 f 为一对一，到Y上并且 $y\in Y$，则 $f^{-1}(y)$ 为使得 $f(x)=y$ 的X的唯一的点 x，而 $f^{-1}[y]=\{x\}$．）

从拓扑空间 $(X, \mathscr{T})$ 到拓扑空间 $(Y, \mathscr{U})$ 内的映射 f 称为**连续**当且仅当每一个开集的逆为开集．更精确地说，f 关于$\mathscr{T}$和$\mathscr{U}$ 为连续或 $\mathscr{T}$-$\mathscr{U}$ 连续当且仅当对每一个$\mathscr{U}$中的U有 $f^{-1}[U]\in\mathscr{T}$．显然这个概念是依赖于值域和定义域空间的拓扑，但当不会引起误解时我们按习惯略去拓扑的陈述．有两个关于连续性的命题，虽然它本身很明显，但却十分重要．第一，若 f 为从X到Y内的连续函数，g 为从Y到Z内的连续函数，则合成 $g\circ f$ 为从X到Z内的连续函数，因为对每一个Z的子集 V 有 $(g\circ f)^{-1}[V]=f^{-1}[g^{-1}[V]]$，所以先利用 g，然后再利用 f 的连续性即可推出若V为开集，则 $(g\circ f)^{-1}[V]$ 也为开集．第二，若 f 为从X到Y内的连续函数，A为X的子集，则 f 在A上的限制 $f|A$ 关于A的相对拓扑为连续，因为若U为Y中的开集，则 $(f|A)^{-1}[U]=A\cap f^{-1}[U]$ 显然即为A中的开集．又我们称使得 $f|A$ 为连续的函数 f 为**在 A 上连续**．实际上还可能出现f为在A上连续，而不在X上连续的情形．

下列的一组条件的每一个都等价于连续性，因为在证明函数的连续性时常常需要，所以是有用的．

1 定理． 若X和Y为拓扑空间，f 为从X到Y的函数，则下列命题等价：

(a) 函数 f 为连续；

(b) 每一个闭集的逆为闭集；

(c) Y 的拓扑的子基的每一个元的逆为开集；

(d) 对 X 中的每一个 x，$f(x)$ 的每一个邻域的逆为 x 的邻域；

(e) 对 X 中的每一个 x 和 $f(x)$ 的每一个邻域 U 有 x 的邻域 V 使得 $f[V]\subset U$；

(f) 对 X 中的每一个收敛于点 s 的网 S（或 $\{S_n, n\in D\}$），合成 $f\circ S(\{f(S_n), n\in D\})$ 收敛于 $f(s)$；

(g) 对 X 的每一个子集 A，闭包的像为像的闭包的子集，即 $f[A^-]\subset f[A]^-$；

(h) 对 Y 的每一个子集 B 有 $f^{-1}[B]^-\subset f^{-1}[B^-]$.

证明. (a)$\longleftrightarrow$(b)：这是下列事实的一个简单推论：函数的逆保持取余集的运算，即对 Y 的每一个子集 B 有 $f^{-1}[Y\sim B]=X\sim f^{-1}[B]$.

(a)$\longleftrightarrow$(c)：若 f 为连续，则子基的元的逆为开集，因为每一个子基的元均为开集. 反之，因为 Y 中的每一个开集 V 为子基的元的有限交的并，故 $f^{-1}[V]$ 为子基的元的逆的有限交的并，因而，若每一个子基的元的逆为开集，则每一个开集的逆亦为开集.

(a)$\rightarrow$(d)：若 f 为连续，$x\in X$，V 为 $f(x)$ 的邻域，则 V 包含 $f(x)$ 的一个开邻域 W 并且 $f^{-1}[W]$ 是 x 的一个开邻域，但它是 $f^{-1}[V]$ 的子集，故 $f^{-1}[V]$ 为 x 的邻域.

(d)$\rightarrow$(e)：假设 (d) 成立，若 U 为 $f(x)$ 的邻域，则 $f^{-1}[U]$ 为 x 的邻域并且使得 $f[f^{-1}[U]]\subset U$.

(e)$\rightarrow$(f)：假设 (e) 成立，命 S 为 X 中收敛于点 s 的网，则当 U 为 $f(s)$ 的邻域时有 s 的邻域 V 使得 $f[V]\subset U$，又因 S 基本上在 V 内，故 $f\circ S$ 基本上在 U 内.

(f)$\rightarrow$(g)：假设 (f) 成立，又设 A 为 X 的子集，s 为 A 的闭包的点，则有 A 中的网 S，它收敛于 s 并且 $f\circ S$ 收敛于 $f(s)$，故 $f(s)$ 为 $f[A]^-$ 中的点. 因而 $f[A^-]\subset f[A]^-$.

(g)$\rightarrow$(h)：假设 (g) 成立，若 $A=f^{-1}[B]$，则 $f[A^-]\subset$

$f[A]^- \subset B^-$，故 $A^- \subset f^{-1}[B^-]$，即 $f^{-1}[B]^- \subset f^{-1}[B^-]$.

(h) → (b)：假设 (h) 成立，若 B 为 Y 的闭子集，则 $f^{-1}[B]^- \subset f^{-1}[B^-] = f^{-1}[B]$，因而 $f^{-1}[B]$ 亦为闭集. |

还有一种连续性的局部形式，它是有用的[1]. 我们称从拓扑空间 X 到拓扑空间 Y 的函数 f **在点 x 处为连续**当且仅当 $f(x)$ 的每一个邻域关于 f 的逆为 x 的邻域. 容易给出在一点处的连续性的形如定理 3.1(e) 和 (f) 的特征. 显然 f 为连续当且仅当它在它的定义域的每一点处为连续.

同胚或**拓扑变换**指的是从拓扑空间 X 到拓扑空间 Y 上的连续的一对一映射，并且 f^{-1} 也要求连续. 若存在从一个空间到另一个空间的同胚，则称这两个空间是**同胚的**并且称其中的每一个**同胚于**另一个. 易见从拓扑空间到它自己上的恒等映射恒为同胚，并且同胚的逆和两个同胚的合成仍为同胚. 因而，拓扑空间的全体可以分成等价类使得每一个拓扑空间同胚于它的等价类的每一个元并且仅同胚于这些空间. 又两个拓扑空间叫做**拓扑等价**当且仅当它们为同胚.

两个离散空间 X 和 Y 为同胚当且仅当存在从 X 到 Y 上的一对一函数，即当且仅当 X 和 Y 有相同的基数. 这是因为定义在离散空间上的每一个函数皆为连续，而与值域空间的拓扑无关. 同样，两个平庸空间(整个空间和开集为仅有的开集)为同胚当且仅当存在从一个空间到另一个空间的一对一映射. 这是因为取值到平庸空间内的每一个函数皆为连续，而与定义域空间的拓扑无关. 一般的说，判断两个拓扑空间是否同胚，可以是十分的困难. 带有通常拓扑的实数集同胚于带有相对拓扑的开区间 (0, 1)，因为容易证明在 (0, 1) 的点 x 处的值为 $(2x-1)/x(x-1)$ 的函数是一个同胚. 然而，(0,1)不同胚于$(0,1)\cup(1,2)$，因为若 f 为从(0,1)到$(0,1)\cup(1,2)$上的同胚(或，事实上为连续函数)，则 $f^{-1}[(0,1)]$ 为(0,1)的既开又闭的真子集，而(0,1) 为连通. 这个简短证明的

1) 若 f 定义在拓扑空间的子集 A 上，则在闭包 A^- 的点的连续性也可以定义(见问题 3.D)；并且也有几个有用的命题.

完成也就是因为空间之一为连通，另一为非连通，而连通空间的同胚空间也为连通空间. 若一个性质当它为某个拓扑空间所具有时它也为每一个同胚的空间所具有，则它就叫做**拓扑不变量**.两个空间不同胚的证明，通常就依赖于找出一个拓扑不变量，它为其中之一，而不为其中之另一所具有. 如果一个性质直接由空间的点和其拓扑所定义，那末它就自然成为一个拓扑不变量. 除连通性外，拓扑具有可数基，每一点的邻域系具有可数基，空间为 T_1-空间或为 Hausdorff 空间等性质也都是拓扑不变量. 拓扑学就是有关拓扑不变量的研究[1].

乘 积 空 间

一族拓扑空间的笛卡儿乘积有一种拓扑化的标准方法，这种构造极为重要，因此我们详细地来考察该拓扑的性质. 设 X 和 Y 为拓扑空间，又设 $\mathscr{B}$ 为所有笛卡儿乘积 $U \times V$ 的族，其中 U 为 X 中的开集，V 为 Y 中的开集，则 $\mathscr{B}$ 的两个元的交仍为 $\mathscr{B}$ 的元，因为 $(U \times V) \cap (R \times S) = (U \cap R) \times (V \cap S)$. 因而根据定理 1.11 $\mathscr{B}$ 是 $X \times Y$ 的某个拓扑的基，该拓扑就叫做 $X \times Y$ 的**乘积拓扑**. $X \times Y$ 的子集 W 关于乘积拓扑为开集当且仅当对 W 的每一个元 (x, y) 有 x 的开邻域 U 和 y 的开邻域 V 使得 $U \times V \subset W$. 又空间 X 和 Y 称为**坐标空间**，将 $X \times Y$ 的点 (x, y) 分别对应于 x 和 y 的函数 P_0 和 P_1 称为到坐标空间内的**射影**. 这些射影为连续函数，因为若 U 为 X 中的开集，则 $P_0^{-1}[U]$ 为 $U \times Y$，从而它也是开集. 利用射影的连续性，我们可给出乘积拓扑在下列意义下的刻划. 假设 $\mathscr{T}$ 是 $X \times Y$ 的拓扑，它使得每一个射影为连续，则当 U 为 X 中的开集，V 为 Y 中的开集时 $U \times V$ 为关于 $\mathscr{T}$ 的开集，因为 $U \times V = P_0^{-1}[U] \cap P_1^{-1}[V]$，又由射影的连续性可知这些集关于 $\mathscr{T}$ 为开集；因此 $\mathscr{T}$ 大于乘积拓扑，从而乘积拓扑为使得到坐标空间内

1) 一个拓扑学者乃是一位不知汽车轮胎与咖啡杯之间的差别的人.

的射影为连续的最小拓扑．

没有任何困难，我们就可以把乘积拓扑的定义推广到任意有限多个坐标空间的笛卡儿乘积．若 $X_0, X_1, \cdots, X_{n-1}$ 中的每一个为拓扑空间，则 $X_0 \times X_1 \times \cdots \times X_{n-1}$ 的乘积拓扑的一个基为所有乘积 $U_0 \times U_1 \times \cdots \times U_{n-1}$ 的族，其中每一个 U_i 为 X_i 中的开集．特别，若每一个 X_i 为带有通常拓扑的实数集，则乘积空间为 n 维欧几里德空间 E_n．E_n 的元为定义在集 $0, 1, \cdots, n-1$ 上的实值函数，又函数 x 在整数 i 处的值为 $x_i(=x(i))$．

现在来定义任意一族拓扑空间的笛卡儿乘积的乘积拓扑．假设对指标集 A 的每一个元 a 给定一个集 X_a，则笛卡儿乘积 $\times\{X_a: a \in A\}$ 定义为所有使得对 A 中的每一个 a 有 $x_a \in X_a$ 的 A 上的函数 x 的集，集 X_a 叫做第 a 个坐标集，而从乘积到第 a 个坐标集内的射影 P_a 定义为 $P_a(x) = x_a$．又假设对每一个坐标集给定一个拓扑 $\mathscr{T}_a$．乘积拓扑的引出是基于使每一个射影为连续这一考虑的[1]．为了得到射影的连续性必须且只须每一个形如 $P_a^{-1}[U]$ 的集皆为开集，其中 U 为 X_a 的开子集．显然所有这种形式的集的族为某个拓扑的子基，并且它还是使得射影为连续的最小拓扑，该拓扑就叫做**乘积拓扑**．这个子基的元为形如 $\{x: x_a \in U\}$ 的集，其中 U 为 X_a 中的开集，直观地说，它们是坐标空间中的开集上的柱；有时也称子基的元由通过“限制第 a 个坐标在第 a 个坐标空间的开子集内”而得到的集所组成．乘积拓扑的一个基为该子基的所有元的有限交组成的族，即该基的元 U 为形如 $\bigcap\{P_a^{-1}[U_a]: a \in F\} = \{x$: 对 F 中的每一个 a 有 $x_a \in U_a\}$ 的集，其中 F 为 A 的有限子集，U_a 为 X_a 中的开集（对 F 中的每一个 a）．应该强调的是这里只是有限交．一般的说，$\times\{U_a: a \in A\}$ 就不是关于乘积拓扑的开集，其中对每一个 a，U_a 为 X_a 中的开集．又**乘积空间**指的是带有乘积拓扑的笛卡儿乘积．

从乘积空间到坐标空间内的射影还有另一个很有用的性质．

1) 乘积拓扑的这种描述方法属于 N. Bourbaki.

我们称从拓扑空间 X 到另一个空间 Y 的函数 f 为**开映射**当且仅当每一个开集的像仍为开集，即若 U 为 X 中的开集，则 $f[U]$ 为 Y 中的开集.

2 定理. 从乘积空间到它的每一个坐标空间内的射影为开映射.

证明. 设 P_c 为从 $\times\{X_a: a\in A\}$ 到 X_c 内的射影，则欲证 P_c 为开映射，只须证乘积中的点 x 的邻域的像为 $P_c(x)$ 的邻域，并且还可以假定乘积空间中的邻域为上面所定义的乘积拓扑的基的元.

假设 $x\in V=\{y:$ 对 F 中的 a 有 $y_a\in U_a\}$，其中 F 是 A 的有限子集并且对 F 中的每一个 a，U_a 是 X_a 中的开集. 我们构造一个 X_c 的"拷贝"，使它包含点 x 如下：对 $z\in X_c$，命 $f(z)_c=z$，又当 $a\neq c$ 时命 $f(z)_a=x_a$，则 $P_c\circ f(z)=z$. 若 $c\notin F$，则易见 $f[X_c]\subset V$ 且 $P_c[V]=X_c$ 为开集. 又若 $c\in F$，则 $f(z)\in V$ 当且仅当 $z\in U_c$ 且 $P_c[V]=U_c$ 也为开集. 定理获证. (事实上，在证明中所定义的函数 f 为同胚这个事实偶而也会用到.) |

我们可能猜想乘积空间中的闭集的射影仍为闭集. 然而，容易看出这是不成立的，欧几里德平面中的集 $\{(x,y): xy=1\}$ 在每一个坐标空间内的射影就不是闭集.

对值域为某个乘积空间的子集的函数，连续性有一个极为有用的刻划.

3 定理. 从某个拓扑空间到乘积空间 $\times\{X_a: a\in A\}$ 的函数 f 为连续当且仅当对每一个射影 P_a，合成 $P_a\circ f$ 为连续.

证明. 若 f 为连续，则 $P_a\circ f$ 也为连续，这是因为 P_a 为连续.

若对每一个 a，$P_a\circ f$ 为连续，则对 X_a 的每一个开集 U，集 $(P_a\circ f)^{-1}[U]=f^{-1}[P_a^{-1}[U]]$ 为开集，由此可见上面所定义的乘积拓扑的子基的每一个元关于 f 的逆为开集，因此根据定理 3.1 (c) f 为连续. |

乘积空间中的收敛，利用射影也能够得到一种很简单的描述.

4 定理. 乘积空间中的网 S 收敛于点 s 当且仅当它在每一个

坐标空间内的射影收敛到 s 的射影.

证明. 因为在每一个坐标空间内的射影为连续，故若 $\{S_n, n\in D\}$ 为笛卡儿乘积 $\times\{X_a: a\in A\}$ 中收敛于点 s 的网，则网 $\{P_a(S_n), n\in D\}$ 必定收敛于 $P_a(s)$.

今证其逆. 设 $\{S_n, n\in D\}$ 为这样的网，它使得对 A 中的每一个 a，$\{P_a(S_n), n\in D\}$ 收敛于 s_a，则对 s_a 的每一个开邻域 U_a，$\{P_a(S_n), n\in D\}$ 基本上在 U_a 内，从而 $\{S_n, n\in D\}$ 基本上在 $P_a^{-1}[U_a]$ 内，于是 $\{S_n, n\in D\}$ 必定基本上在每一个形如 $P_a^{-1}[U_a]$ 的集的有限交内，但所有如此的有限交的族关于乘积拓扑为 s 的邻域系的一个基，故 $\{S_n, n\in D\}$ 收敛于 s. ▎

我们把关于乘积拓扑的收敛叫做**坐标收敛**或**点式收敛**. 后一种说法，在所有坐标空间都相同的情况下用得更多. 对于这种重要的特殊情形，笛卡儿乘积 $\times\{X_a: a\in A\}$ 就简单的成为所有从 A 到 X 的函数的集，通常把它记作 X^A. 又 X^A 中的网 $\{F_n, n\in D\}$ 关于点式收敛拓扑收敛于 f 当且仅当对 A 中的每一个 a，网 $\{F_n(a), n\in D\}$ 收敛于 $f(a)$. 这个事实表明我们采用点式收敛的术语似乎是有理由的. 在这种情况下，乘积拓扑也叫做**简单拓扑**.

我们自然要问拓扑空间的乘积是否能够遗传它的坐标空间所具有的性质. 例如，可以问在每一个坐标空间为 Hausdorff 空间或满足第一或第二可数性公理的情况下，乘积空间是否也具有这些性质. 下面的一些定理将要回答这些问题.

5 定理. Hausdorff 空间的乘积仍为 Hausdorff 空间.

证明. 若 x 和 y 为 $\times\{X_a: a\in A\}$ 的不同的点，则对 A 中的某个 a 有 $x_a\neq y_a$，故当每一个坐标空间为 Hausdorff 空间时存在 x_a 和 y_a 的互不相交的开邻域 U 和 V，因而 $P_a^{-1}[U]$ 和 $P_a^{-1}[V]$ 为乘积中 x 和 y 的互不相交的邻域. ▎

回顾一下，平庸拓扑空间指的是空集和整个空间为仅有的开集的空间.

6 定理. 设对指标集 A 的每一个元 a，X_a 为满足第一可数性公理的拓扑空间，则乘积 $\times\{X_a: a\in A\}$ 满足第一可数性公理当且

仅当除可数多个外所有 X_a 为平庸空间.

证明. 假设 B 为 A 的可数子集并且当 a 属于 $A \sim B$ 时 X_a 为平庸空间,又设 x 为乘积空间中的一个点. 对 A 中的每一个 a, 选择 x_a 在 X_a 中的邻域系的一个可数基 $\mathscr{U}_a$,显然当 a 属于 $A \sim B$ 时 $\mathscr{U}_a = \{X_a\}$. 再考虑所有形如 $P_a^{-1}[U]$ 的集的有限交组成的族,其中 $a \in A$, $U \in \mathscr{U}_a$. 则它为可数集, 这是因为当 $a \in A \sim B$ 时, $P_a^{-1}[U] = \times\{X_b: b \in A\}$; 又这些有限交的族为 x 的邻域系的一个基,因而乘积空间满足第一可数性公理.

今证其逆. 假设 B 为 A 的不可数子集并且使得对 B 中的每一个 a 存在 x_a 在 X_a 中的一个邻域, 它为 X_a 的真子集, 又设存在 x 的邻域系的一个可数基 $\mathscr{U}$. 因为 $\mathscr{U}$ 的每一个元 U 包含上面所定义的乘积拓扑的基的一个元, 故除有限多个 A 中的元 a 外 $P_a[U] = X_a$, 又 B 为不可数集,因而有 B 的一个元 a 使得对 $\mathscr{U}$ 中的每一个 U 有 $P_a[U] = X_a$. 但存在 x_a 的开邻域 V, 它为 X_a 的真子集,从而易见没有 $\mathscr{U}$ 的元,它为 $P_a^{-1}[V]$ 的子集,因为 $\mathscr{U}$ 的每一个元都射影到整个 X_a 上,于是矛盾. ▎

此外, 坐标空间也可以遗传乘积空间的某些性质. 若乘积空间为 Hausdorff 空间,则每一个坐标空间亦为 Hausdorff 空间, 又若乘积空间在每一点处有可数局部基,则每一个坐标空间也如此. 这些命题都很容易得到,因此我们略去它的证明.

7 注记. Tychonoff 在两篇经典论文 (Tychonoff [1] 和 [2]) 中定义了乘积拓扑并且证明了它的最重要的性质. 他的结果现在是一般拓扑中的标准工具 (参看第五章). 在 Tychonoff 的工作之前关于点式收敛拓扑的函数序列的收敛已经有许多研究. 但在这些工作中出现很多困难,因为拓扑不能完全由序列收敛来描述,至少对于最有趣的情形是如此(见问题 3.W).

商 空 间

首先我们简要地回顾一下引导出乘积拓扑定义的方法. 若 f

是定义在集 X 上并且取值于拓扑空间 Y 内的函数，则我们恒可指定 X 的拓扑使得 f 为连续．显然，离散拓扑就具有该性质，然而它并不有趣．一个更为有趣的拓扑是所有形如 $f^{-1}[U]$ 的集组成的族 $\mathscr{T}$，其中 U 是 Y 中的开集，它之所以显然为一个拓扑是因为函数的逆保持并的运算．易见使得 f 为连续的每一个拓扑都包含 $\mathscr{T}$，因而 $\mathscr{T}$ 是使得 f 为连续的最小拓扑．若给定的是一族函数，即对指标集 A 的每一个元 a 都确定了一个函数 f_a，则子基为所有形如 $f_a^{-1}[U]$ 的集的族的拓扑恰好就具有相同的性质，其中 a 属于 A，U 为 f_a 的值域中的开集．这也就是我们定义乘积拓扑所用的方法．

本节的目的是研究反过来的情形．设 f 是定义在拓扑空间 X 上并且取值于集 Y 上的函数，问 Y 是否可拓扑化而使得 f 为连续？若 Y 的子集 U 关于使得 f 为连续的一个拓扑为开集，则 $f^{-1}[U]$ 为 X 中的开集．另一方面，所有使得 $f^{-1}[U]$ 为 X 中的开集的 Y 的子集 U 组成的族 $\mathscr{U}$ 是 Y 的一个拓扑，因为该族的元的交（或并）的逆为逆的交（或并）．因此，拓扑 $\mathscr{U}$ 是使得 f 为连续的 Y 的最大拓扑；它叫做 Y 的**商拓扑**（关于 f 和 X 的拓扑的商拓扑）．Y 的子集 B 关于商拓扑为开集当且仅当 $f^{-1}[Y\sim B]=X\sim f^{-1}[B]$ 为 X 中的开集，因此 B 为闭集当且仅当 $f^{-1}[B]$ 为闭集．

当 f 没有另外一些严格限制时，关于商拓扑我们还很难说些什么．因此我们将只考虑 f 属于以下所给出的两个“对偶”的类之一中的函数．回顾一下，定义在一个拓扑空间上并且取值于另一个拓扑空间内的函数 f 叫做开的当且仅当每一个开集的像仍为开集．又我们称函数 f 为闭的当且仅当每一个闭集的像仍为闭集．上一节我们已经知道从欧几里德平面到它的第一个坐标空间上的射影 $P(x, y)=x$ 为开而非闭的映射，现在再给出它的这样的子空间的例子，它使得 P 在其上为闭而非开，以及为既非开又非闭．设子空间 $X=\{(x, y):\ x=0$ 或 $y=0\}$，即它由两个轴所组成，将它投射到实轴上，亦即令 $P(x, y)=x$，这时 $(0, 1)$ 的小邻域的像为唯一的点 0，因此 P 不是定义在 X 上的开映射，但容易证明它

是闭的．若去掉 $(0,0)$，而留下 $X\sim\{(0,0)\}$，则 P 在该子空间上为既非开又非闭(闭集 $\{(x,y): y=0$ 并且 $x\neq 0\}$ 的象就不是闭集)．

从开或闭映射的定义我们容易看出它是依赖于值域空间的拓扑．然而，当 f 为连续的开或闭映射时，值域的拓扑就由映射 f 和定义域的拓扑所完全确定．

8 定理． 若 f 是从拓扑空间 $(X,\mathscr{T})$ 到拓扑空间 $(Y,\mathscr{U})$ 上的连续映射并且使得它也是开或闭映射，则 $\mathscr{U}$ 为商拓扑．

证明．若 f 为开映射并且 U 为 Y 的子集使得 $f^{-1}[U]$ 为开集，则 $U=f[f^{-1}[U]]$ 为关于 $\mathscr{U}$ 的开集，因此，当 f 为开映射时每一个关于商拓扑的开集也是关于 $\mathscr{U}$ 的开集，即商拓扑小于 $\mathscr{U}$．若 f 为连续的开映射，则又因商拓扑为使得 f 为连续的最大拓扑，故 $\mathscr{U}$ 即为商拓扑．

对闭映射来证明本定理时，只须把上面叙述中的每一处以“闭”来代替“开”．|

我们知道若 f 是从某个拓扑空间到乘积空间的函数，则 f 为连续当且仅当 f 与每一个射影的合成为连续．对于商空间也有一个与此相似的命题．

9 定理． 设 f 是从空间 X 到空间 Y 上的连续映射，又设 Y 带有商拓扑．则从 Y 到空间 Z 的映射 g 为连续当且仅当合成 $g\circ f$ 为连续．

证明．若 U 为 Z 中的开集并且 $g\circ f$ 为连续，则 $(g\circ f)^{-1}[U]=f^{-1}[g^{-1}[U]]$ 为 X 中的开集，故由商拓扑的定义便知 $g^{-1}[U]$ 为 Y 中的开集．

其逆显然．|

商拓扑和开或闭映射的性质与值域空间的关系很小，这几乎是明显的．事实上，若 f 是从拓扑空间 X 到带有商拓扑的空间 Y 上的连续映射，则从 X，它的拓扑和所有形如 $f^{-1}[y](y\in Y)$ 的集组成的族可复制出 Y 的一个拓扑的“拷贝”．具体作法如下：设 $\mathscr{D}$ 为所有形如 $f^{-1}[y](y\in Y)$ 的 X 的子集组成的族，又设 P 为从

X 到 $\mathscr{D}$ 的函数，它在 x 处的值为 $f^{-1}[f(x)]$，再对 Y 的每一个 y，命 $g(y)=f^{-1}(y)$，则 g 是从 Y 到 $\mathscr{D}$ 上的一对一映射并且 $g\circ f=P$，$f=g^{-1}\circ P$，若 $\mathscr{D}$ 指定了商拓扑（关于 P），则由定理 3.9 可知 g 为连续(因为 $g\circ f=P$) 并且 g^{-1} 也为连续(因为 $g^{-1}\circ P=f$)，因而 g 为一个同胚．

上面的注释表明从本质上看值域空间在讨论中是不起什么作用的，本节剩下所要给出的定理也正是为了进一步显示出这个事实．作为准备，我们先简要的讨论一下一个确定的集 X 的子集族的一个性质．所谓 X 的**分解(分划)**是指 X 的这样的互不相交的子集族 $\mathscr{D}$，它使得它的并集为 X．又从 X 到分解 $\mathscr{D}$ 上的**射影(商映射)** 是指在 x 处的值为使得 x 属于它的 $\mathscr{D}$ 的唯一的元的函数 P．这时存在描述分解的一种等价方法．给定 $\mathscr{D}$，定义 X 上的一个关系：点 x 与点 y 满足 R 关系当且仅当 x 和 y 属于分解的同一个元．即定义**分解 $\mathscr{D}$ 的关系** R 为由使得 x 和 y 属于 $\mathscr{D}$ 的同一个元的一切 (x, y) 所组成的 $X\times X$ 的子集，亦即有 $R=\bigcup\{D\times D: D\in\mathscr{D}\}$．若 P 为从 X 到 $\mathscr{D}$ 上的射影，则 $R=\{(x, y): P(x)=P(y)\}$．显然关系 R 为等价关系，即它为反身，对称并且传递(见预备知识)．反之，每一个 X 上的等价关系也定义了一个子集(等价类)族，使得它是 X 的分解．若 R 为 X 上的等价关系，则 $\boldsymbol{X/R}$ 定义为所有等价类的族．又若 A 为 X 的子集，则$R[A]$为所有与 A 的点满足 R 关系的点组成的集，即 $R[A]=\{y$: 对 A 中的某个 x 有 $(x, y)\in R\}$，等价地有 $R[A]=\bigcup\{D$: $D\in X/R$ 且 $D\cap A$ 为非空$\}$．当 x 为 X 的点时我们简记 $R[\{x\}]$ 为 $R[x]$，即集 $R[x]$ 为 x 属于它的等价类，并且若 P 为从 X 到分解上的射影，则 $P(x)=R[x]$．

在以下的讨论中，我们假定 X 为确定的拓扑空间，R 为 X 上的等价关系并且 P 为从 X 到所有等价类的族 X/R 上的射影．而**商空间**就指的是带有商拓扑(关于 P) 的族 X/R．若 $\mathscr{A}\subset X/R$，则 $P^{-1}[\mathscr{A}]=\bigcup\{A: A\in\mathscr{A}\}$，因而 $\mathscr{A}$ 为关于商拓扑的开(闭)集当且仅当 $\bigcup\{A: A\in\mathscr{A}\}$ 为 X 中的开(闭)集．

10 定理． 设 P 为从拓扑空间 X 到商空间 X/R 上的射影，则

下列命题等价：

(a) P 为开映射；

(b) 若 A 为 X 的开子集，则 $R[A]$ 为开集；

(c) 若 A 为 X 的闭子集，则所有为 A 的子集的 X/R 的元的并为闭集.

在 (a)，(b) 和 (c) 中将"开"和"闭"互换后所得的结果仍然等价.

证明. 先证 (a) 等价于 (b).

先注意对每一个 X 的子集 A，集 $R[A] = P^{-1}[P[A]]$. 若 P 为开映射并且 A 为开集，则因 P 为连续，故 $P^{-1}[P[A]]$ 为开集. 反之，若对每一个开集 A，$P^{-1}[P[A]]$ 为开集，则因由商拓扑的定义可知 $P[A]$ 为开集，故 P 为开映射.

再证 (b) 等价于 (c).

注意所有为 A 的子集的 X/R 的元的并为 $X \sim R[X \sim A]$，而该集对每一个闭集 A 为闭集当且仅当对于开集 $X \sim A$，集 $R[X \sim A]$ 是开集.

又对偶命题的证明，只须自始至终将"开"和"闭"加以互换. |

若 X 为 Hausdorff 空间，或满足可数性公理之一，则我们自然要问商空间 X/R 是否也必定遗传这些性质. 然而除带有一些严格限制的情形外，回答是否定的. 例如，若 X 为带有通常拓扑的实数集，R 为所有使得 $x - y$ 为有理数的 (x, y) 组成的集，则商空间为平庸空间并且从 X 到 X/R 上的射影 P 为开映射. 因此开映射可以变 Hausdorff 空间为非 Hausdorff 空间. 至于变 Hausdorff 空间为非 Hausdorff 空间，或变满足第一可数性公理的空间为不满足该公理的空间的闭映射的例子只是稍微更复杂些，但并不困难. (问题 3.R，问题 4.G.)

有一个另外的假设条件，它有时也会用到. 这就是假定序偶的集 R 为乘积空间 $X \times X$ 中的闭集. 这个条件还可以再叙述成：若 x 和 y 为 X 的元，它不满足 R 关系，则有 (x, y) 在乘积空间 $X \times X$ 中的邻域 W，它与 R 互不相交. 但如此的邻域 W 必包含形如

$U\times V$的邻域，其中U和V分别为x和y的邻域，又$U\times V$与R互不相交当且仅当没有U中的点与V中的点满足R关系．因此，R为$X\times X$中的闭集当且仅当若x和y为X中不满足R关系的点，则分别有x和y的邻域U和V使得没有U中的点与V中的点满足R关系．显然这又等价于没有X/R的元与U和V相交．

11 定理． 若商空间 X/R 为 Hausdorff 空间，则R为乘积空间$X\times X$中的闭集．

若从空间X到商空间 X/R 上的射影P为开映射并且R为$X\times X$中的闭集，则X/R为 Hausdorff 空间．

证明．若X/R为 Hausdorff 空间并且$(x,y)\notin R$，则$P(x)\neq P(y)$并且有$P(x)$和$P(y)$的互不相交的开邻域U和V．因$P^{-1}[U]$和$P^{-1}[V]$为开集，又因从它关于P的像互不相交可推出没有$P^{-1}[U]$的点与$P^{-1}[V]$的点满足R关系，故$P^{-1}[U]\times P^{-1}[V]$为与$R$互不相交的$(x,y)$的邻域，即$R$为闭集．

今设P为开映射，R为$X\times X$中的闭集，又设$P(x)$和$P(y)$为X/R的两个不同的元，则x和y不满足R关系．因R为闭集，故有x和y的开邻域U和V使得没有U中的点与V中的点满足R关系，从而U和V的像互不相交，再从P为开映射即可推出它又是$P(x)$和$P(y)$的开邻域．|

闭映射在另外一种不同的名称下也已经被广泛地研究．我们称拓扑空间X的分解$\mathscr{D}$为**上半连续**当且仅当对$\mathscr{D}$中的每一个D和每一个包含D的开集U有开集V使得 $D\subset V\subset U$ 并且V是$\mathscr{D}$的元的并．（上半连续概念的来源见问题 3.F.）

12 定理． 拓扑空间X的分解$\mathscr{D}$为上半连续当且仅当从X到$\mathscr{D}$上的射影P为闭映射．

证明．根据定理 3.10，P为闭映射当且仅当对X的每一个开子集U，所有为U的子集的$\mathscr{D}$的元的并V为开集．因此，若P为闭映射，$D\in\mathscr{D}$，U为包含D的开集，则V即为所需的开集，故$\mathscr{D}$为上半连续．

今证其逆．假设$\mathscr{D}$为上半连续，U为X的开子集，V为所有

为 U 的子集的 $\mathscr{D}$ 的元的并．若 $x\in V$，则 $x\in D\subset U$ 对 $\mathscr{D}$ 中的某个 D 成立，故由上半连续性有开集 W，它是 $\mathscr{D}$ 的元的并，并且使得 $D\subset W\subset U$，于是 W 为 V 的子集，从而 V 为 x 的一个邻域．这表明 V 是开集，因为它是它自己的每一点的邻域，再由定理 3.10 便知 P 为闭映射．|

设 A 和 B 为 X 的互不相交闭子集，定义 X 的分解 $\mathscr{D}$ 为 A，B 和所有集 $\{x\}$，其中 x 属于 $X\sim(A\cup B)$ 时．这个分解的商空间有时叫做"叠合 A 的所有点并且叠合 B 的所有点而得的空间"．容易验证 $\mathscr{D}$ 为上半连续，并且当 X 为 Hausdorff 空间时关系 $R=\bigcup\{D\times D:D\in\mathscr{D}\}$ 为 $X\times X$ 中的闭集．我们自然猜想具有这种简单构造的商空间可能遗传空间 X 的有趣性质．不幸，就在这种情况下，X 可以是 Hausdorff 或满足第一或第二可数性公理的空间，而相应的性质对商空间却不成立．

13 注记． 上半连续族的概念是 R. L. Moore 在二十年代末期引进的；而稍后首先由 Aronszajn 对开映射进行了充分的研究 (Aronszajn [2])． 又本节的许多结果均可在 Whyburn [2] 中找到．

问　　题

A　连通空间

连通空间关于连续映射的像仍为连通空间．

B　关于连续性的定理

设 A 和 B 为拓扑空间 X 的子集，它使得 $X=A\cup B$ 并且 $A\sim B$ 和 $B\sim A$ 分离．若 f 为 X 上的函数，它在 A 上为连续并且也在 B 上为连续，则 f 在 X 上为连续．(见定理 1.19．)

C　关于连续函数的习题

若 f 和 g 为定义在拓扑空间 X 上并且取值于 Hausdorff 空间 Y 内的连续函数，则 X 中所有使得 $f(x)=g(x)$ 的点 x 的集为闭集．因此，若 f 和 g

在 X 的一个稠密子集上相同(对属于 X 的一个稠密子集的 x 有 $f(x)=g(x)$)，则 $f=g$。

D 在一点处的连续性;连续扩张

设 f 定义在拓扑空间 X 的子集 X_0 上并且取值于 Hausdorff 空间 Y 内；则 f 在 x 处为连续当且仅当 x 属于 X_0 的闭包并且对值域的某个元 y，它的每一个邻域的逆为 X_0 与 x 的一个邻域的交。

(a) 函数 f 在 x 处为连续当且仅当若 S 和 T 为收敛于 x 的网，则 $f\circ S$ 和 $f\circ T$ 收敛于 Y 的同一个点。

(b) 设 C 为使 f 在该点处为连续的点的集，又设 f' 为 C 上的函数，它在点 x 处的值为在一点处的连续性的定义中所给出的值域空间 Y 的元 y (更精确的说，f' 的图形为 $C\times Y$ 与 f 的图形的闭包的交)。函数 f' 具有性质：若 U 为 X 的开集，则 $f'[U]\subset f[U]^-$。又函数 f' 为连续，只要 Y 具有性质：Y 的每一点的所有闭邻域的族是该点的邻域系的基。(如此的拓扑空间称为正则空间。此处关于 Y 为正则的要求是实质的，这正如 Bourbaki 和 Dieudonné[1] 所指明的那样。)

E 关于实值连续函数的习题

设 f 和 g 为拓扑空间上的实值函数，并且关于实数的通常拓扑为连续，又设 a 为确定的实数。

(a) 在 x 处的值为 $af(x)$ 的函数 af 为连续。(证明变实数 r 为 ar 的函数为连续，并且利用连续函数的合成仍为连续函数的事实。)

(b) 在 x 处的值为 $|f(x)|$ 的函数 $|f|$ 为连续。

(c) 若 $F(x)=(f(x),g(x))$，则 F 关于欧几里德平面的通常拓扑为连续。(检验 $P\circ F$ 是连续的，这里 P 是到某个坐标空间内的射影。)

(d) 函数 $f+g$，$f-g$ 和 $f\cdot g$ 为连续并且若 g 恒不为零，则 f/g 亦为连续。(首先证明 $+$，$-$ 和 $\cdot$ 为从欧几里德平面到实数内的连续函数。(也见问题 3.S.))

(e) 函数 $\max[f,g]=(|f+g|+|f-g|)/2$ 和 $\min[f,g]=(|f+g|-|f-g|)/2$ 为连续。

F 上半连续函数

拓扑空间 X 上的实值函数 f 称为上半连续当且仅当集 $\{x: f(x)\geqslant a\}$ 对

每一个实数 a 为闭集. 又实数集 R 的上拓扑 $\mathscr{U}$ 指的是由空集, R 和所有形如$\{t: t<a\}$的集所组成的拓扑,此处 $a\in R$. 若 $\{S_n, n\in D\}$为实数的网,则定义 $\lim\sup\{S_n: n\in D\}$ 为 $\lim\{\sup\{S_m: m\in D$ 且 $m\geqslant n\}: n\in D\}$, 其中极限是对实数的通常拓扑来取的.

(a) 实数的网 $\{S_n, n\in D\}$ 关于$\mathscr{U}$收敛于 s 当且仅当 $\lim\sup\{S_n: n\in D\}\leqslant s$.

(b) 若 f 为X上的实值函数,则 f 为上半连续当且仅当 f 关于上拓扑为连续,又当且仅当 $\lim\sup\{f(x_n): n\in D\}\leqslant f(x)$, 只要 $\{x_n, n\in D\}$ 为X中收敛于点 x 的网.

(c) 若 f 和 g 为上半连续并且 t 为非负实数,则 $f+g$ 和 tf 为上半连续.

(d)若 F 为一族上半连续的函数并且使得 $i(x)=\inf\{f(x): f\in F\}$ 对X中的每一个 x 存在,则 i 为上半连续. (注意 $\{x: i(x)\geqslant a\}=\bigcap\{\{x: f(x)\geqslant a\}: f\in F\}$.)

(e) 若 f 为X上的有界实值函数,则有最小的上半连续函数 f^- 使得 $f^-\geqslant a$. 若$\mathscr{V}$为点 x 的邻域系,并且 $S_V=\sup\{f(y): y\in V\}$, 则 $f^-(x)=\lim\{S_V, V\in\mathscr{V}, \subset\}$.

(f) 实值函数 g 称为下半连续当且仅当$-g$为上半连续. 若 f 为有界实值函数,命 $f_-=-(-f)^-$, 又定义 f 的振幅 Q_f 为 $Q_f(x)=f^-(x)-f_-(x)$ $(x\in X)$, 则 Q_f 为上半连续并且 f 为连续当且仅当对X中的一切 x 有 $Q_f(x)=0$.

(g) 设 f 为X上的非负实值函数,又设 R 带有通常拓扑, $G=\{(x, t): 0\leqslant t\leqslant f(x)\}$ 带有 $X\times R$ 的相对乘积拓扑. 命$\mathscr{D}$是由G的所有"垂直条"组成的分解,所谓"垂直条"是指形如 $(\{x\}\times R)\cap G$ 的集. 若分解$\mathscr{D}$为上半连续,则 f 也为上半连续. (其逆亦真,但最简单的证明需要定理 5.12.)

G 关于拓扑等价的习题

(a) 任何两个带有实数的通常拓扑的相对拓扑的开区间为同胚.

(b) 任何两个闭区间为同胚,并且任何两个半开区间为同胚.

(c) 没有开区间同胚于闭区间或半开区间,也没有闭区间同胚于半开区间.

(d) 欧几里德平面的子空间 $\{(x, y): x^2+y^2=1\}$ 不同胚于实数空间的子空间.

(上述的某些空间有一个或多个点 x 使得 $\{x\}$ 的余集为连通.)

H　同胚与一对一的连续映射

给定两个拓扑空间 X 和 Y，一个从 Y 到 X 上的一对一连续映射和一个从 X 到 Y 上的一对一连续映射，这时 X 和 Y 还未必为同胚.（设空间 X 由可数多个互不相交的半开区间和可数多个孤立点（点 x 使得 $\{x\}$ 为开集）所组成．又设 Y 由可数多个开区间和可数多个孤立点所组成．再注意可数多个半开区间可以通过一对一连续的方法映到某个开区间上．我相信这个例子是属于 R. H. Fox 的.）

I　关于两个变量的每一个的连续性

设 X 和 Y 为拓扑空间，$X\times Y$ 为乘积空间，又设 f 为从 $X\times Y$ 到另一个拓扑空间的函数，则 $f(x, y)$ 关于 x 连续当且仅当对每一个 y，函数 $f(\ ,y)$（它在 x 处的值为 $f(x, y)$）为连续．相似地，$f(x, y)$ 关于 y 连续当且仅当对每一个 $x\in X$，函数 $f(x,\ )$ $(f(x,)(y)=f(x,y))$ 为连续．若 f 在乘积空间上为连续，则 f 关于 x 并且关于 y 连续，但其逆不真．（经典的例子为定义在欧几里德平面上的实值函数：$f(x, y)=xy/(x^2+y^2)$ 并且 $f(0,0)=0$.）

J　关于 n 维欧几里德空间的习题

n 维欧几里德空间 E_n 的子集 A 称为凸集当且仅当对 A 的每一对点 x 和 y 和每一个满足 $0\leqslant t\leqslant 1$ 的实数 t，点 $tx+(1-t)y$ 均为 A 的元．（我们定义：$(tx+(1-t)y)_i=tx_i+(1-t)y_i$.）这时 E_n 的任意两个开凸子集必为同胚．又闭凸子集如何？

K　关于乘积空间中闭包，内部和边界的习题

设 X 和 Y 为拓扑空间，$X\times Y$ 为乘积空间．又对每一个集 C，命 C^b 为 C 的边界，则当 A 和 B 分别为 X 和 Y 的子集时有

(a) $(A\times B)^-=A^-\times B^-$;

(b) $(A\times B)^0=A^0\times B^0$;

(c) $(A\times B)^b=(A\times B)^-\sim(A\times B)^0=((A^b\cup A^0)\times(B^b\cup B^0))\sim(A^0\times B^0)=(A^b\times B^b)\cup(A^b\times B^0)\cup(A^0\times B^b)=(A^b\times B^-)\cup(A^-\times B^b)$.

L　关于乘积空间的习题

假设对指标集 A 的每一个元 a，X_a 为拓扑空间．又设 B 和 C 为 A 的互不

相交子集并且使得 $A = B\cup C$，则乘积空间 $\times\{X_b:\ b\in B\}\times\times\{X_c:\ c\in C\}$ 同胚于乘积空间 $\times\{X_a:\ a\in A\}$. 同时对每一个确定的拓扑空间 X，乘积 X^A 同胚于 $X^B\times X^C$ 并且 $(X^B)^C$ 同胚于 $X^{B\times C}$，此时，所有这些空间内的拓扑都取作为乘积拓扑.

M　具有可数基空间的乘积

乘积拓扑具有可数基当且仅当每一个坐标空间的拓扑具有可数基并且除可数多个外所有坐标空间为平庸的.

N　关于乘积和可分性的例子

设 Q 为闭单位区间，X 为乘积空间 Q^Q. 又设 X 的子集 A 由点的特征函数所组成；更精确地说，即 $x\in A$ 当且仅当对 Q 中的某个 q 有 $x(q)=1$ 并且 x 在 $Q\sim\{q\}$ 上为零.

(a) 空间 X 为可分.（所有 X 内具有有限值域的 x（有时叫做梯形函数）组成的集在 X 中稠密. 同时还有该集的一个可数子集也在 X 中稠密.）

(b) 带有相对拓扑的集 A 为离散并且为不可分.

(c) 存在 A 在 X 中的唯一的聚点 x，且若 U 为 x 的邻域，则 $A\sim U$ 为有限集.

O　连通空间的乘积

任意一族连通拓扑空间的乘积仍为连通空间.（取定乘积中的一个点 x，并且设 A 为所有使得有某个连通子集同时包含 x 和 y 的点 y 的集. 然后证明 A 为稠密.）

P　关于 T_1-空间的习题

T_1-空间的乘积仍为 T_1-空间. 若 $\mathscr{D}$ 为拓扑空间的分解，则商空间为 T_1-空间当且仅当 $\mathscr{D}$ 的元恒为闭集.

Q　关于商空间的习题

从拓扑空间 X 到商空间 X/R 内的射影为闭映射当且仅当对 X 的每一个子集 A，$R[A]^-\subset R[A^-]$；又它为开映射当且仅当 $R[A^\circ]\subset R[A]^\circ$ 对每一个子集 A 成立.（$^-$和$^\circ$分别为闭包和内部算子.）

R 关于商空间和对角序列的例子

设 X 为带有通常拓扑的欧几里德平面，A 为所有使得 $y=0$ 的点 (x, y) 的集，又设分解 $\mathscr{D}$ 由 A 和所有使得 $(x, y)\notin A$ 的集 $\{(x, y)\}$ 所组成，则带有商拓扑的 $\mathscr{D}$ 有如下的性质：

(a) 从 X 到商空间上的射影为闭映射.

(b) 存在可数多个 A 的邻域使得它的交为 $\{A\}$.

(c) 对每一个非负整数 m，序列 $\{(m, 1/(n+1)), n\in\omega\}$ 在商空间中收敛于 A. 若 $\{N_n, n\in\omega\}$ 为非负整数序列的子序列，则序列 $\{(n, 1/(N_n+1)), n\in\omega\}$ 不收敛于 A.（后者可以叫做是原来给定的序列族的对角序列.）

(d) 商空间不满足第一可数性公理.

注 这个例子属于 R. S. Novosad.

S 拓扑群

$(G, \cdot, \mathscr{T})$ 叫做**拓扑群**当且仅当 $(G, \cdot)$ 为群，$(G, \mathscr{T})$ 为拓扑空间并且在 $G\times G$ 的元 (x, y) 处取值为 $x\cdot y^{-1}$ 的函数关于 $G\times G$ 的乘积拓扑为连续. 当不会引起误解时我们略去群的运算 $\cdot$ 和拓扑 $\mathscr{T}$，而简称"G 为拓扑群". 若 X 和 Y 为 G 的子集，则 $X\cdot Y$ 为所有 G 中这样的 z 组成的集它使得 $z=x\cdot y$ 对 X 中的某个 x 和 Y 中的某个 y 成立. 若 x 为 G 中的点，则我们分别把 $\{x\}\cdot Y$ 和 $Y\cdot\{x\}$ 简记为 $x\cdot Y$ 和 $Y\cdot x$，又定义 Y^{-1} 为 $\{x: x^{-1}\in Y\}$.

(a) 若 X, Y, Z 为 G 的子集，则 $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$ 并且 $(X\cdot Y)^{-1}=Y^{-1}\cdot X^{-1}$.

(b) 设 $(G, \cdot)$ 为群，$\mathscr{T}$ 为 G 的拓扑，则 $(G, \cdot, \mathscr{T})$ 为拓扑群当且仅当对 G 中的每一对 x 与 y 及 $x\cdot y^{-1}$ 的每一个邻域 W，存在 x 的邻域 U 和 y 的邻域 V 使得 $U\cdot V^{-1}\subset W$. 等价的有 $(G, \cdot, \mathscr{T})$ 为拓扑群当且仅当 i 和 m 为连续，其中 $i(x)=x^{-1}$，$m(x, y)=x\cdot y$.

(c) 若 G 为拓扑群，则 i 为从 G 到 G 上的同胚，其中 $i(x)=x^{-1}$. 又对 G 中的每一个 a，L_a 和 R_a（叫做**关于 a 的左和右平移**）均为同胚，其中 $L_a(x)=a\cdot x$，$R_a(x)=x\cdot a$.

注意拓扑群的拓扑由该群的一个元的邻域系所决定，这个事实（下面有精确的陈述）是很重要的，它使得许多概念可以局部化.

(d) 若 G 为拓扑群，$\mathscr{U}$ 为单位元的邻域系，则 G 的子集 A 为开集当且仅当对 A 中的每一个 x 有 $x^{-1}\cdot A\in\mathscr{U}$，等价地，当且仅当对 A 中的每一个 x 有 $A\cdot x^{-1}\in\mathscr{U}$. 又子集 A 的闭包为 $\bigcap\{U\cdot A: U\in\mathscr{U}\}=\bigcap\{A\cdot U: U\in\mathscr{U}\}$.

(注意 $x\in U\cdot A$ 当且仅当 $(U^{-1}\cdot x)\cap A$ 为非空.)

(e) 拓扑群的单位元 e 的邻域系 $\mathscr{U}$ 具有性质:

(i) 若 U 和 V 属于 $\mathscr{U}$, 则 $U\cap V\in\mathscr{U}$;

(ii) 若 $U\in\mathscr{U}$ 并且 $U\subset V$, 则 $V\in\mathscr{U}$;

(iii) 若 $U\in\mathscr{U}$, 则对某个 $V\in\mathscr{U}$ 有 $V\cdot V^{-1}\subset U$;

(iv) 对 $\mathscr{U}$ 中的每一个 U 和 G 中的每一个 x 有 $x\cdot U\cdot x^{-1}\in\mathscr{U}$.

另一方面, 若给定一个群 G 和一族满足这四个条件的子集, 则存在唯一的 G 的拓扑 $\mathscr{T}$ 使得 $(G,\cdot,\mathscr{T})$ 为一个拓扑群并且 $\mathscr{U}$ 为单位元的邻域系.

(f) 每一个带有离散拓扑或平庸拓扑的群为拓扑群. 若 G 为实数集, 则 $(G,+,\mathscr{T})$ 为拓扑群, 并且 $(G\sim\{0\},\cdot,\mathscr{T})$ 也为拓扑群;其中 $\mathscr{T}$ 为通常拓扑. 若 G 为所有整数的集, p 为一个素数, $\mathscr{U}$ 是所有满足后一条件的 G 的子集 U 所构成的集族: 它是由 p^k 的所有倍数组成的集 (k 为某个正整数), 则 $\mathscr{U}$ 为关于使得 $(G,+,\mathscr{T})$ 为拓扑群的一个拓扑 $\mathscr{T}$ 的 0 点邻域系.

(g) 拓扑群为 Hausdorff 空间当它是 T_0-空间时. (即若 x 和 y 为不同的元, 则或者有 x 的邻域使得 y 不属于它, 或者是反过来的情形. 注意若 $x\notin U\cdot y$, 则 $x\cdot y^{-1}\notin U$ 并且若 $V^{-1}\cdot V\subset U$, 则 $V\cdot x\cap V\cdot y$ 为空集.)

(h) 若 U 为开集并且 X 为拓扑群的任意子集, 则 $U\cdot X$ 和 $X\cdot U$ 为开集. 然而, 当 X 和 Y 均为闭集时 $X\cdot Y$ 可以不是闭集. (考虑带有通常加法的欧几里德平面, 又取 $X=Y=\{(x,y)\colon y=1/x^2\}$.)

(i) 群的笛卡儿乘积 $\times\{G_a\colon a\in A\}$ 关于运算: $(x\cdot y)_a=x_a\cdot y_a$ 对 A 中的每一个 a, 仍为群. 又该乘积关于乘积拓扑为拓扑群并且到每一个坐标空间内的射影为连续的开同态[1].

注 Bourbaki [1], Pontrjagin [1] 和 Weil [2] 是关于拓扑群的基本参考书; 也可以参阅 Chevalley [1].

T 拓扑群的子群

(a) 拓扑群的子群关于相对拓扑仍为拓扑群.

(b) 子群的闭包仍为子群, 不变子群的闭包仍为不变子群 (不变 = 正规).

(c) 每一个具有非空内部的子群为既开又闭. 子群 H 或者为闭, 或者 $H^-\sim H$ 在 H^- 中稠密.

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1) 某些作者利用术语"表示"来代表连续同态, 术语"同态"来代表到它的值域上的连续的开同态.

(d) 包含一个确定开子集的拓扑群的最小子群为既开又闭.

(e) 拓扑群的单位元的连通区为不变子群.

(f) 连通拓扑群的离散(带有相对拓扑)正规子群为中心的子集. (对子群H的确定的元h, 考虑变x为$x^{-1}\cdot h\cdot x$的从G到H内的映射.)

U 商群和同态

设G为拓扑群, H为子群, G/H为所有左陪集(对G中的x的形如$x\cdot H$的集)的族, 则G/H关于商拓扑为齐次空间. 若H为不变子群, 则G/H为群, 它叫做商群.

(a) 从拓扑群G到齐次商空间G/H上的射影为连续的开映射. (证明所有与开集U相交的左陪集的并为$U\cdot H$并且应用定理3.10.)

(b) 若H为不变子群, 则G/H关于商拓扑为拓扑群并且射影为连续的开同态.

(c) 变元A为$a\cdot A$的齐次空间的映射为同胚, 其中a为G的一个确定的元.

(d) 若f为从拓扑群G到另一个群H内的同态, 则f为连续当且仅当H的单位元的邻域的逆为G的单位元的邻域.

(e) 若f为从拓扑群G到拓扑群J内的连续同态, 又设$f[G]$具有商拓扑, 则从G到$f[G]$上的映射为连续的开映射, 并且从$f[G]$到J内的恒等映射为连续. 因此, 每一个连续同态可以分解成一个连续的开同态和一个连续的一对一同态的合成. 又若f为从G到J上的连续的开同态, 则J拓扑同构于G/K, 其中K为f的核.

(f) 若$J\subset H\subset G$并且J和H为G的不变子群, 则H/J为G/J的子群, H/J的商拓扑为G/J的相对商拓扑, 并且变A为$A\cdot H$的从G/J到G/H上的映射为连续的开映射, 因而$(G/J)/(H/J)$拓扑同构于G/H.

V 匣空间

笛卡儿乘积$\times\{X_a: a\in A\}$的匣拓扑指的是以所有集$\times\{U_a: a\in A\}$的族为基的拓扑, 其中U_a是X_a中的开集(对A中的每一个a). 因此, 开集的笛卡儿乘积关于匣拓扑仍为开集.

(a) 到每一个坐标空间内的射影关于匣拓扑为连续的开映射.

(b) 设Y为实数的无穷多次的乘积, 即$Y=R^A$, 其中R为实数集, A为一个无限集, 则匣拓扑不满足第一可数性公理, 并且Y中包含点y的连通区

为所有使得 $\{a: x_a \neq y_a\}$ 为有限集的点组成的集。(设 x 和 y 为 Y 的点，它的坐标在一个 A 的元的无限集 $a_0, a_1, \cdots, a_p, \cdots$ 上为互异。又设 Z 为所有使得对某个 K 有 $p|z(a_p)-x(a_p)|/|x(a_p)-y(a_p)|<K$ 对所有 p 成立的 Y 中的 z 组成的集，则 Z 为既开又闭，$x \in Z$ 并且 $y \notin Z$.)

(c) 证明 (b) 的结果对每一个至少包含两个点的无限多个连通 Hausdorff 拓扑群的乘积仍然成立。先证拓扑群的乘积关于匣拓扑仍为拓扑群。

W 实线性空间上的泛函

设 $(X, +, \cdot)$为实线性空间，则X上的实值线性函数叫做**线性泛函**。所有X上的线性泛函的集Z在加法和数量乘法的自然定义下仍为实线性空间。显然，Z是乘积 $R^X = \times\{R: x \in X\}$ 的子集，其中R是实数集，而相对乘积拓扑就称为**弱*或 w^*-拓扑**(**简单拓扑**)。(空间Z为R^X的子群，并且根据问题3.S (i) 它是一个拓扑群，然而下列结果并不需要关于拓扑群的命题.)

下列命题给出了Z的 w^*-稠密子空间和 w^*-连续线性泛函的刻划.

(a) 若 $f, g_1, \cdots, g_n$ 为Z的元并且当每个 $g_i(x)=0$ 时必有 $f(x)=0$，则存在实数 $a_1, \cdots, a_n$ 使得 $f=\Sigma\{a_i g_i: i=1, \cdots, n\}$. (考虑由 $(G(x))_i = g_i(x)$所定义的从X到E^n内的映射G. 证明有一个诱导映射F(见预备知识)使得 $f=F \circ G$.)

(b) **稠密性引理**. 设Y为Z的线性子空间，它使得对X的每一个非零的元x有Y中的g满足 $g(x) \neq 0$，则Y在Z中 w^*-稠密. (为了证明 $f \in Y^-$，必须对X的每一个有限子集 $x_1, \cdots, x_n$，证明有一个Y的元，它在 $x_1, \cdots, x_n$ 的每一点处逼近f. 再证明有Y中的g使得 $g(x_i)=f(x_i)$ 对每一个 x_i 成立，$i=1, \cdots, n$.)

(c) **计值定理**. Z上的线性泛函F为 w^*-连续当且仅当它是一个计值映射，即对X中的某个x，$F(g)=g(x)$ 对Z中的所有g成立. (若F为 w^*-连续，则对X中某些 $x_1, \cdots, x_n$ 和某些正实数 $r_1, \cdots, r_n$ 有 $|F(g)|<1$ 当 $|g(x_i)|<r_i$ 对每一个i成立时. 证明若对每一个i有 $g(x)=0$，则 $F(g)=0$.)

注 乘积拓扑的概念产生于关于 w^*-拓扑的序列收敛的讨论。后者已有广泛的研究(例如见 Banach [1])，但在其研究中出现了一些棘手的情况，而通过拓扑的进一步发展，它才有所澄清。我们可以定义集的序列闭包为该集和所有它的序列极限点的并，而集为序列闭当且仅当它与它的序列闭包相同。这时不难看出一个集可以是关于 w^*-拓扑的序列闭集，但却不是 w^*-闭集。这还不是把序列收敛作为研究对象的问题所在。而真正有影响的事实，

则是一个集的序列闭包可以不再是序列闭集,即序列闭包不是 Kuratowski 的闭包算子.因此,一般拓扑的工具不能应用于序列闭包算子,并且对每一个结论的专门论证都是需要的.它的进一步的讨论和例子见 Banach [1; 208 页].

X 实线性拓扑空间

实线性拓扑空间 $(r.l.t.s)$ 指的是 $(X,+,\cdot,\mathscr{T})$,它使得 $(X,+,\cdot)$ 为实线性空间,$(X,+,\mathscr{T})$ 为拓扑群,并且数量乘法"$\cdot$"为从 $X\times$(实数)到 X 的连续函数.回顾一下,实线性空间的子集 K 为凸集当且仅当若 $0\leqslant t\leqslant 1$ 并且 x 和 y 为 K 的元,则 $t\cdot x+(1-t)\cdot y\in K$.

(a) 对确定的实数 a, $a\neq 0$, 变实线性拓扑空间的每一个元 x 为 $a\cdot x$ 的函数是一个同胚.

(b) 实线性拓扑空间的笛卡儿乘积关于点式加法和数量乘法以及乘积拓扑仍为 $r.l.t.s.$.

(c) 若 Y 为 $r.l.t.s.X$ 的线性子空间,则 Y 关于相对拓扑为 $r.l.t.s.$,并且 X/Y 关于商拓扑为 $r.l.t.s.$.

(d) 设 K 为 $r.l.t.s.X$ 的凸集,f 为 X 上的线性泛函,则 f 在 K 上连续当且仅当对每一个实数 t,集 $f^{-1}[t]\cap K$ 为 K 中的闭集.(若 K 中的网 $\{x_n, n\in D\}$ 收敛于 K 的元 x,但使得$\{f(x_n), n\in D\}$ 不收敛于 $f(x)$,则对 D 的某个共尾子集中的 n,选在 x_n 与 x 的线段上的点 y_n 使得 $f(y_n)$ 与 $f(x)$ 只差一个常数.)

(e) 若 f 为 $r.l.t.s.X$ 上的实值线性函数(即线性泛函),则 f 为连续当且仅当 $\{x: f(x)=0\}$ 为闭集.

注 线性拓扑空间概念相对说是较新的(Kolmogoroff [1] 和 V.Neumann [1]);它产生于 Banach 空间与其共轭空间的弱和弱* 拓扑的研究.线性拓扑空间的初等理论大部分是拓扑群理论的直接的应用;而与拓扑群理论相区别的结果全部都依赖于凸性的论证(这是完全正常的;因为作为仅有的区别的特征的数量乘法的主要应用就是在凸性的论证中).又在本书问题中所给出的 $r.l.t.$ 空间的少量结果不能作为该理论的合适的引论,因为我们没有列入关于凸性的命题,而这在比较认真的研究中是基本的.下列书籍可以作为学习的参考书: Bourbaki [3], Nachbin [1] 和 Nakano [1];其中的第一本包括了关于(不必交换的)拓扑体上的线性拓扑空间的研究.

第四章 嵌入和度量化

我们知道一般拓扑学也是按照数学中经常出现的一种发展模式而展开的．即从在表面上似乎没有多少相仿的几种情形中考察它们之间的相似性和反复出现的论证入手．然后，再试图把这些不同例子中所共有的概念和方法抽象出来．如果我们分析得已经足够透澈，那末我们就能找到一种包含其中大部分乃至全部例子的理论．这种理论就其自身而言也有研究的价值．也正是依照这样的方法，通过多次实践，拓扑空间的概念才得到了发展．因此，它也就是一个不断整理、抽象和扩充过程的必然结果．如果这些例子包含的抽象方法不止一种，那末我们就必须检验每一种抽象方法，看它们是否包含了这些例子的中心思想．这种检验通常是通过比较我们所抽象出的对象和导出它的那些对象来完成的．因此，我们自然也就要问是不是一个拓扑空间，至少在某些适当的限制下必定是导出这个概念的那些特殊的具体空间中的一个．而用来作为比较空间的"标准"例子是单位区间的笛卡儿乘积和度量空间．本章我们将给出度量空间和伪度量空间的初等性质，同时还将给出空间为度量空间或为单位区间的笛卡儿乘积的子空间的充要条件．

最后，提醒一句：拓扑空间的概念决不能包含度量空间所具有的一切性质．另外，在第六章中我们还将引出度量空间概念的另一种与此不同但更为深刻的抽象．

连续函数的存在

这一节我们证明四个引理，其目的是为了构造出拓扑空间上的实值连续函数．现在先来考察与此有关的一类较为特殊的拓扑

空间. 我们称空间为**正规的**[1] 当且仅当对每一对互不相交的闭集 A 和 B 有互不相交的开集 U 和 V 使得 $A\subset U$ 并且 $B\subset V$. 又 **T_4-空间**是指这样的正规空间,它同时又是 T_1-空间(对每一个 x, $\{x\}$ 为闭集). 如果规定集 U 为**集 A 的邻域**当且仅当 A 是 U 的内部 U^0 的子集,那末正规性的定义又可以叙述成: 空间为正规当且仅当互不相交的闭集有互不相交的邻域. 我们还可以给出正规条件的另一种叙述形式. 一个集的一些邻域的族叫做该**集的邻域系的一个基**当且仅当这个集的每一个邻域包含有该族的一个元. 于是若 W 为正规空间的闭子集 A 的一个邻域, 则有互不相交的开集 U 和 V 使得 $A\subset U$ 并且 $X\sim W^0\subset V$, 因而 A 的任意邻域 W 包含有闭邻域 U^-. 这表明当空间为正规时,闭集 A 的所有闭邻域的族是 A 的邻域系的一个基. 今证, 其逆亦真. 因为若 A 和 B 为互不相交的闭集,而 W 为 A 的一个闭邻域,它包含在 $X\sim B$ 内,则 W^0 和 $X\sim W$ 分别为 A 和 B 的互不相交的开邻域,故获证.

$A\subset U^0$

每一个离散空间和每一个平庸空间均为正规空间, 因此正规空间可以不是 Hausdorff 空间, 也可以不满足第一或第二可数性公理. 但 T_4-空间 (T_1 并且正规)必为 Hausdorff 空间. 正规空间的闭子集对于相对拓扑仍为正规空间. 然而,正规空间的子空间,乘积空间和商空间未必仍为正规空间. (见问题 4.E, 4.F.)

对于 T_1-空间有一个介于 Hausdorff 和正规性之间的条件,并且在某些情况下它还可以推出正规性. 我们称拓扑空间为**正则的**当且仅当对每一个点 x 和 x 的每一个邻域 U 有 x 的一个闭邻域 V 使得 $V\subset U$, 即每一点的所有闭邻域的族为该点邻域系的一个基. 一个等价的说法是: 对每一个点 x 和每一个闭集 A, 若 $x\notin A$, 则有互不相交的开集 U 和 V 使得 $x\in U$ 并且 $A\subset V$. 正则空间同时还是 T_1-空间,就叫做 **T_3-空间**. 再回顾一下,所谓 Lindelöf 空间,是指这样的拓扑空间,它使得每一个开覆盖有可数子覆盖.

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1) 这种命名方式是长期沿用的惯例的一个极好例子, 即称我们不能处理的问题为非正规, 非正则等. 又正规空间类的非正规性的简要的讨论将在本章最后的问题中给出.

1 引理 (Tychonoff). 每一个正则的 Lindelöf 空间为正规空间.

证明. 假设 A 和 B 为 X 的互不相交的闭子集. 因为 X 为正则空间,故对 A 的每一个点有一个邻域,它的闭包与 B 不相交,从而所有闭包与 B 不相交的开集组成的族 $\mathscr{U}$ 为 A 的一个覆盖. 类似地,有闭包与 A 不相交的开集组成的族 $\mathscr{V}$ 为 B 的一个覆盖,并且 $\mathscr{U}\cup\mathscr{V}\cup\{X\sim(A\cup B)\}$ 为 X 的一个覆盖. 因此,有 $\mathscr{U}$ 的元的序列 $\{U_n, n\in\omega\}$ 覆盖 A 和 $\mathscr{V}$ 的元的序列 $\{V_n, n\in\omega\}$ 覆盖 B. 设 $U'_n = U_n \sim \bigcup\{V_p^-: p\leqslant n\}$, $V'_n = V_n \sim \bigcup\{U_p^-: p\leqslant n\}$, 则因 $U'_n\cap V_m$ 当 $m\leqslant n$ 时为空集,故 $U'_n\cap V'_m$ 当 $m\leqslant n$ 时为空集,交换 U 和 V 的地位,再应用同样的讨论便知 $U'_n\cap V'_m$ 对所有的 m 和 n 为空集,于是 $\bigcup\{U'_n: n\in\omega\}$ 与 $\bigcup\{V'_n: n\in\omega\}$ 互不相交. 最后, $V_p^-\cap A$ 和 $U_p^-\cap B$ 对所有的 p 为空集,因而,互不相交的开集 $\bigcup\{U'_n: n\in\omega\}$ 和 $\bigcup\{V'_n: n\in\omega\}$ 分别包含 A 和 B.|

特别,满足第二可数性公理的正则空间必为正规空间.

现在我们开始来构造连续的实值函数. 若 A 和 B 为互不相交的闭集,我们需要作出这样的连续实值函数,它取值于 $[0,1]$ 区间而且它在 A 上为 0 并在 B 上为 1. 代替直接构造函数 f, 我们来构造一些集,它们相应于(近似地)形如 $\{x: f(x)<t\}$ 的集. 下面的两个引理说明了一个子集族和一个实值函数之间的关系.

2 引理. 假设对正实数的稠密子集 D 的每一个元 t, F_t 为集 X 的一个子集, 它合于:

(a) 若 $t<s$, 则 $F_t\subset F_s$;

与

(b) $\bigcup\{F_t: t\in D\}=X$.

对 X 中的 x, 命 $f(x)=\inf\{t: x\in F_t\}$, 则对每一个实数 s 有 $\{x: f(x)<s\}=\bigcup\{F_t: t\in D \text{ 且 } t<s\}$ 并且 $\{x: f(x)\leqslant s\}=\bigcap\{F_t: t\in D \text{ 且 } t>s\}$.

证明. 我们来直接计算. 因为集 $\{x: f(x)<s\}=\{x: \inf\{t: x\in F_t\}<s\}$, 又该下确界小于 s 当且仅当 $\{t: x\in F_t\}$ 的某个元

小于 s，故集 $\{x: f(x)<s\}$ 为所有使得对某个 t 有 $t<s$ 和 $x\in F_t$ 的 x 组成的集，即为 $\bigcup\{F_t: t\in D$ 且 $t<s\}$。这就证明了第一个等式。

今证第二个等式。注意，从 $\inf\{t: x\in F_t\}\leqslant s$ 可推出对每一个大于 s 的 u 有 $t<u$ 使得 $x\in F_t$。反之，若对 D 中满足 $t>s$ 的每一个 t 有 $x\in F_t$，则因 D 在正实数中稠密，故 $\inf\{t: x\in F_t\}\leqslant s$。因此，所有使得 $f(x)=\inf\{t: x\in F_t\}\leqslant s$ 的 x 组成的集为 $\{x$: 若 $t\in D$ 且 $t>s$，则 $x\in F_t\}=\bigcap\{F_t: t\in D$ 且 $t>s\}$。‖

3 引理. 假设对正实数的稠密子集 D 的每一个元 t，F_t 为拓扑空间 X 的一个开子集，它合于：

(a) 若 $t<s$，则 F_t 的闭包为 F_s 的一个子集；

与

(b) $\bigcup\{F_t: t\in D\}=X$。

命 $f(x)=\inf\{t: x\in F_t\}$，则函数 f 连续。

证明. 按照定理 3.1，函数为连续，只要值域空间的拓扑的某个子基的每一个元的逆恒为开集，又对实数 s，所有形如 $\{t: t<s\}$ 或 $\{t: t>s\}$ 的集组成的族为实数集的通常拓扑的一个子基，因此欲证 f 连续，只须证对每一个实数 s，$\{x: f(x)<s\}$ 为开集并且 $\{x: f(x)\leqslant s\}$ 为闭集。

引用上一引理，$\{x: f(x)<s\}$ 为开集 F_t 的并，因而亦为开集。再引用上一引理，$\{x: f(x)\leqslant s\}=\bigcap\{F_t: t\in D$ 且 $t>s\}$，于是，如果我们能够证明这个集与 $\bigcap\{F_t^-: t\in D$ 且 $t>s\}$ 相同，那末证明就全部完成。

因为对每一个 t，$F_t\subset F_t^-$，故 $\bigcap\{F_t: t\in D$ 且 $t>s\}\subset\bigcap\{F_t^-: t\in D$ 且 $t>s\}$。另一方面，对 D 中满足 $t>s$ 的每一个 t 有 $r\in D$ 使得 $s<r<t$，因而 $F_r^-\subset F_t$，从而又得到反过来的包含关系.‖

现在，容易证明这一节的主要结果。

4 引理 (Urysohn). 若 A 和 B 为正规空间 X 的互不相交闭子集，则有从 X 到区间 $[0, 1]$ 的连续函数 f 使得 f 在 A 上为 0 并且

在 B 上为 1.

证明. 设 D 为正二进有理数的集（即所有形如 $p2^{-q}$ 的数的集，其中 p 和 q 为正整数）. 当 $t\in D$ 并且 $t>1$ 时，命 $F(t)=X$，又命 $F(1)=X\sim B$，$F(0)$ 为一个这样的开集，它包含 A 并且使得 $F(0)^-$ 与 B 不相交，又当 $t\in D$ 并且 $0<t<1$ 时，表 t 为 $t=(2m+1)2^{-n}$ 的形式并且对于 n 归纳的选取 $F(t)$ 为一个这样的开集，它包含 $F(2m2^{-n})^-$ 同时使得 $F(t)^-\subset F((2m+2)2^{-n})$. 这种选取是可能的，因为 X 为正规空间. 令 $f(x)=\inf\{t: x\in F(t)\}$，则由上一引理便知 f 连续. 因为对 D 中的每一个 t 有 $A\subset F(t)$，函数 f 在 A 上为 0；又因为对 $t\leqslant 1$ 有 $F(t)\subset X\sim B$ 并且对 $t>1$ 有 $F(t)=X$，函数 f 在 B 上为 1. |

嵌入到立方体内

我们称带有乘积拓扑的一些闭单位区间的笛卡儿乘积为一个立方体. 也就是说，立方体是所有从集 A 到闭单位区间 Q 的函数组成的集 Q^A 并且带有点式，或坐标收敛的拓扑. 若把立方体作为空间的一种标准类型，则我们需要描述同胚于立方体的子空间的那些拓扑空间. 完成这个目的所应用的方法虽然简单，但却值得注意，它在其它问题中还要用到.

假设 F 为一个函数族，它的每一个元 f 是从拓扑空间 X 到拓扑空间 Y_f（对于族中不同的元值域可以是不同的），则有从 X 到乘积 $\times\{Y_f: f\in F\}$ 内的一个自然映射，它由映 X 的点 x 为第 f 个坐标是 $f(x)$ 的乘积中的元所定义. 亦即，**计值映射** e 定义为：$e(x)_f=f(x)$. 现在来说明当 F 的元 f 连续时，e 是连续的，而且在附加上 F 包含有"足够多的函数"之后，e 还是一个同胚. 我们称 X 上的函数族 F 为**分离点**当且仅当对每一对不同的点 x 和 y 有 F 中的 f 使得 $f(x)\neq f(y)$. 又称 F 为**分离点和闭集**当且仅当对 X 的每一个闭子集 A 和 $X\sim A$ 的每一个点 x 有 F 中的 f 使得 $f(x)$ 不属于 $f[A]$ 的闭包.

5 嵌入引理． 设 F 为一个连续函数族，它的每一个元 f 是从拓扑空间 X 到拓扑空间 Y_f，则：

(a) 计值映射 e 是从 X 到乘积空间 $\times\{Y_f: f\in F\}$ 的连续函数．

(b) 函数 e 为从 X 到 $e[X]$ 上的开映射，假如 F 分离点和闭集．

(c) 函数 e 为一对一当且仅当 F 分离点．

证明．映射 e 与到第 f 个坐标空间内的射影 P_f 的合成为连续，这是因为 $P_f\circ e(x)=f(x)$．因而，由定理 3.3 便知 e 连续．

欲证命题 (b)，只须证点 x 的开邻域 U 关于 e 的像包含乘积中的 $e(x)$ 的一个邻域与 $e[X]$ 的交．选 F 的一个元 f 使得 $f(x)$ 不属于 $f[X\sim U]$ 的闭包，此时乘积中所有使得 $y_f\notin f[X\sim U]^-$ 的 y 组成的集为开集，又易见它与 $e[X]$ 的交为 $e[U]$ 的子集．因此，e 为从 X 到 $e[X]$ 上的一个开映射．

命题 (c) 是明显的．|

上一引理将把空间拓扑地嵌入到立方体内的问题化为寻找定义在该空间上的连续实值函数的一个"丰富的"的集的问题．显然有这样的拓扑空间，在其上的每一个连续实值函数皆为常数，例如任何平庸空间就具有这个性质，也还有非不足道的例子，正则 Hausdorff 空间上的每一个连续实值函数就恒为常数[1]．拓扑空间 X 叫做全正则当且仅当对 X 的每一个元 x 和 x 的每一个邻域 U 有从 X 到闭单位区间的连续函数 f 使得 $f(x)=0$ 并且在 $X\sim U$ 上 f 恒等于 1．显然，所有从全正则空间到单位区间 $[0,1]$ 的连续函数的族在上一引理的意义下分离点和闭集(它的逆命题也成立，但此处并不需要)．若全正则空间同时又是 T_1-空间(对每一个 x，$\{x\}$ 为闭集)，则所有从该空间到 $[0,1]$ 的连续函数的族也分离点．我们称全正则的 T_1-空间为 **Tychonoff 空间**．若 X 为 Tychonoff 空间，F 为所有从 X 到 $[0,1]$ 的连续函数的族，则嵌入引理 4.5 说明从 X 到立方体 Q^F 内的赋值映射为一个同胚．于是每

1) 见 Hewitt[1] 和 Novak[1]．关于分离公理的其它事实见 van Est 和 Freudenthal[1]．

一个 Tychonoff 空间都同胚于某个立方体的一个子空间. 这个事实实际上是 Tychonoff 空间的一个特征，我们现在着手给出它的证明.

由于 Urysohn 的引理 4.4，每一个正规的 T_1 空间为 Tychonoff 空间. 又每一个全正则空间为正则空间，因为若 U 为 x 的一个邻域，f 为一个连续函数，它在 x 处等于 0，又在 $X \sim U$ 上等于 1，则 $V = \{y: f(y) < 1/2\}$ 为开集并且它的闭包包含在 U 的子集 $\{y: f(y) \leq 1/2\}$ 内. 对于 T_1-空间有一系列所谓的分离公理：Hausdorff，正则，全正则和正规. 除正规性外这些性质在这样的意义下是可遗传的，即当空间 X 具有某种性质时 X 的每一个子空间也具有该性质. 同样，除正规性外每一种相同类型的空间的乘积仍为同一类型的空间. 这些事实的证明除了现在所需要的下列定理外都作为本章的问题(问题 4. H).

6 定理. Tychonoff 空间的乘积空间仍为 Tychonoff 空间.

证明. 为方便起见，我们约定从拓扑空间 X 到闭单位区间的连续函数 f 为关于偶 (x, U) 的函数当且仅当 x 为 X 的一个点，U 为 x 的一个邻域并且 $f(x)=0$，f 在 $X \sim U$ 上恒等于 1. 若 $f_1, \cdots, f_n$ 分别为关于 $(x, U_1), \cdots, (x, U_n)$ 的函数，其中 n 是正整数，又若 $g(x) = \sup\{f_i(x): i = 1, \cdots, n\}$，则 g 为关于 $(x, \bigcap\{U_i: i = 1, \cdots, n\})$ 的函数. 因此空间为全正则空间，只要对每一个 x 和 x 的每一个属于拓扑的某个子基的邻域 U 有关于 (x, U) 的函数.

设 X 为 Tychonoff 空间的乘积 $\times\{X_a: a \in A\}$ 并且 $x \in X$，若 U_a 为 x_a 在 X_a 中的一个邻域，f 为关于 (x_a, U_a) 的函数，则 $f \circ P_a$ 为关于 $(x, P_a^{-1}[U_a])$ 的函数，其中 P_a 是到第 a 个坐标空间的射影. 而所有形如 $P_a^{-1}[U_a]$ 的集组成的族为乘积拓扑的一个子基，故乘积空间为全正则空间. 又 T_1-空间的乘积仍为 T_1-空间，于是定理获证. |

7 嵌入定理. 为了使得拓扑空间为 Tychonoff 空间必要且充分的条件是它同胚于某个立方体的一个子空间.

证明．因为闭单位区间为 Tychonoff 空间，故由定理 4.6 便知作为闭单位区间乘积的立方体亦为 Tychonoff 空间，从而立方体的每一个子空间仍为 Tychonoff 空间。

又我们已经知道若 X 为 Tychonoff 空间，F 为所有从 X 到闭单位区间 Q 的连续函数组成的集，则（根据嵌入引理 4.5）赋值映射为从 X 到立方体 Q^F 内的一个同胚．|

度量和伪度量空间

有许多拓扑空间，它的拓扑是由距离概念引出的．所谓集 X 的**度量**是指从笛卡儿乘积 $X\times X$ 到非负实数的一个这样的函数 d，它使得对所有 X 的点 x，y 和 z 有

(a) $d(x, y)=d(y, x)$;

(b) （三角不等式）$d(x, y)+d(y, z)\geqslant d(x, z)$;

(c) $d(x, y)=0$ 当 $x=y$ 时;

(d) 若 $d(x,y)=0$，则 $x=y$.

其中最后的一个条件对许多目的而言是非本质的．只满足 (a), (b) 和 (c) 的函数 d 叫做**伪度量**（有时称为**偏差**，虽然偏差也在稍为不同的意义下被应用）．这一节的一切定义是对伪度量给出的，但若以度量来代替伪度量，则也有相同的定义。

伪度量空间指的是 (X, d)，其中 d 是 X 的伪度量．对 X 的元 x 和 y，数 $d(x, y)$ 叫做是从 x 到 y 的**距离**（当可能引起误解时就表为 d-距离）．若 r 为一个正数，则称集 $\{y: d(x, y)<r\}$ 为以 x 为心，r 为 d-半径的**开球**，或简称为点 x 的开 r-球，又 $\{y: d(x, y)\leqslant r\}$ 称为点 x 的**闭 r-球**．虽然两个开球的交可以不是一个开球，然而当 $d(x,y)<r$ 并且 $d(x,z)<s$ 时满足 $d(w,x)<\min[r-d(x, y), s-d(x, z)]$ 的每一个点 w 同时为点 y 的开 r-球和点 z 的开 s-球的元（根据三角不等式），即两个开球的交包含其内每一点的一个开球．因此，所有开球的族为 X 的一个拓扑的基（见定理 1.11），这个拓扑就叫做 X 的**伪度量拓扑**．注意，这时每一

个闭球对于伪度量拓扑为闭集.

设 X 为一个集并且定义当 $x=y$ 时 $d(x, y)$ 是 0，在别处时是 1，则 d 为 X 的一个度量并且每一点 x 的开 1-球为 $\{x\}$，从而 $\{x\}$ 对于度量拓扑为开集并且空间是离散空间. 此时，每一点的闭 1-球恰为 X，这表明开 r-球的闭包可以不同于闭 r-球. 若对所有 $X\times X$ 中的 (x, y)，定义 d 为 0，则 d 不是一个度量，而是一个伪度量，并且每一点的开 r-球均为整个空间，于是 X 的伪度量拓扑为平庸拓扑. 又若 X 为实数集并且 $d(x,y)=|x-y|$，则 d 为一个度量，它叫做实数的**通常度量**，并且它所确定的拓扑恰好就是实数的通常拓扑.

对于伪度量 d，从点 x 到子集 A 的距离定义为 $D(A,x)=\inf\{d(x, y): y\in A\}$.

8 定理. 若 A 为伪度量空间的一个确定子集，则从点 x 到 A 的距离对于伪度量拓扑为 x 的连续函数.

证明. 由 $d(x, z)\leqslant d(x, y)+d(y, z)$ 对 A 中的 z 取下确界即得 $D(A, x)\leqslant d(x, y)+D(A, y)$，交换 x 与 y 又得到一个相似的不等式，因而 $|D(A, x)-D(A, y)|<d(x, y)$. 由此可见，若 y 在点 x 的开 r-球中，则 $|D(A,x)-D(A, y)|<r$，从而推出 $D(A, x)$ 的连续性.|

9 定理. 在伪度量空间中，集 A 的闭包恰为所有与 A 距离为 0 的点组成的集.

证明. 因为 $D(A, x)$ 对于 x 连续，故集 $\{x: D(A,x)=0\}$ 为闭集，又它包含 A，从而也包含 A 的闭包 A^-. 另一方面，若 $y\notin A$，则有 y 的一个邻域(可以取它为一个开 r-球)，它与 A 不相交，于是 $D(A, y)\geqslant r$，因而 $\{x: D(A,x)=0\}\subset A^-$. 总之 $A^-=\{x: D(A, x)=0\}$. |

10 定理. 每一个伪度量空间均为正规空间.

证明. 设 A 和 B 为伪度量空间 X 的互不相交闭子集，又设 $D(A, x)$ 和 $D(B, x)$ 分别为从 x 到 A 和从 x 到 B 的距离，命 $U=\{x: D(A,x)-D(B,x)<0\}$，$V=\{x: D(A, x)-D(B,$

$x) > 0\}$，则因函数 $D(A,x)-D(B,x)$ 对于 x 连续，故 U 和 V 均为开集，又易见 U 和 V 互不相交并且由定理 4.9 即得 $A\subset U$ 和 $B\subset V$. |

11 定理. 每一个伪度量空间满足第一可数性公理. 它满足第二可数性公理当且仅当空间为可分.

证明. 因为一个集对于伪度量拓扑为开集当且仅当它包含有其内每一点的一个开球，故点 x 的所有开球组成的族为 x 的邻域系的一个基，又点 x 的每一个开球包含有具有有理半径的开球，于是就得到 x 的邻域系的一个可数基，从而空间满足第一可数性公理。

由于任何满足第二可数性公理的空间必为可分，所以剩下只要证明可分伪度量空间的拓扑具有可数基. 设 Y 为一个可数的稠密子集，又设 $\mathscr{U}$ 为所有具有有理半径的以 Y 的元为心的开球组成的族，则 $\mathscr{U}$ 自然是可数集. 若 U 为点 x 的一个邻域，则对某个正数 r，有点 x 的开 r-球，它包含在 U 内. 设 s 为小于 r 的正有理数，y 为满足 $d(x,y)<s/3$ 的 Y 的点，又设 V 为点 y 的 $2s/3$-球，则 $x\in V\subset U$，故 $\mathscr{U}$ 为拓扑的一个基. |

12 定理. 在伪度量空间 (X,d) 中，网 $\{S_n, n\in D\}$ 收敛于点 s 当且仅当 $\{d(S_n,s), n\in D\}$ 收敛于 0.

证明. 注意网 $\{S_n, n\in D\}$ 收敛于 s 当且仅当它最终地在点 s 的每一个开 r-球内，而这一点成立又当且仅当 $\{d(S_n,s), n\in D\}$ 最终地在具有通常度量的实数空间的零点的每一个开 r-球内. |

伪度量空间 (X,d) 的子集 A 的**直径**指的是：$\sup\{d(x,y): x\in A$ 且 $y\in A\}$. 如果该上确界不存在，那末就称直径为无限. 值得注意的是，具有有限直径这个性质并不是一个拓扑不变量.

13 定理. 设 (X,d) 为一个伪度量空间，又设 $e(x,y)=\min[1,d(x,y)]$，则 (X,e) 亦为一个伪度量空间并且它的拓扑与 (X,d) 的拓扑相同.

因此，任何伪度量空间同胚于一个直径至多为 1 的伪度量空间.

证明. 首先证明 e 为一个伪度量，我们只须证若 a, b 和 c 为满足 $a+b\geqslant c$ 的非负的数，则 $\min[1, a]+\min[1, b]\geqslant\min[1,c]$,这是因为取 $a=d(x,y)$, $b=d(y, z)$ 和 $c=d(x, z)$, 上述不等式就变成关于 e 的三角不等式. 若 $\min[1, a]$ 或 $\min[1, b]$ 为 1，则因 $\min[1, c]\leqslant 1$，故不等式自然成立；若它们均不为 1，则由 $a+b\geqslant c\geqslant\min[1, c]$ 可知不等式也成立. 因此 e 为 X 的一个伪度量.

其次，因为由所有 r 小于 1 的开 r-球组成的族为伪度量拓扑的一个基，而该族不论是对于 d，还是 e 作为伪度量都完全一样，故这两个伪度量拓扑相同.

最后，显然 X 的 e-直径至多为 1.|

我们知道，不可数多个拓扑空间的乘积，一般不满足第一可数性公理（见定理 3.6)，因此，不可能指望对于任意多个伪度量空间的乘积，可以找到一个伪度量使得伪度量拓扑就是乘积拓扑. 但对于可数多个的乘积，情况是令人满意的. 根据上一定理，我们可以只限于讨论直径至多为 1 的空间.

14 定理. 设 $\{(X_n, d_n), n\in\omega\}$ 为一列每一个直径至多为 1 的伪度量空间，若定义 d 为: $d(x, y)=\Sigma\{2^{-n}d_n(x_n, y_n): n\in\omega\}$, 则 d 为笛卡儿乘积的一个伪度量并且伪度量拓扑就是乘积拓扑.

证明. 关于 d 为一个伪度量的简单证明此处从略（有关可和性的问题 2. G 包含有所必须的工具).

今证这两个拓扑相同. 首先，注意若 V 为乘积中的点 x 的开 2^{-p}-球，$U=\{y: d_n(x_n, y_n)<2^{-p-n-2}$ 当 $n\leqslant p+2$ 时$\}$，则 $U\subset V$，这是因为当 $y\in U$ 时有 $d(x,y)<\Sigma\{2^{-p-n-2}: n=0,\cdots,p+2\}+\Sigma\{2^{-n}: n=p+3,\cdots\}<2^{-p-1}+2^{-p-1}=2^{-p}$; 但 U 对于乘积拓扑为 x 的一个邻域，故每一个关于伪度量拓扑的开集也是关于乘积拓扑的开集.

其次，考虑乘积拓扑所定义的子基的一个元 U，这时 U 的形式为 $\{x: x_n\in W\}$，其中 W 是 X_n 中的开集，对于 U 中的 x，则因有点 x_n 的开 r-球，它是 W 的子集，又 $d(x, y)\geqslant 2^{-n}d_n(x_n, y_n)$，故

点 x 的开 $r2^{-n}$-球为 U 的子集；这表明乘积拓扑所定义的子基的每一个元，从而乘积拓扑的每一个元对于伪度量拓扑均为开集．∎

若 (X, d) 和 (Y, e) 为伪度量空间，f 是从 X 到 Y 上的映射，则称 f 为等距（d-e 等距）当且仅当对 X 的一切点 x 和 y 有 $d(x, y)=e(f(x), f(y))$．每一个等距均为连续的开映射（关于这两个伪度量拓扑），这是因为每一个点 x 的开 r-球的像是点 $f(x)$ 的开 r-球．又两个等距的合成仍为等距并且当等距为一对一时其逆亦为等距．对于度量空间，等距必为一对一，因此，从度量空间到度量空间上的等距恒为同胚．所有度量空间的全体可以分成互相等距的空间的等价类．所谓**度量不变量**是指每一个这样的性质，当它为某个度量空间所具有时它也为每一个等距的度量空间所具有．显然，度量不变量可以不是拓扑不变量（例如考虑直径为无限这个性质）．

每一个伪度量空间在一种意义下与度量空间几乎没有差别．为了便于精确地加以叙述，我们规定伪度量空间的**两个子集 A，B 的距离**为 $D(A, B)=\operatorname{dist}(A, B)=\inf\{d(x, y): x\in A, y\in B\}$．一般地说，$D$ 不是一个伪度量，这是因为空间 X 与每一个非空子集的距离为零并且三角不等式不成立．然而，对于空间的某种分解的元，D 实际上是一个度量，这正是下面我们所要讨论的内容．对伪度量空间 (X, d) 命 $\mathscr{D}$ 为所有形如 $\{x\}^-$ 的集所组成的集族，则由定理 4.9 便知 $\{x\}^-$ 恰为所有使得 $d(x,y)=0$ 的点 y 的集，并且分解 $\mathscr{D}$ 就是商集 X/R，其中 R 是关系：$\{(x, y): d(x, y)=0\}$．

15 定理． 设 (X, d) 为一个伪度量空间，又设 $\mathscr{D}$ 为所有集 $\{x\}^-$ 的族，其中 $x\in X$，并且对 $\mathscr{D}$ 的元 A 和 B，命 $D(A, B)=\operatorname{dist}(A, B)$，则 $(\mathscr{D}, D)$ 为一个度量空间，它的拓扑为关于 $\mathscr{D}$ 的商拓扑，并且从 X 到 $\mathscr{D}$ 上的射影为一个等距．

证明．因为 $u\in\{x\}^-$ 当且仅当 $d(u, x)=0$，又当且仅当 $x\in\{u\}^-$，故若 $u\in\{x\}^-$，$v\in\{y\}^-$，则 $d(u, v)\leqslant d(u, x)+d(x, y)+d(y, v)=d(x, y)$，又此时也有 $x\in\{u\}^-$，$y\in\{v\}^-$，因而 $d(u, v)=d(x, y)$．这表明对 $\mathscr{D}$ 的元 A 和 B，$\mathscr{D}(A, B)$

与 $d(x, y)$ 对 A 中的每一个 x 和 B 中的每一个 y 都相等. 由此可见 $(\mathscr{D}, D)$ 为一个度量空间，并且从 X 到 $\mathscr{D}$ 上的射影为一个等距.

若 U 为 X 中的一个开集并且 $x \in U$，则对某个 $r > 0$，U 包含点 x 的一个开 r-球，从而也包含 $\{x\}^-$，于是由定理 3.10 便知从 X 到 $\mathscr{D}$ 上的射影关于 $\mathscr{D}$ 的商拓扑为一个开映射. 又该射影关于由 D 所导出的度量拓扑亦为一个开映射，因此，根据定理 3.8 即得这两个拓扑相同. ▎

度　量　化

给定拓扑空间 $(X, \mathscr{T})$，我们自然要问是否有 X 的一个度量使得 $\mathscr{T}$ 就是度量拓扑. 我们称这样的度量，它**度量化了**这个拓扑空间，又称该空间为**可度量化**. 相似地，拓扑空间叫做**可伪度量化**，当且仅当有一个伪度量使得拓扑就是伪度量拓扑. 因为一个伪度量为度量当且仅当空间为 T_1-空间(即对每一点 x，$\{x\}$ 为闭集)，故空间为可度量化当且仅当它为 T_1-空间并且可伪度量化. 这一节的一切定理都是对可度量化空间叙述的，而对可伪度量化空间的相应定理自然就是明显的了.

本节的两个主要定理分别给出了拓扑空间为可度量化且可分，和可度量化的充要条件. 其中的第一个是经典的 Urysohn 度量化定理；它的证明的所有片段都已有用并且证明还是简单的，也就是把这些有关事实适当的配合在一起. 而第二个定理则是新近才证明的 (它的历史将在本节最后的注中给出). 虽然通过对 Urysohn 方法的适当变形就可以证明条件的充分性，但必要性还需要一种新的构造. 至于所要引入的这种新的概念的进一步研究将在第五章的末了一节进行. 最后，整个度量化问题在第六章中还将从另一种不同的观点来讨论；然而那里所得到的结果并不包括本节的定理.

空间可度量化证明的模型是很简单的. 根据定理 4.14，可数多

个伪度量空间的乘积仍为伪度量空间. 又根据嵌入引理 4.5,若 F 为 T_1-空间 X 上的一个连续函数族, 其中 F 的元 f 映 X 到空间 Y_f 内,则从 X 到 $\times\{Y_f: f\in F\}$ 内的赋值映射为一个同胚只要 F 分离点和闭集 (即若 A 为 X 的闭子集, x 为 $X\sim A$ 的元,则对 F 的某个元 f 有 $f(x)\notin f[A]^-$). 于是, 度量化 T_1-空间的问题就化为寻找从 X 到某个伪度量空间的可数多个连续函数的族 F 使得 F 分离点和闭集(可伪度量化的 T_1-空间必为可度量化).

为方便起见, 命 Q^ω 表示闭单位区间和它自己的可数多次乘积;即 Q^ω 为所有从非负整数到闭单位区间 Q 的函数组成的集并且带有乘积拓扑.

16 度量化定理 (Urysohn). 拓扑具有可数基的正则 T_1-空间恒同胚于立方体 Q^ω 的一个子空间,因而必为可度量化.

证明. 根据定理前面的注记,我 们只须证存在从 X 到 Q 的连续函数的可数族 F, 它分离点和闭集. 设 $\mathscr{B}$ 为 X 的拓扑的一个可数基,又设 $\mathscr{A}$ 为所有使得 U 和 V 属于 $\mathscr{B}$ 并且 $U^-\subset V$ 的 (U,V) 组成的集,则 $\mathscr{A}$ 显然为可数集. 对 $\mathscr{A}$ 中的每一个偶 (U,V), 选一个从 X 到 Q 的连续函数 f 使得 f 在 U 上为零并且在 $X\sim V$ 上为 1 (因为由 Tychonoff 的引理 4.1 和 Urysohn 的引理 4.4 即知如此的函数必存在), 命 F 为如此所得的函数所组成的族, 则 F 自然也为可数集. 剩下要证的是: F 分离点和闭集. 若 B 为闭集, $x\in X\sim B$, 选 $\mathscr{B}$ 中的元 V 使得 $x\in V\subset X\sim B$, 再选 $\mathscr{B}$ 中的元 U 使得 $x\in U^-\subset V$, 则 $(U,V)\in\mathscr{A}$, 于是,若取 f 为 F 中相应的元,则有 $f(x)=0\notin\{1\}=f[B]^-$. |

我们容易进一步描述出上面的度量化定理所能应用的拓扑空间的类.

17 定理. 若 X 为 T_1-空间,则下列命题等价:

(a) X 为正则空间并且它的拓扑具有可数基;

(b) X 同胚于立方体 Q^ω 的一个子空间;

(c) X 为可度量化并且可分.

证明. 上一定理证明了 (a) $\to$ (b).

因为由定理4.14可知立方体Q^ω为可度量化，再由问题3.M又知它满足第二可数性公理，故它的每一个子空间为可度量化并且满足第二可数性公理，从而亦为可分，即有(b)→(c).（注意：可分空间的子空间未必也恒为可分空间.）

最后，证明(c)→(a). 因为若X为可度量化并且可分，则它必为正则空间，并且由定理4.11它还满足第二可数性公理，故获证.|

对于不可分空间，度量化定理仍然严重依赖于我们所已经用过的想法. 通过对研究方法的简要讨论，我们将会看到前面所用的办法还能得到改进. X的一个度量的作出是通过寻找从X到一些伪度量空间内的映射的族. 但要注意：用来作为值域空间的空间只是闭单位区间Q. 我们再用稍微不同的形式加以叙述，即若f为从X到Q的函数，则可作出X的一个伪度量，只要命$d(x,y)=|f(x)-f(y)|$. 而Urysohn度量化定理证明的完成就是利用了可数多个这种类型的伪度量，现在的问题是来推广这种作法. 若F为一个从X到Q的函数族，则可能选取的一个伪度量为和：$\Sigma\{|f(x)-f(y)|: f\in F\}$，为了使得从$X$到伪度量空间$(X,d)$内的恒等映射为连续，这个和必须对于$x$与$y$连续，有一个比族$F$的有限性要弱得多的条件可以保证这种连续性. 即为了得到这种连续性，只须对X的每一点x有x的一个邻域U使得除有限多个F的元外在U上恒为零；换言之，某种类型的局部有限性就足够了. 这个局部有限性概念就是我们解决问题的关键.

我们称拓扑空间的子集族$\mathscr{A}$为**局部有限的**，当且仅当该空间的每一点有一个邻域，它只与$\mathscr{A}$的有限多个元相交. 由这个定义立即推出一个点为并$\bigcup\{A: A\in\mathscr{A}\}$的一个聚点当且仅当它是$\mathscr{A}$的某个元的一个聚点，从而并的闭包为闭包的并，即$[\bigcup\{A: A\in\mathscr{A}\}]^-=\bigcup\{A^-: A\in\mathscr{A}\}$. 又易见所有$\mathscr{A}$的元的闭包组成的族亦为局部有限. 我们又称族$\mathscr{A}$为**离散的**当且仅当该空间的每一点有一个邻域，它至多只与$\mathscr{A}$的一个元相交. 显然，离散的族为局部有限，并且若$\mathscr{A}$为离散，则所有$\mathscr{A}$的元的闭包组成的族

亦为离散．最后，我们称族 $\mathscr{A}$ 为 σ 局部有限（σ 离散）当且仅当它是可数多个局部有限(相应地，离散)的子族的并．

现在我们可以叙述如下的度量化定理．它的证明则包含在其后的一系列引理之中．

18 度量化定理． 对于任何拓扑空间下列三个条件等价：

(a) 空间为可度量化；

(b) 空间为 T_1 和正则空间，并且拓扑有一个 σ 局部有限基；

(c) 空间为 T_1 和正则空间并且拓扑有一个 σ 离散基．

因为 (c)→(b) 是明显的，故只须依次证明 (b)→(a) 和 (a)→(c)．证明的第一步是给出 Tychonoff 的引理 4.1 的一种变形．

19 引理． 拓扑有一个 σ 局部有限基的正则空间恒为正规空间．

证明．若 A 和 B 为空间 X 的互不相交闭子集，则分别有 A 和 B 的开覆盖 $\mathscr{U}$ 和 $\mathscr{V}$ 使得 $\mathscr{U}$ 的每一个元的闭包与 B 不相交，$\mathscr{V}$ 的每一个元的闭包与 A 不相交，并且 $\mathscr{U}$ 和 $\mathscr{V}$ 是一个 σ 局部有限基 $\mathscr{B}$ 的子族，故由此即得 $\mathscr{U}=\bigcup\{\mathscr{U}_n: n\in\omega\}$，$\mathscr{V}=\bigcup\{\mathscr{V}_n: n\in\omega\}$，其中 $\mathscr{U}_n$，$\mathscr{V}_n$ 均为局部有限的族．命 $U_n=\bigcup\{W: W\in\mathscr{U}_n\}$，又命 $V_n=\bigcup\{W: W\in\mathscr{V}_n\}$，则 $U_n^-=\bigcup\{W^-: W\in\mathscr{U}_n\}$，故 U_n^- 与 B 不相交，类似的有 V_n^- 与 A 不相交．这正好就是引理 4.1 证明中所出现的情形，如同那里一样命 $U_n'=U_n\sim\bigcup\{V_k^-: k\leqslant n\}$，$V_n'=V_n\sim\bigcup\{U_k^-: k\leqslant n\}$ 就完成了证明．事实上，所有集 U_n' 的并和所有集 V_n' 的并就分别是 A 和 B 所需的互不相交的邻域．▌

下面的引理给出了定理 4.18 中所指出的条件为可度量化的充分条件的证明．

20 引理． 拓扑有一个 σ 局部有限基的正则 T_1-空间恒为可度量化．

证明．首先注意如果证明了存在空间 X 上的伪度量的一个可数族 D 使得 D 的每一个元在 $X\times X$ 上为连续并且对 X 的每一个闭子集 A 和 $X\sim A$ 的每一点 x 有 D 中的元 d 使得从 x 到 A 的 d-距

离为正，那末也就证明了X为可度量化，这是因为此时映X到每一个伪度量空间(X, d)内的映射恒为连续并且可以如同 Urysohn 定理一样来应用定理 4.5 和定理 4.14. 因此，现在的问题是具体作出如此的族D.

设$\mathscr{B}$为X的拓扑的σ局部有限基，又设$\mathscr{B}=\{\mathscr{B}_n: n\in\omega\}$，其中每一个$\mathscr{B}_n$均为局部有限. 对整数$m$和$n$的每一个序偶和$\mathscr{B}_m$的每一个元$U$，命$U'$为所有$\mathscr{B}_n$中这样的元的并，它的闭包包含在$U$内. 因为$\mathscr{B}_n$为局部有限，故$U'$的闭包为$U$的一个子集，于是由定理 4.19 和定理 4.4 有一个从X到单位区间的连续函数f_U使得它在U'上为 1 并且在$X\sim U$上为 0. 命$d(x, y)=\Sigma\{|f_U(x)-f_U(y)|: U\in\mathscr{B}_m\}$，则$d$在$X\times X$上的连续性是$\mathscr{B}_m$的局部有限性的一个直接推论. 最后，命$D$为如此得到的伪度量的族，则因每一个伪度量由整数的一个序偶所作出，故D为可数集. 此外，若A为X的一个闭子集并且$x\in X\sim A$，则对某个m和$\mathscr{B}_m$中的某个U有$x\in U\subset X\sim A$，又对某个n和$\mathscr{B}_n$中的某个V有$x\in V$并且$V^-\subset U$，显然，对于这个序偶所作出的d，从x到A的d-距离至少为 1. |

剩下的是度量化定理证明的最有趣的部分，也就是证明每一个度量空间有一个σ离散基. 有一个比它更强的结果，并且这个更强的定理为以后所需要，为此我们先引入一个新概念. 我们称集X的覆盖$\mathscr{B}$为覆盖$\mathscr{A}$的一个**加细**当且仅当$\mathscr{B}$的每一个元为$\mathscr{A}$的某个元的一个子集. 例如，在度量空间内，所有半径为 1/2 的开球组成的族就是所有半径为 1 的开球组成的族的一个加细. 下面的定理表明：伪度量空间的任何开覆盖有一个σ离散的开的加细. 由此即知，每一个伪度量拓扑有一个σ离散基，这是因为我们可以选取由所有半径为$1/n$的开球所组成的覆盖的一个σ离散加细$\mathscr{B}_n$，并且所有族$\mathscr{B}_n$的并又是一个σ离散基. 这个事实也就完成了度量化定理 4.18 的证明.

21 定理. 伪度量空间的每一个开覆盖有一个σ离散的开的加细.

证明. 设 $\mathscr{U}$ 为伪度量空间 (X, d) 的一个开覆盖. 证明的第一步是把 $\mathscr{U}$ 的每一个元 U 分解成"同心圆", 对每一个正整数 n 和 $\mathscr{U}$ 的每一个元 U, 命 U_n 为所有使得 $\operatorname{dist}[x, X \sim U] \geqslant 2^{-n}$ 的 U 的元 x 组成的集, 则由三角不等式显然有 $\operatorname{dist}[U_n, X \sim U_{n+1}] \geqslant 2^{-n} - 2^{-n-1} = 2^{-n-1}$. 选一个关系 $<$ 使得族 $\mathscr{U}$ 为良序(见预备知识定理25(h)), 并且对每一个正整数 n 和 $\mathscr{U}$ 的每一个元 U, 命 $U_n^* = U_n \sim \bigcup\{V_{n+1}: V \in \mathscr{U}$ 且 $V < U\}$, 则对每一对 $\mathscr{U}$ 中的 U 与 V 和每一个正整数 n 有 $U_n^* \subset X \sim V_{n+1}$ 或 $V_n^* \subset X \sim U_{n+1}$, 而这依赖于在序的意义下 U 是在 V 之后, 还是之前, 显然在任何一种情况下均有 $\operatorname{dist}[U_n^*, V_n^*] \geqslant 2^{-n-1}$. 由此即知, 若定义 $U_n^\sim$ 为所有这样的点 x 组成的集, 它使得从 x 到 U_n^* 的距离小于 2^{-n-3}, 则 $\operatorname{dist}[U_n^\sim, V_n^\sim] \geqslant 2^{-n-2}$, 因而, 对每一个确定的 n, 所有形如 $U_n^\sim$ 的集组成的族为离散. 命 $\mathscr{V}$ 为对一切 n 和所有 $\mathscr{U}$ 中的 U 的 $U_n^\sim$ 组成的族, 则 $\mathscr{V}$ 为 X 的一个开覆盖, 这是因为若 U 为 $\mathscr{U}$ 中这样的元中的第一个, 它使得 x 属于它, 则对某个 n 有 $x \in U_n^\sim$. 又易见 $U_n^\sim \subset U$, 因此, $\mathscr{V}$ 为 $\mathscr{U}$ 的一个 σ 离散的开的加细. ∎

22 注记. 实际上是有两个度量化问题, 其中的拓扑问题就是我们所已经讨论过的, 而一致度量化问题将在第六章再来讨论(那里给出了它的详细的叙述和历史). 在这两个问题中, 十分奇怪的是后一问题得到圆满解决要比前一问题早得多. 虽然 Urysohn 定理只讨论了一种特殊情形, 但直到不久以前它还是拓扑问题中最满意的定理. 现在的这种令人满意的情况, 它的关键是由两篇文章完成的. Dieudonné[1] 开始了具有这样性质的空间的研究, 即每一个开覆盖有一个局部有限的开的加细 (仿紧空间; 见第五章). A. H. Stone[1] 证明了每一个可度量化的空间必为仿紧空间(这个定理的一个特殊情形已由 C. H. Dowker[1] 在早些时候证得). 局部有限的刻划由几位数学家发现, 特别是 Nagata[1] 和 Smirnov[1]. 而 σ 离散的刻划则属于 Bing[1]. 可度量化条件的必要性 (定理 4.21)的证明事实上是 Stone 的仿紧性证明的初始片段.

Smirnov[2] 也证明了从仿紧和局部可度量化蕴含可度量化.

最后，给出伪度量空间作用的一个简短陈述。我们知道在分析学中所出现的空间大多数是伪度量空间，而不是度量空间，甚至在度量化问题中通过伪度量的作法也还是方便的。当然，通常我们可以用一个和它相联系的度量空间来代替伪度量空间（定理4.15），然而取商空间的方法有些冗繁，并且对大多数场合来讲要求 $d(x,y)=0$ 当且仅当 $x=y$ 是完全无关的。我们还可以试图只使用伪度量，但这有不便之处，例如，当我们去构造拓扑映射时就会如此。一种可能的出路是把“拓扑映射”重新定义为拓扑之间的一种关系，它诱导出一个一对一保持交和并的映射。

问　　题

A　正则空间

(a) 设 X 为正则空间，又设 $\mathscr{D}$ 为所有形如 $\{x\}^-$ 的子集的族，其中 x 属于 X，则 $\mathscr{D}$ 为 X 的一个分解，从 X 到商空间 $\mathscr{D}$ 上的射影为既开又闭的映射并且该商空间为正则 Hausdorff 空间。（若 A 是 X 的既开又闭的子集，则当 $x\in A$ 时 $\{x\}^-\subset A$。）

(b) 正则空间的乘积仍为正则空间。

B　度量空间上的函数的连续性

从伪度量空间 (X,d) 到伪度量空间 (Y,e) 的函数 f 为连续当且仅当对 X 中的每一个 x 和每一个 $\varepsilon>0$ 有 $\delta>0$ 使得当 $d(x,y)<\delta$ 时 $e(f(x),f(y))<\varepsilon$。

C　关于度量的问题

设 f 为定义在非负实数集上的连续实值函数并且 $f(x)=0$ 当且仅当 $x=0$，又设 f 非降并且对一切非负的 x 与 y 有 $f(x+y)\leqslant f(x)+f(y)$。（满足最后这个条件的函数叫做次可加。）若 (X,d) 为度量空间并且 $e(x,y)=f(d(x,y))$，则 (X,e) 也为度量空间并且空间 (X,e) 的度量拓扑与 (X,d) 的度量拓扑相同。（这个结果的一个特殊情形在文献中时常出现：即 $f(x)=x/(1+x)$。）

D　关于子集的 Hausdorff 度量

设 (X, d) 为具有有限直径的度量空间，又设 $\mathscr{A}$ 为所有闭子集组成的族．对 $r>0$ 和 $\mathscr{A}$ 中的 A，命 $V_r(A)=\{x:\ \mathrm{dist}(x, A)<r\}$，并且对 $\mathscr{A}$ 的 A 和 B，定义 $d'(A,B)=\inf\{r:\ A\subset V_r(B)$ 且 $B\subset V_r(A)\}$．d' 叫做 Hausdorff 度量，它不同于正文中所利用的集之间的距离．

(a) $(\mathscr{A}, d')$ 为度量空间并且变 X 中的 x 为 $\mathscr{A}$ 中的 $\{x\}$ 的映射为从 X 到 $\mathscr{A}$ 的一个子空间上的等距．

(b) $\mathscr{A}$ 的 Hausdorff 度量的拓扑并不能由 X 的度量拓扑所决定．例如，设 X 为正实数集，又设 $d(x, y)=|x/(1+x)-y/(1+x)|$，$e(x, y)=\min[1, |x-y|]$，则 (X, d) 与 (X, e) 的度量拓扑相同，但 $(\mathscr{A}, d')$ 和 $(\mathscr{A}, e')$ 却不相同．(在 $(\mathscr{A}, d')$ 中正整数集为所有它的有限子集组成的族的一个聚点．)

注　关于这个论题的情况和文献见 Michael[2]．

E　关于正规空间的乘积的例子(序数)

一般地说，正规空间的乘积并不是正规空间[1]．设 Ω_0 为所有小于第一个不可数序数 Ω 的序数组成的集，又设 Ω' 为 $\Omega_0\cup\{\Omega\}$，并且每一个都带有序拓扑．

(a) 交错引理．设 $\{x_n,\ n\in\omega\}$ 和 $\{y_n,\ n\in\omega\}$ 为 Ω_0 中的两个序列并且对每一个 n 有 $x_n\leqslant y_n\leqslant x_{n+1}$，则这两个序列收敛于 Ω_0 中的同一个点．

(b) 若 A 和 B 为 Ω_0 的互不相交闭子集，则 Ω 不能同时是 A 和 B 的聚点．

(c) Ω_0 和 Ω' 均为正规空间．(若 A 和 B 为互不相交闭子集并且 $A\cup B$ 的第一个点属于 A，则可找到有限序列 $a_0, b_0, a_1, \cdots, a_n$ (或 b_n) 使得 $a_i\in A$，$b_i\in B$ 并且对每一个 i 没有 A 中的点位于 a_i 和 b_i 之间，也没有 B 中的点位于 b_i 和 a_{i+1} 之间，此时区间 $(a_i, b_i]$ 为既开又闭．)

(d) 若 f 为从 Ω_0 到 Ω_0 的函数，它使得对每一个 x 有 $f(x)\geqslant x$，则对某个 Ω_0 中的 x，点 (x, x) 为 f 的图形的聚点．(归纳地定义序列 $x_{n+1}=f(x_n)$，注意 $x_n\leqslant f(x_n)\leqslant x_{n+1}$ 并且利用交错引理．)

(e) 乘积 $\Omega_0\times\Omega'$ 不是正规空间．(设 A 为所有点 (x, x) 组成的集，又设 $B=\Omega_0\times\{\Omega\}$．若 U 为 A 的邻域，命 $f(x)$ 为大于 x 并且使得 $(x, f(x))\notin U$ 的

1) 利用下一章的方法这个问题的一部分可以得到稍许加强．然而，此处所给出的事实后面将要用到．我相信该例子独立地属于 J. Dieudonné 和 A. P. Morse．

最小序数，则 (d) 可以应用.)

F　关于正规空间的子空间的例子 (Tychonoff 板)

正规空间的子空间可以不是正规空间. 设 Ω' 为不大于第一个不可数序数 Ω 的序数组成的集又设 ω' 为不大于第一个无限序数 ω 的序数组成的集并且每一个都带有序拓扑，则乘积 $\Omega'\times\omega'$ 叫做 Tychonoff 板. 不难直接证明它是正规空间，这个事实也是下一章的一个定理的直接推论. 设 X 为 $(\Omega'\times\omega')\sim\{(\Omega,\omega)\}$，即从 Tychonoff 板中去掉一点，又设 A 为所有 X 中第一个坐标为 Ω 的点组成的集，B 为所有第二个坐标为 ω 的点组成的集，则 A 和 B 不存在互不相交的邻域. (若 U 是 A 的邻域，对 ω' 中的 x，命 $f(x)$ 为使得当 $y>f(x)$ 时有 $(y,x)\in U$ 的第一个序数，则 f 的值的上确界小于 Ω.)

G　商的乘积和非正则的 Hausdorff 空间的例子

设 X 为非正规的正则 Hausdorff 空间，又设 A 和 B 为互不相交的闭集并且使得 A 的每一个邻域与 B 的每一个邻域相交，再设 Δ 为所有 (x,x) 组成的集，其中 $x\in X$ (Δ 为 X 上的恒等关系).

(a) 设 $R=\Delta\cup(A\times A)$，则 R 为 $X\times X$ 中的闭集，并且商空间 X/R 为非正则的 Hausdorff 空间. (商空间的元为 A 和 $\{x\}$，其中 $x\in X\sim A$.)

(b) 设 $S=\Delta\cup(A\times A)\cup(B\times B)$，则 S 为 $X\times X$ 中的闭集，但 X/S 为非 Hausdorff 空间. (X/S 的元为 A，B 和 $\{x\}$，其中 $x\in X\sim(A\cup B)$.)

(c) 存在从 $X\times X$ 到 $(X/S)\times(X/S)$ 上的自然映射，它映 (x,y) 为 $(S[x],S[y])$. 我们自然要问该映射是否为开映射，假如 X/S 给定商拓扑并且 $(X/S)\times(X/S)$ 和 $X\times X$ 给定乘积拓扑. (这等价于问商的乘积是否拓扑等价于乘积的商.) 若 S 为 (b) 中所定义的关系，则该映射不是开映射. (考虑 $A\times B$ 的邻域 $X\times X\sim(A\times A\cup B\times B\cup\Delta)$.)

H　可遗传，可乘和可除的性质

空间的性质 P 叫做可遗传当且仅当具有 P 的空间的每一个子空间也具有 P，叫做可乘当且仅当具有 P 的空间的乘积也具有 P，叫做可除当且仅当每一个具有 P 的空间的商也具有 P. 若考虑性质: T_1，H = Hausdorff，R = 正则，CR = 全正则，T = Tychonoff，N = 正规，C = 连通，S = 可分，C_I = 第一可数性公理，C_{II} = 第二可数性公理，M = 可度量化以及 L = Lindelöf，则下列的表可由 + 或 − 所填满，它是根据每列上端的性质是否为

左端所指出的类型来填的。请通过反例(多数必须的例子已经在各个问题中叙述过)或推证(当成立时)来证明。

	T_1	H	R	CR	T	N	C	S	C_I	C_{II}	M	L
可遗传	+	+	+	+	+	−	−	−	+	+	+	−
可乘	+	+	+	+	+	−	+	−	−	−	−	−
可除	−	−	−	−	−	−	+	+	−	−	−	+

若改变该问题,而只考虑闭子空间或开映射,则得到完全不同的结果。

I 半开区间空间

设 X 为具有半开区间拓扑的实数集(所有半开区间 $[a, b)$ 的族为一个基,见问题 1.K 和 1.L),则

(a) X 为正则空间。

(b) X 为正规空间。(回顾一下,x 的每一个开覆盖有可数子覆盖.)

(c) 乘积空间 $X \times X$ 为非正规空间。(设 $Y = \{(x,y): x + y = 1\}$,又设 A 为所有第一个坐标为无理数的 Y 的元的集,$B = X \sim A$. 假定 U 和 V 为 A 和 B 的互不相交的邻域,又对 A 中的 x,命 $f(x) = \sup\{e: [x, x + e) \times [1 - x, 1 - x + e) \subset U\}$,则 f 为所有无理数的集上的函数并且恒不为零. 这时,矛盾的导出是依赖于如下的事实,即对某个正整数 n 有一个有理数,它是 $\{x: f(x) \geqslant 1/n\}$ 的聚点. 而这个事实则是实数空间(带有通常拓扑)为第二范畴定理(见第七章)的一个明显推论,但要给出它的直接证明似乎还有困难.)

注 这个例子属于 Sorgenfrey[1].

J 实连续函数零点的集

拓扑空间的子集叫做一个 G_δ 当且仅当它是一个可数开集族的元的交。

(a) 若 f 为 X 上的连续实值函数,则 $f^{-1}[0]$ 为一个 G_δ.(集 $\{0\}$ 为实数空间的一个 G_δ.)

(b) 若 A 为正规空间 X 中的一个 G_δ,则存在连续实值函数 f 使得 $A = f^{-1}[0]$。

K 完备正规空间

拓扑空间叫做完备正规当且仅当它为正规空间并且每一个闭子集为一个 G_δ。

(a) 每一个伪度量空间为完备正规空间.

(b) 不可数多个单位区间的乘积不是完备正规空间。(在这种空间内，G_δ 不能由一个单独的点组成.)

L 全正则空间的刻划

拓扑空间为全正则空间当且仅当它同胚于伪度量空间乘积的一个子空间。

M 正规空间的上半连续分解

正规拓扑空间关于闭连续映射的像仍为正规空间。

第五章 紧空间

紧拓扑空间的概念(和本书所研究的每一个概念一样)是实数集的某种重要性质的一种抽象．古典的 Heine-Borel-Lebesgue 定理表明实数空间的闭有界子集的每一个开覆盖有有限子覆盖．这个定理具有非常深远的影响，因此和一些最佳的定理一样，它的结论就变成了定义．我们称拓扑空间为**紧(重紧)**的当且仅当每一个开覆盖有有限子覆盖[1]．拓扑空间的子集 A 叫做紧集当且仅当它关于相对拓扑为紧，等价地说，A 为紧集当且仅当 A 的每一个由 X 的开集组成的覆盖有有限子覆盖．

等价性

本节主要利用闭集，收敛，基和子基来给出紧性的刻划．

我们称集族 $\mathscr{A}$ 具有**有限交性质**当且仅当 $\mathscr{A}$ 的每一个有限子族的元的交为非空．通过 De Morgan 公式(预备知识定理 2)，我们容易建立这个概念和紧性之间的联系．

1 定理．拓扑空间为紧当且仅当每一个具有有限交性质的闭集族有非空的交．

证明．若 $\mathscr{A}$ 为拓扑空间 X 的一个子集族，则由 De Morgan 公式，$X \sim \bigcup\{A: A \in \mathscr{A}\} = \bigcap\{X \sim A: A \in \mathscr{A}\}$，故 $\mathscr{A}$ 为 X 的一个覆盖当且仅当所有 $\mathscr{A}$ 的元的余集的交为空集．但空间 X 为紧当且仅当每一个不存在任何有限子族能够覆盖 X 的开集族都不是 X 的一个覆盖，因而又当且仅当每一个具有有限交性质的闭集族有非空的交．∎

1) "紧"一词也有用来表示"列紧"和"可数紧"(本章最后的问题中的术语)．而 N. Bourbaki 和他的同事们则是对紧 Hausdorff 空间才使用该词的．

2 定理． 拓扑空间 X 为紧当且仅当 X 中的每一个网都有聚点．

因而，X 为紧当且仅当 X 中的每一个网有收敛于 X 中的点的子网．

证明．设 $\{S_n, n\in D\}$ 为紧拓扑空间 X 中的一个网，又对 D 中的每一个 n，命 A_n 为所有 S_m 组成的集其中 $m\geqslant n$，则因 D 关于 $\geqslant$ 为有向集，故所有集 A_n 的族具有有限交性质，从而所有闭包 A_n^- 组成的族也具有有限交性质．由于 X 为紧，所以有一点 s 它属于每一个 A_n^-，于是由定理 2.7 便知这样的点 s 就是网 $\{S_n, n\in D\}$ 的聚点．

今证其逆．设 X 为一个拓扑空间并且其内的每一个网均有聚点，又设 $\mathscr{A}$ 为 X 的一个闭子集族并且具有有限交性质．定义 $\mathscr{B}$ 为所有 $\mathscr{A}$ 的元的有限交的族，则 $\mathscr{B}$ 也具有有限交性质，并且从 $\mathscr{A}\subset\mathscr{B}$ 可推出我们只须证 $\bigcap\{B: B\in\mathscr{B}\}$ 为非空．因为 $\mathscr{B}$ 的两个元的交仍为 $\mathscr{B}$ 的元，故 $\mathscr{B}$ 关于 $\subset$ 为有向集．如果我们对每一个 $\mathscr{B}$ 中的 B 选一个元 S_B，那末 $\{S_B, B\in\mathscr{B}\}$ 即为 X 中的网，从而有聚点 s．若 B 和 C 为 $\mathscr{B}$ 的元并且 $C\subset B$，则 $S_C\in C\subset B$，故网 $\{S_B, B\in\mathscr{B}\}$ 最终地在闭集 B 内，于是聚点 s 属于 B．这表明 s 属于 $\mathscr{B}$ 的每一个元，亦即所有 $\mathscr{B}$ 的元的交为非空．

最后，定理的第二个结论由定理 2.6（点 s 为网 S 的聚点当且仅当 S 有某个子网收敛于 s）即可推出．|

在某些情况下，我们可以通过子集的聚点的存在性来给出紧性的刻划．下面的一系列引理及其随后的定理就指出了这种情况．本章最后的问题又将证明这里所加的限制还是必要的．为方便起见，我们在叙述结果时将利用聚点概念的一种变形．我们称点 x 为集 A 的 **ω 聚点**当且仅当 x 的每一个邻域都包含无限多个 A 的点．显然，一个集的每一个 ω 聚点也是该集的聚点，并且当空间为 T_1 空间时其逆亦真．

3 引理． 拓扑空间的每一个序列有聚点当且仅当每一个无限集有 ω 聚点．

证明．假设每一个序列有聚点，并且 A 为无限子集，则 A 中必有一个不同的点的序列（一个一对一的序列），而这样的序列的每一个聚点显然就是 A 的 ω 聚点．

反之，若拓扑空间的每一个无限子集有 ω 聚点，并且 $\{S_n, n\in\omega\}$ 为空间中的一个序列，则必定出现下列两种情形之一：或者序列的值域为无限，或者序列的值域为有限，在前一种情况下，该无限集的每一个 ω 聚点就是这个序列的聚点，而在后一种情况下，存在空间的某个点 x 使得对无限多个非负整数 n 有 $S_n = x$，即 x 为该序列的聚点．|

4 引理． 若 X 为 Lindelöf 空间并且 X 中的每一个序列都有聚点，则 X 为紧．

证明．按照定义，我们必须证明 X 的每一个开覆盖有有限子覆盖．但根据假设，我们又不妨假定该开覆盖由集 A_0, $A_1, \cdots$, $A_n, \cdots$ 所组成，其中 n 属于 ω．现在，按照归纳方法进行，命 $B_0 = A_0$，并且对每一个 ω 中的 p，命 B_p 为 A 的序列中第一个不能由 $B_0, B_1, \cdots, B_{p-1}$ 覆盖的元．如果这种选法到某一步成为不可能，那末已选出的集就是所需的有限子覆盖．否则，对每一个 ω 中的 p，可以选取 B_p 中的一点 b_p 使得当 $i < p$ 时 $b_p \notin B_i$．设 x 为该序列的聚点，则对某个 p 有 $x \in B_p$，再从 x 为聚点即可推出有某个 $q > p$ 使得 $b_q \in B_p$，故矛盾．|

下一定理概述了关于序列，子序列，聚点和紧性之间的联系．

5 定理． 若 X 为拓扑空间，则下面的条件有如下的关系．对所有的空间有 (a) 等价于 (b) 并且 (d) 可推出 (a)．若 X 满足第一可数性公理，则 (a), (b) 和 (c) 等价．若 X 满足第二可数性公理，则四个条件都等价．若 X 为伪度量空间，则这四个条件中的每一个都蕴含 X 满足第二可数性公理并且四个条件等价．

(a) X 的每一个无限子集有 ω 聚点．

(b) X 中的每一个序列有聚点．

(c) 对 X 中的每一个序列存在子序列收敛于 X 中的点．

(d) 空间 X 为紧．

证明．引理 5.3 表明 (a) 等价于 (b)．因为序列是网的特殊情形，故由定理 5.2 即得 (d) 蕴含 (b)．

若 X 满足第一可数性公理，则由定理 2.8 便知 (b) 和 (c) 等价．

若 X 满足第二可数性公理，则每一个开覆盖有可数子覆盖，因而再利用引理 5.4 这四个命题就都等价．

若 X 为伪度量空间，则 X 满足第一可数性公理，从而前三个条件等价并且其中的每一个均可由紧性推出，于是欲证本定理只须证每一个无限子集都有聚点的伪度量空间必为可分，因为这时该空间就自然满足第二可数性公理．假设 X 为具有上述性质的伪度量空间，对正数 r，考虑所有这样的集 A 组成的族，它使得 A 中任意两个不同的点的距离至少为 r，根据预备知识定理 25 易见该集族有一个极大元 A_r．这个集 A_r 一定是有限集，因为以 X 的每一点为球心的 $r/2$ 球至多包含 A_r 的一个元，亦即 A_r 没有聚点．同时以 X 的每一点 x 为球心的 r 球一定与 A_r 相交，因为 A_r 为极大，否则就可以把 x 添加到 A_r 内．最后，命 A 为所有集 A_r 的并，其中 r 取遍所有正整数的倒数，则 A 为可数集并且它显然在 X 中稠密．|

显然若 $\mathscr{B}$ 是紧空间 X 的拓扑的一个基并且 $\mathscr{A}$ 是由 $\mathscr{B}$ 的元组成的 X 的一个覆盖，则 $\mathscr{A}$ 有有限子覆盖．反之，假设 $\mathscr{B}$ 为拓扑的一个基并且由 $\mathscr{B}$ 的元组成的每一个覆盖有有限子覆盖，此时若 $\mathscr{C}$ 为 X 的任意开覆盖，命 $\mathscr{A}$ 为所有满足后一条件的 $\mathscr{B}$ 的元所构成的族：它是 $\mathscr{C}$ 中某个元的子集，则因 $\mathscr{B}$ 是一个基，故族 $\mathscr{A}$ 是 X 的一个覆盖，从而 $\mathscr{A}$ 有有限子覆盖 $\mathscr{A}'$，再对 $\mathscr{A}'$ 的每一个元，选 $\mathscr{C}$ 的一个元包含它，即得 $\mathscr{C}$ 的一个有限子覆盖．这就证明了若"拓扑的一个基为紧"，则空间亦为紧．这是一个有用，但不很深刻的结果．关于子基的相应定理则既深刻，又有用．

6 定理 (Alexander)．若 $\mathscr{S}$ 为空间 X 的拓扑的一个子基，并且由 $\mathscr{S}$ 的元所组成的 X 的每一个覆盖有有限子覆盖，则 X 为紧．

证明．为简洁起见，我们规定 X 的子集族为不充分的当且仅当它不能覆盖 X，又规定它为有限不充分的，当且仅当它没有任何

有限子族能够覆盖 X. 于是, X 的紧性的定义又可叙述成: 每一个有限不充分的开集族是不充分的. 注意到有限不充分的开集族的类有有限特征, 因此由 Tukey 引理(预备知识定理 25(c)) 便知每一个有限不充分的族包含在一个极大族内. 这样的极大有限不充分的族 $\mathscr{A}$ 有一个特殊性质, 它可以如下确定[1]: 若 $C \notin \mathscr{A}$ 并且 C 为开集, 则由极大性有 $\mathscr{A}$ 的有限子族 $A_1, \cdots, A_m$ 使得 $C \cup A_1 \cup \cdots \cup A_m = X$, 故没有包含 C 的开集能够属于 $\mathscr{A}$, 又若 D 为另一个开集并且 $D \notin \mathscr{A}$, 则有 $\mathscr{A}$ 中的 $B_1, \cdots, B_n$ 使得 $D \cup B_1 \cup \cdots \cup B_n = X$, 从而由简单的集论运算就有 $(C \cap D) \cup A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots \cup B_n = X$, 亦即 $C \cap D \notin \mathscr{A}$. 因此, 若一个有限的开集族不存在属于 $\mathscr{A}$ 的元, 则也不存在包含该有限族的交的开集能够属于 $\mathscr{A}$; 换言之, 若 $\mathscr{A}$ 的一个元包含一个由开集组成的有限交 $C_1 \cap C_2 \cap \cdots \cap C_p$, 则必有某个 $C_i \in \mathscr{A}$.

现在转向定理的证明. 假设 $\mathscr{S}$ 为一个子基并且由 $\mathscr{S}$ 的元所组成的每一个开覆盖有有限子覆盖 (即每一个有限不充分的子族是不充分的), 又假设 $\mathscr{B}$ 为一个由 X 的开子集组成的有限不充分的集族, 则有一个包含 $\mathscr{B}$ 而又和它同样类型的极大族 $\mathscr{A}$, 并且这时我们只须证 $\mathscr{A}$ 是不充分的. 因为所有属于 $\mathscr{S}$ 的 $\mathscr{A}$ 的元的族 $\mathscr{S} \cap \mathscr{A}$ 为有限不充分的, 故 $\mathscr{S} \cap \mathscr{A}$ 不能覆盖 X, 于是欲证本定理又只须证 $\bigcup\{A: A \in \mathscr{A}\}$ 中的每一点都属于 $\bigcup\{A: A \in \mathscr{S} \cap \mathscr{A}\}$. 由于 $\mathscr{S}$ 为一个子基, 所以 $\mathscr{A}$ 的元 A 的每一点 x 属于某个包含在 A 内的 $\mathscr{S}$ 的元的有限交, 再根据上一段的讨论便知该有限族的某个元属于 $\mathscr{A}$, 因而 $\bigcup\{A: A \in \mathscr{A}\} = \bigcup\{A: A \in \mathscr{S} \cap \mathscr{A}\}$, 亦即定理获证. ▌

紧性和分离性

本节我们来考察紧性与所谓的分离公理相联系的一些结论,

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1) 问题 2.1 恰好就是此处所需要的结果.

并且在每一种情况下所证明的定理都假定以"紧集"代替"点"的分离公理（Hausdorff，正则，全正则）．另外还导出了关于从紧空间到 Hausdorff 空间内的连续映射的一个简单而重要的推论．最后证明了 A. D. Wallace 的一个分离定理，它包括在它前面的大多数定理．

不难看出，紧空间 X 的闭子集 A 必为紧集，因为从 A 为闭集可推出对 A 中的每一个网有子网收敛于 A 中的点．（一个在紧性定义基础上的直接证明几乎也同样简单．）其逆不真，因为若 A 为平庸空间 X（只有 X 和空集为开集）的非空真子集，则 A 为紧集，但不是闭的．然而，当 X 为 Hausdorff 空间时这种情况不可能出现．

7 定理． 若 A 为 Hausdorff 空间 X 的紧子集，又 x 为 $X \sim A$ 的点，则有 x 和 A 的互不相交的邻域．

因而，Hausdorff 空间的每一个紧子集必为闭集．

证明．因为 X 为 Hausdorff 空间，故对 A 的每一点 y 有一个邻域 U 使得 x 不属于闭包 U^-，但 A 为紧集，于是有覆盖 A 的有限开集族 $U_0, U_1, \cdots, U_n$ 使得当 $i = 0, 1, \cdots, n$ 时 $x \notin U_i^-$，从而若命 $V = \bigcup\{U_i: i = 0, 1, \cdots, n\}$，则 $A \subset V$ 并且 $x \notin V^-$，因此 $X \sim V^-$ 和 V 是 x 和 A 的互不相交的邻域．|

8 定理． 设 f 为映紧拓扑空间 X 到拓扑空间 Y 上的连续函数，则 Y 为紧，并且当 Y 为 Hausdorff 空间，f 为一对一映射时 f 为一个同胚．

证明．若 $\mathscr{A}$ 为 Y 的开覆盖，则所有由形如 $f^{-1}[A]$ 的集组成的集族为 X 的开覆盖，其中 A 属于 $\mathscr{A}$，从而它有有限子覆盖，显然所有该子覆盖的元的像组成的族为 $\mathscr{A}$ 的有限子族并且它覆盖 Y，于是 Y 为紧．

假设 Y 为 Hausdorff 空间并且 f 为一对一映射，若 A 为 X 的闭子集，则 A 为紧集，故它的像 $f[A]$ 亦为紧集，从而必为闭集，这表明对每一个闭集 A，$(f^{-1})^{-1}[A]$ 恒为闭集，即 f^{-1} 为连续．|

9 定理． 若 A 和 B 为 Hausdorff 空间 X 的互不相交的紧子集，则有 A 和 B 的互不相交的邻域．

因而，每一个紧 Hausdorff 空间必为正规空间.

证明. 因为对 A 中的每一个 x，由定理5.7有 x 的一个邻域和 B 的一个邻域，它们互不相交，即有 x 的邻域 U，它的闭包与 B 不相交，又因 A 为紧集，故有一个有限子族 $U_0, U_1, \cdots, U_n$ 使得 U_i^- 与 B 都不相交 $(i = 0, 1, \cdots, n)$ 并且 $A \subset V = \bigcup\{U_i: i = 0, 1, \cdots, n\}$. 于是 V 为 A 的邻域，并且 $X \sim V^-$ 为 B 的邻域，同时它与 V 不相交. |

10 定理. 若 X 为正则拓扑空间，A 为紧子集，U 为 A 的邻域，则有 A 的闭邻域 V 使得 $V \subset U$.

因而，每一个紧正则空间必为正规空间.

证明. 因为 X 为正则空间，故对每一个 A 中的 x 有一个开邻域 W 使得 $W^- \subset U$，从而再由紧性便知有 A 的一个有限开覆盖 $W_0, W_1, \cdots, W_n$ 使得对每一个 i 都有 $W_i^- \subset U$，于是 $V = \bigcup\{W_i^-: i = 0, 1, \cdots, n\}$ 即为所需要的 A 的邻域. |

11 定理. 若 X 为完全正则空间，A 为紧子集，U 为 A 的邻域，则有从 X 到闭区间 $[0,1]$ 的连续函数 f 使得 f 在 A 上为 1 并且在 $X \sim U$ 上为 0.

证明. 因为对 A 中的每一个 x 有一个连续函数 g，它在 x 处为 1 并且在 $X \sim U$ 上为0，又集 $\{y: g(y) > 1/2\}$ 为 X 中的开集，故若定义 h 为 $h(y) = \min[2g(y), 1]$，则 h 为取值于 $[0, 1]$ 的连续函数，并且在 $X \sim U$ 上为 0，在 x 的某个邻域上为 1. 从而根据 A 为紧集便知有有限多个从 X 到 $[0, 1]$ 的连续函数 $h_0, h_1, \cdots, h_n$ 使得 $A \subset \bigcup\{h_i^{-1}[1]: i = 0, 1, \cdots, n\}$ 并且每一个 h_i 在 $X \sim U$ 上为 0. 于是，在 x 处的值为 $\max\{h_i(x): i = 0, 1, \cdots, n\}$ 的函数 f 即为所需要的函数. |

上述两个定理的每一个都有一种表面上不同的另外叙述方式；即把条件中的"A 为紧集并且 U 为 A 的邻域"换成"A 为紧集并且 B 为与 A 不相交的闭集"，同时也把结论换成相应的形式.

本节的大多数结果都是下一定理的直接推论.

12 定理 (Wallace). 若 X 和 Y 为拓扑空间，A 和 B 分别为 X

和 Y 的紧子集，W 为 $A\times B$ 在乘积空间 $X\times Y$ 中的邻域，则有 A 的邻域 U 和 B 的邻域 V 使得 $U\times V\subset W$.

证明．因为对 $A\times B$ 的每一个元 (x,y) 有 x 的开邻域 R 和 y 的开邻域 S 使得 $R\times S\subset W$，又 B 为紧集，故对 A 中一个确定的 x 有 x 的邻域 R_i 和相应的开集 S_i，其中 $i=0,1,\cdots,n$，使得 $B\subset Q=\bigcup\{S_i:\ i=0,1,\cdots,n\}$，于是，若命 $P=\bigcap\{R_i:i=0,1,\cdots,n\}$，则 P 为 x 的邻域并且与 B 的邻域 Q 满足 $P\times Q\subset W$. 再由于 A 为紧集，从而有 X 中的开集 P_i 和 Y 中的开集 Q_i，其中 $i=0,1,\cdots,m$，使得每一个 Q_i 均为 B 的邻域，$P_i\times Q_i\subset W$ 并且 $A\subset\bigcup\{P_i:\ i=0,1,\cdots,m\}=U$，由此可见，$U$ 和 $V=\bigcap\{Q_i:\ i=0,1,\cdots,m\}$ 分别为 A 和 B 的邻域并且 $U\times V$ 是 W 的子集，即定理获证．|

紧空间的乘积

关于紧空间乘积的经典的 Tychonoff 定理无疑是有关紧性的最有用的定理，并且就单个定理而论，也可以说，它是一般拓扑学中最重要的定理．本节集中讨论 Tychonoff 定理和它的一些推论．

13 定理 (Tychonoff). 一族紧拓扑空间的笛卡儿乘积关于乘积拓扑仍为紧.

证明．设 $Q=\times\{X_a:\ a\in A\}$，其中每一个 X_a 为紧拓扑空间，并且 Q 具有乘积拓扑，又设 $\mathscr{S}$ 为由所有形如 $P_a^{-1}[U]$ 的集所组成的乘积拓扑的子基，其中 P_a 为到第 a 个坐标空间内的射影并且 U 为 X_a 中的开集，则由定理 5.6 可知空间 Q 为紧，只须每一个不存在任何有限子族能够覆盖 Q 的 $\mathscr{S}$ 的子族 $\mathscr{A}$ 都不能覆盖 Q. 对每一个指标 a，命 $\mathscr{B}_a$ 为所有使得 $P_a^{-1}[U]\in\mathscr{A}$ 的 X_a 的开集 U 组成的集族，则 $\mathscr{B}_a$ 的任何有限子族都不能覆盖 X_a，因此由紧性便知存在点 x_a 使得对每一个 $\mathscr{B}_a$ 中的 U 都有 $x_a\in X_a\sim U$，于是第 a 个坐标为 x_a 的点 x 就不能属于 $\mathscr{A}$ 的任何元，从而 $\mathscr{A}$ 不是

Ω的覆盖. ∎

现在我们给出 Tychonoff 定理的另一种证明，这种证明并不依据 Alexander 的定理 5.6.

另一种证明 (Bourbaki). 显然我们只须证若 $\mathscr{B}$ 为乘积中的子集族并且具有有限交性质，则$\bigcap\{B^-: B\in\mathscr{B}\}$ 为非空. 因为所有具有有限交性质的族组成的类具有有限特征，故由 Tukey 引理 (预备知识定理 25(c))我们可以假定$\mathscr{B}$关于该性质为极大. 由于$\mathscr{B}$为极大，所以每一个包含 $\mathscr{B}$ 的一个元的集属于$\mathscr{B}$并且$\mathscr{B}$的两个元的交属于 $\mathscr{B}$. 而且若C与 $\mathscr{B}$ 的每一个元相交，则由极大性 $C\in\mathscr{B}$[1]. 最后，因为所有 $\mathscr{B}$ 的元到坐标空间 X_a 内的射影组成的族具有有限交性质，故可选出一个点 x_a，它属于$\bigcap\{P_a[B]^-: B\in\mathscr{B}\}$，显然第 a 个坐标为 x_a 的点 x 具有性质：x_a 的每一个邻域U与每一个 $P_a[B]$ 相交，其中 $B\in\mathscr{B}$，即对 x_a 的每一个邻域U有 $P_a^{-1}[U]\in\mathscr{B}$，亦即这种类型的集的有限交也属于 $\mathscr{B}$，于是 x 的每一个属于乘积拓扑所定义的基的邻域恒属于$\mathscr{B}$，从而与$\mathscr{B}$的每一个元相交，这表明对每一个 $\mathscr{B}$ 中的B有 x 属于 B^-，因而定理获证. ∎

关于 Tychonoff 定理的一些重要应用，我们将在函数空间那一章来讨论；现在只给出一个很简单的推论. 我们称伪度量空间的子集为**有界的**当且仅当它具有有限的直径. 于是，实数空间的子集为有界当且仅当它同时有上界和下界. 下面的定理就是古典的 Heine-Borel-Lebesgue 定理.

14 定理. n维欧几里德空间的子集为紧集当且仅当它为闭的有界集.

证明. 设A为 E_n 的紧子集，则因 E_n 为 Hausdorff 空间，故A为闭集. 由于紧性，A可以由一个半径为 1 的开球组成的有限族所覆盖，而每一个这样的开球又都有界，所以A也有界.

今证其逆，假设A为 E_n 的闭的有界子集，命 B_i 为A关于到

1) 显然我们是重复证明了问题 2.1 的一部分.

第 i 个坐标空间内的射影的像，则因射影要缩小距离，故每一个 B_i 也都有界. 注意 $A\subset \times\{B_i: i=0,1,\cdots,n-1\}$，并且该集又是实数的闭有界区间的某个乘积的子集，即 A 亦为该乘积的闭子集，又因紧空间的乘积仍为紧空间，故只须证闭区间 $[a,b]$ 关于通常拓扑为紧. 设 $\mathscr{C}$ 为 $[a,b]$ 的一个开覆盖，又设 c 为所有 $[a,b]$ 中这样的元 x 的上确界，它使得有 $\mathscr{C}$ 的某有限子族覆盖 $[a,x]$（该集为非空，因为 a 显然是它的一个元），选 $\mathscr{C}$ 中的 U 使得 $c\in U$，又选开区间 (a,c) 中的元 d 使得 $[d,c]\subset U$，因为有 $\mathscr{C}$ 的一个有限子族覆盖 $[a,d]$，故该族再加上 U 就覆盖了 $[a,c]$. 若 $c=b$ 不成立，则这个有限子族也就覆盖了 c 的右方的某个区间，这与 c 的取法矛盾. 定理获证. |

因为闭单位区间为紧，故每一个立方体（闭单位区间的乘积）也为紧. 于是，Tychonoff 空间（全正则的 T_1-空间）的如下刻划几乎就成为明显的了.

15 定理. 拓扑空间为 Tychonoff 空间当且仅当它同胚于某个紧 Hausdorff 空间的一个子空间.

证明. 由定理 4.7 每一个 Tychonoff 空间同胚于某个立方体的一个子集，而我们又已知任何立方体均为紧Hausdorff 空间.

反之，每一个紧 Hausdorff 空间为正规空间，因而（Urysohn 的引理4.4）为 Tychonoff 空间，于是它的每一个子空间亦为 Tychonoff 空间. |

多于有限多个非紧空间的乘积是在一种有点奇特的状态下不再为紧. 我们称拓扑空间的一个子集在该空间中为**无处稠密**[1)]当且仅当它有空的内部.

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1) 校注. 本书作者在书内对“无处稠密”给了两个不等价的定义，一个在原书 145 页上（即这里陈述的定义），另一个在原书的 201 页上（在译本的 186 页上）. 前一定义比后一定义弱，但后一定义是习用的. 为着避免同一本书内一个名词有两个不同的含义起见，我们可把前者叫做弱无处稠密，后者叫无处稠密. 由于仅在定理 5.16 中和定理 5.19 的证明中出现，我们也可以直接用“内部为空的集”去代替它，这样便可根本不在这里引进无处稠密的概念，而把这一术语留给后者.

16 定理. 若有无限多个坐标空间为非紧,则乘积中的每一个紧子集有空的内部.

证明. 假设$\times\{X_a: a\in A\}$ 有一个紧子集 B, 它有一个内点 x, 则 B 包含有 x 的一个邻域 U, 它是乘积拓扑所定义的基的一个元, 亦即为形如 $\bigcap\{P_a^{-1}[V_a]: a\in F\}$, 其中 F 为 A 的有限子集并且 V_a 为 X_a 的开集. 若 $b\in A\sim F$, 则 $P_b[B]=X_b$, 因为 X_b 是一个紧空间的连续像, 故 X_b 为紧. 因而,除有限多个外所有的坐标空间均为紧.|

局 部 紧 空 间

我们称拓扑空间为**局部紧**当且仅当它的每一点至少有一个紧邻域. 显然, 紧空间为局部紧, 每一个离散空间为局部紧, 并且局部紧空间的闭子空间也为局部紧(注意,闭集与紧集的交为该紧集的闭子集,因而亦为紧集). 局部紧空间同样也具有紧空间的许多良好性质. 下面的命题是研究这种空间的一种方便工具.

17 定理. 若X为局部紧拓扑空间并且它为 Hausdorff 或正则空间, 则每一点的所有闭的紧邻域组成的集族为该点邻域系的一个基.

证明. 设 x 为X的一个点,C 为 x 的一个紧邻域,U 为 x 的任意邻域. 若X为正则空间, 则有 x 的一个闭邻域 V, 它是U和C的内部的子集,并且易见V为闭的紧集.

若X为 Hausdorff 空间,W为 $U\cap C$ 的内部, 则因 W^-为紧 Hausdorff空间,故由定理 5.9W包含有一个闭的紧集,它是 x 在W^-中的邻域,但它也是 x 在W中的邻域(即关于对W的相对拓扑),从而也就是 x 在X 中的一个邻域.|

特别,由此即得每一个局部紧 Hausdorff 空间恒为正则空间; 实际上,还有一个更强的命题成立.

18 定理. 若U为正则局部紧拓扑空间 X 的闭的紧子集A的邻域,则有A的闭的紧邻域V使得 $A\subset V\subset U$.

而且，存在从X到闭单位区间的连续函数f使得f在A上为0并且在$X \sim V$上为1.

证明. 因为对A中的每一点x都有一个邻域W，它是U的闭的紧子集，但A为紧集，故A可由这样的邻域的一个有限族所覆盖，从而它的并V即为A的一个闭的紧邻域.

由于V关于相对拓扑为紧正则空间，所以亦为正规空间(定理5.10)，从而有从V到闭单位区间的连续函数g使得g在A上为零并且在$V \sim V^0$(V^0为V的内部)上为1，于是若命f在V上等于g并且在$X \sim V$上等于1，则因V^0与$X \sim V$分离并且f在V和$X \sim V^0$上为连续，故f为连续(问题3.B). |

由此即得：每一个局部紧正则拓扑空间恒为全正则空间，并且每一个局部紧 Hausdorff 空间恒为 Tychonoff 空间.

一般的说，局部紧空间的连续像未必为局部紧，这只须注意每一个离散空间为局部紧，而每一个拓扑空间又必为某个离散空间的一对一的连续像(利用同一个集的离散拓扑和恒等函数). 若一个函数为连续的开映射，则一个点的紧邻域的像为像点的一个紧邻域，因而，局部紧空间的像为局部紧. 这个简单事实和以前的一个结果也就给出了局部紧的乘积空间的一种确切的描述.

19 定理. 若乘积空间为局部紧，则每一个坐标空间为局部紧并且除有限多个外所有的坐标空间均为紧.

证明. 若乘积空间为局部紧，则因它到坐标空间内的射影为开映射，故每一个坐标空间为局部紧.

若有无限多个坐标空间为非紧，则由定理5.16该乘积的每一个紧子集的内部为空，故不存在具有紧邻域的点. |

商　　空　　间

这一节继续从事在第三章中所开始的关于商空间的研究. 我们有兴趣的是有关紧性的结论，并且本节唯一的一条定理就概括了在附加另外假设下，所推出的若干很好性质. 前面我们已经看到

紧空间的连续像仍为紧空间，然而在不附加另外假设的情况下像空间仍然可以完全不引人注目．例如，若X为具有通常拓扑的闭单位区间，$\mathscr{D}$为由所有形如$\{x: x-a$为有理数$\}$的子集所组成的分解，则商空间为紧并且在商空间上的射影为开映射，但商拓扑为平庸拓扑(只有空间和空集为开集)．我们所得到的结果是：若$\mathscr{D}$的元为紧集，并且该分解为上半连续，则商空间可以遗传X的许多性质．

20 定理． 设X为拓扑空间，$\mathscr{D}$为X的上半连续分解，并且它的元为紧集同时$\mathscr{D}$具有商拓扑，则$\mathscr{D}$分别为 Hausdorff，正则，局部紧，或满足第二可数性公理的空间，只要X具有相应的性质．

证明．为方便起见，我们规定X的子集是容许的当且仅当它是$\mathscr{D}$的元的并．根据上半连续性的定义，$\mathscr{D}$的元A在X中的每一个邻域都包含一个容许的邻域，因而，A在X中的邻域关于射影的像是A在$\mathscr{D}$中的邻域．此外，由定理 3.12 射影还变闭集为闭集．

假设X为 Hausdorff 空间，A和B为$\mathscr{D}$的不同的元，则由定理 5.9 存在A和B在X中的互不相交的邻域，而它又包含有互不相交的容许的邻域，故该容许的邻域关于射影的像即为所需A和B在$\mathscr{D}$中的互不相交邻域．

若X为正则空间，$A\in\mathscr{D}$并且$\mathscr{U}$是A在$\mathscr{D}$中的一个邻域，则所有$\mathscr{U}$的元的并U是A在X中的一个邻域，根据定理 5.10 存在A在X中的一个闭邻域，它包含在U内，显然该邻域关于射影的像就是A在$\mathscr{D}$中所需的邻域．

若X为局部紧，则易见$\mathscr{D}$的每一个元在X中有一个紧邻域，并且它关于射影的像即为在$\mathscr{D}$中的一个紧邻域．

最后，假设X的拓扑有一个可数基$\mathscr{B}$，则由所有$\mathscr{B}$的有限子族的并所组成的集族$\mathscr{U}$仍为可数集．对$\mathscr{U}$的每一个元U，命U'为所有是U的子集的$\mathscr{D}$的元的并，又命$\mathscr{T}$为当U属于$\mathscr{U}$时的所有集U'组成的族，则$\mathscr{T}$的元的像为开集并且可以证明这些像的集就是商拓扑的一个基．事实上，这只须证对每一个$\mathscr{D}$中的

A和每一个A的邻域V有$\mathscr{T}$中的U使得$A \subset U \subset V$. 因为A可以由有限多个$\mathscr{B}$的元所覆盖，并且还可以使得这些元的并W(是$\mathscr{U}$的一个元)含包在V内，故若命$U = W'$，则$U \in \mathscr{T}$并且$A \subset U \subset V$，于是定理获证. ▍

这个定理有一个有趣的推论. 若X为可分度量空间并且它的一个上半连续分解的元恒为紧集，则该商空间为Hausdorff, 正规并且满足第二可数性公理的空间，从而亦为可度量化的空间.

紧 扩 张

在研究非紧拓扑空间X时，作出一个本身为紧并且包含X作为子空间的空间通常是方便的. 例如，对实数空间添加$+\infty$和$-\infty$两点就常常有用.这样所得到的空间有时叫做**扩张了的**实数，当规定$+\infty$为最大元，$-\infty$为最小元时它是线性有序集. 对于这种序(通常序的一种推广)，扩张了的实数的每一个非空子集都有下确界和上确界并且关于它的序拓扑为紧(问题5.C). 这种扩张了的实数,在现在将要精确给出的一种意义下,是实数空间的一个**紧扩张**. 当然，这种作法首先是为了方便. 它并没有为我们增添关于实数的知识. 然而使我们能够运用典型的紧性推理方法且简化了许多证明.

拓扑空间的一种最简单的紧扩张是通过添加一个单独的点而得到的. 它的步骤与分析学里相似，我们知道在函数论中复数球面就是通过对欧几里德平面添加一个单独的点∞而作出的并且规定∞的邻域为该平面的有界子集的余集. 这种作法可以移植到任意的拓扑空间；而在扩张空间内引入拓扑的思路就是依靠在复数球面中∞的开邻域的余集必为紧集这个事实. 拓扑空间X的**单点紧扩张**[1]是指具有这样的拓扑的集$X^* = X \cup \{\infty\}$，它的元为X的开子集和X^*的所有这样的子集U，它使得$X^* \sim U$为X的闭,紧

1) 这个定义实际上直到∞被定义之前是不完备的. 事实上，任何元只要它不是X的元，例如X就均可取做∞.

子集. 自然，我们必须证明它的确给出 X^* 的一个拓扑，这在下列命题的证明中可得到实现.

21 定理 (Alexandroff). 拓扑空间 X 的单点紧扩张 X^* 为紧并且 X 是它的子空间. 空间 X^* 为 Hausdorff 空间当且仅当 X 为局部紧 Hausdorff 空间.

证明. 因为集 U 为 X^* 中的开集当且仅当有 (a) $U \cap X$ 为 X 中的开集与 (b) 当 $\infty \in U$ 时 $X \sim U$ 为紧集，故 X^* 的开集的有限交和任意并与 X 的交仍为开集. 若 ∞ 为 X^* 的两个开子集的交的元，则这个交的余集为 X 的两个闭紧子集的并，从而也为闭，紧集. 又若 ∞ 属于 X^* 的一族开子集的元的并，则 ∞ 属于该族的某个元 U 并且这个并的余集为紧集 $X \sim U$ 的闭子集，于是亦为闭，紧集. 总之 X^* 为拓扑空间并且 X 是它的子空间. 再注意若 $\mathscr{U}$ 为 X^* 的开覆盖，则从 ∞ 必为 $\mathscr{U}$ 中某个 U 的元以及 $X \sim U$ 为紧集即可推出 $\mathscr{U}$ 有有限的子覆盖，于是便知 X^* 为紧.

若 X^* 为 Hausdorff 空间，则它的开子空间 X 自然是局部紧 Hausdorff 空间. 最后，需要证明当 X 为局部紧 Hausdorff 空间时 X^* 为 Hausdorff 空间，这又只须证若 $x \in X$，则 x 和 ∞ 有互不相交的邻域. 因为 X 为局部紧 Hausdorff 空间，故 x 在 X 中有一个闭，紧邻域 U，并且 $X^* \sim U$ 即为所需的 ∞ 的邻域. |

若 X 为紧拓扑空间，则 ∞ 为单点紧扩张的一个孤立点(即 $\{\infty\}$ 为既开又闭). 反之，若 ∞ 为 X^* 的一个孤立点，则 X 为 X^* 中的闭集，从而亦为紧集.

单点紧扩张是一种类型很特殊的紧扩张，因此我们需要考虑将拓扑空间嵌到某个紧空间的其它方法. 为方便起见，显然，与其坚持原始空间必须是所作出的紧空间的真子空间，还不如只要求它能够拓扑地嵌入. 根据这种想法，我们定义拓扑空间 X 的**紧扩张**为 (f, Y)，其中 Y 为紧拓扑空间，f 为从 X 到 Y 的一个稠密子空间上的一个同胚. (这时，X 的单点紧扩张即为 (i, X^*)，其中 i 为恒等映射.) 又称紧扩张 (f, Y) 为 Hausdorff 紧扩张，当且仅当 Y 为 Hausdorff 空间. 对空间 X 的所有紧扩张组成的族我们定义关

系 $(f, Y) \geqslant (g, Z)$ 当且仅当存在从 Y 到 Z 上的连续映射 h 使得 $h \circ f = g$. 等价地有，$(f, Y) \geqslant (g, Z)$ 当且仅当从 $f[X]$ 到 Z 的函数 $g \circ f^{-1}$ 有映 Y 到 Z 内的连续扩张 h. 若函数 h 可以取为同胚，则称 (f, Y) 和 (g, Z) 为**拓扑等价**. 在这种情况下关系 $(f, Y) \geqslant (g, Z)$ 和 $(g, Z) \geqslant (f, Y)$ 同时成立，因为 h^{-1} 为从 Z 到 Y 上的连续映射并且使得 $f = h^{-1} \circ g$.

22 定理. 拓扑空间的所有紧扩张组成的族关于 $\geqslant$ 为半序集. 若 (f, Y) 和 (g, Z) 为某个空间的 Hausdorff 紧扩张并且 $(f, Y) \geqslant (g, Z) \geqslant (f, Y)$，则 (f, Y) 和 (g, Z) 为拓扑等价.

证明. 若 $(f, Y) \geqslant (g, Z) \geqslant (h, U)$，并且其中的每一个均为空间 X 的紧扩张，则分别有从 Y 到 Z 和从 Z 到 U 的连续函数 j 和 k 使得 $g = j \circ f$ 和 $h = k \circ g$ 成立，从而 $h = k \circ j \circ f$ 并且有 $(f, Y) \geqslant (h, U)$. 因而 X 的所有紧扩张组成的族关于 $\geqslant$ 为半序集.

若 (f, Y) 和 (g, Z) 为 Hausdorff 紧扩张，并且其中的每一个关于序 $\geqslant$ 在另一个之后，则 $f \circ g^{-1}$ 和 $g \circ f^{-1}$ 分别有到整个 Z 和 Y 的连续扩张 j 和 k. 因 $k \circ j$ 为 Z 的稠密子集 $g[X]$ 上的恒等映射，Z 为 Hausdorff 空间，故 $k \circ j$ 为从 Z 到它自己上的恒等映射，类似地有 $j \circ k$ 为从 Y 到它自己上的恒等映射，于是 (f, Y) 和 (g, Z) 为拓扑等价.|

显然，紧 Hausdorff 空间 X 的最小紧扩张就是 X 它自己(更精确地说应该是 (i, X)，其中 i 为 X 上的恒等映射). 我们自然期望非紧空间的单点紧扩张关于序 $\geqslant$ 为最小紧扩张. 当我们限制注意力于 Hausdorff 紧扩张时，这确实成立(问题 5.G 的一个推论). 虽容易看出，一般并不存在这样的紧扩张，它能够更小于每一个其它的紧扩张. 另一方面，若 X 为具有 Hausdorff 紧扩张的空间(根据定理 5.15 如此的空间为 Tychonoff 空间)，则它有一个最大的紧扩张，这就是下面我们所要讨论的内容.

对每一个拓扑空间 X，命 $F(X)$ 为所有从 X 到闭单位区间 Q 的连续函数组成的族，则由 Tychonoff 定理便知立方体 $Q^{F(X)}$ (单位区间 Q 的 $F(X)$ 次乘积)为紧. 再命 e 为变 X 的元 x 为 $Q^{F(X)}$ 的

元 $e(x)$ 的计值映射，其中 $e(x)$ 的第 f 个坐标为 $f(x)$ $(f\in F(X))$，则 e 为从 X 到立方体 $Q^{F(X)}$ 内的连续映射，并且当 X 为 Tychonoff 空间时 e 为从 X 到 $Q^{F(X)}$ 的某个子空间上的一个同胚(这些事实由嵌入引理 4.5即得)．所谓 **Stone-Čech 紧扩张**就是指 $(e, \beta(X))$，其中 $\beta(X)$ 为 $e[X]$ 在立方体 $Q^{F(X)}$ 内的闭包．在证明这个紧扩张的重要性质之前，我们先给出一个引理．

23 引理． 若 f 为从集 A 到集 B 内的函数，f^* 为从 Q^B 到 Q^A 内的这样的映射，它对一切 Q^B 中的 y 由 $f^*(y)=y\circ f$ 确定，则 f^* 为连续．

证明．因为由定理 3.3 可知映到乘积空间内的映射为连续当且仅当该映射与每一个射影的合成为连续；又若 a 为 A 的一个元，则 $P_a\circ f^*(y)=P_a(y\circ f)=y(f(a))$，但 $y(f(a))$ 为 y 到 Q^B 的第 $f(a)$ 个坐标空间内的射影，即 $P_a\circ f^*$ 为连续映射．|

在该引理中所提到的作法是值得注意的，因为当讨论函数空间时，它有系统的应用．注意，由 f 所诱导的函数 f^*，在 f 变 A 到 B 内，而 f^* 变 Q^B 到 Q^A 内的意义下，与 f 的方向相反．

借助于这个引理，Stone-Čech 紧扩张的主要定理的导出就变成通过并不繁杂计算的一种惯用手法．

24 定理 (Stone-Čech)．若 X 为 Tychonoff 空间，f 为从 X 到紧 Hausdorff 空间 Y 的连续函数，则有 f 的一个连续扩张，它变紧扩张 $\beta(X)$ 到 Y 内．(更精确地说，若 $(e, \beta(X))$ 为 Stone-Cech 紧扩张，则 $f\circ e^{-1}$ 可以扩张成从 $\beta(X)$ 到 Y 的连续函数．)

证明．给定一个 f，我们通过对每一个 $F(Y)$ 中的 a，命 $f^*(a)=a\circ f$ 来定义从 $F(Y)$ 到 $F(X)$ 的 f^*，又通过对每一个 $Q^{F(X)}$ 中的 q，命 $f^{**}(q)=q\circ f^*$ 来定义从 $Q^{F(X)}$ 到 $Q^{F(Y)}$ 的 f^{**}．再命 e 为从 X 到 $Q^{F(X)}$ 内的计值映射，g 为从 Y 到 $Q^{F(Y)}$ 内的计值映射．于是，利用下列的图解式即可说明我们所要证明的结果．

$$\begin{tikzcd}
\beta(X)\subset Q^{F(X)} \arrow[r, "f^{**}"] & Q^{F(Y)}\supset\beta(Y) \\
X \arrow[u, "e"] \arrow[r, "f"] & Y \arrow[u, "g"']
\end{tikzcd}$$

因为映射 e 为一个同胚，又从 Y 为紧 Hausdorff 空间可推出映射 g 为从 Y 到 $\beta(Y)$ 上的一个同胚，再由引理 5.23 可知映射 f^{**} 为连续，故若能证得 $f^{**}\circ e = g\circ f$，则 $g^{-1}\circ f^{**}$ 即为 $f\circ e^{-1}$ 所需的连续扩张。

事实上，若 x 为 X 的一个元，h 为 $F(Y)$ 的一个元，则分别由 f^{**}，f^*，e 和 g 的定义即得 $(f^{**}\circ e)(x)(h) = (e(x)\circ f^*)(h) = e(x)(h\circ f) = h\circ f(x) = g(f(x))(h) = (g\circ f)(x)(h)$. |

上述定理的扩张性质表明 Stone-Čech 紧扩张 $(e,\beta(X))$ 关于序 $\geqslant$ 在每一个其它的 Hausdorff 紧扩张之后，亦即为最大的 Hausdorff 紧扩张。又若 (f, Y) 也具有这个扩张性质，则 $(f,Y)\geqslant(e,\beta(X))$，从而由定理 5.22 它拓扑等价于 $(e, \beta(X))$。因此，定理 5.24 的扩张性质给出了紧扩张 $(e, \beta(X))$（精确到拓扑等价）的一种刻划。

25 注记. 上述结果（M. H. Stone [6] 和 Čech [1]）提供了一种极大紧扩张。为了各种不同的目的，许多其它较小的紧扩张也已经被作出。关于这个主题有很大数量的文献，我们只可能引用一些作为范例的著作。最古老紧扩张理论之一（Carathéodory 的素端点理论）的新近著作可参阅 Ursell 与 Young [1]. Freudenthal[1] 考察了一种紧扩张，它在比由 $\beta(X)$ 所优有更多限制的类中为极大。紧扩张的一种一般讨论是由 Myškis([1],[2]和[3]) 给出的。他区分了紧扩张的"外部"描述（例如 $\beta(X)$ 以及问题 7.T 所梗概介绍的群的殆周期紧扩张）和"内部"描述（例如 Alexandroff 的单点紧扩张与问题 5. R 的 Wallman 紧扩张）。紧扩张的内部和外部描述之间的关系常常又是这种概念的有效性之关键所在。$\beta(X)$ 内部构造的某些部分已被讨论（见 Nagata [2]，Smirnov [3] 和 Wallace [2]）。紧扩张 $\beta(X)$ 与绝对闭包概念也有联系，例如参看 M. H. Stone [6]，A. D. Alexandroff [1]，Katětov [1] 和 Ramanathan [1].

Lebesgue 覆盖引理

我们知道有一个极为有用的 Lebesgue 引理，即若 $\mathscr{U}$ 为实数集的闭区间的开覆盖，则有正数 r 使得当 $|x-y|<r$ 时 x 和 y 同属于该覆盖的某个元. 也就是说，在某种意义下每一个开覆盖"一致地"覆盖了该区间的点. 本节就来证明这个引理以及它的一种可以应用于任意紧空间的拓扑的变形. 后一结果也可以看成是下一节关于仿紧性概念的一种准备.

26 定理. 若 $\mathscr{U}$ 为伪度量空间 (X, d) 的紧子集 A 的开覆盖，则有正数 r 使得每一个以 A 的点为心的开 r-球都包含在 $\mathscr{U}$ 的某个元内.

证明. 设 $U_1, \cdots, U_n$ 为 A 的开覆盖 $\mathscr{U}$ 的有限子覆盖，$f_i(x)=\operatorname{dist}[x, X-U_i]$，$f(x)=\max[f_i(x): i=1, \cdots, n]$，则每一个 f_i 为连续，从而 f 为连续. 因为 A 的每一点属于某个 U_i，故对 A 中的每一个 x 有 $f(x) \geqslant f_i(x)>0$，又集 $f[A]$ 为正实数集的紧子集，于是有正实数 r 使得对 A 中的所有 x 有 $f(x)>r$. 这表明对 A 中的每一个 x 有 i 使得 $f_i(x)>r$，亦即以 x 为心的开 r-球包含在 U_i 内.|

上一定理有一个有用的推论. 若 A 为伪度量空间的紧子集，U 为 A 的邻域，则有正数 r 使得 U 包含每一个以 A 的点为心的开 r-球；即 A 与 $X \sim U$ 的距离为正.

定理5.26还有一种富有启发性的陈述方法. 因为若 V 为所有使得 $d(x, y)<r$ 的 X 的点偶组成的集，则 $V[x]=\{y: (x, y) \in V\}$ 就是以 x 为心的开 r-球，又集 V 为 $X \times X$ 的开子集并且包含对角线 Δ（所有 (x, x) 的集，其中 x 属于 X），故由上一定理可推出如下的拓扑的结果：若 $\mathscr{U}$ 为紧伪度量空间的开覆盖，则有 $X \times X$ 的对角线的邻域 V 使得对每一点 x，集 $V[x]$ 包含在 $\mathscr{U}$ 的某个元内. Lebesgue 引理的这种变形可以转到任意的紧正则空间.

我们称拓扑空间的覆盖 $\mathscr{U}$ 为**齐-覆盖**当且仅当有 $X\times X$ 的对角线的邻域 V 使得对每一点 x，集 $V[x]$ 包含在 $\mathscr{U}$ 的某个元内．换言之，由所有形如 $V[x]$ 的集组成的集族将 $\mathscr{U}$ 加细了．再回忆一下，覆盖 $\mathscr{A}$ 叫做是 $\mathscr{U}$ 的加细当且仅当 $\mathscr{A}$ 的每一个元是 $\mathscr{U}$ 的某个元的子集；另外，集族 $\mathscr{B}$ 叫做局部有限当且仅当空间的每一点有一个邻域，它只与有限多个 $\mathscr{B}$ 的元相交．又我们称集族为闭当且仅当它的每一个元均为闭集．

27 定理．若空间的开覆盖有闭且局部有限的加细，则它为齐-覆盖．

因而，紧正则空间的每一个开覆盖为齐-覆盖．

证明．设 $\mathscr{U}$ 为拓扑空间 X 的开覆盖，$\mathscr{A}$ 为它的一个闭且局部有限的加细，则对 $\mathscr{A}$ 中的每一个 A 有 $\mathscr{U}$ 的元 U_A 使得 $A\subset U_A$．命 $V_A=(U_A\times U_A)\cup((X\sim A)\times(X\sim A))$，则易见 V_A 为 $X\times X$ 的对角线的一个开邻域并且当 $x\in A$ 时 $V_A[x]=U_A$．于是，若命 $V=\bigcap\{V_A: A\in\mathscr{A}\}$，则对每一点 x，集 $V[x]\subset V_A[x]=U_A$，从而由所有形如 $V[x]$ 的集组成的集族为 $\mathscr{U}$ 的一个加细．剩下要证的是：V 为 $X\times X$ 的对角线的一个邻域．对对角线的每一点 (x,x)，选 x 的邻域 W 使得 W 只与有限多个 $\mathscr{A}$ 的元相交，若 $W\cap A$ 为空集，则 $W\subset X\sim A$ 并且 $W\times W\subset V_A$，这样就推出了 V 包含 $W\times W$ 与有限多个集 V_A 的交，亦即为 (x,x) 的一个邻域．

最后，若 X 为紧正则空间，则每一个开覆盖 $\mathscr{U}$ 有闭且有限的加细（借助这样的开子集覆盖 X，它的闭包加细了 $\mathscr{U}$），因而每一个开覆盖为齐-覆盖．▌

*仿　紧　性

我们称拓扑空间为仿紧当且仅当它为正则[1] 并且每一个开覆

1）仿紧的通常定义是以“Hausdorff”来代替“正则”．然而我们不难证明当每一个开覆盖有开且局部有限的加细时 Hausdorff 空间必为正则空间．

盖有开且局部有限的加细．本节目的是证明仿紧性和一些其它条件的等价性．所用方法与第六章有密切联系．

回忆一下，拓扑空间的子集族$\mathscr{A}$叫做离散当且仅当该空间的每一点都有一个邻域，它至多与这个族的一个元相交；又族$\mathscr{A}$叫做σ离散（σ局部有限）当且仅当它是可数多个离散（局部有限）的子族的并．现在，我们可以来陈述这一节的主要定理；它的证明由随后的一系列引理给出．

28 定理．若X为正则拓扑空间，则下列命题等价：

(a) 空间X为仿紧．

(b) X的每一个开覆盖有局部有限的加细．

(c) X的每一个开覆盖有闭且局部有限的加细．

(d) X的每一个开覆盖为齐一覆盖．

(e) X的每一个开覆盖有开且σ离散的加细．

(f) X的每一个开覆盖有开且σ局部有限的加细．

证明的步骤是：(a) → (b) → (c) → (d) → (e) → (f) → (b) → (a)．其中第一步是明显的，下面的引理证明了第二步．

29 引理．若X为正则空间并且每一个开覆盖有局部有限的加细，则每一个开覆盖有闭且局部有限的加细．

证明．若$\mathscr{U}$为X的开覆盖，则因X为正则空间，故有一个开覆盖$\mathscr{V}$使得由$\mathscr{V}$的所有元的闭包组成的集族为$\mathscr{U}$的加细．（对每一个x，若$x \in U$，则有x的开邻域V使得$V^- \subset U$．）从而，命$\mathscr{A}$为$\mathscr{V}$的局部有限的加细，则由$\mathscr{A}$的所有元的闭包组成的集族$\mathscr{B}$亦为局部有限并且$\mathscr{A}$的每一个元的闭包为$\mathscr{V}$中某个元V的闭包V^-的子集，于是$\mathscr{B}$即为$\mathscr{U}$所需要的闭且局部有限的加细．|

根据定理5.27，对任何拓扑空间，每一个有闭且局部有限的加细的开覆盖为齐一覆盖．因而，从定理中的(c)可推出(d)．在证明下一步之前，我们先证明两个引理，它本身也有一定的兴趣．为方便起见，再回顾一下为此所需要的一些事实（见预备知识关于关系的那一节）．若U为$X \times X$的子集并且$x \in X$，则$U[x]$是指由所有使得$(x, y) \in U$的点y组成的集．若A为X的子集，则$U[A]=$

$\{y:$ 对 A 中的某个 x 有 $(x,y)\in U\}$；显然 $U[A]$ 为所有集 $U[x]$ 的并，其中 x 属于 X. 集 $\{(x,y):(y,x)\in U\}$ 记为 U^{-1}，又 U 叫做是对称的，假如 $U=U^{-1}$；易见 $U\cap U^{-1}$ 恒为对称. 若 U 和 V 为 $X\times X$ 的子集，则 $U\circ V$ 是指由所有使得对 X 中的某个 y 有 $(x,y)\in V$ 和 $(y,z)\in U$ 成立的 (x,z) 组成的集. 换言之，$(x,z)\in U\circ V$ 当且仅当对某个 y 有 $(x,z)\in V^{-1}[y]\times U[y]$，从而 $U\circ V$ 为所有集 $V^{-1}[y]\times U[y]$ 的并，其中 y 属于 X. 特别，若 V 为对称，则 $V\circ V=\bigcup\{V[y]\times V[y]:y\in X\}$. 最后，对 X 的每一个子集 A 有 $U\circ V[A]=U[V[A]]$ 成立.

30 引理. 设 X 为使得每一个开覆盖为齐-覆盖的拓扑空间，若 U 为 $X\times X$ 的对角线的邻域，则有该对角线的对称邻域 V 使得 $V\circ V\subset U$.

证明. 因为 U 为对角线的邻域，故对 X 的每一个点 x 有邻域 $W(x)$ 使得 $W(x)\times W(x)\subset U$，从而由所有形如 $W(x)$ 的集组成的集族 $\mathscr{W}$ 为 X 的一个开覆盖，于是由假设有对角线的邻域 R 使得由所有集 $R[x]$ 组成的集族为 $\mathscr{W}$ 的一个加细，因而对每一个 x 有 $R[x]\times R[x]\subset U$. 最后，命 $V=R\cap R^{-1}$，则 V 为对角线的一个对称邻域并且对一切 x 有 $V[x]\times V[x]\subset U$，但 $V\circ V$ 为所有集 $V[x]\times V[x]$ 的并，这样就推出了 $V\circ V\subset U$. |

上一引理有这样一种直观说法. 即若我们称两点 x 和 y 至多相隔 U-距离，假如 $(x,y)\in U$；则存在 V 使得当 x 和 y，y 和 z 至多相隔 V-距离时 x 和 z 至多相隔 U-距离.

下一引理将证明仿紧空间满足一种很强的正规性条件.

31 引理. 设 X 为使得每一个开覆盖为齐-覆盖的拓扑空间，又设 $\mathscr{A}$ 为局部有限(或离散)的 X 的子集族. 则有 $X\times X$ 的对角线的邻域 V 使得由所有集 $V[A]$ 组成的集族为局部有限(相应地离散)，其中 A 属于 $\mathscr{A}$.

证明. 若 $\mathscr{A}$ 为局部有限的子集族，则有 X 的开覆盖 $\mathscr{U}$ 使得 $\mathscr{U}$ 的每一个元只与族 $\mathscr{A}$ 的有限多个元相交. 又命 U 为对角线的一个邻域，它使得由所有集 $U[x]$ 组成的集族为 $\mathscr{U}$ 的一个加细，

则由上一引理有对角线的一个邻域 V 使得 $V\circ V\subset U$，并且还可以假定 $V=V^{-1}$. 此时，若 $V\circ V[x]\cap A$ 为空集，则 $V[x]$ 必定不与 $V[A]$ 相交，这是因为：若 $y\in V[x]\cap V[A]$，则 $(y,x)\in V^{-1}=V$ 并且对 A 中的某个 z 有 $(z,y)\in V$，即 $(z,x)\in V\circ V$，亦即 $z\in V\circ V[x]$，于是得到矛盾. 因而，若 $V[x]$ 与 $V[A]$ 相交，则 $V\circ V[x]$ 与 A 相交，并且由此即可推出由所有集 $V[A]$ 组成的集族为局部有限，其中 A 属于 $\mathscr{A}$.

若将"有限多个"换成"至多一个"，则得对离散的族相应命题的证明. |

因为 $V[x]$ 是 V 在映 X 的每一点 y 为 (x,y) 的连续映射下的逆像，故若 V 为 $X\times X$ 的开子集，则对 X 的每一点 x，$V[x]$ 为开集. 于是，若 A 为 X 的子集，则 $V[A]$ 为开集，这是因为它是所有集 $V[x]$ 的并，其中 x 属于 A. 从而上一引理表明，我们可以将局部有限或离散的族的每一个元扩张为开集并且仍然保存该族的特征. 特别，若正则空间的每一个开覆盖 $\mathscr{U}$ 有局部有限的加细 $\mathscr{A}$，则该引理可以应用(我们已经证明了定理 5.28 中的 (b) → (c) → (d))，于是有对角线的开邻域 V 使得所有集 $V[A]$ 组成的族为局部有限，其中 A 属于 $\mathscr{A}$. 虽然这样所得到的族可以不是 $\mathscr{U}$ 的加细，但这一点容易补救，只须选 $\mathscr{U}$ 中的 U_A 使得 $A\subset U_A$，然后命 $W_A=U_A\cap V[A]$. 显然按这种方式所作出的集族就是 $\mathscr{U}$ 的一个开且局部有限的加细，这样也就推出了空间为仿紧，即定理 5.28 中的 (b) → (a) 成立.

引理 5.31 有一个明显推论. 因为由两个互不相交的闭子集所组成的集族显然为离散，故得：

32 系. 任何仿紧空间皆为正规空间.

由此可见，如果我们再得到如下的两个事实，那末定理 5.28 的证明就全部完成. 亦即只须证若 X 为正则空间并且每一个开覆盖为齐一覆盖，则每一个开覆盖有开且 σ 离散的加细；以及若 X 的每一个开覆盖有开且 σ 局部有限的加细，则每一个开覆盖有局部有限的加细. (定理 5.28 中的 (e) → (f) 是明显的.)

33 引理. 若 X 为使得每一个开覆盖为齐-覆盖的空间，则 X 的每一个开覆盖有开且 σ 离散的加细.

证明. 如同定理 4.21，它的证明是应用 A. H. Stone 的一种技巧.（该引理从定理 4.21 和第六章的结果亦可导出.）根据引理 5.31，我们只须找出开覆盖 $\mathscr{U}$ 的一个 σ 离散的加细，这是因为这样的 σ 离散的加细可以"扩张"成一个开且 σ 离散的加细.

设 V 为对角线的开邻域，它使得由所有集 $V[x]$ 组成的集族为 $\mathscr{U}$ 的一个加细，其中 x 属于 X，命 $V_0 = V$ 并且对每一个正整数 h，按照归纳方法，选对角线的一个开的对称邻域 V_n 使得 $V_n \circ V_n \subset V_{n-1}$. 再命 $U_1 = V_1$ 并且按照归纳方法，命 $U_{n+1} = V_{n+1} \circ U_n$，则易见对每一个 n 有 $U_n \subset V_0$，并且由此可推出对每一个 n 所有 $U_n[x]$ 组成的族为 $\mathscr{U}$ 的一个加细，其中 x 属于 X.

选关系 $<$ 使得 X 为良序集(见预备知识定理 25)，并且对每一个 n 和每一个 x，命 $U_n^*(x) = U_n[x] \sim \bigcup\{U_{n+1}[y]: y < x\}$，则对每一个确定的 n，由所有集 $U_n^*(x)$ 组成的集族 $\mathscr{U}_n$ 为离散，这可以证明如下：若对 X 中的某个 z，邻域 $V_{n+1}[z]$ 与 $U_n^*(y)$ 相交，则 $z \in V_{n+1}[U_n^*(y)]$，即 $V_{n+1}[U_n^*(y)]$ 为 z 的邻域，又由作法易知当 $x \neq y$ 时 $U_n^*(x)$ 与 $V_{n+1}[U_n^*(y)]$ 互不相交.

剩下要证的是 X 的每一个点都属于某个 $\mathscr{U}_n$ 的某个元. 对 X 中的 x，选 y 为 X 中这样的点中的第一个，它使得对某个 n 有 x 属于 $U_n[y]$，显然，这时对某个 n 有 $x \in U_n^*(y)$. |

34 引理. 若空间的每一个开覆盖有开且 σ 局部有限的加细，则每一个开覆盖有局部有限的加细.

证明. 设 $\mathscr{U}$ 为一个开覆盖，又设 $\mathscr{V}$ 为它的开且 σ 局部有限的加细，并且 $\mathscr{V} = \bigcup\{\mathscr{V}_n: n \in \omega\}$，其中每一个 $\mathscr{V}_n$ 皆为局部有限的开集族. 若对每一个 n 和 $\mathscr{V}_n$ 的每一个元 V，命 $V^* = V \sim \bigcup\{U:$ 对某个 $k < n$ 有 $U \in \mathscr{V}_k\}$，又命 $\mathscr{W}$ 为所有形如 V^* 的集组成的集族，则 $\mathscr{W}$ 为 $\mathscr{U}$ 的一个加细，最后，对 X 中的 x，命 n 为使得 x 属于 $\mathscr{V}_n$ 的某个元 V 的第一个整数，此时 V 为 x 的一个邻域并且与 $\mathscr{W}$ 的每一个这样的元都不相交，它不是由族 $\mathscr{V}_k$ 所作出

的，其中 $k \leqslant n$，即 $\mathscr{W}$ 为局部有限. ∎

定理 4.21 表明伪度量空间的每一个开覆盖有开且 σ 离散的加细. 根据这个事实和本节的定理 5.28 即得:

35 系. 每一个伪度量空间皆为仿紧空间.

最后，我们指出仿紧空间的子空间，商空间和乘积空间一般不再是仿紧空间. 而且，局部可度量化，局部紧，Hausdorff，正规并且满足第一可数性公理的空间仍然可以不是仿紧空间. 所需要的例子在本章最后的问题中给出.

36 注记. 仿紧性还有另外的刻划可以添加到定理 5.28. 这就是正则空间为仿紧当且仅当它为完满正规（见问题 5.V). 这种刻划属于 A. H. Stone[1]. 定理 5.28 中的 (b), (c), (e) 和 (f) 的等价性是属于 E. Michael[1] 的. 此外，就我所知 (d) 的等价性是首先由 J. S. Griffin 和我本人给出的.

仿紧性用 σ 离散的刻划可以看成是可数维数的一种定义（见 Hurewicz 和 Wallman [1;32] 与 Eilenberg[1]). 还有一个 F_σ 定理(也见 Michael[1])，它在维数理论中也有所应用.

问　　题

A　关于紧空间上的实函数的习题

(a) 若 A 为实数空间的非空紧子集，则 A 的上确界和下确界皆属于 A.

(b) 紧空间 X 上的每一个连续函数 f 恒有最大值和最小值，即存在 X 的点 x 和 y 使得 $f(x)$ 和 $f(y)$ 分别为 f 在 X 上的上确界和下确界.

(c) 设 f 为紧空间 X 上的连续实值函数，若 f 恒为正，则 f 在这样意义下偏离零点，即存在 $e>0$ 使得对 X 中的 x 恒有 $f(x)>e$.

B　紧子集

(a) 拓扑空间的两个紧子集的交可以不是紧集. 然而，任意一族闭的紧子集的元的交仍为闭的紧集.（显然，具有非紧的交的两个紧子集必须是某个非 Hausdorff 空间的子集. 又设 X 为实数空间和具有两个元的平庸空间的乘积.)

(b) 拓扑空间的紧子集的闭包可以不是紧集. 然而，正则空间的紧子集的闭包仍为紧集.

(c) 若 A 和 B 为伪度量空间的互不相交闭子集并且 A 为紧集则有 A 的元 x 使得 $\mathrm{dist}(A, B)=\mathrm{dist}(x, B)>0$. (函数 $\mathrm{dist}(x, B)$ 关于 x 为连续并且对 A 中的 x 为正.)

(d) 若 A 和 B 为伪度量空间的互不相交闭的紧子集，则有 A 的元 x 和 B 的元 y 使得 $d(x, y)=\mathrm{dist}(A, B)$.

C 关于序拓扑的紧性

设 X 为一个集，它关于关系 $<$ 为线性有序，又设 X 具有序拓扑（见问题 1.1)，则 X 的每一个闭，序有界子集为紧集当且仅当 X 关于 $<$ 为有序完备.（X 的所有形如 $\{x: a<x\}$ 或 $\{x: x<a\}$ 的子集组成的集族为关于 X 的序拓扑的一个子基，并且应用 Alexander 的子基定理 5.6. 根据定理 5.14 中所使用的论证方法，我们也可以给出一种不利用定理 5.6 的证明.)

D 紧度量空间的等距映射

设 X 和 Y 为度量空间，并且 X 为紧，又设 f 为从 X 到 Y 的一个子空间上的等距映射，g 为从 Y 到 X 的一个子空间上的距映射，则 f 映 X 到 Y 上.（设 h 为从 X 到它自己的一个真子集上的等距映射，并且 $x\in X\sim h[X]$，命 $a=\mathrm{dist}(x, h[X])$，再按照归纳方法，命 $x_0=x$，$x_{n+1}=h(x_n)$，然后证明若 $m\neq n$，则 $d(x_m, x_n)\geqslant a$.)

E 可数紧和列紧空间

拓扑空间叫做**可数紧**当且仅当每一个可数开覆盖有有限子覆盖，又叫做**列紧**当且仅当每一个序列有收敛子序列.

(a) 空间为可数紧当且仅当每一个序列都有聚点.

(b) T_1-空间为可数紧当且仅当每一个无限集都有聚点.（见引理 5.3.)

(c) T_1-空间为可数紧当且仅当每一个无限开覆盖都有真子覆盖.（若 A 为没有聚点的无限集，则 A 的每一个子集皆为闭集. 作开覆盖 $\mathscr{U}$：对 A 的每一点选一个这样的开邻域，它不包含 A 的其它点，另外当有必要时再加上开集 $X\sim A$，则 $\mathscr{U}$ 就没有真子覆盖. 另一方面，若 $\mathscr{V}$ 为没有真子覆盖的开覆盖，则 $\mathscr{V}$ 的每一个元包含有一个这样的点，它不属于 $\mathscr{V}$ 的其它元.)

(d) 满足第一可数性公理的空间为可数紧当且仅当它为列紧（定理

5.5)。

(c) 具有序拓扑的所有小于第一个不可数序数 Ω 的序数的集 Ω_0 为局部紧，Hausdorff，满足第一可数性公理并且列紧的空间，但不是紧空间。

注 命题(c)属于 Arens 和 Dugundji[1]。

F 紧性；紧连通集的交

(a) 设 $\mathscr{A}$ 为一个闭，紧集的族并且使得 $\bigcap\{A: A\in\mathscr{A}\}$ 为某个开集 U 的子集，则有 $\mathscr{A}$ 的有限子族 $\mathscr{F}$ 使得 $\bigcap\{A: A\in\mathscr{F}\}\subset U$。

(b) 若 $\mathscr{A}$ 为 Hausdorff 空间 X 的一个紧子集的族并且使得 $\mathscr{A}$ 的元的有限交为连通，则 $\bigcap\{A: A\in\mathscr{A}\}$ 亦为连通。

G 关于局部紧性的问题

若 X 为 Hausdorff 空间，Y 为稠密的局部紧子空间，则 Y 为开集。

H 紧性的套的特征

拓扑空间 X 为紧当且仅当每一个非空闭集套有一个非空的交。（回顾一下，所谓套乃是指这样的集族，它关于包含关系为线性有序。若每一个非空闭集套有一个非空的交并且 $\mathscr{A}$ 为一个具有有限交性质的闭集族，命 $\mathscr{B}$ 为包含 $\mathscr{A}$ 并且具有有限交性质的一个极大闭集族，又命 $\mathfrak{N}$ 为 $\mathscr{B}$ 中的一个极大套。考察 $\mathscr{B}$ 和 $\mathfrak{N}$ 的性质即可导出它的一个证明。另一个完全不同的证明可以建立在良序的基础上，这要利用下一问题中所提到的方法的部分。）

I 完全聚点

点 x 叫做拓扑空间的子集 A 的一个完全聚点，当且仅当对 x 的每一个邻域 U，集 A 和 $A\cap U$ 有相同的基数。拓扑空间为紧当且仅当每一个无限子集都有完全聚点。（若 X 为非紧，则可选出没有有限子覆盖的开覆盖 $\mathscr{A}$ 使得 $\mathscr{A}$ 的基数 c 尽可能的小。设 C 为基数 c 的良序集，它使得每一个元的所有前趋元的集的基数小于 c（在附录中证明了 C 就是一个这样的集）。又设 f 为从 C 到 $\mathscr{A}$ 上的一对一映射，则对 C 的每一个元 b，并 $\bigcup\{f(a): a<b\}$ 不能覆盖 X，事实上，该并的余集的基数至少和 c 一样大。因而可从该余集选出 x_b 使得当 $a<b$ 时 $x_a\neq x_b$。然后再考虑所有 x_b 的集。）

J 例子：带有字典序的单位方形

设 X 为闭单位区间 Q 和它自己的笛卡儿乘积并且按字典序使其成为有序集(即$(a, b)<(c, d)$当且仅当 $a<c$，或 $a=c$ 并且 $b<d$)，则带有序拓扑 X 的为紧的连通 Hausdorff 空间．它满足第一可数性公理，但不可分，因而也不可度量化．

K　关于正规性和乘积的例子(序数)

局部紧的，正规 Hausdorff 空间和紧 Hausdorff 空间的乘积可以不是正规空间．(困难的部分已经在问题 4.E 中得到，现在只须证 Ω' 和 Ω_0 分别为紧和局部紧的 Hausdorff 空间，其中 Ω' 为所有小于或等于 Ω 的序数的空间，Ω_0 为所有小于 Ω 的序数的集，并且每一个都带有序拓扑．)

L　超穷线

设 A 为一个良序集，又设半开区间 $[0, 1)$ 带有通常序，再设 $A\times[0, 1)$ 带有字典序和序拓扑．讨论该空间的性质．

M　例子：Helly 空间

Helly 空间是指定义在闭单位区间 Q 上并且取值于 Q 内的所有非降函数组成的族． 它是乘积空间 Q^Q 的一个子集并且它的拓扑是相对乘积拓扑．该空间 H 具有如下性质：

(a) H 为紧 Hausdorff 空间．(它是 Q^Q 的一个闭子空间．)

(b) H 满足第一可数性公理，因而为列紧．(H 的每一个元的所有不连续点的集为可数集．利用这一点以及 Q 为可分的事实可作出 H 的一个点 h 的邻域系的一个可数基．)

(c) H 可分．(利用有理数，即可作出一个可数稠密集．)

(d) H 不是度量空间．(对 Q 中的 t，命 $f_t(x)$ 当 $x<t$ 时为 0，当 $x>t$ 时为 1，并且命 $f_t(t)=1/2$，则所有形如 f_t 的函数组成的族 A 为不可数集并且任何 A 的元都不是 A 的聚点．然而，紧度量空间的每一个子空间又必为可分．)

N　关于闭映射和局部紧性的例子

(a) 设 X 为带有通常拓扑的实数空间，又设 I 为整数集，再设 $\mathscr{D}$ 为这样的分解，它的元为 I 和所有集 $\{x\}$，其中 x 属于 $X\sim I$，则 X 到商空间上的射影为闭连续映射，但商空间非局部紧并且也不满足第一可数性公理．

(b) 设 Ω_0 为所有小于 Ω 的序数组成的集并且带有序拓扑，又设 A 为闭的不可数集并且它的余集也不可数，再设 $\mathscr{D}$ 为这样的分解，它的元为 A 和所有集 $\{x\}$，其中 x 属于 $\Omega_0\sim A$，则 Ω_0 到商空间上的射影为闭连续映射，并且商空间为紧，但不满足第一可数性公理。(利用问题 4.E 的交错引理.)

O Cantor 空间

Cantor 不连续统(三分集)是指闭单位区间的所有这样的元组成的集，在它的三进展式中不出现数字 1. (为方便起见，以下我们只利用无理的三进展式，即在展式中不能从某一位起都恒等于 0. 正如预备知识定理 14 所指出的，每一个实数都有一个唯一的无理展式.) 这个不连续统之所以叫做三分集是因为：三等分区间 $[0, 1]$ 所得的(开)中间部分恰为所有这样的数组成的集它的三进展式在小数点后的第一位为 1. 又三等分每一个剩余区间，所得的中间部分是由这样的点组成，它的展式的第二位小数为 1，而第一位不为 1. 继续下去，显然便知该不连续统通过连续删去三等分后的中间部分即可得到。

又乘积空间2^A(即所有从集 A 到仅有 0 和 1 两个元的离散空间的函数组成的族并且具有乘积拓扑) 叫做 Cantor 空间.

(a) Cantor 不连续统同胚于 2^ω. (对 2^ω 中的 x，命 $f(x)$ 为 $[0, 1]$ 中这样的元，它的三进展式在第 p 位为 $2x(p)$.)

(b) 该不连续统的每一点均为聚点并且该不连续统的余集为实数的一个稠密开子集.

(c) 若 A 为 2^ω 的一个非空闭子集，则有从 2^ω 到 A 的连续函数 r 使得对 A 中的 x 有 $r(x) = x$.(如果我们着眼于作为 2^ω 的同胚像的 Cantor 不连续统，那末它的证明就更容易看出一些.)

(d) 每一个紧 Hausdorff 空间为某个 Cantor 空间的一个闭子集的连续像. (设 F 为所有 I_2 上这样的函数 f 组成的族，它使得 $f(0)$ 和 $f(1)$ 为紧 Hausdorff 空间 X 的闭子集并且 $f(0)\cup f(1) = X$. 若 x 为 2^F 的一个元，$f\in F$，则 $f(x_f)$ 为 X 的闭子集. 这时交 $\bigcap\{f(x_f): f\in F\}$ 为空集或由单个点组成，在后一种情况下就定义这个点为 $\phi(x)$. 我们能够证明 ϕ 的定义域为 2^F 的一个闭子集；并且若 U 为 X 的一个子集，则 $\phi^{-1}[U] = \{x: x$ 为 ϕ 的定义域的一个元并且 $\bigcap\{f(x_f): f\in F\}\subset U\}$.)

(e) 每一个紧度量空间 X 为 2^ω 的连续像. (代替上面证明中的族 F，我们作一个更小的族，它起着相同的作用. 若 $U_0, \cdots, U_n, \cdots$ 为 X 的拓扑的一

个基，则命 $f_n(0)=U_n^-$，$f_n(1)=X\sim U_n$.)

(f) 每一个 Cantor 空间 2^A 满足可数链条件；即每一个互不相交的开集族必可数.（若 $\mathscr{U}$ 是 2^A 的一个互不相交的开子集族，则我们可以假定 $\mathscr{U}$ 的元恒属于乘积拓扑所定义的基，即每一个元都在一种自然意义下是有限多个半空间的交. 这时，对某个整数 n 有一个无限（事实上不可数）的互不相交的族，它的每一个元恰为 n 个半空间的交. 再通过对于互不相交性的简单论证也就完成了它的证明.)

有一个较短、但更牵强附会的证明. 一个具有按坐标 mod 2 加法的 Cantor 空间必为紧拓扑群，从而存在一个 Haar 测度（见 Halmos [1;254]). 因为该测度为有限并且对开集为正，故可数链条件显然成立).

(g) 并非每一个紧 Hausdorff 空间均为某个 Cantor 空间的连续像.（不可数的离散空间的单点紧扩张就不满足可数链条件.)

注　命题 (b) 属于 Cantor，(e) 属于 P. Alexandroff 和 Urysohn，而 (f) 和 (g) 属于 J. W. Tukey. 又命题 (g) 是 Szpilrajn[1] 的某些结果的一个推论.

P　Stone-Čech 紧扩张的刻划

设 (f, Y) 为拓扑空间 X 的一个 Hausdorff 紧扩张并且使得对于 X 上的每一个有界连续的实值函数 g，函数 $g\circ f^{-1}$ 恒有连续扩张，则 (f,Y) 拓扑等价于 Stone-Čech 紧扩张 $(e, \beta(X))$.（考虑 $\beta(X)$ 的定义.)

Q　关于紧扩张的例子（序数）

设 Ω' 为所有小于或等于 Ω 的序数组成的集，又设 $\Omega_0=\Omega'\sim\{\Omega\}$，并且每一个都具有序拓扑，则 Stone-Čech 紧扩张 $\beta(\Omega_0)$ 同胚于 Ω'.（这从上一问题即可推出，只要我们证明 Ω_0 上的每一个有界实值连续函数 f 都在下列意义下最终地为常数[1)]，即对某个 Ω_0 中的 x 当 $y>x$ 时有 $f(y)=f(x)$. 若 f 为一个有界连续实值函数，又 r 和 s 为满足 $r>s$ 的实数，则由问题 4.E 的交错引理即可证得集 $\{x: f(x)\geqslant r\}$ 和 $\{x: f(x)\leqslant s\}$ 中之一必为可数集. 利用这个事实就不难看出 f 最终地为常数. 实际上，f 为有界的假设是非本质的.)

注　这个结果属于 Tong[1].

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1) Ω_0 的这个精细的性质，E. Hewitt[1] 在构造一个正则 Hausdorff 空间 X 使得 X 上的每一个连续实值函数为常数时就已经用到.

R　Wallman 紧扩张

设 X 为一个 T_1 空间，又设 $\mathscr{F}$ 为 X 的所有闭子集组成的族，再设 $w(X)$ 为 $\mathscr{F}$ 的所有这样的子族 $\mathscr{A}$ 的全体，它具有有限交性质并且在 $\mathscr{F}$ 中关于该性质为极大.

(a) 若 $\mathscr{A}\in w(X)$，则 $\mathscr{A}$ 的两个元的交仍为 $\mathscr{A}$ 的一个元；对偶地，若 A 和 B 为 $\mathscr{F}\sim\mathscr{A}$ 的元，则 $A\cup B$ 仍为 $\mathscr{F}\sim\mathscr{A}$ 的元.（见问题 2.I.）

(b) 对 X 的每一点 x，命 $\phi(x)=\{A: A\in\mathscr{F}$ 且 $x\in\mathscr{A}\}$，则 ϕ 是从 X 到 $w(X)$ 内的一对一映射.

(c) 对 X 的每一个开子集 U，命 $U^*=\{\mathscr{A}:\mathscr{A}\in w(X)$ 且对 $\mathscr{A}$ 中的某个 A 有 $A\subset U\}$，则 $w(X)\sim U^*=\{\mathscr{A}: X\sim U\in\mathscr{A}\}$. 若 U 和 V 为 X 的开子集，则 $(U\cap V)^*=U^*\cap V^*$ 并且 $(U\cup V)^*=U^*\cup V^*$.

(d) 设 $w(X)$ 具有这样的拓扑，它的一个基为所有形如 U^* 的集组成的族，其中 U 为 X 中的开集，则 $w(X)$ 为紧，映射 ϕ 连续并且 $\phi(X)$ 在 $w(X)$ 中稠密.（证明紧性是通过对这个基的元的余集的有限交性质的论证.）

(e) 若 X 为正规空间，则 $w(X)$ 为 Hausdorff 空间.

(f) 若 f 为 X 上的一个有界连续实值函数，则 $f\circ\phi^{-1}$ 可以连续地扩张到整个 $w(X)$.（若不可能有这样的连续扩张，则通过一些论证可以说明存在实数的互不相交闭子集 R 和 S 使得 $f^{-1}[R]$ 和 $f^{-1}[S]$ 互不相交，而这些集关于 ϕ 的像的闭包却相交. 另一方面，若 A 和 B 为 X 的互不相交闭子集，则 $\{\mathscr{A}: A\in\mathscr{A}\}$ 和 $\{\mathscr{A}:B\in\mathscr{A}\}$ 互不相交并且为 $w(X)$ 中的闭集.）

(g) 若 $w(X)$ 为 Hausdorff 空间，则 Wallman 紧扩张拓扑等价于 Stone-Čech 紧扩张.（见问题 5.P.）

注　Wallman 紧扩张(Wallman[1])的主要价值在于：变 U 为 U^* 的对应保持有限交和并的运算. 另外，通过该对应 X 的拓扑变为 $w(X)$ 的拓扑的一个基，并且从这个事实可推出 X 的维数(在覆盖意义下)和 $w(X)$ 的维数相等，以及 X 和 $w(X)$ 有同构的 Čech 同调群. 还有一种有关的构造见 Samuel[1].

S　Boole 环：Stone 表示定理

设 $(R,+,\cdot)$ 为 Boole 环（见问题 2.K），又设 S' 为所有从 R 到 I_2(＝mod 2 的整数)内的环同态组成的集，再设 $S=S'\sim\{0\}$，其中 0 是恒等于零的同态，则 S' 是乘积 I_2^R 的一个子集. 另外环 R 的 Stone 空间是指具有相对乘积拓扑的 S (I_2 指定了离散拓扑).

Boole 空间是指这样的 Hausdorff 空间，它使得所有紧，开集组成的族为

拓扑的一个基。Boole空间自然为局部紧。又 Boole 空间的特征环是指所有取值于I_2 的这样的连续函数 f 组成的环，它使得 $f^{-1}[1]$ 为紧(即所有在某个紧集外为零的取值于 I_2 的函数；有时也叫做具有紧支集的函数)。

(a) Boole 环 R 的 Stone 空间为 Boole 空间并且当 R 有单位元时它还是紧的。(在这种情况下 $S=\{h:\ h\in S'$ 且 $h(1)=1\}$。)

(b) Stone-Weierstrass mod 2. 设 $\mathscr{F}$ 为 Boole 空间 X 的特征环，又设 $\mathscr{G}$ 为 $\mathscr{F}$ 的子环，它具有两点性质(即对 X 中不同的点 x 和 y 与 I_2 中的 a 和 b 有 $\mathscr{G}$ 中的 g 使得 $g(x)=a$ 并且 $g(y)=b$)，则 $\mathscr{F}=\mathscr{G}$。

(若 X 为紧，则 $\mathscr{G}$ 具有两点性质当 $1\in\mathscr{G}$ 并且 $\mathscr{G}$ 在这样的意义下分离点时，即对 X 中不同的点 x 和 y 有 $\mathscr{G}$ 中的 g 使得 $g(x)\neq g(y)$。通过一种惯用而又有益的紧性论证方法我们即可得到 (b)。开始我们先证明对 X 的紧子集 Y 和 $X\sim Y$ 的点 x 有 $\mathscr{G}$ 中的 g 使得 $g(x)=0$ 并且 g 在 Y 上为 1。)

(c) 表示定理。每一个 Boole 环同构于(关于计值映射) 它的 Stone 空间的特征环。(对 R 中的 r，在 r 处的计值 $e(r)$ 为 S 上的这样的函数，它在 S 的元 s 处的值为 $s(r)$。这个定理的成立是依赖于有足够多同态的存在(问题 2.K) 和上一命题 (b)。)

(d) 若 X 为 Boole 空间，$\mathscr{F}$ 为它的特征环并且 $\mathscr{I}$ 为 $\mathscr{F}$ 中的一个极大真理想，则对 X 中的某个 x 有 $\mathscr{I}=\{f:\ f(x)=0\}$。(首先证明如果没有一个点使得 $\mathscr{I}$ 的一切元在其上为零，那末 $\mathscr{I}=\mathscr{F}$。)

(e) 对偶表示定理。若 X 为 Boole 空间，则 X 同胚于(关于计值映射)它的特征环的 Stone 空间。(每一个极大理想为一个到 I_2 内的唯一同态的零点的集并且每一个如此的零点的集为一个极大理想。上一命题 (d) 实质上证明了计值映射映 X 到 Stone 空间上。)

注　上述结果属于 M. H. Stone[1]。

表示 Boole 空间的方法有一种有趣的变形。若 X 为 Boole 空间，命 $\mathscr{F}$ 为所有从 X 到 I_2 的连续函数的环 (去掉了 $f^{-1}[1]$ 为紧集的要求)，则从 X 到 $\mathscr{F}$ 的 Stone 空间 S 的计值映射仍为一个同胚，但 S 为紧并且事实上它是同胚于 Stone-Čech 紧扩张 $\beta(X)$。我们略去这个事实以及 Boole 环的理想和子环的借助 Stone 空间来刻划的证明。

最后，我们对这个问题的安排是使它能够转到局部紧 Hausdorff 空间 X 上的所有使得对 $e>0$，集 $\{x:\ |f(x)|\geq e\}$ 为紧集的连续实值函数的代数上去。最困难的步骤为问题 7.R 的 Stone-Weierstrass 定理，而上述的 (b) 是它的一种缩影。另外它还可以转到一种与上一段极为相似的情况，即若 X 为

Tychonoff 空间，则 X 上的有界连续函数代数的所有实同态空间同胚于 $\beta(X)$.

T 紧连通空间(链推理)

设 (X, d) 为紧的伪度量空间. 对每一个正数 e，定义从 X 的点 x 到 X 的点 y 的 e-链为一个有限的点列，它的起点为 x，终点为 y 并且相邻两点之间的距离小于 e. 另外对 X 的每一个子集 A，$C_e(A)$ 定义为所有能够通过一个 e-链与 A 的点相连接的点组成的集，又 $C(A)$ 定义为 $\bigcap\{C_e(A): e>0\}$. 一个等价的定义是：命 $V_0(A)=A$，$V_1(A)=\{x: \operatorname{dist}(x, A)<e\}$，再归纳地命 $V_{n+1}(A)=V_1(V_n(A))$，则集 $C_e(A)=\bigcup\{V_n(A): n\in\omega\}$.

(a) 对每一个 $e>0$ 和每一个集 A，集 $C_e(A)$ 为既开又闭.

(b) 若 A 为 X 的连通子集，则 $C(A)$ 为连通. 因此对每一点 x，$C(\{x\})$ 为 X 关于 x 的连通区 C_x.（若 $C(A)$ 为互不相交闭子集 B 和 D 的并，则命 $f=[\operatorname{dist}(B, D)]/3$ 并且利用问题 5.G 证明对某个正的 e 有 $C_e(A)\subset\{x: \operatorname{dist}(x, B\cup D)<f\}$.）

(c) 若 A 为 X 的子集，则 $C(A)=\bigcup\{C_x: x\in A^-\}$.（若 $x\notin C(A)$，则对某个正的 e 有 $x\notin C_e(A)$.）

(d) 将 X 分成连通区的分解为上半连续.

(e) 若 X 为连通，U 为点 x 的一个开邻域，则存在 U 的某个连通区使得它的闭包与 $X\sim U$ 相交.（若不然，则有连通区闭包的一个紧邻域 V，它包含在 U 内. 又 V 关于 x 的连通区包含在 V 的内部 V^0 内，并且利用(c)可证得存在 V 的开和闭子集分别包含 $V\sim V^0$ 和 x.）

(f) 不存在多于一个点的 X 的闭连通子集是可数多个互不相交闭子集的并.（命题(e)在这个证明中起着一种鉴别的作用. 若集 $\bigcup\{A_n: n\in\omega\}$ 为闭连通集，而集 A_n 为闭集并且互不相交，则可找到一个闭连通集，它与 A_1 不相交，但与不止一个的 A_n 相交.）

(g) 设 X 为具有通常度量的欧几里德平面的子集 $\{(x, y): x^2y^2=1\}$，则 X 为局部紧并且对每一个 $e>0$，任何两点可以通过一个 e-链相连接，但 X 不连通.

注 这个问题的结果可以很自然地推广到紧 Hausdorff（或紧正则）空间. 而齐-覆盖定理 5.27 就提供了必要的工具.

为了避免命题(e)造成对连通集性质过于乐观的印象，我们再陈述 Knaster 和 Kuratowski [1] 的经典例子，即存在欧几里德平面的连通子空间

X 和 X 的点 x 使得 $X \sim \{x\}$ 不包含连通集。

U　完满正规空间

若 $\mathscr{U}$ 为集 X 的一个子集族，x 为 X 的一个点，则 $\mathscr{U}$ 在 x 处的星形是指所有使得 x 属于其内的 $\mathscr{U}$ 中这样的元的并。我们称覆盖 $\mathscr{V}$ 为 $\mathscr{U}$ 的星形加细当且仅当所有 $\mathscr{V}$ 在 X 的点处的星形组成的族为 $\mathscr{U}$ 的一个加细。又称拓扑空间为完满正规当且仅当每一个开覆盖有开的星形加细。这时有：正则拓扑空间为完满正规当且仅当它为仿紧。（若 X 为仿紧，则由齐-覆盖性质和引理 5.30 即可给出完满正规性的一个容易的证明。另一方面，若 X 为完满正规，$\mathscr{U}$ 为一个开覆盖，而 $\mathscr{V}$ 为 $\mathscr{U}$ 的一个开的星形加细，则 $\bigcup\{V \times V: V \in \mathscr{V}\}$ 为对角线的一个邻域.）

注　完满正规性的定义属于 J. W. Tukey[1]，他证明了许多有用的性质。至于它与仿紧性的等价，则是由 A. H. Stone[1] 证明的。

V　点有限覆盖与亚紧空间

我们称 X 的子集族为点有限当且仅当 X 的每一点都属于有限多个该族的元。又称拓扑空间为亚紧当且仅当每一个开覆盖有点有限的加细。

(a) 设 $\mathscr{U}$ 为正规空间 X 的一个点有限开覆盖，则对 $\mathscr{U}$ 中的每一个 U 可选到一个开集 $G(U)$ 使得 $G(U)^- \subset U$ 并且所有集 $G(U)$ 组成的族为 X 的一个覆盖.（选满足下列条件的所有函数 F 组成的类中的一个极大元：F 的定义域为 $\mathscr{U}$ 的一个子族，并且对 F 的定义域中的每一个 U，$F(U)$ 为开集并且它的闭包包含在 U 内，同时 $\bigcup\{F(U): U \in F \text{ 的定义域}\} \cup \bigcup\{V: V \in \mathscr{U} \text{ 且 } V \notin F \text{ 的定义域}\} = X$.而从 $\mathscr{U}$ 的点有限性即可推出如此的极大元 F 必定存在.）

(b) 一个集的点有限覆盖恒有一个极小子覆盖（即一个这样的子覆盖，它不存在仍为覆盖的真子族）.

(c) 亚紧 T_1-空间为可数紧(见问题 5.E) 当且仅当它为紧。

注　命题 (b) 和 (c) 直接取之于 Arens 和 Dugundji [1]。

W　单位分解

拓扑空间 X 上的单位分解指的是从 X 到非负实数的所有这样的连续函数组成的族 F，它使得对 X 中的每一个 x 有 $\Sigma\{f(x): f \in F\} = 1$ 并且除有限多个外，所有 F 的元在 X 的每一点的某个邻域外为零。又单位分解 F 叫做从属于 X 的覆盖 $\mathscr{U}$ 当且仅当 F 的每一个元在 $\mathscr{U}$ 的某个元外为零，则：对正规空

间的每一个局部有限的开覆盖 $\mathscr{U}$ 有从属于 $\mathscr{U}$ 的单位分解. 还可以证明一个稍微更强的结果：若 $\mathscr{U}$ 为正规空间的一个局部有限开覆盖，则对 $\mathscr{U}$ 中的每一个 U 可选到一个非负连续函数 f_U 使得 f_U 在 U 外为 0 并且处处小于或等于 1，同时对一切 x 有 $\Sigma\{f_U(x): U\in\mathscr{U}\}=1$. (见上面的问题 5.V(a).)

注 就我所知,这个结果(近似的形式)是独立地属于 Hurewicz, Bochner 和 Dieudonné 的.

X 关于半连续函数的中间定理

设 g 和 h 分别为仿紧空间 X 上的下半和上半连续的实值函数,又设对 X 中的所有 x 有 $h(x)<g(x)$，则有 X 上的连续实值函数 p 使得对每一个 x 有 $h(x)<p(x)<g(x)$. (设 $\mathscr{U}$ 为 X 的所有这样开子集 U 组成的族,它使得 h 在 U 上的上确界小于 g 在 U 上的下确界，又设 F 为从属于 $\mathscr{U}$ 的单位分解. 若对 F 中的每一个 f,选 k_f 使得当 $f(x)\neq 0$ 时有 $h(x)<k_f<g(x)$，并且命 $p(x)=\Sigma\{k_f f(x): f\in F\}$，则 p 在点 x 处的值为位于 $h(x)$ 和 $g(x)$ 之间的数的平均.)

注 通过先寻找族 $\mathscr{U}$ 的一个可数加细，上述结果可以得到改进. 这时该命题对可数仿紧空间(即每一个可数开覆盖都有局部有限的加细)也成立. 而且这个定理的这种加强形式的逆也还是成立的. Dowker[2] 已经证明了下列命题的等价性：(1) X 为可数仿紧并且正规的空间，(2) X 和闭单位区间的乘积为正规空间，(3) 上述命题. Dowker 还证明了完备正规空间（正规并且每一个闭子集为一个 G_δ）恒为可数仿紧空间. 但我们还不知道正规 Hausdorff 空间是否必定为可数仿紧空间[1].

Y 仿紧空间

(a) 每一个正则 Lindelöf 空间为仿紧空间.

(b) 若定义拓扑空间为 σ 紧当且仅当它是可数多个紧子集的并,则每一个 σ 紧空间为 Lindelöf 空间.

(c) 若正则空间为 Lindelöf 子空间的一个开离散族的元之并,则它为仿

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1) 这个问题医在 1951 年由 C. H. Dowker 正式提出而命名为 Dowker 问题. 它的答案是否定的，因为在 1971 年 M. E. Rudin 巧妙地作出了一个反例，即她作出了一个非可数仿紧的正规 Hausdorff 空间（详见 M. E. Rudin, A normal space X for which $X\times I$ is not normal, *Fund. Math.* 73 (1971) 179—186).——校者注

紧空间。因而，每一个局部紧群必为仿紧。(考察所有这样的陪集组成的族，它是以包含单位元的一个确定紧邻域的最小子群为模的。)

(d) 问题1.K和4.I中的半开区间空间为正则 Lindelöf 空间，因而为仿紧。这个空间和它自己的笛卡儿乘积为非正规空间，从而为非仿紧。

(e) 具有序拓扑的所有小于第一个不可数序数的序数组成的集为非仿紧。(考察由所有形如 $\{x: x<a\}$ 的集所组成的覆盖。该覆盖的任何一个加细的每一个元的上确界小于 Ω。)

注 上述的命题(a)属于 Morita[1]。关于仿紧性更进一步的情况(F_σ定理，乘积等)见 Michael[1]。Bing[1] 研究了介于正规性和仿紧性之间的一种正规性条件。从这种联系的角度，还应当强调引理5.31说明了仿紧空间的一种值得注意的正规性的性质。

第六章 一致空间

度量空间有几条非拓扑的性质，但与拓扑的性质有着密切的联系.现在我们给出所考虑的这种联系的例子，而将定义和证明留到以后. Cauchy 序列的性质不是一个拓扑不变量，因为由 $f(x)=1/x$ 所确定的从正实数空间到它自己上的同胚 f 变 Cauchy 序列 $\{1/(n+1): n\in\omega\}$ 为非 Cauchy 序列 $\{n+1: n\in\omega\}$. 但从关于 Cauchy 序列的命题可能导出拓扑的结果，例如所有实数的空间的子集 A 为闭集当且仅当每一个 A 中的 Cauchy 序列收敛于 A 中的某个点. 另外，反过来的情况也可能出现，例如紧度量空间上的每一个连续函数为一致连续. 这时我们从一个拓扑的前提（空间为紧）导出了一个非拓扑的结论（函数为一致连续）. 本章就着重研究这种类型的拟拓扑结果.

在研究一致性性质时，我们所使用的数学构造通常叫做一致空间. 一个简短的讨论将说明这个属于 A. Weil[1] 的概念如何适用.

我们称伪度量空间 (X, d) 中的序列 $\{x_n, n\in\omega\}$ 为 Cauchy 序列当且仅当 m 和 $n\to\infty$ 时 $d(x_m, x_n)$ 收敛于零. 这个概念在任意拓扑空间内是没有意义的. 为了定义 Cauchy 序列，必须知道在某种意义下什么样的点偶 (x, y) 的距离 $d(x, y)$ 为任意小. 这一陈述可以按下列方式使它精确化. 若命 $V_{d,r}=\{(x, y): d(x, y)<r\}$，则 $\{x_n, n\in\omega\}$ 为 Cauchy 序列当且仅当对每一个正的 r 当 m 和 n 充分大时 (x_m, x_n) 为 $V_{d,r}$ 的元. 另外，一致连续的概念也可以通过所有形如 $V_{d,r}$ 的集所构成的集族来加以描述. 这样，就启发我们对集 X 和 $X\times X$ 的一个特殊的子集族的考虑.

若 X 为拓扑群，则我们称序列 $\{x_n, n\in\omega\}$ 为 Cauchy 序列当且仅当 m 和 n 充分大时 $x_m x_n^{-1}$ 接近于群的单元 e. 而且，在叙述这

一定义时所需要的知识仍是关于点偶的知识．这表明我们必须知道什么样的点偶 (x,y) 才能使得 xy^{-1} 接近于单元 e．若对 e 的每一个邻域 U，命 $V_U=\{(x,y): xy^{-1}\in U\}$，则易见所有形如 V_U 的集所组成的集族就决定了什么样的序列为 Cauchy 序列．

我们定义一致空间为一个集 X 与 $X\times X$ 的一个满足某些自然条件的子集族．显然这是来自上面两个例子所提供的模型．然而应当强调指出，这决不是研究一致结构的仅有的框架．我们也可以研究集 X 和由 X 上的伪度量所组成的一个特殊族，或标出 X 的一类一致覆盖(粗糙地说，即在 Lebesgue 覆盖引理 5.26 的意义下）的那种覆盖．我们还可以讨论取值于比实数限制更少的一种结构的“度量”．所有这些概念本质上是等价的，这将在本章末的问题中给出．

最后，还必须指出度量空间有一些一致性质不能推广到限制更少的情形．最末一节将针对其中某些性质进行研究．

一致结构和一致拓扑

现在我们开始从事一个集 X 和它自己的笛卡儿乘积 $X\times X$ 的子集的研究．这些子集都是在预备知识意义下的关系．为方便起见，我们先回顾一下在预备知识中已给出的一些定义和有关的结果．一个关系就是序偶的一个集，并且若 U 为一关系，则其逆关系 U^{-1} 是指所有使得 $(y,x)\in U$ 的 (x,y) 的集．取逆的运算在 $(U^{-1})^{-1}$ 恒为 U 的意义下是对合的．另外若 $U=U^{-1}$，则称 U 为对称的．若 U 和 V 为关系，则合成 $U\circ V$ 指的是所有满足后一条件的 (x,z) 所组成的集：存在某个 y 使 $(x,y)\in V$ 和 $(y,z)\in U$ 成立．易见合成运算满足结合律，即 $U\circ(V\circ W)=(U\circ V)\circ W$，并且恒有 $(U\circ V)^{-1}=V^{-1}\circ U^{-1}$ 成立．对 X 中的 x，所有的 (x,x) 所组成的集叫做恒等关系或对角线，并且记为 $\Delta(X)$，或简记为 Δ．对 X 的每一个子集 A，集 $U[A]$ 定义为 $\{y$: 对 A 中的某个 x 有 $(x,y)\in U\}$，若 x 为 X 的一个点，则 $U[x]$ 就是指 $U[\{x\}]$．对每一个 U，V 和每一

个 A 有 $U\circ V[A]=U[V[A]]$ 成立．最后还需要一个简单的引理．

1 引理． 若 V 为对称的，则 $V\circ U\circ V=\bigcup\{V[x]\times V[y]:(x,y)\in U\}$.

证明．因为由定义 $V\circ U\circ V$ 是所有满足后一条件的 (u,v) 的集：对某个 x 和某个 y 有 $(u,x)\in V$，$(x,y)\in U$ 和 $(y,v)\in V$ 成立，又由于 V 为对称的，故它就是所有满足后一条件的 (u,v)的集：对 U 中的某个 (x,y) 有 $u\in V[x]$ 和 $v\in V[y]$ 成立．但 $u\in V[x]$ 和 $v\in V[y]$ 当且仅当 $(u,v)\in V[x]\times V[y]$，从而 $V\circ U\circ V=\{(u,v)$: 对 U 中的某个 (x,y) 有 $(u,v)\in V[x]\times V[y]\}=\bigcup\{V[x]\times V[y]:(x,y)\in U\}$. |

集 X 的一个**一致结构**是 $X\times X$ 的一个非空子集族 $\mathscr{U}$ 并且它满足：

(a) $\mathscr{U}$ 的每一个元包含对角线 Δ;

(b) 若 $U\in\mathscr{U}$，则 $U^{-1}\in\mathscr{U}$;

(c) 若 $U\in\mathscr{U}$，则对 $\mathscr{U}$ 中的某个 V 有 $V\circ V\subset U$;

(d) 若 U，V 为 $\mathscr{U}$ 的元，则 $U\cap V\in\mathscr{U}$;

(e) 若 $U\in\mathscr{U}$ 并且 $U\subset V\subset X\times X$，则 $V\in\mathscr{U}$.

又 $(X,\mathscr{U})$ 叫做**一致空间**.

根据度量的概念，我们不难看清上述的各个条件的度量根源．其中的第一条是从条件 $d(x,x)=0$ 导出的，而第二条则是从对称条件 $d(x,y)=d(y,x)$ 导出的．第三条是三角不等式的一种退化的形式——粗糙地说，即对 r-球恒有 $r/2$-球．第四条和第五条相似于一点邻域系的公理，并且它将要用来导出关于后面所要定义的一种拓扑的邻域系的相应性质.

对于一个集 X 可以有许多不同的一致结构．其中的最大者是所有包含 Δ 的 $X\times X$ 的子集所组成的集族，而最小者是 $X\times X$ 为仅有的元的集族．若 X 为实数集，则 X 的**通常一致结构**是指所有满足后一条件的子集 U 所组成的集族 $\mathscr{U}$：对某个正数 r 有 $\{(x,y):|x-y|<r\}\subset U$．$\mathscr{U}$ 的每一个元都是对角线 Δ (方程为

$y=x$ 的直线)的邻域，然而，对角线的每一个邻域并不都是 $\mathscr{U}$ 的元．例如，集 $\{(x,y): |x-y|<1/(1+|y|)\}$ 是 Δ 的邻域，但就不是 $\mathscr{U}$ 的元．

显然 X 的两个一致结构的并或交仍为一个一致结构的结论并不普遍成立，但一族一致结构的并可以在一种相当自然的意义下生成一个一致结构．我们称一致结构 $\mathscr{U}$ 的子族 $\mathscr{B}$ 为 $\mathscr{U}$ 的**基**当且仅当 $\mathscr{U}$ 的每一个元都包含有 $\mathscr{B}$ 的一个元．若 $\mathscr{B}$ 为 $\mathscr{U}$ 的一个基，则 $\mathscr{B}$ 就完全决定了 $\mathscr{U}$，因为 $X\times X$ 的子集 U 属于 $\mathscr{U}$ 当且仅当 U 包含有 $\mathscr{B}$ 的一个元．称子族 $\mathscr{S}$ 为 $\mathscr{U}$ 的**子基**当且仅当 $\mathscr{S}$ 的元的所有有限交的族为 $\mathscr{U}$ 的基．这些定义完全相似于拓扑的基和子基的定义．

2 定理． $X\times X$ 的子集族 $\mathscr{B}$ 为 X 的某个一致结构的基当且仅当

(a) $\mathscr{B}$ 的每一个元包含对角线 Δ；

(b) 若 $U\in\mathscr{B}$，则 U^{-1} 包含有 $\mathscr{B}$ 的一个元；

(c) 若 $U\in\mathscr{B}$，则对 $\mathscr{B}$ 中的某个 V 有 $V\circ V\subset U$；

(d) $\mathscr{B}$ 的两个元的交包含 $\mathscr{B}$ 的一个元．

我们略去这个命题的直接的证明．

作为某个一致结构的子基的刻划并不能也这样容易地给出．然而，下列的简单结果可以满足我们的需要．

3 定理． $X\times X$ 的子集族 $\mathscr{S}$ 为 X 的某个一致结构的子基，假如

(a) $\mathscr{S}$ 的每一个元包含对角线 Δ，

(b) 对 $\mathscr{S}$ 中的每一个 U，集 U^{-1} 包含有 $\mathscr{S}$ 的一个元，

(c) 对 $\mathscr{S}$ 中的每一个 U 有 $\mathscr{S}$ 中的 V 使得 $V\circ V\subset U$．

特别，任意一族 X 的一致结构的并是 X 的某个一致结构的子基．

证明．显然，我们只须证 $\mathscr{S}$ 的元的所有有限交组成的族 $\mathscr{B}$ 满足定理 6.2 的条件．而这一点从下列事实容易推得：若 $U_1,\cdots,U_n$ 和 $V_1,\cdots,V_n$ 是 $X\times X$ 的子集，并设 $U=\bigcap\{U_i: i=1,\cdots,$

$n\}$，$V=\bigcap\{V_i: i=1,\cdots,n\}$，则当 $V_i\subset U_i^{-1}$（相应地 $V_i\circ V_i\subset U_i$）对每一个 i 都成立时，有 $V\subset U^{-1}$（相应地 $V\circ V\subset U$）. |

若$(X,\mathscr{U})$为一致空间，则**一致结构$\mathscr{U}$的拓扑**或**一致拓扑$\mathscr{T}$**是指所有满足后一条件的 X 的子集 T 所构成的集族：对 T 中每一个 x 有 $\mathscr{U}$ 中一个元 U 满足 $U[x]\subset T$.（这恰好是度量拓扑的推广，度量拓扑就是所有包含以其内每一点为球心的一个球的那种集所组成的集族.）我们必须证明 $\mathscr{T}$ 的确是一个拓扑，但这是不难的：根据定义 $\mathscr{T}$ 的元的并确实也是 $\mathscr{T}$ 的元；若 T 和 S 为 $\mathscr{T}$ 的元，$x\in T\cap S$，则有 $\mathscr{U}$ 中的 U 和 V 使得 $U[x]\subset T$ 并且 $V[x]\subset S$，从而 $(U\cap V)[x]\subset T\cap S$，于是 $T\cap S\in\mathscr{T}$；即 $\mathscr{T}$ 为一个拓扑.

现在我们来考察一致结构和一致拓扑之间的关系.

4 定理. X 的子集 A 关于一致拓扑的内部是所有符合条件：对 $\mathscr{U}$ 中的某个 U 有 $U[x]\subset A$ 的点 x 的集.

证明. 我们只须证集 $B=\{x$：对 $\mathscr{U}$ 中的某个 U 有 $U[x]\subset A\}$ 关于一致拓扑为开集，因为 B 必定包含 A 的每一个开子集，故若 B 为开集，则它必定就是 A 的内部. 若 $x\in B$，则有 $\mathscr{U}$ 的元 U 使得 $U[x]\subset A$，同时还有 $\mathscr{U}$ 中的 V 使得 $V\circ V\subset U$；于是若 $y\in V[x]$，则 $V[y]\subset V\circ V[x]\subset U[x]\subset A$，即 $y\in B$；从而 $V[x]\subset B$，亦即 B 为开集. |

因为显然对一致结构 $\mathscr{U}$ 中的每一个 U，$U[x]$ 是 x 的邻域，故所有集 $U[x]$，$U\in\mathscr{U}$ 的族是 x 的邻域系的一个基（实际上该族与邻域系相同，但这一点不很重要）. 于是下列命题是明显的.

5 定理. 若 $\mathscr{B}$ 为一致结构 $\mathscr{U}$ 的一个基（相应地子基），则对每一个 x 和 U 属于 $\mathscr{B}$ 所有集 $U[x]$ 的族是 x 的邻域系的一个基（相应地子基）.

X 的一致拓扑可以用来作出 $X\times X$ 的一个乘积拓扑. 正如我们所期望的那样，一致结构的元关于乘积拓扑有一种特殊的结构.

6 定理. 若 U 为一致结构 $\mathscr{U}$ 的一个元，则 U 的内部亦为 $\mathscr{U}$ 的一个元；因而，$\mathscr{U}$ 的所有开对称的元所组成的集族为 $\mathscr{U}$ 的一个

基.

证明. 因为 $X\times X$ 的子集 M 的内部为所有符合条件：对 $\mathscr{U}$ 中的某对 U 和 V，有 $U[x]\times V[y]\subset M$ 的 (x, y) 的集，又因 $U\cap V\in\mathscr{U}$，故 M 的内部为 $\{(x, y)$：对 $\mathscr{U}$ 中的某个 V，有 $V[x]\times V[y]\subset M\}$. 若 $U\in\mathscr{U}$，则有 $\mathscr{U}$ 的对称的元 V 使得 $V\circ V\circ V\subset U$，再根据引理 6.1 有 $V\circ V\circ V=\bigcup\{V[x]\times V[y]: (x,y)\in V\}$. 因此，$V$ 的每一点都是 U 的内点，即 U 的内部包含 V，从而为 $\mathscr{U}$ 的一个元.|

根据上一定理，一致结构的每一个元都是对角线的邻域. 但应该强调指出，该命题的逆并不成立. 实际上，X 可以有许多很不相同的一致结构，而它却具有相同的拓扑，因而对角线也具有相同的邻域系.

7 定理. X 的子集 A 关于一致拓扑的闭包为 $\bigcap\{U[A]: U\in\mathscr{U}\}$. $X\times X$ 的子集 M 的闭包为 $\bigcap\{U\circ M\circ U: U\in\mathscr{U}\}$.

证明. 因为点 x 属于 X 的子集 A 的闭包当且仅当对 $\mathscr{U}$ 中的每一个 U，$U[x]$ 与 A 相交，但 $U[x]$ 与 A 相交当且仅当 $x\in U[A]$，又 $\mathscr{U}$ 的每一个元包含有一个对称的元，故 $x\in A^-$ 当且仅当对 $\mathscr{U}$ 中的每一个 U 有 $x\in U[A]$. 从而第一个命题获证.

类似地，若 U 为 $\mathscr{U}$ 的一个对称的元，则 $U[x]\times U[y]$ 与 $X\times X$ 的子集 M 相交当且仅当对 M 中的某个 (u, v) 有 $(x, y)\in U[u]\times U[v]$，即当且仅当 $(x, y)\in\bigcup\{U[u]\times U[v]: (u, v)\in M\}$. 因为由引理 6.1，最后的一个集为 $U\circ M\circ U$，故得 $(x, y)\in M^-$ 当且仅当 $(x, y)\in\bigcap\{U\circ M\circ U: U\in\mathscr{U}\}$.|

8 定理. 一致结构 $\mathscr{U}$ 的所有闭对称的元所构成的集族为 $\mathscr{U}$ 的一个基.

证明. 若 $U\in\mathscr{U}$，则 $\mathscr{U}$ 有一个元 V 使得 $V\circ V\circ V\subset U$，故由上一定理便知 $V\circ V\circ V$ 包含 V 的闭包；从而 U 包含 $\mathscr{U}$ 的一个闭的元 W 同时 $W\cap W^{-1}$ 即为 $\mathscr{U}$ 的一个闭对称的元.|

稍后，我们将要证明一致空间(更精确地说，就是带有一致拓扑的空间)恒为全正则空间. 而目前容易看出这样的空间必为正则

空间，因为点 x 的每一个邻域包含一个邻域 $V[x]$ 使得 V 为 $\mathscr{U}$ 的一个闭元，从而 $V[x]$ 为闭集. 因此，具有一致拓扑的空间为 Hausdorff 空间当且仅当每一个由单个点所组成的集为闭集. 又由于集 $\{x\}$ 的闭包为 $\bigcap\{U[x]: U\in\mathscr{U}\}$，所以该空间为 Hausdorff 空间当且仅当 $\bigcap\{U: U\in\mathscr{U}\}$ 为对角线 Δ. 在这种情况下 $(X,\mathscr{U})$ 就叫做 **Hausdorff** 或**分离一致空间**.

一致连续性；乘积一致结构

若 f 为定义在一致空间 $(X,\mathscr{U})$ 上并且取值于一致空间 $(Y,\mathscr{V})$ 内的函数，则称 f 关于 $\mathscr{U}$ 和 $\mathscr{V}$ 为**一致连续**当且仅当对 $\mathscr{V}$ 中的每一个 V，集 $\{(x,y): (f(x),f(y))\in V\}$ 为 $\mathscr{U}$ 的一个元. 这个条件还可以叙述成另外的几种形式. 对每一个从 X 到 Y 的函数 f，命 f_2 为由 $f_2(x,y)=(f(x),f(y))$ 所定义的从 $X\times Y$ 到 $X\times Y$ 的函数，则 f 为一致连续当且仅当对 $\mathscr{V}$ 中的每一个 V 有 $\mathscr{U}$ 中的 U 使得 $f_2[U]\subset V$. 我们还有：若 $\mathscr{S}$ 为 $\mathscr{V}$ 的一个子基，则 f 为一致连续当且仅当对 $\mathscr{S}$ 中的每一个 V 有 $f_2^{-1}[V]\in\mathscr{U}$（注意，$f_2^{-1}$ 保存并与交）. 若 Y 为实数集并且 $\mathscr{V}$ 为通常一致结构，则可推出：f 为一致连续当且仅当对每一个正数 r 有 $\mathscr{U}$ 中的 U 使得当 $(x,y)\in U$ 时有 $|f(x)-f(y)|<r$. 若 X 也是带有通常一致结构的实数空间，则 f 为一致连续当且仅当对每一个正数 r 有正数 s 使得当 $|x-y|<s$ 有 $|f(x)-f(y)|<r$.

显然，若 f 为从 X 到 Y 的函数，g 为 Y 上的函数，则 $(g\circ f)_2=g_2\circ f_2$，并且由此可推出两个一致连续函数的合成也为一致连续函数. 若 f 为从 X 到 Y 上的一对一映射并且 f 和 f^{-1} 为一致连续，则称 f 为**一致同构**并且空间 X 和 Y（更精确地说，是 $(X,\mathscr{U})$ 和 $(Y,\mathscr{V})$）叫做**一致等价**. 不难看出，两个一致同构的合成，一个一致同构的逆以及从一个空间到它自己上的恒等映射均为一致同构. 因而所有一致空间的全体可以分成由一致等价空间所组成的等价类. 一个性质，如果当它为一个一致空间所具有时，也为每一个一

致等价空间所具有．那末就叫做是一个一致不变性(量)．而本章除一些个别情形外所研究的性质都是一致不变性(量)．

我们可以预料一致连续性蕴含关于一致拓扑的连续性．这就是

9 定理． 每一个一致连续函数关于一致拓扑为连续，因而每一个一致同构为一个同胚．

证明．设 f 为从 $(X, \mathscr{U})$ 到 $(Y, \mathscr{V})$ 的一个一致连续函数，U 为 $f(x)$ 的邻域，则有 $\mathscr{V}$ 中的 V 使得 $V[f(x)]\subset U$，并且 $f^{-1}[V[f(x)]] = \{y: f(y)\in V[f(x)]\} = \{y: (f(x), f(y))\in V\} = f_2^{-1}[V](x)$，而这是 x 的一个邻域．故 $f^{-1}[U]$ 是 x 的邻域，从而连续性获证．|

若 f 是从集 X 到一致空间 $(Y, \mathscr{V})$ 的函数，则所有集 $f_2^{-1}[V]$，其中 V 属于 $\mathscr{V}$，的族为 X 的一个一致结构的结论并不成立．困难在于可以有 $X\times X$ 的子集，它包含某个集 $f_2^{-1}[V]$，但却不是 $Y\times Y$ 的任何子集的逆．然而这个困难并不很大；我们能够证明所有 $f_2^{-1}[V]$ 的族是 X 的某个一致结构 $\mathscr{U}$ 的基．显然，f_2^{-1} 保存包含关系，交和逆(即 $f_2^{-1}[V^{-1}] = [f_2^{-1}[V]]^{-1}$) 的运算，因而，只须证明：对 $\mathscr{V}$ 的每一个元 U 有 $\mathscr{V}$ 中的 V 使得 $f_2^{-1}[V]\circ f_2^{-1}[V]\subset f_2^{-1}[U]$．但若 $V\circ V\subset U$ 并且 (x, y) 和 (y, z) 属于 $f_2^{-1}[V]$，则 $(f(x), f(y))$ 和 $(f(y), f(z))$ 都属于 V，故 $(f(x), f(z))\in V\circ V$．这样就推出了所有 $\mathscr{V}$ 的元的逆所构成的族的确是 X 的一个一致结构 $\mathscr{U}$ 的基．易见 f 关于 $\mathscr{U}$ 和 $\mathscr{V}$ 为一致连续，并且事实上 $\mathscr{U}$ 还小于每一个使得 f 为一致连续的其它的一致结构．

若 $(X, \mathscr{U})$ 为一致空间，Y 为 X 的一个子集，则由上述讨论便知有一个最小的一致结构 $\mathscr{V}$ 使得从 Y 到 X 内的恒等映射为一致连续．显然，$\mathscr{V}$ 的元就是 $\mathscr{U}$ 的元与 $Y\times Y$ 的交(有时叫做 $\mathscr{U}$ 在 $Y\times Y$ 上的迹)．上述的一致结构 $\mathscr{V}$ 叫做 $\mathscr{U}$ 对 Y 的**相对化**，或 Y 上的**相对一致结构**，而 $(Y, \mathscr{V})$ 就叫做空间 $(X, \mathscr{U})$ 的一致子空间．在这里我们略去相对一致结构 $\mathscr{V}$ 的拓扑恰为 $\mathscr{U}$ 的拓扑的相对化这个事实的简单证明．

上面我们已经看到，总有一个唯一的最小一致结构，它使得从集 X 到某个一致空间内的一个映射为一致连续．这个命题也可以推广到函数族 F 的情形，其中 F 的每一个元 f 映 X 到一致空间 $(Y_f, \mathscr{U}_f)$ 内．即所有形如 $f_2^{-1}[U] = \{(x,y): (f(x),f(y)) \in U\}$ 的集，其中 $f \in F$，$U \in \mathscr{U}_f$，的族是 X 的某个一致结构 $\mathscr{U}$ 的子基，同时 $\mathscr{U}$ 也就是使得每一个映射 f 为一致连续的最小一致结构（定理 6.3 证明了所有形如 $f_2^{-1}[U]$ 的集，其中 $f \in F$，$U \in \mathscr{U}_f$，的族是某个一致结构的子基，又易见 $\mathscr{U}$ 使得每一个 f 为一致连续并且还小于每一个具有这个性质的一致结构）．按照这个方法，我们就可以定义出乘积一致结构．若对指标集 A 的每一个元 a，$(X_a, \mathscr{U}_a)$ 为一致空间，则 $\times\{X_a: a \in A\}$ 的**乘积一致结构**是指使得到每一个坐标空间内的射影为一致连续的最小一致结构．这时所有形如 $\{(x, y): (x_a, y_a) \in U\}$ 的集，其中 a 属于 A，U 属于 $\mathscr{U}_a$，的族是乘积一致结构的一个子基．因为若 x 为乘积空间的一个元，则 x 的邻域系（关于一致拓扑）的一个子基可从乘积一致结构的上述子基作出，故所有形如 $\{y: (x_a, y_a) \in U\}$ 的集组成的族是 x 的邻域系的一个子基．这就推出了 x 关于乘积一致结构的拓扑的邻域系的一个基为所有形如 $\{y: y_a \in U[x_a]\}$ 的集的有限交组成的族，其中 a 属于 A，U 属于 $\mathscr{U}_a$．但该族也是 x 关于乘积拓扑的邻域系的一个基，因而乘积拓扑就是乘积一致结构的拓扑．这也就是下面定理的第一部分．

10 定理．乘积一致结构的拓扑为乘积拓扑．

从一个一致空间到一族一致空间的乘积空间的函数 f 为一致连续当且仅当 f 与每一个到坐标空间内的射影的合成为一致连续．

证明．若取值于乘积 $\times\{X_a: a \in A\}$ 的 f 为一致连续，则每一个射影 P_a 为一致连续，故合成 $P_a \circ f$ 也为一致连续．

若对每一个 A 中的 a，$P_a \circ f$ 为一致连续并且 U 为 X_a 的一致结构的元，则 $\{(u,v): (P_a \circ f(u), P_a \circ f(v)) \in U\}$ 为 f 的定义域的一致结构 $\mathscr{V}$ 的元，但该集可写成 $f_2^{-1}[\{(x, y): (x_a, y_a) \in U\}]$ 的形式，

故乘积一致结构的一个子基的每一个元关于f_i的逆属于$\mathscr{V}$，即f为一致连续.|

从下一命题起，我们开始讨论关于X的一致结构和伪度量之间的关系.

11 定理. 设$(X,\mathscr{U})$为一致空间，d为X的伪度量，则d在$X\times X$上关于乘积一致结构为一致连续当且仅当对每一个正数r，集$\{(x,y): d(x,y)<r\}$是$\mathscr{U}$的一个元.

证明. 命$V_{d,r}=\{(x,y): d(x,y)<r\}$，则只须证对每一个正的$r$有$V_{d,r}\in\mathscr{U}$当且仅当$d$关于$X\times X$的乘积一致结构为一致连续.

若U为$\mathscr{U}$的元，则集$\{((x,y),(u,v)): (x,u)\in U\}$和$\{((x,y),(u,v)): (y,v)\in U\}$属于乘积一致结构，并且容易看出所有形如$\{((x,y),(u,v)): (x,u)\in U$ 且 $(y,v)\in U\}$的集组成的族是乘积一致结构的一个基. 故若d为一致连续，则对每一个正的r有$\mathscr{U}$中的U使得当(x,u)和(y,v)属于U时有$|d(x,y)-d(u,v)|<r$. 特别取$(u,v)=(y,y)$便知当$(x,y)\in U$时有$d(x,y)<r$，即$U\subset V_{d,r}$，因而$V_{d,r}\in\mathscr{U}$.

今证其逆，注意若(x,u)和(y,v)均属于$V_{d,r}$，则$|d(x,y)-d(u,v)|<2r$，这是因为$d(x,y)\leqslant d(x,u)+d(u,v)+d(y,v)$，$d(u,v)\leqslant d(x,u)+d(x,y)+d(y,v)$. 由此即可推出，若对每一个$r$有$V_{d,r}\in\mathscr{U}$，则$d$为一致连续.|

度　量　化

本节的目的是比较一致空间和伪度量空间. 这种比较是检验一种推广的有效性的典型方法的一个例子. 将推广对象与被推广的数学对象进行对比，以期发现这些基本概念被分离的程度. 在此情形(和许多其它的情形一样)从比较就可得到推广了的对象用它的原始对象的一种表示. 集X的每一个伪度量的族都确定一个一致结构. 这一节的主要结果是每一个一致结构均可按这种方式从

它的一致连续伪度量的族导出．另外还将证明一个一致结构能够由单个的伪度量所导出的充要条件为该一致结构有可数基．

集 X 的每一个伪度量 d 按照下面方法都生成一个一致结构．对每一个正数 r，命 $V_{d,r}=\{(x,y): d(x,y)<r\}$，则易见 $(V_{d,r})^{-1}=V_{d,r}$，$V_{d,r}\cap V_{d,s}=V_{d,t}$，其中 $t=\min[r,s]$，并且 $V_{d,r}\circ V_{d,r}\subset V_{d,2r}$，这就推出了所有形如 $V_{d,r}$ 的集的族是 X 的某个一致结构的基，这一致结构叫做**伪度量一致结构或由 d 所生成的一致结构**．我们称一致空间$(X,\mathscr{U})$为**可伪度量化的**(或**可度量化的**)当且仅当有伪度量(或度量) d 使得$\mathscr{U}$为由 d 所生成的一致结构．由伪度量 d 所生成的一致结构也可按另外的方法加以描述．根据定理 6.11，伪度量 d 关于一致结构 $\mathscr{V}$(更精确地说，关于由 $\mathscr{V}$ 所作出的乘积一致结构)为一致连续当且仅当对每一个正的 r 有 $V_{d,r}\in\mathscr{V}$，于是由 d 所生成的一致结构$\mathscr{U}$可以刻划为：使得 d 在 $X\times X$ 上一致连续的最小一致结构．我们应注意到伪度量拓扑与$\mathscr{U}$的一致拓扑相同，因为 $V_{d,r}[x]$ 是以 x 为心的开 r-球并且所有这种形式的集组成的族同时是关于这两个拓扑的 x 的邻域系的基．

下面的引理给出了一致空间度量化定理的决定性的一步．

12 度量化引理．设 $\{U_n, n\in\omega\}$ 为 $X\times X$ 的子集的序列，合于：$U_0=X\times X$，每一个 U_n 都包含对角线，并且对每一个 n 有 $U_{n+1}\circ U_{n+1}\circ U_{n+1}\subset U_n$，则有 $X\times X$ 上的一个非负实值函数 d 使得

(a) 对一切 x，y 和 z 有 $d(x,y)+d(y,z)\geqslant d(x,z)$；

(b) 对每一个正整数有 $U_n\subset\{(x,y): d(x,y)<2^{-n}\}\subset U_{n-1}$．

若每一个 U_n 为对称，则有一个伪度量 d 满足条件 (b)．

证明．若命 $f(x,y)=2^{-n}$ 当 $(x,y)\in U_{n-1}\sim U_n$ 时，$f(x,y)=0$ 当 (x,y) 属于每一个 U_n 时，则 f 为 $X\times X$ 上的一个实值函数．所需的函数 d 可借助于链推理从它的“第一次近似” f 作出．对 X 中的每一个 x 和每一个 y，命 $d(x,y)$ 为 $\sum\{f(x_i,x_{i+1}): i=0,\cdots,n\}$ 在所有使得 $x=x_0$ 且 $y=x_{n+1}$ 的有限序列 x_0，$x_1,\cdots,x_{n+1}$ 上的下确界．显然，d 满足三角不等式并且从 $d(x,

$y) \leqslant f(x, y)$ 可推出 $U_n \subset \{(x, y): d(x, y) < 2^{-n}\}$. 若每一个 U_n 为对称，则对每一对 (x, y) 有 $f(x, y) = f(y, x)$，因而在这种情况下 d 为一个伪度量.

今证 $f(x_0, x_{n+1}) \leqslant 2\sum\{f(x_i, x_{i+1}): i = 0, \cdots, n\}$，从而就完成了证明. 事实上，从它可推出若 $d(x, y) < 2^{-n}$，则 $f(x, y) < 2^{-n+1}$，即 $(x, y) \in U_{n-1}$，故 $\{(x, y): d(x, y) < 2^{-n}\} \subset U_{n-1}$. 而这个事实的证明则是通过关于 n 的归纳法，并且注意当 $n = 0$ 时不等式显然成立. 为方便起见，我们称数 $\Sigma\{f(x_i, x_{i+1}): i = r, \cdots, s\}$ 为链从 r 到 $s + 1$ 的长度，并且以 a 表示链从 0 到 $n + 1$ 的长度. 设 k 为使得链从 0 到 k 的长度至多为 $a/2$ 的最大整数并且注意此时链从 $k + 1$ 到 $n + 1$ 的长度也至多为 $a/2$，则由归纳假设可知 $f(x_0, x_k)$ 和 $f(x_{k+1}, x_{n+1})$ 的每一个都至多为 $2(a/2) = a$，又 $f(x_k, x_{k+1})$ 也至多为 a，故若命 m 为使得 $2^{-m} \leqslant a$ 的最小整数，则 (x_0, x_k)，(x_k, x_{k+1}) 和 (x_{k+1}, x_{n+1}) 都属于 U_m，从而 $(x_0, x_{n+1}) \in U_{m-1}$，即 $f(x_0, x_{n+1}) \leqslant 2^{-m+1} \leqslant 2a$，于是引理获证. ∎

若 X 的一致结构 $\mathscr{U}$ 有可数基 $V_0, V_1, \cdots, V_n, \cdots$，则用归纳法可作出 $U_0, U_1, \cdots, U_n, \cdots$ 使得每一个 U_n 为对称，$U_n \circ U_n \circ U_n \subset U_{n-1}$ 并且对每一个正整数 n 有 $U_n \subset V_n$. 所有集 U_n 的族为 $\mathscr{U}$ 的一个基，并且应用度量化引理可推出一致空间 $(X, \mathscr{U})$ 为可伪度量化的. 因而有：

13 度量化定理. 一致空间为可伪度量化当且仅当它的一致结构有可数基.

这个定理显然蕴含一致空间为度量化当且仅当它为 Hausdorff 空间并且它的一致结构有可数基.

14 注记. 据我所知这个定理首先出现于 Alexandroff 和 Urysohn [2]. 这两位作者的目的是寻找拓扑度量化问题的解决(定理 4.18)，并且他们的结果可(近似地)陈述为：Hausdorff 空间 $(X, \mathscr{T})$ 为可度量化的当且仅当存在一个具有可数基的一致结构使得 $\mathscr{T}$ 为一致拓扑. 这对拓扑度量化问题虽然是一种不够满意的解决，但(带有稍许加强的结论)却恰好是一致空间的度量化定理.

Chittenden[1] 首先证明了定理 6.13 的一种"一致"形式并且它的证明后来由 A. H. Frink[1] 以及 Aronszajn[1] 大大地简化. 上面的证明则是 Bourbaki 对 Frink 的证明所作的一种整理. 定理 6.13 的上述形式首先出现于 André Weil 的经典著作[1],在其中他引进了一致空间的概念.

又集 X 的每一个伪度量的族 P 按照下面方法也都生成一个一致结构. 命 $V_{p,r}=\{(x, y): p(x, y)<r\}$, 则所有形如 $V_{p,r}$ 的集的族是 X 的某个一致结构$\mathscr{U}$的子基, 其中 p 属于 P, r 为正数. 该一致结构$\mathscr{U}$就叫做**由 P 所生成的一致结构**. 它还可以用几种别的有益的方法来加以描述. 根据定理6.11伪度量 p 在 $X\times X$ 上关于由$\mathscr{V}$所生成的乘积一致结构为一致连续当且仅当对每一个正的 r 有 $V_{p,r}\in\mathscr{V}$, 因而由 P 所生成的一致结构就是使得 P 的每一个元 p 在 $X\times X$ 上为一致连续的最小一致结构. 现在再给出它的一种描述. 因为对 P 的确定的元 p, 所有集 $V_{p,r}$ 的族是伪度量空间 (X, p) 的一致结构的一个基,其中 r 为正数,故若$\mathscr{V}$为 X 的一个一致结构,则从 $(X,\mathscr{V})$ 到 (X, p) 内的恒等映射为一致连续当且仅当对每一个正的 r 有 $V_{p,r}\in\mathscr{V}$, 由此便知一致结构$\mathscr{U}$即为使得对 P 中的每一个 p, 从 X 到 (X, p) 内的恒等映射为一致连续的最小一致结构. 从这个事实我们又可引出它的另一种描述. 设 Z 为乘积 $\times\{X: p\in P\}$ (即 X 和它自己的与 P 的元一样多次的乘积), 并且 f 为由 $f(x)_p=x$ 所定义的从 X 到 Z 内的映射, 其中 x 属于 X, p 属于 P, 设该乘积的第 p 个坐标空间带有伪度量 p 的一致结构并且 Z 具有乘积一致结构, 则从 Z 到第 p 个坐标空间内的射影为从 X 到伪度量空间 (X, p) 上的恒等映射,故由定理 6.10 可推出由 P 所生成的一致结构使: 从 X 到 Z 内的映射为一致连续的最小一致结构;但 f 为一对一,从而为从 X 到伪度量空间乘积的某个子空间上的一致同构.

显然,知道什么样的一致结构是由伪度量的族所生成,有一定的重要性, 这也可以称为关于一致空间的广义度量化问题. 而它的解决则是上面的一些结果的一种应用. 设 $(X,\mathscr{U})$ 为一致空

间，P为所有在 $X\times X$ 上为一致连续的X的伪度量的族，则由定理 6.11 可知由 P 所生成的一致结构小于$\mathscr{U}$；但度量化引理 6.12 证明了对$\mathscr{U}$的每一个U有 P的元 p 使得$\{(x,y):p(x,y)<1/4\}$包含在U内，即$\mathscr{U}$小于由 P 所生成的一致结构．于是有：

15 定理． X的每一个一致结构由所有在 $X\times X$ 上为一致连续的伪度量的族所生成．

上一定理有一个有趣的系．我们已经知道若X的一致结构$\mathscr{U}$由伪度量的族 P 所生成，则该空间一致同构于伪度量空间的乘积的某个子空间，并且当$(X,\mathscr{U})$为 Hausdorff 空间时这个结果还可以得到加强．因为一致结构$\mathscr{U}$是使得对 P 中的每一个 p，从X到伪度量空间(X,p)的恒等映射为一致连续的最小一致结构，又由定理 4.15 空间(X,p)关于某个映射h_p等距于度量空间(X_p,p^*)，故$\mathscr{U}$是使得每一个映射h_p为一致连续的最小一致结构，从而若通过$h(x)_p=h_p(x)$定义一个从X到$\times\{X_p:\ p\in P\}$内的映射h，则由定理 6.10 便知$\mathscr{U}$是使得h为一致连续的最小一致结构．特别，当$(X,\mathscr{U})$为 Hausdorff 空间时h必为一对一，于是在这种情况下h就是一个一致同构．因而从上一定理也就推出了下面的结果(Weil[1])．

16 定理． 每一个一致空间都一致同构于伪度量空间的乘积的一个子空间，又每一个一致 Hausdorff 空间都一致同构于度量空间的乘积的一个子空间．

上一定理给出了能够是某个一致结构的一致拓扑的那些拓扑的一种刻划，因为拓扑空间为全正则空间当且仅当它同胚于伪度量空间的乘积的一个子空间(问题 4.L)．

17 系． 集X的拓扑$\mathscr{T}$为X的某个一致结构的一致拓扑当且仅当拓扑空间$(X,\mathscr{T})$为全正则空间．

在这一节剩下的部分，我们来澄清一致结构和伪度量之间的关系．我们称集X的一个伪度量的族 P 为一个**格集**(或**规范族**)当且仅当有X的一个一致结构$\mathscr{U}$使得P恰好就是所有在$X\times X$上关于由$\mathscr{U}$所导出的乘积一致结构为一致连续的伪度量的族．族 P 叫

做一致结构$\mathscr{U}$的格集，而$\mathscr{U}$则叫做 P 的一致结构（按照定理 6.15，$\mathscr{U}$由 P 所生成）．每一个伪度量的族都生成一个一致结构，它也说成生成该一致结构的格集．我们还可以给出由一族伪度量 P 所生成的格集的一种直接的描述．因为所有形如 $V_{p,r}$ 的集的族是该格集的一致结构的子基，其中 p 属于 P，r 为正数，故伪度量 q 在乘积上为一致连续当且仅当对每一个正数 s，集 $V_{q,s}$ 包含集 $V_{p,r}$ 的某个有限交，其中 p 属于 P．通过这段说明建立了下面的命题．

18 定理．设 P 为集 X 的一族伪度量，又设 Q 为由 P 所生成的格集，则伪度量 q 属于 Q 当且仅当对每一个正数 s 有正数 r 和 P 的一个有限子族 $\{p_1, \cdots, p_n\}$ 使得 $\bigcap\{V_{p_i,r}: i=1,\cdots,n\}\subset V_{q,s}$．

注意每一个在一致结构概念的基础上的概念均可借助于一个格集加以描述，因为每一个一致结构由它的格集所完全确定．而下面的定理就是这样的描述的一种汇集．回忆一下：p-$\operatorname{dist}(x, A)=\inf\{p(x, y): y\in A\}$ 是从点 x 到集 A 的 p-距离．

19 定理．设 $(X,\mathscr{U})$ 为一致空间，P 为$\mathscr{U}$的格集，则：

(a) 所有集 $V_{p,r}$ 的族为一致结构$\mathscr{U}$的一个基，其中 p 属于 P，r 为正数．

(b) X 的子集 A 关于一致拓扑的闭包为所有使得对 P 中的每一个 p 有 p-$\operatorname{dist}(x, A)=0$ 的 x 的集．

(c) 集 A 的内部为所有使得对 P 中的某一个 p 和某一个正数 r 有球 $V_{p,r}[x]\subset A$ 的点 x 的集．

(d) 假设P'为 P 的一个子族它生成 P，则 X 中的网$\{S_n, n\in D\}$收敛于点 s 当且仅当对每一个P'中的 p，$\{p(S_n, s), n\in D\}$ 收敛于零．

(e) 从 X 到一致空间 $(Y,\mathscr{V})$ 的函数 f 为一致连续当且仅当对$\mathscr{V}$的格集 Q 的每一个元 q 有 $q\circ f_2\in P$．（回忆一下，$f_2(x, y)=(f(x), f(y))$．）

等价地，f 为一致连续当且仅当对 Q 中的每一个 q 和每一个正数 s 有 P 中的 p 和正数 r 使得当 $P(x,y)<r$ 时 $q(f(x), f(y))<s$．

(f) 若对指标集 A 的每一个元 a，$(X_a, \mathscr{U}_a)$ 为一致空间，P_a 为 $\mathscr{U}_a$ 的格集，则 $\times\{X_a: a \in A\}$ 的乘积一致结构的格集是由所有形如 $q(x, y) = p_a(x_a, y_a)$ 的伪度量所生成，其中 a 属于 A，p_a 属于 P_a.

证明从略．它是以前的结果的一种直接的应用．

完　备　性

本节着重讨论建立在 Cauchy 网概念的基础上的一些初等定理．我们称一致空间为完备当且仅当该空间的每一个 Cauchy 网都收敛于某个点．这一节的两个最有用的结果是：完备空间的乘积仍为完备空间，以及取值于完备的 Hausdorff 空间的一致连续函数 f 必有定义域为 f 的定义域的闭包的一致连续扩张．

以下恒假定 X 是一个集，$\mathscr{U}$ 是 X 的一个一致结构，P 是 $\mathscr{U}$ 的格集(即 P 是所有在 $X \times X$ 上为一致连续的 X 的伪度量的族)．所有定义都同时利用 $\mathscr{U}$ 和 P 两种方式给出，而证明则利用对所论问题最为方便的形式．以 $V_{p,r}$ 表示集 $\{(x, y): p(x, y) < r\}$.

我们称一致空间 $(X, \mathscr{U})$ 中的网 $\{S_n, n \in D\}$ 为 **Cauchy 网当且仅当**对 $\mathscr{U}$ 的每一个元 U 有 D 中的 N 使得当 m 和 n 关于 D 的序都在 N 之后时有 $(S_m, S_n) \in U$．该定义也可借助于 $X \times X$ 中的网来加以描述．在这种方式下，它可陈述为：网 $\{S_n, n \in D\}$ 为 Cauchy 网当且仅当网 $\{(S_m, S_n), (m, n) \in D \times D\}$ 最终地在 $\mathscr{U}$ 的每一个元内(在 $D \times D$ 内具有乘积序)．因为所有形如 $V_{p,r}$ 的集的族是一致结构 $\mathscr{U}$ 的一个基，其中 p 属于格集 P，r 为正数，故又可推出：$\{S_n, n \in D\}$ 为 Cauchy 网当且仅当 $\{(S_m, S_n), (m, n) \in D \times D\}$ 最终地在每一个形如 $V_{p,r}$ 的集内．换言之，$\{S_n, n \in D\}$ 为 Cauchy 网当且仅当对每一个属于格集 P 的伪度量 p 有 $\{p(S_m, S_n), (m, n) \in D \times D\}$ 收敛于零．

有一个关于 Cauchy 网的简单的引理，由于它时常被应用，值得给予一个正式的陈述．

20 引理. 一致空间 $(X,\mathscr{U})$ 中的网 $\{S_n, n\in D\}$ 为 Cauchy 网当且仅当下列结论中任一个成立:

(a) 网 $\{(S_m, S_n), (m, n)\in D\times D\}$ 最终地在一致结构 $\mathscr{U}$ 的某个子基的每一个元内;

(b) 对于生成格集 P 的某个伪度量族的每一个元 p, 网 $\{p(S_m, S_n), (m, n)\in D\times D\}$ 收敛于零.

证明. 若伪度量族 Q 生成 P, 则所有 $V_{p,r}$ 的族是该一致结构的一个子基, 其中 p 属于 Q, r 为正数, 故(b)的证明可归结到(a)的证明.

为了证明(a)只须注意, 若一个网 (例如 $\{(S_m, S_n), (m, n)\in D\times D\}$) 最终地在有限多个集中的每一个内, 则它也最终地在它们的交内. |

下面的命题阐明了 Cauchy 网和关于一致拓扑的收敛性之间的关系.

21 定理. 每一个关于一致拓扑收敛于某个点的网恒为 Cauchy 网. Cauchy 网恒收敛于它的每一个聚点.

证明. 若 $\{S_n, n\in D\}$ 收敛于点 s, 则对于格集 P 的每一个元 d, $\{d(S_n, s), n\in D\}$ 收敛于零, 但从 $d(S_m, S_n)\leqslant d(S_m, s)+d(S_n, s)$ 可推出 $\{d(S_m, S_n), (m, n)\in D\times D\}$ 收敛于零, 故该网为 Cauchy 网.

假设 $\{S_n, n\in D\}$ 为 Cauchy 网, s 为它的一个聚点, 则对 P 中的 d 和正数 r 有 D 中的 N 使得当 $m\geqslant N$ 且 $n\geqslant N$ 时有 $d(S_m, S_n)<r/2$, 又从 s 为聚点可推出有 D 中的 p 使得 $d(S_p, s)\leqslant r/2$ 并且 $p\geqslant N$, 故当 $n\geqslant p$ 时有 $d(S_n, s)\leqslant d(S_n, S_p)+d(S_p, s)<r$, 即该网收敛于 s. |

我们称一致空间为**完备的**当且仅当空间中的每一个 Cauchy 网都收敛于该空间的一个点. 显然, 完备空间的每一个闭子空间仍为完备空间. 又若 $(X, \mathscr{U})$ 为 Hausdorff 空间并且 $(Y, \mathscr{V})$ 为其完备子空间, 则 Y 为 X 中的闭集, 因为 Y 中收敛于 X 的点 x 的网必为 Cauchy 网并且 x 是它的唯一的极限点. 这个明显的结果是关

于完备性的最有用的事实中的一个.

22 定理. 完备空间的闭子空间仍为完备空间，并且 Hausdorff 一致空间的完备子空间必为闭子空间.

在继续进行空间的完备性研究之前有必要先陈述完备空间的几个例子. 若$\mathscr{U}$是X的最大一致结构（即由所有包含对角线的$X\times X$的子集所组成)，则$(X,\mathscr{U})$为完备. X的最小一致结构同样也产生一个完备空间. 若一致空间$(X,\mathscr{U})$关于一致拓扑为紧，则它为完备，因为每一个网都有一个聚点，从而由定理 6.21 可知每一个 Cauchy 网恒收敛于某个点. 实数空间关于通常一致结构为完备，这可通过证明下列事实而得到：每一个 Cauchy 网最终地在实数空间的某个有界子集A内，从而最终地在紧集A^-内.

有一个完备性的刻划，它是由紧性的一个刻划而引起的. 回忆一下，集族叫做具有有限交性质当且仅当该族任意有限多个元的交均不为空集，又拓扑空间为紧当且仅当每一个具有有限交性质的闭集族的一切元的交为非空. 为了描述完备性，我们对这集族加上另外一个限制. 一致空间$(X,\mathscr{U})$的子集族$\mathscr{A}$叫做**含有小集**当且仅当对$\mathscr{U}$中的每一个U有$\mathscr{A}$的一个元A使得对于某个点x，A是$U[x]$的子集. 它的另一表达形式为：对$\mathscr{U}$中的每一个U有$\mathscr{A}$中的一个A使得$A\times A\subset U$. 借助于一致空间的格集P可得族$\mathscr{A}$含有小集当且仅当对每一个正的r和P中的每一个d有$\mathscr{A}$中的一个A使得A的d-直径小于r. 在这里我们略去这三个命题等价的证明.

23 定理[1]**.** 一致空间为完备当且仅当每一个具有有限交性质并且含有小集的闭集族有一个非空的交.

证明. 设$(X,\mathscr{U})$为一致空间，又设$\mathscr{A}$为具有有限交性质并且含有小集的闭集族. 命$\mathscr{F}$为所有$\mathscr{A}$的元的有限交的族，则易见$\mathscr{F}$关于$\subset$为有向集，再对$\mathscr{F}$中的每一个F，选F中的一个点x_F，则网$\{x_F, F\in\mathscr{F}\}$为 Cauchy 网，因为若A和B关于序$\subset$在$\mathscr{F}$的

1）若称含有小集的滤子为 **Cauchy 滤子**，则该定理可陈述成：空间为完备当且仅当每一个 Cauchy 滤子都收敛于某个点.

元 F 之后（即 $A \subset F$，$B \subset F$），则 x_A 和 x_B 属于 F，并且 $\mathscr{F}$ 含有小集．由此可见 $\{x_F,\ F \in \mathscr{F}\}$ 收敛于某个点，又从该网最终地在 $\mathscr{F}$ 的每一个元内可推出该点必须属于 $\mathscr{F}$ 的每一个元．因而交 $\bigcap\{A:\ A \in \mathscr{A}\}$ 为非空．

今证其逆．设 $\{x_n,\ n \in D\}$ 为 Cauchy 网，又对 D 中的每一个 n，命 A_n 为所有使得 $m \geq n$ 的点 x_m 的集，则所有形如 A_n 的集的族 $\mathscr{A}$ 具有有限交性质，因为该网为 Cauchy 网，族 $\mathscr{A}$ 含有小集，故有一个点 y 它属于所有闭包的交 $\bigcap\{A_n^-:\ n \in D\}$，再根据定理 2.7，$y$ 为网 $\{x_n,\ n \in D\}$ 的聚点．因 $\{x_n,\ n \in D\}$ 为一个 Cauchy 网，因而它收敛于 y．|

有人可能会猜疑满足第一可数性公理的一致空间会是完备的，假如空间中的每一个 Cauchy 序列都收敛于空间的某个点．不幸，这个猜疑是没有根据的，然而下面的较弱的结果确实成立．

24 定理． 可伪度量化的一致空间为完备当且仅当空间中的每一个 Cauchy 序列都收敛于一个点．

证明．若一致空间为完备，则每一个 X 中的 Cauchy 网，特别 X 中的每一个 Cauchy 序列都收敛于某个点．

另一方面，假设 $(X,\ d)$ 为使得每一个 Cauchy 序列都收敛于某个点的伪度量空间，又设 $\mathscr{A}$ 为具有有限交性质并且含有小集的 X 的闭子集族．对每一个非负整数 n，选 $\mathscr{A}$ 的一个元 A_n 使得它的直径小于 2^{-n}，又选点 x_n 属于 A_n，则当 m 和 n 充分大时 $d(x_m,\ x_n)$ 充分小，因为 x_m 和 x_n 分别属于 A_m 和 A_n，而这两个集相交并且其中的每一个都有充分小的直径．故 $\{x_n, n \in \omega\}$ 为 Cauchy 序列，从而收敛于 X 的某个点 y．若 B 为 $\mathscr{A}$ 的任意的一个元，则因 B 与 A_n 相交，$\operatorname{dist}(x_n,\ B) < 2^{-n}$，故 y 属于 B 的闭包．因 $\mathscr{A}$ 为闭集的族，y 就属于 $\mathscr{A}$ 的每一个元．|

证明完备性的通常方法是先证明所考虑的空间一致同构于完备空间的乘积的某个闭子空间，然后再利用下列定理．至于这个定理的证明则需要 Cauchy 网关于一致连续映射的像仍为 Cauchy 网这一事实——一个由定义即为明显的事实．

25 定理. 一致空间族的乘积为完备当且仅当每一个坐标空间皆为完备.

乘积中的网为 Cauchy 网当且仅当它到每一个坐标空间内的射影皆为 Cauchy 网.

证明. 假设对指标集 A 的每一个元 a, $(Y_a, \mathscr{U}_a)$ 为完备一致空间,则因对每一个 a, Cauchy 网在 Y_a 内的射影仍为 Cauchy 网,从而收敛于某个点,譬如是 y_a, 故该网在乘积中收敛于第 a 个坐标为 y_a 的点 y, 于是该乘积为完备. 至于其逆的简单的证明则从略.

若 $\{x_n, n\in D\}$ 为乘积中的网并且它在每一个坐标空间内的射影为 Cauchy 网,则对 $\mathscr{U}_a$ 的每一个元 U, 网 $\{(x_m, x_n), (m, n)\in(D\times D)\}$ 最终地在 U 对射影的逆像内,即 $\{(x_m, x_n), (m, n)\in (D\times D)\}$ 最终地在 $\{(x, z): (x_a, z_a)\in U\}$ 内,但所有这种形式的集的族是乘积一致结构的一个子基,故由定理 6.20 便知 $\{x_n, n\in D\}$ 为 Cauchy 网. 至于其逆则是明显的. ∎

我们称函数 f **在一致空间 $(X, \mathscr{U})$ 的子集 A 上为一致连续**当且仅当它在 A 上的限制 $f|A$ 关于相对一致结构为一致连续. 若值域为一个完备的 Hausdorff 空间[1], f 在它的定义域 A 上为一致连续,则有一个唯一的一致连续扩张,它的定义域为 A 的闭包.

26 定理. 设 f 为一个函数,它的定义域为一致空间 $(X, \mathscr{U})$ 的子集 A, 并且它取值于完备的 Hausdorff 一致空间 $(Y, \mathscr{V})$. 若 f 为在 A 上一致连续,则 f 有一个唯一的一致连续扩张 f^-, 它的定义域为 A 的闭包.

证明. 函数 f 为 $X\times Y$ 的子集 (我们对函数和它的图形不加区分),要求的扩张就是 f 在 $X\times Y$ 内的闭包 f^-((x, y) 属于 f^- 当且仅当 A 中有一个收敛于 x 的网使得它的像网收敛于 y). 显然 f^- 的定义域为 A 的闭包. 今证: 若 W 为 $\mathscr{V}$ 的一个元,则有 $\mathscr{U}$ 中的 U 使得当 (x, y) 和 (u, v) 为 f^- 的元并且 $x\in U[u]$ 时有 $y\in W[v]$.

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1) 这个要求对扩张的存在性并不必要,然而对唯一性却是必须的.

由于 Y 为 Hausdorff 空间，这就将证明 f^- 为一个函数并且 f^- 为一致连续.

选 $\mathscr{V}$ 的一个元 V，它为闭，对称并且使得 $V\circ V\subset W$，再选 $\mathscr{U}$ 的一个元 U，它为开，对称并且使得对 A 中的每一个 x 有 $f[U[x]]\subset V[f(x)]$. 假设 (x, y) 和 (u, v) 属于 f^- 并且 $x\in U[u]$. 因为 $U[x]$ 和 $U[u]$ 的交为开集，故有 A 中的 z 使得 x 和 u 都属于 $U[z]$，从而由 f^- 的定义便知 y 和 v 都属于 $f[U[z]]$ 的闭包，于是 y 和 v 都属于 $V[f(z)]$，因此 $(y, v)\in V\circ V\subset W$，即 $y\in W[v]$. ▌

完 备 扩 张

本节的目的是证明每一个一致空间都一致同构于某个完备一致空间的稠密子空间. 因此，我们有可能对一致空间添加"理想元素"而得到一个完备一致空间. 这种处理是受第 5 章的紧扩张方法的启发. 但有一个重要的不同之点：一致空间的完备扩张是(本质上)唯一的.

首先证明对度量空间 X 我们能够找到一个完备度量空间 X^* 使得 X 等距于 X^* 的某个稠密子空间（不恰好是一致同构）. 然后在这个预备性结果的基础上，我们就可以进一步作出一个一致空间的完备扩张.

27 定理. 每一个度量（或伪度量）空间均可通过一对一的等距映射映到一个完备度量(相应地，伪度量)空间的一个稠密子集上.

证明. 我们只须对伪度量空间 (X, d) 证明该定理，因为对度量空间的相应结果，再由定理 4.15 即可推得.

设 X^* 为所有 X 中的 Cauchy 序列的类，并且对 X^* 的元 S 和 T，命 $d^*(S, T)$ 为 $d(S_m, T_m)$ 当 $m\to\infty$ 时的极限(更精确地说，就是 $\{d(S_m, T_m), m\in\omega\}$ 的极限)，则易证 d^* 为 X^* 的伪度量. 命 F 为映 X 的每一点 x 为恒等于 x 的序列的映射，即对一切 n 有

$F(x)_n = x$，则易见 F 为一对一的等距映射，于是剩下要证的是 $F[X]$ 在 X^* 中稠密并且 X^* 为完备.

这两个结论中的第一个几乎是明显的，因为若 $S \in X^*$ 并且 n 增大，则 $F(S_n)$ 接近于 S. 今证 X^* 为完备，首先注意只须证 $F[X]$ 中的每一个 Cauchy 序列收敛于 X^* 的某个点，因为 $F[X]$ 在 X^* 中稠密. 最后，$F[X]$ 中的每一个 Cauchy 序列必定形如 $F \circ S = \{F(S_n), n \in \omega\}$，其中 S 为 X 中的 Cauchy 序列，而 $F \circ S$ 在 X^* 中收敛于 X^* 的元 S.∎

由定理 6.16 可知每一个一致空间一致同构于伪度量空间的乘积的一个子空间，而每一个 Hausdorff 一致空间一致同构于度量空间的乘积的一个子空间. 上一定理蕴含：度量或伪度量空间一致同构于同一类型的完备空间的一个子空间，于是不难得到：

28 定理. 每一个一致空间都一致同构于一个完备一致空间的一个稠密子空间. 每一个 Hausdorff 一致空间都一致同构于一个完备 Hausdorff 一致空间的一个稠密子空间.

一致空间 $(X, \mathscr{U})$ 的一个**完备扩张**是一个偶 $(f, (X^*, \mathscr{U}^*))$，其中 $(X^*, \mathscr{U}^*)$ 是一个完备一致空间，f 是从 X 到 X^* 的一个稠密子空间内的一个一致同构. 一个完备扩张叫做 Hausdorff 的当且仅当 $(X^*, \mathscr{U}^*)$ 是一个 Hausdorff 一致空间. 于是上一定理可以陈述为：每一个 (Hausdorff) 一致空间有一个 (Hausdorff) 完备扩张.

注意 Hausdorff 完备扩张具有唯一性. 若 f 和 g 分别为从 X 到完备 Hausdorff 一致空间 X^* 和 X^{**} 的某个稠密子空间上的一致同构，则由定理 6.26 可知 $g \circ f^{-1}$ 和 $f \circ g^{-1}$ 分别有一个一致连续扩张，它们的定义域分别为 X^* 和 X^{**}，这就推出了 $g \circ f^{-1}$ 的扩张是从 X^* 到 X^{**} 上的一个一致同构. 于是粗糙地可陈述成：Hausdorff 一致空间的 Hausdorff 完备扩张关于一个一致同构是唯一的.

紧 空 间

我们已经知道集 X 的每一个全正则拓扑 $\mathscr{T}$ 必为某个一致结

构$\mathscr{U}$的一致拓扑，但该一致结构通常是不唯一的．然而当$(X,\mathscr{T})$为紧正则空间时，我们可导出拓扑为$\mathscr{T}$的一致结构恰好只有一个．在这种情况下，拓扑就决定了一致结构，拓扑不变量也就是一致不变量，于是我们的理论也就成为一种特别简单的形式．本节就是要着重讨论这个唯一性定理的证明以及两个其它的命题．和以前一样，我们将根据方便上的要求，或者用空间的一致结构，或者用相应的一致连续伪度量的格集．

29 定理． 若$(X,\mathscr{U})$为紧一致空间，则对角线Δ在$X\times X$中的每一个邻域均为$\mathscr{U}$的元，并且每一个在$X\times X$上为连续的伪度量均为$\mathscr{U}$的格集的一个元．

证明．设$\mathscr{B}$为$\mathscr{U}$的所有闭元的族，V为Δ的任意开邻域．若$(x,y)\in\bigcap\{U:U\in\mathscr{B}\}$，则因$\mathscr{B}$为$\mathscr{U}$的一个基，故$y$属于$x$的每一个邻域，从而$(x,y)$属于$\Delta$的每一个邻域．这就推出了$\bigcap\{U:U\in\mathscr{B}\}$为$V$的子集．因为$\mathscr{B}$的每一个元$U$为紧集并且$V$为开集，故$\mathscr{B}$的某个有限子族的交亦为$V$的子集，从而$V\in\mathscr{U}$．

若X的伪度量d在$X\times X$上为连续，则对每一个正数r，集$\{(x,y):d(x,y)<r\}$为对角线的一个邻域，故d为一致连续，因而属于$\mathscr{U}$的格集．▎

因为每一个紧正则拓扑空间为全正则空间，故它的拓扑是某个一致结构的一致拓扑．这个一致结构则由上述定理所证实，从而有

30 系． 若$(X,\mathscr{T})$为紧正则拓扑空间，则对角线Δ的所有邻域的族为X的一致结构并且$\mathscr{T}$是它的一致拓扑．

还有另外的一个系．

31 定理． 从紧一致空间到一致空间的每一个连续函数为一致连续．

证明．若f为从X到Y的连续函数，则f_2为从$X\times X$到$Y\times Y$的连续函数，其中$f_2(x,y)=(f(x),f(y))$，故若d属于Y的格集，则合成$d\circ f_2$在$X\times X$上为连续，从而由定理6.29可知$d\circ f_2$属于X的格集，于是函数f为一致连续．▎

每一个紧一致空间 $(X,\mathscr{U})$ 可在下述意义下表成有限多个小集的并，即对于每一个属于$\mathscr{U}$的格集的伪度量 d 和每一个正数 r，X 有一个由 d-直径小于 r 的集所组成的有限覆盖. 事实上这是紧性的一个直接推论，因为 X 可由有限多个它的点为心的 $r/3$ 球所覆盖，而其中的每一个球的直径自然都小于 r. 一致空间 $(X,\mathscr{U})$ 称为**全有界(或预紧)**当且仅当对$\mathscr{U}$的格集中的每一个伪度量 d 和每一个正的 r，X 为有限多个 d-直径小于 r 的集的并.借助于$\mathscr{U}$，这还可以陈述成：对$\mathscr{U}$中的每一个 U，集 X 为有限多个满足 $B\times B\subset U$ 的集 B 的并，或再等价地陈述成：对$\mathscr{U}$中的每一个 U 有 X 的一个有限子集 F 使得 $U[F]=X$. 一致空间的子集 Y 叫做全有界，当且仅当具有相对一致结构的 Y 为全有界.

紧性和全有界性之间有一个简单而很有用的关系.

32 定理. 一致空间 $(X,\mathscr{U})$ 为全有界的当且仅当 X 中的每一个网均有 Cauchy 子网.

因此，一致空间为紧的当且仅当它为全有界的和完备的.

证明. 假设 S 为全有界一致空间 $(X,\mathscr{U})$ 中的一个网，则它的 Cauchy 子网的存在，实际上就是问题 2.J 的一个明显推论，但我们还可以给出一个不用这个结果的直接证明. 设 $\mathscr{A}$ 为所有使得 S 常常在 A 内的 X 的子集 A 的族，则 $\{X\}\subset\mathscr{A}$ 并且由极大原理(预备知识定理 25) 有 $\mathscr{A}$ 的一个极大子族 $\mathscr{B}$，它包含 $\{X\}$ 并且具有有限交性质.因为根据 $\mathscr{B}$ 的极大性可知若有限多个 $\mathscr{A}$ 的元 $B_1,\cdots,B_n$ 之并属于 $\mathscr{B}$，则对某个 i 有 $B_i\in\mathscr{B}$ (详细情形见问题 2.J)，从 X 为全有界可推出它可由有限多个小集所覆盖，故 $\mathscr{B}$ 含有小集. 最后，从定理 2.5 可推出有 S 的一个子网它最终地在 $\mathscr{B}$ 的每一个元内，并且易见这个子网为一个 Cauchy 子网.

若 $(X,\mathscr{U})$不是全有界，则对某个$\mathscr{U}$中的 U 和每一个 X 的有限子集 F 有 $U[F]\neq X$，于是利用归纳法可找到一个序列 $\{x_n,n\in\omega\}$ 使得当 $p<n$ 时有 $x_n\notin U[x_p]$，显然该序列没有 Cauchy 子网.

最后，若$(X,\mathscr{U})$ 为完备的和全有界的，则每一个网有一个子

网收敛于X的某个点，因而该空间为紧．我们已经知道任何紧空间必为完备的．▎

关于紧空间还有另一个很有用的引理．该命题是 Lebesgue 覆盖引理（定理 5.26）的一种推广．

一致空间 $(X, \mathscr{U})$的子集A的一个覆盖称为**一致覆盖**当且仅当有$\mathscr{U}$的一个元U使得对A中的每一个x,集$U[x]$为该覆盖的某个元的子集（即所有$U[x]$的族为该覆盖的一个加细，其中x属于A）．借助于一致结构$\mathscr{U}$的格集可知A的一个覆盖为一致覆盖当且仅当有该格集的一个元p和一个正数r使得以A的每一点为心的d-半径r的开球包含在该覆盖的某个元内．

33 定理． 一致空间的紧子集的每一个开覆盖为一致覆盖．特别，紧子集A的每一个邻域含有形如$U[A]$的一个邻域其中U为一致结构的元．

证明．设$\mathscr{A}$为一致空间$(X, \mathscr{U})$的紧子集A的一个开覆盖，则对每一个A中的x有$\mathscr{U}$中的U使得$U[x]$为$\mathscr{A}$的某个元的子集，因而有$\mathscr{U}$中的V使得$V\circ V[x]$为$\mathscr{A}$的某个元的子集．选有限多个A的元$x_1, \cdots, x_n$和$\mathscr{U}$的元$V_1, \cdots, V_n$使得集$V_i[x_i]$覆盖A并且对每一个i, $V_i\circ V_i[x_i]$为$\mathscr{A}$的某个元的子集．从而，若命$W = \bigcap\{V_i: i = 1, \cdots, n\}$，则对每一个$A$中的$y$有某个$i$使得$y$属于$V_i[x_i]$，故$W[y]\subset W\circ V_i[x_i]\subset V_i\circ V_i[x_i]$，即$W[y]$为$\mathscr{A}$的某个元的子集．▎

度量空间特有的性质

本节着重讨论关于完备度量空间的两个命题，它们是完备性的最有用的结论中的两个结果，然而，似乎不可能把它们推广到完备一致空间．其中的第一个命题是关于范畴的经典的 Baire 定理，这个定理以及一、两个与此有关的结果就占了这一节的大部分篇幅．本节的最后一个定理是：完备度量空间关于连续一致开映射的像仍为完备空间，假如值域空间为 Hausdorff 空间．而它的证明

则依赖于一个引理，这个引理在讨论中我们将叙述成对证明该命题所需要的更为一般的形式．另外从该引理(本质上是 Banach 的一种论证的形式化）还直接导出了线性赋范空间理论的闭图像定理和开映射定理（见问题 6.R)．

34 定理(Baire)．*设 X 为完备伪度量空间，或为局部紧正则空间，则 X 的可数多个开稠密子集的交也在 X 中稠密．*

证明　我们对局部紧正则空间来证明这个定理，而把对完备伪度量空间所应作的修改放在括弧内．

假设 $\{G_n, n\in\omega\}$ 为 X 的开稠密子集的序列并且 U 为 X 的任意非空开子集，则只须证 $U\cap\bigcap\{G_n, n\in\omega\}$为非空．为此，选开集 V_0 使得 V_0^- 为 $U\cap G_0$ 的紧子集（使得 V_0^- 为 $U\cap G_0$ 的子集并且它的直径小于 1），然后对每一个正整数 n，归纳地选 V_n 使得 V_n^- 为 $V_{n-1}\cap G_n$ 的子集(并且 V_n 的直径小于 $1/n$)．这种选取是可能的，因为 G_n 为开稠密子集．因为所有集 A_n^- 的族具有有限交性质，其中 n 为非负整数，并且 A_0^- 为紧集(该族含有小集)，故 $\bigcap\{A_n^-, n\in\omega\}$ 为非空，从而由 $A_{n+1}^-\subset U\cap G_n$ 可推出 $U\cap\bigcap\{G_n, n\in\omega\}$为非空．∎

值得注意，Baire 定理是从一个非拓扑的前提(空间为完备伪度量空间)导出了一个拓扑的结论(可数多个开稠密集的交仍为稠密集)．还有一个与此相等价的纯拓扑的命题．若 $(X,\mathscr{T})$ 为拓扑空间并且使得对 X 的某个伪度量 d，空间 (X, d) 为完备同时 $\mathscr{T}$ 为伪度量拓扑，则相同的结论成立．(存在这样的完备度量的拓扑空间业已用另一种不同方式所刻划．见问题 6. K.)

在讨论关于 Baire 定理的问题中有一个很方便的术语．拓扑空间的子集 A 叫做在 X 中**无处稠密**，当且仅当 A 的闭包的内部为空集；它的另外的陈述方法是：A 在 X 中为无处稠密当且仅当开集 $X\sim A^-$ 在 X 中稠密．显然，有限多个无处稠密集的并仍为无处稠密集．X 的子集 A 叫做在 X 中为**稀疏**，或在 X 中为**第一范畴**当且仅当 A 为可数多个无处稠密集的并．于是 Baire 定理又可以陈述成：完备度量空间的稀疏子集的余集为稠密集（稀疏集的余集有时叫做**剩余集**)．

集 A 叫做**非稀疏的**，或在 X 中为**第二范畴**当且仅当它在 X 中不是稀疏的集．下面的结果是一种局部化定理．从集 A 为非稀疏，我们可推出存在点 x 使得 A 与 x 的每一个邻域的交为非稀疏集．有时也称 A 在这些点上为第二范畴．

35 定理．设 A 为拓扑空间 X 的一个子集，又设 $M(A)$ 为所有使得 $V\cap A$ 在 X 中为稀疏的开集 V 的并，则 $A\cap M(A)^-$ 在 X 中为稀疏的．

证明．设 $\mathscr{U}$ 为关于下列性质为极大的互不相交的开集族：若 $U\in\mathscr{U}$，则 $U\cap A$ 为稀疏．由极大原理（预备知识定理 25）这样的族 $\mathscr{U}$ 必存在．命 $W=\bigcup\{U: U\in\mathscr{U}\}$，则本定理的证明归结为证明 $W\cap A$ 为稀疏．因为假如这件事成立，则由 $W^-\sim W$ 为无处稠密便知 $A\cap W^-$ 为稀疏，于是从 $\mathscr{U}$ 的极大性可推出 W^- 包含每一个使得 $V\cap A$ 为稀疏的开集 V．现在证明 $W\cap A$ 为稀疏．对每一个 $\mathscr{U}$ 中的 U，将 $U\cap A$ 写成 $\bigcup\{U_n: n\in\omega\}$ 的形式，其中 U_n 为无处稠密，则因族 $\mathscr{U}$ 为互不相交，故对每一个非负整数 n，集 $\bigcup\{U_n: U\in\mathscr{U}\}$ 为无处稠密，而 $W\cap A$ 为稀疏．∎

上一定理的一个重要推论是：若拓扑空间的子集 A 为非稀疏，则有一个非空开集 V 使得 A 与 V^- 的每一点的每一个邻域的交为非稀疏．

本章的最后一个定理是证明完备性在某种映射作用下能够得到保持．从一致空间 $(X,\mathscr{U})$ 到一致空间 $(Y,\mathscr{V})$ 内的映射 f 叫做**一致开**当且仅当对 $\mathscr{U}$ 中的每一个 U 有 $\mathscr{V}$ 中的 V 使得对于 X 中的每一个 x 有 $f[U[x]]\supset V[f(x)]$．一致开映射并不对任意一致空间都保持完备性；Köthe[1] 已经给出一个使得商空间不完备的完备线性拓扑空间和闭子空间的例子．这个定理与 Baire 定理一样也为伪度量空间所特有．

这里所给出的这个定理的证明是依赖于一条它本身还有其它深刻结论（见问题 6.R）的引理．这引理是关于伪度量空间 (X,d) 和一致空间 $(Y,\mathscr{V})$ 的点之间的一个关系 R（即 R 是 $X\times Y$ 的子集）．设 $U_r=\{(x,y): d(x,y)<r\}$，则 $U_r[x]$ 就表示关于 x

的 r-球．

36 引理． 设 R 为完备伪度量空间 (X, d) 与一致空间 $(Y, \mathscr{V})$ 的乘积的闭子集，又设对每一个正的 r 有 $\mathscr{V}$ 中的 V 使得对 R 中的每一个 (x, y) 有 $R[U_r[x]]^-$ 包含 $V[y]$，则对每一个 r 和每一个正的 e 有 $R[U_{r+e}[x]]\supset R[U_r[x]]^-\supset V[y]$ 成立．

证明．在证明中我们需要一个关键性的事实：若 A 为 X 的子集，$v\in R[A]^-$，则有任意小直径的集 B 使得 $v\in R[B]^-$ 并且 $A\cap B$ 为非空．而它的成立则是因为：若 r 为任意数，V 为 $\mathscr{V}$ 的对称的元，合于：对 R 的每一个元 (x, y) 有 $R[U_r[x]]^-\supset V[y]$，又如 v' 为使得 $v'\in V[v]$ 的 $R[A]$ 的点，u 为使得 $(u, v')\in R$ 的 A 的点，则 $v\in V[v']\subset R[U_r[u]]^-$ 并且 $U_r[u]$ 的直径至多为 $2r$．

现在来证明本引理．设 $v\in R[U_r[x]]^-$，则只须证 $v\in R[U_{r+e}[x]]$．设 $A_0=U_r[x]$，并且对每一个正整数 n，归纳地选 X 的子集 A_n 使得 $v\in R[A_n]^-$，$A_n\cap A_{n+1}$ 为非空并且 A_n 的直径小于 $e2^{-n}$．因为 X 为完备，易见存在点 u 使得 u 的每一个邻域 W 都含有某一个 A_n（从而 $v\in R[W]^-$），显然 $d(x, u)<r+e$．对于 u 的每一个邻域 W 和 v 的每一个邻域 Z 有 $R[W]$ 与 Z 相交，因而有 R 中的 (u', v') 使得 u' 属于 W 并且 v' 属于 Z，即 $R\cap(W\times Z)$ 为非空．但 R 为闭，于是 $(u, v)\in R$，因而就完成了证明．|

现在假设 f 为一致开的连续映射，X 为完备伪度量空间，Y 为 Hausdorff 空间并且 Y^* 为 Y 的 Hausdorff 完备扩张，则因 f 为连续，f（的图形）为 $X\times Y^*$ 的闭子集，并且满足上一引理的条件，因为它是从 X 到 Y 内的一致开映射．于是从前引理可推出 f 为从 X 到 Y^* 内的一致开映射．最后，因为对于 $\mathscr{V}$ 中的某个 V，$f[X]$ 包含 $V[f[X]]$，故 $f[X]$ 为 Y^* 的闭（并且开）的子集，从而 $f[X]$ 为完备．

37 系． 设 f 为从完备伪度量空间到 Hausdorff 一致空间内的连续一致开映射，则映射 f 的值域为完备的．

问　　题

A　关于闭关系的习题

设 X 和 Y 为拓扑空间，R 为 $X\times Y$ 的闭子集．若 A 为 X 的紧子集，则 $R[A]$ 为 Y 的闭子集．(若 $y\notin R[A]$，则 $A\times\{y\}$ 包含在开集 $(X\times Y)\sim R$ 内，从而定理 5.12 可以应用．)

B　关于两个一致空间的乘积的习题

设 $(X,\mathscr{U})$ 和 $(Y,\mathscr{V})$ 为一致空间，并且对 $\mathscr{U}$ 中的每一个 U 和 $\mathscr{V}$ 中的每一个 V，命 $W(U,V)=\{((x,y),(u,v))\colon (x,u)\in U$ 且 $(y,v)\in V\}$．

(a) 所有形如 $W(U,V)$ 的集组成的族是 $X\times Y$ 的乘积一致结构的一个基．

(b) 若 R 为 $X\times Y$ 的子集，则 $W(U,V)[R]=V\circ R\circ U^{-1}=\bigcup\{U[x]\times V[y]\colon (x,y)\in R\}$．

(c) $X\times Y$ 的子集 R 的闭包为 $\bigcap\{V\circ R\circ U^{-1}\colon U\in\mathscr{U}$ 且 $V\in\mathscr{V}\}$．

C　一个离散不可度量化的一致空间

注意一致空间 $(X,\mathscr{U})$ 可以为不可度量化，即使 $\mathscr{U}$ 的拓扑为可度量化．设 Ω_0 为所有小于第一个不可数序数 Ω 的序数的集，并且对 Ω_0 的每一个元 a，命 $U_a=\{(x,y)\colon x=y$ 或 $x\leqslant a$ 且 $y\leqslant a\}$，则所有形如 U_a 的集组成的族是 Ω_0 的一致结构 $\mathscr{U}$ 的一个基(注意 $U_a=U_a\circ U_a=U_a^{-1}$)．该一致结构的拓扑为离散拓扑，因而为可度量化，但一致空间 $(\Omega_0,\mathscr{U})$ 为不可度量化．

D　具有套状基的一致空间的习题

设 $(X,\mathscr{U})$ 为 Hausdorff 一致空间，并且假定 $\mathscr{U}$ 的一个基 $\mathscr{B}$ 关于包含关系为线性有序集，则或者 $(X,\mathscr{U})$ 为可度量化，或者每一个 X 的开子集的可数族的交均为开集．

E　例子：一个很不完备的空间(序数)

设 Ω_0 为所有小于第一个不可数序数 Ω 的序数的集，$\mathscr{T}$ 为 Ω_0 的序拓扑，则 Ω_0 有一个唯一的一致结构使得它的拓扑为 $\mathscr{T}$ 并且 Ω_0 关于该一致结构为不完备．(先利用问题 4.E 的方法证明，若 U 为 $\Omega_0\times\Omega_0$ 中包含对角线的一个

开子集，则对某个 x 当 $y>x$ 和 $z>x$ 时有 $(y, z)\in U$ 成立。然后再证明拓扑为 $\mathscr{T}$ 的一致结构必定与紧空间 $\Omega'=\{x: x\leqslant\Omega\}$ 的相对一致结构相同.)

注 Ω_0 的这个性质是由 Dieudonné[5] 发现的。Doss[1] 对于好像 Ω_0 那样具有一个唯一的一致结构的拓扑空间进行了刻划。

F 关于全有界性的子基定理

在一致空间内，与紧空间的子基的 Alexander 定理（定理 5.6）相似的定理是：设 $(X, \mathscr{U})$ 为一致空间，它使得对于 $\mathscr{U}$ 的某个子基的每一个元 U 有 X 的一个有限覆盖 $A_1,\cdots, A_n$ 使得对每一个 i 满足 $A_i\times A_i\subset U$，则空间$(X,\mathscr{U})$为全有界.

因此，一致空间的乘积为全有界当且仅当每一个坐标空间为全有界。

又 Tychonoff 乘积定理（定理 5.13）对全正则空间的相应形式可从上一命题和定理 6.32 推出。

G 某些极端的一致结构

(a) 若 $(X, \mathscr{T})$ 为 Tychonoff 空间，则 X 的 Stone-Čech 紧扩张的一致结构对 X 的相对一致结构是使得每一个有界实值连续函数为一致连续的最小一致结构。

(b) 若$(X,\mathscr{T})$为全正则空间，则 X 有一个最大一致结构 $\mathscr{V}$，其拓扑为 $\mathscr{T}$。这个一致结构也可以描述成使得每一个到度量空间内的连续映射，或每一个到一致空间内的连续映射为一致连续的最小一致结构。显然，V 是 $\mathscr{V}$ 的元当且仅当 V 是 $X\times X$ 中的对角线的邻域并且有该对角线的对称邻域的序列 $\{V_n, n\in\omega\}$ 使得 $V_0\subset V$ 同时 $V_{n+1}\circ V_{n+1}\subset V_n$ 对每一个 ω 中的 n 成立。

注 这两种构造都是以前已经用过的一种方法的例子。若 F 为任意一个 X 上的函数族，它的每一个元 f 映 X 到一致空间 Y_f 内，则有使得每一个 f 为一致连续（或等价地，到 $\times\{Y_f: f\in F\}$ 内的自然映射为一致连续）的最小一致结构。

关于某些极端的一致结构的进一步知识见 Shirota[1]。

H 一致邻域系

集 X 的**一致邻域系**是指满足下列条件的对应 V 和序 $\geqslant$：

(i) 对指标集 A 的每一个元 a 和 X 的每一点 x，$V_a(x)$ 是使得 x 属于它的 X 的子集；

(ii) 关系$\geqslant$使指标集A为有向集;

(iii) 若$a\geqslant b$, 则对一切x有 $V_a(x)\subset V_b(x)$;

(iv) 对A的每一个元a有A中的b使得 $y\in V_a(x)$ 当 $x\in V_b(y)$ 时;

(v) 对A的每一个元a有A中的 b 使得 $z\in V_a(x)$ 当 $y\in V_b(x)$ 且 $z\in V_b(y)$ 时。

(a) 若 $(V, \geqslant)$ 为X的一致邻域系,则所有形如: $\{(x, y): y\in V_a(x)\}$ 的集组成的族是X的某个一致结构$\mathscr{U}$的基,其中a是A的任意元. 称该一致结构为这个一致邻域系的一致结构. 这个一致结构具有性质: 对A中的每一个a和$\mathscr{U}$中的某个U有 $U[x]\subset V_a(x)$ 对一切x成立,又对$\mathscr{U}$中的每一个U和A中的某个a有 $V_a(x)\subset U[x]$ 对一切x成立。

(b) 设$\mathscr{U}$为X的一致结构,并且对$\mathscr{U}$的每一个元U和X的每一个元x,命 $V_U(x)=U[x]$, 则$\mathscr{U}$关于$\subset$为有向集并且 $(V, \subset)$ 为使得一致结构为$\mathscr{U}$的X的一个一致邻域系。

(c) 设P为X的一致结构$\mathscr{U}$的格集,又设A为P和正实数集的笛卡儿乘积。且规定$(p,r)\geqslant(q,s)$当且仅当$r\leqslant s$并且对X中的一切x和y有$p(x,y)\geqslant q(x,y)$. A为有向集. 若 $V_{p,r}(x)=\{y: p(x, y)<r\}$, 则 $(V, \geqslant)$ 为使得其一致结构为$\mathscr{U}$的X的一个一致邻域系。

注 由上易见,“附加指标”的邻域也可以用来讨论一致结构,并且所得到的理论与一致空间理论相等同。这些事实都属于 Weil[1]。

I 偏差和度量

集X的偏差是$X\times X$上的非负实值函数e并且满足:

(i) $e(x, y)=0$ 当且仅当 $x=y$;

(ii) 对每一个正数s有正数r使当 $e(x, y)$ 和 $e(y, z)$ 都小于r时有 $e(x, z)<s$.

若e是X的偏差,则有$X\times X$上的非负函数p使得:

(i) $p(x, y)=0$当且仅当 $x=y$;

(ii) 对一切X中的x, y和z有 $p(x, y)+p(y, z)\geqslant p(x, z)$;

(iii) 对每一个正数s有正数r使当 $e(x, y)<r$ 时有 $p(x, y)<s$, 相似地,当 $p(x, y)<r$ 时有 $e(x, y)<s$.

若 $e(x, y)=e(y, x)$ 对一切x和y成立,则p可以取成度量。

注 这本质上是 Chittenden 的度量化定理(见定理 6.14). 拓扑空间关于满足度量除“$d(y, x)=d(x, y)$”外的一切要求的函数d的“度量化”

问题已经由 Ribeiro[2] 和 Balanzat[1] 研究过。

另外某些作者是将术语偏差用来表示取值于比实数限制更少的一种结构（例如半序集）的度量函数．关于在这种概念的基础上的一致结构的讨论见 Appert[1]，Colmez[1]，Cohen 和 Goffman [1]，Gomes[1]，Kalisch[1] 和 Lasalle[1]。

J　一致覆盖系

设 Φ 为集 X 的一个覆盖的族，它满足：

(i) 若 $\mathscr{A}$ 和 $\mathscr{B}$ 为 Φ 的元，则有 Φ 的一个元，它同时是 $\mathscr{A}$ 和 $\mathscr{B}$ 的加细；

(ii) 若 $\mathscr{A}\in\Phi$，则有 Φ 的一个元，它是 $\mathscr{A}$ 的星形加细；

(iii) 若 $\mathscr{A}$ 为 X 的覆盖并且 $\mathscr{A}$ 的某个加细属于 Φ，则 $\mathscr{A}$ 属于 Φ。

设 $\mathscr{U}$ 为 X 的一致结构使得所有形如 $\bigcup\{A\times A:\ A\in\mathscr{A}\}$ 的集的族是 $\mathscr{U}$ 的一个基，其中 $\mathscr{A}$ 属于 Φ，则 Φ 恰为 X 的所有关于 $\mathscr{U}$ 的一致覆盖的族。

注　一致结构借助于覆盖的描述方式．已由 J. W. Tukey[1] 很有效地使用了；更早就使用这种普遍形式的是 Alexandroff 和 Urysohn[2]。

K　拓扑完备空间：可度量化空间

拓扑空间 $(X,\mathscr{T})$ 叫做度量化地拓扑完备当且仅当有 X 的一个度量 d 使得 (X,d) 为完备并且 $\mathscr{T}$ 就是度量拓扑．拓扑空间 $(X,\mathscr{T})$ 叫做一个绝对 G_δ 当且仅当它可度量化并且在每一个它所拓扑嵌入的度量空间内为一个 G_δ（可数多个开集的交）．于是：拓扑空间为度量化地拓扑完备当且仅当它是一个绝对 G_δ．它的证明依赖于一系列引理。

(a) 设 (X,d) 为完备度量空间，又设 U 为 X 的开子集，对 U 中的 x，命 $f(x)=1/\mathrm{dist}(x,X\sim U)$，再命 $d^*(x,y)=d(x,y)+|f(x)-f(y)|$，则 d^* 为一个度量，U 对于 d^* 为完备并且对于 U，d 和 d^* 的拓扑相同。

(b) 完备度量空间中的 G_δ 同胚于一个完备度量空间．（设 $U=\bigcap\{U_n:\ n\in\omega\}$，考虑从 U 到完备度量空间 (U_n,d_n^*) 的乘积内的映射，其中 d_n^* 是如同 (a) 中的从 d 和 U_n 所作出的度量．）

(c) 若存在从 Hausdorff 空间 X 的稠密子集 Y 到完备度量空间 Z 上的一个同胚，则 Y 为 X 中的一个 G_δ．（对每一个整数 n，命 U_n 为所有使得 x 的某个邻域的像的直径小于 $1/n$ 的 X 的点 x 组成的集，则该同胚可连续扩张成从 $\bigcap\{U_n:\ n\in\omega\}$ 到 Z 上的连续映射 f^- 并且 $f^{-1}\circ f^-$ 为恒等映射．）

注　这些都是古典的结果；(b)属于 Alexandroff[1] 和 Hausdorff[9]，而(c)

属于 Sierpinski[2].

L 拓扑完备空间；可一致化空间

拓扑空间 $(X, \mathscr{T})$ 叫做**拓扑完备**当且仅当有 X 的一致结构 $\mathscr{U}$ 使得 $(X, \mathscr{U})$ 为完备并且 $\mathscr{T}$ 为一致拓扑。

(a) 若 $\mathscr{U}$ 和 $\mathscr{V}$ 为 X 的一致结构，满足 $\mathscr{U} \subset \mathscr{V}$，又若 $(X, \mathscr{U})$ 为完备并且 $\mathscr{U}$ 的拓扑与 $\mathscr{V}$ 的拓扑相同，则 $(X, \mathscr{V})$ 为完备。因此，全正则空间为拓扑完备当且仅当它关于其拓扑为 $\mathscr{T}$ 的最大一致结构为完备。

(b) 设 $(X, \mathscr{U})$ 为完备一致空间，又设 F 为一个 F_σ（可数多个闭集的并），再设 $x \in X \sim F$，则存在 X 上的连续实值函数，它在 F 上为正并且在 x 处为零。因而存在开集 V 和 V 的一致结构 $\mathscr{V}$ 使得 V 包含 F，$x \notin V$，$(V, \mathscr{V})$ 为完备并且 $\mathscr{V}$ 的拓扑与 $\mathscr{U}$ 的相对拓扑相同（回顾一下问题 6.K (a) 中所使用的方法。）

(c) 若 $(X, \mathscr{U})$ 为完备一致空间，Y 为 X 的一个子集并且它是一族 F_σ 的元的交，则具有相对一致拓扑的 Y 为拓扑完备。（见问题 6.K.）

(d) 每一个仿紧空间 X 为拓扑完备。（考虑由对角线的一切邻域所组成的一致结构。注意不收敛于 X 的点的 Cauchy 网必定对于每一点 x 都最终地在 x 的某个邻域的余集内，再应用仿紧空间的齐-覆盖性即导致一个矛盾。）

注 拓扑完备性的问题在 Dieudonné [6] 中已经被研究；特别是，他证明了每一个可度量化空间为拓扑完备（这是上面的 (c) 或 (d) 的推论）。Shirota[1] 在与 Hewitt[2] 的工作相联系的一个方向上证明了拓扑完备性的一些有趣并且深刻的定理。也可参看 Umegaki[1]。

我猜测1)全正则空间 X 为仿紧当且仅当

(i) 对角线的所有邻域的族为一个一致结构；

(ii) X 为拓扑完备。

注意 (i) 或 (ii) 本身都推不出仿紧性。满足 (i) 的非仿紧空间在问题 6.E 中已经给出。从条件 (i) 可推出正规性（设 A 和 B 为互不相交的闭集，选一个对称的 U 使得 $U \circ U \subset (X \sim A) \times (X \sim A) \cup (X \sim B) \times (X \sim B)$ 并且考虑 $U[A]$ 和 $U[B]$；用类似的推理（如同 H. J. Cohen[1] 所证明的）可以得到一种更强的正规性条件）。然而，不可数多个实数空间的乘积为完备，但却不是正规的（A. H. Stone[1]）。

在上面 (c) 中出现的 F_σ 条件是 Smirnov[3] 关于正规性的工作所示的。

1) Issac Namioka 已经证明了这个猜测是不成立的。——译者注

M　离散子空间推理；可数紧性

(a) 若一致空间 $(X, \mathscr{U})$ 的一个子集 A 不是全有界的，则有 $\mathscr{U}$ 的一个元 U 和 A 的一个无限子集 B 使得对于 B 的每一对不同的点 x, y, $U[x]$ 与 $U[y]$ 互不相交；等价地，$\mathscr{U}$ 的格集有一个伪度量 d 使得对 B 的不同的点 x 和 y 有 $d(x, y)\geqslant 1$. (如同 B，这样的集叫做一致离散.)

(b) 拓扑空间 $(X, \mathscr{T})$ 的子集 A 叫做相对可数紧，当且仅当 A 中的每一个序列在 X 中有一个聚点. 全正则空间 $(X, \mathscr{T})$ 的每一个相对可数紧子集对于使得其拓扑为 $\mathscr{T}$ 的最大一致结构为全有界. 若 $(X, \mathscr{T})$ 为拓扑完备，则一个子集为相对可数紧当且仅当它的闭包为紧，闭子集为紧当且仅当它为可数紧.

N　不变度量

集 X 的伪度量 p 称为在某个从 X 到它自己上的一对一映射的族 F 的元的作用下不变，或简称为 F-不变当且仅当对于 X 中的一切 x 与 y 和 F 中的一切 f, $p(x, y) = p(f(x), f(y))$.

X 的一致结构 $\mathscr{U}$ 的元 U 叫做 F-不变，假如 $(x, y)\in U$ 当且仅当对于 F 中的一切 f, $(f(x), f(y))\in U$. 于是：所有在 $X\times X$ 上为一致连续的 F-不变伪度量的族生成一致结构 $\mathscr{U}$ 当且仅当 $\mathscr{U}$ 的所有 F-不变元的族为一个基. (见定理6.12.)

注　这是下一问题所陈述的关于拓扑群的度量化定理的一个直接推广.

O　拓扑群：一致结构和度量化

设 $(G, \mathscr{T})$ 为拓扑群，对于单位元的每一个邻域 U，命 $U_L = \{(x, y): x^{-1}y\in U\}$ 和 $U_R = \{(x, y): xy^{-1}\in U\}$. 考虑 G 的如下的一致结构：以所有集 U_L 的族为基的左一致结构 $\mathscr{L}$，其中 U 为单位元的邻域，以所有的 U_R 为基的右一致结构 $\mathscr{R}$ 和以 $\mathscr{L}\cup\mathscr{R}$ 为子基的双边一致结构 $\mathscr{U}$.

(a) 拓扑 $\mathscr{T}$ 是 $\mathscr{L}$, $\mathscr{R}$ 和 $\mathscr{U}$ 中的每一个的拓扑.

(b) 一致结构 $\mathscr{L}(\mathscr{R})$ 由所有在 $G\times G$ 上连续的左不变(相应地右不变)的伪度量所生成. (见问题 6.N.)

(c) 设 I 为单位元 e 的所有在内自同构作用下不变的邻域的族，则 I 为 e 的邻域系的一个基当且仅当所有在 $G\times G$ 上连续并且同时为左和右不变的伪度量的族所生成的一致结构的拓扑为 $\mathscr{T}$. (若 U 为 e 的不变邻域，则 $U_L = U_R$，并且该集同时在左和右平移作用下为不变. 若 p 为左和右不变，则 $p(e,$

$y) = p(x^{-1}ex, x^{-1}yx)$.)

(d) 设 G 为所有形如 $g(x) = ax + b$ 的实值函数组成的集，其中 $a \neq 0$，则 G 以合成为运算，构成一个群，并且可以这样拓扑化，即规定 g 接近于单位元当且仅当 a 接近于 1 同时 $|b|$ 接近于零．对于这个群，$\mathscr{L} \neq \mathscr{R}$ 并且不存在双边不变度量．($\mathscr{L} \neq \mathscr{R}$ 的事实可直接从所定义的基推出．为了说明不存在不变度量，证明对每一个 g，若 $a \neq 1$，则有 G 中的 f 使得 $f^{-1} \circ g \circ f$ 的常数项为任意大.)

注　G 的左、右或双边不变度量的存在，在 e 的邻域系具有可数基的附加假设下可从前面所述的事实推出．左不变度量的存在属于 Birkhoff[2] 和 Kakutani[1]．双边不变定理属于 Klee[1]．

应当注意，具有双边不变度量的拓扑群为可度量化的要求是很苛刻的．特别，这种类型的局部紧群有一个 Haar 测度，它同时在左和右平移作用下均为不变．

P　拓扑群的几乎开子集

拓扑空间 X 的子集 A 叫做在 X 中几乎开，或满足 Baire 条件当且仅当有稀疏集 B 使得对称差 $(A \sim B) \cup (B \sim A)$ 为开集．

(a) 子集 A 为在 X 中几乎开当且仅当有稀疏集 B 和 C 使得 $(A \sim B) \cup C$ 为开集．几乎开的集的可数并和余集仍为几乎开．每一个 Borel 集恒为几乎开．(Borel 集族是具有下列性质的最小集族 $\mathscr{B}$：$\mathscr{B}$ 包含所有的开集并且 $\mathscr{B}$ 的元的可数并和余集仍属于 $\mathscr{B}$)．

(b) Banach-Kuratowski-Pettis 定理．若 A 包含拓扑群 X 的一个非稀疏的几乎开子集，则 AA^{-1} 为单位元的一个邻域．(若 A 为非稀疏，则 X 也为非稀疏，又因 X 为拓扑群，故每一个非空开子集亦为非稀疏．对 X 的每一个几乎开子集 B，命 B^* 为所有使得 $U \cap (X \sim B)$ 为稀疏的开集 U 的并，则 $(xB)^* = xB^*$ 并且当 C 也为几乎开时 $(B \cap C)^* = B^* \cap C^*$．故 $xA^* \cap A^* = (xA \cap A)^*$ 并且若 $xA^* \cap A^*$ 为非空，则 $xA \cap A$ 也为非空．于是 $A^*(A^*)^{-1} = \{x: xA^* \cap A^*$ 为非空$\} \subset \{x: xA \cap A$ 为非空$\} = AA^{-1}$.)

(c) 非稀疏的拓扑群 X 的几乎开子群或者在 X 中为稀疏，或者在 X 中为既开又闭．

(d) 几乎开的要求从定理中不能删去．存在实数群 X 的子群 Y 使得商群 X/Y 为可数无限，并且因为对于 X/Y 的每一个元 Z 有从 X 到它自己上的一个同胚变 Y 到 Z 上，从而可推出 Y 在 X 中为非稀疏．(设 B 为 X 对于有理

数的 Hamel 基，C 为 B 的可数无限子集并且 Y 为所有 $B\sim C$ 的元的有限线性有理组合的集.）

注　关于定理 (b) 的历史和文献见 Pettis[1]. (d) 中的作法并不为实数所特有；有关现象也出现于许多更一般的情形. 这个问题的基本思想属于 Hausdorff；在此方向上最深刻的已知结果可在 Pettis[2] 中找到，其中也给出了历史和进一步的文献.

Q　拓扑群的完备扩张

设 $(G,\cdot,\mathscr{T})$ 为拓扑群，又设 $\mathscr{L}$ 为它的左一致结构，$\mathscr{R}$ 为它的右一致结构，并且 $\mathscr{U}$ 为它的双边一致结构（$\mathscr{U}$ 是大于 $\mathscr{L}$ 和 $\mathscr{R}$ 的每一个的最小一致结构）. 问题 6.O 已经指出 $\mathscr{T}$ 是 $\mathscr{L}$，$\mathscr{R}$ 和 $\mathscr{U}$ 的每一个的拓扑.

(a) $(G,\mathscr{L})$ 为完备当且仅当 $(G,\mathscr{R})$ 为完备. 又一个网关于 $\mathscr{U}$ 为 Cauchy 网当且仅当它关于 $\mathscr{L}$ 和 $\mathscr{R}$ 的每一个为 Cauchy 网. 于是若 $(G,\mathscr{L})$ 为完备，则 $(G,\mathscr{U})$ 也为完备. 另一方面，一致空间 $(G,\mathscr{L})$ 为完备，假如 $(G,\mathscr{U})$ 为完备并且该群具有性质：若 $\{x_n, n\in D\}$ 为关于 $\mathscr{L}$ 的 Cauchy 网，则 $\{x_n^{-1}, n\in D\}$ 亦为关于 $\mathscr{L}$ 的 Cauchy 网.（等价地，$\mathscr{L}$ 和 $\mathscr{R}$ 有相同的 Cauchy 网.）此外关于该群的确定的元的左平移为 $\mathscr{L}$ 一致连续，右平移为 $\mathscr{R}$ 一致连续，反演（从 x 变到 x^{-1} 的映射）为 $\mathscr{U}$ 一致连续. 而乘法（从 (x,y) 变到 xy 的映射）通常为不一致连续.

(b) **定理.** 设 $(G,\cdot,\mathscr{T})$ 为 Hausdorff 拓扑群，又设 $(H,\mathscr{V})$ 为一致空间 $(G,\mathscr{U})$ 的 Hausdorff 完备扩张，$\mathscr{S}$ 为 $\mathscr{V}$ 的拓扑，则群的运算·可以按一种唯一的方法扩张到 H 上使得 $(H,\cdot,\mathscr{S})$ 成为拓扑群并且 $\mathscr{V}$ 成为它的双边一致结构.

(c) 上述定理给出了拓扑群关于右一致结构的完备扩张，假如 $\mathscr{L}$ 和 $\mathscr{R}$ 有相同的 Cauchy 网. 又由 (a) 可知这个条件对于右完备扩张的存在是必要的. 但该条件并不经常满足. 例如：设 G 为所有从闭单位区间 $[0,1]$ 到它自己上的同胚的群，又群的运算为合成并且群的拓扑由（右不变）度量：$d(f,g)=\sup\{|f(x)-g(x)|: x\in[0,1]\}$ 所确定，则有 G 中的序列 $\{f_n, n\in\omega\}$，它一致收敛于一个非一对一的函数，因而序列 $\{(f_n)^{-1}, n\in\omega\}$ 关于左一致结构不是 Cauchy 序列. 然而群 G 关于双边一致结构 $\mathscr{U}$ 为完备，因为 $\mathscr{U}$ 是度量：$d(x,y)+d(x^{-1},y^{-1})$ 的一致结构.

(d) **定理.** 设 $(G,\cdot,\mathscr{T})$ 为可度量化的拓扑群，d 为度量化 G 的右不变度量，又设 $d^*(x,y)=d(x,y)+d(x^{-1},y^{-1})$，则双边一致结构 $\mathscr{U}$ 为度量

d^* 的一致结构. 又一致空间 $(G, \mathscr{U})$ 为完备当且仅当 G 对于某个其拓扑为 $\mathscr{T}$ 的度量为完备. (等价地, 当且仅当 G 为每一个它所拓扑地嵌入的度量空间中的一个 G_δ.)此外,若$\mathscr{L}$和$\mathscr{R}$有相同的 Cauchy 序列并且 G 对于某个其拓扑为 $\mathscr{T}$ 的度量为完备, 则 G 对于每一个其拓扑为 $\mathscr{T}$ 的右不变度量为完备. (见问题 6.K 和 6.P.)

注 有两种重要的特殊情况,对于它们可达到右完备扩张的目的. 若群的单位元有全有界的邻域,或反演(变 x 为 x^{-1} 的映射)在单位元的某个邻域上为一致连续,则每一个左 Cauchy 网也是右 Cauchy 网并且双边完备扩张就产生了右完备扩张. 这些结果都可以直接证明, 而没有很大的困难; 它们在 Bourbaki [1] 和 Weil [2] 中给出. (c) 的例属于 Dieudonné[3], 而 (d) 属于 Klee[1].

(d)的结果——从度量拓扑的完备性导出完备性——不能推广到不可度量化的群. (见问题 7.M.)

R 同态的连续性和开性: 闭图形定理

在整个问题中,假设 G 和 H 为 Hausdorff 拓扑群, $\mathscr{U}$ 为 G 中单位元的一切邻域的族, $\mathscr{V}$ 为 H 中相应的族.

(a) 闭图形定理. 设 G 为拓扑群, H 为可度量化的拓扑群并且关于它的右一致结构为完备,又设 f 为从 G 到 H 内的一个同态,合于:

(i) f 的图形为 $G\times H$ 的闭子集,

(ii) 当 $V\in\mathscr{V}$ 时有 $f^{-1}[V]$ 的闭包属于 $\mathscr{U}$.

那末 f 为连续.

对偶地,从 H 到 G 内的一个同态 g 为开的,假如

(i)* g 的图形为 $H\times G$ 的闭子集,

(ii)* 当 $V\in\mathscr{V}$ 时有 $g[V]$ 的闭包属于 $\mathscr{U}$.

(定理证明是分别对关系 f^{-1} 和 g 应用引理 6.36 而得到的. 对于 H 用一个右不变度量; H 对于每一个使它度量化的右不变度量为完备.)

(b) 若在上述定理中,假定 H 为 Lindelöf 空间(每一个开覆盖有一个可数子覆盖) 并且 G 为非稀疏, 则条件 (ii) 自然满足; 如果还有 $g(H)=G$, 那末 (ii)* 也自然满足. 若 G 和 H 为线性拓扑空间, f 和 g 为线性函数, $g(H)=G$ 并且 G 为非稀疏, 则 (ii) 和 (ii)* 都自然满足. (若 $V\in\mathscr{V}$, 则 $f[G]^-\subset Vf[G]$, 又若 H 为 Lindelöf 空间, 则 $f[G]$ 可用由 $f[G]$ 的元对 V 所作的平移中的可数多个所覆盖. 注意这些 V 的平移关于 f 的逆的闭包互相

同胚并且当G为非稀疏时有非空的内部. 因而$f^{-1}[V]^-$含有开集并且$(f^{-1}[V^{-1}V])^- \supset (f^{-1}[V^{-1}]f^{-1}[V])^- \supset f^{-1}[V^{-1}]^- f^{-1}[V]^- = (f^{-1}[V]^-)^{-1}(f^{-1}[V]^-)$. 这就推出了对每一个$\mathscr{V}$中的$V$有$f^{-1}[V^-]^- \in \mathscr{U}$, 一种类似的推理对$g$也适用. 在线性拓扑空间的情况下我们可以利用数量积来代替$\mathscr{V}$的元的平移.)

(c) 若H为局部紧拓扑群,则闭图形定理成立,即从(a)的(i)和(ii)可推出连续性,以及对偶的命题.(这是一个比上面更为简单的结果. 它依赖于引理 6.1.)

注 对于完备线性赋范空间,闭图形定理属于 Banach[1,4]. 该定理的每一种已知的形式都需要对H加上很强的可数性或紧性的假定. 许多引人注目的猜测的一个反例可如下作出. 设G为一个任意的无限维完备线性赋范空间,又设H为G并且带有这样的拓扑,它使得所有在每一个方向上都含有一个线段的凸集所构成的族为零点的邻域基,则从H到G上的恒等映射g为连续并且满足上述的(i)*和(ii)*(见问题 6.U (a)). 空间H具有许多有趣的性质: 例如它为完备,并且一致有界性定理(问题 6.U (b))对它成立. 然而g显然不是开映射.

S 可和性

设f为一个函数,它的定义域包含集A并且值域包含在完备 Abel 的 Hausdorff 拓扑群G内,又设$\mathscr{A}$为A的所有有限子集的族,并且对$\mathscr{A}$中的F,命S_F为$f(a)$的和,其中a属于F,则族$\mathscr{A}$关于$\supset$为有向集,并且$\{S_F, F \in \mathscr{A}, \supset\}$为$G$中的网. 若该网收敛于$G$的一个元$s$,则称$f$为在$A$上可和,并且定义$s$为$f$在$A$上的和,记作: $s = \Sigma\{f(a): a \in A\} = \Sigma_A f$.

(a) 关于可和性的 Cauchy 判别法. 函数f为在A上可和当且仅当对G的零点的每一个邻域U有A的有限子集B使得对于$A \sim B$的每一个有限子集C有$\Sigma_C f \in U$. 因而,在A上可和的函数也在A的每一个子集上可和.

(b) 若f和g为在A上可和,则$f+g$(其中$(f+g)(x) = f(x) + g(x)$)也为在A上可和并且$\Sigma_A(f+g) = \Sigma_A f + \Sigma_A g$.

(c) 若f在A上定义并且为可和,$\mathscr{B}$为A的这样的互不相交子集族,它覆盖A,则$\Sigma_A f = \Sigma\{\Sigma\{f(b): b \in B\}: B \in \mathscr{B}\}$. 然而,从累次和的存在并不能推出在$A$上的可和性.(对于一种特殊情况,从累次和的存在可推出在A上的可和性,见问题 2.G.)

T　一致局部紧空间

一致空间 $(X, \mathscr{U})$ 叫做**一致局部紧**当且仅当有$\mathscr{U}$的一个元U使得对X中的每一个x，$U[x]$为紧．特别，每一个局部紧拓扑群关于它的左和右一致结构为一致局部紧．

(a) 设$(X, \mathscr{U})$为一致空间，U为$\mathscr{U}$的一个元，又设$U_0 = U$并且对每一个正整数n，$U_n = U \circ U_{n-1}$，则对X的每一个子集A，集$\bigcup\{U_n[A]: n \in \omega\}$为既开又闭．

(b) 若U为$X \times X$的对角线的一个闭邻域，A为X的一个紧子集，并且对A中的每一个x，$U \circ U[x]$为紧，则$U[A]$为紧集．(由问题6.A可知$U[A]$为闭集．)

(c) 连通一致局部紧空间$(X, \mathscr{U})$为σ-紧(即X为可数多个紧子集的并)．

(d) 每一个一致局部紧空间为一个互不相交的开σ-紧子空间的族的并．因而，每一个这样的空间为仿紧．

(e) 设$(X, \mathscr{T})$为拓扑空间，则有其拓扑为$\mathscr{T}$的一致结构$\mathscr{U}$使得$(X, \mathscr{U})$为一致局部紧当且仅当$(X, \mathscr{T})$为局部紧和仿紧．(见定理5.28.)

注　(a)本质上是问题5.T的链推理．应注意到，问题5.T的关于连通区和连通集的命题不能推广到一致局部紧空间．

U　一致有界性定理

(a) 设X为实线性拓扑空间，它在它自己内为非稀疏，又设K为X的一个闭凸子集，它满足$K = -K$并且在每一个方向上都含有一个线段(即对每一个X中的x有正实数t使得当$0 \leqslant s \leqslant t$时有$sx \in K$)，则$K$为0点的邻域．(证明$K$在$X$中为非稀疏．然后由问题6.P可知$K - K$为0点的邻域并且由凸性可推出$2K$为0点的邻域．)

(b) **定理**．设F为一族从非稀疏线性拓扑空间X到线性赋范空间Y的连续线性函数并且假定对X的每一点x，$\sup\{\|f(x)\|: f \in F\}$为有限，则对于$X$的0点的某个邻域$U$，$\sup\{\|f(x)\|: x \in U$且$f \in F\}$为有限．(利用上述命题来证明，若$S$为$Y$的0点的单位球，则$\bigcap\{f^{-1}[S]: f \in F\}$为$X$的0点的邻域．)

注　命题(b)是经典的Banach-Steinhaus定理(Banach[1; 80])．它的这种叙述形式显然可得到某种推广；而这样推广的基本思想则是命题(a)．利用下一章的术语，(b)的结论还可以叙述成：F在0点处为同等连续．

V　Boole σ-环

一个 Boole 环 $(B,+,\cdot)$ 叫做 Boole σ-环当且仅当每一个可数子集对于 B 的自然序有一个上确界(见问题 2.K)。Boole σ-环的自然例子是:

(i) 环 $(\mathscr{L},\triangle,\cap)$,其中 $\mathscr{L}$ 为所有 $[0,1]$ 的 Lebesgue 可测子集的族,或环 $\mathscr{L}$ 关于所有测度为零的集所组成的族 $\mathfrak{N}$ 的商为 σ-环. (此处 $\triangle$ 为对称差. 族 $\mathfrak{N}$ 在一种明显的意义下实际上为 σ-理想.)

(ii) 环 $(\mathscr{A}/\mathfrak{m},\triangle,\cap)$,其中 $\mathscr{A}$ 为所有 $[0,1]$ 的 Borel 子集的族,$\mathfrak{m}$ 为由稀疏的 Borel 集所组成的子族。

这个问题的目的是给出任意 Boole σ-环的型 (ii) 的一种表示定理. 在整个问题中,假定 $\mathscr{B}$ 为一个局部紧 Boole 空间 X 的一切紧开子集的族. 不失普遍性,我们可以只限于考察型如 $(\mathscr{B},\triangle,\cap)$ 的环. (参看 Stone 表示定理 5.S.)

(a) 若 $(\mathscr{B},\triangle,\cap)$ 为一个 Boole σ-环,则可数多个 $\mathscr{B}$ 的元的并的闭包仍为 $\mathscr{B}$ 的元(即可数多个 X 的紧开子集的并的闭包仍为紧开子集)。

(b) 设 $\mathscr{A}$ 为 X 的子集族,它在这样意义下为最小,即 $\mathscr{B}\subset\mathscr{A}$ 并且 $\mathscr{A}$ 的元的可数并和对称差仍属于 $\mathscr{A}$,又设 $\mathfrak{m}$ 为所有 X 的稀疏子集的族,则对每一个 $\mathscr{A}$ 的元 A 有唯一的 $\mathscr{B}$ 的元 B 使得 $A\triangle B\in\mathfrak{m}$. (见问题 6.P(a).)

(c) 定理 Boole σ-环 $\mathscr{A}$ 为 $\mathscr{B}$ 和 σ-理想 $\mathscr{A}\cap\mathfrak{m}$ 的直接和. 因而 $\mathscr{B}$ 同构于 Boole σ-环 $\mathscr{A}$ 关于 σ-理想 $\mathscr{A}\cap\mathfrak{m}$ 的商.

注　这个问题的结果属于 Loomis[1]. 又具有性质: 开集的闭包仍为开集的空间(满足可数链条件的 Boole σ-环的 Stone 空间就是这样的空间),有时叫做极不连通空间. 在这种类型的紧空间上的实值有界 Borel 函数的空间,关于一种类似于命题 (c) 的方法可以分解成连续函数和在某个稀疏集外为零的函数. 这个事实以及其它结果均可参阅 M. H. Stone[4], 亦可参阅 Dixmier[1].

第七章 函数空间

本章主要讨论函数空间．即空间的元为从一个确定的集 X 到一个确定的拓扑空间或一致空间 Y 的函数．下面几乎都是考虑关于 X 的拓扑为连续的函数所作成的空间．简要地说，我们研究的目的是定义连续函数集的拓扑和一致结构，并且证明所得到的空间的紧性，完备性以及连续性的性质．

这一章结果的大部分在以前的实变数函数理论中有它的原始形式．但关于联合连续性和紧开拓扑的定理则是新近的，它最初属于 Fox[1]．至于函数空间的更进一步的情况可在 Arens [2]，Bourbaki [1]，Myers [2] 以及 Tukey [1] 中找到．

点式收敛

关于函数空间的一种拓扑以前就已经有过一些广泛的研究．设 F 为一个函数族，其中的每一个函数从集 X 到拓扑空间 Y，则 F 包含在乘积 $Y^X = \times\{Y: x \in X\}$ 内．F 的点式收敛（坐标收敛，简单收敛）拓扑或简称为**点式拓扑** $\mathscr{P}$ 是指相对乘积拓扑．于是网 $\{f_n, n \in D\}$ 收敛于 g 当且仅当对 X 中的每一个 x 有 $\{f_n(x), n \in D\}$ 收敛于 $g(x)$（见定理 3.4）．又所有形如 $\{f: f(x) \in U\}$ 的子集的族为 $\mathscr{P}$ 的一个子基，其中 x 为 X 中的点，U 为 Y 内的开集．另外对 X 的每一个点 x 有一个 F 上的函数 e_x，它对 F 中的一切 f 由 $e_x(f) = f(x)$ 所定义，该函数叫做在 x 处的计值映射（或在第 x 个坐标空间内的射影）．在 x 处的计值映射关于 $\mathscr{P}$ 为连续的开映射（定理 3.2），并且 $\mathscr{P}$ 为使得每一个计值映射为连续的 F 的最小拓扑．又从某个拓扑空间到 F 的函数 g 关于 $\mathscr{P}$ 为连续当且仅当对 X 的每一个点 x，$e_x \circ g$ 为连续（定理 3.3）．显然，点式拓扑只

依赖于函数族和 Y 的拓扑．因此，就是 X 给定了拓扑，我们也不需要把它写在定义或定理之中．若 Y 为 Hausdorff 或正则空间，则空间 F 遗传了相同的性质（定理 3.5 和问题 4.A），但当 Y 为局部紧或满足第一或第二可数性公理的空间时 F 可以不具有这些性质（定理 3.6 和 5.19）．

函数空间关于点式拓扑为紧的特征是关于紧空间乘积的 Tychonoff 定理（定理 5.13）的一个明显推论．在陈述这个结果之前，为方便起见，我们规定从集 X 到拓扑空间 Y 的函数族 F 为**点式闭**当且仅当 F 为乘积空间 Y^X 的闭子集．若 A 为 X 的子集，则 $F[A]$ 定义为所有点 $f(x)$ 的集，其中 $x\in A$，$f\in F$．特别，当 $x\in X$ 时 $F[\{x\}]$ 简记为 $F[x]$．若 e_x 为在 x 处的计值映射，则易见 $e_x[F]=F[x]$．

1．定理． 为了使得从集 X 到拓扑空间 Y 的函数族 F 关于点式收敛拓扑为紧，只需：

(a) F 在 Y^X 中为点式闭，

(b) 对 X 的每一个点 x，集 $F[x]$ 有紧的闭包．

若 Y 为 Hausdorff 空间，则条件 (a)，(b) 也是必要的．

证明．因为族 F 不仅为 Y^X 的子族，而且包含在 $\times\{F[x]^-: x\in X\}$ 内，故若条件 (b) 满足，则由 Tychonoff 乘积定理可知该乘积为 Y^X 的紧子集，又若 F 为点式闭，则 F 为紧．从而证明了 (a) 和 (b) 的充分性．

若 Y 为 Hausdorff 空间并且 F 关于点式拓扑为紧，则由定理 5.7 F 为闭集，又集 $F[x]$ 为闭紧集，因为在每一点 x 处的计值映射为从 F 到 Hausdorff 空间 Y 的连续映射．∎

应当指出，上一定理要比关于点式收敛拓扑的一些讨论更为重要．点式拓扑在许多情况下并不自然．例如，设 X 为一个集，又对 X 的每一个有限子集 A，命 C_A 为 A 的特征函数（即当 $x\in A$ 时 $C_A(x)=1$，又当 $x\notin A$ 时 $C_A(x)=0$），则因 X 的所有有限子集的族 $\mathscr{A}$ 关于 $\supset$ 为有向集，故 $\{C_A, A\in\mathscr{A}\}$ 为从 X 到闭单位区间的函数的网，这时该网收敛于恒等于 1 的函数 e，因为对每一点

x 有 $\{x\}\in\mathscr{A}$ 并且当 $A\supset\{x\}$ 时有 $C_A(x)=1$. 显然,使得有限集的特征函数"接近"于单位函数的拓扑对于许多目的来说是不适宜的. 更有趣的拓扑是使得收敛带有更强的限制,即更大的拓扑. 注意: 若 $(F,\mathscr{T})$ 为紧并且 $\mathscr{T}$ 大于点式收敛拓扑 $\mathscr{P}$, 则从 $(F,\mathscr{T})$ 到 $(F,\mathscr{P})$ 上的恒等映射 i 为连续, 并且当 $(F,\mathscr{P})$ 为 Hausdorff 空间时 i 必为同胚. 因而, 若 $(F,\mathscr{T})$ 为紧 Hausdorff 空间并且 $\mathscr{T}$ 大于点式拓扑,则 $\mathscr{T}$ 与点式收敛拓扑相同. 这个简单的注记就指出了证明函数空间 F 关于拓扑 $\mathscr{T}$ 为紧的一种典型方法. 我们首先证明 F 关于点式收敛拓扑为紧, 然后再证明 F 中 $\mathscr{P}$ 收敛的网亦必为 $\mathscr{T}$ 收敛. 若 Y 为 Hausdorff 空间,则只须集中注意力于这两个命题,因为当其中有一不成立时 F 关于 $\mathscr{T}$ 为非紧.

有时我们还需要考察关于定义域空间的某个子集的点的点式收敛. 假设 F 为一个函数族,其中的每一个函数从集 X 到拓扑空间 Y, 又设 A 为 X 的子集, 则有从 F 到乘积空间 Y^A 内的一个自然映射 R, 它由映 F 的每一个元 f 为它在 A 上的限制所定义: 即对 F 中的每一个 f 有 $R(f)=f|A$. 显然,使得 R 为连续的 F 的最小拓扑 $\mathscr{P}_A$, 由所有 Y^A 的开子集关于 R 的逆像组成. 该拓扑叫做**在 A 上点式收敛的拓扑**. 又所有形如 $\{f: f(x)\in U\}$ 的集的族是 $\mathscr{P}_A$ 的一个子基,其中 x 属于 A, U 为 Y 中的开集, 并且 F 中的网 $\{f_n, n\in D\}$ 关于 $\mathscr{P}_A$ 收敛于 g 当且仅当对 A 中的每一个 x, $\{f_n(x), n\in D\}$ 收敛于 $g(x)$. 另外映射 R 为一对一当且仅当对 F 的不同的元 f 与 g 有 A 的某个点 x 使得 $f(x)\neq g(x)$. 若 X 的子集 A 具有这个性质,则称它为**可分离族 F 的元**.

2 定理. 设 F 为一个函数族, 其中的每一个函数从集 X 到 Hausdorff 空间 Y, 又设 A 为 X 的子集,则带有在 A 上点式收敛的拓扑 $\mathscr{P}_A$ 的族 F 为 Hausdorff 空间当且仅当 A 可分离 F 的元. 若 F 关于在 X 上点式收敛的拓扑为紧,并且 A 可分离 F 的元, 则 $\mathscr{P}$ 与 $\mathscr{P}_A$ 一致.

证明. 因为乘积空间 Y^A 为 Hausdorff 空间,故由 $\mathscr{P}_A$ 的定义

可知具有该拓扑的 F 为 Hausdorff 空间当且仅当限制映射 R 为一对一，即当且仅当 A 可分离 F 的元.

因为 $\mathscr{P}_A\subset\mathscr{P}$，故从 $(F,\mathscr{P})$ 到 $(F,\mathscr{P}_A)$ 上的恒等映射 i 为连续，从而若 $(F,\mathscr{P})$ 为紧并且 $(F,\mathscr{P}_A)$ 为 Hausdorff 空间，则 i 必为同胚，即 $\mathscr{P}=\mathscr{P}_A$. |

若值域空间为一致空间，则点式收敛拓扑为某个一致结构的拓扑.

若 F 为从集 X 到一致空间 $(Y,\mathscr{V})$ 的函数族，则 F 为乘积 $\times\{Y;\ x\in X\}$ 的子集，并且我们称相对一致结构为**点式收敛**(或简单收敛)**的一致结构**，有时还简记为 $\mathscr{P}$ 一致结构. 至于它的性质，以前就已经研究过了(例如定理 6.25).

若 A 为 X 的子集，则在 A 上的点式收敛的一致结构，或简记为 $\mathscr{P}_A$ 一致结构，是指使得从 F 到所有从 A 到 Y 的函数的族内的限制映射 R 为一致连续的最小一致结构. 在这里我们不加证明地指出该一致结构的如下的简单的事实.

3 定理. 设 F 为从集 X 到一致空间 $(Y,\mathscr{V})$ 的函数族，又设 A 为 X 的子集，则在 A 上点式收敛的一致结构具有性质:

(a) 所有形如 $\{(f,\ g):\ (f(x),g(x))\in V\}$ 的集组成的族为 $\mathscr{P}_A$ 一致结构的一个子基，其中 V 属于 $\mathscr{V}$，x 属于 A.

(b) $\mathscr{P}_A$ 一致结构的拓扑为在 A 上点式收敛的拓扑.

(c) 网 $\{f_n,\ n\in D\}$ 为 Cauchy 网当且仅当对 A 中的每一个 x，$\{f_n(x),\ n\in D\}$ 为 Cauchy 网.

(d) 若 $(Y,\mathscr{V})$ 为完备并且 $R[F]$ 在 Y^A 中关于在 A 上的点式收敛为闭，则 F 关于 $\mathscr{P}_A$ 一致结构为完备.

紧开拓扑和联合连续性

给定从拓扑空间 X 到拓扑空间 Y 的函数族 F 的一种拓扑，我们自然要问 $f(x)$ 是否关于 f 和关于 x 为联合连续. 稍微更精确地说，也就是对于 F 的什么样拓扑才能使得当 $F\times X$ 给定乘积拓

扑时变 (f, x) 为 $f(x)$ 的从 $F\times X$ 到 Y 的映射为连续? 本节就来进行对这个问题的研究. 结果，我们知道函数空间有一种特殊的拓扑，它与该问题有关，同时我们也就从定义出这个拓扑并且推得它的若干初等性质开始. 这一节完全着眼于拓扑问题，而与此相联系的函数空间的一种一致结构将在以后再来讨论. 另外，在本节中我们恒假定 F 为一个函数族，并且其中的每一个函数从拓扑空间 X 到拓扑空间 Y.

为方便起见，对 X 的每一个子集 K 和 Y 的每一个子集 U，定义 $W(K, U)$ 为所有映 K 到 U 内的 F 的元的集，即 $W(K, U)=\{f: f[K]\subset U\}$. 于是 F 的**紧开拓扑**就是指以所有形如 $W(K,U)$ 的集组成的族为一个子基的拓扑，其中 K 为 X 的紧子集，U 为 Y 的开集. 这时，所有形如 $W(K, U)$ 的集的有限交的族也就是紧开拓扑的一个基，即该基的每一个元为形如：$\bigcap\{W(K_i, U_i): i=0, 1, \cdots, n\}$ 的集其中每一个 K_i 为 X 的紧子集，而每一个 U_i 为 Y 的开子集. 由于由单个点所组成的集为紧集，所以紧开拓扑与点式拓扑是可比较的.

4 定理. 紧开拓扑 $\mathscr{C}$ 包含点式收敛拓扑 $\mathscr{P}$. 空间 $(F, \mathscr{C})$ 为 Hausdorff 空间，假如值域空间 Y 为 Hausdorff 空间；又为正则空间，假如 Y 为正则空间并且 F 的每一个元为连续.

证明. 显然，对 X 中的每一个 x 和 Y 的每一个开子集 U，集 $W(\{x\}, U)=\{f: f(x)\in U\}$ 属于 $\mathscr{C}$，因为 $\{x\}$ 为紧集. 于是 $\mathscr{P}\subset\mathscr{C}$，因为所有这种形式的集的族为点式拓扑的一个子基.

因为若 Y 为 Hausdorff 空间，则由定理 3.5 可知 $(F, \mathscr{P})$ 亦为 Hausdorff 空间，又因若 U 和 V 为 F 的元的互不相交 $\mathscr{P}$ 邻域，则它亦为 $\mathscr{C}$ 邻域，故 $(F, \mathscr{C})$ 为 Hausdorff 空间.

最后，假定 Y 为正则空间，则需要证明 F 的每一个元 f 的每一个邻域都含有闭邻域. 显然这又只须证 f 的每一个属于 $\mathscr{C}$ 的一个子基的邻域含有闭邻域，因为 f 的每一个邻域含有一个属于该子基的邻域的有限交. 今设 $f\in W(K, U)$，其中 K 为 X 的紧子集，U 为 Y 的开子集，则因 $f[K]$ 为紧集，又 Y 为正则空间，故从定理

5.10 可推出有 $f[K]$ 的闭邻域 V 使得 $V\subset U$. 由于 $f\in W(K, V)\subset W(K, U)$ 并且易见 $W(K, V)$ 确为 f 的邻域,所以剩下要证的是 $W(K, V)$ 为闭集. 而这只须注意 $W(K, V)$ 为所有集 $W(\{x\}, V)$ 的交,其中 x 属于 K, 同时每一个集 $W(\{x\}, V)$ 必为 $\mathscr{P}$ 闭集,从而为 $\mathscr{C}$ 闭集. ▌

显然没有希望证明当 Y 为正规或满足第一或第二可数性公理的空间时 $(F, \mathscr{C})$ 也具有这些性质, 因为若 X 为离散空间, 则仅有的紧集为有限集,从而 $\mathscr{C}$ 与点式收敛拓扑相同, 但正规空间或满足某个可数性公理的空间的乘积可以不具有相应的性质, 于是带有拓扑 $\mathscr{C}$ 的 F 也可以不具有该性质.

设 P 为变 (f, x) 为 $f(x)$ 的从 $F\times X$ 到 Y 内的映射,并且对 F 的每一个拓扑在 $F\times X$ 上诱导了乘积拓扑, 现在问映射 P 关于该乘积拓扑是否为连续. 我们称 F 的拓扑为**联合连续**当且仅当从 $F\times X$ 到 Y 内的映射 P 为连续. 容易看出, 点式收敛拓扑通常为不联合连续. 而离散拓扑为联合连续,因为若 U 为 Y 的开子集,则 $P^{-1}[U]=\{(f, x): f(x)\in U\}=\bigcup\{\{f\}\times f^{-1}[U]: f\in F\}$,即它是开集的并(假定 F 为一个连续函数族). 又若 F 的某个拓扑为联合连续,则比它更大的拓扑亦为联合连续. 因而,一个自然的问题是寻找最小的联合连续的拓扑, 假如这样的拓扑存在的话. 一般的说,这样的最小拓扑是不存在的, 然而, 如果把联合连续性的条件稍微放宽, 那末它就能给出紧开拓扑一种确切的描述. 我们称函数族 F 的拓扑为**在集 A 上联合连续**当且仅当映射 P 为在 $F\times A$ 上连续,其中 $P(f, x)=f(x)$ (注意: 这并不表示 P 在 $F\times A$ 的每一点处为连续,而只是说明限制 $P|(F\times A)$ 为连续). 又称 F 的拓扑为**在紧集上联合连续**当且仅当它在定义域空间的每一个紧集上联合连续. 显然, 这样的族的每一个元 f 它在每一个紧集 K 上为连续(即 $f|K$ 为连续).

5 定理. 每一个在紧集上联合连续的拓扑大于紧开拓扑 $\mathscr{C}$. 若 X 为正则或 Hausdorff 空间并且 F 的每一个元为在 X 的每一个紧集上连续,则 $\mathscr{C}$ 为在紧集上联合连续.

证明．假定 F 的拓扑 $\mathscr{T}$ 为在紧集上联合连续，U 为 Y 的开子集，K 为 X 的紧子集，并且 P 为满足 $P(f, x) = f(x)$ 的映射，则需要证明 $W(K, U)$ 为 $\mathscr{T}$ 开集，其中 $W(K, U) = \{f: f[K] \subset U\}$．因为 $\mathscr{T}$ 为在紧集上联合连续，故集 $V = (F \times K) \cap P^{-1}[U]$ 为 $F \times K$ 中的开集，又当 $f \in W(K, U)$ 时有 $\{f\} \times K \subset V$，而 $\{f\} \times K$ 为紧集，于是由定理 5.12 有 f 的 $\mathscr{T}$ 邻域 N 使得 $N \times K \subset P^{-1}[U]$，这表明 f 的 $\mathscr{T}$ 邻域 N 的每一个元均为紧开邻域 $W(K, U)$ 的元，从而 $W(K, U)$ 为 $\mathscr{T}$ 开集，即定理的第一个命题获证．

现在来证明第二个结论．假设 K 为 X 的紧子集，$x \in K$，U 为 Y 的开子集并且 $(f, x) \in P^{-1}[U]$，则因 f 为在 K 上连续，故有紧集 M 使得它是 x 在 K 中的邻域并且有 $f[M] \subset U$（注意 X 为 Hausdorff 或正则空间），于是 $W(M, U) \times M$ 为 (f, x) 在 $F \times K$ 中的邻域并且它包含在 $P^{-1}[U]$ 内，从而得在 K 上的联合连续性．┃

可以指出，若 X 为局部紧，则拓扑为在紧集上联合连续当且仅当它为联合连续．因而，若 X 为局部紧正则空间，则连续函数族的紧开拓扑就是最小的联合连续拓扑．

若族 F 的拓扑 $\mathscr{T}$ 为在紧集上联合连续，则 $\mathscr{T} \supset \mathscr{C} \supset \mathscr{P}$，其中 $\mathscr{C}$ 为紧开拓扑，$\mathscr{P}$ 为点式拓扑．又若 $(F, \mathscr{T})$ 为紧并且值域空间为 Hausdorff 空间，则 $(F, \mathscr{P})$ 为 Hausdorff 空间，从而 $\mathscr{T} = \mathscr{C} = \mathscr{P}$．这个事实说明了在关于 $\mathscr{C}$ 紧性的下一定理中所给出的条件之一的必要性．为了直接应用于以后的问题，该结果按一种稍微特别的形式给出．

6 定理．设 X 为正则或 Hausdorff 空间，Y 为 Hausdorff 空间，C 为所有在 X 的每一个紧集上连续的从 X 到 Y 的函数的族，又设 $\mathscr{C}$ 和 $\mathscr{P}$ 分别为紧开和点式拓扑，则 C 的子族 F 为 $\mathscr{C}$ 紧当且仅当

(a) F 为 C 中的 $\mathscr{C}$ 闭集，

(b) 对 X 的每一个元 x，$F[x]$ 有紧的闭包，

(c) 对于 F 在 Y^X 中的 $\mathscr{P}$ 闭包，拓扑 $\mathscr{P}$ 为在紧集上联合连续．

证明．假设 F 为 $\mathscr{C}$ 紧，则因 Y 为 Hausdorff 空间，故 $(C, \mathscr{C})$

为 Hausdorff 空间，从而 F 为 $\mathscr{C}$ 闭集．由于计值映射在点 x 处为 $\mathscr{P}$ 连续，于是为 $\mathscr{C}$ 连续，所以 F 的像 $F[x]$ 为紧集．因为 F 为 $\mathscr{C}$ 紧并且为 $\mathscr{P}$ Hausdorff 空间，故 F 的拓扑 $\mathscr{C}$ 和 $\mathscr{P}$ 相同，从而 F 为 Y^X 中的 $\mathscr{P}$ 闭集，并且由定理 7.5 可知 F 的拓扑 $\mathscr{C}$（于是 $\mathscr{P}$）为在紧集上联合连续．这就完成了条件 (a)，(b) 和 (c) 的必要性的证明．

假设条件 (a)，(b) 和 (c) 成立．命 F^- 为 F 在 Y^X 中的 $\mathscr{P}$ 闭包，而条件 (b) 表明对每一个 x，$F[x]^-$ 为紧集，即 F^- 为 $\mathscr{P}$ 紧集 $\times\{F[x]^-: x\in X\}$ 的闭子集，故 F^- 为 $\mathscr{P}$ 紧集．又由条件 (c) 可知，对于 F^-，拓扑 $\mathscr{P}$ 为在紧集上联合连续，于是 F^- 的每一个元为在每一个紧集上连续，即 $F^-\subset C$，但从定理 7.5 可推出，对于 F^-，拓扑 $\mathscr{P}$ 大于 $\mathscr{C}$，从而对于 F^-，这两个拓扑相同．再由条件 (a) 可知族 F 为 C 中的 $\mathscr{C}$ 闭集，因而为 C 的子集 F^- 中的 $\mathscr{C}$（并且 $\mathscr{P}$）闭集，即 $F^-=F$，亦即 F 为 $\mathscr{C}$ 紧．|

7 注记． 所有在每一个紧子集上连续的函数的族 C 与所有连续函数的族一致，假如空间为局部紧或满足第一可数性公理（见定理 7.13 和它前面的讨论）．虽然所有连续函数的族通常是有趣的，但一些数学结构中也需要有类 C（而并不是凭一时的兴趣）．事实上在稍后的完备性的讨论中就出现了这个类．

紧开拓扑和联合连续性之间的关系首先是由 Fox[1] 研究的，他证明了对于连续函数族，紧开拓扑小于每一个联合连续拓扑，并且它自己为联合连续，假如定义域空间为局部紧．又最小的联合连续拓扑并不普遍存在的事实的证明见 Arens [2]．

一　致　收　敛

本节致力于从集 X 到一致空间 $(Y,\mathscr{V})$ 的函数族 F 的一种一致结构的研究．该一致结构与集 X 所给定的拓扑无关得到的主要结果之一是：所有关于 X 的拓扑为连续的函数的族为所有从 X 到 Y 的函数的空间的闭子空间．即连续函数的一致极限仍为连续函

数.

我们在这里所要考察的一致收敛的一致结构是最大的一致结构，而点式收敛一致结构则是最小的一致结构. 这两种一致结构都可以看成是在一个集族 $\mathscr{A}$ 的元上一致收敛的一致结构的特殊情形. 而这个概念将在这一节的最后加以扼要的讨论；对每一个 X 的子集族 $\mathscr{A}$ 作出了该一致结构，并且导出了有关的初等性质.

设 F 为从集 X 到一致空间 $(Y,\mathscr{V})$ 的函数族，对 $\mathscr{V}$ 的每一个元 V，命 $W(V)$ 为所有使得对 X 中的每一个 x 有 $(f(x),g(x))\in V$ 的 (f,g) 的集[1]，则 $W(V)[f]$ 为所有使得对 X 中的每一个 x 有 $g(x)\in V[f(x)]$ 的 g 的集. 又易见对 $\mathscr{V}$ 的所有元 U 和 V 有 $W(V^{-1})=(W(V))^{-1}$，$W(U\cap V)=W(U)\cap W(V)$ 和 $W(U\circ V)\supset W(U)\circ W(V)$ 成立. 因而由定理 6.2 可知所有集 $W(V)$ 的族为 F 的某个一致结构 $\mathscr{U}$ 的一个基，其中 V 属于 $\mathscr{V}$. 这个一致结构 $\mathscr{U}$ 就叫做**一致收敛的一致结构**，或简称为 **u.c. 一致结构**. 而 $\mathscr{U}$ 的拓扑则叫做**一致收敛的拓扑**，或简称 **u.c. 拓扑**.

显然 $\mathscr{U}$ 大于点式收敛的一致结构，因为若 y 是 X 的一个任意的元，$V\in\mathscr{V}$，则 $\{(f,g)$: 对 X 中的所有 x 有 $(f(x),g(x))\in V\}\subset\{(f,g):(f(y),g(y))\in V\}$，从而定义 $\mathscr{U}$ 的基的每一个元是定义点式一致结构的子基的某个元的子集. 这也就推出了 u.c. 拓扑大于点式拓扑. 又容易直接看出从一致收敛可推出点式收敛，因为 F 中的网 $\{f_n, n\in D\}$ 关于 u.c. 拓扑收敛于 g 当且仅当对 $\mathscr{V}$ 中的每一个 V，该网最终地在 $W(V)[g]$ 内，而这又当且仅当存在 D 中的某个 m 使得当 $n\geq m$ 时对 X 中的一切 x 有 $f_n(x)\in V[g(x)]$. 下列定理给出了一致结构 $\mathscr{U}$ 的一些其它初等性质.

8 定理. 设 F 为所有从集 X 到一致空间 $(Y,\mathscr{V})$ 的函数的族，$\mathscr{U}$ 为一致收敛的一致结构，则：

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1) 利用通常的关系概念，集 $W(V)$ 可以得到一种很简单的描述：$W(V)=\{(f,g): g\circ f^{-1}\subset V\}$. 该命题是明显的，因为 $g\circ f^{-1}$ 恰好是所有 $(f(x),g(x))$ 的集，其中 x 属于 X. 又易见 $W(V)=\{(f,g): g\subset V\circ f\}$，$W(V)[f]=\{g: g\subset V\circ f\}=\{g$: 对 X 中的每一个 x 有 $g(x)\in V[f(x)]\}$.

(a) 一致结构 $\mathscr{U}$ 由所有形如 $d^*(x, y)=\sup\{d(f(x), g(x)): x \in X\}$ 的伪度量的族所生成，其中 d 为 $(Y, \mathscr{V})$ 的格集的有界元.

(b) F 中的网 $\{f_n, n \in D\}$ 一致收敛于 g 当且仅当它关于 $\mathscr{U}$ 为 Cauchy 网并且对 X 中的每一个 x 有 $\{f_n(x), n \in D\}$ 收敛于 $g(x)$.

(c) 若 $(Y, \mathscr{V})$ 为完备，则一致空间 $(F, \mathscr{U})$ 也为完备.

证明. 为了证明命题 (a) 先注意所有形如 $\{(y, z): d(y, z) \leqslant r\}$ 的集的族是 $\mathscr{V}$ 的一个基，其中 r 为正数，d 为 $\mathscr{V}$ 的格集的有界元. 它之所以成立是因为对每一个伪度量 e，伪度量 $d = \min[1, e]$ 为有界并且具有相同的一致结构. 但 $\{(f, g): d^*(f, g) \leqslant r\} = \{(f, g)$: 对 X 中的每一个 x 有 $d(f(x), g(x)) \leqslant r\} = W(\{(y, z): d(y, z) \leqslant r\})$，其中 W 为定义 u.c. 一致结构时所利用的对应. 因而就推出了 d^* 属于 $\mathscr{U}$ 的格集并且这种形式的伪度量生成该格集.

命题 (b) 的一半是明显的，因此只须证若 Cauchy 网 $\{f_n, n \in D\}$ 点式收敛于 g，则它一致收敛于 g. 设 V 为 $\mathscr{V}$ 的一个任意的闭对称的元，选 D 中的 m 使得当 $n \geqslant m$, $p \geqslant m$ 时对 X 中的每一个 x 有 $f_p(x) \in V[f_n(x)]$；这样的选择是可能的，因为该网关于 $\mathscr{U}$ 为 Cauchy 网. 于是从 $V[f_n(x)]$ 为闭集和 $f_p(x)$ 收敛于 $g(x)$ 可推出 $g(x) \in V[f_n(x)]$，从而对每一个 $n \geqslant m$ 和 X 中的每一个 x 有 $f_n(x) \in V[g(x)]$，即命题 (b) 获证.

最后容易看出命题 (c) 是命题 (b) 和完备空间的乘积仍为完备空间的事实的一个直接推论. |

下列定理陈述了关于连续函数族的 $\mathscr{U}$ 的主要性质.

9 定理. 设 F 为所有从拓扑空间 X 到一致空间 $(Y, \mathscr{V})$ 的连续函数的族，$\mathscr{U}$ 为一致收敛的一致结构，则

(a) 族 F 为所有从 X 到 Y 的函数的空间的闭子空间，因而 $(F, \mathscr{U})$ 为完备，假如 $(Y, \mathscr{V})$ 为完备.

(b) 一致收敛的拓扑为联合连续.

证明. 显然要证命题 (a) 只须证所有不连续函数的集为所有从 X 到 Y 的函数的空间 G 的开子集. 设 f 在 X 的点 x 处为不连

续，则有$\mathscr{V}$的元V使得$f^{-1}[V[f(x)]]$不是x的邻域，再选$\mathscr{V}$的对称的元W使得 $W\circ W\circ W\subset V$．今证若函数g对每一个y满足$(g(y),\ f(y))\in W$，则 $g^{-1}[W[g(x)]]$ 不是x的邻域，从而g亦为不连续函数，这表明$G\sim F$关于一致收敛的拓扑为开子集．事实上，若对每一个y有$(g(y),\ f(y))\in W$，则 $g\subset W\circ f$ 并且$g^{-1}\subset f^{-1}\circ W^{-1}=f^{-1}\circ W$，因而 $g^{-1}\circ W\circ g\subset f^{-1}\circ W\circ W\circ W\circ f\subset f^{-1}\circ V\circ f$，于是 $g^{-1}[W[g(x)]]$ 为$f^{-1}[V[f(x)]]$的子集，即$g^{-1}[W[g(x)]]$不是x的邻域．

剩下的是命题(b)的证明．为了证明从$F\times X$到Y内的映射在点$(f,\ x)$处的连续性，我们只要验证对$\mathscr{V}$中的V，若$y\in f^{-1}[V[f(x)]]$并且对一切z有 $g(z)\in V[f(z)]$，则$g(y)\in V[f(y)]\subset V\circ V[f(x)]$．|

通过考察在定义域空间的子集族$\mathscr{A}$的每一个元上的一致收敛，我们还可以作出一些有用的一致结构．现在设F为从集X到一致空间$(Y,\mathscr{V})$的函数族，$\mathscr{A}$为X的子集族，**则在$\mathscr{A}$的每一个元上一致收敛的一致结构$\mathscr{U}|\mathscr{A}$**是指以所有形如$\{(f,g)$：对A中的所有x有$(f(x),\ g(x))\in V\}$的集组成的族为一个子基的一致结构，其中V属于$\mathscr{V}$，A属于$\mathscr{A}$．这个一致结构也可以通过另外的方法加以描述．若对$\mathscr{A}$中的每一个A，命R_A为变f为f在A上的限制的映射，即$R_A(f)=f|A$，则R_A变F为从A到Y的已给定一致收敛结构的函数族，$\mathscr{U}|\mathscr{A}$就可以看成是使得每一个R_A为一致连续的最小一致结构．

从上述关于一致收敛的命题，我们可导出关于一致结构$\mathscr{U}|\mathscr{A}$的相应的结果，这里略去它的简单证明．

10 定理． 设X为拓扑空间，$(Y,\mathscr{V})$为一致空间，又设$\mathscr{A}$为覆盖X的X的子集族，再设G为所有从X到Y的函数的族F，为所有在$\mathscr{A}$的每一个元上连续的函数的族，则：

(a) 在$\mathscr{A}$的每一个元上一致收敛的一致结构$\mathscr{U}|\mathscr{A}$大于点式收敛的一致结构并且小于在X上一致收敛的一致结构．

(b) 网$\{f_n,\ n\in D\}$关于$\mathscr{U}|\mathscr{A}$的拓扑收敛于g当且仅当它

为 Cauchy 网(关于 $\mathscr{U}|\mathscr{A}$) 并且点式收敛于 g.

(c) 若 $(Y,\mathscr{V})$ 为完备,则 G 关于 $\mathscr{U}|\mathscr{A}$ 为完备.

(d) 族 F 关于 $\mathscr{U}|\mathscr{A}$ 的拓扑为 G 中的闭集，因而若 $(Y,\mathscr{V})$ 为完备,则 $(F,\mathscr{U}|\mathscr{A})$ 也为完备.

(e) F 的 $\mathscr{U}|\mathscr{A}$ 的拓扑在 $\mathscr{A}$ 的每一个元上联合连续.

应当强调指出，所有连续函数的族关于 $\mathscr{U}|\mathscr{A}$ 可以不完备. 若 $\mathscr{A}$ 为所有集 $\{x\}$ 组成的族，其中 x 属于 X,则 $\mathscr{U}|\mathscr{A}$ 就是点式收敛的一致结构并且所有连续函数的族关于该一致结构一般地说就并不完备. 但是，如果 $\mathscr{A}$ 使得从在 $\mathscr{A}$ 的每一个元上的连续性可推出在 X 上的连续性，那末上一命题的 (d) 就证明了所有从 X 到完备空间的连续函数的族关于 $\mathscr{U}|\mathscr{A}$ 的完备性. 特别，当 X 的每一点都有属于 $\mathscr{A}$ 的邻域时就属于这种情形.

在紧集上的一致收敛

在本节中我们把沿着两个不同方向的研究结合起来. 假设 F 为从拓扑空间 X 到一致空间 $(Y,\mathscr{V})$ 的连续函数族，则**在紧集上一致收敛的一致结构**指的是一致结构 $\mathscr{U}|\mathscr{C}$,其中 $\mathscr{C}$ 是所有 X 的紧子集的族. $\mathscr{U}|\mathscr{C}$ 的拓扑有时叫做**紧收敛拓扑**. 现在证明该拓扑与由 X 的拓扑和一致结构 $\mathscr{V}$ 的拓扑所作出的紧开拓扑相同. 于是一致结构 $\mathscr{U}|\mathscr{C}$ 就依赖于 Y 的一致结构 $\mathscr{V}$,但 $\mathscr{U}|\mathscr{C}$ 的拓扑只依赖于 $\mathscr{V}$ 的拓扑. 一致结构 $\mathscr{U}|\mathscr{C}$ 在空间 X 具有“充分多”的紧集的情况下特别有用，并且在这一节的最后再简要讨论一下一类满足这种条件的空间.

11 定理. 设 F 为从拓扑空间 X 到一致空间 $(Y,\mathscr{V})$ 的连续函数族，则在紧集上一致收敛的拓扑就是紧开拓扑.

证明. 设 K 为 X 的紧子集，U 为 Y 的开子集，$f\in F$ 并且假定 $f[K]\subset U$，则 $f[K]$ 为紧集并且由定理 6.33 可知有 $\mathscr{V}$ 中的 V 使得 $V[f[K]]\subset U$，于是易见若 g 为使得对 K 中的每一个 x 有 $g(x)\in V[f(x)]$ 成立的函数，则 $g[K]\subset U$ 也成立. 因而每一个形如 $\{f:$

$f[K]\subset U\}$ 的集关于 $\mathscr{U}|\mathscr{C}$ 的拓扑为开集，亦即紧开拓扑小于 $\mathscr{U}|\mathscr{C}$ 的拓扑.

今证其逆，显然只须证对 X 的每一个紧子集 K，$\mathscr{V}$ 中的每一个 V 以及每一个连续函数 f 有 X 的紧子集 $K_1,\cdots,K_n$ 和 Y 的开子集 $U_1,\cdots,U_n$ 使得 $f[K_i]\subset U_i$ 并且当 $g[K_i]\subset U_i$ 对每一个 i 成立时 $g(x)\in V[f(x)]$ 对每一个 x 成立. 为此，选 $\mathscr{V}$ 的闭对称的元 W 使得 $W\circ W\circ W\subset V$，再选 K 中的 $x_1,\cdots,x_n$ 使得集 $W[f(x_i)]$ 的全体覆盖 $f[K]$，命 $K_i=K\cap f^{-1}[W[f(x_i)]]$，又命 U_i 为 $W\circ W[f(x_i)]$ 的内部. 于是，若对每一个 i 有 $g[K_i]\subset U_i$，则：对每一个 K 中的 x 有 i 使得 $x\in K_i$，从而 $g(x)\in W\circ W[f(x_i)]$，但 $f(x)\in W[f(x_i)]$，即 $(g(x),f(x))\in W\circ W\circ W\subset V$. |

若一致空间 $(Y,\mathscr{V})$ 为完备，$\mathscr{A}$ 为拓扑空间 X 的子集族，则由定理 7.10 可知所有在 $\mathscr{A}$ 的每一个元上连续的从 X 到 Y 的函数的族为 $\mathscr{U}|\mathscr{A}$ 完备. 于是为了使得所有连续函数的族为完备只须 $\mathscr{A}$ 满足条件：当函数在 $\mathscr{A}$ 的每一个元上连续时，则它连续. 若 f 为从 X 到 Y 的函数，B 为 Y 的子集，则上述条件可从下列事实推出：若对 $\mathscr{A}$ 的每一个元 A，$A\cap f^{-1}[B]$ 为闭集，则 $f^{-1}[B]$ 也为闭集. 特别，所有从 X 到 Y 的连续函数的空间关于在紧集上一致收敛为完备，假如 X 满足条件：若 X 的子集 A 与每一个闭紧集的交为闭集，则 A 为闭集. 我们称这样的拓扑空间为 **k 空间**. 显然，所有闭紧集的族 $\mathscr{C}$ 完全决定了 k 空间的拓扑，因为 A 为闭集当且仅当对 $\mathscr{C}$ 中的每一个 C 有 $A\cap C\in\mathscr{C}$. 又取余集即得 k 空间的子集 U 为开集当且仅当对每一个闭紧集 C，$U\cap C$ 为 C 中的开集.

由 k 空间定义和上述讨论便知下列定理成立.

12 定理. 所有从 k 空间到完备一致空间的连续函数的族关于在紧集上一致收敛为完备.

下面再给出 k 空间的两类最重要的例子.

13 定理. 若 X 为局部紧或满足第一可数性公理的 Hausdorff 空间，则 X 为 k 空间.

证明. 在每一种情况的证明之前，均假设 B 为 X 的非闭子集，

x 为不属于 B 的 B 的聚点，并且要证的是存在某个闭紧集 C 使得 $B\cap C$ 不是闭集。

若 X 为局部紧，则有 x 的紧邻域 U，交 $B\cap U$ 不是闭集，因为 x 是 B 的聚点，但不是 B 的元。

若 X 满足第一可数性公理，则有 $B\sim\{x\}$ 中的序列 $\{y_n, n\in\omega\}$，它收敛于 x，显然 $\{x\}$ 和所有点 y_n 组成的集的并为紧集，但它与 B 的交不是闭集。|

紧性和同等连续性

本节是关于寻找函数族对紧开拓扑为紧的条件的两节中的第一节．所期望的结论是拓扑的，并且最深刻的结果也是在纯拓扑的前提下得到的．然而，对一致结构，论证更为简单，因此这一节就先讨论映到一致空间内的映射．

至于纯拓扑的问题则留到本章的最后一节再来讨论．

设 F 为一族从拓扑空间 X 到一致空间 $(Y, \mathscr{V})$ 内的映射，则族 F **在点 x 处为同等连续**当且仅当对 $\mathscr{V}$ 的每一个元 V 有 x 的邻域 U 使得 $f[U]\subset V[f(x)]$ 对 F 的每一个元 f 成立．等价地有，F 在 x 处为同等连续当且仅当对 $\mathscr{V}$ 中的每一个 V，$\bigcap\{f^{-1}[V[f(x)]]: f\in F\}$ 为 x 的邻域．另外，粗略地说，F 在 x 处为同等连续当且仅当有 x 的邻域，它关于 F 的每一个元的像均为充分小．

14 定理． 若 F 在 x 处为同等连续，则 F 关于点式收敛拓扑 $\mathscr{P}$ 的闭包在 x 处也为同等连续．

证明．若 V 为 Y 的一致结构的一个闭的元，则所有满足条件 $f[U]\subset V[f(x)]$ 的函数 f 的类显然关于点式收敛拓扑为闭集，因为它与 $\bigcap\{\{f: (f(y), f(x))\in V\}: y\in U\}$ 相同．这样也就推出了 F 的点式闭包在 x 处为同等连续．|

我们称函数族 F 为**同等连续**当且仅当它在每一点处为同等连续．根据上一定理，同等连续函数族关于点式收敛拓扑的闭包仍为同等连续；特别，该闭包的元必为连续函数．对于同等连续函数

族，点式收敛拓扑还有其它值得注意的性质.

15 定理. 若 F 为同等连续，则点式收敛拓扑为联合连续，从而与在紧集上一致收敛的拓扑相同.

证明. 为了证明从 $F\times X$ 到 Y 内的映射在 (f,x) 处为连续，设 V 为 Y 的一致结构的元，U 为 x 的邻域，它使得对 F 中的所有 g 有 $g[U]\subset V[g(x)]$. 于是，若 g 为 f 的 $\mathscr{P}$ 邻域 $\{h:\ h(x)\in V[f(x)]\}$ 的元，$y\in U$，则有 $g(y)\in V[g(x)]$ 和 $g(x)\in V[f(x)]$，即 $g(y)\in V\circ V[f(x)]$，从而同等连续性获证.

因为由定理 7.5 可知每一个联合连续拓扑大于紧开拓扑，又由定理 7.11 可知紧开拓扑与在紧集上一致收敛的拓扑相同，故点式收敛拓扑与在紧集上一致收敛的拓扑相同. ▮

由上一定理我们可推出同等连续函数族关于在紧集上一致收敛的拓扑为紧，假如它关于点式拓扑 $\mathscr{P}$ 为紧，又 Tychonoff 乘积定理给出了 $\mathscr{P}$ 为紧的充分条件. 按照这个方法，从同等连续性和某些其它条件即可推出函数族的紧性. 另一方面，下列定理又给出了从紧性推到同等连续性的结果.

16 定理. 若从拓扑空间 X 到一致空间 $(Y,\mathscr{V})$ 的函数族 F 关于联合连续拓扑为紧，则 F 为同等连续.

证明. 假设 x 为 X 的一个确定的点，V 为 $\mathscr{V}$ 的一个对称的元，显然若能证得：存在 x 的邻域 U 使得对 F 中的每一个 g 有 $g[U]\subset V\circ V[g(x)]$，则定理就已获证. 因为 F 的拓扑为联合连续，故对 F 的每一个元 f 有 f 的邻域 G 和 x 的邻域 W 使得 $G\times W$ 映到 $V[f(x)]$ 内，于是当 $g\in G$ 并且 $w\in W$ 时有 $g(x)$ 和 $g(w)$ 属于 $V[f(x)]$，从而 $g(w)\in V\circ V[f(x)]$，即对 G 中的每一个 g 有 $g[W]\subset V\circ V[g(x)]$. 又因为 F 为紧，故有覆盖 F 的有限族 $G_1,\cdots,G_n$ 以及相应的 x 的邻域 $W_1,\cdots,W_n$ 使得对 G_i 中的每一个 g 有 $g[W_i]\subset V\circ V[g(x)]$. 于是，若命 U 为 x 的邻域 W_i 的交，则易见对 G 中的每一个 g 有 $g[U]\subset V\circ V[g(x)]$. ▮

关于局部紧空间的 Ascoli 定理是前面的一些结果的一个直接推论. 事实上，由定理 7.6 便知我们只须把条件"F 的 $\mathscr{P}$ 闭包上的

点式拓扑$\mathscr{P}$在紧集上联合连续”换为条件“族F为同等连续”。又由定理 7.14 和 7.15，从后一条件可推出前一条件，而由定理 7.16 从紧性又可推出同等连续性（我们也可以简单地作出不依赖于定理 7.6 的一种证明）。

17 Ascoli 定理．设C为所有从正则局部紧空间到 Hausdorff 一致空间的连续函数的族并且带有在紧集上一致收敛的拓扑，则C的子族F为紧当且仅当

(a) F为C中的闭集；

(b) 对X的每一个元x，$F[x]$有紧的闭包；

(c) 族F为同等连续．

对于k空间（即与每一个闭紧集的交恒为闭集的集也必为闭集的空间）上的函数族也有某种形式的 Ascoli 定理成立．这时需要一种稍加改变了的同等连续性概念．我们称函数族F为**在集A上同等连续**当且仅当所有F的元在A上的限制的族为同等连续．注意，在A的每一点处为同等连续的函数族必为在A上同等连续，但其逆不真．然而仍然有：在A上同等连续的函数族必在A的内部的每一点处为同等连续．

我们略去下一定理的证明．它是定理 7.6，这一节的结果以及在每一个紧集上连续的k空间上的函数必为连续的事实的一个直接推论[1]．

18 Ascoli 定理．设C为所有从 Hausdorff 或正则的k空间X到 Hausdorff 一致空间的连续函数的族并且带有在紧集上一致收敛的拓扑，则C的子族F为紧当且仅当

(a) F为C中的闭集；

(b) 对X中的每一个x，$F[x]$的闭包为紧；

(c) F在X的每一个紧子集上为同等连续．

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1) 显然“X为k空间”的条件可从定理的假设中删去，假如把所有连续函数的族换成是所有在每一个紧集上为连续的函数的族．而且，同一个结果通过对具有使得：集A为$\mathscr{T}$闭集当且仅当对每一个闭紧集B，$A\cap B$为闭集的拓扑$\mathscr{T}$的X，应用已给的定理也可得到．

*齐-连续性

本节着重讨论对拓扑空间的一种 Ascoli 定理的证明. 它的处理方法,除了以一种拓扑概念来代替同等连续性的(一致)概念外,和以前差不多一样. 至于这两种概念之间的联系则在这一节的最后再简要地加以讨论.

设 F 为一个函数族, 其中的每一个是从拓扑空间 X 到拓扑空间 Y 的函数, 则齐-连续性概念可直观地描述如下: 对每一个 X 中的 x, Y 中的 y 和 F 中的 f, 当 $f(x)$ 接近 y 时 f 映接近 x 的点为接近 y 的点. 显然, 我们称族 F 为**齐-连续**, 当且仅当对每一个 X 中的 x, Y 中的 y 和每一个 y 的邻域 U 有 x 的邻域 V 和 y 的邻域 W 使得当 $f(x)\in W$ 时有 $f[V]\subset U$. 下面的结论就着重指出了这个定义与联合连续性之间的紧密联系: F 为齐-连续当且仅当对每一个 X 中的 x, Y 中的 y 和每一个 y 的邻域 U 有 x 的邻域 V 和 y 的邻域 W 使得 $\{f: f\in F$ 且 $f(x)\in W\}\times V$ 关于自然映射的像包含在 U 内. 至于齐-连续函数族的这个重要性质的证明则是容易的.

19 定理. 设 F 为从拓扑空间 X 到正则空间 Y 的齐-连续的函数族, $\mathscr{P}$ 为点式收敛拓扑, 则 F 的 $\mathscr{P}$ 闭包 F^- 为齐-连续并且在 $F^-\mathscr{P}$ 上为联合连续.

证明. 定理的后一结论由齐-连续性定义的第二种形式即可推得,因为当 W 为 Y 的开集时 $\{f: f\in F$ 且 $f(x)\in W\}$ 为 $\mathscr{P}$ 开集.

今证 F 的 $\mathscr{P}$ 闭包为齐-连续. 假设 $x\in X$, $y\in Y$ 并且 U 为 y 的邻域, 则因 Y 为正则空间, 故不妨设 U 为闭集. 设 V 为 x 的邻域, W 为 y 的开邻域, 它使得当 $f\in F$ 且 $f(x)\in W$ 时有 $f[V]\subset U$, 又设 $\{g_n, n\in D\}$ 为 F 中的网,它点式收敛于 g 并且 $g(x)\in W$, 则 $\{g_n(x), n\in D\}$ 最终地在 W 内,从而对 V 中的每一个 z 有 $\{g_n(z), n\in D\}$ 最终地在 U 内,于是 $g(z)\in U$, 这就证明了 $g[V]\subset U$. ∎

根据上一结果和定理 7.6, 关于齐-连续函数族的紧性的充分

条件就几乎是明显的了．而下一命题则证明了在 Ascoli 定理中所给出的这个条件的必要性．

20 定理． 若从拓扑空间 X 到正则 Hausdorff 空间 Y 的连续函数族 F 关于联合连续拓扑为紧，则 F 为齐-连续．

证明．因为从紧空间 F 到带有点式收敛拓扑的 F 内的恒等映射为连续，又后一拓扑为 Hausdorff 拓扑，故这两个拓扑相同，于是 F 的点式收敛拓扑为联合连续．假设 $x\in X$，$y\in Y$ 并且 U 为 y 的开邻域，取 y 的闭邻域 W 使得 $W\subset U$，则所有使得 $f(x)\in W$ 的 F 的元 f 的集 K 为点式闭，从而为紧集．命 P 为使得 $P(f,x)=f(x)$ 的函数，则紧集 $K\times\{x\}$ 包含在 $P^{-1}[U]$ 内，但 P 为连续，故由定理 5.12 有 x 的邻域 V 使得 $K\times V\subset P^{-1}[U]$，即当 $v\in V$ 且 $f(x)\in W$ 时有 $f(v)\in U$．|

21 Ascoli 定理． 设 C 为所有从正则局部紧空间 X 到正则 Hausdorff 空间 Y 的连续函数的族并且带有紧开拓扑，则 C 的子集 F 为紧当且仅当

(a) F 为 C 中的闭集；

(b) 对 X 中的每一个 x，$F[x]$ 的闭包为紧；

(c) F 为齐-连续．

证明．若 F 关于紧开拓扑为紧，则由定理 7.6 和 7.20 可推出条件 (a)，(b)，(c)．

若 F 满足条件(a)，(b)，(c)，则 F 的点式闭包为一个齐-连续的函数族，但由定理 7.19 可知关于它点式拓扑为联合连续，故再由定理 7.6 即可推出紧性．|

上一定理可以在定理 7.17 所已经推广了的相同方式下推广到 k 空间．我们称函数族 F 为**在集 A 上齐-连续**当且仅当所有 F 的元在 A 上的限制的族为齐-连续．根据这个定义，自然就能够对 k 空间 X 证明 Ascoli 定理(定理 7.21)，只要将条件 (c) 换成“F 为在 X 的每一个紧子集上齐-连续”．至于该事实的直接证明则从略．

在这一节的最后，我们叙述两个命题，它阐明了齐-连续性和同等连续性之间的关系．

22 定理. 从拓扑空间到一致空间的同等连续的函数族为齐-连续.

证明. 假设 F 为从 X 到 Y 的同等连续的函数族，$x \in X$，$y \in Y$ 并且 U 为 y 的邻域，则我们可以假定 U 为以 y 为心，d-半径为 r 的球，其中 d 为 Y 的格集的伪度量并且 $r > 0$. 因为 F 在 x 处为同等连续，故有 x 的邻域 V 使得当 $z \in V$ 时对所有 F 中的 f 有 $d(f(x), f(z)) < r/2$，从而当 $z \in V$ 并且 $f(x)$ 属于以 y 为心，d-半径为 $r/2$ 的球时有 $f(z) \in U$. |

在某种意义下，同等连续性又是关于值域空间"一致化了"的齐-连续性，并且正如我们所期望的那样，在一种适当的紧性条件的前提下，从齐-连续性即可推出同等连续性.

23 定理1). 若 F 为从拓扑空间 X 到一致空间 Y 的齐-连续的函数族，并且对 X 的点 x，$F[x]$ 有紧的闭包，则 F 在 x 处为同等连续.

证明. 假设 d 为 Y 的格集的元并且 $r > 0$，则对 $F[x]^-$ 中的每一个 y 有 y 的邻域 W 和 x 的邻域 V 使得当 $f(x) \in W$ 时 $f[V]$ 包含在以 y 为心，d-半径为 $r/2$ 的球内，但 $F[x]^-$ 为紧集，故有有限多个 $F[x]^-$ 的点 y_i 的邻域 W_i 和相应的 x 的邻域 V_i，其中 $i = 1, \cdots n$，使得所有 W_i 的族覆盖 $F[x]^-$ 并且当 $f(x) \in W_i$ 时 $f[V_i]$ 为以 y_i 为心，d-半径为 $r/2$ 的球的子集. 因而，若 $z \in T = \bigcap\{V_i: i = 0, 1, \cdots, n\}$ 并且 $f \in F$，则 $f(x)$ 属于某个 W_i，但 $f[T]$ 为某个 d-半径为 $r/2$ 的球的子集，故对 T 中的每一个 y 有 $d(f(x), f(y)) < r$，即 F 为同等连续. |

24 注记. 本节的结果属于 A. P. Morse 和我自己. 对拓扑空间的 Ascoli 定理的其它形式已由 Gale[1] 得到.

1) 条件"$F[x]$ 有紧的闭包"换成"$F[x]$ 为全有界"时该定理不成立.

问 题

A 关于点式收敛拓扑的习题

Tychonoff 空间 X 上的所有连续实值函数组成的集关于点式收敛拓扑在 X 上的所有实值函数组成的集中稠密.

B 关于函数的收敛的习题

设 f 为闭单位区间 $[0, 1]$ 上的连续实值函数，它使得 $f(0)=f(1)=0$ 并且 f 不恒等于零.对每一个非负整数 n，命 $g_n(x)=f(x^n)$，则 $\{g_n, n\in\omega\}$ 点式收敛（但不一致收敛）于恒等于零的函数 h. 又 $\{h\}$ 与所有 g_n 组成的集的并关于点式收敛拓扑为紧集，但关于一致收敛拓扑为非紧集.

C 在稠密子集上的点式收敛

设 F 为从拓扑空间 X 到一个一致空间的同等连续的函数族，A 为 X 的稠密子集，则 X 上的点式收敛的一致结构与 A 上的点式收敛的一致结构相同.

D 对角线方法和列紧性

比 Tychonoff 乘积定理证明本身更为重要的是如下的对角线方法，它是证明函数族紧性的典型方法. 回忆一下，拓扑空间叫做序列紧，假如空间的每一个序列都有收敛于该空间的点的子序列.

(a) 可数多个序列紧拓扑空间的乘积仍为列紧.（假设 $\{Y_m, m\in\omega\}$ 为列紧空间组成的序列，又 $\{f_n, n\in\omega\}$ 为乘积 $\times\{Y_m, m\in\omega\}$ 中的序列. 选 ω 的无限子集 A_0 使得 $\{f_n(0), n\in A_0\}$ 收敛于 Y_0 的一个点，又归纳的选 A_k 的无限子集 A_{k+1} 使得 $\{f_n(k+1), n\in A_{k+1}\}$ 收敛于 Y_{k+1} 的一个点. 若 N_k 为 A_k 的第 k 个元，则 $\{f_{N_k}, k\in\omega\}$ 即为所需的子序列.）

(b) 设 Y 为序列紧的一致空间，X 为可分拓扑空间. 又设 F 为从 X 到 Y 的同等连续的函数族并且关于点式收敛拓扑为 Y^X 中的闭集，则 F 关于点式收敛拓扑（或紧开拓扑）为序列紧.（利用问题 7.C 并且注意 Y 中的每一个 Cauchy 网都有极限点.）

注 关于函数空间的可数紧性的若干很好的结果，最近由 Grothendieck[1] 得到. 他的结果可以直接应用于线性拓扑空间的一些有趣的问题.

E Dini 定理

若拓扑空间 X 上的连续实值函数的单调增加的网 $\{f_n, n\in D\}$ 点式收敛于连续函数 f，则该网在紧集上一致收敛于 f.（这是一种简单的紧性论证．对 X 的紧子集 C，命 $A_n=\{(x,y):\ x\in C$ 且 $f_n(x)\leq y\leq f(x)\}$ 并且注意所有集 A_n 的交就是 $f|C$ 的图形，其中 n 属于 D.）

F 一种诱导映射的连续性

设 X 和 Y 为集，$\mathscr{A}$ 和 $\mathscr{B}$ 分别为 X 和 Y 的子集族，又设 F 为所有从 X 到一致空间 $(Z,\mathscr{U})$ 的函数组成的族，G 为所有从 Y 到 $(Z,\mathscr{U})$ 的函数组成的族．若 T 为从 X 到 Y 内的映射，则从 G 到 F 内的诱导映射 T^* 定义为：当 g 属于 G 时 $T^*(g)=g\circ T$．若对 $\mathscr{A}$ 的每一个元 A 集 $T[A]$ 包含在 $\mathscr{B}$ 的某个元内，则 T^* 关于 F 的一致结构 $\mathscr{U}|\mathscr{A}$ 和 G 的一致结构 $\mathscr{U}|\mathscr{B}$（分别在 $\mathscr{A}$ 和 $\mathscr{B}$ 的元上一致收敛）为一致连续．特别，T^* 关于一致收敛的一致结构为一致连续并且关于点式收敛的一致结构为连续，假如 $\mathscr{B}$ 覆盖 Y．若 X 和 Y 为拓扑空间并且 T 为连续，则 T^* 关于在紧集上的一致收敛为一致连续．

注　某些其它种类的自然诱导映射已经由 Arens 和 Dugundji[2] 研究过．

G 一致同等连续性

从一致空间 $(X,\mathscr{U})$ 到 $(Y,\mathscr{V})$ 一致空间的函数族 F 叫做**一致同等连续**当且仅当对 $\mathscr{V}$ 的每一个元 V 有 $\mathscr{U}$ 中的 U 使得当 $f\in F$ 并且 $(x,y)\in U$ 时有 $(f(x),f(y))\in V$．

(a) 族 F 为一致同等连续当且仅当它在下列意义下为一致联合连续，即当 F 的一致结构为一致收敛的一致结构并且 $F\times X$ 带有乘积一致结构时，从 $F\times X$ 到 Y 内的自然映射为一致连续．

(b) 一致同等连续的函数族的点式闭包仍为一致同等连续．

(c) 若 X 为紧并且 F 为同等连续，则 F 为一致同等连续．

注　上述命题的证明不需要新的方法．这一方面的更详细的论述已在 Arens [2] 和 Bourbaki [1] 中给出．

H 关于一致结构 $\mathscr{U}|\mathscr{A}$ 的习题

设 X 为一个集，$\mathscr{A}$ 为 X 的覆盖，它关于 $\supset$ 为有向集（即对 $\mathscr{A}$ 中的 A 和 B 有 $\mathscr{A}$ 中的 C 使得 $C\supset A\cup B$），又设 $(Y,\mathscr{V})$ 为一致空间，再设 F 为从 X 到 Y 的函数族，它带有在 $\mathscr{A}$ 的元上一致收敛的一致结构 $\mathscr{U}|\mathscr{A}$．最后假设 S 为 F 中

的网并且对$\mathscr{A}$的每一个元A有S的一个给定的子网$\{S\circ T_A(m),\ m\in E_A\}$，它在$A$上一致收敛于$F$的元$s$. 给出关于$\mathscr{U}|\mathscr{A}$的拓扑收敛于$s$的$S$的子网的明显表达式.

I　计值映射的连续性

若F为从集X到集Y的函数族，则计值映射映X到从F到Y的函数族G内；这里在X的点x处的计值映射$E(x)$是定义为$E(x)(f)=f(x)$，其中f属于F. 设$(X,\mathscr{U})$和$(Y,\mathscr{V})$为一致空间，又设G带有在F的子集族$\mathscr{A}$的元上一致收敛的一致结构，则从X到G内的计值映射E为连续，假如$\mathscr{A}$的每一个元为同等连续，又计值映射为一致连续，假如$\mathscr{A}$的每一个元为一致同等连续.

J　k空间的子空间，乘积空间和商空间

(a) 存在 Tychonoff 空间它不是k空间，且每一个 Tychonoff 空间可嵌入到某个紧 Hausdorff 空间内，由此可推出并不是k空间的每一个子空间都仍为k空间. (见问题 2.E 的例.)

(b) 不可数多个实直线的乘积不是k空间. (设A为由所有这样的元x所组成的该乘积的子集，它使得对某个非负整数n，除在一个至多n个指标的集外，x的每一个坐标等于n，并且在该集上x的坐标为零，则A不是闭集，但对每一个紧集C，$A\cap C$为紧集.)

(c) 设X为k空间，R为X上的等价关系，并且X/R带有商拓扑. 若X/R为 Hausdorff 空间，则它也为k空间.

K　拓扑的k扩张

设$(X,\mathscr{T})$为 Hausdorff 空间，则$\mathscr{T}$的k扩张定义为所有使得：对每一个紧集C，$U\cap C$为C中的开集的X的子集U组成的族$\mathscr{T}_k$(等价地有，A为$\mathscr{T}_k$闭集当且仅当对每一个$\mathscr{T}$紧集C，$A\cap C$为$\mathscr{T}$紧集).

(a) 若C为X的$\mathscr{T}$紧子集，则$\mathscr{T}$关于C的相对拓扑与$\mathscr{T}_k$的相对拓扑相同. 因此集为$\mathscr{T}$紧集当且仅当它为$\mathscr{T}_k$紧集.

(b) 空间$(X,\mathscr{T}_k)$为k空间.

(c) X上的函数为$\mathscr{T}_k$连续当且仅当它在X的每一个紧子集上为$\mathscr{T}$连续.

(d) 拓扑$\mathscr{T}_k$为在紧集上与$\mathscr{T}$相同(在对紧集的相对拓扑与$\mathscr{T}$的相对拓

扑相同的意义下)的最大拓扑.

L　齐-连续性的刻划

从拓扑空间 X 到拓扑空间 Y 的函数族 F 为齐-连续当且仅当对 $F\times X$ 中每一个使得 $\{x_n,\ n\in D\}$ 收敛于 x 并且 $\{f_n(x),\ n\in D\}$ 收敛于 y 的网 $\{(f_n,\ x_n),\ n\in D\}$ 有 $\{f_n(x_n),\ n\in D\}$ 收敛于 y.

M　连续收敛

设 F 为从空间 X 到空间 Y 的连续函数族. 网 $\{f_n,\ n\in D\}$ 叫做连续收敛于 F 的元 f 当且仅当 $\{x_n,\ n\in D\}$ 为收敛于点 x 的 X 中的网时有 $\{f_n(x_n),\ n\in D\}$ 收敛于 $f(x)$.

(a) F 的拓扑 $\mathscr{T}$ 为联合连续当且仅当对 F 中的网,只要它 $\mathscr{T}$ 收敛于元 f,则也连续收敛于 f.

(b) 若 F 中的序列关于紧开拓扑收敛于 f,则它连续收敛于 f.

(c) 假设 X 满足第一可数性公理,又假设带有紧开拓扑 $\mathscr{C}$ 的 F 也满足该公理,则 $\mathscr{C}$ 为联合连续并且 F 中的序列 $\mathscr{C}$ 收敛于元 f 当且仅当它连续收敛于 f.

N　线性赋范空间的共轭空间

设 X 为实线性赋范空间,X^* 为它的共轭空间,即所有 X 上的实值连续线性函数的空间.定义 X^* 的范数拓扑为: $\|f\| = \sup\{|f(x)| : \|x\|\leqslant 1\}$. 又称 X^* 的点式收敛拓扑为 w^* 拓扑. 另外,X^* 的子集 F 叫做 w^* 有界当且仅当对 X 的每一个元 x,所有 $f(x)$ 的集为有界,其中 f 属于 F.

(a) 空间 X^* 关于 w^* 拓扑为不完备, 除非 X 上的每一个线性函数均为连续.(见问题 3.W. 假定有足够多的 X 上连续线性泛函可以分离点——这个事实是 Hahn-Banach 定理的一个推论, Banach [1;27].)

(b) 定理 (Alaoglu). X^* 的单位球关于 w^* 拓扑为紧集. 因而,X^* 的每一个范数有界的 w^* 闭子集为 w^* 紧集.(单位球为乘积 $\times\{[-\|x\|,\ \|x\|]: x\in X\}$ 中的闭子集.)

(c) 带有 w^* 拓扑的空间 X^* 为仿紧,从而为拓扑完备.(见问题 5.Y 和问题 6.L.)

(d) 若 X^* 的子集 F 为同等连续,则它的 w^* 闭包亦为同等连续. 若 F 为同等连续,则 F 的 w^* 闭包为 w^* 紧. 若 F 的 w^* 闭包为 w^* 紧,则 F 为 w^* 有

界.(注意 F 为同等连续当且仅当它为范数有界.)

(e) 若 X 非稀疏,特别若 X 为完备,则 X^* 的每一个 w^* 有界的子集 F 为同等连续.(对集 $\{x$: 对 F 中的每一个 f 有 $|f(x)|\leqslant 1\}$ 应用问题 6.U(b),或6.U(a).)

(f) "X 非稀疏"的假设不能从 (e) 中删去.(考察所有除在一个有限的指标集外皆为零的实序列的空间 X,其范数为 $\|x\| = \sum\{|x_n|: n\in\omega\}$. 若 $f_n(x) = nx_n$, 则序列 $\{f_n, n\in\omega\}$ 关于 w^* 拓扑收敛于零.)

注 这个问题的主要结果多少有点古典了,并且其中的某些可以明显地推广到更少限制的情形.然而,与 (d) 和 (e) 相等价的结果却不对任意的完备线性拓扑空间都成立.与(f)相联系的一个有趣的事实是线性赋范空间 X 的共轭空间的 w^* 紧凸子集恒为同等连续,而它的证明则并不是完全显然的.

O Tietze 扩张定理[1)]

(a) 设 X 为正规空间,A 为闭子集,f 为从 A 到闭区间 $[-1, 1]$ 的连续函数,则 f 有一个连续扩张 g,它变 X 到 $[-1, 1]$ 内.(设 $C = \{x: f(x)\leqslant -1/3\}$, $D = \{x: f(x)\geqslant 1/3\}$, 则由 Urysohn 引理有从 X 到 $[-1/3, 1/3]$ 的 f_1 使得 f_1 在 C 上为 $-1/3$, 在 D 上为 $1/3$. 显然,对 A 中的一切 x 有 $|f(x) - f_1(x)|\leqslant 2/3$. 同样的论证也可以应用于函数 $f - f_1$.)

注 Dugundji[1], Dowker[3] 和 Hanner[1] 已经证明了 Tietze 定理的一些有趣的推广.

P 关于 $C(X)$ 的线性子空间的稠密性引理

设 X 为拓扑空间,又设 $C(X)$ 为 X 上的所有有界连续实值函数的空间,并且 $C(X)$ 带有一致收敛拓扑(等价地,$C(X)$ 关于 $\|f\| = \sup\{|f(x)|: x\in X\}$ 为赋范空间). 若称 $C(X)$ 的子集 L 为具有两集性质当且仅当对 X 的互不相交闭子集 A 和 B,以及每一个闭区间 $[a, b]$,有 L 的一个元 f 使得 f 映 X 到 $[a,b]$ 内,并且 f 在 A 上为 a,在 B 上为 b,则 $C(X)$ 的每一个具有两集性质的线性子空间在 $C(X)$ 中稠密.(设 g 为 $C(X)$ 的一个任意的元并且 $\mathrm{dist}(g, L)$

1) 这个定理之所以在此处出现是因为它的证明需要连续函数的一致极限仍为连续函数的事实. 必须老实承认,在以前各章中已经有三个问题其中用到了这一个事实.

>0，选 L 中的 h 使得 $\|g-h\|$ 渐近于 $\operatorname{dist}(g,L)$。若 $k=g-h$，则 $\operatorname{dist}(k,L)=\operatorname{dist}(g,L)$ 并且 $\|k\|$ 渐近于它。证明存在 L 的元 f 使得 $\|k-f\|\leqslant 2\|k\|/3$。）

Q 关于 Banach 代数的平方根引理[1]。

实（或复）Banach 代数是指实（或复）数上的一个这样的代数 A，它带有一个范数使得 A 为完备线性赋范空间并且乘法满足条件：$\|xy\|\leqslant\|x\|\|y\|$。（借助于通常的算子范数，代数 A 可以看成是具有这样的结合乘法的 Banach 空间，它使得关于确定的元 x 的左乘为范数至多 $\|x\|$ 的线性算子。）以下恒假定 A 为一个确定的（实或复）Banach 代数。

从 D 到线性赋范空间的函数 f 叫做绝对可和当且仅当 $\sum\{\|f(n)\|: n\in D\}$ 存在。

(a) A 中每一个绝对可和函数为可和。若 $\{x_n, n\in\omega\}$ 和 $\{y_m, m\in\omega\}$ 为绝对可和，则 $\{x_n y_m, (m,n)\in\omega\times\omega\}$ 也为绝对可和并且

$\sum\{x_n: n\in\omega\}\sum\{y_m: m\in\omega\}=\sum\{x_n y_m: (m,n)\in\omega\times\omega\}$。（这个结果的有效性就在于最后一个和式可通过任意方式归并被加数来加以计算。见问题 6.S.）

(b) 设 a_n 为 $(1-t)^{1/2}$ 关于 $t=0$ 的展开式中的第 n 个二项式系数，则 $a_0=1$ 并且 a_n 为负当 n 为正时，$\sum\{a_n: n\in\omega\}=0$ 并且 $\sum\{a_n a_{p-n}: n\in\omega \text{ 且 } n\leqslant p\}$ 为 1，-1 和 0 分别当 $p=0$，$p=1$ 和 $p>1$ 时。（交替地，我们可以递推地定义 a_n 使得所陈述的最后一个关系式成立。在证明了当 n 为正时 $a_n<0$ 之后，注意部分和 $\sum\{a_n t^n: n<p\}$ 关于 n 单调下降，并且对 $0\leqslant t<1$，因而也对 $t=1$，以 $(1-t)^{1/2}$ 为下界。）

(c) 若该代数有单位元 u，并且 $\|x-u\|\leqslant 1$，则有此代数的元 y 使得 $y=x^{\frac{1}{2}}$。显然，y 可以取为 $\sum\{a_n(u-x)^n: n\in\omega\}$，其中 a_n 如同 (b) 中所定义的。（这里假定 $x^0=u$。元 y 也可以写成 $y=\sum\{a_n[(u-x)^n-u]: n\geqslant 1\}$ 的形式。在这种形式下，显然 y 是 x 的多项式的极限并且这些多项式还可以取成没有常数项。）

注 显然通过上面所述的方法还可以得到许多事实（例如，若 $\|x\|<1$，则 $\sum\{x^n: n\in\omega\}$ 为 $u-x$ 的乘法的逆）。至于 Banach 代数的系统讨论见

1) 这个命题在此处给出，本质上是作为 Stone-Weierstrass 定理的一种准备。然而，该引理对更一般的情况也有某些重要性，因此，我们就任意 Banach 代数的情形来加以陈述。

Loomis [2] 和 Hille [1]。

R Stone-Weierstrass 定理

(a) 设 X 为紧拓扑空间，$C(X)$ 为所有 X 上连续实值函数的代数并且 $C(X)$ 带有范数：$\|f\| = \sup\{|f(x)|: x \in X\}$，则 $C(X)$ 的子代数 R 在 $C(X)$ 中稠密，假如它具有两点性质：对 X 的不同的点 x 和 y，以及每一对实数 a 和 b，有 R 中的 f 使得 $f(x) = a$ 并且 $f(y) = b$。

特别，R 为稠密，假如常数函数属于 R 并且 R 分离点(在从 $x \neq y$ 可推出有某个 R 中的 f 使得 $f(x) \neq f(y)$ 的意义下)。

它的证明是通过如下的一系列引理完成的。

(i) 若 $f \in R$，则 $|f|$ 属于 R 的闭包 R^-，其中 $|f|(x) = |f(x)|$。(取 f^2 的平方根并利用问题 7.P。)

(ii) 若 f 和 g 属于某个子代数，则 $\max[f, g]$ 和 $\min[f, g]$ 属于该子代数的闭包。(这里 $\max[f, g](x) = \max[f(x), g(x)]$。注意 $\max[a, b] = [(a + b) + |a - b|]/2$ 以及 $\min[a, b] = [(a + b) - |a - b|]/2$。)

(iii) 若子代数具有两点性质，$f \in C(X)$，$x \in X$ 并且 $\epsilon > 0$，则有 R^- 中的 g 使得 $g(x) = f(x)$ 并且 $g(y) < f(y) + \epsilon$ 对 X 中的一切 y 成立。(利用 X 的紧性，取一个适当选择的有限的函数族的最小值。)

该定理现在通过取一个适当选择的有限的函数族的最大值从 (iii) 即可推出。

(b) 若 X 为拓扑空间并且所有 X 上的连续实值函数的族 $C(X)$ 给定在紧集上一致收敛的拓扑，则 $C(X)$ 的每一个具有两点性质的子代数在 $C(X)$ 中稠密。

注 毫无问题，这是 $C(X)$ 的最有用的已知结果。但关于复值函数相应的定理是不成立的(例如，考虑在单位圆域上连续并且在其内部解析的函数)。至于更详细的讨论见 M. H. Stone [5]。

S $C(X)$的构造

在整个问题中设 X, Y 和 Z 为紧 Hausdorff 空间，$C(X)$，$C(Y)$ 和 $C(Z)$ 分别为 X, Y 和 Z 上的一切连续实值函数的代数。又代数的实同态是指到实数内的同态。

(a) 对每一个从 X 到 Y 的连续函数 F，命 F^* 为由 $F^*(h) = h \circ F$ 所定义的从 $C(Y)$ 到 $C(X)$ 内的诱导映射，其中 h 属于 $C(Y)$，则

(i) F^* 为从 $C(Y)$ 到 $C(X)$ 内的同态.

(ii) F 映 X 到 Y 上当且仅当 F^* 为从 $C(Y)$ 到 $C(X)$ 的一个包含单位元的子代数上的同构.

(iii) F 为一对一当且仅当 F^* 映 $C(Y)$ 到 $C(X)$ 上.

(iv) 若 G 为从 Y 到 Z 内的连续映射,则 $(G\circ F)^* = F^*\circ G^*$.

(v) 若 F 为从 X 到 Y 上的拓扑映射,则 $(F^{-1})^* = (F^*)^{-1}$.

(b) $C(X)$ 的拓扑由它的代数运算所完全决定. 更详细地说: 假设当 $f-g$ 为 $C(X)$ 的某个元的平方时命 $f\geq g$, 又命 $\|f\| = \inf\{k: -ku\leq f\leq ku\}$, 其中 u 为恒等于 1 的函数. 那末当 ϕ 为 $C(X)$ 的实同态时有 $|\phi(f)|\leq\|f\|$ 并且除 ϕ 为恒等于零的情形外有 $\phi(u)=1$.

(c) 设 S 为所有使得 $\phi(u)=1$ 的 $C(X)$ 的实同态 ϕ 组成的集并且具有点式收敛拓扑,又设 E 为从 X 到 S 内的计值映射(即 $E(x)(f)=f(x)$), 则 E 为从 X 到 S 上的拓扑映射. (证明 S 为紧, 利用 Stone-Weierstrass 定理来证明从 $C(X)$ 到 $C(S)$ 内的计值映射 D 为从 $C(X)$ 到 $C(S)$ 上的同构,再证明 $E^* = D^{-1}$, 并且利用 (a).)

(d) 空间 X 为可度量化当且仅当 $C(X)$ 为可分. (这个结果在该问题的其余部分中用不到;可以作为应用 (c) 的一个简单练习而给出.)

(e) 若 H 为从 $C(Y)$ 到 $C(X)$ 的同态,它变 $C(Y)$ 的单位元为 $C(X)$ 的单位元,则有唯一的从 X 到 Y 内的连续映射 F 使得 $H=F^*$. (该同态 H 诱导了从所有 $C(X)$ 上的实同态的集到所有 $C(Y)$ 上的实同态的集内的一个映射.)

(f) 设 R 为使得 $u\in R$ 的 $C(X)$ 的闭子代数, 又设 F 为由 $F(x)_f = f(x)$ 所定义的从 X 到 $\times\{f[X]: f\in R\}$ 内的映射,再设 Y 为 F 的值域,则 R 为所诱导的从 $C(Y)$ 到 $C(X)$ 内的同构 F^* 的值域.

(g) 设 I 为 $C(X)$ 内的闭理想,又设 $Z=\{x$: 对 I 中的所有 f 有 $f(x)=0\}$, 则 I 为所有在 Z 上恒等于零的 $C(X)$ 的元的集. (若 Z 为空集, 则有 I 的一个元,它不在任何点上为零,因而有逆. 考察子代数 $C+I$, 其中 C 为所有常数函数的集. 因为 Z 为非空, $C+I$ 为闭,故 (f) 可以应用.)

注 关于 $C(X)$ 的构造已经相当的清楚. 更进一步的情况和文献在 S. B. Myers[1] 关于该论题的一篇评论性文章中给出. 也可参阅 Hewitt [2].

上述问题中所略述的处理方法不是仅可能的一种方法——从那些基本事实 (Stone-Weierstrass 定理, Tychonoff 乘积定理和 Tietze 定理) 我们可用不同方法来导出所期望的结果. 然而,上面所用的方法部分地可看成是一种一般方法的例子. 即对某个对象的类的每一个元 (在现在的情况下是紧

Hausdorff 空间 X）相应有另一个对象（在现在的情况下是 Banach 代数 $C(X)$）．而且对原始对象的一个特殊的映射类的每一个元（在所述的情况下是连续映射）确定了一个满足某些条件（例如（a）的（iv）和（v））的诱导映射．在这种情况下，诱导映射与导出它的映射反向——如此的对应叫做反变．又 Tychonoff 空间的 Stone-Čech 紧扩张与其明显的诱导映射给出了诱导映射与原始映射同向的一个例子——一个协变的对应．

这种一般方法的研究，已经由 Eilenberg 和 Steenrod[1] 最成功地用于它们的同调理论的公理化处理．而该方法本身首先由 Eilenberg 和 Maclane 所研究．如果对象与映射理论的研究称作银河系范围的，那按此类比，拓扑空间的研究只能叫作地球范围的研究了．

T 群的紧扩张；殆周期函数

我们自然试图按照将 Tychonoff 空间嵌到它的 Stone-Čech 紧扩张内的多少有点相同的方法，将拓扑群映成紧拓扑群的稠密子群．一般地说，拓扑地嵌入是不可能的——拓扑并且同构地嵌到每一个 Hausdorff 群内的完备群在其内必为闭．然而，我们仍然可以得到许多有趣的结果；下面来作一个简单的介绍．它的进一步发展，是由如下事实所推动：若 ϕ 为从拓扑群 G 到紧群 H 内的连续同态，g 为 H 上的连续实值函数，则 $g\circ\phi$ 具有所有左位移的集（关于一致收敛的一致结构）为全有界的性质．

在整个问题中，假定 G 为确定的拓扑群．对群 G 上的每一个有界实值函数 f 和 G 中的每一个 x，定义 f 关于 x 的左平移 $L_x(f)$ 为：$L_x(f)(y)=f(x^{-1}y)$．又所有有界实值函数的函数空间关于 $d(f,g)=\sup\{|f(x)-g(x)|: x\in X\}$ 为可度量化，并且我们定义函数 f 的左轨道 X_f 为所有 f 的左平移的集关于度量拓扑的闭包．再定义 f 为左殆周期当且仅当 X_f 为紧集．

设 A 为所有 G 上连续左殆周期函数的集，则对 G 中的每一个 x，左平移 L_x 映 A 到 A 内．我们以点式收敛来拓扑化所有从 A 到 A 内的映射的空间，并且命 $\alpha[G]$ 为所有左平移的集关于该拓扑的闭包．

(a) 引理．设 (X,d) 为紧度量空间，K 为所有从 X 到它自己内的等距映射（关于合成）的群，则在 X 上一致收敛的拓扑（对 K）为度量：$d^*(R,S)=\sup\{d(R(x),S(x)): x\in X\}$ 的拓扑并且即为在 X 上点式收敛的拓扑．又群 K 关于该拓扑为紧拓扑群．

(b) $\alpha[G]$ 为紧．（注意 $\alpha[G]\subset\times\{X_f: f\in A\}$．）

(c) $\alpha[G]$ 的每一个元为变每一个左轨道 X_f 到它自己上的等距映射．

又从 $\alpha[G]$ 到乘积空间 $\times\{K_f: f\in A\}$ 内的自然映射为拓扑同构，其中 K_f 为所有 X_f 的等距映射组成的群. 因而 $\alpha[G]$ 为一个拓扑群.

(d) 若 A 给定在 G 上点式收敛的拓扑并且 $\alpha[G]$ (A^A 的子集) 带有所得到的乘积拓扑，则对于 $\alpha[G]$，这两种拓扑相同. 因而，在 $\alpha[G]$ 中 $R_n\to R$ 当且仅当对 A 中的所有 f 和 G 中的所有 x 有 $R_n(f)(x)\to R(f)(x)$.

(e) 变 G 的元 x 为 L_x 的从 G 到 $\alpha[G]$ 内的映射 L 为连续同态. 对于 G，使得 L 为连续的最小拓扑与使得 A 的每一个元 f 为连续的最小拓扑相同. ($\alpha[G]$ 也可以描述为 Gmod 所有用 A 的元都不能分离单位元的 G 的元组成的子群所生成的商群，关于使得 A 中的每一个 f 为一致连续的最小一致结构的完备扩张.)

(f) 若 g 为 $\alpha[G]$ 上的连续实函数，则 $g\circ L\in A$. 又若 $f\in A$ 并且定义 $\alpha[G]$ 上的函数 g 为 $g(R)=R^{-1}(f)(e)$，则 $f=g\circ L$ 并且 g 为连续. 于是所有 $\alpha[G]$ 上的连续实函数的族等距(并且同构)于 A.

(g) 若 ϕ 为从 G 到紧拓扑群 H 内的连续同态，则有从 $\alpha[G]$ 到 H 内的连续同态 θ 使得 $\phi=\theta\circ L$. (更一般地，对于 H，任意的同态 ϕ 都诱导了从 $\alpha[G]$ 到 $\alpha[H]$ 内的自然同态 θ 使得 $\theta\circ L=L\circ\phi$. 见 α 的定义.)

上述这些结果有许多明显的系；例如函数为左周期当且仅当它为右周期，以及类 A 为同构于所有紧群 $\alpha[G]$ 上连续函数的代数的 Banach 代数.

(h) 术语"殆周期"是从类 A 的一种另外的描述导出的. 我们称 G 的元 x 为实函数 f 的左 e 周期当且仅当对所有 G 中的 y 有 $|f(x^{-1}y)-f(y)|<e$. 命 A_e 为连续函数 f 的所有左 e 周期的集，则下列命题等价:

(i) 有从 G 到紧群 H 内的同态 ϕ 和 H 上的连续实值函数 h 使得 $g=h\circ\phi$;

(ii) f 的所有左平移的集关于一致收敛的一致结构为全有界;

(iii) 对每一个正数 e 有 G 的有限子集 B 使得 $G=BA_e$.

(注意到 $|L_x(f)(z)-L_y(f)(z)|<e$ 对所有 z 成立当且仅当 $y^{-1}x$ 为左 e 周期，即可阐明 (ii) 与 (iii) 之间的联系.)

注 上述结果最早属于 Weil[2]. 而 (h) 的 (ii) 与 (iii) 的等价性则是 Bochner 的一个经典定理. Loomis[2] 也研究了殆周期函数，他先证明群上的所有左殆周期函数的集满足刻划函数组成的 Banach 代数的特征的条件，然后再定义 $\alpha[G]$ 为该 Banach 代数的一切实同态的集.

命题 (a) 提出了怎样拓扑化一个同胚群使之成为拓扑群的一般问题. 关于该方向的结果和文献参看 Arens[3] 与 Dieudonné[4].

附录 初等集论

本附录集中研究初等集论．同时构造了序数和基数，而且大多数常用定理都给出了证明． 此外还定义了非负整数，并把 Peano 公设当作定理给予了证明．

我们假定读者知晓初等逻辑的一些实用知识，但并不必要熟悉形式逻辑．无论如何，对数学体系本质的理解（在技术的意义下)有助于弄清和推进某些研讨． Tarski 在 [1] 中高超的解释很清晰地描述了这样的体系，至于一般的背景，我们推荐上文．

本文集论的这种叙述方法可以毫不困难地翻译成一种完全形式的语言[1]．为了便于形式的或者非形式的处理，我们把材料分成两部分，第二部分实质上是第一部分精确的重述． 因此可以把它删去而不会损害连贯性． 我们采用的公理体系是 Skolem 和 A. P. Morse 体系的变形，且更接近于由 Gödel 所系统叙述的 Hilbert-Bernays-von Neumans 体系． 这里采用的公理化是用来迅速而又自然地给出一个数学基础，其中摆脱了较明显的悖论．由于这种缘故，有限的公理体系被遗弃，而把整个理论建筑在八个公理和一个公理图式之上[2]．（也就是说，在某种指定的形式下的一切语句都被认作公理．）

1) 也就是说，这些定理用逻辑常项、逻辑变元和体系常项的术语来叙述是可能的，并且这些证明可以借助于推论的规则由公理推出．当然在展开这种理论时，需要有形式逻辑的基础知识，我曾在这种形式理论的课程中，（本质地）使用了 Quine 的逻辑元公理[1]．

2) 实际上，用作定义的公理图式也假定为不含明显的语句，即某种形式的语句，它们涉及一个新的常项，等价关系或恒同关系的常项，它们可认作定义，或恰恰一样地也可以认作定理．这种公理图式在下面的意义上，可以认为是合理的，即若定义与规定的法则一致，则理论中就不会产生新的矛盾，也不会有实质上新的推论．而这些结果都是属于 S. Leśniewski 的．

把许多命题作为定理来叙述是很方便的．实质上，它们都是所期望结果的预备知识．这样作虽然搞乱了定理与命题，但它容许省略许多证明和简缩另一些证明．大多数这样的做法从定义与定理的陈述看来是较明显的．

分类公理图式

相等恒用在逻辑上恒等的意义之下；“1＋1＝2”意指“1＋1”与“2”是同一事物的两个名字．除了通常的相等公理外，还要假设一个代换规则：特别在一个定理中，用一个对象代替与它相等的对象，结果仍是一个定理．

除了“＝”与其它逻辑常项之外，还有两个基本的常项（未定义）．第一个是“∈”，它读作“是…的一个元”，或者“属于”．第二个常项颇有点古怪，它被写成“{··：···}”并读作“{所有··的集使得···}”．所以它是分类．一个关于术语“类”的应用的注解可以搞清许多事情，这个术语不在任何公理、定义或定理中出现，但这些语句的最初解释[1]都是关于类（集系、集族）的结论．因此“类”这个术语用在提出这种解释的讨论中．

小写拉丁字母都是（逻辑上的）变元、一个常项与一个变元之间的差别完全表现在代换律中．例如，在一个定理中用其它不出现在该定理里的变元来替换一个变元其结果仍是一个定理．但是对于常项就没有这样的替换存在．

I 外延公理[2]．对于每个 x 与 y，$x=y$ 成立之充分必要条件是对每一个 z 当且仅当 $z\in x$ 时，$z\in y$．

于是两个类恒同当且仅当每个类的任一元，也是另一个的元．我们在定义或定理的叙述中常常省略“对于每个x”或者“对于每

1）或许用别的解释方法也是可能的．

2）有人曾尝试用此作为相等的定义，从而减去一条公理，省略所有关于相等的逻辑前提，这是完全可以办到的．但是，对于等式，这时再也没有无限制的代换规则了．且必须假设一条公理：若 $x\in z$，$y=x$，则 $y\in z$．

个 y". 如果一个变元，比如说"x"出现时且前面没有"对于每个 x"或者"对于某一 x"的字样，我们就理解有关的定义与定理是对"每一个 x 的".

最初的定义给那些仍为某些类元的类起了一个特殊的名字. 为什么要区别这两种类的理由，将在稍后来讨论.

1 定义. x 为一集当且仅当对于某一 y, $x \in y$.

下面的任务是描述这种分类的用处，在分类常项的括号中第一个位置由变元占据，第二个位置由一个公式占据，例如 $\{x: x \in y\}$. 我们把 $u \in \{x: x \in y\}$ 当且仅当 u 是一个集同时 $u \in y$ 当作公理来接受，更一般地，把每个下面形式的说法都假定为一个公理：$u \in \{x: \cdots x \cdots\}$ 当且仅当 u 是一个集并且 $\cdots u \cdots$. 这里"$\cdots x \cdots$"被假定为一个公式，而"$\cdots u \cdots$"被假定为每个"x"出现处用"u"来代替时所得之公式. 于是 $u \in \{x: x \in y$ 且 $z \in x\}$ 当且仅当 u 是一个集并且 $u \in y$ 和 $z \in u$.

这种公理图式除了要求"u 是一个集"外，恰恰是通常的类的直观上的构造. 这种要求显然很不自然，并且在直观上也是十分难达到的. 然而没有它，单在外延公理的基础上就可造出一个矛盾来.（参看定理 39 和它前面的讨论.）这种复杂化（它在论证集的存在性大量技术性的工作上需要）是为了避免明显的矛盾所要付出的一点代价. 不过不太明显的矛盾很可能还存在.

分类公理图式(续)

分类公理图式的严格论述要用一种公式的描述法，而它适合[1]：

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1) 这种迂廻的语言偏偷是需要的. 我们遵循命名利用引号的规定，例如"Bosten"是 Bosten 的名字，如果 $\mathscr{A}$ 为一公式同时 $\mathscr{B}$ 为一公式，则"$\mathscr{A} \to \mathscr{B}$"不是一个公式，例如，如果 $\mathscr{A}$ 为"$x = y$"同时 $\mathscr{B}$ 为"$y = z$"，则"$x = y$"$\to$"$y = z$"不是一个公式. 公式（例如"$x = y$"）不含有引号. 代替"$\mathscr{A} \to \mathscr{B}$"我们想要讨论的是在"$\alpha \to \beta$"中用 $\mathscr{A}$ 代替"α"和用 $\mathscr{B}$ 代替"β"所得的结果. 这种迂廻曲折的说法能用 Quine 的角规定所避免.

(a) 对于下面的每一个用变元替换"α"与"β"所得的结果是一个公式.

$$\alpha=\beta;\qquad \alpha\in\beta.$$

(b) 对于下面的每一个用变元替换"α"与"β"和用公式替换"A"与"B"所得的结果是一公式.

如果 A，则 B；A 当且仅当 B；A 不真；

A 与 B；A 或者 B；

对于每一个 α，A；对于某一 α，A；

$\beta\in\{\alpha:A\}$；$\{\alpha:A\}\in\beta$；$\{\alpha:A\}\in\{\beta:B\}$.

从 (a) 中的原始公式开始，按 (b) 中所允许的构造，递归地构造出来的东西叫公式.

II 分类公理图式 一个公理的许多结论，如果下面的"α"与"β"都用变元来代替，"A"用一个公式 $\mathscr{A}$ 来代替且"B"由这样的公式来代替，即 $\mathscr{A}$ 中用代替 β 的变元来换每个曾代替 α 的变元所得的公式：

对于每一个 β，$\beta\in\{\alpha:\ A\}$ 的充分必要条件是"β 是集"和

类的初等代数

已经叙述的公理使得我们能直接由形式逻辑的结果来推演许多定理. 由于这种演绎是简单易明的，所以仅只在必要时才给出它的证明.

2 定义. $x\cup y=\{z:\ z\in x$ 或者 $z\in y\}$.

3 定义. $x\cap y=\{z:\ z\in x$ 同时 $z\in y\}$.

类 $x\cup y$ 是 x 与 y 的并，而 $x\cap y$ 是 x 与 y 的交.

4 定理. $z\in x\cup y$ 当且仅当 $z\in x$ 或者 $z\in y$，而 $z\in x\cap y$ 当且仅当 $z\in x$ 同时 $z\in y$.

证明. 由分类公理 $z\in x\cup y$ 当且仅当 $z\in x$ 或者 $z\in y$ 同时 z 是一个集. 但是鉴于集的定义 1，$z\in x$ 或者 $z\in y$. 并且 z 为一

个集当且仅当 $z\in x$ 或者 $z\in y$. 类似的论述可以用来证明关于交的相应结果.▌

5 定理. $x\cup x=x$ 同时 $x\cap x=x$.

6 定理. $x\cup y=y\cup x$ 同时 $x\cap y=y\cap x$.

7 定理[1]. $(x\cup y)\cup z=x\cup(y\cup z)$ 同时 $(x\cap y)\cap z=x\cap(y\cap z)$.

这些定理说明并与交在通常的意义下是可交换与可结合的运算. 而下面是分配律.

8 定理. $x\cap(y\cup z)=(x\cap y)\cup(x\cap z)$ 同时 $x\cup(y\cap z)=(x\cup y)\cap(x\cup z)$.

9 定义. $x\notin y$ 当且仅当 $x\in y$ 不真.

10 定义. $\sim x=\{y:\ y\notin x\}$.

类 $\sim x$ 是 x 的余.

11 定理. $\sim(\sim x)=x$.

12 定理(De Morgan). $\sim(x\cup y)=(\sim x)\cap(\sim y)$, 同时 $\sim(x\cap y)=(\sim x)\cup(\sim y)$.

证明. 这两个论断我们仅只证头一个. 根据分类公理和余的定义 10, 对于每个 z, $z\in\sim(x\cup y)$ 当且仅当 z 为一集同时 $z\in x\cup y$ 不真. 利用定理 4, $z\in x\cup y$ 是指 $z\in x$ 或者 $z\in y$. 因而 $z\in\sim(x\cup y)$ 是指 z 是一个集同时 $z\notin x$ 和 $z\notin y$; 也就是说 $z\in\sim x$ 同时 $z\in\sim y$. 再利用定理 4, $z\in\sim(x\cup y)$ 当且仅当 $z\in(\sim x)\cap(\sim y)$, 故根据外延公理推得 $\sim(x\cup y)=(\sim x)\cap(\sim y)$. ▌

13 定义. $x\sim y=x\cap(\sim y)$.

类 $x\sim y$ 是 x 与 y 之差, 或者 y 相对于 x 的余.

14 定理. $x\cap(y\sim z)=(x\cap y)\sim z$.

命题"$x\cup(y\sim z)=(x\cup y)\sim z$"似乎不太可靠, 虽然在现阶段举出一个反例还不可能, 稍微确切地讲, 即它在如今已假定的公理之基础上是不可能被证明的. 可以构造本体系的初始部分的

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1) 如果常项"$\cup$"在定义的开头出现, 在那里将无需用括号; 即用 $\cup xy$ 来代替"$x\cup y$". 在这种情况下, 定理的第一部分将要读成: $\cup\cup xyz=\cup x\cup yz$.

一个模型，使得对于每个 x 与 y 都有 $x \not\in y$（不存在集）。这个命题的否定法也能在目前将要假定的公理之基础上加以证明。|

15 定义. $0 = \{x: x \neq x\}$.

类 0 为空类，或者零。

16 定理. $x \not\in 0$.

17 定理. $0 \cup x = x$ 同时 $0 \cap x = 0$.

18 定义. $\mathscr{U} = \{x: x = x\}$.

类 $\mathscr{U}$ 是全域。

19 定理. $x \in \mathscr{U}$ 当且仅当 x 是一个集.

20 定理. $x \cup \mathscr{U} = \mathscr{U}$ 同时 $x \cap \mathscr{U} = x$.

21 定理. $\sim 0 = \mathscr{U}$ 同时 $\sim \mathscr{U} = 0$.

22 定义[1]. $\bigcap x = \{z$: 对于每个 y，如果 $y \in x$，则 $z \in y\}$.

23 定义. $\bigcup x = \{z$: 对于某一 y，$z \in y$ 同时 $y \in x\}$.

类 $\bigcap x$ 是 x 的元的交。注意 $\bigcap x$ 的元均为 x 的元的元，而它可以属于 x，也可以不属于 x。类 $\bigcup x$ 是 x 的元的并。研究一个集 z 属于 $\bigcap x$（或者属于 $\bigcup x$）是指 z 属于 x 的每个（相应地，属于某一个）x 的元。

24 定理. $\bigcap 0 = \mathscr{U}$ 同时 $\bigcup 0 = 0$.

证明. $z \in \bigcap 0$ 当且仅当 z 为一个集，同时 z 属于每一个 0 的元。由于（定理 16）0 中不存在元，又 $z \in \bigcap 0$ 当且仅当 z 是一个集，故由定理 19 和外延公理 $\bigcap 0 = \mathscr{U}$。第二个论断也是很容易证明的。|

25 定义. $x \subset y$ 当且仅当对于每个 z，如果 $z \in x$，则 $z \in y$.

一个类 x 是 y 的一个子类，或者说被包含在 y 中，或者说 y 包含 x 当且仅当 $x \subset y$。"$\subset$"绝对不要与"$\in$"相混淆。例如，$0 \subset 0$ 但 $0 \in 0$ 不真。

26 定理. $0 \subset x$ 同时 $x \subset \mathscr{U}$.

27 定理. $x = y$ 当且仅当 $x \subset y$ 同时 $y \subset x$.

1) 关于一个族中元之交的约束变项的记号，在本附录中是不需要的. 所以用的记号要比本书的其它部分简单一些.

28 定理. 如果 $x\subset y$ 且 $y\subset z$, 则 $x\subset z$.

29 定理. $x\subset y$ 当且仅当 $x\cup y=y$.

30 定理. $x\subset y$ 当且仅当 $x\cap y=x$.

31 定理. 如果 $x\subset y$, 则 $\bigcup x\subset\bigcup y$ 同时 $\bigcap y\subset\bigcap x$.

32 定理. 如果 $x\in y$,则 $x\subset\bigcup y$ 同时 $\bigcap y\subset x$.

上面的定义和定理都经常使用——但经常没明确的指出来.

集的存在性

这一节论及集的存在性和函数的构造的最初几步以及其它集论中的基本关系.

III 子集公理 如果 x 是一个集,存在一个集 y 使得对于每个 z, 假定 $z\subset x$, 则 $z\in y$.

33 定理. 如果 x 是一个集同时 $z\subset x$, 则 z 是一个集.

证明. 依照子集公理, 如果 x 是一个集存在 y 使得: 假定 $z\subset x$, 则 $z\in y$. 从而由定义 1 z 是一个集. (注意这个证明并没用到子集公理的全部内容,因为论证并没有要求 y 是一个集.)∎

34 定理. $0=\bigcap\mathscr{U}$ 同时 $\mathscr{U}=\bigcup\mathscr{U}$.

证明. 如果 $x\in\mathscr{U}$,则 x 是一个集,同时由 $0\subset x$ 自 33 推出 0 是一个集. 于是 $0\in\mathscr{U}$ 同时 $\bigcap\mathscr{U}$ 的每个元属于 0. 故得知 $\bigcap\mathscr{U}$ 没有元,显然(即定理 26) $\bigcup\mathscr{U}\subset\mathscr{U}$. 如果 $x\in\mathscr{U}$,则 x 是一个集同时由子集公理存在一个集 y 使得: 如果 $z\subset x$, 则 $z\in y$. 特别是 $x\in y$, 又由于 $y\in\mathscr{U}$, 从而推得 $x\in\bigcup\mathscr{U}$. 所以 $\mathscr{U}\subset\bigcup\mathscr{U}$, 故得等式.∎

35 定理. 如果 $x\neq 0$,则 $\bigcap x$ 是一个集.

证明. 如果 $x\neq 0$, 则一定有 y, 而 $y\in x$, 但是 y 是一个集, 从而利用定理 32 知 $\bigcap x\subset y$. 故自定理 33 推出 $\bigcap x$ 是一个集.∎

36 定义. $2^x=\{y: y\subset x\}$.

37 定理. $\mathscr{U}=2^{\mathscr{U}}$.

证明．每一个 $2^{\mathscr{U}}$ 的元是一个集，所以属于 $\mathscr{U}$．$\mathscr{U}$ 的每个元是一个集并且被包含在 $\mathscr{U}$ 中(定理 26)，故属于 $2^{\mathscr{U}}$．|

38 定理．如果 x 是一个集，则 2^x 是一个集，同时对每个 y，$y \subset x$ 当且仅当 $y \in 2^x$．

值得注意的是集的存在性在目前已指明的这些公理之基础上尚不能证明的．但是想证明存在一个不是集的类，这是可以办到的．令 R 等于 $\{x: x \notin x\}$．由分类公理，$R \in R$ 当且仅当 $R \notin R$ 同时 R 是一个集，于是推得 R 不是一个集．注意分类公理如果不包含这"是一个集"的限制，则导致一个明显的矛盾结果：$R \in R$ 当且仅当 $R \notin R$．这是 Russell 谬论．这个论证的推论是 $\mathscr{U}$ 不是一个集，此因应用了 $R \subset \mathscr{U}$ 和定理 33．(这个正则性公理将推出 $R = \mathscr{U}$；并且这个公理同时又提供了 $\mathscr{U}$ 不是一个集的不同证法．)

39 定理．$\mathscr{U}$ 不是一个集．

40 定义．$\{x\} = \{z: 如果 x \in \mathscr{U}，则 z = x\}$．

单点 x 是 $\{x\}$．

这个定义是技术上设计的一个例子，它用起来十分方便．如果 x 是一个集，则 $\{x\}$ 是一个类，而它仅只有元 x．然而，如果 x 不是一个集，则 $\{x\} = \mathscr{U}$ (这些叙述是定理 41 和 43)．实际上，兴趣在于这里的 x 是一个集的情况，并且对于这种情况用较自然的定义 $\{z: z = x\}$ 可给出同样的结果．然而，如果这些运算被设置以至 $\mathscr{U}$ 为应用这种运算到它适当的范围之外所得的结果，它大大简化许多结果的论述．

41 定理．如果 x 是一个集，则对于每个 y，$y \in \{x\}$ 当且仅当 $y = x$．

42 定理．如果 x 是一个集，则 $\{x\}$ 是一个集．

证明．如果 x 是一个集，则 $\{x\} \subset 2^x$，又 2^x 是一个集．|

43 定理．$\{x\} = \mathscr{U}$ 当且仅当 x 不是一个集．

证明．如果 x 是一个集，则 $\{x\}$ 是一个集．因而不等于 $\mathscr{U}$．如果 x 不是一个集，则 $x \notin \mathscr{U}$ 再由定义得 $\{x\} = \mathscr{U}$．|

44 定理. 如果 x 是一个集，则 $\bigcap\{x\}=x$ 同时 $\bigcup\{x\}=x$；如果 x 不是一个集，则 $\bigcap\{x\}=0$ 同时 $\bigcup\{x\}=\mathscr{U}$.

证明. 利用定理 34 和 41. |

IV 并的公理: 如果 x 是一个集同时 y 是一个集，则 $x\cup y$ 也是一个集.

45 定义. $\{xy\}=\{x\}\cup\{y\}$.

类 $\{xy\}$ 是一个无序偶.

46 定理. 如果 x 是一个集同时 y 是一个集，则 $\{xy\}$ 是一个集，同时 $z\in\{xy\}$ 当且仅当 $z=x$，或者 $z=y$；$\{xy\}=\mathscr{U}$ 当且仅当 x 不是一个集或者 y 不是一个集.

47 定理. 如果 x 与 y 是两个集，则 $\bigcap\{xy\}=x\cap y$ 同时 $\bigcup\{xy\}=x\cup y$；如果 x 或者 y 不是一个集，则 $\bigcap\{xy\}=0$ 同时 $\bigcup\{xy\}=\mathscr{U}$.

序偶：关系

本节集中研究序偶的性质和关系. 关于序偶重要的性质是定理55：如果 x 与 y 均为集，则 $(x,y)=(u,v)$ 当且仅当 $x=u$ 同时 $y=v$.

48 定义. $(x,y)=\{\{x\}\{xy\}\}$.

类 (x,y) 是一序偶.

49 定理. (x,y) 是一个集当且仅当 x 是一个集，并且 y 是一个集；如果 (x,y) 不是一个集，则 $(x,y)=\mathscr{U}$.

50 定理. 如果 x 与 y 均为集，则 $\bigcup(x,y)=\{xy\}$，$\bigcap(x,y)=\{x\}$，$\bigcup\bigcap(x,y)=x$，$\bigcap\bigcap(x,y)=x$，$\bigcup\bigcup(x,y)=x\cup y$ 同时 $\bigcap\bigcup(x,y)=x\cap y$.

如果 x 或者 y 不是一个集，则 $\bigcup\bigcap(x,y)=0$；$\bigcap\bigcap(x,y)=\mathscr{U}$，$\bigcup\bigcup(x,y)=\mathscr{U}$ 同时 $\bigcup\bigcap(x,y)=0$.

51 定义. z 的 1^{st} 坐标 $=\bigcap\bigcap z$.

52 定义. z 的 2^{nd} 坐标 $=(\bigcap\bigcup z)\cup((\bigcup\bigcup z)\sim(\bigcup\bigcap z))$.

除去一种情况之外，这些定义仅被使用在这种情况下，即这里的 z 是一个序偶. z 的第一个坐标是 z 的 1^{st} 坐标. 同时 z 的第二个坐标是 z 的 2^{nd} 坐标.

53 定理. $\mathscr{U}$ 的 2^{nd} 坐标 $=\mathscr{U}$.

54 定理. 如果 x 与 y 均为集，(x, y) 的 1^{st} 坐标 $=x$ 同时 (x, y) 的 2^{nd} 坐标 $=y$. 如果 x 或 y 不是一个集，则 (x, y) 的 1^{st} 坐标 $=\mathscr{U}$ 同时 (x, y) 的 2^{nd} 坐标 $=\mathscr{U}$.

证明. 如果 x 与 y 均为集，则对于 1^{st} 坐标的等式由定理 50 和定义51立得. 对于 2^{nd} 坐标的等式借助定理 50 和定义 52 来证明 $y=(x\cap y)\cup((x\cup y)\sim x)$. 容易看出 $(x\cup y)\sim x=y\sim x$，于是利用分配律 $(y\cap x)\cup(y\cap\sim x)$ 等于 $y\cap(x\cup\sim x)=y\cap\mathscr{U}=y$. 如果 x 或者 y 不是一个集，则用 50 计算 (x, y) 的 1^{st} 坐标和 (x, y) 的 2^{nd} 坐标是很容易的. |

55 定理. 如果 x 与 y 均为集，同时 $(x,y)=(u,v)$，则 $x=u$ 同时 $y=v$.

56 定义. r 是一个关系当且仅当对于 r 的每个元 z 存在 x 与 y 使得 $z=(x, y)$.

一个关系是一个类，它的元为序偶.

57 定义. $r\circ s=\{u$: 对于某个 x，某个 y 及某个 z，$u=(x, z)$，$(x, y)\in s$ 同时 $(y, z)\in r\}$.

类 $r\circ s$ 是 r 与 s 的合成.

为了避免过多的记号，我们认定 $\{(x, y):\cdots\}$ 与 $\{u$: 对于某个 x，某个 z，$u=(x, z)$ 同时$\cdots\}$ 相同，于是 $r\circ s=\{(x, z)$: 存在某个 y，使得 $(x, y)\in s$ 同时 $(y, z)\in r\}$.

58 定理. $(r\circ s)\circ t=r\circ(s\circ t)$.

59 定理. $r\circ(s\cup t)=r\circ s\cup r\circ t$，同时 $r\circ(s\cap t)\subset(r\circ s)\cap(r\circ t)$.

60 定义. $r^{-1}=\{(x, y):(y, x)\in r\}$.

如果 r 是一个关系，r^{-1} 叫作关于 r 之逆关系.

61 定理. $(r^{-1})^{-1}=r$.

62 定理. $(r\circ s)^{-1}=s^{-1}\circ r^{-1}$.

函　　数

直观上,一个函数是看作与一个叫作其图象的序偶类恒等的.所有函数均为单值的.因而两个不同的序偶属于同一个函数时,必需有不同的第一个坐标.

63 定义. f 是一个函数当且仅当 f 是一个关系同时对每个 x、每个 y 和每个 z,如果 $(x,y)\in f$ 且 $(x,z)\in f$,则 $y=z$.

64 定理. 如果 f 是一个函数同时 g 是一个函数,则 $f\circ g$ 也是一个函数.

65 定义. f 的定义域 $=\{x$: 对于某个 $y,(x,y)\in f\}$.

66 定义. f 的值域 $=\{y$: 对于某个 $x,(x,y)\in f\}$.

67 定理. $\mathscr{U}$ 的定义域 $=\mathscr{U}$ 同时 $\mathscr{U}$ 的值域 $=\mathscr{U}$.

证明. 如果 $x\in\mathscr{U}$,则 $(x,0)$ 与 $(0,x)$ 属于 $\mathscr{U}$,故 x 属于 $\mathscr{U}$ 的定义域和 $\mathscr{U}$ 的值域.|

68 定义. $f(x)=\bigcap\{y:(x,y)\in f\}$.

如果 z 属于 f 的每个元之第二个坐标,而 f 的第一个坐标是 x,则 $z\in f(x)$.

类 $f(x)$ 是 f 在 x 处的值,或者在 f 的映射下 x 的象.所以应注意:如果 x 是 f 定义域的一个子集,则 $f(x)$ 不等于$\{y$: 对于某个 $z,z\in x$ 同时 $y=f(z)\}$.

69 定理. 如果 $x\notin f$ 的定义域,则 $f(x)=\mathscr{U}$; 如果 $x\in f$ 的定义域,则 $f(x)\in\mathscr{U}$.

证明. 如果 $x\notin f$ 的定义域,则 $\{y:(x,y)\in f\}=0$ 同时 $f(x)=\mathscr{U}$ (定理 24). 如果 $x\in f$ 的定义域,则 $\{y:(x,y)\in f\}\neq 0$ 同时(由定理 35) $f(x)$ 是一个集.|

上面的定理并没要求 f 是一个函数.

70 定理. 如果 f 是一个函数,则 $f=\{(x,y):y=f(x)\}$.

71 定理[1]. 如果 f 与 g 都是函数，则 $f=g$ 的充要条件是对于每个 x，$f(x)=g(x)$.

下面的两个公理[2]，进一步描述了所有集的类.

V 代换公理. 如果 f 是一个函数同时 f 的定义域是一个集，则 f 的值域是一个集.

VI 合并公理. 如果 x 是一个集，则 $\bigcup x$ 也是一个集.

72 定义. $x\times y=\{(u,v)\colon u\in x$ 同时 $v\in y\}$.

类 $x\times y$ 是 x 与 y 的笛卡儿乘积.

73 定理. 如果 u 与 y 均为集，则 $\{u\}\times y$ 也是集.

证明. 显然能造一个函数 (即 $\{(w,z)\colon w\in y$ 且 $z=(u,w)\}$)，它的定义域是 y，值域是 $\{u\}\times y$，于是应用代换公理得证. |

74 定理. 如果 x 与 y 均为集，则 $x\times y$ 也是集.

证明. 设 f 是使得 f 的定义域 $=x$ 的函数，同时对于在 x 中的 u，$f(u)=\{u\}\times y$. (存在唯一的这种函数；即 $f=\{(u,z)\colon u\in x$ 同时 $z=\{u\}\times y\}$.) 根据代换公理，f 的值域是一个集. 由直接计算，f 的值域 $=\{z\colon$ 对于 u，$u\in x$ 同时 $z=\{u\}\times y\}$. 从而 $\bigcup$(f 的值域)由合并公理得知是一个集，并等于 $x\times y$. |

75 定理. 如果 f 是一个函数同时 f 的定义域是一个集，则 f 是一个集.

证明. 因为 $f\subset$(f 的定义域)$\times$(f 的值域). |

76 定义. $y^x=\{f\colon f$ 是一个函数，f 的定义域 $=x$ 同时 f 的值域 $\subset y\}$.

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1) 如果 $f(x)$ 被定义成以 x 为第一个坐标的 f 之元的第二个坐标的并，这个定理不真. 因为这时如果 $y\in\mathscr{U}$ 且 $y\notin f$ 的定义域，则 $f(y)=0$. 而且如果 $g=f\cup\{(y,0)\}$，则对于每个 x，$g(x)=f(x)$，但是 f 不等于 g.

2) 这两个公理也可以用一个公理来代替：如果 f 是一个函数同时 f 的定义域是一个集，则 $\bigcup$(f 的值域)是一个集. (施用在本书前面已使用过的约束变元记号，这便能很自然地叙述成：如果 d 是一个集，同时对于每个在 d 中的 a，$x(a)$ 是一个集，则 $\bigcup\{x(a)\colon a\in d\}$ 是一个集.) 要想从上面得到公理 V 和公理 VI，大体上可以如下进行：关于公理 V，对给定的 f，造一个其元形如 $(x,\{f(x)\})$ 的新函数. 对于公理 VI，对给定的 x 研究其元都是形如 (u,v) 且 u 在 x 中的函数.

77 定理. 如果 x 与 y 均为集，则 y^x 也是集.

证明. 如果 $f\in y^x$，则 $f\subset x\times y$，f 是一个集. 所以 $f\in 2^{x\times y}$ (定理 38) 同时 $2^{x\times y}$ 是一个集. 由于 $y^x\subset 2^{x\times y}$，故从子集公理推得 y^x 是一个集. |

为了方便，进而给出三个定义.

78 定义. f 在 x 上，当且仅当 f 为一函数同时 $x=f$ 的定义域.

79 定义. f 到 y，当且仅当 f 是一个函数同时 f 的值域 $\subset y$.

80 定义. f 到 y 上，当且仅当 f 是一个函数同时 f 的值域 $=y$.

良　　序

本节的许多结果在下面展开整数、序数与基数等理论中是不必要的. 而它们之所以被包含在这里是因为它们自身是很有趣的，并且这些方法是今后要用到的构造法的一种简化形式.

由于基本的构造性的结论已经被证明过了，所以我们假定省略几步是可以的.

81 定义. xry 当且仅当 $(x,y)\in r$.

如果 xry，则 x 是 r-关系于 y，或者 x 是 r-前于 y.

82 定义. r 连接 x 当且仅当 u 与 v 属于 x 时，不是 urv 便是 vru.

83 定义. r 在 x 中是传递的当且仅当 u、v 与 w 均为 x 的元而且 urv 和 vrw 时，则 urw.

如果 x 在 r 中是传递的，则称 r 序 x. 如果 u 与 v 属于 x 并且 r 序 x，特别有术语“ur-前于 v”.

84 定义. r 在 x 中是非对称的当且仅当 u 与 v 均为 x 的元并且 urv，则 vru 不真.

这也就是说，如果 $u\in x$，$v\in x$ 并且 ur-前于 v，则 v 不 r-前于 u.

85 定义. $x \neq y$ 当且仅当 $x = y$ 不真.

86 定义. z 是 x 的一个 r-首元当且仅当 $z \in x$,并假定 $y \in x$,和 $z \neq y$,则 yrz 不真.

87 定义. r 良序 x,当且仅当 r 连接 x 并且如果 $y \subset x$ 和 $y \neq 0$,则存在一个 y 的 r-首元.

88 定理. 如果 r 良序 x,则 r 在 x 中是传递的,并且 r 在 x 中是非对称的.

证明. 如果 $u \in x$, $v \in x$, urv 同时 vru, 则 $\{uv\} \subset x$, 从而存在一个 $\{uv\}$ 的 r-首元 z. 不是 $z = u$ 便是 $z = v$. 所以不是 vru 不真便是 urv 不真. 这个矛盾证明了 r 在 x 中是非对称的. 如果 r 在 x 中为传递的不真,则对于 x 的元 u、v 与 w, urv, vrw 和 wru 成立,此因 r 连接 x. 可是这样 $\{u\} \cup \{v\} \cup \{w\}$ 便没有一个 r-首元了.|

89 定义. y 是 x 的 r-截片当且仅当 $y \subset x$, r 良序 x 同时对于每个 u 与 v 使得 $u \in x$, $v \in y$ 且 urv,则 $u \in y$ 成立.

这也就是说,x 的一个子集 y 是 x 的一个 r-截片是指 r 良序 x,同时没有 $x \sim y$ 的元 r-前于 y 的元.

90 定理. 如果 $n \neq 0$ 同时 n 的每个元是 x 的一个 r-截片;则 $\bigcup n$ 与 $\bigcap n$ 都是 x 的 r-截片.

91 定理. 如果 y 是 x 的 r-截片且 $y \neq x$,则存在 x 中的某个 v, 使得 $y = \{u: u \in x \text{ 且 } urv\}$ 成立.

证明. 如果 y 是 x 的一个 r-截片,同时 $y \neq x$, 则存在一个 $x \sim y$ 的 r-首元 v, 如果 $u \in x$ 和 urv, 于是由 v 是 $x \sim y$ 的 r-首元, $u \notin x \sim y$, 所以 $u \in y$. 因此 $\{u: u \in x \text{ 且 } urv\} \subset y$.

另一方面,如果 $u \in y$, 于是由 $v \notin y$ 和 y 是一个 r-截片,所以 vru 不真,故 urv 成立,这样便推出了要求的等式.|

92 定理. 如果 x 和 y 都是 z 的 r-截片,则 $x \subset y$ 或者 $y \subset x$.

93 定义[1]**.** f 是 r-s 保序的,当且仅当 f 是一个函数,r 良序

1) 在此附录中无需研究(像在预备知识中)其定义域和值域均非良序的保序函数. 为了简单化这个原先的术语已经被修改了.

f 的定义域，s 良序 f 的值域，而且只要 u 与 v 都是 f 定义域中使得 urv 的元便有 $f(u)sf(v)$.

94 定理. 如果 $x\subset y$ 且 f 是一个在 x 上到 y 的 $r-r$ 保序函数，则对于在 x 中的每个 u，$f(u)ru$ 不真.

证明．为了得此定理必须证明 $\{u: u\in x$ 且 $f(u)ru\}$ 是空的．如果不存在一个此类的 r–首元 v，则 $f(v)rv$，并且如果 urv，则 $urf(u)$ 或者 $u=f(u)$．由于 $f(v)rv$，于是 $f(v)r(f(v))$ 或者 $f(v)=f(f(v))$．但是由于 f 是 $r-r$ 保序的，所以 $f(f(v))rf(v)$，从而推得一矛盾.∎

于是 $r-r$ 保序函数不能把它定义域的元映成一个 r–前趋.

象定理 94 这样的证明是依据使定理不成立的首元的研究．这种证明叫归纳法证明.

95 定义. f 是一个1–1函数当且仅当 f 与 f^{-1} 同时都是函数.

这等价于，f 是一个函数同时假定 x 与 y 是 f 定义域中的不同元，则 $f(x)\neq f(y)$.

96 定理. 如果 f 是 $r-s$ 保序的，则 f 是一个 1–1 函数同时 f^{-1} 是 $s-r$ 保序的.

证明．如果 $f(u)=f(v)$，那末对于在 urv 或者 vru 的情况下，$f(u)sf(v)$ 或者 $f(v)sf(u)$ 是不可能的．故 $u=v$ 并且 f 是 1–1 的．如果 $f(u)sf(v)$，则 $u\neq v$，并且如果 vru，则 $f(v)sf(u)$．从而得一矛盾．因此 f^{-1} 是 $r-s$ 保序的.∎

97 定理. 如果 f 与 g 都是 $r-s$ 保序的，f 的定义域与 g 的定义域均为 x 的 r–截片，同时 f 的值域与 g 的值域均为 y 的 s–截片，则 $f\subset g$ 或者 $g\subset f$.

证明．由定理 92 知：不是 f 的定义域 $\subset g$ 的定义域便是 g 的定义域$\subset f$ 的定义域，并且如果证明了对于同时属于 f 和 g 的一切 u，$f(u)=g(u)$，此定理将得以证明.

如果类 $\{z: z\in(f$ 的定义域$)\cap(g$ 的定义域$)$ 且 $g(z)\neq f(z)\}$ 非空，存在一个 r–首元 u，则 $f(u)\neq g(u)$ 并且还可以假定 $f(u)sg(u)$．由于 g 的值域是一个 s–截片，对于在 x 中的某个 v 有

$g(v)=f(u)$ 和 vru，此因 f^{-1} 是保序的．但是 u 是使 $f\neq g$ 的 r-首元，所以 $f(v)=g(v)=f(u)$．从而得一矛盾．❙

98 定义．f 在 x 和 y 中是 r-s 保序的当且仅当 r 良序 x，s 良序 y，f 是 r-s 保序的，f 的定义域是 x 的一个 r-截片，同时 f 的值域是 y 的一个 s-截片．

按照定理 97，如果 f 和 g 在 x 和 y 中都是 r-s 保序的，则 $f\subset g$ 或者 $g\subset f$．

99 定理．如果 r 良序 x 且 s 良序 y，则存在一个函数 f 它在 x 和 y 中是 r-s 保序的，并使得不是 f 的定义域 $=x$ 便是 f 的值域 $=y$．

证明．令 $f=\{(u,v):u\in x$，且对某一函数 g，它在 x 和 y 中是 r-s 保序的，$u\in g$ 的定义域且 $(u,v)\in g\}$．根据前面的定理 f 是一个函数，同时容易看出它的定义域是 x 的一个 r-截片并且它的值域是 y 的一个 s-截片．故 f 在 x 和 y 中是 r-s 保序的，而剩下的只要证明不是 f 的定义域 $=x$ 便是 f 的值域 $=y$．

倘若不然，存在 $x\sim(f$ 的定义域$)$的一个 r-首元 u 和 $y\sim(f$ 的值域$)$的一个 s-首元 v，同时也容易看出函数 $f\cup\{(u,v)\}$ 在 x 和 y 中是 r-s 保序的．于是由 f 的定义 $(u,v)\in f$．所以 $u\in f$ 的定义域．矛盾．❙

在某种情况下，可以说出上面定理的结论中的两种结果会出现哪一种，因为如果 x 是一个集而 y 不是一个集，则根据代换公理 f 的值域 $=y$ 是不可能．

100 定理．如果 r 良序 x，s 良序 y，x 是一个集，而 y 不是一个集，则在 x 和 y 中存在唯一的 r-s 保序函数，它的定义域是 x．

序　　数

在这一节里定义序数，同时建立其基本性质，在开始讨论序数之前将提出另一公理，先验地可能存在两个类 x 与 y 使得 x 为 y 仅有的元，同时 y 为 x 仅有的元．更一般地，存在类 z 它的元可以

相互归属. 在这种情况下，z 的每一个元由 x 的元所组成. 下面的公理断然的否定了这个可能性，而要求每一个非空类 x 至少有一个元，而它的元不属于 x.

VII 正则性公理. 如果 $x \neq 0$，则存在 x 的元 y 使得 $x \cap y = 0$.

101 定理. $x \notin x$.

证明. 如果 $x \in x$，则 x 是一个非空集，同时是 $\{x\}$ 仅有的元. 由正则性公理得知在 $\{x\}$ 中存在 y 使得 $y \cap \{x\} = 0$，于是必然得 $y = x$，可是这样一来 $y \in y \cap \{x\}$，从而得一矛盾.|

102 定理. $x \in y$ 同时 $y \in x$ 不真.

证明. 如果 $x \in y$ 同时 $y \in x$，则 x 与 y 都是集，并且是 $\{z: z = x$ 或者 $z = y\}$ 仅有的元. 应用正则性公理于后面这个集，那末正象上面定理的证明一样将导致矛盾.|

当然这个定理可以推广到多于两个集上，正则性公理实际上推出另一更强的结果，直观的叙述如下：不可能存在这样的序列，对于每个 n 使得 $x_{n+1} \in x_n$. 这个结果的严格叙述必须向后放一放.

103 定义. $E = \{(x, y): x \in y\}$.

类 E 是 E–关系. 注意如果 $x \in y$ 同时 y 不是一个集，则由定理 54，$(x, y) = \mathscr{U}$ 且 $(x, y) \notin E$.

104 定理. E 不是一个集.

证明. 如果 $E \in \mathscr{U}$，则 $\{E\} \in \mathscr{U}$ 并且 $(E, \{E\}) \in E$. 我们记得 $(x, y) = \{\{x\}\{xy\}\}$，并且如果 (x, y) 是一个集，$z \in (x, y)$ 当且仅当 $z = \{x\}$ 或者 $z = \{xy\}$. 从而 $E \in \{E\} \in \{\{E\}\{E\{E\}\}\} \in E$. 但如果 $a \in b \in c \in a$，那末把正则性公理应用到 $\{x: x = a$ 或者 $x = b$ 或者 $x = c\}$ 上便得一矛盾的结果.|

从开头几个序数结构的非形式的讨论中，可以获得概念上的启发[1]. 第一个序数将是 0，紧接着 $1 = 0 \cup \{0\}$，紧接着 $2 = 1 \cup \{1\}$，

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1) 这种讨论不是很确切的. 因为它没有证明 0 是一个集. 事实上，依照我们所设置的这些公理，这一点是无法证明的. 集的存在性(由此事实 0 是一个集)由无限性公理获得，而它在下节之始才加以叙述.

紧接着 $3=2\cup\{2\}$. 显然 0 是 1 仅有的元，0 与 1 又是 2 仅有的元，而 0，1 与 2 是 3 仅有的元. 3 前面的每个序数不仅是一个元而且也是 3 的子集. 序数就是这样定义以便得到这种很特殊的结构.

105 定义[1]. x 为充满的当且仅当每个 x 的元是 x 的子集.

换句话说，x 为充满的当且仅当每个 x 的元的元是 x 的元.

另一个等价说法：x 为充满的当且仅当 E 在 x 中是传递的.

下面的定义是属于 R. M. Robinson 的.

106 定义. x 是一个序数当且仅当 E 连接 x，同时 x 是充满的.

也就是说，已给 x 的两个元一个是另一个的元，同时每个 x 的元的元属于 x.

107 定理. 如果 x 是一个序数，则 E 良序 x.

证明. 如果 u 与 v 都是 x 的元，并且 uEv，则（由定理 102）vEu 不真，故 E 在 x 中是非对称的. 如果 y 是 x 的一个非空子集，则由正则性公理在 y 中存在 u 使得 $u\cap y=0$. 于是 y 没有元属于 u，故 u 是 y 的 E-首元. |

108 定理. 如果 x 是一个序数，$y\subset x$，$y\neq x$，同时 y 是充满的，则 $y\in x$.

证明. 如果 uEv 并且 vEy，于是根据 y 是充满的得 uEy. 所以 y 是 x 的一个 E-截片. 从而依据定理 91 存在 x 的元 v 使得 $y=\{u: u\in x$ 且 $uEv\}$. 由于 v 的每个元都是 x 的元，$y=\{u: u\in v\}$ 和 $y=v$. 故得证. |

109 定理. 如果 x 是一个序数同时 y 也是一个序数，则 $x\subset y$ 或者 $y\subset x$.

证明. 类 $x\cap y$ 是充满的，于是由前面的定理知：不是 $x\cap y=x$ 便是 $x\cap y\in x$. 在第一种情况下 $x\subset y$. 如果 $x\cap y\in x$，则 $x\cap y\notin y$，因为在这种情况下有 $x\cap y\in x\cap y$. 由于 $x\cap y\notin y$，所以由上面定理推得 $x\cap y=y$. 故 $y\subset x$. |

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1) "完全的"这个术语习惯上用来代替"充满的"，但是"完全的"早已被用于不同的意义之下了.

110 定理. 如果 x 是一个序数，并且 y 也是一个序数，则 $x\in y$ 或者 $y\in x$ 或者 $x=y$.

111 定理. 如果 x 是一序数并且 $y\in x$; 则 y 是一个序数.

证明. 因为 x 是充满的，同时 E 连接 x，所以 E 连接 y 是显然的. 又由 E 良序 x 同时 $y\subset x$，故关系 E 在 y 上是传递的. 从而如果 uEv 且 vEy，则 uEy. 所以 y 是充满的. |

112 定义. $R=\{x:\ x$ 是一序数$\}$.

113 定理[1]. R 是一个序数，但不是一个集.

证明. 最后两个定理证明了 E 连接 R，并且 R 是充满的；所以 R 是一个序数.

如果 R 是一个集，则 $R\in R$. 而这是不可能的. |

由于定理 110，R 是仅有的非集的序数.

114 定理. R 的每个 E-截片是一个序数.

证明. 如果 R 的 E-截片不等于 R，则由定理 91 知存在 R 的元 v 使得 $x=\{u:\ u\in R$ 且 $u\in v\}$. 由于 v 的每个元是一个序数，$x=\{u:\ u\in v\}=v$. |

115 定义. x 是一个序数当且仅当 $x\in R$.

116 定义. $x<y$ 当且仅当 $x\in y$.

117 定义. $x\leqq y$ 当且仅当 $x\in y$，或者 $x=y$.

118 定理. 如果 x 与 y 均为序数，那末当且仅当 $x\subset y$ 时，$x\leqq y$ 成立.

119 定理. 如果 x 是一个序数，则 $x=\{y:\ y\in R$ 且 $y<x\}$.

120 定理. 如果 $x\subset R$，则 $\bigcup x$ 是一个序数.

证明. 由定理 110 和 111 推得 E 连接 $\bigcup x$，再由 x 的元均为充满的事实推出 $\bigcup x$ 是充满的. |

不难看出，如果 x 是 R 的子集，则序数 $\bigcup x$ 是大于等于 x 的每个元的第一个序数，同时 $\bigcup x$ 是一个集当且仅当 x 为一个集. 然而这些结果并不太需要.

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1) 这个定理实质上是 Burali-Forti 悖论的叙述——在历史上是直观集论的第一个悖论.

121 定理. 如果 $x \subset R$ 且 $x \neq 0$,则 $\bigcap x \in x$.

诚然,在这种情况下,$\bigcap x$ 是 x 的 E-首元.

122 定义. $x+1 = x \cup \{x\}$.

123 定理. 如果 $x \in R$,则 $x+1$ 是 $\{y: y \in R$ 且 $x < y\}$ 的 E-首元.

证明. 容易验证 E 连接 $x+1$,同时 $x+1$ 是充满的同时是一序数. 如果存在 u 使得 $x < u$ 和 $u < x+1$, 则由 x 是一个集和 $u \in x \cup \{x\}$ 不是 $u \in x$ 同时 $x \in u$ 便是 $u = x$ 和 $x \in u$. 而这两个结果都是不可能的(定理 101 和 102). 于是获得欲证的结论. ▎

124 定理. 如果 $x \in R$, 则 $\bigcup(x+1) = x$.

125 定义. $f|x = f \cap (x \times \mathscr{U})$.

这个定义仅只在 f 是一个关系时才使用. 在这种情况下,$f|x$ 是一个关系同时被称为 f 在 x 上的限制.

126 定理. 如果 f 是一个函数,则 $f|x$ 也是一个函数,它的定义域为 $x \cap (f$ 的定义域), 并且对于每个在 $f|x$ 定义域中的 y, $(f|x)(y) = f(y)$.

关于序数最后面的这个定理断言(直观上),在一个序数上用下面的方法定义一个函数是可能的. 即它在其定义域内任意元上的值可对已得到的函数值应用预先确定的规律所给出. 稍微确切地说,已给函数 g 可能求得在一个序数上唯一的函数 f 使得对于每个序数 x,$f(x) = g(f|x)$. 于是 $f(x)$ 的值完全由 g 和 f 在 x 前面的序数处的值所决定.

这个定理的应用即是用超穷归纳法来定义一个函数. 此证明类似于定理 99,而且同样类型的预备引理也是需要的.

127 定理. 令 f 是一个使得 f 的定义域为一序数的函数,同时对于在 f 定义域中的 u,$f(u) = g(f(u))$. 如果 h 也是一个使得 h 的定义域为一序数的函数,同时对于在 h 定义域中的 u, $h(u) = g(h|u)$, 则 $h \subset f$, 或者 $f \subset h$.

证明. 由 f 的定义域和 h 的定义域都是序数, 所以可以假定 f 的定义域 $\subset h$ 的定义域.(由定理 109 推出不是这种情况便是相

反的情况.)剩下的只需证明对于在 f 定义域中的 u, $f(u)=h(u)$.

倘若不然,令 u 为 f 定义域中使得 $f(u)\neq h(u)$ 的 E-首元,则对于在 u 前面的每个序数 v, $f(v)=h(v)$, 所以 $f|u=h|u$. 于是 $f(u)=g(f|u)=h(u)$. 从而得一矛盾. ❙

128 定理. 对于每个 g 存在唯一的函数 f 使得 f 的定义域是一个序数,并且对于每个序数 x, $f(x)=g(f|x)$.

证明. 令 $f=\{(u,v): u\in R$ 且存在一个函数 h 使得 h 的定义域是一个序数,对于在 h 定义域中的 z, $h(z)=g(h|z)$ 同时 $(u,v)\in h\}$.

由前面的定理推得 f 是一个函数, 而 f 的定义域是 R 的一个 E-截片是显然的,所以它是一个序数. 何况,如果 h 是在一个序数上定义的函数使得对于在 h 定义域中的 z, $h(z)=g(h|z)$ 成立,则 $h\subset f$. 如果 $z\in f$ 的定义域,则 $f(z)=g(h|z)$.

最后假定 $x\in R\sim(f$ 的定义域), 则由定理 69 $f(x)=\mathscr{U}$. 又由于 f 的定义域是一个集, 所以 f 是一个集 (定理 75). 如果 $g(f|x)=g(f)=\mathscr{U}$, 则得等式 $f(x)=g(f|x)$. 否则 $g(f)$ 是一个集(仍依据定理 69). 在这种情况下,如果 y 是 $R\sim(f$ 的定义域)的 E-首元且 $h=f\cup\{(y,g(f))\}$, 则 h 的定义域是一个序数,并且对于 h 定义域中的 z, $h(z)=g(h|z)$, 故 $h\subset f$ 同时 $y\in f$ 的定义域. 从而得一矛盾. 所以 $g(f)=\mathscr{U}$, 于是定理得证. ❙

对这个定理的技巧应作一点注解. 如果 f 的定义域不是 R, 则对于每个使得 f 的定义域 $\leqq x$ 的序数 x, $g(f)=\mathscr{U}$ 且 $f(x)=\mathscr{U}$. 如果 $g(0)=\mathscr{U}$, 则 $f=0$.

整　　数[1]

在这一节里定义整数, 并且 Peano 公理将作为定理推出. 实数可以利用这些公理由整数和下面两点事实来构造(参看 Landau [1]):

1) 非负整数.

整数类是一个集（定理 138），同时利用归纳法在整数上定义一个函数是可能的.（预备知识定理 13；这个事实也可以作为定理 128 的一个系推出.）此外还需要另一个公理.

VIII 无限性公理. 对于某一集 y, $0 \in y$ 同时只要 $x \in y$，则 $x \cup \{x\} \in y$.

特别是 0 为一个集，此因 0 被包含在一个集内.

129 定义. x 是一整数当且仅当 x 是一个序数同时 E^{-1} 良序 x.

130 定义. x 是 y 的一个 E-末元就是说 x 是 y 的一个 E^{-1} 首元.

131 定义. $\omega = \{x: x$ 是整数$\}$.

132 定理. 一个整数的元是一个整数.

证明. 一个整数 x 的元是一个序数并且是 x 的一个子集，同时 x 被 E^{-1} 所良序. |

133 定理. 如果 $y \in R$ 且 x 是 y 的一个 E-末元，则 $y = x + 1$.

证明. 由定理 123，$x + 1$ 是 $\{z: z \in R$ 且 $x < z\}$ 的 E-首元，于是 $x + 1 \leq y$，此因 $y \in R$ 和 $x < y$. 由于 x 是 y 的 E-末元且 $x < x + 1$，所以 $x + 1 < y$ 不真. |

134 定理. 如果 $x \in \omega$ 则 $x + 1 \in \omega$.

135 定理. $0 \in \omega$ 并且如果 $x \in \omega$，则 $0 \neq x + 1$.

也就是说，0 非整数的后继.

136 定理. 如果 x 和 y 均为 ω 的元，且 $x + 1 = y + 1$，则 $x = y$.

证明. 由定理 124，如果 $x \in R$，则 $\bigcup(x + 1) = x$. |

下面的定理是数学归纳原理.

137 定理. 如果 $x \subset \omega$, $0 \in x$ 并且只要 $u \in x$，就有 $u + 1 \in x$，则 $x = \omega$.

证明. 如果 $x \neq \omega$，令 y 为 $\omega \sim x$ 的 E-首元同时注意 $y \neq 0$. 由于 $y \subset y + 1$ 并且 $y + 1$ 是一个整数，所以存在一个 y 的 E-

末元 u，而且显然 $u\in x$. 于是由定理 123 得 $y=u+1$，故 $y\in x$. 从而得一矛盾. ▌

定理 134、135、136 和 137 均为关于整数的 Peano 公理. 下面的定理推出 ω 是一个集.

138 定理. $\omega\in R$.

证明. 由无限性公理知存在一个集 y 使得 $0\in y$，并且如果 $x\in y$，则 $x+1\in y$. 又由数学归纳法(也就是上面的定理) $\omega\cap y=\omega$. 所以 ω 是一个集，此因 $\omega\subset y$. 由于 ω 由序数组成，E 连接 ω 并且 ω 是充满的，此因每个整数的元是一个整数. ▌

选 择 公 理

现在我们叙述最后一个公理，并导出两个有力的推论.

139 定义. c 是一个选择函数的充要条件是 c 为一函数，并且对于 c 定义域的每个元 x，$c(x)\in x$.

直观上，一个选择函数是由 c 的定义域的每个集中同时选一个元的这种选取法.

下面是 Zermelo 公设的一个强化形式，或称为选择公理.

IX 选择公理. 存在一个选择函数 c，它的定义域是 $\mathscr{U}\sim\{0\}$.

函数 c 从每个非空集中选取一个元.

140 定理. 如果 x 是一个集，存在 1-1 函数，它的值域是 x 而它的定义域是一个序数.

证明. 证明的路线是用超穷归纳法构造一个满足这个定理要求的函数. 令 g 是使得对于每个集 h，$g(h)=c(x\sim h$ 的值域$)$的函数，这里 c 是一个满足选择公理的选择函数. 应用定理 128 存在函数 f 使得 f 的定义域是一个序数，同时对每个序数 u，$f(u)=$ $=g(f|u)$. 于是 $f(u)=c(x\sim(f|u)$ 的值域$)$，并且如果 $u\in f$ 的定义域，则 $f(u)\in(x\sim(f|u)$ 的值域$)$. 现在 f 是 1-1 的，因为若对于 $f(u)=f(v)$ 且 $u<v$，则有 $f(v)\in(f|v)$ 的值域，而它与

$f(v)\in(x\sim(f|v)$ 的值域）的事实相矛盾．由于 f 是 1-1 的所以 f 的定义域 $=R$ 是不可能的．因为在这种情况下 f^{-1} 是一个函数，它的定义域是 x 的子类，故是一个集，于是根据代换公理 f^{-1} 的值域是一个集，而 R 不是一个集，从而 f 的定义域 $\neq R$． 因为 f 的定义域 $\in f$ 的定义域，$f(f$ 的定义域$)=\mathscr{U}$，所以 $c(x\sim f$ 的值域$)=\mathscr{U}$．由于 c 的定义域是 $\mathscr{U}\sim\{0\}$，故 $x\sim f$ 的值域 $=0$．这样一来立刻推得 f 是满足定理要求的函数．|

141 定义．n 是一个套的充分必要条件是只要 x 与 y 均为 n 的元，则 $x\subset y$ 或者 $y\subset x$．

下面的结论是一个引理，它是证明定理 143 所需要的．

142 定理．如果 n 是一个套，同时 n 的每个元是一个套，则 $\bigcup n$ 也是一个套．

证明．如果 $x\in m$，$m\in n$，$y\in p$ 且 $p\in n$，则不是 $m\subset p$ 便是 $p\subset m$，因为 n 是一个套．假定 $m\subset p$，则 $x\in p$ 且 $y\in p$，同时由于 p 是一个套，$x\subset y$ 或者 $y\subset x$．|

下面的定理称作 Hausdorff 极大原理，它断言在任何集中极大套的存在性．这个证明与定理 140 的证明有密切关系．

143 定理．如果 x 是一个集，则存在一个套 n 使得 $n\subset x$，并且如果 m 是一个套，$m\subset x$ 且 $n\subset m$，则 $m=n$．

证明．此证明是利用超穷归纳法；直观上，我们选一个套，然后再选一个较大的，继续进行下去，我们知道因为 R 非集，所有包含在 x 中的套组成的集将不会取完 R 中的序数．对每一 h，令 $g(h)=c(\{m$: m 是一个套，$m\subset x$，且对 h 值域中的 p，$p\subset m$ 同时 $p\neq m\})$，这 c 是满足选择公理的一个选择函数．（直观上所选的 $g(h)$ 是在 x 中的套，而它真正被包含在每个前面所选的套中．）由定理 128 存在函数 f 使得 f 的定义域是一个序数，并且对于每个序数 u，$f(u)=g(f|u)$．由 g 的定义推出，如果 $u\in f$ 的定义域，则 $f(u)\subset x$ 同时 $f(u)$ 是一个套，而且如果 u 与 v 均为 f 定义域的元，并且 $u<v$，则 $f(u)\subset f(v)$ 同时 $f(u)\neq f(v)$．从而 f 是 1-1 的，f^{-1} 是一个函数．又由于 x 是一个集，所以 f 的定义域 $\in R$．

因为 f(f 的定义域)$=\mathscr{U}$[1], $g(f)=\mathscr{U}$；从而不存在套 m 被包含在 x 中，并真正包含 f 值域的每个元.

最后，$\bigcup$(f的值域)是一个套，它包含 f 值域的所有元，从而不存在套 m 被包含在 x 中，同时真正包含 $\bigcup$(f的值域). ▎

基　　数

在这一节里定义基数并证明通常用到的大部分性质. 这些证明紧密地依赖着早先的结果.

144 定义. $x\approx y$ 当且仅当存在一个 1-1 函数 f，f 的定义域$=x$ 而 f 的值域 $=y$.

如果 $x\approx y$，则称 x 等势于 y，或者 x 与 y 是等势的.

145 定理. $x\approx x$.

146 定理. 如果 $x\approx y$，则 $y\approx x$.

147 定理. 如果 $x\approx y$ 同时 $y\approx z$，则 $x\approx z$.

148 定义. x 是一个基数就是说 x 是一个序数，并且如果 $y\in R$ 和 $y<x$，则 $x\approx y$ 不真.

也就是说，一个基数是一个序数，而它不等势于任何较小的序数.

149 定义. $C=\{x:\ x$ 是基数$\}$.

150 定理. E 良序 C.

151 定义. $P=\{(x,y):\ x\approx y$ 且 $y\in C\}$.

类 P 是由所有使得 x 是一个集且 y 是等势于 x 的基数之偶对 (x,y) 所组成. 对于每个集 x，基数 $P(x)$ 是 x 的势，或者 x 的基数.

推出下面一系列结果所需要的基本事实已经被论证.

152 定理. P 是一个函数，P 的定义域$=\mathscr{U}$ 且 P 的值域$=C$.

证明. 定理 140 是其证明的主要步骤. ▎

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1) 因为序数 (f 的定义域) $\in f$ 的定义域，由定理 69，f(f 的定义域)$=\mathscr{U}$.

153 定理. 如果 x 是一个集，则 $P(x) \approx x$.

154 定理. 如果 x 与 y 均为集，则当且仅当 $P(x) = P(y)$ 时，$x \approx y$.

155 定理. $P(P(x)) = P(x)$.

证明. 如果 x 不是一个集，则由定理 69 $P(x)=\mathscr{U}$ 且 $P(\mathscr{U})=\mathscr{U}$. |

156 定理. 当且仅当 x 是一个集并且 $P(x) = x$ 时，$x \in C$ 成立.

157 定理. 如果 $y \in R$ 且 $x \subset y$，则 $P(x) \leqq y$.

证明. 由定理 99 存在一个 1-1 函数 f，它在 x 与 R 中是 $E-E$ 保序的，并且使得不是 f 的定义域 $=x$ 便是 f 的值域 $=R$. 由于 x 是一个集，R 不是一个集，故 f 的定义域 $=x$. 由定理 94 对于 x 中的 u，$f(u) \leqq u$，从而 x 等势于一个小于等于 y 的序数. |

158 定理. 如果 y 是一个集且 $x \subset y$，则 $P(x) \leqq P(y)$.

下面是 Schroeder-Bernstein 定理. 它能不用选择公理而直接证明(预备知识定理 20).

159 定理. 如果 x 与 y 均为集，$u \subset x$，$v \subset y$，$x \approx v$ 且 $y \approx u$，则 $x \approx y$.

证明. 利用定理 157，$P(x)=P(v) \leqq P(y)=P(u) \leqq P(x)$. |

160 定理. 如果 f 是一个函数同时 f 是一个集，则 P(f 的值域)$\leqq P$(f 的定义域).

证明. 如果 f 是自 x 上到 y 上的一个函数，并且 c 是满足选择公理的一个选择函数，则存在函数 g 使得 g 的定义域 $=y$，同时对于 y 中的 v，$g(v) = c(\{u: v = f(u)\})$. 从而 y 等势于 x 的子集. |

下面经典的定理是属于 Cantor 的.

161 定理. 如果 x 是一个集，则 $P(x) < P(2^x)$.

证明. 定义域为 x 且在 x 的元 u 处计值为 $\{u\}$ 的函数是 1-1 的. 所以 x 等势于 2^x 的子集且 $P(x) \leqq P(2^x)$. 如果 $P(x) = P(2^x)$，则存在 1-1 函数 f，它的定义域是 x 且值域是 2^x. 于是存

在 x 的元 u 使得 $f(u)=\{v: v\in x$ 且 $v\notin f(v)\}$. 这么一来 $u\in f(u)$, 就是指 $u\notin f(u)$. 从而得一矛盾.▌

上面就其结构而言类似于 Russell 悖论.

162 定理. C 不是一个集.

证明. 如果 C 是一个集,则 $\bigcup C$ 也是一个集, $P(2^{\bigcup C})\in C$. 故 $P(2^{\bigcup C})\subset\bigcup C$, 所以 $P(2^{\bigcup C})\leq P(\bigcup C)$. 从而得一矛盾.▌

有了一些准备之后,我们把基数分成两类,有限基数与无限基数,并对每一类证明一些特殊的性质.

163 定理. 如果 $x\in\omega$, $y\in\omega$ 且 $x+1\approx y+1$, 则 $x\approx y$.

证明. 如果 f 是 $x+1$ 上到 $y+1$ 上的 1-1 函数, 则存在 $x+1$ 上到 $y+1$ 上的一个 1-1 函数 g 使得 $g(x)=y$; 例如, 令 g 为 $(f\sim\{(x, f(x))\}\cup\{(f^{-1}(y), y)\})\cup\{(f^{-1}(y), f(x))\}\cup\{(x,y)\}$, 则 $g|x$ 是 x 上到 y 上的 1-1 函数.▌

164 定理. $\omega\subset C$.

证明. 这个定理的证明是利用归纳法. 把前面的定理应用到等势于一个较小整数的那种整数的第一整数上,便得一个矛盾,于是证明了每个整数是一个基数.▌

165 定理. $\omega\in C$.

证明. 如果 $\omega\approx x$ 并且 $x\in\omega$, 则 $x\subset x+1\subset\omega$, 故 $P(x+1)=P(x)$. 这与前面定理所叙述的结论每个整数是一个基数相矛盾.▌

166 定义. x 是有限的当且仅当 $P(x)\in\omega$.

167 定理. x 是有限的当且仅当存在 r 使得 r 良序 x 并且 r^{-1} 也良序 x.

证明. 如果 $P(x)\in\omega$, 则 E 与 E^{-1} 良序 $P(x)$, 由于 $x\approx P(x)$, 所以不难求得 r 使得 r 与 r^{-1} 都良序 x. 反之如果 r 与 r^{-1} 都良序 x,则由定理 99 知存在一个 1-1 函数 f, 它在 x 与 R 中是 $r-E$ 保序的,并使得不是 f 的定义域 $=x$ 便是 f 的值域 $=R$. 如果 $\omega\subset f$ 的值域, 则 r^{-1} 非良序 x, 此因 ω 没有 E-末元素. 所以 f 的值域 $\in\omega$, f 的定义域 $=x$. 于是定理得证.▌

下面这些关于有限集的定理都能对集的势进行归纳证明，或者用构造一个良序和应用定理 167 加以证明．这两类证明的例子都将给出．

168 定理．如果 x 与 y 均有限，则 $x\cup y$ 也是有限的．

证明．如果 r 与 r^{-1} 同时都良序 x，并且 s 与 s^{-1} 同时都良序 y，则对于 x 中的点应用 r，对于 $y\sim x$ 中的点应用 s，并令 $y\sim x$ 中的每个元跟在 x 的所有点之后，而它能构成欲求类型 $x\cup y$ 的一个序．▌

169 定理．如果 x 有限且 x 的每个元有限，则 $\bigcup x$ 也有限．

证明．可以对 $P(x)$ 进行归纳法证明．很明显考虑所有使得如果 $P(x)=u$ 且 x 的每个元有限则 $\bigcup x$ 有限之整数 u 的集 s，于是显然 0 属于此集．如果 $u\in s$，$P(x)=u+1$ 且 x 的每个元是有限的，则 x 可分成两个子集：其一有势 u，其二为单点．由归纳假设以及前面的定理，则证明了 $\bigcup x$ 是有限的．故 $s=\omega$．▌

170 定理．如果 x 与 y 有限，则 $x\times y$ 也有限．

证明．类 $x\times y$ 是一有限类中元的并．这些元形为 $\{v\}\times y$，这里 v 属于 x．▌

171 定理．如果 x 是有限的，则 2^x 也是有限的．

证明．如果 y 是一个整数，则 $y+1$ 的子集能够被分成两类：其一是那些 y 的子集，其二是那些 y 的子集与 $\{y\}$ 之并．这样便给出了定理进行归纳证明所必须的基础．▌

172 定理．如果 x 是有限的，$y\subset x$ 且 $P(y)=P(x)$，则 $x=y$．

证明．只要考虑 x 为整数的情况就足够了．假定 $y\subset x$，$y\neq x$，$P(y)=x$ 且 $x\in\omega$，于是 $x\neq 0$．所以对某一整数 u，$x=u+1$．因为 $y\neq x$，所以存在一个 u 的子集等势于 y，故 $P(y)\leq u$．但是 $P(y)=x=u+1$．所以这与每个整数是一个基数的事实相矛盾．▌

定理 172 是关于有限集不能等势于它的真子集的性质，实际上，它描述了有限集的特征．

173 定理. 如果 x 是一个集，而非有限，则存在一个 x 的子集 y 使得 $y \neq x$ 并且 $x \approx y$.

证明. 由于 x 是一个集，而非有限，$\omega \subset P(x)$，所以在 $P(x)$ 上存在一个函数 f 使得对于 ω 中的 u, $f(u) = u + 1$, 并且对于 $P(x) \sim \omega$ 中的 u, $f(u) = u$. 此函数 f 是 1-1 的，并且 f 的值域 $= P(x) \sim \{0\}$. 由于 $P(x) \approx x$，从而定理得证. ▌

174 定理. 如果 $x \in R \sim \omega$，则 $P(x + 1) = P(x)$.

证明. 显然 $P(x) \leqq P(x) + 1$，由于 x 非有限，所以存在 x 的子集 u 使得 $u \neq x$ 并且 $u \approx x$. 从而在 $x + 1$ 上存在 1-1 函数 f 使得对于在 x 中的 y, $f(y) \in u$ 和 $f(x) \in x \sim u$. 故 $P(x + 1) \leqq P(x)$. ▌

剩下的主要定理依赖于笛卡儿乘积 $R \times R$ 上的一个序关系. 给这个序一种直观的描述法可能是有益的. 它作为一个良序，并在 $\omega \times \omega$ 上具有性质：$\omega \times \omega$ 的元 (x, y) 之所有前趋元之类为有限的.（这个事实与推广是说明此序有效性的关键.）$\omega \times \omega$ 的图形是作为欧几里得平面的子集并把 $\omega \times \omega$ 分类，而使得 x 与 y 的最大值和 u 与 v 最大值相同的偶对 (x, y) 和 (u, v) 放在同一类. 于是每一个类由正方形的两边组成，并且这个序把较小正方形上的点排在大正方形上点之前. 对于在同一正方形边界上的那些点，这个序沿着上边缘向右进行，一直达到但不包含角点，再沿着右边缘向上结束于角点.

如果 x 与 y 都是序数，它们之中较大的是 $x \cup y$. 这样引导出下面的定义.

175 定义. $\max[x, y] = x \cup y$.

176 定义. $\ll = \{z$: 对于在 $R \times R$ 中的某个 (u, v) 与在 $R \times R$ 中的某个 (x, y), $z = ((u, v), (x, y))$，且 $\max[u, v] < \max[x, y]$ 或者 $\max[u, v] = \max[x, y]$ 且 $u < x$，或者 $\max[u, v] = \max[x, y]$ 且 $u = x$ 和 $v < y\}$.

177 定理. $\ll$ 良序 $R \times R$.

这个定理的证明虽很直接，但要繁琐地应用定义和 $<$ 良序 R

的事实.

178 定理. 如果 $(u, v) \ll (x, y)$,则 $(u, v) \in (\max[x, y]+1) \times (\max[x, y]+1)$.

证明. 无疑 $\max[u, v] \leqq \max[x, y]$, 故 $\max[u, v] \subset \max[x, y]$. 由于序数 u 与 v 均为 $\max[x, y]$ 的子集, 所以它们都是 $\max[x,y]+1$ 的元.▌

179 定理. 如果 $x \in C \sim \omega$, 则 $P(x \times x) = x$.

证明. 我们用归纳法来进行此证明. 设 x 为使得定理不真的 $C \sim \omega$ 的首元,于是由定理 99 知存在一个函数 f 在 $x \times x$ 与 R 中是 $\ll$-E 保序的,并使得不是 f 的定义域 $= x \times x$ 便是 f 的值域 $= R$. 由于 $x \times x$ 是一个集而 R 不是一个集, f 的定义域 $= x \times x$. 我们证明如果 $(u, v) \in x \times x$, 则 $f((u, v)) < x$, 继而推出本定理. 由前面的定理 (u, v) 所有前趋元的类是 $(\max[u, v]+1) \times (\max[u, v]+1)$ 的子集. 如果 $x = \omega$, 则 u 与 v 同时为有限的. 因为 $\max[u, v] < x$, 由定理 170, $(\max[u, v]+1) \times (\max[u, v]+1)$有限,故 $f((u, v))$ 仅有有限个前趋元并且 $f((u, v)) < x$. 如果 $x \neq \omega$、且 $\max[u, v]$ 非有限,则由定理 174, $P(\max[u,v]+1) = P(\max[u, v]) < x$, 故 $P(f((u, v))) < x$ 且 $f((u, v)) < x$.▌

180 定理. 如果 x 与 y 都是 C 的元,而其中一个不属于 ω,则 $P(x \times y) = \max[P(x), P(y)]$.

$C \sim \omega$ 的元被称为无限基数,或者超穷基数.

关于基数许多重要而又有用的定理在前面所列举的定理中尚有些未曾给出; 而进一步的情况和参考资料譬如可以看 Fraenkel[1]. 最后我们以简单地叙述一个古典集论所未曾解决的问题作为这方面讨论的终结.

181 定理. 存在唯一的 $<$-$<$ 保序函数以 R 为定义域, 并以 $C \sim \omega$ 为值域.

证明. 由 99, 在 R 与 $C \sim \omega$ 中存在唯一的 $<$-$<$ 保序函数 f, 使得 f 的定义域 $= R$ 或者 f 的值域 $= C \sim \omega$. 由于 R 与 $C \sim \omega$ 的每个截片是一个集,并且 R 与 $C \sim \omega$ 都不是集, 所以 f 的定义

域$\neq R$或者f的值域$\neq C\sim\omega$是不可能的. ∎

这个唯一的$<$-$<$保序函数的存在性是由前面的定理所保证，而它通常用$\aleph$来表示. 于是$\aleph(0)$(或者$\aleph_0$)为ω. 紧接的下一个基数$\aleph_1$也用Ω来表示；因此它是第一个不可数序数. 由于$P(2^{\aleph_0})>\aleph_0$推得$P(2^{\aleph_0})\geq\aleph_1$. 这两个基数的相等是极有吸引力的猜测. 它被称为连续统假设. 广义连续统假设是这样叙述的：如果x是一个序数，则$P(2^{\aleph_x})=\aleph_{x+1}$. 此假设既不曾被证明也不曾被否定1). 然而 Gödel[1] 曾证明了一个很妙的元数学定理：

如果在连续统假设的基础上产生了一个矛盾，则矛盾也可以在不假定连续统假设的情况下被找到. 对广义连续统假设和选择公理也是一样.

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译者为本书增添的附录

不分明拓扑学介绍

1965 年 L. A. Zadeh[1] 首先引入了不分明集，在此基础上 1968 年 C. L. Chang[1] 进一步展开了对不分明拓扑空间的研究。十年来经过 C. K. Wong[1-3], R. Lowen[1-4], R. H. Warren[1-3], B. Hutton[1-3] 以及我国蒲保明、刘应明[1,2] 等人的工作，已经把本书的大部分内容转移到了不分明拓扑空间，形成了不分明拓扑学.

自从本书原著出版以来，一般拓扑学取得了许多重要进展，考虑到不分明拓扑学乃是它的一种全面（而非个别问题）的推广，同时对数学的不少领域都起着一定的影响，因此，译者冒昧写此附录，以期求得广大读者的批评与指正.

§0. 不分明集与不分明点

我们恒设 $X=\{x\}$ 为非空集.

定义 1. 我们称 A 为 X 上的不分明集当且仅当

$$A=\{(x,\ \mu_A(x))\,|\,x\in X,\ \mu_A\in[0,\ 1]^X\},$$

这里 $\mu_A(x)$ 叫做 A 的程度函数或隶属函数.

由定义 1 容易看出不分明集 A 完全由它的程度函数 $\mu_A(x)$ 所刻划.

若 $\mu_A(x)$ 仅取 0 或 1，则不把它和 X 的子集 A 加以区别.

定义 2. 设 A 和 B 为 X 的不分明集，则规定它们之间的相等和运算为：

$A=B \Longleftrightarrow \mu_A(x)=\mu_B(x)$, 对一切 $x\in X$;

$A\subset B \Longleftrightarrow \mu_A(x)\leqslant\mu_B(x)$, 对一切 $x\in X$;

$$C=A\cup B\Longleftrightarrow \mu_C(x)=\max[\mu_A(x),\mu_B(x)],\quad 对一切\ x\in X;$$
$$D=A\cap B\Longleftrightarrow \mu_D(x)=\min[\mu_A(x),\mu_B(x)],\quad 对一切\ x\in X;$$
$$E=A'\Longleftrightarrow \mu_E(x)=1-\mu_A(x),\quad 对一切\ x\in X.$$

类似地，一族不分明集 $\{A_i\}, i\in I$ 的并 $C=\bigcup_{i\in I}A_i$ 与交 $D=\bigcap_{i\in I}A_i$ 的程度函数分别由

$$\mu_C(x)=\sup_{i\in I}\mu_{A_i}(x),\quad (x\in X)$$

和

$$\mu_D(x)=\inf_{i\in I}\mu_{A_i}(x),\quad (x\in X)$$

确定.

另外，对不分明集同样也有 De Morgan 公式成立.

定义 3. 设 f 为从 X 到 Y 的函数，B 为 Y 的不分明集，则 $f^{-1}[B]$ 表示由程度函数

$$\mu_{f^{-1}[B]}(x)=\mu_B(f(x)),\quad (x\in X)$$

所确定的 X 的不分明集.

反之，设 A 为 X 的不分明集，则 $f[A]$ 表示由程度函数

$$\mu_{f[A]}(y)=\begin{cases}\sup\limits_{x\in f^{-1}[y]}\mu_A(x), & 当\ f^{-1}[y]\ 为非空时,\\ 0, & 在别处\end{cases}$$

$(y\in Y)$ 所确定的 Y 的不分明集.

定理 1. 设 f 为从 X 到 Y 的函数，则有：

(1) $f^{-1}[B']=(f^{-1}[B])'$;

(2) $f[A']\supset(f[A])'$，这里 f 是满映射；

(3) 从 $B_1\subset B_2$ 可推出 $f^{-1}[B_1]\subset f^{-1}[B_2]$;

(4) 从 $A_1\subset A_2$ 可推出 $f[A_1]\subset f[A_2]$;

(5) $B\supset f[f^{-1}[B]]$;

(6) $A\subset f^{-1}[f[A]]$;

(7) 再设 g 为从 Y 到 Z 的函数，则对任何 Z 的不分明集 C 有

$$(g\circ f)^{-1}[C]=f^{-1}[g^{-1}[C]],$$

其中 $g\circ f$ 为 g 与 f 的合成.

定理中的 A 和 B 分别为 X 和 Y 的不分明集.

证明．我们只证明 (1) 与 (2)，其余的证明留给读者．

(1) 只须注意对一切 $x\in X$ 有

$$\mu_{f^{-1}[B']}(x)=\mu_{B'}(f(x))=1-\mu_B(f(x))=1-\mu_{f^{-1}[B]}(x)$$
$$=\mu_{(f^{-1}[B])'}(x).$$

(2) 于 $y\in Y$，由于 f 是满映射，$f^{-1}[y]$ 为非空，于是有

$$\mu_{f[A']}(y)=\sup_{x\in f^{-1}[y]}\mu_{A'}(x)=\sup_{x\in f^{-1}[y]}(1-\mu_A(x))$$
$$=1-\inf_{x\in f^{-1}[y]}\mu_A(x)$$

和

$$\mu_{(f[A])'}(y)=1-\mu_{f[A]}(y)=1-\sup_{x\in f^{-1}[y]}\mu_A(x),$$

故

$$\mu_{f[A']}(y)\geqslant\mu_{(f[A])'}(y),$$

从而获证. ▎

定义 4. 若 X 的不分明集的程度函数仅在 $x\in X$ 处取异于 0 的值 $(=\lambda)$，则称该不分明集为 X 的不分明点，记作 x_λ，而 x 叫做它的支点.

同样，当 $\lambda=1$ 时我们就不再区别 x_λ 和 X 的点 x.

为了更好地刻划两个不分明集的相交程度，除推广通常的“相交”概念外，再引用一种“相重”概念.

显然，在下述意义下不分明集 A 与 A' 可以相交，但并不相重.

定义 5. 设 A 和 B 为 X 的不分明集，若

$$\min[\mu_A(x),\ \mu_B(x)]>0$$

则称 A 和 B 相交于点 x；A 与 B 叫做相交当且仅当存在 $x\in X$ 使得 A 和 B 相交于 x. 又若 $\mu_A(x)+\mu_B(x)>1$，则称 A 和 B 相重于点 x；A 与 B 叫做相重当且仅当存在 $x\in X$ 使得 A 和 B 相重于 x.

另外，当 $0<\lambda\leqslant\mu_A(x)$ 时，就称 x_λ 属于 A，记为 $x_\lambda\in A$；而当 $\lambda+\mu_A(x)>1$ 时，则称 x_λ 重于 A.

易见当 A 与 B 相重于点 x 时它们也必相交于 x.

注．“相重”是蒲保明、刘应明[1]所引入的一个十分重要的概念，本附录所有有关这方面的结果均属于他们的[1].

§1. 不分明拓扑空间

定义 1. 若 X 的不分明集族 $\mathscr{T}=\{A\}$ 满足：

1° $X, \Phi \in \mathscr{T}$,

2° 若 $A, B \in \mathscr{T}$，则 $A \cap B \in \mathscr{T}$,

3° 若 $A_i \in \mathscr{T}(i \in I)$，则 $\bigcup_{i \in I} A_i \in \mathscr{T}$,

则称它为 X 的一个不分明拓扑，又称偶对 $(X, \mathscr{T})$ 为不分明拓扑空间，而 $\mathscr{T}$ 中的元 A 叫做 $\mathscr{T}$-开集，其余集 A' 叫做 $\mathscr{T}$-闭集.

例 1. 如同一般拓扑学一样，X 的平庸不分明拓扑仅含 X 与 Φ，而离散不分明拓扑则由 X 的所有不分明集组成.

例 2. 由拓扑空间 $(X, \mathscr{U})$ 诱出的不分明拓扑空间 $(X, F(\mathscr{U}))$（或称半连续不分明拓扑空间，见 C. K. Wong[1]，对此 M. D. Weiss[1] 还曾作过细致的讨论）.

于 X 的不分明集 A 和 $\alpha \in [0, 1)$，命

$$\Gamma_{A,\alpha} = \{x \mid \mu_A(x) > \alpha,\ x \in X\},$$

再命

$$F(\mathscr{U}) = \{A \mid A \text{ 为 } X \text{ 的不分明集并且对任何 } \alpha \in [0, 1),\ \Gamma_{A,\alpha} \text{ 为 } \mathscr{U}\text{-开集}\},$$

则易证 $F(\mathscr{U})$ 为 X 的不分明拓扑（自然我们把它叫做半连续不分明拓扑）.

事实上，我们有

1° $X, \Phi \in F(\mathscr{U})$ 是明显的；

2° 于 $A, B \in F(\mathscr{U})$，则 $C = A \cap B \in F(\mathscr{U})$，这只须注意从 $\mu_C(x) = \min[\mu_A(x), \mu_B(x)]$ 可推出对任何 $\alpha \in [0, 1)$ 有

$$\Gamma_{C,\alpha} = \Gamma_{A,\alpha} \cap \Gamma_{B,\alpha};$$

3° 于 $A_i \in F(\mathscr{U})$，$i \in I$，则因从 $\sup_{i \in I} \mu_{A_i}(x) > \alpha$ 可推出至少有一个 A_i 满足 $\mu_{A_i}(x) > \alpha$，故对任何 $\alpha \in [0, 1)$ 有

$$\Gamma_{\bigcup_{i \in I} A_i, \alpha} = \bigcup_{i \in I} \Gamma_{A_i, \alpha}.$$

注. 不分明拓扑的上述定义属于 C. L. Chang[1]. 而 R. Lowen[3] 又提出了另一种不同的定义,他把定义 1 中的 1° 改为更强的

1* 对任何 $\alpha\in[0, 1]$ 有 $C_\alpha\in\mathscr{T}$, 其中 C_α 相应的程度函数为

$$\mu_{C_\alpha}(x)=\alpha,\quad \text{当}\ x\in X\ \text{时}.$$

这种不分明拓扑空间叫做满层空间, 它虽不以分明拓扑空间为特款,却有许多应用,见蒲保明、刘应明[2].

吴从炘[4] 为了考虑不分明拓扑线性空间的需要,又把 1* 减弱为:

1** 若存在 $\alpha_0\in(0, 1)$, $A^{(0)}\in\mathscr{T}$, $x_0\in X$ 使得 $A^{(0)}(x_0)=\alpha_0$, 则对任何 $\alpha\in[0, 1]$ 有 $C_\alpha\in\mathscr{T}$; 否则就只有 $C_0, C_1\in\mathscr{T}$.

这样就又包括分明拓扑空间作为特款.

定义 2. 我们称不分明集 U 为 x_λ 的邻域 (重域) 当且仅当有 $V\in\mathscr{T}$ 使得 $V\subset U$ 并且 $x_\lambda\in V$ (x_λ 重于 V).

定理 1. 设 $(X, \mathscr{T})$ 为不分明拓扑空间,若以 $\mathscr{U}_e$ 表示不分明点 e 的所有邻域构成的邻域系(相应地重域系),则

(1) 若 $U\in\mathscr{U}_e$, 则 e 属于(相应地重于) U,

(2) 若 $U, V\in\mathscr{U}_e$, 则 $U\cap V\in\mathscr{U}_e$,

(3) 若 $U\in\mathscr{U}_e$, 不分明集 $V\supset U$, 则 $V\in\mathscr{U}_e$,

(4) 若 $U\in\mathscr{U}_e$, 则有 $V\in\mathscr{U}_e$ 使得 $V\subset U$ 并且对每一个属于(相应地重于) V 的不分明点 d 有 $V\in\mathscr{U}_d$.

反之, 若对 X 的每一个不分明点 e 都对应有一个满足 (1)—(3)的不分明集族 $\mathscr{U}_e$, 则所有使对任何重于不分明集 U 的不分明点 d 有 $U\in\mathscr{U}_d$ 的 U 构成 X 的一个不分明拓扑 $\mathscr{T}$; 当 $\mathscr{U}_e$ 还满足 (4) 时, $\mathscr{U}_e$ 就是 e 关于不分明拓扑 $\mathscr{T}$ 的重域系. (注意: 对邻域系情形尚需增添条件.)

证明. 参看本书问题 1.2 即可. |

注. 我们也可以在不分明拓扑空间 $(X, \mathscr{T})$ 中引入不分明集 A 的邻域:

不分明集 U 叫做 A 的邻域是指存在 $V\in\mathscr{T}$ 满足 $A\subset V\subset U$。至于它的有关性质，见 C. L. Chang[1] 等人的工作。

定义 3. 我们称不分明拓扑空间 $(X,\mathscr{T})$ 中所有包含不分明集 A 的 $\mathscr{T}$-闭集的交为 A 的 $\mathscr{T}$-闭包，记作 $\overline{A}$.

显然，$\overline{A}$ 是包含 A 的最小 $\mathscr{T}$-闭集.

定理 2. 不分明点 $x_\lambda\in\overline{A}$ 当且仅当 x_λ 的每一个重域均与 A 相重。

证明. $x_\lambda\in\overline{A}$

$\Longleftrightarrow$ 对任何 $\mathscr{T}$-闭集 $B\supset A$ 恒有 $x_\lambda\in B$，即 $\mu_B(x)\geqslant\lambda$

$\Longleftrightarrow$ 对任何 $\mathscr{T}$-开集 $C\subset A'$ 恒有 $\mu_C(x)\leqslant 1-\lambda$

$\Longleftrightarrow$ 对任何满足 $\mu_C(x)>1-\lambda$ 的 $\mathscr{T}$-开集 C，C 不含于 A'，即 C 与 $(A')'=A$ 相重

$\Longleftrightarrow$ 对 x_λ 的任何 $\mathscr{T}$-开的重域 C，它与 A 总相重

$\Longleftrightarrow$ x_λ 的每一个重域均与 A 相重.|

定义 4. ^{c} 叫做 X 上的一个不分明闭包算子当且仅当 c:

$$\begin{array}{ccc}[0,1]^X & \to & [0,1]^X\\ \downarrow & & \downarrow\\ A & \to & A^c\end{array}$$

且满足 Kuratowski 的四条公理:

1° $\Phi^c=\Phi$,

2° $A\subset A^c$,

3° $(A^c)^c=A^c$,

4° $(A\cup B)^c=A^c\cup B^c$.

定理 3. 若 $(X,\mathscr{T})$ 为不分明拓扑空间，则映射$^-$:

$$\begin{array}{ccc}[0,1]^X & \to & [0,1]^X\\ \downarrow & & \downarrow\\ A & \to & \overline{A}\end{array}$$

为 X 的一个不分明闭包算子.

反之，若 ^{c} 为 X 的一个不分明闭包算子，则所有满足 $A^c=A$ 的不分明集 A 的余集的全体构成 X 的一个不分明拓扑 $\mathscr{T}$，并且 A^c 就是 A 关于不分明拓扑 $\mathscr{T}$ 的 $\mathscr{T}$-闭包 $\overline{A}$.

证明．如同本书第一章定理 8 之证，只须注意所要用到的一个简单事实：

当 $A\subset B$ 时有 $A^c\subset B^c$，

需改证为：

由 $A\subset B$ 有 $B=A\cup B$，从而 $B^c=A^c\cup B^c\supset A^c$．▌

定义 5. 我们称不分明拓扑空间 $(X,\mathscr{T})$ 中所有包含于不分明集 A 的 $\mathscr{T}$-开集的并为 A 的 $\mathscr{T}$-内部，记作 A°．

显然，A° 是包含于 A 的最大 $\mathscr{T}$-开集．

定义 6. 在不分明拓扑空间 $(X,\mathscr{T})$ 中，规定不分明集 A 的 F-边界 $b(A)=\overline{A}\cap(\overline{A'})$

显然，$\overline{A}\supset A\cup b(A)$，但等号一般并不成立，这是与一般拓扑学的一个不同之处．

例．对 X，令其不分明拓扑为

$$\mathscr{T}=\{X,\Phi,x_{\frac{1}{2}}\},$$

取 $A=x_{\frac{2}{3}}$，$e=x_{\frac{1}{2}}$，则 e 的 $\mathscr{T}$-开的重域为 X 或 $x_{\frac{1}{2}}$，皆重于 A，故由定理 2 便知 $e\in\overline{A}$，而另一方面 $e\bar{\in}A$，又从 e 的重域 $x_{\frac{1}{2}}$ 与 A' 不相重可推出有 $e\bar{\in}(\overline{A'})$，即 $e\bar{\in}b(A)$，总之 $e\bar{\in}A\cup b(A)$．

注．R. H. Warren[2,3] 给出不分明集的 F-边界的另一种不同定义，他定义不分明集 A 的 F-边界为：

$\partial A=\Phi$，当 $\overline{A}\cap(\overline{A'})=\Phi$ 时，

$\partial A=\cap\{B\mid B$ 为 $\mathscr{T}$-闭集并且当 $x\in\overline{A}\cap(\overline{A'})$ 时有

$\mu_B(x)=\mu_A(x)\}$，在别处．

易见 $\partial A\supset b(A)$，另外有 $\overline{A}=A\cup\partial A$（见 R. H. Warren[2] 命题 4.3)．

定义 7. 设 $(X,\mathscr{T})$ 为不分明拓扑空间，若 X 的不分明点 e 的邻域系（相应地重域系）$\mathscr{U}_e$ 的子族 $\mathscr{B}$ 具有性质：对任何 $A\in\mathscr{U}_e$ 有 $B\in\mathscr{B}$ 使得 $B\subset A$，则称 $\mathscr{B}$ 为 e 的一个邻域基（相应地重域基）．

当 X 的每一个不分明点都有可数的邻域基（相应地重域基）时，就称 X 满足第一可数公理（相应地 Q-第一可数公理），或称为

C_1 空间(相应地 $Q\text{-}C_1$ 空间)。

定义 8. 设 $(X, \mathscr{T})$ 为不分明拓扑空间，$\mathscr{T}$ 的子族 $\mathscr{B}$ 叫做 $\mathscr{T}$ 的一个基当且仅当 $\mathscr{T}$ 中的每一个元均可表为若干个 $\mathscr{B}$ 中的元之并；又 $\mathscr{T}$ 的子族 $\mathscr{S}$ 叫做 $\mathscr{T}$ 的一个子基当且仅当 $\mathscr{S}$ 中的元之有限交的全体构成 $\mathscr{T}$ 的一个基.

当 $\mathscr{T}$ 具有可数基时，就称 X 满足第二可数公理，或称为 $C_{\rm II}$ 空间。

定理 4. 若 $(X, \mathscr{T})$ 为 C_1 空间，则必为 $Q\text{-}C_1$ 空间.

证明. 对 X 的任何不分明点 $e = x_\lambda$，取

$$\mu_n \to 1 - \lambda \quad (\mu_n \in (1 - \lambda, 1], n = 1, 2, \cdots),$$

又记 $e_n = x_{\mu_n}$，设 $\mathscr{B}_n$ 为 e_n 的可数 $\mathscr{T}$-开的邻域基，则因从 $B \in \mathscr{B}_n$ 可推出 $\mu_B(x) \geqslant \mu_n > 1 - \lambda$，即 B 亦为 e 的重域，故所有 $\{\mathscr{B}_n\}$ $(n = 1, 2, \cdots)$ 中的元的全体构成 e 的一个可数 $\mathscr{T}$-开的重域族 $\mathscr{B}$.

今证 $\mathscr{B}$ 为 e 的重域基.

事实上，对于 e 的 $\mathscr{T}$-开的重域 A，则有 $\mu_A(x) > 1 - \lambda$，从而有 μ_m 使

$$\mu_A(x) \geqslant \mu_m > 1 - \lambda,$$

即 $e_m \in A$，亦即 A 为 e_m 的 $\mathscr{T}$-开的邻域，于是有 $\mathscr{B}_m$(也就是 $\mathscr{B}$)中的元 $B \subset A$ 使得

$$\mu_B(x) \geqslant \mu_m > 1 - \lambda,$$

这表明 B 为 e 的 $\mathscr{T}$-开的重域. |

定理 5. 若 $(X, \mathscr{T})$ 为 $C_{\rm II}$ 空间，则必为 $Q\text{-}C_1$ 空间.

证明. 设 $\mathscr{B}$ 为 $\mathscr{T}$ 的可数基，于 X 的不分明点 e，记 $\mathscr{B}$ 中所有与 e 相重的元的族为 $\tilde{\mathscr{B}}$，则 $\tilde{\mathscr{B}}$ 就是 e 的一个可数 $\mathscr{T}$-开的重域基.

事实上，于 e 的 $\mathscr{T}$-开的重域 A，则因 A 可表为若干个 $\mathscr{B}$ 中的元之并，故易见 e 必和其中的某一个 $B(\subset A)$ 相重. |

注. 蒲保明、刘应明[2] 利用半连续不分明拓扑空间作出了是 $Q\text{-}C_1$ 空间而不是 C_1 以及是 $C_{\rm II}$ 空间而不是 C_1 的例子.

在不分明拓扑学中，我们也可以如同一般拓扑学一样引入分离性，隔离性以及连通性等概念.

定义 9. 给定不分明拓扑空间 $(X, \mathscr{T})$，它称为 T_0 空间，假如对任何两个不分明点 e 和 d，$e \neq d$，恒有 $e \bar{\in} \bar{d}$ 或 $d \bar{\in} \bar{e}$；又当每一个不分明点均为 $\mathscr{T}$-闭集时，就叫做 T_1 空间；如果对任何两个支点不同的不分明点 e 和 d 都存在各自的重域 B 和 C 使得 $B \cap C = \Phi$，那末就称之为 T_2 空间.

容易看出若不分明拓扑空间 $(X, \mathscr{T})$ 为 T_1 空间，则它亦为 T_0 空间；但从 $(X, \mathscr{T})$ 为 T_2 空间却推不出它为 T_0 空间.

例. 设 $X = \{y, z\}$，其中 $y \neq z$，又设它的不分明拓扑 $\mathscr{T}$ 的一个基由所有 y_λ，$\lambda \in (2/3, 1]$，z_λ，$\lambda \in (0, 1]$ 和 Φ 构成，则 $(X, \mathscr{T})$ 显然为 T_2 空间.

另一方面，取不分明点 $y_{\frac{1}{2}}$ 和 $y_{\frac{2}{3}}$，则易见

$$\bar{y}_{\frac{1}{2}} = \bar{y}_{\frac{2}{3}} = X,$$

故 $(X, \mathscr{T})$ 非 T_0 空间.

定理 6. 若不分明拓扑空间 $(X, \mathscr{T})$ 为 T_0 与 T_2 空间，则它也是 T_1 空间.

证明. 如若不然，即有不分明点 y_λ 使

$$\bar{y}_\lambda \supset y_\lambda, \quad 而 \quad \bar{y}_\lambda \neq y_\lambda,$$

则只有两种可能：

(1) 存在不分明点 $x_\mu \in \bar{y}_\lambda$ 使得 $x \neq y$，

(2) 存在不分明点 $y_\nu \in \bar{y}_\lambda$ 使得 $\nu > \lambda$，

但由此均不难推出矛盾.

事实上，在(1)的情况下，由于 $(X, \mathscr{T})$ 为 T_2 空间，所以有 x_μ 的重域 B 和 y_λ 的重域 C 使 $B \cap C = \Phi$，从而 $\lambda + \mu_C(y) > 1$，即 $\mu_C(y) > 1 - \lambda \geqslant 0$，亦即 $\mu_B(y) = 0$，这表明 x_μ 的重域 B 与 y_λ 不相重，于是由定理 2 便知 $x_\mu \bar{\in} \bar{y}_\lambda$，矛盾.

又在(2)的情况下，由于 $(X, \mathscr{T})$ 为 T_0 空间，而 $y_\nu \in \bar{y}_\lambda$，所以有 $y_\lambda \bar{\in} \bar{y}_\nu$，但另一方面有

$$\mu_{y_\lambda}(y) = \lambda < \nu = \mu_{y_\nu}(y) \leqslant \mu_{\bar{y}_\nu}(y),$$

即 $y_1 \in \bar{y}_p$，矛盾.▌

注. B. Hutton[2] 还引进了不分明拓扑空间的正规性.

定义 10. 我们称不分明拓扑空间 $(X, \mathscr{T})$ 的不分明集 A_1 和 A_2 为分离的当且仅当

$$\overline{A}_1 \cap A_2 = A_1 \cap \overline{A}_2 = \Phi.$$

定义 11. 不分明拓扑空间 $(X, \mathscr{T})$ 叫做连通的当且仅当它不可以表成两个非分离的不分明集 A 和 B 的并，其中 $A \neq \Phi$，$B \neq \Phi$.

注. 蒲保明、刘应明[3] 对分离性和连通性也得到一系列结果. 他们还指出本书第一章仅定理 17 才完全不能移植于不分明拓扑学，其反例如下：

设 $X = X_1 \cup X_2$（X_1 与 X_2 为互不相交的非空集），又设它的不分明拓扑 $\mathscr{T}$ 由所有 $[\lambda_1; \lambda_2]$，$0 \leqslant \lambda_1, \lambda_2 \leqslant 1/2$ 和 X 构成，其中 $[\lambda_1; \lambda_2]$ 表示由程度函数

$$\mu_A(x) = \begin{cases} \lambda_1, & \text{当 } x \in X_1 \text{ 时,} \\ \lambda_2, & \text{当 } x \in X_2 \text{ 时} \end{cases}$$

所确定的不分明集 A，令

$$Y = \left[\frac{2}{3}; 1\right], \quad Z = \left[1; \frac{2}{3}\right],$$

则易见 Y 和 Z 均为 $\mathscr{T}$-闭集，但 $Y \sim Z = [0; 1]$ 和 $Z \sim Y = [1; 0]$ 却不是分离的（因为 $\overline{(Y \sim Z)} = [1/2; 1]$，所以 $\overline{(Y \sim Z)} \cap (Z \sim Y) \neq \Phi$），此处我们以 $A \sim B$ 表示由程度函数

$$\mu_{A \sim B}(x) = \begin{cases} \mu_A(x), & \text{当 } \mu_A(x) > \mu_B(x) \text{ 时,} \\ 0, & \text{在别处} \end{cases}$$

所确定的不分明集.

§2. 紧不分明拓扑空间

定义 1. 我们称不分明集族 $\{A_i | i \in I\}$ 为不分明集 B 的覆盖当且仅当 $B \subset \bigcup_{i \in I} A_i$；若每一个 A_i 均为 $\mathscr{T}$-开集，则称之为 $\mathscr{T}$-开

覆盖；当其某个子族仍为覆盖时就称之为子覆盖.

定义 2. 不分明拓扑空间$(X,\mathscr{T})$叫做紧当且仅当X的每一个$\mathscr{T}$-开覆盖有有限子覆盖.

定义 3. 不分明拓扑空间$(X,\mathscr{T})$叫做可数紧当且仅当X的每一个可数$\mathscr{T}$-开覆盖有有限子覆盖.

定义 4. 我们称不分明集族$\{A_i|i\in I\}$具有有限交性质当且仅当它的每一个有限子族的交为非空.

定理 1. 不分明拓扑空间$(X,\mathscr{T})$为紧(可数紧)当且仅当每一个具有有限交性质的$\mathscr{T}$-闭集族(可数族)均有非空的交.

证明. 因为由 De Morgan 公式可知不分明集族$\{A_i|i\in I\}$为X的覆盖当且仅当$\bigcap\limits_{i\in I}A_i'=\Phi$，故从$X$为紧(可数紧)当且仅当$X$的任何没有有限子族覆盖$X$的$\mathscr{T}$-开集族(可数族)均非$X$的覆盖即可推出$X$为紧(可数紧)当且仅当$X$的任何具有有限交性质的$\mathscr{T}$-闭集族(可数族)均有非空的交.▌

定义 5. 设不分明集族$\{A_i|i\in I\}$为X的覆盖，如果对任何$x\in X$存在A_i使得$\mu_{A_i}(x)=1$，那末$\{\Gamma_{i,0}|i\in I\}$就是X的一个分划，叫做该覆盖有X的一个0-分划，其中

$$\Gamma_{i,0}=\{x|x\in X,\ \mu_{A_i}(x)=1\}.$$

定理 2. 不分明拓扑空间$(X,\mathscr{T})$为紧(可数紧)当且仅当X的每一个$\mathscr{T}$-开覆盖(可数$\mathscr{T}$-开覆盖)有X的一个有限0-分划.

证明. 必要性.

设$\{A_i|i\in I\}$为X的$\mathscr{T}$-开覆盖(可数$\mathscr{T}$-开覆盖)，则由假设可知它有有限子覆盖$\{A_{i_k}|k=1,2,\cdots,n\}$，即对一切$x\in X$有

$$\max_{1\leqslant k\leqslant n}\mu_{A_{i_k}}(x)=1,$$

因此$\{A_{i_k}|k=1,2,\cdots,n\}$有$X$的一个有限0-分划，也就是$\{A_i|i\in I\}$有$X$的一个有限0-分划.

充分性.

只须注意若X的$\mathscr{T}$-开覆盖(可数$\mathscr{T}$-开覆盖)$\{A_i|i\in I\}$有X的有限0-分划$\{\Gamma_{k,0}|k=1,2,\cdots,n\}$，则$\{A_{i_k}|k=1,2,\cdots,$

$n\}$ 为 $\{A_i|i\in I\}$ 的子覆盖，其中 A_{i_k} 为相应于 $\Gamma_{k,0}$ 的 $\{A_i|i\in I\}$ 中的元. ∎

例. 考察由拓扑空间 $(X, \mathscr{U})$ 诱出的不分明拓扑空间 $(X, F(\mathscr{U}))$. 特别取 $(X, \mathscr{U})$ 为带有通常拓扑的所有实数的全体，即 R^1.

将 R^1 表为可数多个互不相交的半开区间 $V_1, V_2, \cdots, V_n, \cdots$, 之并，再命 A_n 是由满足

$$\mu_{A_n}(x)=\begin{cases}1, & \text{当 } x\in \overline{V}_n \text{ 时},\\ <1, & \text{在别处}\end{cases}$$

的连续程度函数所确定的不分明集，则不难看出 $\{A_n|n=1, 2,\cdots\}$ 为 $(X, F(\mathscr{U}))$ 的一个 $F(\mathscr{U})$-开覆盖，但它没有有限子覆盖，故 $(X, F(\mathscr{U}))$ 非可数紧，从而自然更非紧.

注. C. L. Chang[1] 所引进的紧性定义还有一些缺点，例如，对于它，一般的 Tychonoff 乘积定理并不成立（参看本附录 §4），因此，在不分明拓扑学中，紧性概念尚未取得公认的最好形式，围绕着紧性应适合一般的 Tychonoff 乘积定理这一问题为中心，最近又出现了不分明紧性的一些新定义（见 R. Lowen[3], [4], [6], T. E. Gantner and R. C. Steinlage[1] 和刘应明[1]），这一概念似乎已逐渐接近于完善的境地.

另外，C. K. Wong[3] 引进了不分明拓扑空间 $(X, \mathscr{T})$ 在不分明点 e 处的局部紧性：即

有紧集 $A\in\mathscr{T}$ 使得 $e\in A$.

F. T. Christoph[1] 又把它减弱为：

有 $B\in\mathscr{T}$ 和紧集 A 使得 $e\in B\subset A$.

例. 设 $X=(0, 1)$，其不分明拓扑 $\mathscr{T}$ 就取为通常拓扑，于不分明点 $e=x_\lambda$，则显然有

$$B=(x-\varepsilon, x+\varepsilon)\in\mathscr{T}, e\in B,$$

其中 $x-\varepsilon>\delta>0$, $x+\varepsilon<1-\delta$，再注意到 $\overline{B}=[x-\varepsilon, x+\varepsilon]$ 为紧便知按 Christoph 意义在 e 处为局部紧.

另一方面，从任何 $A\in\mathscr{T}$ 皆非紧集又可推出在 Wong 意义下

于 e 处非局部紧.

§3. 不分明连续函数

定义 1. 从不分明拓扑空间 $(X,\mathscr{T})$ 到不分明拓扑空间 $(Y,\mathscr{U})$ 的函数 f 叫做 F-连续当且仅当 $f^{-1}[B]\in\mathscr{T}$，当 $B\in\mathscr{U}$ 时.

显然，若再设 g 是从 $(Y,\mathscr{U})$ 到不分明拓扑空间 $(Z,\mathscr{V})$ 的 F-连续函数，则其合成 $g\circ f$ 从 X 到 Z F-连续，这只须注意当 $V\in\mathscr{V}$ 时有

$$(g\circ f)^{-1}[V]=f^{-1}[g^{-1}[V]]\in\mathscr{T}.$$

定理 1. 设 f 为从紧（可数紧）不分明拓扑空间 $(X,\mathscr{T})$ 到不分明拓扑空间 $(Y,\mathscr{U})$ 上的 F-连续函数，则 $(Y,\mathscr{U})$ 也是紧（可数紧）的.

证明. 设 $\{B_i|i\in I\}$ 为 Y 的 $\mathscr{U}$-开覆盖（可数 $\mathscr{U}$-开覆盖），则因对 $x\in X$ 恒有

$$\mu_{\bigcup\limits_{i\in I}f^{-1}[B_i]}(x)=\sup_{i\in I}\mu_{f^{-1}[B_i]}(x)=\sup_{i\in I}\mu_{B_i}(f(x))=1,$$

故 $\{f^{-1}[B_i]|i\in I\}$ 为 X 的 $\mathscr{T}$-开覆盖（可数 $\mathscr{T}$-开覆盖），从而有有限子覆盖 $\{f^{-1}[B_{i_k}]|k=1,2,\cdots,n\}$，注意到 f 是从 X 到 Y 上，易知此时对 Y 的不分明集 B 恒有 $f[f^{-1}[B]]=B$，于是 $\{B_{i_k}|k=1,2,\cdots\}$ 也就是 Y 的覆盖，即 $(Y,\mathscr{U})$ 亦为紧（可数紧）❙

如同定理 1 之证，我们还可以得到

定理 2. 设 f 为从 Lindelöf 不分明拓扑空间 $(X,\mathscr{T})$ 到不分明拓扑空间 $(Y,\mathscr{U})$ 上的 F-连续函数，则 $(Y,\mathscr{U})$ 也是 Lindelöf 的.

所谓 $(X,\mathscr{T})$ 是 Lindelöf 的，是指它的每一个 $\mathscr{T}$-开覆盖均有可数的子覆盖.

定义 2. 我们称从不分明拓扑空间 $(X,\mathscr{T})$ 到不分明拓扑空间 $(Y,\mathscr{U})$ 的一对一 F-连续函数为 F-同胚当且仅当它的逆

也 F-连续，这时又称 X 和 Y 为 F-同胚，或 F-拓扑等价.

注. 本书第三章定理 1（连续映射的刻划定理）是一条很基本而有用的定理，这个定理已被推广到不分明拓扑空间（见蒲保明、刘应明[2]，蒋继光[1]也再次得到该定理）.

§4. 乘积与商不分明拓扑空间

定义 1. 设 $(X_i, \mathscr{T}_i)$, $i \in I$ 为一族不分明拓扑空间，$X = \prod_{i \in I} X_i$ 为通常的笛卡儿乘积，P_i 为从 X 到 X_i 上的射影，命

$$\mathscr{S} = \{P_i^{-1}[B] \mid B \in \mathscr{T}_i,\ i \in I\},$$

则所有 $\mathscr{S}$ 中的元的有限交的并的全体构成 X 的一个不分明拓扑 $\mathscr{T}$（它以 $\mathscr{S}$ 为子基），我们称之为 X 的乘积不分明拓扑，并且把 $(X, \mathscr{T})$ 叫做乘积不分明拓扑空间.

定理 1. 设 $(X_i, \mathscr{T}_i)$, $i \in I$ 为一族不分明拓扑空间，则有

(1) 乘积不分明拓扑 $\mathscr{T}$ 为使得 P_i 为 F-连续 $(i \in I)$ 的 $X = \prod_{i \in I} X_i$ 的最小不分明拓扑；

(2) 设 $(Y, \mathscr{U})$ 亦为不分明拓扑空间，f 为从 Y 到 X 的函数，则 f 为 F-连续当且仅当对每一个 $i \in I$，$P_i \circ f$ 为 F-连续.

证明. (1) 显然.

(2) 只须证充分性. 设 $B \in \mathscr{T}_i$，则

$$(f^{-1} \circ P_i^{-1})[B] = (P_i \circ f)^{-1}[B] \in \mathscr{U},$$

注意 $\mathscr{T}$ 中的元均为 $\mathscr{S} = \{P_i^{-1}[B] \mid B \in \mathscr{T}_i,\ i \in I\}$ 的元的有限交的并，而 f^{-1} 又可保持不分明集的交与并的运算，因而 f^{-1} 映 $\mathscr{T}$ 的元为 $\mathscr{U}$ 的元，即 f 为 F-连续.∎

定理 2. 设 $(X_k, \mathscr{T}_k)$, $k = 1, 2, \cdots, n$ 为有限多个紧不分明拓扑空间，则乘积不分明拓扑空间 $(X, \mathscr{T})$ 也是紧的.

证明. 显然只须证 $n = 2$ 的情形.

此时 X 的乘积不分明拓扑 $\mathscr{T}$ 的一个基为：

$$\mathscr{T}_0 = \{A_1 \times A_2 \mid A_1 \in \mathscr{T}_1,\ A_2 \in \mathscr{T}_2\},$$

其中 $A_1\times A_2$ 相应的程度函数为

$$\mu_{A_1\times A_2}(x)=\min\{\mu_{A_1}(x_1),\ \mu_{A_2}(x_2)\}.$$

设 $\{B_i|i\in I\}$ 为 $(X,\mathscr{T})$ 的 $\mathscr{T}$-开覆盖，其中

$$B_i=A_1^{(i)}\times A_2^{(i)}(A_1^{(i)}\in\mathscr{T}_1,A_2^{(i)}\in\mathscr{T}_2,\ i\in I),$$

又对任何 $y\in X_2$，$\delta>0$，命

$$S_y=X_1\times\{y\},$$

$$V_{y,\delta}=\{B_i|\mu_{A_2^{(i)}}(y)>1-\delta \text{ 并且至少有一个 } x\in X_1 \text{ 使得 } \mu_{A_1^{(i)}}(x)>1-\delta,\ i\in I\},$$

则 $V_{y,\delta}$ 为 X 的子集 S_y 的 $\mathscr{T}$-开覆盖.

事实上，若 $(x,y)\in S_y$，则因有 $\{B_i|i\in I\}$ 的可数子族 $\{B_{i_k}|k=1,2,\cdots\}$ 使得

$$\lim_{k\to\infty}\mu_{A_1^{(i_k)}\times A_2^{(i_k)}}(x,y)=1,$$

故当 k 充分大时 $A_1^{(i_k)}\times A_2^{(i_k)}\in V_{y,\delta}$，不妨设这对一切 k 均成立，于是对一切 $(x,y)\in S_y$ 作同样的处理就得到 $V_{y,\delta}$ 的一个子族，它覆盖 S_y.

设

$$W_{y,\delta}=\{A_1^{(i)}|A_1^{(i)}\times A_2^{(i)}\in V_{y,\delta}\},$$

则因对任何 $x\in X_1$ 有 $V_{y,\delta}$ 的可数子族 $\{B_{i_k}|k=1,2,\cdots\}$ 使得

$$\lim_{k\to\infty}\mu_{A_1^{(i_k)}\times A_2^{(i_k)}}(x,y)=1,$$

即 $\lim\limits_{k\to\infty}\mu_{A_1^{(i_k)}}(x)=1$，故 $W_{y,\delta}$ 为 $(X_1,\mathscr{T}_1)$ 的 $\mathscr{T}_1$-开覆盖，从而由 $(X_1,\mathscr{T}_1)$ 为紧便知 $W_{y,\delta}$ 有有限子覆盖 $Z_{y,\delta}$. 今对每一个 $A_1^{(i)}\in Z_{y,\delta}$，选 $A_2^{(i)}$ 使 $A_1^{(i)}\times A_2^{(i)}\in V_{y,\delta}$，记如此得到的有限族 $\{A_1^{(i)}\times A_2^{(i)}\}$ 为 $H_{y,\delta}$，而把其中的 $A_2^{(i)}$ 的族记作 $G_{y,\delta}$，于是易知所有 $G_{y,\delta}$ 中的元 $(y\in X_2,\ \delta>0)$ 的全体就构成 $(X_2,\mathscr{T}_2)$ 的一个 $\mathscr{T}_2$-开覆盖，因而从 $(X_2,\mathscr{T}_2)$ 为紧，它又有有限子覆盖 G_{y_i,δ_i}，$i=1,2,\cdots,m$.

今证 H_{y_i,δ_i}，$i=1,2,\cdots,m$ 为 $\{B_i|i\in I\}$ 的有限子覆盖.

事实上，只须证它是 X 的一个覆盖. 于 $(x,y)\in X$，则有某个 G_{y_i,δ_i} 中的元 A_2 使得 $\mu_{A_2}(y)=1$，另一方面又存在 Z_{y_i,δ_i} 中的元

A_1 使得 $\mu_{A_1}(x) = 1$，因而 $A_1 \times A_2$ 为 H_{y_1,θ_1} 中的元并且 $\mu_{A_1\times A_2}(x, y) = 1$. ▎

注. 该定理对可数紧情形是不成立的，C. K. Wong[2] 对此的证明有错误(参看王国俊，两个可数紧的不分明拓扑空间的积空间不必是可数紧的(尚未发表)).

还值得注意的是上述定理对紧不分明拓扑空间的可数族也不再成立.

例. 对任何非空集 Y，命 $X_n = Y$，它的不分明拓扑 $\mathscr{T}_n = \{\Phi, Y, A_n\}(n = 1, 2, \cdots)$，其中 A_n 是由程度函数

$$\mu_{A_n}(y) = 1 - \frac{1}{n}(y \in Y)$$

所确定的不分明集，则易见所有 $(X_n, \mathscr{T}_n)$ 均为紧(可数紧)，但其乘积不分明拓扑空间 $(X, \mathscr{T})$ 就不再是紧的.

事实上，因为对一切 n，$P_n^{-1}[A_n] \in \mathscr{T}$($P_n$ 为从 X 到 X_n 上的射影)，又当 $x \in X$ 时恒有

$$\mu_{P_n^{-1}[A_n]}(x) = 1 - \frac{1}{n},$$

故 $\{P_n^{-1}[A_n] | n = 1, 2, \cdots\}$ 为 X 的 $\mathscr{T}$-开覆盖(可数 $\mathscr{T}$-开覆盖)，而它显然没有有限子覆盖.

注. 有关不分明拓扑空间相应的 Tychonoff 型定理在 J. A. Goguen[1], R. Lowen[4], T. E. Gantner and R. C. Steinlage[1] 和刘应明[1] 等人的工作中还有更进一步的研究.

定义 2. 设 $(X, \mathscr{T})$ 为不分明拓扑空间，R 为 X 上的等价关系，X/R 为通常的商集，P 为从 X 到 X/R 上的射影，命

$$\mathscr{U} = \{B | P^{-1}[B] \in \mathscr{T}\},$$

则 $\mathscr{U}$ 为 X/R 上的一个不分明拓扑，我们称之为 X/R 上的商不分明拓扑，并且把 $(X/R, \mathscr{U})$ 叫做商不分明拓扑空间.

定理 3. 设 $(X, \mathscr{T})$ 为不分明拓扑空间，R 为 X 上的等价关系，则有

(1) 商不分明拓扑 $\mathscr{U}$ 为使得 P 为 F-连续的 X/R 上的最大

不分明拓扑；

(2) 设 $(Y,\mathscr{V})$ 亦为不分明拓扑空间，g 为从 X/R 到 Y 的函数，则 g 为 F-连续当且仅当 $g\circ P$ 为 F-连续。

证明．(1) 显然．

(2) 只须证充分性．设 $V\in\mathscr{V}$，则

$$(g\circ P)^{-1}[V]=P^{-1}[g^{-1}[V]]\in\mathscr{T},$$

再由 $\mathscr{U}$ 的定义便知 $g^{-1}[V]\in\mathscr{U}$，故 g 为 F-连续．▎

定理 4. 若 $(X,\mathscr{T})$ 为紧(可数紧)不分明拓扑空间，R 为 X 上的等价关系，则商不分明拓扑空间 $(X/R,\mathscr{U})$ 亦为紧(可数紧)．

证明　由 §3 定理 1 和从 X 到 X/R 上的射影 P 为 F-连续即得．▎

注．F. T. Christoph[1] 将商不分明拓扑空间又作了推广．蒲保明、刘应明[2] 结合 C. L. Chang[1]，J. A. Goguen[1]，C. K. Wong[2,3] 等人的工作还将本书第三章的定理推广到了不分明拓扑空间．

§5. 不分明网的 Moore-Smith 收敛

蒲保明、刘应明[1] 首次引入了不分明网，并且将本书第二章关于 Moore-Smith 收敛的全部定理都推广到了不分明拓扑空间，现仅陈述其中的一部分．

定义 1. 设 D 为非空集，$\geqslant$ 是 D 上的一个半序，若对任何 m，$n\in D$ 有 $P\in D$ 使得 $P\geqslant m$，$P\geqslant n$，则称 $(D,\geqslant)$ 为由半序 $\geqslant$ 定向的定向集．

定义 2. 设 $(D,\geqslant)$ 为定向集，ϕ 为所有 X 的不分明点组成的集，我们称从 D 到 ϕ 的函数 S 为 X 的不分明网，对 $n\in D$，$S(n)$ 也常记作 S_n，于是网 S 也常表为 $\{S_n, n\in D\}$，D 叫做网 S 的定义域．

定义 3. 设 $\{S_n,\ n\in D\}$ 为 X 的不分明网，A 为 X 的不分明

集．若对所有 n，S_n 皆重于 A，则称网 S 重于 A；若有 $m\in D$ 使当 $n\geqslant m$ 时 S_n 皆重于 A，则称网 S 最终地重于 A；若对任何 $m\in D$ 都有 $n\in D$，$n\geqslant m$ 使得 S_n 重于 A，则称网 S 常常重于 A；若对一切 n 有 $S_n\in A$，则称网 S 在 A 中．

定义 4. 设 $(X,\mathscr{T})$ 为不分明拓扑空间，S 为 X 的不分明网，e 为 X 的不分明点，若对 e 的每一个重域 B，网 S 最终地重于 B，则称网 S 收敛于 e．

定理 1. 若 $(X,\mathscr{T})$ 为不分明拓扑空间，A 为 X 的不分明集，则 X 的不分明点 $e\in\overline{A}$ 当且仅当存在网 S 在 A 中并且收敛于 e．

证明．必要性．

根据 §1 定理 1，e 的重域系以包含关系为半序成为一个定向集，记作 D，于是由 §1 定理 2 可知对任何 $B\in D$，B 与 A 相重，设重于某个 $z\in X$，则 $\mu_B(z)+\mu_A(z)>1$，即 $\mu_A(z)-\lambda>0$，这表明不分明点 $z_\lambda\in A$ 并且重于 B，显然由 B 对应 z_λ 所给出的网就满足我们的要求．

充分性．

设不分明网 $S=\{S_n,n\in D\}$ 在 A 中并且收敛于不分明点 e，则对 e 的任何重域 B，从 S 最终地重于 B 可推出有某个 $S_n=z_\lambda\in A$ 并且重于 B，即 $\lambda\leqslant\mu_A(z)$ 并且 $\lambda+\mu_B(z)>1$，从而 $\mu_A(z)+\mu_B(z)>1$，亦即 B 与 A 相重，再根据 §1 定理 2 便知 $e\in\overline{A}$．▮

定义 5. X 的不分明网 $T=\{T_m,m\in E\}$ 叫做不分明网 $S=\{S_n,n\in D\}$ 的子网当且仅当存在从 E 到 D 的函数 N 满足

1° $T=S\circ N$，即对每一个 $m\in E$，$T_m=S_{N(m)}$；

2° 对任何 $n\in D$ 存在 $m\in E$ 使当 $P\in E$，$P\geqslant m$ 时有 $N(P)\geqslant n$．

定理 2. 设 $S=\{S_n,n\in D\}$ 为 X 的不分明网，$\mathscr{B}$ 为 X 的不分明集族，使得任何两个 $\mathscr{B}$ 中的元的交仍含有 $\mathscr{B}$ 中的一个元，又设 S 常常重于 $\mathscr{B}$ 中的每一个元，则 S 有子网 T 使得 T 最终地重于

$\mathscr{B}$ 中的每一个元.

证明. 设 D_1 为 $\mathscr{B}$ 由包含关系所定向的定向集，令

$$E=\{(m, A)\mid m\in D, A\in D_1 \text{ 并且 } S_m \text{ 重于 } A\},$$

则易见 E 是 $D\times D_1$ 的子集，而 $D\times D_1$ 的乘积半序（$(m_1, A_1)\geqslant(m_2, A_2)$ 表示在 D 和 D_1 中分别有 $m_1\geqslant m_2$ 和 $A_1\geqslant A_2$）在 E 上的限制又给出了 E 上的一个半序 $\geqslant$，今证 E 由该半序所定向.

事实上，于 (m, A), $(n, B)\in E$，取 $G\in\mathscr{B}$, $G\subset A\cap B$，则因 S 常常重于 G，故有 $P\in D$ 使得 $P\geqslant m$, $P\geqslant n$ 并且 S_P 重于 G，于是 $(P, G)\in E$ 并且有 $(P, G)\geqslant(m, A)$ 和 $(P, G)\geqslant(n, B)$.

作从 E 到 D 的函数 N:

$$N(m, A)=m,$$

显然它满足定义 5 的 2°，从而 $T=S\circ N$ 是 S 的一个子网，又对任何 $A\in\mathscr{B}$ 有 $m\in D$ 使得 S_m 重于 A，于是 $(m, A)\in E$ 并且当在 E 中有 $(n, B)\geqslant(m, A)$ 时 $T(n, B)=S\circ N(n, B)=S_n$ 重于 B，从而也重于 A，即 T 最终地重于 A. |

定义 6. 不分明拓扑空间 $(X, \mathscr{T})$ 的不分明点 e 叫做不分明网 S 的聚点当且仅当对 e 的每一个重域 B，S 常常重于 B.

定理 3. 不分明拓扑空间 $(X, \mathscr{T})$ 的不分明点 e 为不分明网 S 的聚点当且仅当有 S 的子网 T 收敛于 e.

证明. 充分性由定义 5，6 即得. 至于必要性则只须注意 e 的重域系 $\mathscr{B}$ 显然满足定理 2 的条件. |

当 X 的不分明网 $\{S_n, n\in D\}$ 的定义域 D 由所有正整数组成（其半序由正整数的大小关系 $\geqslant$ 给出）时，就称这种网为 X 的不分明序列，类似于子网的定义，也可以从序列出发来定义子序列.

定理 4. 设不分明拓扑空间 $(X, \mathscr{T})$ 为 C_1 或 Q-C_1，A 为 X 的不分明集，e 为 X 的不分明点，则有

(1) $e\in\overline{A}$ 当且仅当有 A 的不分明序列收敛于 e;

(2) e 为不分明序列 S 的聚点当且仅当 S 有子序列收敛于 e.

证明. 由 §1 定理 4 我们只须就 $(X, \mathscr{T})$ 为 Q-C_1 空间的情

形加以证明．此时可设 $\{B_k|k=1, 2,\cdots\}$ 为 e 的可数 $\mathscr{T}$-开的重域基，并且还可设 $B_k \supset B_{k+1}$，称之为单调重域基．今以它代替 e 的重域系．

于是，(1) 仿定理 1 之证即得．至于 (2)，充分性显然，其必要性可以比定理 3 更简单地予以证明：

设 e 为 $S=\{S_n|n=1, 2,\cdots\}$ 的聚点，对每一个 B_k 取 S_{n_k} 使 S_{n_k} 重于 B_k 并且 $n_k > n_{k-1}$（规定 $n_0=0$），于是 $T=\{S_{n_k}|k=1, 2,\cdots\}$ 就是收敛于 e 的 S 的子序列．▎

另外，我们还顺便指出 B. Hutton[3] 研究了不分明拓扑空间的一致性结构．

最后译者对关肇直和蒲保明两先生的支持与鼓励表示衷心的感谢！

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本附录写于三年前，现在这个领域又有很大发展[如可参看金长泽，不分明拓扑学的某些发展概况，东北师大学报（自然科学版），**2** (1981)，53—67]，不过作为基础知识，内容还是可以的，因此其它就不再赘述了。——译者注

索　　引

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二　画

三　画

四　画

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八 画

九 画

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十 二 画

十 三 画

十 四 画

十 六 画

十 八 画