Gauge Field Theories, Second Edition

STEFAN POKORSKI

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Quantum field theory forms the present theoretical framework for our understanding of the fundamental interactions of particle physics. This up-dated and expanded text examines gauge theories and their symmetries with an emphasis on their physical and technical aspects.

Beginning with a new chapter giving a systematic introduction to classical field theories and a short discussion of their canonical quantization and the discrete symmetries C, P and T, the book provides a brief exposition of perturbation theory, the renormalization programme and the use of the renormalization group equation. It then explores topics of current research interest including chiral symmetry and its breaking, anomalies, and low energy effective lagrangians and some basics of supersymmetry. A chapter on the basics of the electroweak theory is now included.

PROFESSOR POKORSKI, a distinguished theoretical physicist, has presented here a self-contained text for graduate courses in physics, the only prerequisite is some grounding in quantum field theory.

Born in 1942, Professor Stefan Pokorski received his PhD in theoretical physics in 1967 from Warsaw University, where he is now holder of the Chair in Theoretical Particle Physics. He has been a member of the Polish Academy of Science since 1991 and was Director of the Institute for Theoretical Physics, University of Warsaw, from 1984 to 1994 and President of the Polish Physical Society from 1992 to 1994. A visiting scientist at CERN and the Max-Planck Institute for Physics, Munich, for periods of time totalling almost ten years, Professor Pokorski has also made regular visits to universities and laboratories around the world. Professor Pokorski has had many scientific articles published in leading international journals and in recent years has concentrated on physics beyond the standard model and supersymmetry.

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Second Edition

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In memory of Osterns – my mother's family

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Preface to the First Edition

This book has its origin in a long series of lectures given at the Institute for Theoretical Physics, Warsaw University. It is addressed to graduate students and to young research workers in theoretical physics who have some knowledge of quantum field theory in its canonical formulation, for instance at the level of two volumes by Bjorken & Drell (1964, 1965). The book is intended to be a relatively concise reference to some of the field theoretical tools used in contemporary research in the theory of fundamental interactions. It is a technical book and not easy reading. Physical problems are discussed only as illustrations of certain theoretical ideas and of computational methods. No attempt has been made to review systematically the present status of the theory of fundamental interactions.

I am grateful to Wojciech Królikowski, Maurice Jacob and Peter Landshoff for their interest in this work and strong encouragement. My warm thanks go to Antonio Bassetto, Wilfried Buchmüller, Wojciech Królikowski, Heinrich Leutwyler, Peter Minkowski, Olivier Piguet, Jacek Prentki, Marco Roncadelli, Henri Ruegg and Wojtek Zakrzewski for reading various chapters of this book and for many useful comments, and especially to Peter Landshoff for reading most of the preliminary manuscript.

I am also grateful to several of my younger colleagues at the Institute for Theoretical Physics in Warsaw for their stimulating interactions. My thanks go to Andrzej Czechowski for his collaboration at the early stage of this project and for numerous useful discussions. I am grateful to Wojciech Dębski, Marek Olechowski, Jacek Pawełczyk, Andrzej Turski, Robert Budzynski, Krzysztof Meissner and Michał Spalinski, and particularly to Paweł Krawczyk for checking a large part of the calculations contained in this book.

Finally my thanks go to Zofia Ziółkowska for her contribution to the preparation of the manuscript.

Stefan Pokorski
Warsaw, 1985

Preface to the Second Edition

This new edition offers a substantial extension of topics covered by the book. The main additions are Chapter 1 and Chapter 12 and extended Appendices. Chapter 1 makes the book more self-contained. It gives a systematic introduction to classical field theories and a brief discussion of their canonical quantization, as some intuition based on canonical quantization proves to be very useful even if the main emphasis is on the path integral approach. Also in Chapter 1 the reader can find a thorough discussion of discrete symmetries C, P and T.

Chapter 12 gives a concise but systematic and self-contained introduction to the electroweak theory. This is an important completion for a modern book on quantum field theory and fundamental interactions, which was missing in the first edition. Appendices A, C and D are new. In particular, Appendix C contains the complete set of Feynman rules for the Standard Model, including counterterms, which is not easily available in the literature. The new Appendix A is a substantial extension of the previous Appendix C. Several smaller changes and corrections have been made in a number of places in the text. An important addition is Section 7.7, in which the modern approach to effective field theories is presented.

I hope that the additions leave intact the main feature of the book: it is still not easy reading!

I am grateful to many people for their help in the completion of this second edition. Very special thanks go to Piotr Chankowski. His help and collaboration in writing Chapters 1 and 12 and the Appendices was absolutely invaluable. Thanks, Piotr. I also thank Mikolaj Misiak and Janusz Rosiek for their collaboration on Sections 7.7 and 15.6, respectively.

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thank Peter Landshoff for his constant encouragement and patience during quite a few years before this edition finally materialized.

Comments and corrections are welcome by e-mail at pokorski@fuw.edu.pl They will be made accessible at the www page: http://www.fuw.edu.pl/∼pokorski/

Stefan Pokorski
Warsaw, 1999

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Introduction

0.1 Gauge invariance

It seems appropriate to begin this book by quoting the following experimental information (Review of Particle Properties 1996):

electron life-time	$> 4.3 \times 10^{23}$ years (68% C.L.)
neutron life-time for the electric charge non-conserving decays ($\mathrm{n} \rightarrow \mathrm{p}$ + neutrals)	$\gtrsim 10^{19}$ years
proton life-time	$> 10^{32}$ years
photon mass	$< 6 \times 10^{-22}$ MeV
neutrino (ν_e) mass	$< 15 \times 10^{-6}$ MeV

From the first three lines of this table we see that the conservation of the electric charge is proved experimentally much more poorly than the conservation of the baryonic charge. Nevertheless, nobody is seriously contesting electric charge conservation whereas experiments searching for proton decay belong to the present frontiers of physics. The reason for this lies in the general conviction that the theory of electromagnetism has gauge symmetry† whereas no gauge invariance principle can be invoked to protect baryonic charge conservation. Exact gauge invariance protects the conservation of the electric charge. It also implies the masslessness of the photon and as seen from the table the present experimental limit on the photon mass is indeed many orders of magnitude better than for the other 'massless' particle: the electron neutrino.

It should be stressed at this point that $U(1)$ gauge invariance implies global $U(1)$ invariance but, of course, the opposite is not true. A tiny mass for the photon would destroy the gauge invariance of electrodynamics but leave unaffected all its Earth-bound effects, including the quantum ones, as long as the electric charge was conserved. In particular, such a theory is also renormalizable (Matthews 1949,

† The reader who is not familiar with notions of global symmetry and gauge symmetry is advised to read Chapter 1 first.

Boulware 1970, Salam & Strathdee 1970) since longitudinally polarized photons decouple from the conserved current.

Thus one may ask what are the virtues of gauge invariance? We trust electric charge conservation, because we expect that gauge invariance is behind it, and we doubt baryon charge conservation, though this has been much better proved experimentally than electric charge conservation. We do not trust global symmetries as candidates for underlying first principles! A global symmetry gives us a freedom of convention: choice of a reference frame (the phase of the electron wave-function for the $U(1)$ symmetry of electrodynamics). It can be redefined freely, provided that all observers in the universe redefine it in exactly the same way. This sounds unphysical and we are led to propose that this freedom of convention is present independently at every space-time point or is not present at all as an exact law of nature. (Approximate global symmetries may, nevertheless, be and are very useful in describing the fundamental interactions.) This aesthetical argument may not convince everybody. Those who remain sceptical should then remember that gauge theories give an economical description of the laws of nature based on well-defined underlying principles which has been phenomenologically successful. As we know at present, this statement accounts not only for electrodynamics with its $U(1)$ abelian gauge symmetry, but also for weak and strong interactions successfully described by gauge field theories with non-abelian symmetry groups: $SU(2) \times U(1)$ and $SU(3)$, respectively. And non-abelian gauge symmetries are more restrictive and more profound than the $U(1)$ symmetry. In particular, non-abelian gauge bosons carry the group charges and their mass terms in the lagrangian would in general, unless introduced by spontaneous symmetry breaking, destroy not only the gauge symmetry but also the current conservation and therefore the renormalizability of the theory. The standard experimental evidence for gauge theories of weak and strong interactions is briefly summarized in the next section.

We end this section with a short historical 'footnote' (Pauli 1933). The terminology 'gauge invariance' can be traced back to Weyl's studies (Weyl 1919) of invariance under space-time-dependent changes of gauge (scale) in an attempt to unify gravity and electromagnetism. This attempt proved, however, unsuccessful. In 1926, Fock observed (Fock 1926) that one could base quantum electrodynamics (QED) of scalar particles on the operator

$$-\mathrm{i}\hbar\frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu$$

where A_μ is the electromagnetic four-potential and that the equations were invariant under the transformation

$$A_\mu \to A_\mu + \frac{\partial f(x)}{\partial x^\mu}, \qquad \Phi \to \Phi \exp[\mathrm{i}ef(x)/\hbar c]$$

which he called gradient transformation. London (1927) pointed out the similarity of Fock's to Weyl's earlier work: instead of Weyl's scale change a local phase change was considered by Fock. In 1929, Weyl studied invariance under this phase change but he kept unchanged his earlier terminology 'gauge invariance' (Weyl 1929). The concept of gauge transformations was generalized to non-abelian gauge groups by Yang & Mills (1954). Similar ideas were also proposed much earlier by Klein (1939) and by Shaw (1955).

0.2 Reasons for gauge theories of strong and electroweak interactions

We summarize very briefly the standard arguments in favour of gauge theories in elementary particle physics. Both quantum chromodynamics (QCD) and the Glashow–Salam–Weinberg theory are syntheses of our understanding of fundamental interactions progressing over many past years.

QCD

QCD emerged as a development of the Gell-Mann–Zweig quark model for hadrons (Gell-Mann 1964, Zweig 1964). The latter was postulated as a rationale for the successful $SU(3)$ classification of hadrons (today one should say flavour $SU(3)$). Assigning quarks q to the fundamental representation of $SU(3)$, not realized by any known hadrons, and giving them spin one-half one obtains the phenomenologically successful $SU(3)$ and $SU(6)$ schemes. $SU(6)$ is obtained by adjoining the group $SU(2)$ of spin rotation to the internal symmetry group $SU(3)$ for baryons (qqq) and mesons ($q\bar{q}$). In particular, the known hadrons indeed realize only those representations of $SU(3)$ which are given by the composite model. The quark model for hadrons, successful as it was, appeared, however, to have difficulties in reconciling the Fermi statistics for quarks with the most natural assumption that in the lowest-lying hadronic states all the relative angular momenta among constituent quarks vanish (s-wave states). Thus, baryon wave-functions should be antisymmetric in spin and flavour degrees of freedom. This is not the case in the original quark model as can be immediately seen from inspection of the $\Delta^{++}(\frac{3}{2}^+)$ wave-function which must be $u\!\uparrow u\!\uparrow u\!\uparrow$; u denotes the quark with electric charge $Q = \frac{3}{2}$, the arrow denotes spin $S_z = \frac{1}{2}$ for each quark.

The difficulty can be resolved by postulating a new internal quantum number for quarks which has been called colour (Greenberg 1964, Han & Nambu 1965, Nambu 1966, Bardeen, Fritzsch & Gell-Mann 1973). If a quark of each flavour has three, otherwise indistinguishable, colour states, Fermi statistics is saved by using a totally antisymmetric colour wave-function $\varepsilon_{abc}u_a\!\uparrow u_b\!\uparrow u_c\!\uparrow$. Assuming furthermore that (i) strong interactions are invariant under global $SU(3)_{\text{colour}}$

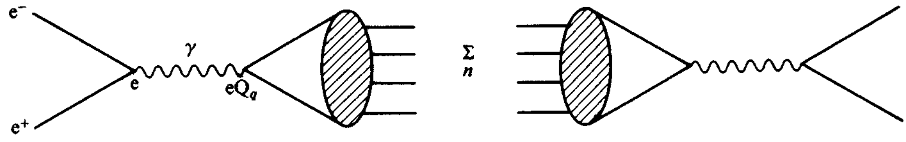


Fig. 0.1. The process $e^+e^- \to \gamma \to$ hadrons in the parton model. The sum is taken over all hadronic states in the reaction $e^+e^- \to \gamma \to \bar{q}q \to$ hadrons.

transformations (the states may then be classified by their $SU(3)_{\text{colour}}$ representation) and (ii) physical hadrons are colourless, i.e. they are singlets under $SU(3)_{\text{colour}}$ (quark confinement), we can understand why only qqq and $q\bar{q}$ states, and not qq or qqqq etc., exist in nature: the singlet representation appears only in the $3 \times 3 \times 3$ and $3 \times \bar{3}$ products.

The concept of colour is also supported by at least two other, strong arguments. One is based on the parton model (Feynman 1972) approach to the reaction $e^+e^- \to$ hadrons. The total cross section for this process is then given by the diagram in Fig. 0.1 and the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{0.1}$$

is predicted to be

$$\begin{aligned} R = \frac{e^2 \sum_q Q_q^2}{e^2} &= \sum_q Q_q^2 = 3 \times \left(\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9} + \cdots\right) \\ &= \tfrac{11}{3} \quad \text{(including quarks up to b)} \end{aligned} \tag{0.2}$$

The experimental value of R is in good agreement with this prediction and in poor agreement with the colourless prediction $\frac{11}{9}$.

Yet another reason for colour is provided by the decay $\pi^0 \to 2\gamma$. Here again the number of quark states matters in explaining the width of $\pi^0 \to 2\gamma$. This problem will be discussed in more detail in Chapter 13.

The concept of colour certainly underlies what we believe to be the true theory of strong interactions, namely QCD. However, the theory also has several other basic features which are partly suggested by experimental observations and partly required by theoretical consistency. Firstly, it is assumed that strong interactions act on the colour quantum numbers and only on them. Experimentally there is no evidence for any flavour dependence of strong forces; all flavour-dependent effects can be explained by quark mass differences and the origin of the quark masses, though not satisfactorily understood yet, is expected to be outside of QCD. In addition, only colour symmetry can be assumed to be an exact symmetry (flavour symmetry is evidently broken) and this, combined with the assumption that it is a gauge symmetry (Han & Nambu 1965, Fritzsch, Gell-Mann & Leutwyler 1973), has profound implications: asymptotic freedom (Gross & Wilczek 1973, Politzer 1973) and presumably, though not proven, confinement of quarks. Both are welcome features. Asymptotic freedom means that the forces become negligible at short distances and consequently the interaction between quarks by exchange of non-abelian gauge fields (gluons) is consistent with the successful, as the first approximation, description of the deep inelastic scattering in the framework of the parton model. It has been shown that only non-abelian gauge theories are asymptotically free (Coleman & Gross 1973). Confinement of the colour quantum numbers, i.e. of quarks and gluons, has not yet been proved to follow from QCD but it is likely to be true, reflecting strongly singular structure of the non-abelian gauge theory in the IR region. Once we assume colourful quarks as elementary objects in hadrons, confinement of colour is desirable in view of the so far unsuccessful experimental search for free quarks and to avoid a proliferation of unwanted states.

An important line of argument in favour of gluons as vector bosons begins with approximate chiral symmetry of strong interactions (Chapter 9). Coupling of fermions with vector and axial-vector fields, but not with scalars or pseudoscalars, is chirally invariant. A theory with axial-vector gluons based on the $SU(3)$ group cannot be consistently renormalized because of anomalies (Chapter 13). Thus we arrive at vector gluon interaction.

In recent years there has been a lot of research in QCD perturbation theory neglecting the unsolved confinement problem. Of course, this cannot be fully satisfactory since experimenters collide hadrons and not quarks and gluons. One can nevertheless argue that it is a justifiable approximation at short distances. Thus, at its present stage the theory provides us with calculable corrections to the free-field behaviour of quarks and gluons in the parton model and can be tested in the deep inelastic region. Given the accuracy of the calculations and of the experimental data one cannot claim yet to have strongly positive experimental verification of perturbative QCD predictions. However, experimental results are certainly consistent with QCD (in particular jet physics already provides us with good evidence for the vector nature of gluons) and in view of its elegance and self-consistency there are few sceptical of its chance of being the theory of strong interactions.

Electroweak theory

There is, at present, impressive experimental evidence for the electroweak gauge theory with the gauge symmetry spontaneously broken. To introduce the Glashow–Salam–Weinberg theory we recall first that the effective Fermi lagrangian for the charged-current weak interactions, valid at low energies, has conventionally been taken to be (for a systematic account of the weak interactions phenomenology see, for example, Gasiorowicz (1966) and more recently Abers & Lee (1973) and Taylor (1976))

$$\mathcal{L}_{\mathrm{eff}}(x) = -2\sqrt{2}\, G_\mu J_\mu^\dagger(x) J^\mu(x) \tag{0.3}$$

where the Fermi β-decay constant $G_\mu = 1.165 \times 10^{-5}$ GeV^{-2} ($\hbar = c = 1$) and the charged current $J_\mu(x)$ is composed of several pieces, each with V–A structure. In terms of the lepton and quark fields it can be written as follows

$$J^\mu = \sum_f \bar{\Psi}_{\mathrm{L}}^f \gamma^\mu T^- \Psi_{\mathrm{L}}^f \tag{0.4}$$

with

$$T^\pm = \tfrac{1}{2}(\sigma^1 \pm \sigma^2) = T^1 \pm \mathrm{i}T^2$$

where σ^a are Pauli matrices and for weak interactions, as known in the 1960s,

$$\Psi_{\mathrm{L}}^f = \tfrac{1}{2}(1-\gamma_5)\left\{ \begin{pmatrix} \nu_{\mathrm{e}} \\ \mathrm{e}^- \end{pmatrix}, \begin{pmatrix} \nu^\mu \\ \mu^- \end{pmatrix}, \begin{pmatrix} u \\ d' \end{pmatrix} \right\} \tag{0.5}$$

The subscript L stands for the left-handed fermions. The prime superscript indicates the existence of mixing of the quark fields observed in strong interactions (mass eigenstates):

$$d' = d \cos\Theta_{\mathrm{C}} + s \sin\Theta_{\mathrm{C}}$$

where the angle Θ_{C} is known as the Cabibbo angle and has been measured in weak decays of strange particles.

We now extend the Fermi–Cabibbo theory by several additional assumptions. Firstly, we postulate the existence of a symmetry group, global for the time being, for the weak interactions. Having the current (0.4) it is natural to postulate $SU(2)$ symmetry and consequently the existence of the neutral current corresponding to the third generator of the $SU(2)$

$$[T^+, T^-] = 2T^3 \tag{0.6}$$

which would induce transitions, such as, for instance, those in Fig. 0.2 occurring with a similar strength to the charged-current reactions. Neutral-current weak transitions with the expected strength have been discovered at CERN (Hasert *et al.* 1973). However, they (i) are not of purely V–A character as expected from the $SU(2)$ model, and (ii) always conserve strangeness to a very good accuracy. According to the existing experimental limits the strangeness non-conserving neutral-current transitions (like $\mathrm{K}_{\mathrm{L}}^0 \to \mu^+\mu^-$ or $\mathrm{K}^0 \leftrightarrow \bar{\mathrm{K}}^0$) are suppressed by many orders of magnitude as compared to the standard weak processes. Both factors

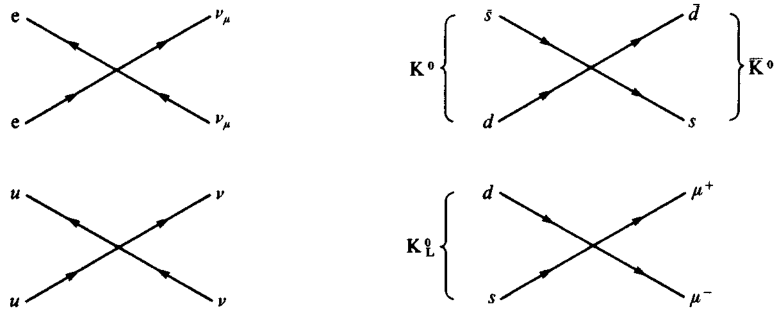


Fig. 0.2. Some neutral-current weak transitions.

call for further invention in searching for a realistic theory of weak interactions. Glashow, Iliopoulos & Maiani (1970) have discovered that the problem of the strangeness non-conserving neutral current is solved if the set of fermionic doublets Ψ_{L}^{f} is completed with a fourth one

$$\begin{pmatrix} c \\ s' \end{pmatrix}, \qquad s' = -d \sin \Theta_{\mathrm{C}} + s \cos \Theta_{\mathrm{C}}$$

One can immediately check that with the s' orthogonal to d' the neutral current is diagonal in flavour. Thus, they have predicted the existence of the charm quark discovered later at SLAC (Aubert *et al.* 1974, Augustin *et al.* 1974). Also the doublet classification of the left-handed fermions with an equal number of lepton and quark doublets, now further confirmed by the experimental discovery of the

$$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ b \end{pmatrix}$$

doublets,† has emerged as an important property of weak interactions. This has profound implications for a successful extension of the effective model into a non-abelian local gauge field theory: it ensures the cancellation of the chiral anomaly and consequently the renormalizability of the theory. A highly consistent scheme begins to be expected.

The next step towards the final form of the Glashow–Salam–Weinberg theory is a ‘unification’ of weak and electromagnetic interactions (Schwinger 1957, Glashow 1961, Salam & Ward 1964). Thus, we want the electric charge Q also to be the generator of the symmetry group of our theory. To achieve this in the most economical way we notice that for the left-handed doublets of fermions we can

† The t quark was discovered by the CDF and D0 groups in Fermilab in 1994.

define a new quantum number Y (weak hypercharge)

$$Y = Q - T^3 \tag{0.7}$$

such that each doublet of the left-handed fermions is an eigenvector of the operator Y (e.g. $Y_{\nu_L} = Y_{e_L} = -\frac{1}{2}$). Therefore Y commutes with the generators of $SU(2)$ and including the right-handed fermions by the prescription $Y = Q$ (they are singlets with respect to $SU(2)$) we arrive at the $SU(2) \times U(1)$ symmetry group for the electroweak interactions. The $U(1)$ current reads

$$J_Y^\mu = \sum_f \bar{\Psi}_L^f \gamma^\mu Y \Psi_L^f + \sum_f \bar{\Psi}_R^f \gamma^\mu Y \Psi_R^f \tag{0.8}$$

where $\Psi_{L(R)}^f$ are left- (right-)handed chiral lepton and quark fields. According to (0.7), the electromagnetic current can be written as follows

$$\begin{aligned} J_{em}^\mu &= J_Y^\mu + \sum_f \bar{\Psi}_L^f \gamma^\mu T^3 \Psi_L^f \\ &= \sum_f \Psi^f Q \gamma^\mu \bar{\Psi}^f \end{aligned} \tag{0.9}$$

The $SU(2) \times U(1)$ group is the minimal one which contains the electromagnetic and weak currents. With electromagnetism being described by a gauge field A_μ our minimal model 'unifying' electromagnetic and weak interactions requires the Yang–Mills gauge fields W_μ^α and B_μ to couple to the $SU(2) \times U(1)$ currents giving the interaction

$$g J_\mu^\alpha W^{\alpha\mu} + g' J_Y^\mu B_\mu \tag{0.10}$$

Since we are dealing with a direct group product the couplings g and g' are independent. Therefore the significance of the electroweak unification comes from the choice of a minimal group structure which takes account of both currents. Even with two independent coupling constants, this 'unification' has a predictive power: it determines the structure of the parity violation in the neutral weak current (see (0.15) below) and gives the W^3–γ mixing in terms of g' and g, which as we will see, reduces the number of free parameters of the theory.

The charged gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2) \tag{0.11}$$

couple directly to the experimentally observed weak currents. As follows from the formula

$$Q = \frac{1}{g}(gT^3) + \frac{1}{g'}(g'Y) \tag{0.12}$$

the photon field A_μ (determined by the fact that it couples to the electromagnetic current) must be a combination of the W^3 and B bosons

$$A_\mu = \left(\frac{1}{g^2} + \frac{1}{g'^2}\right)^{-1/2} \left(\frac{1}{g} W_\mu^3 + \frac{1}{g'} B_\mu\right) \tag{0.13}$$

The orthogonal combination

$$Z_\mu = \left(\frac{1}{g^2} + \frac{1}{g'^2}\right)^{-1/2} \left(\frac{1}{g'} W_\mu^3 - \frac{1}{g} B_\mu\right) \tag{0.14}$$

couples to the neutral weak current J_μ^z

$$J_\mu^z = \frac{g}{\cos\Theta_\mathrm{W}} (J_\mu^3 - \sin^2\Theta_\mathrm{W} J_\mu^\mathrm{em}) \tag{0.15}$$

where J_em^μ is given by (0.9) and we have introduced the standard notation in terms of the so-called Weinberg angle (Glashow 1961, Weinberg 1967*b*)

$$\sin^2\Theta_\mathrm{W} = \frac{g'^2}{g^2 + g'^2} \quad \text{or} \quad \tan\Theta_\mathrm{W} = \frac{g'}{g} \quad \text{or} \quad e = g\sin\Theta_\mathrm{W} = g'\cos\Theta_\mathrm{W} \tag{0.16}$$

(the last relation follows from the fact that the electromagnetic coupling constant is electric charge e, $e > 0$).

If initially all vector fields are massless, we must now introduce masses for intermediate vector boson fields $W^\pm$ and Z to account for the weakness of the weak interactions as compared to electromagnetism. One knows that the massive Yang–Mills theory, although less divergent than a theory with arbitrary couplings, is not renormalizable unless the vector boson masses are introduced by means of the Higgs mechanism which breaks the non-abelian gauge invariance spontaneously ('t Hooft 1971*a*, *b*). In the minimal model this is achieved (Weinberg 1967*b*, Salam 1968) with one complex $SU(2)$ doublet of scalar fields whose neutral component has a non-vanishing vacuum expectation value v. Then the boson masses are (at the tree level)

$$M_W = \tfrac{1}{2}vg, \quad M_Z = \tfrac{1}{2}v(g'^2 + g^2)^{1/2}$$

so that

$$M_W/M_Z\cos\Theta_\mathrm{W} = 1 \tag{0.17}$$

Thus, apart from the fermion mass matrix and the scalar self-coupling we have a three-parameter (g, g', v) electroweak theory. For instance, knowing the electromagnetic coupling $\alpha = e^2/4\pi$, the Fermi constant G_μ and the Weinberg angle (parity non-conservation in all neutral-current transitions is consistent with the form (0.15) and with $\sin^2\Theta_{\mathrm{W}} \approx 0.23$ one predicts, at the tree level,

$$M_W = M_Z \cos\Theta_{\mathrm{W}} = \left(\frac{\pi\alpha}{\sqrt{2}G_\mu}\right)^{1/2} \frac{1}{\sin\Theta_{\mathrm{W}}} \approx 80 \text{ GeV} \tag{0.18}$$

Vector bosons with the expected masses were discovered at CERN in 1982! This triumph of the spontaneously broken Yang–Mills theory has since been confirmed by comparing higher accuracy data with higher order calculations. However, the discovery of the Higgs particle still remains (in 1999) a challenge for future experiments. We would like to stress the role of gauge symmetry and the unification idea, apparently coupled to each other, in this very successful description of the electroweak interactions.

Non-abelian gauge theories are considered at present to be the most promising theoretical framework for fundamental interactions. They are also extensively studied in the hope of providing the unified description of all interactions (grand unification, supergravity) going beyond the so-called standard $SU(3) \times SU(2) \times U(1)$ model summarized in this section.

1

Classical fields, symmetries and their breaking

Classically, we distinguish particles and forces which are responsible for interaction between particles. The forces are described by classical fields. The motion of particles in force fields is subject to the laws of classical mechanics. Quantization converts classical mechanics into quantum mechanics which describes the behaviour of particles at the quantum level. A state of a particle is described by a vector $|\Phi(t)\rangle$ in the Hilbert space or by its concrete representation, e.g. the wave-function $\Phi(x) = \langle \mathbf{x}|\Phi(t)\rangle$ $(x = (t, \mathbf{x}))$ whose modulus squared is interpreted as the density of probability of finding the particle in point $\mathbf{x}$ at the time t. As a next step, the wave-function is interpreted as a physical field and quantized. We then arrive at quantum field theory as the universal physical scheme for fundamental interactions. It is also reached by quantization of fields describing classical forces, such as electromagnetic forces. The basic physical concept which underlies quantum field theory is the equivalence of particles and forces. This logical structure of theories for fundamental interactions is illustrated by the diagram shown below:

PARTICLES	FORCES
Classical mechanics (quantization)	Classical field theory (quantization)
Quantum mechanics (wave-function $\Phi(x)$ is interpreted as a physical field and quantized)	

QUANTUM FIELD THEORY

This chapter is devoted to a brief summary of classical field theory. The reader should not be surprised by such formulations as, for example, the 'classical' Dirac field which describes the spin $\frac{1}{2}$ particle. It is interpreted as a wave function for the Dirac particle. Our ultimate goal is quantum field theory and our classical fields in this chapter are not necessarily only the fields which describe the classical forces observed in Nature.

1.1 The action, equations of motion, symmetries and conservation laws

Equations of motion

All fundamental laws of physics can be understood in terms of a mathematical construct: *the action*. An ansatz for the action $S = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}$ can be regarded as a formulation of a theory. In classical field theory the lagrangian density $\mathcal{L}$ is a function of fields Φ and their derivatives. In general, the fields Φ are multiplets under Lorentz transformations and in a space of internal degrees of freedom. It is our experience so far that in physically relevant theories the action satisfies several general principles such as: (i) Poincaré invariance (or general covariance for theories which take gravity into consideration), (ii) locality of $\mathcal{L}$ and its, at most, bilinearity in the derivatives $\partial_\nu \Phi(x)$ (to get at most second order differential equations of motion),† (iii) invariance under all symmetry transformations which characterize the considered physical system, (iv) S has to be real to account for the absence of absorption in classical physics and for conservation of probability in quantum physics. The action is then the most general functional which satisfies the above constraints.

From the action we get:

equations of motion (invoking Hamilton's principle);
conservations laws (from Noether's theorem);
transition from classical to quantum physics (by using path integrals or canonical quantization).

In this section we briefly recall the first two results. Path integrals are discussed in detail in Chapter 2.

The equations of motion for a system described by the action

$$S = \int_{t_0}^{t_1} \mathrm{d}t \int_V \mathrm{d}^3x\, \mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) \tag{1.1}$$

(the volume V may be finite or infinite; in the latter case we assume the system to be localized in space‡) are obtained from Hamilton's principle. This ansatz (whose physical sense becomes clearer when classical theory is considered as a limit of quantum theory formulated in terms of path integrals) says: the dynamics of the system evolving from the initial state $\Phi(t_0, \mathbf{x})$ to the final state $\Phi(t_1, \mathbf{x})$ is

† Later we shall encounter effective lagrangians that may contain terms with more derivatives. Such terms can usually be interpreted as a perturbation.

‡ Although we shall often use plane wave solutions to the equations of motion, we always assume implicitly that real physical systems are described by wave packets localized in space.

such that the action (taken as a functional of the fields and their first derivatives) remains stationary during the evolution, i.e.

$$\delta S = 0 \tag{1.2}$$

for arbitrary variations

$$\Phi(x) \to \Phi(x) + \delta\Phi(x) \tag{1.3}$$

which vanish on the boundary of $\Omega \equiv (\Delta t, V)$.

Explicit calculation of the variation δS gives:

$$\delta S = \int_\Omega \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \delta(\partial_\nu \Phi) \right] \tag{1.4}$$

(summation over all fields Φ and for each field Φ over its Lorentz and 'internal' indices is always understood). Since in the variation the coordinates x do not change we have:

$$\delta(\partial_\nu \Phi) = \partial_\nu \delta\Phi \tag{1.5}$$

and (1.4) can be rewritten as follows:

$$\delta S = \int_\Omega \mathrm{d}^4 x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \right] \delta\Phi + \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \delta\Phi \right] \right\} \tag{1.6}$$

The second term in (1.6) is a surface term which vanishes and, therefore, the condition $\delta S = 0$ gives us the Euler–Lagrange equations of motion for the classical fields:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i^\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i^\mu)} = 0 \qquad \begin{array}{l} \nu,\ \mu = 0, 1, 2, 3 \\ i = 1, \ldots, n \end{array} \tag{1.7}$$

(here we keep the indices explicitly, with Φ taken as a Lorentz vector; is are internal quantum number indices). It is important to notice that lagrangian densities which differ from each other by a total derivative of an arbitrary function of fields

$$\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu(\Phi) \tag{1.8}$$

give the same classical equations of motion (due to the vanishing of variations of fields on the boundary of Ω).

Global symmetries

Eq. (1.6) can also be used to derive conservation laws which follow from a certain class of symmetries of the physical system. Let us consider a Lie group of continuous global (x-independent) infinitesimal transformations (here we restrict ourselves to unitary transformations; conformal transformations are discussed in

Chapter 7) acting in the space of internal degrees of freedom of the fields Φ_i. Under infinitesimal rotation of the reference frame (passive view) or of the physical system (active view) in that space:

$$\Phi_i(x) \to \Phi_i'(x) = \Phi_i(x) + \delta_0\Phi_i(x) \tag{1.9}$$

where

$$\delta_0\Phi_i(x) = -\mathrm{i}\Theta^a T^a_{ij}\Phi_j(x) \tag{1.10}$$

and $\Phi_i'(x)$ is understood as the ith component of the field $\Phi(x)$ in the new reference frame or of the rotated field in the original frame (active). Note that a rotation of the reference frame by angle Θ corresponds to active rotation by $(-\Theta)$. The T^as are a set of hermitean matrices T^a_{ij} satisfying the Lie algebra of the group G

$$\left[T^a, T^b\right] = \mathrm{i}c^{abc}T^c, \qquad \mathrm{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab} \tag{1.11}$$

and the Θ^as are x-independent.

Under the change of variables $\Phi \to \Phi'(x)$ given by (1.9) the lagrangian density is transformed into

$$\mathcal{L}'\big(\Phi'(x), \partial_\mu\Phi'(x)\big) \equiv \mathcal{L}\big(\Phi(x), \partial_\mu\Phi(x)\big) \tag{1.12}$$

and, of course, the action remains numerically unchanged:

$$\delta_\Theta S = S' - S = \int_\Omega \mathrm{d}^4x\,\left[\mathcal{L}'(\Phi', \partial_\mu\Phi') - \mathcal{L}(\Phi, \partial_\mu\Phi)\right] = 0 \tag{1.13}$$

(we use the symbol $\delta_\Theta S$ for the change of the action under transformation (1.9) on the fields to distinguish it from the variations $\delta S'$ and δS). For the variations we have $\delta S' = \delta S$. Thus, if $\Phi(x)$ describes a motion of a system, $\Phi'(x)$ is also a solution of the equations of motion for the transformed fields which, however, in general are different in form from the original ones.

Transformations (1.9) are symmetry transformations for a physical system if its equations of motion remain form-invariant in the transformed fields. In other words, a solution to the equations of motion after being transformed according to (1.9) remains a solution of the same equations. This is ensured if the density $\mathcal{L}$ is invariant under transformations (1.9) (for scale transformations see Chapter 7):

$$\mathcal{L}'(\Phi'(x), \partial_\mu\Phi'(x)) = \mathcal{L}(\Phi'(x), \partial_\mu\Phi'(x)) \tag{1.14}$$

or, equivalently,

$$\mathcal{L}(\Phi'(x), \partial_\mu\Phi'(x)) = \mathcal{L}(\Phi(x), \partial_\mu\Phi(x)).$$

Indeed, variation of the action generated by arbitrary variations of the fields $\Phi'(x)$ with boundary conditions (1.3) is again given by (1.6) (with $\Phi \to \Phi'$ etc.)

and we derive the same equations for $\Phi'(x)$ as for $\Phi(x)$. Moreover, the change (1.14) of the lagrangian density under the symmetry transformations:

$$\mathcal{L}'(\Phi', \partial_\mu \Phi') - \mathcal{L}(\Phi, \partial_\mu \Phi) = \mathcal{L}(\Phi', \partial_\mu \Phi') - \mathcal{L}(\Phi, \partial_\mu \Phi) = 0 \tag{1.15}$$

is formally given by the integrand in (1.6) with $\delta\Phi$s given by (1.9). It follows from (1.15) and the equations of motion that the currents

$$j^a_\mu(x) = -\mathrm{i}\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi_i)} T^a_{ij} \Phi_j \tag{1.16}$$

are conserved and the charges

$$Q^a(t) = \int \mathrm{d}^3x \; j^a_0(t, \mathbf{x}) \tag{1.17}$$

are constants of motion, provided the currents fall off sufficiently rapidly at the space boundary of Ω.

It is very important to notice that the charges defined by (1.17) (even if they are not conserved) are the generators of the transformation (1.10). Firstly, they satisfy (in an obvious way) the same commutation relation as the matrices T^a. Secondly, they indeed generate transformations (1.10) through Poisson brackets. Let us introduce conjugate momenta for the fields $\Phi(x)$:

$$\Pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi(t, \mathbf{x}))} \tag{1.18}$$

(we assume here that $\Pi \neq 0$; there are interesting exceptions, with $\Pi = 0$, such as, for example, classical electrodynamics; these have to be discussed separately). Again, the Lorentz and 'internal space' indices of the Πs and Φs are hidden. The hamiltonian of the system reads

$$H = \int \mathrm{d}^3x \, \mathcal{H}(\Pi, \Phi), \qquad \mathcal{H} = \Pi \partial_0 \Phi - \mathcal{L} \tag{1.19}$$

and the Poisson bracket of two functionals F_1 and F_2 of the fields Π and Φ is defined as

$$\{F_1(t, \mathbf{x}), F_2(t, \mathbf{y})\} = \int \mathrm{d}^3z \left[\frac{\partial F_1(t, \mathbf{x})}{\partial \Phi(t, \mathbf{z})} \frac{\partial F_2(t, \mathbf{y})}{\partial \Pi(t, \mathbf{z})} - \frac{\partial F_1(t, \mathbf{x})}{\partial \Pi(t, \mathbf{z})} \frac{\partial F_2(t, \mathbf{y})}{\partial \Phi(t, \mathbf{z})} \right] \tag{1.20}$$

We get, in particular,

$$\left. \begin{aligned} &\{\Pi(t, \mathbf{x}), \Phi(t, \mathbf{y})\} = -\delta(\mathbf{x} - \mathbf{y}) \\ &\{\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = \{\Phi(t, \mathbf{x}), \Phi(t, \mathbf{y})\} = 0 \end{aligned} \right\} \tag{1.21}$$

The charge (1.17) can be written as

$$Q^a(t) = \int \mathrm{d}^3x \; \Pi(t, \mathbf{x})(-\mathrm{i}T^a)\Phi(t, \mathbf{x}) \tag{1.22}$$

and for its Poisson bracket with the field Φ we get:

$$\left\{Q^a(t), \Phi(t, \mathbf{x})\right\} = (\mathrm{i}T^a)\Phi(t, \mathbf{x}) \tag{1.23}$$

i.e. indeed the transformation (1.10). Thus, generators of symmetry transformations are conserved charges and vice versa.

Space-time transformations

Finally, we consider transformations (changes of reference frame) which act simultaneously on the coordinates and the fields. The well-known examples are, for instance, translations, spatial rotations and Lorentz boosts. In this case our formalism has to be slightly generalized. We consider an infinitesimal transformation

$$x'^\mu = x^\mu + \delta x^\mu = x^\mu + \varepsilon^\mu(x) \tag{1.24}$$

and suppose that the fields $\Phi(x)$ seen in the first frame transform under some representation $T(\varepsilon)$ of the transformation (1.24) (which acts on their Lorentz structure) into $\Phi'(x')$ in the transformed frame:

$$\Phi'(x') = \exp[-\mathrm{i}T(\varepsilon)]\Phi(x) \tag{1.25}$$

(the Lorentz indices are implicit and $\Phi'_\alpha(x')$ is understood as the αth component of the field $\Phi(x')$ in the new reference frame). For infinitesimal transformations we have (∂' denotes differentiation with respect to x'):

$$\left.\begin{aligned} \Phi'(x') &= \Phi'(x) + \delta x^\mu \partial_\mu \Phi(x) + \mathcal{O}(\varepsilon^2) \\ \partial'_\mu \Phi'(x') &= \partial_\mu \Phi'(x) + \delta x^\nu \partial_\mu \partial_\nu \Phi(x) + \mathcal{O}(\varepsilon^2) \end{aligned}\right\} \tag{1.26}$$

The lagrangian density $\mathcal{L}'$ after the transformation is defined by the equation (see, for example, Trautman (1962, 1996))

$$\begin{aligned} S' &= \int_{\Omega'} \mathrm{d}^4x'\, \mathcal{L}'\left(\Phi'(x'), \partial'_\mu \Phi'(x')\right) \\ &= \int_\Omega \mathrm{d}^4x \left[\mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) - \partial_\mu \delta\Lambda^\mu\left(\Phi(x)\right)\right] \end{aligned} \tag{1.27}$$

(Ω' denotes the image of Ω under the transformation (1.24) and $\delta\Lambda^\mu$ is an arbitrary function of the fields) which is a sufficient condition for the new equations of motion to be equivalent to the old ones. By this we mean that a change of the reference frame has no implications for the motion of the system, i.e. a transformed solution to the original equations remains a solution to the new equations (since (1.27) implies $\delta S' = \delta S$). In general $\det(\partial x'/\partial x) \neq 1$ and also $\mathcal{L}'\big(\Phi'(x), \partial'_\mu \Phi'(x')\big) \neq \mathcal{L}\big(\Phi(x), \partial_\mu \Phi(x)\big)$. Moreover, there is an arbitrariness in the choice of $\mathcal{L}'$ due to the presence in (1.27) of the total derivative of an arbitrary function $\delta\Lambda^\mu(\Phi(x))$.

Symmetry transformations are again defined as transformations which leave equations of motion form-invariant. A sufficient condition is that, for a certain choice of $\delta\Lambda$, the $\mathcal{L}'$ defined by (1.27) satisfies the equation:

$$\mathcal{L}'(\Phi'(x'), \partial'_\mu \Phi'(x')) = \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x')) \tag{1.28}$$

or, equivalently,

$$\mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x'))\, \mathrm{d}^4x' = \left[\mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) - \partial_\mu \delta\Lambda^\mu(\Phi(x))\right] \mathrm{d}^4x \tag{1.28a}$$

Note that, if $\det(\partial x'/\partial x) = 1$, we recover condition (1.14) up to the total derivative. The most famous example of symmetry transformations up to a non-vanishing total derivative is supersymmetry (see Chapter 15).

The change of the action under symmetry transformations can be calculated in terms of

$$\delta\mathcal{L} = \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x')) - \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) \tag{1.29}$$

where x and x' are connected by the transformation (1.24). Using (1.26) we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi}\delta_0\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0(\partial_\mu\Phi) + \delta x^\mu \partial_\mu\mathcal{L} + \mathcal{O}(\varepsilon^2) \tag{1.30}$$

where

$$\delta_0\Phi(x) = \Phi'(x) - \Phi(x) \tag{1.31}$$

and, therefore, since δ_0 is a functional change,

$$\delta_0\partial_\mu\Phi = \partial_\mu\delta_0\Phi \tag{1.32}$$

Eq. (1.30) can be rewritten in the following form:

$$\delta\mathcal{L} = \delta x^\mu\partial_\mu\mathcal{L} + \left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]\delta_0\Phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi\right) \tag{1.33}$$

Since

$$\det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right) = 1 + \partial_\mu\delta x^\mu \tag{1.34}$$

we finally get (using (1.28a), (1.33) and the equations of motion):

$$\begin{aligned} 0 &= \mathcal{L}\partial_\mu\delta x^\mu + \delta x^\mu\partial_\mu\mathcal{L} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi\right) + \partial_\mu\delta\Lambda^\mu + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi + \delta\Lambda^\mu + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{1.35}$$

Re-expressing $\delta_0\Phi$ in terms of $\delta\Phi = \Phi'(x') - \Phi(x) = \delta_0\Phi + \delta x^\mu \partial_\mu\Phi$ we conclude that

$$\partial_\mu j^\mu \equiv \partial_\mu \left\{ \left[\mathcal{L} g^\mu{}_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\rho\Phi \right] \delta x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta\Phi + \delta\Lambda^\mu \right\} = 0 \tag{1.36}$$

This is a generalization of the result (1.16) to space-time symmetries and both results are the content of the Noether's theorem which states that symmetries of a physical system imply conservation laws.

Examples

We now consider several examples. Invariance of a physical system under translations

$$x'^\mu = x^\mu + \varepsilon^\mu \tag{1.37}$$

implies the conservation law

$$\partial_\mu \Theta^{\mu\nu}(x) = 0 \tag{1.38}$$

where $\Theta^{\mu\nu}$ is the energy–momentum tensor

$$\Theta^{\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi - g^{\mu\nu}\mathcal{L} \tag{1.39}$$

(we have assumed that the considered system is such that $\mathcal{L}'(x')$ defined by (1.27) with $\Lambda \equiv 0$ satisfies $\mathcal{L}'(x') = \mathcal{L}(x')$; note also that $\det(\partial x'/\partial x) = 1$). The four constants of motion

$$P^\nu = \int \mathrm{d}^3x\, \Theta^{0\nu}(x) \tag{1.40}$$

are the total energy of the system ($\nu = 0$) and its momentum vector ($\nu = 1, 2, 3$). It is easy to check that transformations of the field $\Phi(x)$ under translations in time and space are given by

$$\{P^\mu, \Phi(x)\} = -\frac{\partial}{\partial x_\mu}\Phi(x) \tag{1.41}$$

so that $\Phi'(x) = \Phi(x) + \{P^\mu, \Phi(x)\}\,\varepsilon_\mu$.

The energy–momentum tensor defined by (1.39) is the so-called canonical energy–momentum tensor. It is important to notice that the physical interpretation of the energy–momentum tensor remains unchanged under the redefinition

$$\Theta^{\mu\nu}(x) \to \Theta^{\mu\nu}(x) + f^{\mu\nu}(x) \tag{1.42}$$

where $f^{\mu\nu}(x)$ are arbitrary functions satisfying the conservation law

$$\partial_\mu f^{\mu\nu}(x) = 0 \tag{1.43}$$

with vanishing charges

$$\int \mathrm{d}^3x\, f^{0\nu}(x) = 0 \tag{1.44}$$

Then (1.38) and (1.40) remain unchanged. A solution to these constraints is

$$f^{\mu\nu}(x) = \partial_\rho f^{\mu\nu\rho} \tag{1.45}$$

where $f^{\mu\nu\rho}(x)$ is antisymmetric in indices $(\mu\rho)$. This freedom can be used to make the energy–momentum tensor symmetric and gauge-invariant (see Problem 1.3).

An infinitesimal Lorentz transformation reads

$$x'^\mu \equiv \Lambda^\mu{}_\nu(\omega)x^\nu \approx x^\mu + \omega^\mu{}_\nu x^\nu \tag{1.46}$$

(with $\omega_{\mu\nu}$ antisymmetric) and it is accompanied by the transformation on the fields

$$\Phi'(x') = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Phi(x) \approx \left(1 + \tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Phi(x) \tag{1.47}$$

where $\Sigma^{\mu\nu}$ is a spin matrix. For Lorentz scalars $\Sigma = 0$, for Dirac spinors $\Sigma^{\mu\nu} = (-\mathrm{i}/2)\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ (for the notation see Appendix A). Inserting (1.46) and (1.47) into (1.36) (using (1.38) and replacing the canonical energy–momentum tensor by a symmetric one and finally using the antisymmetry of $\omega^{\mu\nu}$), we get the following conserved currents:

for a scalar field:

$$M^{\mu,\nu\rho}(x) = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu} \tag{1.48}$$

for a field with a non-zero spin:

$$M^{\mu,\nu\rho}(x) = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Sigma^{\nu\rho}\Phi \tag{1.49}$$

The conserved charges

$$M^{\nu\rho}(t) = \int \mathrm{d}^3x\, M^{0,\nu\rho}(t, \mathbf{x}) \tag{1.50}$$

are identified with the total angular momentum tensor of the considered physical system. The charges $M^{\mu\nu}$ are the generators of Lorentz transformations on the fields $\Phi(x)$ with

$$\{M^{\mu\nu}, \Phi(x)\} = -(x^\mu\partial^\nu - x^\nu\partial^\mu + \Sigma^{\mu\nu})\Phi(x) \tag{1.51}$$

so that $\Phi'(x) = \Phi(x) - \frac{1}{2}\omega_{\mu\nu}\{M^{\mu\nu}, \Phi(x)\}$.

The tensor $M^{\mu\nu}$ is not translationally invariant. The total angular momentum three-vector is obtained by constructing the Pauli–Lubański vector

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\kappa}M^{\nu\rho}P^\kappa \tag{1.52}$$

(where $\varepsilon_{\mu\nu\rho\kappa}$ is the totally antisymmetric tensor defined by $\varepsilon_{0123} = -\varepsilon^{0123} = 1$) which reduces in the rest frame $P^\mu = (m, \mathbf{0})$ to the three-dimensional total angular momentum $M^k \equiv \frac{1}{2}\varepsilon^{ijk}M^{ij}$: $M^k = W^k/m$.

1.2 Classical field equations

Scalar field theory and spontaneous breaking of global symmetries

In this section we illustrate our general considerations with the simplest example: the classical theory of scalar fields $\Phi(x)$. We take these to be complex, in order to be able to discuss a theory which is invariant under continuous global symmetry of phase transformations ($U(1)$ symmetry group of one-parameter, unitary, unimodular transformations):

$$\Phi'(x) = \exp(-\mathrm{i}q\Theta)\Phi(x) \approx (1 - \mathrm{i}q\Theta)\Phi(x) \tag{1.53}$$

Thus, the field $\Phi(x)$ (its complex conjugate $\Phi^\star(x)$) carries the internal quantum number q $(-q)$. It is sometimes also useful to introduce two real fields

$$\phi = \frac{1}{\sqrt{2}}(\Phi + \Phi^\star), \qquad \chi = \frac{-\mathrm{i}}{\sqrt{2}}(\Phi - \Phi^\star) \tag{1.54}$$

The lagrangian density which satisfies all the constraints of the previous section reads:

$$\mathcal{L} = \partial_\mu\Phi\partial^\mu\Phi^\star - m^2\Phi\Phi^\star - \frac{1}{2}\lambda(\Phi\Phi^\star)^2 - \frac{1}{3m_1^2}(\Phi\Phi^\star)^3 + \cdots = T - V \tag{1.55}$$

where T is the kinetic energy density and V is the potential energy density. The terms which are higher than second powers of the bilinear combination $\Phi\Phi^\star$ require on dimensional grounds couplings with dimension of negative powers of mass.†

Using the formalism of the previous section we derive the equation of motion (since we have two physical degrees of freedom, the variations in Φ and $\Phi^\star$ are independent):

$$(\Box + m^2)\Phi + \lambda\Phi\Phi^\star\Phi + \frac{1}{m_1^2}(\Phi\Phi^\star)^2\Phi + \cdots = 0 \tag{1.56}$$

In the limit of vanishing couplings (free fields) we get the Klein–Gordon equation. The conserved $U(1)$ current

$$j_\mu(x) = \mathrm{i}q(\Phi^\star\partial_\mu\Phi - \Phi\partial_\mu\Phi^\star) \tag{1.57}$$

† In units $c = 1$, the action has the dimension $[S] = \mathrm{g}\cdot\mathrm{cm} = [\hbar]$. Thus $[\mathcal{L}] = \mathrm{g}\cdot\mathrm{cm}^{-3}$, $[\Phi] = (\mathrm{g}/\mathrm{cm})^{1/2}$ and the coefficients $[m^2] = \mathrm{cm}^{-2}$, $[\lambda] = (\mathrm{g}\cdot\mathrm{cm})^{-1}$, $[m_1^2] = \mathrm{g}^2$. In units $c = \hbar = 1$ we have $[\mathcal{L}] = \mathrm{g}^4$, $[\Phi] = \mathrm{g}$, $[m^2] = [m_1^2] = \mathrm{g}^2$ and S and λ are dimensionless.

and the canonical energy–momentum tensor

$$\Theta_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi^\star + \partial_\nu \Phi \partial_\mu \Phi^\star - g_{\mu\nu}(\partial_\rho \Phi \partial^\rho \Phi^\star) + g_{\mu\nu} m^2 \Phi \Phi^\star + \tfrac{1}{2}\lambda g_{\mu\nu}(\Phi\Phi^\star)^2 + \cdots \tag{1.58}$$

are obtained from (1.16) and (1.33), respectively. This tensor can be written in other forms, for example, by adding the term $\partial_\rho(g_{\mu\nu}\Phi\partial^\rho\Phi^\star - g_{\rho\nu}\Phi\partial^\mu\Phi^\star)$ and using the equation of motion. Invariance under Lorentz transformation gives the conserved current (1.48).

The $U(1)$ symmetry of our scalar field theory can be realized in the so-called Wigner mode or in the Goldstone–Nambu mode, i.e. it can be spontaneously broken. The phenomenon of spontaneous breaking of global symmetries is well known in condensed matter physics. The standard example of a physical theory with spontaneous symmetry breakdown is the Heisenberg ferromagnet, an infinite array of spin $\frac{1}{2}$ magnetic dipoles. The spin–spin interactions between neighbouring dipoles cause them to align. The hamiltonian is rotationally invariant but the ground state is not: it is a state in which all the dipoles are aligned in some arbitrary direction. So for an infinite ferromagnet there is an infinite degeneracy of the vacuum.

We say that the symmetry G is spontaneously broken if the ground state (usually called the vacuum) of a physical system with symmetry G (i.e. described by a lagrangian which is symmetric under G) is not invariant under the transformations G. Intuitively speaking, the vacuum is filled with scalars (with zero four-momenta) carrying the quantum number (s) of the broken symmetry. Since we do not want spontaneously to break the Lorentz invariance, spontaneous symmetry breaking can be realized only in the presence of scalar fields (fundamental or composite).

Let us return to the question of spontaneous breaking of the $U(1)$ symmetry in the theory defined by the lagrangian (1.55). The minimum of the potential occurs for the classical field configurations such that

$$\frac{\partial V}{\partial \Phi} = \frac{\partial V}{\partial \Phi^\star} = 0 \tag{1.59}$$

For $m^2 > 0$ the minimum of the potential exists for $\Phi = \Phi_0 = 0$ and $V(\Phi_0) = 0$. However, for $m^2 < 0$ (and $m_1^2 > 0$; study the potential for $m_1^2 < 0$!) the solutions to these equations are (note that λ must be positive for the potential to be bounded from below)

$$\Phi_0(\Theta) = \exp(\mathrm{i}\Theta)\left(\frac{-m^2}{\lambda}\right)^{1/2}\left[1 + \mathcal{O}\left(\frac{m^2}{\lambda^2 m_1^2}\right)\right] \tag{1.60}$$

with $V(\Phi_0) = -(m^4/2\lambda)$. The dependence of the potential on the field Φ is shown in Fig. 1.1. Thus, the $U(1)$ symmetric state is no longer the ground state of this

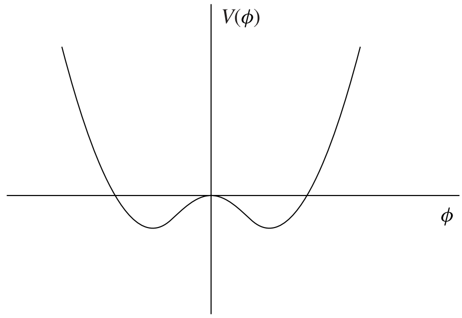


Fig. 1.1.

theory. Instead, the potential has its minimum for an infinite, degenerate set of states (one often calls it a 'flat direction') connected to each other by the $U(1)$ rotations. In this example we can choose any one of these vacua as the ground state of our theory with spontaneously broken $U(1)$ symmetry.

Physical interpretation of this model (and of the negative mass parameter m^2) is obtained if we expand the lagrangian (1.55) around the true ground state, i.e. rewrite it in terms of the fields (we choose $\Theta = 0$ and put $m_1^2 = \infty$): $\tilde{\Phi}(x) = \Phi(x) - \Phi_0$. After a short calculation we see that the two dynamical degrees of freedom, $\tilde{\phi}(x)$ and $\tilde{\chi}(x)$ (see (1.54)), have the mass parameters $(-m^2)$ (i.e. positive) and zero, respectively (see Chapter 11 for more details). The $\tilde{\chi}(x)$ describes the so-called Goldstone boson – a massless mode whose presence is related to the spontaneous breaking of a continuous global symmetry. The degenerate set of vacua (1.60) requires massless excitations to transform one vacuum into another one. We shall return to this subject later (Chapter 9).

In theories with spontaneously broken symmetries the currents remain conserved but the charges corresponding to the broken generators of the symmetry group are only formal in the sense that, generically, the integral (1.17) taken over all space does not exist. Still, the Poisson brackets (or the commutators) of a charge with a field can be defined provided we integrate over the space after performing the operation at the level of the current (see, for example, Bernstein (1974)).

Spinor fields

The basic objects which describe spin $\frac{1}{2}$ particles are two-component anticommuting Weyl spinors λ_α and $\bar{\chi}^{\dot{\alpha}}$, transforming, respectively, as representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group or, in other words, as two inequivalent representations of the $SL(2, C)$ group of complex two-dimensional matrices M

of determinant 1:

$$\lambda'_\alpha(x') = M_\alpha{}^\beta \lambda_\beta(x) \tag{1.61}$$

$$\bar{\chi}'^{\dot{\alpha}}(x') = \left(M^{\dagger-1}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\chi}^{\dot{\beta}}(x) \tag{1.62}$$

where x' and x are connected by the transformation (1.46),

$$M = \exp\left[-(\mathrm{i}/4)\omega_{\mu\nu}\sigma^{\mu\nu}\right] \tag{1.63}$$

and the two-dimensional matrices $\sigma^{\mu\nu}$ are defined in Appendix A. We use the same symbol for two- and four-dimensional matrices $\sigma^{\mu\nu}$ since it should not lead to any confusion in every concrete context. Two other possible representations of the $SL(2, C)$ group denoted as λ^α and $\bar{\chi}_{\dot{\alpha}}$ and transforming through matrices $(M^{-1})^T$ and M^* are unitarily equivalent to the representations (1.61) and (1.62), respectively. The corresponding unitary transformations are given by:

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \qquad \bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} \tag{1.64}$$

where the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ are defined in Appendix A. Since transition from the representation (1.61) to (1.62) involves complex conjugation it is natural to denote one of them with a bar. The distinction between the undotted and dotted indices is made to stress that the two representations are inequivalent. The pairwise equivalent representations correspond to lower and upper indices. The choice of (1.61) and (1.62) as fundamental representations is dictated by our convention which identifies (A.32) with the matrix M in (A.16). Those unfamiliar with two-component spinors and their properties should read Appendix A before starting this subsection.

Scalars of $SL(2, C)$ can now be constructed as antisymmetric products $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$ and $(0, \frac{1}{2}) \otimes (0, \frac{1}{2})$. They have, for example, the structure $\varepsilon^{\alpha\beta}\lambda_\beta\phi_\alpha = \lambda^\alpha\phi_\alpha = -\lambda_\alpha\phi^\alpha \equiv \lambda\cdot\phi$ or $\bar{\chi}^{\dot{\beta}}\bar{\eta}^{\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\eta^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}} \equiv \bar{\chi}\cdot\bar{\eta}$. Thus, the tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ are metric tensors for the spinor representations. A four-vector is a product of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, for example, $\varepsilon^{\alpha\beta}\lambda_\beta(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}} = \lambda^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}$ or $\bar{\chi}_{\dot{\alpha}}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,\lambda_\beta$ (see Appendix A for the definitions of σ^μ and $\bar{\sigma}^\mu$).

We suppose now that a physical system can be described by a Weyl field (or a set of Weyl fields) λ_α which transforms as $(\frac{1}{2}, 0)$ under the Lorentz group. Its complex conjugates are $\bar{\lambda}_{\dot{\alpha}} \equiv (\lambda_\alpha)^\star$ (see the text above (1.64)) and $\bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}$ which transforms as $(0, \frac{1}{2})$ representation. (Note, that numerically $\left\{\varepsilon^{\alpha\beta}\right\} = \{\varepsilon^{\dot{\alpha}\dot{\beta}}\} = \mathrm{i}\sigma^2$.) It is clear that $SL(2, C)$-invariant kinetic terms read

$$\mathcal{L} = \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar{\lambda} = \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda \tag{1.65}$$

where the two terms are equal up to a total derivative. The factor i has been introduced for hermiticity of the lagrangian since one assumes that the spinor components are anticommuting c-numbers. Indeed, this is what we have to assume for these classical fields when we follow the path integral approach to field quantization (see Chapter 2).

Furthermore, if the field λ carries no internal charges (or transforms as a real representation of an internal symmetry group), one can construct the Lorentz scalars $\lambda\lambda = \lambda^\alpha\lambda_\alpha$ and $\bar{\lambda}\bar{\lambda} = \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$ and introduce the so-called Majorana mass term

$$\mathcal{L}_m = -\tfrac{1}{2}m\left(\lambda\lambda + \bar{\lambda}\bar{\lambda}\right) \tag{1.66}$$

The most general hermitean combination, $-\frac{1}{2}m_1(\lambda\lambda + \bar{\lambda}\bar{\lambda}) - (\mathrm{i}/2)m_2(\lambda\lambda - \bar{\lambda}\bar{\lambda})$, can be recast in the above form with $m = (m_1^2 + m_2^2)^{1/2}$ by the transformation $\lambda \to \exp(\mathrm{i}\varphi)\lambda$ with $\exp(2\mathrm{i}\varphi) = (m_1 - \mathrm{i}m_2)/(m_1^2 + m_2^2)^{1/2}$. The term $\mathcal{L}_m$ does not vanish since λ_αs are anticommuting c-numbers. Finally, we remark that a system described by a single Weyl spinor has two degrees of freedom, with the equation of motion obtained from the principle of minimal action (variations with respect to λ and $\bar{\lambda}$ are independent):

$$\left(\mathrm{i}\bar{\sigma}^\mu\partial_\mu\lambda\right)_{\dot{\alpha}} = m\bar{\lambda}_{\dot{\alpha}} \tag{1.67}$$

We now consider the case of a system described by a set of Weyl spinors λ^i_α which transform as a complex representation R (see Appendix E for the definition of complex and real representations) of a group of internal symmetry. Such theory is called chiral. Then, the kinetic lagrangian remains as in (1.65) but no invariant mass term can be constructed since Lorentz invariants $\lambda^i\lambda^j$ and $\bar{\lambda}^i\bar{\lambda}^j$ are not invariant under the symmetry group. Note that in this case the spinors $\bar{\lambda}^{\dot{\alpha}}$ (and $\bar{\lambda}_{\dot{\alpha}}$), i.e. the spinors transforming as $(0, \frac{1}{2})$ (or its unitary equivalent) under the Lorentz transformations, form the representation $R^\star$, i.e. complex conjugate to R.

There are also interesting physical systems that consist of pairs of independent sets of Weyl fields λ_α and λ^c_α (the internal symmetry indices i, for example, λ^i_α, and the summation over them are always implicit), both being $(\frac{1}{2}, 0)$ representations of the Lorentz group but with λ^c_α transforming as the representation $R^\star$, complex conjugate to R. Such theory is called vector-like. We call such spinors charge conjugate to each other. The operation of charge conjugation will be discussed in more detail in Section 1.5. The mass term can then be introduced and the full lagrangian of such a system then reads:

$$\mathcal{L} = \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda + \mathrm{i}\lambda^c\sigma^\mu\partial_\mu\bar{\lambda}^c - m\left(\lambda\lambda^c + \bar{\lambda}\bar{\lambda}^c\right) \tag{1.68}$$

Similarly to the case of Majorana mass, the mass term $-\mathrm{i}m'(\lambda\lambda^c - \bar{\lambda}\bar{\lambda}^c)$, if present, can be rotated away by the appropriate phase transformation. Remember that $\lambda^c\sigma^\mu\partial_\mu\bar{\lambda}^c = \bar{\lambda}^c\bar{\sigma}^\mu\partial_\mu\lambda^c$ (up to a total derivative), so both fields λ and λ^c enter the lagrangian in a fully symmetric way. The mass term, the so-called Dirac mass, is now invariant under the internal symmetry group. The equations of motion which follow from (1.68) are:

$$\left(\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu}\lambda\right)_{\dot{\alpha}} = m\bar{\lambda}^{c}_{\dot{\alpha}} \tag{1.69}$$

and similarly for λ^c. Thus, the mass term mixes positive energy solutions for λ with negative ones for λ^c (which are represented by positive energy solutions for $\bar{\lambda}^c_{\dot{\alpha}}$) and vice versa. In accord with the discussion of Appendix A, this can be interpreted as a mixing between different helicity states of the same massless particle.

As we shall discuss in more detail later, the electroweak theory is chiral and quantum chromodynamics is vector-like. In both cases, it is convenient to take as the fundamental fields of the theory Weyl spinors which transform as $(\frac{1}{2}, 0)$ representations of $SL(2, C)$. We recall that (as discussed in Appendix A) positive energy classical solutions for spinors λ^i_αs describe massless fermion states with helicity $-\frac{1}{2}$ and, therefore, λ^i_αs are called left-handed chiral fields. In the electroweak theory (see Chapter 12) all the left-handed physical fields are included in the set of λ^i_αs which transforms then as a reducible, complex representation R of the symmetry group $G = SU(2) \times U(1)$ ($R \neq R^\star$). To be more specific, in this case the set λ^i_α includes pairs of left-handed particles with opposite electric charges such as, for example, e^-, e^+ and quarks with electric charges $\mp\frac{1}{3}$, $\pm\frac{2}{3}$, i.e. this set forms a real representation of the electromagnetic $U(1)$ gauge group, but transforms as a complex representation R under the full $SU(2) \times U(1)$ group. Negative energy classical solutions for λ^i_αs, in the Dirac sea interpretation, describe massless helicity $+\frac{1}{2}$ states transforming as $R^\star$. Each pair of states described by λ^i_α is connected by the CP transformation (see later) and can therefore be termed particle and antiparticle. These pairs consist of the states with opposite helicities. (Whether within a given reducible representation contained in R the $-\frac{1}{2}$ or $+\frac{1}{2}$ helicity states are called particles is, of course, a matter of convention.)

An equivalent description of the same theory in terms of the fields $\bar{\lambda}^{i\dot{\alpha}}$ transforming as $(0, \frac{1}{2})$ representations of $SL(2, C)$ and as $R^\star$ under the internal symmetry group can be given. In this formulation positive (negative) energy solutions for $\bar{\lambda}^{i\dot{\alpha}}$s describe all $+\frac{1}{2}$ $(-\frac{1}{2})$ helicity massless states of the theory (formerly described as negative (positive) energy solutions for λ^i_αs).

In the second case (QCD) the fundamental fields of the theory, the left-handed λ^i_αs and λ^{ci}_αs, transform as $\mathbf{3}$ and $\mathbf{3}^\star$ of $SU(3)$, respectively, and are related by charge conjugation. Positive energy solutions for λ^i_αs are identified with $-\frac{1}{2}$ helicity states of massless quarks and positive energy solutions for λ^{ci}_αs describe $-\frac{1}{2}$ helicity states of massless antiquarks. Negative energy solutions for λ^i_αs and λ^{ci}_αs describe, respectively, antiquarks and quarks with helicity $+\frac{1}{2}$. Since the representation R is real, mass terms are possible.

For a vector-like theory, like QCD, with charge conjugate pairs of Weyl spinors λ and λ^c, both transforming as $(\frac{1}{2}, 0)$ of the Lorentz group but as R and $R^\star$ representations of an internal symmetry group, it is convenient to use the Dirac four-component spinors (in the presence of the mass term in (1.68) λ and $\bar{\lambda}^c$ can be interpreted as describing opposite helicity (chirality) states of the same massless particle which mix with each other due to the mass term)

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{c\dot{\alpha}} \end{pmatrix} \qquad \text{or} \qquad \Psi^c = \begin{pmatrix} \lambda^c_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{1.70}$$

transforming as R and $R^\star$, respectively, under the group of internal symmetries. The lagrangian (1.68) can be rewritten as

$$\mathcal{L} = \mathrm{i}\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \tag{1.71}$$

where we have introduced $\bar{\Psi} \equiv (\lambda^{c\alpha}, \bar{\lambda}_{\dot{\alpha}})$ to recover the mass term of (1.68). Writing $\bar{\Psi} = \Psi^\dagger \gamma^0$ and comparing the kinetic terms of (1.68) and (1.71) we define the matrices γ^μ. They are called Dirac matrices in the chiral representation and are given in Appendix A. Also in Appendix A we derive the Lorentz transformations for the Dirac spinors. Another possible hermitean mass term, $-\mathrm{i}m\bar{\Psi}\gamma_5\Psi$, can be rotated away by the appropriate phase transformation of the spinors λ and λ^c.

Four-component notation can also be introduced and, in fact, is quite convenient for Weyl spinors λ_α and $\bar{\chi}^{\dot{\alpha}}$ transforming as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ under the Lorentz group. We define four-component chiral fermion fields as follows

$$\Psi_{\mathrm{L}} = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \qquad \Psi_{\mathrm{R}} = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{1.72}$$

They satisfy the equations

$$\left.\begin{aligned} \Psi_{\mathrm{L}} &= \frac{1-\gamma_5}{2}\Psi_{\mathrm{L}} \equiv P_{\mathrm{L}}\Psi_{\mathrm{L}}, \qquad & P_{\mathrm{R}}\Psi_{\mathrm{L}} = 0 \\ \Psi_{\mathrm{R}} &= \frac{1+\gamma_5}{2}\Psi_{\mathrm{R}} \equiv P_{\mathrm{R}}\Psi_{\mathrm{R}}, \qquad & P_{\mathrm{L}}\Psi_{\mathrm{R}} = 0 \end{aligned}\right\} \tag{1.73}$$

which define the matrix γ_5 in the chiral representation. By analogy with the nomenclature introduced for λ_α and $\bar{\chi}^{\dot{\alpha}}$ the fields Ψ_{L} and Ψ_{R} are called left- and right-handed chiral fields, respectively. Possible Lorentz invariants are of the form $\bar{\Psi}_{\mathrm{R}}\Psi_{\mathrm{L}} \equiv \chi^\alpha\lambda_\alpha$ and $\bar{\Psi}_{\mathrm{L}}\Psi_{\mathrm{R}} \equiv \bar{\lambda}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$. The invariants $\lambda^\alpha\lambda_\alpha$ and $\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$ can be written as $\Psi_{\mathrm{L}}^T C \Psi_{\mathrm{L}}$ and $\bar{\Psi}_{\mathrm{L}} C \bar{\Psi}_{\mathrm{L}}^T$, respectively, with the matrix C defined in (A.104). In particular, for a set of Weyl spinors λ_α and their complex conjugate

$\bar{\lambda}^{\dot{\alpha}}$, transforming as R and $R^\star$, respectively, under the internal symmetry group, the sets†

$$\Psi^i_{\mathrm{L}} = \begin{pmatrix} \lambda^i_\alpha \\ 0 \end{pmatrix} \qquad \Psi^{ci}_{\mathrm{R}} = \begin{pmatrix} 0 \\ \bar{\lambda}^{i\dot{\alpha}} \end{pmatrix} \tag{1.74}$$

also transform as R and $R^\star$, respectively. For instance, in ordinary QED if Ψ_{L} has a charge -1 $(+1)$ under the $U(1)$ symmetry group, we can identify it with the left-handed chiral electron (positron) field. The Ψ^c_{R} is then the chiral right-handed positron (electron) field.

Lorentz invariants $\overline{\Psi^i_{\mathrm{L}}}\Psi^{cj}_{\mathrm{R}}$ and $\overline{\Psi^{ci}_{\mathrm{R}}}\Psi^j_{\mathrm{L}}$ (or a combination of them) are generically not invariant under the internal symmetry group but such invariants can be constructed by coupling them, for example, to scalar fields transforming properly under this group (see later). The kinetic part of the lagrangian for the chiral fields is the same as in (1.71) but no mass term is allowed.

We note that $\gamma^5 = \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ and that the following algebra is satisfied:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{1.75}$$

Finally, for objects represented by a Weyl field λ_α with no internal charges or transforming as a real representation of an internal symmetry group we can introduce the so-called Majorana spinor

$$\Psi_{\mathrm{M}} = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{1.76}$$

In this notation the lagrangian given by (1.65) and (1.66) reads

$$\mathcal{L}_{\mathrm{M}} = \frac{\mathrm{i}}{2}\bar{\Psi}_{\mathrm{M}}\gamma^\mu\partial_\mu\Psi_{\mathrm{M}} - \frac{m}{2}\bar{\Psi}_{\mathrm{M}}\Psi_{\mathrm{M}} \tag{1.77}$$

The equations of motion for free four-dimensional spinors have the form

$$\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\Psi = 0 \tag{1.78}$$

for the Dirac spinors and with $m = 0$ for the chiral spinors. In the latter case it is supplemented by the conditions (1.73). The solutions to the equations of motion for the Weyl spinors and for the four-component spinors in the chiral and Dirac representations for the γ^μ matrices are given in Appendix A.

† The notation in (1.74) has been introduced by analogy with (1.70). Note, however, that the superscript 'c' in the spinors Ψ^{ci}_{R} is now used even though λ_α and $\bar{\lambda}^{\dot{\alpha}}$ are not charge conjugate to each other (since they are in different Lorentz group representations). Thus, unlike the spinors in (1.70) the spinors in (1.74) are defined for both chiral and vector-like spectra of the fundamental fields. For a vector-like spectrum we have $\Psi^{ci}_{\mathrm{R}} = (\Psi^{ci})_{\mathrm{R}}$.

A few more remarks on the free Dirac spinors should be added here. The momentum conjugate to Ψ is

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi_\alpha)} = \mathrm{i}\Psi_\alpha^\dagger \tag{1.79}$$

For the hamiltonian density we get

$$\mathcal{H} = \Pi \partial_0 \Psi - \mathcal{L} = \Psi^\dagger \gamma^0 (-\mathrm{i}\boldsymbol{\gamma} \cdot \boldsymbol{\partial} + m)\Psi \tag{1.80}$$

and the energy–momentum tensor density reads

$$\Theta^{\mu\nu} = \mathrm{i}\bar{\Psi}\gamma^\mu \frac{\partial}{\partial x_\nu}\Psi \tag{1.81}$$

((1.81) is obtained after using the equations of motion). The angular momentum density

$$M^{\kappa,\mu\nu} = \mathrm{i}\bar{\Psi}\gamma^\kappa \left(x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu} - \frac{\mathrm{i}}{2}\sigma^{\mu\nu} \right) \Psi \tag{1.82}$$

(where four-dimensional $\sigma_{\mu\nu} = (\mathrm{i}/2)[\gamma_\mu, \gamma_\nu]$) gives the conserved angular momentum tensor

$$M^{\mu\nu} = \int \mathrm{d}^3 x \, M^{0\mu\nu} \tag{1.83}$$

The lagrangian (1.65) can be extended to include various interaction terms. For instance, for a neutral (with respect to the internal quantum numbers) Weyl spinor λ interacting with a complex scalar field ϕ (also neutral) the dimension four ($[m]^4$ in units $\hbar = c = 1$) Yukawa interaction terms are:

$$\lambda\lambda\phi, \qquad \lambda\lambda\phi^\star, \qquad \bar{\lambda}\bar{\lambda}\phi^\star, \qquad \bar{\lambda}\bar{\lambda}\phi \tag{1.84}$$

Similar terms can be written down for several spinor fields ($\lambda, \chi, \ldots$) interacting with scalar fields. Additional symmetries, if assumed, would further constrain the allowed terms. These may be internal symmetries or, for instance, supersymmetry (see Chapter 15). The simplest supersymmetric theory, the so-called Wess–Zumino model, describing interactions of one complex scalar field $\phi(x)$ with one chiral fermion $\lambda(x)$ has the following lagrangian†

$$\begin{aligned}\mathcal{L}_{\mathrm{WZ}} = {} & \mathrm{i}\bar{\lambda}\bar{\sigma} \cdot \partial\lambda - \tfrac{1}{2}m(\lambda\lambda + \bar{\lambda}\bar{\lambda}) + \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi \\ & - g(\phi\lambda\lambda + \phi^\star\bar{\lambda}\bar{\lambda}) - gm\phi^\star\phi(\phi^\star + \phi) - g^2(\phi^\star\phi)^2 \end{aligned} \tag{1.85}$$

The absence of otherwise perfectly allowed coupling $-g'(\phi^\star\lambda\lambda + \phi\bar{\lambda}\bar{\lambda})$ is due to supersymmetry. In the four-component notation we get $(\phi\lambda\lambda + \phi^\star\bar{\lambda}\bar{\lambda}) = \bar{\Psi}_{\mathrm{M}} P_{\mathrm{L}} \Psi_{\mathrm{M}}\phi + \bar{\Psi}_{\mathrm{M}} P_{\mathrm{R}} \Psi_{\mathrm{M}}\phi^\star$, where Ψ_{M} is given by (1.76).

† This form of the lagrangian follows after eliminating auxiliary fields via their (algebraic) equations of motion, see Chapter 15.

In general, in the four-component notation the Yukawa terms have the generic structure $(\bar{\Psi}^i_{\mathrm{R}}\Psi^j_{\mathrm{L}} \pm \bar{\Psi}^j_{\mathrm{L}}\Psi^i_{\mathrm{R}})\Phi^k(\Phi^{k\star})$ where the chiral fields are defined by (1.72) and the internal symmetry indices are properly contracted.

If λ^is transform as a representation R (with hermitean generators T^a) of an internal symmetry group, the interactions with vector fields A^a_μ in the adjoint representation of the group (as for the gauge fields, see Section 1.3) take the form:

$$\bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu = \overline{\Psi^i_{\mathrm{L}}}\gamma^\mu(T^a)_i{}^j\Psi_{\mathrm{L}j}A^a_\mu$$

with Ψ_{L} given by (1.72).

For a vector-like theory, the lagrangian (1.68) in which λ^{ci} transforms as a representation R^* of the group (with generators $-T^{a*}$, see Appendix E) can be supplemented by terms like:

$$\begin{aligned}
&\bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu + \bar{\lambda}^c_i\bar{\sigma}^\mu(-T^{a*})^i{}_j\lambda^{cj}A^a_\mu \\
&\qquad = \bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu + \lambda^{cj}\sigma^\mu(T^a)_j{}^i\bar{\lambda}^c_j A^a_\mu \\
&\qquad = \bar{\Psi}^i\gamma^\mu(T^a)_i{}^j\Psi_j A^a_\mu
\end{aligned}$$

In the last form (in which $\Psi(x)$ is given by (1.70)) this interaction can be added to the lagrangian (1.71). For the abelian field A_μ this is the same as the minimal coupling of the electron to the electromagnetic field ($p_\mu \to p_\mu + eA_\mu$). Other interaction terms of a Dirac field (1.70), for example, with a scalar field $\Psi(x)$ or a vector field $A_\mu(x)$, are possible as well. A Lorentz-invariant and hermitean interaction part of the lagrangian density may consist, for example, of the operators:

$$\bar{\Psi}\Psi\Phi, \qquad \bar{\Psi}\gamma^5\Psi\Phi, \qquad \bar{\Psi}\gamma^5\gamma_\mu A^\mu\Psi \tag{1.86}$$

$$\frac{1}{M}\bar{\Psi}\Psi\Phi\Phi, \qquad \frac{1}{M}\bar{\Psi}\gamma_\mu A^\mu\Psi\Phi, \qquad \frac{1}{M}\bar{\Phi}\Phi A_\mu A^\mu, \qquad \frac{1}{M^2}\bar{\Psi}\Psi\bar{\Psi}\Psi \qquad \text{etc.} \tag{1.87}$$

The first two terms are Yukawa couplings (scalar and pseudoscalar). The terms in (1.87) require dimensionful coupling constants, with the mass scale denoted by M. The last term, describing the self-interactions of a Dirac field, is called the Fermi interaction. It is straightforward to include the interaction terms in the equation of motion.

1.3 Gauge field theories

$U(1)$ *gauge symmetry*

As a well-known example, we consider gauge invariance in electrodynamics. It is often introduced as follows: consider a free-field theory of n Dirac particles with the lagrangian density

$$\mathcal{L} = \sum_{i=1}^{n} \left(\bar{\Psi}_i \mathrm{i}\gamma^\mu \partial_\mu \Psi_i - m\bar{\Psi}_i \Psi_i \right) \tag{1.88}$$

Then define a $U(1)$ group of transformations on the fields by

$$\Psi_i'(x) = \exp(-\mathrm{i}q_i \Theta)\Psi_i(x) \tag{1.89}$$

where the parameter q_i is an eigenvalue of the generator Q of $U(1)$ and numbers the representation to which the field Ψ_i belongs. The lagrangian (1.88) is invariant under that group of transformations.

By Noether's theorem (see Section 1.1), the $U(1)$ symmetry of the lagrangian (1.88) implies the existence of the conserved current

$$j_\mu(x) = \sum_i q_i \bar{\Psi}_i \gamma_\mu \Psi_i \tag{1.90}$$

and therefore conservation of the corresponding charge (1.17). We now consider gauge transformations (local phase transformations in which Θ is allowed to vary with x)

$$\Psi_i'(x) = \exp[-\mathrm{i}q_i \Theta(x)]\Psi_i(x) \tag{1.91}$$

It is straightforward to verify that lagrangian (1.88) is not invariant under gauge transformations because transformation of the derivatives of fields gives extra terms proportional to $\partial_\mu \Theta(x)$. To make the lagrangian invariant one must introduce a new term which can compensate for the extra terms. Equivalently, one should find a modified derivative $D_\mu \Psi_i(x)$ which transforms like $\Psi_i(x)$

$$[D_\mu \Psi_i(x)]' = \exp[-\mathrm{i}q_i \Theta(x)] D_\mu \Psi_i(x) \tag{1.92}$$

and replace ∂_μ by D_μ in the lagrangian (1.88).

The derivative D_μ is called a covariant derivative. The covariant derivative is constructed by introducing a vector (gauge) field $A_\mu(x)$ and defining†

$$D_\mu \Psi_i(x) = [\partial_\mu + \mathrm{i}q_i e A_\mu(x)]\Psi_i(x) \tag{1.93}$$

where e is an arbitrary positive constant. The transformation rule (1.92) is ensured if the gauge field $A_\mu(x)$ transforms as:

$$A_\mu'(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \Theta(x) \tag{1.94}$$

Covariant derivatives play an important role in gauge theories. In particular

† The sign convention in the covariant derivative is consistent with the standard minimal coupling in the classical limit (Bjorken and Drell (1964); note that $e < 0$ in this reference).

one may construct new covariant objects by repeated application of covariant derivatives. For the antisymmetric product of two derivatives

$$[D_\mu, D_\nu]\Psi_i = D_\mu(D_\nu\Psi_i) - D_\nu(D_\mu\Psi_i)$$

one gets

$$[D_\mu, D_\nu]\Psi_i = \mathrm{i}q_i e[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)]\Psi_i \tag{1.95}$$

By comparing the gauge transformation properties of both sides in (1.95) we conclude that

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{1.96}$$

is gauge-invariant. The field strength tensor $F_{\mu\nu}$ can be used to complete the lagrangian with the gauge-invariant kinetic energy term for the gauge field itself (assume one fermion field for definiteness)

$$\mathcal{L} = \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi - \tfrac{1}{4}F_{\mu\nu}^2 \tag{1.97}$$

The Euler–Lagrange equations of motion are now

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= eq\bar{\Psi}\gamma^\nu\Psi \\ (\mathrm{i}\not{\partial} - m)\Psi &= eq\not{A}\Psi \end{aligned}$$

or

$$(\mathrm{i}\not{D} - m)\Psi = 0$$

and identifying e ($e > 0$) with electric charge we recognize in our theory the Maxwell–Dirac electrodynamics.

Of course, $U(1)$ gauge invariance implies global $U(1)$ invariance and the conservation of current (1.90) and the corresponding charge (1.17). It also implies absence of the gauge field mass term $m^2 A_\mu A^\mu$. But such a term does not break the global $U(1)$ symmetry.

Exactly analogous considerations apply to scalar field theories with $U(1)$ gauge symmetry and to theories with Weyl fermions.

Non-abelian gauge symmetry

To construct a non-abelian gauge field lagrangian we repeat the same steps. We start with the free-field lagrangian for Dirac fields Ψ which transform according to a representation of some non-abelian Lie group

$$\Psi'(x) = \exp(-\mathrm{i}\Theta^\alpha T^\alpha)\Psi(x) \tag{1.98}$$

where the T^α are the hermitean matrix representations of the generators of the group, appropriate for the fields Ψ and satisfying relations (1.11).

The free-field lagrangian is invariant under global group transformations (1.98). Let us now consider the extension of the group G to a group of local gauge transformations. Generalizing the $U(1)$ case we seek a covariant derivative such that

$$[D_\mu \Psi(x)]' = \exp[-\mathrm{i}\Theta^\alpha(x)T^\alpha] D_\mu \Psi(x) = U(x) D_\mu \Psi(x) \tag{1.99}$$

By analogy with (1.93) we expect it to be given by a combination of the normal derivative and a transformation on the fields Ψ

$$D_\mu \Psi(x) = [\partial_\mu + A_\mu(x)]\Psi(x) \tag{1.100}$$

where A_μ is a (antihermitean) element of the Lie algebra†

$$A_\mu(x) = +\mathrm{i}g A^\alpha_\mu(x) T^\alpha \tag{1.101}$$

Thus we need gauge fields in the number given by the number of generators of the group. The constant g is arbitrary. In classical physics, g is a dimensionful parameter and therefore can always be scaled to one (similar arguments apply to (1.93)). At the quantum level g is relevant since quantum theory contains the Planck constant $\hbar$ and g is dimensionless in units $\hbar = 1$. Equivalently, at the quantum level the normalization of the field is fixed by the normalization of the single particle state. With the form (1.100) the condition (1.99) is satisfied if the gauge transformation rule for $A_\mu(x)$ is as follows:

$$\begin{aligned} A'_\mu(x) &= U(x)A_\mu(x)U^{-1}(x) - [\partial_\mu U(x)]U^{-1}(x) \\ &= U(x)[A_\mu(x) + \partial_\mu]U^{-1}(x) \end{aligned} \tag{1.102}$$

or for an infinitesimal transformation

$$\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = \partial_\mu \Theta(x) + [A_\mu(x), \Theta(x)] \tag{1.103}$$

where

$$\Theta(x) = +\mathrm{i}\Theta^\alpha(x)T^\alpha \tag{1.104}$$

is an infinitesimal gauge parameter in the matrix notation. The corresponding transformation for the gauge fields $A^\alpha_\mu(x)$ reads

$$\delta A^\alpha_\mu(x) = (1/g)\partial_\mu \Theta^\alpha(x) + c_{\alpha\beta\gamma}\Theta^\beta(x)A^\gamma_\mu(x) \tag{1.105}$$

It can be checked that transformations (1.103) and (1.105) form a group. It is also seen from (1.105) that under global transformations ($\partial_\mu\Theta = 0$) the gauge fields transform according to the adjoint representation of the group with $(T^\alpha)_{\beta\gamma} = -\mathrm{i}c_{\alpha\beta\gamma}$.

† The sign convention for the non-abelian case is consistent with the convention (1.93); this is convenient for embedding electromagnetism into the electroweak theory.

We similarly can construct new covariant quantities by repeated application of the covariant derivative. We get, for instance,

$$[D_\mu, D_\nu]\Psi(x) = (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)])\Psi(x) \tag{1.106}$$

Thus, the antisymmetric tensor (the field strength)

$$G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{1.107}$$

is a covariant quantity transforming under gauge transformations as follows:

$$G'_{\mu\nu}(x) = U(x)G_{\mu\nu}(x)U^{-1}(x) \tag{1.108}$$

For an infinitesimal transformation we get

$$\delta G_{\mu\nu}(x) = [G_{\mu\nu}(x), \Theta(x)] \tag{1.109}$$

It is obvious from (1.107) that the tensor $G_{\mu\nu}$ can be decomposed in terms of group generators

$$G_{\mu\nu} = \mathrm{i}gG^\alpha_{\mu\nu}T^\alpha \tag{1.110}$$

where

$$G^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - gc^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu \tag{1.111}$$

The transformation rule for $G^\alpha_{\mu\nu}(x)$ follows from (1.108) and (1.109).

Finally, we can generalize the definition of the covariant derivative to apply it to any Lie algebra element $\xi(x) = +\mathrm{i}\xi^\alpha(x)T^\alpha$. By applying the covariant derivative $D_\rho = \partial_\rho + A_\rho(x)$ to the object $\xi\Psi$ and insisting on the Leibnitz rule

$$D_\rho(\xi\Psi) = (D_\rho\xi)\Psi + \xi(D_\rho\Psi)$$

we get

$$D_\mu\xi = \partial_\mu\xi + [A_\mu, \xi] \tag{1.112}$$

Then the gauge transformation (1.103) can be written simply as

$$\delta A_\mu(x) = D_\mu\Theta(x) \tag{1.113}$$

Moreover, we have

$$[D_\mu, D_\nu]G_{\rho\sigma} = [G_{\mu\nu}, G_{\rho\sigma}] \tag{1.113a}$$

We are ready to write down the gauge-invariant lagrangian for a non-abelian gauge field theory with fermions. It reads

$$\mathcal{L} = +\frac{1}{2g^2}\,\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] + \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi \tag{1.114}$$

($\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$ is gauge-invariant because $\mathrm{Tr}[UG_{\mu\nu}G^{\mu\nu}U^{-1}] = \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$).

Terms of higher order in fields are not allowed for quantum field theory if it is to be renormalizable (Chapter 4). The term $G_\mu{}^\mu$ is zero and the possibility of a parity non-conserving term $\varepsilon^{\mu\nu\rho\delta}G_{\mu\nu}G_{\rho\delta}$ in the lagrangian will be discussed later. The pure gauge field term must be present in the lagrangian because we need the kinetic energy term to be quadratic in the gauge fields. Its gauge-invariant form then implies the presence of two gauge self-interaction terms of the order g and g^2, respectively

$$\frac{1}{2g^2}\operatorname{Tr}[G_{\mu\nu}G^{\mu\nu}] = -\tfrac{1}{4}(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu)^2 + g c^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \partial^\mu A^{\nu\alpha} - \tfrac{1}{4} g^2 c^{\alpha\beta\gamma} c^{\alpha\sigma\delta} A^\beta_\mu A^\gamma_\nu A^{\mu\sigma} A^{\nu\delta} \tag{1.115}$$

(we recall our normalization $\operatorname{Tr}[T^\alpha T^\beta] = \frac{1}{2}\delta^{\alpha\beta}$).

We end this section with a derivation of the Euler–Lagrange equations for gauge fields. One requires the action to be stationary with respect to small variations of the gauge fields

$$A_\mu \to A_\mu + \delta A_\mu \tag{1.116}$$

and its derivative

$$\delta(\partial_\nu A_\mu) = \partial_\nu(\delta A_\mu)$$

where we formally treat δA_μ as a covariant quantity transforming in an adjoint representation of the group

$$\delta A'_\mu(x) = U(x)\delta A_\mu(x) U^{-1}(x) \tag{1.117}$$

The change of the field strength (1.107) under the variation (1.116) can be expressed as follows:

$$G_{\mu\nu} \to G_{\mu\nu} + D_\mu(\delta A_\nu) - D_\nu(\delta A_\mu) \tag{1.118}$$

Here we have used definition (1.112) of a covariant derivative acting on a covariant object δA_μ. The variation of the gauge field action is

$$\delta \int \mathrm{d}^4 x \, \frac{1}{2g^2}\operatorname{Tr}[G_{\mu\nu}G^{\mu\nu}] = \frac{2}{g^2}\int \mathrm{d}^4 x \operatorname{Tr}[G^{\mu\nu}(D_\mu \delta A_\nu)] \tag{1.119}$$

(antisymmetry of $G_{\mu\nu}$ in μ and ν has been used). Using the relation

$$D_\mu \operatorname{Tr}[(G^{\mu\nu}\delta A_\nu)] = \operatorname{Tr}[(D_\mu G^{\mu\nu})\delta A_\nu] + \operatorname{Tr}[G^{\mu\nu}(D_\mu \delta A_\nu)] \tag{1.120}$$

and the invariance of $\operatorname{Tr}[G_{\mu\nu}\delta A^\nu]$ under gauge transformations (therefore, on the l.h.s. of (1.120) we may replace the covariant derivative by the ordinary derivative; this remark also helps to check relation (1.120)) we conclude, by integrating (1.120) and neglecting the surface terms at infinity on the l.h.s., that

$$\delta \int \mathrm{d}^4x \, \frac{1}{2g^2} \, \mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] = -\frac{2}{g^2} \int \mathrm{d}^4x \, \mathrm{Tr}[(D_\mu G^{\mu\nu})\delta A_\nu] \tag{1.121}$$

The remaining term

$$\delta \int \mathrm{d}^4x \, \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi$$

can be written as

$$\delta \int \mathrm{d}^4x \, \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi = 2 \int \mathrm{d}^4x \, \mathrm{Tr}[j_\mu \delta A^\mu] \tag{1.122}$$

where the current j_μ defined by the variation (1.122) reads

$$j_\mu(x) = \mathrm{i} j_\mu^\alpha(x) T^\alpha$$

with

$$j_\mu^\alpha = \bar{\Psi}\gamma_\mu T^\alpha \Psi \tag{1.123}$$

Combining (1.121) and (1.122) we get the equation of motion

$$(1/g^2) D_\mu G^{\mu\nu} - j^\nu = 0 \tag{1.124}$$

Applying the covariant derivative to (1.124) one derives the covariant divergence equation for the current (1.123)

$$D_\mu j^\mu(x) = 0 \tag{1.125}$$

Thus, gauge fields can only couple consistently to currents which are covariantly conserved. We also see from (1.125) that the non-abelian charge associated with the fermionic current is not a constant of motion; since gauge fields are not neutral their contribution must be included to find conserved charges.

1.4 From classical to quantum fields (canonical quantization)

In this book we use the path integral approach to the quantization of fields (see Chapter 2). However, for the sake of completeness and easy reference, in this section we briefly recall the canonical quantization, procedure (for more details see Bjorken & Drell (1964), Itzykson & Zuber (1980)).

Scalar fields

We begin with a scalar field theory and consider only free fields. The canonical quantization of interacting fields can be performed along similar lines in the so-called interaction picture (we refer the reader to the standard textbooks, for example, Bjorken & Drell (1965); see also Section 2.7). An arbitrary solution to the free Klein–Gordon equation (1.56) may be expanded as a Fourier integral over plane-wave solutions:

$$\Phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \left[a(\mathbf{k}) \exp(-\mathrm{i}k \cdot x) + a^\dagger(\mathbf{k}) \exp(\mathrm{i}k \cdot x)\right] \tag{1.126}$$

where

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2\pi \delta(k^2 - m^2)\theta(k_0) \tag{1.127}$$

is a Lorentz-invariant measure, $k_0 = E = +(\mathbf{k}^2 + m^2)^{1/2}$ and, for a real field $\Phi(x)$, the $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are complex conjugate to each other. In the canonical quantization, the fields and their conjugate momenta $\Pi = \partial_0\Phi$ are promoted to hermitean operators in the Heisenberg picture (in a Hilbert space of state vectors) by postulating for them canonical equal-time commutation relations, i.e. replacing the Poisson brackets in (1.21) by commutators, $\{\ ,\ \} \to -\mathrm{i}[\ ,\ ]$:

$$\left.\begin{aligned} &[\Pi_i(t, \mathbf{x}), \Phi_j(t, \mathbf{y})] = -\mathrm{i}\delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \\ &[\Phi_i(t, \mathbf{x}), \Phi_j(t, \mathbf{y})] = [\Pi_i(t, \mathbf{x}), \Pi_j(t, \mathbf{y})] = 0 \end{aligned}\right\} \tag{1.128}$$

(the indices (i, j) are internal symmetry indices).

It follows from the postulated relations (1.128) that the (hermitean conjugate to each other) operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ satisfy the following algebra:

$$\left.\begin{aligned} &\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = (2\pi)^3 2k_0\delta(\mathbf{k} - \mathbf{k}') \\ &[a(\mathbf{k}), a(\mathbf{k}')] = \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0 \end{aligned}\right\} \tag{1.129}$$

The ground state (the vacuum) is defined by the relations:

$$a(\mathbf{k})|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \tag{1.130}$$

The energy–momentum operator (1.40) can be rewritten in terms of the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$:

$$P^\mu = \int \mathrm{d}^3x\, \Theta^{0\mu}(\mathbf{x}) = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} k^\mu \left[a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})\right] \tag{1.131}$$

Since $a(\mathbf{k})a^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) + (2\pi)^3 2k_0\delta(\mathbf{0})$, we see that the vacuum energy is infinite. For most physical questions (apart from the problem of the cosmological constant) the vacuum energy is unobservable. Therefore, we redefine the hamiltonian as

$$P^0 \to P^0 - \langle 0|(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}))|0\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} k^0 a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.132}$$

The prescription

$$\tfrac{1}{2}(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})) \to :\tfrac{1}{2}(a^\dagger(\mathbf{k})a(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k})): \equiv a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.133}$$

is called 'normal (or Wick's) ordering' and is denoted by a double-dot symbol. Its only effect is to subtract from the original expressions the infinite vacuum energy. For a product of the fields $\Phi(x) = \Phi^{(+)}(x) + \Phi^{(-)}(x)$ we get, for example, $:\Phi\Phi: = \Phi^{(-)}\Phi^{(-)} + 2\Phi^{(-)}\Phi^{(+)} + \Phi^{(+)}\Phi^{(+)}$, where $\Phi^{(+)}(\Phi^{(-)})$ are positive (negative) frequency parts of $\Phi(x)$ given by (1.126). It follows from (1.129) and (1.132) that

$$\left[P^\mu, a^\dagger(\mathbf{k})\right] = k^\mu a^\dagger(\mathbf{k}) \tag{1.134}$$

and, consequently,

$$P^\mu a^\dagger(\mathbf{k})|0\rangle = k^\mu a^\dagger(\mathbf{k})|0\rangle \tag{1.135}$$

Thus, the state $|\mathbf{k}\rangle \equiv a^\dagger(\mathbf{k})|0\rangle$ is an energy–momentum eigenstate. Hence, the $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$ are the creation and annihilation operators, respectively, for a quantum of four-momentum k^μ. The field–particle duality now becomes evident. Also, we get

$$\langle \mathbf{k}|\Phi(x)|0\rangle = \langle 0|a(\mathbf{k})\Phi(x)|0\rangle = \exp(\mathrm{i}k \cdot x) \tag{1.136}$$

i.e. the field $\Phi(x)$ (its negative frequency part) acting on the vacuum creates a particle at point x, with continuous momentum spectrum and propagating as a plane wave (with positive energy).† When acting on a particle at the point x, the positive frequency part of the field $\Phi(x)$ annihilates that state into the vacuum and the negative frequency part creates two-particle state.

Simultaneous measurement of the field strength Φ at points x and y is possible only if

$$[\Phi(x), \Phi(y)] = 0 \tag{1.137}$$

† Note that in quantum mechanics, the wave-function $\Phi(x)$ describes a state in the configuration representation, $\Phi(x) = \langle \mathbf{x}|\Phi(t)\rangle$, whereas in quantum field theory $\Phi(x)|0\rangle$ is a state $|x\rangle$ of a particle localized at the space-time point x, with its momentum representation $\langle \mathbf{k}|\Phi(x)|0\rangle = \langle \mathbf{x}|\mathbf{k}\rangle^\dagger$. Hence, in quantum mechanics the positive frequency part of $\Phi(x)$ describes propagation with positive energy but, interpreted as a quantum field, it is its negative frequency part which creates a particle with positive energy.

By direct calculation we get

$$[\Phi(x), \Phi(y)] = \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\varepsilon(k_0) \exp[-\mathrm{i}k \cdot (x - y)] \equiv \mathrm{i}\Delta(x - y) \tag{1.138}$$

where

$$\varepsilon(k_0) = \begin{cases} +1 & \text{for} \quad k_0 > 0 \\ -1 & \phantom{\text{for}} \quad k_0 < 0 \end{cases}$$

One checks that Δ is a solution of a free Klein–Gordon equation

$$\left.\begin{aligned} &\left(\Box_x + m^2\right) \Delta(x - y) = 0 \\ &\Delta(x - y) = -\Delta(y - x) \end{aligned}\right\} \tag{1.139}$$

Since $\Delta(y - x)$ vanishes for $t_1 - t_2 = 0$, from its Lorentz invariance we conclude that

$$\Delta(x - y) = 0 \qquad \text{for all} \quad (x - y)^2 < 0 \tag{1.140}$$

Thus the field strength is simultaneously measurable in two points with a space-like interval, i.e. consistently with the notion of locality and causality (two points with space-like intervals cannot be connected in a causal way). It follows from the postulated commutation relations (1.128) that the quanta of a scalar field obey Bose–Einstein statistics.

Quantization of a complex scalar field $\Phi(x)$ proceeds in quite an analogous way. In fact, the procedure is exactly the same if we work with the two real fields ϕ and χ defined in (1.54). In terms of the complex field and its conjugate momenta

$$\Pi_i = \partial_0 \Phi_i^\dagger, \qquad \Pi_i^\dagger = \partial_0 \Phi_i \tag{1.141}$$

we postulate

$$\left[\Pi_i(t, \mathbf{x}), \Phi_j(t, \mathbf{y})\right] = \left[\Pi_i^\dagger(t, \mathbf{x}), \Phi_j^\dagger(t, \mathbf{y})\right] = -\mathrm{i}\delta_{ij}\delta(\mathbf{x} - \mathbf{y}) \tag{1.142}$$

and the other commutators vanish. Fourier integrals, analogous to (1.126), now read:

$$\left.\begin{aligned} \Phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \left[a(\mathbf{k}) \exp(-\mathrm{i}k \cdot x) + b^\dagger(\mathbf{k}) \exp(\mathrm{i}k \cdot x)\right] \\ \Phi^\dagger(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0} \left[b(\mathbf{k}) \exp(-\mathrm{i}k \cdot x) + a^\dagger(\mathbf{k}) \exp(\mathrm{i}k \cdot x)\right] \end{aligned}\right\} \tag{1.143}$$

where $a(\mathbf{k})$ and $b(\mathbf{k})$ are two independent, non-hermitean, operator-valued functions. They can easily be expressed in terms of the hermitean annihilation and creations operators $a_1(\mathbf{k})$, $a_2(\mathbf{k})$ and $a_1^\dagger(\mathbf{k})$, $a_2^\dagger(\mathbf{k})$ for the quanta of the real fields

ϕ and χ. From (1.142) there follow the commutation relations for operators $a(\mathbf{k})$ and $b(\mathbf{k})$:

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{k}')\right] = (2\pi)^3 2k_0 \delta(\mathbf{k} - \mathbf{k}') \tag{1.144}$$

Thus, $a(\mathbf{k})$, $b(\mathbf{k})$ and $a^\dagger(\mathbf{k})$, $b^\dagger(\mathbf{k})$ are also annihilation and creation operators and it is clear from (1.143) that the field $\Phi(x)$ ($\Phi^\dagger(x)$) creates quanta of type b (a) and annihilates those of type a (b). A transparent interpretation of the quanta a and b is obtained by noting that the classical lagrangian (1.55) is invariant under abelian global $U(1)$ symmetry: $\Phi \to \exp(-\mathrm{i}q\Theta)\Phi$, $\Phi^\dagger \to \exp(\mathrm{i}q\Theta)\Phi^\dagger$, i.e. the fields Φ and $\Phi^\dagger$ carry opposite $U(1)$ charges.† The conserved charge Q (1.17) can be expressed in terms of the operators $a(\mathbf{k})$ and $b(\mathbf{k})$:

$$\begin{aligned} Q &= \mathrm{i}q \int \mathrm{d}^3x \, :\!\left[\Phi^\dagger(x)\partial_0\Phi(x) - \partial_0\Phi^\dagger(x)\Phi(x)\right]\!: \\ &= q \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}\left[a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})\right] \equiv q(N_a - N_b) \end{aligned} \tag{1.145}$$

This equation defines the operators N_a and N_b which are the number operators for the quanta a and b carrying opposite charges:

$$\left.\begin{aligned} Qa^\dagger(\mathbf{k})|0\rangle &= qa^\dagger(\mathbf{k})|0\rangle \\ Qb^\dagger(\mathbf{k})|0\rangle &= -qb^\dagger(\mathbf{k})|0\rangle \end{aligned}\right\} \tag{1.146}$$

Note that the charge of the state created by the field $\Phi(x)$ transforming as $\Phi(x) \to \exp(-\mathrm{i}q\Theta)\Phi(x)$ is $Q = -q$. Moreover, using (1.145) we get $[Q, \Phi(x)] = -q\Phi(x)$, i.e. (1.23) with { , } replaced by $-\mathrm{i}$[,]. We can introduce the unitary operator $U = \exp(\mathrm{i}\Theta Q) \approx 1 + \mathrm{i}\Theta Q$ such that $Ub^\dagger(\mathbf{k})|0\rangle = \exp(-\mathrm{i}\Theta q)\, b^\dagger(\mathbf{k})|0\rangle$ and $U\Phi(x)U^{-1} = \exp(-\mathrm{i}\Theta q)\Phi(x)$ (we have used (1.146)). A generalization to non-abelian global symmetries is straightforward. These equations can also be interpreted in the following way. We consider the wave-function in the configuration space of the one-particle state $b^\dagger(\mathbf{k})|0\rangle$: $\phi(x) = \langle x|b^\dagger(\mathbf{k})|0\rangle = \langle 0|\Phi^\dagger(x)b^\dagger(\mathbf{k})|0\rangle$ and in the $U(1)$ rotated basis (or of the rotated physical system in the same basis) $\phi'(x) = \langle 0|\Phi^\dagger(x)Ub^\dagger(\mathbf{k})|0\rangle$. Using (1.146) we get $\phi'(x) = \exp(-\mathrm{i}q\Theta)\phi(x)$. Therefore, the symmetry transformation on the quantum fields is the analogue of the transformation on the classical fields interpreted as the wave-functions of one-particle states.

The Feynman propagator

For the sake of definiteness, we again consider scalar field theory with abelian $U(1)$ symmetry and $q = 1$. One-particle states a ($Q = +1$) and b ($Q = -1$) at

† The value of the charge q relative to the charges of other fields can only be fixed in the presence of interactions.

the space-time point $(t, \mathbf{x})$ are created by the operators

$$\left.\begin{aligned} \Phi^\dagger(t,\mathbf{x})|0\rangle &\equiv \Phi^{\dagger(-)}(t,\mathbf{x})|0\rangle \\ \Phi(t,\mathbf{x})|0\rangle &\equiv \Phi^{(-)}(t,\mathbf{x})|0\rangle \end{aligned}\right\} \tag{1.147}$$

where

$$\left.\begin{aligned} \Phi^{\dagger(-)}(t,\mathbf{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} a^\dagger(\mathbf{k}) \exp(\mathrm{i}k\cdot x) \\ \Phi^{(-)}(t,\mathbf{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} b^\dagger(\mathbf{k}) \exp(\mathrm{i}k\cdot x) \end{aligned}\right\} \tag{1.148}$$

Let the particle a propagate to $t' > t$: the probability amplitude to find it at $(t', \mathbf{x}')$ is given by

$$\begin{aligned} &\theta(t'-t)\langle 0|\Phi(t',\mathbf{x}')\Phi^\dagger(t,\mathbf{x})|0\rangle \\ &= \theta(t'-t)\langle 0|\Phi^{(+)}(t',\mathbf{x}')\Phi^{\dagger(-)}(t,\mathbf{x})|0\rangle \end{aligned} \tag{1.149}$$

This amplitude can also be interpreted as the creation of the particle a at $(t, \mathbf{x})$ and its reabsorption into the vacuum at $(t', \mathbf{x}')$. Another way of increasing the charge by $+1$ at $(t, \mathbf{x})$ and lowering it by -1 at $(t', \mathbf{x}')$ is to create a quantum of charge -1 at $(t', \mathbf{x}')$ and to propagate it to $(t, \mathbf{x})$ with $t > t'$:

$$\begin{aligned} &\theta(t-t')\langle 0|\Phi^\dagger(t,\mathbf{x})\Phi(t',\mathbf{x}')|0\rangle \\ &= \theta(t-t')\langle 0|\Phi^{\dagger(+)}(t,\mathbf{x})\Phi^{(-)}(t',\mathbf{x}')|0\rangle \end{aligned} \tag{1.150}$$

The Feynman propagator is defined as superposition of the two amplitudes:

$$\begin{aligned} G_0^{(2)}(x',x) &\equiv \theta(t'-t)\langle 0|\Phi(t',\mathbf{x}')\Phi^\dagger(t,\mathbf{x})|0\rangle \\ &\quad + \theta(t-t')\langle 0|\Phi^\dagger(t,\mathbf{x})\Phi(t',\mathbf{x}')|0\rangle \\ &\equiv \langle 0|T\left(\Phi^\dagger(t,\mathbf{x})\Phi(t',\mathbf{x}')\right)|0\rangle \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\big[\theta(t'-t)\exp[-\mathrm{i}k\cdot(x'-x)] \\ &\quad + \theta(t-t')\exp[\mathrm{i}k\cdot(x'-x)]\big] \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\varepsilon}\exp[-\mathrm{i}k\cdot(x'-x)] \end{aligned} \tag{1.151}$$

where T denotes the so-called chronological ordering of the field operators.

Spinor fields

Quantization of a Weyl or Dirac field can be based on the phenomenological Pauli's exclusion principle for spin $\frac{1}{2}$ particles or on the requirement of locality (i.e. commutativity at the space-like intervals for observables constructed from the Dirac spinors; this is the content of the theorem on the connection between spin and statistics). It can be also introduced by postulating supersymmetry (a certain symmetry between bosons and fermions, see Chapter 15). We are always consistently led to the conclusion that the Weyl and Dirac fields must be quantized by assigning for them certain anticommutation relations.

Using the classical spinors introduced in Section 1.2 and Appendix A, we can write down the general wave expansion for solutions of the free Weyl equations for the massless left-handed field $\lambda_\alpha(x)$

$$\lambda_\alpha(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[b_{\mathrm{L}}(\mathbf{k})a_\alpha(\mathbf{k})\exp(-\mathrm{i}k\cdot x) + d^\dagger_{\mathrm{R}}(\mathbf{k})b_\alpha(\mathbf{k})\exp(\mathrm{i}k\cdot x)\right] \tag{1.152}$$

and for its hermitean conjugation $\lambda^\dagger_{\dot\alpha}(x)$ (related to the right-handed field through $\bar\lambda^{\dot\beta} = \lambda^\dagger_{\dot\alpha}\varepsilon^{\dot\beta\dot\alpha}$):

$$\lambda^\dagger_{\dot\alpha}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[b^\dagger_{\mathrm{L}}(\mathbf{k})a^\star_{\dot\alpha}(\mathbf{k})\exp(\mathrm{i}k\cdot x) + d_{\mathrm{R}}(\mathbf{k})b^\star_{\dot\alpha}(\mathbf{k})\exp(-\mathrm{i}k\cdot x)\right] \tag{1.153}$$

where $a_\alpha(\mathbf{k})$ and $b_\alpha(\mathbf{k})$ are positive and negative energy chiral solutions to a free Weyl equation $\mathrm{i}\bar\sigma^\mu\partial_\mu\lambda(x) = 0$ (normalized as in (A.68)) and b_{L}, d_{R} are expansion coefficients. As for scalar fields, the classical Weyl fields are now promoted to operators but instead of relations (1.128), we postulate (the canonical momentum conjugate to $\lambda_\alpha(x)$ is $\Pi^\alpha = \mathrm{i}\lambda^\dagger_{\dot\beta}(\bar\sigma^0)^{\dot\beta\alpha}$)

$$\left\{\lambda_\alpha(t,\mathbf{x}), \lambda^\dagger_{\dot\beta}(t,\mathbf{y})\right\} = (\sigma^0)_{\alpha\dot\beta}\delta(\mathbf{x}-\mathbf{y}) \tag{1.154}$$

Other anticommutators vanish. The expansion coefficients become operators with the following anticommutation relations:

$$\left\{b_{\mathrm{L}}(\mathbf{k}), b^\dagger_{\mathrm{L}}(\mathbf{k}')\right\} = \left\{d_{\mathrm{R}}(\mathbf{k}), d^\dagger_{\mathrm{R}}(\mathbf{k}')\right\} = (2\pi)^3 2k^0\delta(\mathbf{k}-\mathbf{k}') \tag{1.155}$$

and all other anticommutators vanish. Inserting (1.152) into the analogue of (1.81) for the left-handed Weyl fields,

$$\Theta^{\mu\nu} = \mathrm{i}\bar\lambda\bar\sigma^\mu\partial^\nu\lambda \tag{1.156}$$

we find

$$\begin{aligned} P^\mu &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} k^\mu\left[b^\dagger_{\mathrm{L}}(\mathbf{k})b_{\mathrm{L}}(\mathbf{k}) - d_{\mathrm{R}}(\mathbf{k})d^\dagger_{\mathrm{R}}(\mathbf{k})\right] \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} k^\mu\left[b^\dagger_{\mathrm{L}}(\mathbf{k})b_{\mathrm{L}}(\mathbf{k}) + d^\dagger_{\mathrm{R}}(\mathbf{k})d_{\mathrm{R}}(\mathbf{k}) - \left\{d_{\mathrm{R}}(\mathbf{k}), d^\dagger_{\mathrm{R}}(\mathbf{k})\right\}\right] \end{aligned} \tag{1.157}$$

It follows from (1.157) and from the anticommutation relations (1.155) that, with the vacuum defined by $b_{\mathrm{L}}(\mathbf{k})|0\rangle = d_{\mathrm{R}}(\mathbf{k})|0\rangle = 0$, its energy is not only infinite (as for the scalar fields) but also is not positive definite. We redefine the vacuum by subtracting this infinite c-number. This is accomplished by the normal ordering. The normal ordering for the Weyl fields is defined as:

$$:\lambda^{\dagger}_{\dot{\alpha}}\lambda_{\beta}: \equiv \lambda^{\dagger(+)}_{\dot{\alpha}}\lambda^{(+)}_{\beta} + \lambda^{\dagger(-)}_{\dot{\alpha}}\lambda^{(+)}_{\beta} + \lambda^{\dagger(-)}_{\dot{\alpha}}\lambda^{(-)}_{\beta} - \lambda^{(-)}_{\beta}\lambda^{\dagger(+)}_{\dot{\alpha}} \tag{1.158}$$

where the subscript '+' ('−') denotes positive (negative) frequency parts of λ and $\lambda^{\dagger}$ given by (1.152) and (1.153). In consistency with the anticommutation relations (1.155), there is a minus sign for each interchange of λ and $\lambda^{\dagger}$ required by the normal ordering. After the normal ordering we get

$$\begin{aligned} P^{\mu} &= \int \mathrm{d}^3x \, :\mathrm{i}\bar{\lambda}\bar{\sigma}^0\partial^{\mu}\lambda(x): \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} k^{\mu}\left[b^{\dagger}_{\mathrm{L}}(\mathbf{k})b_{\mathrm{L}}(\mathbf{k}) + d^{\dagger}_{\mathrm{R}}(\mathbf{k})d_{\mathrm{R}}(\mathbf{k})\right] \end{aligned} \tag{1.159}$$

i.e. we have redefined the vacuum so that its energy is zero. It follows from the relation (1.159) that operators $b^{\dagger}_{\mathrm{L}}(\mathbf{k})$ and $d^{\dagger}_{\mathrm{R}}(\mathbf{k})$ have the interpretation of creation operators of positive-energy states with four-momentum $(E, \mathbf{k})$ which, however, have opposite charges under the global $U(1)$ phase transformations. Indeed, the lagrangian for a single Weyl field has global $U(1)$ symmetry generated by the transformations $\lambda' = \exp(-\mathrm{i}q\Theta)\lambda$. The conserved current $j^{\mu} = q\bar{\lambda}\bar{\sigma}^{\mu}\lambda$ $(\partial j^{\mu}/\partial x^{\mu} = 0)$ and the corresponding charge $Q = \int \mathrm{d}^3x \, j^0(t, \mathbf{x})$ can be expressed in terms of the creation and annihilation operators. After the normal ordering we get

$$\begin{aligned} Q &= q\int \mathrm{d}^3x \, :\lambda^{\dagger}\bar{\sigma}^0\lambda: = q\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} :b^{\dagger}_{\mathrm{L}}(\mathbf{k})b_{\mathrm{L}}(\mathbf{k}) + d_{\mathrm{R}}(\mathbf{k})d^{\dagger}_{\mathrm{R}}(\mathbf{k}): \\ &= q\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[N^{+}_{\mathrm{L}}(\mathbf{k}) - N^{-}_{\mathrm{R}}(\mathbf{k})\right] \end{aligned} \tag{1.160}$$

where

$$N^{+}_{\mathrm{L}}(\mathbf{k}) = b^{\dagger}_{\mathrm{L}}(\mathbf{k})b_{\mathrm{L}}(\mathbf{k}), \qquad N^{-}_{\mathrm{R}}(\mathbf{k}) = d^{\dagger}_{\mathrm{R}}(\mathbf{k})d_{\mathrm{R}}(\mathbf{k})$$

can be interpreted as the number operators for positive-energy state with $U(1)$ charges $+q$ and $-q$, respectively. Acting with the field $\lambda(x)$ on the vacuum and projecting on the momentum eigenstates $d^{\dagger}_{\mathrm{R}}(\mathbf{k})|0\rangle$, we see that $\lambda^{\alpha}(x)$ creates a helicity $h = +\frac{1}{2}$ state with charge $-q$ at the point x, described by a superposition of the plane wave solutions

$$\begin{aligned} {}^{\dot{\alpha}}\langle\mathbf{x}|\mathbf{k}, h = +\tfrac{1}{2}\rangle &= \langle\mathbf{k}, h = +\tfrac{1}{2}|\lambda^{\alpha}(x)|0\rangle^{\dagger} \\ &= \langle 0|d_{\mathrm{R}}(\mathbf{k})\lambda^{\alpha}(x)|0\rangle^{\dagger} = b^{\star\dot{\alpha}}(\mathbf{k})\exp(-\mathrm{i}k\cdot x) \end{aligned}$$

with four-momenta k^μ and positive energy (note, that $b^{\star\dot{\alpha}}(\mathbf{k}) \sim a'^{\dot{\alpha}}(\mathbf{k})$ is the solution of (A.60) with positive energy and positive helicity). The field $\lambda^\dagger_{\dot{\alpha}}(x)$ creates a helicity $-\frac{1}{2}$ state with charge $+q$ at the point x – a state which is a superposition of the plane wave solutions ${}_\alpha\langle \mathbf{x}|\mathbf{k}, h = -\frac{1}{2}\rangle = \langle \mathbf{k}, h = -\frac{1}{2}|\lambda^\dagger_{\dot{\alpha}}(x)|0\rangle^\dagger = a_\alpha(\mathbf{k})\exp(-\mathrm{i}k\cdot x)$.

For the Dirac field the procedure described above leads to the field operators

$$\begin{aligned}\Psi(x) &= \sum_{s=1,2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\big[b(\mathbf{k},s)u(\mathbf{k},s)\exp(-\mathrm{i}k\cdot x)\\ &\quad + d^\dagger(\mathbf{k},s)v(\mathbf{k},s)\exp(\mathrm{i}k\cdot x)\big]\\ \Psi^\dagger(x) &= \sum_{s=1,2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\big[b^\dagger(\mathbf{k},s)\bar{u}(\mathbf{k},s)\gamma_0\exp(\mathrm{i}k\cdot x)\\ &\quad + d(\mathbf{k},s)\bar{v}(\mathbf{k},s)\gamma_0\exp(-\mathrm{i}k\cdot x)\big]\end{aligned} \tag{1.161}$$

satisfying anticommutation relations

$$\left\{\Psi_\alpha(t,\mathbf{x}), \Psi^\dagger_\beta(t,\mathbf{y})\right\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) \tag{1.162}$$

(α and β are spinor indices) with the other anticommutators vanishing. The functions $u(\mathbf{k},s)$ and $v(\mathbf{k},s)$ are positive and negative energy solutions for the four-component Dirac equation; $s = 1, 2$ numbers the $+\frac{1}{2}$ and $-\frac{1}{2}$ spin projections along the chosen quantization axis (see Appendix A). Similar expansions can be written down in terms of the helicity amplitudes. For the operators $b(\mathbf{k},s)$ and $d(\mathbf{k},s)$ we get

$$\left\{b(\mathbf{k},s), b^\dagger(\mathbf{k}',s')\right\} = \left\{d(\mathbf{k},s), d^\dagger(\mathbf{k}',s')\right\} = (2\pi)^3 2k^0\delta_{ss'}\delta(\mathbf{k}-\mathbf{k}')$$

and all other anticommutators vanish. Defining the normal ordering for the Dirac fields

$$:\bar{\Psi}_\alpha\Psi_\beta: = \bar{\Psi}^{(+)}_\alpha\Psi^{(+)}_\beta + \bar{\Psi}^{(-)}_\alpha\Psi^{(+)}_\beta + \bar{\Psi}^{(-)}_\alpha\Psi^{(-)}_\beta - \Psi^{(-)}_\beta\bar{\Psi}^{(+)}_\alpha \tag{1.163}$$

where the superscript '+' ('−') denotes positive (negative) frequency parts of Ψ and $\bar{\Psi}$ given by (1.161), we get for the momentum operator the expression

$$P^\mu = \int \mathrm{d}^3x :\Theta^{0\mu}(x): = \sum_{s=1,2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}k^\mu\left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s) + d^\dagger(\mathbf{k},s)d(\mathbf{k},s)\right]$$

The operators $b^\dagger(\mathbf{k},s)$ and $d^\dagger(\mathbf{k},s)$ have the interpretation of creation operators of positive-energy states with four-momentum $(E,\mathbf{k})$ with opposite charges under the global $U(1)$ phase transformations $\Psi' = \exp(-\mathrm{i}q\Theta)\Psi$. The conserved current

$j^\mu = q\bar{\Psi}\gamma^\mu\Psi$ $(\partial j^\mu/\partial x^\mu = 0)$ and the corresponding charge $Q = \int \mathrm{d}^3x\, j^0(t, \mathbf{x})$ can be expressed in terms of the creation and annihilation operators

$$
\begin{aligned}
Q = q\int \mathrm{d}^3x :\Psi^\dagger\Psi: &= q\sum_{s=1,2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E} :b^\dagger(\mathbf{k}, s)b(\mathbf{k}, s) + d(\mathbf{k}, s)d^\dagger(\mathbf{k}, s): \\
&= q\sum_{s=1,2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[N^+(\mathbf{k}, s) - N^-(\mathbf{k}, s)\right] \qquad (1.164)
\end{aligned}
$$

where

$$
N^+(\mathbf{k}, s) = b^\dagger(\mathbf{k}, s)b(\mathbf{k}, s), \qquad N^-(\mathbf{k}, s) = d^\dagger(\mathbf{k}, s)d(\mathbf{k}, s)
$$

are interpreted as the number operators for positive-energy states with $+q$ and $-q$ $U(1)$ charges, respectively. As for the scalar fields, one can introduce the unitary operator U which realizes the $U(1)$ transformations on the quantum states and on the quantum Weyl and Dirac fields.

Similarly to the case of Weyl fields, acting with the field operator $\Psi^\dagger_\alpha(x)$ on the vacuum and projecting on the momentum eigenstates $|b(\mathbf{k}, s)\rangle = b^\dagger(\mathbf{k}, s)|0\rangle$, we see that $\Psi^\dagger_\alpha(x)$ creates a state of particle b with the spin s and charge $+q$ at the point x, described by a superposition of the plane wave solutions

$$
\begin{aligned}
{}_\alpha\langle\mathbf{x}|b(\mathbf{k}, s)\rangle &= \langle b(\mathbf{k}, s)|\Psi^\dagger_\alpha(x)|0\rangle^\dagger \\
&= \langle 0|b(\mathbf{k}, s)\Psi^\dagger_\alpha(x)|0\rangle^\dagger = u_\alpha(\mathbf{k}, s)\exp(-\mathrm{i}k\cdot x)
\end{aligned}
$$

with four-momenta k^μ, spin s and positive energy. The operator $\Psi_\alpha(x)$, on the other hand, creates a state of particle d with the wave-function given by

$$
\begin{aligned}
{}_\alpha\langle\mathbf{x}|d(\mathbf{k}, s)\rangle &= \langle d(\mathbf{k}, s)|\Psi_\alpha(x)|0\rangle^\dagger \\
&= \langle 0|d(\mathbf{k}, s)\Psi_\alpha(x)|0\rangle^\dagger = v^\star_\alpha(\mathbf{k}, s)\exp(-\mathrm{i}k\cdot x)
\end{aligned}
$$

For chiral fields in the four-component notation one has

$$
\Psi_{\mathrm{L,R}}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[b_{\mathrm{L,R}}(\mathbf{k})u_{\mathrm{L,R}}(\mathbf{k})\exp(-\mathrm{i}k\cdot x) + d^\dagger_{\mathrm{R,L}}(\mathbf{k})v_{\mathrm{R,L}}(\mathbf{k})\exp(\mathrm{i}k\cdot x)\right] \qquad (1.165)
$$

where $u_{\mathrm{L,R}}$ ($v_{\mathrm{R,L}}$) are positive (negative) energy solutions to the Dirac equation (1.78), which satisfy the chirality conditions $P_{\mathrm{L,R}}u_{\mathrm{L,R}} = u_{\mathrm{L,R}}$, $P_{\mathrm{L,R}}v_{\mathrm{R,L}} = v_{\mathrm{R,L}}$. Obviously, in the chiral representation for the Dirac matrices we have

$$
u_{\mathrm{L}}(\mathbf{k}) = \begin{pmatrix} a(\mathbf{k}) \\ 0 \\ 0 \end{pmatrix}, \qquad u_{\mathrm{R}}(\mathbf{k}) = \begin{pmatrix} 0 \\ 0 \\ a'(\mathbf{k}) \end{pmatrix} \qquad (1.166)
$$

and similar expressions for v_{R} and v_{L} with $a(\mathbf{k})$ and $a'(\mathbf{k})$ replaced by $b(\mathbf{k})$ and $b'(\mathbf{k})$, respectively. (We recall that $a(k)$, $a'(k)$, $b(k)$, $b'(k)$ are solutions to the Weyl equations.) Thus, a chiral left-handed massless field Ψ_L creates a particle with charge $(-q)$ and positive helicity and $\Psi_L^\dagger$ creates a particle with charge $(+q)$ and negative helicity (see the remarks below (A.62)).

A self-conjugate four-component field, (1.76) (neutral under global symmetry transformations) has the following expansion

$$\begin{aligned}\Psi_M(x) = \sum_{s=1,2} \int \frac{d^3k}{(2\pi)^3 2E} \big[& b(\mathbf{k}, s) u(\mathbf{k}, s) \exp(-ik \cdot x) \\ & + \exp(i\rho) b^\dagger(\mathbf{k}, s) v(\mathbf{k}, s) \exp(ik \cdot x)\big] \\ \Psi_M^\dagger(x) = \sum_{s=1,2} \int \frac{d^3k}{(2\pi)^3 2E} \big[& b^\dagger(\mathbf{k}, s) \bar{u}(\mathbf{k}, s) \gamma_0 \exp(ik \cdot x) \\ & + \exp(-i\rho) b(\mathbf{k}, s) \bar{v}(\mathbf{k}, s) \gamma_0 \exp(-ik \cdot x)\big]\end{aligned} \tag{1.167}$$

$\exp(i\rho)$ is a phase factor depending on the choice of the phase for one-particle states (Kayser 1994).

Symmetry transformations for quantum fields

One way to discuss symmetry transformations for quantum fields is to interpret solutions to the classical field equations as wave-functions of quantum states (see the remark below (1.146)). The following results apply to free and interacting fields. We begin with the Lorentz transformations and consider two observers in two different reference frames $\mathcal{O}$ and $\mathcal{O}'$, and by analogy between classical and quantum physics we assume the transformation law (1.25) for wave-functions (in configuration space) of quantum states. First of all, it is obvious that a given (the same) physical quantum system is seen by both observers. It is described by them by the state vectors which form two isomorphic to each other but, in general, different Hilbert spaces. We shall make here the following strong assumption which is a necessary (but not sufficient) condition for the change of the reference frame to be a symmetry transformation: the two Hilbert spaces are identical and can be identified with each other (in consequence, the operators acting in one of them can also be identified with suitable operators acting in the other space). Therefore the Hilbert space used by observer $\mathcal{O}$ is mapped onto itself by a unitary operator U which is defined by the equation:

$$|v'\rangle = U|v\rangle \tag{1.168}$$

In the frame $\mathcal{O}'$ the state vector $|v'\rangle$ describes the same physical state as does the state vector $|v\rangle$ in $\mathcal{O}$. The state vectors form a representation of the Lorentz group. We note already here that the assumption about the identity of the two

Hilbert spaces is not necessarily satisfied for such physically relevant changes of the reference frame as, for example, space reflection (to be discussed later) which is not a symmetry of many physical systems.

Our goal is to find the action of the operator U on the field operators $\Phi(x)$ which may carry a Lorentz index α. For Poincaré transformations $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \approx x^{\mu} + \varepsilon^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu}$, (1.47) interpreted as the equation for the one-particle wave-functions reads:

$$\phi'_{\alpha}(x') \equiv {}_{\alpha}\langle x'|v'\rangle = \left(\exp(\tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\right)_{\alpha}{}^{\beta}\ {}_{\beta}\langle x|v\rangle \tag{1.169}$$

where

$$\left.\begin{aligned} {}_{\alpha}\langle x| &= \langle 0|\Phi_{\alpha}(x) \\ {}_{\alpha}\langle x'| &= \langle 0|\Phi_{\alpha}(x') \\ |v'\rangle &= U(\varepsilon,\omega)|v\rangle \end{aligned}\right\} \tag{1.170}$$

(the index α refers to the original or to the new reference frame; in the first and the second equation of (1.170) the same field operator Φ appears since the Hilbert spaces of both observers are identical). Therefore,

$$U\Phi_{\alpha}(x)U^{-1} = \left(\exp(-\tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\right)_{\alpha}{}^{\beta}\ \Phi_{\beta}(x') \tag{1.171}$$

(Internal quantum number indices may be implicit.†)

For translations, (1.171) simplifies to:

$$U\Phi_{\alpha}(x)U^{-1} = \Phi_{\alpha}(x') \tag{1.172}$$

Writing

$$\left.\begin{aligned} U(\varepsilon) &= \exp\left(\mathrm{i}\varepsilon_{\mu}P^{\mu}\right) \\ U(\omega) &= \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \end{aligned}\right\} \tag{1.173}$$

we obtain the infinitesimal form of these transformations

$$\mathrm{i}[P_{\mu}, \Phi(x)] = \frac{\partial\Phi(x)}{\partial x^{\mu}} \tag{1.174}$$

and

$$\mathrm{i}\left[M^{\mu\nu}, \Phi(x)\right] = \left(x^{\mu}\frac{\partial}{\partial x_{\nu}} - x^{\nu}\frac{\partial}{\partial x_{\mu}} + \Sigma^{\mu\nu}\right)\Phi(x) \tag{1.175}$$

which replace the Poisson brackets (1.41) and (1.51). These equations suggest

† Also, one often defines the operator $\Phi'_{\alpha}(x') = U^{-1}\Phi_{\alpha}(x)U$ such that the matrices satisfy the equation $\langle v_2|\Phi'_{\alpha}(x')|v_1\rangle = \left(\exp(\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu})\right)_{\alpha}{}^{\beta}\langle v_2|\Phi_{\beta}(x)|v_1\rangle$ or the operator $\Phi'_{\alpha}(x) = U\Phi(x)U^{-1}$ such that $\langle v'_2|\Phi'(x)|v'_1\rangle = \langle v_2|\Phi(x)|v_1\rangle$.

therefore the identification of the generators P^μ and $M^{\mu\nu}$ with the four-momentum vector and the angular momentum tensor, respectively. In every concrete theory those operators can be constructed using equations like (1.131) or (1.132) or (1.157) and one can check explicitly if (1.174) and (1.175) follow from the commutation relations imposed in the quantization procedure. If so, then the theory is Lorentz-invariant.

An interesting exception to relations (1.174) and (1.175) is the quantization of the gauge fields in non-covariant gauges (for example, in the Coulomb gauge). Nevertheless, the unitary operators $U(\varepsilon, \omega)$ can be found and the theory, although not manifestly so, is Lorentz-invariant. The Lorentz boosts must be supplemented by additional gauge transformations (to recover the non-covariant gauge in the new frame). See, for example, Bjorken & Drell (1965).

The $U(1)$ transformations in the internal quantum number space on the free quantum fields have already been discussed in the first subsection of this section. The results obtained there remain generally valid for non-abelian groups of transformations and for interacting fields. We may follow the strategy of interpreting (1.9) as the equation relating the wave-functions of one-particle states in the original and rotated basis built by the action of the field operator $\Phi(x)$ on the vacuum: $|i\rangle = \Phi_i(x)|0\rangle$ (the index i refers to the original or rotated basis). We have then:

$$\phi_i(x) \equiv \langle i|v'\rangle = \left(\exp(-\mathrm{i}\Theta^a T^a)\right)_{ij} \langle j|v\rangle \tag{1.176}$$

where the states $|v\rangle$ and $|v'\rangle = U|v\rangle$ describe the same physical system before and after the rotation of the basis (we again assume that the two Hilbert spaces can be identified with each other). Therefore, using (1.176), we find the transformation law for the operators

$$U\Phi_i(x)U^{-1} = \left(\exp(-\mathrm{i}\Theta^a T^a)\right)_{ij} \Phi_j(x) \tag{1.177}$$

In deriving the last equation we have assumed that the vacuum is invariant under the considered group of transformations: $U|0\rangle = |0\rangle$. Later we shall also discuss theories with spontaneously broken global symmetries (see Chapter 9) where the ground state of the system (the physical vacuum determined by the equations of motion) is not invariant under the symmetry group of the lagrangian. Eq. (1.177) then remains valid for the original fields of the lagrangian but not for the physical fields defined as perturbation around the true vacuum.

With

$$U(\Theta) = \exp(\mathrm{i}\Theta^a Q^a),$$

where Q^a are generators of G, we get

$$[Q^a, \Phi] = -T^a \Phi \tag{1.178}$$

i.e. (1.23) with $\{\ ,\ \}$ replaced by $-\mathrm{i}[\ ,\ ]$.

A group G of transformations in the space of internal quantum numbers is a symmetry group provided the action is invariant $U \int \mathcal{L}(x) U^{-1} = \int \mathcal{L}(x)$.

1.5 Discrete symmetries

Space reflection

Under the transformation of space reflection $x^\mu \to x'^\mu$ with

$$t' = t, \qquad \mathbf{x}' = -\mathbf{x}$$

the transformation laws for classical physical observables such as energy E, the three-momentum $\mathbf{p}$, total angular momentum $\mathbf{J}$, helicity $h = (\mathbf{J} \cdot \mathbf{p})/|\mathbf{p}|$ etc. are obvious:

$$E' = E, \qquad \mathbf{p}' = -\mathbf{p}, \qquad \mathbf{J}' = \mathbf{J}, \qquad h' = -h \tag{1.179}$$

All internal charges are invariant with respect to space inversion.

We may also address the question of the transformation law for classical fields. For scalar fields, the most general form of transformation under space reflection is:

$$\Phi_P(x') = \begin{cases} \pm\Phi(x) & \text{for a real field} \\ \exp(\mathrm{i}\gamma)\Phi(x) & \text{for a complex field} \end{cases} \tag{1.180}$$

where the phase γ is arbitrary for a free field (a free scalar complex field has no well-defined parity). Clearly, the Klein–Gordon equations of motion for scalar fields remain form-invariant under those transformations, i.e. they have the same form in the original and in the primed reference frames.

The Weyl spinors transform under space inversion from the one in the $(\frac{1}{2}, 0)$ representation of the Lorentz group to the one in the $(0, \frac{1}{2})$ representation and vice versa. Indeed, if two right-handed coordinate frames $\mathcal{O}_1$ and $\mathcal{O}_2$ are connected by the Lorentz transformation with parameters η_i, ξ_i (see Appendix A for the definition of the parameters η_i and ξ_i) then their images under space reflection, $\mathcal{O}'_1$ and $\mathcal{O}'_2$ are connected by the Lorentz transformation with parameters $\eta_i, -\xi_i$. Correspondingly, a spinor λ_α transforming as $\lambda'_\beta = M(\boldsymbol{\eta}, \boldsymbol{\xi})_\beta{}^\alpha \lambda_\alpha$, seen from the reflected frame, is a dotted spinor $\bar{\lambda}^{\dot{\alpha}}_P$ transforming under Lorentz transformations as:

$$\bar{\lambda}'^{\dot{\alpha}}_P = [M^{\dagger -1}(\boldsymbol{\eta}, -\boldsymbol{\xi})]^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\lambda}^{\dot{\beta}}_P, \tag{1.181}$$

since the matrices $M^{\dagger-1}(\boldsymbol{\eta}, -\boldsymbol{\xi})$ and $M(\boldsymbol{\eta}, \boldsymbol{\xi})$ are numerically equal. Thus:

$$\lambda_\alpha(x) \to \bar{\lambda}_P^{\dot{\alpha}}(x') \equiv \exp(\mathrm{i}\gamma)\bar{P}^{\dot{\alpha}\beta}\lambda_\beta(x) \tag{1.182}$$

where the only role of $\bar{P} \equiv \bar{\sigma}^0$ is to change the indices from undotted to dotted ones. The transformation (1.182) satisfies the necessary relation:

$$[M^{\dagger-1}(\boldsymbol{\eta}, -\boldsymbol{\xi})]^{\dot{\alpha}}{}_{\dot{\beta}}\bar{P}^{\dot{\beta}\sigma}\lambda_\sigma(x) = \bar{P}^{\dot{\alpha}\beta}[M(\boldsymbol{\eta}, \boldsymbol{\xi})]_\beta{}^\sigma\lambda_\sigma(x) \tag{1.183}$$

because $\bar{P}$ is numerically equal to the unit matrix. Similarly, we get

$$\bar{\chi}^{\dot{\alpha}}(x) \to \chi_{P\alpha}(x') \equiv \exp(\mathrm{i}\bar{\gamma})P_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}}(x) = \exp(\mathrm{i}\bar{\gamma})(\sigma^0)_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}}(x) \tag{1.184}$$

and under Lorentz transformations $\chi_{P\alpha}(x')$ transforms with $M(\boldsymbol{\eta}, -\boldsymbol{\xi}) = M^{\dagger-1}(\boldsymbol{\eta}, \boldsymbol{\xi})$.

The Weyl equation is not invariant under space reflection. The positive (negative) energy solution of the Weyl equation $\mathrm{i}\bar{\sigma}^\mu\partial_\mu\lambda = 0$ (see Appendix A): $\sim\exp(-\mathrm{i}k\cdot x)a(\mathbf{k})$ $(\sim\exp(\mathrm{i}k\cdot x)b(\mathbf{k}))$ upon substituting $\mathbf{x} = -\mathbf{x}'$ becomes the solution with positive (negative) energy of the equation $\mathrm{i}\sigma^\mu\partial'_\mu\lambda = 0$. Thus, in the reflected coordinate frame it describes, in accord with (1.179), opposite helicity state, $h = +\frac{1}{2}$ $(-\frac{1}{2})$, and transforms, therefore, according to the representation $(0, \frac{1}{2})$.

It is very important to notice that the parity transformation on the Weyl spinors transforms fields $(\frac{1}{2}, 0)$ in representation R of a group G acting in the space of internal quantum numbers into $(0, \frac{1}{2})$ again in representation R (and not $R^\star$). Therefore parity transformation of the Weyl spinors can be a symmetry of a physical system only if it contains the corresponding pairs of fundamental fields.

For a four-component Dirac spinor in the chiral representation of the Dirac matrices, $\Psi = \begin{pmatrix}\lambda_\alpha \\ \bar{\lambda}^{c\dot{\beta}}\end{pmatrix}$, the transformation law follows directly from the transformation of the Weyl spinors (1.182), (1.184).

$$\Psi_P(x') = \begin{pmatrix}\exp(\mathrm{i}\bar{\gamma})(\sigma^0\bar{\lambda}^c)_\alpha \\ \exp(\mathrm{i}\gamma)(\bar{\sigma}^0\lambda)^{\dot{\beta}}\end{pmatrix}(x) \tag{1.185}$$

Such transformation preserves the form of the massive Dirac equation (i.e. $\Psi_P(x')$ defined in (1.185) satisfies the Dirac's equation in the reflected frame) provided we choose $\gamma = \bar{\gamma}$. In this case

$$\Psi_P(x') = \exp(\mathrm{i}\gamma)\gamma_0\Psi(x) \tag{1.186}$$

where the phase γ is still arbitrary. The space reflection is a symmetry of the free Dirac fermion. Moreover, bilinear forms like $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma_5\Psi$, $\bar{\Psi}\gamma_\mu\Psi$, $\bar{\Psi}\gamma_\mu\gamma_5\Psi$ etc. have well-defined properties under the parity transformation.

The interaction fixes the relative parities of the different fields provided that space inversion is a good symmetry. If parity is not a symmetry of the lagrangian, there is no choice of the phases γ such that $\mathcal{L}_{\mathcal{P}}(\Phi_P) = \mathcal{L}(\Phi_P)$. For instance, the pion–nucleon interaction can be described, assuming spin zero for the pion, by two Lorentz invariants $\bar{\Psi}\Psi\Phi$ and $\bar{\Psi}\gamma_5\Psi\Phi$. Thus, if parity is conserved in strong interactions, the pion field must be a scalar or a pseudoscalar (as seen in strong interactions). From more detailed study of strong interactions it has been determined experimentally that the pion field indeed has spin zero and negative parity (and space inversion is a symmetry of strong interactions; see, for example, Werle (1966)).

In quantum theory, the Hilbert spaces $\mathcal{H}$ of the observer $\mathcal{O}$ and $\mathcal{H}_P$ of the observer $\mathcal{O}_P$ (in the reflected coordinate frame) are isomorphic but can, in general, be different. For space reflection to be a symmetry of the physical system, it is necessary that these two Hilbert spaces are identical and can be identified with each other. This also means that the operators $\Phi_P(x)$ acting in $\mathcal{H}_P$ can be identified with some operators acting in the original Hilbert space. We can then introduce a unitary linear operator $\mathcal{P}$ which transforms the states from one reference frame to another, i.e. maps the Hilbert space onto itself, and study its action on the field operators. For complex scalar field theory, interpreting the transformation on the classical fields as the transformation on the wave-functions, the analogue of (1.169) reads:

$$\langle 0|\Phi_P(x')|v_P\rangle = \exp(\mathrm{i}\gamma)\langle 0|\Phi(x)|v\rangle \tag{1.187}$$

(for a real scalar field $\exp(\mathrm{i}\gamma) = \pm 1$) where $|v_P\rangle = \mathcal{P}|v\rangle$.

Identifying $\Phi_P(x') \equiv \Phi(x')$, this equation is satisfied with

$$\mathcal{P}\Phi(x)\mathcal{P}^{-1} = \exp(-\mathrm{i}\gamma)\Phi(x') \tag{1.188}$$

(we assume the vacuum to be invariant under space reflection; if the invariance under space reflection is spontaneously broken we face the same situation as for the spontaneously broken global symmetries) and for the creation and annihilation operators we get

$$\left.\begin{aligned} \mathcal{P}a(\mathbf{k})\mathcal{P}^{-1} &= \exp(-\mathrm{i}\gamma)a(-\mathbf{k}) \\ \mathcal{P}b^\dagger(\mathbf{k})\mathcal{P}^{-1} &= \exp(-\mathrm{i}\gamma)b^\dagger(-\mathbf{k}) \end{aligned}\right\} \tag{1.189}$$

Thus, for one-particle states we have

$$\left.\begin{aligned} \mathcal{P}|a,\mathbf{k}\rangle &\equiv \mathcal{P}a^\dagger(\mathbf{k})|0\rangle = \exp(\mathrm{i}\gamma)|a,-\mathbf{k}\rangle \\ \mathcal{P}|b,\mathbf{k}\rangle &\equiv \mathcal{P}b^\dagger(\mathbf{k})|0\rangle = \exp(-\mathrm{i}\gamma)|b,-\mathbf{k}\rangle \end{aligned}\right\} \tag{1.190}$$

The operator $\mathcal{P}$ can be expressed in terms of the creation and annihilation operators (see, for example, Bjorken & Drell (1965)).

For a chiral theory containing left-handed Weyl fields λ^i_α transforming as a complex representation R of the internal symmetry group the Hilbert spaces $\mathcal{H}_P$ and $\mathcal{H}$ mapped into each other by the parity transformation are, in general, different (the states in $\mathcal{H}$ and in $\mathcal{H}_\mathcal{P}$ have opposite helicities $-\frac{1}{2}$ and $+\frac{1}{2}$, respectively). Thus, space reflection cannot be a symmetry of such a theory. For this to be the case, the system must also contain the right-handed field operators $\bar{\lambda}^{ci\dot{\alpha}}(x)$, transforming as $(0, \frac{1}{2})$ and R under the Lorentz and internal symmetry group transformations, respectively, with the expansions analogous to the ones given in (1.152) and (1.153):

$$\left.\begin{aligned} \bar{\lambda}^{c\dot{\alpha}}(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[b_\mathrm{R}(\mathbf{k})a'^{\dot{\alpha}}(\mathbf{k})\exp(-\mathrm{i}k\cdot x) + d^\dagger_\mathrm{L}(\mathbf{k})b'^{\dot{\alpha}}(\mathbf{k})\exp(\mathrm{i}k\cdot x)\right] \\ \bar{\lambda}^{c\dagger\alpha}(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E}\left[b^\dagger_\mathrm{R}(\mathbf{k})a'^{\star\alpha}(\mathbf{k})\exp(\mathrm{i}k\cdot x) + d_\mathrm{L}(\mathbf{k})b'^{\star\alpha}(\mathbf{k})\exp(-\mathrm{i}k\cdot x)\right] \end{aligned}\right\} \tag{1.191}$$

where $a'(\mathbf{k})$ and $b'(\mathbf{k})$ are positive and negative energy chiral solutions to the free Weyl equation $\mathrm{i}\sigma^\mu\partial_\mu\bar{\chi}(x) = 0$ and b_R and $d^\dagger_\mathrm{L}$ are the annihilation and creation operators. In other words, the Weyl fields must be in a real representation. For instance, $R = R' \oplus R'^\star$ with, for example, λ^i_α transforming as R' and λ^{ci}_α transforming as $R'^\star$. The action of the parity operator $\mathcal{P}$ on the left-handed Weyl field $\lambda_\alpha(x)$ can now be defined. Identifying $\lambda^{\dot{\alpha}}_P(x') \equiv \bar{\lambda}^{c\dot{\alpha}}(x')$ and following the same arguments as for the scalar fields (see (1.187)), with (1.182) (or (1.184)) taken as the transformation law for the wave-functions, we get†

$$\mathcal{P}\lambda_\alpha(x)\mathcal{P}^{-1} = \exp(-\mathrm{i}\gamma)\sigma^0_{\alpha\dot{\beta}}\bar{\lambda}^{c\dot{\beta}}(x') \tag{1.192}$$

$$\mathcal{P}\bar{\lambda}^{c\dot{\alpha}}(x)\mathcal{P}^{-1} = \exp(-\mathrm{i}\bar{\gamma})(\bar{\sigma}^0)^{\dot{\alpha}\beta}\lambda_\beta(x') \tag{1.193}$$

For the creation and annihilation operators, using the properties (A.87) of the classical solutions of the Weyl equations, we obtain

$$\mathcal{P}b_\mathrm{L}(\mathbf{k})\mathcal{P}^{-1} = b_\mathrm{R}(-\mathbf{k}), \qquad \mathcal{P}d^\dagger_\mathrm{R}(\mathbf{k})\mathcal{P}^{-1} = d^\dagger_\mathrm{L}(-\mathbf{k}) \tag{1.194}$$

up to arbitrary and uncorrelated phases present in the classical solutions of the Weyl equations (A.61), (A.64).

For the four-component Dirac fields the parity operation can be realized as a mapping of $\mathcal{H}$ onto itself. In this case we can identify the operator Ψ_P with Ψ (they have the same transformation properties under the internal symmetry group)

† If the representation R is irreducible and real (like, for example, adjoint representation of $SU(N)$) then one can simply identify $\lambda^{\dot{\alpha}}_P(x')$ with $S_P\bar{\lambda}^{\dot{\alpha}}(x')$, where S_P is a constant matrix and parity can be a symmetry of the theory.

so that

$$\mathcal{P}\Psi(x)\mathcal{P}^{-1} = \exp(-\mathrm{i}\gamma)\gamma^0\Psi(x') \tag{1.195}$$

and

$$\mathcal{P}b(\mathbf{p}, s)\mathcal{P}^{-1} = \exp(-\mathrm{i}\gamma)b(-\mathbf{p}, s) \tag{1.196}$$

$$\mathcal{P}d^\dagger(\mathbf{p}, s)\mathcal{P}^{-1} = -\exp(-\mathrm{i}\gamma)d^\dagger(-\mathbf{p}, s) \tag{1.197}$$

and similarly for $b^\dagger$ and d but with the complex conjugate phase $\exp(\mathrm{i}\gamma)$. The spinor properties $\gamma^0 u(-\mathbf{p}, s) = u(\mathbf{p}, s)$, $\gamma^0 v(-\mathbf{p}, s) = -v(\mathbf{p}, s)$ have been used (see Appendix A). Analogous relations for helicity creation and annihilation operators can be derived using (A.89) Although the phase factor $\exp(-\mathrm{i}\gamma)$ is arbitrary, the (massive) particle and its antiparticle always have negative relative parity (assuming vacuum to be symmetric under the parity operation, $\mathcal{P}|0\rangle = +|0\rangle$):

$$\begin{aligned}\mathcal{P}|b\mathbf{k}_1, s_1, d\mathbf{k}_2, s_2\rangle &\equiv \mathcal{P}b^\dagger(\mathbf{k}_1, s_1)d^\dagger(\mathbf{k}_2, s_2)|0\rangle \\ &= -|b - \mathbf{k}_1, s_1, d - \mathbf{k}_2, s_2\rangle\end{aligned} \tag{1.198}$$

Note that this conclusion does not hold for massless Weyl states because the phases in (1.194) are arbitrary and uncorrelated (the states with opposite helicities are not related by Lorentz transformations).

For a massive Majorana field operator with the expansion given by (1.167) the application of the rule (1.195) gives two relations

$$\left.\begin{aligned}\mathcal{P}b(\mathbf{k}, s)\mathcal{P}^{-1} &= \exp(-\mathrm{i}\gamma)b(-\mathbf{k}, s) \\ \mathcal{P}b^\dagger(\mathbf{k}, s)\mathcal{P}^{-1} &= -\exp(-\mathrm{i}\gamma)b^\dagger(-\mathbf{k}, s)\end{aligned}\right\} \tag{1.199}$$

which are compatible only if $\exp(-\mathrm{i}\gamma) = \pm\mathrm{i}$. In this case the theory containing the massive Majorana spinor may be parity-invariant. It is also worth noting that for such a particle

$$\mathcal{P}|\mathbf{k}, s\rangle = \pm\mathrm{i}|-\mathbf{k}, s\rangle$$

i.e. a Majorana particle at rest ($\mathbf{k} = 0$) is a parity eigenstate with purely imaginary eigenvalue.

One can also check that the free Dirac lagrangian is invariant under space reflection: $\mathcal{P}\mathcal{L}(x)\mathcal{P}^{-1} = \mathcal{L}(x')$.

The parity of the electromagnetic field is determined by its coupling to the classical current, $j_\mu(x) = q\bar{\Psi}(x)\gamma_\mu\Psi(x)$. The requirement of the invariance of the classical lagrangian $\mathcal{L}_{\text{int}} = -ej_\mu(x)A^\mu(x)$ describing interaction of the charged four-component Dirac field Ψ with the electromagnetic field $A_\mu(x)$

together with the property $\Psi_P(x') = \exp(\mathrm{i}\gamma)\gamma_0\Psi(x)$ leads to the conclusion that $\bar{\Psi}_P(x')\gamma_\mu\Psi_P(x')A^\mu_P(x') = \bar{\Psi}(x)\gamma_\mu\Psi(x)A^\mu(x)$, provided

$$A^0_P(x') = A^0(x), \qquad \mathbf{A}_P(x') = -\mathbf{A}(x) \tag{1.200}$$

Thus, in quantum theory we identify

$$\left.\begin{aligned} A^0_P(x') &\equiv A^0(x') = \mathcal{P}A^0(x)\mathcal{P}^{-1} \\ \mathbf{A}_P(x') &\equiv \mathbf{A}(x') = -\mathcal{P}\mathbf{A}(x)\mathcal{P}^{-1} \end{aligned}\right\} \tag{1.201}$$

Time reversal

The classical sense of time inversion is clear: the velocities of a system in some final state are reversed and it is moving backwards. If the dynamics is invariant under such transformation, the system will eventually reach the original initial state. Formally, time reversal can be described by the change of coordinates† $x^\mu \to x'^\mu$ with

$$t' = -t, \qquad \mathbf{x}' = \mathbf{x} \tag{1.202}$$

Classical physical observables behave as follows (compare with (1.179)):

$$E' = E, \quad \mathbf{p}' = -\mathbf{p}, \quad \mathbf{J}' = -\mathbf{J}, \quad h' = h \tag{1.203}$$

(all time derivatives change sign). All charges remain, of course, unchanged. We consider now a classical complex scalar field $\Phi(x)$. Its transformation under time reversal can be inferred from the physical requirement that the charges (i.e. $j^a_0(t, \mathbf{x})$) do not change whereas for the space coordinates of the current $j^a_i(-t, \mathbf{x})_T = -j^a_i(t, \mathbf{x})$ (since the velocities are reversed). From (1.16) (or (1.57) for the abelian case) we get

$$\Phi_T(x') = \exp(\mathrm{i}\gamma)\Phi^\star(x) \tag{1.204}$$

where the phase factor can actually be only ± 1 (since the real component fields cannot change). Clearly, a positive energy plane wave with momentum $\mathbf{p}$ transforms into a positive energy plane wave in the reflected frame with momentum $\mathbf{p}' = -\mathbf{p}$, in accord with (1.203). The Klein–Gordon equation is form-invariant under time reversal, i.e. if $(\partial^2 + m^2)\Phi(x) = 0$ then also $(\partial'^2 + m^2)\Phi_T(x') = 0$. If the fields carry internal quantum number index i and transform as $\Phi'(x) = \exp(-\mathrm{i}\Theta^a T^a)\Phi(x)$ then the fields $\Phi_T(x')$ transform the same way in spite of the complex conjugation because the change $t \to -t$ assures invariance of the charges. The coupling with the electromagnetic field is invariant under the transformation

† Physically, it is however easier to think in terms of the active transformation.

(1.204) and the simultaneous transformation of the potential four-vector: $A_0(x') = A_0(x)$, $\mathbf{A}_i(x') = -\mathbf{A}_i(x)$.

Since helicity does not change under time reversal, the Lorentz transformation properties for the Weyl fields should not change. Because positive energy solutions with with momenta $\mathbf{p}$ must transform into positive energy solutions with $-\mathbf{p}$, the transformation on the Weyl fields must involve their complex conjugation. Moreover, from the additional physical requirement that charges are invariant and the space components of the current $j_\mu(x) = \bar{\lambda}\bar{\sigma}^\mu\lambda$ change sign we conclude that the transformation $\lambda_\alpha \to \lambda_{T\alpha}$ must include the matrix σ^2. Taking proper account of the Lorentz indices we finally conclude that

$$\lambda_{T\alpha}(x') = \exp(\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar{\sigma}^3)_\alpha{}^\beta(\lambda^\star(x))_\beta \tag{1.205}$$

where the phase factor $\exp(\mathrm{i}\gamma)$ is arbitrary. Furthermore, similarly to the space inversion, one can argue that a Lorentz transformation described by parameters η_i, ξ_i, in the 'time-reflected' frame is described by the parameters $\eta_i, -\xi_i$. Thus, the necessary relation

$$\exp(\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar{\sigma}^3)_\alpha{}^\beta \left(M(\boldsymbol{\eta},\boldsymbol{\xi})_\beta{}^\kappa\lambda_\kappa\right)^\star = M(\boldsymbol{\eta}, -\boldsymbol{\xi})_\alpha{}^\beta \exp(\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar{\sigma}^3)_\beta{}^\kappa(\lambda_\kappa)^\star \tag{1.206}$$

is fulfilled because $\sigma^1\bar{\sigma}^3 M^\star(\boldsymbol{\eta},\boldsymbol{\xi}) = M(\boldsymbol{\eta}, -\boldsymbol{\xi})\sigma^1\bar{\sigma}^3$. It is obvious that $\lambda_{T\alpha}(x')$ satisfies the Weyl equation for the left-handed spinors in the primed variables: $\mathrm{i}\bar{\sigma}\cdot\partial'\lambda_T(x') = 0$.

Similarly, the right-handed spinors transform into

$$\bar{\chi}^{\dot{\alpha}} \to \bar{\chi}_T^{\dot{\alpha}}(x') = \exp(\mathrm{i}\bar{\gamma})(\mathrm{i}\bar{\sigma}^1\sigma^3)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{\chi}^{\dot{\beta}}(x))^\star \tag{1.207}$$

and $\bar{\chi}_T^{\dot{\alpha}}(x')$ transform again as $(0, \frac{1}{2})$ under the Lorentz transformations.

The transformation law for the four-component Dirac spinors follows directly from the transformation properties of the left- and right-handed Weyl spinors. In the chiral representation of the Dirac matrices we have

$$\Psi(x) \to \Psi_T(x') = \begin{pmatrix} \exp(\mathrm{i}\gamma)\mathrm{i}\sigma^1\bar{\sigma}^3\lambda^\star(x) \\ \exp(\mathrm{i}\bar{\gamma})\mathrm{i}\bar{\sigma}^1\sigma^3\bar{\lambda}^{c\star}(x) \end{pmatrix} \tag{1.208}$$

Such transformation preserves the form of the massive Dirac equation (i.e. $\Psi_T(x')$ satisfies the Dirac equation in the primed variables) provided $\gamma = \bar{\gamma}$. In this case (1.208) can be rewritten as

$$\left.\begin{aligned} \Psi_T(x') &= \exp(\mathrm{i}\gamma)\mathrm{i}\gamma^1\gamma^3\Psi^\star(x) \\ \Psi_T^\dagger(x') &= -\exp(\mathrm{i}\gamma)\mathrm{i}\Psi^{\dagger\star}(x)\gamma^3\gamma^1 \end{aligned}\right\} \tag{1.209}$$

with the phase γ still arbitrary.

In the quantum case we define an operator $\mathcal{T}$ such that $|v_T\rangle = \mathcal{T}|v\rangle$. As for the space inversion, the time reflection can be a symmetry of the physical system only if the two Hilbert spaces can be identified with each other.

Interpreting the classical transformation (1.204) as the transformation on the wave-function $\langle 0|\Phi(x)|v\rangle$ for a state $|v\rangle$ we must have:

$$\langle 0_T|\Phi_T(x')|v_T\rangle = \exp(\mathrm{i}\gamma)\langle 0|\Phi(x)|v\rangle^\star \tag{1.210}$$

It follows that the operator $\mathcal{T}$ must be antilinear (antiunitary): $\mathcal{T}^\dagger\mathcal{T} = 1$ and $\mathcal{T}(\alpha_1|v\rangle + \alpha_2|w\rangle) = \alpha_1^\star\mathcal{T}|v\rangle + \alpha_2^\star\mathcal{T}|w\rangle$. This can be seen by applying (1.210) to a state vector $|v\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle$, with complex coefficients α_i.† By applying the antiunitarity condition successively to $|v\rangle$, $|v_1\rangle + |v_2\rangle$ and to $|w\rangle = |v_1\rangle + \mathrm{i}|v_2\rangle$ one can show that (Wigner 1931)

$$\langle w_T|v_T\rangle = \langle v|w\rangle = \langle w|v\rangle^\star \tag{1.211}$$

The l.h.s. of (1.210) can now be written as

$$\langle 0_T|\Phi_T(x')|v_T\rangle = \langle 0_T|\mathcal{T}\mathcal{T}^{-1}\Phi_T(x')\mathcal{T}|v\rangle = \langle 0|\mathcal{T}^{-1}\Phi_T(x')\mathcal{T}|v\rangle^\star \tag{1.212}$$

where we have applied‡ the rule (1.211) to the state $\mathcal{T}(\mathcal{T}^{-1}\Phi_T(x')\mathcal{T}|v\rangle)$. Therefore, identifying $\Phi_T(x')$ with $\Phi(x')$ and comparing with the r.h.s. of (1.210), we conclude that

$$\mathcal{T}\Phi(x)\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma)\Phi(x') \tag{1.213}$$

(for a real scalar field we must have $\exp(-\mathrm{i}\gamma) = \pm 1$). The transformation on the quantum fields does not involve any complex conjugation and the consistency with the classical transformation is ensured due to the antilinearity of $\mathcal{T}$. From (1.213) for the creation and annihilation operators we get:

$$\mathcal{T}a(\mathbf{k})\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma)a(-\mathbf{k}), \qquad \mathcal{T}b^\dagger(\mathbf{k})\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma)b^\dagger(-\mathbf{k}) \tag{1.214}$$

and similarly for their hermitean conjugates.

For the Weyl fields we can identify $\mathcal{H}_T$ with $\mathcal{H}$ and the operators $\lambda_{T\alpha}$ with λ_α. The equation for the matrix elements similar to (1.212) leads to

$$\left.\begin{aligned} \mathcal{T}\lambda_\alpha(x)\mathcal{T}^{-1} &= \exp(-\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar\sigma^3)_\alpha{}^\beta\lambda_\beta(x') \\ \mathcal{T}\bar\lambda^{\dot\alpha}(x)\mathcal{T}^{-1} &= \exp(-\mathrm{i}\bar\gamma)(\mathrm{i}\bar\sigma^1\sigma^3)^{\dot\alpha}{}_{\dot\beta}\bar\lambda^{\dot\beta}(x') \end{aligned}\right\} \tag{1.215}$$

† Antiunitarity of the $\mathcal{T}$ operator can be justified also by applying it to the time component of (1.174):

$$\mathcal{T}[H, \Phi(x)]\mathcal{T}^{-1} = \mathcal{T}\left(-\mathrm{i}\frac{\partial\Phi(x)}{\partial t}\right)\mathcal{T}^{-1}$$

The physically motivated requirement that $\mathcal{T}H(t)\mathcal{T}^{-1} = H(t')$ again leads to the conclusion that if $\Phi(x')$ is to satisfy (1.174) with primed variables then $\mathcal{T}$ must be antiunitary.

‡ Notice that since $\mathcal{T}$ is antiunitary $\langle 0_T| \neq \langle 0|\mathcal{T}^{-1}$.

Using the properties (A.91) of the solutions to the Weyl equations we get for the creation and annihilation operators

$$\mathcal{T} b_{\mathrm{L}}(\mathbf{k})\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma) b_{\mathrm{L}}(-\mathbf{k}), \qquad \mathcal{T} d_{\mathrm{R}}^{\dagger}(\mathbf{k})\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma) d_{\mathrm{R}}^{\dagger}(-\mathbf{k}) \quad (1.216)$$

(up to some arbitrary phase factors associated with the classical solutions for the Weyl spinors) and similar relations for the hermitean conjugate operators. It is clear that the operator $\mathcal{T}$ changes the direction of the momentum of the state leaving its helicity unchanged.

For the four-component Dirac fields one gets

$$\mathcal{T}\Psi(x)\mathcal{T}^{-1} = \exp(-\mathrm{i}\gamma)(\mathrm{i}\gamma^1\gamma^3)\Psi(x') \quad (1.217)$$

leading, upon using the properties of the classical solutions to the Dirac equation (summarized in (A.93)), to the relations

$$\left.\begin{aligned} \mathcal{T} b(\mathbf{k}, s)\mathcal{T}^{-1} &= -\mathrm{i}(-1)^s \exp(-\mathrm{i}\gamma) b(-\mathbf{k}, s') \\ \mathcal{T} d^{\dagger}(\mathbf{k}, s)\mathcal{T}^{-1} &= \mathrm{i}(-1)^s \exp(-\mathrm{i}\gamma) d^{\dagger}(-\mathbf{k}, s') \end{aligned}\right\} \quad (1.218)$$

($s' = 2(1)$ for $s = 1(2)$; $s = 1(2)$ denotes the spin projection equal $+\frac{1}{2}(-\frac{1}{2})$ with respect to the chosen quantization axis). For the massive Majorana field (1.167) the application of rule (1.217) leads to the first relation (1.218) and the consistency condition $\exp(-2\mathrm{i}\gamma) = \exp(2\mathrm{i}\rho)$.

The transformation properties of the electromagnetic field under the operation of time reversal (already discussed) can be determined by its coupling to the classical current, $j_\mu(x) = q\bar{\Psi}(x)\gamma_\mu\Psi(x)$. As in the case of the parity transformation, the requirement of the invariance of the lagrangian $\mathcal{L}_{\mathrm{int}} = -ej_\mu(x)A^\mu(x)$ together with the properties (1.209) leads to the conclusion† that $\bar{\Psi}_T(x')\gamma_\mu\Psi_T(x')A^\mu_T(x')$ $= \bar{\Psi}(x)\gamma_\mu\Psi(x)A^\mu(x)$ provided

$$A^0_T(x') = A^0(x), \qquad \mathbf{A}_T(x') = -\mathbf{A}(x) \quad (1.219)$$

Thus, in quantum theory

$$\left.\begin{aligned} A^0_T(x') &\equiv A^0(x') = \mathcal{T} A^0(x)\mathcal{T}^{-1} \\ \mathbf{A}_T(x') &\equiv \mathbf{A}(x') = -\mathcal{T}\mathbf{A}(x)\mathcal{T}^{-1} \end{aligned}\right\} \quad (1.220)$$

Charge conjugation

The transformation of charge conjugation is genuinely connected to quantum physics. It can be defined as a transformation on wave-functions (positive energy solutions to the classical wave equations) of quantum systems. We consider a

† In verifying the invariance of the quantum lagrangian by using (1.218) one should also remember that $\mathcal{T}\gamma^\mu\mathcal{T}^{-1} = \gamma^{\mu\star}$.

physical system which carries some internal quantum numbers (charges) and is described by the wave-function $\Phi_i(x)$ transforming as a representation R of a group of transformations G:

$$\text{abelian} \quad \Phi'_{\text{pos}}(x) = \exp(-\mathrm{i}q\Theta)\Phi_{\text{pos}}(x) \tag{1.221}$$

or

$$\text{non-abelian} \quad \Phi^{i\prime}_{\text{pos}}(x) = \left(\exp(-\mathrm{i}\Theta^a T^a)\right)^i_{\ j} \Phi^j_{\text{pos}}(x) \tag{1.222}$$

We assume that the group G is a global symmetry group of this system. One can introduce a 'charge conjugate' reference frame (a 'reflection' in the space of internal charges). For an observer using this reference frame the same physical system is described by another wave-function $\Phi^c_i(x)$ which transforms as representation $R^\star$ of G, complex conjugate to R but has the same space-time Lorentz transformation properties and the same dependence on the space-time variables as the wave-function $\Phi_i(x)$

$$\Phi^{c\prime}_{\text{pos}}(x) = \exp[-\mathrm{i}(-q)\Theta]\Phi^c_{\text{pos}}(x) \tag{1.223}$$

or

$$\Phi^{ci\prime}_{\text{pos}}(x) = \left(\exp[-\mathrm{i}\Theta^a(-T^{a\star})]\right)^i_{\ j} \Phi^{cj}_{\text{pos}}(x) \tag{1.224}$$

(see (E.10) for the definition of complex and real representations). Thus, 'opposite' internal charges are assigned to the same system. This change of the 'reference frame' is called the transformation of charge conjugation (we could as well take the active view and replace the original system by a new one with 'reversed' charges). This transformation may or may not be a symmetry of a given system.

Charge conjugation is a symmetry of the system if the charge conjugate wave function $\Phi^c_i(x)$ can be identified with a solution of the relativistic equation of motion of the system, obtained by a transformation on its negative energy solutions and usually also called charge conjugation. One should stress that from the fundamental fact of the existence of the negative energy solutions to the relativistic wave equations follows the existence of particle and antiparticle pairs (i.e. the states whose wave-functions transform as R and $R^\star$, respectively): the negative energy solutions can always be reinterpreted as antiparticle (positive energy) wave-functions. However, after such reinterpretation (often and somewhat misleadingly also called charge conjugation) the antiparticle wave-functions may satisfy different equations of motion than the negative energy solutions did and cannot be identified with the 'charge conjugate' wave-function of the system. This means that the system seen by the observer in the new reference frame satisfies other equations of motion than in the original reference frame and the transformation of charge conjugation is not a symmetry of this particular system.

Again, this does not change the fact that particles and antiparticles do exist in pairs, as predicted by the presence of the negative energy solutions to the equations of motion.

We now consider several examples. We begin with a set of free complex scalar fields $\Phi_i(x)$ which satisfy the Klein–Gordon equation. This equation is invariant under the operation

$$\Phi_i(x) \to \Phi_i^c(x) \equiv \exp(\mathrm{i}\gamma)\Phi_i^\star(x) \tag{1.225}$$

Thus, if $\Phi_i(x)$ are solutions of the Klein–Gordon equation the same is true for the functions $\Phi_i^c(x)$. It is trivial but important to notice that the operation (1.225) transforms negative energy solutions into positive energy ones and vice versa, i.e. transforms solutions $\Phi_i^{\text{neg}}(x)$ which have no wave-function interpretation into $\Phi_i^{c\,\text{pos}}(x)$ which can be interpreted as wave-functions of antiparticles (with reversed global quantum numbers). Moreover, the space-time dependences of the functions $\Phi_i^{\text{pos}}(x)$ and $\Phi_i^{c\,\text{pos}}(x)$ are the same. Thus, the existence of the symmetry operation (1.225) for the relativistic scalar wave equation provides the interpretation of its negative energy solutions and one predicts the existence of particle–antiparticle pairs described by wave-functions with the same space-time form but opposite internal charges.

We may extend our considerations to scalar fields with gauge symmetries, i.e. coupled in a gauge-invariant way to (abelian or non-abelian) gauge vector fields. The equations of motion follow then from the lagrangian

$$\begin{aligned}\mathcal{L} &= \left|\left(\partial^\mu + \mathrm{i}gA^{a\mu}T^a\right)\Phi\right|^2 \\ &= \Phi^*\left(\overleftarrow{\partial}_\mu - \mathrm{i}gA_\mu^a T^a\right)\left(\overrightarrow{\partial}^\mu + \mathrm{i}gA^{a\mu}T^a\right)\Phi\end{aligned}$$

(in going from the first to the second line the hermiticity of the generators T^a has been used). It is easy to check that they are invariant under the operation

$$\left.\begin{aligned}&\Phi_i(x) \to \Phi_i^c(x) = \exp(\mathrm{i}\gamma)(\Phi_i(x))^\star \\ &A_\mu^a(x) \to A_\mu^{ca}(x) : A_\mu^{ca}(x)(T^a)^* = -A_\mu^a(x)T^a\end{aligned}\right\} \tag{1.226}$$

i.e. the wave-function of the antiparticle in the charge conjugate vector potential $A_\mu^{ca}(x)$ has the same form as the particle wave-function in the potential $A_\mu^a(x)$.

For the Weyl equation there is no symmetry operation which transforms its negative energy solutions $\lambda_{i\alpha}^{\text{neg}}(x)$ ($\bar{\lambda}_i^{\dot\alpha\,\text{neg}}(x)$) into solutions $\lambda_{i\alpha}^{c\,\text{pos}}(x)$ ($\bar{\lambda}_i^{c\dot\alpha\,\text{pos}}(x)$) of the same equation with the same Lorentz transformation properties and the same space-time dependence as $\lambda_{i\alpha}^{\text{pos}}(x)$ ($\bar{\lambda}_i^{\dot\alpha\,\text{pos}}(x)$) and with the internal charges reversed compared to the charges of $\lambda_{i\alpha}^{\text{pos}}(x)$. Instead, the operation $\lambda_{i\alpha}(x) \to \bar{\lambda}_i^{\dot\alpha}(x)$ transforms the negative energy solution $\lambda_{i\alpha}^{\text{neg}}(x)$ (transforming as R and $(\frac{1}{2}, 0)$ under internal and Lorentz groups, respectively) into $\bar{\lambda}_i^{\dot{\alpha}\,\mathrm{pos}}(x)$ which satisfies the Weyl equation for right-handed spinors (i.e. transforming as $R^\star$ and $(0, \frac{1}{2})$).

The same conclusion remains valid for the Weyl fields coupled to the abelian or non-abelian gauge fields. Therefore the positive energy solution to a Weyl equation is a wave-function of a particle with $h = -\frac{1}{2}(+\frac{1}{2})$ and the 'charge conjugate' form of the negative energy solution is the wave-function of its antiparticle with $h = +\frac{1}{2}(-\frac{1}{2})$.† However, for a quantum system described by a set of left- and right-handed Weyl equations with solutions $\lambda_{i\alpha}^{\mathrm{pos}}(x)$, $\bar{\lambda}_i^{c\dot{\alpha}\,\mathrm{pos}}(x)$ transforming as R, $(\frac{1}{2}, 0)$ and $R^\star$, $(0, \frac{1}{2})$ there exists the symmetry operation $\lambda_{i\alpha} \to \bar{\lambda}_i^{\dot{\alpha}}$, $\bar{\lambda}_i^{c\dot{\alpha}} \to \lambda_{i\alpha}^c$ which transforms the negative energy solutions of each of the equations into the positive energy solutions of the other equations, with the same Lorentz and space-time dependence as $\bar{\lambda}_i^{\dot{\alpha}\,\mathrm{pos}}(x)$ and $\lambda_{i\alpha}^{\mathrm{pos}}(x)$, respectively. Such a system is equivalent to a Dirac field

$$\Psi(x) = \begin{pmatrix} \lambda_{i\alpha}(x) \\ \bar{\lambda}_i^{c\dot{\beta}}(x) \end{pmatrix} \tag{1.227}$$

which satisfies the Dirac equation. The operation

$$\Psi(x) \to \Psi^c(x) = \exp(\mathrm{i}\gamma) \begin{pmatrix} \lambda_{i\alpha}^c(x) \\ \bar{\lambda}_i^{\dot{\beta}}(x) \end{pmatrix} \tag{1.228}$$

transforms the negative energy solutions $\Psi^{\mathrm{neg}}(x)$ into positive energy $\Psi^{c\,\mathrm{pos}}(x)$ which transform as $R^\star$ and have the same space-time dependence as $\Psi^{\mathrm{pos}}(x)$. For a general Dirac field (a solution to the massive Dirac equation) there exists the symmetry operation

$$\left.\begin{aligned} \Psi^c &= \exp(\mathrm{i}\gamma) C\gamma^0\Psi^\star = \exp(\mathrm{i}\gamma) C\bar{\Psi}^T, \\ \bar{\Psi}^c &= -\exp(-\mathrm{i}\gamma)\Psi^T C^{-1} \end{aligned}\right\} \tag{1.229}$$

where the 4×4 matrix C (its explicit form and properties are given in Appendix A) satisfies $(C\gamma_0)\gamma^{\mu\star}(C\gamma_0)^{-1} = -\gamma^\mu$, so that Ψ^c indeed satisfies the Dirac equation. For the four-component spinors we have $C\bar{u}^T(\mathbf{p}, s) = v(\mathbf{p}, s)$ and $C\bar{v}^T(\mathbf{p}, s) = u(\mathbf{p}, s)$.

For a Dirac field coupled to (abelian or non-abelian) gauge fields the lagrangian

$$\mathcal{L} = \bar{\Psi}\left(\mathrm{i}\not{\partial} - g\not{A}^a T^a - m\right)\Psi$$

is invariant under simultaneous transformation

$$\Psi \to \Psi^c, \qquad A_\mu^a \to A_\mu^{ca}(x) \tag{1.230}$$

with $A_\mu^{ca}(x)$ defined in (1.226) (remember that Ψs are anticommuting c-numbers;

† Note that the helicity of the solution $\bar{\lambda}_i^{\dot{\alpha}\,\mathrm{pos}}(x)$ is the same as the helicity of $\lambda_{i\alpha}^{\mathrm{neg}}(x)$.

the Dirac lagrangian is invariant up to a total derivative). After such transformations, the space-time dependence of the solutions $\Psi(x)$ and $\Psi^c(x)$ remains the same.

With few exceptions (see (1.243)), the arbitrary phase γ in (1.225) and (1.229) is unphysical and can be taken as zero. The transformation properties of the spinor bilinears follow from (1.229): $\bar{\Psi}_1^c\Psi_2^c = \bar{\Psi}_2\Psi_1$, $\bar{\Psi}_1^c\gamma_\mu\Psi_2^c = -\bar{\Psi}_2\gamma_\mu\Psi_1$ etc.

Moving now to the description of the physical system in terms of the state vectors (rather than their representation in the configuration space) the operation of charge conjugation maps the original Hilbert space $\mathcal{H}$ onto the Hilbert space $\mathcal{H}^c$ which describes the physical system in the 'charge conjugate' frame. The states in $\mathcal{H}^c$ have the same space-time description but opposite internal charges. This operation can be a symmetry only if $\mathcal{H}$ and $\mathcal{H}^c$ are identical and can be identified with each other (in consequence, the operators acting in $\mathcal{H}$ can be identified with some operators acting in $\mathcal{H}^c$). We can then define a linear (unitary) operator $\mathcal{C}$, $|v^c\rangle = \mathcal{C}|v\rangle$. Next, we would like to find the action of $\mathcal{C}$ on the field operators. We use the fact that for the wave-functions $\langle 0|\Phi^c(x)|v^c\rangle = \exp(\mathrm{i}\gamma)\langle 0|\Phi(x)|v\rangle$, i.e. they have the same space-time dependence. For scalar field theory, we see that, in agreement with the transformation properties under internal symmetry group, we can identify the operators

$$\Phi^c(x) \equiv \Phi^\dagger(x) \tag{1.231}$$

getting†

$$\mathcal{C}\Phi_i(x)\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)\Phi_i^\dagger(x) \tag{1.232}$$

The presence in (1.232) of an arbitrary phase reflects the freedom to redefine any quantum state by multiplying it by a phase factor. In the presence of interactions this freedom can be used to define the maximal discrete symmetry of the system. The creation and annihilation operators defined in (1.126) and (1.143) satisfy the following relations

$$\mathcal{C}a(\mathbf{k})\mathcal{C}^{-1} = \pm a(\mathbf{k}) \tag{1.233}$$

for real scalar fields (1.126) and

$$\mathcal{C}a(\mathbf{k})\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)b(\mathbf{k}), \qquad \mathcal{C}b^\dagger(\mathbf{k})\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)a^\dagger(\mathbf{k}) \tag{1.234}$$

for complex scalar fields (1.143). Remembering our discussion in Section 1.4 we see that, consistently with its definition, charge conjugation transforms a particle state with four-momentum $(E, \mathbf{k})$ into the antiparticle state with the same four-momentum.

† It is worth stressing the difference between charge conjugation and time reversal, compare (1.225) with (1.204) and (1.213) with (1.232). This difference follows from the antilinearity of $\mathcal{T}$.

It is sometimes very useful to consider the eigenstates of the charge conjugation operator $\mathcal{C}$ (similar comments apply to the combined transformation CP). For instance, when C (and/or CP) is a good symmetry and the system is neutral in all conserved charges (the latter is necessary since, for example, for abelian charges $\mathcal{C}Q = -Q\mathcal{C}$, so no common eigenstates exist) then the description in terms of the $\mathcal{C}$ (or $\mathcal{CP}$) eigenstates is convenient. Important physical examples are the positronium state (electromagnetic interactions conserve both C and Q but positronium is electrically neutral) and the K^0–$\bar{\mathrm{K}}^0$ (or B^0–$\bar{\mathrm{B}}^0$) mixing by electroweak interactions ($\mathcal{CP}$ is almost conserved but the flavour charges like strangeness or beauty are not). The K^0–$\bar{\mathrm{K}}^0$ mixing will be discussed in more detail at the end of this section and in Section 12.7. Assuming the vacuum to be invariant under C, it then follows that the states given by the linear superposition of the particle and antiparticle states

$$|k_C\rangle = \frac{\exp(\mathrm{i}\phi_\pm)}{\sqrt{2}}\left(a^\dagger(\mathbf{k})|0\rangle \pm \exp(\mathrm{i}\gamma)b^\dagger(\mathbf{k})|0\rangle\right) \tag{1.235}$$

are eigenvectors of the charge conjugation operator, with eigenvalues ± 1, respectively. The presence in this equation of the phase factor $\exp(\mathrm{i}\gamma)$ is crucial for a phase-convention-free conclusion. Note also that the state $|k_C\rangle$ can be written as

$$|k_C\rangle = \frac{\exp(\mathrm{i}\phi_\pm)}{\sqrt{2}}\left(a^\dagger(\mathbf{k})|0\rangle \pm \mathcal{C}a^\dagger(\mathbf{k})|0\rangle\right) \tag{1.236}$$

In a very similar manner one can construct the states at rest invariant under the CP transformation even when C and P are not conserved separately. For the states with momentum $\mathbf{k}$ the CP acts as follows: $\mathcal{CP}|a,\mathbf{k}\rangle = \exp(\mathrm{i}\gamma)|b,-\mathbf{k}\rangle$.

For the Weyl fields, the Hilbert space $\mathcal{H}_C$ can be identified with $\mathcal{H}$ (and C can be a symmetry operations) only if the set of those fields transforms as a real representation of the group G. With λ_α and λ^c_α transforming as R and $R^\star$, respectively, and following the same arguments as for the (1.232) we get

$$\mathcal{C}\lambda_\alpha\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)\lambda^c_\alpha \tag{1.237}$$

and a similar equation for $\lambda^\dagger_{\dot\alpha}$ with the opposite phase. Using the expansions of the fields λ_α and λ^c_α given in (1.152) and (1.153) we arrive at:

$$\mathcal{C}b_\mathrm{L}(\mathbf{k})\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)d_\mathrm{L}(\mathbf{k}), \tag{1.238}$$

$$\mathcal{C}d^\dagger_\mathrm{R}(\mathbf{k})\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)b^\dagger_\mathrm{R}(\mathbf{k}), \tag{1.239}$$

up to arbitrary phases present in the classical solutions for the Weyl equations. It is clear that a particle state with momentum $\mathbf{k}$ and helicity h is transformed under charge conjugation into an antiparticle state with the same momentum and helicity.

Following the same arguments, the action of $\mathcal{C}$ on the four-component Dirac field operators is given by

$$\Psi^c(x) \equiv C\bar{\Psi}^\dagger = \exp(\mathrm{i}\gamma)\mathcal{C}\Psi(x)\mathcal{C}^{-1} \tag{1.240}$$

which leads to

$$\mathcal{C}b(\mathbf{k}, s)\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)d(\mathbf{k}, s) \tag{1.241}$$

$$\mathcal{C}d^\dagger(\mathbf{k}, s)\mathcal{C}^{-1} = \exp(-\mathrm{i}\gamma)b^\dagger(\mathbf{k}, s) \tag{1.242}$$

for the creation and annihilation operators. We have used the transformation properties (A.46) for spinors $u(\mathbf{k}, s)$ and $v(\mathbf{k}, s)$.

For a Majorana field with expansion (1.167), using the definition (1.229) we obtain

$$\left.\begin{aligned} \mathcal{C}b^\dagger(\mathbf{k}, s)\mathcal{C}^{-1} &= \exp(-\mathrm{i}\rho)\exp(-\mathrm{i}\gamma)b^\dagger(\mathbf{k}, s) \\ \mathcal{C}b(\mathbf{k}, s)\mathcal{C}^{-1} &= \exp(-\mathrm{i}\rho)\exp(-\mathrm{i}\gamma)b(\mathbf{k}, s) \end{aligned}\right\} \tag{1.243}$$

Consistency of these two equations requires

$$\exp(-\mathrm{i}\rho)\exp(-\mathrm{i}\gamma) = \pm 1 \tag{1.244}$$

Thus, the Majorana field satisfies

$$\mathcal{C}\Psi_\mathrm{M}(x)\mathcal{C}^{-1} = \pm\Psi_\mathrm{M}(x) \tag{1.245}$$

It is its own charge conjugate up to a sign. Also, it follows that a Majorana particle is an eigenstate of the charge conjugation operator with eigenvalue ± 1. Thus, a state of such a particle at rest may be a $\mathcal{CP}$ eigenstate with eigenvalues $\pm\mathrm{i}$ (see (1.199)).

Summary and the CPT transformation

We have discussed the action of the discrete transformations (space, time and charge inversions) on classical fields. We have identified the requirements which are necessary for these transformations to be symmetries of free (or interacting with the gauge fields) quantum systems of scalar, Weyl and Dirac particles described by state vectors in some Hilbert space. For those transformations which are symmetries of the system one can introduce unitary operators acting on quantum fields.

It is interesting that even free particle systems are not all invariant under the three discrete transformations. We have learned that a system of free Weyl particles which transforms as a complex representation R of some group of internal symmetries is invariant under the time reflection but not under separate transformations of space reflection and charge conjugation. It is natural to ask what happens to such a system under simultaneous C and P transformations. It turns out that the Weyl equation is invariant under the replacement $\lambda_\alpha(x) \to \lambda_{CP}(x) = \exp(-i\gamma)\bar{\lambda}^{\dot{\alpha}}(t, -\mathbf{x})$, i.e. the field $\lambda_{CP}(x)$ satisfies the same equation as the field $\lambda_\alpha(x)$. For quantized fields $\mathcal{CP}\lambda_\alpha(x)(\mathcal{CP})^{-1} = \exp(-\gamma)\bar{\lambda}^{\dot{\alpha}}(t, -\mathbf{x})$ and then using (1.152) and (1.153) we can infer the transformation properties of the annihilation and creation operators:

$$\left.\begin{aligned} \mathcal{CP}b_{\mathrm{L}}(\mathbf{k})(\mathcal{CP})^{-1} &= \exp(i\gamma)d_{\mathrm{R}}(-\mathbf{k}) \\ CPd_{\mathrm{R}}^{\dagger}(\mathbf{k})CP^{-1} &= \exp(i\gamma)b_{\mathrm{L}}^{\dagger}(-\mathbf{k}) \end{aligned}\right\} \tag{1.246}$$

A particle state with momentum $\mathbf{k}$ and helicity h is transformed under CP transformation into an antiparticle state with opposite momentum and helicity (as expected from the Dirac sea interpretation of the negative energy solutions to the Weyl equation). We conclude that all the considered systems are invariant under the CP and CPT transformations.

The discrete symmetries can easily be broken by interactions or even mass terms. Consider the theory of Weyl fermions in which each set of λs transforming as R has its counterpart λ^c transforming as $R^\star$ of the internal symmetry group. The most general hermitean mass term of the lagrangian is

$$\mathcal{L}_{\text{mass}} = -m_{AB}\lambda_A\lambda_B^c - m_{AB}^\star\bar{\lambda}_A\bar{\lambda}_B^c \tag{1.247}$$

It is straightforward to check that if $\mathcal{L}_{\text{mass}}$ is to conserve P we must have $m_{AB} = m_{BA}^\star$, i.e. the mass matrix m_{AB} must be hermitean. Similarly, for C to be conserved the mass matrix must be symmetric. For CP to be conserved m_{AB} must be real. Finally, it can be checked that $\mathcal{L}_{\text{mass}}$ remains invariant under the CPT transformation

$$\mathcal{CPT}\mathcal{L}_{\text{mass}}(\mathcal{CPT})^{-1} = \mathcal{L}_{\text{mass}} \tag{1.248}$$

independently of the form of the matrix m_{AB}. Similar considerations apply to complex Yukawa couplings. One should also stress that often there exists some arbitrariness in the field phase conventions, which can be used to make the maximal symmetry of the lagrangian explicit.

The importance of the CPT transformation follows from a fundamental theorem of quantum field theory which says that CPT invariance follows from the basic requirements of locality, Lorentz invariance and unitarity. Among the consequences of the CPT theorem are equal masses for particles and their antiparticles, equal lifetimes, equal and opposite electric charges, and equal magnetic moments.

A number of experimental upper limits on violation of the CPT theorem can be found in the Particle Data Book (1996). They include several measurements on electrons and positrons, muons and protons. The most stringent bound exists for the mass difference of the K^0 and $\bar{\mathrm{K}}^0$: $\Delta m_{\mathrm{K}}/m_{\mathrm{K}} < 9 \times 10^{-19}$. In spite of the tremendous accuracy with which the CPT theorem has been verified, it is of great interest if the theorem remains valid at even smaller scales of length, down to the Planck scale. (For a review of speculations on the possibility of violation of the CPT theorem, see Ellis, Mavromatos & Nanopoulos (1996).)

CP *violation in the neutral* K^0–$\bar{\mathrm{K}}^0$*-system*

In this subsection, we shall discuss the effects of the CP violation in the K^0–$\bar{\mathrm{K}}^0$ mixing. The physical processes under consideration here will be neutral kaon decays to two pions. They were the first processes in which violation of the CP symmetry was observed (Christenson *et al.* 1964). More details about CP violation in the neutral kaon system can be found in Jarlskog (1989) and Bernabeu (1997).

We recall that both pions and neutral kaons belong to the lightest quark–antiquark bound states (mesons) which are stable with respect to strong interactions. They decay only due to electroweak interactions. Their flavour structure is the following:

$$|\pi^+\rangle = |u\bar{d}\rangle \tag{1.249}$$

$$|\pi^-\rangle = |d\bar{u}\rangle \tag{1.250}$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left[|u\bar{u}\rangle - |d\bar{d}\rangle\right] \tag{1.251}$$

$$|\mathrm{K}^0\rangle = |d\bar{s}\rangle \tag{1.252}$$

$$|\bar{\mathrm{K}}^0\rangle = |s\bar{d}\rangle \tag{1.253}$$

While the pion states π^+, π^- and π^0 are physically observable particles, the neutral kaon states K^0 and $\bar{\mathrm{K}}^0$ combine to the following physical ones:

$$|\mathrm{K_L}\rangle = \frac{1}{\left[2(1+|\bar{\varepsilon}|^2)\right]^{1/2}}\left[(1+\bar{\varepsilon})|\mathrm{K}^0\rangle - (1-\bar{\varepsilon})|\bar{\mathrm{K}}^0\rangle\right] \tag{1.254}$$

$$|\mathrm{K_S}\rangle = \frac{1}{\left[2(1+|\bar{\varepsilon}|^2)\right]^{1/2}}\left[(1+\bar{\varepsilon})|\mathrm{K}^0\rangle + (1-\bar{\varepsilon})|\bar{\mathrm{K}}^0\rangle\right] \tag{1.255}$$

We choose our phase convention for normalization of the neutral kaon states so that $\mathcal{CP}|\mathrm{K}^0\rangle = |\bar{\mathrm{K}}^0\rangle$ and $\mathcal{CPT}|\mathrm{K}^0\rangle = |\bar{\mathrm{K}}^0\rangle$.

The names of $\mathrm{K_{L(ong)}}$ and $\mathrm{K_{S(hort)}}$ originate from their drastically different lifetimes $\tau_{\mathrm{L}} \simeq 5 \times 10^{-8}$ s and $\tau_{\mathrm{S}} \simeq 9 \times 10^{-11}$ s, respectively. On the other hand, their masses $m_{\mathrm{L}} \simeq m_{\mathrm{S}} \simeq 498$ MeV are almost equal: $\Delta m \equiv m_{\mathrm{L}} - m_{\mathrm{S}} \simeq 3.5 \times 10^{-12}$ MeV.

If CP was conserved in Nature, the parameter $\bar{\varepsilon}$ would vanish. Then, K_L and K_S would be CP-odd and CP-even, respectively. In such a case, K_L could not decay to two pions which are CP-even in the S-wave. In reality, K_L indeed decays much more often to three than to two pions. This explains its long lifetime, because decays to three pions are strongly phase-space suppressed.

Let us consider a state which at the initial time t_0 is a neutral kaon at rest. For $t > t_0$, we write it as a linear combination of eigenstates of the strong interaction hamiltonian H_s

$$|\Psi(t)\rangle = a(t)|K^0\rangle + \bar{a}(t)|\bar{K}^0\rangle + \sum_n b_n(t)|n\rangle \tag{1.256}$$

where

$$H_s|K^0\rangle = m_K|K^0\rangle, \qquad H_s|\bar{K}^0\rangle = m_K|\bar{K}^0\rangle, \qquad H_s|n\rangle = E_n|n\rangle \tag{1.257}$$

and $b_n(t = t_0) = 0$. The states $|n\rangle$ stand for all the eigenstates of H_s which are different from $|K^0\rangle$ and $|\bar{K}^0\rangle$.

The state $|\Psi(t)\rangle$ satisfies the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)\rangle = (H_s + H_w)|\Psi(t)\rangle \tag{1.258}$$

where H_w is the electroweak interaction hamiltonian. Solving this equation perturbatively in H_w, we find the time evolution equation for $a(t)$ and $\bar{a}(t)$

$$i\frac{d}{dt}\begin{bmatrix} a(t) \\ \bar{a}(t) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} a(t) \\ \bar{a}(t) \end{bmatrix} \tag{1.259}$$

expressed in terms of matrix elements of H_w between strong interaction eigenstates:

$$H_{ij} = M_{ij} - \frac{i}{2}\Gamma_{ij} \qquad (M = M^\dagger,\ \Gamma = \Gamma^\dagger) \tag{1.260}$$

$$M_{ij} = m_K\delta_{ij} + \langle i|H_w|j\rangle + \mathrm{P}\sum_n \frac{\langle i|H_w|n\rangle\langle n|H_w|j\rangle}{m_K - E_n} + \mathcal{O}(H_w^3) \tag{1.261}$$

$$\Gamma_{ij} = 2\pi\sum_n \langle i|H_w|n\rangle\langle n|H_w|j\rangle\delta(m_K - E_n) + \mathcal{O}(H_w^3) \tag{1.262}$$

Here, the sum over n denotes the sum-integral over all states different from $|K^0\rangle \equiv |1\rangle$ and $|\bar{K}^0\rangle \equiv |2\rangle$, while P stands for the principal value.†

The matrix H_{ij} in (1.259) is not hermitean, because the coefficients $b_n(t)$ have been removed from the evolution equation. This is the reason why the absorptive part Γ_{ij} arises in H_{ij}. It parametrizes the decay rates of the kaons to other strong interaction eigenstates.

† Eqs. (1.261) and (1.262) are obtained by sending t_0 to $(-\infty)$ and turning H_w adiabatically off at $t \to \pm\infty$ (see, for example, Landau & Lifshitz, (1959) §43).

It is important to note that the $\mathcal{O}(H_\mathrm{w})$ and $\mathcal{O}(H_\mathrm{w}^2)$ parts of H_{ij} are actually of the same order in the electroweak coupling constants. This is because the leading contribution to $\langle i|H_\mathrm{w}|j\rangle$ originates from double $W^\pm$ boson exchange (Fig. 12.17), while kaon decays can proceed via only single $W^\pm$ boson exchange.

The CPT symmetry of electroweak interactions implies that the diagonal elements of the matrix H_{ij} are equal, i.e.

$$H_{11} = H_{22} \equiv H \tag{1.263}$$

If electroweak interactions were symmetric under CP, then the off-diagonal elements H_{12} and H_{21} would be equal, too. Although the latter equality is violated in Nature, it is usually convenient to verify what happens in this limit with the relations we shall discuss below.

The eigenvalues of the matrix H_{ij} are

$$\lambda_{\mathrm{L,S}} = m_{\mathrm{L,S}} - \frac{\mathrm{i}}{2}\Gamma_{\mathrm{L,S}} = H \pm H_{12}^{1/2} H_{21}^{1/2} \tag{1.264}$$

while the corresponding eigenvectors $|\mathrm{K_L}\rangle$ and $|\mathrm{K_S}\rangle$ are given in (1.254) and (1.255), respectively. The parameter $\bar{\varepsilon}$ which occurs in these equations is related to the matrix H_{ij} as follows:

$$\bar{\varepsilon} = \frac{H_{12}^{1/2} - H_{21}^{1/2}}{H_{12}^{1/2} + H_{21}^{1/2}}. \tag{1.265}$$

Here, we have assumed that $H_{21}^{1/2}$ tends continuously to $H_{12}^{1/2}$ when H_{21} tends to H_{12} in the limit of no CP violation.†

Once we know the eigenbasis of H_{ij}, solving (1.259) is trivial. The considered state $|\Psi(t)\rangle$ can now be written as follows:

$$\begin{aligned} |\Psi(t)\rangle = {} & c_\mathrm{L} \exp\left(-\mathrm{i}m_\mathrm{L}t - \tfrac{1}{2}\Gamma_\mathrm{L}t\right)|\mathrm{K_L}\rangle \\ & + c_\mathrm{S} \exp\left(-\mathrm{i}m_\mathrm{S}t - \tfrac{1}{2}\Gamma_\mathrm{S}t\right)|\mathrm{K_S}\rangle + \sum_n b_n(t)|n\rangle \end{aligned} \tag{1.266}$$

with some time-independent coefficients c_L and c_S. Thus, $m_{\mathrm{L.S}}$ and $\Gamma_{\mathrm{L,S}}$ are, respectively, masses and decay widths of the physical neutral kaons $\mathrm{K_L}$ and $\mathrm{K_S}$.

The observable we would like to consider is the ratio of decay amplitudes of $\mathrm{K_L}$ and $\mathrm{K_S}$ into two-pion states with vanishing total isospin

$$\varepsilon_\mathrm{K} = \frac{A[\mathrm{K_L} \to (\pi\pi)_{I=0}]}{A[\mathrm{K_S} \to (\pi\pi)_{I=0}]} \tag{1.267}$$

† This may not hold only in some really peculiar phase conventions for the parametrization of the Cabibbo–Kobayashi–Maskawa matrix and/or for the complex square root.

where

$$|(\pi\pi)_{I=0}\rangle = \frac{1}{\sqrt{3}}\left(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle - |\pi^0\pi^0\rangle\right) \tag{1.268}$$

It is convenient to express ε_{K} in terms of the decay amplitudes

$$\left.\begin{aligned} A[\mathrm{K}^0 \to (\pi\pi)_{I=0}] &\equiv A_0 \exp(\mathrm{i}\delta_0) \\ A[\bar{\mathrm{K}}^0 \to (\pi\pi)_{I=0}] &\equiv \bar{A}_0 \exp(\mathrm{i}\delta_0) \end{aligned}\right\} \tag{1.269}$$

where δ_0 is the strong phase shift in elastic scattering of two pions in the isospin zero state. The CPT symmetry implies that $\bar{A}_0 = A_0^*$. Using this fact together with (1.254), (1.255), (1.265), (1.267) and (1.269), one finds after a short calculation that

$$\varepsilon_{\mathrm{K}} = \frac{H_{12}^{1/2} A_0 - H_{21}^{1/2} A_0^*}{H_{12}^{1/2} A_0 + H_{21}^{1/2} A_0^*} \tag{1.270}$$

which can be equivalently rewritten as

$$\varepsilon_{\mathrm{K}} = \frac{H_{12} A_0^2 - H_{21} A_0^{*2}}{4 H_{12}^{1/2} H_{21}^{1/2} |A_0|^2}(1 - \varepsilon_{\mathrm{K}}^2) = \frac{H_{12} A_0^2 - H_{21} A_0^{*2}}{[2(m_{\mathrm{L}} - m_{\mathrm{S}}) - \mathrm{i}(\Gamma_{\mathrm{L}} - \Gamma_{\mathrm{S}})]|A_0|^2}(1 - \varepsilon_{\mathrm{K}}^2) \tag{1.271}$$

The advantage of the latter formula is that it is manifestly independent of sign conventions used for complex square roots and of phase conventions in the parametrization of the Cabibbo–Kobayashi–Maskawa (CKM) matrix.

Now, it is time to make some approximations. We know from experiment that $\varepsilon_{\mathrm{K}} \ll 1$ and $\Gamma_{\mathrm{L}} \ll \Gamma_{\mathrm{S}} \simeq 2(m_{\mathrm{L}} - m_{\mathrm{S}}) \equiv 2\Delta m$. Consequently,

$$\varepsilon_{\mathrm{K}} \simeq \frac{2\mathrm{i}\,\mathrm{Im}(M_{12} A_0^2) + \mathrm{Im}(\Gamma_{12} A_0^2)}{2(1+\mathrm{i})\Delta m |A_0|^2}, \tag{1.272}$$

where we have used the decomposition (1.260) of H_{ij} into hermitean and antihermitean parts M_{ij} and Γ_{ij}, respectively.

Next, we take into account the experimental fact that $\mathrm{K_S}$ decays dominantly to isospin-zero two-pion states. This means that the sum representing Γ_{ij} in (1.262) is dominated by $(\pi\pi)_{I=0}$ intermediate states. Consequently, Γ_{12} is approximately proportional to $A_0^* \bar{A}_0 = A_0^{*2}$, and $\mathrm{Im}(\Gamma_{12} A_0^2)$ can be neglected. Therefore,

$$\varepsilon_{\mathrm{K}} \simeq \frac{\mathrm{Im}(M_{12} A_0^2)}{\sqrt{2}\Delta m |A_0|^2} \exp(\mathrm{i}\pi/4). \tag{1.273}$$

In general, the phase of the amplitude A_0 can take any value between 0 and 2π, depending on phase conventions in the parametrization of the CKM matrix. In the standard phase convention (see Chapter 12), A_0 is approximately real, and we can write

$$\bar{\varepsilon} \simeq \varepsilon_{\mathrm{K}} \simeq \frac{\operatorname{Im} M_{12}}{\sqrt{2}\,\Delta m} \exp(\mathrm{i}\pi/4). \tag{1.274}$$

The equality of $\bar{\varepsilon}$ and ε_{K} for real A_0 follows directly from comparing (1.270) and (1.265). However, the parameter $\bar{\varepsilon}$ is, in general, dependent on phase conventions. Therefore $\bar{\varepsilon}$ is not an observable, unlike ε_{K}.

The value of ε_{K} is measured quite precisely nowadays, and it is of order 10^{-3} (for the description of some of the methods of measuring ε_{K} see Marshak, Riazuddin & Ryan (1969). On the other hand, $\bar{\varepsilon}$ does not even need to be small. When it is small, we can write in analogy to (1.272)

$$\varepsilon_{\mathrm{K}} \simeq \frac{2\mathrm{i}\operatorname{Im} M_{12} + \operatorname{Im}\Gamma_{12}}{2(1+\mathrm{i})\Delta m} \tag{1.275}$$

However, we cannot neglect $\operatorname{Im}\Gamma_{12}$ before specifying the phase convention. As we have observed before, the phase of Γ_{12} is approximately opposite to the phase of A_0^2. On the other hand, as follows from (1.273), the quantity $\operatorname{Im}(M_{12}A_0^2)$ is a measure of CP violation in the neutral kaon mixing, Thus, if ε_{K} differs from zero, then either $\operatorname{Im} M_{12}$ or $\operatorname{Im}\Gamma_{12}$ must be non-vanishing. Consequently, the value of $\bar{\varepsilon}$ is bounded from below when CP is violated. In other words, when CP is violated, then the mass eigenstates $\mathrm{K_L}$ and $\mathrm{K_S}$ in (1.254) and (1.255) are not CP eigenstates.

Problems

1.1 Consider the action for a motion of a particle in a potential $V(x)$:

$$S([\mathbf{x}], t_1, t_2) = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{1}{2} m \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) - V(\mathbf{x})\right]$$

where $[\mathbf{x}]$ denotes a functional of the path $\mathbf{x}(t)$.

Calculate the variation S under the transformation

$$\mathbf{x}(t) \to \mathbf{x}(t) + \delta\mathbf{x}(t)$$

Derive the Newton equation $m\,\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2 = -\boldsymbol{\nabla} V$ from the principle of the minimal action (subject to the constraint $\delta\mathbf{x}(t_1) = \delta\mathbf{x}(t_2) = 0$). Prove that energy E, momentum $\mathbf{p}$ and orbital momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ are constants of motion if the potential is symmetric under translations and rotations.

1.2 Write down the Newton equation $\mathrm{d}^2x/\mathrm{d}t^2 + \partial V/\partial x = 0$ in the new reference frame, corresponding to the transformation $t' = t + \varepsilon t^2$. Construct the lagrangians which give both equations and compare with (1.27).

1.3 Use (1.35) to study conservation laws in classical electrodynamics (in the absence of sources) with $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Derive the canonical energy–momentum tensor and bring into the symmetric and gauge-invariant form

$$\Theta^{\mu\nu} = \left(F^{\mu\alpha}F_\alpha^\nu + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

Check that it is traceless. Express $\Theta^{\mu\nu}$ in terms of the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields. In particular show that the energy density is $\Theta^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ and the momentum density $\Theta^{0i} = (\mathbf{E}\times\mathbf{B})^i$ is the Poynting vector. The conservation law $\partial_0\Theta^{00} + \partial_i\Theta^{0i} = 0$ connects the change in the energy density to the Poynting vector. Finally, derive the angular momentum vector for the electromagnetic field and split it into orbital and intrinsic orbital momentum parts.

1.4 Consider a closed system of a charged classical particle interacting with an electromagnetic field. Write down the action in terms of the electromagnetic potential A_μ and derive the equations of motion and the energy–momentum conservation laws.

1.5 Show that for an abelian gauge theory the Noether current corresponding to the gauge symmetry transformation (1.91) and (1.94) is

$$S^\mu = \partial_\rho(F^{\rho\mu}\Theta(x))$$

and that the associated conserved charge

$$Q_s = \int \mathrm{d}^3x\, S^0(\mathbf{x}, t)$$

either vanishes or reduces to the electric charge associated with global $U(1)$ invariance provided Θ approaches an angle-independent limit at spatial infinity. Thus gauge invariance leads to no new conservation laws as compared to global invariance.

1.6 In abelian gauge theory the dual field strength tensor $\tilde{F}^{\mu\nu}$ is defined as

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \qquad \varepsilon^{0123} = -1$$

Show that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total divergence

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K^\mu$$

where

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$$

In a non-abelian case a matrix-valued dual field strength is

$$\tilde{G}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$$

and

$$\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] = \partial_\mu K^\mu$$

where

$$K^\rho = \varepsilon^{\rho\sigma\mu\nu}\,\mathrm{Tr}[G_{\sigma\mu}A_\nu + \tfrac{2}{3}A_\sigma A_\mu A_\nu]$$

1.7 Check the Jacobi identity for the covariant derivatives

$$[D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0$$

and the Bianchi identity

$$D_\mu G_{\nu\sigma} + D_\nu G_{\sigma\mu} + D_\sigma G_{\mu\nu} = 0$$

or

$$D_\mu \tilde{G}^{\mu\nu} = 0$$

1.8 In pure Yang–Mills theory the Noether current for global symmetry transformations is

$$^{\mathrm{Y}}j^a_\mu = c_{abc} A^b_\nu G^{c\nu}_{\ \ \mu}$$

and for gauge transformations

$$^{\mathrm{Y}}j^\mu_\Theta = \frac{2}{g^2}\,\mathrm{Tr}[G^{\nu\mu} D_\nu \Theta]$$

Using field equations $D_\mu G^{\mu\nu} = 0$ show that both currents are conserved. Show that the charge

$$Q^a = \int \mathrm{d}^3x\, {}^{\mathrm{Y}}j^a_0(\mathbf{x}, t)$$

is time-independent, provided ${}^{\mathrm{Y}}\mathbf{j}^a$ falls off sufficiently rapidly at large $|\mathbf{x}|$. Since ${}^{\mathrm{Y}}\mathbf{j}^a$ is not gauge-invariant, the fall-off requirement restricts the large $|\mathbf{x}|$ behaviour of gauge transformation. Show that Q^a is gauge-covariant against gauge transformations which approach a definite angle-independent limit as $|\mathbf{x}| \to \infty$. Show that the current ${}^{\mathrm{Y}}j^\mu_\Theta$ generates no new charges provided Θ approaches an angle-independent limit at spatial infinity.

Include fermion fields. Using Noether's theorem derive the conserved symmetry current:

$$J^a_\mu = j^a_\mu + c_{abc} A^b_\nu G^{c\nu}_{\ \ \mu}$$

where j^a_μ is given by (1.123). Show that the ordinary conservation of J^a_μ is equivalent to the covariant conservation of j^a_μ and the gauge field equation (1.124). Show that

$$\partial_\mu G^{\mu\nu}_a = g J^\nu_a$$

2

Path integral formulation of quantum field theory

2.1 Path integrals in quantum mechanics

Our first aim is to show that the matrix elements of quantum mechanical operators can be written as functional (path) integrals over all trajectories, with the integrand dependent on the action integral. Intuitively, the need for integration is obvious: the position operator does not commute with the momentum or the hamiltonian. Consequently, time evolution changes the position eigenstate into one in which position is not determined. The quantum system has no definite trajectory and it is necessary to take a sum over all possible ones, according to the superposition principle.

Transition matrix elements as path integrals

Consider a quantum mechanical system with one degree of freedom. The *eigenstates* of the position operator are introduced as follows

$$\begin{aligned} X_{\mathrm{H}}(t)|x, t\rangle &= x|x, t\rangle \qquad && \text{Heisenberg picture} \\ X_{\mathrm{S}}|x\rangle &= x|x\rangle && \text{Schrödinger picture} \end{aligned}$$

with the relation

$$|x\rangle = \exp[-(\mathrm{i}/\hbar)Ht]|x, t\rangle$$

where H denotes the hamiltonian of the system. The matrix element

$$\langle x', t'|x, t\rangle = \langle x'|\exp[-(\mathrm{i}/\hbar)H(t'-t)]|x\rangle \tag{2.1}$$

corresponds to the transition from the eigenstate $|x\rangle$ at the moment of time t to the state $|x'\rangle$ at the time t', and is a Green's function: define $|t\rangle$ by $H|t\rangle = \mathrm{i}\hbar(\partial/\partial t)|t\rangle$, then

$$\langle x|t'\rangle = \int \mathrm{d}x' \,\langle x|\exp[-(\mathrm{i}/\hbar)H(t'-t)]|x'\rangle\langle x'|t\rangle$$

The matrix element (2.1) we shall first represent as a multiple integral which will then be used to define the functional integral by a limiting procedure. First we divide the time interval $(t' - t)$ into $(n+1)$ equal parts of length ε

$$\begin{aligned} t' &= (n+1)\varepsilon + t \\ t_j &= j\varepsilon + t \qquad (j = 1, \ldots, n) \end{aligned}$$

Next, we use the completeness relation at each of the times t_j:

$$\int \mathrm{d}x_j \, |x_j, t_j\rangle\langle x_j, t_j| = 1 \tag{2.2}$$

together with

$$\begin{aligned} \langle x_j, t_j | x_{j-1}, t_{j-1}\rangle &= \left\langle x_j \middle| \exp\left(-\frac{\mathrm{i}}{\hbar}\varepsilon H\right) \middle| x_{j-1} \right\rangle \\ &= \langle x_j | x_{j-1}\rangle - \frac{\mathrm{i}\varepsilon}{\hbar}\langle x_j | H | x_{j-1}\rangle + O\big(\varepsilon^2\big) \end{aligned} \tag{2.3}$$

where $x_0, x_{n+1}, t_0, t_{n+1}$ are to be understood as x, x', t, t', respectively. Choosing the hamiltonian $H = H(P, X)$ to be of the form $H = f(P) + g(X)$ we can write

$$\begin{aligned} \langle x_j | H | x_{j-1}\rangle &= \int \mathrm{d}p_j \, \langle x_j | p_j\rangle\langle p_j | H | x_{j-1}\rangle \\ &= \int \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\left[\frac{\mathrm{i}}{\hbar} p_j (x_j - x_{j-1})\right] H(p_j, x_{j-1}) \end{aligned} \tag{2.4}$$

where $H(p, x)$ is now the classical c-number hamiltonian. Using (2.4), (2.3) becomes

$$\begin{aligned} \langle x_j, t_j | x_{j-1}, t_{j-1}\rangle &= \int \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\left[\frac{\mathrm{i}}{\hbar} p_j (x_j - x_{j-1})\right]\left[1 - \frac{\mathrm{i}}{\hbar}\varepsilon H(p_j, x_{j-1})\right] + O\big(\varepsilon^2\big) \\ &= \int \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\left[\frac{\mathrm{i}}{\hbar} p_j (x_j - x_{j-1}) - \frac{\mathrm{i}}{\hbar}\varepsilon H(p_j, x_{j-1})\right] + O\big(\varepsilon^2\big) \end{aligned} \tag{2.5}$$

and we obtain the following expression for the matrix element (2.1):

$$\begin{aligned} \langle x', t' | x, t\rangle = \lim_{n\to\infty} \int \prod_{j=1}^{n} \mathrm{d}x_j \int \prod_{j=1}^{n+1} \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\Bigg\{ \frac{\mathrm{i}}{\hbar} \sum_{j=1}^{n+1} [p_j (x_j - x_{j-1}) \\ - H(p_j, x_{j-1})(t_j - t_{j-1})] \Bigg\} \end{aligned} \tag{2.6}$$

where the limit $n \to \infty$ ($\varepsilon \to 0$) has been taken and the $O(\varepsilon^2)$ terms neglected. This result we shall write in the compact form

$$\langle x', t'|x, t\rangle = \int \frac{\mathcal{D}x\,\mathcal{D}p}{2\pi\hbar} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} [p\dot{x} - H(p, x)]\,\mathrm{d}\tau\right\} \tag{2.7}$$

$\int(\mathcal{D}x\,\mathcal{D}p/2\pi\hbar) \equiv \int \prod_\tau [\mathrm{d}x(\tau)\,\mathrm{d}p(\tau)/2\pi\hbar]$ is called a functional integration over all phase space, with the boundary conditions $x(t) = x$, $x(t') = x'$ implied in this case. Eq. (2.7) is the promised path integral representation of $\langle x', t'|x, t\rangle$.

If the hamiltonian is of a simple form

$$H = (1/2m)P^2 + V(X) \tag{2.8}$$

it is convenient to perform the momentum integrations in (2.6). Shifting the integration variables: $p_j \to p_j - m(\Delta x_j/\varepsilon)$ we obtain

$$\int \frac{\mathrm{d}p_j}{2\pi\hbar} \exp\left[\frac{\mathrm{i}}{\hbar}\left(p_j \Delta x_j - \frac{p_j^2}{2m}\varepsilon\right)\right] = \frac{1}{N_j}\frac{1}{2\pi\hbar} \exp\left[\frac{\mathrm{i}}{\hbar}\varepsilon\frac{m}{2}\left(\frac{\Delta x_j}{\varepsilon}\right)^2\right] \tag{2.9}$$

where $\Delta x_j = x_j - x_{j-1}$ and

$$\frac{1}{N_j} = \int \mathrm{d}p_j \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{p_j^2}{2m}\varepsilon\right)$$

The final result has the form of a functional integral over configuration space:

$$\langle x', t'|x, t\rangle = \frac{1}{N}\int \frac{\mathcal{D}x}{2\pi\hbar} \exp\left\{\frac{\mathrm{i}}{\hbar}S[x]\right\} \tag{2.10}$$

Here $S[x] = \int_t^{t'} L(x, \dot{x})\,\mathrm{d}\tau$ is the action integral over the trajectory $x(\tau)$, where $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$ is the Lagrange function and the normalization factor N is given by

$$\frac{1}{N} = \int \mathcal{D}p \exp\left(-\frac{\mathrm{i}}{\hbar}\int_t^{t'} \frac{p}{2m}\mathrm{d}\tau\right)$$

Starting with the canonically quantized theory described by hamiltonian (2.8) we have derived path integral representation (2.10). We can use another approach, namely, to define the quantum theory by functional integral (2.10), i.e. we can choose the path integral formulation as the quantization prescription for a system with the classical hamiltonian in the form (2.8). Then our derivation proves the equivalence of the path integral and canonical quantization methods for systems described by hamiltonian (2.8). There are, however, systems for which canonical quantization is ambiguous due to non-commutativity of operators P and X. Here belong, for instance, theories where the lagrangian is

$$L = \tfrac{1}{2}f(x)\dot{x}^2 + g(x)\dot{x} - V(x) \tag{2.11}$$

On the other hand, quantum theory, which in the classical limit gives the theory (2.11), is unambiguously defined by the path integral ansatz in configuration space, insensitive to the ordering of P and X. It can be shown (Cheng 1972) that the appropriately generalized path integral reads

$$\langle x', t'|x, t\rangle \sim \lim_{j\to\infty} \int \prod_{t_j} \frac{\mathrm{d}x(t_j)}{2\pi\hbar} f^{1/2}(x(t_j)) \exp\left\{\frac{\mathrm{i}}{\hbar} S[x(t_j)]\right\} \tag{2.12}$$

Equivalently, one can postulate (2.7) with the c-number hamiltonian corresponding to lagrangian (2.11), supplemented by the rule that, whenever there is an ambiguity, the integrals over p_j are to be performed before the x-integration. Integrating over p_j one derives (2.12) (see Problem 2.1). The modification of the functional measure $\mathcal{D}x$ into

$$\mathcal{D}x = \lim_{i\to\infty} \int \prod_i \mathrm{d}x_i \, f^{1/2}(x_i)$$

comes from the integral

$$\frac{1}{N_j} = \int \mathrm{d}p_j \exp\left[-\frac{\mathrm{i}}{2\hbar} f^{-1}(x_j) p_j^2 \varepsilon\right]$$

which now depends on x_j. The functional integral (2.12) defines the same dynamics as the Schrödinger equation with one particular ordering of the operators P and X, namely the symmetric ordering, for example, $[f(x)p]_{\mathrm{S}} = \frac{1}{2}[f(x)p + pf(x)]$.

Theories like (2.11) are of physical interest. A lagrangian of this kind appears, for instance, for a system of two interacting particles

$$L = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 - V(x, y) \tag{2.13}$$

subject to a constraint equation $y = c(x)$. Therefore

$$L = \frac{m}{2}\left[\left(1 + \frac{\mathrm{d}c}{\mathrm{d}x}\right)^2 \dot{x}^2\right] - V(x, c(x)) \tag{2.14}$$

We shall encounter a similar situation in Chapter 3 when quantizing gauge field theories.

In summary, in the following we shall consider quantum theories defined by the path integral formulation. Their correspondence to canonically quantized theories, if of interest, must be verified case by case. For field theories considered in this book this, in fact, can be done at least in the framework of perturbation theory.

If $S[x(\tau)] \gg \hbar$ we can evaluate (2.10) by use of the saddle point approximation. The integral is then dominated by trajectories close to the classical one $x_{\mathrm{c}}(t)$, which satisfies $\delta S[x_{\mathrm{c}}(\tau)] = 0$: i.e. the path integral formulation allows a relatively simple understanding of the classical and semi-classical limit.

Matrix elements of position operators

The matrix element $\langle x', t'|x, t\rangle$ determines all transition probabilities between quantum mechanical states. In view of further applications of functional formalism to quantum field theories it is also important to know the path integral representation of the matrix elements of the position operators, corresponding to the field operator of quantum field theory. For the time-ordered product of N such operators the following expression can be shown to hold

$$\langle x', t'|TX(t_1)\dots X(t_N)|x, t\rangle = \int \frac{\mathcal{D}x\,\mathcal{D}p}{2\pi\hbar} x(t_1)\dots x(t_N) \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} [p\dot{x} - H(p, x)]\,\mathrm{d}\tau\right\} \tag{2.15}$$

Let us check (2.15) for the product of two operators: $X(\tau_1)X(\tau_2)$ at $\tau_1 > \tau_2$. Again we divide the time axis into small intervals, choosing $t_1 \dots t_n$ in such a way that

$$\tau_1 = t_{i_1}, \qquad \tau_2 = t_{i_2}$$

and apply the completeness relation at each t_i. We obtain

$$\begin{aligned}\langle x', t'|X(\tau_1)X(\tau_2)|x, t\rangle &= \int \prod_i \mathrm{d}x_i\, \langle x', t'|x_n, t_n\rangle \cdots \langle x_{i_1}, t_{i_1}|X(\tau_1)|x_{i_1-1}, t_{i_1-1}\rangle \cdots \\ &\quad\times \langle x_{i_2}, t_{i_2}|X(\tau_2)|x_{i_2-1}, t_{i_2-1}\rangle \cdots \langle x_1, t_1|x, t\rangle \\ &= \int \prod_i \mathrm{d}x_i\, x_{i_1}\, x_{i_2}\, \langle x', t'|x_n, t_n\rangle \cdots \langle x_1, t_1|x, t\rangle \end{aligned} \tag{2.16}$$

Proceeding exactly as when deriving (2.6) we obtain (2.15). Note that (2.16) is true for $\tau_1 > \tau_2$. When $\tau_1 < \tau_2$, the r.h.s. of (2.16) corresponds to the matrix element $\langle x', t'|X(\tau_2)X(\tau_1)|x, t\rangle$. Therefore the path integral, like (2.15), defines the matrix element of the time-ordered product of the position operators

$$\int \frac{\mathcal{D}x\,\mathcal{D}p}{2\pi\hbar} x(t_1)\,x(t_2) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} [p\dot{x} - H]\,\mathrm{d}\tau\right\} = \begin{cases} \langle x', t'|X(t_1)X(t_2)|x, t\rangle, & t_1 > t_2 \\ \langle x', t'|X(t_2)X(t_1)|x, t\rangle, & t_1 < t_2 \end{cases} \tag{2.17}$$

As before, it is also possible to change from phase space path integrals to path integrals over configuration space.

Also, let us note that the transition amplitude in the presence of an external source $J(\tau)$

$$\langle x', t'|x, t\rangle^J = \int \frac{\mathcal{D}x\,\mathcal{D}p}{2\pi\hbar} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} [p\dot{x} - H(p, x) + \hbar J(\tau)x(\tau)]\,\mathrm{d}\tau\right\} \tag{2.18}$$

corresponding to the usual transition amplitude with the hamiltonian modified by a source term: $H \to H - \hbar Jx$, can be used as a generating functional of the matrix elements of the position operators, which are then given by its functional derivatives with respect to $J(\tau)$

$$\langle x', t'|TX(t_1)\dots X(t_N)|x,t\rangle = \left(\frac{1}{\mathrm{i}}\right)^N \frac{\delta^N}{\delta J(t_1)\dots\delta J(t_N)}\bigg|_{J=0} \langle x', t'|x,t\rangle^J \tag{2.19}$$

Instead of a formal definition of the functional derivative, for our purpose it is sufficient to know that, in the case of the functional $F[J]$ defined as

$$F[J] = \int \mathrm{d}q_1 \dots \int \mathrm{d}q_n\, f(q_1\dots q_n)\, J(q_1)\dots J(q_n) \tag{2.20}$$

the functional derivative with respect to J is simply (f can be chosen to be symmetric in all variables)

$$\frac{\delta F[J]}{\delta J(q)} = \int \mathrm{d}q_1\dots\mathrm{d}q_{n-1} J(q_1)\dots J(q_{n-1})\, nf(q_1\dots q_{n-1}q) \tag{2.21}$$

This corresponds to the usual rule of differentiating the monomials. If the functional is defined instead by the series

$$\Phi[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}q_1\dots\mathrm{d}q_n\, \phi_n(q_1\dots q_n)\, J(q_1)\dots J(q_n) \tag{2.22}$$

then

$$\phi_n(q_1\dots q_n) = \frac{\delta^n \Phi[J]}{\delta J(q_1)\dots\delta J(q_n)}\bigg|_{J=0} \tag{2.23}$$

which compares with the Taylor expansion of the usual functions.

2.2 Vacuum-to-vacuum transitions and the imaginary time formalism

General discussion

In field theoretical applications we shall mainly deal with the matrix elements of the field operators taken between the vacuum states: the Green's functions. Let us first consider the analogous problem in quantum mechanics. Assume that the lagrangian L of the system is time-independent. The energy eigenstates correspond to the wave-functions $\Phi_n(x) = \langle x|n\rangle$. In particular, the ground state or *the vacuum* is described by the function $\Phi_0(x) = \langle x|0\rangle$. It will be convenient to use $\Phi_0(x,t)$ defined as

$$\Phi_0(x,t) = \exp[-(\mathrm{i}/\hbar)E_0 t]\langle x|0\rangle = \langle x|\exp[-(\mathrm{i}/\hbar)Ht]|0\rangle = \langle x,t|0\rangle \tag{2.24}$$

We are interested in the matrix element $\langle 0|TX(t_1)\dots X(t_n)|0\rangle$. It reads

$$\langle 0|TX(t_1)\dots X(t_n)|0\rangle = \int \mathrm{d}x'\,\mathrm{d}x\;\Phi_0^*(x',t')\,\langle x',t'|TX(t_1)\dots X(t_n)|x,t\rangle\Phi_0(x,t) \tag{2.25}$$

and for the matrix element $\langle x',t'|TX(t_1)\dots X(t_n)|x,t\rangle$ the functional form (2.15) can be used. The considered vacuum expectation value can also be obtained from a generating functional

$$\langle 0|TX(t_1)\dots X(t_N)|0\rangle = \left(\frac{1}{\mathrm{i}}\right)^N \frac{\delta^N}{\delta J(t_1)\dots\delta J(t_N)}\bigg|_{J=0} W[J] \tag{2.26}$$

where

$$W[J] = \langle 0|0\rangle^J = \int \mathrm{d}x'\,\mathrm{d}x\;\Phi_0^*(x',t')\,\langle x',t'|x,t\rangle^J\,\Phi_0(x,t) \tag{2.27}$$

with $\langle x',t'|x,t\rangle^J$ given by (2.18). It is very important that the generating functional $W[J]$ can also be derived in another way. We shall show that

$$W[J] = \lim_{\substack{T_1\to+\mathrm{i}\infty\\ T_2\to-\mathrm{i}\infty}} \frac{\exp[(\mathrm{i}/\hbar)E_0(T_2-T_1)]}{\Phi_0^*(x_1)\Phi_0(x_2)}\langle x_2,T_2|x_1,T_1\rangle^J \tag{2.28}$$

This implies that $W[J]$ is, in fact, determined by the transition amplitude $\langle x_2,T_2|x_1,T_1\rangle^J$ at any given x_1, x_2 (for instance $x_1 = x_2 = 0$) provided that the analytic continuation to the imaginary values of T_1 and T_2 is performed. To derive (2.28) let us choose that the source $J(t)$ vanishes outside the time interval (t,t') with $T_2 > t' > t > T_1$. Then we can write

$$\langle x_2,T_2|x_1,T_1\rangle^J = \int \mathrm{d}x'\,\mathrm{d}x\;\langle x_2,T_2|x',t'\rangle\,\langle x',t'|x,t\rangle^J\,\langle x,t|x_1,T_1\rangle \tag{2.29}$$

where

$$\begin{aligned}\langle x,t|x_1,T_1\rangle &= \langle x|\exp[-(\mathrm{i}/\hbar)H(t-T_1)]|x_1\rangle\\ &= \sum_n \Phi_n(x)\Phi_n^*(x_1)\exp[-(\mathrm{i}/\hbar)E_n(t-T_1)]\end{aligned}$$

and similarly for $\langle x_2,T_2|x',t'\rangle$. The only T-dependent terms are now the factors $\exp[-(\mathrm{i}/\hbar)E_n(t-T_1)]$ and we can continue to $T_1 \to +\mathrm{i}\infty$ explicitly. Recalling that E_0 is the lowest energy eigenvalue we get

$$\begin{aligned}\lim_{T_1\to+\mathrm{i}\infty}\exp[-(\mathrm{i}/\hbar)E_0T_1]\langle x,t|x_1,T_1\rangle &= \Phi_0(x)\exp[-(\mathrm{i}/\hbar)E_0t]\Phi_0^*(x_1)\\ &= \Phi_0(x,t)\Phi_0^*(x_1)\end{aligned}$$

and, in the same way

$$\lim_{T_2\to-\mathrm{i}\infty}\exp[(\mathrm{i}/\hbar)E_0T_2]\langle x_2,T_2|x',t'\rangle = \Phi_0^*(x',t')\Phi_0(x_2)$$

After employing (2.27) and (2.29), (2.28) follows. Notice also that (2.28) and (2.26) imply

$$\langle 0|TX(t_1)\ldots X(t_N)|0\rangle = \lim_{\substack{T_1\to+\mathrm{i}\infty\\ T_2\to-\mathrm{i}\infty}} \frac{\exp[(\mathrm{i}/\hbar)E_0(T_2-T_1)]}{\Phi_0^*(x_1)\Phi_0(x_2)} \times\langle x_2, T_2|TX(t_1)\ldots X(t_N)|x_1, T_1\rangle \tag{2.30}$$

We conclude that the vacuum matrix elements can be calculated by taking functional derivatives of the generating functional $W[J]$, given by (2.28). The J-independent factors are irrelevant, because we can always consider quantities like

$$\frac{1}{\langle 0|0\rangle}\langle 0|TX(t_1)\ldots X(t_N)|0\rangle = \left(\frac{1}{\mathrm{i}}\right)^N \frac{1}{W[0]}\frac{\delta^N}{\delta J(t_1)\ldots\delta J(t_N)}\bigg|_{J\equiv 0} W[J] \tag{2.31}$$

Consequently, instead of (2.28) we can write

$$W[J] = \lim_{\substack{T_1\to+\mathrm{i}\infty\\ T_2\to-\mathrm{i}\infty}} \int_{\substack{x(T_1)=x_1\\ x(T_2)=x_2}} \mathcal{D}x \exp\left\{(\mathrm{i}/\hbar)\int_{T_1}^{T_2}[L(x,\dot{x})+\hbar Jx]\,\mathrm{d}t\right\} \tag{2.32}$$

where x_1, x_2, are arbitrary.

Harmonic oscillator

As an example of particular interest in view of further field theoretical applications we shall consider the case of the harmonic oscillator

$$L = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2x^2 = \tfrac{1}{2}(\mathrm{d}/\mathrm{d}t)(x\dot{x}) - \tfrac{1}{2}x\ddot{x} - \tfrac{1}{2}\omega^2x^2$$

The action integral, including the source term, can be written as follows

$$S^J[x] = \int_{T_1}^{T_2}[L+\hbar Jx]\,\mathrm{d}t = \tfrac{1}{2}x\dot{x}\Big|_{T_1}^{T_2} + \tfrac{1}{2}(x, Ax) + (\hbar J, x) \tag{2.33}$$

where (f, g) denotes the 'scalar product'

$$(f, g) = \int_{T_1}^{T_2}\mathrm{d}t\; f(t)\,g(t)$$

and the operator A is defined by the equation

$$\int_{T_1}^{T_2}\mathrm{d}t\,\mathrm{d}t'\,x(t)\,A(t,t')\,x(t') = \int_{T_1}^{T_2}\left(-x\ddot{x}-\omega^2x^2\right)\mathrm{d}t \tag{2.34}$$

so that

$$A(t,t') = -\left(\mathrm{d}^2\big/\mathrm{d}t'^2+\omega^2\right)\delta(t'-t) \tag{2.35}$$

and

$$Ax \equiv \int_{T_1}^{T_2} \mathrm{d}t'\, A(t,t')\, x(t')$$

Let us now change the functional integration variable in (2.32) by introducing $z(t)$

$$x(t) = x_{\mathrm{c}}(t) + z(t)$$

where $x_{\mathrm{c}}(t)$ is a solution of the classical equation of motion (in the presence of the source)

$$\ddot{x} + \omega^2 x = \hbar J$$

Requiring that $x_{\mathrm{c}}(t)$ satisfies the boundary conditions $x_{\mathrm{c}}(T_1) = x_1$, $x_{\mathrm{c}}(T_2) = x_2$, we have $z(T_1) = z(T_2) = 0$. The classical trajectory $x_{\mathrm{c}}(t)$ can be written as a superposition of a solution $x_0(t)$ of the homogeneous equation, and the special solution of the complete equation

$$x_{\mathrm{c}}(t) = x_0(t) + \hbar \int \mathrm{d}t'\, G(t-t')\, J(t') \tag{2.36}$$

Here $G(t-t')$ is the Green's function of the classical equation of motion

$$\left(\mathrm{d}^2/\mathrm{d}t^2 + \omega^2\right) G(t-t') = \delta(t-t') \tag{2.37}$$

so that $G = -A^{-1}$.† Re-expressing the action integral (2.33) in terms of the classical trajectory and the new integration variable $z(t)$ we get

$$S^J[x_{\mathrm{c}} + z] = \tfrac{1}{2} x_{\mathrm{c}} \dot{x}_{\mathrm{c}}\big|_{T_1}^{T_2} + \tfrac{1}{2}(z, Az) + \tfrac{1}{2}\hbar (J, x_0) + \tfrac{1}{2}\hbar^2 (J, GJ) \tag{2.38}$$

where we have used the relations

$$Ax_{\mathrm{c}} = -\hbar J, \qquad (x_{\mathrm{c}}, Az) = (Ax_{\mathrm{c}}, z) - \dot{z} x_{\mathrm{c}}\big|_{T_1}^{T_2}$$

together with (2.36).

The resulting $W[J]$ is of the following form

$$W[J] = \lim_{\substack{T_1 \to +\mathrm{i}\infty \\ T_2 \to -\mathrm{i}\infty}} \exp\left[(\mathrm{i}/2\hbar)\hbar^2 (J, GJ)\right] \exp\left\{(\mathrm{i}/2\hbar)\left[x_{\mathrm{c}}\dot{x}_{\mathrm{c}}\big|_{T_1}^{T_2} + \hbar (J, x_0)\right]\right\}$$
$$\times \int_{\substack{z(T_1)=0 \\ z(T_2)=0}} \mathcal{D}z \exp\left[(\mathrm{i}/\hbar) \int_{T_1}^{T_2} L(z)\, \mathrm{d}t\right] \tag{2.39}$$

The important observation is that if the Green's function (2.37) is chosen to satisfy the conditions

$$G(t) \underset{t \to \pm \mathrm{i}\infty}{\longrightarrow} 0, \qquad \mathrm{d}G(t)/\mathrm{d}t \underset{t \to \pm \mathrm{i}\infty}{\longrightarrow} 0 \tag{2.40}$$

† In the space of functions $x(t)$ with zero modes $x_0(t)$ of A excluded.

the only functional dependence of $W[J]$ on J is given by the factor

$$W[J] \sim \exp\left[(\mathrm{i}/2\hbar)\hbar^2(J, GJ)\right] \tag{2.41}$$

To prove this assertion we should show that $x_\mathrm{c}\dot{x}_\mathrm{c}\big|_{T_1}^{T_2}$ and (J, x_0) are J-independent whenever the boundary conditions (2.40) are satisfied. In fact (2.36) and (2.40) imply

$$x_\mathrm{c}\dot{x}_\mathrm{c}\Big|_{T_1}^{T_2} \underset{\substack{T_1\to+\mathrm{i}\infty \\ T_2\to-\mathrm{i}\infty}}{\longrightarrow} x_0\dot{x}_0\Big|_{T_1}^{T_2}$$

with x_0 obviously J-independent. For (J, x_0) the argument is somewhat more involved. Consider the solution of the homogeneous equation

$$x_0(t) = A\exp(\mathrm{i}\omega t) + B\exp(-\mathrm{i}\omega t)$$

with A, B chosen to satisfy the boundary conditions $x_0(T_1) = x_1$, $x_0(T_2) = x_2$

$$A = \frac{x_2\exp(-\mathrm{i}\omega T_1) - x_1\exp(-\mathrm{i}\omega T_2)}{\exp[\mathrm{i}\omega(T_2 - T_1)] - \exp[-\mathrm{i}\omega(T_2 - T_1)]},$$
$$B = \frac{x_1\exp(\mathrm{i}\omega T_2) - x_2\exp(\mathrm{i}\omega T_1)}{\exp[\mathrm{i}\omega(T_2 - T_1)] - \exp[-\mathrm{i}\omega(T_2 - T_1)]}$$

In the limit $T_1 \to +\mathrm{i}\infty$, $T_2 \to -\mathrm{i}\infty$ we have

$$A \approx x_2\exp(-\omega|T_2|), \qquad B \approx x_1\exp(-\omega|T_1|)$$

implying that (J, x_0) also vanishes in this limit.

Note that in (2.41) the limit $T_1 \to +\mathrm{i}\infty$, $T_2 \to -\mathrm{i}\infty$ has in fact been omitted. This could be done because the integral

$$\int_{T_1}^{T_2} \mathrm{d}t \int_{T_1}^{T_2} \mathrm{d}t'\, J(t)\, G(t - t')\, J(t') \equiv (J, GJ)$$

is independent of the limits of integration due to the requirement that $J(t)$ vanishes outside some finite time interval (t_1, t_2) such that $T_2 > t_2 > t_1 > T_1$. The dependence on T_1 and T_2 can now occur only in the factors which are J-independent because of the boundary conditions (2.40), and therefore irrelevant for our discussion.

To complete the case of the harmonic oscillator we must find the Green's function $G(t - t')$ consistent with the boundary conditions specified by (2.40). Introducing the Fourier transform $\tilde{G}(\nu)$

$$G(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\tilde{G}(\nu)\exp(-\mathrm{i}\nu t)$$

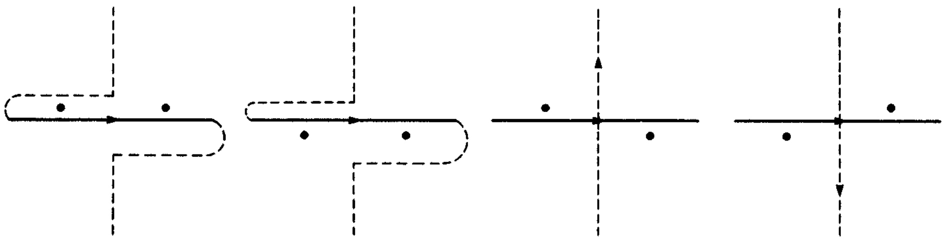


Fig. 2.1. Possible prescriptions for bypassing the singularity at $\nu = \pm\omega$ in the integral (2.42) in the complex ν plane.

we obtain the formal solution of (2.37)

$$G(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{1}{\nu^2 - \omega^2} \exp(-\mathrm{i}\nu t) \tag{2.42}$$

This result represents, in fact, four distinct solutions depending on the prescription for bypassing the singularity at $\nu = \pm\omega$, which may be one of those shown in Fig. 2.1. Cases (a) and (b) are easily seen to be in conflict with the boundary conditions (2.40). It suffices to perform the analytic continuation of (2.42) by deforming the contour as shown in the picture and then to integrate for imaginary t. The two remaining solutions agree with (2.40): we again continue analytically to imaginary t by deforming the contour of the ν-integration from the real to the imaginary axis. They correspond to adding to the denominator of the integrand in (2.42) ($+\mathrm{i}\varepsilon$) and ($-\mathrm{i}\varepsilon$), respectively

$$G(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{1}{\nu^2 - \omega^2 \pm \mathrm{i}\varepsilon} \exp(-\mathrm{i}\nu t) \tag{2.43}$$

Adding ($+\mathrm{i}\varepsilon$) to the Green's function denominator is equivalent to introducing the extra term $\left(\pm\mathrm{i}\varepsilon x^2\right)$ to the lagrangian function. Only for $\left(+\mathrm{i}\varepsilon x^2\right)$ is the functional integral well defined, so that the final solution for the Green's function is specified uniquely

$$G(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{1}{\nu^2 - \omega^2 + \mathrm{i}\varepsilon} \exp(-\mathrm{i}\nu t) \tag{2.44}$$

After the contour integration this becomes (for real t)

$$G(t) = \frac{\mathrm{i}}{2\omega}[\exp(-\mathrm{i}\omega t)\Theta(t) + \exp(\mathrm{i}\omega t)\Theta(-t)] \tag{2.45}$$

The physical interpretation of the ($+\mathrm{i}\varepsilon$) prescription is thus made clear: it corresponds to the positive frequency solutions propagating into the future.

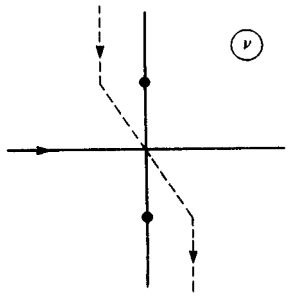


Fig. 2.2. Rotation of the integration contour from the real to the imaginary axis in the complex ν-plane which defines the analytic continuation of $G_E(\tau)$ to $\tau = it$.

Euclidean Green's functions

Now let us observe that the same Green's function can also be derived in another way, i.e. by the analytic continuation from the imaginary time region. To this end we first define the 'Euclidean' (i.e. imaginary time) Green's function $G_E(\tau)$, which obeys the equation

$$\left[-\left(d^2/d\tau^2\right) + \omega^2\right] G_E(\tau) = \delta(\tau) \tag{2.46}$$

Consequently,

$$G_E(\tau) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{1}{\nu^2 + \omega^2} \exp(-i\nu\tau) \tag{2.47}$$

As the singularities of the integrand appear at $\nu = \pm i\omega$, away from the integration contour, $G_E(\tau)$ is uniquely specified to be

$$G_E(\tau) = -\left(\frac{1}{2\omega}\right) \exp(-\omega|\tau|) \tag{2.48}$$

in agreement with the boundary conditions specified for $G(t)$. We shall now show that the real time Green's function $G(t)$ is given by the analytic continuation of $G_E(\tau)$ to $\tau = it$

$$G(t) = -iG_E(it) \tag{2.49}$$

In order to perform this continuation the integration contour in (2.47) should be deformed to keep the integral convergent during the $\tau = it$; such a rotation is illustrated in Fig. 2.2. The result is

$$G(t) = -\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{1}{\nu^2 - \omega^2 + i\varepsilon} \exp(-i\nu t)$$

which coincides with (2.44).

It is also possible to continue to $\tau = -it$. In this case, however, the resulting real time Green's function appears with the opposite, $(-\mathrm{i}\varepsilon)$, prescription. We conclude that the analytic continuation is correct for $\tau = \mathrm{i}t$. One way to see the difference is to make the substitution $t = \mp\mathrm{i}\tau$ under the functional integral. The action changes as follows:

$$\begin{aligned}\frac{\mathrm{i}}{\hbar}S[x] = \frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x)\right] &\underset{t=\pm\mathrm{i}\tau}{\longrightarrow} \frac{\mathrm{i}}{\hbar}(\pm\mathrm{i})\int \mathrm{d}\tau\left[-\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - V(x)\right] \\ &= \pm\frac{1}{\hbar}S_{\mathrm{E}}[x]\end{aligned} \tag{2.50}$$

where $S_{\mathrm{E}}[x]$ is the 'Euclidean' action. The resulting functional integral is of the form

$$\int \mathcal{D}x \exp\{\pm(1/\hbar)S_{\mathrm{E}}[x]\} \tag{2.51}$$

and only for the $(-)$ sign is it well defined.

For the generating functional $W_{\mathrm{E}}[J]$ one gets

$$W_{\mathrm{E}}[J] = \int \mathcal{D}x \exp\left\{-(1/\hbar)S_{\mathrm{E}}[x] + \int \mathrm{d}\tau\, J(\tau)\,x(\tau)\right\} \sim \exp\left[\tfrac{1}{2}\hbar(J, G_{\mathrm{E}}J)\right]$$

where $G_{\mathrm{E}}(\tau)$ is defined by (2.46).

2.3 Path integral formulation of quantum field theory

Green's functions as path integrals

The results of Sections 2.1 and 2.2 are easily generalized to the case of more than one degree of freedom. If the number of degrees of freedom is to be n, the coordinate x should be replaced by an n-component vector. The functional integral would now correspond to the sum over all trajectories in the n-dimensional configuration space, satisfying appropriate boundary conditions.

In the field theory, the trajectory $x(t)$ is replaced by a field function $\Phi(\mathbf{x}, t)$. The degrees of freedom are now labelled by the continuous index $\mathbf{x}$; the number of degrees of freedom is obviously infinite. To define the appropriate path integral one can start from a multiple integral on a discrete and, for a beginning, finite lattice of space-time points. This amounts to defining the quantum field theory as a limit of a theory with only a finite number of degrees of freedom.

The limit of an infinite lattice, related to the thermodynamical limit of statistical mechanics, already defines a theory with an infinite number of degrees of freedom. However, this lattice theory does not have enough space-time invariance and a continuous theory must be defined. The latter limit is accompanied by infinities, the 'UV divergences' of quantum field theory. The definition of the functional integral in quantum field theory is thus more ambiguous than in the case of quantum mechanics. Nevertheless, the functional formalism in quantum field theory is of great heuristic value. It is a very convenient tool for studying perturbation theory and allows a natural description of some non-perturbative phenomena.

The quantum field theory is usually formulated in terms of the vacuum expectation values of the chronologically ordered products of the field operators, the Green's functions:

$$G^{(n)}(x_1, \dots, x_n) = \langle 0|T\Phi(x_1)\dots\Phi(x_n)|0\rangle \tag{2.52}$$

Using our experience from the previous section we shall write down the path integral representation for them. In particular, it is important to remember the role played by (2.28) and (2.30) in getting rid of the vacuum wave functions originally present in (2.25) as the boundary conditions. Thus, by analogy with the results of the previous section, we postulate the following path integral representation

$$G^{(n)}(x_1, \dots, x_n) \sim \int \mathcal{D}\Phi\, \Phi(x_1)\dots\Phi(x_n) \exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\, \mathcal{L}\right] \tag{2.53}$$

($\mathcal{D}\Phi$ denotes integration over all functions $\Phi(\mathbf{x}, t)$ of space and time, because, for each value of $\mathbf{x}$, $\Phi(\mathbf{x}, t)$ corresponds to a separate degree of freedom; $\mathcal{L}$ is the lagrangian density). Eq. (2.30) suggests that any boundary conditions are, in fact, irrelevant here, provided that we take the imaginary time limit. In particular, the trajectories may be not constrained at all.

In analogy with (2.30), the rotation of the time contour away from the real axis should start somewhere at large absolute values of time, so that the arguments of the Green's functions, $t_1 \dots t_n$, stay real. If instead the whole time axis is rotated, $t_i = -\mathrm{i}\tau_i$, the result is a Euclidean Green's function (see the end of Section 2.2). The latter has a particularly convenient path integral representation, because the weight factor in the integrand, $\exp(-S_{\mathrm{E}}/\hbar)$, is then non-negative. This Euclidean path integral formalism can be used to define the Minkowski space Green's functions by an analytic continuation of the Euclidean ones. An expression like (2.53) would then be understood as an analytic continuation in the variables $t_1, \dots, t_n$ of the analogous Euclidean formula. Equivalently one can work in Minkowski space and use the $(+\mathrm{i}\varepsilon)$ prescription (see (2.62)).

Eq. (2.53) should be regarded as the formulation of the theory. What is the relation between the path integral and the usual (canonical) operator formulation of quantum field theory based on the same lagrangian $\mathcal{L}$? For our purposes these are equivalent if they imply the same perturbation theory Feynman rules. This has to be checked in each case of interest. However, the derivation of the perturbation theory rules is much simpler in the functional framework, particularly in the case of gauge field theories. In the following we shall often use the path integral formulation, referring to the operator formalism only if it helps to make the presentation more concise, like in the case of the scattering operator discussed in Section 2.7.

It is convenient to normalize the Green's functions by factorizing out the vacuum amplitude

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n) &= \langle 0|T\Phi(x_1)\ldots\Phi(x_n)|0\rangle/\langle 0|0\rangle \\ &= N\int \mathcal{D}\Phi\, \Phi(x_1)\ldots\Phi(x_n)\exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\, \mathcal{L}\right] \end{aligned} \tag{2.54}$$

where

$$1/N = \int \mathcal{D}\Phi \exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\, \mathcal{L}\right] \sim \langle 0|0\rangle \tag{2.55}$$

This removes the extra factors like those appearing in (2.30). The T-product defined by (2.54) is a covariant quantity (see Section 10.1 on ambiguities of T-products). The derivatives of the Green's functions are functional integrals of derivatives of fields, e.g.

$$\Box_x G^{(2)}(x, y) \sim \int \mathcal{D}\Phi\, \Box_x\, \Phi(x)\, \Phi(y) \exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\, \mathcal{L}\right] \tag{2.56}$$

The Green's functions (2.54) satisfy the standard equations of motion which can be derived using the invariance of the functional integral under a shift of variables $\Phi(x) \to \Phi(x) + \varepsilon f(x)$ (see Problem 2.2): for example, in a scalar massless field theory defined by the lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)$$

we have

$$\Box_x G^{(2)}(x, y) = -\langle 0|TV'(\Phi(x))\Phi(y)|0\rangle - \mathrm{i}\delta(x - y) \tag{2.56a}$$

We observe also that given a theory formulated in terms of the Green's functions (2.54) quantities like $\langle 0|T\Box_x\Phi(x)\Phi(y)|0\rangle$ remain *a priori* undefined. We can define them by requiring that they satisfy certain constraints like, for example, for a scalar massless field theory

$$\Box_x\langle 0|T\Phi(x)\Phi(y)|0\rangle = \langle 0|T\Box_x\Phi(x)\Phi(y)|0\rangle - \mathrm{i}\delta(x - y)$$

where the l.h.s. is given by (2.56a), which amounts to specifying a regularization prescription for them.

The Green's functions (2.54) are given by the functional derivatives of the functional $W[J]$ equivalent to the vacuum transition amplitude in the presence of the external source $J(x)$

$$W[J] = N\int \mathcal{D}\Phi \exp\left\{(\mathrm{i}/\hbar)\int \mathrm{d}^4x\, [\mathcal{L} + \hbar J(x)\Phi(x)]\right\} \tag{2.57}$$

Expanding in powers of J and using (2.54) we can rewrite $W[J]$ as follows:

$$W[J] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, G^{(n)}(x_1, \ldots, x_n) \, J(x_1) \ldots J(x_n)$$

Consequently,

$$G^{(n)}(x_1, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J \equiv 0} W[J] \tag{2.58}$$

The Green's functions can also be considered as the analytic continuation of those obtained from the generating functional defined in the Euclidean space with $x_0 = -\mathrm{i}\hat{x}_4$, where $\hat{x}_4$ is real:

$$W_{\mathrm{E}}[J] = N \int \mathcal{D}\Phi \exp\left\{-(1/\hbar) S_{\mathrm{E}}[\Phi(\hat{x})] + \int \mathrm{d}^3x \, \mathrm{d}\hat{x}_4 \, J\Phi\right\}$$

where, for example, for a free scalar field theory

$$S_{\mathrm{E}}[\Phi(\hat{x})] = \tfrac{1}{2} \int \mathrm{d}^3x \, \mathrm{d}\hat{x}_4 \left[\left(\frac{\partial \Phi}{\partial \hat{x}_4}\right)^2 + (\boldsymbol{\nabla}\Phi)^2 + m^2\Phi^2\right].$$

The Euclidean Green's functions are given by

$$G_{\mathrm{E}}^{(n)}(\hat{x}_1, \ldots, \hat{x}_n) = \frac{\delta}{\delta J(\hat{x}_1)} \cdots \frac{\delta}{\delta J(\hat{x}_n)}\bigg|_{J \equiv 0} W_{\mathrm{E}}[J]$$

Using path integral formalism one can derive equations of motion for them (Problem 2.2): for example, in free scalar field theory

$$\left(-\frac{\partial^2}{\partial \hat{x}_4^2} - \boldsymbol{\nabla}^2 + m^2\right) G_{\mathrm{E}}^{(2)}(\hat{x}, \hat{y}) = \delta(\hat{x}_4 - \hat{y}_4)\delta(\mathbf{x} - \mathbf{y})$$

analogous to those in Minkowski space.

The Minkowski space Green's functions are given by analytic continuation

$$G^{(n)}(x_1, \ldots, x_n) = (\mathrm{i})^n G_{\mathrm{E}}^{(n)}(\hat{x}_1, \ldots, \hat{x}_n)$$

where

$$\mathbf{x} = \hat{\mathbf{x}}, \qquad x_0 = -\mathrm{i}\hat{x}_4$$

Since in Euclidean space the exponent in the generating functional $W_{\mathrm{E}}[J]$ is bounded from above we can evaluate the functional integral by the saddle point method (see Problem 2.8). We recall that the saddle point approximation rests on the fact that the integral

$$I = \int \mathrm{d}x \exp[-a(x)]$$

can be successfully approximated by

$$I \cong \exp[-a(x_0)] \int \mathrm{d}x \exp\left[-\tfrac{1}{2}(x-x_0)^2 a''(x_0)\right]$$

where x_0 satisfies $a'(x_0) = 0$, if $a''(x_0) > 0$ and if the points away from the minimum do not contribute much.

We remark also that we can define the field operator evolution for imaginary time

$$\Phi(x) = \exp(\mathrm{i}Hx_0)\Phi(0, \mathbf{x}) \exp(-\mathrm{i}Hx_0) = \exp(H\hat{x}_4)\Phi(0, \mathbf{x}) \exp(-H\hat{x}_4)$$

and also the operator ordering $\hat{T}$ with respect to $\hat{x}_4$ analogously to the T-product definition, for example,

$$\hat{T}\Phi(\hat{x})\Phi(\hat{y}) = \Theta(\hat{x}_4 - \hat{y}_4)\Phi(\hat{x})\Phi(\hat{y}) + \Theta(\hat{y}_4 - \hat{x}_4)\Phi(\hat{y})\Phi(\hat{x})$$

whose vacuum expectation value then satisfies the same equation of motion as $G_{\mathrm{E}}^{(2)}(\hat{x}, \hat{y})$ defined by the generating functional in Euclidean space.

Action quadratic in fields

In the case in which the classical action is quadratic in the field variable $\Phi(x)$

$$S = \tfrac{1}{2} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \Phi(x)\, A(x, y)\, \Phi(y) \tag{2.59}$$

the generating functional $W_0[J]$ (index 0 corresponds to the specific form (2.59) of the action) is easily obtained in a closed form. Repeating the steps that led us to the derivation of formula (2.41) we obtain

$$W_0[J] = \exp\left[\left(\frac{\mathrm{i}}{2\hbar}\right)\hbar^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\, G(x, y)\, J(y)\right] \tag{2.60}$$

where $G(x, y)$ is the Green's function of the classical field equation

$$\int A(x, y)\, \Phi(y)\, \mathrm{d}^4y = -\hbar J(x) \tag{2.61}$$

satisfying conditions (2.40).

One example of a theory with quadratic action is the theory of a free field, with the lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2\right) + \tfrac{1}{2}\mathrm{i}\varepsilon\Phi^2 \tag{2.62}$$

The extra term $\frac{1}{2}\mathrm{i}\varepsilon\Phi^2$ has been added in accordance with the discussion following

(2.43): it makes the functional integral (2.57) well-defined and simultaneously ensures the correct boundary conditions for the Green's function $G(x, y)$. As

$$\partial_\mu \Phi \partial^\mu \Phi = \partial_\mu \big(\Phi \partial^\mu \Phi\big) - \Phi \partial_\mu \partial^\mu \Phi \tag{2.63}$$

and neglecting in the action the surface term at infinity, the action can be written in the form

$$S = \tfrac{1}{2} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \Phi(x) \big[-\partial_\mu \partial^\mu - m^2 + \mathrm{i}\varepsilon\big]\, \delta(x-y)\, \Phi(y) \tag{2.64}$$

i.e.

$$A(x, y) = -\big[\partial_\mu \partial^\mu + m^2 - \mathrm{i}\varepsilon\big] \delta(x-y) \tag{2.65}$$

The classical field equation

$$\big(\partial_\mu \partial^\mu + m^2 - \mathrm{i}\varepsilon\big) \Phi(x) = \hbar J(x) \tag{2.66}$$

is then solved by

$$\Phi(x) = \hbar \int \mathrm{d}^4y\, G(x-y)\, J(y) \tag{2.67}$$

with the Green's function $G(x-y)$ satisfying

$$\big[\partial_\mu \partial^\mu_{(x)} + m^2 - \mathrm{i}\varepsilon\big] G(x-y) = \delta(x-y) \tag{2.68}$$

Introducing the Fourier transform $\tilde{G}(k)$

$$G(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k\, \tilde{G}(k) \exp[-\mathrm{i}k(x-y)] \tag{2.69}$$

we obtain from (2.68)

$$\tilde{G}(k) = -\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} \tag{2.70}$$

and therefore

$$G(x-y) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} \exp[-\mathrm{i}k(x-y)] \tag{2.71}$$

Again, the Green's function agrees with the analytic continuation from the Euclidean region.

Gaussian integration

We end this section with a comment on gaussian integration. Having defined functional integrals as an appropriate limit of multiple integrals, one encounters integrals of the type

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-b\varepsilon x^2 + \mathrm{i}bx^2\right) = 2\int_0^{\infty} \mathrm{d}x \exp\left[-bx^2(\varepsilon - \mathrm{i})\right] \tag{2.72}$$

with $\varepsilon \to 0$, where the ε-term corresponds to the $(+\mathrm{i}\varepsilon)$ prescription. Rotating the contour of integration into $x' = x\exp(\mathrm{i}\varphi)$ such that $\exp(2\mathrm{i}\varphi)(\varepsilon - \mathrm{i}) = 1$ one easily finds

$$I = \exp\left[\mathrm{i}\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\varepsilon\right)\right](\pi/b)^{1/2} \underset{\varepsilon\to 0}{\longrightarrow} (\mathrm{i}\pi/b)^{1/2} \tag{2.73}$$

which is the analytic continuation of the gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-ax^2\right) = (\pi/a)^{1/2} \tag{2.74}$$

for complex a (and $\mathrm{Re}\, a > 0$). A similar result holds for integration over the complex variable $z = x + \mathrm{i}y$

$$\int \mathrm{d}z^* \,\mathrm{d}z \exp(-az^*z) \equiv 2\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[-a\left(x^2 + y^2\right)\right] = 2\pi/a \tag{2.75}$$

for complex a (and $\mathrm{Re}\, a > 0$). We write $\int \mathrm{d}z^* \,\mathrm{d}z$ for $2\int \mathrm{d}\,\mathrm{Re}\, z \int \mathrm{d}\,\mathrm{Im}\, z$ by convention. For many degrees of freedom $z_1, \ldots, z_n$ we can define the complex vector z

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

and denote by (z^*, Az) the scalar product of vectors z and Az, where A is a $(n \times n)$-dimensional complex matrix. Then we have the following generalization of relation (2.75):

$$\int \mathrm{d}z_1^* \,\mathrm{d}z_1 \ldots \int \mathrm{d}z_n^* \,\mathrm{d}z_n \exp[\mathrm{i}(z^*, Az)] = (2\pi)^n \mathrm{i}^n / \det A \tag{2.76}$$

for any positive-definite matrix A which can be diagonalized by unitary transformations. The result is obvious for a diagonal matrix A (it follows from multiple use of (2.75)). For the general case one has to show that the integration measure is invariant under the unitary transformation diagonalizing A (the scalar product (z^*, Az) is obviously invariant). Denote the unitary transformation by $U = A + \mathrm{i}B$, with A and B real matrices. From the unitarity of U: $UU^\dagger = 1$ it follows that

$$AA^{\mathrm{T}} + BB^{\mathrm{T}} = 1, \qquad BA^{\mathrm{T}} - AB^{\mathrm{T}} = 0 \tag{2.77}$$

The change of integration variables $z' = Uz$ can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.78}$$

and using (2.77) we conclude that the transformation matrix M is orthogonal ($MM^{\mathrm{T}} = 1$). Therefore the integration measure is indeed invariant under the unitary transformation U. For integration over n real variables $x \equiv (x_1, \ldots, x_n)$, one has

$$\int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp[\mathrm{i}(x, Ax)] = \pi^{n/2}\mathrm{i}^{n/2}/[\det A]^{1/2} \tag{2.79}$$

In the limit $n \to \infty$ integral (2.79) is the J-independent path integral in (2.39). Although we do not have to evaluate it explicitly, it is worth remembering its compact form (2.79). Another point is that in addition to results (2.76) and (2.79) we can also easily calculate integrals in which the product (z^*, Az) is replaced by an expression

$$(z^*, Az) + (b^*, z) + (z^*, b) \equiv F(z) \tag{2.80}$$

where b is a constant vector, and matrix A is now hermitean. Indeed, expression (2.80) can be written as

$$F(z) = (\omega^*, A\omega) - \left(b^*, A^{-1}b\right) \tag{2.81}$$

where $\omega = z - z_0$ and $z_0 = -A^{-1}b$ is the minimum of $F(z)$. Actually, (2.81) has been used in the derivation of (2.41) and (2.60) for the generating functional $W_0[J]$.

Finally, we recall that for any matrix L which can be diagonalized by a unitary transformation

$$\det(1 - L) = \exp \operatorname{Tr} \ln(1 - L)$$

where

$$\operatorname{Tr} \ln(1 - L) = -\operatorname{Tr}\left[L + \tfrac{1}{2}L^2 + \tfrac{1}{3}L^3 + \cdots\right]$$

2.4 Introduction to perturbation theory

Perturbation theory and the generating functional

We shall discuss first a simple case of a scalar field theory described by the lagrangian

$$\mathcal{L} = \tfrac{1}{2}\left[\partial_\mu \Phi(x) \partial^\mu \Phi(x) - m^2 \Phi^2(x)\right] - \frac{\lambda}{4!}\Phi^4(x) \tag{2.82}$$

where x denotes four coordinates in Minkowski space. We seek a method for calculating an arbitrary Green's function in such a theory. For that purpose it is convenient to use the functional formulation of the theory in which the Green's functions are given by functional derivatives of the generating functional $W[J]$

$$G^{(n)}(x_1, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)}\bigg|_{J\equiv 0} W[J] \tag{2.83}$$

where

$$W[J] = N \int \mathcal{D}\Phi \exp\left\{(\mathrm{i}/\hbar) \int \mathrm{d}^4x\, [\mathcal{L} + \hbar J(x)\Phi(x)]\right\} \tag{2.84}$$

and factor N is defined by (2.55).

We recall at this point that, so far, we have been able to calculate exactly the generating functional $W_0[J]$ for a theory of non-interacting scalar fields with the action S_0 given by

$$S_0 = \int \mathrm{d}^4x\, \mathcal{L}_{\text{free}} = \int \mathrm{d}^4x\, \tfrac{1}{2}\left(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2\right) \tag{2.85}$$

or, more generally, containing terms at most quadratic in fields. The analogous, exact, solution is not known for the full theory (2.82) and the method to be used for calculating the Green's functions (2.83) is a perturbative expansion in terms of powers of S_I defined as follows

$$S_\mathrm{I} = S - S_0 \tag{2.86}$$

where

$$S = \int \mathrm{d}^4x\, \mathcal{L}$$

Such an expansion should be understood as an expansion in a neighbourhood of the vacuum state for which $\Phi(x) = 0$, so that the action S_I can be regarded as a small parameter. It is convenient to note the following functional identity

$$\begin{aligned}&\int \mathcal{D}\Phi \exp\left(\frac{\mathrm{i}}{\hbar}\left\{S_0[\Phi] + S_\mathrm{I}[\Phi] + \hbar \int J(x)\,\Phi(x)\,\mathrm{d}^4x\right\}\right)\\ &\quad = \exp\left\{\frac{\mathrm{i}}{\hbar} S_\mathrm{I}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\right\} \int \mathcal{D}\Phi \exp\left(\frac{\mathrm{i}}{\hbar}\left\{S_0[\Phi] + \hbar \int J(x)\,\Phi(x)\,\mathrm{d}^4x\right\}\right)\end{aligned} \tag{2.87}$$

so that (2.84) can be rewritten as

$$W[J] = \frac{\exp\left\{\frac{\mathrm{i}}{\hbar} S_\mathrm{I}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\right\} W_0[J]}{\exp\left\{\frac{\mathrm{i}}{\hbar} S_\mathrm{I}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\right\} W_0[J]\Big|_{J\equiv 0}} = N \exp\left\{\frac{\mathrm{i}}{\hbar} S_\mathrm{I}\left[\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right]\right\} W_0[J] \tag{2.88}$$

The perturbation series is generated by expanding the exponential factor

$\exp\{(\mathrm{i}/\hbar)\ S_\mathrm{I}[(1/\mathrm{i})(\delta/\delta J)]\}$ in powers of S_I and performing the functional differentiations as indicated. This is equivalent to expanding $\exp\{(\mathrm{i}/\hbar)S_\mathrm{I}[\Phi]\}$ under the path integral. In perturbation theory one gets, therefore, the following general formula for the Green's functions (2.83)

$$G^{(n)}(x_1,\ldots,x_n) = \frac{\displaystyle\int \mathcal{D}\Phi\,\Phi(x_1)\ldots\Phi(x_n)\left[\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{\mathrm{i}}{\hbar}S_\mathrm{I}\right)^N\right]\exp\left(\frac{\mathrm{i}}{\hbar}S_0\right)}{\displaystyle\int \mathcal{D}\Phi\left[\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{\mathrm{i}}{\hbar}S_\mathrm{I}\right)^N\right]\exp\left(\frac{\mathrm{i}}{\hbar}S_0\right)} \tag{2.89}$$

If S_I can be written as an integral over polynomials in fields (as in our example) any Green's function can be calculated term by term using (2.89). Each term can be written as a respective derivative of the functional $W_0[J]$.

Wick's theorem

As an introduction to the detailed exposition of perturbative rules we consider again the case of non-interacting scalar fields and prove Wick's theorem. The n-point Green's function $G_0^{(n)}(x_1,\ldots,x_n)$ is given by (2.83) with $W[J]$ replaced by $W_0[J]$. From the previous considerations we know that

$$W_0[J] = \exp\left[\tfrac{1}{2}\mathrm{i}\hbar\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\,G(x-y)\,J(y)\right] \tag{2.90}$$

up to the overall J-independent normalization factor which cancels out in the definition of the Green's functions, where $G(x-y)$ is the Green's function for the free classical Klein–Gordon equation

$$G(x-y) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\,\frac{\exp[-\mathrm{i}k(x-y)]}{k^2-m^2+\mathrm{i}\varepsilon} \tag{2.91}$$

Referring again to (2.83) one observes that the n-point Green's function is given by the $\left(\frac{1}{2}n\right)$th term of the Taylor expansion of exponent (2.90). For the free propagator one immediately gets

$$G_0^{(2)}(x_1,x_2) = -\mathrm{i}\hbar G(x_1-x_2) \tag{2.92}$$

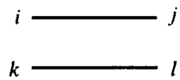
(the symmetry $G(x_1-x_2) = G(x_2-x_1)$ has been taken into account). The case $n=4$ is somewhat more complicated. (Notice that all $G^{(n)}$-functions with n odd equal zero.) It is convenient to convert the integrals in (2.90) into sums over discrete points and write

$$\mathrm{i}^4 G_0^{(4)}(x_1,\ldots,x_4) = \left.\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}\right|_{J=0}\left(\frac{\mathrm{i}}{2}\hbar\right)^2\frac{1}{2!}\sum_{i,j,k,l} J_iJ_jJ_kJ_lG_{ij}G_{kl} \tag{2.93}$$

where

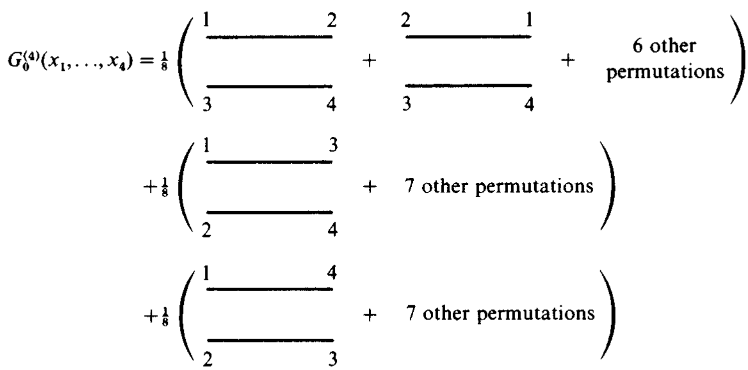
$$J_i = J(x_i), \qquad G_{ij} = G(x_i - y_j)$$

The structure of (2.93) can be represented diagrammatically by the 'prototype' diagram



$$\equiv (-\mathrm{i}\hbar G_{ij}) \times (-\mathrm{i}\hbar G_{kl})$$

and the differentiation with respect to $J_1, \ldots, J_4$ leads to all possible assignments of subscripts 1–4 to subscripts i, j, k, l. One gets



So finally

$$G_0^{(4)}(x_1, \ldots, x_4) = G_0^{(2)}(x_1, x_2) G_0^{(2)}(x_3, x_4) + G_0^{(2)}(x_1, x_3) G_0^{(2)}(x_2, x_4) + G_0^{(2)}(x_1, x_4) G_0^{(2)}(x_2, x_3) \tag{2.94}$$

This is the content of Wick's theorem (see, for example, Bjorken & Drell (1965)) which states that any n-point free Green's function can be written as a sum over all possible products of $\frac{1}{2}n$ two-point Green's function $G_0^{(2)}(x_i, x_j)$.

An example: four-point Green's function in $\lambda\Phi^4$

Coming back to our main problem we shall now discuss several examples in order to derive general rules for perturbative calculations in the theory of interacting fields. Let us consider the four-point Green's function and discuss the numerator of (2.89) in the first order in S_{I} which reads

$$\int \mathcal{D}\Phi\, \Phi(x_1) \ldots \Phi(x_4)\, (-\mathrm{i}\lambda) \frac{1}{4!\hbar} \int \mathrm{d}^4 y\, \Phi^4(y) \exp\left(\frac{\mathrm{i}}{\hbar} S_0\right) \tag{2.95}$$

Fig. 2.3.

or, using the generating functional $W_0[J]$,

$$(-\mathrm{i}\lambda)\frac{1}{4!\hbar}\int \mathrm{d}^4y \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_4)}\frac{\delta^4}{\delta J(y)}\bigg|_{J\equiv 0} W_0[J] \tag{2.96}$$

It is clear that the non-zero contribution to (2.96) comes from the term in the Taylor expansion of (2.90) which contains four propagators

$$\left(\tfrac{1}{2}\mathrm{i}\hbar\right)^4\frac{1}{4!}\left[\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\, G(x-y)\, J(y)\right]^4 \tag{2.97}$$

and can be represented graphically by the prototype diagram in Fig. 2.3. To make the combinatorics transparent we have again replaced the integrations in (2.97) by discrete sums. The differentiations of (2.97) now provide all possible assignments of points x_1, x_2, x_3, x_4 and four points y (it is convenient to split them into y_1, y_2, y_3, y_4, remembering that the limit $y_1 = y_2 = y_3 = y_4$ should be understood) to the points i, j, k, l, m, n, o, p. It is easy to discuss them in a systematic fashion. Let us first assume that pairs of points joined by the propagators are specified, for example, (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Then, we have an additional combinatorial factor following from the 'horizontal' symmetry (for example, $x_1 = i$, $y_1 = j$ and $x_1 = j$, $y_1 = i$) which is 2^n for n pairs, and another factor from the 'vertical' symmetry ((x_1, y_1) can be assigned to (i, j) or (k, l) etc.) which is $n!$ (again for n pairs). These factors cancel with the factor $1/(2^n n!)$ present in (2.97) and this happens for any term of expansion (2.89). So we have to remember about $1/N!$ present in (2.89) ($N = 1$ in our examples) and about $1/4!$ present in the definition of the coupling constant and, on the other hand, we have to take account of all possible choices of pairs of points x_i and y_i joined by propagators. The result is as shown in Fig. 2.4 with self-evident classification of different possibilities. In the limit $y_1 = y_2 = y_3 = y_4 = y$ we get the diagrams shown in Fig. 2.5 where the dot represents the integral of a product of two propagators taken at the same space-time points y (notice that the combinatorial factors in front of the second and the third diagrams include different

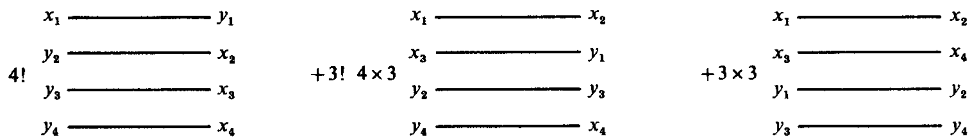


Fig. 2.4.

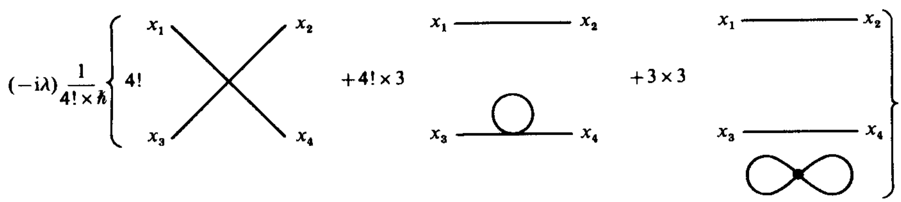


Fig. 2.5.

permutations of x_1, x_2, x_3, x_4). The analytic expression for the first diagram reads

$$(-i\lambda/\hbar)\int d^4y\, G_0^{(2)}(x_1, y)\, G_0^{(2)}(x_2, y)\, G_0^{(2)}(x_3, y)\, G_0^{(2)}(x_4, y) \tag{2.98}$$

It remains to extend our considerations to terms S_{I}^N, $N > 1$ and to discuss the role of the denominator in (2.89). These problems are related since $G_N^{(n)}$ involves in principle a quotient of terms of arbitrary order. We observe that, for example, the S_{I}^2 term in the numerator is given by the diagrams (*a*)–(*j*) in Fig. 2.6. In general, one can convince oneself that the numerator of (2.89) can be represented by a product of two infinite series of amplitudes as shown in Fig. 2.7. The second bracket contains the sum of the so-called vacuum amplitudes which do not depend on the Green's function considered and, most importantly, it happens to be just the denominator of (2.89). (The proof of this again requires some attention to the combinatorial factors.) So we may ignore the denominator of (2.89) and simultaneously all the diagrams containing vacuum subgraphs. Actually, for a given Green's function we shall be interested only in connected diagrams like those in Fig. 2.8 which give non-trivial S-matrix elements for $G^{(4)}(x_1, \ldots, x_4)$. They define the connected Green's functions $G_{\mathrm{conn}}^{(n)}$. The amplitude of, for example, the central diagram can be easily read off from the numerator of (2.89). In the following we shall understand the central diagram to be the one in which the pairs of external points, which are joined by propagators to each of the interaction points, have been specified. In momentum space this corresponds to assigning to each vertex definite external momenta. In this sense we have three different central diagrams, each of which can be discussed as follows. There are twelve fields under the functional integral,

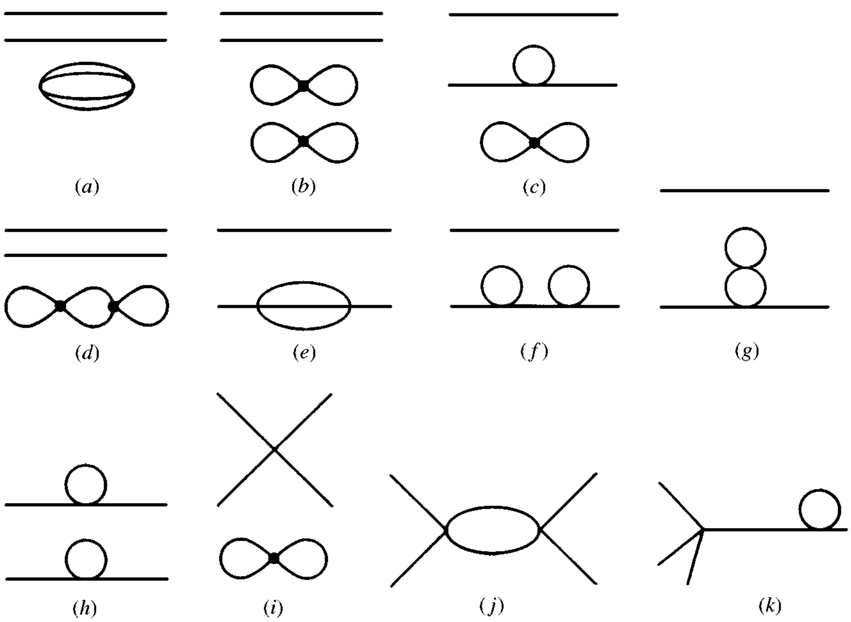


Fig. 2.6.

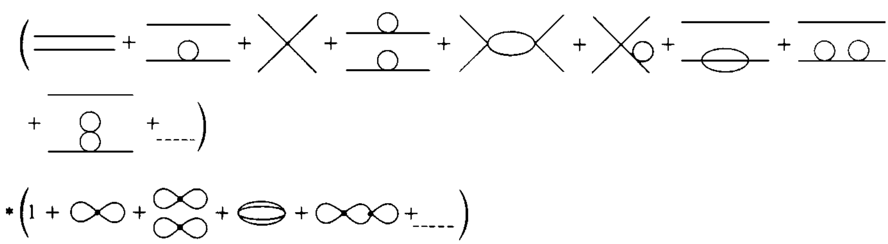


Fig. 2.7.

so the ‘prototype’ diagram with six propagators is relevant for combinatorics. No two external points can be joined by a propagator. Two external points must be joined with the interaction point at y and the other two with the interaction point at z. In addition two propagators must connect y and z. We can have therefore the situation depicted in Fig. 2.9 and, in addition, y_1 can be replaced by any y_i (factor 4), y_2 by any $y_j \neq y_i$ (factor 3), the same for z_4 and z_3 (factor 4×3), there are two combinations for the remaining y_l joining z_k (factor 2) and finally y can be interchanged with z in connections with external points (factor 2). We get $4! \times 4!$ to cancel $1/4! \times 4!$ from the definition of the coupling constant. (The factors from the ‘vertical’ and ‘horizontal’ symmetries cancel as usual.) Since $N = 2$ there is

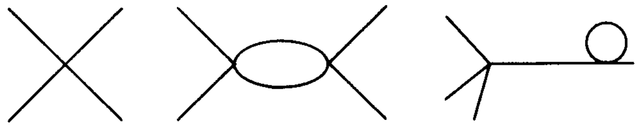


Fig. 2.8.

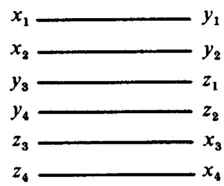


Fig. 2.9.

an extra factor $\frac{1}{2}$ (we will call it the symmetry factor) and the amplitude reads

$$\frac{1}{2}(-\mathrm{i}\lambda/\hbar)^2 \int \mathrm{d}^4 y\, \mathrm{d}^4 z\, G_0^{(2)}(x_1, y)\, G_0^{(2)}(x_2, y)\, G_0^{(2)}(y, z) \\ \times G_0^{(2)}(y, z)\, G_0^{(2)}(x_3, z)\, G_0^{(2)}(x_4, z) \tag{2.99}$$

Our examples can be generalized to the following rules. To calculate $G_{\mathrm{conn}}^{(n)}(x_1, \ldots, x_n)$ to order N in S_{I} draw all connected diagrams consistent with the structure of the numerator in (2.89). For each diagram calculate the combinatorial (symmetry) factor. Attach $(-\mathrm{i}\lambda/\hbar)$ to each vertex and $G_0^{(2)}$ to each line.

Momentum space

One is usually interested in Green's functions in momentum space defined as Fourier transforms of the respective Green's functions in configuration space. We define $\tilde{G}^{(n)}(p_1, \ldots, p_n)$ by the relation

$$G^{(n)}(x_1, \ldots, x_n) = \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \cdots \frac{\mathrm{d}^4 p_n}{(2\pi)^4} \exp(-\mathrm{i} p_1 x_1) \cdots \exp(-\mathrm{i} p_n x_n) \\ \times \tilde{G}^{(n)}(p_1, \ldots, p_n) \tag{2.100}$$

so that all momenta are treated as outgoing (due to $(+\mathrm{i}\varepsilon)$) in propagators, positive frequencies propagate into future $x_n^0 \to +\infty$ (see (2.45)). In actual physical transitions

$$x_1^0 \ldots x_k^0 \to -\infty$$
$$x_{k+1}^0 \ldots x_n^0 \to +\infty$$

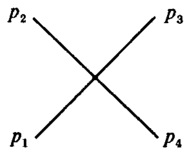


Fig. 2.10.

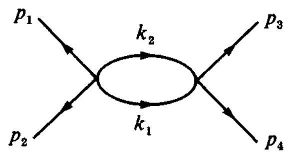


Fig. 2.11.

and therefore the momenta $p_1, \ldots, p_n$ defined by (2.100) are related to the physical momenta $\tilde{p}_i$ ($\tilde{p}_i^0 > 0$) of the initial and final particles as follows

$$\begin{aligned} p_i &= -\tilde{p}_i \qquad && i = 1, \ldots, k \\ p_i &= \tilde{p}_i && i = k+1, \ldots, n \end{aligned}$$

In momentum space we get from (2.92) and (2.91)

$$\tilde{G}_0^{(2)}(p_1, p_2) = (2\pi)^4 \delta(p_1 + p_2) \hbar \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\varepsilon} \tag{2.101}$$

For the diagram shown in Fig. 2.10, represented by expression (2.98) the momentum space amplitude reads

$$(2\pi)^4 \delta\left(\sum_{i=1}^4 p_i \right) (-\mathrm{i}\lambda/\hbar) \prod_{i=1}^4 \hbar \frac{\mathrm{i}}{p_i^2 - m^2 + \mathrm{i}\varepsilon} \tag{2.102}$$

and for the diagram shown in Fig. 2.11 we get from (2.99)

$$\begin{aligned} \tfrac{1}{2}(-\mathrm{i}\lambda/\hbar)^2 \prod_{i=1}^4 \hbar \frac{\mathrm{i}}{p_i^2 - m^2 + \mathrm{i}\varepsilon} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + k_1 + k_2) \\ \times (2\pi)^4 \delta(p_3 + p_4 - k_1 - k_2) \hbar \frac{\mathrm{i}}{k_1^2 - m^2 + \mathrm{i}\varepsilon} \hbar \frac{\mathrm{i}}{k_2^2 - m^2 + \mathrm{i}\varepsilon} \end{aligned} \tag{2.103}$$

The Feynman rules in momentum space are now obvious. Usually, the diagrams as above are assumed to represent amplitudes without the propagators on external lines (Feynman diagrams). Observe that each vertex contributes a factor $\hbar^{-1}$ and

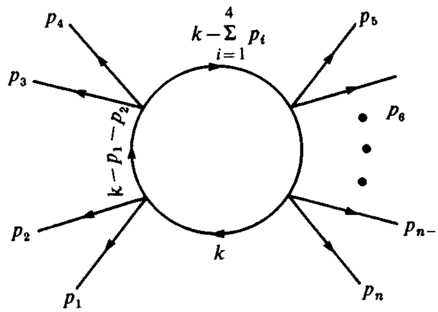


Fig. 2.12.

each internal line gives a factor $\hbar$. Thus a Feynman diagram with L loops is of the order $O\left(\hbar^{L-1}\right)$, because $L =$ (number of internal lines − number of vertices + 1).

It is easy to extend these considerations to a loop with N vertices. Again, assume that the pairs of external points which are joined to each vertex as well as the ordering of vertices along the loop are specified; this corresponds to a well-defined situation in momentum space. We have the factor $(1/N!)(1/4!)^N$ from the general formula (2.89) and an extension of our discussion of the $N = 2$ case gives $(3 \times 4)^N \times 2^{N-1} \times N!$. So we again get the symmetry factor $\frac{1}{2}$ for a Feynman diagram in momentum space, with momenta specified along all the internal lines, for example, as in Fig. 2.12. For N vertices there are $(2N-1)(2N-3)\ldots 1 \times (N-1)!$ different graphs in momentum space; the factor $(2N-1)\ldots 1$ gives the number of different groupings of $2N$ momenta in N pairs and $(N-1)!$ gives the number of their different orderings along the loop.

An important modification of the lagrangian (2.82) follows if we specify that all products of the field operators in it are normal-ordered

$$\mathcal{L} = \tfrac{1}{2} :\partial_\mu \Phi(x) \partial^\mu \Phi(x): - \tfrac{1}{2} m^2 :\Phi^2(x): - (\lambda/4!) :\Phi^4(x): \tag{2.104}$$

This can be rewritten in a form similar to (2.82), but with a number of counterterms; remember that under the path integral the fields are c-numbers and the only way of specifying their order is by introducing appropriate counterterms. From Wick's theorem it follows that (see Section 2.7):

$$\Phi^2(x) = :\Phi^2(x): + \text{const.} \tag{2.105}$$

$$\Phi^4(x) = :\Phi^4(x): + 6G_0^{(2)}(x,x) :\Phi^2(x): + \text{const.} \tag{2.106}$$

so that

$$:\Phi^4(x): = \Phi^4(x) - 6G_0^{(2)}(x,x)\Phi^2(x) - \text{const.} \tag{2.107}$$

and the other counterterms are constants. In the resulting perturbation expansion

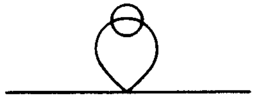


Fig. 2.13.

the diagram  is now cancelled by the $-6G_0^{(2)}(x,x)\Phi^2(x)$ counterterm. Note, however, that the higher order corrections to this diagram, like that shown in Fig. 2.13, are not cancelled. The constant counterterms cancel some vacuum diagrams.

2.5 Path integrals for fermions; Grassmann algebra

Anticommuting c-numbers

The path integral quantization method involves classical fields only and the theory is formulated directly in terms of Green's functions given by path integrals like (2.53). Since for Fermi fields we must have, for example,

$$\langle 0|T\Psi(x_1)\bar{\Psi}(x_2)|0\rangle = -\langle 0|T\bar{\Psi}(x_2)\Psi(x_1)|0\rangle \tag{2.108}$$

the fermion classical fields in the path integral must be taken to be anticommuting quantities. The anticommuting c-number variables, or functions, are said to form Grassmann algebra.

There follows a brief introduction to the subject of Grassmann algebra. Let us take η to be a Grassmann variable,

$$\{\eta, \eta\} = 0 \qquad \text{or} \qquad \eta^2 = 0 \tag{2.109}$$

Any function $f(\eta)$ can be written as

$$f(\eta) = f_0 + \eta f_1 \tag{2.110}$$

For instance if $f(\eta)$ is taken to be an ordinary number, then f_0 and f_1 are ordinary and Grassmann numbers, respectively. One defines the left and right derivatives by the equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\eta = \eta\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\eta} = 1 \tag{2.111}$$

Therefore, when f_1 is a Grassmann number,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}f(\eta) = f_1 = -f(\eta)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\eta} \tag{2.112}$$

Furthermore, we define the integration by the requirement of linearity and the following relations

$$\int \mathrm{d}\eta\, 1 = 0, \qquad \int \mathrm{d}\eta\, \eta = 1 \tag{2.113}$$

The first relation follows from the requirement that the property of a convergent integral over commuting numbers

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x+a)$$

valid for any finite a, holds for an integral over anticommuting numbers as well. The second relation is the normalization convention. We also add a specification that rule (2.113) applies when $\mathrm{d}\eta$ and η are next to each other. Comparing (2.113) with (2.111) we see that the integrals and the left derivatives are identical

$$\int \mathrm{d}\eta\, f(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta} f(\eta) = f_1 \tag{2.114}$$

The integral of the derivative vanishes:

$$\int \mathrm{d}\eta \frac{\mathrm{d}}{\mathrm{d}\eta} f(\eta) = \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} f(\eta) = 0 \tag{2.115}$$

Consider now the change of integration variable $\eta \to \eta' = a + b\eta$. One gets

$$\int \mathrm{d}\eta\, f(\eta) = \int \mathrm{d}\eta' \left(\frac{\mathrm{d}\eta}{\mathrm{d}\eta'}\right)^{-1} f(\eta(\eta')) \tag{2.116}$$

i.e. the standard Jacobian appears inverted. All the rules can be easily generalized to the case of n real Grassmann variables. For a complex Grassmann variable the real and imaginary parts can be replaced by η and η^* as independent generators of Grassmann algebra.

In the following chapter the Gaussian integral for complex Grassmann variables will be shown to be very useful. It can be shown that instead of (2.76) we now have

$$\int \mathrm{d}\eta_1\, \mathrm{d}\eta_1^* \ldots \mathrm{d}\eta_n\, \mathrm{d}\eta_n^* \exp(\eta^*, A\eta) = \det A \tag{2.117}$$

where

$$(\eta^*, A\eta) = \sum_{i,j} \eta_i^* A_{ij} \eta_j$$

Eq. (2.117) is valid for an arbitrary A. For a diagonal A the proof of (2.117) follows immediately from the integration rules for Grassmann variables. The corresponding formula also holds for an integral over real variables (see Problem 2.4).

Gaussian integrals can be used to define the determinant of a matrix acting in superspace. Coordinates in superspace are commuting and anticommuting variables, z and η, respectively. We can define a linear transformation in superspace

$$\begin{pmatrix} z' \\ \eta' \end{pmatrix} = \begin{bmatrix} A & D \\ C & B \end{bmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix} \equiv M \begin{pmatrix} z \\ \eta \end{pmatrix}$$

where matrices A and B are commuting, and C and D are anticommuting. Vectors in superspace are decomposed in terms of the variables z and η.

The superdeterminant of a superspace matrix M can be constructed by calculating a generalized Gaussian integral

$$(\det M)^{-1} = \int \frac{\mathrm{d}z\,\mathrm{d}z^*}{2\pi} \mathrm{d}\eta^*\,\mathrm{d}\eta \exp[-(z^*, Az) - (z^*, D\eta) - (\eta^*, Cz) - (\eta^*, B\eta)]$$

To evaluate the integral we make a shift in integration variables

$$\eta = \eta' - B^{-1}Cz$$
$$\eta^* = \eta^{*\prime} - z^* D B^{-1}$$

and using (2.76) and (2.117) we get

$$\det M = \frac{\det\left(A - DB^{-1}C\right)}{\det B}$$

If we shift z and z^* we get

$$\det M = \frac{\det A}{\det\left(B - CA^{-1}D\right)}$$

These results for the superdeterminant are not surprising if we write

$$M = \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} 1 & A^{-1}D \\ 0 & B - CA^{-1}D \end{bmatrix}$$

or

$$M = \begin{bmatrix} 1 & D \\ 0 & B \end{bmatrix} \begin{bmatrix} A - DB^{-1}C & 0 \\ B^{-1}C & 1 \end{bmatrix}$$

Several other properties of the superdeterminant are listed in Problem 2.9.

Dirac propagator

The classical Dirac fields $\Psi(x)$ and $\bar{\Psi}(x)$ are taken to be elements of an infinite-dimensional Grassmann algebra. The generating functional $W_0^{\Psi}[\alpha]$ for a free-fermion field described by the lagrangian

$$\mathcal{L}_\Psi = \bar{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi$$

is

$$W_0^\Psi[\alpha,\bar{\alpha}] = N\int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\exp\left\{(\mathrm{i}/\hbar)\int \mathrm{d}^4x\,[\mathcal{L}_\Psi(x) + \hbar\bar{\alpha}(x)\Psi(x) + \hbar\bar{\Psi}(x)\alpha(x)]\right\} \tag{2.118}$$

where

$$N^{-1} = \langle 0|0\rangle = W_0^\Psi[0,0]$$

and $\alpha(x)$ and $\bar{\alpha}(x)$ are sources of fermion fields. Using a formula analogous to (2.81) but for Grassmann fields

$$\int \mathcal{D}\eta\,\mathcal{D}\eta^*\exp[\mathrm{i}(\eta^*, A\eta) + \mathrm{i}(\beta^*,\eta) + \mathrm{i}(\eta^*,\beta)] = \det(\mathrm{i}A)\exp\left[\mathrm{i}\big(\beta^*, -A^{-1}\beta\big)\right] \tag{2.119}$$

we can rewrite $W_0^\Psi[\alpha,\bar{\alpha}]$ as follows:

$$W_0^\Psi[\alpha,\bar{\alpha}] = \exp\left[\mathrm{i}\hbar\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{\alpha}(x)\,S(x-y)\,\alpha(y)\right] \tag{2.120}$$

where

$$(\mathrm{i}\not{\partial} - m)S(x-y) = -\mathbb{1}\delta(x-y) \tag{2.121}$$

Therefore

$$S(x-y) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathbb{1}\frac{\not{k}+m}{k^2-m^2+\mathrm{i}\varepsilon}\exp[-\mathrm{i}k(x-y)] \tag{2.122}$$

The Feynman propagator defined as

$$\begin{aligned} S_\mathrm{F}(x-y) = \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle/\langle 0|0\rangle = \int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\,\Psi(x)\,\bar{\Psi}(y) \\ \times\exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\,\mathcal{L}_\Psi(x)\right]\Big/\int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\exp\left[(\mathrm{i}/\hbar)\int \mathrm{d}^4x\,\mathcal{L}_\Psi(x)\right] \end{aligned} \tag{2.123}$$

reads

$$S_\mathrm{F}(x-y) = \left(\frac{1}{\mathrm{i}}\right)^2\frac{\delta}{\delta\bar{\alpha}(x)}W_0^\Psi[\alpha,\bar{\alpha}]\frac{\overleftarrow{\delta}}{\delta\alpha(y)}\bigg|_{\alpha=\bar{\alpha}=0} \tag{2.124}$$

and using (2.120) one gets

$$S_\mathrm{F}(x-y) = -\mathrm{i}\hbar S(x-y) \tag{2.125}$$

Imagine now a theory of interacting fermions with the action

$$\begin{aligned} S &= \int \mathrm{d}^4x\,\mathcal{L}_\Psi(x) + \int \mathrm{d}^4x\,\bar{\Psi}(x)\,V(x)\,\Psi(x) \\ &= S_0 + S_\mathrm{I} \end{aligned} \tag{2.126}$$

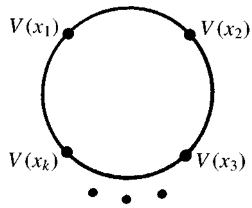


Fig. 2.14.

Following Section 2.4 we can develop perturbation theory with the interaction term as the perturbation. In the expansion in powers of S_{I} of, for example, the vacuum-to-vacuum amplitude (the denominator in (2.89)) we find under the path integral the following products of the Ψ and $\bar{\Psi}$ fields:

$$\int \mathrm{d}x_1 \ldots \mathrm{d}x_k\, \bar{\Psi}(x_1)\, V(x_1)\, \Psi(x_1)\, \bar{\Psi}(x_2)\, V(x_2)\, \Psi(x_2) \ldots \bar{\Psi}(x_k)\, V(x_k)\, \Psi(x_k) \tag{2.127}$$

which correspond to fermion loops like that shown in Fig. 2.14. The path integral with insertion (2.127) can be calculated explicitly by differentiating $W_0^{\Psi}[\alpha, \bar{\alpha}]$ given by (2.120) with respect to α and $\bar{\alpha}$. Before making the substitution

$$\Psi(x_i) \to \frac{1}{\mathrm{i}} \frac{\overrightarrow{\delta}}{\delta\bar{\alpha}(x_i)}, \qquad \bar{\Psi}(x_i) \to \frac{1}{\mathrm{i}} \frac{\overleftarrow{\delta}}{\delta\alpha(x_i)} \tag{2.128}$$

we must, however, rearrange the fields as follows

$$V(x_1)\Psi(x_1)\bar{\Psi}(x_2)V(x_2)\Psi(x_2)\bar{\Psi}(x_3)\ldots V(x_k)\Psi(x_k)\bar{\Psi}(x_1)$$

This reordering introduces the familiar minus sign for a closed fermion loop. From (2.120) and (2.128) we obtain for the closed fermion loop the following result

$$(-)V(x_1)S_{\mathrm{F}}(x_1 - x_2)V(x_2)S_{\mathrm{F}}(x_2 - x_3)\ldots V(x_k)S_{\mathrm{F}}(x_k - x_1) \tag{2.129}$$

2.6 Generating functionals for Green's functions and proper vertices; effective potential

Classification of Green's functions and generating functionals

There are three basic sorts of Green's functions. The full Green's functions are defined by (2.54)

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n) &= \langle 0|T\Phi(x_1)\ldots\Phi(x_n)|0\rangle/\langle 0|0\rangle \\ &= \int \mathcal{D}\Phi\,\Phi(x_1)\ldots\Phi(x_n)\exp\{(\mathrm{i}/\hbar)S[\Phi]\} \\ &\quad\times\left(\int \mathcal{D}\Phi\exp\{(\mathrm{i}/\hbar)S[\Phi]\}\right)^{-1} \end{aligned} \tag{2.54}$$

with appropriate generalization to the case of other fields; for simplicity, in this section we shall always consider the case of a single field. In perturbation theory the full Green's functions $G^{(n)}$ are given by the sum of all diagrams with n external legs, including disconnected ones, with the exception of the vacuum diagrams which are cancelled by the normalization factor.

The connected Green's functions $G^{(n)}_{\text{conn}}(x_1, \ldots, x_n)$ are defined as the connected part of $G^{(n)}$. A connected Green's function is obtained by disregarding all terms which factorize into two or more functions with no overlapping arguments. For a Fourier transform this implies that no subset of n external momenta is conserved separately. The Feynman diagrams contributing to $G^{(n)}_{\text{conn}}$ are all connected.

The connected proper vertex functions $\Gamma^{(n)}(x_1, \ldots, x_n)$, also called the one-particle-irreducible (1PI) Green's functions, are given by the Feynman diagrams which are one-particle-irreducible; i.e. they remain connected after an arbitrary internal line is cut. In the lowest order (no loops) the connected proper vertex functions coincide when appropriately normalized with the vertices of the original lagrangian.

To each type of the Green's functions we assign the corresponding generating functional. These are defined as follows

$$W[J] = \sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\, G^{(n)}(x_1, \ldots, x_n)\, J(x_1)\ldots J(x_n) \tag{2.130}$$

$$Z[J] = \hbar\sum_{n=1}^{\infty}\frac{\mathrm{i}^{n-1}}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\, G^{(n)}_{\text{conn}}(x_1, \ldots, x_n)\, J(x_1)\ldots J(x_n) \tag{2.131}$$

and

$$\Gamma[\Phi] = \sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\,\Gamma^{(n)}(x_1, \ldots, x_n)\,\Phi(x_1)\ldots\Phi(x_n) \tag{2.132}$$

Observe that in the generating functional $\Gamma[\Phi]$ we denote the arguments as $\Phi(x)$, not $J(x)$: this is convenient because of the relationship between $\Gamma[\Phi]$ and the action integral $S[\Phi]$ which we shall discuss below. It follows from definitions (2.130)–(2.132) that

$$G^{(n)}(x_1, \ldots, x_n) = \left(\frac{1}{\mathrm{i}}\right)^n \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J\equiv 0} \tag{2.133}$$

$$G^{(n)}_{\text{conn}}(x_1, \ldots, x_n) = \frac{1}{\hbar}\left(\frac{1}{\mathrm{i}}\right)^{n-1} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)}\bigg|_{J\equiv 0} \tag{2.134}$$

and

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n \Gamma[\Phi]}{\delta \Phi(x_1) \ldots \delta \Phi(x_n)}\bigg|_{\Phi\equiv 0} \tag{2.135}$$

The generating functional of the full Green's functions has been studied in Sections 2.4 and 2.5. It is given by the path integral formula

$$W[J] = \int \mathcal{D}\Phi \exp\left[\frac{\mathrm{i}}{\hbar}\left\{S[\Phi] + \hbar \int \mathrm{d}^4 x\, J(x)\, \Phi(x)\right\}\right] \Big/ \int \mathcal{D}\Phi \exp\left\{\frac{\mathrm{i}}{\hbar} S[\Phi]\right\} \tag{2.136}$$

We want to show that the functionals $Z[J]$ and $\Gamma[\Phi]$ are given by the following relations:

$$W[J] = \exp\{(\mathrm{i}/\hbar) Z[J]\} \tag{2.137}$$

and

$$\Gamma[\Phi_{\text{CL}}] = Z[J] - \hbar \int \mathrm{d}^4 x\, J(x)\, \Phi_{\text{CL}}(x) \tag{2.138}$$

where the 'classical' field $\Phi_{\text{CL}}(x)$ is defined as the field which minimizes the combination

$$\Gamma[\Phi] + \hbar \int \mathrm{d}^4 x\, J(x)\, \Phi(x)$$

or by the equation

$$\frac{\delta \Gamma[\Phi]}{\delta \Phi(x)}\bigg|_{\Phi=\Phi_{\text{CL}}} = -\hbar J(x) \tag{2.139}$$

By differentiating (2.138) with respect to $J(x)$ we also find

$$\frac{\delta Z[J]}{\delta J(x)} = \int \mathrm{d}^4 y \frac{\delta \Gamma[\Phi_{\text{CL}}]}{\delta \Phi_{\text{CL}}(y)} \frac{\delta \Phi_{\text{CL}}(y)}{\delta J(x)} + \hbar \Phi_{\text{CL}}(x) + \hbar \int \mathrm{d}^4 y \frac{\delta \Phi_{\text{CL}}(y)}{\delta J(x)} J(y) \tag{2.140}$$

or, with (2.139),

$$\Phi_{\text{CL}}(x) = \frac{1}{\hbar} \frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\Phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \tag{2.141}$$

Eq. (2.138) is the Legendre transform. If $Z[J]$ is known we can use (2.141) to determine $J(x)$ in terms of $\Phi_{\mathrm{CL}}(x)$ so that the r.h.s. of (2.138) can be written as a functional of $\Phi_{\mathrm{CL}}(x)$ which determines $\Gamma[\Phi_{\mathrm{CL}}]$. The value of (2.141) when the external source is turned off ($J(x) \equiv 0$) is the vacuum expectation value of the field $\Phi(x)$ which, for the time being, we assume to be zero:

$$\left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J\equiv 0} = 0 \tag{2.142}$$

The case of a non-zero field vacuum expectation value reflecting spontaneous symmetry breaking will be discussed later in this section.

Eq. (2.139) expresses $J(x)$ in terms of $\Phi_{\mathrm{CL}}(x)$ and in this sense it is the inverse of (2.141). In particular, together with (2.141) and (2.142), it implies that

$$\left.\frac{\delta\Gamma[\Phi_{\mathrm{CL}}]}{\delta\Phi_{\mathrm{CL}}(x)}\right|_{\Phi_{\mathrm{CL}}\equiv 0} = 0 \tag{2.143}$$

because when $J \equiv 0$, Φ_{CL} takes the value 0 and vice versa.

It remains to prove that (2.137) and (2.138) hold for the generating functionals defined by (2.130)–(2.132). This can be checked by induction (see, for example, Abers & Lee (1973)). Here we shall use another argument which allows a simple understanding of the relationship between $\Gamma[\Phi]$ and $S[\Phi]$.

Effective action

Consider the classical field equation (we take the case of the $\lambda\Phi^4$ theory for definiteness)

$$\left(\partial_\mu\partial^\mu + m^2\right)\Phi_{\mathrm{c}} = -(\lambda/3!)\Phi_{\mathrm{c}}^3 + \hbar J(x), \qquad \lambda > 0 \tag{2.144}$$

Eq. (2.144) follows from the action principle

$$\left.\frac{\delta}{\delta\Phi}\right|_{\Phi=\Phi_{\mathrm{c}}} \left\{S[\Phi] + \hbar\int \mathrm{d}^4x\, J(x)\,\Phi(x)\right\} = 0 \tag{2.145}$$

(do not confuse Φ_{c} and Φ_{CL}), where

$$S[\Phi] = \int \mathrm{d}^4x \left[-\tfrac{1}{2}\Phi\left(\partial_\mu\partial^\mu + m^2\right)\Phi - (\lambda/4!)\Phi^4\right]$$

Eq. (2.144) can be solved perturbatively. In the zeroth order in λ we obtain

$$\Phi_{\mathrm{c}}^{(0)}(x) = \int \mathrm{d}^4y\, [-\mathrm{i}\hbar G(x-y)]\mathrm{i}J(y) \tag{2.146}$$

where $G(x-y)$ is defined by (2.68)

$$\left(\partial_\mu\partial^\mu + m^2\right)G(x-y) = \delta(x-y) \tag{2.68}$$

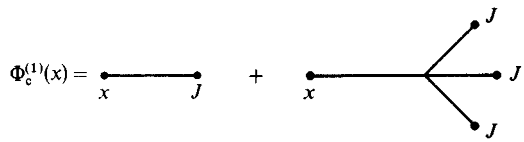


Fig. 2.15.

(the iε term should always be remembered). To obtain the first order correction we insert $\Phi_c^{(0)}$ in the r.h.s. of (2.144). The result reads

$$\Phi_c^{(1)}(x) = \Phi_c^{(0)}(x) + \int d^4y\,[-i\hbar G(x-y)](-i\lambda/\hbar 3!)\left\{\int d^4z\,[-iG(y-z)]iJ(z)\right\}^3 \tag{2.147}$$

which can be represented diagrammatically as shown in Fig. 2.15. Repeating the iterations we obtain the perturbation series for the classical field $\Phi_c(x)$. It is easy to see that only tree diagrams appear and that they are all connected.

Consider now the generating functional $W[J]$, given by (2.136), in the classical approximation. This corresponds to replacing $\Phi(x)$ in the integrand of (2.136) by the classical solution $\Phi_c(x)$. Only the numerator then contributes because $J \equiv 0$ in the denominator, and we get

$$W[J] = \exp\left((i/\hbar)\left\{S[\Phi_c] + \hbar\int d^4x\, J(x)\,\Phi_c(x)\right\}\right) \tag{2.148}$$

Thus, if we assume (2.137), in the classical approximation $Z_c[J]$ reads

$$Z_c[J] = S[\Phi_c] + \hbar\int d^4x\, J(x)\,\Phi_c(x) \tag{2.149}$$

and it is indeed given by the connected tree diagrams. This can be seen by calculating $\delta^n Z/\delta J(x_1)\ldots\delta J(x_n)|_{J\equiv 0}$ using (2.149) and the expansion (2.147) for the $\Phi_c(x)$.

Moreover, from (2.138) and (2.141), in the classical approximation $\Gamma[\Phi_{CL}] = S[\Phi_c]$ so in this approximation $\Gamma[\Phi_c]$ as given by (2.138) is indeed the generating function for the 1PI Green's functions (because $\delta^n S[\Phi_c]/\delta\Phi_c(x_1)\ldots\delta\Phi_c(x_n)|_{\Phi_c=0}$ generates proper vertex functions in the tree approximation).

When the loop corrections are taken into account the connected Green's functions as well as the generating functional $Z[J]$ can still be understood as given by the tree diagrams but with the elementary vertices replaced by 1PI Green's functions. If in (2.136) we replace the action $S[\Phi]$ by the functional $\Gamma[\Phi]$ and instead of doing the path integration exactly, again make in the integrand the 'classical' approximation $\Phi(x) \to \Phi_{CL}(x)$ with $\Phi_{CL}(x)$ defined by (2.139), we shall obtain the exact result. This is because all loop corrections are already included in $\Gamma[\Phi]$. In this sense $\Gamma[\Phi]$ can be treated as the effective action including the loop corrections. $\Gamma^{(n)}(x_1, \ldots, x_n)$s are n-point non-local vertices of this effective action. We conclude that to all orders

$$W[J] = \exp\left[(\mathrm{i}/\hbar)\left\{\Gamma[\Phi_{\mathrm{CL}}] + \hbar \int \mathrm{d}^4x\, J(x)\, \Phi_{\mathrm{CL}}(x)\right\}\right] \tag{2.150}$$

and therefore, as $\Gamma[\Phi_{\mathrm{CL}}]$ and Φ_{CL} are both given by connected diagrams, $Z[J]$ given by (2.137) is indeed connected (the connected diagrams exponentiate). Also (2.138) holds to all orders.

It is worth observing explicitly that differentiating (2.141) with respect to $\Phi_{\mathrm{CL}}(y)$, using (2.139) and setting $\Phi_{\mathrm{CL}}(x) = 0$ one gets

$$\int \mathrm{d}^4z\, \Gamma^{(2)}(x-z)\, G^{(2)}_{\mathrm{conn}}(z-x) = \mathrm{i}\hbar\delta(x-y) \tag{2.151}$$

or, in momentum space,

$$\tilde{\Gamma}^{(2)}(p)\tilde{G}^{(2)}(p) = \mathrm{i}\hbar \tag{2.152}$$

(we draw attention to the factor i). For higher n, strictly speaking, $\mathrm{i}\Gamma^{(n)}$s are the 1PI functions given directly in terms of Feynman diagrams.

Spontaneous symmetry breaking and effective action

The formalism of the effective action can be extended to the case of a non-vanishing field vacuum expectation value which corresponds to spontaneous symmetry breaking (the latter is discussed in detail in Chapter 9). The functionals $W[J]$ and $Z[J]$ are taken as given by (2.136) and (2.137), respectively. However, we now suppose that (2.142) is replaced by

$$\frac{\langle 0|\Phi(x)|0\rangle}{\langle 0|0\rangle} = \frac{1}{\hbar}\frac{\delta Z[J]}{\delta J(x)}\bigg|_{J\equiv 0} = v \tag{2.153}$$

where $v =$ const. from translational invariance. It can be proved, for example, by induction, that higher derivatives of $Z[J]$ at $J \equiv 0$ are connected Green's functions of the field $\bar{\Phi} = \Phi - v$ whose vacuum expectation value vanishes:

$$\frac{1}{\hbar}\left(\frac{1}{\mathrm{i}}\right)^{n-1} \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J\equiv 0} = \langle 0|T\bar{\Phi}(x_1)\ldots\bar{\Phi}(x_n)|0\rangle \tag{2.154}$$

The effective action is defined by (2.138) and (2.139), so that

$$\frac{\delta\Gamma[\Phi_{\mathrm{CL}}]}{\delta\Phi_{\mathrm{CL}}(x)}\bigg|_{\Phi_{\mathrm{CL}}=v} = 0 \tag{2.155}$$

This follows from (2.139), (2.141) and (2.153) because when $J \equiv 0$, Φ_{CL} takes the value v and vice versa. Also, we can prove, for example, by induction, that $\Gamma[\Phi_{CL}]$ given by (2.138) generates 1PI vertices for the field $\bar{\Phi} = \Phi - v$

$$\left.\frac{\delta^n \Gamma[\Phi_{CL}]}{\delta\Phi_{CL}(x_1)\ldots\delta\Phi_{CL}(x_n)}\right|_{\Phi_{CL}=v} = \Gamma^{(n)}_{\bar{\Phi}=\Phi-v}(x_1,\ldots,x_n) \tag{2.156}$$

and

$$\Gamma[\Phi_{CL}] = \sum_{n=1}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\,\Gamma^{(n)}_{\bar{\Phi}=\Phi-v}(x_1,\ldots,x_n) \\ \times(\Phi_{CL}(x_1)-v)\ldots(\Phi_{CL}(x_n)-v) \tag{2.157}$$

It is important to observe that even for $\langle 0|\Phi(x)|0\rangle = v \neq 0$ the effective action $\Gamma[\Phi_{CL}]$ can also be expanded in terms of the original fields $\Phi(x)$, see (2.132). The coefficients of such an expansion

$$\left.\frac{\delta^n \Gamma[\Phi_{CL}]}{\delta\Phi_{CL}(x_1)\ldots\delta\Phi_{CL}(x_n)}\right|_{\Phi_{CL}\equiv 0} = \Gamma^{(n)}_{\Phi}(x_1,\ldots,x_n) \tag{2.158}$$

are the 1PI Green's functions for the original fields $\Phi(x)$. This can be proved, for example, by repeating the inductive arguments leading to (2.156) but for $\Phi_{CL} = 0$ and correspondingly for $J(x) = J_0$, where

$$\left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J\equiv J_0} = 0 \tag{2.159}$$

One can first check by induction that higher derivatives of $Z[J]$ taken for $J \equiv J_0$ give connected Green's functions $\langle 0|T\Phi\ldots\Phi|0\rangle_{\text{conn}}$. Then we differentiate the appropriate number of times the relation $\delta Z[J]/\delta J(x) = \Phi_{CL}(x)$ with respect to Φ_{CL} and express the derivatives $\delta^n J(x)/\delta\Phi_{CL}(x_1)\ldots\Phi_{CL}(x_n)$ in terms of $\Gamma[\Phi_{CL}]$ defined by (2.138) and (2.139). In the limit $J \equiv J_0$ and $\Phi_{CL} \equiv 0$ we get the desired result. The relation between the Green's functions $\Gamma^{(n)}_{\bar{\Phi}=\Phi-v}$ and $\Gamma^{(n)}_{\Phi}$ is given by the analogue of the relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x)|_{x=a} = \sum_{m=0}^{\infty}\frac{a^m}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n+m} f(x)|_{x=0}$$

Thus we have

$$\Gamma^{(n)}_{\bar{\Phi}=\Phi-v}(x_1,\ldots,x_n) = \sum_{m=0}^{\infty}(1/m!)v^m\int \mathrm{d}^4x_{n+1}\ldots\mathrm{d}^4x_{n+m} \\ \times\Gamma^{(n+m)}_{\Phi}(x_1,\ldots,x_n,\ldots,x_{n+m}) \tag{2.160}$$

or, in momentum space,

$$\tilde{\Gamma}^{(n)}_{\bar{\Phi}}(p_1, \dots, p_n) = \sum_{m=0}^{\infty} (1/m!) v^m \tilde{\Gamma}^{(n+m)}_{\Phi}(p_1, \dots, p_n, \underbrace{0, \dots, 0}_{m}) \tag{2.161}$$

where $\tilde{\Gamma}^{(n)}(p_1, \dots, p_n)$ is the Fourier transform of the proper vertex $\Gamma^{(n)}(x_1, \dots, x_n)$

$$\tilde{\Gamma}^{(n)}(p_1, \dots, p_n)(2\pi)^4 \delta(p_1 + \dots + p_n) = \int \prod_{i=1}^{n} \mathrm{d}^4 x \exp(\mathrm{i} p_i x_i)\, \Gamma^{(n)}(x_1, \dots, x_n)$$

Eq. (2.161) expresses the 1PI Green's functions in a theory with spontaneously broken symmetry in terms of the 1PI Green's functions calculated as in the symmetric mode.

Effective potential

Eq. (2.155) has very important implications in studies of spontaneous symmetry breaking: it tells us that the field vacuum expectation value is the value of $\Phi_{\mathrm{CL}}(x)$ which extremizes $\Gamma[\Phi_{\mathrm{CL}}]$. These implications can be discussed more conveniently if we introduce the so-called effective potential $V(\varphi)$. It is defined by the equation

$$\Gamma[\Phi(x) = \varphi = \text{const.}] = -(2\pi)^4 \delta(0) V(\varphi) \tag{2.162}$$

($V(\varphi)$ is a function, not a functional). Condition (2.155) now translates into

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\varphi}\right|_{\varphi = v} = 0 \tag{2.163}$$

which is more useful than the original one because the effective potential $V(\varphi)$ can be calculated perturbatively in a systematic way. Using representation (2.132) for the $\Gamma[\Phi]$ we get

$$V(\varphi) = -\sum_{n=2}^{\infty} (1/n!) \tilde{\Gamma}^{(n)}_{\Phi}(0, 0, \dots, 0) \varphi \dots \varphi \tag{2.164}$$

so that the nth derivative of V at $\varphi = 0$ is the sum of all 1PI diagrams for the *original* fields of the lagrangian, with n vanishing external momenta. (In the tree approximation V is just the ordinary potential, the negative sum of all non-derivative terms in the lagrangian density.) Alternatively, one can rewrite the lagrangian of the theory in terms of shifted fields $\bar{\Phi} = \Phi - v$ considering v as an extra free parameter, develop the appropriate Feynman rules and fix v directly from (2.155) and (2.156), i.e. by demanding that all tadpole diagrams sum to zero. This method of calculation is equivalent to computing directly the derivative of V and demanding that it vanishes, without computing V first, but it makes use of the

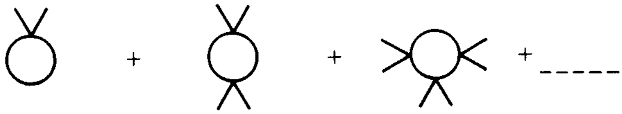


Fig. 2.16.

formulation of the theory in which the underlying spontaneously broken symmetry is not explicit (see Problem 4.1).

Knowledge of the effective potential means that one knows the structure of spontaneous symmetry breaking. In principle, using (2.164), we can calculate the effective potential in perturbation theory but the calculation involves a double sum: over all 1PI Green's functions and for each 1PI Green's function there is the expansion in powers of the coupling constant. A sensible way of organizing this double series is the loop expansion. It has been seen at the end of Section 2.4 that loop expansion is an expansion in powers of Planck's constant $\hbar$: a Feynman diagram with L loops is $O\left(\hbar^{L-1}\right)$. Thus, the loop expansion preserves any symmetry of the lagrangian and it is unaffected by shifts of fields and by the redefinition of the division of the lagrangian into free and interacting parts associated with such shifts. Indeed expansion in powers of $\hbar$ is an expansion in a parameter that multiplies the total Lagrange density; $(1/\hbar)\mathcal{L} \to \mathcal{L}$ corresponds to a change of units into $\hbar = 1$.

The calculation of the effective potential for the $\lambda\Phi^4$ theory in the one-loop approximation is simple. To the lowest order (tree approximation) the only non-vanishing 1PI Green's functions are $\tilde{\Gamma}^{(2)}(p, p) = p^2 - m^2$ and $\tilde{\Gamma}^{(4)} = -\lambda$, and we get

$$V_0(\varphi) = +\tfrac{1}{2}m^2\varphi^2 + (\lambda/4!)\varphi^4 \tag{2.165}$$

To the next order (one-loop approximation) we have the infinite series of diagrams shown in Fig. 2.16 which gives the following contribution

$$\begin{aligned} V_1(\varphi) &= \mathrm{i}\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\lambda\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon}\frac{\varphi^2}{2}\right]^n \\ &= -\tfrac{1}{2}\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(1 - \frac{\frac{1}{2}\lambda\varphi^2}{k^2 - m^2 + \mathrm{i}\varepsilon}\right) \end{aligned} \tag{2.166}$$

The factor $(1/2n)\left(\frac{1}{2}\right)^n$ can be easily understood by recalling the discussion at the end of Section 2.4: each diagram represents $[(2n-1)(2n-3)\ldots 1](n-1)!$ Feynman diagrams each with the symmetry factor $\frac{1}{2}$; thus we get $(1/2n!)[(2n-1)(2n-3)\ldots 1](n-1)!\frac{1}{2} = (1/2n)\left(\frac{1}{2}\right)^n$. The integral in (2.166) exhibits the UV divergence typical for perturbative calculations in quantum field theories.

Its evaluation requires setting up the renormalization programme and for further discussion of V_1 given by (2.166) we refer the reader to Section 11.2.

Our final, but very important, remark in this section deals with the physical meaning of the effective potential. It can be shown (see, for example, Coleman (1974)) that $V(\varphi)$ is the expectation value of the energy density in a certain state for which the expectation value of the field is φ. This interpretation of $V(\varphi)$ is not surprising: it is already suggested by the classical limit of $V(\varphi)$ and by, for example, (2.163) for the field vacuum expectation value $\langle 0|\Phi(x)|0\rangle = v$. If $V(\varphi)$ has several local minima, it is only the absolute minimum that corresponds to the true ground state of the theory.

2.7 Green's functions and the scattering operator

The Green's functions of quantum field theory can be used to determine the matrix elements of the operator S which are directly related to scattering amplitudes and therefore experimental results. Formally, the operator S can be obtained as an infinite time limit of the evolution operator in the interaction picture. In the interaction picture the time evolution of operators is governed by the free, H_0, part of the complete hamiltonian operator $H = H_0 + H_\text{I}$ of the system (from now on we put $\hbar = 1$)

$$\Phi(\mathbf{x}, t) = \exp(\mathrm{i}H_0 t)\Phi(\mathbf{x}, 0)\exp(-\mathrm{i}H_0 t) \tag{2.167}$$

while the state vectors obey the equation

$$\mathrm{i}\frac{\partial}{\partial t}\left| t\right\rangle = H_\text{I}(t)|t\rangle; \qquad h_\text{I}(t) = \exp(\mathrm{i}H_0 t)H_\text{I}\exp(-\mathrm{i}H_0 t) \tag{2.168}$$

with the formal solution

$$|t\rangle = U(t, t')|t'\rangle \tag{2.169}$$

$$U(t, t') = T\exp\left[-\mathrm{i}\int_{t'}^{t} \mathrm{d}t\; H_\text{I}(t)\right] \tag{2.170}$$

We then define the scattering operator to be

$$S = U(\infty, -\infty) = T\exp\left[-\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}t\; H_\text{I}(t)\right] \tag{2.171}$$

It transforms the 'incoming' ($t \to -\infty$) states into the 'outgoing' ($t \to +\infty$) ones.

The interaction hamiltonian $H_\text{I}(t)$ in (2.170) can be expressed in terms of the interaction picture field operators. For simplicity, let us consider the case of a single scalar field, described by the lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu\Phi\partial^\mu\Phi - m^2\Phi^2\right) - V(\Phi) \tag{2.172}$$

or, equivalently, by the hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3x \left\{\Pi^2(\mathbf{x}, t) + [\nabla\Phi(x)]^2 + m^2\Phi^2(x)\right\} + \int \mathrm{d}^3x\, V(\Phi(x)) = H_0 + H_{\mathrm{I}} \tag{2.173}$$

where

$$\Pi(\mathbf{x}, t) = \partial\mathcal{L}/\partial\dot{\Phi}(\mathbf{x}, t).$$

Eq. (2.167) implies that the interaction picture field operator obeys the Klein–Gordon equation of a free field

$$\left(\Box + m^2\right)\Phi(x) = 0 \tag{2.174}$$

so that at any moment of time we can write

$$\Phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}[a(\mathbf{k}) f_k(\mathbf{x}, t) + a^\dagger(\mathbf{k}) f_k^*(\mathbf{x}, t)] \tag{2.175}$$

where f_k are the plane wave solutions of the Klein–Gordon equation

$$f_k(\mathbf{x}, t) = \exp[-\mathrm{i}(k_0 t_0 - \mathbf{k}\cdot\mathbf{x})], \qquad k_0 = +\left(\mathbf{k}^2 + m^2\right)^{1/2} \tag{2.176}$$

and $a(\mathbf{k}), a^\dagger(\mathbf{k})$ are the annihilation and creation operators of scalar particles with three-momentum $\mathbf{k}$ satisfying $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2k_0\delta(\mathbf{k} - \mathbf{k}')$ (see Section 1.4). We can now use (2.171), (2.173) and (2.175) to express S in terms of the annihilation and creation operators and calculate the matrix elements between states containing scalar particles.

When calculating the matrix elements it is convenient to deal with normal-ordered products (in the interaction picture), with all creation operators standing to the left of the annihilation operators.

This is done by using Wick's theorem. When applied to the T-ordered product of m field operators Wick's theorem has the well-known form (Bjorken & Drell 1965)

$$\begin{aligned} T\Phi(x_1)\ldots\Phi(x_m) = {}&{:}\Phi(x_1)\ldots\Phi(x_m){:} + \sum_{\substack{i,j\\ i\neq j}}(-\mathrm{i})G(x_i - x_j){:}\prod_{k\neq i,j}\Phi(x_k){:} + \cdots \\ &+ \sum_{i_1,j_1}\cdots\sum_{i_n,j_n}(-\mathrm{i})G(x_{i_1} - x_{j_1})\ldots(-\mathrm{i})G(x_{i_n} - x_{j_n}) \\ &\times\begin{cases} 1 & \text{when} \quad m = 2n \\ {:}\Phi(x_l){:} & \text{when} \quad m = 2n+1,\ l \neq i_1 \neq \cdots \neq j_n \end{cases} \end{aligned} \tag{2.177}$$

A more general and compact formulation is

$$T\,F[\Phi] = \exp\left[-\tfrac{1}{2}\mathrm{i}\int \frac{\delta}{\delta\Phi(x)} G(x-y) \frac{\delta}{\delta\Phi(y)} \mathrm{d}^4x\,\mathrm{d}^4y\right] :F[\Phi]: \tag{2.178}$$

where $F[\Phi]$ is some functional of the field operator Φ and $G(x-y)$ is the Green's function given by (2.91). Functional differentiation with respect to the operator Φ has been introduced here. It is a straightforward generalization of the usual functional differentiation: the only difference is that one must keep in mind that the field operators for different values of time do not commute. If we assume $F[\Phi]$ to be a product of four field operators, $\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)$, and take the matrix element of (2.178) between the vacuum states, the result is exactly (2.94), the corollary of Wick's theorem used to derive the perturbation theory Feynman rules in the previous section.

Now let us take the interaction hamiltonian to be

$$H_{\mathrm{I}}(t) = \int \mathrm{d}^3x\, V(\Phi) = -\int \mathrm{d}^3x\, J(x)\,\Phi(x) \tag{2.179}$$

where $J(x)$ is a c-number function. This corresponds to the case of scattering from an external source. We obtain, after performing the functional differentiation in (2.178):

$$\begin{aligned} T\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J\Phi\right) &= \exp\left[\tfrac{1}{2}\mathrm{i}\int \mathrm{d}^4y\,\mathrm{d}^4x\, J(x)\,G(x-y)\,J(y)\right] \\ &\quad\times :\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J\Phi\right): \\ &= W_0[J] :\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J\Phi\right): \end{aligned} \tag{2.180}$$

where $W_0[J]$ is the previously introduced generating functional of the free-field Green's functions. We observe that

$$J(x)W_0[J] = -\mathrm{i}\int \mathrm{d}^4y\, G^{-1}(x-y)\frac{\delta}{\delta J(y)} W_0[J] \tag{2.181}$$

so that (2.180) can be rewritten as follows

$$T\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J\Phi\right) = :\exp\left[\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Phi(x)\,G^{-1}(x-y)\frac{\delta}{\delta J(y)}\right]: W_0[J] \tag{2.182}$$

Here $G^{-1}(x-y)$ is the inverse operator to $G(x-y)$

$$\int \mathrm{d}^4y\, G^{-1}(x-y)\,G(y-z) = \delta(x-z) \tag{2.183}$$

i.e. the Klein–Gordon operator.

The generalization to arbitrary $V(\Phi)$ follows from the formula

$$T\exp\left[-\mathrm{i}\int \mathrm{d}^4x\, V(\Phi)\right] = \exp\left[-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right] T\exp\left(\mathrm{i}\int \mathrm{d}^4x\, J\Phi\right)\Bigg|_{J\equiv 0} \tag{2.184}$$

Consequently, the S operator of the interacting scalar field can be written in the form

$$\begin{aligned} S = T\exp\left[-\mathrm{i}\int \mathrm{d}^4x\, V(\Phi)\right] &= :\exp\left[\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Phi(x)\,G^{-1}(x-y)\frac{\delta}{\delta J(y)}\right]: \\ &\quad\times \exp\left[-\mathrm{i}\int \mathrm{d}^4x\, V\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right] W_0[J]\Bigg|_{J\equiv 0} \\ &= N^{-1} :\exp\left[\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Phi(x)\,G^{-1}(x-y)\frac{\delta}{\delta J(y)}\right]: W[J]\Bigg|_{J\equiv 0} \end{aligned} \tag{2.185}$$

We have been dealing with fields in the interaction picture but in $W[J]$ we can recognize, by (2.88), the generating functional of the previously introduced Green's functions of the interacting theory, defined as the vacuum matrix elements of the T-ordered products of the field operator in the *Heisenberg* picture; see (2.54). Expanding the exponential we obtain

$$\begin{aligned} S = N^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\,\mathrm{d}^4y_1\ldots \mathrm{d}^4x_n\,\mathrm{d}^4y_n\, G^{(n)}(y_1,\ldots,y_n) \\ \times\, \mathrm{i}G^{-1}(x_1-y_1)\ldots \mathrm{i}G^{-1}(x_n-y_n) :\Phi(x_1)\ldots\Phi(x_n): \end{aligned} \tag{2.186}$$

The factor N^{-1} contains all the vacuum-to-vacuum transitions and will be omitted in the following.

The calculation of matrix elements is now straightforward. Let the initial state involve n particles and the final state m particles. We have (c, c' are the normalization constants)

$$\left.\begin{aligned} |\mathbf{k}_1\ldots\mathbf{k}_n\rangle &= ca^+(\mathbf{k}_1)\ldots a^+(\mathbf{k}_n)|0\rangle \\ \langle\mathbf{k}'_1\ldots\mathbf{k}'_m| &= c'\langle 0|a(\mathbf{k}'_1)\ldots a(\mathbf{k}'_m) \end{aligned}\right\} \tag{2.187}$$

It is convenient to assume that $\mathbf{k}'_i \neq \mathbf{k}_j$ for all i, j pairs. Only the term in the series with $:\Phi(x_1)\ldots\Phi(x_{n+m}):$ will then contribute. Of the $(n+m)$ field operators n will act as the annihilation and m as the creation operators: this gives a combinatorial factor $\binom{n+m}{m}$. Finally, there are $n!m!$ ways of matching the annihilation and creation operators with the external particles. The result is

$$\langle \mathbf{k}'_1 \ldots \mathbf{k}'_m | S | \mathbf{k}_1 \ldots \mathbf{k}_n \rangle = cc' \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_{n+m} \, f^{*\prime}_{k_1}(x_1) \ldots f_{k_n}(x_{n+m}) \\ \times \int \mathrm{d}^4 y_1 \ldots \mathrm{d}^4 y_{n+m} \, \mathrm{i} G^{-1}(x_1 - y_1) \cdots \\ \times \mathrm{i} G^{-1}(x_{n+m} - y_{n+m}) G^{(n+m)}(y_1, \ldots, y_{n+m}) \quad (2.188)$$

With the usual invariant normalization

$$|\mathbf{k}_1 \ldots \mathbf{k}_n\rangle = a^\dagger(\mathbf{k}_1) \ldots a^\dagger(\mathbf{k}_n)|0\rangle \quad (2.189)$$

this is just the on-shell limit of the Fourier transformed function

$$\langle \mathbf{k}'_1 \ldots \mathbf{k}'_m | S | \mathbf{k}_1 \ldots \mathbf{k}_n \rangle = \lim_{k_i^2 \to m^2} \prod_i \left[\frac{1}{\mathrm{i}} \left(k_i^2 - m^2 \right) \right] \\ \times \tilde{G}^{(n+m)}(k'_1, \ldots, k'_m, -k_1, \ldots, -k_n) \quad (2.190)$$

where

$$\tilde{G}^{(n)}(k_1, \ldots, k_n) = \int \mathrm{d}^4 y_1 \exp(\mathrm{i} k_1 y_1) \ldots \int \mathrm{d}^4 y_n \exp(\mathrm{i} k_n y_n) \, G^{(n)}(y_1 \ldots y_n) \quad (2.191)$$

We see that the S matrix element is equal to the multiple on-shell residue of the Fourier transformed Green's function. Its connected part is proportional to the on-shell limit of the so-called connected truncated Green's function, obtained from $\tilde{G}^{(n)}_{\mathrm{conn}}$ by factorizing out the exact two-point functions $\tilde{G}^{(2)}(k_i)$ for each of the external lines of $\tilde{G}^{(n)}_{\mathrm{conn}}$.

At this point we must observe that in the derivation of (2.190) we have made the assumption (following from (2.175)) that the one-particle state created by $\Phi(x)$ from the vacuum has the standard normalization

$$\langle \mathbf{k} | \Phi(x) | 0 \rangle = \exp[\mathrm{i}(k_0 x_0 - \mathbf{k} \cdot \mathbf{x})] \quad (2.192)$$

with $\langle \mathbf{k}|$ normalized as in (2.189), i.e.

$$\langle \mathbf{k}| = \langle 0 | a(\mathbf{k}) \quad (2.193)$$

or

$$\langle \mathbf{k} | \mathbf{k}' \rangle = (2\pi)^3 2 k_0 \delta(\mathbf{k} - \mathbf{k}')$$

Normalization (2.192) holds for the field operators both in the interaction and in the Heisenberg pictures. Fields satisfying (2.192) we shall call the 'physical' fields. More general cases will be discussed in Chapter 4.

The above formal considerations leading to the Lehman–Symanzik–Zimmermann reduction theorem (2.190) can be supplemented by the following arguments, which by themselves can actually be taken as a proof of (2.190). Let us consider the Fourier transform (2.191) of the Green's function $G^{(n)}(y_1, \ldots, y_n) = \langle 0|T\Phi(y_1)\ldots\Phi(y_n)|0\rangle$, where we have taken $\langle 0|0\rangle = 1$, and assume that the field $\Phi(x)$ is normalized according to (2.192). Among the various terms which contribute to $\tilde{G}^{(n)}(k_1, \ldots, k_n)$ there is a time-ordering in which $\Phi(y_1)$ stands furthest to the left. We can insert a complete set of intermediate states between $\Phi(y_1)$ and the remaining time-ordered product getting

$$\tilde{G}^{(n)}(k_1, \ldots, k_n) = \sum\!\!\!\!\!\!\int_m \mathrm{d}^4 y_1 \ldots \mathrm{d}^4 y_n \exp(\mathrm{i}k_1 y_1)\ldots\exp(\mathrm{i}k_n y_n) \\ \times\langle 0|\Phi(y_1)|m\rangle\langle m|T\Phi(y_2)\ldots\Phi(y_n)|0\rangle\Theta\big(y_1^0 - \max\left\{y_2^0\ldots y_n^0\right\}\big) + \cdots \tag{2.194}$$

The y_1-integration can now be carried out using

$$\langle 0|\Phi(y_1)|n\rangle = \langle 0|\Phi(0)|n\rangle\exp(-\mathrm{i}p_n y_1)$$

and the integral representation for the Θ-function

$$\Theta\big(y_1^0 - t\big) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\exp\left[-\mathrm{i}\omega\big(y_1^0 - t\big)\right]}{\omega + \mathrm{i}\varepsilon}$$

One then gets the following

$$\begin{aligned}\int \mathrm{d}^4 y_1 \exp[\mathrm{i}(k_1 - p_n)y_1]\,\Theta\big(y_1^0 - t\big)& \\ &= \mathrm{i}\delta(\mathbf{k}_1 - \mathbf{p}_n)(2\pi)^3\frac{1}{k_1^0 - p_n^0 + \mathrm{i}\varepsilon}\exp\left[\mathrm{i}\big(k_1^0 - p_n^0\big)t\right] \\ &= \mathrm{i}(2\pi)^3\delta(\mathbf{k}_1 - \mathbf{p}_n)\frac{k_1^0 + p_n^0}{k_1^2 - p_n^2 + \mathrm{i}\varepsilon}\exp\left[\mathrm{i}\big(k_1^0 - p_n^0\big)t\right]\end{aligned} \tag{2.195}$$

Thus, a single particle state of mass m contributes a pole at $k_1^2 = m^2$ while no other state $|n\rangle$ and none of the remaining terms arising from other time-orderings can have such a pole. It is important to observe that the pole arises due to the limit $y_1^0 \to +\infty$ in the integral (2.195). We conclude therefore that

$$\lim_{k_1^2\to m^2}\tilde{G}^{(n)}(k_1, \ldots, k_n) = \frac{\mathrm{i}}{k_1^2 - m^2}\langle\mathbf{k}_1, t = +\infty|T\Phi(y_2)\ldots\Phi(y_n)|0\rangle \tag{2.196}$$

$$\left(\int |1\rangle\langle 1| \equiv \int\frac{\mathrm{d}^3 p}{(2\pi)^3 2p_0}|\mathbf{p}\rangle\langle\mathbf{p}|\right)$$

Similarly,

$$\lim_{k_1^2 \to m^2} \tilde{G}^{(n)}(-k_1, \dots, k_n) = \frac{\mathrm{i}}{k_1^2 - m^2} \langle 0|T\Phi(y_2)\dots\Phi(y_n)|\mathbf{k}_1, t = -\infty\rangle \tag{2.197}$$

By repeating these steps we prove (2.190), remembering that

$$\langle \mathbf{k}'_1 \dots \mathbf{k}'_m|S|\mathbf{k}_1 \dots \mathbf{k}_n\rangle = \langle \mathbf{k}'_1 \dots \mathbf{k}'_m, t = +\infty|\mathbf{k}_1 \dots \mathbf{k}_n, t = -\infty\rangle$$

As an explicit example illustrating (2.190) consider elastic scattering of two identical scalar particles in the Φ^4 theory. In the lowest order, using (2.102) we get

$$(S - 1)_{2\to 2} = -\mathrm{i}\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2) \tag{2.198}$$

The basic result, (2.188) and (2.190), is valid for both the path integral and canonical formulations of quantum field theory. Perturbation theory for calculating the Green's functions present in (2.188) and (2.190) can also be developed either in the path integral approach, as in Section 2.4, or in the (canonical) operator formalism (see, for instance, Bjorken & Drell (1965), Weinberg (1995)). The latter approach is sometimes quite convenient, in particular for quick tree level manipulations. The S-matrix elements can be directly calculated as matrix elements of the operator in (2.171) (expressed in terms of fields satisfying free equations of motion) taken between free particle states. For instance, for $2 \to 2$ scattering in the Φ^4 theory we have:

$$\begin{aligned} S_{2\to 2} &= \langle \mathbf{p}_1, \mathbf{p}_2|S|\mathbf{k}_1, \mathbf{k}_2\rangle \\ &= \langle 0|a(\mathbf{p}_1)a(\mathbf{p}_2)T\exp\left\{-\mathrm{i}\frac{\lambda}{4!}\int \mathrm{d}^4x :\Phi^4:\right\}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle \end{aligned} \tag{2.199}$$

In the lowest order the contribution to the transition matrix is

$$\begin{aligned} \langle \mathbf{p}_1, \mathbf{p}_2|S - 1|\mathbf{k}_1, \mathbf{k}_2\rangle &= -\mathrm{i}\frac{\lambda}{4!}\int \mathrm{d}^4x\, \langle 0|a(\mathbf{p}_1)a(\mathbf{p}_2) :\Phi^4(x): a^\dagger(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)|0\rangle \\ &= -\mathrm{i}\frac{\lambda}{4}\int \mathrm{d}^4x\, \langle 0|a(\mathbf{p}_1)a(\mathbf{p}_2)\Phi^+(x)\Phi^+(x)\Phi^-(x)\Phi^-(x) \\ &\qquad \times a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle \end{aligned} \tag{2.200}$$

where Φ^+ (Φ^-) stands for positive (negative) frequency parts of the field operators (1.126). In this order the terms in (2.177) that contain contractions do not contribute to the connected amputated part of the matrix element. Commuting the creation and annihilation operators (using (1.129), (1.130)) and performing the integral over d^4x as well as the integrals over momenta in the field operators (1.126) we rederive (2.198).

As a second example consider the so-called four-fermion interaction lagrangian of the generic type

$$\mathcal{L}_{\rm I} = -g :(\bar{\Psi}_{f_4}(x)\Gamma_A\Psi_{f_3}(x))(\bar{\Psi}_{f_2}(x)\Gamma_B\Psi_{f_1}(x)): \tag{2.201}$$

(where $\Gamma_{A(B)}$ are some constant matrices in the spinor space) and compute the matrix elements of $S-1$ for the process $f_1\bar{f}_2 \to \bar{f}_3 f_4$. In the lowest order we have

$$\begin{aligned}&\langle f_4(\mathbf{p}_4, s_4),\ \bar{f}_3(\mathbf{p}_3, s_3)|S-1|\bar{f}_2(\mathbf{p}_2, s_2),\ f_1(\mathbf{p}_1, s_1)\rangle\\ &\quad = -\mathrm{i}g\int \mathrm{d}^4x\ \langle 0|b_4 d_3 :\big(\Psi^\dagger_{f_4}(x)\gamma^0\Gamma_A\Psi_{f_3}(x)\big)\big(\Psi^\dagger_{f_2}(x)\gamma^0\Gamma_B\Psi_{f_1}(x)\big): d_2^\dagger, b_1^\dagger|0\rangle\end{aligned} \tag{2.202}$$

where we have used the abbreviations $b_4 \equiv b(\mathbf{p}_4, s_4)$ etc. Using the Wick theorem (2.177) and repeating the same steps as in the previous example one arrives at

$$\begin{aligned}(S-1)_{f_1\bar{f}_2\to\bar{f}_3 f_4} &= -\mathrm{i}g\sum_{s_i'}\int\frac{\mathrm{d}^3q_4}{(2\pi)^3 2E_{q_4}}\cdots\int\frac{\mathrm{d}^3q_1}{(2\pi)^3 2E_{q_1}}\\ &\quad\times\int \mathrm{d}^4x \exp[-\mathrm{i}x\cdot(q_1+q_2-q_3-q_4)]\\ &\quad\times\langle 0|b_4 d_3 :[\bar{u}(\mathbf{q}_4, s_4')b^\dagger(\mathbf{q}_4, s_4')\Gamma_B v(\mathbf{q}_3, s_3')d^\dagger(\mathbf{q}_3, s_3')]\\ &\quad\times[\bar{v}(\mathbf{q}_2, s_2')d(\mathbf{q}_2, s_2')\Gamma_A u(\mathbf{q}_1, s_1')b(\mathbf{q}_1, s_1')]: d_2^\dagger, b_1^\dagger|0\rangle\\ &= -\mathrm{i}g(2\pi)^4\delta(p_4+p_3-p_2-p_1)\\ &\quad\times[\bar{u}(\mathbf{p}_4, s_4)\Gamma_B v(\mathbf{p}_3, s_3)][\bar{v}(\mathbf{p}_2, s_2)\Gamma_A u(\mathbf{p}_1, s_1)]\end{aligned} \tag{2.203}$$

It follows from this example that the initial (final) Dirac fermion with momentum $\mathbf{k}$ and spin s is described by the wave-function $u(\mathbf{k}, s)$ ($\bar{u}(\mathbf{k}, s)$) and the initial (final) antifermion with momentum $\mathbf{k}$ and spin s by $\bar{v}(\mathbf{k}, s)$ ($v(\mathbf{k}, s)$). The operator method is convenient to reach this conclusion quickly. Also, it is clear that the absolute sign of the amplitude containing more than one fermion (or antifermion) in the initial and/or final states is purely conventional as it depends on the arbitrary ordering of fermionic creation operators defining these states. The relative signs of several different contributions to the transition amplitudes (if present) and various combinatorial factors are easy to fix by operator manipulations.

Problems

2.1 Derive (2.12) from (2.7), integrating over p_is first.

2.2 For a scalar field theory defined by a lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\Phi(x)\partial_\mu\Phi(x) - V(\Phi(x))$$

and the corresponding equation of motion

$$\Box\Phi + V'(\Phi) = 0$$

derive the equation of motion for the Green's function $\langle 0|T\Phi(x)\Phi(y)|0\rangle$:

$$\Box_x \langle 0|T\Phi(x)\Phi(y)|0\rangle = -\langle 0|TV'(\Phi(x))\Phi(y)|0\rangle - \mathrm{i}\delta^{(4)}(x-y)$$

using definitions (2.54) and (2.56) and the invariance of the generating functional (2.57) under an infinitesimal change of variables

$$\Phi(x) \to \Phi(x) + \varepsilon f(x)$$

where $f(x)$ is a function of x

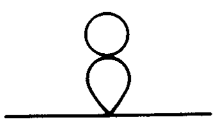
$$\int \mathcal{D}\Phi \exp\left\{\mathrm{i}S[\Phi+\varepsilon f] + \mathrm{i}\int \mathrm{d}^4x\,(\Phi+\varepsilon f)\,J\right\} = \int \mathcal{D}\Phi \exp\left\{\mathrm{i}S[\Phi] + \mathrm{i}\int \mathrm{d}^4x\,\Phi J\right\}$$

Repeat the same in Euclidean space.

2.3 Using the rules of Section 2.4 calculate the following symmetry factors S in the $\lambda\Phi^4$ theory

$S = \frac{1}{6}$

$S = \frac{1}{2}$



$S = \frac{1}{4}$

2.4 For real Grassmann variables η_i prove the formula

$$\int \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n \exp\left[\tfrac{1}{2}(\eta\, A\eta)\right] = (\det A)^{1/2}$$

for any antisymmetric matrix A, where $(\eta, A\eta) = \sum_{i,j} \eta_i A_{ij}\eta_j$.

2.5 Prove (2.154), (2.156) and (2.158).

2.6 Prove (2.178).

2.7 Using representation (2.10) calculate the Green's function for the harmonic oscillator

$$\langle x'|\exp[-\mathrm{i}(t'-t)H]|x\rangle = \left(\frac{\omega}{2\mathrm{i}\pi\sin\omega(t'-t)}\right)^{1/2} \exp[\mathrm{i}S(t'-t)]$$

where

$$H = -\tfrac{1}{2}\left(\partial^2/\partial x^2\right) + \tfrac{1}{2}\omega^2 x^2$$

$$S(\tau) = \frac{\omega}{2\sin\omega\tau}\left[\left(x'^2 + x^2\right)\cos\omega\tau - 2x'x\right]$$

Note that $S(\tau)$ is the classical action (along the classical path).

2.8 For a scalar field theory as in Problem 2.2 with $V(\Phi) = \lambda\Phi^4$ calculate the effective potential $\Gamma[\Phi]$ by evaluating the generating functional $W_E[J]$ in Euclidean space

$$W_E[J] = N_E \int \mathcal{D}\Phi \exp\left\{-S_E[\Phi] + \int J\Phi \, d^4\hat{x}\right\}$$

by the method of steepest descent. The steps are:

(a) Expand the exponent around Φ_0 (often called the background field) which is the solution of the classical equation of motion in the presence of the external source $J(x)$

$$(-\hat{\partial}_\mu\hat{\partial}^\mu + m^2)\Phi_0 + V'(\Phi_0) - J = 0 \tag{2.204}$$

to write

$$\begin{aligned} W_E[J] = N_E \exp&\left\{-S_E[\Phi_0] + \int J\Phi_0 \, d^4\hat{x}\right\} \\ &\times \int \mathcal{D}\Phi \exp\left\{-\frac{1}{2}\iint d^4\hat{x}\, d^4\hat{y} \frac{\delta^2 S_E[\Phi_0]}{\delta\Phi(\hat{x})\delta\Phi(\hat{y})}\right. \\ &\left.\times [\Phi(\hat{x}) - \Phi_0(\hat{x})][\Phi(\hat{y}) - \Phi_0(\hat{y})] + \cdots\right\} \end{aligned} \tag{2.205}$$

(b) In the classical approximation

$$W_E[J] = N_E \exp\left\{-S_E[\Phi_0] + \int d^4\hat{x}\, J\Phi_0\right\}$$

Evaluate $Z[J]$ and $\Gamma[\Phi]$ explicitly by solving (2.204) perturbatively in powers of λ. Check that in this approximation $\Gamma[\Phi]$ is the classical action.

(c) Calculate $\Gamma[\Phi]$ in the saddle point approximation. Evaluate the Gaussian integral in (2.205) by using $\det A = \exp(\mathrm{Tr}\ln A)$ and expanding the logarithm in powers of λ. Calculate the effective potential and prove (2.166).

2.9 For a matrix M acting in superspace

$$\begin{pmatrix} z \\ \eta \end{pmatrix}' = M\begin{pmatrix} z \\ \eta \end{pmatrix} = \begin{pmatrix} A & D \\ C & B \end{pmatrix}\begin{pmatrix} z \\ \eta \end{pmatrix}$$

where z and η are commuting and anticommuting coordinates, respectively:

(a) Write $\det M$ instead of $(\det M)^{-1}$ as a Gaussian integral.

(b) Show that the supertrace defined as

$$\mathrm{Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, B$$

satisfies the cyclicity property

$$\mathrm{Tr}[M_1 M_2] = \mathrm{Tr}[M_2 M_1]$$

(*c*) Show that

$$\operatorname{Tr}\ln M_1 M_2 = \operatorname{Tr}\ln M_1 + \operatorname{Tr}\ln M_2$$

Hint: use the Campbell–Backer–Hausdorff formula for ordinary matrices

$$\exp(A)\exp(B) = \exp\left\{A + B + \frac{1}{2!}[A, B] + \frac{1}{3!}\left(\tfrac{1}{2}[[A, B], B] + \tfrac{1}{2}[A, [A, B]]\right)\right.$$
$$\left. + \frac{1}{4!}[[A, [A, B]], B] + \cdots\right\}$$

The exponent and the logarithm of a matrix in superspace are defined by a series expansion, as for ordinary matrices.

(*d*) Writing

$$M = \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix}\begin{bmatrix} 1 & A^{-1}D \\ 0 & B - CA^{-1}D \end{bmatrix}$$

and using property (*c*), prove that

$$\ln\det M = \operatorname{Tr}\ln M$$

and

$$\det(M_1 M_2) = \det M_1 \det M_2$$

(*e*) Write the Jacobian of a transformation in superspace.

3

Feynman rules for Yang–Mills theories

3.1 The Faddeev–Popov determinant

Gauge invariance and the path integral

As discussed in the Introduction gauge quantum field theories are of special physical interest in our attempts to describe fundamental interactions. In this chapter we shall derive Feynman rules for gauge theories. Consider a path integral over gauge field A_μ, corresponding to a physical, i.e. gauge-invariant, quantity $f(A_\mu)$

$$\int \mathcal{D}A_\mu \, f(A_\mu) \, \exp\left(\mathrm{i}\int \mathrm{d}^4x \, \mathcal{L}\right) \tag{3.1}$$

For brevity we write A_μ instead of A_μ^α, where α is the gauge group index; $\mathcal{L}$ is the lagrangian density. The integration measure $\mathcal{D}A_\mu$ we assume to be invariant under gauge transformations: i.e. it must have the following property:

$$\mathcal{D}A_\mu = \mathcal{D}A_\mu^g$$

where g is an arbitrary transformation from the gauge group. A_μ^g denotes the result of this transformation when applied to A_μ. The simplest formal ansatz for the integration measure which has this property is $\mathcal{D}A_\mu = \prod_{\mu,\alpha,x} \mathrm{d}A_\mu^\alpha(x)$, the straightforward generalization of the integration measure introduced previously for scalar fields.

As explained before, the path integral in (3.1) runs over all possible configurations $A_\mu(x)$, which implies multiple counting of the physically equivalent configurations (equivalent up to a gauge transformation). Let us divide the configuration space $\{A_\mu(x)\}$ into the equivalence classes $\{A_\mu^g(x)\}$ called the orbits of the gauge group. An orbit of the group includes all the field configurations which result when all possible transformations g from the gauge group $\mathcal{G}$ are applied to a given initial field configuration $A_\mu(x)$. This construction can be represented graphically as shown in Fig. 3.1. The integrand of (3.1) is constant along any orbit

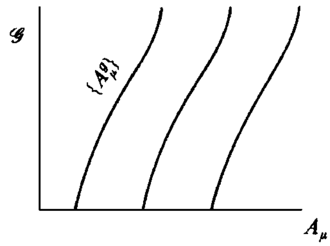


Fig. 3.1.

of the gauge group. Consequently, the integral as it stands is proportional to an infinite constant (the volume of the total gauge group $\mathcal{G}$). This is not an important difficulty in itself, because the infinity can always be cancelled by a normalization constant. The problem appears when we want to calculate (3.1) perturbatively because local gauge symmetry implies that the quadratic part of the gauge field action density has zero modes and therefore cannot be inverted in the configuration space $\{A_\mu(x)\}$ so that the propagator of the gauge field cannot be defined. One way to resolve this problem is to apply perturbation theory to the functional integral over the coset space of the orbits of the group (in other words, over the physically distinct field configurations) which follows from (3.1) after the infinite constant has been factorized out. We shall now describe a procedure by means of which this factorization can be achieved.

Let $\mathcal{D}g$ denote an invariant measure on the gauge group $\mathcal{G}$

$$\mathcal{D}g = \mathcal{D}(gg'); \qquad \mathcal{D}g = \prod_x \mathrm{d}g(x) \tag{3.2}$$

and let us introduce a functional $\Delta[A_\mu]$ defined by the following equation

$$1 = \Delta[A_\mu] \int \mathcal{D}g\, \delta\big[F\big[A_\mu^g\big]\big] \tag{3.3}$$

Here $\delta[f(x)]$ represents the product of the usual Dirac δ-functions: $\prod_x \delta(f(x))$, one at each space-time point. As for the functional $F[A_\mu]$, we assume that the equation

$$F\big[A_\mu^g\big] = 0 \tag{3.4}$$

has exactly one solution, g_0, for any initial field A_μ. In the configuration space $\{A_\mu(x)\}$ the equation $F[A_\mu] = 0$ must then define a surface that crosses any of the orbits of the group exactly once, i.e. (3.4) defines a 'gauge'. Let us observe that the functional $F[A_\mu]$ should be of the general form $F^\alpha(x, [A_\mu])$, where α is the group index of the adjoint representation and x a space-time point. In fact to fix a gauge we need (at each space-time point) one equation for each parameter of the group.

$\Delta[A_\mu]$ is invariant under gauge transformations. This can be shown as follows

$$\Delta^{-1}\left[A_\mu^g\right] = \int \mathcal{D}g'\,\delta\left[F\left[A_\mu^{gg'}\right]\right] = \int \mathcal{D}(gg')\,\delta\left[F\left[A_\mu^{gg'}\right]\right]$$
$$= \int \mathcal{D}g''\,\delta\left[F\left[A_\mu^{g''}\right]\right] = \Delta^{-1}[A_\mu] \tag{3.5}$$

where we have used the invariance property of the group measure $\mathcal{D}g$. In effect, the functional $\Delta[A_\mu]$ depends only on the orbit to which A_μ belongs.

Our aim is to replace integration over all field configurations by integration restricted to the hypersurface $F[A_\mu] = 0$. In this case each orbit will contribute only one field configuration and we shall have an integration over physically distinct fields. We start by inserting (3.3) (which is equal to one) under the path integral in (3.1). Next we change the order of integrations. The result is

$$\int \mathcal{D}g \int \mathcal{D}A_\mu\,\Delta[A_\mu]\,f(A_\mu)\,\delta\left[F\left[A_\mu^g\right]\right]\,\exp\{\mathrm{i}S[A_\mu]\} \tag{3.6}$$

An important observation is that the complete expression under the $\int \mathcal{D}g$ integral is in fact independent of g. To show this we can use the gauge invariance of $\int \mathcal{D}A_\mu$, $\Delta[A_\mu]$, $f(A_\mu)$ and $S[A_\mu]$ to replace them by $\int \mathcal{D}A_\mu^g$, $\Delta\left[A_\mu^g\right]$, $f\left(A_\mu^g\right)$ and $S\left[A_\mu^g\right]$: the result

$$\int \mathcal{D}A_\mu^g\,\Delta\left[A_\mu^g\right]\,f\left(A_\mu^g\right)\,\delta\left[F\left[A_\mu^g\right]\right]\,\exp\left\{\mathrm{i}S\left[A_\mu^g\right]\right\}$$

can be made manifestly g-independent by a change in notation: $A_\mu^g \to A_\mu$. Consequently, the group integration $\int \mathcal{D}g$ factorizes out to produce an infinite constant: the volume of the full gauge group. We obtain finally

$$\left(\int \mathcal{D}g\right)\int \mathcal{D}A_\mu\,\Delta[A_\mu]\,\delta[F[A_\mu]]\,f(A_\mu)\,\exp\{\mathrm{i}S[A_\mu]\} \tag{3.7}$$

which defines the theory as formulated in the $F[A_\mu] = 0$ gauge. Unlike (3.1), (3.7) can be used as a starting point for perturbative calculations.

Faddeev–Popov determinant

We have still to calculate $\Delta[A_\mu]$. From (3.3) we obtain, after formal manipulations

$$\Delta^{-1}[A_\mu] = \int \mathcal{D}F\left(\det\frac{\delta F\left[A_\mu^g\right]}{\delta g}\right)^{-1}\delta[F]$$

i.e.

$$\Delta[A_\mu] = \det\frac{\delta F\left[A_\mu^g\right]}{\delta g}\bigg|_{F[A_\mu^g]=0} \tag{3.8}$$

$\Delta[A_\mu]$ is usually called the Faddeev–Popov determinant. It is convenient to use the gauge invariance property of $\Delta[A_\mu]$ to choose A_μ which already satisfies the gauge condition $F[A_\mu] = 0$. Then in (3.8) we can replace the constraint $F\left[A_\mu^g\right] = 0$ by $g = 1$, which simplifies the practical calculations

$$\Delta[A_\mu] = \det \left. \frac{\delta F\left[A_\mu^g\right]}{\delta g} \right|_{g=1}; \qquad F[A_\mu] = 0 \tag{3.9}$$

Near $g = 1$ we only have to deal with the infinitesimal transformations: $U(\Theta) = 1 - \mathrm{i}T^\alpha \Theta^\alpha(x)$ (where $\Theta^\alpha(x) \ll 1$) and the invariant group measure $\mathcal{D}g$ takes the simple form $\mathcal{D}\Theta \equiv \prod_{\alpha,x} \mathrm{d}\Theta^\alpha(x)$. It is easy to check the invariance property of $\mathcal{D}\Theta$

$$D\Theta = D\Theta'' \tag{3.10}$$

where $U(\Theta'') = U(\Theta)U(\Theta')$ and Θ' is arbitrary. For infinitesimal transformations $\Theta'' \approx \Theta' + \Theta$ and therefore $\det(\mathrm{d}\Theta''/\mathrm{d}\Theta) = 1$, so that (3.10) is proved.

We can now rewrite (3.9) in a more explicit form, with all relevant indices

$$\Delta[A_\mu] = \det \left. \frac{\delta F^\alpha\left(x, \left[A_\mu^\Theta\right]\right)}{\delta \Theta^\beta(y)} \right|_{\Theta=0}; \qquad F^\alpha(x, [A_\mu]) = 0 \tag{3.11}$$

Here A_μ^Θ stands for A_μ^g and α, β are the gauge group indices. We have to calculate a determinant of a matrix in both space-time and the group indices $M^{\alpha\beta}(x, y)$. This matrix appears in the expansion of $F^\alpha\left(x, \left[A_\mu^\Theta\right]\right)$ in powers of the infinitesimal parameters $\Theta^\beta(y)$

$$F^\alpha\left(x, \left[A_\mu^\Theta\right]\right) = F^\alpha(x, [A_\mu]) + \int \mathrm{d}^4 y\, M^{\alpha\beta}(x, y)\, \Theta^\beta(y) + \cdots \tag{3.12}$$

so that

$$M^{\alpha\beta}(x, y) = \left. \frac{\delta F^\alpha\left(x, \left[A_\mu^\Theta\right]\right)}{\delta \Theta^\beta(y)} \right|_{\Theta=0} \tag{3.13}$$

and

$$\Delta[A_\mu] = \det M; \qquad F[A_\mu] = 0 \tag{3.14}$$

Recalling (1.105) for the infinitesimal transformation of the gauge field, we can also write

$$\begin{aligned} M^{\alpha\beta}(x, y) &= \int \mathrm{d}^4 z \left. \frac{\delta F^\alpha(x, [A_\mu])}{\delta A_\nu^\rho(z)} \frac{\delta A_\nu^\rho(z)}{\delta \Theta^\beta(y)} \right|_{\theta=0} \\ &= \int \mathrm{d}^4 z \frac{\delta F^\alpha(x, [A_\mu])}{\delta A_\nu^\rho(z)} \left(c_{\rho\beta\gamma} A_\nu^\gamma(z) + \frac{1}{g} \delta_{\rho\beta} \partial_\nu^{(z)} \right) \delta(z - y) \end{aligned} \tag{3.15}$$

For the subsequent applications the following observations will be helpful. First let us note that $\Delta[A_\mu]$ under a path integral like (3.7) can be replaced by $\det M$, because the condition $F[A_\mu] = 0$ is already enforced by the δ-functional which fixes the gauge. Consider a class of gauge conditions of the form $F[A_\mu] - C(x) = 0$, where $C(x)$ is an arbitrary function of a space-time point. For all gauge conditions belonging to this class $\det M$ is the same because $C(x)$ is unaffected by the gauge transformation from A_μ to A_μ^g. We can make use of this feature in order to replace the δ-functional in (3.7) by some other functional of the gauge condition, which may be more convenient for practical calculations. In the $F[A_\mu] - C(x) = 0$ gauge (3.7) becomes

$$\left(\int \mathcal{D}g\right) \int \mathcal{D}A_\mu \ \det M \, \delta[F[A_\mu] - C(x)] \, f(A_\mu) \ \exp\{\mathrm{i}S[A_\mu]\} \tag{3.16}$$

Being gauge-invariant, this is obviously independent of $C(x)$. We can integrate (3.16) functionally over $\int \mathcal{D}C$ with an arbitrary weight functional $G[C]$; the result will differ from (3.16) only by an overall normalization constant. Observing that the δ-functional in (3.16) is the only C-dependent term, we obtain

$$\begin{aligned} &\int \mathcal{D}A_\mu \ \det M \ f(A_\mu) \ \exp\{\mathrm{i}S[A_\mu]\} \int \mathcal{D}C \, \delta[F[A_\mu] - C] \, G[C] \\ &\quad = \int \mathcal{D}A_\mu \ \det M \ f(A_\mu) \ \exp\{\mathrm{i}S[A_\mu]\} G[F[A_\mu]] \end{aligned} \tag{3.17}$$

A popular choice for $G[C]$ is

$$G[C] = \exp\left\{-\frac{\mathrm{i}}{2\alpha} \int \mathrm{d}^4x \, [C(x)]^2\right\} \tag{3.18}$$

where α is a real constant. This is equivalent to replacing the original lagrangian density by

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\alpha}(F[A_\mu])^2 \tag{3.19}$$

which is no longer gauge-invariant; consequently, perturbation theory may apply in this case. Notice also that in the limit $\alpha \to 0$ (3.18) approaches the δ-functional (up to the α-dependent normalization factor which for our purposes is irrelevant). The derivation of Feynman rules in gauges fixed by δ-functionals can often be simplified by taking this limit.

We have shown how a path integral representation, (3.1), of a gauge-invariant quantity can be rewritten in a fixed gauge, so that the perturbation theory may be set up. The rules of this perturbation theory will depend on the gauge, but the final results will not, because the quantity we calculate is gauge-invariant by assumption. In the following we shall also deal with gauge-dependent quantities. In particular the generating functional $W[J]$ in the $F[A_\mu] = 0$ gauge is given by the following expression:

$$W[J] = N \int \mathcal{D}A_\mu \, \Delta[A_\mu] \, \delta(F[A_\mu]) \, \exp\left[\mathrm{i} \int \mathrm{d}^4 x \left(\mathcal{L} + J^\alpha_\mu A^\mu_\alpha \right) \right] \tag{3.20}$$

Due to the presence of fixed external sources J^α_μ, this is not a gauge-invariant object and the same applies to its functional derivatives with respect to J, the Green's functions of the gauge field. Although in a sense unphysical, the Green's functions are often useful at intermediate stages of calculations and physical quantities may be conveniently defined in terms of them. In particular, as we know from Section 2.7, an element of the S-matrix is obtained from the corresponding Green's functions by removing single-particle propagators corresponding to external lines, taking the Fourier transform of the resulting 'amputated' Green's function and placing external momenta on the mass-shell. Although we have not discussed the renormalization programme yet, it is in order to mention here that the renormalized S-matrix elements are gauge-independent. The equivalence of different gauges can be explicitly checked for various cases (see, for example, Abers & Lee (1973), Slavnov & Faddeev (1980), 't Hooft & Veltman (1973)). One can show that only the renormalization constants attached to each external line depend on the gauge but that is of no consequence for the renormalized S-matrix. The other gauge-dependent terms do not contribute to the poles of the Green's functions at $p_i^2 = m^2$. These formal arguments are rigorous except for the fact that in a gauge theory, if not with a spontaneously broken gauge symmetry, the S-matrix suffers from IR divergences and may not even be defined. In such cases one can discuss IR regularized theory but the regulator must not break gauge invariance. IR dimensional regularization can be used as an intermediate step to prove gauge independence of the physical, finite quantities.

Examples

As a simple example let us consider the case of quantum electrodynamics (QED). We shall calculate the free photon propagators $G_0^{\mu\nu}(x_1, x_2)$ corresponding to different gauge conditions. First we assume the so-called Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{3.21}$$

The $M(x, y)$-matrix defined by (3.13) is then of the form

$$M(x, y) = +\frac{1}{e} \partial^2 \delta(x - y)$$

where $\partial^2 = \partial_\mu \partial^\mu$ and the Faddeev–Popov determinant is independent of the field A_μ; consequently it can be included into the overall normalization factor, which is irrelevant. To deal with the δ-functional $\delta\left[\partial_\mu A^\mu\right]$ which fixes the gauge we shall make use of a decomposition of the field A_μ into transverse A^{T}_μ and longitudinal A^{L}_μ parts

$$A^{\mathrm{T}}_\mu = P_{\mu\nu} A^\nu, \qquad A^{\mathrm{L}}_\mu = (g_{\mu\nu} - P_{\mu\nu}) A^\nu$$

where $P_{\mu\nu} = g_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2$ is the projection operator. Clearly we have

$$\partial_\mu A^\mu_{\mathrm{T}} = 0$$

so that the gauge-fixing δ-functional does not affect the integration over the transverse component of the gauge field. The longitudinal fields are of the form $A^{\mathrm{L}}_\mu = \partial_\mu \Theta$ ('pure gauge'). The Lorentz gauge condition does not set them exactly to zero; instead, it requires that $\partial^2 \Theta = 0$. This is the residual gauge freedom allowed by (3.21).

The existence of this residual gauge freedom contradicts our previous assumption that the gauge condition $F\left[A^g_\mu\right] = 0$ has exactly one solution g_0 for any A_μ. However, let us observe that the residual gauge freedom can be removed by an appropriate choice of the boundary conditions. In Euclidean space we can do it by assuming that the gauge field A_μ vanishes at Euclidean infinity. Such boundary conditions are in fact implicit in the usual derivation of perturbation theory rules from path integrals. There is also another sort of gauge ambiguity (the Gribov ambiguity) with more physical implications. In the case of the Lorentz gauge it appears only in non-abelian gauge theories. This is the case when the points of intersection of a group orbit with the $F = 0$ surface are at a finite distance from each other, i.e. are separated by finite gauge transformations. Perturbation theory is, however, unaffected by this ambiguity.

Taking this all into account we obtain the following path integral formula for $W_0[J]$ in the Lorentz gauge (in the following the fermion degrees of freedom are always suppressed):

$$W_0[J] \sim \int \mathcal{D}A^{\mathrm{T}}_\mu \exp\left[\mathrm{i} \int \mathrm{d}^4x \left(\tfrac{1}{2} A^{\mathrm{T}}_\mu \partial^2 g^{\mu\nu} A^{\mathrm{T}}_\nu + J^{\mathrm{T}}_\mu A^\mu_{\mathrm{T}}\right)\right] \tag{3.22}$$

(the extra term $\frac{1}{2}\mathrm{i}\varepsilon A^{\mathrm{T}}_\mu g^{\mu\nu} A^{\mathrm{T}}_\nu$ in the action should always be kept in mind). Here we have made use of the fact that the lagrangian density of a free electromagnetic field can be written as follows

$$-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2} A_\mu \partial^2 \left(g^{\mu\nu} - \partial^\mu \partial^\nu / \partial^2\right) A_\nu = \tfrac{1}{2} A^{\mathrm{T}}_\mu \partial^2 g^{\mu\nu} A^{\mathrm{T}}_\nu \tag{3.23}$$

Thus the longitudinal A^μ_{L} does not appear in the action and in the Lorentz gauge $\delta\left(\partial_\mu A^\mu_{\mathrm{L}}\right)$ makes the integration over $\mathcal{D}A_{\mathrm{L}}$ trivial. We can simply set $A^\mu_{\mathrm{L}} = 0$. We know that the J-dependence of a path integral like (3.22) is given by

$$W_0[J] = \exp\left[\tfrac{1}{2}\mathrm{i} \int \mathrm{d}^4x\, \mathrm{d}^4y\, J^{\mathrm{T}}_\mu(x)\, D^{\mu\nu}(x-y)\, J^{\mathrm{T}}_\nu(y)\right] \tag{3.24}$$

where $-D_{\mu\nu}(x-y)$ is the inverse of the quadratic action. In transverse space the unit operator is equivalent to $P_{\mu\nu}\delta(x-y)$, so that we have the following equation for $D_{\mu\nu}$:

$$\big(\partial^2 - \mathrm{i}\varepsilon\big)D_{\mu\nu}(x-y) = -\big(g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2\big)\delta(x-y) \tag{3.25}$$

After a Fourier transformation this becomes

$$\big(-k^2 - \mathrm{i}\varepsilon\big)\tilde{D}_{\mu\nu}(k) = -\big(g_{\mu\nu} - k_\mu k_\nu/k^2\big)$$

so that

$$\tilde{D}_{\mu\nu}(k) = \frac{1}{k^2+\mathrm{i}\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) = \frac{\tilde{P}_{\mu\nu}}{k^2+\mathrm{i}\varepsilon} \tag{3.26}$$

and

$$D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2+\mathrm{i}\varepsilon}\exp[-\mathrm{i}k(x-y)] \tag{3.27}$$

The prescription of the k^2 pole in the $k_\mu k_\nu/k^2$ term can be arbitrary. Note also that in transverse space, as long as we deal only with transverse sources, we can always replace $P_{\mu\nu}$ by $g_{\mu\nu}$.

We can now write the formula for the free electromagnetic field propagator in the Lorentz gauge

$$G_0^{\mu\nu}(x_1, x_2) = \left(\frac{1}{\mathrm{i}}\right)^2 \frac{\delta^2 W_0[J]}{\delta J_\mu(x_1)\delta J_\nu(x_2)}\bigg|_{J\equiv 0} = -\mathrm{i}D^{\mu\nu}(x_1 - x_2) \tag{3.28}$$

where $D_{\mu\nu}(x_1 - x_2)$ is given by (3.27).

A generalization of this result follows if we make use of another procedure for fixing the gauge, summarized in (3.17)–(3.19). With $F[A_\mu] = \partial_\mu A^\mu$ we obtain the effective lagrangian

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\big(\partial_\mu A^\mu\big)^2 = \tfrac{1}{2}A^\mu\left[g_{\mu\nu}\partial^2 - \left(1 - \frac{1}{\alpha}\right)\partial_\mu\partial_\nu\right]A^\nu \tag{3.29}$$

The quadratic action is now no longer proportional to a projection operator and therefore can be inverted over all configuration space. Again we can write

$$W_0[J] = \exp\left[\tfrac{1}{2}\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\; J^\mu(x)\,D_{\mu\nu}(x-y)\,J^\nu(y)\right] \tag{3.30}$$

where $D_{\mu\nu}(x-y)$ satisfies the equation

$$\big[\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu(1 - 1/\alpha) - \mathrm{i}\varepsilon\big]D^\nu_\lambda(x-y) = -g_{\mu\lambda}\delta(x-y) \tag{3.31}$$

with the solution

$$D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+\mathrm{i}\varepsilon}\right) + \alpha\frac{k_\mu k_\nu}{k^2+\mathrm{i}\varepsilon}\right]\frac{1}{k^2+\mathrm{i}\varepsilon}\exp[-\mathrm{i}k(x-y)] \tag{3.32}$$

In the limit $\alpha \to 0$ this coincides with (3.27) (the Lorentz or Landau gauge). The case $\alpha = 1$ corresponds to the Feynman gauge.

The free propagators for non-abelian gauge fields in covariant gauges are also given by (3.28) and (3.32). The only modification is that they are diagonal matrices in the non-abelian quantum number space.

Non-covariant gauges

Non-covariant gauges, depending on an arbitrary four-vector n_μ, are also frequently used. For instance the standard canonical quantization of QED uses the Coulomb gauge, also called the radiation gauge (Bjorken & Drell 1965), defined by the condition

$$\partial_\mu A^\mu - \left(n_\mu \partial^\mu\right)\left(n_\mu A^\mu\right) = 0, \qquad n_\mu = (1,0,0,0) \tag{3.33}$$

In QCD one often uses axial gauges

$$n_\mu A^\mu_\alpha = 0, \qquad n^2 = 0 \quad \text{or} \quad n^2 < 0, \qquad \alpha = 1,\ldots,8 \tag{3.34}$$

which allow the quanta of the vector fields to be interpreted as partons. It is easy to check that in axial gauges the Faddeev–Popov determinant is A^α_μ independent even in a non-abelian theory.

Free propagators of the gauge field in the Coulomb and axial gauges can be formally obtained following the procedure summarized in (3.17)–(3.19), inverting the term of the action quadratic in the gauge fields and taking the limit $\alpha \to 0$. Only in this limit do the propagators behave as k^{-2}. For the Coulomb gauge one gets

$$\tilde{D}^{\alpha\beta}_{\mu\nu} = \frac{\delta^{\alpha\beta}}{k^2+\mathrm{i}\varepsilon}\left[g_{\mu\nu} - \frac{k\cdot n(k_\mu n_\nu + k_\nu n_\mu) - k_\mu k_\nu}{(k\cdot n)^2 - k^2}\right] \tag{3.35}$$

and for axial gauges

$$\tilde{D}^{\alpha\beta}_{\mu\nu} = \frac{\delta^{\alpha\beta}}{k^2+\mathrm{i}\varepsilon}\left[g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} + \frac{n^2}{(k\cdot n)^2}k_\mu k_\nu\right] \tag{3.36}$$

Propagator (3.35) can be split into the transverse propagator and the static Coulomb interaction term (Bjorken & Drell 1965). Our formal derivation of (3.36) neglects subtle problems raised by the presence of the $(n\cdot k)$ singularity. A more complete treatment of $n^2 < 0$ and $n^2 = 0$ gauges has been given in the framework of canonical quantization (Bassetto, Lazzizzera & Soldati 1985, Bassetto, Dalbosco, Lazzizzera & Soldati 1985). The $(n \cdot k)$ singularities are related to the residual gauge freedom still present after the axial gauge condition is imposed. In the space-like case $n^2 < 0$ one can show (Bassetto, Lazzizzera & Soldati 1985) that this residual gauge symmetry is directly connected to the spatial asymptotic behaviour of the potentials and can be eliminated by imposing a specific boundary condition. Choice of this condition gives rise to a specific prescription for the spurious singularity in the free boson propagator.

The treatment of the light-cone gauge $n^2 = 0$ is quite different (Bassetto *et al.* 1985). This time the residual gauge freedom involving functions which do not depend on the variable $n_\mu x^\mu$ cannot be eliminated: we are not allowed to impose boundary conditions which would interfere with the time evolution of the system. Nevertheless a hamiltonian canonical quantization leads to the well-defined prescription for the singularity $(n \cdot k)$:

$$\frac{1}{n \cdot k} = \frac{k_0 + k_3}{k^2 + k_\perp^2 + \mathrm{i}\varepsilon}, \qquad n = (1, 0, 0, 1) \tag{3.37}$$

This prescription has been proposed earlier without any canonical justification as one which allows for Wick rotation (Mandelstam 1983, Leibbrandt 1984). In QCD calculations the principal value prescription has also been extensively used. With both kinds of prescription dimensional regularization is usually adopted (Pritchard & Stirling 1979 and Appendix B). However, the general proof of the renormalizability of the theory in the light-cone gauge is still lacking.

3.2 Feynman rules for QCD

Calculation of the Faddeev–Popov determinant

In order to derive Feynman rules for a non-abelian gauge theory we have to take proper account of the Faddeev–Popov determinant whose presence under the functional integral (3.20) is a consequence of quantization with constraints: gauge-fixing conditions. For the sake of definiteness we choose a class of covariant gauges $\partial_\mu A^\mu = c(x)$ or, equivalently, assume (3.19). It then follows from (3.15) that the Faddeev–Popov matrix reads

$$M^{\alpha\beta}(x, y) = (1/g)\big(\delta^{\alpha\beta}\partial^2 + gc^{\alpha\beta\gamma}A^\mu_\gamma(x)\partial_\mu\big)\delta(x-y) = (1/g)\big(D_\mu\partial^\mu\big)^{\alpha\beta}\delta(x-y) \tag{3.38}$$

Relation (3.38) follows from (3.15), not only on the hypersurface $\partial_\mu A^\mu = 0$ but also for $\partial_\mu A^\mu = c(x)$ because $\det\big(\partial_\mu D^\mu\big) = \det\big(D_\mu\partial^\mu\big)$. The matrix $M^{\alpha\beta}$, depends on gauge fields A^γ_μ.

The standard method of dealing with the Faddeev–Popov determinant $\det M^{\alpha\beta}(x, y)$ is to replace it by an additional functional integration over some auxiliary complex fields $\eta(x)$ (ghost fields) which are Grassmann variables. Using the results of Section 2.5 we can write

$$\det M^{\alpha\beta}(x, y) = C \int \mathcal{D}\eta\, \mathcal{D}\eta^* \, \exp\left[\mathrm{i} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \eta^{*\alpha}(x)\, M^{\alpha\beta}(x, y)\, \eta^{\beta}(y) \right] \tag{3.39}$$

where C is some unimportant constant (the factor $(1/g)$ in relation (3.38) is also included in it) and the integration over η and η^* is equivalent to integration over the real and the imaginary parts of the field η. Having (3.39) and the explicit form (3.38) of the matrix $M^{\alpha\beta}(x, y)$ we can take account of $\det M^{\alpha\beta}(x, y)$ in the framework of our general perturbative strategy for calculating various Green's functions. The Feynman rules for ghost fields follow in a straightforward way from (3.39) and (3.38). The ghost field propagator can be calculated from the generating functional $W_0^{\eta}[\beta]$

$$\begin{aligned} W_0^{\eta}[\beta] \sim \int \mathcal{D}\eta\, \mathcal{D}\eta^* \, \exp\Big\{ \mathrm{i} \int \mathrm{d}^4x\, \big[&\eta^{*\alpha}(x) \delta^{\alpha\beta} \partial^2 \eta^{\beta}(x) \\ &+ \beta^{*\alpha}(x)\eta^{\alpha}(x) + \eta^{*\alpha}(x)\beta^{\alpha}(x) \big] \Big\} \end{aligned} \tag{3.40}$$

Using standard methods, the β-dependence of $W_0^{\eta}[\beta]$ reads explicitly

$$W_0^{\eta}[\beta] = \exp\left[\mathrm{i} \int \mathrm{d}^4x\, \mathrm{d}^4y\, \beta^{*\alpha}(x)\, \Delta^{\alpha\beta}(x-y)\, \beta^{\beta}(y) \right] \tag{3.41}$$

where

$$\Delta^{\alpha\beta}(x-y) = \delta^{\alpha\beta} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2 + \mathrm{i}\varepsilon} \exp[-\mathrm{i}k(x-y)] \tag{3.42}$$

One can define the 'propagator' of the ghost field η

$$G_0^{\eta\alpha\beta}(x, y) = \left(\frac{1}{\mathrm{i}}\right)^2 \frac{\delta}{\delta\beta^{*\alpha}(x)} W_0^{\eta}[\beta] \frac{\overleftarrow{\delta}}{\delta\beta^{\beta}(y)} \bigg|_{\beta=\beta^*=0} \tag{3.43}$$

and expanding (3.41) we get

$$G_0^{\eta\alpha\beta}(x, y) = -\mathrm{i}\Delta^{\alpha\beta}(x-y) \tag{3.44}$$

Graphically, propagator $G_0^{\eta\alpha\beta}(x, y)$ will be denoted by

with an arrow pointing towards the source of η.

Feynman rules

The full generating functional $W[J, \alpha, \beta]$ for QCD reads as follows:

$$W[J, \alpha, \beta] = N \int \mathcal{D}A^\alpha_\mu\, \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\, \mathcal{D}\eta\, \mathcal{D}\eta^*\, \exp\Big\{ \mathrm{i}[S[A, \Psi] + S^\eta[A, \eta] + S_\mathrm{G}] + \mathrm{i}\int \mathrm{d}^4x \left[J^\alpha_\mu(x)\, A^\mu_\alpha(x) + \beta^{*\alpha}(x)\,\eta^\alpha(x) + \eta^{*\alpha}(x)\,\beta^\alpha(x) + \bar{\alpha}(x)\,\Psi(x) + \bar{\Psi}(x)\,\alpha(x)\right]\Big\} \tag{3.45}$$

where

$$\left.\begin{aligned} S[A, \Psi] &= \int \mathrm{d}^4x \left(-\tfrac{1}{4}G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha + \bar{\Psi}\mathrm{i}\gamma_\mu D^\mu\Psi - m\bar{\Psi}\Psi\right) \\ S^\eta[A, \eta] &= \int \mathrm{d}^4x\, \eta^{*\alpha}(x)\left[\delta^{\alpha\beta}\partial^2 + gc^{\alpha\beta\gamma}A^\mu_\gamma(x)\partial_\mu\right]\eta^\beta(x) \\ S_\mathrm{G} &= -(1/2\alpha)\int \mathrm{d}^4x \left[\partial^\mu A^\alpha_\mu\right]^2 \end{aligned}\right\} \tag{3.46}$$

Dirac spinors Ψ, $\bar{\Psi}$ describe quarks and are vectors in the three-dimensional colour space. We split the action into the term S_0 describing non-interacting theory (bilinear in fields and including S_G)

$$S_0[A, \Psi, \eta] = \int \mathrm{d}^4x \left[\tfrac{1}{2}A^\alpha_\mu(x)\delta^{\alpha\beta}\partial^2\left(g^{\mu\nu} - \partial^\mu\partial^\nu/\partial^2\right)A^\beta_\nu(x) + \bar{\Psi}(x)\left(\mathrm{i}\gamma_\mu\partial^\mu - m\right)\Psi(x) + \eta^{*\alpha}(x)\delta^{\alpha\beta}\partial^2\eta^\beta(x)\right] + S_\mathrm{G} \tag{3.47}$$

and into the interaction term S_I

$$S_\mathrm{I}[A, \Psi, \eta] = \int \mathrm{d}^4x \left[gc_{\alpha\beta\gamma}\left(\partial_\mu A^\alpha_\nu\right)A^{\mu\beta}A^{\nu\gamma} - \tfrac{1}{4}g^2c_{\alpha\beta\gamma}c_{\alpha\rho\sigma}A^\beta_\mu A^\gamma_\nu A^{\mu\rho}A^{\nu\sigma} - g\bar{\Psi}\gamma_\mu A^{\mu\alpha}T^\alpha\Psi + gc_{\alpha\beta\gamma}A^\gamma_\mu\eta^{*\alpha}\left(\partial^\mu\eta^\beta\right)\right] \tag{3.48}$$

Matrices T^α are $SU(3)$ colour generators in the fundamental representation. The 'free' generating functional reads

$$W_0 = \exp\Big\{\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}^4y \left[\tfrac{1}{2}J^\alpha_\mu(x)D^{\mu\nu}_{\alpha\beta}(x-y)J^\beta_\nu(y) + \bar{\alpha}(x)_\alpha S_{\alpha\beta}(x-y)\alpha_\beta(y) + \beta^{*\alpha}(x)\Delta^{\alpha\beta}(x-y)\beta^\beta(y)\right]\Big\} \tag{3.49}$$

where the Green's functions $D^{\mu\nu}_{\alpha\beta}$, $S_{\alpha\beta}$ and $\Delta^{\alpha\beta}$ are given by (3.32), (2.122) and (3.42), respectively. The propagators $\langle 0|TA^\alpha_\mu(x)A^\beta_\nu(y)|0\rangle$, $\langle 0|T\Psi^\alpha(x)\bar{\Psi}^\beta(y)|0\rangle$ and $\langle 0|T\eta^\alpha(x)\eta^{*\beta}(y)|0\rangle$ are obtained by multiplying the respective D-, S- and Δ-functions by $(-\mathrm{i})$.

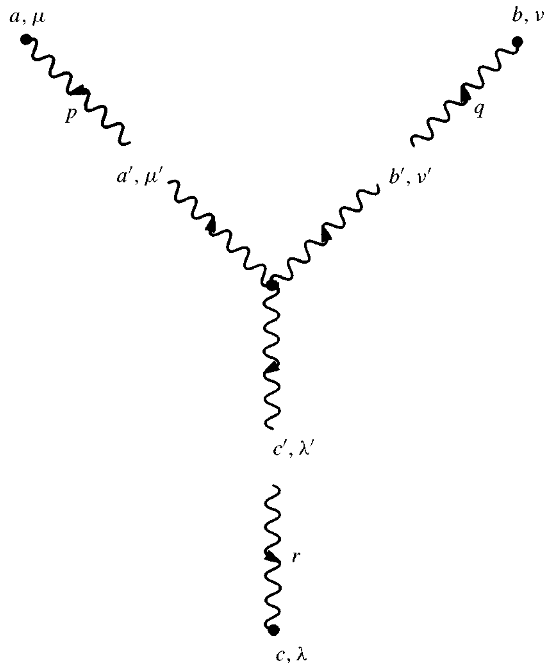


Fig. 3.2.

Any Green's function in QCD can be calculated perturbatively from the expansion

$$
\begin{aligned}
&\langle 0|T A_\mu^\alpha \ldots A_\nu^\beta \Psi \ldots \bar{\Psi}|0\rangle \\
&\quad = \frac{\displaystyle\int \mathcal{D}[A\Psi\bar{\Psi}\eta\eta^*]\big(A_\mu^\alpha \ldots A_\nu^\beta \Psi \ldots \bar{\Psi}\big) \sum_N \frac{(\mathrm{i}S_\mathrm{I})^N}{N!} \exp(\mathrm{i}S_0)}{\displaystyle\int \mathcal{D}[A\Psi\bar{\Psi}\eta\eta^*] \sum_N \frac{(\mathrm{i}S_\mathrm{I})^N}{N!} \exp(\mathrm{i}S_0)}
\end{aligned}
\tag{3.50}
$$

We recall that the presence of the denominator in (3.50) eliminates all vacuum-to-vacuum diagrams. One is usually interested in Green's functions in momentum space.

We shall now consider several Green's functions in the first order in S_I to derive Feynman rules for vertices in QCD. We begin with

$$
G_{\mu\nu\lambda}^{(3g)abc}(x_1, x_2, x_3) \equiv \langle 0|T A_\mu^a(x_1) A_\nu^b(x_2) A_\lambda^c(x_3)|0\rangle \tag{3.51}
$$

which defines the full symmetrized triple gluon ($3g$) vertex by means of the relation

$$G^{(3g)abc}_{\mu\nu\lambda}(x_1, x_2, x_3) = \int \mathrm{d}^4y \left[-\mathrm{i}D^{aa'}_{\mu\mu'}(x_1 - y)\right]\left[-\mathrm{i}D^{bb'}_{\nu\nu'}(x_2 - y)\right] \times \left[-\mathrm{i}D^{cc'}_{\lambda\lambda'}(x_3 - y)\right] V^{\mu'\nu'\lambda'}_{a'b'c'}(y) \tag{3.52}$$

illustrated in Fig. 3.2. In the first order in S_I one has

$$G^{(3g)abc}_{\mu\nu\lambda}(x_1, x_2, x_3) = \frac{1}{\mathrm{i}^6} \frac{\delta^3}{\delta J^a_\mu(x_1)\delta J^b_\nu(x_2)\delta J^c_\lambda(x_3)} \mathrm{i} \int \mathrm{d}^4y \left(g c_{def} \partial^{(d)}_\rho\right) \times \frac{\delta^3}{\delta J^d_\kappa(y)\delta J^e_\rho(y)\delta J^{f\kappa}(y)} W_0[J, \beta, \alpha]|_{J=\beta=\alpha=0} \tag{3.53}$$

(the superscript 'd' in $\partial^{(d)}_\rho$ means that the component A^d_κ is differentiated with respect to y) and the cubic term in the expansion of W_0 contributes to the final result. Following the standard method we assign points x_1, x_2, x_3 and $y_1, y_2, y_3 \to y$ to the prototype diagram

For a connected diagram there are six different orderings in pairs joined by propagators. For each assignment there is a $2^3 3!$ symmetry factor which cancels with factors coming from the expansion of W_0. The factor $\left(1/\mathrm{i}^6\right)\mathrm{i}^3 = (-\mathrm{i})^3$ is included in the three propagators. The final result reads:

$$G^{(3g)abc}_{\mu\nu\lambda}(x_1, x_2, x_3) = \mathrm{i} g c_{def} \times \int \mathrm{d}^4y\, \partial^{(d)}_\rho \sum \left[-\mathrm{i}D^{ad}_{\mu\kappa}(x_1 - y)\right]\left[-\mathrm{i}D^{be}_{\nu\rho}(x_2 - y)\right]\left[-\mathrm{i}D^{cf\kappa}_{\lambda}(x_3 - y)\right] \tag{3.54}$$

where the sum denotes all permutations of sets of indices (d, κ), (e, ρ) and (f, κ) and the derivative is with respect to y. Writing the result (3.54) in the form (3.52) we get

$$V^{abc}_{\mu\nu\lambda} = \mathrm{i} g c_{def} \big(\delta^{ad}\delta^{be}\delta^{cf} g_{\mu\lambda}\partial^{(d)}_\nu + \delta^{af}\delta^{bd}\delta^{ce} g_{\mu\nu}\partial^{(d)}_\lambda + \delta^{ae}\delta^{bf}\delta^{cd} g_{\nu\lambda}\partial^{(d)}_\mu + \delta^{ad}\delta^{bf}\delta^{ce} g_{\mu\nu}\partial^{(d)}_\lambda + \delta^{ae}\delta^{bd}\delta^{cf} g_{\nu\lambda}\partial^{(d)}_\mu + \delta^{af}\delta^{be}\delta^{cd} g_{\mu\lambda}\partial^{(d)}_\nu\big) \tag{3.55}$$

In momentum space we define $\tilde{G}^{(3g)abc}_{\mu\nu\lambda}(p, q, r)$ as

$$G^{(3g)abc}_{\mu\nu\lambda}(x_1, x_2, x_3) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{\mathrm{d}^4r}{(2\pi)^4} \exp(-\mathrm{i}p \cdot x_1) \exp(-\mathrm{i}q \cdot x_2) \times \exp(-\mathrm{i}r \cdot x_3)\, \tilde{G}^{(3g)abc}_{\mu\nu\lambda}(p, q, r) \tag{3.56}$$

and the $3g$ vertex $V^{\mu\nu\lambda}_{abc}(p,q,r)$ by the relation

$$\tilde{G}^{(3g)abc}_{\mu\nu\lambda}(p,q,r)=\tilde{G}^{(2g)aa'}_{\mu\mu'}(p)\tilde{G}^{(2g)bb'}_{\nu\nu'}(q)\tilde{G}^{(2g)cc'}_{\lambda\lambda'}(r)\tilde{V}^{\mu'\nu'\lambda'}_{a'b'c'}(p,q,r) \tag{3.57}$$

where

$$\tilde{G}^{(2g)aa'}_{\mu\mu'}(p)=\frac{-\mathrm{i}}{p^2+\mathrm{i}\varepsilon}\left[g_{\mu\mu'}-(1-\alpha)\frac{p_\mu p_{\mu'}}{p^2}\right]\delta^{aa'}$$

etc. Writing the propagators in relation (3.52) in terms of their Fourier transforms (3.57), differentiating according to (3.55), integrating over d^4y and finally comparing with (3.56) and (3.57) one gets the final result

$$\begin{aligned}\tilde{V}^{abc}_{\mu\nu\lambda}(p,q,r)=&-(2\pi)^4\delta(p+q+r)gc_{abc}\\&\times[(r-p)_\nu g_{\mu\lambda}+(q-p)_\lambda g_{\mu\nu}+(r-q)_\mu g_{\nu\lambda}]\end{aligned} \tag{3.58}$$

(where all momenta are outgoing). Similarly, one can derive the expression for the $4g$ vertex. The starting point is the Green's function

$$G^{(4g)abcd}_{\mu\nu\lambda\sigma}(x_1,x_2,x_3,x_4)\equiv\langle 0|TA^a_\mu(x_1)A^b_\nu(x_2)A^c_\lambda(x_3)A^d_\sigma(x_4)|0\rangle$$

which in the first order in S_{I} defines the $4g$ vertex according to the following relation:

$$\begin{aligned}G^{(4g)abcd}_{\mu\nu\lambda\sigma}(x_1,x_2,x_3,x_4)=&(-\mathrm{i})^4\int\mathrm{d}^4y\,D^{aa'}_{\mu\mu'}(x_1-y)\,D^{bb'}_{\nu\nu'}(x_2-y)\\&\times D^{cc'}_{\lambda\lambda'}(x_3-y)\,D^{dd'}_{\sigma\sigma'}(x_4-y)\,V^{\mu'\nu'\lambda'\sigma'}_{a'b'c'd'}\end{aligned} \tag{3.59}$$

Writing

$$\begin{aligned}G^{(4g)abcd}_{\mu\nu\lambda\sigma}(x_1,x_2,x_3,x_4)=&\frac{1}{\mathrm{i}^8}\frac{\delta^4}{\delta J^a_\mu(x_1)\delta J^b_\nu(x_2)\delta J^c_\lambda(x_3)\delta J^d_\sigma(x_4)}\\&\times\mathrm{i}\int\mathrm{d}^4y\left(-\tfrac{1}{4}g^2c_{efh}c_{emn}\right)\frac{\delta^4}{\delta J^f_\kappa\,\delta J^h_\rho\,\delta J^{m\kappa}\,\delta J^{n\rho}(y)}W_0|_{J=\beta=\alpha=0}\end{aligned} \tag{3.60}$$

one can calculate $V^{abcd}_{\mu\nu\lambda\sigma}$. There are 24 different assignments of pairs of points $x_1\ldots x_4$, $y_1\ldots y_4$ to the prototype diagram. The additional $4!2^4$ symmetry factor is cancelled by the same factor coming from the expansion of W_0. Remembering that the Lorentz contraction in the $4g$ vertex always occurs between colour indices f and m and h and n one gets the final result:

$$\begin{aligned}V^{abcd}_{\mu\nu\lambda\sigma}=&-\mathrm{i}g^2[c_{abe}c_{cde}(g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\lambda})+c_{ace}c_{bde}(g_{\mu\nu}g_{\lambda\sigma}-g_{\mu\sigma}g_{\nu\lambda})\\&+c_{ade}c_{bce}(g_{\mu\nu}g_{\lambda\sigma}-g_{\mu\lambda}g_{\nu\sigma})]\end{aligned} \tag{3.61}$$

which is obviously also true in momentum space.

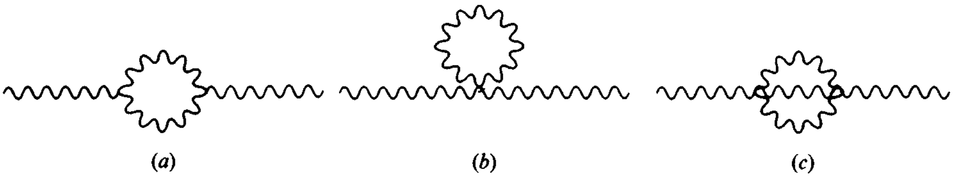


Fig. 3.3.

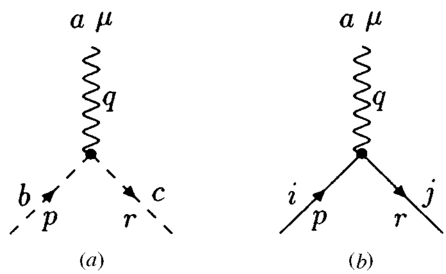


Fig. 3.4.

Following the functional approach one can also easily find that loop diagrams in Fig. 3.3 contribute with additional factors $\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{6}$, respectively. In particular diagrams (a) and (b) occur in the second order in S_{I}; e.g. the expression corresponding to the first diagram is of the form

$$\frac{(\mathrm{i}S_{\mathrm{I}})^2}{2!}\frac{1}{4!}\frac{1}{2^4}JDJ\ JDJ\ JDJ\ JDJ \tag{3.62}$$

The number of different assignments to the prototype diagram is $3 \times 2 \times 3 \times 2 \times 4!2^4$. Noticing that the product of two vertices gives us according to (3.58) 36 combinations we are left with an additional factor $\frac{1}{2}$. In a similar way one can obtain the other factors.

The final task is to find expressions for the ghost and fermion vertices in Fig. 3.4. They can be obtained from the Green's functions $\langle 0|T\eta^c(x_3)\eta^{b\star}(x_2)A^a_\mu(x_1)|0\rangle$ and $\langle 0|T\Psi_j(x_3)\bar{\Psi}_i(x_2)A^a_\mu(x_1)|0\rangle$, respectively, written as

$$\int \mathrm{d}^4y\,\left[-\mathrm{i}D^{aa'}_{\mu\mu'}(x_1-y)\right]\left[-\mathrm{i}\Delta^{bb'}(x_2-y)\right]\left[-\mathrm{i}\Delta^{cc'}(x_3-y)\right]V^{\mu'}_{a'b'c'}(y) \tag{3.63}$$

and similarly for the $\bar{\Psi}\Psi A$ vertex. Following the previous calculations closely we obtain in momentum space

$$\text{Diagram 3.4}(a): \qquad \tilde{V}^{abc}_{\mu} = -gc_{cab}r_{\mu}(2\pi)^4\delta(r+q-p) \tag{3.64}$$

$$\text{Diagram 3.4}(b): \qquad \tilde{V}^{abc}_{\mu} = -\mathrm{i}g\gamma_{\mu}\left(T^a\right)_{ji}(2\pi)^4\delta(r+q-p) \tag{3.65}$$

Of course, Feynman rules for QED in the covariant gauge can be derived analogously. All Feynman rules are given in Appendices B and C.

3.3 Unitarity, ghosts, Becchi–Rouet–Stora transformation

Unitarity and ghosts

A spin-one massless particle has only two physical polarization states $\varepsilon_{\mu}(\lambda, k)$, $\lambda = 1, 2$. Since the three four-vectors k_{μ}, $\varepsilon_{\mu}(\lambda, k)$ do not span four-dimensional space, the usual on-shell Lorentz condition $k \cdot \varepsilon = 0$ for spin-one massive particles is not enough to determine the photon polarization vectors uniquely. To make them unique we must sacrifice manifest Lorentz covariance and, for instance, choose another vector n_{μ} such that

$$k \cdot n \neq 0, \qquad n \cdot \varepsilon = 0 \tag{3.66}$$

Together with equations

$$k \cdot \varepsilon = 0, \qquad \varepsilon^*(\lambda_1, k)\varepsilon(\lambda_2, k) = -\delta_{\lambda_1\lambda_2} \tag{3.67}$$

conditions (3.66) specify the vectors $\varepsilon(\lambda, k)$ uniquely. The polarization sum reads

$$\sum_{\lambda=1,2} \varepsilon^*_{\mu}(\lambda, k)\varepsilon_{\nu}(\lambda, k) = -g_{\mu\nu} + \frac{k_{\mu}n_{\nu} + k_{\nu}n_{\mu}}{n \cdot k} - \frac{n^2 k_{\mu}k_{\nu}}{(k \cdot n)^2} \equiv P_{\mu\nu} \tag{3.68}$$

In field theory every particle is associated with a field with definite transformation properties under the homogeneous Lorentz group. Thus the four-vector gauge field $A_{\mu}(x)$ actually represents only two physical degrees of freedom. An important point is that fixing the gauge is usually not sufficient to totally remove the unphysical degrees of freedom from the theory (an exception is the axial gauge with $n^2 < 0$). A well-known example of their presence in the theory is the Gupta–Bleuler formulation of QED in the covariant gauge $\partial_{\mu}A^{\mu} = 0$. In general, such degrees of freedom can be seen explicitly if we construct the Fock space of the theory: a longitudinal photon state and a negative norm scalar photon state are present. A physically sensible theory can be defined in Hilbert space restricted to the transverse photons (see, for example, de Rafael (1979)). However, this restriction does not prevent one from having unphysical states propagating as virtual states in the intermediate steps of perturbative calculations. Indeed, the

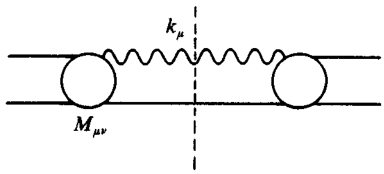


Fig. 3.5.

gauge boson propagators are gauge-dependent. The question then arises about the unitarity of the theory.

The requirement that the S-matrix is unitary implies that the scattering amplitude T, defined as

$$S_{\text{if}} = \delta_{\text{if}} + \text{i}(2\pi)^4\delta(p_{\text{i}} - p_{\text{f}})T_{\text{if}} \tag{3.69}$$

satisfies the relation

$$T_{\text{if}} - T^*_{\text{fi}} = \text{i}\sum_n T_{\text{i}n}T^*_{\text{f}n}(2\pi)^4\delta(p_{\text{i}} - p_n) \tag{3.70}$$

where the sum is taken over all physical states for which transitions to the initial and final states are allowed by quantum number conservation laws. For the forward elastic scattering the l.h.s. of (3.70) is just the imaginary part of the amplitude T_{ii} and in the general case it is the absorptive part of T_{if} (see, for example, Bjorken & Drell (1965), Section 18.12). The absorptive part of the scattering amplitude can be calculated perturbatively by means of the so-called Landau–Cutkosky rule: the contribution due to a Feynman diagram with a given intermediate state, Fig. 3.5, is obtained from the Feynman amplitude by replacing the propagators in this intermediate state by their imaginary parts. For instance, for the gauge boson propagators in the Feynman gauge (see Section 3.1)

$$D^{\alpha\beta}_{\mu\nu} = \delta^{\alpha\beta}\frac{g_{\mu\nu}}{k^2 + \text{i}\varepsilon} \to -2\pi\text{i}\delta^{\alpha\beta}g_{\mu\nu}\delta\big(k^2\big)\Theta(k_0) \tag{3.71}$$

It is clear that in gauges which do not remove the unphysical degrees of freedom, like in the Feynman gauge, the unphysical intermediate states are included in our calculation. On the other hand we can calculate the absorptive part of the scattering amplitude from the r.h.s. of the unitarity relation (3.70). Here we sum over physical states only. Thus the unitarity condition demands that the unphysical intermediate states do not contribute to the amplitude when also calculated by means of the Landau–Cutkosky rule. In fact one can expect that gauge invariance, which reflects the presence of the redundant degrees of freedom, ensures at the same time their decoupling. An explicit verification of this point may, however, be quite subtle as

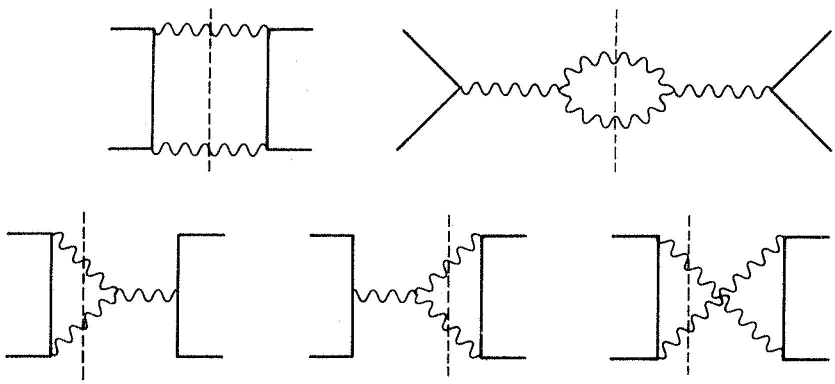


Fig. 3.6.

for solving the equations of motion we need a gauge-fixing term which breaks the manifest invariance. Nevertheless in QED there is no problem due to the fact that photons always couple to conserved currents

$$k_\mu M^{\mu\nu} = 0 \tag{3.72}$$

Therefore $P_{\mu\nu}$ given by (3.68) can be replaced by

$$P_{\mu\nu} \to -g_{\mu\nu} + \alpha k_\mu k_\nu \tag{3.73}$$

where α is an arbitrary quantity.

In non-abelian gauge theories, the situation is more complicated as can be illustrated by the standard example of the fermion–antifermion scattering in the lowest non-trivial order in the gauge-coupling constant (Feynman 1977, Aitchison & Hey 1982). The sum of the diagrams with only gauge particles in the intermediate state (Fig. 3.6) calculated in a covariant gauge (3.19) does not satisfy the unitarity condition (3.70), where the r.h.s. amplitudes $T_{\mathrm{i}n}$ and $T^*_{\mathrm{f}n}$ are given by the diagrams in Fig. 3.7 with only physical gluons as the state n (Problem 3.1). However, we have learned in Sections 3.1 and 3.2 that a consistent quantization of a non-abelian gauge theory requires the addition to the lagrangian of an extra term, the so-called ghost term (3.39). It turns out that these additional unphysical degrees of freedom cancel the gauge field unphysical polarization states exactly, leading to a perfectly unitary theory. In our specific example the cancellation occurs due to the contribution of the diagram with the ghost loop, shown in Fig. 3.8. An explicit perturbative calculation (Problem 3.1) shows that certain relations between the $f\bar{f} \to A^\alpha_\mu A^\beta_\nu$ and $f\bar{f} \to \eta^\alpha\eta^\beta$ amplitudes, where A_μ and η are the gauge and ghost fields, respectively, are responsible for the cancellation. These are examples of Ward identities for a non-abelian gauge theory.

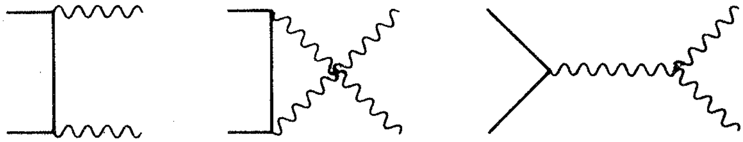


Fig. 3.7.

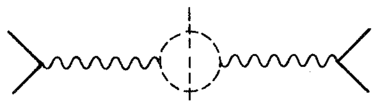


Fig. 3.8.

BRS and anti-BRS symmetry

We have introduced ghosts as a technical device to express the modification of the functional integral measure required to compensate for gauge-dependence introduced by the gauge-fixing term. The discussion in the previous subsection suggests, however, that it is very natural to adopt the point of view that ghosts, i.e. fields with unphysical statistics, are necessary to compensate for effects due to the quantum propagation of the unphysical states of the gauge fields. Provided one is able to express the gauge symmetry in a form involving both the classical and ghost fields one can introduce both fields from the very beginning as fundamental fields of a gauge theory and construct its lagrangian requiring invariance under this extended symmetry principle. Such a symmetry, the so-called BRS and anti-BRS symmetry, has indeed been discovered, originally as a symmetry of the Faddeev–Popov lagrangians with specifically chosen gauge-fixing conditions. We shall first follow this historical path and later comment on the extension which is independent of the notion of a lagrangian and remains in the spirit of the above outlined programme. It turns out that given any gauge symmetry one can always find symmetry transformations acting on gauge and ghost fields such that invariance under this symmetry is equivalent to gauge invariance of physical quantities.

To introduce BRS symmetry (Becchi, Rouet & Stora 1974, 1976) let us consider the action including the gauge-fixing term and the ghost term

$$S_{\text{eff}} = S + \int \mathrm{d}^4 x \left[-\frac{1}{2\alpha} \left(F^{\alpha}[x, A(x)] \right)^2 + \eta^* M \eta \right] = S + S_{\text{G}} + S_{\eta} \tag{3.74}$$

where

$$(M\eta)_\alpha = \int \mathrm{d}^4y\, M_{\alpha\beta}(x,y)\,\eta_\beta(y)$$

and

$$M^{\alpha\beta}(x,y) = \int \mathrm{d}^4z \frac{\delta F^\alpha[x, A(x)]}{\delta A^\rho_\mu(z)} D^{\rho\beta}_\mu(z)\delta(z-y) \tag{3.75}$$

and

$$D^{\rho\beta}_\mu = \partial_\mu \delta^{\rho\beta} + gc^{\rho\beta\gamma}A^\gamma_\mu$$

Thus

$$(M\eta)_\alpha = \delta F_\alpha[x, A(x)]$$

where δF_α is the variation of the functional F_α under a gauge transformation with $\Theta^\beta(x)$ replaced by $\big(g\eta^\beta(x)\big)$. The full action (3.74) is invariant under the global transformation

$$\left.\begin{aligned} \delta_{\mathrm{BRS}}A^\alpha_\mu &= D^{\alpha\beta}_\mu \Theta\eta^\beta \equiv \Theta\big(sA^\alpha_\mu\big) \\ \delta_{\mathrm{BRS}}\Psi &= -\mathrm{i}gT^\alpha\Theta\eta^\alpha\Psi \equiv \Theta(s\Psi) \\ \delta_{\mathrm{BRS}}\eta^\alpha &= +\tfrac{1}{2}gc^{\alpha\beta\gamma}\eta^\beta\eta^\gamma\Theta \equiv \Theta\big(s\eta^\alpha\big) \\ \delta_{\mathrm{BRS}}\eta^{*\alpha} &= +(1/\alpha)\Theta F^\alpha \equiv \Theta\big(s\eta^{*\alpha}\big) \end{aligned}\right\} \tag{3.76}$$

where Θ is a space-time-independent infinitesimal Grassmann parameter (note that it is not necessary to think of η and η^* in (3.39) as hermitean conjugates: see Problem 3.4). We observe that for the classical fields A_μ and Ψ the BRS transformation has the form of a gauge transformation with

$$\Theta^\alpha(x) = +g\Theta\eta^\alpha(x) \tag{3.77}$$

Therefore the BRS transformation leaves the original action invariant without the gauge-fixing and the ghost terms. Thus it remains to show that

$$\delta_{\mathrm{BRS}}(S_{\mathrm{G}} + S_\eta) = 0 \tag{3.78}$$

This can be checked by a straightforward calculation. As intermediate steps we record the relation

$$\delta_{\mathrm{BRS}}(M\eta) = 0 \tag{3.79}$$

and the fact that the transformations on A^α_μ, Ψ and η are nilpotent (see Problem 3.2)

$$s^2A^\alpha_\mu = s^2\Psi = s^2\eta^\alpha = 0 \tag{3.80}$$

(but $s^2\eta^{*\alpha} = (M\eta)^\alpha$). We note also that the measure $\mathcal{D}(A\Psi\eta\eta^*)$ is invariant under the BRS transformations.

The BRS symmetric lagrangian can also be written down in QED. In this case the ghost field is a free real scalar field and the covariant derivative in the first equation in (3.76) is replaced by the normal derivative (Problem 3.3).

The BRS transformation can be written down in a form more symmetric with respect to the fields η and η^* if we introduce an auxiliary field b. This also allows formulation of the notion of BRS symmetry in a lagrangian-independent way (transformations (3.76) depend on the gauge-fixing term of the lagrangian). This step is very important for promoting BRS symmetry to the fundamental symmetry of gauge theories. To be specific let us discuss the case of a covariant gauge. Using the freedom we have to choose any gauge condition without affecting the S-matrix we can take

$$\mathcal{L}_{\mathrm{G}} = b^\alpha \partial_\mu A^\mu_\alpha + \tfrac{1}{2}\alpha b^\alpha b_\alpha \tag{3.81}$$

as our gauge-fixing condition. After taking into account the equation of motion for the field b we see that (3.81) is equivalent to the standard gauge fixing term $-(1/2\alpha)\left(\partial^\mu A^\alpha_\mu\right)^2$. Using the auxiliary field b the BRS transformation for η^* and b reads

$$s\eta^{*\alpha} = b^\alpha, \qquad sb^\alpha = 0 \tag{3.82}$$

The action of s on an arbitrary function of the fields follows from their action on field polynomials

$$s(AB) = (sA)B \pm AsB \tag{3.83}$$

where the minus sign occurs if there is an odd number of ghosts and antighosts in A. The nilpotency of BRS symmetry can now be expressed as

$$s^2 = 0 \tag{3.84}$$

It has also been discovered (Curci & Ferrari 1976, Ojima 1980) that in addition to BRS symmetry there is another symmetry which leaves the quantum Yang–Mills action (3.74) invariant provided the gauge-fixing functional F^α is linear in fields. These anti-BRS transformations read

$$\left.\begin{aligned}
\bar{s}A^\alpha_\mu &= D^{\alpha\beta}_\mu \eta^{*\beta} \\
\bar{s}\eta^{*\alpha} &= \tfrac{1}{2} g c^{\alpha\beta\gamma}\eta^{*\beta}\eta^{*\gamma} \\
\bar{s}\eta^\alpha &= -b^\alpha + g c^{\alpha\beta\gamma}\eta^{*\beta}\eta^\gamma \\
\bar{s}b^\alpha &= g c^{\alpha\beta\gamma}\eta^{*\beta} b^\gamma \\
\bar{s}\Psi &= -\mathrm{i} g T^\alpha \eta^{*\alpha}\Psi
\end{aligned}\right\} \tag{3.85}$$

and

$$s\bar{s} + \bar{s}s = \bar{s}^2 = 0 \tag{3.86}$$

We have introduced the BRS and anti-BRS transformations as symmetry transformations of a Yang–Mills lagrangian with a gauge-fixing term linear in fields. With the auxiliary field b interpreted as a Lagrange multiplier for the gauge-fixing condition, the symmetry generators s and $\bar{s}$ become independent of the notion of the lagrangian and satisfy the nilpotency relations (3.84) and (3.86). In terms of the fields, which are matrix-valued in the Lie algebra of the gauge group, we have

$$\left.\begin{array}{ll} sA_\mu = D_\mu \eta & \bar{s}A_\mu = D_\mu \eta^* \\ s\Psi = -\eta\Psi & \bar{s}\Psi = -\eta^*\Psi \\ s\eta = -\frac{1}{2}[\eta, \eta] & \bar{s}\eta^* = -\frac{1}{2}[\eta^*, \eta^*] \\ s\eta^* = b & \bar{s}\eta = -[\eta^*, \eta] - b \\ sb = 0 & \bar{s}b = -[\eta^*, b] \end{array}\right\} \tag{3.87}$$

An important next step is to promote full BRS (BRS or/and anti-BRS) symmetry to the fundamental symmetry of a gauge theory (Alvarez-Gaumé & Baulieu 1983). In fact, one can show that starting from a set of infinitesimal gauge transformations building up a closed algebra (possibly with field-dependent structure functions) with a Jacobi identity one can always build an associated nilpotent full BRS algebra. One can also prove that full BRS invariance of the lagrangian leads to the gauge independence of physics. Thus we are led to construct the quantum lagrangian of a gauge theory as the most general s- or/and $\bar{s}$-invariant and Lorentz-invariant function (with ghost number zero) of physical and ghost fields. This construction simultaneously generates gauge-fixing and ghost terms in the lagrangian. It is not just a formal improvement of the Faddeev–Popov approach since it also gives consistent quantum theories in cases for which the Faddeev–Popov method is inapplicable. Actually, the latter is a consistent approach only for four-dimensional Yang–Mills theories when one chooses a linear gauge condition. Specifying the general BRS-invariant lagrangian to this special case we indeed recover the lagrangian of Faddeev and Popov.†

In the general case, in a BRS-invariant lagrangian four-ghost or higher ghost interactions are present which cannot be obtained from the ordinary Faddeev–Popov procedure. Such couplings are in fact necessary for a consistent renormalizable Yang–Mills theory if the gauge-fixing functional F^α is chosen to be non-linear in fields. Indeed, in such cases, one-loop calculation, based on the Faddeev–Popov lagrangian generating only quadratic ghost interactions, shows that the four-point ghost function is divergent. Thus, four-ghost interaction terms should have been introduced at the tree level. Also, in supergravity, quartic ghost interactions, absent in the Faddeev–Popov approach, have been shown to be necessary (Kallosch 1978,

† In Problem 3.5 this is discussed for a less general lagrangian, which is simultaneously BRS *and* anti-BRS invariant.

Nielsen 1978, Sterman, Townsend & van Nieuwenhuizen 1978). They are naturally generated by determining the quantum lagrangian from the requirement of its BRS invariance. Thus, full BRS invariance indeed emerges as the fundamental symmetry of the quantum theory associated with any given underlying gauge invariance.

Problems

3.1 Calculate in a covariant gauge the absorptive part of the amplitude corresponding to the sum of diagrams in Fig. 3.6 and Fig. 3.8 and check the unitarity condition (3.70).

3.2 Prove the invariance of (3.74) under transformation (3.76). Check (3.79) and (3.80). (Use the relation for the structure constants which follows from the commutation relations $\left[T^a, T^b\right] = \mathrm{i}c^{abc}T^c$ with $\left(T^a\right)_{bc} = -\mathrm{i}c^{abc}$.)

3.3 Check the invariance of the generalized QED lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(\mathrm{i}\not{D} - m)\Psi - (1/2\alpha)\left(\partial_\mu A^\mu\right)^2 - \tfrac{1}{2}(\partial_\mu\omega)\left(\partial^\mu\omega\right)$$

where $\omega(x)$ is a free scalar field, under the transformation

$$\begin{aligned}\delta\Psi &= +\mathrm{i}e\Theta\omega(x)\Psi(x)\\ \delta A_\mu &= \Theta\partial_\mu\omega(x)\\ \partial\omega &= (1/\alpha)\Theta\partial_\mu A^\mu\end{aligned}$$

3.4 Note that the fields η and η^* in (3.39) can be any anticommuting fields. Check that, in particular, they can be taken as real and imaginary parts of η, i.e. as hermitean fields. Check that due to the phase arbitrariness of the exponent in (3.39) the Faddeev–Popov term $\mathcal{L}_\eta$ can be made hermitean.

3.5 Check that the most general Yang–Mills and Lorentz scalar lagrangian which is BRS- and anti-BRS-invariant and has ghost number zero can be written as follows (Baulieu & Thierry Mieg 1982)

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(F_{\mu\nu})^2 - \tfrac{1}{2}(D_\mu\Phi)^2 + \bar{\Psi}\mathrm{i}\not{D}\Psi + \text{Yukawa terms} + \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\ &+ V_{\mathrm{inv}}(\Phi) + s\bar{s}\left(\tfrac{1}{2}A_\mu^2 + \beta\eta^*\eta + \gamma\langle\Phi\rangle\Phi\right) + \tfrac{1}{2}\alpha b^2\end{aligned}$$

where α, β, γ are arbitrary gauge parameters and $\langle\Phi\rangle$ is the vacuum expectation value of the boson field Φ. When $\beta = 0$ one recovers the usual Faddeev–Popov lagrangian for linear gauges.

3.6 Use the superspace formulation of the Yang–Mills theory (Bonora & Tonin 1981, Hirshfeld & Leschke 1981) to construct the BRS- and anti-BRS-invariant lagrangians. Follow the same method in supersymmetric Yang–Mills theory (Falck, Hirshfeld & Kubo 1983) and in supergravity.

4

Introduction to the theory of renormalization

4.1 Physical sense of renormalization and its arbitrariness

Bare and 'physical' quantities

Let us consider a theory of interacting fields defined by a given lagrangian density; for definiteness, let us take the case of a scalar field with quartic coupling

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \Phi_\mathrm{B})^2 - \tfrac{1}{2}m_\mathrm{B}^2\Phi_\mathrm{B}^2 - \frac{\lambda_\mathrm{B}}{4!}\Phi_\mathrm{B}^4 \tag{4.1}$$

We know already how to derive the Feynman rules which allow us to calculate the Green's functions of this theory in the form of the perturbative expansion in powers of the coupling constant λ_B. However, some of the loop momentum integrations in the resulting Feynman diagrams are divergent in the UV, reflecting the fact that, most likely, our theory does not properly describe the physics at very short distances. Consequently, the coefficients of the perturbative series are, in general, infinite. Does this mean that the lagrangian (4.1) cannot be used to define a theory?

In the following we shall show that this conclusion is not necessarily true. Two basic steps in interpreting a theory like (4.1) are the regularization of divergent integrals and the subsequent renormalization of the parameters and the Green's functions. A regularization procedure is necessary to make sense of the intermediate steps of perturbative calculations. The renormalization procedure accounts for the fact that, for instance, the 'bare' coupling constant λ_B is not a correct expansion parameter. This is not so unnatural as it seems: the 'bare' parameters of the theory, like λ_B and the 'bare' mess m_B (as well as the Green's functions of the 'bare' field Φ_B) are not directly related to observable quantities.

After renormalization, the observable quantities are expressed in terms of experimentally measured parameters and our ignorance about the short-distance physics is hidden in those free parameters of the theory determined by experiment. We shall now discuss this problem in some detail.

First let us recall the definition of the 'physical' field Φ_{F} given in Section 2.7:

$$\langle \mathbf{k}|\Phi_{\mathrm{F}}(x)|0\rangle = \exp[\mathrm{i}(k_0 x_0 - \mathbf{k}\cdot\mathbf{x})]; \qquad k_0 = (\mathbf{k}^2 + m_{\mathrm{F}}^2)^{1/2} \tag{2.192}$$

The 'physical' field Φ_{F} is thus defined by a normalization condition on the one-particle state created by Φ_{F}. The on-shell truncated Green's functions of the physical field have a straightforward physical interpretation: they are equal to the appropriate S-matrix elements. Off-shell the equality also holds for analytic continuations.

The normalization of a field operator $\Phi(x)$ is directly related to the on-shell residue of its two-point Green's function in momentum space. This is convenient because we prefer to deal with Green's functions and not field operators throughout. Consider thus the two-point Green's function of the field Φ

$$G^{(2)}(x, y) = \langle 0|T\Phi(x)\Phi(y)|0\rangle \tag{4.2}$$

and insert the complete set of states (2.189) between the Φs. The contribution of the one-particle intermediate state is then of the form

$$\begin{aligned}
&\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k_0}[\langle 0|\Phi(x)|\mathbf{k}\rangle\langle\mathbf{k}|\Phi(y)|0\rangle\Theta(x_0 - y_0) \\
&\quad + \langle 0|\Phi(y)|\mathbf{k}\rangle\langle\mathbf{k}|\Phi(x)|0\rangle(y_0 - x_0)] \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^3}\Theta(k_0)\delta(k^2 - m_{\mathrm{F}}^2)\{\Theta(x_0 - y_0)\exp[-\mathrm{i}k(x - y)] \\
&\quad + \Theta(y_0 - x_0)\exp[\mathrm{i}k(x - y)]\}|\langle 0|\Phi(0)|\mathbf{k}\rangle|^2 \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k_2 - m_{\mathrm{F}}^2 + \mathrm{i}\varepsilon}\exp[-\mathrm{i}k(x - y)]|\langle 0|\Phi(0)|\mathbf{k}\rangle|^2
\end{aligned} \tag{4.3}$$

(the last step can be easily checked by calculating the r.h.s. first). Therefore, the Fourier-transformed two-point function near the mass-shell behaves as

$$\tilde{G}^{(2)}(k) \underset{k^2\to m_{\mathrm{F}}^2}{\approx} \frac{\mathrm{i}}{k^2 - m_{\mathrm{F}}^2 + \mathrm{i}\varepsilon}|\langle 0|\Phi(0)|\mathbf{k}\rangle|^2 \tag{4.4}$$

note that $\langle 0|\Phi(0)|\mathbf{k}\rangle$ is a constant from relativistic invariance. If $\Phi(x) = \Phi_{\mathrm{F}}(x)$ it now follows from (2.192) that

$$\tilde{G}_{\mathrm{F}}^{(2)}(k) \underset{k^2\to m_{\mathrm{F}}^2}{\approx} \frac{\mathrm{i}}{k^2 - m_{\mathrm{F}}^2 + \mathrm{i}\varepsilon} \tag{4.5}$$

The result (4.5) for the 'physical' field Φ_{F} does not, in general, agree with the result of the perturbation theory calculation for 'bare' Green's function: $G_{\mathrm{B}}^{(2)}(x, y) = \langle 0|T\Phi_{\mathrm{B}}(x)\Phi_{\mathrm{B}}(y)|0\rangle$ (with some regularization introduced to keep the loop integrals finite). We shall find instead that

$$\tilde{G}_{\mathrm{B}}^{(2)}(k) \underset{k^2 \to m_{\mathrm{F}}^2}{\approx} \frac{Z_3}{k^2 - m_{\mathrm{F}}^2 + \mathrm{i}\varepsilon} \tag{4.6}$$

where

$$Z_3 \neq 1$$

and is actually divergent in the absence of regularization. The 'bare' field Φ_{B} is thus not 'physical'.

We can, however, express the original field Φ_{B} in terms of the physical field Φ_{F}

$$\Phi_{\mathrm{B}} = Z_3^{1/2} \Phi_{\mathrm{F}} \tag{4.7}$$

which implies a simple relation between the Green's functions (connected or not):

$$G_{\mathrm{F}}^{(n)}(x_1, \ldots, x_n) = Z_3^{-n/2} G_{\mathrm{B}}^{(n)}(x_1, \ldots, x_n) \tag{4.8}$$

and its counterpart for the connected proper vertex functions

$$\Gamma_{\mathrm{F}}^{(n)}(x_1, \ldots, x_n) = Z_3^{n/2} \Gamma_{\mathrm{B}}^{(n)}(x_1, \ldots, x_n) \tag{4.9}$$

which also holds for the truncated Green's functions that appear in the expression (2.190) for the S-matrix elements. Consequently, the relation between the S-matrix and 'bare' Green's functions involves the renormalization constant Z_3. The 'bare' Green's functions by themselves are not observable. On the other hand, the 'physical' Green's functions are observable and must come out finite in perturbation theory. This is not yet the case in the 'bare' perturbation expansion in λ_{B}. Suppose that we define the coupling constant λ_{F} as the value of the connected proper vertex function $\tilde{\Gamma}_{\mathrm{F}}^{(4)}(p_i)$ at some specified point, the 'renormalization point', in momentum space. We can then attempt to express other physical Green's functions as perturbation series in powers of λ_{F}. In this way we would obtain the expression of one observable quantity, the physical Green's function, in terms of other observable quantities: λ_{F} and the physical mass. We expect that a relation between observables should be free from infinities. Consequently, the coefficients of the resulting perturbation expansion should be finite for observable quantities if the perturbation theory applies.

Let us denote the renormalization point by μ. By definition we have

$$\tilde{\Gamma}_{\mathrm{F}}^{(4)}|_{\mu} = \lambda_{\mathrm{F}} \tag{4.10}$$

We can use the regularized bare perturbation expansion to calculate $\tilde{\Gamma}_{\mathrm{B}}^{(4)}$ at the same point. The result we shall write as

$$\tilde{\Gamma}_{\mathrm{B}}^{(4)}|_{\mu} = \frac{\lambda_{\mathrm{B}}}{Z_1} \tag{4.11}$$

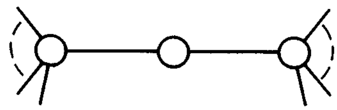


Fig. 4.1.

which defines the new renormalization constant Z_1. Using (4.9) we now obtain

$$\lambda_{\mathrm{B}} = (Z_1/Z_3^2)\lambda_{\mathrm{F}} \tag{4.12}$$

The physical mass m_{F}^2 is defined as the square of the four-momentum of a freely propagating particle (see (2.192)). From the unitarity condition for the S-matrix, expressed in terms of the scattering amplitude T_{if}, (3.69) and (3.70), it then follows that the amplitude T_{if} for the process $i \to f$ which can proceed via a one-particle intermediate state has a pole at $p^2 = m_{\mathrm{F}}^2$. In perturbation theory the general structure of the amplitude with a one-particle intermediate state corresponds to the diagram in Fig. 4.1, where the central bubble denotes the full propagator. Therefore the full propagator must have a pole at $p^2 = m_{\mathrm{F}}^2$

$$\tilde{G}^{(2)}(p) = \frac{R}{p^2 - m_{\mathrm{F}}^2} + \cdots$$

and the physical mass can also be defined as the value of p^2 for which the full propagator has a pole. Defined as a pole of the scattering amplitude, m_{F}^2 is an experimentally measurable parameter.

Equivalently, we shall define m_{F}^2 as the position of the zero of the inverse propagator (see (2.152))

$$\tilde{\Gamma}^{(2)}(p^2 = m_{\mathrm{F}}^2) = 0 \tag{4.13}$$

which can be either 'bare' or 'physical'. This definition is convenient because it avoids multiple poles present in $\tilde{G}^{(2)}(p)$ calculated to any finite order. Again, we can calculate m_{F}^2 in the regularized bare perturbation expansion and define the mass renormalization constant Z_0 by the relation

$$m_{\mathrm{F}}^2 = (Z_3/Z_0)m_{\mathrm{B}}^2 \tag{4.14}$$

Altogether, we have three independent renormalization constants: Z_0, Z_1 and Z_3 corresponding to three relations, (4.14), (4.12) and (4.7), between 'bare' and 'physical' quantities. Using these relations we can now rewrite the original lagrangian (4.1) in terms of new fields Φ_F and parameters m_F, λ_F

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}(\partial_\mu \Phi_B)^2 - \tfrac{1}{2} m_B^2 \Phi_B^2 - (\lambda_B/4!)\Phi_B^4 = \tfrac{1}{2} Z_3 (\partial_\mu \Phi_F)^2 - \tfrac{1}{2} Z_0 m_F^2 \Phi_F^2 \\ - Z_1(\lambda_F/4!)\Phi_F^4 = \tfrac{1}{2}(\partial_\mu \Phi_F)^2 - \tfrac{1}{2} m_F^2 \Phi_F^2 - (\lambda_F/4!)\Phi_F^4 \\ + \tfrac{1}{2}(Z_3 - 1)(\partial_\mu \Phi_F)^2 - \tfrac{1}{2}(Z_0 - 1) m_F^2 \Phi_F^2 - (Z_1 - 1)(\lambda_F/4!)\Phi_F^4 \end{aligned} \quad (4.15)$$

where

$$\Phi_B = Z_3^{1/2}\Phi_F, \quad m_B^2 = Z_0 Z_3^{-1} m_F^2, \quad \lambda_B = Z_1 Z_3^{-2}\lambda_F \quad (4.16)$$

The resulting 'renormalized' lagrangian we shall use as the basis for the formulation of the new perturbation expansion parametrized by the 'physical' quantities: m_F and λ_F. It is this 'physical' perturbation theory and its generalizations described below that will be used in calculations: we shall not refer to the 'bare' lagrangian explicitly.

The final important remark is the following. We have introduced the renormalization constants as multiplicative constants. This is not necessary, and for the parameters of the lagrangian (couplings and masses) it is not always convenient. A physically important example is, for instance, a scalar field theory with two fields, ϕ and Φ, with the masses m and M, respectively. In this case, the renormalization of m^2 requires an additive term δm^2 which has a part proportional to M^2 and vice versa. Another important example is a theory of scalar and fermion fields, Φ and Ψ, with the couplings $\lambda\Phi^4$ and $g\overline{\Psi}\Psi\Phi$) (Yukawa coupling). The former is, for example, renormalized by the fermion loop contributing to the proper 4Φ vertex function and this contribution is proportional to g^4 but not to λ. Thus, an additive renormalization of λ, $\lambda_B = Z_3^{-2}(\lambda_F + \delta\lambda)$, is more appropriate. The latter notation, $m_B^2 = Z_3^{-1}(m_F^2 + \delta m^2)$ and $\lambda_B = Z_3^{-2}(\lambda_F + \delta\lambda)$ is, in fact, general enough to include both cases.

Counterterms and the renormalization conditions

In (4.15) we have divided the lagrangian into two parts. The first part is identical to (4.1) with the original parameters replaced by physical ones. The perturbation theory based on this part of the lagrangian alone, unless subject to some regularization procedure, would give rise to divergences analogous to those of the 'bare' perturbation expansion. We expect that these divergences will be cancelled order-by-order if we take into account the second, Z_i-dependent part of the lagrangian: the 'counterterms'. The counterterms can be properly defined only in a regularized theory. After the cancellation of the regulator-dependent terms, the regulator can be removed. If this procedure works in any order of perturbation theory with finite number of counterterms, we say that the theory is renormalizable.

The theory defined by (4.1) is renormalizable if considered in the space-time of dimension $d \leq 4$.

The renormalization constants Z_0, Z_1 and Z_3 are, of course, not observable quantities and even if expressed in the new 'physical' perturbation expansion in terms of λ_{F} and m_{F} are divergent order-by-order after the regularization is removed.

Now let us make the following important observation concerning the practical application of our results. We shall always use the 'physical' perturbation expansion and never refer to the 'bare' parameters at any stage of calculations. From this point of view to define Z_3, Z_1 and Z_0 by relations like (4.7), (4.12) and (4.14) is inconvenient because these relations involve 'bare' quantities. Instead, we shall fix the counterterms by imposing the 'renormalization conditions' on the physical Green's functions. We have, in fact, already used one such condition to define λ_{F}: it is (4.10). To fix all counterterms we need two more conditions, which we choose as follows

$$\frac{\partial}{\partial p^2}\tilde{\Gamma}_{\mathrm{F}}^{(2)}(p)|_{p^2=m_{\mathrm{F}}^2} = 1 \tag{4.17}$$

which implies

$$\tilde{\Gamma}_{\mathrm{F}}^{(2)}(p) \underset{p^2=m_{\mathrm{F}}^2}{\longrightarrow} p^2 - m_{\mathrm{F}}^2$$

in accord with the fact that field Φ_{F} in (4.15) is assumed to be the 'physical' field; and

$$\Sigma_{\mathrm{F}}(p)|_{p^2=m_{\mathrm{F}}^2} = 0 \tag{4.18}$$

where

$$\tilde{\Gamma}_{\mathrm{F}}^{(2)}(p) = p^2 - m_{\mathrm{F}}^2 - \Sigma_{\mathrm{F}}(p) \tag{4.19}$$

This choice is exactly equivalent to our previous definitions. The counterterms, present in the Feynman rules of the 'physical' perturbation expansion, can be determined from the renormalization conditions order-by-order. Observe that we now use m_{F} and λ_{F} as free parameters rather than m_{B} and λ_{B}: this is also more natural physically.

Arbitrariness of renormalization

We have outlined the procedure which replaces the original 'bare' perturbation expansion with divergent coefficients by a new perturbation expansion in which the coefficients are finite. How unique is this procedure? We have used the physical observability of new parameters and Green's functions as a guide in order to ensure that they are finite and in finite relation to each other. However, if the perturbation theory parametrized by m_{F} and λ_{F} is finite we would not expect to lose this property if we change the parametrization and use parameters which differ from m_F and λ_F by a finite amount. In other words once we have specified the counterterms which cancel the infinities we can make a finite change in them. We thus have an infinite, n-parameter class of different 'renormalization prescriptions' (n is the number of counterterms). This renormalization symmetry of the theory is very important, and not only because it allows one to choose the most convenient renormalization prescription; it is the basis of the renormalization group and the renormalization group equations.

Consider some examples. First of all we can change the renormalization point μ: $\mu \to \mu'$. As a result, the value of λ_F changes: $\lambda_F \to \lambda'_F$. As the bare parameters are fixed, also the renormalization constant Z_1, defined by (4.11), must change: $Z_1 \to Z'_1$. It is possible to do this transformation without affecting Z_3 or Z_0: the field Φ_F is still given by (4.7) and therefore does not change; however, it will now have to be expressed in terms of the new parameter λ'_F. The Green's functions are still 'physical' but have a new perturbation expansion, in powers of λ'_F.

So far we have kept to the original, 'physical' scheme. Let us now change the value of the mass renormalization constant Z_0: $Z_0 \to Z'_0$ keeping Z_3 unchanged. As m_B^2 is fixed and the physical content of the theory should not change we must conclude that the new mass parameter, the 'renormalized mass', defined as

$$m_R^2 = (Z_3/Z'_0)m_B^2 \tag{4.20}$$

is no longer the physical mass. In terms of the 'renormalization conditions' discussed before this may correspond to replacing (4.18) by

$$\Sigma_R(p)|_\mu = 0 \tag{4.21}$$

where now

$$\tilde{\Gamma}_R^{(2)} = p^2 - m_R^2 - \Sigma_R(p) \tag{4.22}$$

and μ is some renormalization point.

Similarly, we do not have to require that the coupling constant parameter is equal to the value of $\tilde{\Gamma}_F^{(4)}$ at any renormalization point. We can use any definition as long as the transition from λ_F to the new parameter λ_R is perturbatively finite: i.e. λ_R can be expressed in terms of λ_F in the λ_F-perturbation expansion, or vice versa. One important example of a renormalization procedure which makes use of this generalization is the minimal subtraction scheme, which we shall discuss in detail in the following sections.

Finally we can relax the requirement of the 'physical' renormalization of the two-point renormalized Green's function. Instead of Φ_F we can use the 'renormalized' field Φ_R provided that their relative normalization is perturbatively finite, i.e. we have

$$\Phi_{\mathrm{R}} = Z_{\mathrm{R}}^{1/2} \Phi_{\mathrm{F}} \tag{4.23}$$

where $Z_{\mathrm{R}} = Z_3^{\mathrm{F}}/Z_3^{\mathrm{R}}$ is finite in the perturbation expansion. The value of Z_{R} can be determined if we know the on-shell residue of the two-point renormalized Green's function

$$\tilde{G}_{\mathrm{R}}^{(2)}(k) \underset{k^2 \to m_{\mathrm{F}}^2}{\approx} \frac{\mathrm{i} Z_{\mathrm{R}}}{k^2 - m_{\mathrm{F}}^2 + \mathrm{i}\varepsilon} \tag{4.24}$$

or, equivalently

$$\frac{\partial}{\partial k^2} \tilde{\Gamma}_{\mathrm{R}}^{(2)}(k)|_{k^2 = m_{\mathrm{F}}^2} = \frac{1}{Z_{\mathrm{R}}} \tag{4.25}$$

Due to the relation

$$G_{\mathrm{F}}^{(n)}(x_1, \ldots, x_n) = Z_{\mathrm{R}}^{-n/2} G_{\mathrm{R}}^{(n)}(x_1, \ldots, x_n) \tag{4.26}$$

the factors of Z_{R} will also appear in the relation between the renormalized Green's functions and the S-matrix elements. This does not make the renormalized Green's functions very 'unphysical' because the Z_{R} factors are finite order-by-order.

The relation between the S-matrix and the renormalized Green's functions has the following form

$$\begin{aligned}\langle \mathbf{k}_1' \ldots \mathbf{k}_m' | S | \mathbf{k}_1 \ldots \mathbf{k}_n \rangle = \lim_{k_i^2 \to m_{\mathrm{F}}^2} \prod_i \left[\frac{1}{\mathrm{i}} (k_i^2 - m_{\mathrm{F}}^2) \right] \\ \times Z_{\mathrm{R}}^{-(n+m)/2} \tilde{G}_{\mathrm{R}}^{(n+m)}(k_1', \ldots, k_m', -k_1, \ldots, -k_n)\end{aligned} \tag{4.27}$$

which generalizes (2.190).

Of course, the lagrangian (4.15) can be rewritten in terms of the new renormalized fields and parameters

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(Z_3 \Phi_{\mathrm{R}}, (Z_0/Z_3) m_{\mathrm{R}}^2, (Z_1/Z_3^2) \lambda_{\mathrm{R}}, \ldots) \\ &= \mathcal{L}(\Phi_{\mathrm{R}}, m_{\mathrm{R}}^2, \lambda_{\mathrm{R}}, \ldots) + \Delta \mathcal{L}\end{aligned} \tag{4.28}$$

where in the $\lambda\Phi^4$ theory

$$\Delta \mathcal{L} = \tfrac{1}{2}(Z_3 - 1)(\partial_\mu \Phi_{\mathrm{R}})^2 - \tfrac{1}{2}(Z_0 - 1) m_{\mathrm{R}}^2 \Phi_{\mathrm{R}}^2 - 1/4!(Z_1 - 1)\lambda_{\mathrm{R}} \Phi_{\mathrm{R}}^4 \tag{4.29}$$

The Feynman rules of the renormalized perturbation theory follow from the renormalized lagrangian (4.28). The counterterms provide vertices, with Z-dependent coupling constants: $(Z_3 - 1)p^2$, $(Z_0 - 1)m_{\mathrm{R}}^2$, $(Z_1 - 1)\lambda_{\mathrm{R}}$. Note that the counterterms are treated strictly perturbatively: when calculating the nth order correction it is enough to know the counterterms up to this order only.

The counterterms are constructed according to a renormalization prescription. One possibility is the 'physical' renormalization prescription described before. A more general case corresponds to specifying finite values for some renormalized Green's functions at a renormalization point. For instance (Σ_{R} is defined by (4.22))

$$\frac{\partial}{\partial k^2}\tilde{\Gamma}_{\mathrm{R}}^{(2)}(k)|_\mu = 1, \quad \Sigma_{\mathrm{R}}(k)|_\mu = 0, \quad \tilde{\Gamma}_{\mathrm{R}}^{(4)}(k)|_\mu = \lambda_{\mathrm{R}} \tag{4.30}$$

Clearly the number of the equations must be equal to the number of the renormalization constants.

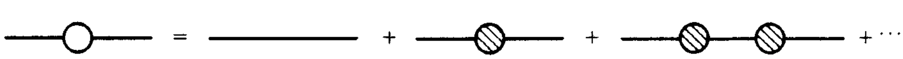
Another possibility is to assume some trial form for the counterterms and adjust the parameters to make the renormalized Green's functions finite order-by-order. This is the case in the minimal subtraction scheme. This procedure is particularly convenient because the counterterms can be determined from the divergent parts of the diagrams alone. We shall discuss this scheme in detail in Section 4.3, after the introduction of dimensional regularization.

Final remarks

Let us finish this section with some remarks concerning the calculation of the scattering amplitudes in the renormalized perturbation expansion. As explained before, we determine the physical mass m_{F}^2 from the requirement that the inverse propagator $\tilde{\Gamma}_{\mathrm{R}}^{(2)}(k) = k^2 - m_{\mathrm{R}}^2 - \Sigma_{\mathrm{R}}(k)$ vanishes at $k^2 = m_{\mathrm{F}}^2$. We use this condition in preference to the one involving the propagator $\tilde{G}_{\mathrm{R}}^{(2)}(k)$. The reason is that the perturbation expansion for $\tilde{G}_{\mathrm{R}}^{(2)}(k)$ has the form

$$\begin{aligned}\tilde{G}_{\mathrm{R}}^{(2)}(k) = {}& \frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2} + \frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2}[-\mathrm{i}\Sigma_{\mathrm{R}}(k)]\frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2} + \frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2}[-\mathrm{i}\Sigma_{\mathrm{R}}(k)]\frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2} \\ & \times [-\mathrm{i}\Sigma_{\mathrm{R}}(k)]\frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2} + \cdots = \frac{\mathrm{i}}{k^2 - m_{\mathrm{R}}^2 - \Sigma_{\mathrm{R}}(k)}\end{aligned} \tag{4.31}$$

corresponding to the diagrams



with the blobs on the r.h.s. denoting $-\mathrm{i}\Sigma_{\mathrm{R}}(k)$. We see that, to any finite order, $\tilde{G}^{(2)}(k)$ has multiple poles at $k^2 = m_{\mathrm{R}}^2$ instead of a single pole at $k^2 = m_{\mathrm{F}}^2$ required by unitarity. To obtain a single pole one must sum the whole geometric series (4.31).

Assume that $\tilde{\Gamma}_{\mathrm{R}}^{(2)}(k)$ calculated up to order n in some renormalization scheme is regular at $k^2 = m_{\mathrm{F}}^2$ and expand

$$\Sigma_{\mathrm{R}}(k^2) = \Sigma_0 + (k^2 - m_{\mathrm{F}}^2)\Sigma_1 + (k^2 - m_{\mathrm{F}}^2)^2\Sigma_2 + \cdots \tag{4.32}$$

where m_{F}^2 is understood as the value of k^2 at which the nth order approximation to

$\tilde{\Gamma}_{\mathrm{R}}^{(2)}(k)$ vanishes; hence $\Sigma_0 = m_{\mathrm{F}}^2 - m_{\mathrm{R}}^2$. We obtain

$$\tilde{\Gamma}_{\mathrm{R}}^{(2)}(k) = (k^2 - m_{\mathrm{F}}^2)[1 - \Sigma_1 - (k^2 - m_{\mathrm{F}}^2)\Sigma_2 - \cdots] \tag{4.33}$$

and, using (4.25)

$$Z_{\mathrm{R}} = \frac{1}{1 - \Sigma_1} \tag{4.34}$$

Consequently, (4.27) becomes

$$\begin{aligned}
&\langle \mathbf{k}_1' \ldots \mathbf{k}_m' | S | \mathbf{k}_1 \ldots \mathbf{k}_n \rangle \\
&\quad = \lim_{k_i^2 \to m_{\mathrm{F}}^2} \prod_i \left(\frac{1}{\mathrm{i}} (k_i^2 - m_{\mathrm{F}}^2) \right) (1 - \Sigma_1)^{(n+m)/2} \\
&\qquad \times \tilde{G}_{\mathrm{R}}^{(n+m)}(k_1', \ldots, k_m', -k_1, \ldots, -k_n) \\
&\quad = (1 - \Sigma_1)^{-(n+m)/2} \tilde{G}_{\mathrm{R(trunc)}}^{(n+m)}
\end{aligned} \tag{4.35}$$

In the lowest non-trivial order (one-loop approximation) we can equivalently replace $(1 - \Sigma_1)^{-(n+m)/2}$ by $(1 + \frac{1}{2}(n + m)\Sigma_1)$ so that we then have the correction $(+\frac{1}{2}\Sigma_1)$ for each external line. We shall often refer to this rule in one-loop calculations.

4.2 Classification of the divergent diagrams

Structure of the UV divergences by momentum power counting

In the previous section we have described the procedure of renormalization. However, we have not given the proof that this procedure actually leads to finite results: we still have to show that a theory like $\lambda\Phi^4$ is renormalizable. In other words, we should still prove that the counterterms induced by the renormalization, which are of the same general structure as vertices of the original lagrangian, are enough to cancel UV divergences due to loop momentum integrations. Here we shall give only a general idea of the proof. We shall concentrate on the structure of UV divergences which appear in Feynman diagrams of $\lambda\Phi^4$ perturbation theory (with counterterms not yet included) and use the results to determine the form of counterterms from the requirement that these divergences are cancelled. They appear to have the same form as the counterterms induced by the renormalization described in Section 4.1. Some examples from QED and $\lambda\Phi^3$ theories will also be considered.

The basic blocks of a renormalization programme are the connected proper vertex functions $\Gamma^{(n)}$, also called the 1PI Green's functions. Let us recall that these are truncated Green's functions which remain connected when an arbitrary internal line is cut. In a renormalization programme these are important for two reasons. First, any other Green's function is made up of 1PI Green's functions joined together to form a tree, each loop included entirely in one of the 1PI parts, so that the momentum integrations in the two distinct parts are independent of each other. Consequently, all loops and therefore all divergences are present already in the 1PI Green's functions. The second reason is that the identification of the necessary counterterms which are to be interpreted as additional vertices in the renormalized lagrangian is particularly simple if one looks at the divergences of the 1PI functions. This is because the connected proper vertex functions can themselves be understood as generalized vertices of an effective lagrangian which includes all loop corrections: the generating functional $\Gamma[\Phi]$.

The UV divergences we are concerned with come from the region of integration where some of the integration momenta are large. Some information about the possible structure of these divergences can be obtained by the simple procedure of counting the powers of momentum in the Feynman integrand. Suppose that all integration momenta become large simultaneously: $k_i \rightarrow \Lambda k_i$, $\Lambda \rightarrow \infty$. The integrand will then behave as a power of Λ, say Λ^{-N}. One defines the so-called superficial degree of divergence of a Feynman diagram

$$D = 4L - N \tag{4.36}$$

where L is the number of independent loops, each corresponding to a $\int \mathrm{d}^4 k_i$ momentum integration. For a diagram with V vertices and I internal lines the number of independent loops is $L = I - V + 1$.

Let us next assume that only some subset S of the integration momenta is large and of the order Λ. The integrand will now be proportional to another power of Λ: Λ^{-N_S}. We can introduce the superficial degree of divergence D_S of this subintegration: $D_S = 4L_S - N_S$, where L_S is the number of integration momenta belonging to S. It is easy to see that D_S can also be regarded as a superficial degree of divergence of a subdiagram built up of all the internal lines of the complete Feynman diagram with momenta depending on those in S: L_S is then the number of loops in this subdiagram. In this way we can assign a subdiagram and a degree of divergence to each loop subintegration. For example, in the case of the two-loop diagram (a) in Fig. 4.2 the subdiagrams (b) and (c) correspond to the subintegrations over l and k, respectively.

A theorem due to Weinberg, states that a Feynman integral converges if the degree of divergence of the diagram as well as the degree of divergence associated with each possible subintegration over loop momenta is negative. This fundamental result will be the basis of our discussion. The first thing to note is that in $\lambda\Phi^4$ theory (and any other interacting quantum field theory in four dimensions) diagrams with $D \geq 0$ are present. In the above example diagram (a) has $D = 2$ while (b) and (c) both have $D = 0$.

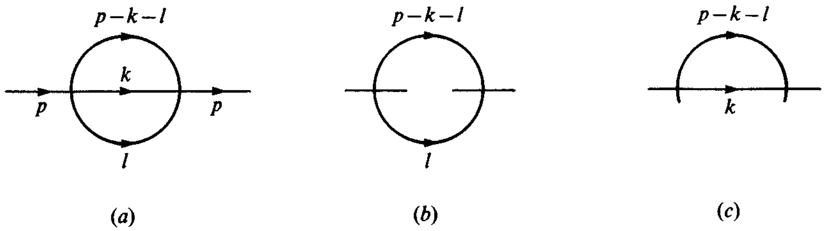


Fig. 4.2.

Classification of divergent diagrams

To find out what counterterms are required we must examine all 1PI Green's functions with $D \geq 0$ which appear in the theory. Consider the case of a scalar field theory without derivative terms in the interaction ($\lambda\Phi^4$ is one example). Then, for any 1PI diagram (4.36) reads

$$D = 4L - N = 4L - 2I \tag{4.37}$$

where I is the number of internal lines. In this formula the restriction to 1PI diagrams is due to the fact that propagators corresponding to the internal lines which do not belong to loops are independent of the integration momenta and consequently do not contribute to N: in 1PI diagrams such lines are absent. We can also write a generalization of (4.37) which is applicable to the case of fermion QED, again with restriction to 1PI diagrams

$$D = 4L - 2I_{\mathrm{B}} - I_{\mathrm{F}} \tag{4.38}$$

Here I_{B}, I_{F} are the numbers of photon and fermion internal lines, respectively. Note that by replacing $4L$ with nL we obtain a generalization of (4.36), (4.37) and (4.38) to the case of an n-dimensional space.

We shall see that D is determined by the number of external lines of the diagram. The argument is based on dimensional analysis so let us first recall the dimensions of some quantities in quantum field theory. We set $\hbar = c = 1$ so that the action integral must be dimensionless, while length and mass are measured in inverse units: $[x]^{-1} = [m]$. The canonical dimensions of Bose and Fermi fields are determined from the requirement that the kinetic terms in the action, which are, respectively, of the form $\int \mathrm{d}^n x\, \Phi\partial^2\Phi$ and $\int \mathrm{d}^n x\, \bar{\Psi}\mathrm{i}\not{\partial}\Psi$, are dimensionless. It follows that

$$[\Phi] = [m]^{(n-2)/2}, \qquad [\Psi] = [m]^{(n-1)/2} \tag{4.39}$$

In the same manner the dimensions of a coupling constant follow from the requirement that the corresponding term in the action is dimensionless. Some examples are

$$\left.\begin{array}{lll} \lambda\Phi^3: & [m]^{-n}[\lambda][m]^{3(n-2)/2} = 1; & [\lambda] = [m]^{3-n/2} \\ \lambda\Phi^4: & [m]^{-n}[\lambda][m]^{4(n-2)/2} = 1; & [\lambda] = [m]^{4-n} \\ g\bar{\Psi}\not{A}\Psi: & [m]^{-n}[g][m]^{2(n-1)/2+(n-2)/2} = 1; & [g] = [m]^{2-n/2} \end{array}\right\} \tag{4.40}$$

where $[m]^{-n}$ is the dimension of the $\int \mathrm{d}^n x$ integration measure. In four dimensions the quartic ($\lambda\Phi^4$) and QED ($g\bar{\Psi}\not{A}\Psi$) coupling constants are dimensionless while the cubic ($\lambda\Phi^3$) coupling constant has the dimension of mass. Assume that at large momenta the Bose and Fermi propagators behave, respectively, as $1/k^2$ and $1/k$. Then, for any 1PI Feynman diagram built up of vertices with dimensionless coupling constants only, the superficial degree of divergence coincides with the dimension of the diagram in the unit of mass. The exception is the possible lowest order no-loop contribution, i.e. the single elementary vertex for which the superficial degree of divergence is undefined. As all contributions to a given Green's function must be of the same dimension it then follows that with dimensionless couplings, the superficial degree of divergence of a diagram is determined solely by the external lines. The canonical dimension of a 1PI diagram with B external boson lines and F external fermion lines is equal to the dimension of a coupling constant corresponding to the vertex of this structure, which is $[m]^{4-B-3F/2}$ in four dimensions. Consequently

$$D = 4 - B - \tfrac{3}{2}F \tag{4.41}$$

As explained, this applies to 1PI diagrams with dimensionless couplings only. If the diagram also contains vertices with dimensional coupling constants of total dimension Δ in the units of mass (for example, $\Delta = K$ for K fermion mass insertions or cubic scalar couplings)

$$D = 4 - B - \tfrac{3}{2}F - \Delta \tag{4.42}$$

(also see Problem 4.4). This formula has some important consequences. One is that in any theory with coupling constants all of positive dimension in the units of mass ($\lambda\Phi^3$ for example) the number of 1PI diagrams with $D \geq 0$ is finite. A theory like this is called super-renormalizable. In the case of the $\lambda\Phi^3$ theory the diagrams with $D \geq 0$ are as shown in Fig. 4.3. Here (*a*), (*b*) and (*c*) are irrelevant vacuum diagrams: they cancel with the normalization factor of our Green's functions, (*d*) is a constant 'tadpole' which contributes to the vacuum expectation value $\langle\Phi\rangle$ of the Φ field and can be disregarded in the $\langle\Phi\rangle = 0$ case (diagrams of this kind will be important in theories with spontaneous symmetry breaking) and (*e*) produces an infinite mass renormalization counterterm which is the only divergent counterterm in $\lambda\Phi^3$ theory with $\langle\Phi\rangle = 0$. In particular, there are no $D \geq 0$ 1PI

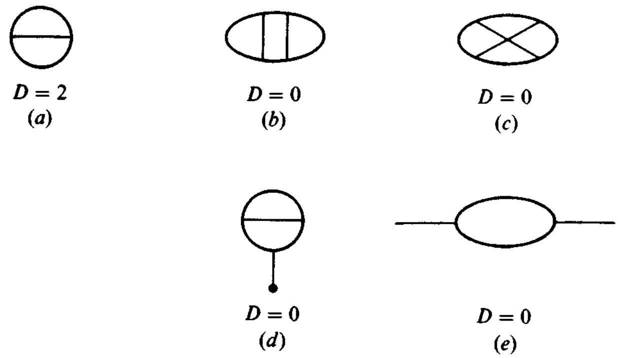


Fig. 4.3.

diagrams with three external lines and consequently no divergent coupling constant renormalization counterterm.

On the other hand in a theory which involves, among others, any coupling constant of negative dimension, like $\lambda\Phi^4$ in more than four space-time dimensions or $G(\bar{\Psi}\Psi)^2$ in four-dimensional space, the perturbation expansion of any 1PI Green's function must include $D \geq 0$ diagrams when the order is high enough. Each 1PI Green's function requires new counterterms and new renormalization conditions: consequently, the theory is non-renormalizable. An important example is Fermi's theory of weak interactions.

Let us now return to theories with dimensionless coupling constants, so that (4.41) applies: all diagrams contributing to a specific 1PI Green's function have the same degree of divergence. For the $\lambda\Phi^4$ theory and for QED we have the $D \geq 0$ connected proper vertex functions shown in Fig. 4.4 (the triple photon proper vertex also has $D = 1$ but it vanishes because the photon has odd charge conjugation).

Necessary counterterms

To make the divergent diagrams finite we must find a way to lower the associated degrees of divergence. This can be achieved by making a subtraction in the Feynman integrand. Consider a 1PI Feynman diagram with $D \geq 0$. Let $\{f\}_D$ denote the sum of first $(D + 1)$ terms of a Taylor expansion of the Feynman integrand f in powers of the external momenta with D equal to the superficial degree of divergence of the diagram considered. If we replace the integrand of this diagram by a subtracted expression

$$f - \{f\}_D \tag{4.43}$$

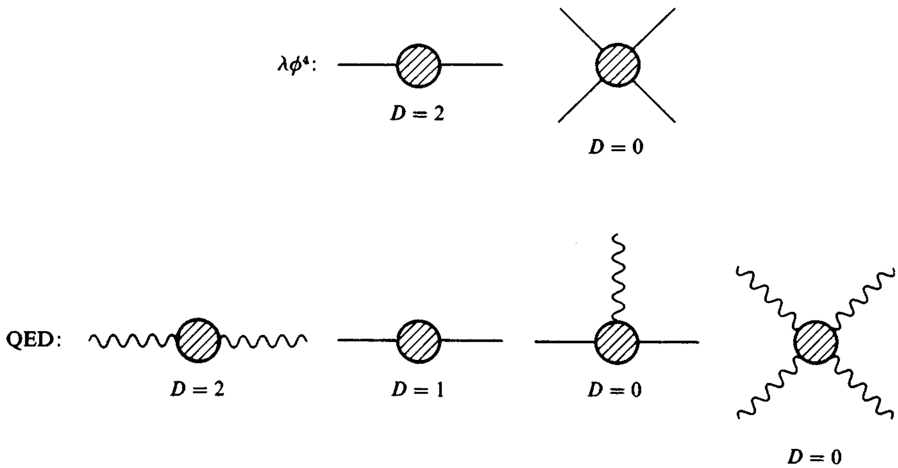


Fig. 4.4.

the superficial degree of divergence will become negative. A similar procedure should be applied to divergent subdiagrams.

In a renormalizable theory we should be able to understand these subtractions as due to the inclusion of counterterms. With some procedure for the regularization of divergent integrals we can calculate the Feynman integrals of the two terms in (4.43) separately. As the second term is a polynomial in the external momenta, after the integration it will produce a polynomial 'counterterm', or a combination of counterterms. Obviously the counterterms will diverge when the intermediate regularization is removed; however, from Weinberg's theorem, they must combine with the other part of (4.43) to ultimately give a finite result provided that we have dealt with all divergent subdiagrams.

Consider the case of the $\lambda\Phi^4$ scalar field theory. We then have two $D \geq 0$ Green's functions: $\tilde{\Gamma}^{(2)}(p)$ with $D = 2$ and $\tilde{\Gamma}^{(4)}(p_1, \ldots, p_4)$ with $D = 0$. The two-point function has $D = 2$ and consequently requires subtraction of the first three terms of the Taylor expansion of the integrand. This produces two counterterms

$$Ap^2 + B$$

The third one, proportional to p_μ, vanishes because of relativistic invariance: all counterterms must be scalars. We recognize the counterparts of the wave-function renormalization constant (p^2 is equivalent to $-\partial^2$) and the mass counterterm. The four-point function has $D = 0$ and only one (constant) counterterm which corresponds to the coupling constant counterterm. We conclude that the subtractions necessary to lower the degree of divergence of the $D \geq 0$ Green's functions are

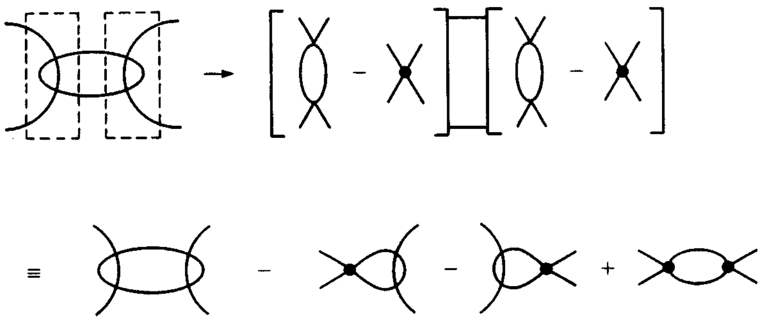


Fig. 4.5.

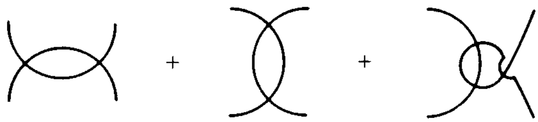


Fig. 4.6.

equivalent to including the counterterms of structure as implied by renormalization of the original lagrangian.

Can these counterterms, if included in the interaction lagrangian, also deal with all divergences due to the $D \geq 0$ subdiagrams? Again (in the $\lambda\Phi^4$ case) these are the subdiagrams with two ($D = 2$) or four ($D = 0$) external lines, so that no new types of counterterm are apparently required. We can envisage the following procedure. Consider a diagram with no counterterm vertices which includes some $D \geq 0$ subdiagrams. We first identify all lowest order (one-loop) divergent subdiagrams. In each of these we make the appropriate subtractions. The effect of these subtractions should be equivalent to adding to the original diagram all diagrams obtained by replacing some of the considered lowest order subdiagrams by the corresponding lowest order counterterms. One example is shown in Fig. 4.5. The boxes indicate the divergent subdiagrams we are considering. Note that in this case the dot  is not the complete lowest order coupling constant counterterm. This is because we make the subtraction in the indicated subdiagrams, and not in complete one-loop vertex functions which are shown in Fig. 4.6. This procedure should then be repeated with respect to higher order $D \geq 0$ subdiagrams and counterterms until all divergences are removed.

Unfortunately, this argument runs into difficulties: one problem is that we cannot always make subtractions independently in different subdiagrams. For instance, subdiagrams (b) and (c) in Fig. 4.2 overlap each other. Another example of such 'overlapping divergences' is the QED diagram shown in Fig. 4.7, where

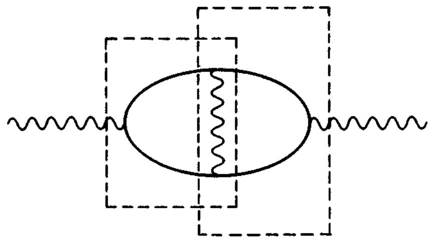


Fig. 4.7.

the $D = 0$ divergent subdiagram can be identified in two distinct ways. The problem of overlapping divergences requires more detailed treatment and we shall not discuss it here. However, the ultimate answer is positive: the counterterms in the renormalized lagrangian do indeed cancel all UV divergences.

We must also note another approach, the BPHZ construction (Bogoliubov–Parasiuk–Hepp and Zimmermann). In this case the subtractions are performed systematically by means of an 'R-operation' which is based on a somewhat different identification of subdiagrams.

4.3 $\lambda\Phi^4$: low order renormalization

Feynman rules including counterterms

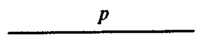
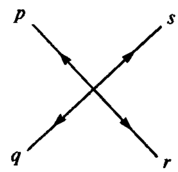
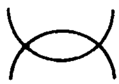
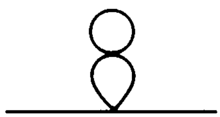
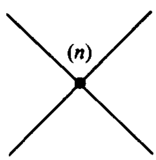
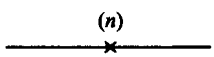
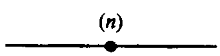
In this section we introduce in some detail the dimensional regularization method ('t Hooft & Veltman 1972, Bollini & Giambiagi 1972) and the minimal subtraction renormalization scheme using the $\lambda\Phi^4$ theory as an example. We choose the minimal subtraction scheme to illustrate our general consideration of Section 4.1 since this method of renormalization is particularly convenient in applications of the renormalization group equation: this is the so-called mass-independent renormalization, namely the renormalization constants Z_i do not depend on mass parameters.

The Feynman rules for the $\lambda\Phi^4$ theory are collected in Table 4.1. It also shows diagrams corresponding to the counterterms present in lagrangian (4.28). All counterterms are treated as interaction terms since we want to use counterterms order-by-order in the perturbation theory.

Renormalization constants Z_i will be determined by the renormalization conditions in each order of the perturbation theory

$$Z_i = 1 + \sum_n (Z_i - 1)^{(n)}$$

Table 4.1.

propagator		$\dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$
loop integration		$\displaystyle\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$
vertex		$-\mathrm{i}\lambda;\ p + q + r + s = 0$
symmetry factors S		$S = \frac{1}{2}$
		$S = \frac{1}{6}$
		$S = \frac{1}{2}$
		$S = \frac{1}{4}$
vertex counterterm		$-\mathrm{i}\lambda(Z_1 - 1)^{(n)}$
mass counterterm		$-\mathrm{i}(Z_0 - 1)^{(n)} m^2$
wave-function renormalization counterterm		$+\mathrm{i}(Z_3 - 1)^{(n)} p^2$

where

$$(Z_i - 1)^{(n)} = f_i^{(n)} \lambda^n$$

It is clear from the previous section that the main task for the lowest order renormalization is to calculate the proper (1PI) Green's functions corresponding to the three divergent diagrams of Fig. 4.8. The calculation will be based on the dimensional regularization method and we introduce it now briefly. We begin with vertex renormalization, i.e. with Fig. 4.8(b). In four space-time dimensions this

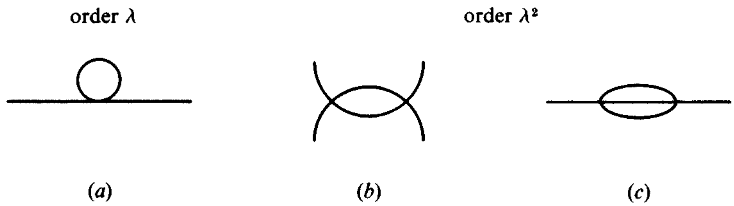


Fig. 4.8.

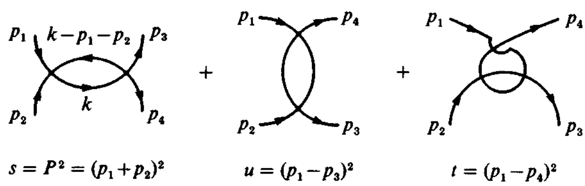


Fig. 4.9.

is logarithmically divergent $\sim \int d^4k/k^4$. However, if we do the integration in $4 - \varepsilon$ dimensions the result is finite. This suggests the possibility of considering the analytic continuation of the amplitude to $4 - \varepsilon$ dimensions (regularization), cancelling all terms divergent when $\varepsilon \to 0$ by properly adjusting the renormalization constants Z_i (renormalization) and continuing the result analytically to four dimensions. Thus, the dimensionally regularized theory (4.1) is defined by the action

$$S = \int d^n x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 - \frac{\lambda \mu^\varepsilon}{4!} \phi^4 + \text{counterterms} \right]$$

where $n = 4 - \varepsilon$. In $n = 4 - \varepsilon$ dimensions the coupling constant λ is dimensionful, so it is convenient to write it as $\mu^\varepsilon \lambda$, where μ is an arbitrary mass scale and λ is dimensionless. The Feynman rules in Table 4.1 are changed accordingly.

Calculation of Fig. 4.8(b)

We first calculate the diagram shown in Fig. 4.8(b) in $4 - \varepsilon$ dimensions and then discuss some subtle points of the method. The kinematics are defined in Fig. 4.9. The first diagram of Fig. 4.9 gives the following expression for the amplitude:

$$A_1 = \frac{1}{2} \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k - P)^2 - m^2 + i\varepsilon} (-i\lambda\mu^\varepsilon)^2 \tag{4.44}$$

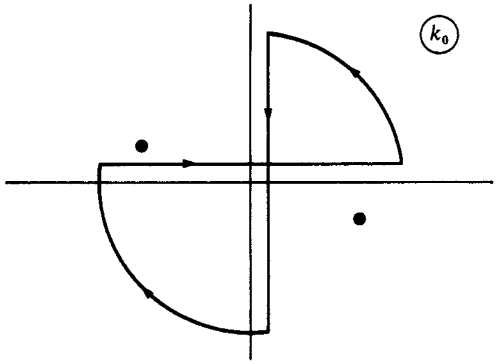


Fig. 4.10.

We introduce the standard Feynman parameters

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}x}{[ax + b(1-x)]^2} \tag{4.45}$$

and change the variables as follows

$$k - Px \to k \tag{4.46}$$

which is legitimate for a convergent integral. We then get

$$A_1 = \tfrac{1}{2}\lambda^2\mu^{2\varepsilon}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{1}{[k^2 + sx(1-x) - m^2 + \mathrm{i}\varepsilon]^2} \tag{4.47}$$

The integration over $\mathrm{d}k_0\,\mathrm{d}^{n-1}k$ can be easily done after the Wick rotation. We note that the poles of the integrand in the k_0-plane are located as shown in Fig. 4.10, since $k_0 = \pm[\mathbf{k}^2 - sx(1-x) + m^2 - \mathrm{i}\varepsilon]^{1/2}$. We can change the contour to integrate over the imaginary axis. Changing variables

$$\begin{aligned}
&\mathrm{i}k_0 \to k_0\\
&\mathrm{d}k_0 \to -\mathrm{i}\,\mathrm{d}k_0\\
&k_0^2 - \mathbf{k}^2 \to -k_0^2 - \mathbf{k}^2\\
&\int_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}k_0 \to +\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}k_0
\end{aligned}$$

one gets the following expression

$$A_1 = \tfrac{1}{2}\lambda^2\mu^{2\varepsilon}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{(+\mathrm{i})}{[k^2 - sx(1-x) + m^2 - \mathrm{i}\varepsilon]^2} \tag{4.48}$$

where k is a vector in n-dimensional Euclidean space. The next step is to use the formula (in Euclidean space)

$$\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2+b)^\alpha} = \frac{1}{(4\pi)^{n/2}} \frac{b^{n/2-\alpha}\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)} \tag{4.49}$$

which leads us to an almost final result

$$A_1 = \tfrac{1}{2}(\lambda\mu^\varepsilon)\mathrm{i}\frac{\lambda}{(4\pi)^{n/2}} \frac{\Gamma(\frac{1}{2}\varepsilon)}{\Gamma(2)} \int_0^1 \mathrm{d}x \int \left[\frac{m^2 - sx(1-x) - \mathrm{i}\varepsilon}{\mu^2}\right]^{-\varepsilon/2} \tag{4.50}$$

To proceed we require the expression

$$\Gamma(x) \underset{x\to 0}{\longrightarrow} 1/x - \gamma + O(x)$$

where $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + 1/n - \ln n) = 0.577$ is Euler's constant, and we also use the expansion

$$a^\varepsilon = \exp(\varepsilon \ln a) = 1 + \varepsilon \ln a + O(\varepsilon^2)$$

In the limit $\varepsilon \to 0$ we get

$$A_1 = \mathrm{i}\lambda\mu^\varepsilon \frac{\lambda}{16\pi^2}\frac{1}{\varepsilon} - \mathrm{i}\lambda\mu^\varepsilon \frac{\lambda}{2(4\pi)^2}\left[\int_0^1 \mathrm{d}x \ln \frac{m^2 - sx(1-x)}{\mu^2} + \gamma - \ln 4\pi\right] + O(\varepsilon) \tag{4.51}$$

The presence of the pole term $1/\varepsilon$ reflects the divergence of the integral in four dimensions. Note also the presence in (4.50) and (4.51) of the factor $\lambda\mu^\varepsilon$ (not expanded in ε) since this is the coupling constant in n dimensions and it enters into the definition of the counterterm $-\mathrm{i}\lambda\mu^\varepsilon(Z_1 - 1)$. The remaining contribution is finite and reads, after integration,

$$\begin{aligned} A_1(s) = -\mathrm{i}\frac{(\lambda\mu^\varepsilon)\lambda}{2(4\pi)^2}\Bigg\{&\gamma - \ln 4\pi - 2 + \ln\frac{m^2}{\mu^2} \\ &+ 2\left(\tfrac{1}{4} - m^2/s\right)^{1/2} \ln\left[\frac{\frac{1}{2} + \left(\frac{1}{4} - m^2/s\right)^{1/2}}{\frac{1}{2} - \left(\frac{1}{4} - m^2/s\right)^{1/2}}\right]\Bigg\} \end{aligned} \tag{4.52}$$

The inclusion of the crossed diagrams gives the final result

$$A_1 = \mathrm{i}(\lambda\mu^\varepsilon)\lambda\frac{3}{16\pi^2\varepsilon}\frac{1}{\varepsilon} + A_1(s) + A_1(t) + A_1(u) \tag{4.53}$$

Comments on analytic continuation to $n \neq 4$ dimensions

Before proceeding further a comment on the explored analytic continuation to $n \neq 4$ dimensions should be made. First, consider the space with one time-like dimension and $n-1 \neq 3$ space-like dimensions. The basic integral

$$I_n = \int \mathrm{d}^n k \frac{1}{(k^2+b)^2}$$

makes sense for $n = 1, 2, 3$ and can be written as

$$I_n = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k_0 \int_0^{\infty} \mathrm{d}r^2\, (r^2)^{(n-3)/2} \frac{1}{(k^2+b)^2} \int \mathrm{d}^{n-1}\Omega, \quad k^2 = k_0^2 - r^2 \tag{4.54}$$

where $\mathrm{d}^{n-1}\Omega$ is an element of the solid angle in $(n-1)$ dimensions

$$\begin{aligned}\Omega^{n-1} &= \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{\pi} \sin\theta_2\, \mathrm{d}\theta_2 \int_0^{\pi} \sin^2\theta_3\, \mathrm{d}\theta_3 \ldots \int_0^{\pi} \sin^{n-3}\theta_{n-2}\, \mathrm{d}\theta_{n-2} \\ &= \frac{2\pi^{(n-1)/2}}{\Gamma\big(\frac{1}{2}(n-1)\big)}\end{aligned}$$

Integral (4.54) can be used to define an analytic function I_n in the range $1 < n < 4$ which coincides with the original I_n for $n = 2, 3$ (the solid angle for a non-integer n is the analytic continuation of Ω^{n-1}, n integer). Furthermore, by partially integrating N times over $\mathrm{d}r^2$ one can find explicit expressions for the analytic continuation of I_n to $(1-2N) < n < 4$ dimensions. In addition we would like to be able to continue I_n to values $n > 4$. This can be achieved using the identity

$$I = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}k_0}k_0 + \frac{\mathrm{d}}{\mathrm{d}r}r\right)$$

and by integrating each term by parts over k_0 and r, respectively. For $1 < n < 4$ the surface terms are zero and one gets

$$\begin{aligned}I_n &\sim \int_{-\infty}^{\infty} \mathrm{d}k_0 \int_0^{\infty} \mathrm{d}r\, r^{n-2} \frac{(-2k_0^2 + 2r^2)}{(k^2+b)^3} \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k_0 \int_0^{\infty} \mathrm{d}r\, r^{n-2} \frac{(n-2)}{(k^2+b)^2} = I_n' + \tfrac{1}{2}(n-2)I_n\end{aligned} \tag{4.55}$$

so that

$$I_n = -\frac{1}{n-4} I_n'$$

where I_n' exists for $1 < n < 5$. The above procedure may be repeated to show that I_n is of the form

$$I_n = \Gamma\big(\tfrac{1}{2}(4-n)\big)\tilde{I}_n \tag{4.56}$$

where $\tilde{I}_n$ is well behaved for arbitrarily large n. So one has constructed an analytic function I_n with simple poles at $n = 4, 6, 8, \ldots$. Dimensional regularization assumes the physical theory to be given by I_n for $n \to 4$.

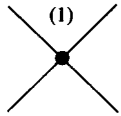


Fig. 4.11.

Lowest order renormalization

Coming back to the result (4.53) one can now renormalize the amplitude by including counterterms. It is clear from the structure of the counterterm that the divergences of (4.53) can be cancelled by the vertex counterterm in Fig. 4.11, i.e. by properly adjusting Z_1. The minimal subtraction (MS) prescription assumes that Z_i have only (multiple) pole terms in ε and the residua are fixed to cancel all the divergences for $\varepsilon \to 0$. With such a prescription one then gets to order λ

$$(Z_1 - 1)^{(1)} = \frac{1}{\varepsilon}\frac{3\lambda}{16\pi^2} \tag{4.57}$$

and the renormalized (finite) amplitude of Fig. 4.8(b) is given by $A_1(s) + A_1(t) + A_1(u)$ of (4.53). The so-called $\overline{\text{MS}}$ scheme in which

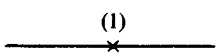
$$(Z_1 - 1)^{(1)} = \frac{1}{2}\frac{3\lambda}{16\pi^2}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right)$$

is also often used.

The other two diagrams of Fig. 4.8 require mass and wave-function renormalization. The amplitude of Fig. 4.8(a) reads

$$\begin{aligned}
&\tfrac{1}{2}(-\mathrm{i}\lambda)\mu^{\varepsilon}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon} \\
&\quad = \tfrac{1}{2}(-\mathrm{i}\lambda)\mu^{\varepsilon}\frac{1}{(4\pi)^2}(4\pi)^{\varepsilon/2}(m^2)^{1-\varepsilon/2}\Gamma\left(\tfrac{1}{2}\varepsilon - 1\right) \\
&\quad = \tfrac{1}{2}\mathrm{i}\frac{2\lambda m^2}{(4\pi)^2}\frac{1}{\varepsilon} - \frac{1}{2}\mathrm{i}\frac{\lambda m^2}{(4\pi)^2}\left(-1 + \gamma - \ln 4\pi + \ln\frac{m^2}{\mu^2}\right)
\end{aligned} \tag{4.58}$$

The divergence is cancelled by the mass counterterm with



$$\equiv (-\mathrm{i})(Z_0 - 1)^{(1)}m^2$$

$$(Z_0 - 1)^{(1)}\frac{\lambda}{16\pi^2} = \frac{1}{\varepsilon} \tag{4.59}$$

in the minimal subtraction renormalization scheme.

The amplitude of Fig. 4.8(c) requires both mass and wave-function renormalization. It is of order λ^2 and therefore in order-by-order renormalization one must include simultaneously the other diagrams of order λ^2 given in Fig. 4.12.

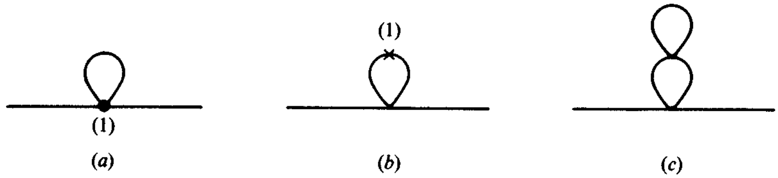
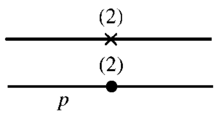


Fig. 4.12.

The divergences which remain after adding the amplitude of Fig. 4.8(c) and those of Fig. 4.12(a)–(c) should be then cancelled by new λ^2 terms of the mass and wave-function renormalization counterterms



$$\equiv \quad -\mathrm{i}(Z_0 - 1)^{(2)} m^2$$
$$\equiv \quad +\mathrm{i}(Z_3 - 1)^{(2)} p^2$$

(notice that in $\lambda\Phi^4$ theory $(Z_3 - 1)^{(1)} = 0$).

We tabulate first the results for Fig. 4.12(a)–(c). Using (4.57) and (4.59) it is easy to get the following (we write down explicitly only pole terms):

Diagram 4.12(a):

$$\mathrm{i}\lambda^2 \frac{m^2}{(16\pi^2)^2} \frac{3}{2} \left(\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon} \ln \frac{m^2}{4\pi\mu^2} + \frac{1}{\varepsilon} - \frac{\gamma}{\varepsilon} \right) + O(\varepsilon^0) \tag{4.60a}$$

Diagram 4.12(b):

$$\mathrm{i}\lambda^2 \frac{m^2}{(16\pi^2)^2} \frac{1}{2} \left(\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon} \ln \frac{m^2}{4\pi\mu^2} - \frac{\gamma}{\varepsilon} \right) + O(\varepsilon^0) \tag{4.60b}$$

Diagram 4.12(c):

$$(-\mathrm{i})\lambda^2 \frac{m^2}{(16\pi^2)^2} \frac{1}{2} \left(\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} \ln \frac{m^2}{4\pi\mu^2} + \frac{1}{\varepsilon} - \frac{2\gamma}{\varepsilon} \right) + O(\varepsilon^0) \tag{4.60c}$$

The only non-trivial contribution is due to Fig. 4.8(c). The amplitude reads

$$I = \tfrac{1}{6}\lambda^2 \mu^{2\varepsilon} \mathrm{i} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{d}^n l}{(2\pi)^n} \frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} \frac{1}{l^2 - m^2 + \mathrm{i}\varepsilon} \frac{1}{(p + l - k)^2 - m^2 + \mathrm{i}\varepsilon} \tag{4.61}$$

which is divergent for (i) l fixed, k large, (ii) k fixed, l large, (iii) both k and l large. This is what is called an overlapping divergence because several overlapping loop subintegrations lead to a divergent result. The calculation of amplitude (4.61) is lengthy (Collins 1974). The structure of pole terms turns out to be the following

$$I = \mathrm{i}\frac{\lambda^2}{(16\pi^2)^2} \left\{ -\frac{m^2}{\varepsilon^2} + \frac{1}{\varepsilon^2} \left[m^2 \ln \frac{m^2}{4\pi\mu^2} + \frac{1}{12} p^2 + (\gamma - \tfrac{3}{2}) m^2 \right] \right\}$$

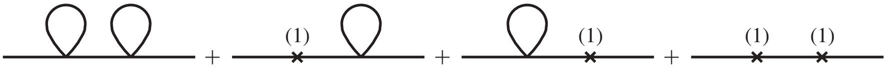


Fig. 4.13.

Adding (4.60) one gets

$$\mathrm{i}\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{1}{\varepsilon^2}2m^2+\frac{1}{\varepsilon}\frac{1}{12}p^2-\frac{1}{\varepsilon}\frac{1}{2}m^2\right)$$

so that

$$\left.\begin{aligned}(Z_0-1)^{(2)} &= \frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\varepsilon^2}-\frac{1}{2}\frac{1}{\varepsilon}\right)\\ (Z_3-1)^{(2)} &= -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\varepsilon}\end{aligned}\right\}\tag{4.62}$$

Combining (4.59) with (4.62) one gets the following results for Z_0 and Z_3 (to order λ^2)

$$Z_0 = 1+\frac{\lambda}{16\pi^2}\frac{1}{\varepsilon}+\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\varepsilon^2}-\frac{1}{2}\frac{1}{\varepsilon}\right)\tag{4.63}$$

$$Z_3 = 1-\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\varepsilon}\tag{4.64}$$

In the literature one can also often find the constants Z_m and Z_λ defined by the relations

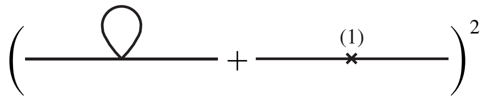
$$\left.\begin{aligned}m_{\mathrm{B}}^2 &= Z_m m_{\mathrm{R}}^2, \quad \text{i.e.} \quad Z_m = Z_0/Z_3\\ \lambda_{\mathrm{B}} &= Z_\lambda\lambda_{\mathrm{R}}, \quad\ \ \text{i.e.} \quad Z_\lambda = Z_1/Z_3^2\end{aligned}\right\}\tag{4.65}$$

We see in particular that to order λ^2

$$Z_m = 1+\frac{1}{\varepsilon}\left[\frac{\lambda}{16\pi^2}-\frac{5}{12}\frac{\lambda^2}{(16\pi^2)^2}\right]+\frac{1}{\varepsilon^2}\frac{2\lambda^2}{(16\pi^2)^2}\tag{4.66}$$

Notice that there are logarithms of mass in the residua of the poles in each diagram. However, they cancel in the final result for constants Z_i and we see explicitly that at least in that order of perturbation theory the minimal subtraction scheme is indeed a mass-independent scheme.

We end this section with two comments. Firstly, it should be remembered that one-particle reducible diagrams do not require new counterterms. Let us illustrate that fact with one example. Consider the diagrams of Fig. 4.13. We observe that the following relation holds.

Diagrams of Fig. 4.13 = 

which illustrates our statement.

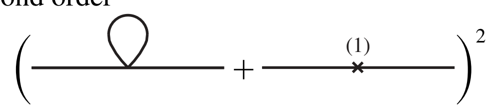
The second remark is devoted to the fact that the finite parts of Feynman integrals are not, in general, obtained by simply discarding poles and multiple poles at $n = 4$ but have to be worked out by order-by-order renormalization procedure.

Consider, for example, mass loop diagrams

 $\equiv (1/\varepsilon) f_1 + f_2 + \varepsilon f_3$

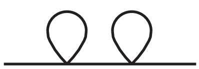
$\equiv -(1/\varepsilon) f_1$

so that in the second order



$\equiv f_2^2$

However, for the two-loop diagram alone we have

 $\equiv (1/\varepsilon^2) f_1^2 + f_2^2 + \varepsilon^2 f_3^2 + 2(1/\varepsilon) f_1 f_2 + 2 f_1 f_3 + 2 f_2 f_3 \varepsilon$

Discarding poles in the last result one obtains

$$f_2^2 + 2 f_1 f_3 \neq f_2^2$$

which is wrong because we insist on unitarity being satisfied order-by-order in perturbation theory and therefore the finite part of the two-loop diagram should read f_2^2 as can be seen by considering the imaginary part of the amplitude, see (3.70).

4.4 Effective field theories

The basic tool for calculations in quantum field theory is perturbation theory. In perturbative calculations we face the problem of divergent results, typical for local field theories. In fact, this problem is also present in classical field theories. The predictive power of a theory is retained due to the procedure of renormalization: divergent contributions are absorbed into free parameters of the theory, which are taken from experiment. Thus, we are still able to predict results of measurements in terms of other measurable quantities.

The number of free parameters is related to the number of so-called counterterms in the lagrangian, which are required for the procedure of renormalization, i.e. to absorb all infinities. It is very important to realize that this number may be finite or infinite! Traditionally, a theory is called renormalizable only if a finite number of counterterms is sufficient to renormalize it. This certainly makes such a theory very predictive: all results are obtained in terms of a finite (usually small) number of measurable parameters. For a theory to be renormalizable in this traditional sense, the corresponding lagrangian may contain only operators of dimension four, or less, i.e. it cannot contain couplings with the dimension of negative powers of mass. We know that such theories are often very good approximations to the real world in some energy range and therefore very interesting to study. However, we also know now that they should be regarded as only effective theories, valid in a limited energy range.

Effective field theories appear naturally in the physical situation characterized by the presence of two hierarchical mass scales $M \gg m$. Suppose that physics at scale $M \gg m$ is described by a renormalizable theory. At the energy scale m, after integrating out heavy degrees of freedom (see Chapters 7 and 12), one obtains an effective theory whose lagrangian may consist of operators of dimension four or less (the 'renormalizable' part) and higher dimension operators suppressed by inverse powers of large mass M. If we neglect all physical effects of those corrections, then physics at m is described by the 'renormalizable' part. A well-known example is, for instance, quantum electrodynamics (QED) considered as the effective low energy part of the electroweak theory (Chapter 12). It is the content of the Appelquist–Carrazone decoupling theorem (Appelquist & Carrazone 1976) that all effects depending on positive powers of M or logarithmic in M can be absorbed into the redefinition of the bare parameters of the effective renormalizable theory.

As we shall see, we may, however, be interested in the $\mathcal{O}(m/M)$ corrections, too. We encounter the latter case, for example, if we want to calculate QED and QCD corrections to low energy, $E \sim 1$ GeV, weak processes (Chapter 12). These are described by the weak effective lagrangian with dimensionful coupling. Since $E/M_{\mathrm{W}} \ll 1$, it is sufficient to use it only in the Born approximation, with no need for higher order perturbation in it. In this case, the virtue of the effective theory is not only that it offers a quickly truncated expansion in powers of E/M_{W} but, first of all, that it provides a tool for easy resummation of large logarithmic corrections $\mathcal{O}(\alpha^n \ln^n(E/M_{\mathrm{W}}))$ (Chapter 12) to weak processes.

We have described here a situation in which the complete renormalizable theory at the scale M is known and the effective theory can be derived from it by perturbative methods. A different situation occurs when we try to describe strong interactions of hadrons at low momentum transfer. Perturbative QCD is not applicable in that region but we may construct, for instance, an effective pion nucleon theory which respects the chiral symmetry of QCD but is not derivable perturbatively from it. Moreover, the physical cut-off Λ for that theory is of similar order of magnitude as the maximal energy of its applicability so that the E/Λ expansion cannot be so quickly truncated and higher order iteration of the effective lagrangian is necessary. Therefore, it must be stressed that there is no fundamental difference between a theory which is renormalizable with a finite or infinite number of counterterms (but only those consistent with the symmetry of the bare action). The latter still has a lot of predictive power (based on such principles as analyticity, unitarity, symmetries), although the number of free parameters increases with the order of perturbation theory.

Finally, we may consider the physical situation when we know about (or expect) the existence of a large physical scale (for example, Planck scale) but we do not know the correct theory at that scale. It is then a very natural hypothesis that physics at lower scales is described by an effective theory which contains all higher dimension operators consistent with symmetries of the low energy theory and suppressed by powers of the large scale.

It is of fundamental importance that Nature can be described by field theories that obey the decoupling theorem. This makes it possible to understand fundamental laws of physics step by step, without the (unrealistic) quest for the theory of everything. On the other hand, it makes it difficult to learn about new physics at higher scales, as such effects are always strongly suppressed.

It should be emphasized that, although a renormalizable theory can formally be used for calculations at any energy scale (since the dependence on larger scales is totally absent), this does not mean that it is correct at any scale. A good example is again QED. At the scale M_Z it must be embedded into the electroweak theory. However, we cannot learn about that by studying electromagnetic processes at low energy, $E \ll M_Z$, unless we have reached the precision of sensitivity to $\mathcal{O}(E/M_Z)$ corrections.

The above comments on effective field theories anticipate a lot of knowledge which the reader is going to acquire in the following chapters. However, they may be a useful guide to further reading.

Problems

4.1 Discuss a model with two scalar fields ϕ and Φ, with the masses m and M, respectively, such that $m \ll M$.

- (a) Construct the most general renormalizable lagrangian invariant under the symmetry $\phi \to -\phi, \phi \to -\Phi$.
- (b) Study spontaneous breaking of this discrete symmetry at the classical level.
- (c) Assuming that the discrete symmetry is spontaneously broken calculate the vacuum expectation value v at one-loop level; calculate the one-loop effective potential by integrating the tadpole amplitudes. Hint: expand $\tilde{\phi} = \phi - v$ where

$\langle\phi\rangle = v$ is to be found from the minimum of the effective potential; rewrite the lagrangian in terms of the $\tilde{\phi}$; the presence of the $\tilde{\phi}^3$ coupling generates tree level (———×) and one-loop (———○) tadpole diagrams; the vacuum expectation value v is fixed by the equation (———× + ———○ = 0); calculate the one-loop contribution ΔV_1 to the effective potential by integrating the equation $\Delta V_1/\mathrm{d}v = \mathrm{i}$ ———○ and identifying $\phi \equiv v$.

(*d*) Single out from the effective potential the corrections to the mass m and to the self-coupling of the field ϕ which depend on the heavy field Φ and compare them with the result of the direct diagrammatical calculation.

(*e*) Formulate the effective theory at the scale $\mu \ll M$; pay attention to the structure of quantum corrections from the exchange of the field Φ.

4.2 Discuss a model with a scalar field ϕ and a fermion field ψ, with the masses M and m, respectively, such that $m < M$.

(*a*) Construct the most general renormalizable lagrangian.

(*b*) Calculate one-loop self-energy diagrams for the field ψ and compare the corrections to its mass with the analogous corrections to the mass of the field in Problem 4.1.

4.3 It is convenient to perform multi-loop calculations in Euclidean space. Instead of performing Wick's rotation in the Feynman integrals it is more convenient to begin with the theory formulated in the Euclidean space. For the Φ^4 theory we then have the generating functional

$$W[J] = \int [\mathrm{d}\Phi] \exp\left[-S_E - \int \mathrm{d}_E^4 x\, J(x)\Phi(x)\right] \tag{4.67}$$

where

$$S_E = \int \mathrm{d}_E^4 x \left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{24}\Phi^4\right] \tag{4.68}$$

Check that the n-point Green's functions in the Minkowski space are obtained from those in the Euclidean space by analytic continuation of their momentum variables: $\hat{p}_i^4 \to -\mathrm{i}p_i^0$, where $\hat{p}^4$ is the 'time' component of the momentum four-vector in the Euclidean space.

4.4 Calculate the superficial degree of divergence in n dimensions of a Feynman diagram with I_f (E_f) internal (external) lines of type f and N_i vertices of type i, with d_i derivatives. Show that $D = \sum_f I_f(2s_f - 2 + n) + \sum_i N_i(d_i - n) + n$, where $s_f = 0$ ($1/2$) for spin 0 (1/2) particles. Using $2I_F + E_f = \sum_i n_{if} N_i$, where n_{if} is the number of lines of type f in the vertex of type i, one gets $D = n - \sum_f E_f(s_f - 1 + n/2) - \sum_i N_i \Delta_i$ with $\Delta_i = n - d_i - \sum_f n_{if}(s_f - 1 - n/2)$. Only the last term in the equation for D depends on the internal structure of diagrams. If all $\Delta_i \geq 0$ then $D \leq D_{\max}$ and the theory is renormalizable with a finite number of counterterms.

5
Quantum electrodynamics

We begin our discussion of gauge theories with the simplest and the best-known one: QED. So far most of the physical applications of gauge theories rely on perturbation theory, crucial for which is the renormalization programme described in Chapter 4 which is usually formulated in terms of the Green's functions. As we know from Chapter 3, to derive the Feynman rules for a gauge theory it is necessary to fix the gauge by adding some gauge-fixing conditions. All physical quantities, and the S-matrix elements in particular, are gauge-independent. This fact, which is necessary for a consistent theory, has to be checked in each case. The Green's functions are not, however, physical objects and they are gauge-dependent. Nevertheless, the gauge symmetry of the theory leads to certain constraints on the Green's functions: there exist relations between them known as the Ward–Takahashi identities. In terms of the Green's functions these are exactly the conditions which protect the physical equivalence of different gauges.

The QED lagrangian is based on the minimal coupling ansatz and in terms of the bare fields and parameters it reads

$$\mathcal{L} = -\tfrac{1}{4} F^{\mathrm{B}}_{\mu\nu} + F^{\mu\nu}_{\mathrm{B}} + \bar{\Psi}_{\mathrm{B}}(\mathrm{i}\not{\partial} - q e_{\mathrm{B}} \not{A}_{\mathrm{B}})\Psi_{\mathrm{B}} - m_{\mathrm{B}} \bar{\Psi}_{\mathrm{B}} \Psi_{\mathrm{B}} + \mathcal{L}_{\mathrm{G}} \tag{5.1}$$

where $e_{\mathrm{B}} > 0$ and $\mathcal{L}_{\mathrm{G}}$ is a gauge-fixing term. We will work in the class of covariant gauges, so that

$$\mathcal{L}_{\mathrm{G}} = -\frac{1}{2a_{\mathrm{B}}} (\partial_\mu A^\mu_{\mathrm{B}})^2$$

To begin, we consider the bare, regularized (finite) Green's functions. It is important that there exist regularization procedures which preserve the gauge symmetry of the theory. The 't Hooft–Veltman dimensional regularization defined for perturbative expansion (we can think, for instance, about perturbative expansion for the bare Green's function) belongs to this category. Thus, the results of our formal manipulations should always be understood as valid order-by-order in perturbation theory.

From the lagrangian (5.1) we can derive the Ward–Takahashi identities for the bare, regularized Green's functions of our theory. Using these identities one can then prove, allowing all parameters and fields to become renormalized, that the theory is renormalizable without destroying its gauge invariance. Leaving aside these formal considerations (consult, for example, Bogoliubov & Shirkov (1959), Slavnov & Faddeev (1980), Collins (1984)) we simply rewrite the lagrangian (5.1) in terms of the renormalized quantities assuming the most general form consistent with gauge invariance after renormalization and we derive Ward identities for the renormalized Green's functions (actually, using these Ward identities one can also prove *a posteriori* that the renormalized Green's functions are finite).

The gauge-invariant structure of the QED lagrangian is not destroyed when each gauge-invariant piece of it is multiplied by a constant. In addition the parameter e can be arbitrarily changed. Thus the most general gauge-invariant form for the lagrangian (5.1) written in terms of the renormalized quantities reads†

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\Psi}[\mathrm{i}\not{\partial} - qe(Z_1/Z_2)\not{A}]\Psi - Z_0 m \bar{\Psi}\Psi - (1/2a)(\partial_\mu A^\mu)^2 \quad (5.2)$$

(as we shall see shortly, there is no need for a gauge-fixing counterterm). The relations between bare and renormalized quantities are as follows

$$\begin{aligned} A_\mu^{\mathrm{B}} &= Z_3^{1/2} A_\mu, \quad \Psi_{\mathrm{B}} = Z_2^{1/2}\Psi, \quad e_{\mathrm{B}} = (Z_1/Z_2 Z_3^{1/2})e \\ m_{\mathrm{B}} &= (Z_0/Z_2)m = (m - \delta m)/Z_2, \quad a_{\mathrm{B}} = Z_3 a \end{aligned}$$

Apart from the gauge-fixing term, lagrangian (5.2) is invariant under the gauge transformations (1.91) and (1.94)

$$\left.\begin{aligned} \Psi'(x) &= \exp[-\mathrm{i}q\Theta(x)]\Psi(x) \\ A'_\mu(x) &= A_\mu(x) + \frac{1}{e}(Z_2/Z_1)\partial_\mu\Theta(x) \end{aligned}\right\} \quad (5.3)$$

If a free scalar ghost field is included the full lagrangian is invariant under BRS symmetry transformation (see Problem 3.3) on renormalized fields; the ghost field, being free, does not undergo renormalization. The renormalization constants Z_i must be fixed by imposing certain renormalization conditions. Our discussion in Chapter 4 of the renormalization programme applies to the present case. In addition it is clear that Z_1/Z_2 must be finite as all other quantities in (5.3) are finite (make this argument precise by considering BRS invariance and taking into account that the ghost field is not renormalized; compare with QCD, Section 8.1). Therefore $Z_1 = Z_2$ up to finite terms. This result will be derived formally using the Ward–Takahashi identities. Actually, in most popular renormalization schemes $Z_1 = Z_2$ including finite terms.

† Strictly speaking one should include the scalar ghost field and refer to the BRS invariance (see Problem 3.3).

Mainly for historical reasons, the so-called on-shell renormalization scheme is the one most commonly used in QED calculations. In this scheme we choose to interpret the coupling constant e, the mass parameter m and the renormalized fields which appear in the lagrangian (5.2) as the electric charge measured, for instance, in the Thomson scattering (see (12.86)), the physical mass of the fermion and the 'physical' fields in the sense of Chapter 4, respectively. The renormalization conditions are then imposed so as to ensure this interpretation of the renormalized quantities in any order of perturbation theory. Generally speaking, this can be achieved by demanding that in the on-shell limit, when all the four-momenta approach their on-shell values, in any order of perturbation expansion, the fermion and the photon propagator as well as the fermion–fermion–photon vertex function approach their zeroth order form. However, an infinite number of other renormalization prescriptions is possible. The renormalized quantities lose their previous interpretation but the theory can be formulated in terms of them in a totally equivalent way. In addition, in each order of perturbation theory there exist definite relations between sets of parameters corresponding to different renormalization schemes.

5.1 Ward–Takahashi identities

General derivation by the functional technique

We derive the Ward–Takahashi identities in QED using the functional technique. The generating functional for the full Green's functions is the following

$$W[J, \alpha, \bar{\alpha}] = \int \mathcal{D}(A_\mu \bar{\Psi}\Psi) \exp(\mathrm{i}S_{\mathrm{eff}}) \tag{5.4}$$

where

$$S_{\mathrm{eff}} = \int \mathrm{d}^4x \, [\mathcal{L}(A_\mu, \Psi) + J_\mu A^\mu + \bar{\alpha}\Psi + \bar{\Psi}\alpha] \tag{5.5}$$

and the lagrangian $\mathcal{L}(A_\mu, \Psi)$ is given by (5.2). Under the functional integral one can perform an infinitesimal gauge transformation

$$\left.\begin{aligned} \Psi' &= [1 - \mathrm{i}q\Theta]\Psi \\ A'_\mu &= A_\mu + \frac{1}{\bar{e}}\partial_\mu\Theta \end{aligned}\right\} \tag{5.6}$$

where $\bar{e} = eZ_1/Z_2$. The integration measure is invariant under infinitesimal gauge transformation and the only gauge-dependent terms in the integrand are source terms and the gauge-fixing term. On the other hand, a change of the integration variables in (5.4) cannot change the value of the integral. Therefore we conclude that

$$\delta W/\delta\Theta(y)|_{\theta=0} = 0 \tag{5.7}$$

where

$$\delta W[J,\alpha,\bar{\alpha},\Theta] = \mathrm{i}\int \mathrm{d}^4x \int \mathcal{D}(A_\mu\bar{\Psi}\Psi)\exp(\mathrm{i}S_{\mathrm{eff}})$$
$$\times\left[\frac{1}{\bar{e}}J_\mu(x)\partial^\mu\Theta(x) - \mathrm{i}q\Theta(x)\bar{\alpha}(x)\Psi(x) + \mathrm{i}q\Theta(x)\bar{\Psi}(x)\alpha(x)\right.$$
$$\left. - \frac{1}{\bar{e}}(1/a)\partial_\mu^{(x)}A^\mu(x)\partial^2\Theta(x)\right] \tag{5.8}$$

Integrating expression (5.8) by parts, neglecting the surface terms at infinity and remembering that functional integrals of derivatives of fields are derivatives of the Green's functions (see Section 2.3) we get

$$\frac{1}{a}\partial^2\partial_\mu\frac{1}{\mathrm{i}}\frac{\delta W}{\delta J_\mu(y)} + \partial_\mu J^\mu(y)W + q\bar{e}\bar{\alpha}(y)\frac{\delta W}{\delta\bar{\alpha}(y)} + q\bar{e}\frac{\delta W}{\delta\alpha(y)}\alpha(y) = 0 \tag{5.9}$$

(remember that $(\vec{\delta}/\delta\alpha)W = -W(\overleftarrow{\delta}/\delta\alpha)$ or in terms of the generating functional $Z[J,\alpha,\bar{\alpha}]$ for the connected Green's functions ($W = \exp(\mathrm{i}Z)$)

$$\frac{1}{a}\partial^2\partial_\mu\frac{\delta Z}{\delta J_\mu(y)} + \partial_\mu J^\mu(y) + \mathrm{i}q\bar{e}\bar{\alpha}(y)\frac{\delta Z}{\delta\bar{\alpha}(y)} + \mathrm{i}q\bar{e}\frac{\delta Z}{\delta\alpha(y)}\alpha(y) = 0 \tag{5.10}$$

Eq. (5.10) is the general form of the Ward–Takahashi identities in QED, in a class of covariant gauges, for the connected Green's functions. It is also very useful to have the analogous relation for the 1PI Green's functions. The generating functional for the latter has been defined in Section 2.6. In the present case its arguments are the 'classical' boson field $A_\mu(x)$ and the 'classical' fermion fields $\Psi(x)$ and $\bar{\Psi}(x)$ (it is convenient to use the same notation for the arguments of Γ as for the fields in the lagrangian; see Section 2.6) so that

$$\Gamma[A_\mu(x),\Psi(x),\bar{\Psi}(x)]$$
$$= Z[J,\alpha\bar{\alpha}] - \int \mathrm{d}^4x\,[J_\mu(x)A^\mu(x) + \bar{\alpha}(x)\Psi(x) + \bar{\Psi}(x)\alpha(x)] \tag{5.11}$$

where (left derivatives only)

$$\left.\begin{array}{lll} \dfrac{\delta Z}{\delta J_\mu} = A_\mu, & \Psi = \dfrac{\delta Z}{\delta\bar{\alpha}}, & \bar{\Psi} = -\dfrac{\delta Z}{\delta\alpha} \\[2ex] J_\mu = -\dfrac{\delta\Gamma}{\delta A^\mu}, & \bar{\alpha} = \dfrac{\delta\Gamma}{\delta\Psi}, & \alpha = \dfrac{\delta\Gamma}{\delta\bar{\Psi}} \end{array}\right\} \tag{5.12}$$

In terms of the functional $\Gamma[A_\mu,\Psi,\bar{\Psi}]$ the identity (5.10) can be rewritten as follows:

$$\frac{1}{a}\partial^2_{(y)}\partial_\mu^{(y)}A^\mu(y) - \partial_\mu^{(y)}\frac{\delta\Gamma}{\delta A_\mu(y)} + \mathrm{i}q\bar{e}\frac{\delta\Gamma}{\delta\Psi(y)}\Psi(y) + \mathrm{i}q\bar{e}\bar{\Psi}(y)\frac{\delta\Gamma}{\delta\bar{\Psi}(y)} = 0 \tag{5.13}$$

In the following we shall illustrate the general relations (5.10) and (5.13) with several examples.

Examples

As the first example we consider the Ward identity for the longitudinal part of the connected photon propagator. Differentiating relation (5.10) with respect to $J_\nu(z)$ and putting $J = \alpha = \bar{\alpha} = 0$ one gets

$$-\frac{1}{a}\partial^2_{(y)}\partial^\mu_{(y)}\frac{\delta^2 Z}{\delta J^\mu(y)\delta J_\nu(z)}\bigg|_{J=0} = \partial^\nu_{(y)}\delta(y-z) \tag{5.14}$$

Using (2.134) and Fourier transforming into momentum space we obtain the following relation:

$$\mathrm{i}(1/a)k^2 k^\mu \tilde{G}_{\mu\nu}(k) = k_\nu \tag{5.15}$$

Here $\tilde{G}_{\mu\nu}(k)$ is the Fourier transform of the connected part of the photon propagator $G_{\mu\nu}(x_1 - x_2)$. The solution to (5.15) reads

$$\tilde{G}_{\mu\nu}(k) = -\mathrm{i}a(k_\mu k_\nu/k^4) + \tilde{G}^{\mathrm{T}}_{\mu\nu}(k) \tag{5.16}$$

where the transverse projection

$$\tilde{G}^{\mathrm{T}}_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu/k^2)f(k^2) \tag{5.17}$$

gives zero contribution to the relation (5.15). The Ward identity (5.15) constrains the full renormalized propagator to the form (5.16). We observe that the longitudinal (gauge-dependent) part is not altered by interactions (compare with (3.32)). This means, among other things, that the renormalization programme does not require any counterterms for the gauge-fixing term.

Our next example is the well-known relation between the vertex and the fermion propagators. Differentiating the general identity (5.10) with respect to

$$\frac{\delta}{\delta\bar{\alpha}(x)}\frac{\delta}{\delta\alpha(z)} = -\frac{\delta}{\delta\alpha(z)}\frac{\delta}{\delta\bar{\alpha}(z)}$$

and setting $J = \bar{\alpha} = \alpha = 0$, one gets

$$\begin{aligned}&\frac{1}{a}\partial^2_{(y)}\partial^{(y)}_\mu\frac{\delta^2}{\delta\bar{\alpha}(x)\delta\alpha(z)}\frac{\delta}{\delta J_\mu(Y)}Z\\&\qquad = \mathrm{i}q\bar{e}\frac{\delta^2 Z}{\delta\alpha(z)\delta\bar{\alpha}(y)}\delta(y-x) + \mathrm{i}q\bar{e}\frac{\delta^2}{\delta\bar{\alpha}(x)\delta\alpha(y)}\delta(y-x)\end{aligned} \tag{5.18}$$

Since

$$\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle = \mathrm{i}\frac{\delta^2 Z}{\delta\bar{\alpha}(x)\delta\alpha(y)}\bigg|_{J=\alpha=\bar{\alpha}=0}$$

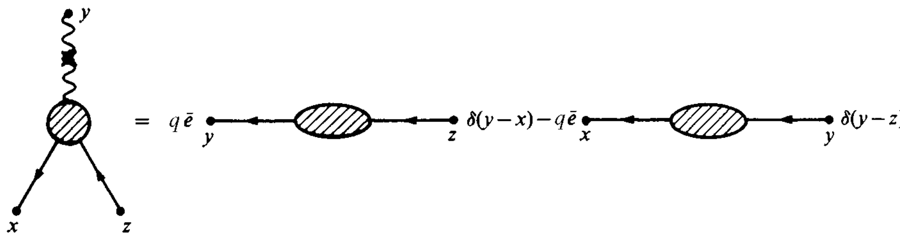


Fig. 5.1.

and

$$\langle 0|TA_\mu(y)\Psi(x)\bar{\Psi}(z)|0\rangle = \left.\frac{\delta^3 Z}{\delta J_\mu(y)\delta\bar{\alpha}(x)\delta\alpha(z)}\right|_{J=\alpha=\bar{\alpha}=0}$$

relation (5.18) can be rewritten as follows

$$\begin{aligned}-(1/a)&\partial^2_{(y)}\partial^\mu_{(y)}\langle 0|TA_\mu(y)\Psi(x)\bar{\Psi}(z)|0\rangle\\ &= q\bar{e}\langle 0|T\Psi(y)\bar{\Psi}(z)|0\rangle\delta(y-z) - q\bar{e}\langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle\delta(y-z)\end{aligned}\tag{5.19}$$

This is the Ward identity in configuration space. Using translational invariance and Fourier transforming into momentum space

$$\left.\begin{aligned}\langle 0|TA_\mu(y)\Psi(x)\bar{\Psi}(z)|0\rangle &= \langle 0|TA_\mu(y-z)\Psi(x-z)\bar{\Psi}(0)|0\rangle\\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\exp[-\mathrm{i}p(x-z)]\\ &\quad\times\exp[-\mathrm{i}q(y-z)]V_\mu(p,q)\\ \langle 0|T\Psi(y-z)\bar{\Psi}(0)|0\rangle &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(y-z)]\mathrm{i}S(k)\\ \langle 0|T\Psi(x-y)\bar{\Psi}(0)|0\rangle &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp[-\mathrm{i}k(x-y)]\mathrm{i}S(k)\end{aligned}\right\}\tag{5.20}$$

then inserting (5.20) into (5.19) and integrating over $\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\exp(\mathrm{i}p'x)\times\exp(\mathrm{i}q'y)\exp(\mathrm{i}k'z)$ to get rid of the δ-functions, we obtain the Ward identity in momentum space

$$-(1/a)q^2q^\mu V_\mu(p,q) = q\bar{e}S(p+q) - q\bar{e}S(p)\tag{5.21}$$

Graphically, relation (5.19) can be represented by Fig. 5.1 and relation (5.21) in momentum space by Fig. 5.2, where the cross denotes $-(1/a)\partial^2\partial^\mu$ or $-(1/a)q^2q^\mu$, respectively. Notice that the form (5.21) in momentum space can be immediately deduced from the graphical representation of relation (5.19) in configuration space.

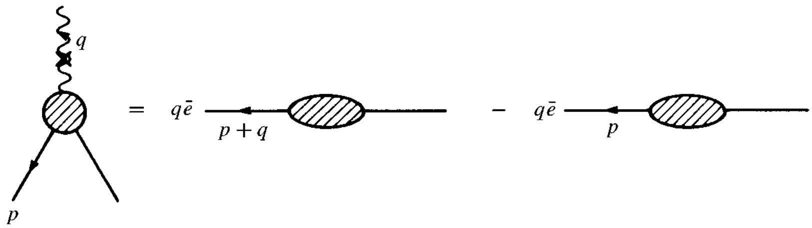


Fig. 5.2.

The Ward identity (5.21) can also be written in terms of the 1PI Green's functions. We take the derivative of (5.13) with respect to

$$\frac{\delta}{\delta\bar{\Psi}(x)}\frac{\delta}{\delta\Psi(z)}$$

set $A_\mu = \bar{\Psi} = \Psi = 0$ and use the definitions (2.135) generalized to include fermions to get the following:

$$\partial^\mu_{(x)}\Gamma^{(3)}_\mu(y, x, z) - \mathrm{i}q\bar{e}\Gamma^{(2)}(x, y)\delta(y - z) + \mathrm{i}q\bar{e}\Gamma^{(2)}(y, z)\delta(y - x) = 0 \quad (5.22)$$

By analogy with (5.21) and (5.19), in momentum space relation (5.22) reads

$$q_\mu\tilde{\Gamma}^{(3)}(p, q) = -\mathrm{i}q\bar{e}\tilde{\Gamma}^{(2)}(p + q) + \mathrm{i}q\bar{e}\tilde{\Gamma}^{(2)}(p) \quad (5.23)$$

where the $\tilde{\Gamma}^{(n)}$s are Fourier transforms of the $\Gamma^{(n)}$s. We recall (2.152): $\tilde{\Gamma}^{(2)} = \mathrm{i}[\tilde{G}^{(2)}(p)]^{-1} = +S^{-1}(k)$ where, for example, for the free propagator, $S(k) = 1/(\not{k} - m)$.

As an interesting consequence of the Ward identities (5.21) or (5.23) we notice that the gauge parameter $\bar{e} = eZ_1/Z_2$ and consequently the ratio Z_1/Z_2 must be finite if the theory is renormalizable. Indeed all other quantities, including e, in these equations are renormalized quantities. In perturbative calculations, using a regularization procedure which preserves gauge invariance, the Z_is are power series in the coupling constant $\alpha = e^2/4\pi$: $(Z_i - 1) = \sum_{n=1}^{\infty} f_i^{(n)}\alpha^n$. The coefficients $f_i^{(n)}$ contain pieces which are divergent in the limit where regularization is absent (for example, when $\varepsilon \to 0$ in dimensional regularization) and possibly also finite pieces. Thus the finiteness of the ratio Z_1/Z_2 implies that order-by-order in perturbation theory $Z_1 = Z_2$ up to the finite terms. In practice, in all the commonly used renormalization schemes $Z_1 = Z_2$ including the finite terms. This is convenient (but not necessary) because the parameter e is then universal for all fermions in the theory. It is also worth remembering that only in the on-shell renormalization scheme is e the electric charge, i.e. $e^2/4\pi = 1/137$. This point will be discussed in more detail in Section 6.4. In addition, when $Z_1 = Z_2$ the counterterms form a gauge-invariant set and the Ward identities for the renormalized Green's functions are identical to those for the bare regularized Green's functions. The latter can, of course, be derived in a similar way to the renormalized Ward identities. The only difference is that one must start with the lagrangian expressed in terms of the bare quantities.

5.2 Lowest order QED radiative corrections by the dimensional regularization technique

General introduction

We recall the QED lagrangian ($q = -1$, $e > 0$) in a covariant gauge

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(\mathrm{i}\not{\partial} + e\not{A})\Psi - m\bar{\Psi}\Psi - (1/2a)(\partial_\mu A^\mu)^2 - (Z_3 - 1)\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ + (Z_2 - 1)\bar{\Psi}\mathrm{i}\not{\partial}\Psi - (Z_0 - 1)\bar{\Psi}m\Psi + (Z_1 - 1)e\bar{\Psi}\not{A}\Psi \qquad (5.24)\end{aligned}$$

(there is no counterterm to the gauge-fixing term: see Section 5.1). The Feynman rules are given in Appendix B.

Dimensional regularization in QED in the presence of γ-matrices requires an extension of previous rules. We notice that, for example, in the relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{1}$$

the only space-time-dependent object is the tensor $g_{\mu\nu}$. In n dimensions we can assume

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] = \tfrac{1}{2}\,\mathrm{Tr}[\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu] = g_{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} = 4g_{\mu\nu} \qquad (5.25)$$

i.e. we still consider γ_μs as 4×4 matrices in the fermion and antifermion spin space, and define

$$g^\mu{}_\mu = g^{\mu\nu}g_{\mu\nu} = n$$

This convention is the one most frequently used in calculations but other conventions are also possible (Collins 1984). In particular, in supersymmetric theories a dimensional regularization method called dimensional reduction (Siegel 1979) is used which preserves supersymmetry. This is not the case for the standard dimensional regularization convention used here. Several other relations and integrals useful in the subsequent calculations are collected in Appendix B. We shall calculate now all the elements which are necessary to discuss the lowest order radiative corrections to QED processes such as, for example, the electron scattering (or e^+e^- pair production) in a static Coulomb potential. An important exercise is to calculate the Thomson limit of the Compton scattering (Problem 5.2) in the on-shell and minimal subtraction renormalization schemes.

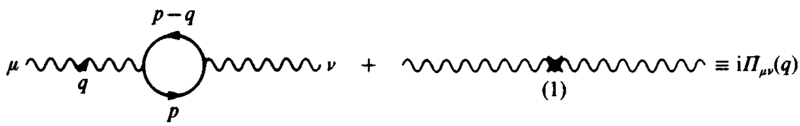


Fig. 5.3.

Vacuum polarization

We calculate the amplitude corresponding to the diagrams of Fig. 5.3. This is a one-loop correction to the free photon propagator. From consideration of Section 5.1 we know that only the transverse part of the photon propagator has higher order corrections. On general grounds one can write it as follows:

$$\begin{aligned} \mathrm{i}\Pi_{\mu\nu}(q) &= \mathrm{i}P^{\mu\nu}\Pi^{\mathrm{T}}(q^2) \\ &= \mathrm{i}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)q^2\Pi(q^2) \end{aligned} \tag{5.26}$$

In the one-loop approximation the transverse part of the photon propagator reads

$$\tilde{G}^{\mathrm{T}}_{\mu\nu}(q) = \frac{-\mathrm{i}P_{\mu\nu}}{q^2} + \frac{(-\mathrm{i}P_{\mu\rho})}{q^2}\mathrm{i}\Pi^{\rho\sigma}(q)\frac{(-\mathrm{i}P_{\sigma\nu})}{q^2} = \frac{-\mathrm{i}P_{\mu\nu}}{q^2}[1 + \Pi(q^2)]$$

Summing over all one-particle reducible diagrams one gets

$$\tilde{G}^{\mathrm{T}}_{\mu\nu}(q) = \frac{-\mathrm{i}P_{\mu\nu}}{q^2[1 - \Pi(q^2)]}$$

The scalar function $\Pi(q^2)$ is given by

$$g^{\mu\nu}\Pi_{\mu\nu}(q) = (n-1)q^2\Pi(q^2) \tag{5.27}$$

where from the Feynman rules we obtain

$$\begin{aligned} \mathrm{i}\Pi_{\mu\nu}(q) = -(\mathrm{i}e)^2 \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\mathrm{i}}{p^2 - m^2} \frac{\mathrm{i}}{(p-q)^2 - m^2} \mathrm{Tr}[\gamma_\mu(\not{p} + m) \\ \times \gamma_\nu(\not{p} - \not{q} + m)] + \mathrm{i}(Z_3 - 1)^{(1)}(q_\mu q_\nu - q^2 g_{\mu\nu}) \end{aligned} \tag{5.28}$$

Therefore, after taking the trace

$$\mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta] = 4(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta})$$

and using the Feynman parametrization $1/ab = \int_0^1 \mathrm{d}x\,(1/[xa + (1-x)b]^2)$ one gets

$$\begin{aligned} (n-1)q^2\Pi(q^2) = \mathrm{i}e^2 \int \frac{\mathrm{d}^n p}{(2\pi)^n} \int_0^1 \mathrm{d}x \frac{4(2-n)(p^2 - p\cdot q) + 4nm^2}{\{x(p^2 - m^2) + (1-x)[(p-q)^2 - m^2]\}^2} \\ + \text{counterterm} \end{aligned} \tag{5.29}$$

Using the integrals (B.18)–(B.20) we get finally

$$\begin{aligned}\Pi(q^2) &= -\frac{\alpha}{\pi}(4\pi)^{\varepsilon/2}\Gamma\left(\tfrac{1}{2}\varepsilon\right)\int_0^1 \mathrm{d}x\, 2x(1-x)\left[\frac{m^2 - x(1-x)q^2 - \mathrm{i}\varepsilon}{\mu^2}\right]^{-\varepsilon/2} \\ &= -\frac{\alpha}{\pi}\left[\frac{1}{\varepsilon}\frac{2}{3} + \frac{1}{3}(\ln 4\pi - \gamma) - \int_0^1 \mathrm{d}x\, 2x(1-x)\ln\frac{m^2 - x(1-x)q^2 - \mathrm{i}\varepsilon}{\mu^2}\right] \\ &\quad + O(\varepsilon) - (Z_3 - 1)^{(1)} \end{aligned} \tag{5.30}$$

(we have taken $n = 4-\varepsilon$, $\alpha = e^2/4\pi$, and α is dimensionless in $4-\varepsilon$ dimensions: $\alpha \to \alpha\mu^{\varepsilon}$, where μ is an arbitrary mass parameter). The pole in ε reflects the UV divergence in four dimensions. It is cancelled by adding the Z_3 counterterm. In the minimal subtraction scheme

$$(Z_3 - 1)^{(1)} = -\frac{\alpha}{\pi}\frac{2}{3}\frac{1}{\varepsilon} \tag{5.31}$$

and the finite part of the vacuum polarization, given by (5.30), can easily be evaluated in the small q^2 and large q^2 limits and reads (up to constant terms)†

$$\left.\begin{array}{ll} \dfrac{\alpha}{3\pi}\ln\dfrac{m^2}{\mu^2} - \dfrac{\alpha}{\pi}\dfrac{q^2}{m^2}\dfrac{1}{15} + O\left(\dfrac{q^4}{m^4}\right) & q^2 \to 0 \\ \dfrac{\alpha}{3\pi}\ln\dfrac{|q^2|}{\mu^2} & |q^2| \gg m^2 \end{array}\right\} \tag{5.32}$$

The renormalization scheme most frequently used in QED is the on-shell renormalization. In this case one demands:

$$\Pi^{\mathrm{ON}}(q^2) = 0 \quad \text{for} \quad q^2 = 0$$

Therefore, from (5.30)

$$(Z_3 - 1)^{(1)} = -\frac{\alpha}{\pi}\left[\frac{2}{3}\frac{1}{\varepsilon} + \tfrac{1}{3}(\ln 4\pi - \gamma) - \tfrac{1}{3}\ln\frac{m^2}{\mu^2}\right] \tag{5.33}$$

and

$$\Pi^{\mathrm{ON}}(q^2) = \frac{\alpha}{\pi}2\int_0^1 \mathrm{d}x\, x(1-x)\ln\frac{m^2 - x(1-x)q^2 - \mathrm{i}\varepsilon}{m^2} \tag{5.34}$$

† The so-called $\overline{\mathrm{MS}}$ scheme in which the counterterms include the constant terms: $2/\varepsilon \to \eta = 2/\varepsilon - \gamma + \ln 4\pi$ is also often used. It is useful to notice that in the $\overline{\mathrm{MS}}$, $\Pi(q^2 = 0) = 0$ for $\mu^2 = m^2$. Hence $\alpha^{\mathrm{ON}} = \alpha(m^2)$ (see also Chapters 6 and 12). In the large q^2 limit, in the $\overline{\mathrm{MS}}$ scheme the sum over all one-loop one particle reducible diagrams gives

$$\tilde{G}^{\mathrm{T}}_{\mu\nu}(q) = \frac{-\mathrm{i}P_{\mu\nu}}{q^2\left(1 - \frac{\alpha}{3\pi}\ln\frac{|q^2|}{\mu^2}\right)}$$

This is the so-called leading logarithmic approximation for the photon propagator.

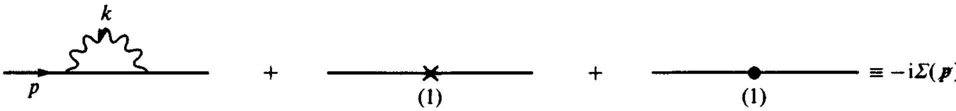


Fig. 5.4.

Asymptotically one obtains

$$\left.\begin{aligned} \Pi^{\mathrm{ON}}(q^2) &\underset{q^2\to 0}{\longrightarrow} -\frac{\alpha}{\pi}\frac{q^2}{m^2}\frac{1}{15} + O\left(\frac{q^4}{m^4}\right) \\ \Pi^{\mathrm{ON}}(q^2) &\underset{-q^2/m^2\to\infty}{\longrightarrow} \frac{\alpha}{\pi}\left[\tfrac{1}{3}\ln\frac{|q^2|}{m^2} - \frac{5}{9} + O\left(\frac{m^2}{q^2}\right)\right] \end{aligned}\right\} \tag{5.35}$$

As a final comment we rewrite $\Pi^{\mathrm{ON}}(q^2)$ in the form of a dispersion integral (the dispersion technique will be discussed in some detail later). Firstly, we change variables in the integral (5.34) into $y = 1 - 2x$. Taking into account the symmetry $y \to -y$ of the integrand we can write

$$\Pi^{\mathrm{ON}}(q^2) = \frac{\alpha}{\pi}\frac{1}{2}\int_0^1 \mathrm{d}y \left[\frac{\partial}{\partial y}(y - \tfrac{1}{3}y^3)\right] \ln\left[1 - \frac{q^2(1-y^2)}{4m^2 - \mathrm{i}\varepsilon}\right] \tag{5.36}$$

and integrate it by parts to get

$$\Pi^{\mathrm{ON}}(q^2) = -\frac{\alpha}{\pi}\frac{1}{2}\int_0^1 \mathrm{d}y\, 2y(y - \tfrac{1}{3}y^3)\frac{q^2}{4m^2 - q^2(1-y^2) - \mathrm{i}\varepsilon} \tag{5.37}$$

Another change of variable $t = 4m^2/(1-y^2)$ gives the result

$$\Pi^{\mathrm{ON}}(q^2) = -\frac{\alpha}{\pi}q^2\int_{4m^2}^{\infty}\frac{\mathrm{d}t}{t}\frac{1}{t - q^2 - \mathrm{i}\varepsilon}\frac{1}{3}\left(1 + \frac{2m^2}{t}\right)\left(1 - \frac{4m^2}{t}\right)^{1/2} \tag{5.38}$$

which is a so-called once subtracted dispersion relation.

Electron self-energy correction

We consider the electron self-energy truncated diagram and the necessary counterterms which are shown in Fig. 5.4. The proper two-point function $\Gamma^{(2)}(\not p)$ reads

$$\Gamma^{(2)}(\not p) = \not p - m - \Sigma(\not p, m) \tag{5.39}$$

The propagator is the inverse of the proper two-point function so we get

$$\hat{G}^{(2)}(\not p) = \frac{\mathrm{i}}{\not p - m - \Sigma} \tag{5.40}$$

The general structure of the electron self-energy correction is obviously the following

$$\Sigma(\not p, m) = \Sigma_V(p^2)\not p + \Sigma_m(p^2)I \tag{5.41}$$

It is often convenient to write it as an expansion

$$\Sigma(\not p, m) = \Sigma_0(\not p = m_{\mathrm{F}}, m) + (\not p - m_{\mathrm{F}})\Sigma_1(\not p = m_{\mathrm{F}}, m) + (\not p - m_{\mathrm{F}})^2\Sigma_2(\not p, m) \tag{5.42}$$

where m_{F}^2 is the zero of the inverse propagator $m_{\mathrm{F}}^2 = \{[m + \Sigma_m(m_{\mathrm{F}})]/[1 - \Sigma_V(m_{\mathrm{F}})]\}^2$ to the order considered. We recall that Σ_1 (actually $\frac{1}{2}\Sigma_1$: see Section 4.1) appears as a correction to an external line of the on-shell scattering amplitude. It can be calculated as ($p^2 = \not p^2$)

$$\Sigma_1 = \left.\frac{\partial\Sigma}{\partial\not p}\right|_{\not p = m_{\mathrm{F}}} = \left.\frac{\partial\Sigma_m}{\partial p^2}\right|_{p^2 = m_{\mathrm{F}}} 2m_{\mathrm{F}} + \Sigma_V(m_{\mathrm{F}}^2) + 2m_{\mathrm{F}}^2\left.\frac{\partial\Sigma_V(p^2)}{\partial p^2}\right|_{p^2 = m_{\mathrm{F}}^2} \tag{5.43}$$

In this section we shall calculate $\Sigma(\not p)$ in the Feynman gauge (unlike $\Pi_{\mu\nu}(q)$, the electron self-energy is gauge-dependent). We can then find the one-loop relation between the pole mass m_{F} and the renormalized mass parameter m as well as the correction Σ_1. The calculation proceeds as follows. The Feynman integral reads.

$$\begin{aligned}-\mathrm{i}\Sigma(\not p) = (\mathrm{i}e)^2 \int \frac{\mathrm{d}^n k}{(2\pi)^n}\gamma^\mu \frac{-\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}\frac{\mathrm{i}}{\not p + \not k - m + \mathrm{i}\varepsilon}\gamma^\nu \\ - \mathrm{i}m(Z_0 - 1)^{(1)} + \mathrm{i}(Z_2 - 1)^{(1)}\not p\end{aligned} \tag{5.44}$$

As before we use dimensional regularization: $n = 4 - \varepsilon$. The functions $\Sigma_m(p^2)$ and $\Sigma_V(p^2)$ are given by

$$\left.\begin{aligned}\mathrm{Tr}\,\Sigma(\not p) &= 4\Sigma_m(p^2) \\ \mathrm{Tr}[\not p\Sigma(\not p)] &= 4p^2\Sigma_V(p^2)\end{aligned}\right\} \tag{5.45}$$

Introducing

$$N = \gamma^\mu(\not p + \not k + m)\gamma^\nu g_{\mu\nu}$$

we find (after some algebra)

$$\left.\begin{aligned}\mathrm{Tr}\,N &= 4nm \\ \mathrm{Tr}[\not p N] &= 4[p^2(2 - n) + (p\cdot k)(2 - n)]\end{aligned}\right\} \tag{5.46}$$

Next, we use Feynman parametrization for the denominators

$$\frac{1}{k^2}\frac{1}{(p + k)^2 m^2} = \int_0^1 \mathrm{d}x \frac{1}{[k^2 + xp\cdot 2k + x(p^2 - m^2)]^2} \tag{5.47}$$

and (B.18)–(B.20) to perform the integration over momentum k to get the following results:

$$\left.\begin{aligned}\Sigma_m &= nmC\int_0^1 \mathrm{d}x\, X - (Z_0-1)^{(1)}m\\ \Sigma_V &= (2-n)C\int_0^1 \mathrm{d}x\,(1-x)X - (Z_2-1)^{(1)}\end{aligned}\right\}\tag{5.48}$$

where

$$C \equiv \frac{\alpha}{4\pi}\left(\frac{-1}{4\pi}\right)^{-\varepsilon/2}\Gamma\left(\tfrac{1}{2}\varepsilon\right)$$
$$X = \left[\frac{x(p^2-m^2)-x^2p^2}{\mu^2}\right]^{-\varepsilon/2}$$

Using standard tricks to single out the pole terms in ε we immediately find the renormalization constants in the minimal subtraction scheme

$$\left.\begin{aligned}(Z_0-1)^{(1)} &= \frac{2\alpha}{\pi}\frac{1}{\varepsilon}\\ (Z_0-1)^{(1)} &= -\frac{\alpha}{2\pi}\frac{1}{\varepsilon}\end{aligned}\right\}\tag{5.49}$$

The final (finite) expressions for Σ_m and Σ_V can then be easily written down, for example,

$$\Sigma_m(p^2) = -\frac{\alpha}{\pi}m\left[\int \mathrm{d}x\,\ln\frac{xm^2-x(1-x)p^2}{4\pi\mu^2}+\gamma+\tfrac{1}{2}\right]\tag{5.50}$$

and similarly for $\Sigma_V(p^2)$. We leave it to the reader to check that in the $\overline{\mathrm{MS}}$ scheme at the one-loop level

$$m_\mathrm{F} = m(\mu)\left(1+\frac{\alpha(\mu)}{\pi}\right)$$

We observe that Σ_m and Σ_V are IR finite (when $p^2 \to m^2$). However, their derivatives contributing to Σ_1 are not. Indeed, differentiating the regularized expression (5.48) with respect to p^2 one obtains the following result

$$\frac{\partial\Sigma_m}{\partial p^2} = nmC\left(-\tfrac{1}{2}\varepsilon\right)\int_0^1 \mathrm{d}x\frac{x(1-x)}{\mu^2}X^{1+2/\varepsilon}\tag{5.51}$$

which is, of course, UV finite (Z_i are constants) and in the limit $p^2 \to m^2$ gives (we can take $m_\mathrm{F}^2 = m^2$ because we work in the first order in α)

$$\left.\frac{\partial\Sigma_m}{\partial p^2}\right|_{p^2=m^2} = n\frac{1}{m}\frac{\alpha}{4\pi}\left(1-\gamma\frac{\varepsilon}{2}\right)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\frac{\Gamma(-\varepsilon)\Gamma(2)}{\Gamma(2-\varepsilon)}\tag{5.52}$$

The integration over Feynman parameters has introduced a new (IR) singularity which has again been regularized by keeping $n = 4 - \varepsilon$, but this time with $\varepsilon < 0$ (changing ε from a negative to a positive value is legitimate since we are already dealing with a UV finite quantity). In order to keep trace of the origin of the singular terms here and in the following we shall often distinguish between ε_{UV} ($n = 4 - \varepsilon_{\text{UV}}, \varepsilon_{\text{UV}} > 0$) and ε_{IR} ($n = 4 + \varepsilon_{\text{IR}}, \varepsilon_{\text{IR}} > 0$).

The final result reads ($\varepsilon_{\text{IR}} > 0$)

$$\left.\frac{\partial \Sigma_m}{\partial p^2}\right|_{p^2=m^2} = \frac{1}{m}\frac{\alpha}{4\pi}\left[4\left(\frac{1}{\varepsilon_{\text{IR}}} + \tfrac{1}{2}\gamma - 1 + \tfrac{1}{2}\ln\frac{m^2}{4\pi\mu^2}\right) + 1\right] \tag{5.53}$$

and for $\partial\Sigma_V/\partial p^2$

$$\left.\frac{\partial \Sigma_V}{\partial p^2}\right|_{p^2=m^2} = -\frac{1}{2m^2}\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon_{\text{IR}}} - 1 + \tfrac{1}{2}\gamma + \tfrac{1}{2}\ln\frac{m^2}{4\pi\mu^2}\right) \tag{5.54}$$

In addition

$$\Sigma_V(p^2 = m^2) = \frac{\alpha}{4\pi}\left(\ln\frac{m^2}{4\pi\mu^2} + \gamma - 2\right) \tag{5.55}$$

so for Σ_1, we get

$$\Sigma_1 = \left.\frac{\partial \Sigma}{\partial \not{p}}\right|_{\not{p}=m} = \frac{\alpha}{\pi}\left(\frac{1}{\varepsilon_{\text{IR}}} + \tfrac{3}{4}\ln\frac{m^2}{4\pi\mu^2} - 1 + \tfrac{3}{4}\gamma\right) \tag{5.56}$$

Our final remark is that although $\Sigma_V(p^2)$ does not have IR singularities (when $p^2 \to m^2$), it nevertheless has the so-called mass singularities (when $p^2 \to m^2 \to 0$).

Electron self-energy: IR singularities regularized by photon mass

The IR singularities encountered above are due to the zero mass of the photon. Instead of using the dimensional method one can regularize the IR singularities by giving the photon a small mass λ, where eventually $\lambda \to 0$ (actually this is the most standard method).

For the electron self-energy correction we now have

$$-\mathrm{i}\Sigma(\not{p}) = -e^2 \int \frac{\mathrm{d}^n k}{(2\pi)^n}\gamma_n(\not{p} + \not{k} + m)\gamma^\mu \frac{1}{k^2 - \lambda^2}\frac{1}{(p+k)^2 - m^2}$$

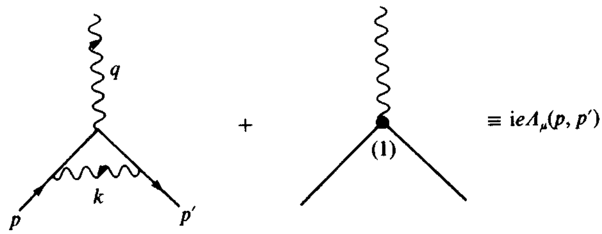


Fig. 5.5.

Following the previous calculation, instead of (5.50) one gets:

$$\left.\begin{aligned}\Sigma_m(p^2) &= -m\frac{\alpha}{\pi}\int \mathrm{d}x \ln\frac{x^2p^2 - x(p^2 - m^2 + \lambda^2) + \lambda^2 - \mathrm{i}\varepsilon}{4\pi\mu^2} + \text{const.}\\ \Sigma_V(p^2) &= \frac{\alpha}{2\pi}\int \mathrm{d}x\,(1-x)\ln\frac{x^2p^2 - x(p^2 - m^2 + \lambda^2) + \lambda^2 - \mathrm{i}\varepsilon}{4\pi\mu^2} + \text{const.}\end{aligned}\right\} \tag{5.57}$$

These integrations can be done by writing the logarithm as $\ln p^2(x - x_1)(x - x_2)$, where x_1 and x_2 are the roots of the quadratic expressions. The derivatives of Σ_m and Σ_V can now be calculated in an elementary way. We get in particular

$$\left.\frac{\partial\Sigma_m}{\partial p^2}\right|_{p^2=m^2} = m\frac{\alpha}{\pi}\frac{1}{2m^2}\ln\frac{m^2}{\lambda^2} + \text{const.} \tag{5.58}$$

$$\left.\frac{\partial\Sigma_V}{\partial p^2}\right|_{p^2=m^2} = \frac{\alpha}{\pi}\left(-\frac{1}{4m^2}\right)\ln\frac{m^2}{\lambda^2} + \text{const.} \tag{5.59}$$

and finally

$$\Sigma_1 = \frac{\alpha}{\pi}\frac{1}{2}\ln\frac{m^2}{\lambda^2} + \frac{\alpha}{\pi}\frac{1}{4}\ln\frac{m^2}{4\pi\mu^2} + \text{const.} \tag{5.60}$$

Comparing with the dimensional regularization of IR singularities we observe the correspondence $2/\varepsilon_{\mathrm{IR}} \to \ln(m^2/\lambda^2)$.

On-shell vertex correction

We consider Fig. 5.5

$$\Lambda_\mu(p, p') = (\mathrm{i}e)^2\int\frac{\mathrm{d}^nk}{(2\pi)^n}\gamma_\alpha\frac{\mathrm{i}}{\not p' + \not k - m}\gamma_\mu\frac{\mathrm{i}}{\not p + \not k - m}\gamma_\beta(-\mathrm{i}g^{\alpha\beta})\frac{1}{k^2}(Z_1 - 1)^{(1)}\gamma_\mu \tag{5.61}$$

We work in the Feynman gauge and assume that the external electron lines are on-shell $p'^2 = p^2 = m^2$. Using the identity $\gamma^\mu \not{a} \gamma_\mu = (2-n)\not{a}$, we can write the integral as

$$\Lambda_\mu(p, p') - (Z_1 - 1)^{(1)}\gamma_\mu = -\mathrm{i}e^2 \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{N}{[(p'+k)^2 - m^2][(p+k)^2 - m^2]k^2} \tag{5.62}$$

where

$$\begin{aligned} N = \gamma_\mu[4p\cdot p' + (n-2)k^2 + 4(p+p')\cdot k] \\ -4(p+p')_\mu \not{k} + k_\mu[4m - 2(n-2)\not{k}] \end{aligned} \tag{5.63}$$

We now introduce Feynman parametrization for the propagators and then have to perform the following integrations

$$2\int_0^1 \mathrm{d}x\,\mathrm{d}y\, y \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{[1, k_\alpha, k_\alpha k_\beta]}{\{k^2 + [p'xy + p(1-x)y]2k\}^3} \tag{5.64}$$

Only the integral with two powers of k in the numerator is UV divergent (but IR finite). We regularize it by taking $n = 4 - \varepsilon_{\mathrm{UV}}$. The other integrals are at most only IR divergent, so we take $n = 4 + \varepsilon_{\mathrm{IR}}$ there. To simplify notation we write the final results in terms of the integrals

$$\left.\begin{aligned} I_1 &= \int_0^1 \frac{\mathrm{d}x}{-q^2x(1-x)+m^2} \qquad I_3 = \int_0^1 \frac{\mathrm{d}x\, x}{-q^2x(1-x)+m^2} \\ I_2 &= \int_0^1 \frac{\mathrm{d}x}{-q^2x(1-x)+m^2} \ln\frac{-q^2x(1-x)+m^2}{4\pi\mu^2} \end{aligned}\right\} \tag{5.65}$$

where $q = p - p'$ (we assume $q^2 \le 0$) and μ^2 is an arbitrary mass scale. We see that $I_3 = \frac{1}{2}I_1$. We get finally

$$\Lambda_\mu = \Lambda_\mu^{\mathrm{UV}} + \Lambda_\mu^{\mathrm{IR}}$$

where (in $n = 4 - \varepsilon_{\mathrm{UV}}$ dimensions)

$$\begin{aligned} \Lambda_\mu^{\mathrm{UV}} = \gamma_\mu \frac{\alpha}{\pi}\left[\frac{1}{2\varepsilon_{\mathrm{UV}}} - \frac{1}{4}(2+\gamma) - \tfrac{1}{4}\int \mathrm{d}x \ln\frac{-q^2x(1-x)+m^2}{4\pi\mu^2}\right] \\ + \frac{\alpha}{2\pi}\frac{m}{2}(p+p')_\mu I_1 + \gamma_\mu(Z_1-1)^{(1)} \end{aligned} \tag{5.66}$$

and (in $n = 4 + \varepsilon_{\mathrm{IR}}$ dimensions)

$$\begin{aligned} \Lambda_\mu^{\mathrm{IR}} = \gamma_\mu \frac{\alpha}{4\pi}(2q^2 - 4m^2)\left[\left(\frac{1}{\varepsilon_{\mathrm{IR}}} + \tfrac{1}{2}\gamma\right)I_1 + \tfrac{1}{2}I_2\right] + \frac{\alpha}{2\pi}\gamma_\mu(p+p')^2 I_1 \\ - \frac{\alpha}{2\pi}(p+p')_\mu m I_1 \end{aligned} \tag{5.66a}$$

The above result is often rewritten in terms of the Dirac and Pauli formfactors

$$\Lambda_\mu(p, p') = \gamma_\mu F_1(q^2) + \tfrac{1}{2}(\mathrm{i}/m)\sigma_{\mu\nu}q^\nu F_2(q^2), \quad \sigma_{\mu\nu} = \tfrac{1}{2}\mathrm{i}[\gamma_\mu, \gamma_\nu] \tag{5.67}$$

Using the so-called Gordon decomposition

$$\bar{u}(p')\gamma_\mu u(p) - \tfrac{1}{2}(\mathrm{i}/m)\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p) = \tfrac{1}{2}(1/m)\bar{u}(p')(p+p')^\mu u(p) \tag{5.68}$$

and the relation $(p+p')^2 = 4m^2 - q^2$, one gets the following:

$$\begin{aligned} F_1(q^2) = {} & \frac{\alpha}{\pi}\left[\frac{1}{2\varepsilon_{\mathrm{UV}}} - \frac{1}{4}(2+\gamma) - \frac{1}{4}\int \mathrm{d}x \ln\frac{-q^2 x(1-x) + m^2}{4\pi\mu^2} + m^3 I_3\right] \\ & + (Z_1 - 1)^{(1)} + \frac{\alpha}{\pi}\left\{\left(\tfrac{1}{2}q^2 - m^2\right)\left[\left(\frac{1}{\varepsilon_{\mathrm{IR}}} + \tfrac{1}{2}\gamma\right)I_1 + \tfrac{1}{2}I_2\right]\right. \\ & \left. + \left(m^2 - \tfrac{1}{2}q^2\right)I_1\right\} \end{aligned} \tag{5.69}$$

$$F_2(q^2)\frac{\alpha}{2\pi}m^2 I_1 \tag{5.70}$$

We can now make several observations. In the minimal subtraction scheme

$$(Z_1 - 1)^{(1)} = -\frac{\alpha}{2\pi}\frac{1}{\varepsilon_{\mathrm{UV}}} \tag{5.71}$$

and, comparing with (5.49), we have $Z_1 = Z_2$ to this order. One can check that for the on-shell renormalization given by the conditions

$$\left.\frac{\partial\Sigma}{\partial \not{p}}\right|_{\not{p}=m} = 0; \qquad F_1(q^2 = 0) = 1$$

the relation $Z_1 = Z_2$ also holds. Indeed from (5.56) and (5.69) it follows that

$$Z_1 = Z_2 = 1 - \frac{\alpha}{\pi}\left(\frac{1}{2\varepsilon_{\mathrm{UV}}} + \tfrac{3}{4}\gamma + 1 - \tfrac{3}{4}\ln\frac{m^2}{4\pi\mu^2} - \frac{1}{\varepsilon_{\mathrm{IR}}}\right) \tag{5.72}$$

in this case (the renormalization constants now become IR divergent). At this point we encourage the reader to calculate the Thomson cross-section in the on-shell renormalization scheme (Problem 5.2).

In our calculation we have reproduced the famous Schwinger result (Schwinger 1949) for the anomalous magnetic moment of the electron

$$F_2(q^2 = 0) = \alpha/2\pi \tag{5.73}$$

Notice that the radiative correction (5.73) for the anomalous magnetic moment is finite before renormalization. It depends on the renormalization scheme through the value of $\alpha(\mu^2)$ but that dependence is of the order $O(\alpha^2)$. Correction (5.73) is finite because the operator $\bar{\Psi}\sigma_{\mu\nu}\Psi F^{\mu\nu}$ has dimension five and cannot be present in the original renormalizable lagrangian. It is generated by radiative corrections as a non-local term which becomes local in the low energy limit $q^2 = 0$ (see also Chapter 12).

As the next point we discuss the structure of the leading logarithmic terms in our result for the vertex corrections in the limit $|-q^2| \gg m^2$. In the minimal subtraction scheme we get

$$-\frac{\alpha}{4\pi}\ln\frac{-q^2}{4\pi\mu^2}$$

as a finite contribution to F_1 from Λ_{UV} and†

$$\frac{\alpha}{\pi}\left[\frac{1}{\varepsilon_{\mathrm{IR}}}\ln\frac{m^2}{-q^2} - \frac{1}{4}\ln^2\frac{m^2}{-q^2} + \left(\tfrac{1}{2}\gamma - 1\right)\ln\frac{m^2}{-q^2} + \tfrac{1}{2}\ln\frac{-q^2}{4\pi\mu^2}\ln\frac{m^2}{-q^2}\right] \tag{5.74}$$

from Λ_{IR}. Here the first two terms are the dimensional regularization form of the famous double logarithms due to the simultaneous presence of IR and mass singularities.

The IR singularities can also be regularized by λ, the photon mass. A straightforward calculation of the leading double logarithms gives a result which corresponds to the replacement $2/\varepsilon_{\mathrm{IR}} \to \ln(m^2/\lambda^2)$. As a final remark we notice that the double logarithm terms do not depend on the renormalization scheme (Z_i, in the on-shell renormalization, for instance, contains only single logarithms).

5.3 Massless QED

For some high energy reactions it is useful to have results in the $m = 0$ approximation. We recalculate now in the Feynman gauge the mass and vertex corrections in this approximation. Consider first the self-energy correction. We have

$$-\mathrm{i}\Sigma(\not{p}) = -e^2\int\frac{\mathrm{d}^n k}{(2\pi)^n}N\frac{1}{k^2}\frac{1}{(p+k)^2} \tag{5.75}$$

where

$$N = (2-n)(\not{p} + \not{k})$$

† It is easy to see that

$$I_1 \approx \frac{-1}{-q^2}2\ln\frac{m^2}{-q^2}$$

$$I_2 \approx \frac{-1}{-q^2}2\ln\frac{-q^2}{4\pi\mu^2}\ln\frac{m^2}{-q^2} + \frac{1}{-q^2}\left(\ln^2\frac{-q^2}{m^2} + \tfrac{1}{3}\pi^3\right)O\left(\frac{1}{q^4}\right)$$

when $|q^2| \gg m^2$. In particular I_2 can be written in terms of the Spence functions. The logarithms can also be obtained simply by writing $\int_0^1 = \int_0^{\xi} + \int_{\xi}^1$ and approximating the integrands in both integrals.

(remember $\gamma^\mu \not{a} \gamma_\mu = (2-n)\not{a}$). Using Feynman parametrization and the integrals (B.18)–(B.20) one gets ($n = 4 - \varepsilon$ and $\alpha \to \alpha\mu^\varepsilon$)

$$\Sigma(\not{p}) = \frac{\alpha}{\pi}\frac{1}{4}\not{p}(2-n)\left(\frac{1}{4\pi}\right)^{-\varepsilon/2}\left(\frac{-p^2}{\mu^2}\right)^{-\varepsilon/2}\Gamma\left(\tfrac{1}{2}\varepsilon\right)\int_0^1 x^{-\varepsilon/2}(1-x)^{1-\varepsilon/2}\,\mathrm{d}x$$
$$- (Z_2 - 1)^{(1)}\not{p} \equiv \Sigma_V(p^2)\not{p} \tag{5.76}$$

We can now proceed in either of two ways. One is to keep external lines off-shell $p^2 < 0$ to regularize the IR singularity. Then, in the minimal subtraction scheme

$$\Sigma(\not{p}) = \not{p}\left(\frac{\alpha}{\pi}\frac{1}{4}\ln\frac{-p^2}{4\pi\mu^2} + \text{const.}\right) \tag{5.77}$$

The other way is to take external lines on-shell $p^2 = 0$ and to use dimensional regularization of the IR singularity. Note that for massless QED the finite wave-function renormalization Σ_1 is given solely by $\Sigma_V(p^2 = 0)$ and not its derivative. We rewrite (5.76) in the following form

$$\Sigma(\not{p}) = \frac{\alpha}{4\pi}(2-n)\not{p}\int_{-p^2}^{\infty}\frac{\mathrm{d}q^2}{q^2}\left(\frac{q^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\Gamma\left(1+\tfrac{1}{2}\varepsilon\right)$$
$$\times\int_0^1 \mathrm{d}x\, x^{-\varepsilon/2}(1-x)^{1-\varepsilon/2} - (Z_2-1)^{(1)}\not{p} \tag{5.78}$$

For $p^2 = 0$ we split the integral into two: $\int_0^{\mu^2} + \int_{\mu^2}^{\infty}$ which are IR and UV divergent, respectively, when $\varepsilon \to 0$ (to regularize the first integral we work in $n = 4 + \varepsilon_{\mathrm{IR}}$ dimensions, so we take $\varepsilon \to -\varepsilon_{\mathrm{IR}}$). Finally, after cancelling the UV divergence by Z_2, we get† in the minimal subtraction scheme

$$\Sigma_V(p^2 = 0) = -\frac{\alpha}{4\pi}\frac{2}{\varepsilon_{\mathrm{IR}}} \tag{5.79}$$

where the pole in $\varepsilon_{\mathrm{IR}}$ reflects the IR singularity.

It is also straightforward to repeat the calculation of the vertex correction in the massless electron case. We have

$$\Lambda_\mu = -\mathrm{i}e^2\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{N}{(p'+k)^2(p+k)^2k^2} \tag{5.80}$$

where

$$N = \gamma_\mu[4p\cdot p' + (n-2)k^2 + 4(p+p')\cdot k] - (p+p')_\mu 4\not{k} - 2(n-2)k_\mu\not{k}$$

† This result is immediate if we note that $\int_{\mu^2}^{\infty} -(Z_2 - 1)$ is already UV finite and we can write it in $(4+\varepsilon_{\mathrm{IR}})$ dimensions by changing $\varepsilon \to -\varepsilon_{\mathrm{IR}}$. Then the $\int_{\mu^2}^{\infty}$ part cancels the $\int_0^{\mu^2}$ part and the final result is simply $-(Z_2 - 1)$ written in terms of $\varepsilon_{\mathrm{IR}}$.

Repeating the steps which led us to the result (5.66), we now get ($n = 4 - \varepsilon_{\mathrm{UV}}$)

$$\begin{aligned}\Lambda_\mu^{\mathrm{UV}} &= \gamma_\mu \frac{\alpha}{2\pi} \Gamma\big(1 + \tfrac{1}{2}\varepsilon_{\mathrm{UV}}\big)(2 - \varepsilon_{\mathrm{UV}}) \left(\frac{1}{\varepsilon_{\mathrm{UV}}} - 1\right) \left(\frac{-q^2}{4\pi\mu^2}\right)^{-\varepsilon_{\mathrm{UV}}/2} \\ &\quad \times \int \mathrm{d}x\, \mathrm{d}y\, y^{1-\varepsilon_{\mathrm{UV}}} [x(1-x)]^{-\varepsilon_{\mathrm{UV}}/2} + \gamma_\mu (Z_1 - 1)^{(1)} \\ &= \gamma_\mu \frac{\alpha}{2\pi} \left(\frac{-q^2}{4\pi\mu^2}\right)^{-\varepsilon_{\mathrm{UV}}/2} \frac{\Gamma\big(1 - \frac{1}{2}\varepsilon_{\mathrm{UV}}\big)}{\Gamma(1 - \varepsilon_{\mathrm{UV}})} \frac{1}{\varepsilon_{\mathrm{UV}}} + \gamma_\mu (Z_1 - 1)^{(1)} \qquad (5.81)\end{aligned}$$

and† ($n = 4 + \varepsilon_{\mathrm{IR}}$)

$$\begin{aligned}\Lambda_\mu^{\mathrm{IR}} &= \gamma_\mu \left(-\frac{\alpha}{2\pi}\right) \left(\frac{-q^2}{4\pi\mu^2}\right)^{\varepsilon_{\mathrm{IR}}/2} \Gamma\big(1 - \tfrac{1}{2}\varepsilon_{\mathrm{IR}}\big) \\ &\quad \times \int \mathrm{d}y\, y^{\varepsilon_{\mathrm{IR}}-1}(1-y) \int \mathrm{d}x\, [x(1-x)]^{\varepsilon_{\mathrm{IR}}/2-1} \\ &= \gamma_\mu \left(-\frac{\alpha}{2\pi}\right) \left(\frac{-q^2}{4\pi\mu^2}\right)^{\varepsilon_{\mathrm{IR}}/2} \frac{\Gamma\big(1 + \frac{1}{2}\varepsilon_{\mathrm{IR}}\big)}{\Gamma(2 + \varepsilon_{\mathrm{IR}})} \left(\frac{2}{\varepsilon_{\mathrm{IR}}}\right) \left[1 + \frac{\pi^2}{6}\big(\tfrac{1}{2}\varepsilon_{\mathrm{IR}}\big)^2\right] \qquad (5.82)\end{aligned}$$

The double pole structure in $\Lambda_\mu^{\mathrm{IR}}$ is due to the simultaneous occurrence of IR (massless photons) and mass (massless electrons) singularities. Both have been regularized by working in $n = 4 + \varepsilon_{\mathrm{IR}}$ dimensions. We note also that, once $(Z_1 - 1)^{(1)}$ is explicitly determined, for example, in the minimal subtraction scheme, we can rewrite $\Lambda_\mu^{\mathrm{UV}}$ (which is, of course, finite) in $n = 4 + \varepsilon_{\mathrm{IR}}$ dimensions by changing $\varepsilon_{\mathrm{UV}} \to -\varepsilon_{\mathrm{IR}}$. Then the full vertex is written in $(4 + \varepsilon_{\mathrm{IR}})$ dimensions.

5.4 Dispersion calculation of $O(\alpha)$ virtual corrections in massless QED, in $(4 \mp \varepsilon)$ dimensions

It is instructive to repeat the calculation of the electron self-energy and vertex corrections using dispersion relations. The dispersive technique has been extensively used in particle physics. We do not attempt its systematic discussion here, referring the interested reader to other books (Bjorken & Drell 1965, Eden, Landshoff, Olive & Polkinghorne 1966). Our purpose is mainly a technical one: to show yet another method of calculating Feynman diagrams which in some cases happens to be instructive.

As we know, in $4 - \varepsilon$ dimensions Feynman amplitudes are UV convergent. So we can always write for them the dispersion relation without subtraction. However, the calculated amplitude, of course, has poles in ε and we have to perform next the standard renormalization procedure, i.e. to add counterterms.

† $\Gamma(1-\varepsilon) \cong \Gamma(1) - \Gamma'(1)\varepsilon + \frac{1}{2}\Gamma''(1)\varepsilon^2 = 1 + \gamma\varepsilon + \frac{1}{2}(\gamma^2 + \frac{1}{6}\pi^2)\varepsilon^2$.

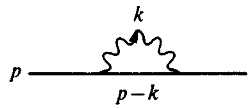


Fig. 5.6.

Self-energy calculation

The dispersion relation for the function $\Sigma_V(p^2)$ defined by (5.41) reads

$$\Sigma_V(q^2 = 0) = \frac{1}{\pi}\int \frac{\mathrm{d}p^2}{p^2}\,\mathrm{Im}\,\Sigma_V(p^2) \tag{5.83}$$

The imaginary part of a Feynman diagram amplitude is obtained by the replacement

$$\frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} \to -2\pi\mathrm{i}\delta(p^2 - m^2)\Theta(p_0) \equiv -2\pi\mathrm{i}\delta_+(p^2 - m^2)$$

in the Feynman integral (Landau–Cutkosky rule). Therefore for the diagram shown in Fig. 5.6 we can write

$$\mathrm{Im}\,\Sigma_V(p^2) = -\tfrac{1}{2}e^2(2-n)\Theta(p^2)\int \mathrm{d}\Phi_2\,(1 - p\cdot k/p^2) \tag{5.84}$$

where

$$\begin{aligned}\mathrm{d}\Phi_2 &= \frac{\mathrm{d}^n k}{(2\pi)^n}2\pi\delta_+(k^2)\frac{\mathrm{d}^n l}{(2\pi)^n}2\pi\delta_+(l^2)(2\pi)^n\delta^{(n)}(k+l-p)\\ &= \frac{\mathrm{d}^n k}{(2\pi)^n}2\pi\delta_+(k^2)2\pi\delta_+((p-k)^2)\end{aligned} \tag{5.85}$$

is the two-body phase space element in n dimensions. In the (k, l) centre of mass system it can be written as follows in terms of momenta $p = k+l$ and $r = \frac{1}{2}(k-l)$:

$$\begin{aligned}\mathrm{d}\Phi_2 &= \frac{1}{(2\pi)^{n-2}}\mathrm{d}^n r\,\delta(\tfrac{1}{4}p^2 + pr + r^2)\delta(\tfrac{1}{4}p^2 - pr + r^2)\\ &= \frac{1}{(2\pi)^{n-2}}\mathrm{d}^n r\,\tfrac{1}{2}\delta(\tfrac{1}{4}p^2 + r^2)\delta(pr)\\ &= \frac{1}{(2\pi)^{n-2}}\mathrm{d}^{n-1}\mathbf{r}\frac{1}{2p_0}\delta(\tfrac{1}{4}p^2 - \mathbf{r}^2)\end{aligned} \tag{5.86}$$

We now introduce polar coordinates for the Euclidean vector $\mathbf{r}$ in $n-1$ dimensions

$$\mathrm{d}^{n-1}\mathbf{r} = |\mathbf{r}|^{n-2}\,\mathrm{d}|\mathbf{r}|\,\mathrm{d}\Omega_{n-1} = |\mathbf{r}|^{n-2}\,\mathrm{d}|\mathbf{r}|\,\mathrm{d}\theta\,(\sin\theta)^{n-3}\,\mathrm{d}\Omega_{n-2} \tag{5.87}$$

where $0 \leq \theta \leq \pi$ and due to an azimuthal symmetry of the problem we integrate over all but one angle θ to get

$$\int \mathrm{d}^{n-1}\mathbf{r} = (\mathbf{r}^2)^{(n-3)/2}\,\mathrm{d}\mathbf{r}^2\,(\sin\theta)^{n-3}\,\mathrm{d}\theta\,\frac{\pi^{(n-2)/2}}{\Gamma(\frac{1}{2}(n-2))} \tag{5.88}$$

For the phase space element integrated over angles one then gets

$$\mathrm{d}\Phi_2 = \frac{\pi^{1-\varepsilon/2}}{(2\pi)^{2-\varepsilon}}\frac{1}{\Gamma(1-\frac{1}{2}\varepsilon)}\left(\frac{p^2}{4}\right)^{-\varepsilon/2}(\sin\theta)^{n-3}\,\mathrm{d}\theta \tag{5.89}$$

(we specify $n = 4-\varepsilon$ and remember that $r_0^2 = 0$ and $p_0 = (p_2)^{1/2}$ in cms). Finally one can write

$$\mathrm{d}\theta\,(\sin\theta)^{n-3} = \mathrm{d}z\,(1-z^2)^{-\varepsilon/2} = \mathrm{d}y\,2[4y(1-y)]^{-\varepsilon/2}$$

where $z = \cos\theta$ and $y = \frac{1}{2}(1+z)$ to get

$$\mathrm{d}\Phi_2 = \frac{1}{8\pi}\left(\frac{p^2}{4\pi}\right)^{-\varepsilon/2}\frac{1}{\Gamma(1-\frac{1}{2}\varepsilon)}\mathrm{d}y\,[y(1-y)]^{-\varepsilon/2} \tag{5.90}$$

We are now ready to calculate $\mathrm{Im}\,\Sigma_V(p^2)$ and $\Sigma_V(q^2 = 0)$. Since in cms $p\cdot k = \frac{1}{2}p^2$ we obtain (as usual, in $n = 4-\varepsilon$ dimensions, we define dimensionless α by $\alpha \to \alpha\mu^\varepsilon$)

$$\Sigma_V(q^2=0) = -\frac{\alpha}{4\pi}\frac{(2-n)}{\Gamma(1-\frac{1}{2}\varepsilon)}\frac{1}{2}\int_0^\infty \frac{\mathrm{d}p^2}{p^2}\left(\frac{p^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\int_0^1 \mathrm{d}y\,[y(1-y)]^{-\varepsilon/2} \tag{5.91}$$

which is equivalent to (5.78).

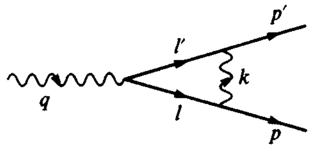
Vertex calculation

Calculation of the vertex correction proceeds similarly. We shall limit ourselves to the UV finite part $\Lambda_\mu^{\mathrm{IR}}$ to understand better the origin of the IR and mass singularities. Writing

$$\Lambda_\mu^{\mathrm{IR}} = \gamma_\mu 2q^2\Lambda(q^2) \tag{5.92}$$

and using the kinematical variables defined in Fig. 5.7 we have

$$\mathrm{Im}\,\Lambda(q^2) = -e^2\int \mathrm{d}\Phi_2\left[1+\frac{4k\cdot(p-p')}{2q^2}\right]\frac{1}{k^2} \tag{5.93}$$



$$k = l - p = p' - l'$$

$$k^2 = -2l\cdot p = -2l'\cdot p'$$

$$p\cdot k = p\cdot l = -\tfrac{1}{2}k^2 = -p'\cdot k$$

Fig. 5.7.

(the term proportional to $\not{k}$ when integrated must give $A\not{p} + B\not{p}'$, so it does not contribute to the on-shell limit), where

$$\begin{aligned} \mathrm{d}\Phi_2 &= \frac{\mathrm{d}^n k}{(2\pi)^n} 2\pi\delta_+((p+k)^2) 2\pi\delta_+((p'+k)^2) \\ &= \frac{\mathrm{d}^n l}{(2\pi)^n} 2\pi\delta_+(l^2) \frac{\mathrm{d}^n l'}{(2\pi)^n} 2\pi\delta_+(l'^2)(2\pi)^n \delta^{(n)}(q + l + l') \end{aligned} \tag{5.94}$$

Therefore

$$\operatorname{Im}\Lambda(q^2) = -e^2 \int \mathrm{d}\Phi_2 \left(\frac{1}{k^2} + \frac{2}{q^2}\right) \tag{5.95}$$

The next step is to use the expression (5.90) for $\mathrm{d}\Phi_2$. In the present case $r = \frac{1}{2}(l - l')$ and in the (l, l') cms system ($\mathbf{l} + \mathbf{l}' = \mathbf{p} + \mathbf{p}' = 0$) $\mathbf{r} = \mathbf{l}$. Taking the z axis in the direction $\mathbf{p}$ we have

$$k^2 = -2l\cdot p = -2l_0 p_0 + 2\mathbf{l}\cdot\mathbf{p} = -2(\tfrac{1}{2}q_0)^2 + 2(\tfrac{1}{2}q_0)z = -\tfrac{1}{2}q^2(1 - z)$$

where $z = \cos\theta$ and finally ($n = 4 + \varepsilon$)

$$\begin{aligned} \Lambda(Q^2) = \frac{\alpha}{4\pi}\frac{1}{4\pi\mu^2}\frac{1}{\Gamma(1 + \frac{1}{2}\varepsilon)} \int_0^\infty \frac{\mathrm{d}q^2}{q^2 - Q^2}\left(\frac{q^2}{4\pi\mu^2}\right)^{\varepsilon/2-1} \\ \times \int_0^1 \mathrm{d}y \left(\frac{1}{1-y} - 2\right)[y(1-y)]^{\varepsilon/2} \end{aligned} \tag{5.96}$$

The integral over $\mathrm{d}y$ gives $[\Gamma(1 + \frac{1}{2}\varepsilon)\Gamma(\frac{1}{2}\varepsilon)/\Gamma(1 + \frac{1}{2}\varepsilon) - \Gamma^2(1 + \frac{1}{2}\varepsilon)/\Gamma(2 + \varepsilon)]$ and it is divergent for $\varepsilon \to 0$. The divergence comes from the region $y \approx 1$ and therefore $z \approx 1$. In our specific reference frame $z = 1$ corresponds to $\mathbf{l} \parallel \mathbf{p}$, i.e. the exchanged photon has to have zero momentum. To integrate over q^2 (for $Q^2 < 0$) we change variables twice: $q^2 = -Q^2 x$ and $u = 1/(1 + x)$. We then get the final result

$$\int_0^\infty \frac{\mathrm{d}q^2}{q^2 - Q^2}(q^2)^{\varepsilon/2-1} = (-Q^2)^{\varepsilon/2-1}\Gamma(1 - \tfrac{1}{2}\varepsilon)\Gamma(\tfrac{1}{2}\varepsilon) \tag{5.97}$$

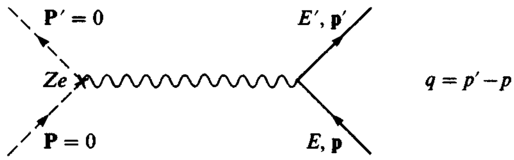


Fig. 5.8.

The pole in ε is due to the lower limit of integration over q^2, i.e. due to a massless electron–positron intermediate state (mass singularity). Finally, using (5.92) we get ($n = 4 + \varepsilon$)

$$\Lambda_\mu^{\mathrm{IR}} = \gamma_\mu \left(-\frac{\alpha}{2\pi}\right)\left(\frac{-Q^2}{4\pi\mu^2}\right)^{\varepsilon/2} \frac{2}{\varepsilon}\Gamma(1 - \tfrac{1}{2}\varepsilon)\left[\frac{\Gamma(1+\frac{1}{2}\varepsilon)\Gamma(\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)} - \frac{\Gamma^2(1+\frac{1}{2}\varepsilon)}{\Gamma(2+\varepsilon)}\right] \tag{5.98}$$

which can be seen to coincide with (5.82).

5.5 Coulomb scattering and the IR problem

Corrections of order α

In the Born approximation the scattering of an electron from a static Coulomb potential

$$A_0(x) = Ze/4\pi|\mathbf{x}|, \quad A_i(x) = 0$$

is described by the diagram in Fig. 5.8. According to the general rules the cross section reads

$$\mathrm{d}\sigma_0 = \frac{1}{2|\mathbf{p}|}|M_0^{(0)}|^2 2\pi\delta(E' - E)\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2E'} \tag{5.99}$$

where

$$|M_0^{(0)}|^2 = \frac{1}{2}\sum_{\mathrm{pol}} |\bar{u}(p')\gamma^0 u(p)|^2 \frac{Z^2 e^4}{|\mathbf{q}|^4} \tag{5.100}$$

and $Ze^2/|\mathbf{q}|^2$ is the Fourier transform of the $A_0(x)$. We use the static approximation to the general formulae (see, for example, Halzen & Martin (1984)):

$$\mathrm{flux}^{-1} = \frac{1}{2E_p 2E_P|\mathbf{v} - \mathbf{V}|} \to \frac{1}{2E_p|\mathbf{v}|}\frac{1}{2|\mathbf{p}|}$$

$$(2\pi)^4\delta(P + p - P' - p')\frac{\mathrm{d}^3\mathbf{P}'}{2E_{P'}(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}'}{2E_{p'}(2\pi)^3} \to 2\pi\delta(E' - E)\frac{\mathrm{d}^3\mathbf{p}'}{2E_{p'}(2\pi)^3}$$

for $\mathbf{P} = \mathbf{P}' = 0$ (remember, we use the relativistic normalization of $2E$ particles in a unit volume $V = 1$ – see Appendix A). As usual

$$\frac{1}{2}\sum_{\text{pol}} |\bar{u}(p')\gamma^0 u(p)|^2 = \frac{1}{2}(\not{p}' + m)_{\alpha\beta}(\gamma^0)_{\beta\delta}(\not{p} + m)_{\delta\gamma}(\gamma^0)_{\gamma\alpha} = \frac{1}{2}\,\mathrm{Tr}[\cdots] \tag{5.101}$$

Performing the necessary calculations we find

$$\mathrm{d}\sigma_0^{(0)} = \frac{1}{4}Z^2\alpha^2 \frac{1}{|\mathbf{p}|^4 \sin^4(\frac{1}{2}\theta)/E^2}[1 - \beta^2 \sin^2(\tfrac{1}{2}\theta)]\,\mathrm{d}\Omega\,\mathrm{d}E'\,\delta(E - E') \tag{5.102}$$

where $\beta = |\mathbf{p}|/E$ and $\mathbf{p}\cdot\mathbf{p}' = 2|\mathbf{p}|^2\cos\theta$. We shall now study real and virtual corrections of order α to the Born cross section.

Virtual corrections of order α are easily included using our results for the electron self-energy, vacuum polarization and the vertex. The amplitude now reads

$$\begin{aligned} M_0^{(0)} &= \bar{u}(p')(1 + \tfrac{1}{2}\Sigma_1)[\gamma_0 + \Lambda_0(p, p')](1 + \tfrac{1}{2}\Sigma_1)u(p)[1 + \Pi(q^2)] \\ &\approx \bar{u}(p')\gamma_0 u(p)[1 + \Sigma_1 + F_1(q^2)] \end{aligned} \tag{5.103}$$

In the final expression (5.103) we have neglected the terms $O(\alpha^2)$ and the corrections which are regular in the IR limit: $\Pi(q^2)$ and the Pauli formfactor $F_2(q^2)$. (In the amplitude for *bremsstrahlung* of a real photon we shall only be interested in the limit of a soft photon.)

Including virtual corrections to order α we obtain therefore

$$\mathrm{d}\sigma_0^{(1)} = \mathrm{d}\sigma_0^{(0)}\,[1 + 2\Sigma_1 + 2F_1(q^2)] \tag{5.104}$$

where using the results of Section 5.3

$$2\Sigma_1 = 2\frac{\alpha}{\pi}\left(\frac{1}{\varepsilon_{\mathrm{IR}}} + \frac{3}{4}\ln\frac{m^2}{4\pi\mu^2} - 1 + \tfrac{3}{4}\gamma\right) \tag{5.105}$$

(one-quarter (one-half) of the $\ln(m^2/4\pi\mu^2)$ comes from the dimensional regularization of the UV (IR) divergence) and

$$\begin{aligned} 2F_1(q^2) = -2\frac{\alpha}{\pi}\frac{1}{4}\ln\frac{-q^2}{4\pi\mu^2} - 2\frac{\alpha}{\pi}\bigg[&\frac{1}{\varepsilon_{\mathrm{IR}}}\ln\frac{-q^2}{m^2} + \frac{1}{4}\ln^2\frac{m^2}{-q^2} \\ &+ (\tfrac{1}{2}\gamma - 1)\ln\frac{-q^2}{m^2} + \frac{1}{2}\ln\frac{-q^2}{4\pi\mu^2}\ln\frac{-q^2}{m^2}\bigg] \end{aligned} \tag{5.106}$$

The above expressions are valid for $|q^2| \gg m^2$ and in the minimal subtraction scheme. The first term in $F_1(q^2)$ comes from the dimensional regularization of the UV divergences. Together with the analogous term in Σ_1 it gives $(\alpha/4\pi)\ln(-q^2/m^2)$. The remaining μ^2 dependence is due only to our method of regularization of the IR singularity and should cancel out in every cross section free of such singularities.

Fig. 5.9.

The origin of the IR singularity encountered in our cross section $d\sigma_0^{(1)}$ is in the masslessness of the photon, or in other words in the long range nature of the electromagnetic interactions. In consequence the system has a high degree of degeneracy: a final state consisting of a single electron state is not distinguishable by any measurement from an $e+n$ soft $(k \to 0)$ γs state. Therefore the previously defined cross section $d\sigma_0$ is not a physically meaningful (measurable) quantity. One feels that a summation over experimentally indistinguishable final states is necessary. Following Bloch and Nordsieck and working to order α we introduce a 'measurable' cross section σ as follows

$$\sigma = \int_0^{\Delta E} d\mathcal{E} \frac{d}{d\mathcal{E}} (\sigma_0^{(1)} + \sigma_{1\gamma}^{(1)}) \tag{5.107}$$

where ΔE is the energy resolution of the detection equipment, $\mathcal{E} = E' - E$ is the energy of the undetected real photon, E and E' are, as before, the energies of the initial and final electrons in the laboratory system, and $d\sigma_{1\gamma}$ is the differential cross section with a photon in the final state. We shall explicitly check that the cross section σ is free of the IR singularity. Later we shall discuss this problem in more general terms.

Working to order α we have to include $d\sigma_0^{(1)}$ given by (5.104) and $d\sigma_{1\gamma}^{(1)}$ which is the cross section for a single photon emission given by the Feynman diagrams shown in Fig. 5.9. Using the standard rules we get the following:

$$d\sigma_{1\gamma}^{(1)} = |M_{1\gamma}^{(1)}|^2 \frac{1}{|\mathbf{v}|} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{d^3\mathbf{k}}{2k_0(2\pi)^3} 2\pi\delta(E - E' - k_0) \tag{5.108}$$

where in the soft photon approximation (momentum k neglected in the numerators of the Feynman expressions)

$$|M_{1\gamma}^{(1)}|^2 = |M_0^{(0)}|^2 e^2 \left[\frac{2p \cdot p'}{k \cdot p\, k \cdot p'} - \frac{m^2}{(k \cdot p)^2} - \frac{m^2}{(k \cdot p')^2} \right] \tag{5.109}$$

In (5.109) the amplitude $|M_0^{(0)}|^2$ is defined by (5.100) and the sum over photon polarizations has been taken. We observe that the corrections due to the soft photon emission factorize. The first term comes from the interference of the two diagrams shown in Fig. 5.9, the next two are just the respective amplitudes squared. We now proceed as follows

$$\int_0^{\Delta E} \mathrm{d}\mathcal{E} \frac{\mathrm{d}\sigma_{1\gamma}^{(1)}}{\mathrm{d}\mathcal{E}} = \int_0^{\Delta E} \mathrm{d}\mathcal{E} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} 2\pi\delta(E - E' - k_0) \frac{E}{|\mathbf{p}|} \frac{|\mathbf{p}'|^2\,\mathrm{d}\Omega}{(2\pi)^3} |M_{1\gamma}^{(1)}|^2$$
$$\cong \int_0^{\Delta E} \mathrm{d}\mathcal{E} \frac{\mathrm{d}\sigma_0^{(0)}}{\mathrm{d}\mathcal{E}} \int_0^{|k|<\Delta E} \frac{\mathrm{d}^3\mathbf{k}}{2|k|(2\pi)^3} e^2$$
$$\times \left[\frac{2p\cdot p'}{k\cdot p\, k\cdot p'} - \frac{m^2}{(k\cdot p)^2} - \frac{m^2}{(k\cdot p')^2} \right] \tag{5.110}$$

where the last step is legitimate in the soft photon limit (we denote $|k| = |\mathbf{k}| = k_0$). At this point we get for the cross section, σ, the result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_0^{(0)}}{\mathrm{d}\Omega} [1 + 2\Sigma_1 + 2F_1(q^2) + I_1 + I_2 + I_3] \tag{5.111}$$

where, for example,

$$I_1 = e^2 \int_0^{|k|<\Delta E} \frac{\mathrm{d}^3\mathbf{k}}{2|k|(2\pi)^3} \frac{2p\cdot p'}{k\cdot p\, k\cdot p'} \tag{5.112}$$

and I_2 and I_3 are self-evident. Our last task is to calculate the corrections I_i. They are IR divergent and for consistency with the calculation of virtual corrections we must regularize them by working in $4 + \varepsilon_{\mathrm{IR}}$ dimensions. We replace

$$\frac{\mathrm{d}^3\mathbf{k}}{2|k|(2\pi)^3} \to \frac{\mathrm{d}^{n-1}\mathbf{k}}{2|k|(2\pi)^{n-1}}$$

where $|k| = (k_1^2 + \cdots + k_{n-1}^2)^{1/2}$ and introduce spherical coordinates in n dimensions (see (5.88)):

$$\int \mathrm{d}^{n-1}k = \int \mathrm{d}|k|\,|k|^{n-2} \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \ldots \sin\theta_{n-3}\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{n-2}$$
$$= 2\frac{\pi^{n/2-1}}{\Gamma(\frac{1}{2}n - 1)} \int \mathrm{d}|k|\,|k|^{n-2} \int_{-1}^{1} \mathrm{d}z\,(1 - z^2)^{n/2-2} \tag{5.113}$$

where $z = \cos\theta_1$ and the integration over the remaining angles has been done in the last step (assuming the integrand depends only on θ_1). Let us consider I_2 first. Writing $(k\cdot p)^2 = |k|^2(E - |\mathbf{p}|z)^2$ we can easily perform the necessary integrations. We see that the probability of a soft photon emission into an interval $\mathrm{d}|k|$ is proportional to $\mathrm{d}|k|/|k|$ which is easy to understand on dimensional grounds (soft emission does not depend on the electron momentum p). We see also that the dominant contribution to I_2 comes from that region of phase space where $\mathbf{k} \parallel \mathbf{p}$. So, that part of the radiation tends to be collimated around the direction of motion of the initial electron. The final result reads

$$I_2 = -\frac{\alpha}{\pi}\left[\frac{1}{\varepsilon_{\mathrm{IR}}} + \frac{1}{2}\ln\frac{(\Delta E)^2}{4\pi\mu^2} - \frac{1}{2}\ln\frac{4E^2}{m^2} + \text{const.}\right] \tag{5.114}$$

For I_3 we get the same result with E changed into E'. To calculate the term I_1 it is convenient to choose the reference frame in which $\mathbf{p} + \mathbf{p}' = 0$ (the Breit frame). In this frame

$$p \cdot k = |\tilde{k}|(\tilde{E} - |\tilde{\mathbf{p}}| \cos\theta)$$
$$p' \cdot k = |\tilde{k}|(\tilde{E} + |\tilde{\mathbf{p}}| \cos\theta)$$

where the sign $\tilde{}$ denotes quantities in the Breit frame and the condition $|k| < \Delta E$ in the laboratory system translates into $|\tilde{k}| < \Delta E \sin(\frac{1}{2}\theta)$, where θ is the scattering angle in the laboratory (indeed $p_{\mathrm{i}} \cdot p_{\mathrm{f}} = p \cdot p' = 2E^2 \sin^2(\frac{1}{2}\theta)$ in the laboratory and $p_{\mathrm{i}} \cdot p_{\mathrm{f}} = 2\tilde{E}^2$ in the Breit frame; so $\tilde{E} = E \sin(\frac{1}{2}\theta)$). In the Breit system the integration over the angle can be performed with the help of Spence's functions and we get

$$I_1 \cong 2\frac{\alpha}{\pi}\left[\frac{1}{\varepsilon_{\mathrm{IR}}}\ln\frac{-q^2}{m^2} + \frac{1}{2}\ln\frac{(\Delta E)^2}{4\pi\mu^2}\ln\frac{-q^2}{m^2} + \frac{1}{4}\ln^2\frac{m^2}{-q^2} + \tfrac{1}{2}\gamma\ln\frac{-q^2}{m^2} + f(\theta)\right] \tag{5.115}$$

for $|q^2| \gg m^2$. Adding all the contributions we obtain the final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma_0^{(0)}}{\mathrm{d}\Omega}\left\{1 + \frac{\alpha}{\pi}\left[\ln\frac{-q^2}{(\Delta E)^2} + \ln\frac{(\Delta E)^2}{-q^2}\ln\frac{-q^2}{m^2} + \frac{3}{2}\ln\frac{-q^2}{m^2} + f(\theta) + \text{const.}\right]\right\} \tag{5.116}$$

(remember that $-q^2 \cong 4E^2$). The cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is indeed free of the IR singularity and depends on the experimental resolution ΔE. The transition probability decreases with decreasing ΔE. Our calculation is valid for $|(\alpha/\pi)\ln[-q^2/(\Delta E)^2]| \ll 1$. Otherwise the correction becomes large and we must include higher order contributions which, as we shall see later, exponentiate. The results presented here are for $|q^2| \gg m^2$. The limit $|q^2| \ll m^2$ can also be easily obtained. It should be noted that although the final result (5.116) does not, of course, depend on the method of regularization of the IR singularity, the splitting into real and virtual contributions does. The coupling constant α is defined by the minimal subtraction renormalization scheme and $\alpha = \alpha^{\mathrm{ON}}[1 + O(\alpha^{\mathrm{ON}})]$, where $\alpha^{\mathrm{ON}} = 1/137$. So working to order α we can neglect the difference.

Finally we observe certain systematics in the cancellation of the IR singularities between virtual and real corrections. This can best be formulated if we represent

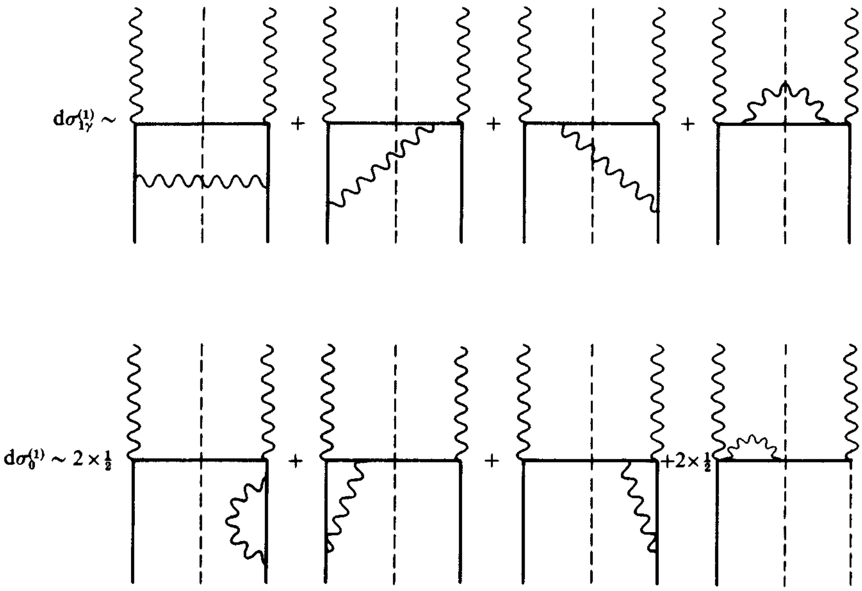


Fig. 5.10.

the cross section for real emissions by cut diagrams as shown in Fig. 5.10, where the factor $\frac{1}{2}$ follows from the rule of Section 4.1 and the factor 2 takes account of two identical interference terms. The cancellation occurs between contributions to the cross section represented by the diagrams in each column, separately.

IR problem to all orders in α

The previous results can be generalized to all orders in the electromagnetic coupling constant α (Yennie, Frautschi & Suura 1961). An important fact is that in QED the IR divergent terms due to virtual corrections and to real soft emissions exponentiate and cancel each other in the physically meaningful (observable) cross section.

Let us consider a process in which there are some electrons and non-soft photons in the initial and final states and we represent their momenta by p and p', respectively. Let the lowest order matrix element for our process be $M_0^{(0)}(p, p')$ and let $M_0^{(n)}(p, p')$ be the contribution corresponding to all diagrams in which there are in addition n virtual photons. The complete matrix element is then

$$M_0(p, p') = \sum_{n=0}^{\infty} M_0^{(n)}(p, p') \tag{5.117}$$

It has been shown (Yennie *et al.* 1961) that the $M^{(n)}$s have the following structure

$$\left.\begin{aligned} M_0^{(0)} &= m_0 \\ M_0^{(1)} &= m_0\alpha B + m_1 \\ M_0^{(2)} &= m_0(\alpha B)^2/2 + m_1\alpha B + m^2 \\ &\vdots \\ M_0^{(n)} &= \sum_{i=0}^{n} m_{n-i}(\alpha B)^i/i! \end{aligned}\right\} \tag{5.118}$$

where m_j is an IR finite function (independent of n) of order α^j relative to $M_0^{(0)}$ and each factor αB contains the IR contribution from one virtual photon. Thus we get

$$M_0 = \exp(\alpha B)\sum_{j=0}^{\infty} m_j = \exp(\alpha B)\hat{M}_0 \tag{5.119}$$

and the IR divergent exponent can be read off from our lowest order calculation, (5.104)–(5.106). An important intermediate step in proving (5.119) is to convince oneself that IR divergences originate from only those virtual photons which have both ends terminating on external lines. Actually, this is to be expected on physical grounds since long wave photons should not see charge distribution at short distance.

For the Coulomb elastic scattering we get from (5.119) and (5.106), in the so-called double logarithmic approximation,

$$M_{\mathrm{el}} = \exp\left(-\frac{\alpha}{\pi}\ln\frac{-q^2}{m^2}\ln\frac{-q^2}{\lambda^2}\right)\hat{M}_{\mathrm{el}} \tag{5.120}$$

where $\hat{M}$ is free of IR singularities. To write (5.120) we have changed the dimensional IR regularization into a cut-off by a small photon mass λ: $1/\varepsilon_{\mathrm{IR}} \to \frac{1}{2}\ln(m^2/\lambda^2)$ (see Section 5.2) in order to follow the traditional notation and in the exponent we have only retained terms quadratic in large logarithms, $\ln(-q^2/m^2)$ and $\ln(-q^2/\lambda^2)$. The exponent in (5.120) is called the on-shell electron formfactor in the double logarithmic approximation. We observe that due to the IR divergences encountered in order-by-order calculation the complete amplitude for purely elastic scattering is zero in the limit $\lambda \to 0$. This is just a reflection of the fact that electron scattering is always accompanied by soft photon radiation and the cross section for elastic scattering is not a physical observable.

Let us now consider an inclusive cross section summed over any number of undetected real photons in the final state, with total energy $\mathcal{E}$ such that $\mathcal{E} \leq \Delta E$, where ΔE is the energy resolution of the detectors. The differential cross section for emission of m undetected real photons with total energy $\mathcal{E}$ reads

$$\frac{\mathrm{d}\sigma_m}{\mathrm{d}\mathcal{E}} = \exp(2\alpha B)\frac{1}{m!}\int \prod_{i=1}^{m} \frac{\mathrm{d}^3\mathbf{k}_i}{2k_{i0}}\delta\left(\mathcal{E} - \sum_i k_{i0}\right)\rho_m(p, p', k_1 \ldots k_m) \tag{5.121}$$

where p and p' represent, as before, momenta of all particles in the initial state and of the detected particles in the final state. Virtual corrections to all orders in α are included in (5.121) and, as mentioned before, each σ_m is zero due to the IR divergences. However, an inclusive cross section defined as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{E}} = \sum_{m=0}^{\infty} \frac{\mathrm{d}\sigma_m}{\mathrm{d}\mathcal{E}} \tag{5.122}$$

turns out to be free of IR singularities and thus it is a finite, physical quantity.

Soft real photons must terminate exclusively on external lines in order to contribute IR singularities. Writing (Yennie *et al.* 1961)

$$\delta\left(\mathcal{E} - \sum_i k_{i0}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y \exp\left(\mathrm{i}y\mathcal{E} - \mathrm{i}y\sum_i k_{i0}\right) \tag{5.123}$$

using a generalization of (5.109)

$$\rho_m(p, p', k_1 \ldots k_m) = \hat{\sigma}_0\, \mathrm{p}(k_1)\, \mathrm{p}(k_m) \tag{5.124}$$

where $\mathrm{p}(k_1)$ is given by the expression in the square brackets in (5.109) and, as in our order α calculation, neglecting for soft photon radiation the energy–momentum conservation constraints one gets the following result:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{E}} = \exp[2\alpha(B + \tilde{B})]\hat{\sigma}_0\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y \exp[\mathrm{i}y\mathcal{E} + D(y)] \tag{5.125}$$

where

$$2\alpha\tilde{B} = \int^{|k|<\mathcal{E}} \frac{\mathrm{d}^3\mathbf{k}_i}{2k_{i0}}\mathrm{p}(k_i) \tag{5.126}$$

and

$$D = \int^{|k|<\mathcal{E}} \frac{\mathrm{d}^3\mathbf{k}_i}{2k_{i0}}\mathrm{p}(k_i)[\exp(-\mathrm{i}yk_{i0}) - 1] \tag{5.127}$$

Thus all IR singularities due to real emissions are collected in the exponential factor $\exp(2\alpha\tilde{B})$, where $\tilde{B}$ is given by our order α calculation, and they cancel with virtual IR singularities in B giving together (see (5.116)):

$$2\alpha(B + \tilde{B}) = \frac{\alpha}{\pi}\left(\ln\frac{-q^2}{\mathcal{E}^2} + \ln\frac{\mathcal{E}^2}{-q^2}\ln\frac{-q^2}{m^2}\right) \tag{5.128}$$

The cross section $d\sigma/d\mathcal{E}$ is finite, as expected. For a detector resolution ΔE the integration over $\mathcal{E}$ up to ΔE is necessary. One should also remember that the IR finite part $\hat{\sigma}_0$ contains, in general, terms like $\ln(q^2/\mu^2)$, where q^2 is some characteristic for the process four-momentum squared and μ^2 is the renormalization point. The coupling constant α in (5.121) is $\alpha = \alpha(\mu^2)$.

Problems

5.1 Derive the Ward identities (5.15) and (5.21) using the invariance of the QED Green's functions under the BRS transformations (Problem 3.3). Hint: apply the BRS transformation to the Green's functions

$$\langle 0|T A_\mu(x)\omega(y)|0\rangle = 0$$

and

$$\langle 0|T \Psi(x)\bar{\Psi}(y)\omega(z)|0\rangle = 0$$

where $\omega(x)$ is the free auxiliary scalar field.

5.2 Derive the Ward identity for the truncated connected Green's function of the Compton amplitude. For the forward Compton amplitude and in the limit of the photon momentum $k \to 0$ (the Thomson limit) express the amplitude in terms of the electron propagator. Show that in the on-shell renormalization scheme the coupling constant e can be defined by means of the Compton amplitude in the Thomson limit.

5.3 Study the singularity structure of the photon propagator in the one-loop approximation in the complex p^2-plane and derive the Landau–Cutkosky rule. Hint: calculate the difference $P(p^2 + i\Delta) - P(p^2 - i\Delta)$, where $P(p^2) = p^2\Pi(p^2)$ and $\Pi(p^2)$ is defined by (5.26): choose the cms frame $p = (p_0, 0)$, study the pole structure of the integrand as a function of k_0 and integrate over dk_0 first.

5.4 Calculate the off-shell electron formfactor (Sudakov formfactor) in the double logarithmic approximation

$$\Gamma_\mu(p_1, p_2, q) = \gamma_\mu \exp\left(-\frac{\alpha}{2\pi}\ln\frac{|q^2|}{|p_1^2|}\ln\frac{|q^2|}{|p_2^2|}\right) \qquad |q^2| \gg |p_1^2|,\ |p_2^2| \gg m^2$$

5.5 Calculate the on-shell (see (5.120)) and the off-shell electron formfactors in the double logarithmic approximation using the light-like gauge $n \cdot A = 0$ where $n^2 = 0$. Compare with the calculation in the Feynman gauge.

6

Renormalization group

6.1 Renormalization group equation (RGE)

Derivation of the RGE

The spirit of the renormalization group approach lies in the observation discussed in Chapter 4 that in a specific theory the renormalized constants such as the couplings or the masses are mathematical parameters which can be varied by arbitrarily changing the renormalization prescription. Once the infinities of the theory have been subtracted out by a renormalization prescription R one is still free to perform further finite renormalizations resulting in each case in a different effective renormalization R'. Each renormalization prescription can be interpreted as a particular reordering of the perturbative expansion and expressing it in terms of the new renormalized constants. The latter are, of course, in each case related differently to physical constants (for example, the mass defined as the pole of the propagator is a physical constant) which are directly measurable and therefore renormalization-invariant. A change in renormalization prescription is compensated by simultaneous changes of the parameters of the theory so as to leave, by construction, all exact physical results renormalization-invariant. (In practice, there is a residual renormalization scheme dependence in each order of perturbation theory.)

Most often, and also in this section, only subsets of arbitrary renormalization prescription transformations which can be parametrized by a single mass scale parameter μ are discussed. The parameter can be, for example, the value of the subtraction point in the μ-subtraction schemes or the dimensionful parameter μ in the minimal subtraction prescription. For a single-mass-scale-dependent subset of renormalization transformations we derive in the following a differential equation which controls the changes in the renormalized parameters induced by changing μ: the RGE. Actually, the set of renormalization prescription transformations $\{R\}$ does not always have a group structure so the name is partly due to historical reasons.

Let us assume a particular renormalization scheme R and call Γ_R the R-renormalized 1PI Green's functions. Their relation to the bare Green's functions Γ_{B} will be

$$\Gamma_R = Z(R)\Gamma_{\mathrm{B}}$$

where $Z(R)$ denotes the appropriate product of renormalization constants defined within the R scheme. If we choose another renormalization scheme R' then,

$$\Gamma_{R'} = Z(R')\Gamma_{\mathrm{B}}$$

Obviously, there exists a relation between both renormalized Green's functions, namely

$$\Gamma_{R'} = Z(R', R)\Gamma_R$$

where

$$Z(R', R) = Z(R')/Z(R)$$

Let us now consider the set of all possible $Z(R', R)$ for arbitrary R and R'. Among the elements of this set there is the composition law

$$Z(R'', R) = Z(R'', R')Z(R', R) \tag{6.1}$$

and, in addition, to each element $Z(R', R)$ there is an inverse

$$Z^{-1}(R', R) = Z(R, R')$$

and we can define the unit element as

$$Z(R, R) = 1$$

The composition law (6.1) is not always defined, however, for arbitrary pairs of Zs: the product

$$Z(R_i, R_j)Z(R_k, R_l) \tag{6.2}$$

is not, in general, an element of the set $Z(R', R)$, unless $R_j = R_k$. A simple example of the above considerations is provided by the photon wave-function renormalization constant Z_3 (de Rafael 1979).

Suppose we work in the μ-subtraction scheme and we want to relate two renormalizations performed at subtraction points μ_1 and μ_2. The relevant quantity will be $Z(\mu_1, \mu_2)$ which in the one-electron loop approximation reads

$$Z(\mu_2, \mu_1) = 1 + \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\, 2x(1-x)\ln\frac{\mu_2^2 x(1-x) + m^2}{\mu_1^2 x(1-x) + m^2} \tag{6.3}$$

This representation of Z obeys the composition law (6.1) but the group multiplication law (6.2) is not obeyed, unless $m = 0$.

In this section we shall discuss the RGE in renormalization schemes such that renormalization constants Z_i are mass-independent (mass-independent renormalization schemes). These include the minimal subtraction prescription and, of course, in a trivial way any prescription for a massless theory.

The RGE in, for example, $\lambda\Phi^4$ theory can be derived as follows. We begin with the relation between the renormalized and bare 1PI Green's functions in the theory regularized by the dimensional method, i.e. in $4-\varepsilon$ dimensions

$$\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu, \varepsilon) = Z_3^{n/2}\Gamma_{\mathrm{B}}^{(n)}(p_i, \lambda_{\mathrm{B}}, m_{\mathrm{B}}, \varepsilon) \tag{6.4}$$

(for the definition of the renormalization constants Z_i see Section 4.1). The finite limit (for $\varepsilon \to 0$) of (6.4) exists since the theory is renormalizable. The bare Green's function $\Gamma_{\mathrm{B}}^{(n)}(p_i, \lambda_{\mathrm{B}}, m_{\mathrm{B}}, \varepsilon)$ is in an obvious way independent of the renormalization scale μ. The RGE will be derived by differentiating relation (6.4) with respect to the renormalization scale μ for fixed λ_{B}, dimensionful in $4-\varepsilon$ dimensions, m_{B} and ε. In the mass-independent renormalization scheme we have

$$Z_i = Z_i(\lambda_R, \varepsilon)$$

where λ_R is defined as dimensionless, i.e.

$$\lambda_{\mathrm{B}}\mu^{-\varepsilon} = Z_3^{-2}Z_1\lambda_R$$

Similarly, we have

$$m_{\mathrm{B}}^2 = Z_3^{-1}Z_0 m_R^2$$

Differentiating (6.4) and using $\mathrm{d}\Gamma_{\mathrm{B}}/\mathrm{d}\mu = 0$, one obtains

$$\begin{aligned}
\mu\left(\frac{\partial}{\partial\mu} + \frac{\mathrm{d}\lambda_R}{\mathrm{d}\mu}\frac{\partial}{\partial\lambda_R} + \frac{\mathrm{d}m_R}{\mathrm{d}\mu}\frac{\partial}{\partial m_R}\right)&\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu, \varepsilon) \\
&= \left(\tfrac{1}{2}nZ_3^{n/2-1}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_3\right)\Gamma_{\mathrm{B}}^{(n)}(p_i, \lambda_{\mathrm{B}}, m_{\mathrm{B}}, \varepsilon) \\
&= \left(\tfrac{1}{2}n\frac{1}{Z_3}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_3\right)\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu, \varepsilon)
\end{aligned} \tag{6.5}$$

We can define now the following functions:†

$$\beta(\lambda_R, \varepsilon) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda_R = \lambda_R\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_3^3Z_1^{-1}\right) - \varepsilon\lambda_R \xrightarrow[\varepsilon\to 0]{} \beta(\lambda_R) \tag{6.6}$$

$$\gamma(\lambda_R, \varepsilon) = \tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_3 \xrightarrow[\varepsilon\to 0]{} \gamma(\lambda_R) \tag{6.7}$$

$$\gamma_{\mathrm{m}}(\lambda_R, \varepsilon) = \frac{\mu}{m_R}\frac{\mathrm{d}m_R}{\mathrm{d}\mu} = \tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(Z_3Z_0^{-1}) \xrightarrow[\varepsilon\to 0]{} \gamma_{\mathrm{m}}(\lambda_R) \tag{6.8}$$

† Similar considerations apply to additive renormalization.

The differential equations (6.6) and (6.8) simply express the μ-independence of the bare parameters λ_{B} and m_{B}. Taking the limit $\varepsilon \to 0$ in (6.5) (finite limits of relations (6.5)–(6.8) exist since the theory is renormalizable) we obtain the final result

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_{\mathrm{m}}(\lambda_R)m_R\frac{\partial}{\partial m_R} - n\gamma(\lambda_R)\right]\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu) = 0 \tag{6.9}$$

It is important to remember that only in mass-independent renormalization schemes or for $m = 0$ do the functions $\beta, \gamma_{\mathrm{m}}$ and γ depend on λ_R only. In general they also can depend on the ratio m_R/μ. This happens, for instance, in the μ-subtraction scheme.

Solving the RGE

Eq. (6.9) describes the change of renormalized parameters and multiplicative scaling factors for the renormalized Green's functions which compensates a change of the renormalization scale μ. To solve (6.9) suppose for a moment that $\gamma(\lambda_R) \equiv 0$. We then have a homogeneous equation $\mu(\mathrm{d}/\mathrm{d}\mu)\Gamma_R^{(n)})(p_i, \lambda_R, m_R, \mu) = 0$ which can be written in the form (subscript R in λ_R and m_R will be omitted from now on):

$$\left[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_{\mathrm{m}}(\lambda)m\frac{\partial}{\partial m}\right]\Gamma_R^{(n)}(p_i, \lambda, m, \mu_0\exp(-t)) = 0 \tag{6.10}$$

where $\mu = \exp(-t)\mu_0$, and μ_0 is some fixed scale. We recall also that the parameters λ and m are renormalized at μ. As we shall see the meaning of this equation is that the dependence of the function $\Gamma_R^{(n)}$ on t and hence on μ can be expressed solely through the functions $\bar{\lambda}(t, \lambda)$, $\bar{m}(t, \lambda, m)$ defined by the equations

$$\left.\begin{aligned} &\int_{\lambda}^{\bar{\lambda}(t,\lambda)} \mathrm{d}x/\beta(x) = t \\ &\bar{m}(t, \lambda, m) = m\exp\left[\int_{\lambda}^{\bar{\lambda}(t,\lambda)} \mathrm{d}x\ (\gamma_{\mathrm{m}}(x)/\beta(x))\right] = m\exp\left[\int_0^1 \gamma_{\mathrm{m}}(\bar{\lambda}(t'))\,\mathrm{d}t'\right] \end{aligned}\right\} \tag{6.11}$$

Taking the partial derivative $\partial/\partial t$ one can rewrite (6.11) as differential equations

$$\left.\begin{aligned} \partial\bar{\lambda}(t, \lambda)/\partial t &= \beta(\bar{\lambda}) \\ \partial\bar{m}(t, \lambda, m)/\partial t &= \gamma_{\mathrm{m}}(\bar{\lambda})\bar{m} \end{aligned}\right\} \tag{6.12}$$

in accord with (6.6) and (6.8), with the boundary conditions

$$\bar{\lambda}(0, \lambda) = \lambda, \qquad \bar{m}(0, \lambda, m) = m$$

We also observe that (6.11) can be written in the form of the RGEs; for instance, differentiating $\bar{\lambda}(t, \lambda)$ given by the first equation of (6.11) with respect to λ one gets

$$\left[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda}\right]\bar{\lambda}(t, \lambda) = 0 \tag{6.13}$$

Using (6.13) we can easily check that any function $f(\lambda, t, \mu_0) = f(\bar{\lambda}(\lambda, t), \mu_0)$ is μ-independent and satisfies the equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} f = \left(-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda}\right) f(\bar{\lambda}(t, \lambda), \mu_0) = 0 \tag{6.14}$$

Therefore the solution to (6.10) can be written as follows

$$\Gamma_R^{(n)}(p_i, \lambda, m, \underbrace{\mu_0 \exp(-t)}_{\mu}) = \Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \underbrace{\mu_0}_{\mu \exp(t)}) \tag{6.15}$$

where $\bar{\lambda}(t)$ stands for $\bar{\lambda}(t, \lambda)$. The solution to the inhomogeneous (6.9) reads

$$\Gamma_R^{(n)}(p_i, \lambda, m, \mu) = \Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \mu \exp(t)) \exp\left[-n \int_0^1 \gamma(\bar{\lambda}(t'))\, \mathrm{d}t'\right] \tag{6.16}$$

as can be checked by applying to it the differential operator of (6.9). Eq. (6.16) relates a Green's function $\Gamma_R^{(n)}(p_i, \lambda, m, \mu)$, with the renormalized parameters equal to λ and m at the renormalization scale μ, to the Green's function $\Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \mu \exp(t))$, where $\bar{\lambda}(t)$ and $\bar{m}(t)$ can be understood as the renormalized parameters corresponding to the scale $\mu \exp(t)$.

The renormalized parameters are renormalization-scale dependent. Any physical quantity P (renormalization-scheme-independent) can be calculated in terms of them: $P = P(m, \lambda, \mu)$. Being by definition renormalization-scheme-independent P must satisfy the homogeneous RGE:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} p = \left[\mu\frac{\partial}{\partial \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_{\mathrm{m}}(\lambda) m \frac{\partial}{\partial m}\right] P(\lambda, m, \mu) \equiv \mathcal{R} P(\lambda, m, \mu) = 0 \tag{6.17}$$

We shall now check, using (6.9), that this is indeed the case for the physical mass defined as the position of a pole in a propagator and for renormalized S-matrix elements. Let us consider the scalar propagator $\Delta(p^2)$ which satisfies the equation

$$[\mathcal{R} + 2\gamma(\lambda)]\Delta(p^2, \lambda, m, \mu) = 0 \tag{6.18}$$

Assuming that Δ has a pole at $p^2 = m_{\mathrm{F}}^2$, we can write a Laurent expansion for Δ:

$$\Delta(p^2, \lambda, m, \mu) = \frac{R}{p^2 - m_{\mathrm{F}}^2} + \tilde{\Delta}(p^2, \lambda, m, \mu) \tag{6.19}$$

The position of the pole, $m_{\mathrm{F}}(\lambda, m, \mu)$ and the residue of the pole $R(\lambda, m, \mu)$ themselves satisfy RGEs which can be derived by using (6.19) in (6.18) and equating the residua of the poles at $p^2 = m_{\mathrm{F}}^2$

$$\left.\begin{aligned} &\mathcal{R} m_{\mathrm{F}}^2(m, \lambda, \mu) = 0 \\ &[\mathcal{R} + 2\gamma(\lambda)] R(\lambda, m, \mu) = 0 \end{aligned}\right\} \tag{6.20}$$

We consider next an arbitrary n-scalar particle S-matrix element. This is obtained from the 1PI Green's function $\Gamma^{(n)}$ by going to the physical poles $p_i^2 = m_i^2$ and multiplying by $R^{n/2}$ (see (4.27)). Since $\Gamma^{(n)}$ satisfies (6.9) and m_i^2 and R satisfy (6.20), we have

$$\mathcal{R} \lim_{p_i^2 \to m_i^2} R^{n/2} \Gamma^{(n)} = \lim_{p_i^2 \to m_i^2} \mathcal{R} R^{n/2} \Gamma^{(n)} = \lim_{p_i^2 \to m_i^2} [n\gamma(\lambda) - n\gamma(\lambda)] R^{n/2} \Gamma^{(n)} = 0 \tag{6.21}$$

One must stress that it is an exact S-matrix element which is renormalization-scheme-independent. In perturbation theory truncated at order N there is always a residual renormalization scheme dependence $O(\lambda^{N+1})$.

Green's functions for rescaled momenta

The RGE is used most frequently in a somewhat different context, namely to discuss the change in the Green's functions when all momenta are rescaled $p_i \to \rho p_i$ (but for fixed μ). Let us take a Green's function of dimension $D : \Gamma_R^{(n)} \sim [M]^D$. On dimensional grounds it can be therefore written as follows

$$\Gamma_R^{(n)}(\rho p_i, m, \lambda, \mu) = \mu^D f(\rho^2 p_i \cdot p_j/\mu^2, m/\mu, \lambda) \tag{6.22}$$

We observe that $\Gamma_R^{(n)}$, is a homogeneous function of order D of variables m, ρ and μ. Using the basic property of such functions we can write for it the following equation:

$$\left(\rho\frac{\partial}{\partial\rho} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - D\right)\Gamma_R^{(n)}(\rho p_i, m, \lambda, \mu) = 0 \tag{6.23}$$

Combining (6.9) and (6.23) one gets ($t = \ln\rho$)

$$\left[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial\lambda} + (\gamma_{\mathrm{m}} - 1)m\frac{\partial}{\partial m} - n\gamma(\lambda) + D\right]\Gamma_R^{(n)}(\rho p_i, m, \lambda, \mu) = 0 \tag{6.24}$$

which is the result we are looking for.

We notice first that for $m = 0$ and for a non-interacting theory ($\beta \equiv \gamma \equiv 0$) the solution to (6.24) reads

$$\Gamma_R^{(n)}(\rho p_i, \lambda, \mu) = \rho^D \Gamma_R^{(n)}(p_i, m, \lambda, \mu) \tag{6.25}$$

i.e. the Green's functions exhibit canonical scaling (see Chapter 7). The general solution to (6.24) can be written using arguments similar to those used to solve (6.9) and it reads

$$\Gamma_R^{(n)}(\exp(t)p_i, \lambda, m, \mu) = \Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{\bar{m}}(t), \mu)\exp\left[Dt - n\int_0^t \gamma(\bar{\lambda}(t'))\,\mathrm{d}t'\right] \tag{6.26}$$

where

$$\bar{\bar{m}}(t) = \bar{m}(t)/\rho$$

and $\bar{\lambda}(t)$ and $\bar{m}(t)$ satisfy (6.11). This solution can also be obtained directly by the following steps

$$\begin{aligned}\Gamma_R^{(n)}(\exp(t)p_i, \lambda, m, \mu) &= \Gamma_R^{(n)}(\exp(t)p_i, \bar{\lambda}(t), \bar{m}(t), \exp(t)\mu)\exp\left[-n\int\gamma(t')\,\mathrm{d}t'\right] \\ &= \Gamma_R^{(n)}(\exp(t)p_i, \bar{\lambda}(t), \exp(t)\exp(-t)\bar{m}(t), \exp(t)\mu) \\ &\quad\times\exp\left(-n\int\gamma\,\mathrm{d}t'\right) \\ &= \rho^D\Gamma_R^{(n)}(p_i, \bar{\lambda}(t), \bar{\bar{m}}(t), \mu)\exp\left(-n\int\gamma\,\mathrm{d}t'\right)\end{aligned}$$

The first step corresponds to a change of the renormalization point according to (6.16) and the last step is a change of the mass scale by a factor ρ. We notice that (6.26), unlike (6.16), relates Green's functions which both have the same renormalization scale μ, but different values of the renormalized couplings: λ, m in the l.h.s. and $\bar{\lambda}, \bar{\bar{m}}(t)$ in the r.h.s. One can say, therefore, that they are Green's functions of different theories.

Result (6.26) gives us Green's functions for the rescaled momenta $\exp(t)p_i$ (off-mass shell) in terms of the 'effective' parameters $\bar{\lambda}(t)$ and $\bar{m}(t)$ which are solutions of (6.11). To use these results in physical applications one must know the functions $\beta(\lambda)$, $\gamma(\lambda)$ and $\gamma_{\mathrm{m}}(\lambda)$ which can only be calculated in perturbation theory.

Strictly speaking the RGE solution (6.26) applies in the deep Euclidean region, $p_i^2 < 0$, away from possible IR problems and new thresholds associated with time-like momenta.

RGE in QED

We choose to work in the set of mass-independent renormalization schemes and in a covariant gauge defined by the gauge-fixing term: $-(1/a)(\partial_\mu A^\mu)^2$. The RGE can then be derived following the steps outlined previously and we get

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha,a)\frac{\partial}{\partial\alpha}+\gamma_m(\alpha,a)m\frac{\partial}{\partial m}+\delta(\alpha,a)\frac{\partial}{\partial a}-n_\gamma\gamma_\gamma(\alpha,a)\right.$$
$$\left.-n_{\mathrm{F}}\gamma_{\mathrm{F}}(\alpha,a)\right]\Gamma^{(n_\gamma,n_{\mathrm{F}})}(p_i,\alpha,m,\mu,a)=0 \tag{6.27}$$

where in analogy to (6.6)–(6.8)

$$\left.\begin{aligned}\beta(\alpha,a)&=\lim_{\varepsilon\to0}\beta(\alpha,a,\varepsilon)=\lim_{\varepsilon\to0}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha\\ \gamma_{\mathrm{m}}(\alpha,a)&=\lim_{\varepsilon\to0}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_{\mathrm{m}}\\ \gamma_\gamma(\alpha,a)&=\lim_{\varepsilon\to0}\tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_3\\ \gamma_{\mathrm{F}}(\alpha,a)&=\lim_{\varepsilon\to0}\tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_2\end{aligned}\right\} \tag{6.28}$$

Derivatives are taken for fixed α_{B} and ε, where $\alpha = e^2/4\pi$, $\alpha_{\mathrm{B}} = \mu^\varepsilon Z_3^{-1}\alpha$ in $n = 4-\varepsilon$ dimensions, $m_{\mathrm{B}} = Z_{\mathrm{m}}^{-1}m$, $A_{\mu\mathrm{B}} = Z_3^{1/2}A_\mu$, $\Psi_{\mathrm{B}} = Z_2^{1/2}\Psi$ (for definitions of the renormalization constants, see Chapter 5). The only new element is the μ-dependence of the gauge-fixing parameter $\alpha = Z_3^{-1}a_{\mathrm{B}}$ which is responsible for the term $\delta(\alpha,a)\partial/\partial a$ with

$$\delta(\alpha,a)=\lim_{\varepsilon\to0}\mu\frac{\mathrm{d}a}{\mathrm{d}\mu}=\lim_{\varepsilon\to0}\left(-a\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_3\right) \tag{6.29}$$

The solution to (6.27) is obtained by the obvious modification of (6.16) with the running parameters given by (6.11) and, in addition,

$$\partial\bar{a}(t,\alpha,a)\partial t=\delta(\bar{\alpha},\bar{a}),\quad \bar{a}(0,\alpha)=a \tag{6.30}$$

Using our previous experience we can also derive the RGE analogous to (6.26) for QED.

6.2 Calculation of the renormalization group functions β, γ, γ_{m}

We address ourselves to the problem of calculating functions, $\beta(\lambda)$, $\gamma(\lambda)$ and $\gamma_{\mathrm{m}}(\lambda)$ in a mass-independent renormalization scheme. We recall the definitions (6.6)–(6.8), for example,

$$\beta(\lambda)=\lim_{\varepsilon\to0}\beta(\lambda,\varepsilon)=\lim_{\varepsilon\to0}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda\bigg|_{\text{fixed }\lambda_{\mathrm{B}},\varepsilon} \tag{6.31}$$

In perturbation theory we can calculate the renormalization constants Z_i which relate λ to λ_B

$$\lambda_B = \mu^\varepsilon Z_\lambda \lambda = \mu^\varepsilon \left[1 + \sum_\nu \frac{a_\nu(\lambda)}{\varepsilon^\nu}\right]\lambda \tag{6.32}$$

where, for instance, $Z_\lambda = Z_3^{-2} Z_1$ for the $\lambda\Phi^4$ theory. The coefficients $a_\nu(\lambda)$ are the series in λ

$$\begin{aligned} a_1(\lambda) &= a_{11}\lambda + a_{12}\lambda^2 + \cdots + a_{1n}\lambda^n \\ &\vdots \\ a_m(\lambda) &= a_{m1}\lambda + a_{m2}\lambda^2 + \cdots + a_{mn}\lambda^n \end{aligned}$$

Using (6.6) we get

$$\beta(\lambda, \varepsilon) = -\varepsilon\lambda - \lambda Z_\lambda^{-1} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_\lambda \tag{6.33}$$

Since

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_\lambda = \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} = \beta(\lambda, \varepsilon) \frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda}$$

we can rewrite (6.33) in the following form

$$\beta(\lambda, \varepsilon) Z_\lambda + \varepsilon\lambda Z_\lambda + \beta(\lambda, \varepsilon)\lambda\, \mathrm{d}Z_\lambda/\mathrm{d}\lambda = 0 \tag{6.34}$$

Substituting expansion (6.32) in (6.34) we look for the solution for $\beta(\lambda, \varepsilon)$ in the form of a series $\beta(\lambda, \varepsilon) = \sum_\nu \beta_\nu \varepsilon^\nu$ which is regular at $\varepsilon = 0$ (since our theory is renormalizable a finite limit of $\beta(\lambda, \varepsilon)$, for $\varepsilon \to 0$, exists). Equating the residua of the poles in ε we obtain the following final result

$$\left.\begin{aligned} \beta(\lambda, \varepsilon) &= -\varepsilon\lambda + \beta(\lambda) \\ \beta(\lambda) &= \lambda^2 \frac{\mathrm{d}a_1}{\mathrm{d}\lambda} \\ \lambda^2 \frac{\mathrm{d}a_{\nu+1}}{\mathrm{d}\lambda} &= \beta(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda a_\nu) \end{aligned}\right\} \tag{6.35}$$

We arrive, therefore, at two important conclusions

(i) $\beta(\lambda)$ is totally determined by the residua of the simple poles in ε of Z_λ.

(ii) The residua of the higher order poles of Z_λ are totally determined by $a_1(\lambda)$. This reflects an important aspect of renormalization theory. In each successive order of perturbation theory only one new divergence arises, i.e. the simple pole in ε. There will, of course, be multiple poles, up to $1/\varepsilon^N$ in the Nth order but their residua are determined by lower order perturbation expansion. One can easily check in particular that $a_{mn} = 0$ for $m < n$.

In a quite analogous way we also find the results for $\gamma(\lambda)$ and $\gamma_{\mathrm{m}}(\lambda)$:

$$\gamma(\lambda) = -\tfrac{1}{2}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} Z_3^{(1)}(\lambda) \tag{6.36}$$

where

$$Z_3 = 1 + \sum_\nu \frac{Z_3^{(\nu)}}{\varepsilon^\nu}$$

and

$$\gamma_{\mathrm{m}}(\lambda) = \tfrac{1}{2}\lambda \frac{\mathrm{d}Z_{\mathrm{m}}^{(1)}}{\mathrm{d}\lambda} \tag{6.37}$$

where

$$m_{\mathrm{B}}^2 = Z_{\mathrm{m}} m^2, \qquad Z_{\mathrm{m}} = 1 + \sum_\nu \frac{Z_{\mathrm{m}}^{(\nu)}}{\varepsilon^\nu}$$

($Z_{\mathrm{m}} = Z_0 Z_3^{-1}$ in the $\lambda\Phi^4$ theory).

Using the results of Section 4.3 for the renormalization constants Z_i in the $\lambda\Phi^4$ theory

$$\left.\begin{aligned} Z_1 &= 1 + \frac{3\lambda}{16\pi^2\varepsilon} \\ Z_0 &= 1 + \frac{\lambda}{16\pi^2\varepsilon} - \frac{\lambda^2}{2(16\pi^2)\varepsilon} + \frac{2\lambda^2}{(16\pi^2)^2\varepsilon^2} \\ Z_3 &= 1 - \frac{\lambda^2}{12(16\pi^2)\varepsilon} \end{aligned}\right\} \tag{6.38}$$

and with help of (6.35), (6.36) and (6.37) one obtains the following renormalization group functions in the $\lambda\Phi^4$ theory:

$$\left.\begin{aligned} \beta(\lambda) &= \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \\ \gamma_{\mathrm{m}}(\lambda) &= \frac{1}{2}\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{(16\pi^2)^2} + O(\lambda^3) \\ \gamma(\lambda) &= \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + O(\lambda^3) \end{aligned}\right\} \tag{6.39}$$

For QED relation (6.32) is replaced by $\alpha_{\mathrm{B}} = \mu^\varepsilon Z_\alpha \alpha$ with $Z_\alpha = Z_3^{-1}$. Using (5.31) we get

$$Z_\alpha = 1 + \frac{\alpha}{\pi}\frac{2}{3}\frac{1}{\varepsilon} + \left(\frac{\alpha}{\pi}\right)^2 b_2 + \cdots$$

and therefore, from (6.35)

$$\beta(\alpha) = \alpha\left[\frac{\alpha}{\pi}\frac{2}{3} + \left(\frac{\alpha}{\pi}\right)^2 b_2 + \cdots\right] \tag{6.40}$$

Though not calculated explicitly, the second order contribution $(\alpha/\pi)^2 b_2$ has been singled out for the sake of the discussion later. Note also that $\beta(e)$ defined as $\beta(e) = \mu(\mathrm{d}/\mathrm{d}\mu)e$ is given by $\beta(e) = 2\pi\beta(\alpha)/e$. Therefore

$$\beta(e) = \frac{e^3}{16\pi^2}\frac{4}{3} + \frac{e^5}{(16\pi^2)^2}8b_2 + \cdots \tag{6.41}$$

6.3 Fixed points; effective coupling constant

Fixed points

The main use of the renormalization group is in discussing the large, or small, momentum behaviour of a quantum field theory which is determined, as follows from (6.26) by the behaviour of the 'effective' coupling constant $\bar{\lambda}(t)$ as $t \to \pm\infty$. The latter is determined by

$$\int_{\lambda}^{\bar{\lambda}(t,\lambda)} \mathrm{d}x/\beta(x) = t \tag{6.11}$$

where $\beta(\lambda)$ can be calculated perturbatively from (6.35). Eq. (6.11) provides a basis for a physically very important classification of different theories. We shall discuss here only theories with one coupling constant. First we assume (6.11) to be valid in the whole range $-\infty < t < \infty$. Otherwise the renormalization scale μ could not vary arbitrarily and the theory would need some natural cut-off Λ. A renormalizable theory with perturbative expansion valid only in a certain energy range cut-off by a scale parameter Λ (if, for example, new interactions switch on at a mass scale Λ) is not necessarily uninteresting physically and we shall come back to this point later.

The solution $\bar{\lambda}(t, \lambda)$ of (6.11) must for $t \to \infty$ and $t \to -\infty$ approach the zero of $\beta(x)$ nearest to λ or go to infinity if there is no zero to approach. A zero of $\beta(x)$ is called a fixed point. Let $\tilde{\lambda}$ be the fixed point nearest to λ. If

$$\lim_{t\to\infty} \bar{\lambda}(t, \lambda) = \tilde{\lambda} \equiv \lambda_+ \tag{6.42}$$

it is called a UV stable fixed point; if

$$\lim_{t\to-\infty} \bar{\lambda}(t, \lambda) = \tilde{\lambda} \equiv \lambda_- \tag{6.43}$$

it is called an IR stable fixed point.

In both cases one says that λ is in the domain of attraction of the fixed point $\tilde{\lambda}$. A

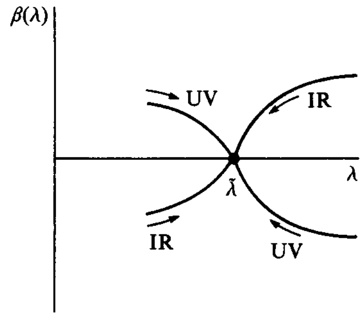


Fig. 6.1.

domain is a region which lies between two zeros of β. Of course, for two different values of the coupling λ lying in two different domains one, in general, has two different theories (or two different phases of one theory). The nature of the fixed point is determined, if it is a simple zero, by the sign of the derivative $\beta'(\lambda)$ at $\lambda = \tilde{\lambda}$. From (6.11) and Fig. 6.1 it is clear that for

$$\beta'(\lambda)|_{\tilde{\lambda}} < 0, \ \tilde{\lambda} \text{ is a UV stable fixed point;}$$
$$\beta'(\lambda)|_{\tilde{\lambda}} > 0, \ \tilde{\lambda} \text{ is an IR stable fixed point.}$$

In other words, both for $\lambda < \tilde{\lambda}$ and $\lambda > \tilde{\lambda}$, to satisfy (6.11) the effective coupling $\bar{\lambda}(t, \lambda) \rightarrow \tilde{\lambda}$ for $t \rightarrow \infty$ if $\beta'(\lambda)|_{\tilde{\lambda}} < 0$, and $\bar{\lambda}(t, \lambda) \rightarrow \tilde{\lambda}$ for $t \rightarrow -\infty$ if $\beta'(\lambda)|_{\tilde{\lambda}} > 0$. (If $\tilde{\lambda}$ is a multiple zero of $\beta(\lambda)$, of odd order M, then the $\tilde{\lambda}$ is a UV (IR) stable fixed point for $\beta(\lambda)/(\lambda - \tilde{\lambda})^M < 0$ (> 0).)

It is obvious but important to notice that for all single coupling theories the value $\lambda = 0$ is a fixed point ($\beta(0) = 0$ in perturbation theory). A theory is said to be asymptotically free if $\lambda = 0$ is a UV stable fixed point and to be IR stable if it is an IR stable fixed point. All single coupling theories are either asymptotically free or IR stable. In an asymptotically free theory the effective coupling vanishes for infinite momenta. For theories with several couplings the origin may be UV stable for some couplings and IR stable for others. The $\lambda\Phi^4$ theory (with $\lambda \geq 0$) and QED are IR stable ($\beta(\lambda)$ is positive for small λ) whereas QCD is asymptotically free ($\beta(\lambda)$ is negative for small λ). Of course, for a given theory, the only place that we can compute $\beta(\lambda)$ in perturbation theory is near $\lambda = 0$ and its behaviour for increasing λ is essentially a speculation. Consequently, the behaviour of the coupling constant $\bar{\lambda}(t, \lambda)$ for $t \rightarrow \infty$ in $\lambda\Phi^4$ and QED, and for $t \rightarrow -\infty$ in QCD also remains, strictly speaking, a speculation. In Fig. 6.2 we show some examples of what $\beta(\lambda)$ might look like.

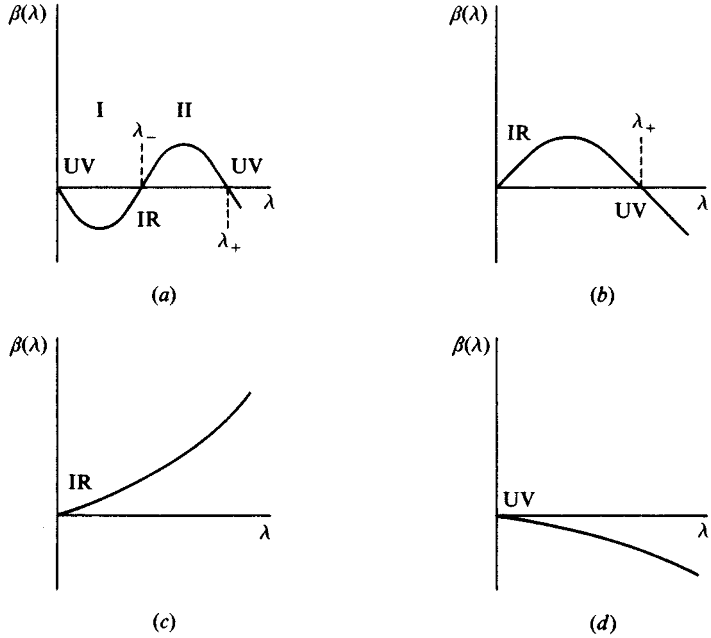


Fig. 6.2.

In Fig. 6.2(a) we have distinguished two domains of attraction. If λ is in region I, the theory is asymptotically free and for $t \to -\infty$ (low momentum) λ approaches the fixed point λ_-. On the other hand, if λ is in the region II then the theory is not asymptotically free; instead $\bar{\lambda}(t, \lambda) \to \lambda_+$ when $t \to \infty$. Fig. 6.2(b) shows what might happen in QED: IR stability and fixed point λ_+ for $t \to \infty$. Fig. 6.2(c) shows another alternative for QED and Fig. 6.2(d) shows what we hope happens in QCD. In Fig. 6.2(d) there is no IR stable fixed point. To satisfy (6.11)

$$\int_{\lambda}^{\bar{\lambda}(-\infty,\lambda)} \mathrm{d}x/\beta(x) = -\infty$$

$\bar{\lambda}(t, \lambda)$ must diverge when $t \to -\infty$ and $\beta(x)$ must be such that $\int_{\lambda}^{\infty} \mathrm{d}x/\beta(x)$ is divergent. If the effective coupling becomes infinite in the IR region one often refers to this situation as IR slavery. It is generally believed (but not proved) that IR slavery in QCD is responsible for confinement.

Finally, suppose that the β-function is always positive (as in Fig. 6.2(c)) and that it diverges at infinity in such a way that

$$\int_{\lambda}^{\infty} \mathrm{d}x/\beta(x) = t_c < \infty$$

It is likely that this situation occurs in the $\lambda\Phi^4$ theory (see (6.39)). For such a theory the renormalized perturbative expansion cannot be consistent for all energies (unless $\lambda(\mu_0) = 0$ for any μ_0). The theory must be modified by new interactions which switch on at some mass scale Λ_c which provides a natural cut-off or must undergo a phase transition at Λ_c. For the theory to be valid up to Λ_c the following equation must be then satisfied

$$\int_{\lambda(\mu_0)}^{\bar{\lambda}(\Lambda_c,\lambda)} \mathrm{d}x/\beta(x) = \ln(\Lambda_c, \mu_0) \tag{6.44}$$

For a given Λ_c the equation gives upper bounds for the low energy coupling constants (assuming $\bar{\lambda}(\Lambda_c, \lambda) = \infty$) which, in turn, can be transformed into an upper bound for the number of elementary particles in the theory or their masses (see Problem 6.4).

Effective coupling constant

We will now study the explicit behaviour of the effective coupling constant in perturbation theory. With $\beta(\alpha)$, where $\alpha = \lambda$ for $\lambda\Phi^4$ and $\alpha = e^2/4\pi$ for QED and QCD, given by an expansion

$$\beta(\alpha) = \alpha[(\alpha/\pi)b_1 + (\alpha/\pi)^2 b_2 + (\alpha/\pi)^3 b_3 + \cdots] \tag{6.45}$$

we can calculate $\alpha(t)$ integrating (6.12) order-by-order in α. We expand

$$\alpha(t) = \sum_{m=1} a_m(t)\alpha^m, \quad a_1 = 1 \tag{6.46}$$

and

$$\beta(\alpha(t)) = \sum_{n=1} \alpha^{n+1}(t) b_n/\pi^n \tag{6.47}$$

Therefore

$$\sum_{m=2} \frac{\mathrm{d}a_m(t)}{\mathrm{d}t}\alpha^m = \sum_{n=1} \frac{b_n}{\pi^n}\left[\alpha + \sum_{m=2} a_m(t)\alpha^m\right]^{n+1} \tag{6.48}$$

Comparing the coefficients at the same powers of α and integrating over t one gets

$$\begin{aligned} a_2(t) &= b_1 t/\pi \\ a_3(t) &= (b_1 t/\pi)^2 + b_2 t/\pi \\ &\vdots \\ a_n(t) &= (b_1 t/\pi)^{n-1} + O(t^{n-2}) \end{aligned} \tag{6.49}$$

so that

$$\alpha(t) = \alpha\left[1 + \sum_{n=1} (b_1 t\alpha/\pi)^n + O(t^{n-1}\alpha^n)\right] \tag{6.50}$$

For large $t = \frac{1}{2}\ln(\mu^2/\mu_0^2)$ or $t = \frac{1}{2}\ln(p^2/p_0^2)$, depending on whether we use (6.16) or (6.26), we can consider the so-called leading logarithm approximation in which we neglect terms $O(t^{n-1}\alpha^n)$ as compared to the terms $O(t^n\alpha^n)$. In this approximation the effective coupling constant reads

$$\alpha(t) = \frac{\alpha}{1-(\alpha/\pi)b_1 t}, \quad \alpha = \alpha(t=0) \tag{6.51}$$

The same result can be obtained by expanding $1/\beta(x)$ in the integral (6.11) in powers of x and directly integrating over x. In this way one gets the series

$$b_1 t = \left[\frac{\pi}{\alpha} - \frac{\pi}{\alpha(t)}\right] + B_1 \ln\frac{\alpha}{\alpha(t)} + \sum_{n=1} \frac{C_n}{n}\left\{\left[\frac{\alpha(t)}{\pi}\right]^n - \left(\frac{\alpha}{\pi}\right)^n\right\} \tag{6.52}$$

where

$$B_n = b_{n+1}/b_1$$

and

$$C_n = (-1)^{n+1}\begin{vmatrix} B_1 & B_2 & \cdots & & B_{n+1} \\ 1 & B_1 & \cdots & & B_n \\ 0 & 1 & B_1 & \cdots & B_{n-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & B_1 \end{vmatrix}$$

Truncating $1/\beta(x)$ at a given order in x gives us the corresponding approximate solution for $\alpha(t)$. In the first approximation ($b_1 \neq 0$, $b_n = 0$ for $n > 1$) we recover the leading-logarithm-result (6.51). In the second approximation ($b_1 \neq 0$, $b_2 \neq 0$, $b_n = 0$ for $n > 2$) we get the following

$$\alpha(t) = \alpha_1(t)\left[1 - \frac{\alpha_1(t)}{\pi}\frac{b_2}{b_1}\ln\left(1 - \frac{\alpha}{\pi}b_1 t\right)\right] \tag{6.53}$$

where $a_1(t)$ is called the leading-logarithm-result (6.51).

In QED $b_1 = +\frac{2}{3}$ (Section 5.2) and $b_2 = \frac{1}{2}$ (de Rafael & Rosner 1974). Taking the limit $t \to -\infty$ in (6.53),

$$\alpha(t) \underset{t\to-\infty}{=} -\frac{\pi}{b_1 t} + \frac{\pi b_2}{b_1^3 t^2}\ln\frac{1}{t} + O\left(\frac{1}{t^2}\right) \tag{6.54}$$

we see explicitly that the point $\alpha = 0$ is an IR stable fixed point. In QCD $b_1 < 0$ and $b_2 < 0$, therefore, $\alpha = 0$ is a UV stable fixed point.

Another renormalization group parameter is the running mass $\bar{m}(t, \mu, \lambda)$

$$\bar{m}(t, m, \lambda) = m \exp\left\{\int_{\lambda}^{\bar{\lambda}(t,\lambda)} [\gamma_{\mathrm{m}}(x)/\beta(x)]\,\mathrm{d}x\right\} \tag{6.55}$$

The behaviour of the Green's functions for rescaled momenta $\exp(+t)p_i$ is controlled by $\exp(-t)\bar{m}(t)$. In particular, in a theory with the UV fixed point λ_+ one gets

$$\exp(-t)\bar{m}(t) = m\exp\{-t[1-\gamma_{\mathrm{m}}(\lambda_+)]\}\exp\left\{-\int_{\lambda}^{\bar{\lambda}(t)}[\mathrm{d}x/\beta(x)][\gamma_{\mathrm{m}}(\lambda_+)-\gamma_{\mathrm{m}}(x)]\right\} \tag{6.56}$$

Whether in the high energy limit masses can be neglected or not depends on $\gamma_{\mathrm{m}}(\lambda_+)$ and on the integral in the exponent. For asymptotically free theories $\lambda_+ = 0$ (and $\gamma_{\mathrm{m}}(0) = 0$) so only the integral is responsible for the large t limit of $\bar{m}(t)$, and, for example, for QCD, mass corrections are suppressed by $\exp(-t)t^p$ as $t \to \infty$, with $p > 0$.

6.4 Renormalization scheme and gauge dependence of the RGE parameters

Renormalization scheme dependence

The last problem we would like to discuss in this chapter is the renormalization-scheme dependence within the class of mass-independent renormalization procedures and the gauge dependence of the perturbative results for the functions β, γ_{m} and γ. The parameters of one renormalization scheme can be determined in terms of those in another scheme. If λ is the coupling in one scheme, another one will give a coupling λ'

$$\left.\begin{aligned} \lambda' = F(\lambda) &= \lambda + O(\lambda^2) \quad \text{for } \lambda\Phi^4 \\ &= \lambda + O(\lambda^3) \quad \text{for trilinear coupling theories} \end{aligned}\right\} \tag{6.57}$$

In the tree approximation the coupling is scheme-independent. The wave-function renormalization Z_3 and mass renormalization Z_{m} will be given in the new scheme by

$$Z'_{\mathrm{m}}(\lambda') = Z_{\mathrm{m}}(\lambda)F_{\mathrm{m}}(\lambda), \quad Z'_3(\lambda') = Z_3(\lambda)F_3(\lambda) \tag{6.58}$$

where $F_i(\lambda) = 1 + O(\lambda^2)$ (for the sake of definiteness we restrict our discussion to the case of the trilinear coupling theories (QED, QCD); modifications for $\lambda\Phi^4$ are trivial). For small enough λ we can assume that $\mathrm{d}F/\mathrm{d}\lambda \neq 0$ and $F_i \neq 0$. We get therefore

$$\left.\begin{aligned}
\beta'(\lambda') &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\lambda_{\mathrm{B},\varepsilon}}\lambda' = (\mathrm{d}F/\mathrm{d}\lambda)\beta(\lambda)\\
\gamma'(\lambda') &= \tfrac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\bigg|_{\lambda_{\mathrm{B},\varepsilon}}\ln Z_3'(\lambda') = \gamma(\lambda) + \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F_3\right)\beta(\lambda)\\
\gamma_{\mathrm{m}}'(\lambda') &= \gamma_{\mathrm{m}}(\lambda) + \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln F_{\mathrm{m}}\right)\beta(\lambda)
\end{aligned}\right\} \tag{6.59}$$

Using (6.59) the following results can easily be obtained (let $\tilde{\lambda}$ and $\tilde{\lambda}' = F(\tilde{\lambda})$ be fixed points in both schemes)

(i)

$$\frac{\mathrm{d}\beta'}{\mathrm{d}\lambda'}\bigg|_{\tilde{\lambda}'} = \frac{\mathrm{d}\beta}{\mathrm{d}\lambda}\bigg|_{\tilde{\lambda}} \tag{6.60}$$

(ii) If $\beta(\lambda) = -2b_0\lambda^3 - 2b_1\lambda^5 + \cdots$ then

$$\beta'(\lambda') = -2b_0\lambda'^3 - 2b_1\lambda'^5 + \cdots \tag{6.61}$$

i.e. the first two coefficients of the expansion for β are for mass-independent schemes, renormalization-scale-independent. Properties (i) and (ii) are important to ensure that the nature of the fixed points (asymptotic freedom, IR stability) is scheme-independent.

(iii)

$$\gamma'(\tilde{\lambda}') = \gamma(\tilde{\lambda}), \quad \gamma_{\mathrm{m}}'(\tilde{\lambda}') = \gamma_{\mathrm{m}}(\tilde{\lambda})$$

This again must be so since the values of γ and γ_{m} at a fixed point determine the scaling behaviour and masses, and have physical significance.

(iv) If $\gamma(\lambda) = \gamma_0\lambda^2 + O(\lambda^4)$ then $\gamma'(\lambda') = \gamma_0\lambda'^2 + O(\lambda'^4)$ and similarly for $\gamma_{\mathrm{m}}(\lambda)$.

This is again important since γ_0 and γ_{m}^0 determine the leading behaviour for $t \to \pm\infty$ of the exponents in (6.26) and (6.11), respectively (scale factors for the field and the mass).

We stress that these properties hold only in the class of mass-independent renormalization schemes. In particular within this class of renormalization schemes and in the leading logarithm approximation, $\alpha(q^2)$ is scheme-independent. In mass-dependent schemes the renormalization constants Z_i and consequently the coefficients b_i, γ_i depend on the ratio m/μ and obviously cannot be equal to those in the mass-independent schemes.

Effective α in QED

It is of some interest to find a link between the effective coupling constant in QED and the 'on-shell' $a^{\mathrm{ON}} = 1/137$ defined by (12.86). To this end we consider the photon propagator renormalized in some mass-independent renormalization scheme

$$\alpha D^{\mu\nu}(q) = -\mathrm{i}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{\alpha d(q^2/\mu^2, \alpha)}{q^2} - \alpha a \mathrm{i}\frac{q^\mu q^\nu}{(q^2)^2} \tag{6.62}$$

where $\alpha = \alpha(\mu)$ and

$$d(q^2/\mu^2, \alpha) = \frac{1}{1 - \Pi(q^2/\mu^2, \alpha)}$$

The function Π is the scalar factor of the vacuum polarization tensor $\Pi_{\mu\nu}(q)$ defined in (5.26). An important observation is that in QED the quantity

$$\alpha_{\mathrm{inv}}(q^2) = \alpha d(q^2/\mu^2, \alpha) = (\alpha/Z_3)Z_3 d(q^2/\mu^2, \alpha) = \alpha_{\mathrm{B}} d_{\mathrm{B}}(q^2) \tag{6.63}$$

is renormalization-scheme-independent because the bare quantities are. Incidentally, we see that in QED the product $\alpha_{\mathrm{B}} d_{\mathrm{B}}(q^2)$ is finite. Relation (6.63) does not hold in QCD because then the Ward identities do not imply $Z_1 = Z_2$. Thus, in QCD the relation between $\alpha d(q^2, \mu^2, \alpha)$ and $\alpha_{\mathrm{B}} d_{\mathrm{B}}(q^2)$ involves uncancelled renormalization constants; $\alpha d(q^2, \mu^2, \alpha)$ is then renormalization-scheme-dependent and $\alpha_{\mathrm{B}} d_{\mathrm{B}}(q^2)$ is not finite.

Turning again to QED we observe that

$$\alpha^{\mathrm{ON}} = \lim_{q^2 \to 0} \alpha_{\mathrm{inv}}(q^2) \tag{6.64}$$

Indeed, in the on-shell renormalization scheme $\Pi^{\mathrm{ON}}(q^2 = 0) = 0$ and given the scheme independence of α_{inv} defined by (6.63) we arrive at the result (6.64). Using (6.63) and (5.32) we also see that $\alpha^{\mathrm{ON}} = \alpha(m^2)$, where $\alpha(m^2)$ is defined in the $\overline{\mathrm{MS}}$ scheme.

It remains to find the relation between the $\alpha_{\mathrm{inv}}(q^2)$ and the running $\alpha(q^2)$ defined as the solution to (6.11) We recall first that

$$\alpha(q^2) \underset{q^2 \to 0}{=} 0$$

(it is an IR fixed point). Since the product $\alpha(\mu)d(q^2/\mu^2, \alpha(\mu))$ is renormalization-scheme-independent we can change $\mu^2 \to q^2$ to get

$$\alpha_{\mathrm{inv}}(q^2) = \alpha(q^2) d(1, \alpha(q^2)) \tag{6.65}$$

Using (5.32) for $\Pi(q^2/\mu^2, \alpha(\mu^2))$ in the first order in α and with $\mu^2 = q^2$ one gets

$$\alpha_{\mathrm{inv}}(q^2) \underset{q^2 \to \infty}{=} \alpha(q^2) \tag{6.66}$$

This result actually holds to any order in α in the leading approximation because (see Problem 6.5)

$$\Pi(q^2/\mu^2, \alpha(\mu^2)) = A_1\alpha(\mu^2)\ln(|q^2|/\mu^2) + \sum_{n=2} A_n\alpha^n(\mu^2)\ln^{n-1}(|q^2|/\mu^2)$$
$$+ \text{ non-leading terms}$$

The value of $\alpha_{\text{inv}}(q^2)$ in the limit $q^2 = 0$ can be obtained in terms of $\alpha(q^2)$ from knowledge of the first terms of the perturbation expansion again using (5.32) as the finite limit of the expression

$$\alpha_{\text{inv}}(q^2) \underset{q^2\to 0}{=} \frac{\alpha(q^2)}{1 - [\alpha(q^2)/3\pi]\ln(|q^2|/m^2)} = \frac{0}{0}. \tag{6.67}$$

The inverse relation to first order in $\alpha^{\text{ON}} = \alpha_{\text{inv}}(0)$ reads

$$\alpha(q^2) \underset{\substack{q^2\to 0 \\ \alpha^{\text{ON}}\ln(|q^2|/m^2)\ll 1}}{=} \alpha^{\text{ON}}\left(1 + \frac{\alpha^{\text{ON}}}{3\pi}\ln\frac{|q^2|}{m^2}\right) \tag{6.68}$$

Gauge dependence of the β-function

In QED the $\beta(\alpha)$-function is gauge-independent. This follows immediately from the fact that the renormalization constant Z_3 is gauge-independent. It renormalizes the gauge-invariant operator $F_{\mu\nu}F^{\mu\nu}$ or, in other words, the transverse part of the propagator, which is gauge-invariant (see also Problem 6.6).

In non-abelian gauge theories the renormalization programme is more complicated (see Section 8.1) and $\beta(\alpha)$ is, in general, gauge-dependent. Nevertheless one can prove that: (i) in any renormalization scheme the coefficient b_1 in the expansion

$$\beta(\alpha) = \alpha\left[b_1\frac{\alpha}{\pi} + b_2\left(\frac{\alpha}{\pi}\right)^2 + \cdots\right] \tag{6.69}$$

is gauge-independent, (ii) in the minimal subtraction scheme $\beta(\alpha)$ is gauge-independent.

To prove the first theorem we note that if we change the gauge we might change $\alpha(\mu)$ to $\alpha'(\mu)$; both can be defined, for example, as the value of the three-point function at some value μ^2 of the kinematical invariants or according to the minimal subtraction scheme. However, we can expand

$$\alpha'(\mu) = \alpha(\mu) + O(\alpha^2) \tag{6.70}$$

$\alpha' = \alpha$ in the zeroth order but the coefficients of the higher order terms might depend, in general, on the gauge parameter a. Now we consider the change of both

$\alpha'(\mu)$ and $\alpha(\mu)$ under a change of the renormalization point μ to calculate $\beta'(\alpha')$ and $\beta(\alpha)$

$$\beta'(\alpha') = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\alpha'\bigg|_{\alpha_B, a_B, \varepsilon \text{ fixed}} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}[\alpha(\mu) + O(\alpha^2)] = \beta(\alpha) + O(a^3) \quad (6.71)$$

because $\alpha(\mu + \mathrm{d}\mu) = \alpha(\mu) + O(\alpha^2)$. This proves theorem (i).

Theorem (ii) can be proved as follows: consider the bare coupling constant α_B of the theory. By its definition it is a fixed, gauge-independent parameter. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}a}\alpha_B\bigg|_{\varepsilon \text{ fixed}} = \frac{\mathrm{d}a_B}{\mathrm{d}a}\frac{\mathrm{d}}{\mathrm{d}a_B}\alpha_B = 0 \quad (6.72)$$

and using the relation $\alpha_B = \mu^\varepsilon Z_\alpha \alpha$ we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}a}(Z_\alpha \alpha)\bigg|_{\alpha_B, \varepsilon \text{ fixed}} = Z_\alpha \frac{\mathrm{d}}{\mathrm{d}a}\alpha + \alpha \frac{\mathrm{d}}{\mathrm{d}a} Z_\alpha \quad (6.73)$$

In the minimal subtraction scheme

$$Z_\alpha = 1 + Z_\alpha^{(1)}(\alpha, a)/\varepsilon + Z_\alpha^{(2)}(\alpha, a)/\varepsilon^2 + \cdots \quad (6.74)$$

and in particular in this scheme Z_α does not contain any constant term $C(\alpha, a)$ which would destroy the following argument. Inserting the expansion (6.74) into (6.73) we get

$$\frac{\mathrm{d}\alpha}{\mathrm{d}a} + \frac{1}{\varepsilon}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}a}Z_\alpha^{(1)} + \alpha \frac{\mathrm{d}Z_\alpha^{(1)}}{\mathrm{d}a}\right) + O\left(\frac{1}{\varepsilon^2}\right) = 0 \quad (6.75)$$

The only way this equation can be satisfied is if $\mathrm{d}\alpha/\mathrm{d}a = 0$ and $\mathrm{d}Z_\alpha^{(1)}/\mathrm{d}a = 0$. Thus in the minimal subtraction scheme the renormalized coupling constant $\alpha(\mu)$ is gauge-independent and consequently the same is true for the $\beta(\alpha) = \mu(\mathrm{d}/\mathrm{d}\mu)\alpha$.

Problems

6.1 Calculate the renormalization group functions β, γ, γ_m in QED in the momentum-subtraction renormalization scheme.

6.2 Consider a two-coupling theory: a non-abelian gauge theory with a scalar multiplet transforming according to the representation R of the gauge group. Write down the system of coupled RGEs for the gauge coupling g and for the meson self-coupling λ, to the lowest order in perturbation theory, and study the effect of the meson self-interaction on the asymptotic freedom of the theory. (See also Gross (1976).)

6.3 Consider the QED of n_h 'heavy' fermions Ψ_h, each with non-vanishing mass M, and n_l 'light' fermions Ψ_l each with zero mass. Obtain the effective field theory for the light fields A_μ and Ψ_l, in the one-loop approximation by 'integrating out' the heavy fermions from the Green's functions with light field external legs: take the photon propagator as an example and check the validity of the Appelquist–Carazzone decoupling theorem in the on-shell renormalization scheme (Appelquist & Carazzone 1975) and in the minimal subtraction scheme (Ovrut & Schnitzer 1981). In the latter case understand the role of a finite renormalization of the coupling constant in absorbing the effect of the heavy mass. Study the RGE for the Green's functions of the effective theory.

6.4 Consider the standard electroweak theory based on the gauge group $SU(2) \times U(1)$ with the gauge symmetry spontaneously broken (see Chapter 12) by one complex scalar doublet. Adopting the point of view expressed by (6.44) and using the lowest order result for the β-function estimate the upper bound for the Higgs particle mass (Dasher & Neuberger 1983). Assume that for consistency the relation $m_{\mathrm{H}} \leq \Lambda_{\mathrm{c}}$ must be satisfied.

6.5 Using the RGE in a mass-independent renormalization scheme show that the vacuum polarization in QED, (5.26), reads

$$\Pi(q^2/\mu^2, \alpha(\mu^2)) = A_1\alpha(\mu^2)\ln(|q^2|/\mu^2) + \sum_{n=2} A_n\alpha^n(\mu^2)\ln^{n-1}(|q^2|/\mu^2) + \text{non-leading terms}$$

where the coefficients A_n are determined in terms of the b_1 and b_2 of the β-function.

6.6 Using the functional integral representation for the bare full two-photon Green's function in QED, show by taking the derivative with respect to the gauge parameter a that the transverse part of the propagator is gauge-independent.

6.7 The model discussed in Problem 4.1 (taken now for $m = M$) has two coupling constants. What symmetry does it possess when two couplings are equal? Discuss the fixed point structure for these couplings.

6.8 Consider $\lambda\Phi^4$ theory in the Euclidean space (Problem 4.3). Calculate the two-loop β-function.

7

Scale invariance and operator product expansion

7.1 Scale invariance

Scale transformations

An important concept in quantum field theory is that of scale invariance. Among other things, it is closely related to the famous Bjorken scaling in the deep inelastic lepton–hadron scattering. Scale transformation for the coordinates is defined as follows

$$x \to x' = \exp(\varepsilon)x \tag{7.1}$$

where ε is a real number. Obviously, all such transformations form an abelian group. Scale invariance is the invariance under the group of scale transformations.

If the configuration variables are scaled down as in (7.1) local fields $\Phi(x)$ (in this section we denote by $\Phi(x)$ any scalar field $\varphi(x)$ or spinor field $\Psi(x)$) will be, in general, subject to a unitary transformation $U(\varepsilon)$ (compare with Section 1.4)

$$U(\varepsilon)\Phi(x)U^{-1}(\varepsilon) = T^{-1}(\varepsilon)\Phi(e^{\varepsilon}x) \tag{7.2}$$

where $T(\varepsilon)$ is a finite-dimensional representation of the group (for classical fields $\Phi'(x') = T(\varepsilon)\Phi(x)$). We assume it to be fully reducible and therefore we can write

$$T(\varepsilon) = \exp(-d_\Phi \varepsilon) \tag{7.3}$$

where the constant d_Φ is called the scale dimension of the field $\Phi(x)$.

Let us consider first a free massless field theory. Its lagrangian does not contain dimensionful constants and should be invariant under the transformation (7.2). For free fields one has the canonical equal-time commutation relations

$$[\varphi(\mathbf{x}, t), \dot{\varphi}(\mathbf{y}, t)] = \mathrm{i}\delta(\mathbf{x} - \mathbf{y}) \tag{7.4}$$

$$\{\Psi(\mathbf{x}, t), \Psi^\dagger(\mathbf{y}, t)\} = \delta(\mathbf{x} - \mathbf{y}) \tag{7.5}$$

and the scale dimension d_Φ is then defined so that the commutation relations (7.4) and (7.5) remain invariant under the scale transformation. Multiplying relations (7.4) and (7.5) by operators U and U^{-1} and using (7.2) one sees that invariance of (7.4) and (7.5) implies that these so-called canonical scale dimensions are $d_\varphi = 1$ for scalar fields and $d_\Psi = \frac{3}{2}$ for spinor fields. We see that the canonical scale dimensions of fields coincide with their ordinary dimensions defined on purely dimensional grounds (see Section 4.2).

In an infinitesimal form the scale transformation (7.2) is as follows:

$$U(\varepsilon)\Phi(x)U^{-1}(\varepsilon) = \Phi(x) + \varepsilon(d_\Phi + x_\mu(\partial/\partial x_\mu))\Phi(x) + O(\varepsilon^2) \tag{7.6}$$

For interacting field theories one still has the canonical commutation relations (7.4) and (7.5) for bare fields. Defining scale dimensions for bare fields as canonical ones, one can easily check that a lagrangian which does not contain dimensionful constants is invariant under the scale transformation (7.2) (of course, strictly speaking, it is the action integral which is scale-invariant). Take, for instance, the following lagrangian density

$$\mathcal{L} = \mathrm{i}\bar{\Psi}\not{\partial}\Psi + \tfrac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) + g\bar{\Psi}\gamma_5\Psi\varphi - (\lambda/4!)\varphi^4 \tag{7.7}$$

Using (7.6) and the values $d_\varphi = 1$, $d_\Psi = \frac{3}{2}$ it is easy to calculate (see (1.29))

$$\begin{aligned}\delta\mathcal{L} &= U\mathcal{L}U^{-1} - \mathcal{L}\\ &= \mathcal{L}(\exp(d_\phi\varepsilon)\phi(x')) - \mathcal{L}(\phi(x)) = \sum_{\Phi=\varphi,\Psi}\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial\Phi/\partial x_\mu)}\delta\left(\frac{\partial\Phi}{\partial x_\mu}\right)\\ &= \varepsilon(4 + x_\mu(\partial/\partial x_\mu))\mathcal{L}(x)\end{aligned} \tag{7.8}$$

where we have used the relation

$$\delta\left(\frac{\partial}{\partial x_\nu}\Phi(x)\right) = \frac{\partial}{\partial x_\nu}\delta\phi(x) = \varepsilon\left(d_\Phi + 1 + x\cdot\frac{\partial}{\partial x}\right)(\partial/\partial x_\nu)\Phi(x)$$

This result is expected on general grounds since in a lagrangian which does not contain dimensionful constants each term has a canonical dimension equal to its ordinary dimension which is four. Therefore the corresponding action integral is invariant under scale transformations

$$\delta I = \int \mathrm{d}^4x\,\delta\mathcal{L}(x) = 0 \tag{7.9}$$

(one can also integrate explicitly using (7.8) and neglecting, as usual, the surface terms at infinity). Terms like $m^2\Phi^2(x)$, if added to (7.7), would, of course, break the scale invariance of the lagrangian as their scale dimension is two. Notice the difference between scale dimension and ordinary dimension which is four for the term $m^2\Phi^2$; scale transformation is a transformation of dynamical variables (the fields) but not of the dimensionful parameters such as masses since we want to stay in the framework of the same physical theory. Terms like $m^2\Phi^2(x)$ break scale invariance explicitly. However, it is already worth recognizing at this point that in quantum field theory described by a scale-invariant lagrangian the scale invariance is, in general, also broken. This is due to the necessity of renormalization which introduces a dimensionful parameter.

Turning back to an exactly scale-invariant world we can introduce the generator D of the scale transformation

$$U(\varepsilon) = \exp(\mathrm{i}\varepsilon D) \tag{7.10}$$

and study its commutation relations with generators of the Poincaré group (for conventions see Problem 7.1)

$$\left.\begin{aligned} U(a) &= \exp(\mathrm{i}a^\mu P_\mu) \\ \text{and} \qquad U(\omega) &= \exp(-\tfrac{1}{2}\mathrm{i}\omega^{\mu\nu} M_{\mu\nu}) \end{aligned}\right\} \tag{7.11}$$

By comparing two sequences of dilatation and translation

$$\Phi(x) \underset{\varepsilon}{\to} \exp(d_\Phi \varepsilon)\Phi(\exp(\varepsilon)x) \underset{a_\mu}{\to} \exp(d_\Phi \varepsilon)\Phi(\exp(\varepsilon)x + a_\mu)$$

and

$$\Phi(x) \underset{a_\mu}{\to} \Phi(x + a_\mu) \underset{\varepsilon}{\to} \exp(d_\Phi \varepsilon)\Phi(\exp(\varepsilon)(x + a_\mu))$$

we conclude that

$$U(\varepsilon)U(a) = U(\exp(\varepsilon)a)U(\varepsilon)$$

or

$$\mathrm{i}[D, P_\mu] = P_\mu \tag{7.12}$$

Similarly, one gets

$$U(\omega)U(\varepsilon) = U(\varepsilon)U(\omega)$$

and

$$[D, M_{\mu\nu}] = 0 \tag{7.13}$$

From (7.6) we get the following differential representation of the generator D

$$\mathrm{i}[D, \Phi(x)] = (d_\Phi + x_\mu(\partial/\partial x_\mu))\Phi(x) \tag{7.14}$$

Dilatation current

The dilatation generator D can be written in terms of the Noether dilatation current $D_\mu(x)$. The conserved dilatation current can be deduced by comparing an infinitesimal change, obtained without the use of the Euler–Lagrange equations of motion, under the scale transformation of the lagrangian density of a scale-invariant theory (7.8)

$$\delta\mathcal{L} = \varepsilon\left(4 + x_\mu \frac{\partial}{\partial x_\mu}\right)\mathcal{L} = \varepsilon \frac{\partial}{\partial x_\mu}(x_\mu \mathcal{L})$$

with an alternate expression for $\delta\mathcal{L}$ which does use Euler–Lagrange equations

$$\delta\mathcal{L} = (\partial/\partial x_\mu)(\Pi_\mu \delta\Phi) \qquad \Phi = (\varphi, \Psi) \tag{7.15}$$

where Π_μ is the conjugate momentum. The Noether dilatation current is therefore the following

$$D_\mu(x) = \Pi_\mu(d_\Phi + x_\nu(\partial/\partial x_\nu))\Phi(x) - x_\mu \mathcal{L}$$

or

$$D_\mu(x) = \Pi_\mu d_\Phi \Phi(x) + x^\nu \Theta^{\text{can}}_{\mu\nu} \tag{7.16}$$

where

$$\Theta^{\text{can}}_{\mu\nu} = \Pi_\mu \frac{\partial}{\partial x^\nu}\Phi(x) - g_{\mu\nu}\mathcal{L}$$

is the canonical energy–momentum tensor. The current is conserved in the presence of the symmetry, but it may, of course, be defined even in the absence of scale invariance.

The dilatation current can be written in a more compact way if we introduce a modified energy–momentum tensor $\Theta_{\mu\nu}$. Let us consider first a scalar field theory and define $\Theta_{\mu\nu}$ as follows:

$$\Theta_{\mu\nu} = \Theta^{\text{can}}_{\mu\nu} - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi^2 \tag{7.17}$$

The new tensor has the following properties

$$\Theta_{\mu\nu} = \Theta_{\nu\mu}, \quad \partial^\mu \Theta_{\mu\nu} = 0$$

and

$$P_\mu = \int \mathrm{d}^3x\, \Theta^{\text{can}}_{0\mu} = \int \mathrm{d}^3x\, \Theta_{0\mu}$$
$$M_{\mu\nu} = \int \mathrm{d}^3x\, (x_\mu \Theta_{0\nu} - x_\nu \Theta_{0\mu})$$

i.e. $\Theta_{\mu\nu}$ is a legitimate energy–momentum tensor of our theory. Using the relation $\Pi_\mu = \partial_\mu\varphi$ it is easy to check that in terms of $\Theta_{\mu\nu}$ the D_μ of (7.16) ($d_\Phi = 1$) reads

$$D_\mu = x^\nu\Theta_{\mu\nu} - \tfrac{1}{6}\partial_\nu(x_\mu\partial^\nu - x^\nu\partial_\mu)\varphi^2$$

The last term is the divergence of an antisymmetric tensor, so it does not contribute to $\partial^\mu D_\mu$ and to D. Thus we redefine

$$D_\mu(x) = x^\nu\Theta_{\mu\nu}(x) \tag{7.18}$$

and

$$\partial^\mu D_\mu(x) = \Theta_\mu{}^\mu \tag{7.19}$$

An important property of the new tensor is that its trace vanishes

$$\Theta_\mu{}^\mu = 0 \tag{7.20}$$

for a scale-invariant scalar field theory (and

$$\Theta_\mu{}^\mu = m^2\varphi^2 \tag{7.20a}$$

if a mass term is added) while the trace of $\Theta^{\text{can}}_{\mu\nu}$ contains singular derivative terms. The new tensor was invented by Callan, Coleman & Jackiw (1970) when looking for an object with the ‘soft’ divergences (7.20a) in the context of searching for an energy–momentum tensor whose matrix elements between physical states are finite. In a renormalizable field theory the finiteness of matrix elements of the T-ordered products of local currents is an additional requirement, which, in general, must be separately verified (see, for instance, Section 10.1). Actually, the proof of Callan, Coleman & Jackiw of the finiteness of matrix elements of the $\Theta_{\mu\nu}$ is incomplete because it ignores the non-soft contribution (7.46) to the $\Theta_\mu{}^\mu$ from the anomalous scale invariance breaking; see Adler, Collins & Duncan (1977) for the complete discussion.

Eq. (7.17) can be generalized to include fields with non-zero spin (Callan *et al.* 1970):

$$\Theta_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi^2 \tag{7.21}$$

where

$$T_{\mu\nu} = \Pi_\mu\partial_\nu\Phi - g_{\mu\nu}\mathcal{L} + \tfrac{1}{2}\partial^\lambda(\Pi_\lambda\Sigma_{\mu\nu}\Phi - \Pi_\mu\Sigma_{\lambda\nu}\Phi - \Pi_\nu\Sigma_{\lambda\mu}\Phi) \tag{7.22}$$

is the conventional symmetric energy–momentum tensor† and where all fields of the theory are assembled into a vector Φ whereas the scalar fields are denoted by φ. The spin matrix $\Sigma_{\mu\nu}$ is defined by the transformation properties of the fields under an infinitesimal Lorentz transformation

$$\delta\Phi = -\tfrac{1}{2}(x_\mu \partial_\nu \Phi - x_\nu \partial_\mu \Phi + \Sigma_{\mu\nu}\Phi)\omega^{\mu\nu} \tag{7.23}$$

and reads, for example, $\Sigma_{\mu\nu} = 0$ for spin 0 fields, $\Sigma_{\mu\nu} = -\frac{1}{2}\mathrm{i}\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ for spin one-half fields. In terms of $T_{\mu\nu}$ the generators of the Lorentz transformation are

$$M_{\mu\nu} = \int \mathrm{d}^3x\, (x_\mu T_{0\nu} - x_\nu T_{0\mu})$$

The tensor $\Theta_{\mu\nu}$ defined by (7.21) has the same properties as the tensor (7.17) defined for scalar fields only. In particular its trace is a measure of scale invariance breaking.

Conformal transformations

It turns out that many scale-invariant theories are invariant under a larger symmetry group of transformations, the so-called conformal group, which in addition to translations, Lorentz transformations and scale transformation also contains special conformal transformations. A conformal (i.e. angle-preserving) transformation must leave invariant the ratio

$$\frac{\mathrm{d}x^\alpha\, \mathrm{d}y_\alpha}{|\mathrm{d}x|\, |\mathrm{d}y|}$$

So it must have the property that

$$\begin{aligned} \mathrm{d}s'^2 &= g_{\alpha\beta}\, \mathrm{d}x'^\alpha\, \mathrm{d}x'^\beta = g_{\alpha\beta}(\partial x'^\alpha/\partial x^\gamma)(\partial x'^\beta/\partial x^\delta)\, \mathrm{d}x^\gamma\, \mathrm{d}x^\delta \\ &= f(x)\, g_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta \equiv f(x)\, \mathrm{d}s^2 \end{aligned} \tag{7.24}$$

where $f(x)$ is a certain scalar function.

For infinitesimal transformations

$$x'^\mu = x^\mu + \varepsilon^\mu(x)$$

† It can be constructed by noting that from conservation of the angular momentum current $\partial_\lambda \mathcal{M}^{\lambda\mu\nu} = 0$, where

$$\mathcal{M}^{\lambda\mu\nu} = x^\mu T^{\lambda\nu}_{\text{can}} - x^\nu T^{\lambda\mu}_{\text{can}} + \Pi^\lambda \Sigma^{\mu\nu}\Phi$$

and

$$M^{\mu\nu} = \int \mathrm{d}^3x\, \mathcal{M}^{0\mu\nu}$$

we get

$$T^{\mu\nu}_{\text{can}} - T^{\nu\mu}_{\text{can}} + \partial_\lambda(\Pi^\lambda \Sigma^{\mu\nu}\Phi) = 0$$

Eq. (7.22) follows if we require $\partial_\mu T^{\mu\nu} = \partial_\nu T^{\mu\nu} = 0$.

one therefore obtains the following condition

$$\partial_\nu \varepsilon_\mu(x) + \partial_\mu \varepsilon_\nu(x) = h(x) g_{\mu\nu} \tag{7.25}$$

where $h(x) = f(x) - 1$. This equation gives

$$(n-2)\partial_\alpha \partial_\beta h(x) = 0$$

$$\partial^\mu \partial_\mu \varepsilon_\alpha(x) = \tfrac{1}{2}(n-2)\partial_\alpha h(x)$$

where n is the dimension of the space-time. We see that for $n > 2$,† $h(x)$ is at most linear in x and, consequently, $\varepsilon_\mu(x)$ is at most quadratic in x. The general solution for $\varepsilon_\mu(x)$ satisfying our constraints reads

$$\varepsilon^\mu(x) = a^\mu + \varepsilon x^\mu + \omega^{\mu\nu} x_\nu + (-2b \cdot x x^\mu + b^\mu x^2);\ \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{7.26}$$

It includes (listed are the finite transformations)

translations	$x'^\mu = x^\mu + a^\mu,$	$h(x) = 0$
scale transformation	$x'^\mu = e^{+\varepsilon} x^\mu,$	$h(x) = 2\varepsilon$
Lorentz transformations	$x'^\mu = \Lambda^{\mu\nu}(\omega) x_\nu,$	$h(x) = 0$
special conformal transformations	$x'^\mu = \dfrac{x^\mu + b^\mu x^2}{1 - 2b \cdot x + b^2 x^2},$	$h(x) = 4b \cdot x$

which in n dimensions form together the $(n + 1 + n(n-1)/2 + n)$-parameter conformal group (see Problem 7.1). Conformal invariance implies scale invariance since the commutator of an infinitesimal conformal transformation and an infinitesimal translation contains a scale transformation. However, the opposite is not true and examples of field theories which are scale-invariant but not conformally invariant can be constructed. Scale invariance implies conformal invariance if, and only if, there exists in the considered theory a symmetric traceless conserved energy–momentum tensor so that $\partial^\mu D_\mu = \Theta^\mu{}_\mu = 0$. It can be checked that such a $\Theta_{\mu\nu}$ exists in all renormalizable field theories of fields of spin ≤ 1. In this case the currents

$$(j^\varepsilon)^\mu(x) = \Theta^{\mu\nu}(x)\varepsilon_\nu(x) \tag{7.27}$$

are conserved for any $\varepsilon_\nu = \varepsilon_\nu(x, a^\alpha, \varepsilon, \omega^{\alpha\beta}, b^\alpha)$ satisfying (7.25) (check it) and indeed scale invariance goes together with the special conformal symmetry. From (7.26) and (7.27) we find explicitly the following four generators of the special conformal transformations

$$K^\alpha = \int \mathrm{d}^3x\, (j^\alpha)^0 = \int \mathrm{d}^3x\, (x^2 \Theta^{\alpha 0} - 2x^\alpha x_\nu \Theta^{\nu 0}) \tag{7.28}$$

For the transformation of fields under special conformal transformations see Problem 7.1.

† The case $n = 2$, interesting for statistical mechanics and string theories, requires a separate discussion; see Problem 7.2.

7.2 Broken scale invariance

General discussion

Scale invariance cannot be an exact symmetry of the real world. If it were, all particles would have to be massless or their mass spectra continuous. Indeed, it follows from commutation relation (7.12) that

$$[D, P^2] = -2\mathrm{i}P^2$$

or, exponentiating,

$$\exp(\mathrm{i}\varepsilon D)P^2\exp(-\mathrm{i}\varepsilon D) = \exp(2\varepsilon)P^2 \tag{7.29}$$

Acting on a single particle state $|p\rangle$ with four-momentum $p(P^2|p\rangle = p^2|p\rangle)$ one gets

$$P^2\exp(-\mathrm{i}\varepsilon D)|p\rangle = \exp(2\varepsilon)p^2\exp(-\mathrm{i}\varepsilon D)|p\rangle \tag{7.30}$$

i.e. the state $\exp(-\mathrm{i}\varepsilon D)|p\rangle$ is an eigenstate of P^2 with eigenvalue $\exp(2\varepsilon)p^2$. If we assume in addition that the vacuum is unique under scale transformations (i.e. if scale invariance is not spontaneously broken)

$$\exp(-\mathrm{i}\varepsilon D)|0\rangle = |0\rangle$$

then we conclude that

$$\exp(-\mathrm{i}\varepsilon D)|p\rangle = \exp(-\mathrm{i}\varepsilon D)a^\dagger(p)|0\rangle = \exp(d_a)a^\dagger(e^\varepsilon p)|0\rangle \tag{7.31}$$

where $a^\dagger(p)$ is the creation operator for the considered particle with momentum p and d_a is its dimension. Result (7.31) means that the state $\exp(-\mathrm{i}\varepsilon D)|p\rangle$ is a quantum of the same field as the state $|p\rangle$ but with a rescaled momentum (if the vacuum was not unique then the state $\exp(-\mathrm{i}\varepsilon D)|p\rangle$ would belong to a different Hilbert space than the state $|p\rangle$ and our final conclusion would be avoided) and therefore, by (7.30), all particles must be massless or the mass spectrum must be continuous.

We will learn in the next chapters that there are several ways to account for the symmetry breaking. The simplest idea for symmetry breaking is that of 'approximate' symmetries. One assumes that there are terms in the lagrangian (e.g. mass terms in the case of scale invariance) which violate the symmetry, but that they are 'small'. Another concept is that of spontaneous symmetry breaking. Here the dynamical equations are completely symmetric, the Noether currents associated with the symmetry are conserved and the Ward identities are still valid, but the ground state is asymmetric. We believe this idea to be realized for chiral symmetry in strong interactions. Finally there is a third way: anomalous breaking of symmetries. Whenever a classical theory possesses a symmetry, but there is no way of quantizing the theory so as to preserve that symmetry, we say that there are anomalies in the conservation equations for the symmetry currents. More technically, quantum field theoretical calculations must be regularized to avoid infinities and anomalies occur when there does not exist a regularization procedure which respects the classical symmetry and the corresponding Ward identity. Of course, a given symmetry may be broken by several of these mechanisms, simultaneously.

In quantum field theory, even with a scale-invariant lagrangian, the anomalous scale invariance breaking is quite expected. Scale invariance requires that there be no dimensionful parameters whereas the regularization of the quantum theory is effected by introducing a dimensionful cut-off or dimensionful coupling constants in the dimensional regularization procedure. Equivalently, an unavoidable renormalization procedure necessarily introduces a scale at which the theory is renormalized and this breaks scale invariance.

In the rest of this section we shall study the 'anomalous' response of the theory to scale transformations by comparing the naive Ward identities with the renormalization group approach. For a discussion of the spontaneous breaking of scale invariance we refer the reader to the literature. In QCD the effects of the anomalous explicit breaking of scale symmetry are dominant, and there is no remaining consequence of the original classical scale symmetry.

Anomalous breaking of scale invariance

Ward identities for the dilatation current (trace identities) can be obtained from the general formula (10.11)

$$\begin{aligned}\frac{\partial}{\partial y_\mu}\langle 0|T D_\mu(y)\Phi(x_1)\dots\Phi(x_n)|0\rangle &= \langle 0|T\Theta_\mu{}^\mu(y)\Phi(x_1)\dots\Phi(x_n)|0\rangle \\ &\quad -\mathrm{i}\sum_i \delta(x_i-y)\langle 0|T\Phi(x_1)\dots\delta\Phi(x_i)\dots\Phi(x_n)|0\rangle \qquad (7.32)\end{aligned}$$

where $\delta\Phi$s are given by (7.14). By integrating with respect to variable y the l.h.s. vanishes: it gives surface terms which can be neglected if there are no zero mass particles coupled to $\Theta_\mu{}^\mu$; in reality the scale invariance may be spontaneously broken but simultaneously the corresponding Goldstone bosons (dilatons) get masses due to anomalies; integrating with respect to y is equivalent to taking the Fourier transform and letting $k\to 0$. We then obtain

$$\begin{aligned}\int \mathrm{d}^4y\,\langle 0|T\Theta_\mu{}^\mu(y)\Phi(x_1)\dots\Phi(x_n)|0\rangle &= \mathrm{i}\sum_i\langle 0|T\Phi(x_1)\dots\delta\Phi(x_i)\dots\Phi(x_n)|0\rangle \\ &= \mathrm{i}\left(nd_\Phi + x_1^\mu\frac{\partial}{\partial x_1^\mu}+\dots+x_n^\mu\frac{\partial}{\partial x_n^\mu}\right)\langle 0|T\Phi(x_1)\dots\Phi(x_n)|0\rangle \qquad (7.33)\end{aligned}$$

In the last step of (7.33) we have brought the derivatives outside the T-product which results in further equal-time commutator terms which, however, cancel in pairs. For example, for $n = 2$ we have

$$
\begin{aligned}
&\langle 0|T\Phi(x_1)x_2^\mu \frac{\partial}{\partial x_2^\mu}\Phi(x_2)|0\rangle + \langle 0|Tx_1^\mu \frac{\partial}{\partial x_1^\mu}\Phi(x_1)\Phi(x_2)|0\rangle \\
&= \left(x_1^\mu \frac{\partial}{\partial x_1^\mu} + x_2^\mu \frac{\partial}{\partial x_2^\mu}\right)\langle 0|T\Phi(x_1)\Phi(x_2)|0\rangle + x_1^0\delta(x_1^0 - x_2^0)\langle 0|[\Phi(x_1), \Phi(x_2)]|0\rangle \\
&\quad + x_2^0\delta(x_1^0 - x_2^0)\langle 0|[\Phi(x_2), \Phi(x_1)]|0\rangle
\end{aligned}
\tag{7.34}
$$

Defining the momentum-dependent Green's functions

$$
\begin{aligned}
&(2\pi)^4\delta\left(\sum_1^n p_i\right)G^{(n)}(p_1, \ldots, p_{n-1}) \\
&\quad = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp\left(\mathrm{i}\sum_i p_i x_i\right)\langle 0|T\Phi(x_1)\ldots\Phi(x_n)|0\rangle
\end{aligned}
\tag{7.35}
$$

and analogously for $G_\Theta^{(n)}(k, p_1, \ldots, p_{n-1})$ as the Fourier transform of the $\langle 0|T\Theta_\mu{}^\mu(y)\Phi(x_1)\ldots\Phi(x_n)|0\rangle$ we get from (7.33) the following Ward identity

$$
\left[-\sum_{i=1}^{n-1} p_i \frac{\partial}{\partial p_i} + n(d_\Phi - 4) + 4\right]G^{(n)}(p_1, \ldots, p_{n-1}) = -\mathrm{i}G_\Theta^{(n)}(0, p_1, \ldots, p_{n-1})
\tag{7.36}
$$

Summation over $n-1$ momenta and the additional term 4 in the square bracket are due to the fact that, effectively, we have only $n-1$ four-dimensional integrations in (7.35). We observe that for $d_\Phi = d^{\text{can}}$ the $nd_\Phi - 4n + 4$ is just the dimension D of the Green's function $G^{(n)}(p_1, \ldots, p_{n-1})(G^{(n)} \sim M^D)$. In addition, if we parametrize all momenta by the scaling factor $p_i = \rho\hat{p}_i$ so that $\Sigma_i p_i\partial/\partial p_i = \rho\partial/\partial\rho$ we can rewrite (7.36) as follows ($t = \ln\rho$)

$$
\begin{aligned}
&[-\partial/\partial t + D]G^{(n)}(\exp(t)p_1, \ldots, \exp(t)p_{n-1}) \\
&\quad = -\mathrm{i}G_\Theta^{(n)}(0, \exp(t)p_1, \ldots, \exp(t)p_{n-1})
\end{aligned}
\tag{7.37}
$$

From (7.37) one can derive analogous equations for the 1PI Green's functions $\Gamma^{(n)}(p_1, \ldots, p_{n-1})$ and $\Gamma_\Theta^{(n)}(0, p_1, \ldots, p_{n-1})$

$$
[-\partial/\partial t + D]\Gamma^{(n)}(\exp(t)p_i) = -\mathrm{i}\Gamma_\Theta^{(n)}(0, \exp(t)p_i)
\tag{7.38}
$$

where $D = 4 - nd_\Phi$ is the dimension of the 1PI Green's function $\Gamma^{(n)}$.

Eq. (7.37) and (7.38) are the Ward identities following from the Noether relation $\partial_\mu D^\mu = \Theta_\mu{}^\mu$ and from the canonical field transformation rule (7.14). In particular, for $\Theta_\mu{}^\mu = 0$, i.e. for a scale-invariant lagrangian, the solution to (7.38) reads

$$
\Gamma^{(n)}(\rho p_i) = \rho^D\Gamma^{(n)}(p_i)
\tag{7.39}
$$

In this case the Green's functions exhibit canonical scaling. However, we have already mentioned that there is no reason to expect these identities to be valid in quantum field theory since the scale invariance is always broken by a dimensionful cut-off. The response of a quantum field theory to the scale transformation is given by the RGE (6.24).

Comparing (6.24) with the Ward identity (7.38) we see that the previously mentioned anomalous breaking of canonical scale invariance occurs in quantum field theory. Either the dilatation current has an anomalous divergence, or the field transforms anomalously, or both. For easy reference we recall the solution to (6.24)

$$\Gamma^{(n)}(\exp(t)p_i, \lambda, m, \mu) \\ = \exp(Dt)\Gamma^{(n)}(p_i, \bar{\lambda}(t), \bar{m}(t), \mu)\exp\left[-n\int_0^t \gamma(\bar{\lambda}(t'))\,\mathrm{d}t'\right] \tag{7.40}$$

where

$$\int_\lambda^{\bar{\lambda}(t)} \mathrm{d}x/\beta(x) = t \tag{7.41}$$

and

$$\bar{m}(t) = (1/\rho)m\exp\left[\int_0^t \gamma_m(\bar{\lambda}(t'))\,\mathrm{d}t'\right] \\ = (1/\rho)m\exp\left\{\int_\lambda^{\bar{\lambda}(t)} \mathrm{d}x\,[\gamma_m(x)/\beta(x)]\right\} \tag{7.41a}$$

Consider first a massless $m = 0$ theory with a conserved Noether dilatation current $\partial_\mu D^\mu = 0$. As mentioned in Section 6.1 only for $\beta(\lambda) = \gamma(\lambda) = 0$, i.e. for a non-interacting theory, do we recover the canonical scaling, namely solution (7.39) to (7.38). Imagine now a theory with a UV fixed point λ_+. The exponential factor in (7.40) can then be written as

$$\exp\left[-n\int_0^t \gamma(\bar{\lambda}(t'))\,\mathrm{d}t'\right] = \rho^{-n\gamma(\lambda_+)+\varepsilon(t)}$$

where

$$\varepsilon(t) = -(1/t)\int_\lambda^{\bar{\lambda}(t)} \mathrm{d}x\, n[\gamma(x) - \gamma(\lambda_+)]/\beta(x) \tag{7.42}$$

If the integral defining $\varepsilon(t)$ is convergent, then $\varepsilon(t) = O(1/t)$ and the theory is asymptotically, for $t \to \infty$, scale-invariant. Apart from a trivial case when λ is exactly equal to λ_+ and the RGE can then be solved directly and gives (7.39) with $D \to D - n\gamma(\lambda_+)$, this happens when λ_+ is a simple zero of $\beta(x) : \beta(\lambda_+) = 0$, $\beta'(\lambda_+) < 0$ (we assume that $\gamma(x)$ is differentiable). The asymptotic behaviour of

$\Gamma^{(n)}(\exp(t)p_i, \lambda, \mu)$ is obtained by expanding $\bar{\lambda}(t)$ about λ_+. The leading term is

$$\Gamma^{(n)}(\exp(t)p_i, \lambda, \mu) \sim \rho^{D-n\gamma(\lambda_+)}\Gamma^{(n)}(p_i, \lambda_+, \mu) \tag{7.43}$$

The function $\gamma(\lambda_+)$ is called the anomalous dimension of the field. The leading corrections to (7.43) arise from $O(\bar{\lambda}(t) - \lambda_+)$ terms in the expansion of $\Gamma^{(n)}(p_i, \bar{\lambda}(t), \mu)$ and of the exponential factor (7.42). More specifically one gets

$$\begin{aligned}\Gamma^{(n)}(\rho p_i, \lambda, \mu) = \rho^{D-n\gamma(\lambda_+)} \exp\left\{-\int_{\lambda}^{\lambda_+} \mathrm{d}x\, [\gamma(x) - \gamma(\lambda_+)]n/\beta(x)\right\}\\ \times \Gamma^{(n)}(p_i, \lambda_+, \mu)[1 + O(\rho^{-|\beta'(\lambda_+)|})]\end{aligned} \tag{7.44}$$

Asymptotic scale invariance, in particular, is not a feature of asymptotically free ($\lambda_+ = 0$) gauge theories. In this case $\beta(x) = -bx^3 + O(x^5)$ and for most of the interesting operators $\gamma(x) - \gamma(0) = 2cx^2 + O(x^4)$. So the integral (7.42) is logarithmically divergent and it causes logarithmic deviations from the asymptotic scale invariance (7.43) (remember that $\bar{\lambda}^2(t) \sim 1/2bt$)

$$\exp\left(\int_{\lambda}^{\bar{\lambda}(t)} \frac{2c}{b}\frac{\mathrm{d}x}{x}\right) \sim (2b\lambda^2 t)^{-c/b} \tag{7.45}$$

We also note an operator equation for the anomalous divergence of the dilatation current in gauge theories reflecting the anomalous scale invariance breaking

$$\Theta_\mu{}^\mu = (\beta(\lambda)/2\lambda^3)G^a_{\mu\nu}G^{\mu\nu}_a \tag{7.46}$$

Rigorous proof of relation (7.46) is rather lengthy (Adler, Collins & Duncan 1977, Collins, Duncan & Joglekar 1977, Nielsen 1977). We see again that only for $\lambda = \lambda_+ \neq 0$ do we recover exact scale invariance, possibly with anomalous dimensions.

We now extend our discussion to theories with scale invariance explicitly broken by mass terms. The identity (for bosons, to be specific)

$$m\frac{\partial}{\partial m}\frac{\mathrm{i}}{p^2 - m^2} = \frac{\mathrm{i}}{p^2 - m^2}(-2\mathrm{i}m^2)\frac{\mathrm{i}}{p^2 - m^2} \tag{7.47}$$

implies that the operation $m\partial/\partial m$ with external momenta and the coupling constant λ fixed (we assume that there is only one mass and one dimensionless coupling constant) is equivalent to the insertion of a new vertex $-\mathrm{i}m^2\Phi^2$ at zero-momentum transfer. Thus we again see that for $\beta = \gamma_m = \gamma = 0$ (6.24) and (7.37) are identical, with $\Theta_\mu{}^\mu = m^2\Phi^2$.

The mass effects in general are summarized in the solutions (7.40) and (7.41) to the RGE. The $\rho \to \infty$ limit of (7.40) is controlled by the asymptotic behaviour of both $\bar{\lambda}(t)$ and $\bar{m}(t)$ (we recall that $t = \ln \rho$)

$$\begin{aligned} \bar{m}(\rho) &= m\rho^{\gamma_m(\lambda_+)-1} \exp\left\{\int_{\lambda}^{\bar{\lambda}} \mathrm{d}x\, [\gamma_m(x) - \gamma_m(\lambda_+)]/\beta(x)\right\} \\ &\underset{\rho\to\infty}{\longrightarrow} m\rho^{\gamma_m(\lambda_+)-1} \times \{\text{logs of } \rho\} \end{aligned} \tag{7.48}$$

Now, for the mass effects to be asymptotically non-leading, it is essential that

$$\bar{m}(\rho) \underset{\rho\to\infty}{\longrightarrow} 0 \quad \text{or} \quad \gamma_m(\lambda_+) < 1$$

Introducing the scale dimension $d_{\Delta m}$ of the mass operator Δm

$$d_{\Delta m} = \begin{cases} 3 + \gamma_m(\lambda_+) & \Delta m = m\bar{\Psi}\Psi \\ 2 + 2\gamma_m(\lambda_+) & \Delta m = \Phi^2 m^2 \end{cases}$$

we get

$$d_{\Delta m} < 4 \tag{7.49}$$

as the condition for 'softly' broken scale invariance (Wilson 1969*a*). Otherwise the mass insertion would effectively correspond to a hard operator and would influence the asymptotic behaviour of the Green's functions. Eq. (7.49) is automatically satisfied if the theory is asymptotically free because the relevant anomalous dimension γ_m vanishes.

7.3 Dimensional transmutation

The phenomenon of dimensional transmutation is closely related to scale invariance broken by the renormalization of a quantum field theory. To introduce this concept let us consider any dimensionless observable quantity $A(Q)$ which can depend only on one dimensionful variable Q, for example, $A(Q) = Q^2 \cdot \sigma_{\text{tot}}$. If we assume our theory to be described by a scale-invariant lagrangian (no dimensionful parameters) then from dimensional analysis we must conclude that

$$A(Q) = \text{const.} \tag{7.50}$$

Indeed, dimensional analysis tells us that, for example, a function $f(x, y)$ which depends on two massive variables x, y and which is dimensionless and whose definition does not involve any massive constants must be a function of the ratio x/y only. So if it does not depend on y it must be a constant. As we know from our experience with quantum field theories the conclusion (7.50) is, in general, wrong. The reason why pure dimensional analysis breaks down in quantum field theory is this: our quantity A actually also depends on the dimensionless free parameter(s) of the theory, for example, bare coupling constant(s): $A = A(Q, g_{\mathrm{B}})$. Moreover, the predictions of the theory cannot be expressed directly in terms of g_{B} because the theory requires renormalization. The physics is independent of the details of the renormalization prescription but not of the necessity of renormalization which produces an effective scale. Dimensional transmutation is a process which exploits the above to introduce dimensionful parameters into the predictions of the theory.

An important illustration of these general considerations is provided by the behaviour of the running coupling constant $\alpha(Q^2)$ in QCD (we forget here about a subtlety: $\alpha(Q^2)$ is not really an observable; it depends on the definition of the renormalization scheme). One-parameter ambiguity for $\alpha(Q^2)$ is reflected in the fact that the theory specifies the first derivative of $\alpha(Q^2)$ rather than $\alpha(Q^2)$ itself

$$\mathrm{d}\alpha/\mathrm{d}t = \beta(\alpha) \tag{7.51}$$

where $Q = \exp(t)Q_0$, $\alpha = g^2/4\pi$ and $\beta(\alpha) = (b_1/\pi)a^2 + O(\alpha^3)$. Integrating (7.51) we get

$$\ln(Q/Q_0) = F(\alpha(Q)) - F(\alpha(Q_0))$$

where

$$F(x) = \int \mathrm{d}x/\beta(x)$$

Thus, an effective scale Q_0 appears as a constant of integration of the RGE (7.51). If we choose Q_0 so that $F(\alpha(Q_0)) = 0$ and call it Λ then

$$\alpha(Q) = F^{-1}[\ln(Q/\Lambda)] \tag{7.52}$$

and in the leading logarithm approximation

$$\alpha(Q) = -\pi/[b_1 \ln(Q/\Lambda)] \tag{7.53}$$

Λ, defined by the condition $F(\alpha(\Lambda)) = 0$, is the confinement scale: $\alpha(\Lambda) = \infty$. Eq. (7.53) can be also rewritten in the following form

$$\Lambda/Q = \exp\{-\pi/[|b_1|\alpha(Q)]\} \tag{7.53a}$$

which tells us that given the strong coupling constant $\alpha(Q)$ at some Q the theory predicts the confinement scale Λ. Dimensionless $\alpha(Q)$ and dimensionful Λ can be traded for each other.

7.4 Operator product expansion (OPE)

Short distance expansion

The OPE technique originates in studies of the relevance of scale invariance in quantum field theory (Wilson 1969*a*). In the previous section we have discussed how mass terms and renormalizable interactions break this invariance which may, however, be re-established, at least up to logarithmic corrections, in the limit of momenta going to infinity. In position space this corresponds to short distances, i.e. to vanishing separation for the operators in the matrix elements. OPE and scale invariance arguments allow us to determine the short distance singularity structure of operator products. The physical importance of studying operator products at short distance will be discussed later. Here we shall introduce the technique of the OPE.

According to Wilson's hypothesis a product of two local operators $A(x)$ and $B(y)$, when x_μ is near to y_μ, may be written as†

$$A(x)B(y) = \sum_n C^n_{AB}(x-y)O_n(\tfrac{1}{2}(x+y)) \tag{7.54}$$

where $\{O_n(x)\}$ is a complete set of hermitean local normal-ordered operators and $\{C^n_{AB}(x)\}$ is a set of c-number functions. The expansion is valid in the weak sense, i.e. when sandwiched between physical states. Relation (7.54) is true in free-field theory (see Problem 7.4) and in any order of perturbation theory for interacting fields (Zimmermann 1970). Assuming scale invariance to be an approximate symmetry at short distance one gets the leading behaviour of the Cs for small values of their arguments, hopefully up to at most logarithmic corrections, in terms of the scale dimensions of the operators. Indeed, commuting the dilatation generator D with (7.54) we get

$$\left[D, A(x)B(y) - \sum_n C^n_{AB}(x-y)O_n((x+y)/2)\right] = 0 \tag{7.55}$$

and in an exactly scale-invariant theory

$$\sum_n \left[d_n - d_A - d_B - (x-y)\frac{\partial}{\partial(x-y)}\right]C^n_{AB}(x-y)O_n(\tfrac{1}{2}(x+y)) = 0 \tag{7.56}$$

where d_n, d_A and d_B are scale dimensions of the operators O_n, A and B, respectively. Since the operators O_n can be chosen to be independent we deduce that

$$(x-y)\frac{\partial}{\partial(x-y)}C^n_{AB}(x-y) = (d_n - d_A - d_B)C^n_{AB}(x-y) \tag{7.57}$$

and therefore C^n_{AB} is homogeneous of degree $d_n - d_A - d_B$ in its argument

$$C^n_{AB}(x) \sim x^{d_n - d_A - d_B} \tag{7.58}$$

Asymptotic scale invariance at short distance implies that relation (7.58) holds at least in the limit $x \to 0$. In QCD asymptotic scale invariance is broken by logarithmic corrections (see Section 7.2) so relation (7.58) is modified but only by logarithmic terms. We see that the most singular contribution to $A(x)B(y)$

† The product also can be written in terms of, say, $O_n(y)$ by expanding $O_n(\frac{1}{2}(x+y))$ in a Taylor series about y.

as $x \to y$ is given by the operator O_n having the lowest dimension. We have learned in Section 7.2 that the scale dimensions d_A, d_B and d_n are not, in general, equal to the canonical dimensions of the corresponding operators. Anomalous dimensions (see (7.43)) can be calculated in perturbation theory. However, for several interesting operators like conserved currents or currents of softly broken symmetries, i.e. broken by operators with scale dimensions $d < 4$, the anomalous dimensions vanish (see Section 10.1).

The Lorentz transformation properties of the Wilson coefficients $C^n_{AB}(x)$ require them to be polynomials in the components of x times scalar functions of x^2: $C^n_{AB} = C^n_{AB}(x, x^2)$. We have to determine prescriptions for handling light-cone singularities of the coefficients C^n_{AB} for different types of products: time-ordered products, simple products, commutators etc. To do this let us refer to free scalar field theory. For instance, for the T-product of two currents $J(x) = :\Phi(x)\Phi(x):$ in free scalar field theory we have (see Problem 7.4)

$$\begin{aligned} TJ(x)J(0) &= -2\Delta_{\mathrm{F}}^2(x, m^2)I + 4\mathrm{i}\Delta_{\mathrm{F}}(x, m^2):\Phi(x)\Phi(0): + :\Phi^2(x)\Phi^2(0): \\ &\underset{x\to 0}{=} -2\Delta_{\mathrm{F}}^2(x, m^2)I + 4\mathrm{i}\Delta_{\mathrm{F}}(x, m^2)J(0) \\ &\quad + 4\mathrm{i}\Delta_{\mathrm{F}}(x, m^2)x_\mu :\Phi(0)(\partial^\mu\Phi)(0): + \cdots \end{aligned} \tag{7.59}$$

where

$$\begin{aligned} \Delta_{\mathrm{F}} = -\mathrm{i}\langle 0|T\Phi(x)\Phi(0)|0\rangle &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k \exp(-\mathrm{i}kx)\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} \\ &\underset{x^2\to 0}{=} -\frac{\mathrm{i}}{4\pi^2}\frac{1}{x^2 - \mathrm{i}\varepsilon} \end{aligned} \tag{7.60}$$

is the free particle propagator in position space. Notice that $:\Phi(x)\Phi(0):$ may be expanded in a Taylor series since it has no singularity. Thus for T-products we conclude that scalar factors in Cs depend on $x^2 - \mathrm{i}\varepsilon$. Since

$$TA(x)B(0) = A(x)B(0)\Theta(x_0) + B(0)A(x)\Theta(-x_0)$$

and

$$\frac{1}{-x^2 + \mathrm{i}\varepsilon x_0}\Theta(x_0) + \frac{1}{-x^2 - \mathrm{i}\varepsilon x_0}\Theta(-x_0) = -\frac{1}{x^2 - \mathrm{i}\varepsilon} \tag{7.61}$$

for simple products we have the following prescription $C^n_{AB}(x, x^2) \equiv C^n_{AB}(x, x^2 - \mathrm{i}\varepsilon x_0)$. For commutators $[A(x), B(0)]$ we then get $(1/(x \pm \mathrm{i}\varepsilon) = P(1/x) \mp \mathrm{i}\pi\delta(x))$†

$$\frac{1}{x^2 - \mathrm{i}\varepsilon x_0} - \frac{1}{x^2 + \mathrm{i}\varepsilon x_0} = 2\pi\mathrm{i}\delta(x^2)\varepsilon(x_0) = \varepsilon(x_0)2\mathrm{i}\,\mathrm{Im}\,\frac{1}{x^2 - \mathrm{i}\varepsilon} \tag{7.62}$$

† Note in particular that:

$$\Delta = -\mathrm{i}\langle 0|[\Phi(x), \Phi(0)]|0\rangle = \varepsilon(x_0)2\,\mathrm{Im}(\mathrm{i}\Delta_{\mathrm{F}}) = -(1/2\pi)\varepsilon(x_0)\delta(x^2)$$

and by differentiating both sides with respect to x^2

$$\left(\frac{1}{-x^2 + \mathrm{i}\varepsilon x_0}\right)^n - \left(\frac{1}{-x^2 - \mathrm{i}\varepsilon x_0}\right)^n = -\frac{2\pi\mathrm{i}}{(n-1)!}\delta^{(n-1)}(x^2)\varepsilon(x_0) \tag{7.63}$$

where

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

In a general case in which the light-cone singularity (7.63) is $(x^2)^{-p}$ with some non-integer power p, the difference

$$(-x^2 + \mathrm{i}\varepsilon x_0)^{-p} - (-x^2 - \mathrm{i}\varepsilon x_0)^{-p}$$

appearing in a commutator $[A(x), B(0)]$ vanishes for space-like separation $x^2 < 0$: the prescription $x^2 \to x^2 - \mathrm{i}\varepsilon x_0$, by placing the cut along the positive real axis, is consistent with causality.

Using the expansion of the form

$$[A(x), B(0)] = \sum_n \tilde{C}^n_{AB} O_n(0) \tag{7.64}$$

where

$$\tilde{C}^n_{AB} = C^n_{AB}(x, x^2 - \mathrm{i}\varepsilon x_0) - C^n_{AB}(x, x^2 + \mathrm{i}\varepsilon x_0)$$

and letting $x_0 \to 0$ one can study the equal-time commutators. This is interesting because the appearance in the commutator of the so-called Schwinger terms (proportional to derivatives of the δ-functions) is linked to the singularity structure of operator products at short distances. Following Wilson (1969*a*) we can see that the coefficients $\tilde{C}^n_{AB}$ are equivalent to a sum of δ-functions

$$\tilde{C}^n_{AB} = {}_0F^n_{AB}(x_0)\delta^{(3)}(\mathbf{x}) + {}_1\mathbf{F}^n_{AB}(x_0) \cdot \boldsymbol{\nabla}\delta^{(3)}(\mathbf{x}) + \cdots \tag{7.65}$$

where

$$\begin{aligned} {}_0F^n_{AB}(x_0) &= \int \mathrm{d}^3x\, \tilde{C}^n_{AB}(x_0, \mathbf{x}) \\ {}_1\mathbf{F}^n_{AB}(x_0) &= -\int \mathrm{d}^3x\, \tilde{C}^n_{AB}(x_0, \mathbf{x})\mathbf{x} \end{aligned}$$

etc. To get (7.65) one considers an integral

$$\int \mathrm{d}^3x\, \tilde{C}^n_{AB}(x_0, \mathbf{x})\rho(\mathbf{x}) = \int \mathrm{d}^3x\, \tilde{C}^n_{AB}(x_0, \mathbf{x})(\rho(0) + \mathbf{x} \cdot \nabla\rho(0) + \cdots) \tag{7.66}$$

where $\rho(\mathbf{x})$ is a differentiable trial function and the Taylor expansion is justified for small x_0 because $\tilde{C}^n_{AB}$ vanishes for $|\mathbf{x}| \geq |x_0|$. Eq. (7.65) follows immediately from (7.66) and from the definitions of the δ-function and its derivatives. The dependence of the functions ${}_iF^n_{AB}(x_0)$ on x_0 is determined by dimensional analysis to be

$$ {}_iF^n_{AB}(x_0) \sim x_0^{-2p_n+3+i} \tag{7.67} $$

when

$$ \tilde{C}^n_{AB} \sim x^{-2p_n} $$

The equal-time commutator is obtained in the limit $x_0 \to 0$. Thus for $p > 1.5$ the coefficient of the δ-function is infinite and for $p > 2$ derivatives of the δ-functions appear in the commutator. Some terms may vanish due to rotational invariance. As an example let us take the product of two electromagnetic currents

$$ J_\mu(x) = :\bar{\Psi}(x)\gamma_\mu\Psi(x): \tag{7.68} $$

In free fermion field theory one has (see Problem 7.4)

$$ J_\mu(x)J_\nu(0) \sim \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4(x^2 - \mathrm{i}\varepsilon x_0)^4}\mathbb{1} + O(x^{-3}) \tag{7.69} $$

The first term gives the familiar Schwinger term in the commutator $[J_0(x), J_i(0)]$ (but not for two J_0s):

$$ [J_0(\mathbf{x}, 0), J_i(0)] \sim C_{ij}\partial_j\delta^{(3)}(\mathbf{x}) \tag{7.70} $$

where C_{ij} is quadratically divergent. We will see that the vacuum expectation value of this term is related to the e^+e^- annihilation cross-section calculated under the assumption that the leading short distance singularities are those of the free-field theory. Such terms are usually not revealed by direct calculation of the commutator of two operators based on equal-time canonical commutation relations between fields and their conjugate momenta.

Light-cone expansion

The OPE can be extended to the light-cone region as well. It is obvious, however, that terms in (7.54) which are finite in the limit $x_\mu \to 0$ may be singular in the limit $x^2 \to 0$ with x_μ finite. Therefore we must expect an infinity of terms to contribute to the leading singularity on the light-cone, in contrast to the situation at $x = 0$ where the leading behaviour is given by one term.

Let us look in more detail at the expansion of a product of two Lorentz scalar operators, for instance, two currents in free scalar field theory, (7.59). As mentioned before, the Lorentz tensor operators $(\partial_\mu\Phi)(0)$, $(\partial_\mu\partial_\nu\Phi)(0)$ etc. cannot be neglected on the light-cone and the general structure of the expansion is as follows:

$$A(\tfrac{1}{2}x)B(-\tfrac{1}{2}x) = \sum_n C^n_{AB}(x^2)x^{\mu_1}\dots x^{\mu_j}O^n_{\mu_1\dots\mu_j}(0) \tag{7.71}$$

We have taken the bases $O^n_{\mu_1\dots\mu_j}$ to be symmetric traceless tensors with j Lorentz indices; they are irreducible tensors of spin j. We see that the behaviour of the expansion for small x^2 is determined by scalar functions of x^2, the $C^n_{AB}(x^2)$ in the neighbourhood of $x = 0$. (Note that we use the same notation for complete Wilson coefficients in (7.54) and for their scalar part in (7.71). This should not lead to any confusion.) As long as there are no extra divergences introduced by the summation over n, short distance behaviour and light-cone behaviour are connected

$$C^n_{AB}(x^2) = (x^2)^{(d_n - j_n - d_A - d_B)/2} + \text{higher orders in } x^2 \tag{7.72}$$

so that the degree of singularity on the light-cone is determined by

$$\tau_n = d_n - j_n \tag{7.73}$$

The difference (dimension of an operator – its spin) is called twist. The leading singularity on the light-cone comes from operators with the lowest twist. In free-field theory there exist infinite series of operators with fixed twist, since operating on a field with a derivative raises the spin and the dimension by one unit simultaneously. The scalar field Φ and the fermion field Ψ as well as their derivatives have twist one. Examples of twist two operators are

$$\Phi^* \overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\dots\overleftrightarrow{\partial}_{\mu_j}\Phi$$

and

$$\bar{\Psi}\gamma_\mu\overleftrightarrow{\partial}_{\mu_1}\dots\overleftrightarrow{\partial}_{\mu_j}\Psi$$

When dimensions are modified by the interactions it is not obvious that they are correlated with the spin in this way. Nevertheless, one usually takes the free-field theory classification of operators as a guide in writing down the OPE.

As an exercise let us expand

$$\begin{aligned}\Phi^*\left(-\frac{x}{2}\right)\Phi\left(\frac{x}{2}\right) &= \Phi^*(0)\left[1 - \overleftarrow{\partial}_{\mu_1}\frac{x^{\mu_1}}{2} + \frac{1}{2!}\overleftarrow{\partial}_{\mu_1}\overleftarrow{\partial}_{\mu_2}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2} - \cdots\right] \\ &\quad\times\left[1 + \frac{x^{\nu_1}}{2}\overrightarrow{\partial}_{\nu_1} + \frac{1}{2!}\frac{x^{\nu_1}}{2}\frac{x^{\nu_2}}{2}\overrightarrow{\partial}_{\nu_1}\overrightarrow{\partial}_{\nu_2} + \cdots\right]\Phi(0) \\ &= \sum_n \frac{1}{n!}\frac{x^{\mu_1}}{2}\frac{x^{\mu_2}}{2}\cdots\frac{x^{\mu_n}}{2}\Phi^*(0)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\Phi(0)\end{aligned} \tag{7.74}$$

Inserting (7.74) into (7.59) we get the leading terms in the light-cone expansion of the product of two scalar currents in free-field theory. In a similar way we can obtain the light-cone expansion for two electromagnetic currents.

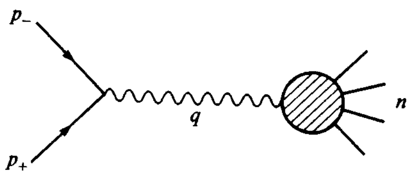


Fig. 7.1.

7.5 The relevance of the light-cone

Electron–positron annihilation

We briefly discuss now the relevance of the light-cone for high energy phenomena. Consider first the process of the electron-positron annihilation into hadrons. It is a process widely investigated experimentally and interesting in many respects as a laboratory for important discoveries in particle physics. We focus only on the total cross section which is of the order of a few tens of nanobarns and is comparable to the cross section for the $e^+e^- \to \mu^+\mu^-$. In the one-photon exchange approximation (see Fig. 7.1) the amplitude is

$$T_n = e^2\bar{v}(p_+)\gamma_\mu u(p_-)(1/q^2)\langle n|J_\mu(0)|0\rangle \tag{7.75}$$

where J_μ is the hadronic electromagnetic current. Thus the total cross section is determined by the tensor

$$\begin{aligned} W_{\mu\nu}(q) &= \sum_n (2\pi)^4\delta(q - p_n)\langle 0|J_\mu(0)|n\rangle\langle n|J_\nu(0)|0\rangle \\ &= \int \mathrm{d}^4x \exp(\mathrm{i}qx)\langle 0|J_\mu(x)J_\nu(0)|0\rangle \end{aligned} \tag{7.76}$$

J_μ is conserved and therefore one may write

$$W_{\mu\nu}(q) = \rho(q^2)(q^2 g_{\mu\nu} - q_\mu q_\nu) \tag{7.77}$$

It can be checked that the total annihilation cross section for unpolarized initial leptons reads

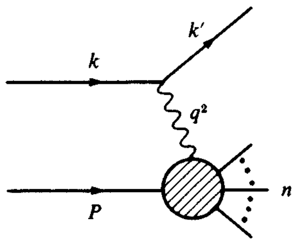
$$\sigma(q^2) = 8\pi^2\alpha^2\rho(q^2)/q^2 \tag{7.78}$$

To see what happens when $q^2 \to \infty$, we observe first that $W_{\mu\nu}(q)$ can be written in terms of the current commutator

$$W_{\mu\nu}(q) = \int \mathrm{d}^4x \exp(\mathrm{i}qx)\langle 0|[J_\mu(x), J_\nu(0)]|0\rangle \tag{7.79}$$

Eq. (7.79) differs from (7.76) by the term

$$-(2\pi)^4\delta(q + p_n)\langle 0|J_\nu(0)|n\rangle\langle n|J_\mu(0)|0\rangle$$



$$q^2 = (k'-k)^2 = -4EE' \sin^2(\tfrac{1}{2}\theta)$$

$$\nu = P \cdot q = m(E'-E)$$

$$s = (P+q)^2 = m^2 + 2\nu + q^2$$

Fig. 7.2.

which is zero because q_0 and p_n^0 are positive. Since $q^2 > 0$ we may choose a frame in which q has only a time component and use the fact that the commutator vanishes for $x^2 < 0$. Then in this frame

$$W_{\mu\nu}(q) = \int_{-\infty}^{\infty} \mathrm{d}x_0 \exp[\mathrm{i}(q^2)^{1/2} x_0] \int_{|\mathbf{x}|<x_0} \mathrm{d}^3x \, \langle 0|[J_\mu(x), J_\nu(0)]|0\rangle \tag{7.80}$$

Since the dominant contribution to the integral over x_0 comes from regions with the least rapid oscillations this Fourier transform is determined by the singularity of the commutator at $x = 0$. In a free-field theory the leading singularity of the commutator originates from the term (7.69) which is responsible for the Schwinger term (7.70). Using (7.63) and the relation

$$\int \mathrm{d}^4x \exp(\mathrm{i}kx)\delta^{(n)}(x^2)\varepsilon(x_0) = \frac{2^{2-2n}}{(n-1)!}\pi^2 \mathrm{i}(k^2)^{n-1}\Theta(k^2)\varepsilon(k_0) \tag{7.81}$$

we easily find (for a single fermion with charge ± 1)

$$W_\mu{}^\mu(q) \underset{q^2\to\infty}{=} (1/2\pi)q^2 \tag{7.82}$$

and consequently

$$\sigma(q^2) \to 4\pi\alpha^2/3q^2 \tag{7.83}$$

This result is not surprising: a free-field theory is asymptotically scale-invariant and the cross section must behave like $1/q^2$ on dimensional grounds.

Deep inelastic hadron leptoproduction

Let us now consider another famous process: the production of hadrons in lepton–hadron collisions with very large momentum transfer. It is shown in Fig. 7.2 which also contains some definitions of kinematical variables: k, k' are lepton four-momenta; P is the four-momentum of the hadronic target (usually a proton); E, E' and θ are the lepton energies and lepton scattering angle in the laboratory frame, respectively. As we shall see it is also very useful to define the so-called Bjorken variable

$$x_{\mathrm{BJ}} = -q^2/2P \cdot q \tag{7.84}$$

For elastic scattering

$$s = m^2 \Rightarrow 2P \cdot q + q^2 = 0 \Rightarrow x_{\mathrm{BJ}} = 1$$

In terms of the Bjorken variable, for fixed q^2, the physical region for the process is therefore

$$0 < x_{\mathrm{BJ}} < 1 \tag{7.85}$$

To be specific we consider the process of electroproduction. As in case of the $\mathrm{e}^+\mathrm{e}^-$ annihilation into hadrons in the one-photon exchange approximation the cross section for this process is determined by the tensor

$$\begin{aligned} W_{\mu\nu}(P, q) &= \sum_n (2\pi)^4 \delta(q + P - p_n) \langle P | J_\mu(0) | n \rangle \langle n | J_\nu(0) | P \rangle \\ &= \int \mathrm{d}^4 x \exp(\mathrm{i}qx) \langle P | J_\mu(x) J_\nu(0) | P \rangle \\ &= \int \mathrm{d}^4 x \exp(\mathrm{i}qx) \langle P | [J_\mu(x), J_\nu(0)] | P \rangle \end{aligned} \tag{7.86}$$

where J_μ is the hadron electromagnetic current. The passage from product to commutator can be justified in a way similar to that for the annihilation reaction. We collect together some properties of the tensor $W_{\mu\nu}(P, q)$, which are important for further discussion. If we introduce the forward Compton scattering amplitude for photons with mass q^2,

$$T_{\mu\nu}(q^2, P \cdot q) = \mathrm{i} \int \mathrm{d}^4 x \exp(\mathrm{i}qx) \langle P | T J_\mu(x) J_\nu(0) | P \rangle \tag{7.87}$$

then it follows from the spectral representations (see also (7.62)) that

$$W_{\mu\nu}(q^2, P \cdot q) = 2 \,\mathrm{Disc}\, T_{\mu\nu}(q^2, P \cdot q) = 2 \,\mathrm{Im}\, T_{\mu\nu} \tag{7.88}$$

where $\mathrm{Disc}\, T_{\mu\nu}$ is the discontinuity of the $T_{\mu\nu}$ along its branch cuts. The amplitude $T_{\mu\nu}$ has normal threshold cuts in $s = (P + q)^2$ and $u = (P - q)^2$ for $m^2 < s < \infty$ and $m^2 < u < \infty$. They correspond to a cut in x_{BJ} for $|x_{\mathrm{BJ}}| < 1$. Note that only for $0 < x_{\mathrm{BJ}} < 1$ is the $W_{\mu\nu}(q^2, P \cdot q)$ given by the deep inelastic cross section.

Now we are ready to discuss the relevance of the light-cone in our process. Let us choose a frame in which

$$P = (m, 0, 0, 0), \quad q = (q_0, 0, 0, q_3)$$

and introduce the so-called light-cone variables

$$x_{\pm} = x_0 \pm x_3, \quad \mathbf{x}_{\perp} = (x_1, x_2)$$

$$q_{\pm} = \tfrac{1}{2}(q_0 \pm q_3) = (\nu/2m)[1 \pm (1 - m^2q^2/\nu^2)^{1/2}], \quad \nu = q_0 m$$

Then

$$\begin{aligned} W_{\mu\nu}(P, q) = \int \mathrm{d}x_- \exp(\mathrm{i}q_+x_-) \int \mathrm{d}x_+ \exp(\mathrm{i}q_-x_+) \\ \times \int_{x_\perp^2 \le x_+x_-} \mathrm{d}^2x_\perp \, \langle P|[J_\mu(x), J_\nu(0)]|P\rangle \end{aligned} \tag{7.89}$$

where the limit on the $\mathbf{x}_\perp$-integration comes from the vanishing of the commutator for $x^2 = x_+x_- - x_\perp^2 < 0$. We see that in the limit

$$\left. \begin{array}{l} q_+ \to \infty \\ q_- \text{ fixed} \end{array} \right\} \tag{7.90}$$

the behaviour of $W_{\mu\nu}$ is determined by the behaviour of the integrand for $x_- \to 0$, x_+ finite, i.e. in the region with the least oscillations. Since the integrand is non-vanishing only for $x_\perp^2 < x_+x_-$ (causality), $x_- \to 0$ for x_+ finite corresponds to $x^2 \to 0$ (but not to $x_\mu \to 0$) and $W_{\mu\nu}$ is determined by the singularity of the current commutator on the light-cone. In the limit (7.90) $q^2 \to -\infty$, $\nu \to \infty$ and x_{BJ} is fixed ($x_{\mathrm{BJ}} = -q^2/2\nu$). The idea of studying the limit (7.90) is due to Bjorken (1969) and it has turned out to be one of the most creative ideas in modern elementary particle physics.

For simplicity let us study the scalar current analogue of the electroproduction tensor

$$W(q^2, P\cdot q) = \int \mathrm{d}^4x \exp(\mathrm{i}qx)\langle P|[J(x), J(0)]|P\rangle \tag{7.91}$$

in the Bjorken limit. Spin complications are not essential for general orientation. Apart from the c-number contribution which is irrelevant for the considered matrix element (first term in (7.59)) the leading singularities of the operator products on the light-cone are given by the twist two operators

$$\begin{aligned} \langle P|[J(x), J(0)]|P\rangle = \frac{1}{\pi^2} 2\pi \mathrm{i}\delta(x^2)\varepsilon(x_0) \sum_n \frac{1}{n!} x_{\mu_1} \ldots x_{\mu_n} \\ \times \langle P|\Phi(0)\partial_{\mu_1} \ldots \partial_{\mu_n}\Phi(0)|P\rangle + \text{less singular terms} \end{aligned} \tag{7.92}$$

where we have used (7.63) and (7.74). For the realistic case of particles with spin, twist two operators are again the leading ones. The general form of the matrix element in (7.92) is

$$\langle P|O_{\mu_1\ldots\mu_n}|P\rangle = A_n P_{\mu_1}\ldots P_{\mu_n} + \delta_{\mu_1\mu_2}P_{\mu_3}\ldots P_{\mu_n} + \cdots \tag{7.93}$$

Terms having at least one Kronecker delta may be ignored because when contracted with the x_μs in (7.92) they give additional factors x^2 and therefore are less singular. So we have

$$\langle P|[J(x), J(0)]|P\rangle = \frac{2\mathrm{i}}{\pi}\delta(x^2)\varepsilon(x_0)f(xP) + \text{less singular terms} \tag{7.94}$$

where

$$f(xP) = \sum_n \frac{1}{n!}(x\cdot P)^n A_n$$

Defining the Fourier transform

$$f(xP) = (1/2\pi)\int \mathrm{d}\xi\,\exp(\mathrm{i}\xi x\cdot P)\tilde{f}(\xi)$$

and using the identity†

$$\mathrm{i}\int \mathrm{d}^4x\,\exp(\mathrm{i}kx)\delta(x^2)\varepsilon(x_0) = (2\pi)^2\varepsilon(k_0)\delta(k^2) \tag{7.95}$$

we get

$$W(q^2, P\cdot q) = 4\int \mathrm{d}\xi\,\delta((\xi P + q)^2)\varepsilon(q_0 + \xi P_0)\tilde{f}(\xi)$$

The roots of the argument of the δ-function are

$$\xi_\pm = -\frac{\nu}{m^2} \pm \frac{\nu}{m^2}\left(1 - \frac{q^2m^2}{\nu^2}\right)^{1/2} \xrightarrow[\mathrm{BJ}]{} \frac{-q^2}{2\nu}, -\frac{2\nu}{m^2} \tag{7.96}$$

where ‘BJ’ means Bjorken limit. As has already been explained, $W(q^2, P\cdot q)$, being the imaginary part of the amplitude for forward off-mass-shell Compton scattering, is non-vanishing for $|-q^2/2\nu| < 1$. Thus only the ξ_+ is relevant and in terms of the Bjorken variable our result reads

$$\nu W(q^2, P\cdot q) = 2\tilde{f}(x_{\mathrm{BJ}}) \tag{7.97}$$

In the free-field theory with twist two operators giving the leading singularity on

† Identity (7.95) follows from the spectral representation for the free-field commutator (Bjorken & Drell 1965) and the relation (7.62):

$$\begin{aligned}\langle 0|[\Phi(x), \Phi(0)]|0\rangle &= -\frac{1}{(2\pi)^3}\int \mathrm{d}^4k\,\exp(\mathrm{i}kx)\delta(k^2 - m^2)\varepsilon(k_0)\\ &= \left(\frac{\mathrm{i}}{2\pi}\right)\delta(x^2)\varepsilon(x_0)\end{aligned}$$

the light-cone, we obtain the famous Bjorken scaling for the structure function $W(q^2, P \cdot q)$.

An important final observation is that only even spin operators are allowed in the expansion (7.92). Indeed W is real (it is a cross section) so $\tilde{f}(\xi)$ must be real and therefore $f(xP)$ must be even in x.

Wilson coefficients and moments of the structure function

Use of the OPE is by no means limited to the free-field theory case. Its validity for renormalized operators has been proved in perturbation theory and therefore it can be used to obtain further information about the scaling limit of the structure functions, beyond that contained in the hypothesis of canonical dimensions. An important step in this direction is to establish some relation between the Wilson coefficients and measurable quantities which is not limited to the canonical scaling case. The standard procedure is as follows. Consider first the forward scattering amplitude for scalar currents

$$T(q^2, P \cdot q) = \mathrm{i} \int \mathrm{d}^4x \exp(\mathrm{i}qx) \langle P|TJ(\tfrac{1}{2}x)J(-\tfrac{1}{2}x)|P\rangle \tag{7.98}$$

The structure function (7.91) is the discontinuity of T, see (7.88). Instead of (7.92) let us consider

$$TJ(\tfrac{1}{2}x)J(-\tfrac{1}{2}x) \underset{x^2 \to 0}{\approx} \sum_{n,k} C_n^k(x^2) x_{\mu_1} \ldots x_{\mu_n} O_k^{\mu_1 \ldots \mu_n} \tag{7.99}$$

where the sum is taken over all twist two operators O_k and all even spins n. Using (7.93) and defining

$$\frac{2^n q^{\mu_1} \ldots q^{\mu_n}}{(-q^2)^{n+1}} \tilde{C}_n^k(q^2) = \mathrm{i} \int \mathrm{d}^4x \exp(\mathrm{i}qx) x^{\mu_1} \ldots x^{\mu_n} C_n^k(x^2) \tag{7.100}$$

the leading contribution to $T(q^2, P \cdot q)$ in the Bjorken limit can be written as

$$T(q^2, P \cdot q) = \frac{1}{-q^2} \sum_{n,k} x_{\mathrm{BJ}}^{-n} \tilde{C}_n^k(q^2) A_n^k \tag{7.101}$$

Note that $\tilde{C}_n^k(q^2)$ are defined so that they are dimensionless for twist two operators in the expansion of the product of two currents with dimension $d_J = 3$. Therefore in the case of canonical scaling $\tilde{C}_n^k(q^2) = \tilde{C}_n^k$ are constants. The coefficients $\tilde{C}_n^k(q^2)$ can be related to the moments of the structure function $W(q^2, P \cdot q)$ if we integrate $T(q^2, x_{\mathrm{BJ}})$ over x_{BJ}, for fixed q^2. Remembering that it is an analytic function of x_{BJ} in the complex x_{BJ}-plane with a branch cut along the real axis for

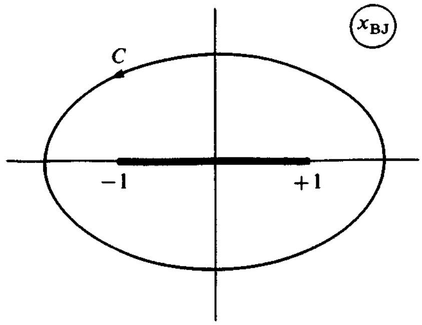


Fig. 7.3.

$|x_{BJ}| < 1$ we integrate over the contour shown in Fig. 7.3. Since the discontinuity of the T is just $\frac{1}{2}W(q^2, P \cdot q)$ we have ($x \equiv x_{BJ}$)

$$\frac{1}{2\pi i} \oint_C \mathrm{d}x\, x^{m-1} T(q^2, x) = \frac{1}{4\pi} \int_{-1}^{1} \mathrm{d}x\, x^{m-1} W(q^2, x) \tag{7.102}$$

and on the other hand

$$\frac{1}{2\pi i} \oint_C \mathrm{d}x\, x^{m-1} T(q^2, x) = \frac{1}{-q^2} \sum_k \tilde{C}_m^k(q^2) A_m^k \tag{7.103}$$

Finally we get

$$M_m(q^2) = (1/2\pi) \int_{-1}^{1} \mathrm{d}x\, x^m \nu W(q^2, x) = \sum_k \tilde{C}_m^k(q^2, \mu^2, g) A_m^k(\mu^2, g) \tag{7.104}$$

In (7.104) we have written down explicitly the dependence on the renormalization point and the coupling constant.

Thus we get a relationship between moments of the deep inelastic structure function and the coefficient functions. Using the crossing properties of the structure function under $q \to -q$ or $x \to -x$

$$W(q^2, x) = -W(q^2, -x) \tag{7.105}$$

we can rewrite (7.104) for even m in terms of the integral over the physical region in the electroproduction process, $0 < x < 1$ (for odd m the integral (7.104) vanishes). We also note that (7.104) is meaningful for values of m such that the integral converges. This may not be the case for low moments: for fixed q^2 the limit $x \to 0$ corresponds to $\nu \to \infty$, where $\nu = P \cdot q \approx s$ and s is the cms energy squared in the forward Compton amplitude. At high energy the imaginary part of this amplitude is expected to behave as s^α, $\alpha \approx 1$, so that $W(q^2, x) \sim x^{-1}$.

An important virtue of the OPE is that it allows us to factorize out the q^2 dependence in quantities like M_m which depend on q^2 and $P^2 = m^2$. This dependence can then be studied by means of the RGE. Similar analysis is not possible directly for $M_n(q^2, P^2 = m^2)$ because the scale transformation $q \to \rho q$, $P \to \rho P$ relates M_ns for different q^2s and m^2s.

As we know, exact Bjorken scaling gives

$$\tilde{C}_n^k(q^2) = \text{const.} \tag{7.106}$$

Interactions can modify this behaviour. In the next section we shall study the q^2 dependence of the Wilson coefficients using RGE.

7.6 Renormalization group and OPE

Renormalization of local composite operators

We are used by now to working with Green's functions involving not only elementary fields but also composite operators, for example,

$$G_{\mathbf{O}}^{(n)}(x_1, \ldots, x_n, x) = \langle 0|T\Phi(x_1)\ldots\Phi(x_n)\mathbf{O}(x)|0\rangle \tag{7.107}$$

or in momentum space

$$(2\pi)^4\delta(p_1 + \cdots + p_n + p)\tilde{G}_{\mathbf{O}}^{(n)}(p_1, \ldots, p_n, p)$$
$$= \int \mathrm{d}^4x \exp(\mathrm{i}px)\prod_{i=1}^{n}\mathrm{d}^4x_i \exp(\mathrm{i}p_ix_i)G_{\mathbf{O}}^{(n)}(x_1, \ldots, x_n, x) \tag{7.108}$$

The generating functional formalism and perturbation theory rules can be extended to such cases if we introduce sources $\Delta(x)$ coupled to operators $\mathbf{O}(x)$. The functional

$$W[J, \Delta] = \int \mathcal{D}\Phi \exp\left[\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\,(J\Phi + \Delta\mathbf{O})\right] \tag{7.109}$$

generates these new Green's functions. Connected Green's functions are obtained from the functional Z and the Legendre transformation performed on sources J (only) generates 1PI Green's functions with $\mathbf{O}$ insertions. As usual one has to take derivatives with respect to Δ and then set $\Delta = 0$. Feynman rules for vertices involving elementary fields and the composite operators $\mathbf{O}(x)$ can be derived in exactly the same way as in Chapter 2 or Chapter 3. The new Green's functions calculated perturbatively, in general, require renormalization. The counterterms present in the lagrangian for the renormalization of the Green's functions involving only elementary fields are not sufficient for eliminating divergences in the Green's functions with composite operator insertions. Additional operator renormalization constants are needed

$$\mathbf{O}^{\mathrm{B}}(x) = Z_{\mathbf{O}} \mathbf{O}^{\mathbf{R}}(x) \tag{7.110}$$

If we want to express $\mathbf{O}^{\mathbf{R}}$ in terms of the renormalized fields the wave-function renormalization constants will explicitly appear, for example,

$$(\Phi_{\mathrm{B}}^2)_{\mathrm{B}} = Z_{\Phi^2}(\Phi_{\mathrm{B}}^2)_{\mathrm{R}} = Z_{\Phi^2} Z_3 (\Phi_{\mathrm{R}}^2)_{\mathrm{R}} \tag{7.111}$$

Only in exceptional cases like conserved currents or currents of softly broken symmetries does $Z_{\mathbf{O}} = 1$ (see Section 10.1). Relations (7.110) and (7.111) correspond to a replacement

$$\Delta(x)\mathbf{O}^{\mathrm{B}}(\Phi_{\mathrm{B}}^i(x)) = \Delta \mathbf{O}^{\mathrm{R}}(\Phi_{\mathrm{R}}^i) + \Delta(Z_{\mathbf{O}} Z_3^{i/2} - 1)\mathbf{O}^{\mathrm{R}}(\Phi_{\mathrm{R}}^i) \tag{7.112}$$

in the generating functional, (7.109). We have taken $\mathbf{O}(x)$ to be of the ith order in the elementary fields $\Phi(x)$. The last term in (7.112) is the new counterterm. The $Z_{\mathbf{O}}$ can be calculated in perturbation theory by calculating the divergent Green's functions with operator insertions. The superficial degree of divergence (see Section 4.2) $D_{\mathbf{O}}$ of $\Gamma_{\mathbf{O}}^{(n)}$ differs from D of $\Gamma^{(n)}$ and reads

$$D_{\mathbf{O}} = D + (d_{\mathbf{O}}^{\mathrm{can}} - 4) \tag{7.113}$$

where $d_{\mathbf{O}}^{\mathrm{can}}$ is the canonical dimension of $\mathbf{O}$. As an example let us consider $\lambda\Phi^4$ theory and discuss a single insertion of $\mathbf{O}(x) = \frac{1}{2}\Phi^2(x)$ (Itzykson & Zuber 1980) or an insertion of $\mathbf{O}(x) = \bar{\Psi}(x)\Psi(x)$ in QCD. The divergent Green's functions with this insertion are $\Gamma_{\Phi^2}^{(2)}$ and $\Gamma_{\bar{\Psi}\Psi}^{(2)}$ with $D_{\mathbf{O}} = 0$. It is an easy exercise to calculate $Z_{\mathbf{O}}$ in these simple cases in the lowest order of perturbation theory. The relevant Feynman diagrams are shown in Fig. 7.4, where the vertex $\otimes$ is in both cases just one and for simplicity we can take the momenta of the external legs to be equal.

Of course, new renormalization constants require new renormalization conditions. For instance, we may impose

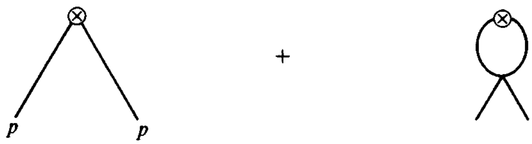
$$\Gamma_{\Phi^2}^{(2)}(0,0) = 1 \qquad \text{and} \qquad \Gamma_{\bar{\Psi}\Psi}^{(2)}(0,0) = 1 \tag{7.114}$$

in agreement with the lowest order.

If an operator $\mathbf{O}$ is multiplicatively renormalized as in (7.110) then the 1PI n-point Green's functions with a single $\mathbf{O}$ insertion satisfy

$$\Gamma_{\mathbf{O},\mathrm{R}}^{(n)}(p_1, \ldots, p_n, p, \lambda, \mu, m) = Z_{\mathbf{O}}^{-1} Z_3^{n/2} \Gamma_{\mathbf{O},\mathrm{B}}^{(n)}(p_1, \ldots, p_n, p, \lambda_{\mathrm{B}}, m_{\mathrm{B}}) \tag{7.115}$$

Simple multiplicative renormalization, (7.110), is actually not always sufficient. For instance, it may not eliminate divergences from some Green's functions involving more then one insertion of $\mathbf{O}$. A well-known example is the vacuum



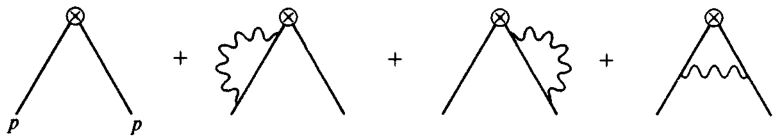


Fig. 7.4.

expectation value of two electromagnetic currents $J_\mu(x)$ which requires a subtraction although $Z_J = 1$ (see Section 10.1).

Another complication arises for the following reason. Using the arguments of Section 4.2 concerning the necessary counterterms in the standard renormalization programme we can convince ourselves that an operator in the lagrangian with dimension d requires, in general, all possible counterterms with dimensions $\leq d$ allowed by the symmetries of the bare lagrangian. If we limit our discussion to Green's functions involving only one insertion of an operator **O** the same is true for the renormalization of such composite operators. Therefore if there exist several operators with the same quantum numbers as the operator **O** and with their canonical dimensions lower or equal to the dimension of the operator **O**, they will mix with **O** under the renormalization. Thus, (7.110) takes a matrix form

$$\mathbf{O}_n^{\mathrm{B}} = (Z_{\mathbf{O}})_{nk}\mathbf{O}_k \tag{7.116}$$

and correspondingly

$$\Gamma^{(n)}_{\mathbf{O}_{\mathrm{n}},\mathrm{R}}(p_1, \ldots, p_n, p, \lambda, \mu, m) = (Z_{\mathbf{O}}^{-1})_{nk} Z_3^{n/2} \Gamma^{(n)}_{\mathbf{O}_{\mathrm{k}},\mathrm{B}}(p_1, \ldots, p_n, p, \lambda_{\mathrm{B}}, m_{\mathrm{B}}) \tag{7.117}$$

Note that the matrix Z_{ij} is not symmetric: $Z_{ij} = 0$ if $\dim \mathbf{O}_i < \dim \mathbf{O}_j$.

From (7.117) we can derive for Green's functions with an operator insertion the RGE analogous to (6.9) and (6.24), for instance,

$$\left\{\left[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m \frac{\partial}{\partial m} - n\gamma + D\right]\delta_{ij} + \gamma_{\mathbf{O}}^{ij}\right\}\Gamma^{(n)}_{\mathbf{O}_j}(\mathrm{e}^t p_i, \lambda(\mu), \mu, m) = 0 \tag{7.118}$$

where

$$\gamma_{\mathbf{O}}^{ij} = -\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}(Z_{\mathbf{O}})^{ik}\right)(Z_{\mathbf{O}}^{-1})^{kj}$$

is the anomalous dimension matrix for the operators $\mathbf{O}_k$ and D is the dimension of the Green's function (equal to the canonical dimension of all operators involved) and the mass dependence has been neglected.

At this point it is worthwhile to observe that in perturbation theory there is no fundamental difference between the renormalization of the original lagrangian, (4.1), and of the composite operators. Both can be formulated in the same language. For instance, the renormalization constants Z_0 and Z_1, defined in Chapter 4 as renormalization of the lagrangian parameters m_B and λ_B, could as well be defined as the renormalization constants for the operators Φ^2 and Φ^4. According to (2.89), the n-particle Green's function is, in fact, a superposition of Green's functions with the vertex and mass operator insertions (we can treat the mass as one of the parameters in perturbation theory). The derivatives $m^2\partial/\partial m^2$ and $\lambda\partial/\partial\lambda$ count then the number of such insertions (see also (7.47)) and we arrive to the same RG equations (6.9) and (6.24), with, for example, γ_m interpreted now as the anomalous dimension of the operator Φ^2 (7.49).

In the opposite direction, the renormalization of local composite operators discussed here can be phrased in the language of Chapter 4. We can interpret the operators $\mathbf{O}_n$ as a perturbation of the lagrangian (4.1):

$$\mathcal{L}^\mathrm{B}_{\Phi^4} \to \mathcal{L}^\mathrm{B}_{\Phi^4} + \sum_n \int \mathrm{d}^4x \, C^\mathrm{B}_n \mathbf{O}^\mathrm{B}_n$$

and we can renormalize the new coupling constants: $C^\mathrm{B}_n = Z^n_\mathbf{O} C_n$. We then derive the RGE for the $\Gamma^{(n)}_{\mathbf{O}_n}$ by a complete analogy with the derivation of (6.9) and (6.24) and, putting $C_n = 1$, we get, for example, (7.118).

RGE for Wilson coefficients

We now turn back to the OPE and study the q^2 dependence of the Wilson coefficient functions by means of the RGE. Let us recall that in general

$$A(\tfrac{1}{2}x)B(-\tfrac{1}{2}x) \underset{x^2\to 0}{\approx} \sum_i C^{AB}_i(x, g(\mu), \mu)\mathbf{O}_i(0) \tag{7.119}$$

and therefore for any Green's function with insertion of the product AB we have

$$\Gamma^{(n)}_{AB}(q, p_1, \ldots, p_n, g(\mu), \mu) = \sum_i \tilde{C}^{AB}_i(q, \mu)\Gamma^{(n)}_{\mathbf{O}_i}(0, p_1, \ldots, p_n, g(\mu), \mu) \tag{7.120}$$

By applying the differential operator

$$\mathcal{D} = -\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}$$

to (7.120) with all momenta rescaled by $\exp(t)$ and using (7.118) we get

$$\left\{\left[-\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}+D_A+D_B-D_{\mathbf{O}}+\gamma_A+\gamma_B\right]\delta_{ij}-\gamma_{\mathbf{O}}^{ij}\right\}\tilde{C}_i^{AB}(\mathrm{e}^t q, g(\mu), \mu) = 0 \tag{7.121}$$

where D_A, D_B and $D_{\mathbf{O}}$ are canonical dimensions of A, B and $\mathbf{O}$ respectively and γ_A, γ_B and $\gamma_{\mathbf{O}}^{ij}$ are their anomalous dimensions. We have assumed here that operators A and B do not mix with other operators when undergoing renormalization. This is in particular trivially true for the electromagnetic currents. The solution to the RGE for the Wilson coefficient is analogous to (6.26). Let us write it down for some specific cases. Firstly take $\tilde{C}_i$ dimensionless, i.e. $D_A+D_B-D_{\mathbf{O}}=0$. In addition put $\gamma_A=\gamma_B=0$ as for conserved currents. For the no-mixing case $\gamma_{\mathbf{O}}^{ij}=\delta^{ij}\gamma_i$ we then have

$$\tilde{C}_i(\mathrm{e}^t q, g(\mu), \mu) = \exp[-\gamma_i(g_+)t]\exp\left\{-\int_g^{\bar{g}(t)}\mathrm{d}x\,[\gamma_i(x)-\gamma_i(g_+)]/\beta(x)\right\} \times \tilde{C}_i(q, \bar{g}(t), \mu) \tag{7.122}$$

To proceed further one has to calculate $\tilde{C}_i(q, \bar{g}(t), \mu)$, $\gamma_i(g)$ and $\beta(g)$ in perturbation theory. In asymptotically free theories the UV fixed point $g_+=0$, $\beta(x)=(2/16\pi^2)b_1x^3+\cdots$ and $\gamma_i(x)=(4/16\pi^2)\gamma_i^0x^2+\cdots$. Putting $q^2=\mu^2$, $t=\frac{1}{2}\ln(q^2/\mu^2)$ and expanding

$$\tilde{C}_i(\mu, \bar{g}(t), \mu) = \tilde{C}_i(\mu, 0, \mu)[1+C_i^1\bar{g}^2(t)/4\pi+\cdots]$$

in the leading order in g^2 we get

$$\tilde{C}_i(q, g(\mu), \mu) = \tilde{C}_i(\mu, 0, \mu)[\bar{g}^2(t)/g^2(\mu^2)]^{-\gamma_i^0/b_1} \tag{7.123}$$

Thus for the moments of the structure function we have (from (7.104) when $k=1$, i.e. if there is only one twist two operator in expansion (7.99))

$$\frac{M_i(q^2)}{M_i(q_0^2)} = \left[\frac{\bar{g}^2(q^2)}{\bar{g}^2(q_0^2)}\right]^{-\gamma_i^0/b_1} = \left[\frac{\ln(q^2/\Lambda^2)}{\ln(q_0^2/\Lambda^2)}\right]^{+\gamma_i^0/b_1} \tag{7.124}$$

In the last step we have used (7.53). Eq. (7.124) is valid for both q_0^2 and q^2 large as compared to Λ^2. For a non-zero UV fixed point $g_+\neq 0$ (7.122) gives a power-like violation of scaling

$$\tilde{C}_i(q, g(\mu), \mu) \approx (q^2/\mu^2)^{-\gamma_i(g_+)}\tilde{C}_i(\mu, g_+, \mu) \tag{7.125}$$

In the case of operator mixing the solution to the RGE can be written in the matrix form (take $g_+ = 0$)

$$\hat{C}(q, g(\mu), \mu) = \left\{ T_g \exp\left[\int_{\bar{g}(q^2)}^{g(\mu^2)} \mathrm{d}x\, \hat{\gamma}(x)/\beta(x) \right] \right\} \hat{C}(\mu, \bar{g}(q^2), \mu) \tag{7.126}$$

where

$$\hat{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}, \qquad \hat{\gamma} = [\gamma_{ij}]$$

and the T_g ordering, necessary since $[\hat{\gamma}(x_1), \hat{\gamma}(x_2)] \neq 0$, is defined as follows

$$\begin{aligned} T_g \exp\left[\int_{\bar{g}}^{g} \mathrm{d}x\, \frac{\hat{\gamma}(x)}{\beta(x)} \right] = 1 &+ \int_{\bar{g}}^{g} \mathrm{d}x \frac{\hat{\gamma}(x)}{\beta(x)} \\ &+ \frac{1}{2!} \int_{\bar{g}}^{g} \mathrm{d}x \int_{\bar{g}}^{x} \mathrm{d}y \frac{\hat{\gamma}(x)}{\beta(x)} \frac{\hat{\gamma}(y)}{\beta(y)} + \cdots \end{aligned} \tag{7.127}$$

OPE beyond perturbation theory

Strictly speaking, the use of the OPE to study the moments of the structure function in deep inelastic lepton–hadron scattering, which is discussed in this section, goes beyond perturbation theory. This is because in the realistic case, when the strong interactions are described by QCD, the matrix elements of the operators taken between hadronic states contain non-perturbative effects responsible for the binding of quarks and gluons into hadrons. Nevertheless we assume the validity of the OPE and, moreover, that the coefficient functions C_n^{AB} are calculable in QCD perturbation theory. This approach is implicitly based on the parton model assumption discussed in Section 8.4. It is also in agreement with the possibility, very suggestive as the physical sense of the OPE formalism, that the contribution from short distances can be separated from that of large distances. One can then expect that to a good approximation the coefficient functions are calculable in perturbation theory while the non-perturbative effects are accounted for by the matrix elements of the operators O_n.

The same idea underlies the use of the OPE to account for the non-perturbative structure of the QCD vacuum, pioneered by Shifman, Vainshtein & Zakharov (1979) and extensively discussed in the recent years. Non-perturbative effects manifest themselves through non-trivial vacuum expectation values of various operators which vanish in perturbation theory

$$\langle 0| :O_n: |0\rangle = 0 \qquad \text{if} \quad O_n \neq I$$

The best-known examples are quark and gluon condensates

$$\langle 0|:\bar{\Psi}\Psi:|0\rangle \quad \text{and} \quad \langle 0|:G^a_{\mu\nu}G_a^{\mu\nu}:|0\rangle$$

the first one is responsible for the spontaneous breaking of chiral symmetry in QCD (Chapter 9). This use of the OPE has been successful phenomenologically but its theoretical status and possible limitations are still subject to some discussion (see, for instance, Novikov *et al.* (1984) and references therein).

The coefficient functions $C_n^{AB}(q)$ can be found by calculating the appropriate Feynman diagrams (Problem 7.5). Important for this programme is the theorem that the coefficient functions in the OPE manifestly reflect the symmetries of the lagrangian whether or not they are symmetries of the vacuum (Bernard *et al.* 1975).

7.7 OPE and effective field theories

OPE is a convenient tool for discussing effective field theories. Let us consider a renormalizable action containing some light field(s) φ with mass m and some heavy field(s) Φ with mass M (such that $m \ll M$)

$$S(\varphi, \Phi) = \int \mathrm{d}x \left[\mathcal{L}_\varphi(x) + \mathcal{L}_\Phi(x) + \mathcal{L}_{\varphi\Phi}(x)\right] \tag{7.128}$$

We are interested in calculating amplitudes involving only the light particles φ with external momenta much smaller than M. The Green's functions for processes involving only light particles are generated by the functional (recall (2.84))

$$\begin{aligned} W[J] &= N\int \mathcal{D}\varphi\,\mathcal{D}\Phi \exp\left\{\mathrm{i}S(\varphi, \Phi) + \mathrm{i}\int \mathrm{d}x\; J(x)\varphi(x)\right\} \\ &= N\int \mathcal{D}\varphi\,\mathcal{D}\Phi \exp\left\{\mathrm{i}\int \mathrm{d}x\left[\mathcal{L}_\varphi(x) + J(x)\varphi(x)\right]\right\} \\ &\quad\times \sum_{k=0}^{\infty}\frac{\mathrm{i}^k}{k!}\int \mathrm{d}y_1\ldots\mathrm{d}y_k\,\mathcal{L}_{\varphi\Phi}(y_1)\ldots\mathcal{L}_{\varphi\Phi}(y_k)\exp\left\{\mathrm{i}\int \mathrm{d}x\,\mathcal{L}_\Phi(x)\right\}. \end{aligned} \tag{7.129}$$

From now on, we assume for simplicity that the interaction lagrangian $\mathcal{L}_{\varphi\Phi}$ is linear in Φ, i.e. $\mathcal{L}_{\varphi\Phi} = \tilde{\mathcal{L}}(\varphi)\Phi$. (The derivatives acting on Φ can be removed by integration by parts.) In such a case, we can write

$$
\begin{aligned}
W[J] = N \int \mathcal{D}\varphi \exp &\left\{ \mathrm{i} \int \mathrm{d}x \left[\mathcal{L}_\varphi(x) + J(x)\varphi(x) \right] \right\} \\
&\times \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_k \, \tilde{\mathcal{L}}(y_1) \ldots \tilde{\mathcal{L}}(y_k) \\
&\times \int \mathcal{D}\Phi \, \Phi(y_1) \ldots \Phi(y_k) \exp \left\{ \mathrm{i} \int \mathrm{d}x \, \mathcal{L}_\Phi(x) \right\} \qquad (7.130) \\
= N' \int \mathcal{D}\varphi \exp &\left\{ \mathrm{i} \int \mathrm{d}x \left[\mathcal{L}_\varphi(x) + J(x)\varphi(x) \right] \right\} \\
&\times \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_k \tilde{\mathcal{L}}(y_1) \ldots \tilde{\mathcal{L}}(y_k) G^\Phi(y_1, \ldots, y_k)
\end{aligned}
$$

Comparison with (2.130), (2.131) and (2.137) allows us to 'exponentiate' with simultaneous replacement of ordinary Green's functions by the connected ones

$$
\begin{aligned}
W[J] = N' \int \mathcal{D}\varphi \exp &\left\{ \mathrm{i} \int \mathrm{d}x \left[\mathcal{L}_\varphi(x) + J(x)\varphi(x) \right] \right. \\
&\left. + \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_k \, \tilde{\mathcal{L}}(y_1) \ldots \tilde{\mathcal{L}}(y_k) G^\Phi_{conn}(y_1, \ldots, y_k) \right\} \qquad (7.130a)
\end{aligned}
$$

We have thus arrived at the effective theory described by a non-local action. As far as the processes involving only external φ particles are concerned, it is perfectly equivalent to the the underlying theory (7.128). The important point is that the effective region of integration over y involves distances much shorter than the region of integration over x.

Now, the OPE has to be performed. The OPE hypothesis (7.54) can be formulated in the path integral language as follows. Let $A(x)$ and $B(x)$ be some monomials in the fields $\varphi(x)$ and their derivatives, belonging to a complete basis of such monomials $\{O_k(x)\}$. Then there exists a set of Wilson coefficients $\{C^k_{AB}(x)\}$ such that†

$$
\begin{aligned}
&\int \mathcal{D}\varphi A(x) B(y) \exp \left\{ \mathrm{i}S[\varphi] + \mathrm{i} \int \mathrm{d}x \, J(x)\varphi(x) \right\} \\
&\quad = \sum_k C^k_{AB}(x-y) \int \mathcal{D}\varphi \, O_k\left(\frac{x+y}{2} \right) \exp \left\{ \mathrm{i}S(\varphi) + \mathrm{i} \int \mathrm{d}x \, J(x)\varphi(x) \right\}
\end{aligned}
\qquad (7.131)
$$

† Strictly speaking, it corresponds only to OPE for T-products. All the local field products here are tacitly assumed to involve counterterms which in the operator language would correspond to their normal ordering (see (2.105)–(2.107)). Moreover, if A or B involve derivatives, the series in the r.h.s. of (7.131) is not in a one-to-one correspondence with that of (7.54), which is due to the features described below (2.56).

One can prove the validity of a similar expansion for multiple products like $A(x)B(y)C(z)$ by appropriate differentiations of (7.131) with respect to $J(z)$ and z (according to the structure of $C(z)$), and applying (7.131) to each term on the r.h.s. In a similar way, one can show that if $A(x)B(y)$ corresponds to a series $\sum C^k_{AB} O_k$ and $C(x)D(y)$ corresponds to a series $\sum C^k_{CD} O_k$, then $A(x)B(y)C(z)D(w)$ corresponds to the product

$$\left[\sum_k C^k_{AB}(x-y) O_k\left(\frac{x+y}{2}\right)\right]\left[\sum_n C^n_{CD}(z-w) O_n\left(\frac{z+w}{2}\right)\right] \tag{7.132}$$

Finally, one arrives at the conclusion that it is possible to replace the products $\tilde{\mathcal{L}}(y_1)\ldots\tilde{\mathcal{L}}(y_n)$ in (7.130a) by sums of the form

$$\tilde{\mathcal{L}}(y_1)\ldots\tilde{\mathcal{L}}(y_k) \longrightarrow \sum_l C^{(k)}_l(y_1,\ldots,y_k) O_l(y_{\text{mean}}) \tag{7.133}$$

in spite of the fact that they stand under the exponent. This latter conclusion would be rather obvious in the operator language. In the next step, we integrate the coefficients $C^{(k)}_l$ with the connected Green's functions keeping all the y_{mean}s fixed, i.e. performing only $k-1$ integrations for each k-point function. The results of this integration are independent of y_{mean}, which follows from translational invariance. In this way we obtain

$$W[J] = N' \int \mathcal{D}\varphi \exp\left\{\mathrm{i}S_{\text{eff}} + \mathrm{i}\int \mathrm{d}x\,[\Delta\mathcal{L}_\varphi(x) + J(x)\varphi(x)]\right\} \tag{7.134}$$

with S_{eff} given by

$$S_{\text{eff}} = \int \mathrm{d}x\left[\mathcal{L}_\varphi(x) + \sum_l \frac{1}{M^{n(l)}} C_l O_l(x)\right], \qquad n(l) \geq 1. \tag{7.135}$$

and

$$C_l = M^{n(l)} \sum_k \frac{\mathrm{i}^k}{k!}\int \mathrm{d}y_1\ldots\mathrm{d}y_k\, C^{(k)}_l(y_1,\ldots,y_k) G^{\Phi}_{conn}(y_1,\ldots,y_k)\delta(y_{\text{mean}}) \tag{7.136}$$

The factors $M^{n(l)}$ have been extracted above just to keep C_l dimensionless. The coefficients C_l are customarily also called Wilson coefficients. However, one should stress the important difference with respect to the original (true) Wilson coefficients defined by (7.133): in (7.136) the latter are integrated with the Green's functions for heavy particles. Therefore, the coefficients C_l are constants, which can be interpreted as additional independent coupling constants for the effective interaction vertices O_l. $\Delta\mathcal{L}_\varphi$ stands for terms which would have $n(l) \leq 0$.

It remains to show that $\Delta\mathcal{L}_\varphi$ (which may contain M independent terms, logarithms and positive powers of large mass M) can be absorbed into $\mathcal{L}_\varphi$ by renormalization of its fields and parameters, and that C_ls are convergent (or behaving at most logarithmically) in the limit $M \to \infty$. This requires the introduction of some restrictions on the action S in (7.128). We assume that the theory described by $\mathcal{L}_\varphi$ is

(i) renormalizable, and the symmetries defining† $\mathcal{L}_\varphi$ are also the symmetries of S including the possible gauge fixing terms for Φ, but not for φ (see Weinberg (1980));
(ii) asymptotically scale-invariant, up to logarithmic corrections.

By assumption (i) $\Delta\mathcal{L}_\varphi$ contains only terms allowed by symmetries of S (we assume the absence of anomalies). This guarantees that $\Delta\mathcal{L}_\varphi$ can be absorbed into $\mathcal{L}_\varphi$ by appropriate redefinition of fields and parameters in $\mathcal{L}_\varphi$.

Determining the behaviour of the C_ls in the limit $M \to \infty$ is based on the fact that the (Euclidean) connected Green's functions for the Φ particles exponentially vanish when splitting among their arguments becomes much larger than $1/M$. Thus, the behaviour of the C_ls at $M \to \infty$ is determined by the leading behaviour of (Euclidean versions of) $C_l^{(k)}(y_1, \ldots, y_k)$ at points where all the y_i almost coincide. Assumption (ii) tells us that the latter behaviour is independent of dimensionful parameters present in $\mathcal{L}_\varphi$, up to logarithmic corrections. Hence, the leading behaviour of the C_ls at $M \to \infty$ has the same property.‡ This is equivalent to saying that they behave at most logarithmically, because powers of $1/M$ have been already extracted in (7.135) to make the C_ls dimensionless.

An important point omitted in the above discussion is checking whether the integrals present in (7.136) are not divergent in spite of their proper behaviour at $M \to \infty$. In fact, they are UV divergent, but these divergences are expected to cancel among different integrals when counterterms for the full theory defined by the lagrangian (7.128) are properly included. Checking this would require expressing $C_l^{(k)}$ in terms of the φ-particle Green's functions (and their derivatives) with the help of (7.131). We are going to bypass this point here.

The action S_{eff} in (7.135) is, in principle, non-renormalizable because it contains terms of arbitrary dimension. However, as we will see below, the renormalization programme can be effectively carried out. Moreover, since the contributions to the amplitudes from higher-dimensional O_ls are suppressed§ by powers of (external momenta)$/M$, only a finite number of O_ls needs to be considered in practice. The parameters (couplings) C_l undergo renormalization and also evolve according to their RGEs which do not depend on the anomalous dimensions of the original

† A renormalizable theory can be defined by specifying its field content and its symmetries. The action is built from all possible terms of dimension ≤ 4 that are invariant under the symmetries. Of course, not every set of fields and symmetries can give us a renormalizable theory.

‡ Provided the interaction $\mathcal{L}_{\Phi\varphi}$ does not contain coupling constants growing with M. Such a situation occurs when Higgs–fermion couplings in the Standard Model are considered in the process of heavy fermion decoupling.

§ This can be easily shown on dimensional grounds.

operators in the OPE (7.133). They can be considered to be new free parameters to be determined from experiment or, if the full theory is known, they can be recovered by requiring the equality of amplitudes generated by (7.128) and (7.135). Only in this sense are they dependent on m/M and other parameters present in (7.128).

The reason for introducing the effective action (7.135) is the fact that the perturbative expansion of low energy amplitudes generated by (7.128) is in practice an expansion in powers of (coupling constants)$\cdot$ ln[(external momenta)$/M$] which may be too large to be expansion parameters. Replacing (7.128) by (7.135) combined with the RGE evolution allows us to extract large logarithms from all orders of the perturbation series. There are four main steps of this procedure:

First, one recovers the renormalization constants in S_{eff} determined by the requirement of finiteness of amplitudes generated by S_{eff}. The renormalization constants in $\mathcal{L}_\varphi$ may be different than in the underlying theory (7.128). The sum over l in (7.135), when expressed in terms of renormalized quantities, takes the form

$$\sum_l \frac{1}{M^{n(l)}} \left[\sum_{l_1} Z_{l_1 l} C_{l_1} + \sum_{l_1 l_2} Z_{l_1 l_2 l} C_{l_1} C_{l_2} + \cdots \right] \tilde{Z}_l O_l \tag{7.137}$$

where the $\tilde{Z}_l$s are products of renormalization constants of the fields present in O_ls, and $Z_{l_1 \ldots l_j l}$ are additional renormalization constants necessary to cancel divergences in diagrams with j O_l-vertices. All the Zs in the above expression depend on m/M and the couplings present in $\mathcal{L}_\varphi$, but not on the coefficients C_l. It is easy to convince oneself that $Z_{l_1 \ldots l_j l}$ is proportional to M^{-p}, where $p = n(l_1) + \cdots + n(l_j) - n(l)$. It vanishes for $p < 0$ which can be justified† using the arguments of Section 4.2. Therefore, only a finite number of renormalization constants has to be found when one works up to a given order in $1/M$. This is why the renormalization programme can, in practice, be carried on in the theory described by (7.135).

Next, one finds the coefficients C_l (and, in principle, also the renormalized parameters in $\mathcal{L}_\varphi$) by requiring equality of amplitudes generated by (7.128) and (7.135). The renormalization scale μ has then to be set equal to $\mu \sim M$, in order to avoid large logarithms.‡

The third step is deriving and solving the RGEs for the effective action parameters, which results in expressing these parameters renormalized at the scale $\mu_{\text{F}} \sim$

† According to (4.42), the superficial degree of divergence of a diagram with $O_{l_1}, \ldots, O_{l_j}$-vertices is equal to $D = 4 - B - \frac{3}{2}F + n(l_1) + \cdots + n(l_j)$. After removing subdivergences, it may require counterterms which are polynomials in external momenta only of order $\leq D$. Such counterterms are generated by O_ls involving B bosonic fields, F fermionic fields and no more than D derivatives. Such O_ls have $n(l) \leq D + B + (3/2)F - 4$, and therefore $n(l_1) + \cdots + n(l_j) - n(l) \geq 0$.

‡ We restrict ourselves to the μ-dependent renormalization schemes such as $\overline{\text{MS}}$ or the μ-subtraction scheme.

(external momenta) in terms of their counterparts renormalized at $\mu_{\mathrm{I}} \sim M$. The latter have been already found by 'matching' in the previous step and serve as initial conditions for the RGEs. As long as one works in the framework of dimensional regularization, the renormalized C_ls should be rescaled by appropriate powers of μ^{ϵ} (as has been done with λ_{R} in Section 6.1) in order to keep them dimensionless. The RGEs are then derived, for example, from the equation $\mathrm{d}/\mathrm{d}\mu\, S_{\mathrm{eff}} = 0$, and the methods† of Section 6.2 are used for determining the μ-derivatives of renormalization constants. For simplicity, let us restrict our attention to the leading order in $1/M$. Then, only the first of the sums in the bracket of (7.137) needs to be considered and, interpreting C_ls as new coupling constants, we get

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_l(\mu) = \sum_k \gamma_{kl}(g(\mu)) C_k(\mu), \tag{7.138}$$

where the anomalous dimension matrix $\hat{\gamma}$ is determined by

$$\gamma_{kl}(g) = -\sum_j \left(\mu \frac{d}{d\mu} Z_{kj} \right) (Z^{-1})_{jl}. \tag{7.139}$$

The last step is calculating the desired amplitudes with help of S_{eff} renormalized at $\mu_{\mathrm{F}} \sim$ (external momenta). This is done either perturbatively or with use of some non-perturbative methods (Buchalla, Buras & Harlander 1990).

In principle, all possible terms allowed by symmetries of (7.128) are expected to appear in the expansion (7.135). Even at the leading order in $1/M$, their number is usually very large. Therefore, at the beginning, elements of all the four steps are performed simultaneously in order to select the set of operators O_l that may contribute to the desired amplitudes at the desired order (even by their influence on the RGE evolution of some directly contributing O_l). One should also remember that even the coefficients C_l vanishing at μ_{I} can acquire non-zero values at μ_{F}, due to the non-diagonal character of the mixing in (7.137).

As already mentioned, the outlined procedure allows us to extract large logarithms from all orders of the perturbation series. In the following, the meaning of this statement will be made more precise. Let us assume that $\mathcal{L}_{\varphi}$ contains only one (dimensionless) coupling constant g. Then, it is convenient to solve the RGEs using the variable $v = g^2 \ln(\mu_{\mathrm{I}}^2/\mu^2)$ instead of μ itself. The variable v is treated as independent of $g(\mu_{\mathrm{I}})$, and all the RGEs are solved perturbatively in $g(\mu_{\mathrm{I}})$. The coupling constant $g(\mu)$ is therefore determined, up to a given order in $g(\mu_{\mathrm{I}})$, by

† It may not always be possible to define a mass-independent renormalization scheme in a theory described by S_{eff}. Then the methods of Section 6.2 have to be modified. Here we consider only those cases in which it is possible to avoid the appearance of dimensionful parameters. Sometimes this requires changing the normalization of some of the O_ls, for example, multiplying them by m.

the series

$$g(\mu) = g(\mu_{\mathrm{I}}) \sum_{n=0}^{\infty} g^{2n}(\mu_{\mathrm{I}}) f_n(v), \tag{7.140}$$

where $f_n(v)$ are certain functions found with help of the equation (see (6.6))

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g(\mu) = \beta(g). \tag{7.141}$$

The coupling $g(\mu)$ from (7.140) is then inserted into the RGE for the coefficients C_l, (7.138). Since $\mu \mathrm{d}/\mathrm{d}\mu = -2g^2(\mu_{\mathrm{I}})\mathrm{d}/\mathrm{d}v$, knowledge of $\hat{\gamma}$ and β up to $(n+1)$th order in g^2 is necessary for determining the C_ls up to nth order in $g^2(\mu_{\mathrm{I}})$ and exactly in v. Thus, if

- the renormalization constants (first step) are found up to $(n + 1)$th order in g^2,
- the matching at μ_{I} (second step) is performed up to nth order in $g^2(\mu_{\mathrm{I}})$,
- the final amplitude (last step) is calculated up to nth order in $g^2(\mu_{\mathrm{F}})$ and expanded with help of (7.140),

we obtain the final answer up to nth order in $g^2(\mu_{\mathrm{I}})$ and exactly in $v_{\mathrm{F}} = 2g^2 \ln(\mu_{\mathrm{I}}/\mu_{\mathrm{F}})$. This is the precise meaning of the statement that all large logarithms are extracted from the perturbation series. Such a result could never be obtained with help of perturbative calculus based on the action (7.128) itself. The price we have to pay is the necessity of truncating the series in $1/M$, but this is often a much better approximation than truncating the series in $g(\mu_{\mathrm{I}})$ which has to be done anyway.

There are two main domains of applications of the described technique. First of them covers low energy electroweak processes involving hadrons (see, for example, Grinstein, Springer & Wise (1990)). There, the role of heavy particles is played by the $\mathrm{W}^{\pm}$ and Z^0 bosons, the top quark and the Higgs bosons. The coupling which becomes too large after multiplication by the logarithm is the QCD gauge coupling. In some processes, the b and c quarks are also treated as heavy. Then the decoupling is done successively and the matching between several 'effective theories' is performed (see, for example, Gilman & Wise (1979)). (More details of the effective theory approach are presented in Sections 12.6 and 12.7). The other domain covers phenomena taking place much above the electroweak scale (see, for example, Wilczek & Zee (1979)). There, the coupling constants are smaller but the evolution is usually longer. In fact, even the coupling constant splitting after the spontaneous symmetry breakdown in GUTs can be most easily understood in terms of the effective theory arising after the heavy gauge boson decoupling (Weinberg 1980). It is important to notice that after decoupling the resulting effective theory usually has less symmetries than the original one (as in the examples mentioned above).

Problems

7.1 (*a*) Check that the set of 15 infinitesimal transformations defined by (7.26) is closed under commutation. Check the differential representation of the generators

$$
\begin{aligned}
\mathrm{i}[P_\mu, f(x)] &= \partial_\mu f \\
\mathrm{i}[M^{\mu\nu}, f(x)] &= (x^\mu \partial^\nu - x^\nu \partial^\mu) f \\
\mathrm{i}[D, f(x)] &= x^\mu \partial_\mu f \\
\mathrm{i}[K_\mu, f(x)] &= (x^2 g_{\mu\nu} - 2x_\nu x_\mu)\partial^\nu f
\end{aligned}
$$

where $f(x)$ is a dimensionless scalar function and

$$f(x') = U f(x) U^{-1}$$

where

$$
\begin{aligned}
U(a) &= \exp(\mathrm{i}a_\mu P^\mu), & U(\varepsilon) &= \exp(\mathrm{i}\varepsilon D), \\
U(\omega) &= \exp(-\tfrac{1}{2}\mathrm{i}\omega^{\mu\nu} M_{\mu\nu}), & U(b) &= \exp(\mathrm{i}b^\mu K_\mu)
\end{aligned}
$$

for $x'^\mu = x^\mu + \varepsilon^\mu(x)$.

(*b*) Check the algebra

$$
\begin{aligned}
[M_{\mu\nu}, D] = [P_\mu, P_\nu] &= [K_\mu, K_\nu] = [D, D] = 0 \\
[P_\mu, D] &= \mathrm{i}P_\mu \\
[K_\mu, D] &= \mathrm{i}K_\mu \\
[M_{\mu\nu}, K_\lambda] &= \mathrm{i}(g_{\mu\lambda} K_\nu - g_{\nu\lambda} K_\mu) \\
[K_\mu, P_\nu] &= 2\mathrm{i}(-g_{\mu\nu} D + M_{\mu\nu}) \\
[M^{\mu\nu}, P^\lambda] &= -\mathrm{i}(g^{\mu\lambda} P^\nu - g^{\nu\lambda} P^\mu) \\
[M^{\alpha\beta}, M^{\mu\nu}] &= -\mathrm{i}(g^{\alpha\mu} M^{\beta\nu} - g^{\beta\mu} M^{\alpha\nu} + g^{\alpha\nu} M^{\mu\beta} - g^{\beta\nu} M^{\mu\alpha})
\end{aligned}
$$

(*c*) Check the infinitesimal transformation law for a field Φ under the special conformal transformation

$$\delta_\mu \Phi(x) = (x^2 \partial_\mu - 2x_\mu x_\nu \partial^\nu - 2x_\mu d_\Phi - 2x^\nu \Sigma_{\mu\nu})\Phi(x)$$

7.2 Study conformal symmetry in two-dimensional quantum field theories (Belavin, Polyakov & Zamolodchikov 1984).

7.3 When the dilatation current is not conserved one may still define a time-dependent dilatation charge by

$$D(t) = \int \mathrm{d}^3x\, x_\mu \Theta^{0\mu}(x) = tP^0 + \int \mathrm{d}^3x\, x_i \Theta^{0i}(x)$$

Check that $D(t)$ generates the same dilatation transformation (7.14) on the fields as in the scale-invariant theory (use the canonical formula (7.16) for $D_\mu(x)$). Obtain the commutators of $D(t)$ with the Poincaré generators

$$\mathrm{i}[D(t), P^\alpha] = P^\alpha - g^{\alpha 0}\int \mathrm{d}^3x\, \partial_\mu D^\mu(x)$$
$$\mathrm{i}[D(t), M^{\alpha\beta}] = \int \mathrm{d}^3x\, (g^{\alpha 0}x^\beta - g^{\beta 0}x^\alpha)\partial_\mu D^\mu(x)$$

7.4 (*a*) Using the Wick expansion and then the Taylor expansion of the well-defined normal-ordered operator products, prove (7.59) in the free scalar field theory.
(*b*) In the free quark model show that

$$T J_\mu^a(x) J_\nu^b(0) \underset{x_\mu \to 0}{\sim} \frac{3\delta_{ab}(g_{\mu\nu}x^2 - 2x_\mu x_\nu)}{(2\pi)^4(x^2 - \mathrm{i}\varepsilon)^4} I + \frac{d_{abc}}{2\pi^2}\frac{\varepsilon_{\mu\nu\alpha\beta}x^\alpha A_c^\beta(0)}{(x^2-\mathrm{i}\varepsilon)^2}$$
$$+ \frac{c_{abc}}{2\pi^2}\frac{x^\rho S_{\mu\rho\nu\sigma}J_c^\sigma(0)}{(x^2-\mathrm{i}\varepsilon)^2} + \cdots$$
$$T J_\mu^a(x) A_\nu^b(0) \underset{x_\mu \to 0}{\sim} \frac{d_{abc}}{2\pi^2}\frac{\varepsilon_{\mu\nu\alpha\beta}x^\alpha J_c^\beta(0)}{(x^2-\mathrm{i}\varepsilon)^2} + \frac{c_{abc}}{2\pi^2}\frac{x^\rho S_{\mu\rho\nu\sigma}A_c^\sigma(0)}{(x^2-\mathrm{i}\varepsilon)^2} + \cdots$$

where

$$J_\mu^a(x) = :\bar{\Psi}_\mathrm{c}(x)\gamma_\mu \tfrac{1}{2}\lambda_a \Psi_\mathrm{c}(x):$$
$$A_\mu^a(x) = :\bar{\Psi}_\mathrm{c}(x)\gamma_\mu\gamma_5 \tfrac{1}{2}\lambda_a \Psi_\mathrm{c}(x):$$

the repeated index c implies a sum over colours; λ_a are $SU(3)$ flavour matrices

$$\tfrac{1}{4}\lambda_a\lambda_b = \tfrac{1}{2}(d_{abc} + \mathrm{i}c_{abc})\lambda_c$$
$$\gamma_\mu\gamma_\rho\gamma_\nu = (S_{\mu\nu\rho\sigma} + \mathrm{i}\varepsilon_{\mu\rho\nu\sigma}\gamma_5)\gamma^\sigma$$
$$S_{\mu\rho\nu\sigma} = (g_{\mu\rho}g_{\nu\sigma} + g_{\rho\nu}g_{\mu\sigma} - g_{\mu\nu}g_{\rho\sigma})$$

Also derive the light-cone expansion for commutators of these currents (for example, Ellis (1977)).

7.5 Obtain the expansion

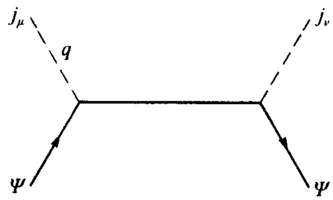
$$\mathrm{i}\int \mathrm{d}^4x \exp(\mathrm{i}qx) T j_\mu(x) j_\nu(0) = (q_\mu q_\nu - q^2 g_{\mu\nu})\left(-\frac{1}{4\pi^2}\ln\frac{-q^2}{\mu^2} + \frac{2m_{\bar{\Psi}}}{q^4}\bar{\Psi}\Psi + \cdots\right)$$

where $j_\mu = \bar{\Psi}\gamma_\mu\Psi$. The first term is given by the diagram

$\langle 0|j_\mu j_\nu|0\rangle =$  $= C_I^{(0)}\langle 0|I|0\rangle$

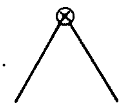
the second by

$$\langle \bar{\Psi} | j_\mu j_\nu | \Psi \rangle =$$



$$= C^{(0)}_{\bar{\Psi}\Psi} \langle \bar{\Psi} | \bar{\Psi}\Psi | \Psi \rangle$$

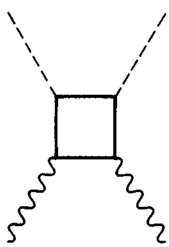
$$= C^{(0)}_{\bar{\Psi}\Psi} \cdot$$



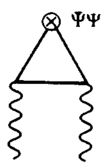
$$= C^{(0)}_{\bar{\Psi}\Psi}$$

Observe that to this order in α taking the appropriate matrix element singles out one particular contribution to the OPE. Extend the calculation to first order in α; in addition to $C^{(1)}_{\mathrm{I}}$ and $C^{(1)}_{\bar{\Psi}\Psi}$ given by the same matrix elements (now taken to the first order in α) calculate also the two-gluon matrix element

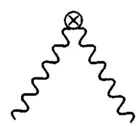
$$\langle g | j_\mu j_\nu | g \rangle = C_{\bar{\Psi}\Psi} \langle g | \bar{\Psi}\Psi | g \rangle + C_G \langle g | G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha | g \rangle$$



$$+ \text{perm} = C^{(0)}_{\Psi\Psi} \cdot$$



$$+ C^{(1)}_G \cdot$$



where $C^{(0)}_{\bar{\Psi}\Psi}$, is given by the zeroth order calculation, and find $C^{(1)}_G$.

8

Quantum chromodynamics

8.1 General introduction

Renormalization and BRS invariance; counterterms

QCD is a theory of interactions of quarks and gluons. Its lagrangian density has been discussed in Sections 1.3 and 3.2. In terms of bare quantities it reads

$$\mathcal{L} = -\tfrac{1}{4} G^{\alpha}_{\mu\nu} G^{\alpha\mu\nu} + \sum_f \bar{\Psi}_f (\mathrm{i}\not{D} - m_f)\Psi_f$$
$$+ \text{gauge-fixing term} + \text{Faddeev–Popov term} \tag{8.1}$$

where the sum is over all quark flavours and the last two terms are given in (3.46) for the class of covariant gauges. The quarks are assumed to transform according to the fundamental representation: each flavour of quark is a triplet of the colour group $SU(3)$. Gauge bosons transform according to the adjoint representation, so that there are eight gluons.

In this section we discuss the theory formulation in covariant gauges. The use of the axial gauge $n \cdot A^{\alpha} = 0$, $n^2 < 0$, or light-like gauge, $n^2 = 0$, where n is in each case some fixed four-vector, is also often very convenient in perturbative QCD calculations. QCD formulation in the axial gauge $n^2 < 0$ can be given in a similarly precise form to that in covariant gauges (for example, Bassetto *et al.* (1985)). In light-like gauges the general proof of renormalizability is still lacking but order-by-order calculations have been consistently performed.

Let us concentrate on the renormalization programme. As in QED, the crucial issue is the gauge invariance or strictly speaking BRS invariance of the theory. It can be proved (for example, Collins (1984)) that the lagrangian is BRS-invariant after renormalization. Let us therefore rewrite lagrangian (8.1) in terms of the renormalized quantities assuming the most general form consistent with BRS invariance.† We get (for one flavour)

$$
\begin{aligned}
\mathcal{L} = &-\tfrac{1}{4}Z_3[\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha - g(Z_{1\mathrm{YM}}/Z_3)c^{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma]^2 \\
&+ Z_2\bar{\Psi}\mathrm{i}\gamma_\mu[\partial^\mu + \mathrm{i}\tilde{g}(Z_1/Z_2)A^{\alpha\mu}T^\alpha]\Psi - mZ_0\bar{\Psi}\Psi \\
&+ \tilde{Z}_2\eta^{*\alpha}[\delta^{\alpha\beta}\partial^2 + \tilde{\tilde{g}}(\tilde{Z}_1/\tilde{Z}_2)c^{\alpha\beta\gamma}A_\mu^\gamma\partial^\mu]\eta^\beta - (1/2a)(\partial_\mu A^{\beta\mu})^2
\end{aligned}
\tag{8.2}
$$

where

$$
gZ_{1\mathrm{YM}}/Z_3 = \tilde{g}Z_1/Z_2 = \tilde{\tilde{g}}\tilde{Z}_1/\tilde{Z}_2 \tag{8.3}
$$

If we use a renormalization scheme such as the minimal subtraction scheme in which the renormalized couplings satisfy

$$
g = \tilde{g} = \tilde{\tilde{g}}
$$

then (8.3) implies the corresponding equality for the ratios of the renormalization constants. As in QED there is no need for a gauge-fixing counterterm.

Relation (8.3) follows from the requirement of the BRS invariance for the lagrangian (8.2). Formally it can be derived from the Slavnov–Taylor identities (Ward identities for a non-abelian gauge theory) which themselves follow from the BRS invariance.

The correspondence between the renormalized quantities and the bare quantities of lagrangian (8.1) is the following

$$
\left.
\begin{array}{lll}
A_{\mathrm{B}\mu}^\alpha = Z_3^{1/2}A_\mu^\alpha & \Psi_\mathrm{B} = Z_2^{1/2}\Psi & a_\mathrm{B} = Z_3 a \\
\eta_\mathrm{B} = \tilde{Z}_2^{1/2}\eta & \eta_\mathrm{B}^* = \tilde{Z}_2^{1/2}\eta^* & m_\mathrm{B} = mZ_0/Z_2
\end{array}
\right\}
\tag{8.4}
$$

$$
g_\mathrm{B} = gZ_{1\mathrm{YM}}/Z_3^{3/2} = \tilde{g}Z_1/Z_2Z_3^{1/2} = \tilde{\tilde{g}}\tilde{Z}_1/\tilde{Z}_2Z_3^{1/2} \tag{8.5}
$$

Lagrangian (8.1) is invariant under the BRS transformation (3.76)

$$
\left.
\begin{aligned}
\delta_{\mathrm{BRS}}\Psi_\mathrm{B} &= -\mathrm{i}g_\mathrm{B}T^\alpha\Theta_\mathrm{B}\eta_\mathrm{B}^\alpha\Psi_\mathrm{B} \equiv \Theta_\mathrm{B}(s\Psi_\mathrm{B}) \\
\delta_{\mathrm{BRS}}\bar{\Psi}_\mathrm{B} &= \mathrm{i}g_\mathrm{B}\bar{\Psi}_\mathrm{B}T^\alpha\Theta_\mathrm{B}\eta_\mathrm{B}^\alpha \equiv \Theta_\mathrm{B}(s\bar{\Psi}_\mathrm{B}) \\
\delta_{\mathrm{BRS}}A_{\mathrm{B}\mu}^\alpha &= \Theta_\mathrm{B}(\partial_\mu\eta_\mathrm{B}^\alpha + g_\mathrm{B}c^{\alpha\beta\gamma}\eta_\mathrm{B}^\beta A_{\mathrm{B}\mu}^\gamma) \equiv \Theta_\mathrm{B}(sA_{\mathrm{B}\mu}^\alpha) \\
\delta_{\mathrm{BRS}}\eta_\mathrm{B}^\alpha &= \tfrac{1}{2}g_\mathrm{B}c^{\alpha\beta\gamma}\eta_\mathrm{B}^\beta\eta_\mathrm{B}^\gamma\Theta_\mathrm{B} \equiv \Theta_\mathrm{B}(s\eta_\mathrm{B}^\alpha) \\
\delta_{\mathrm{BRS}}\eta_\mathrm{B}^{*\alpha} &= -(1/a)\Theta_\mathrm{B}\partial_\mu A_\mathrm{B}^{\mu\alpha} \equiv \Theta_\mathrm{B}(s\eta_\mathrm{B}^{*\alpha})
\end{aligned}
\right\}
\tag{8.6}
$$

† See also the beginning of Chapter 5.

and lagrangian (8.2) under

$$\left.\begin{aligned}
\delta_{\mathrm{BRS}}\Psi &= -\mathrm{i}g\tilde{Z}_1 T^\alpha \Theta \eta^\alpha \Psi \\
\delta_{\mathrm{BRS}}\bar{\Psi} &= \mathrm{i}g\tilde{Z}_1 \bar{\Psi} T^\alpha \Theta \eta^\alpha \\
\delta_{\mathrm{BRS}} A^\alpha_\mu &= \Theta(\partial_\mu \eta^\alpha \tilde{Z}_2 + g\tilde{Z}_1 c^{\alpha\beta\gamma}\eta^\beta A^\gamma_\mu) \\
\delta_{\mathrm{BRS}}\eta^\alpha &= \tfrac{1}{2} g\tilde{Z}_1 c^{\alpha\beta\gamma}\eta^\beta\eta^\gamma\Theta \\
\delta_{\mathrm{BRS}}\eta^{*\alpha} &= -(1/a)\Theta\partial_\mu A^{\mu\alpha}
\end{aligned}\right\} \tag{8.7}$$

Invariance of the lagrangian (8.2) under (8.7) can be checked explicitly but it also follows immediately from the invariance of bare lagrangian (8.1) under (8.6) if relations (8.3), (8.4) and (8.5) are inserted into (8.6) and if Θ is identified as

$$\Theta = (1/Z_3^{1/2}\tilde{Z}_2^{1/2})\Theta_{\mathrm{B}}$$

We note the presence of the divergent renormalization constants $\tilde{Z}_i$ in the BRS transformation (8.7). This is actually what is necessary for finiteness of the composite operators $\delta\Psi$, $\delta\bar{\Psi}$, δA^α_μ and $\delta\eta^\alpha$ (Collins 1984).

Lagrangian (8.2) can be rewritten introducing counterterms (we take $g = \tilde{g} = \tilde{\tilde{g}}$)

$$\begin{aligned}
\mathcal{L} = &-\tfrac{1}{4}G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha + \bar{\Psi}\mathrm{i}\not{D}\Psi - m\bar{\Psi}\Psi - (1/2a)(\partial_\mu A^\mu_\beta)^2 \\
&+ \eta^{*\alpha}(\delta^{\alpha\beta}\partial^2 + gc^{\alpha\beta\gamma}A^\mu_\gamma\partial_\mu)\eta^\beta \\
&+ (Z_2 - 1)\bar{\Psi}\mathrm{i}\not{\partial}\Psi - (Z_1 - 1)g\bar{\Psi}\gamma^\mu A^\alpha_\mu T^\alpha\Psi - m(Z_0 - 1)\bar{\Psi}\Psi \\
&+ (\tilde{Z}_2 - 1)\eta^{*\alpha}\partial^{\alpha\beta}\partial^2\eta^\beta + (\tilde{Z}_1 - 1)g\eta^{*\alpha}c^{\alpha\beta\gamma}A^\mu_\gamma\partial_\mu\eta^\beta \\
&- \tfrac{1}{4}(Z_3 - 1)(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu)^2 + \tfrac{1}{2}(Z_{1\mathrm{YM}} - 1)g(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu)c^{\alpha\beta\gamma}A^\mu_\beta A^\nu_\gamma \\
&- \tfrac{1}{4}g^2(Z^2_{1\mathrm{YM}}Z_3^{-1} - 1)c^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu c^{\alpha\rho\sigma}A^{\rho\mu}A^{\sigma\nu}
\end{aligned} \tag{8.8}$$

The Feynman rules for the vertices of (8.8) have been derived in Section 3.2. They are given in Appendix B. The complete set which includes the counterterms is given in Appendix C (in the notation of Chapter 12).

Asymptotic freedom of QCD

The RGE, the notion of the effective coupling constant and its role in determining the behaviour of a quantum field theory at different momentum scales have been extensively discussed in Chapter 6. In this context the behaviour of observable cross sections has been discussed in Chapter 7 in the framework of the OPE which provides a systematic way of factorizing effects at different mass scales.

QCD is an asymptotically free theory: the coupling constant decreases as the scale at which it is defined is increased. Thus, the behaviour of the Green's functions when all the momenta are scaled up with a common factor is governed by a theory where $g \to 0$. This can explain the success of the parton model in describing the scaling phenomena in the deep inelastic processes. Only theories with a non-abelian gauge symmetry can be asymptotically free in four space-time dimensions (Coleman & Gross 1973).

The β-function in QCD defined as

$$\beta(\alpha) = \mu(\mathrm{d}/\mathrm{d}\mu)\alpha(\mu^2)|_{\alpha_{\mathrm{B}}\ \text{fixed}}, \quad \alpha = g^2/4\pi$$

is given by expansion (6.45)

$$\beta(\alpha) = \alpha\left[\frac{\alpha}{\pi}b_1 + \left(\frac{\alpha}{\pi}\right)^2 b_2 + \left(\frac{\alpha}{\pi}\right)^3 b_3 + \cdots\right] \tag{6.45}$$

and the coefficients b_i are calculable perturbatively. Using dimensional regularization and a mass-independent renormalization scheme we have (6.33) and (6.35): $\beta(\alpha)$ is totally determined by the residua of the simple UV pole of Z_α (beyond the one-loop order we need to calculate a non-leading UV divergence) where according to (8.5) one can use one of the relations

$$Z_\alpha = Z_{1\mathrm{YM}}^2/Z_3^3 = Z_1^2/Z_2^2 Z_3 = \tilde{Z}_1^2/\tilde{Z}_2^2 Z_3 \tag{8.9}$$

Diagrams contributing to different renormalization constants are determined by the structure of counterterms in lagrangian (8.8). For instance at the one-loop level we have the contributions shown in Fig. 8.1 (sums over flavours and over different permutations are omitted). Tadpole diagrams have been ignored because they do not contribute when dimensional regularization is used. One can use the most convenient way of calculating Z_α.

As compared to QED (Section 5.2) the new features in QCD Feynman diagram computation are the group theory factors present for each diagram. The basic ingredients are $SU(3)$ generators in the fundamental three-dimensional representation conventionally taken as

$$(T^a)_{bc} = \tfrac{1}{2}(\lambda^a)_{bc}$$

where λs are the Gell-Mann matrices, $\mathrm{Tr}[\lambda^a\lambda^b] = 2\delta^{ab}$ and in the regular (adjoint) *eight*-dimensional representation

$$(T^a)_{bc} = -\mathrm{i}c_{abc}$$

where c_{abc}s are the $SU(3)$ structure constants. Both appear in the QCD vertices, and in the diagram calculation one often encounters the following combinations:

$$\sum_{a,c}(T^a_{\mathrm{R}})_{bc}(T^a_{\mathrm{R}})_{cd} = (T^2_{\mathrm{R}})_{bd} = C_{\mathrm{R}}\delta_{bd} \tag{8.10}$$

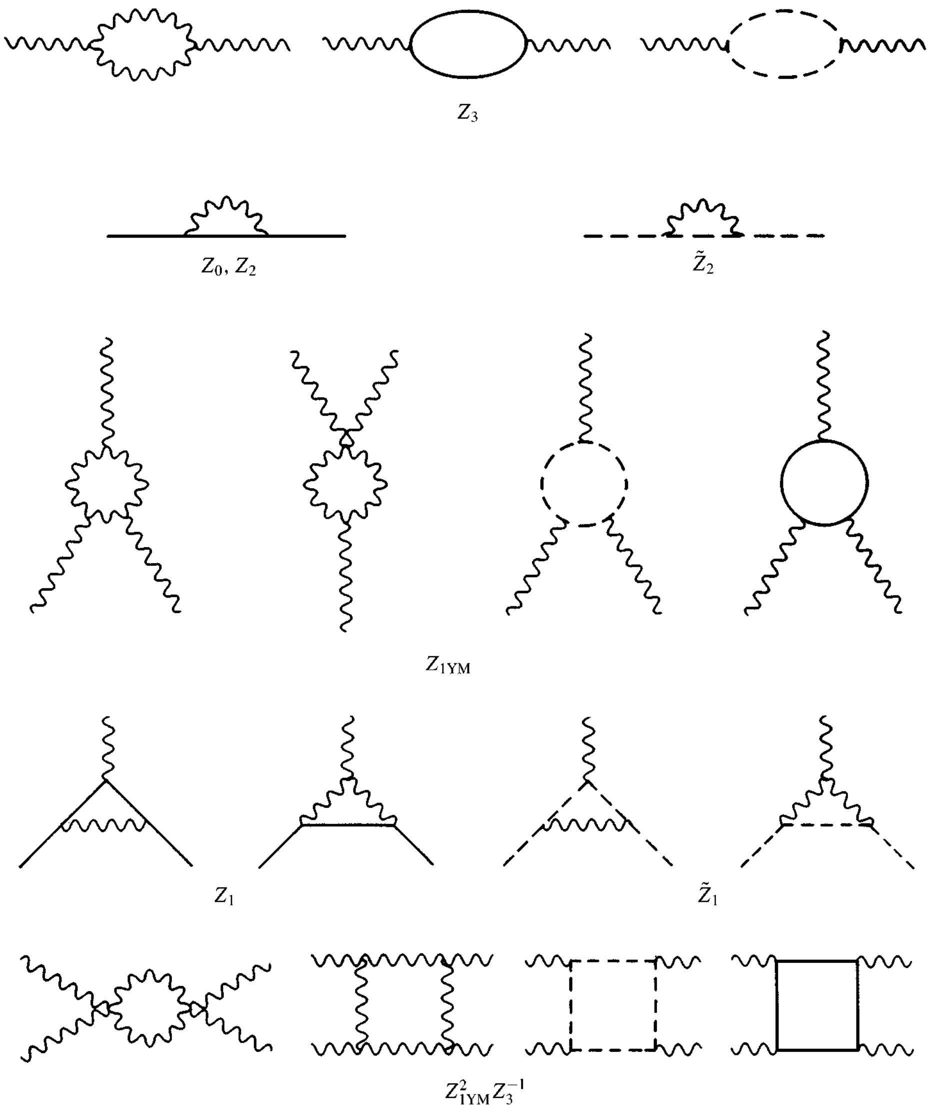


Fig. 8.1.

and

$$\sum_{c,d}(T_R^a)_{dc}(T_R^b)_{cd} = \mathrm{Tr}[T_R^a T_R^b] = T_R \delta^{ab} \tag{8.11}$$

C_R is the eigenvalue of the quadratic Casimir operator in the representation R and for $SU(N)$

$$
\begin{array}{ll}
C_{\mathrm{F}} = (N^2 - 1)/2N & \mathrm{F} \equiv \text{fundamental} \\
C_{\mathrm{A}} = N & \mathrm{A} \equiv \text{adjoint}
\end{array}
$$

The trace T_{R} is

$$
\begin{aligned}
T_{\mathrm{F}} &= \tfrac{1}{2} \\
T_{\mathrm{A}} &= N
\end{aligned}
$$

So far the first three terms in the expansion (6.45) have been calculated and in the minimal subtraction scheme they are (for $SU(N)$ and n flavours)†

$$
\left.
\begin{aligned}
-b_1 &= \tfrac{11}{6} C_{\mathrm{A}} - \tfrac{2}{3} T_{\mathrm{F}} n \\
-b_2 &= \tfrac{17}{12} C_{\mathrm{A}}^2 - \tfrac{1}{2} C_{\mathrm{F}} T_{\mathrm{F}} n - \tfrac{5}{6} C_{\mathrm{A}} T_{\mathrm{F}} n \\
-b_3 &= \tfrac{2857}{1728} C_{\mathrm{A}}^3 - \tfrac{1415}{864} C_{\mathrm{A}}^2 T_{\mathrm{F}} n + \tfrac{79}{432} C_{\mathrm{A}} T_{\mathrm{F}}^2 n^2 - \tfrac{205}{288} C_{\mathrm{A}} C_{\mathrm{F}} T_{\mathrm{F}} n \\
&\quad + \tfrac{44}{288} C_{\mathrm{F}} T_{\mathrm{F}}^2 n^2 + \tfrac{1}{16} C_{\mathrm{F}}^2 T_{\mathrm{F}} n
\end{aligned}
\right\} \tag{8.12}
$$

The QCD effective coupling constant is given by expression (6.51) in the leading logarithm approximation, expression (6.53) in the next-to-leading approximation, and generally by (6.52). It is often convenient to express the result of a QCD calculation in terms of the dimensional parameter Λ instead of the dimensionless α. This has been discussed in Section 7.3.

The Slavnov–Taylor identities

As with the Ward identities in QED, their analogue in a non-abelian gauge theory – the Slavnov–Taylor identities for bare regularized or for renormalized Green's functions – can be derived in several different ways. One can proceed as in Section 5.1: apply gauge transformation to the generating functional for full or connected Green's functions which is written as a functional of the physical field sources only. Then one derives the non-abelian analogue of (5.11) (Abers & Lee 1973, Marciano & Pagels 1978). This form generates all the identities for the connected Green's functions but, because of the presence of ghost fields it is very complicated to obtain the analogous relation for the 1PI generating functional $\Gamma(A, \bar{\Psi}, \Psi)$. A more elegant method is based on the BRS transformation (Lee 1976). In addition to sources $J, \alpha, \bar{\alpha}$ for the fields A, $\bar{\Psi}$ and Ψ, respectively, one introduces sources c^{α} and $c^{*\alpha}$ for the ghost field $\eta^{*\alpha}$ and η^{α} as well as sources K, L, $\bar{L}$ and M for the composite operators sA, $s\bar{\Psi}$, $s\Psi$ and $s\eta$ that appear in the BRS

† The coefficient b_1 was first calculated by Politzer (1973) and Gross & Wilczek (1973), the coefficient b_2 by Caswell (1974) and Jones (1974) and the coefficient b_3 by Tarasov, Vladimirov & Zharkov (1980).

transformation (8.6). This is to obtain identities that are linear in derivatives with respect to the sources. Next one defines the generating functional

$$\begin{aligned} W[J, \alpha, \bar{\alpha}, c, c^*, K, L, \bar{L}, M] \\ = \int \mathcal{D}(A\Psi\bar{\Psi}\eta\eta^*) \exp\Big[\mathrm{i}S_{\mathrm{eff}} + \mathrm{i}\int \mathrm{d}^4x\, (JA + \bar{\alpha}\Psi + \bar{\Psi}\alpha \\ + c^*\eta + \eta^* c + KsA + s\bar{\Psi}L + \bar{L}s\Psi + Ms\eta) \Big] \end{aligned} \tag{8.13}$$

Changing variables in the functional integral according to the BRS transformation and using the invariance

$$\delta S_{\mathrm{eff}}/\delta\Theta = 0$$

as well as the nilpotent relations

$$s^2 A = s^2\eta = s^2\Psi = s^2\bar{\Psi} = 0$$

one obtains the desired general relation generating the Slavnov–Taylor identities. Since the ghost fields are treated on equal footing with the physical fields it is now straightforward to obtain similar relation-generating identities for the 1PI Green's functions. As usual this is achieved by means of the Legendre transform

$$\begin{aligned} \Gamma[A, \Psi, \bar{\Psi}, \eta, \eta^*, K, L, \bar{L}, M] = Z[J, \alpha, \bar{\alpha}, c, c^*, K, L, \bar{L}, M] \\ - \int \mathrm{d}^4x\, (JA + \bar{\alpha}\Psi + \bar{\Psi}\alpha + c^*\eta + \eta^* c) \end{aligned} \tag{8.14}$$

where $\eta^\alpha = \delta Z/\delta c^{*\alpha}$, $c^{*\alpha} = -\delta\Gamma/\delta\eta^\alpha$ etc. One can convince oneself that in the presence of additional sources $K, L, \bar{L}$ and M the functional Γ still generates 1PI Green's functions.

The general relations whose derivation has just been briefly described contain all the information about BRS invariance and are useful in formal manipulations. The simplest way, however, to get specific identities for the Green's functions of interest is based on the observation that since BRS invariance is an exact symmetry of the theory all the Green's functions are BRS invariant (for example, Llewellyn Smith (1980)). For instance, let us consider the regularized Green's function $\langle 0|TA^\alpha_\mu(x)\eta^{*\beta}(y)|0\rangle$. Using (8.6) we have

$$\begin{aligned} \delta_{\mathrm{BRS}}\langle 0|TA^\alpha_\mu(x)\eta^{*\beta}(y)|0\rangle = \langle 0|T(\partial_\mu\eta^\alpha(x) + gc^{\alpha\beta\gamma}\eta^\beta(x)A^\gamma_\mu(x))\eta^{*\beta}(y)|0\rangle \\ - (1/a)\langle 0|TA^\alpha_\mu(x)\partial_\nu A^{\nu\beta}(y)|0\rangle = 0 \end{aligned} \tag{8.15}$$

Differentiating (8.15) with respect to x_μ and using the equation of motion for the Green's function $\langle 0|T\eta^\alpha(x)\eta^{*\beta}(y)|0\rangle$ which can be derived analogously to (2.56a) we conclude that

$$(1/a)\partial^\mu_{(x)}\partial^\nu_{(y)}\langle 0|TA^\alpha_\mu(x)A^\beta_\nu(y)|0\rangle = -\mathrm{i}\delta(x-y)\delta^{\alpha\beta} \tag{8.16}$$

Result (8.16) implies that as in QED (see (5.14)) the longitudinal part of the propagator is unchanged by the higher order corrections.

8.2 The background field method

This method is based on the idea of quantizing field theory in the presence of a background field chosen so that certain symmetries of the generating functionals are restored. Let us consider gauge theories. We start with the gauge-fixed functional integral (the dependence on other fields and on the ghost sources is not indicated explicitly)

$$W[J] = \int \mathcal{D}A'_\mu \exp(\mathrm{i}S_{\mathrm{eff}} + \mathrm{i}J\cdot A'), \qquad J\cdot A' = \int \mathrm{d}^4x\, J^\alpha_\mu A'^\mu_\alpha \tag{8.17}$$

and define, as usual,

$$Z[J] = -\mathrm{i}\ln W[J] \tag{8.18}$$

and the effective action

$$F[\bar{A}] = Z[J] - J\cdot\bar{A}, \qquad \bar{A}^\mu_\alpha = \delta Z/\delta J^\alpha_\mu \tag{8.19}$$

Manifest gauge invariance is lost because of the gauge-fixing procedure. However, we can restore it by introducing a background field with properly chosen transformation properties. Let us consider first the gauge-invariant part of the action and introduce the background field by the replacement

$$A'_\mu = A_\mu + \mathbf{A}_\mu \tag{8.20}$$

where $\mathbf{A}^\alpha_\mu$ is the background field and A^α_μ is the quantum field (we recall that $A_\mu(x) = \mathrm{i}gA^\alpha_\mu(x)T^\alpha$, $\Theta(x) = \mathrm{i}\Theta^\alpha(x)T^\alpha$ and we define $\mathbf{A}_\mu(x) = \mathrm{i}g\mathbf{A}^\alpha_\mu(x)T^\alpha$). Invariance of the action under the transformation (1.113)

$$\delta A'_\mu = D_\mu\Theta(x) = \partial_\mu\Theta(x) + [A'_\mu(x), \Theta(x)] \tag{8.21}$$

implies invariance under two kinds of transformations on A_μ and $\mathbf{A}_\mu$ which give the same $\delta A'_\mu$

quantum

$$\left.\begin{aligned} &\delta\mathbf{A}_\mu = 0 \\ &\delta A_\mu = D_\mu\Theta(x) + [\mathbf{A}_\mu, \Theta(x)], \quad D_\mu = \partial_\mu + [A_\mu, \ldots] \end{aligned}\right\} \tag{8.22}$$

background

$$\left.\begin{aligned} &\delta \mathbf{A}_\mu = \mathbf{D}_\mu \Theta(x), \quad \mathbf{D}_\mu = \partial_\mu + [\mathbf{A}_\mu, \ldots] \\ &\delta A_\mu = [A_\mu(x), \Theta(x)] \end{aligned}\right\} \tag{8.23}$$

We can maintain the manifest gauge invariance of S_{eff} with respect to the *background* transformations if the gauge-fixing function $\tilde{F}^\alpha[A, \mathbf{A}]$ transforms covariantly under these transformations. For instance, choose

$$(\tilde{F}^\alpha(A_\mu, \mathbf{A}_\mu))^2 = -(1/2a)(\mathbf{D}_\mu A^\mu)^2_\alpha = -(1/2a)(\partial^\mu A^\alpha_\mu - g c^{\alpha\beta\gamma} \mathbf{A}^\mu_\beta A^\gamma_\mu)^2 \tag{8.24}$$

The Faddeev–Popov ghost term then takes account of

$$\det M_{\alpha\beta} = \det(\delta \tilde{F}^\alpha / \delta \Theta^\beta)$$

where $\delta \tilde{F}^\alpha$ corresponds to the quantum gauge transformation

$$\delta A^\alpha_\mu = (1/g)\partial_\mu \Theta^\alpha + c^{\alpha\beta\gamma} \Theta^\beta (A^\gamma_\mu + \mathbf{A}^\gamma_\mu)$$

and it also transforms covariantly under the background gauge transformation.

The generating functional

$$\tilde{W}[J, \mathbf{A}_\mu] = \int \mathcal{D} A_\mu \exp\{\mathrm{i} S_{\text{eff}}[A_\mu + \mathbf{A}_\mu] + \mathrm{i} J \cdot A\} \tag{8.25}$$

is manifestly invariant under the background gauge transformations if we complete them with the transformation

$$\delta J^\alpha_\mu = c^{\alpha\beta\gamma} \Theta^\beta J^\gamma_\mu \tag{8.26}$$

The same is true for

$$\tilde{Z}[J, \mathbf{A}_\mu] = -\mathrm{i} \ln \tilde{W}[J, \mathbf{A}_\mu] \tag{8.27}$$

and for

$$\begin{aligned} \tilde{\Gamma}[\tilde{A}_\mu, \mathbf{A}_\mu] &= \tilde{Z}[J, \mathbf{A}_\mu] - J \cdot \tilde{A} \\ \tilde{A}^\alpha_\mu &= \delta \tilde{Z}[J, \mathbf{A}_\mu] / \delta J^a_\mu \end{aligned} \tag{8.28}$$

if

$$\delta \tilde{A}^\alpha_\mu = c^{\alpha\beta\gamma} \Theta^\beta \tilde{A}^\gamma_\mu$$

The next step is to find the relationship between the original $\Gamma[\bar{A}]$ and the $\tilde{\Gamma}[\tilde{A}_\mu, \mathbf{A}_\mu]$ in the presence of the background field. Changing the integration variables in (8.25) for $\tilde{W}[J, \mathbf{A}_\mu]$: $A_\mu \to A_\mu + \mathbf{A}_\mu$ one gets

$$\tilde{W}[J, \mathbf{A}_\mu] = W[J] \exp(-\mathrm{i} J \cdot \mathbf{A}_\mu) \tag{8.29}$$

and correspondingly

$$\tilde{Z}[J, \mathbf{A}_\mu] = Z[J] - J \cdot \mathbf{A}_\mu \tag{8.30}$$

Therefore

$$\tilde{\Gamma}[\tilde{A}_\mu, \mathbf{A}_\mu] = Z[J] - J \cdot \mathbf{A}_\mu - J \cdot \tilde{A}_\mu = \Gamma[\mathbf{A}_\mu + \tilde{A}_\mu] \tag{8.31}$$

and we conclude that the effective action $\tilde{\Gamma}[\tilde{A}_\mu, \mathbf{A}_\mu]$ is the usual effective action $\Gamma[\bar{A}_\mu]$ evaluated at $\bar{A}_\mu = \mathbf{A}_\mu + \tilde{A}_\mu$. It should also be remembered that this $\Gamma[\bar{A}_\mu]$ corresponds to the gauge-fixing term $F^\alpha[A'] = \tilde{F}^\alpha[A' - \mathbf{A}, \mathbf{A}]$ which may be an unusual gauge in the standard approach.

In particular it is convenient to calculate $\tilde{\Gamma}[\tilde{A}_\mu, \mathbf{A}_\mu]$ for $\tilde{A}_\mu = 0$, i.e. restricting ourselves to diagrams with no external $\tilde{A}_\mu$ lines. These '$\tilde{A}_\mu$-vacuum' diagrams are the 1PI subset of diagrams obtained from $\tilde{W}[0, \mathbf{A}_\mu]$ given by (8.25), i.e. diagrams with only internal A_μ lines and external $\mathbf{A}_\mu$, ghost and other non-gauge field lines. We get

$$\Gamma[\mathbf{A}_\mu] = \tilde{\Gamma}[0, \mathbf{A}_\mu] \tag{8.32}$$

A very important fact is that $\tilde{\Gamma}[0, \mathbf{A}_\mu]$ is invariant under the transformation $\delta\mathbf{A}_\mu = \mathbf{D}_\mu\Theta(x)$, i.e. it is a gauge-invariant functional of $\mathbf{A}_\mu$. This property is maintained in a perturbative loop-by-loop calculation of $\tilde{\Gamma}[0, \mathbf{A}_\mu]$ (check it). It implies, in particular, that the divergent terms and the counterterms are manifestly invariant with respect to the background gauge transformation. Thus the infinities appearing in $\tilde{\Gamma}[0, \mathbf{A}_\mu]$ must take the form of a divergent constant times $(\mathbf{G}^\alpha_{\mu\nu})^2$, where

$$\mathbf{G}^\alpha_{\mu\nu} = \partial_\mu \mathbf{A}^\alpha_\nu - \partial_\nu \mathbf{A}^\alpha_\mu - g c^{\alpha\beta\gamma} \mathbf{A}^\beta_\mu \mathbf{A}^\gamma_\nu$$

It is then straightforward to show (Problem 8.3) that the renormalization constants

$$\left.\begin{aligned} (\mathbf{A}_\mu)_\mathrm{B} &= \mathbf{Z}_A^{1/2}\mathbf{A}_\mu \\ g_\mathrm{B} &= Z_g g \end{aligned}\right\} \tag{8.33}$$

satisfy

$$Z_g = \mathbf{Z}_A^{-1/2} \tag{8.34}$$

and, for instance, the calculation of the β-function in the one-loop approximation simplifies to the calculation of the diagrams in Fig. 8.2.

It is recommended that the reader derives the Feynman rules necessary for the calculation of $\tilde{\Gamma}[0, \mathbf{A}_\mu]$ and completes the evaluation of the β-function in the one-loop approximation (Problem 8.3). The background field method is extensively used in supergravity theories (Gates, Grisaru, Roček & Siegel 1983).

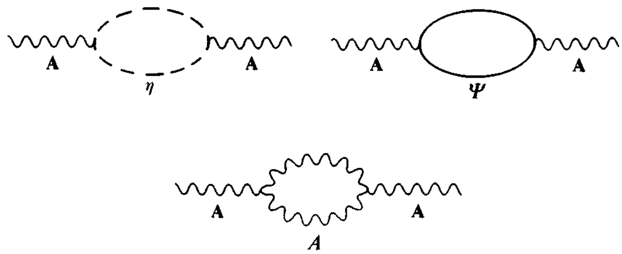


Fig. 8.2.

8.3 The structure of the vacuum in non-abelian gauge theories

Homotopy classes and topological vacua

For the moment we shall consider the pure Yang–Mills gauge theory with $SU(2)$ as the gauge group. As we know, the so-called gauge-fixing procedure used in perturbation theory usually leaves some residual gauge freedom. For instance we can use the gauge $A_0 = 0$ (for the present discussion this is convenient but not necessary; see, for example, Coleman (1979)) which obviously leaves room for further gauge transformations $U(\mathbf{r})$ depending on spatial variables only. Thus the vacuum defined by $G_{\mu\nu} = 0$ restricts the potentials to 'pure gauge' configurations

$$A_i(\mathbf{r}) = -U^{-1}(\mathbf{r})\partial_i U(\mathbf{r}) \tag{8.35}$$

Further discussion relies on an additional restriction that

$$\lim_{r\to\infty} U(\mathbf{r}) = U_\infty \tag{8.36}$$

where U_∞ is a global (position-independent) gauge transformation. Equivalently, we assume that all vector potentials fall faster than $1/r$ at large distances. In pure Yang–Mills theory one can argue (Jackiw 1980), but not prove, that physically admissible gauge transformations satisfy (8.36) which ensures that the non-abelian charge is well defined (see Problem 1.8).

As far as the gauge functions $U(\mathbf{r})$ are concerned the three-dimensional spatial manifold with points at infinity identified is topologically equivalent to S^3 – the surface of a four-dimensional Euclidean sphere labelled by three angles. The matrix functions U provide a mapping of S^3 into the manifold of the $SU(2)$ gauge group. Since any element M in the $SU(2)$ group can be written as

$$M = a + \mathrm{i}\mathbf{b}\cdot\boldsymbol{\sigma}$$

where $\boldsymbol{\sigma}$s are Pauli matrices and where real a and $\mathbf{b}$ satisfy

$$a^2 + \mathbf{b}^2 = 1$$

we conclude that the manifold of the $SU(2)$ group elements is topologically equivalent to S^3. Thus, gauge transformations $U(\mathbf{r})$ provide mappings $S^3 \to S^3$. They can be classified into homotopy classes.

Let X and Y be two topological spaces and f_0, f_1 be two continuous mappings from X into Y. They are said to be homotopic if they are continuously deformable into each other, i.e. if, and only if, there exists a continuous deformation of maps $F(x, t), 0 \leq t \leq 1$, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. The function $F(x, t)$ is called the homotopy. We can divide all mappings of X into Y into homotopic classes: two mappings are in the same class if they are homotopic. A group structure can be defined on the set of homotopy classes.

Let us consider a simple example. Let S^1 be a unit circle parametrized by the angle θ, with θ and $\theta + 2\pi$ identified. We are interested in mappings from S^1 into the manifold of a Lie group G. The group of homotopy classes of mappings from S^1 into the manifold of G is called the first homotopy group of G and is denoted by $\Pi_1(G)$. Take the mappings from S^1 into $G \equiv U(1)$ represented by a set of unimodular complex numbers $z = \exp(\mathrm{i}u)$ which is itself topologically equivalent to S^1. Thus $S^1 \to S^1$ and the continuous functions

$$u(\theta) = \exp[\mathrm{i}(N\theta + a)] \tag{8.37}$$

form a homotopic class for different values of a and a fixed integer N. One can think of $u(\theta)$ as a mapping of a circle into another circle such that N points of the first circle are mapped into one point of the second circle (winding N times around the second circle). Hence the integer N is called the winding number and each homotopy class is characterized by its winding number. For this example

$$\Pi_1(U(1)) = \Pi_1(S^1) = Z$$

where Z denotes an additive group of integers. We observe that in our example the winding number N can be written as

$$N = -\mathrm{i} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \left[\frac{1}{u(\theta)} \frac{\mathrm{d}u}{\mathrm{d}\theta} \right] \tag{8.38}$$

The mappings of any winding number can be obtained by taking powers of

$$u^{(1)}(\theta) = \exp(\mathrm{i}\theta) \tag{8.39}$$

Instead of the unit circle S^1 we can consider the whole real axis $-\infty < x < \infty$ with the points $x = \pm\infty$ identified which is topologically equivalent to S^1 (see Problem 8.1).

This discussion can be generalized by taking $X = S^n$ (the n-dimensional sphere) or the topologically equivalent n-dimensional cube with all its boundaries identified and equivalent to some x_0 of S^n. The classes of mappings $S^n \to S^n$ with one fixed point $f(x_0) = y_0$ form a group called the nth homotopy group of S^m and designated by $\Pi_n(S^m)$. We have

$$\Pi_n(S^n) = Z \tag{8.40}$$

i.e. mappings $S^n \to S^n$ are classified by the number of times one n-sphere covers the other. Ordinary space R^3 with all points at ∞ identified is equivalent to S^3. As we have already mentioned the group $SU(2)$ is also equivalent to S^3. Thus

$$\Pi_3(SU(2)) = \Pi_3(S^3) = Z \tag{8.41}$$

and this together with relations (8.35) and (8.36) implies an infinity of topologically distinct vacua for the $SU(2)$ Yang–Mills theory. The same result holds for an arbitrary simple Lie group (for $SU(N)$ in particular) due to a theorem (Bott 1956) which says that any continuous mapping of S^3 into G can be continuously deformed into a mapping into an $SU(2)$ subgroup of G. Only if the group is $U(1)$, is every mapping of S^3 into $U(1)$ continuously deformable into the constant map $f(x) = y_0$ corresponding to $N = 0$.

It can be shown that analogously to (8.38) and (8.168) in Problem 8.1 the winding number for gauge transformations providing the mapping $S^3 \to G$ is given by

$$N = -(1/24\pi^2) \int \mathrm{d}^3x \, \mathrm{Tr}[\varepsilon_{ijk} A_i A_j A_k] \tag{8.42}$$

where the A_is are given by (8.35). For prototype mapping with $N = 1$, see Problem 8.1.

Physical vacuum

The inequivalence of topologically distinct vacua can be understood as a consequence of Gauss' law

$$D_i^{ab} G_{0i}^b = 0 \tag{8.43}$$

To see this let us first recall the canonical formalism. In the $A_0^a = 0$ gauge the canonical variables are the A_i^as and their conjugate momenta

$$\Pi_i^a = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_a^i)} = G_{0i}^a = \partial_0 A_i^a = -E_i^a \tag{8.44}$$

The hamiltonian

$$H = \int \mathrm{d}^3x \, \mathcal{H}(x), \quad \mathcal{H} = \Pi_i^a \partial_0 A_i^a - \mathcal{L} \tag{8.45}$$

is the energy

$$H = \frac{1}{2}\int \mathrm{d}^3x\,(E_a^2 + B_a^2) = \int \mathrm{d}^3x\,\Theta^{00} \tag{8.46}$$

$$E_i^a = G_{i0}^a = -\Pi_i^a, \quad B_i^a = -\tfrac{1}{2}\varepsilon_{ijk}G_{jk}^a \tag{8.47}$$

When the canonical commutation relations are imposed

$$\left.\begin{aligned}&[A_i^a(\mathbf{x},t), A_j^b(\mathbf{y},t)] = [E_i^a(\mathbf{x},t), E_j^b(\mathbf{y},t)] = 0\\ &[E_i^a(\mathbf{x},t), A_j^b(\mathbf{y},t)] = \mathrm{i}\delta^{ab}\delta_{ij}\delta(\mathbf{x}-\mathbf{y})\end{aligned}\right\} \tag{8.48}$$

the hamiltonian equations give

$$\partial_0 A_i^a = \mathrm{i}[H, A_i^a] = -E_i^a \tag{8.49}$$

$$\partial_0 E_i^a = \mathrm{i}[H, E_i^a] = \varepsilon_{ijk}D_j^{ab}B_k^b \tag{8.50}$$

but Gauss' law (8.43) is not found. We cannot set $D_i^{ab}E_i^b$ to zero as this operator does not commute with the canonical variables. Actually, as seen from (8.53), it generates infinitesimal space-dependent gauge transformations. Thus, although we obtain a consistent quantum theory, it is different from the desired gauge theory. Gauss' law can be incorporated into the theory by demanding that the physical states are only those which are annihilated by $D_i^{ab}G_{0i}^b$

$$D_i^{ab}G_{0i}^b|\Psi\rangle = 0 \tag{8.51}$$

Having (8.51) we can return to our main question and recognize, as the next step, that only gauge transformations which belong to the homotopy class $N = 0$ can be built up by iterating the infinitesimal gauge transformations (see Problem 8.2). The generator of these gauge transformations is (see Problem 1.8)

$$Q_G = \int \mathrm{d}^3x\, G_{0i}^a D_i^{ab}\Theta^b(\mathbf{x}) \tag{8.52}$$

and $\hat{U} = \exp(\mathrm{i}Q_G)$. Integrating by parts and noting that $\Theta^a(\mathbf{x})$ vanishes at $\mathbf{x} \to \infty$ for the $N = 0$ class one gets

$$Q_G = \int \mathrm{d}^3x\,[-D_i^{ab}G_{0i}^a\Theta^b(\mathbf{x})] \tag{8.53}$$

Using (8.51) and (8.53) we conclude that such transformations leave the vacuum state $|N\rangle$ invariant:

$$T^{(0)}|N\rangle = |N\rangle \tag{8.54}$$

where $T^{(0)}$ is the unitary transformation corresponding to $\hat{U} = \exp(\mathrm{i}Q_G)$ in the class $N = 0$. However, for a gauge transformation in the class $N = 1$ we have

$$T^{(1)}|N\rangle = |N+1\rangle \tag{8.55}$$

Since the states $|N\rangle$ are functionals of A_μ, (8.55) follows from the transformation properties of the winding number $N(A)$ given by (8.42)

$$N(U_n^{-1}\mathbf{A}U_n - U_n^{-1}\partial_i U_n) = N(\mathbf{A}) + n \tag{8.56}$$

where U_n is in the class $N = n$ and A_μ is pure gauge: $\mathbf{A} = H^{-1}\partial_\mu H$. Now the true vacuum must be at least phase-invariant under all gauge transformations because they commute with observables. Because T is unitary its eigenvalues are $\exp(-\mathrm{i}\Theta)$, $0 \le \Theta \le 2\pi$, and the physical vacuum is given by

$$|\Theta\rangle = \sum_{N=-\infty}^{\infty} \exp(\mathrm{i}N\Theta)|N\rangle \tag{8.57}$$

so that

$$T^{(M)}|\Theta\rangle = \exp(-\mathrm{i}M\Theta)|\Theta\rangle \tag{8.58}$$

The physical vacuum is characterized by Θ which is a constant of motion and each $|\Theta\rangle$ vacuum is the ground state of an independent sector of the Hilbert space. Indeed, if $\hat{O}$ is any gauge-invariant operator

$$[\hat{O}, \hat{U}_N] = 0$$

then

$$0 = \langle\Theta|[\hat{O}, \hat{U}_N]|\Theta'\rangle = [\exp(-\mathrm{i}N\Theta') - \exp(-\mathrm{i}N\Theta)]\langle\Theta|\hat{O}|\Theta'\rangle$$

and

$$\langle\Theta|\hat{O}|\Theta'\rangle = 0 \quad \text{if} \quad \Theta' \neq \Theta \tag{8.59}$$

We want to stress that we have been working in a fixed gauge $A_0 = 0$. Thus the definitions of the winding number as well as of the topological vacua are gauge-dependent. In our gauge, pure gauge configurations for which **E** and **B** are zero are analogous to infinitely degenerate zero-energy configurations in a quantum mechanical problem with periodic potential. Gauge invariance led us to the notion of the physical $|\Theta\rangle$ vacua as superpositions of the topological vacua (see also Callan, Dashen & Gross (1976)). As in the quantum mechanical problem, the degeneracy between different Θ-vacua is lifted if there exists tunnelling between topological vacua. In pure Yang–Mills theories such tunnelling is due to instantons. In the absence of tunnelling, for example, when there are massless fermions in the theory, the Θ-vacua remain degenerate and the angle Θ has no physical meaning.

This interpretation of the $|\Theta\rangle$-vacuum is certainly gauge-dependent (Bernard & Weinberg 1977). For instance, one could choose a physical gauge where A_μ is uniquely determined in terms of $G_{\mu\nu}$ with no residual gauge freedom at all. The classical vacuum will be unique. However, the physical $|\Theta\rangle$-vacuum is, of course, a gauge-invariant concept. This will become clear in the next subsection where we discuss the functional integral representation for vacuum-to-vacuum transitions.

Θ-vacuum and the functional integral formalism

Let us consider a vacuum-to-vacuum transition given by the functional integral

$$\langle 0|\exp(-\mathrm{i}Ht)|0\rangle \sim \int \mathcal{D}A_\mu \exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{FP}})\right] \tag{8.60}$$

It remains to be understood what one means by 'vacuum' for the transition given by (8.60). Firstly, we must pay some attention to the possible boundary conditions on the fields in the integral (8.60). We are interested in transitions from one classical vacuum to another so we require that the initial $t \to -\infty$ and final $t \to +\infty$ field configurations are of the pure gauge form. Actually, we shall require that gauge potentials tend to a pure gauge at large distances in all four directions

$$A_\mu \underset{|x_\mu|\to\infty}{\longrightarrow} U^{-1}\partial_\mu U \tag{8.61}$$

For the time being we leave the gauge unspecified.

Let us now consider the integral

$$Q = -(1/16\pi^2)\int \mathrm{d}^4x\,\mathrm{Tr}[\tilde{G}^{\mu\nu}G_{\mu\nu}] \tag{8.62}$$

where

$$\tilde{G}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$$

Q is gauge-invariant and is called the topological charge or the Pontryagin index. It is stationary with respect to any local variation δA_μ whether or not the equation of motion is satisfied

$$\begin{aligned}
\delta Q &\sim \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{Tr}[\tilde{G}_{\mu\nu}(D^\mu\delta A^\nu - D^\nu\delta A^\mu)] \\
&= \int \mathrm{d}^4x\,\mathrm{Tr}[\tilde{G}_{\mu\nu}D^\mu\delta A^\nu] \\
&= -\int \mathrm{d}^4x\,\mathrm{Tr}[D_\mu\tilde{G}^{\mu\nu}\delta A_\nu] + \int \mathrm{d}^4x\,\partial^\mu\,\mathrm{Tr}[\tilde{G}_{\mu\nu}\delta A^\nu]
\end{aligned} \tag{8.63}$$

The first term vanishes from the Bianchi identities (Problem 1.7) and no surface terms arise from the second term; Q is a topological invariant. Since (see Problem 1.6)

$$-(1/16\pi^2)\,\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] = \partial_\mu K^\mu \tag{8.64}$$

where

$$K_\mu = -(1/16\pi^2)\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}[G_{\alpha\beta}A_\gamma + \tfrac{2}{3}A_\alpha A_\beta A_\gamma]$$

we can write Q as an integral over the surface S at infinity

$$Q = \int \mathrm{d}S_\mu\, K^\mu \tag{8.65}$$

For potentials A_μ satisfying (8.61) Q may be written in terms of U by inserting the asymptotic form of K_μ in (8.65)

$$Q = -(1/24\pi^2)\int \mathrm{d}S_\mu\,\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}[U^{-1}\partial_\alpha U U^{-1}\partial_\beta U U^{-1}\partial_\gamma U] \tag{8.66}$$

To make contact with the discussion earlier in this section let us take the gauge $A_0 = 0$. Then from (8.42) and 8.66

$$Q = \int \mathrm{d}^3x\, K^0\big|_{-\infty}^{\infty} = N(+\infty) - N(-\infty) \tag{8.67}$$

Using the remaining gauge freedom we can set $N(-\infty) = 0$. Therefore in this gauge the topological charge Q can be interpreted as the winding number of the pure gauge configuration to which A_i, tends (we always also assume (8.36) which is necessary for the integral in (8.67) to be finite) and the functional integral (8.60), with the integral restricted to the Q sector defined by (8.62), as the transition amplitude between topological vacua $|N = 0\rangle$ and $|N = Q\rangle$ or more generally between $|N\rangle$ and $|N + Q\rangle$. In the gauge $A_0 = 0$ we are now ready to construct the functional integral corresponding to the $|\Theta\rangle \to |\Theta\rangle$ transition. The result turns out to have the gauge-invariant form for which we are looking. We have

$$\begin{aligned}
\langle\Theta|\exp(-\mathrm{i}Ht)|\Theta'\rangle &= \sum_{m,n}\exp[\mathrm{i}(n-m)\Theta]\exp[\mathrm{i}n(\Theta'-\Theta)]\langle m|\exp(-\mathrm{i}Ht)|n\rangle \\
&= 2\pi\delta(\Theta-\Theta')\sum_Q \exp(-\mathrm{i}Q\Theta)\langle N+Q|\exp(-\mathrm{i}Ht)|N\rangle \\
&\sim \sum_Q \exp(-\mathrm{i}Q\Theta)\int \mathcal{D}A_Q \exp\left[\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{G}}+\mathcal{L}_{\mathrm{FP}})\right] \\
&= \int \mathcal{D}A_\mu \exp\Big(\mathrm{i}\int \mathrm{d}^4x\,\{\mathcal{L}_{\mathrm{YM}}+\mathcal{L}_{\mathrm{G}}+\mathcal{L}_{\mathrm{FP}} \\
&\quad + (1/16\pi^2)\Theta\,\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]\}\Big)
\end{aligned} \tag{8.68}$$

where we have used (8.62). As promised the final result is gauge-independent and it defines the theory having a well-defined vacuum state $|\Theta\rangle$ The $|\Theta\rangle$-vacuum is defined by the vacuum expectation values of operators which in turn are determined by the functional integral with a given value of Θ. The gauge-dependent notion of the topological vacua does not appear in this formulation. However, using the path integral representation (8.68) we can always write the physical vacuum as a superposition

$$|\Theta\rangle = \sum\!\!\!\!\!\!\int \mathrm{d}\nu \exp(\mathrm{i}\nu\Theta)|\nu\rangle$$

where $|\nu\rangle$ is an eigenstate of the operator $\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]$ with the eigenvalue ν. The new term in the lagrangian is a total divergence so it does not influence the classical equations of motion. Also it does not contribute to the energy–momentum tensor. However, it modifies quantum dynamics which depends on the action. Explicit solutions for field configurations giving $Q = n \neq 0$ are known (instantons). It is convenient to work in Euclidean space and to look at Minkowski Green's functions as the analytic continuation of the Euclidean ones. The instantons are classical $E = 0$ trajectories in Euclidean space-time.†

Fermions must be included in a realistic theory. New interesting aspects then appear which cannot be considered in detail before discussion of chiral symmetry and anomalies. Let us summarize briefly only the most important facts (see also Chapter 13). Firstly, take fermions to be massless. Then the Θ-vacua turn out to be degenerate: they are the degenerate vacua of spontaneously broken axial $U_{\mathrm{A}}(1)$ global symmetry of a theory with massless fermions. The spontaneous breakdown of the $U_{\mathrm{A}}(1)$ is the combined effect of the anomaly in the divergence of the axial current and of the existence of field configurations with a non-zero topological charge Q. This mechanism does not imply the existence of the Goldstone boson coupled to the gauge-invariant $U_{\mathrm{A}}(1)$ current and it offers a prototype solution to the so-called $U_{\mathrm{A}}(1)$ problem.

With massless fermions Θ changes with chiral rotations and can be rotated away. This is no longer true when fermions are massive since then chiral symmetry is also broken explicitly. We face a situation where on one hand the field configurations with $Q \neq 0$ are welcome to solve the $U_{\mathrm{A}}(1)$ problem and on the other hand the experimentally allowed value of Θ is surprisingly small as for a strong interaction parameter: from the upper limit on the neutron dipole moment $\Theta < 10^{-19}$. The Θ-term violates P and CP conservation so one could argue that we should set $\Theta = 0$ in the strong interaction theory. However, this does not help since CP-violating weak interactions renormalize Θ to a non-zero value. So far no convincing explanation exists for the smallness of Θ (Peccei & Quinn 1977, Weinberg 1978, Wilczek 1978, Dine, Fischler & Srednicki 1981). Finally in the presence of fermions the spontaneous breakdown of the $U_{\mathrm{A}}(1)$ and of the chiral

† By a gauge transformation which removes A_4 (we are in the gauge $A_0 = 0$, hence in Euclidean space $A_4 = 0$) one can cast the instanton solution into a form which tends to pure gauge configurations satisfying (8.36) and having different winding numbers as $x_4 \to +\infty$ and $x_4 \to -\infty$. Thus tunnelling takes place and the Θ-vacua are not degenerate.

$SU(n) \times SU(n)$ are phenomena which are connected to each other by the chiral Ward identities based on the spontaneous breaking of the $SU(n) \times SU(n)$ and on the anomalous non-conservation of the $U_{\mathrm{A}}(1)$ current.†

The structure of the QCD vacuum briefly discussed in this section is of fundamental physical importance. However, non-perturbative in nature, it is usually neglected in applications of perturbative QCD to short distance phenomena. One then simply sets $\Theta = 0$.

8.4 Perturbative QCD and hard collisions

Historically, the light-cone expansion discussed in Chapter 7 was the original foundation for perturbative QCD applications. However, this OPE combined with the renormalization group techniques are applicable to a limited number of problems and a more general approach based on direct summation of Feynman diagrams has been widely developed. It can be used for any hard scattering process, not necessarily light-cone dominated, in which there is a large momentum scale q^2. If in a given process there are several large momenta $O(q^2)$ then we assume the ratio of these variables to be fixed as $q^2 \to \infty$. This is, for instance, the Bjorken limit in the deep inelastic lepton–hadron scattering discussed in Section 7.5.

Parton picture

There are two basic ingredients in applications of perturbative QCD to hadronic processes. The first one is to assume the parton picture which provides a connection between QCD and hadrons. No such connection has so far been derived from QCD itself. By the parton picture one means, speaking most generally, the assumption that in processes with large momentum transfer q hadrons can be pictured as ensembles of partons (quarks and gluons) and that there is no interference between the long time and distance physics responsible for the binding of quarks and gluons into hadrons and the short distance effects relevant for the large momentum transfer process under consideration and occurring at the parton level. The usual argument to support this conjecture is as follows: view the scattering in a frame in which the hadron (or hadrons) is fast: $P \to \infty$. The time scale of the bound state effects is dilated as P rises. The life-time of a virtual state with quasi-free constituents is $P/(\Delta E)^2$ and it becomes long compared to the collision interaction time which is $O(1/q_0)$ (here ΔE is the scale of binding effects: $\Delta E \sim O(1/r)$ where r is the radius of the hadron).

The parton picture is implicit in the OPE analysis of the deep inelastic lepton–hadron scattering in Section 7.5 and, for example, in the QCD calculation of the

† This point has been particularly emphasized by Crewther (1979).

total cross section for the e^-e^+ annihilation into hadrons which is identified with the total cross section $e^-e^+ \to$ quarks and gluons. In most applications, however, this parton model assumption takes a more concrete form. We assume that the hard scattering cross section can be written as a convolution of soft hadronic wave functions, which are process-independent, with the cross section for the hard subprocess which involves only partons and can be studied perturbatively. To be specific let us consider the deep inelastic electron–proton scattering depicted in Fig. 7.2. We call P and p the proton and parton momenta, respectively, and use the so-called light-cone parametrization of the four-momenta

$$\left.\begin{aligned} &P = (P + m^2/4P, \mathbf{O}, P - m^2/4P) \\ &P = (zP + (p_\perp^2 + p_\perp^2)/4zP, \mathbf{p}_\perp, zP - (p_\perp^2 + p^2)/4zP) \\ &\mathrm{d}^4 p = \tfrac{1}{2}(\mathrm{d}z/|z|)\, \mathrm{d}^2 p_\perp\, \mathrm{d}p^2 \end{aligned}\right\} \tag{8.69}$$

which defines z as $z = (p_0 + p_3)/2P$ and is very convenient for this discussion. We assume that in the Bjorken limit

$$\mathrm{d}\sigma^{\mathrm{had}}(P, q) = \sum_i \int \mathrm{d}z\, \mathrm{d}^2 p_\perp\, \mathrm{d}p^2\, f_i(z, \mathbf{p}_\perp, p^2)\sigma_i^{\mathrm{QCD}}(zP, \mathbf{p}_\perp, p^2, q) \tag{8.70}$$

where σ_i^{QCD} is the cross section for deep inelastic scattering on the parton 'i' as the target and f_i is the four-momentum distribution of partons of type 'i' in the proton, i.e. it is related to the square of the soft wave-function. We expect that both $p_\perp$ and p^2 are determined by soft binding effects. In the limit $P \to \infty$, $p_\perp$ and p^2 finite (infinite momentum frame) we see that $p = zP$ up to corrections $O(p_\perp/zP)$. In this limit we envisage the proton as a collection of massless, parallelly moving partons and $f_i(z)\,\mathrm{d}z$ as the average number of partons of type 'i' having momentum fraction $z, z + \mathrm{d}z$ of the total proton momentum. We shall see shortly that in the infinite momentum frame (8.70) has a very interesting physical interpretation (see (8.129) and the discussion following it).

Factorization theorem

The second central issue for perturbative QCD calculations is the factorization theorem (Ellis *et al.* 1979, Sachrajda 1983*b*). It is a generalization of the factorization encountered in the OPE approach and applies to the $\mathrm{d}\sigma^{\mathrm{QCD}}$ which under the parton model assumption is to be calculated perturbatively as the sum of appropriate quark–gluon Feynman diagrams. The problem is that, as we expect from our QED examples, the perturbative expression for $\mathrm{d}\sigma^{\mathrm{QCD}}$ contains singular contributions coming from certain regions of phase space in which the quark and gluon momenta are small or parallel. A typical example, but not the only one

(Sachrajda 1983*b*), is large logarithms like $\alpha \ln(q^2/p^2)$ coming from collinear parton radiation where q^2 is the large scale of the process and p^2 is the parton mass. The idea of factorization can be illustrated by the identity

$$[1 + \alpha \ln(q^2/p^2) + \cdots] = [1 + \alpha \ln(M^2/p^2) + \cdots][1 + \alpha \ln(q^2/M^2) + \cdots] \tag{8.71}$$

and the theorem which has been proved to all orders in perturbation theory up to power corrections $O(p^2/q^2)$ states that, for example, again for the lepton–parton deep inelastic scattering

$$\mathrm{d}\sigma_i^{\mathrm{QCD}}(p, q) = \int \mathrm{d}y\, F_i(y, p^2/M^2)\, \mathrm{d}\hat{\sigma}_i(yp, q^2/M^2) + O(p^2/M^2) \tag{8.72}$$

where $0 < y < 1$ and the F_i are universal process-independent functions containing all the logarithms singular in the limit $p^2 \to 0$ (mass singularities or more generally all long distance singularities). The cross sections $\mathrm{d}\hat{\sigma}_i$ are finite in the limit $p^2 \to 0$ and M^2 is an arbitrary mass introduced to factorize out the long distance effects of perturbation theory. Using (8.70) and (8.72) the hadronic cross section can be written as a convolution

$$\mathrm{d}\sigma^{\mathrm{had}} = \sum_i \int f_i * F_i\, \mathrm{d}\hat{\sigma}_i = \sum_i \int \tilde{F}_i\, \mathrm{d}\hat{\sigma}_i \tag{8.73}$$

In the following we explicitly get (8.72) and (8.73) in the first order perturbation theory. It is, however, absolutely crucial for any relevance of perturbative QCD to hard hadronic processes that the factorization theorem holds asymptotically to all orders in perturbation theory. Then, although we cannot calculate the cross sections $\mathrm{d}\sigma^{\mathrm{had}}$ or even $\mathrm{d}\sigma^{\mathrm{QCD}}$ from first principles, the theory has a clear predictive power by relating one cross section to another. This may concern the same process at different momentum transfers, for example, scaling violation in the deep inelastic lepton–hadron scattering or two different processes that involve the same distribution functions $\tilde{F}_i$. In this second case it is usually convenient to let M^2 be the scale q^2 itself and simultaneously be the renormalization scale. For instance, one can compute the cross section for lepton pair production in $p\bar{p}$ collisions using distribution functions defined and measured in the inclusive lepto-production.

In the following we present a sample of lowest order QCD calculations and then briefly mention possible strategies for all order extensions. For more details both technical and phenomenological the reader should consult some of the excellent review articles (e.g. Buras 1980, Sachrajda 1983*a*, Reya 1981, Dokshitzer, Dyakonov & Troyan 1980) on this subject.

8.5 Deep inelastic electron–nucleon scattering in first order QCD (Feynman gauge)

Structure functions and Born approximation

We consider this process in the one-photon exchange approximation. The kinematical variables are defined in Fig. 7.2. The unpolarized cross section for the process $eN \to eX$ has the form

$$\begin{aligned} \mathrm{d}\sigma = & \frac{1}{\text{Flux}} \frac{\mathrm{d}^3\mathbf{k}'}{2E_{k'}(2\pi)^3} \sum_N \int \prod_{n=1}^N \frac{\mathrm{d}^3\mathbf{p}_n}{2E_{p_n}(2\pi)^3} \\ & \times (2\pi)^4 \delta^{(4)}\left(k + P - k' - \sum_n p_n\right) \frac{e^4}{q^4} \frac{1}{4} \sum_{\text{pol}} |M_X|^4 \end{aligned} \quad (8.74)$$

where p_n are four-momenta of the final particles and X includes all the necessary spin indices for a given final state (we neglect all particle masses with the exception of the initial nucleon mass). Writing

$$\tfrac{1}{4} \sum_{\text{pol}} |M_X|^4 = L_{\mu\nu} \tilde{W}^{\mu\nu} \quad (8.75)$$

where the tensor

$$L_{\mu\nu} \equiv \tfrac{1}{2} \operatorname{Tr}(\not{k}\gamma_\mu \not{k}'\gamma_\nu) = 2(k_\mu k'_\nu + k_\nu k'_\mu - k \cdot k' g_{\mu\nu}) \quad (8.76)$$

comes from the lepton vertex we can rewrite the cross section as follows ($q \equiv k - k'$):

$$\mathrm{d}\sigma = \frac{e^4}{q^4} \frac{1}{\text{Flux}} \frac{\mathrm{d}^3\mathbf{k}'}{2E_{k'}(2\pi)^3} 4\pi L_{\mu\nu} W^{\mu\nu} \quad (8.77)$$

where we define

$$\begin{aligned} 4\pi W^{\mu\nu} & \equiv \sum_N \int \prod_{n=1}^N \frac{\mathrm{d}^3\mathbf{p}_n}{2E_{p_n}(2\pi)^3} (2\pi)^4 \delta^{(4)}\left(P + q - \sum_n p_n\right) \tilde{W}^{\mu\nu} \\ & = \sum_{\text{pol}} \int \mathrm{d}^4 x \exp(\mathrm{i} q \cdot x) \langle P | J_\mu(x) J_\nu(0) | P \rangle \end{aligned} \quad (8.78)$$

The most general Lorentz and parity-invariant form of the tensor $W_{\mu\nu}$ reads

$$W_{\mu\nu} = -(g_{\mu\nu} - q_\mu q_\nu / q^2) W_1 + [P_\mu - q_\mu(P \cdot q/q^2)][P_\nu - q_\nu(P \cdot q/q^2)] \frac{W_2}{m^2} \quad (8.79)$$

where $W_i = W_i(q^2, \nu)$, i.e. they are Lorentz scalars. The cross section in the laboratory frame then reads (Flux $= 4(P \cdot k) = 4Em$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega\, \mathrm{d}E'} = \frac{mEE'}{\pi} \frac{\mathrm{d}\sigma}{\mathrm{d}|q^2|\, \mathrm{d}\nu} = \frac{\alpha^2}{q^4} \frac{16E'^2}{4m} [\cos^2(\tfrac{1}{2}\theta) W_2 + 2\sin^2(\tfrac{1}{2}\theta) W_1] \quad (8.80)$$

This result follows directly from relations (8.74)–(8.79). From the expression (8.80) we see that the structure functions W_1 and W_2 defined by (8.75) and (8.78) are real dimensionless functions which are measurable experimentally.

We also introduce another often used notation

$$F_1 = W_1, \qquad F_2 = P \cdot q W_2/m^2 \tag{8.81}$$

and express the structure functions F_1 and F_2 in terms of the tensor $W_{\mu\nu}$ as follows:

$$\left.\begin{aligned} g^{\mu\nu} W_{\mu\nu} &= -(n-1)F_1 + \frac{1}{2x} F_2 \\ 2x \frac{P_\mu P_\nu}{P \cdot q} W_{\mu\nu} &= -F_1 + \frac{1}{2x} F_2 \end{aligned}\right\} \tag{8.82}$$

where $x = -q^2/2P \cdot q$ is the Bjorken variable. Relations (8.82) are written for the general case of n space-time dimensions. Calculating the structure functions F_1 and F_2 we also neglect the nucleon mass as we are interested in the kinematical region $|q^2| \sim s \gg m^2$. Structure functions F_i defined by (8.75), (8.79) and (8.81) are free of kinematical singularities when $m \to 0$. We get finally

$$\left.\begin{aligned} F_1 &= \frac{1}{n-2}\left(-g^{\mu\nu} + 2x \frac{P^\mu P^\nu}{P \cdot q}\right) W_{\mu\nu} \\ \frac{1}{x} F_2 &= 2F_1 + 4x \frac{P^\mu P^\nu}{P \cdot q} W_{\mu\nu} \end{aligned}\right\} \tag{8.83}$$

We want to calculate the structure functions F_i in perturbative QCD under the parton model assumption (8.70).

We assume that partons in a nucleon have some momentum distributions $f_i(z)\,\mathrm{d}z$, where $p = zP$. According to (8.70) the lepton–hadron scattering is an incoherent mixture of elementary scatterings

$$\mathrm{d}\sigma^{\mathrm{had}} = \int_0^1 \mathrm{d}z \sum_i f_i(z)\, \mathrm{d}\sigma_i^{\mathrm{QCD}}(p = zP, k, k') \tag{8.84}$$

where $\mathrm{d}\sigma_i^{\mathrm{QCD}}$ is the cross section for the lepton scattering on a parton of type 'i'.

In the actual physical applications of (8.84) we must, of course, remember that hadrons consist of quarks with different flavours and of gluons. Eq. (8.84) is a short-hand notation for:

in the case of the proton as a target:

$$\sum_i f_i(z)\, \mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}(u_{\mathrm{p}} + \bar{u}_{\mathrm{p}}) + \tfrac{1}{9}(d_{\mathrm{p}} + \bar{d}_{\mathrm{p}}) + \tfrac{1}{9}(s_{\mathrm{p}} + \bar{s}_{\mathrm{p}})]\, \mathrm{d}\sigma^q + G_{\mathrm{p}}(z)\, \mathrm{d}\sigma^G \tag{8.85}$$

in the case of the neutron as a target:

$$\sum_i f_i(z)\,\mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}(u_\mathrm{n}+\bar{u}_\mathrm{n}) + \tfrac{1}{9}(d_\mathrm{n}+\bar{d}_\mathrm{n}) + \tfrac{1}{9}(s_\mathrm{n}+\bar{s}_\mathrm{n})]\,\mathrm{d}\sigma^q + G_\mathrm{n}(z)\,\mathrm{d}\sigma^G \tag{8.86}$$

where the numerical coefficients are the squared electric charges of quarks, $q_N = q_N(z)$ denotes the momentum distribution of quark q in hadron N and $G_N(z)$ denotes the gluon distribution in hadron N. Once the squared electric charges of quarks are factorized out the $\mathrm{d}\sigma^q$ is flavour-independent (and averaged over colours of the initial and summed over colours of the final partons). The $\mathrm{d}\sigma^G$ vanishes in the zeroth order in the strong coupling constant α but contributes in higher orders.

There are several obvious constraints on the $q_N(z)$ determined by the quantum numbers of the nucleon. For instance,

$$\int_0^1 [s_\mathrm{p}(z) - \bar{s}_\mathrm{p}(z)]\,\mathrm{d}z = 0 \qquad \text{(strangeness)} \tag{8.87}$$

$$\int_0^1 \{\tfrac{2}{3}[u_\mathrm{p}(z) - \bar{u}_\mathrm{p}(z)] - \tfrac{1}{3}[d_\mathrm{p}(z) - \bar{d}_\mathrm{p}(z)]\}\,\mathrm{d}z = 1 \qquad \text{(charge)} \tag{8.88}$$

etc. Isospin symmetry ($\mathrm{p} \leftrightarrow \mathrm{n}$, $u \leftrightarrow d$) gives

$$\left.\begin{aligned} u_\mathrm{p}(z) &= d_\mathrm{n}(z) \equiv u(z) \\ d_\mathrm{p}(z) &= u_\mathrm{n}(z) \equiv d(z) \\ s_\mathrm{p}(z) &= s_\mathrm{n}(z) \equiv s(z) \\ G_\mathrm{p}(z) &= G_\mathrm{n}(z) \equiv G(z) \end{aligned}\right\} \tag{8.89}$$

and then for protons

$$\sum_i f_i(z)\,\mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}(u+\bar{u}) + \tfrac{1}{9}(d+\bar{d}) + \tfrac{1}{9}(s+\bar{s})]\,\mathrm{d}\sigma^q + G(z)\,\mathrm{d}\sigma^G \tag{8.90}$$

and for neutrons

$$\sum_i f_i(z)\,\mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}(d+\bar{d}) + \tfrac{1}{9}(u+\bar{u}) + \tfrac{1}{9}(s+\bar{s})]\,\mathrm{d}\sigma^q + G(z)\,\mathrm{d}\sigma^G \tag{8.91}$$

It is convenient to decompose the quark distribution functions into valence quark and sea quark distributions

$$q(z) = q_\mathrm{v}(z) + q_\mathrm{s}(z) \tag{8.92}$$

where we assume

$$u_\mathrm{s} = \bar{u}_\mathrm{s} = d_\mathrm{s} = \bar{d}_\mathrm{s} = s_\mathrm{s} = \bar{s}_\mathrm{s} \equiv S(z) \tag{8.93}$$

Then for protons

$$\sum_i f_i(z)\,\mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}u_{\mathrm{v}} + \tfrac{1}{9}d_{\mathrm{v}} + \tfrac{4}{3}S(z)]\,\mathrm{d}\sigma^q + G(z)\,\mathrm{d}\sigma^G \tag{8.94}$$

and for neutrons

$$\sum_i f_i(z)\,\mathrm{d}\sigma_i^{\mathrm{QCD}} = [\tfrac{4}{9}d_{\mathrm{v}} + \tfrac{1}{9}u_{\mathrm{v}} + \tfrac{4}{3}S(z)]\,\mathrm{d}\sigma^q + G(z)\,\mathrm{d}\sigma^G \tag{8.95}$$

One also often talks about flavour non-singlet cross sections sensitive only to some combinations of the valence quark distributions, for example,

$$\mathrm{d}\sigma^{\mathrm{ep}}(x, q^2) - \mathrm{d}\sigma^{\mathrm{en}}(x, q^2) = \int \mathrm{d}z\,\tfrac{1}{3}[u_{\mathrm{v}}(z) - d_{\mathrm{v}}(z)]\,\mathrm{d}\sigma^q(zP, q^2) \tag{8.96}$$

and flavour singlet cross sections sensitive to the distributions $\frac{1}{2}(q_{\mathrm{s}} + \bar{q}_{\mathrm{s}}) = S(z)$ and $G(z)$. In the following we calculate $\mathrm{d}\sigma^q$ in the zeroth and first orders in the strong coupling α and study flavour non-singlet hadron structure functions.

We can define the quark structure functions for the elementary subprocess again by (8.79) and (8.81)

$$\begin{aligned} W^q_{\mu\nu} = &-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_1^q(y, q^2) \\ &+ \frac{1}{p\cdot q}\left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right)F_2^q(y, q^2) \end{aligned} \tag{8.97}$$

where

$$y = -q^2/2p\cdot q = x/z \tag{8.98}$$

is the Bjorken variable defined for electron–quark scattering. Using (8.84), (8.77) and remembering about the normalizing factor $1/zP_0$ in $\mathrm{d}\sigma^q$ we can express the hadron structure functions in terms of $F^q_{1,2}(y, q^2)$ as follows:

$$F^{\mathrm{had}}(x, q^2) = \int_0^1 (\mathrm{d}z/z) \sum_i f_i(z) F^q(x/z, q^2) \tag{8.99}$$

where $F(y, q^2)$ stands for $F_1(y, q^2)$ or $(1/y)F_2(y, q^2)$.

In the Born approximation the quark structure functions are determined by elastic electron–quark scattering. We have

$$4\pi W^q_{\mu\nu} = \int \frac{\mathrm{d}^{n-1}\mathbf{p}_1}{(2\pi)^{n-1}2p_1^0}(2\pi)^n\delta^{(n)}(p + q - p_1)\tfrac{1}{2}\,\mathrm{Tr}[\not{p}\gamma_\mu\not{p}_1\gamma_\nu] \tag{8.100}$$

We can also use relation (8.105) to calculate the cut diagram shown in Fig. 8.3.

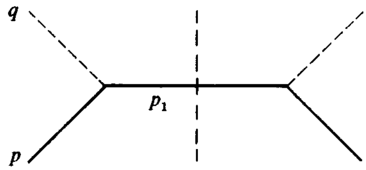


Fig. 8.3.

Then

$$W^q_{\mu\nu} = -\frac{1}{4\pi}\int \frac{\mathrm{d}^n p_1}{(2\pi)^n}(2\pi)^n\delta^{(n)}(p_1 - p - q)(-2\pi\mathrm{i})\delta_+(p_1^2)(-\mathrm{i})^2\tfrac{1}{2}\,\mathrm{Tr}[\gamma_\mu \mathrm{i}\not{p}_1\gamma_\nu\not{p}] \tag{8.101}$$

(we have used the identity $1 = \int[(\mathrm{d}^n p_1)/(2\pi)^n](2\pi)^n\delta^{(n)}(p_1 - p - q)$) which indeed coincides with (8.100). Using (8.83) we easily get the quark structure functions F_i in the Born approximation

$$(1/y)F_2^q = 2F_1^q = 2p\cdot q\delta(2pq + q^2) = \delta(1-y) \tag{8.102}$$

where $y = -q^2/2p\cdot q$. Indeed, we know already that $y = 1$ for elastic scattering. We encounter here the so-called Bjorken scaling law: structure functions, which in general can depend on y and q^2, depend on y only. This will no longer be the case when we include radiative corrections. In terms of the Bjorken variable x defined for the hadronic process we have

$$\frac{1}{x/z}F_2^q\left(\frac{x}{z}\right) = 2F_1^q\left(\frac{x}{z}\right) = \delta\left(1 - \frac{x}{z}\right) \tag{8.103}$$

where $x = -q^2/2P\cdot q$ and $p = zP$. For the hadronic structure functions we get finally

$$(1/x)F_2^{\mathrm{had}}(x, q^2) = 2F_1^{\mathrm{had}}(x, q^2) = \int(\mathrm{d}z/z)\sum_i f_i(z)\delta(1 - x/z) = \sum_i f_i(x) \tag{8.104}$$

The Bjorken variable x in (8.104) is defined, as before, in terms of the total nucleon momentum P, which together with $q = k' - k$, is the kinematic parameter of the reaction. The Bjorken parameter is therefore determined by the specific kinematic configuration in our experiment. We arrive at a very interesting conclusion: it follows from (8.104) that in measuring the hadron structure functions as a function of x we effectively study the momentum distribution of the hadron constituents. In particular we have learned from experiment this way that half of the nucleon momentum is carried by gluons. It is clear from calculation of the elastic cross section (see (8.100)) that our result (8.104) holds up to corrections $O(p_\perp/zP)$,

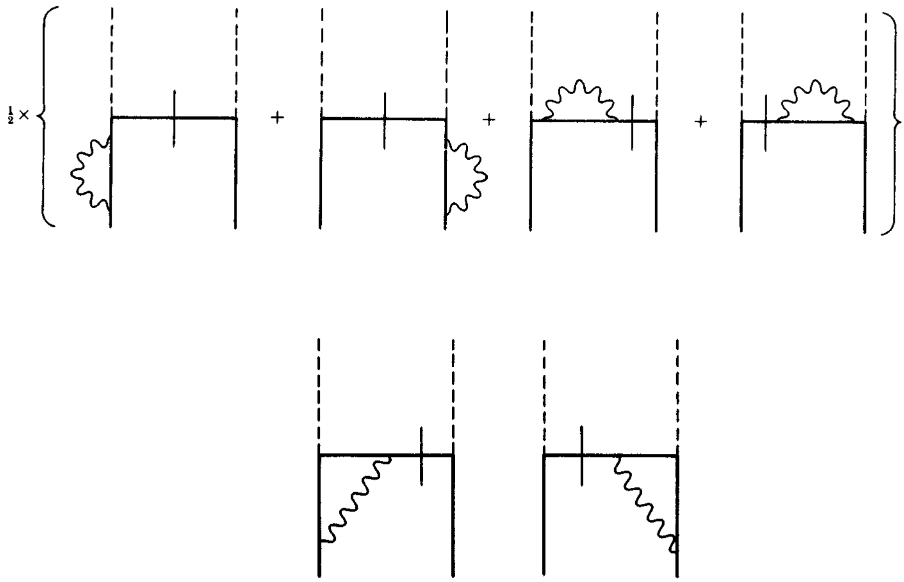


Fig. 8.4.

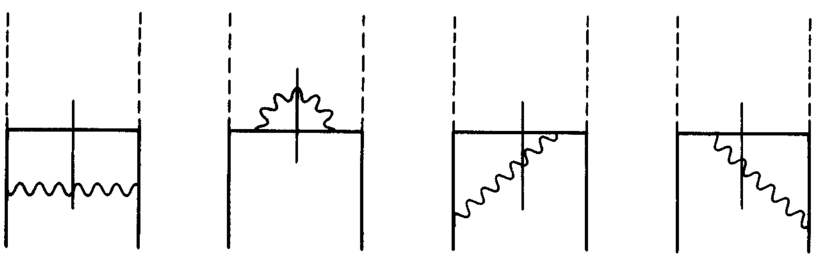


Fig. 8.5.

where $p_\perp$ is the typical transverse momentum of partons in the hadron. So the corrections are frame-dependent and the transparent physical interpretation (8.104) of the hadron structure functions and of the Bjorken variable x (for $x \neq 0$) is valid in a so-called infinite momentum frame in which $P \to \infty$, $p_\perp$ finite.

Deep inelastic quark structure functions in the first order in the strong coupling constant

We now calculate the $\mathrm{d}\sigma^q$ in the first order in α^{strong}. Figs. 8.4 and 8.5 show the diagrams for the virtual and real corrections. We use the notation of cut diagrams,

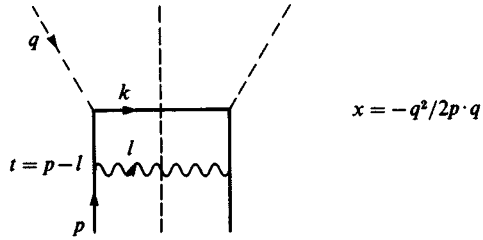
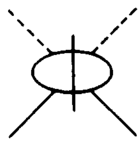


Fig. 8.6.

where

$$W_{\mu\nu} = \frac{1}{4\pi}$$  (8.105)

The calculation is performed in the Feynman gauge. The presence for each diagram of the group theoretical factor $C_{\mathrm{F}} = \frac{3}{4}$ should be kept in mind. It is omitted in most of the following expressions. All the quantities are for lepton–quark scattering but the superscript 'q' will also be omitted.

Here and in the following we shall be mainly concerned with the quantity

$$\sigma = -\frac{1}{1+\frac{1}{2}\varepsilon} g_{\mu\nu} W_{\mu\nu} \quad (n = 4 + \varepsilon) \tag{8.106}$$

The term $p^\mu p^\nu W_{\mu\nu}$ in (8.83) is much simpler to calculate.

The ladder diagram in Fig. 8.6 gives the following contribution to the cross section σ

$$\sigma_{\mathrm{L}} = \frac{1}{1+\frac{1}{2}\varepsilon} \alpha\mu^{-\varepsilon} \int \mathrm{d}\Phi_2 \, \tfrac{1}{2} \, \mathrm{Tr}[\not{p}\gamma_\alpha(\not{p} - \not{l})\gamma_\mu \not{k} \gamma^\mu (\not{p} - \not{l})\gamma^\alpha] \left[\frac{1}{(p-l)^2}\right]^2 \tag{8.107}$$

with

$$\mathrm{d}\Phi_2 = \frac{\mathrm{d}^n l}{(2\pi)^n} 2\pi\delta_+(l^2) \frac{\mathrm{d}^n k}{(2\pi)^n} 2\pi\delta_+(k^2)(2\pi)^n \delta^{(n)}(q + p - l - k)$$

(contrary to the calculation of Section 5.5 we do not assume the gluon to be soft). The trace gives

$$\tfrac{1}{2} \, \mathrm{Tr}[\cdots] = 4(1 + \tfrac{1}{2}\varepsilon)^2 W^2[-(p-l)^2] = 4(1 + \tfrac{1}{2}\varepsilon)^2 \frac{1-x}{x} Q^2[-(p-l)^2] \tag{8.108}$$

where $W = q + p$, $Q^2 = -q^2$ and $x = -q^2/2p \cdot q$ is the Bjorken variable for the lepton–quark scattering. Using expression (5.90) for the two-body phase space in the (p, q) cms we get the following:

$$\sigma_{\mathrm{L}} = \alpha \frac{1}{8\pi} \frac{1}{\Gamma(1+\frac{1}{2}\varepsilon)} (1+\tfrac{1}{2}\varepsilon) 4 \left(\frac{Q^2}{4\pi\mu^2} \frac{1-x}{x} \right)^{\varepsilon/2} \times \int \mathrm{d}y \, [y(1-y)]^{\varepsilon/2} \frac{1-x}{x} Q^2 \left[\frac{-1}{(p-l)^2} \right] \tag{8.109}$$

with $y = \frac{1}{2}(1-z)$ and $z = \cos\theta$, where θ is the polar angle between the three-vectors $\mathbf{l}$ and $\mathbf{p}$. We observe that in the (p, q) cms $2p \cdot l = 2p_0 l_0 (1-z) = p \cdot W(1-z) = (Q^2/x)y$ and the denominator in (8.107) gives a singularity at $y = 0$, i.e. for $\mathbf{l} \parallel \mathbf{p}$ and any value of the momentum $|\mathbf{l}|$ allowed by the conservation law $q+p = k+l$ $(4|\mathbf{l}|^2 = -q^2(1-x)/x$ in cms). This is the so-called mass singularity. We finally get

$$\sigma_{\mathrm{L}} = \left(\frac{\alpha}{2\pi} \right) \left(\frac{Q^2}{4\pi\mu^2} \right)^{\varepsilon/2} \frac{(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\frac{1}{2}\varepsilon)} \frac{\Gamma(\frac{1}{2}\varepsilon)\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)} \left[(1-x) \left(\frac{1-x}{x} \right)^{\varepsilon/2} \right] \tag{8.110}$$

The pole in ε reflects the mass singularity regularized by working in $n = 4 + \varepsilon$ dimensions. The behaviour near $x = 1$ is IR behaviour ($|\mathbf{l}| \to 0$ when $x \to 1$). The cut ladder diagram in the Feynman gauge exhibits no IR singularity.† However, we expect that the full cross section for a single photon production is IR divergent. As we know from Section 5.5 in the properly defined physical cross section the IR divergence is cancelled by virtual corrections to the elastic scattering, i.e. $\int A\delta(1-x)\,\mathrm{d}x + \int \sigma(x, Q^2)\,\mathrm{d}x$ is finite. In order to demonstrate this cancellation explicitly we write the cross section σ in terms of the so-called regularized distribution. For any function $f(x)$ we define the distribution $f_+(x)$ as follows

$$f_+(x) = f(x) - \delta(1-x) \int_0^1 \mathrm{d}y \, f(y) \tag{8.111}$$

i.e. we explicitly single out a possibly singular behaviour of the function $f(x)$ as $x \to 1$. We can then write

$$\sigma_{\mathrm{L}} = \left(\frac{\alpha}{2\pi} \right) \left(\frac{Q^2}{4\pi\mu^2} \right)^{\varepsilon/2} \left\{ \left(\frac{1}{\varepsilon} + 1 + \tfrac{1}{2}\gamma \right) \delta(1-x) + \left(\frac{2}{\varepsilon} + 1 + \gamma \right) \left[(1-x) \left(\frac{1-x}{x} \right)^{\varepsilon/2} \right]_+ \right\} \tag{8.112}$$

The contribution of the longitudinal term $p_\mu p_\nu W^{\mu\nu}$ can also be easily included.

† Note the difference with Section 5.5: now $m = 0$.

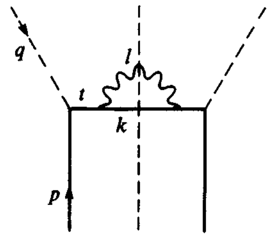


Fig. 8.7.

The ladder diagram gives

$$\left(\frac{1}{x}F_2\right)^{\text{long}} = \frac{\alpha}{2\pi}3x = \frac{\alpha}{2\pi}[\tfrac{3}{2}\delta(1-x) + (3x)_+] \tag{8.113}$$

and the contribution of the rest of the diagrams to $p_\mu p_\nu W^{\mu\nu}$ vanishes.

We proceed to calculate cut self-energy corrections. The contribution of the diagram in Fig. 8.7 reads

$$\sigma_{\Sigma_c} = \frac{1}{1+\frac{1}{2}\varepsilon}\alpha\mu^{-\varepsilon}\int \mathrm{d}\Phi_2\left(\frac{1}{t^2}\right)^2 \tfrac{1}{2}\,\mathrm{Tr}[\not{p}\gamma^\mu \not{t}\gamma^\alpha \not{k}\gamma_\alpha \not{t}\gamma_\mu] \tag{8.114}$$

As before, $\mathrm{d}\Phi^2$ is the two-body (l, k) phase space and we work in $n = 4 + \varepsilon$ dimensions. The trace gives $2(1+\frac{1}{2}\varepsilon)^2 2(2p\cdot l)2k\cdot l = 2(1+\frac{1}{2}\varepsilon)^2 2(t^2)^2 y/(1-x)$, where $y = \frac{1}{2}(1+\cos\theta)$ and θ is the angle between vectors **p** and **k** in the (k, l) cm frame. We see that the propagators are cancelled and we get finally

$$\begin{aligned}\sigma_{\Sigma_c} &= \alpha\frac{1}{8\pi}\left(\frac{Q^2}{4\pi\mu^2}\frac{1-x}{x}\right)^{\varepsilon/2} 4(1+\tfrac{1}{2}\varepsilon)\frac{1}{\Gamma(1+\frac{1}{2}\varepsilon)}\int_0^1 \mathrm{d}y\,[y(1-y)]^{\varepsilon/2}\frac{y}{1-x} \\ &= \frac{\alpha}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}(1+\tfrac{1}{2}\varepsilon)\left[\left(\frac{1-x}{x}\right)^{\varepsilon/2}\frac{1}{1-x}\right]\frac{\Gamma(2+\frac{1}{2}\varepsilon)}{\Gamma(3+\frac{1}{2}\varepsilon)}\end{aligned} \tag{8.115}$$

In the present case the integration over y is finite: there is no mass singularity. The result is, however, divergent for $x \to 1$ (IR divergence). Using the regularized distribution we get:

$$\sigma_{\Sigma_c} = \frac{\alpha}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\left\{\left[\frac{1}{\varepsilon} + \tfrac{1}{2}(\gamma - 1)\right]\delta(1-x) + \frac{1}{2}\left(\frac{1}{1-x}\right)_+\right\} \tag{8.116}$$

Finally we consider the diagrams in Fig. 8.8 and the corresponding cross section

$$\sigma_{\Gamma_c} = 2\alpha\mu^{-\varepsilon}\frac{1}{1+\frac{1}{2}\varepsilon}\int \mathrm{d}\Phi_2\frac{1}{t^2(p-l)^2}\tfrac{1}{2}\,\mathrm{Tr}[\not{p}\gamma^\mu \not{t}\gamma^\alpha \not{k}\gamma_\mu(\not{p} - \not{l})\gamma_\alpha] \tag{8.117}$$

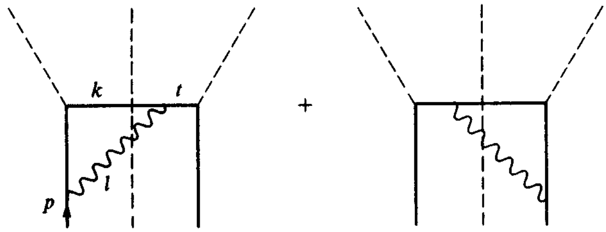


Fig. 8.8.

where $d\Phi_2$ is as before. Using $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} - \varepsilon\not{a}\not{b}\not{c}$ and $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4ab + \varepsilon\not{a}\not{b}$ we evaluate the trace and get

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{Tr}[\cdots] = -\{&4(2p\cdot k)(2p\cdot t - 2t\cdot l) + \tfrac{1}{2}\varepsilon 4(2p\cdot t)(2k\cdot p - 2k\cdot l)\\ &+ \varepsilon\,\mathrm{Tr}[\not{p}\not{k}\not{t}(\not{p}-\not{l})] + \tfrac{1}{2}\varepsilon^2\,\mathrm{Tr}[\not{p}\not{t}\not{k}(\not{p}-\not{l})]\}\end{aligned} \tag{8.118}$$

The last two terms do not contribute (one is exactly zero and the second gives $\varepsilon^2(-8)(p\cdot l)(k\cdot l)$, i.e. both propagators are cancelled and there is no divergence to cancel ε^2). In the (k, l) cm frame we have

$$\begin{gathered}2p\cdot t = Q^2/x, \quad 2p\cdot k = 2p\cdot(t-l) = Q^2/x + (p-l)^2,\\ 2t\cdot l = 2k\cdot l = (Q^2/x)(1-x)\end{gathered}$$

and

$$\sigma_{\Gamma_c} = 2\alpha\frac{1}{1+\frac{1}{2}\varepsilon}\int d\Phi_2\, 4\left[\frac{x}{1-x}\frac{1}{y}(1+\tfrac{1}{2}\varepsilon) - \frac{x}{1-x}\left(1+\frac{\varepsilon}{2x}\right)\right] \tag{8.119}$$

We see that the cut vertex has (in the Feynman gauge) both IR and mass singularity. Performing the integration over $d\Phi_2$ we obtain the following

$$\begin{aligned}\sigma_{\Gamma_c} = 2\left(\frac{\alpha}{2\pi}\right)\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\frac{\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)}&\left\{\frac{1}{1+\varepsilon}\left[\left(\frac{2}{\varepsilon}\right)^2 + \frac{\pi^2}{6} - 1\right]\delta(1-x)\right.\\ &\left.+\left(\frac{2}{\varepsilon}-1\right)\left[\frac{x}{1-x}\left(\frac{1-x}{x}\right)^{\varepsilon/2}\right]_+\right\}\end{aligned} \tag{8.120}$$

Final result for the quark structure functions

Virtual corrections to the considered cross section can be easily calculated using the results of Chapter 5 for the electron self-energy correction $\Sigma(p^2)$ and the vertex correction $\Lambda(q^2)$. For instance, the first two diagrams in Fig. 8.4 give together (in

$n = 4 + \varepsilon$ dimensions)

$$\begin{aligned}\sigma &= -\frac{g_{\mu\nu}W^{\mu\nu}}{1+\frac{1}{2}\varepsilon} = \frac{-1}{1+\frac{1}{2}\varepsilon}\frac{1}{4\pi}\int\frac{\mathrm{d}^n k}{(2\pi)^n}(2\pi)^n\delta^{(n)}(k-p-q)2\pi\delta_+(k^2)(-\mathrm{i})^2\\ &\quad\times\tfrac{1}{2}\,\mathrm{Tr}\left[\not{p}\gamma^\mu\not{k}\gamma_\mu\frac{\mathrm{i}}{\not{p}}[-\mathrm{i}\Sigma(p^2=0)\not{p}]\right]\\ &= \Sigma(p^2=0)\delta(1-x) = -\frac{\alpha}{2\pi}\frac{1}{\varepsilon}\delta(1-x)\end{aligned} \tag{8.121}$$

The same result holds for the next two diagrams of Fig. 8.4. Also, with $\Sigma(p^2)$ changed into $\Lambda(q^2)$, each of the vertex correction diagrams can be calculated in a similar way.

We are now ready to collect all the results ($n = 4 + \varepsilon$ everywhere):

Real

$$\left.\begin{aligned}\sigma_{\mathrm{L}} &= \left(\frac{\alpha}{2\pi}\right)\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\left\{\left(\frac{1}{\varepsilon}+1+\tfrac{1}{2}\gamma\right)\delta(1-x)+\left(\frac{2}{\varepsilon}+1+\gamma\right)\right.\\ &\qquad\left.\times\left[(1-x)\left(\frac{1-x}{x}\right)^{\varepsilon/2}\right]_+\right\}\\ \sigma_{\Sigma_{\mathrm{c}}} &= \left(\frac{\alpha}{2\pi}\right)\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\left[\left(\frac{1}{\varepsilon}+\tfrac{1}{2}(\gamma-1)\right)\delta(1-x)+\frac{1}{2}\left(\frac{1}{1-x}\right)_+\right]\\ \sigma_{\Gamma_{\mathrm{c}}} &= 2\left(\frac{\alpha}{2\pi}\right)\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\frac{\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)}\left\{\left[\left(\frac{2}{\varepsilon}\right)^2+\left(\frac{\pi^2}{6}-1\right)\right]\frac{1}{1+\varepsilon}\delta(1-x)\right.\\ &\qquad\left.+\left(\frac{2}{\varepsilon}-1\right)\left[\frac{x}{1-x}\left(\frac{1-x}{x}\right)^{\varepsilon/2}\right]_+\right\}\\ \left(\frac{1}{x}F^2\right)^{\mathrm{long}} &= \frac{\alpha}{2\pi}[\tfrac{3}{2}\delta(1-x)+(3x)_+]\end{aligned}\right\} \tag{8.122}$$

Virtual

$$\left.\begin{aligned}\sigma_\Sigma &= 2\left(-\frac{\alpha}{2\pi}\right)\frac{1}{\varepsilon}\delta(1-x)\\ \sigma_\Gamma &= 2\frac{\alpha}{2\pi}\left\{\frac{1}{\varepsilon}-\frac{1}{\varepsilon}\frac{\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}+\left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\frac{\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)}\frac{(-1)}{1+\varepsilon}\right.\\ &\qquad\left.\times\left[\left(\frac{2}{\varepsilon}\right)^2+\frac{\pi^2}{6}\right]\right\}\delta(1-x)\end{aligned}\right\} \tag{8.123}$$

The first two terms in σ_Γ correspond to the renormalized Λ^{UV} (Section 5.3) written in $n = 4+\varepsilon$ dimensions. The final result for $(1/x)F_2$ in the lepton–quark scattering reads

$$\frac{1}{x}F_2 = C_{\mathrm{F}}\left(\frac{\alpha}{2\pi}\right)\left[\left(\frac{2}{\varepsilon}+\gamma\right)\left(\frac{1+x^2}{1-x}\right) + \frac{1+x^2}{1-x}\left(\ln\frac{1-x}{x} - 1\right) + 2(1-x) \right.$$
$$\left. + \frac{1}{2}\left(\frac{1}{1-x}\right) + 3x\right]_+ \left(\frac{Q^2}{4\pi\mu^2}\right)^{\varepsilon/2} + \delta(1-x) + O(\alpha^2) \tag{8.124}$$

We now add several important comments on the result obtained to clarify its physical sense. As we already know from Section 5.5, the IR singularities cancel between real and virtual corrections and they are not present in (8.124). However, this is not the case with mass singularities (due to the decay of a massless particle into two parallelly moving massless particles) which explicitly reflect themselves in the final result as poles in ε. We certainly have to cut them off by some physical parameter before inserting result (8.124) into (8.99) to compare our calculation with experiment. The trick is to 'factorize out' the mass singularities by writing the result (8.124) in the limit $\varepsilon \to 0$ as (see (8.71))

$$\frac{1}{x}F_2(x, Q^2) = \int_0^1 \frac{\mathrm{d}z}{z}\Phi\left(\frac{x}{z}, Q^2, Q_0^2\right)\frac{1}{z}F_2(z, Q_0^2) + O(\alpha^2) \tag{8.125}$$

where Q_0^2 is an arbitrary parameter and from (8.124)

$$\Phi(t, Q^2, Q_0^2) = \delta(1-t) + C_{\mathrm{F}}\frac{\alpha}{2\pi}\left(\ln\frac{Q^2}{Q_0^2}\right)\left(\frac{1+t^2}{1-t}\right)_+ + O(\alpha) \tag{8.126}$$

where the $O(\alpha)$ terms are finite when $Q^2 \to \infty$. The meaning of (8.125) is clear: the structure function for any Q^2 is expressed in terms of the structure function for Q_0^2 and of the theoretically calculated function $\Phi(x/z, Q^2, Q_0^2)$ free of mass singularity.

Hadron structure functions; probabilistic interpretation

To calculate hadronic structure functions we use the convolution (8.99) and get

$$\begin{aligned}\frac{1}{x}F_2^{\mathrm{had}}(x, Q^2) &= \int_x^1 \frac{\mathrm{d}z}{z} f(z)\frac{z}{x}F_2\left(\frac{x}{z}, Q^2\right) \\ &= \frac{1}{x}\int_x^1 \mathrm{d}t\, \Phi(t, Q^2, Q_0^2)\int_{x/t}^1 \mathrm{d}z\, f(z)F_2\left(\frac{x}{zt}, Q_0^2\right)\end{aligned} \tag{8.127}$$

where $x = Q^2/2Pq$ is the Bjorken variable for the lepton–hadron scattering. Consequently

$$\frac{1}{x}F_2^{\mathrm{had}}(x, Q^2) = \int_x^1 \frac{\mathrm{d}z}{z}\Phi\left(\frac{x}{z}, Q^2, Q_0^2\right)\frac{1}{z}F_2^{\mathrm{had}}(z, Q_0^2) \tag{8.128}$$

and interestingly enough we again recover the result (8.125). The effect of the intrinsic quark distribution in the nucleon is included in $F_2^{\mathrm{had}}(z, Q_0^2)$. We do not have to know it theoretically to compare the theory with experiment as far as Q^2 evolution of the structure functions is concerned.

By introducing an intermediate scale Q_0^2 we escape two problems: the intrinsic parton distribution which originates from non-perturbative effects need not be known theoretically and the perturbative mass singularities are not present in our final formula. They have been absorbed into $F_2^{\mathrm{had}}(z, Q_0^2)$, which we take as the experimentally measured structure function. Our theoretical prediction is the evolution function $\Phi(x/z, Q^2, Q_0^2)$ and we do not worry that $F_2^{\mathrm{had}}(z, Q_0^2)$ is actually singular in our calculation since this is due to our ignorance of how to include correctly the soft binding effects.

It is also worth observing that the leading Q^2 dependence of the result (8.124) for the quark structure functions and therefore also for the hadron structure functions (8.128) can be interpreted in a very transparent probabilistic parton-like way. To this end let us formally rewrite (8.124) as follows

$$\frac{1}{x}F_2(x, Q^2) = \int_0^1 \frac{\mathrm{d}z}{z}\frac{1}{z}F_2(z, Q^2)\delta\left(1 - \frac{x}{z}\right) \tag{8.129}$$

Formally, we obtain an expression for F_2 which is the same as (8.104) for the structure function of a composite object consisting of a set of partons which scatter elastically at large Q^2. We are tempted to interpret the quantity $(1/z)F(z, Q^2)$ as the average number of partons, with momentum in the interval zp, $(z + \mathrm{d}z)p$, seen in the initial quark with momentum p by a probe (electron) in the process of deep inelastic scattering with momentum transfer Q^2. The theory predicts its evolution with Q^2. Let us assume for the moment that our interpretation is sensible. The hadronic structure function $(1/x)F_2^{\mathrm{had}}(x, Q^2)$ is then a convolution of the intrinsic parton momentum distribution in a hadron (Q^2-independent) and the parton momentum distribution in a parton when probed with large Q^2. Technically, the second effect is present in perturbation theory, whereas the first one has its origin in non-perturbative binding effects.

Is our interpretation of (8.129) sensible? It is for the leading Q^2 dependence. This point will become clearer in the next section when we repeat our calculation in a so-called axial gauge. The reason is that interference diagrams with a gluon emitted or absorbed by the final quark contribute to the final result in the present calculation. It turns out, however, that with the proper choice of gauge the leading effect comes from the ladder diagram and so our interpretation is possible. It applies, however, only to the leading Q^2 dependence because to interpret (8.129) in terms of the elastic scattering of a set of partons with momentum distribution $(1/z)F_2(z, Q^2)$ we have to worry, as before, about corrections $O(p_\perp/zp)$. But now the transverse momentum of a parton is limited only by kinematics and can be as large as $O((Q^2)^{1/2})$. For the probabilistic interpretation we have not only to stay in an infinite momentum frame but also in a limited region of phase space ($p_\perp$ small). This we can do by writing $\ln(Q^2/Q_0^2) = \ln(\varepsilon Q^2/Q_0^2) + \ln(Q^2/\varepsilon Q^2)$, where ε is some fixed arbitrarily small number. The first logarithm comes from the region where $p_\perp \lesssim (\varepsilon Q^2)^{1/2}$ is small and at the same time it gives the leading Q^2 dependence correctly (for $Q^2 \to \infty$). The function

$$P(z) = \left(\frac{1+z^2}{1-z}\right)_+ \tag{8.130}$$

is called the Altarelli–Parisi function. One often considers the Q^2 evolution of moments of the structure function, e.g.

$$M_N(Q^2) = \int_0^1 \mathrm{d}x\, x^{N-1} F(x, Q^2) \tag{8.131}$$

where again $F(x) = F_1(x)$ or $F(x) = (1/x)F_2(x)$. From (8.128) and (8.126) one gets

$$M_N = \int_0^1 \mathrm{d}x\, x^{N-1} \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; \Phi(t, Q^2, Q_0^2)\delta(x - zt)(1/z)F_2(z, Q_0^2) \tag{8.132}$$

or in the leading Q^2 dependence approximation

$$M_N(Q^2) = M_N(Q_0^2)\left[C_\mathrm{F}\frac{\alpha}{2\pi}P_N \ln\left(\frac{Q^2}{Q_0^2}\right) + 1\right] \tag{8.133}$$

where P_N is the Nth moment of the Altarelli–Parisi function:

$$P_N = \int_0^1 \mathrm{d}x\, x^{N-1}\left(\frac{1+x^2}{1-x}\right)_+ = \int_0^1 \mathrm{d}x \frac{1+x^2}{1-x}(x^{N-1} - 1) \tag{8.134}$$

8.6 Light-cone variables, light-like gauge

In this section we describe briefly the calculation of $\mathrm{d}\sigma^q$ given by the diagrams in Figs. 8.4 and 8.5 using the light-cone parametrization of the momenta and the light-like gauge. To be specific let us consider first the ladder diagram in Fig. 8.6. The gluon propagator in the $n^2 = 0$ gauge is

$$-\mathrm{i}\left(g_{\mu\nu} - \frac{l_\mu n_\nu + l_\nu n_\mu}{l \cdot n}\right)\delta_{\alpha\beta} \tag{8.135}$$

and we parametrize the momenta as follows

$$\left.\begin{aligned}
&p = (P + p^2/4P, \mathbf{O}_\perp, P - p^2/4P) \\
&l = (zP + l_\perp^2/4zP, \mathbf{l}_\perp, zP - l_\perp^2/4zP) \\
&t = (yP + (t^2 + t_\perp^2)/4yP, \mathbf{t}_\perp, yP - (t^2 + t_\perp^2)/4yP) \\
&q = (\eta P + q^2/4\eta P, \mathbf{O}_\perp, \eta P - q^2/4\eta P) \\
&y = 1 - z, \quad t^2 z = -l_\perp^2
\end{aligned}\right\} \tag{8.136}$$

For the gauge four-vector n we take

$$n = (p \cdot n/2P, \mathbf{O}, -p \cdot n/2P) \tag{8.137}$$

We can interpret (8.136) and (8.137) noticing that P is an arbitrary boost parameter defining the frame (invariant quantities are P-independent) so that we may start, for example, from the p rest frame ($4P^2 = p^2$) and make an infinite boost $P \to \infty$ in the direction opposite to n. We observe that:

(i) $l \cdot n/p \cdot n = z$;
(ii) $\eta = -x$ for $p^2 = 0$ because $2p \cdot q = q^2/\eta$;
(iii) $1 - z = x + O(l_\perp^2(-q^2))$ because $(p - l)^2 = -l_\perp^2/z$ and $(p - l + q)^2 = 0$.

The two-body phase space

$$\mathrm{d}\Phi_2 = \frac{\mathrm{d}^n l}{(2\pi)^n} 2\pi\delta_+(l^2) \frac{\mathrm{d}^n k}{(2\pi)^n} 2\pi\delta_+(k^2)(2\pi)^n \delta^{(n)}(q + p - l - k) \tag{8.138}$$

can be written as

$$\mathrm{d}\Phi_2 = \frac{\mathrm{d}^n t}{(2\pi)^{n-2}} \delta_+((p - t)^2)\delta_+((q + t)^2) \tag{8.139}$$

From

$$\left.\begin{aligned}
&\delta_+((p - t)^2) = \delta(-(1 - y)t^2/y - t_\perp^2/y), && (p - t)_0 > 0 \\
&\delta_+((q + t)^2) = \delta((y - x)(t^2/y - q^2/x) - x t_\perp^2/y), && (q + t)_0 > 0
\end{aligned}\right\} \tag{8.140}$$

one concludes that

$$\begin{aligned}
&x < y < 1 \\
&t^2 = \frac{y - x}{1 - x} \frac{q^2}{x}, \quad \frac{q^2}{x} < t^2 < 0
\end{aligned} \tag{8.141}$$

Furthermore,

$$\frac{\mathrm{d}^n t}{(2\pi)^{n-2}} = \frac{1}{(2\pi)^{n-2}}\,\mathrm{d}t^2\,\frac{\mathrm{d}y}{2|y|}\,\mathrm{d}^{n-2}\mathbf{t}_\perp \tag{8.142}$$

and

$$\int \mathrm{d}^{n-2}\mathbf{t}_\perp = \int t_\perp^{n-3}\,\mathrm{d}t_\perp\,\mathrm{d}\Omega_{n-2} = \frac{\pi^{(n-2)/2}}{\Gamma\big(\frac{1}{2}(n-2)\big)}\int \mathrm{d}t_\perp^2\,(t_\perp^2)^{(n-4)/2} \tag{8.143}$$

So finally, integrating over $\mathrm{d}(t_\perp^2/y)$ by means of the $\delta((p-t)^2)$, we get $(n=4+\varepsilon)$

$$\Phi_2 = \frac{1}{8\pi}\frac{1}{\Gamma(1+\frac{1}{2}\varepsilon)}\int_{q^2/x}^{0}\mathrm{d}t^2\int_x^1\frac{\mathrm{d}y}{y}y\left[\frac{(1-y)|t^2|}{4\pi}\right]^{\varepsilon/2}\delta\left(t^2(1-x)-\frac{(y-x)q^2}{x}\right) \tag{8.144}$$

$$= \frac{1}{8\pi}\frac{1}{\Gamma(1+\frac{1}{2}\varepsilon)}\frac{1}{1-x}\int_x^1\mathrm{d}y\left[\frac{(1-y)(y-x)}{x(1-x)}\frac{-q^2}{4\pi}\right]^{\varepsilon/2} \tag{8.145}$$

For the Feynman part of the ladder diagram, using (8.107) and (8.108) together with (8.144) we recover the previous result (8.110).

The axial part of the ladder diagram

$$\sigma_{\mathrm{L}}^{\mathrm{A}} = C_{\mathrm{F}}\frac{1}{1+\frac{1}{2}\varepsilon}\alpha\mu^{-\varepsilon} \times\int \mathrm{d}\Phi_2\,\tfrac{1}{2}\,\mathrm{Tr}\left[\not p\not l\frac{1}{\not t}\gamma_\mu(\not t+\not q)\gamma^\mu\frac{1}{\not t}\not n+\not p\not n\frac{1}{\not t}\gamma_\mu(\not t+\not q)\gamma^\mu\frac{1}{\not t}\not l\right]\left(-\frac{1}{l\cdot n}\right) \tag{8.146}$$

can be calculated using the expansion

$$\left.\begin{aligned} \not t &= y\not p + \frac{t^2+t_\perp^2}{2y}\frac{\not n}{p\cdot n}+\not t_\perp \\ \not q &= -x\not p + \frac{-q^2}{x}\frac{\not n}{2p\cdot n} \end{aligned}\right\} \tag{8.147}$$

and (8.144). After some effort one gets

$$\sigma_{\mathrm{L}}^{\mathrm{A}} = C_{\mathrm{F}}\left(\frac{\alpha}{2\pi}\right)\left(\frac{-q^2}{4\pi\mu^2}\right)^{\varepsilon/2}\left\{2\frac{\Gamma(1+\frac{1}{2}\varepsilon)}{\Gamma(1+\varepsilon)}\left[\left(\frac{2}{\varepsilon}\right)^2-\frac{2}{\varepsilon}+\frac{\pi^2}{6}\right]\right. \left.\times\delta(1-x)\left(\frac{2}{\varepsilon}+\gamma\right)\left[\frac{2x}{1-x}\left(\frac{1-x}{x}\right)^{\varepsilon/2}\right]_+\right\} \tag{8.148}$$

We see that

$$\sigma_{\mathrm{L}}^{\mathrm{F}}+\sigma_{\mathrm{L}}^{\mathrm{A}}=C_{\mathrm{F}}\left(\frac{\alpha}{2\pi}\right)\left[\left(\frac{2}{\varepsilon}+\gamma\right)\left(\frac{1+x^2}{1-x}\right)_+ + A(\varepsilon)\delta(1-x)+\cdots\right]\left(\frac{-q^2}{4\pi\mu^2}\right)^{\varepsilon/2} \tag{8.149}$$

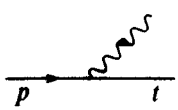
where the neglected terms do not have a pole in ε and are polynomials in x and logarithms in x and $1-x$. Thus in the axial gauge the ladder diagram by itself gives the leading logarithmic contribution (8.126) to the quark structure function. (Remember that we are interested in the finite $x \neq 0$ region so $\ln x$ and $\ln(1-x)$ can be dropped.) We expect therefore that the contribution from the remaining diagrams with real emissions is non-leading. By explicit evaluation of the axial parts of these amplitudes one gets:

$$\sigma_{\Sigma_{\mathrm{c}}}^{\mathrm{A}}=\left(\frac{1}{x}F^2\right)^{\mathrm{long}}=0, \quad \sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{A}}=-\sigma_{\mathrm{L}}^{\mathrm{A}}$$

so that

$$\sigma_{\mathrm{L}}^{\mathrm{F}}+\sigma_{\mathrm{L}}^{\mathrm{A}}+\sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{F}}+\sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{A}}+\sigma_{\Sigma_{\mathrm{c}}}^{\mathrm{F}}+\sigma_{\Sigma_{\mathrm{c}}}^{\mathrm{A}}=\sigma_{\mathrm{L}}^{\mathrm{F}}+\sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{F}}+\sigma_{\Sigma_{\mathrm{c}}}^{\mathrm{F}} \tag{8.150}$$

and $(\sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{F}}+\sigma_{\Gamma_{\mathrm{c}}}^{\mathrm{A}})$ does not contribute to the leading logarithm, i.e. it is free of mass singularities. Actually, one can support this conclusion by the following general argument. The leading logarithm reflecting the presence of mass singularity obviously comes from the integration over t^2 in the region $\varepsilon q^2 < t^2 < 0$, i.e. from the region where t and l are almost collinear with p: $t \cong yp$. Part of the Dirac algebra involves



$$\not{t}\gamma_\mu\not{p} \cong y\not{p}\gamma_\mu\not{p}=y2p_\mu\not{p}=\frac{2y}{1-y}l_\mu\not{p} \tag{8.151}$$

and since in the axial gauge the on-shell gluons have only physical polarization the contraction with the gluon tensor $\mathrm{d}^{\mu\nu}(l)=\sum_{\lambda=1,2}\varepsilon^{*\mu}(\lambda,l)\varepsilon^{\nu}(\lambda,l)$, $l^2=0$ gives

$$l_\mu\,\mathrm{d}^{\mu\nu}(l)=0 \tag{8.152}$$

Actually, one can check that the vertex vanishes linearly in θ, where θ is the gluon emission angle. So the ladder diagram gives

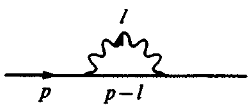
$$\int \mathrm{d}\Phi_2\,(1/t^2)^2\theta^2 \sim \int \mathrm{d}\theta\,\theta\theta^2/(\theta^2)^2 \sim \ln\theta \tag{8.153}$$

whereas the cut-vertex diagram gives

$$\int \mathrm{d}\Phi_2\,(1/t^2)\theta \sim \int \mathrm{d}\theta\,\theta(\theta/\theta^2) \tag{8.154}$$

and hence no logarithm.

The poles in ε in the coefficient of the $\delta(1-x)$ are cancelled by the virtual corrections. Correspondingly the situation with real emissions in the axial gauge, mass singularity is present only in the quark self-energy correction. It is instructive to calculate the virtual corrections in the light-like gauge and using the light-cone parametrization of the momenta. This we leave as an exercise for the reader limiting ourselves to the following two remarks. Take, for instance, the self-energy correction



$= -\mathrm{i}\Sigma$

where Σ can be decomposed as follows

$$\Sigma = A\not p + B\frac{p^2}{2p\cdot n}\not n \tag{8.155}$$

with

$$A = \tfrac{1}{2}\,\mathrm{Tr}\left[\frac{\not n}{2p\cdot n}\Sigma\right] \tag{8.156}$$

$$A + B = \tfrac{1}{2}\,\mathrm{Tr}\left[\frac{\not p\not n\not p}{2p\cdot n}\Sigma\right]\frac{1}{p^2} \tag{8.157}$$

The full propagator reads

$$\begin{aligned} G(p^2) &= \frac{\mathrm{i}}{\not p - \Sigma} = \frac{\mathrm{i}}{1-(A+B)}\left[\frac{\not p}{p^2} - \frac{B}{1-A}\frac{\not n}{2p\cdot n}\right] \\ &= [1 + (A+B) + \cdots]\left(\frac{\mathrm{i}\not p}{p^2} + \cdots\right) \end{aligned} \tag{8.158}$$

Apart from IR and mass singularities Σ is also UV divergent, i.e. contains poles in $\varepsilon_{\mathrm{UV}}$ where $n = 4 - \varepsilon_{\mathrm{UV}}$. The necessary counterterms are of the form

$$a\not p + \not b\not n$$

The term proportional to $\not n$ reflects the pathology of the $n^2 = 0$ gauge because

$$\int \mathrm{d}^4 l \frac{1}{(n\cdot l)l^2(p-l)^2}$$

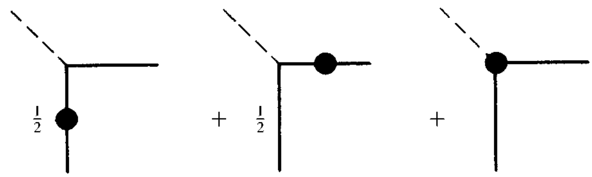


Fig. 8.9.

should be convergent on dimensional grounds,† and it makes the general proof of the renormalizability in this gauge difficult. However, in order-by-order calculation one can check explicitly that such terms cancel out in the full vertex function, see Fig. 8.9.

Our second remark is that an explicit calculation of the virtual corrections using the light-cone parametrization of the momenta is called the Sudakov technique. It consists of noting that the integral

$$\int \frac{\mathrm{d}^n l}{(2\pi)^n} \frac{1}{l^2 + \mathrm{i}\varepsilon} \frac{1}{(p-l)^2 + \mathrm{i}\varepsilon} \sim \int \frac{\mathrm{d}l^2}{l^2} \frac{\mathrm{d}z}{2|z|} \, \mathrm{d}l_\perp^2 (l_\perp^2)^{(n-4)/2} \times \frac{z}{z(1-z)(p^2 - l^2/z) - l_\perp^2 + \mathrm{i}\varepsilon} \tag{8.159}$$

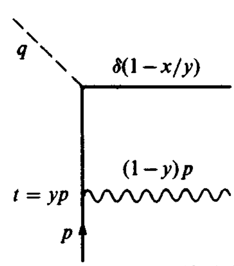
can be evaluated by integrating over $\mathrm{d}l_\perp^2$ first. Going to the complex $l_\perp^2$-plane, since there is a pole at

$$l_\perp^2 = z(1-z)(p^2 - l^2/z) + \mathrm{i}\varepsilon \tag{8.160}$$

and the integral over the semi-circle, which is at infinity, vanishes, we can replace the denominator by $\delta(z(1-z)(p^2 - l^2/z) - l_\perp^2)$.

Since in the leading logarithmic approximation, only the ladder diagram contributes to the final result, its probabilistic interpretation is quite natural. In fact this result has the following structure

$$\sigma_{\mathrm{L}}(p, q^2) = \frac{\alpha}{2\pi} C_{\mathrm{F}} \int_{p^2}^{q^2} \frac{\mathrm{d}t^2}{t^2} \int_x^1 \frac{\mathrm{d}y}{y} \left(\frac{1+y^2}{1-y}\right)_+ \sigma_{\mathrm{L}}(yp, q) \tag{8.161}$$



where

$$\sigma_{\mathrm{L}}(yp, q) = \delta(1 - x/y)$$

† In the Feynman parametrization such integrals are discussed in Appendix B.

and we have chosen to regularize the mass singularity with $p^2 \neq 0$. This coincides with (8.129) and

$$P_{q\to q}(y)\left(\frac{1+y^2}{1-y}\right)_+$$

can be interpreted as the variation of the probability density per unit t^2 of finding a quark in a quark with fraction y of its momentum. Of course, our previous discussion concerning the hadron structure functions, (8.127)–(8.129), is still valid.

8.7 Beyond the one-loop approximation

Since $\alpha_{\text{strong}} \sim O(1)$ the relevance of perturbative QCD to the description of the actual experimental data often relies on our ability to go beyond the one-loop approximation. In addition we must also include in considerations the singlet and the gluon structure and fragmentation functions. This vast area of technical and phenomenological activity goes far beyond the scope of the present book and can by itself be a subject of an extended monograph. Let us only briefly indicate the techniques that have been developed to include higher order effects in α_{strong}. One approach is based on the OPE and on the RGE for the Wilson coefficient functions (Chapter 7). Another one, applicable to a wider class of problems, consists of summing infinite sets of Feynman diagrams by cleverly picking out the leading contribution, the next-to-leading one and so on (Gribov & Lipatov 1972, Llewellyn Smith 1978, Dokshitzer, Dyakonov & Troyan 1980). For the regions $x \neq 0$ and $x \neq 1$, i.e. neglecting $\ln x$ and $\ln(1-x)$ terms as compared to $\ln(q^2/p^2)$ terms, the summation of the so-called leading logarithms $\alpha^n \ln^n(q^2/p^2)$ and also of the next-to-leading $\alpha^n \ln^{n-1}(q^2/p^2)$ terms has been explicitly performed and compared, whenever possible, with the OPE. It is very interesting that these results can also be given a probabilistic interpretation. In the leading logarithm approximation it is concisely formulated by means of the Altarelli–Parisi equations (Altarelli & Parisi 1977).

Further very interesting and more recent progress consists of extending the QCD perturbative techniques to processes where there are two types of large logarithm (for example, Dokshitzer, Dyakonov & Troyan (1980), Parisi & Petronzio (1979)) and to the so-called doubly-logarithmic region of the structure and fragmentation functions ($x \cong 0$). In the context of QCD pioneered by Furmański, Petronzio & Pokorski (1979) and Bassetto, Ciafaloni & Marchesini (1980) and fully clarified by Bassetto, Ciafaloni, Marchesini & Mueller (1982), this extension is, for instance, important for studying new particle production in hard collisions in very high energy accelerators.

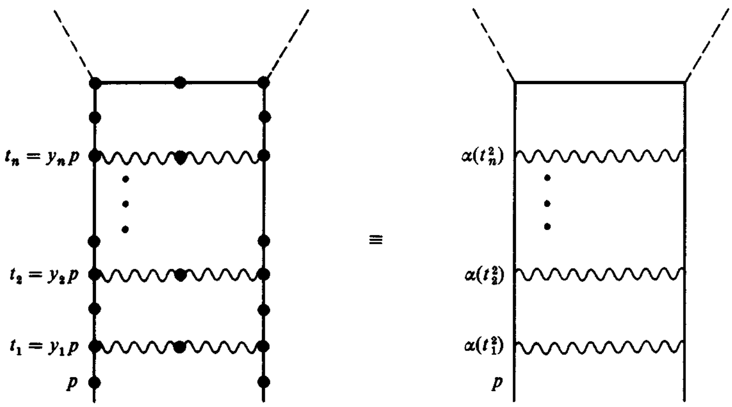


Fig. 8.10.

Let us be a little bit more explicit concerning the summation of the leading logarithm terms $\alpha^n \ln^n(q^2/p^2)$ generated by mass singularities in the quark structure function. It turns out that the dominant diagrams in the axial gauges are the generalized ladder diagrams, Fig. 8.10, i.e. ladder diagrams with vertex and self-energy insertions. The dominant region of integration is where

$$|p^2| \ll |t_1^2| \ll |t_2^2| \ll \cdots \ll |t_n^2| \ll |q^2|$$

Furthermore, it can be shown that the generalized ladder can be replaced by a ladder with the coupling constant $\alpha(t_i^2)$ at each vertex. Thus in the leading logarithm approximation the cross section is a generalization of (8.161) and reads

$$\begin{aligned}
\sigma_{\mathrm{L}}^n = &\int_{p^2}^{|q^2|} \frac{\mathrm{d}t_n^2}{t_n^2} C_{\mathrm{F}} \frac{\alpha(t_n^2)}{2\pi} \int_{p^2}^{|t_n^2|} \frac{\mathrm{d}t_{n-1}^2}{t_{n-1}^2} C_{\mathrm{F}} \frac{\alpha(t_{n-1}^2)}{2\pi} \cdots \\
&\times \int_{p^2}^{|t_2^2|} \frac{\mathrm{d}t_1^2}{t_1^2} C_{\mathrm{F}} \frac{\alpha(t_1^2)}{2\pi} \int_x^1 \mathrm{d}y_1 P_{q\to q}(y_1) \int_x^{y_1} \frac{\mathrm{d}y_2}{y_1} P_{q\to q}\left(\frac{y_2}{y_1}\right) \cdots \\
&\times \int_x^{y_{n-2}} \frac{\mathrm{d}y_{n-1}}{y_{n-2}} P_{q\to q}\left(\frac{y_{n-1}}{y_{n-2}}\right) \int_x^{y_{n-1}} \frac{\mathrm{d}y_n}{y_{n-1}} P_{q\to q}\left(\frac{y_n}{y_{n-1}}\right) \delta(y_n - x) \frac{1}{y_n}
\end{aligned} \tag{8.162}$$

Using (8.162) we can easily calculate the moments of $(1/x)F_2$. First we note that

$$\begin{aligned}
&\left(\frac{C_{\mathrm{F}}}{2\pi}\right)^n \int_{p^2}^{|q^2|} \frac{\mathrm{d}t_n^2}{t_n^2} \alpha(t_n^2) \int_{p^2}^{|t_n^2|} \frac{\mathrm{d}t_{n-1}^2}{t_{n-1}^2} \alpha(t_{n-1}^2) \cdots \to \frac{1}{n!} (C_{\mathrm{F}}/2)^n \\
&\quad \times \int_{p^2}^{|q^2|} \frac{\mathrm{d}t_n^2}{t_n^2} \alpha(t_n^2) \int_{p^2}^{|q^2|} \frac{\mathrm{d}t_{n-1}^2}{t_{n-1}^2} \alpha(t_{n-1}^2) \cdots = \frac{1}{n!} \left[\ln \frac{\ln(-q^2/\Lambda^2)}{\ln(-p^2/\Lambda^2)}\right]^n \left(-\frac{C_{\mathrm{F}}}{b_1}\right)^n
\end{aligned} \tag{8.163}$$

where $\alpha(t^2) = -2\pi/[b_1 \ln(t^2/\Lambda^2)]$, $-b_1 = \frac{11}{2} - \frac{1}{3}n_f$. Changing variables

$$z_1 = y_1, \quad z_2 = y_2/y_1, \quad \ldots, \quad z_n = y_n/y_{n-1}$$
$$z_1 z_2 \ldots z_n = y_n$$

we get

$$\int_0^1 \mathrm{d}x\, x^N \prod_i \int \mathrm{d}z_i\, P_{q\to q}(z_i) \frac{1}{z_1 \ldots z_n} \delta(x - z_1 \ldots z_n) = \prod_i \int_0^1 \mathrm{d}z_i\, z_i^{N-1} P_{q\to q}(z_i) \tag{8.164}$$

Therefore, finally

$$\begin{aligned} M_N &= \int_0^1 \mathrm{d}x\, x^N \frac{1}{x} F_2(x, q^2) \\ &= \sum_n \frac{1}{n!} \left\{ \left[\ln \frac{\ln(-q^2/\Lambda^2)}{\ln(-p^2/\Lambda^2)} \right] \left[-\frac{C_\mathrm{F}}{b_1} \int_0^1 \mathrm{d}z\, z^{N-1} P_{q\to q}(z) \right] \right\}^n \\ &= \left[\frac{\ln(-q^2/\Lambda^2)}{\ln(-p^2/\Lambda^2)} \right]^{-(C_\mathrm{F}/b_1) \int_0^1 \mathrm{d}z\, z^{N-1} P_{q\to q}(z)} \end{aligned} \tag{8.165}$$

and

$$\frac{M_N(Q^2)}{M_N(Q_0^2)} = \left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right]^{\gamma_N^0/b_1}, \quad Q^2 = -q^2 \tag{8.166}$$

where

$$-\gamma_N^0 = C_\mathrm{F} \int_0^1 \mathrm{d}z\, z^{N-1} \left(\frac{1+z^2}{1-z} \right)_+ = -\frac{2}{3} \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^{N-2} \frac{1}{j} \right] \tag{8.167}$$

The same result holds, of course, for the moments of the hadron structure function and it coincides with (7.124) obtained using the OPE. We see that γ_N^0 are the g^2-coefficients in the expansion of the anomalous dimensions of the operators in (7.99) in the coupling constant.

Comments on the IR problem in QCD

The Bloch–Nordsieck theorem in QED states that IR divergences cancel for physical processes, i.e. for processes with an arbitrary number of undetectable soft photons. In QCD it has been demonstrated that IR divergences cancel for processes with none or one coloured particle in the initial state (for a review see, for example, Sachrajda (1983*b*)). However, it has been shown by means of an explicit example that such a cancellation does not occur for processes with two coloured particles in the initial state even if an average over different colour states is taken. For instance, in the process qq $\to$ $\mathrm{l}^+\mathrm{l}^-$X these divergences arise at the two-loop level. Of course, in physical processes coloured partons never appear as free on-shell particles and these divergences are regulated by a mass-scale characteristic of the hadronic wave function which is of the order of the inverse hadronic radius. However, residual large logarithms remain and could, in principle, spoil the factorization property discussed in Section 8.4 which states that all long distance effects can be absorbed into the quark and gluon distribution functions and is crucial for the sensible use of perturbative QCD. Fortunately enough it has been shown that such large IR logarithms are suppressed by at least one power of Q^2 and asymptotically the factorization property is still maintained.

In the context of the IR problem in QCD it is worth recalling the Kinoshita–Lee–Nauenberg (KLN) theorem (Kinoshita 1962, Lee & Nauenberg 1964). The KLN theorem is a mathematical theorem saying that, as a consequence of unitarity, transition probabilities are finite when the sum over all degenerate states (final and initial) is taken. This is true order-by-order in perturbation theory in bare quantities or if the minimal subtraction renormalization is used (to avoid IR or mass singularities in the renormalization constants). The physical meaning of this theorem seems, however, to be unclear. In QED the cancellation of IR singularities occurs separately in the final state and there is no need to invoke the KLN theorem. In non-abelian gauge theories this does not happen but the initial state is determined by the non-perturbative confinement effects and the relevance of the KLN theorem is doubtful.

Problems

8.1 (*a*) Consider mappings of the real axis with the end-points identified into a set of unimodular complex numbers $z = \exp[\mathrm{i}\alpha(x)]$. Check that the winding number can be written as

$$N = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{-\mathrm{i}}{\alpha(x)}\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right] \tag{8.168}$$

Check that $N = 1$ for $\alpha(x) = \exp[\mathrm{i}\pi x/(x^2 + a^2)^{1/2}]$.

(*b*) For $S^3 \to SU(2)$ the transformation

$$A_\mu \to \exp(-\tfrac{1}{2}\mathrm{i}\Theta^a\sigma^a)(A_\mu + \partial_\mu)\exp(\tfrac{1}{2}\mathrm{i}\Theta^a\sigma^a)$$

with $\Theta^a(\mathbf{x}) \equiv 0$ or $\Theta^a(\mathbf{x})$ continuously deformable to $\Theta^a(\mathbf{x}) \equiv 0$, belongs to $N = 0$. Clearly, then $\Theta^a(\infty) = 0$. Check that with

$$\Theta^a(\mathbf{x}) = 2\pi x^a/(\mathbf{x}^2 + a^2)^{1/2}$$

or, with Θ^a continuously deformable to it, $U(\mathbf{x})$ belongs to $N = 1$ class. What is U_∞? Can such a transformation be built up from infinitesimal transformations $\Theta^a(\mathbf{x}) \to \delta\Theta^a(\mathbf{x})$ for all $\mathbf{x}$?

8.2 Show that gauge transformations which belong to the homotopy class $N \neq 0$ cannot be built up by iterating the infinitesimal gauge transformations. For this assume that a 'large' ($N \neq 0$) gauge transformation

$$U = \exp(-\tfrac{1}{2}\mathrm{i}\Theta^a \sigma^a)$$

where $\Theta^a(\mathbf{x})$ approaches a non-zero limit at spatial infinity is given by the operator

$$\exp(\mathrm{i}Q_G)\mathbf{A}\exp(-\mathrm{i}Q_G) = U^{-1}\mathbf{A}U - U^{-1}\boldsymbol{\nabla}U$$

where

$$\exp(\mathrm{i}Q_G) = \exp\left(-\mathrm{i}\int \mathrm{d}^3x\, D^i_{ab}\Theta_b \cdot \Pi^i_a\right)$$

Applying it to

$$W(\mathbf{A}) = \int \mathrm{d}^3x\, K^0$$

where K^0 is given by (8.64), show that

$$\exp(\mathrm{i}Q_G)W(\mathbf{A})\exp(-\mathrm{i}Q_G)W(\mathbf{A})$$

contradicting the correct result

$$\exp(\mathrm{i}Q_G)W(\mathbf{A})\exp(-\mathrm{i}Q_G) = W(U^{-1}\mathbf{A}U - U^{-1}\boldsymbol{\nabla}U) = W(\mathbf{A}) + N$$

(Use $\delta W(\mathbf{A})/\delta A^i_a = (1/8\pi^2)B^a_i$.)

8.3 Derive the Feynman rules for calculating the 1PI Green's functions generated by $\tilde{\Gamma}[0, \mathbf{A}_\mu]$ defined in Section 8.2. Use the shifted action $S[A_\mu + \mathbf{A}_\mu]$, the gauge-fixing term (8.24) and the Faddeev–Popov term obtained from (3.15) and (3.39) with $F^\alpha \to \tilde{F}^\alpha[A_\mu, \mathbf{A}_\mu]$ and δA_μ given by (8.22). Prove (8.34) and complete the calculation of the β-function in Yang–Mills theories in the one-loop approximation (Abbott 1982).

9

Chiral symmetry; spontaneous symmetry breaking

Chiral symmetry plays an important role in the dynamics of fermions coupled to gauge fields. Historically, its importance for particle physics originates in the idea that $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ is an approximate symmetry of the strong interactions, realized as spontaneously broken symmetry with pions being Goldstone bosons in the symmetry limit. Much of the progress of particle physics since 1960 is related to understanding the phenomenological aspects of spontaneous and explicit chiral symmetry breaking. The successes of gauge theories in describing the fundamental interactions further stimulate interest in chiral symmetry and this is for at least two reasons: one may hope firstly to understand the underlying mechanism of spontaneous chiral symmetry breaking in the strong interactions and in gauge theories in general and, secondly, to find the dynamical theory of the fermion mass matrix which might eventually lead to the true theory of fundamental interactions. This chapter serves as an introduction to the subject of chiral symmetry and to techniques used in its exploration.

9.1 Chiral symmetry of the QCD lagrangian

We begin our discussion by considering the QCD lagrangian with two quarks u and d in the theoretical limit in which the quark mass parameters in the lagrangian are put equal to zero. In fact, the masses of the light quarks are small in a sense which will be specified later on and we shall assume in the following that the theory with the quark masses neglected is a correct first approximation.

Consider the general massless Dirac lagrangian of fermion fields Ψ_{ri} coupled to gauge fields A^{α}_{μ}

$$\mathcal{L} = \sum_{r} \sum_{i}^{n(r)} \bar{\Psi}_{ri} \mathrm{i}\gamma^{\mu} D_{\mu} \Psi_{ri}, \qquad \Psi = \Psi(x) \tag{9.1}$$

where

$$D_\mu = \partial_\mu + \mathrm{i}gA^\alpha_\mu t^\alpha_r$$

and the index α runs over the generators of the gauge group, the matrices t^α_r represent these generators in the representation r of the gauge group to which the fermions are assigned and the index i runs over all $n(r)$ flavours belonging to the same representation r. For QCD with two flavours $r \equiv 3$, $i = 1, 2$ and Ψ_i (we now omit the subscript r) denotes a vector in the colour space. In this case lagrangian (9.1) is invariant under the general unitary transformation in the two- dimensional flavour space. Introducing $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ we can write

$$\mathcal{L} = \bar{\Psi} \begin{pmatrix} \mathrm{i}\not{D} & \\ & \mathrm{i}\not{D} \end{pmatrix} \Psi \tag{9.2}$$

which is invariant under $SU(2) \times U(1)$ transformations

$$\left.\begin{aligned} \Psi &\to \exp(-\mathrm{i}\alpha^a \tfrac{1}{2}\sigma^a)\Psi \\ \bar{\Psi} &\to \bar{\Psi} \exp(+\mathrm{i}\alpha^a \tfrac{1}{2}\sigma^a) \end{aligned}\right\} \tag{9.3}$$

where $a = 0, 1, 2, 3$ and σ^a are Pauli matrices for $a = 1, 2, 3$ and $\sigma^0 = 1$. The corresponding conserved currents are

$$\mathcal{V}^a_\mu = \bar{\Psi}\gamma_\mu \tfrac{1}{2}\sigma^a \Psi \tag{9.4}$$

and the $SU(2) \times U(1)$ invariance gives the isospin and the baryon number

$$Q^a = \int \mathrm{d}^3x\, \bar{\Psi}(x)\gamma_0 \tfrac{1}{2}\sigma^a \Psi(x), \quad a = 0, 1, 2, 3 \tag{9.5}$$

conservation.

However, the massless theory (9.2) also has another global symmetry, involving γ_5

$$\left.\begin{aligned} \Psi &\to \exp(-\mathrm{i}\alpha^a \tfrac{1}{2}\sigma^a \gamma_5)\Psi \\ \bar{\Psi} &\to \bar{\Psi} \exp(-\mathrm{i}\alpha^a \tfrac{1}{2}\sigma^a \gamma_5) \end{aligned}\right\} \tag{9.6}$$

where $a = 0, 1, 2, 3$. Indeed, since $\{\gamma_5, \gamma_\mu\} = 0$ the exponentials again cancel when (9.6) is introduced into lagrangian (9.2). The axial currents read

$$\mathcal{A}^a_\mu = \bar{\Psi}\gamma_\mu\gamma_5 \tfrac{1}{2}\sigma^a \Psi \tag{9.7}$$

and the axial charges†

$$Q_5^a = \int d^3x\, \bar{\Psi}(x)\gamma_0\gamma_5 \tfrac{1}{2}\sigma^a \Psi(x) \tag{9.8}$$

together with the charges (9.5) form the chiral $SU_L(2) \times SU_R(2) \times U(1) \times U_A(1)$ algebra (as can be seen using (9.5) and (9.8) and the equal-time canonical commutation relations between the fields)

$$\left.\begin{aligned} &Q_L^a = \tfrac{1}{2}(Q^a - Q_5^a), \quad Q_R^a = \tfrac{1}{2}(Q^a + Q_5^a), \quad a = 0, 1, 2, 3 \\ &[Q_{L,R}^a, Q_{L,R}^b] = i\varepsilon^{abc} Q_{L,R}^c, \quad a, b, c = 1, 2, 3 \\ &[Q_{L,R}^a, Q_R^b] = 0, \quad a, b = 0, 1, 2, 3 \end{aligned}\right\} \tag{9.9}$$

(for completeness: $[Q^a, Q^b] = [Q_5^a, Q_5^b] = i\varepsilon^{abc}Q^c; [Q_5^a, Q^b] = i\varepsilon^{abc}Q_5^c$). An important property of the generators Q_L^a and Q_R^a is that they transform into each other under the parity operation: $PQ_{R,L}^a P^{-1} = Q_{L,R}^a$; hence the name chiral transformations.

Before we proceed to explore the chiral symmetry of the massless QCD lagrangian (9.2) we introduce some useful notation. Any spinor field Ψ can be decomposed into 'chiral' fields‡ as follows

$$\Psi = \Psi_L + \Psi_R = \tfrac{1}{2}(1 - \gamma_5)\Psi + \tfrac{1}{2}(1 + \gamma_5)\Psi = P_L\Psi + P_R\Psi \tag{9.10}$$

where

$$P_L^2 + P_R^2 = 1, \quad P_L P_R = 0$$

and the lagrangian (9.2) can then be rewritten as

$$\mathcal{L} = \bar{\Psi}_L i\not{D}\mathbb{1}\Psi_L + \bar{\Psi}_R i\not{D}\mathbb{1}\Psi_R \tag{9.11}$$

($\mathbb{1}$ is unit matrix in the flavour space).

Chiral symmetry of the lagrangian (9.11) means invariance under the transformations

$$\left.\begin{aligned} \Psi_L^i \to \Psi_L'^i = (U_L)^i_j \Psi_L^j \\ \Psi_R^i \to \Psi_R'^i = (U_R)^i_j \Psi_R^j \end{aligned}\right\} \tag{9.12}$$

where indices i, j are flavour indices and the unitary transformations U_L and U_R read

$$\left.\begin{aligned} U_L = \exp(-i\alpha_L^a T_L^a) \\ U_R = \exp(-i\alpha_R^a T_R^a) \end{aligned}\right\} \tag{9.13}$$

† Charges may be formal in the sense that the integral over all space may not exist. Even so, their commutators exist provided we first commute under the space integrals and then do the integrations. This remark is relevant for the case of spontaneous symmetry breakdown.

‡ Decomposition (9.10) has a clear interpretation in terms of the free particle solutions of the massless Dirac equation. This is discussed in Appendix A.

with $T^a_{\mathrm{L}} = T^a \otimes P_{\mathrm{L}}$ and $T^a_{\mathrm{R}} = T^a \otimes P_{\mathrm{R}}$. The matrices T form a representation of the corresponding generators Q in the space of fermion multiplets Ψ. We see that multiplet $\Psi_{\mathrm{L}}(\Psi_{\mathrm{R}})$ is a singlet with respect to $SU_{\mathrm{R}}(2)(SU_{\mathrm{L}}(2))$ transformations. At this level we should also conclude that because of the $U(1) \times U_{\mathrm{A}}(1)$ invariance the fermion numbers of Ψ_{R} and Ψ_{L} (corresponding to $U(1) \pm U_{\mathrm{A}}(1)$ charges) are separately conserved. The problem of the $U_{\mathrm{A}}(1)$ symmetry will be briefly discussed in Chapter 13. For the time being we will be concerned with the chiral $SU(2) \times SU(2)$ only. Some useful properties of the chiral fields Ψ_{L} and Ψ_{R} are given in Appendix A.

In many problems we work with one type of chiral field Ψ_{R} or Ψ_{L} only. For instance in 'chiral' gauge theories such that Ψ_{R} and Ψ_{L} transform differently under the gauge group (for example, in grand unification schemes) only Ψ_{L} (or only Ψ_{R}) fields can be placed in irreducible multiples of the gauge group. Indeed, $\bar{\Psi}_{\mathrm{R}}\gamma_\mu\Psi_{\mathrm{L}}$ currents vanish and we would not be able to introduce symmetry operations between Ψ_{R} and Ψ_{L}. The description in terms of, for example, the left-handed fields can be obtained if we replace Ψ_{R} by their charge conjugate partners

$$(\Psi_{\mathrm{R}})^{\mathrm{C}} = (\Psi^{\mathrm{C}})_{\mathrm{L}} \underset{\mathrm{df}}{=} \tilde{\Psi}_{\mathrm{L}} \tag{9.14}$$

Then we can formulate our theory in terms of Ψ_{L} and $\tilde{\Psi}_{\mathrm{L}}$. Another notation worth mentioning is the one using the two-component Weyl spinors. We refer the reader to Appendix A for further details on both subjects.

9.2 Hypothesis of spontaneous chiral symmetry breaking in strong interactions

We have assumed that the massless QCD with its chiral $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2)$ global symmetry is a good approximation to the real world. Now let us imagine in addition that the physical vacuum defined by the minimum of the expectation values of the hamiltonian $\langle 0|H|0\rangle = \langle H\rangle_{\mathrm{min}}$ is invariant under the chiral transformations, i.e.

$$Q^a_{\mathrm{R}}|0\rangle = Q^a_{\mathrm{L}}|0\rangle = 0 \tag{9.15}$$

We can invoke Coleman's theorem (see, for example, Itzykson & Zuber (1980), p. 513): 'a symmetry of the vacuum is a symmetry of the world' or

$$Q_i|0\rangle = 0 \quad \Rightarrow \quad [Q_i, H] = 0 \tag{9.16}$$

to conclude that the physical states in the spectrum of H can be classified according to the irreducible representations of the chiral group generated by $Q^a_{\mathrm{R,L}}$ (as can be seen by constructing them from the vacuum by means of the field operators and then using the properties (9.16) and the transformation properties of the field operators under the chiral group). It is easy then to see that all isospin multiplets would have to have at least one degenerate partner of opposite parity. Let the state $|\Psi\rangle$ be an energy and parity eigenstate:

$$H|\Psi\rangle = E|\Psi\rangle \quad \text{and} \quad P|\Psi\rangle = |\Psi\rangle$$

Since Q_L and Q_R commute with the hamiltonian we also have

$$HQ_\mathrm{L}|\Psi\rangle = EQ_\mathrm{L}|\Psi\rangle \quad \text{and} \quad HQ_\mathrm{R}|\Psi\rangle = EQ_\mathrm{R}|\Psi\rangle$$

In addition

$$PQ_\mathrm{R}^\mathrm{L}|\Psi\rangle = PQ_\mathrm{R}^\mathrm{L}P^+P|\Psi\rangle = Q_\mathrm{L}^\mathrm{R}|\Psi\rangle$$

and therefore the state

$$|\Psi'\rangle = \frac{1}{\sqrt{2}}(Q_\mathrm{R} - Q_\mathrm{L})|\Psi\rangle$$

degenerate with $|\Psi\rangle$ is also a parity eigenstate

$$P|\Psi'\rangle = -|\Psi'\rangle$$

The presence of parity degenerate states is not a general feature of the known hadron spectrum. Therefore, if we wish to maintain our assumption about massless QCD being a good approximation to the strong interaction we must give up assumption (9.15). Instead we assume that the physical vacuum is not invariant (but the hamiltonian remains invariant) under the full chiral group and

$$Q_5^a|0\rangle \neq 0, \qquad Q^a|0\rangle = 0 \tag{9.17}$$

One then talks about spontaneous breakdown, or Nambu–Goldstone realization (Nambu 1960, Goldstone 1961), of the chiral symmetry $SU_\mathrm{L}(2) \times SU_\mathrm{R}(2)$ into isospin symmetry $SU(2)$ generated by Q^a. Of course, given a theory like massless QCD the choice between (9.15) and (9.17) is no longer free for us. The theory itself should, in principle, tell us whether the symmetry or part of it is spontaneously broken or not. Unfortunately, the problem of calculating the spontaneous chiral symmetry breaking in QCD has not yet been satisfactorily solved. It is then interesting to begin as we do: assume the spontaneous breakdown (9.17) and explore its consequences in the hope of finding support for the assumption.

The most important consequence of the spontaneous symmetry breakdown is summarized in Goldstone's theorem: if a theory has a global continuous symmetry of the lagrangian which is not a symmetry of the vacuum there must be a massless (Goldstone) boson, scalar or pseudoscalar, corresponding to each generator (with its quantum numbers) which does not leave the vacuum invariant. The proof of the theorem will be given later. A very simple heuristic argument is as follows:

if $Q_5^a|0\rangle \neq 0$ then there is a state $Q_5^a|0\rangle$ degenerate with the vacuum† because $[Q_5^a, H] = 0$. By successive application of Q_5^a one can construct an infinite number of degenerate states. A massless boson with internal quantum numbers, spin and parity of Q_5^a must exist to account for the degeneracy.

Very plausible candidates for Goldstone bosons in the strong interactions with spontaneously broken chiral symmetry are pseudoscalar mesons. Restricting ourselves to the $SU(2) \times SU(2)$ case we see that the triplet of pions has the right quantum numbers for such an interpretation. That they are not exactly massless (but nevertheless much lighter than other hadrons) is attributed to an explicit chiral symmetry breaking by the quark mass terms in the lagrangian.

Another consequence of spontaneous chiral symmetry breaking which agrees with the experimental data is the Goldberger–Treiman relation. This is based on the identification of the axial current of the two-flavour QCD with the axial current of the weak interactions for the first generation of quarks. This identification is explicit in the $SU(2) \times U(1)$ Glashow–Salam–Weinberg theory of weak interactions. The derivation of the Goldberger–Treiman relation in the limit of exact chiral symmetry (with quark masses and pion mass neglected) which is spontaneously broken, then goes as follows: consider the matrix element

$$\langle 0|\mathcal{A}_\mu^a(x)|\pi^b(q)\rangle = \exp(-\mathrm{i}qx)\langle 0|\mathcal{A}_\mu^a(0)|\pi^b(q)\rangle \tag{9.18}$$

where, from Lorentz covariance,

$$\langle 0|\mathcal{A}_\mu^a(0)|\pi^b(q)\rangle = \mathrm{i}f_\pi q_\mu \delta^{ab} \tag{9.19}$$

The constant f_π is called the pion decay constant and $\pi^{(1)} = (1/\sqrt{2})(\pi^+ + \pi^-), \pi^{(2)} = -(\mathrm{i}/\sqrt{2})(\pi^+ - \pi^-), \pi^3 = \pi^0$ (therefore, the $\pi^\pm$ decay constant is $\sqrt{2}f_\pi$). In the Born approximation for the electroweak decays (see Chapter 12) the pion decay, $\pi \to l\nu$, amplitude is proportional to the matrix element (9.18) (we neglect the Cabbibo angle). We assume the axial charge does not annihilate the vacuum, therefore $f_\pi \neq 0$. In our approximation the axial current is conserved: $\partial^\mu \mathcal{A}_\mu^a(x) = 0$ and from (9.18) and (9.19) we get

$$\langle 0|\partial^\mu \mathcal{A}_\mu^a(x)|\pi^b(q)\rangle = \exp(-\mathrm{i}qx) f_\pi m_\pi^2 \delta^{ab} \tag{9.20}$$

This is Goldstone's theorem: $f_\pi \neq 0$ therefore $m_\pi^2 = 0$. Next consider nucleon matrix elements of the axial current (β-decay). Assuming C invariance and isospin

† This interpretation requires some caution (Bernstein 1974) since $Q_5^a|0\rangle$ is not a normalizable state. It means that $U = \exp(-\mathrm{i}\alpha^a Q_5^a)$ is not a unitary operator. Nevertheless, expressions like UAU^+ can be meaningful. If the exponentials are expanded, the resulting commutators may be well defined if we first commute currents and then integrate over space.

invariance one has

$$\langle \mathrm{N}(p')|\mathcal{A}_\mu^-(x)|\mathrm{P}(p)\rangle = \exp(\mathrm{i}qx)\bar{u}(p')\tfrac{1}{2}\sigma^-[\gamma_\mu\gamma_5 g_\mathrm{A}(q^2) + q_\mu\gamma_5 h(q^2)]u(p) \tag{9.21}$$

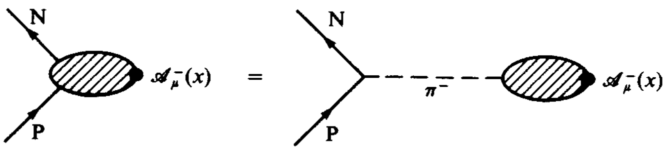
where

$$\sigma^- = (1/\sqrt{2})(\sigma^1 - \mathrm{i}\sigma^2), \quad q = p' - p$$

Current conservation implies

$$2m_\mathrm{N} g_\mathrm{A}(q^2) + q^2 h(q^2) = 0 \tag{9.22}$$

where m_N is the nucleon mass. For $q^2 = 0$ this relation can be satisfied either for $g_\mathrm{A}(0) = 0$ or if $h(q^2)$ has a pole for $q^2 = 0$. If pions are Goldstone bosons they contribute such a pole as can be seen from (9.23)



$$= \sqrt{2}g_{\pi\mathrm{N}}(\mathrm{i}/q^2)\mathrm{i}\sqrt{2}f_\pi q_\pi \bar{u}\gamma_5 u \exp(\mathrm{i}qx) \tag{9.23}$$

where $g_{\pi\mathrm{N}}$ is the pion–nucleon coupling constant with the same normalization convention as for f_π. Hence

$$m_\mathrm{N} g_\mathrm{A}(0) = g_{\pi\mathrm{N}} f_\pi \tag{9.24}$$

which is the Goldberger–Treiman relation derived from the assumption of spontaneously broken chiral symmetry of strong interactions. One can now ask whether the relation (9.24) is a good approximation to the real world where $m_\pi^2 \neq 0$. Experimentally, from πN scattering $g_{\pi\mathrm{N}}^2/4\pi = 14.6$, from (physical) $\pi \to l\nu$ $|f_\pi| = 93$ MeV, from β-decay $g_\mathrm{A}(0) = 1.24$ and relation (9.24) happens to be satisfied within 7%. This agreement supports our main assumption: massless QCD with spontaneously broken chiral symmetry is a good approximation for the strong interactions.

Relating physical quantities to the results obtained in the exact symmetry limit $m_\pi^2 = 0$ is called the PCAC (partially conserved axial vector current) approximation. Originally, the PCAC approximation has been formulated in another, equivalent, way as an operator equation

$$\partial^\mu \mathcal{A}_\mu^a(x) = m_\pi^2 f_\pi \pi^a(x) \tag{9.25}$$

where m_π^2 is the physical pion mass, $\pi^a(x)$ is the pion field normalized as follows: $\langle\pi^a(p)|\pi^b(x)|0\rangle = \delta^{ab}\exp(\mathrm{i}px)$. Relation (9.25) is an identity on the pion mass-shell, as can be seen from (9.20), and it is assumed that the pion field so defined is a good interpolating field off-shell at $p^2 = 0$. For a more extensive discussion of the derivation of the Goldberger–Treiman relation based on relation (9.25) and its connection to the chiral derivation see, for example, Pagels (1975).

9.3 Phenomenological chirally symmetric model of the strong interactions (σ-model)

As we have already mentioned, calculating spontaneous chiral symmetry breaking in QCD is a difficult problem which has not yet been solved. It has, therefore, proved instructive in several aspects to study a phenomenological chirally symmetric field theory model of the strong interactions, the so called σ-model. In this model the scalar fields are introduced as elementary fields and their interactions are arranged to produce the spontaneous breakdown of the symmetry.

The elementary fields in the σ-model are: the nucleon doublet, massless to start with so in fact we have two doublets N_R and N_L transforming like (1, 2) and (2, 1) under $SU_\mathrm{L}(2) \times SU_\mathrm{R}(2)$; the triplet pion field $\boldsymbol{\pi}(0^-)$ and σ'-a scalar 0^+ meson.

The postulated lagrangian is as follows

$$\begin{aligned} \mathcal{L} = \bar{N} \mathrm{i} \not{\partial} N + g \bar{N}(\sigma' + \mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \gamma_5) N + \tfrac{1}{2}[(\partial_\mu \boldsymbol{\pi})^2 + (\partial_\mu \sigma')^2] \\ - \tfrac{1}{2}\mu^2(\sigma'^2 + \boldsymbol{\pi}^2) - \tfrac{1}{4}\gamma(\sigma'^2 + \boldsymbol{\pi}^2)^2 \end{aligned} \tag{9.26}$$

where

$$\boldsymbol{\pi} = (\pi^1, \pi^2, \pi^3), \quad \boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$$

This is the usual pseudoscalar pion–nucleon coupling. The fourth field σ' has been introduced to get a chirally symmetric theory. We rewrite

$$\left.\begin{aligned} &\bar{N} \mathrm{i} \not{\partial} N = \bar{N}_\mathrm{R} \mathrm{i} \not{\partial} N_\mathrm{R} + \bar{N}_\mathrm{L} \mathrm{i} \not{\partial} N_\mathrm{L} \\ &\bar{N}(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}\gamma_5) N = \bar{N}_\mathrm{L}(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) N_\mathrm{R} + \bar{N}_\mathrm{R}(\sigma' - \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) N_\mathrm{L} \end{aligned}\right\} \tag{9.27}$$

to see that for chiral invariance of the lagrangian the $(\sigma + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma})$ must transform as the (2, 2) representation of the $SU(2) \times SU(2)$ (remember that $2^* \equiv 2$). This corresponds to the following transformations (see Appendix E):

$$\left.\begin{array}{ll} \text{under vector } SU(2) \text{, i.e. isotopic spin group} & \begin{aligned} &\boldsymbol{\pi} \to \boldsymbol{\pi} + \delta\boldsymbol{\alpha} \times \boldsymbol{\pi} \\ &\sigma' \to \sigma' \end{aligned} \\ \text{under axial vector rotations} & \begin{aligned} &\boldsymbol{\pi} \to \boldsymbol{\pi} + \delta\boldsymbol{\alpha}\sigma' \\ &\sigma' \to \sigma' - \delta\boldsymbol{\alpha} \cdot \boldsymbol{\pi} \end{aligned} \end{array}\right\} \tag{9.28}$$

($\delta\boldsymbol{\alpha}$ is an infinitesimal vector in the isospin space). Chiral symmetry implies conserved axial and vector currents. They read

$$\left.\begin{aligned} \mathcal{V}^a_\mu(x) &= \bar{N}(x)\gamma_\mu\tfrac{1}{2}\sigma^a N(x) + [\boldsymbol{\pi}(x)\times\partial_\mu\pi(x)]^a \\ \mathcal{A}^a_\mu(x) &= \bar{N}(x)\gamma_\mu\gamma_5\tfrac{1}{2}\sigma^a N(x) + \sigma'(x)\partial_\mu\pi^a(x) - \pi^a(x)\partial_\mu\sigma'(x) \end{aligned}\right\} \tag{9.29}$$

Constructing then charges Q^a, Q^a_5, Q_{L}, Q_{R} and using canonical commutation relations for fields one gets

$$\left.\begin{aligned} [Q^a_{\substack{\mathrm{L}\\ \mathrm{R}}}, \sigma'] &= \mp\tfrac{1}{2}\mathrm{i}\pi^a \\ [Q^a_{\substack{\mathrm{L}\\ \mathrm{R}}}, \pi^b] &= \tfrac{1}{2}\mathrm{i}\varepsilon^{abc}\pi^c \pm \tfrac{1}{2}\mathrm{i}\delta^{ab}\sigma' \end{aligned}\right\} \tag{9.30}$$

Note (Appendix E) that in general if a field multiplet Φ transforms as $\Phi_i \to \Phi'_i = (\delta_{ij} - \mathrm{i}\delta\alpha^a T^a_{ij})\Phi_j$ under linear transformations generated by generators Q^a then, since on the other hand the operator relation

$$\begin{aligned} \Phi_i \to \Phi'_i &= \exp(\mathrm{i}\alpha^a Q^a)\Phi_i(-\mathrm{i}\alpha^a Q^a) \\ &\cong \Phi_i + \mathrm{i}\delta\alpha^a[Q^a, \Phi_i] \end{aligned} \tag{9.31}$$

holds, it must be that

$$[Q^a, \Phi_i(x)] = -T^a_{ij}\Phi_j(x) \tag{9.32}$$

(matrices T^a form a representation of generators Q^a).

It is now necessary to establish the correct ground state of the model. The vacuum is defined by the minimum of the expectation values of the hamiltonian. As we know, for example, from the effective potential formalism, in the tree approximation the vacuum expectation values of the fields are equal to the values of the classical fields in the minimum of the potential $V(\sigma', \boldsymbol{\pi})$. We can rewrite the potential as

$$V(\sigma', \boldsymbol{\pi}) = -\tfrac{1}{4}\lambda(\sigma'^2 + \boldsymbol{\pi}^2 + \mu^2/\lambda)^2 \tag{9.33}$$

to see that for $\mu^2/\lambda > 0$ the minimum occurs for $\sigma' = \boldsymbol{\pi} = 0$, whereas for $\mu^2/\lambda < 0$ it exists for

$$\sigma'^2 + \boldsymbol{\pi}^2 = |\mu^2/\lambda| \tag{9.34}$$

With this second choice the model accounts for the spontaneous breakdown of chiral symmetry. The physical vacuum is chosen (defined) from the degenerate ground states given by the condition (9.34). In particular we would like to preserve unbroken isospin symmetry. The physical vacuum must then satisfy (9.34) and also

$$Q^a|0\rangle = 0, \quad Q^a_5|0\rangle \neq 0 \tag{9.35}$$

These conditions are satisfied with the choice

$$\langle 0|\boldsymbol{\pi}|0\rangle = 0, \quad \langle 0|\sigma'|0\rangle = -(|\mu^2/\lambda|)^{1/2} \equiv v \tag{9.36}$$

Indeed the vacuum must be an isospin singlet (hence $\langle 0|\boldsymbol{\pi}|0\rangle = 0$) but not a chiral singlet (hence $\langle 0|\sigma'|0\rangle \neq 0$, where σ' belongs to the chiral doublet). Formally, one can see that $\langle 0|\sigma'|0\rangle \neq 0$ implies $Q_5^a|0\rangle \neq 0$ (and vice versa) by taking the vacuum expectation value of the commutator $[Q_5^a, \pi^b] = -\mathrm{i}\delta^{ab}\sigma'$. Another way is to consider $\langle 0|\exp(\mathrm{i}\alpha^a Q_5^a)\sigma' \exp(-\mathrm{i}\alpha^a Q_5^a)|0\rangle$ to conclude that generators of the transformations which do not leave $\langle 0|\sigma'|0\rangle$ invariant are spontaneously broken. It is worth stressing at this point that in our discussion of the σ-model we have fixed at the very beginning the physical interpretation of the group generators and of the fields. In the basis so defined the choice of the physical vacuum among the degenerate ground states given by the condition (9.34) is no longer arbitrary if we want to keep the isospin unbroken. However, we could also proceed in another way: define any of the *a priori* equivalent degenerate ground states as the physical vacuum, find the broken and unbroken generators of the theory, give to them and to the fields the desired physical interpretation. The pattern of the spontaneous symmetry breaking from such a general viewpoint will be discussed in Section 9.5.

In view of (9.36) we must, to ensure the orthogonality of the vacuum to the one-particle state, define the physical σ field as

$$\sigma = \sigma' - v \tag{9.37}$$

The lagrangian (9.26) can now be rewritten in terms of the field σ

$$\begin{aligned}\mathcal{L} = {} & \bar{N}\mathrm{i}\not\partial N + g\bar{N}(\sigma + \mathrm{i}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\gamma_5)N + gv\bar{N}N + \tfrac{1}{2}[(\partial_\mu\boldsymbol{\pi})^2 + (\partial_\mu\sigma)^2] \\ & - |\mu^2|\sigma^2 - \tfrac{1}{4}\lambda(\sigma^2 + \boldsymbol{\pi}^2)^2 - \lambda v(\sigma^3 + \sigma\boldsymbol{\pi}^2) + \text{const.}\end{aligned} \tag{9.38}$$

It describes nucleons of mass $m_{\mathrm{N}} = g(|\mu^2/\lambda|)^{1/2} = -gv$, a σ meson of mass $m_\sigma = 2(-\mu^2/2)^{1/2} = (2v^2\lambda)^{1/2}$ and the isospin triplet of massless pions. In addition we observe that in the tree approximation, from (9.29)

$$\begin{aligned}\langle 0|\mathcal{A}_\mu^a(x)|\pi^b(q)\rangle &= \langle 0|\sigma'|0\rangle\langle 0|\partial_\mu\pi^a(x)|\pi^b(q)\rangle \\ &= -\langle 0|\sigma'|0\rangle \mathrm{i}q_\mu\delta^{ab}\exp(-\mathrm{i}qx)\end{aligned}$$

In the tree approximation the other terms in the axial current (9.29) give a vanishing contribution to this matrix element. Comparing with (9.19) one gets

$$f_\pi = -\langle 0|\sigma'|0\rangle$$

The v or f_π is the only mass scale of the model.

Our last remark in this section is as follows: the model can be extended to account for an explicit breakdown of the chiral symmetry. To do this in an isotopically invariant manner one adds to $\mathcal{L}$ a term $-\varepsilon\sigma'$. Then minimizing the potential (9.33) plus the extra term one finds instead of (9.36) the following equation for the vacuum expectation value v:

$$-\varepsilon - \lambda v^3 - \mu^2 v = 0$$

With the shift (9.37) one now gets

$$m_\pi^2 = \mu^2 + \lambda v^2 = -\varepsilon/v = \varepsilon/f_\pi$$

The axial current is no longer conserved

$$\partial^\mu \mathcal{A}_\mu^a(x) = \varepsilon\pi^a(x) \tag{9.39}$$

and finally we get

$$\partial^\mu \mathcal{A}_\mu^a(x) = m_\pi^2 f_\pi \pi^a(x) \tag{9.40}$$

which is the operator PCAC relation (9.25) introduced earlier. Another effect of the $\varepsilon\sigma'$ term is that it defines the unique physical vacuum. This vacuum alignment problem will be discussed in more detail later on.

9.4 Goldstone bosons as eigenvectors of the mass matrix and poles of Green's functions in theories with elementary scalars

We have learned in the previous section that spontaneous breakdown of the chiral symmetry in the σ-model gives massless Goldstone bosons in the physical spectrum. Here we study the problem of Goldstone bosons in theories with elementary scalars (QCD-like theories are discussed in Section 9.6) in a more general context.

Goldstone bosons as eigenvectors of the mass matrix

First we generalize the discussion of Section 9.3 and show in the tree approximation that in a theory with spontaneously broken continuous global symmetry the mass matrix of the scalar sector has eigenvalues zero. The corresponding eigenvectors are Goldstone boson fields. Next we give the general proof of Goldstone's theorem (Goldstone, Salam & Weinberg 1962) which does not rely on perturbation theory. This proof can also be easily extended to theories without elementary scalars.

Consider a theory described by a lagrangian $\mathcal{L}$ which has some global continuous symmetry G generated by charges Q^a. Among other things, the theory contains elementary scalar fields Φ_i, $i = 1, \ldots, n$, which we take as real; any complex representation can always be turned into a real one by doubling the number of the basis vectors. The piece $\mathcal{L}_s$ of the lagrangian containing the scalar fields Φ_i is

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - V(\Phi_i) + \cdots \tag{9.41}$$

where the unspecified terms, irrelevant for our discussion in the tree approximation, are possible couplings of the scalar fields to other fields of the theory. The $V(\Phi_i)$ is a real polynomial in fields Φ_i, of the fourth order at most, if the theory is to be renormalizable. We assume furthermore that the set Φ_i forms a multiplet Φ transforming under some irreducible representation of the group G of the global continuous symmetry of the lagrangian $\mathcal{L}$ (if the original fields form a reducible multiplet we change the basis to write it as a direct sum of irreducible multiplets)

$$\Phi_i \to \Phi'_i \approx \Phi_i - \mathrm{i}\Theta^a T^a_{ij} \Phi_i \tag{9.42}$$

Matrices T^a form the representation of charges Q^a in the n-dimensional space of scalar fields Φ_i. Because $\mathrm{i}T^a$ must be real, T^a is purely imaginary and being hermitean it is antisymmetric. The transformation rule (9.42) is equivalent to the following commutation relation between charges Q^a and fields Φ_i (see (9.32)):

$$[Q^a(t), \Phi_i(\mathbf{x}, t)] = -T^a_{ij}\Phi_j(\mathbf{x}, t) \tag{9.43}$$

We are interested in theories in which the global symmetry G is spontaneously broken by the vacuum which is such that some of the fields Φ_i have non-vanishing vacuum expectation values

$$\langle 0|\Phi_i|0\rangle = v_i \neq 0 \tag{9.44}$$

In the tree or classical approximation, when the energy density of the vacuum state is just given by the minimum of the potential $V(\Phi_i)$, one then concludes that for the spontaneous breakdown of the symmetry G the condition for the minimum of $V(\Phi_i)$

$$0 = \delta V(\Phi_i) = \frac{\partial V}{\partial \Phi_i}\delta\Phi_i = -\mathrm{i}\frac{\partial V}{\partial \Phi_i}\Theta^a T^a_{ij}\Phi_j \tag{9.45}$$

must have solutions with $\Phi_i = v_i \neq 0$ for some of the Φ_is. Notice that if the vector $v = (v_1 \ldots v_n)^{\mathrm{T}}$ is a solution of (9.45) then it follows from the G invariance of the potential V that the vector

$$v' = \exp(-\mathrm{i}\Theta^a T^a)v \tag{9.46}$$

is another solution to this equation. Thus, for a given potential V which gives the spontaneous breakdown of the symmetry G the vacuum is infinitely degenerate and any choice of it is physically equivalent (a choice of a solution v is a choice of the vacuum). Of course, different potentials may, in principle, give physically different, non-degenerate, vacua. Possible patterns of spontaneous symmetry breaking are discussed in more detail in the next section.

Choosing some solution v of (9.45) as the physical vacuum of our theory it is convenient to define the physical fields Φ'

$$\Phi'_i = \Phi_i - v_i \tag{9.47}$$

and to rewrite in terms of them the potential V

$$V(\Phi_i) = \text{const} + \tfrac{1}{2} M^2_{ik} (\Phi - v)_i (\Phi - v)_k + \cdots \tag{9.48}$$

where

$$M^2_{ik} = \left(\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_k} \right)_{\Phi = v} \tag{9.49}$$

is the mass matrix for the physical scalar fields. The term linear in fields does not appear in (9.48) because of (9.45). Differentiating (9.45) one gets the following result:

$$M^2_{ik} \mathrm{i} T^a_{ij} v_j = 0 \tag{9.50}$$

which suggests that the mass matrix has eigenvectors $T^a v$ corresponding to zero eigenvalues. To discuss this point more precisely we divide the generators of G into two groups. The first one consists of those which remain unbroken with our choice of the vacuum. Denoting their matrix representation by Y^i, $i = 1, \ldots, n$, one has $Y^I v = 0$; the vacuum is invariant with respect to Y^i and (9.50) does not give any new information. To the second group belong generators which are spontaneously broken. Denoting their matrix representation by X^i, $i = n + 1, \ldots, N$ (note the values of the index i) we have $X^i v \neq 0$ and (9.50) implies the existence, for each broken generator of G, of the eigenvectors $X^i v$ corresponding to zero eigenvalues of the mass matrix. The massless boson fields (normal coordinates) are then the following:

$$\Pi^l = \mathrm{i}(X^l v)_i \Phi_i \tag{9.51}$$

We recall at this point that our discussion is for spontaneously broken global symmetries. For spontaneously broken gauge symmetries the Goldstone boson fields can be gauged away and do not appear in the physical spectrum. These degrees of freedom are used for longitudinal components of the massive gauge boson. This so-called Higgs mechanism is discussed in Section 11.1.

We conclude these considerations with several useful remarks. Obviously, if some of the fields have non-vanishing vacuum expectation values, some generators of G must be broken. The generators which remain unbroken must form the algebra of some subgroup H of group G (it may, of course, be that $H \equiv 1$). Indeed, if it is not the case then

$$[Y^i, Y^j] = \mathrm{i}c_{ijk}Y^k + \mathrm{i}c_{ijl}X^l, \quad i, j, k \le n, \quad l > n \tag{9.52}$$

with some of the c_{ijl} different from zero. Acting with both sides of (9.52) on the vector v we get a contradiction. Therefore

$$c_{ijl} = 0, \quad i, j \le n, \quad l > n \tag{9.53}$$

and the unbroken generators form the algebra of a group. Furthermore, it is easy to see that broken generators transform under some representation of the subgroup H

$$[Y^i, X^j] = \mathrm{i}c_{ijk}X^k, \quad i \le n, \quad j, k > n \tag{9.54}$$

This follows from the property (9.53) and from the antisymmetry of the structure constants in indices i, j, k. Thus, the r.h.s. of (9.54) cannot contain a term $c_{ijl}Y^l$, where $i, l \le n$, $j > n$. Eqs. (9.54) and (9.51) taken together tell us that Goldstone bosons transform under some representation of the unbroken subgroup H. Finally, in many physical problems one can define a parity operation P which leaves the Lie algebra of the group G invariant and such that

$$PY^iP^{-1} = Y^i, \quad PX^iP^{-1} = -X^i \tag{9.55}$$

This is the case, for example, for chiral symmetry; see Section 9.1. Then there is one more useful and obvious relation

$$[X^i, X^j] = \mathrm{i}c_{ijk}Y^k, \quad i, j > n, \quad k \le n \tag{9.56}$$

General proof of Goldstone's theorem

We turn now to the general proof of Goldstone's theorem (Goldstone, Salam & Weinberg 1962) in the class of theories specified in the beginning of this section. One can show that spontaneous breakdown of some global continuous symmetry G of the lagrangian implies poles at $p^2 = 0$ in certain Green's functions and consequently implies the existence of massless bosons in the physical spectrum.

Let us consider the Green's function

$$G^a_{\mu.k}(x - y) = \langle 0|Tj^a_\mu(x)\Phi_k(y)|0\rangle \tag{9.57}$$

where $j^a_\mu(x)$ is the current corresponding to a generator Q^a of the symmetry G and $\Phi_k(y)$ belongs to an irreducible multiplet of real scalar fields. The Green's function $G^a_{\mu,k}(x - y)$ satisfies a Ward identity obtained by differentiating (9.57) with proper account of the Θ-functions involved in the definition of the T-product

$$\partial^\mu_{(x)}G^a_{\mu,k}(x - y) = \delta(x^0 - y^0)\langle 0|[j^a_0(x), \Phi_k(y)]|0\rangle \tag{9.58}$$

This is an example of the so-called non-anomalous Ward identity (see Section 10.1); for the discussion of anomalies in theories with fermions see Chapter 13. Using the relation

$$[j_0^a(\mathbf{x}, t), \Phi_k(\mathbf{y}, t)] = -T^a_{kj}\Phi_j(\mathbf{y}, t)\delta(\mathbf{x} - \mathbf{y}) \tag{9.59}$$

which follows from the assumed transformation properties, see (9.43), of fields Φ_i under the group G, one gets the following Ward identity (translational invariance is assumed)

$$\partial^\mu_{(x)} G^a_{\mu,k}(x - y) = -\partial(x - y)T^a_{kj}\langle 0|\Phi_j(0)|0\rangle \tag{9.60}$$

A few words are worthwhile here about renormalization properties of (9.60) (see also the next chapter). The Ward identity is, in principle, derived for bare currents and fields for which transformation properties (9.59), are assumed. However, the conserved currents are not subject to renormalization and the field renormalization constants $\Phi_i^{\mathrm{B}} = Z_i^{1/2}\Phi_i^{\mathrm{R}} = Z^{1/2}\Phi_i^{\mathrm{R}}$ do not carry the group index because the lagrangian is G invariant. Thus, the renormalized fields transform under the group G in the same way as the bare fields and the same Ward identity (9.60) holds both for bare and renormalized Green's functions. Let us take it as a relation for renormalized quantities. Introducing the Fourier transform $\tilde{G}^a_{\mu,k}(p)$

$$G^a_{\mu,k}(x - y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp[-\mathrm{i}p(x - y)]\tilde{G}^a_{\mu,k}(p) \tag{9.61}$$

one gets from (9.60) the following:

$$\mathrm{i}p^\mu \tilde{G}^a_{\mu,k}(p) = T^a_{kj}\langle 0|\Phi_j(0)|0\rangle \tag{9.62}$$

This last equation implies that if some of the vacuum expectation values $\langle 0|\Phi_j|0\rangle$ do not vanish then there are poles at $p^2 = 0$ in the Green's functions $\tilde{G}^a_{\mu,k}(p)$, related to these $\langle 0|\Phi_j|0\rangle$ by (9.62). Indeed, from Lorentz invariance the general structure of $\tilde{G}^a_{\mu,k}(p)$ can be written as

$$\tilde{G}^a_{\mu,k}(p) = p_\mu F^a_k(p^2) \tag{9.63}$$

and therefore

$$F^a_k(p^2) = -\mathrm{i}T^a_{kj}\langle 0|\Phi_j(0)|0\rangle(1/p^2) \tag{9.64}$$

Our last step is to show that poles at $p^2 = 0$ imply the existence of massless bosons in the physical spectrum. To this end we consider the matrix element

$$\langle 0|j^a_\mu(x)|\pi^k(p)\rangle = \mathrm{i}f^a_k p_\mu \exp(-\mathrm{i}px) \tag{9.65}$$

where the vector $|\pi^k(p)\rangle$ describes a particle of mass m_k which is a quantum of the field Φ_k. Using the reduction formula (see Section 2.7) one can relate the matrix element $\langle 0|j_\mu^a(x)|\pi^k(p)\rangle$ to the Green's function $\tilde{G}^a_{\mu,k}(p)$

$$\langle 0|j_\mu^a(x)|\pi^k(p)\rangle = \int \mathrm{d}^4y\,\mathrm{d}^4z\, f_p(z) G^a_{\mu,k'}(x-y)\mathrm{i}G^{-1}_{k'k}(y-z) \tag{9.66}$$

where

$$f_p(z) = \exp(-\mathrm{i}pz)$$

and

$$\left.\begin{aligned} G_{k'k}(y-z) &= \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{-\delta_{k'k}}{q^2-m_k^2+\mathrm{i}\varepsilon}\exp[-\mathrm{i}q(y-z)] \\ \int \mathrm{d}^4y\, G^{-1}(x-y)G(y-z) &= \delta(x-z) \end{aligned}\right\} \tag{9.67}$$

Writing $\exp(-\mathrm{i}pz)$ as $\exp(-\mathrm{i}px)\exp[\mathrm{i}p(x-y)]\exp[\mathrm{i}p(y-z)]$ we get from (9.66) the following result:

$$\begin{aligned} \langle 0|j_\mu^a(x)|\pi^k(p)\rangle &= \lim_{p^2\to m_k^2} \exp(-\mathrm{i}px)\tilde{G}^a_{\mu,k'}(p)\mathrm{i}\tilde{G}^{-1}_{kk}(p) \\ &= \lim_{p^2\to m_k^2} -\mathrm{i}\exp(-\mathrm{i}px)\tilde{G}^a_{\mu,k}(p)(p^2-m_k^2) \end{aligned} \tag{9.68}$$

Comparison with (9.65) gives us our final equation:

$$\lim_{p^2\to m_k^2} \tilde{G}^a_{\mu,k}(p)(p^2-m_k^2) = -f_k^a p_\mu \tag{9.69}$$

If $\tilde{G}^a_{\mu,k}(p)$ is one of those Green's functions which (according to (9.64)) have for spontaneously broken symmetry a pole at $p^2=0$ then (9.69) implies $m_k^2=0$ and $f_k^a \neq 0$:

$$f_k^a = \mathrm{i}T^a_{kj}\langle 0|\Phi_j(0)|0\rangle \tag{9.70}$$

Thus there must be massless bosons $|\Pi^a(p)\rangle = \mathrm{i}T^a_{kj}\langle 0|\Phi_j(0)|0\rangle|\pi^k(p)\rangle$ in the physical spectrum corresponding to each broken generator which does not leave the vacuum invariant $T^a_{kj}\langle 0|\Phi_j(0)|0\rangle \neq 0$. If the broken generators and the corresponding currents form a single real irreducible representation of the unbroken subgroup H which leaves the vacuum invariant then from Schur's lemma

$$\langle 0|j_\mu^a|\pi^k(p)\rangle \sim \delta^{ak} \tag{9.71}$$

and

$$f_k^a = f^a\delta_{ak}$$

For instance, for the σ-model with $\langle 0|\sigma'|0\rangle \neq 0$ we get from (9.70), with matrices T^a_{kj} from Appendix E,

$$f_\pi = -\langle 0|\sigma'|0\rangle \tag{9.72}$$

as the generalization of the result obtained in the tree approximation.

Goldstone's theorem does not hold for spontaneously broken gauge symmetries (see Section 11.1). The breakdown of the presented proof is most easily seen if we work in a 'physical' gauge in which the gauge field has only physical degrees of freedom. Gauge fixing then requires a specification of some four-vector n_μ and the general form of the matrix element involving the gauge source current is no longer given by (9.63): a term proportional to n_μ can be present. For a discussion in a covariant gauge see, for example, Bernstein (1974).

9.5 Patterns of spontaneous symmetry breaking

For a theory with some global symmetry G and a given multiplet content of fields, patterns of spontaneous symmetry breaking are determined by the properties of the vacuum defined by the non-vanishing of the vacuum expectation values of some fields. In theories with elementary scalars the non-vanishing vacuum expectation values may already appear in the tree approximation as a solution minimizing the assumed potential $V(\Phi)$ or may be generated by radiative corrections (Section 11.2). In theories without elementary scalars the symmetry is spontaneously broken by some condensates (for example, $\langle 0|\bar{\Psi}\Psi|0\rangle$) which, in the absence of reliable methods for calculating non-perturbative effects, are usually introduced by assumption. A choice of the vacuum specifies the pattern of the spontaneous symmetry breaking which may, in general, be such that some subgroup H of G remains unbroken. As we have already mentioned in the previous section for any choice of the vacuum specified by the non-vanishing vacuum expectation values there exists a set of physically equivalent vacua, which also minimize the same potential $V(\Phi)$, obtained from the original one by the set of unitary transformations belonging to G. Different choices of the physical vacuum among the degenerate vacua give the same pattern of symmetry breaking; they simply correspond to different orientations of the unbroken subgroup H in the group G which can be transformed one into the other by a redefinition of the basis.

The question we would like to discuss here is what are the possible patterns of spontaneous breaking of a given symmetry G. Although different patterns, if they exist, must, in general, correspond to different potentials $V(\Phi)$, no explicit reference to the potential is necessary for our discussion.

Our first example is a group of orthogonal transformations, $O(3)$ for definiteness, and the scalar fields are assumed to form a real multiplet $\Phi = (\Phi_1, \Phi_2, \Phi_3)^{\mathrm{T}}$

transforming under the vector representation. The matrix representation of generators Q_i in the space of Φs is

$$T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad T_y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \quad T_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.73}$$

A possible choice for the vacuum expectation value of the multiplet Φ is, for instance,

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \equiv \Phi_0$$

The vacuum defined this way is invariant under the transformations generated by Q_z: taking the transformed vacuum $\exp(-\mathrm{i}\alpha Q_z)|0\rangle$ and using (9.32) we get

$$\langle O|\exp(\mathrm{i}\alpha Q_z)\Phi \exp(-\mathrm{i}\alpha Q_z)|0\rangle \cong \Phi_0 + \mathrm{i}\alpha\langle 0|[Q_z, \Phi]|0\rangle = \Phi_0 - \mathrm{i}\alpha T_z \Phi_0 = \Phi_0 \tag{9.74}$$

However, the vacuum defined by (9.74) breaks spontaneously the other two generators:

$$\mathrm{i}T_x\Phi_0 = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad \mathrm{i}T_y\Phi_0 = \begin{pmatrix} -v \\ 0 \\ 0 \end{pmatrix} \tag{9.75}$$

Thus the group $O(3)$ is broken to the group $O(2)$ of rotations around the z axis. There are two Goldstone bosons corresponding to two linearly independent vectors $\mathrm{i}T_x\Phi_0$ and $\mathrm{i}T_y\Phi_0$. Any other choice of the vacuum expectation value

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{9.76}$$

which can be obtained from the original one by unitary transformations belonging to G gives the same pattern of symmetry breaking but this time with the combination $v_1T_x + v_2T_y + v_3T_z$ unbroken. Both theories are physically equivalent and transform one into the other by a redefinition of the basis. This would not be the case if there was, in space, a physically distinguished direction, that is if the $O(3)$ symmetry was also broken explicitly. This point will be discussed in detail in the next chapter.

Consider now unitary groups with scalars in a real representation. Any complex field Φ can be written in a real basis by defining two real fields

$$\left.\begin{aligned} \varphi_i &= (1/\sqrt{2})(\Phi_i + \Phi_i^*) \\ \chi_i &= (1/\sqrt{2}\mathrm{i})(\Phi_i - \Phi_i^*) \end{aligned}\right\} \tag{9.77}$$

For the group $U(1)$ of the unitary transformations

$$\Phi \to \Phi' = \exp(-\mathrm{i}\Theta)\Phi$$

the real representation is

$$\Phi^{\mathrm{r}} \to \Phi'^{\mathrm{r}} = \exp(-\mathrm{i}\Theta T)\Phi^{\mathrm{r}}$$

where

$$\Phi^{\mathrm{r}} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{9.78}$$

and T is purely imaginary

$$T = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$$

because Φ^{r} must be real. Obviously, in this case, there is just one pattern of spontaneous symmetry breaking and any non-zero choice of the vacuum expectation value $\langle 0|\Phi^{\mathrm{r}}|0\rangle = (v_1 v_2)^{\mathrm{T}}$ is equally good.

Our next example is the group $SU(2)$ with a doublet of complex fields Φ_1 and Φ_2. In the real representation

$$\Phi^{\mathrm{r}} = \begin{pmatrix} \varphi_1 \\ \chi_1 \\ \varphi_2 \\ \chi_2 \end{pmatrix}$$

the generators of the $SU(2)$ transformations are

$$T^1 = \begin{pmatrix} & & & \mathrm{i} \\ & & -\mathrm{i} & \\ & \mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix}, \qquad T^2 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -\mathrm{i} & \\ & & & -\mathrm{i} \\ \hline \mathrm{i} & & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ & \mathrm{i} & & \end{array}\right),$$

$$T^3 = \left(\begin{array}{cc|cc} & \mathrm{i} & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\mathrm{i} & & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & & -\mathrm{i} \\ & & \mathrm{i} & \end{array}\right) \tag{9.79}$$

Let us take, for instance, $\langle 0|\Phi^{\mathrm{r}}|0\rangle = (0, 0, v, 0)^{\mathrm{T}}$. Following the discussion of our first example we see that the group is broken completely because the three vectors $\mathrm{i}T^a\langle 0|\Phi^{\mathrm{r}}|0\rangle \neq 0$ are linearly independent. All other possible non-vanishing vacuum expectation values can be obtained from the one above by unitary $SU(2)$ transformations and, thus, give the same pattern of symmetry breaking.

It is easy to extend the analysis to the $SU(2) \times U(1)$ group with one complex doublet of scalar fields (Glashow–Salam–Weinberg model). In this case we have one more generator

$$T = \left(\begin{array}{cc|cc} & \mathrm{i} & & \\ -\mathrm{i} & & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & & \mathrm{i} \\ & & -\mathrm{i} & \end{array}\right)$$

but the vector $\mathrm{i}T\langle 0|\Phi^{\mathrm{r}}|0\rangle$ is not linearly independent from the other three vectors $\mathrm{i}T^a\langle 0|\Phi^{\mathrm{r}}|0\rangle$. For instance, for $\langle 0|\Phi^{\mathrm{r}}|0\rangle = (0, 0, v, 0)^{\mathrm{T}}$ we have $T\langle 0|\Phi^{\mathrm{r}}|0\rangle = -T^3\langle 0|\Phi^{\mathrm{r}}|0\rangle$ and the combination $T + T^3$ remains unbroken. Other choices of the vacuum give only different unbroken combinations of the generators T and T^a but a redefinition of the basis transforms one case into another. Thus, again only one pattern of symmetry breaking is possible.

The spontaneous breaking of chiral symmetry in the σ-model can be discussed in a similar manner. Using the real basis $\Phi^{\mathrm{r}} = (\pi_1\pi_2\pi_3\sigma')^{\mathrm{T}}$ and the matrices T^a and ${}_5T^a$ collected in Appendix E we see that, for example, for $\langle 0|\Phi^{\mathrm{r}}|0\rangle = (0, 0, 0, v)^{\mathrm{T}}$ the generators ${}_5T^a$ are broken and the T^as remain unbroken: $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2) \rightarrow SU_{\mathrm{L+R}}(2)$. This corresponds to the standard choice $\langle 0|\sigma'|0\rangle \neq 0$ and to the πs forming a triplet under the $SU(2)$. However, any other choice of vacuum among the $SU(2) \times SU(2)$ degenerate set gives the same pattern of symmetry breaking. Take, for example, $\langle 0|\Phi^{\mathrm{r}}|0\rangle = (v, 0, 0, 0)^{\mathrm{T}}$. Then the generators ${}_5T^2$, ${}_5T^3$and T^1 remain unbroken and we see from the commutation relations (9.9) that they also form an $SU(2)$ algebra. We can call this group the isospin group and interpret as physical pions those combinations of the fields $\pi^i$and σ which transform as a triplet under this new $SU(2)$, thus obtaining the same theory as before.

So far we have been discussing examples with only one possible pattern of spontaneous symmetry breaking. This is, of course, not always the case. As our last example we consider the group $SU(3)$ with a real multiplet Φ of scalar fields transforming under the adjoint representation of the $SU(3)$

$$[Q^a, \Phi_b] = -T^a_{bc}\Phi_c$$

where

$$T^a_{bc} \equiv -\mathrm{i}c_{abc} \tag{9.80}$$

It is convenient to write the scalar multiplet in the matrix form

$$\hat{\Phi} = \tfrac{1}{2}\Phi^b\lambda^b, \qquad b = 1, \ldots, 8 \tag{9.81}$$

and λ^b are the Gell-Mann matrices. The matrix $\hat{\Phi}$ is hermitean and $\hat{\Phi}_{ii} = 0$ so it has eight independent elements. The transformation rule for the matrix $\hat{\Phi}$ is

$$\hat{\Phi} \to \hat{\Phi}' = \exp(-\mathrm{i}\Theta^a \tfrac{1}{2}\lambda^a)\hat{\Phi}\exp(\mathrm{i}\Theta^a \tfrac{1}{2}\lambda^a) \cong \hat{\Phi} - \mathrm{i}\Theta^a[\tfrac{1}{2}\lambda^a, \hat{\Phi}] \tag{9.82}$$

and because of (9.81) it is equivalent to

$$\Phi \to \Phi' = \exp(-\mathrm{i}\Theta^a T^a)\Phi$$

with T^a given by (9.80), for the multiplet Φ. Because the matrix Φ is hermitean it can be diagonalized by unitary transformations and written in terms of commuting generators of $SU(3)$. Taking them as λ^3 and λ^8 we can write the most general form of the vacuum expectation values of the scalar fields as follows:

$$\langle 0|\hat{\Phi}|0\rangle = \tfrac{1}{2}v_3\lambda^3 + \tfrac{1}{2}v_8\lambda^8 \tag{9.83}$$

First of all we notice that in our example the maximal possible breaking is $SU(3) \to U_3(1) \times U_8(1)$ because it is clear from (9.83) and (9.82) that λ^3 and λ^8 remain unbroken for any choice of v_3 and v_8. In this case we have six Goldstone bosons. However, another pattern of symmetry breaking is also possible, namely $SU(3) \to SU(2) \times U(1)$. If we assume $v_3 = 0$, $v_8 \neq 0$ the generators $\lambda^1, \lambda^2, \lambda^3$ $(SU(2))$ and $\lambda^8(U(1))$ remain unbroken; if $v_3 = +\sqrt{3}v_8$ the $\lambda^6, \lambda^7, \frac{1}{2}(\lambda^3 - \sqrt{3}\lambda^8)(SU(2))$ and $\frac{1}{2}(\sqrt{3}\lambda^3 + \lambda^8)$ $(U(1))$ are unbroken; if $v_3 = -\sqrt{3}v_8$ the $\lambda^4, \lambda^5, \frac{1}{2}(\lambda^3 + \sqrt{3}\lambda^8)$ $(SU(2))$ and $\frac{1}{2}(\sqrt{3}\lambda^3 - \lambda^8)$ $(U(1))$ are unbroken. It is also worth noticing that a vacuum expectation value giving $SU(3) \to SU(2) \times U(1)$ cannot be transformed by unitary transformations belonging to $SU(3)$ to one which gives $SU(3) \to U(1) \times U(1)$ and they must correspond to different potentials $V(\Phi)$. However, for each pattern there is an infinitely degenerate set of possible vacua obtained from (9.83) (with specified v_3 and v_8) by unitary $SU(3)$ transformations.

For a given pattern of spontaneous symmetry breaking one usually works from the beginning in a convenient basis with a specified physical interpretation of the group generators and of the fields, as we do for the σ-model in Section 9.3 and for QCD in Section 9.6.

9.6 Goldstone bosons in QCD

The purpose of this section is to extend the proof of Goldstone's theorem to theories like QCD, with no elementary scalars. Chiral symmetry of the massless QCD lagrangian has been discussed in Section 9.1. Considerable phenomenological evidence has then been summarized that the group $SU(2) \times SU(2)$ is an approximate symmetry of the strong interactions which is spontaneously broken by the vacuum. One should be aware, however, of the fact, that our understanding of the underlying mechanism of spontaneous symmetry breaking in theories without elementary scalars is not yet satisfactory. It has at least been realized that only asymptotically free theories allow spontaneous chiral symmetry breaking at reasonable momentum scales (Lane 1974, Gross & Neveu 1974). It is not our aim here to review partial results suggesting that this indeed happens; we shall assume the spontaneous breakdown of chiral symmetry and then show that it implies the existence of massless bosons (composite states) in the physical spectrum. It is nevertheless useful to give at least an intuitive argument in favour of the spontaneous breaking of chiral symmetries in asymptotically free theories. This relies on the fact that the gauge coupling in QCD becomes strong at large distances. Assuming that it becomes arbitrarily strong we expect that the ground state of the theory has an indefinite number of massless fermion pairs which can be created and annihilated by the strong coupling. We still expect the ground state to be invariant under Lorentz transformations so these pairs must have zero total momentum and angular momentum. Thus we find that the vacuum $|0\rangle$ has the property that operators which destroy or create such a fermion pair have non-zero vacuum expectation values $\langle 0|\bar{\Psi}_{\mathrm{R}}\Psi_{\mathrm{L}}|0\rangle \neq 0$, $\langle 0|\bar{\Psi}_{\mathrm{L}}\Psi_{\mathrm{R}}|0\rangle \neq 0$; we omit indices here. In the following we assume that the chiral symmetry of QCD is indeed spontaneously broken by non-vanishing vacuum expectation values of some fermion condensates.

We consider QCD with two flavours and the chiral symmetry $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2)$. Let us introduce scalar operators Φ and $\Phi^\dagger$

$$\Phi_{ij} = \bar{\Psi}_{\mathrm{L}j}\Psi_{\mathrm{R}i}, \quad (\Phi^\dagger)_{ij} = \bar{\Psi}_{\mathrm{R}j}\Psi_{\mathrm{L}i} \tag{9.84}$$

($i, j = 1, 2$ are flavour indices). From the transformation properties of the fields $\Psi_{\mathrm{L}i}$ and $\Psi_{\mathrm{R}j}$ under the chiral group

$$\begin{aligned} \Psi'_{\mathrm{R}i} &= (U_{\mathrm{R}})_{ik}\Psi_{\mathrm{R}k} \\ \Psi'^{\dagger}_{\mathrm{L}j} &= \Psi^{\dagger}_{\mathrm{L}k}(U^{\dagger}_{\mathrm{L}})_{kj} \end{aligned}$$

it follows that

$$\left.\begin{aligned} \Phi' &= U_{\mathrm{R}}\Phi U^{\dagger}_{\mathrm{L}} \\ (\Phi^\dagger)' &= U_{\mathrm{L}}\Phi^\dagger U^{\dagger}_{\mathrm{R}} \end{aligned}\right\} \tag{9.85}$$

We assume

$$\langle 0|\Phi_{ij}|0\rangle \neq 0, \quad \langle 0|\Phi^{\dagger}_{ij}|0\rangle \neq 0$$

and in addition

$$\langle 0|\Phi_{ij}|0\rangle = \langle 0|\Phi^{\dagger}_{ij}|0\rangle$$

The latter equality ensures that parity is not spontaneously broken since then

$$\langle 0|\bar{\Psi}_i \gamma_5 \Psi_j|0\rangle = \langle 0|(\Phi - \Phi^\dagger)_{ji}|0\rangle = 0 \tag{9.86}$$

(we always work in a specified basis of chiral fields).

The vacuum expectation value $\langle 0|\Phi|0\rangle$, being a hermitean 2×2 matrix, can be most generally written as (see the discussion preceding (9.83))

$$\langle 0|\Phi|0\rangle = v\mathbb{1} = v_3\sigma^3 \tag{9.87}$$

where σ^3 is the diagonal Pauli matrix. We want the chiral $SU_{\rm L}(2) \times SU_{\rm R}(2)$ symmetry to be broken to the $SU_{\rm L+R}(2)$ symmetry which is the isospin symmetry in our basis. Therefore we assume $v \neq 0$, $v_3 = 0$ in (9.87): it is clear from (9.85) that the vacuum expectation value $v\mathbb{1}$ is invariant under the simultaneous $U_{\rm L} = U_{\rm R}$ transformation. Notice the difference with the σ-model with one complex doublet, or four real fields, where the $SU(2) \times SU(2) \to SU(2)$ is the only possible pattern of symmetry breaking; now the $\langle 0|\Phi|0\rangle$ is a 2×2 complex hermitean matrix. We can also write

$$\langle 0|\Phi_{ij}|0\rangle = \tfrac{1}{2}\langle 0|\bar{\Psi}\Psi|0\rangle\delta_{ij} \tag{9.87a}$$

where

$$\bar{\Psi}\Psi = \bar{\Psi}_1\Psi_1 + \bar{\Psi}_2\Psi_2$$

We are ready to prove Goldstone's theorem. The proof is similar to that in Section 9.4 and it is based on Ward identities for certain appropriate Green's functions. First we collect the useful equal-time commutation relations following from (9.59) and (9.85)

$$\left.\begin{aligned}
[j^a_{0\rm L}(\mathbf{x}, t), \Phi(\mathbf{y}, t)] &= \Phi(\mathbf{x}, t)T^a\delta(\mathbf{x} - \mathbf{y})\\
[j^a_{0\rm R}(\mathbf{x}, t), \Phi(\mathbf{y}, t)] &= -T^a\Phi(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y})\\
[j^a_{0\rm L}(\mathbf{x}, t), \Phi^\dagger(\mathbf{y}, t)] &= -T^a\Phi^\dagger(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y})\\
[j^a_{0\rm R}(\mathbf{x}, t), \Phi^\dagger(\mathbf{y}, t)] &= \Phi^\dagger(\mathbf{x}, t)T^a\delta(\mathbf{x} - \mathbf{y})
\end{aligned}\right\} \tag{9.88}$$

where the $j^a_{0\rm L}$ ($j^a_{0\rm R}$) are the zeroth components of the currents of the $SU_{\rm L}(2)$ ($SU_{\rm R}(2)$) transformations and the matrices T^a represent the generators of the $SU_{\rm L}(2)$ (when to the right of the operator Φ) and of the $SU_{\rm R}(2)$ (when to the left of the Φ) in the space of chiral doublets $\Psi_{\rm L}$ and $\Psi_{\rm R}$, respectively. The above commutation relations are valid both for the bare and the renormalized quantities because chiral symmetry is an exact symmetry of the lagrangian (see also the next chapter). From now on we always refer to renormalized quantities. For the proof of Goldstone's theorem we use a Ward identity for Green's functions involving composite operator $\Pi^a(x)$ defined as follows:

$$\Pi^a(x) = \mathrm{Tr}[\Pi(x) \cdot T^a]$$

where

$$\Pi(x) = \mathrm{i}(\Phi - \Phi^\dagger), \quad \Pi^\dagger = \Pi \tag{9.89}$$

and therefore

$$\Pi^a(x) = \mathrm{i}(\Phi_{ij} - \Phi^\dagger_{ij})T^a_{ji} = \mathrm{i}\bar{\Psi}T^a\gamma_5\Psi$$

($T^a = \frac{1}{2}\sigma^a$, σ^a are Pauli matrices). Using the commutation relations (9.88) we get the following result for the axial currents:

$$[\mathcal{A}^a_0(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = -\mathrm{i}\{T^a, \Phi(\mathbf{y}, t) + \Phi^\dagger(\mathbf{y}, t)\}\delta(\mathbf{x} - \mathbf{y}) \tag{9.90}$$

where the curly bracket on the r.h.s. means anticommutator. Using this last result we finally arrive at the desired Ward identity:

$$\partial^\mu_{(x)}\langle 0|T\mathcal{A}^a_\mu(x)\Pi^b(y)|0\rangle = -\mathrm{i}\delta(x-y)\langle 0|\operatorname{Tr}[\{T^a, \Phi(y)+\Phi^\dagger(y)\}T^b]|0\rangle \tag{9.91}$$

(it is unaltered by the chiral anomalies discussed in Chapter 13). Using $\operatorname{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$ and (9.87) one gets

$$\partial^\mu_{(x)}\langle 0|T\mathcal{A}^a_\mu(x)\Pi^b(y)|0\rangle = -2v\mathrm{i}\delta(x - y)\delta^{ab} \tag{9.92}$$

This result is analogous to (9.60) and it implies a pole at $p^2 = 0$ in the Fourier transform $\tilde{G}^{ab}_{\mu,\Pi}(p)$ of the Green's function $G^{ab}_{\mu,\Pi}(x - y)$

$$G^{ab}_{\mu,\Pi}(x - y) = \langle 0|T\mathcal{A}^a_\mu(x)\Pi^b(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\exp[-\mathrm{i}p(x - y)]\tilde{G}^{ab}_{\mu,\Pi}(p) \tag{9.93}$$

Following the steps (9.62)–(9.64) we get

$$\tilde{G}^{ab}_{\mu,\Pi}(p) = p_\mu\frac{2v}{p^2}\delta^{ab} = p_\mu\frac{\langle\bar{\Psi}\Psi\rangle}{p^2}\delta^{ab} \tag{9.94}$$

The pole at $p^2 = 0$ in the Green's function $\tilde{G}^{ab}_{\mu,\Pi}(p)$ implies the existence in the physical spectrum of the massless Goldstone boson with the quantum numbers of the operators Π^a. The proof goes as in Section 9.4. The only modification is that some care must be taken about the interpretation of the operator $\Pi(x)$. Indeed, consider the Green's function

$$G^{ab}(x - y) = \langle 0|T\Pi^a(x)\Pi^b(y)|0\rangle \tag{9.95}$$

A single particle state with appropriate quantum numbers gives a pole at $p^2 = m^2$ in the Fourier transform of (9.95)

$$\int \mathrm{d}^4x\, G^{ab}(x)\exp(\mathrm{i}px) \underset{p^2\to m^2}{\approx} \frac{\mathrm{i}Z^2}{p^2 - m^2}\delta^{ab} \tag{9.96}$$

where Z is a finite dimensionful constant (Π^a is a renormalized operator). It does not carry the group index because G is an exact symmetry of the lagrangian.

Introducing the physical, in the sense of Chapter 2, field $\pi^a(x)$:

$$\pi^a(z) = Z^{-1}\Pi^a(x) \tag{9.97}$$

we repeat the step (9.65)

$$\langle 0|\mathcal{A}^a_\mu(x)|\pi^b(p)\rangle = \mathrm{i} f_\pi \delta^{ab} p_\mu \exp(-\mathrm{i}px) \tag{9.98}$$

and the steps (9.66)–(9.68)

$$\langle 0|\mathcal{A}^a_\mu(x)|\pi^b(p)\rangle \underset{p^2\to m^2}{\approx} -\mathrm{i}\exp(-\mathrm{i}px)Z^{-1}\tilde{G}^{ab}_{\mu,\Pi}(p)(p^2 - m^2) \tag{9.99}$$

to get

$$\lim_{p^2\to m^2} \tilde{G}^{ab}_{\mu,\Pi}(p)(p^2 - m^2) = -Zf_\pi\delta^{ab}p_\mu \tag{9.100}$$

Since $\tilde{G}^{ab}_{\mu,\Pi}(p)$ has the pole (9.94) due to the spontaneous breakdown of chiral symmetry, (9.100) implies $m^2 = 0$ and

$$Zf_\pi = -\langle\bar{\Psi}\Psi\rangle \tag{9.101}$$

This completes the proof of Goldstone's theorem. In this case we have two low energy parameters: f_π and $\langle\bar{\Psi}\Psi\rangle$.

10

Spontaneous and explicit global symmetry breaking

So far we have been discussing theories described by lagrangians having some exact global symmetry G spontaneously broken by the vacuum. Now we would like to introduce the possibility that the symmetry G of the lagrangian is also broken explicitly by a small perturbation. Several interesting physical problems fall into this category. We begin with a systematic discussion of Ward identities between Green's functions involving global symmetry currents. This will also be useful as a supplement to our use of Ward identities in the earlier chapters and as an introduction to Chapter 13 devoted to the problem of anomalies.

10.1 Internal symmetries and Ward identities

Preliminaries

We are interested in theories described by lagrangians $\mathcal{L}$ such that

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

(or $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$), where $\mathcal{L}_0$ is invariant under global internal symmetry G and $\mathcal{L}_1$ explicitly breaks this symmetry. In addition the symmetry G may or may not be spontaneously broken. We assume that $\mathcal{L}_1$ can be treated as a perturbation whose effect can be calculated as a perturbative expansion in an appropriate small parameter. Furthermore, let the bare fields in the lagrangian $\mathcal{L}$ form a basis for some definite representation R of the symmetry group G of the lagrangian $\mathcal{L}_0$

$$\delta\Phi_i = -\mathrm{i}\Theta^a T^a_{ij}\Phi_j \equiv \Theta^a \delta^a \Phi_i \tag{10.1}$$

(Φ stands for boson or fermion fields).

The bare currents $j_\mu^a(x)$ constructed according to Noether's theorem

$$j_\mu^a(x) = -\mathrm{i}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)} T_{ij}^a \Phi_j \tag{10.2}$$

which are conserved currents of the G symmetric classical theory described by $\mathcal{L}_0$, remain unchanged when $\mathcal{L}_0 \to \mathcal{L}$ if $\mathcal{L}_1$ does not contain derivatives of the fields present in the $j_\mu^a(x)$; we assume this to be the case.

Under an infinitesimal global symmetry transformation

$$\Phi_i'(x) = \Phi_i(x) + \Theta^a \delta^a \Phi_i(x) \tag{10.3}$$

the lagrangian $\mathcal{L}$ transforms as follows:

$$\mathcal{L}'(x) = \mathcal{L}(x) + \Theta^a \delta^a \mathcal{L}(x) = \mathcal{L}(x) + \Theta^a \delta^a \mathcal{L}_1(x) \tag{10.4}$$

and, using the classical equations of motion,

$$\partial^\mu j_\mu^a(x) = \delta^a \mathcal{L}_1(x) \tag{10.5}$$

In the following we shall also need the variation of the same lagrangian $\mathcal{L}$ under local transformations

$$\Phi_i'(x) = \Phi_i(x) + \Theta^a(x) \delta^a \Phi_i(x) \tag{10.6}$$

Remembering that $\mathcal{L}_0$ is invariant under global transformations (10.3) and that $\mathcal{L}_1$ does not contain derivatives of the fields we get

$$\delta \mathcal{L}(x) = \Theta^a(x) \delta^a \mathcal{L}(x) + j_\mu^a(x) \partial^\mu \Theta^a(x) \tag{10.7}$$

We want to derive Ward identities for bare regularized or for renormalized Green's functions following from the symmetries of the lagrangian. We assume in this section that the UV regularization of the theory does preserve its classical symmetries. As we will see in Chapter 13 this is not true in theories with chiral fermions. In such theories some of the Ward identities derived here are afflicted with anomalies.

Let us first consider the case in which the symmetry G is an exact symmetry of the lagrangian, i.e. $\mathcal{L}_1 = 0$. If the regularization preserves this symmetry then the necessary counterterms are also G-invariant. Both bare regularized and renormalized Green's functions are G symmetric. The renormalized fields transform under the same representation R of G as the bare fields: the renormalization constants are G-invariant. The same discussion applies when the symmetry G is spontaneously broken since in this case also only symmetric counterterms are needed: this follows from (2.161). Whether G is spontaneously broken or not we are able to derive the same Ward identities.

With $\mathcal{L}_1$ present we consider the case in which symmetry breaking operators have dimension $d < 4$ ($\mathcal{L}_1$ then involves dimensionful constants). This is so-called soft explicit symmetry breaking. Then the counterterms required by $\mathcal{L}_1$, being of dimension lower than or equal to d (see the discussion in Section 4.2) do not affect counterterms of dimension 4. Thus, for such a 'soft' breaking the wave-function renormalization constants, included in counterterms of dimension 4, remain symmetric under G (strictly speaking there exists a renormalization scheme such that they remain symmetric) and the renormalized fields of the theory with softly broken symmetry G transform under G in the same way as the bare fields. Our general remarks can be illustrated with the minimal subtraction renormalization scheme used for QCD with the mass term $m\bar{\Psi}\Psi$; the wave-function renormalization constants are mass-independent and thus the same as in the chirally symmetric theory.

Ward identities from the path integral

Given all these preliminary constraints we can proceed to derive Ward identities using the functional integral formulation of quantum theory. Consider the generating functional $W[J]$, for example, for bare regularized Green's functions

$$W[J] = N \int \mathcal{D}\Phi_i \exp\left(\mathrm{i}\left\{\int \mathrm{d}^4x\, [\mathcal{L}(\Phi_i) + J^i\Phi_i]\right\}\right) \tag{10.8}$$

The integral is invariant under the change of integration variables defined by (10.6). If the integration measure is invariant,† using (10.7) we get

$$0 = \int \mathrm{d}^4x \int \mathcal{D}\Phi_i \exp\left(\mathrm{i}S + \mathrm{i}\int \mathrm{d}^4x\, J^i\Phi_i\right) \\ \times [\Theta^a(x)\delta^a\mathcal{L} + j^a_\mu(x)\partial^\mu\Theta^a(x) + J^i(x)\Theta^a(x)\delta^a\Phi_i(x)] \tag{10.9}$$

The crucial point in deriving (10.9) is the invariance of the integration measure under the transformation (10.6). It breaks down for chiral transformations in theories with chiral fermions (see Chapter 13). In other words, in this case there is no regularization of the path integral and no renormalization prescription for the Green's functions which preserve the full chiral symmetry and anomalies are present. We now define the Green's functions

$$\langle 0|TA(x)\prod_i \Phi_i(y_i)|0\rangle \sim \int \mathcal{D}\Phi_i A(x)\prod_i \Phi_i(y_i)\exp(\mathrm{i}S) \tag{10.10}$$

where $A(x)$ denotes one of the $j^a_\mu(x)$, $\delta^a\mathcal{L}(x)$ and $\delta^a\Phi_i(x)$. One should remember that the quantity on the l.h.s. of (10.10) is just a short-hand notation for its r.h.s.

† We consider cases in which $\det(\mathcal{D}\Phi'_i/\mathcal{D}\Phi_j) = 1 + O(\Theta^2)$. This is true when generators T^a in (10.1) are traceless. We also assume the absence of anomalies.

The notation has been chosen, however, to make easy contact with the operator language. Integrating (10.9) by parts (remember the remarks following (2.55)) differentiating it functionally with respect to the sources J_i and then setting them to zero we get the following general Ward identity:

$$(\partial/\partial x_\mu)\langle 0|T j^a_\mu(x)\prod_i \Phi_i(y_i)|0\rangle = \langle 0|T\delta^a\mathcal{L}(x)\prod_i \Phi_i(y_i)|0\rangle \\ -\mathrm{i}\sum_i \delta(x-y_i)\langle 0|T\delta^a\Phi_i(y_i)\prod_{j\neq i}\Phi_j(y_j)|0\rangle \tag{10.11}$$

Ward identity (10.11) is valid irrespective of whether the symmetry is spontaneously broken or not. If the symmetry G is an exact global symmetry of the lagrangian the first term on the r.h.s. of (10.11) vanishes. Under the constraint of explicit breaking being only soft the Ward identity is valid both for bare regularized and for the renormalized Green's functions.

We may also be interested in Ward identities for Green's functions involving several symmetry currents j^a_μ and/or the symmetry breaking operator $\mathcal{L}_1$. We take the latter to have well-defined transformation properties under the symmetry group. The most compact way to derive such Ward identities is based on the general idea (see Chapter 8) of using background fields to get, by construction, an action which is exactly invariant under certain symmetry transformations. In our case the background fields are the sources j^i, A^a_μ and K for the operators Φ^i, j^a_μ and $\mathcal{L}_1$, respectively. We want to introduce them in such a way that the action $S[\Phi, J, A, K]$ is invariant under the *local* transformation (10.6). This is achieved if

$$S[\Phi, J, A, K] = S_0[\Phi, A] + \int \mathrm{d}^4x\,(J^i\Phi_i + K\cdot\mathcal{L}_1) \tag{10.12}$$

where $S_0[\Phi]$ is the part of the action symmetric under global transformation (10.1) and the background fields A^a_μ have been introduced as gauge fields by changing the normal derivatives in S_0 into covariant derivatives $\partial_\mu \to \partial_\mu + \mathrm{i}A^a_\mu T^a$. Action (10.12) is invariant under (10.6) if simultaneous transformations on A, J and K defined as

$$\left.\begin{aligned} &\delta A^a_\mu = +\partial_\mu\Theta^a(x) + c^{abc}\Theta^b A^c \\ &\delta J^i\Phi_i = -J^i\delta\Phi_i, \quad \delta K\cdot\mathcal{L}_1 = -K\cdot\delta\mathcal{L}_1 \end{aligned}\right\} \tag{10.13}$$

The generating functional

$$W[A, J, K] = N\int \mathcal{D}\Phi_i \exp\{\mathrm{i}S[\Phi, J, A, K]\}$$

is then invariant under transformations (10.13) of the background fields

$$W[A, J, K] = W[A', J', K']$$

The last statement contains all the information equivalent to Ward identities which follow from the global symmetry of the original theory. For infinitesimal transformations the variation of the functional $W[A, J, K]$ reads

$$0 = \int \mathrm{d}^4 x \left[\frac{\delta W}{\delta J_i(x)} \delta J^i(x) + \frac{\delta W}{\delta A^a_\mu(x)} \delta A^a_\mu(x) + \frac{\delta W}{\delta K(x)} \delta K(x) \right] \tag{10.14}$$

or specifying j^a_μ, for instance, to flavour fermionic currents,

$$0 = \int \mathrm{d}^4 x \int \mathcal{D}\Phi_i \exp\{\mathrm{i}S[A, J, K, \Phi]\}(J^i \delta^a \Phi_i + K \cdot \delta^a \mathcal{L}_1 - \partial_\mu j^{\mu a} + c^{abc} A^b_\mu j^{\mu c})\Theta^a(x) \tag{10.15}$$

Eq. (10.15) generates the desired Ward identities valid to any order in $\mathcal{L}_1$ by differentiating with respect to sources J^i, K and A^a_μ and setting $J^i = A^a_\mu = 0$, $K = 1$. The method can be generalized to generate Ward identities involving other composite operators with well-defined transformation properties under the global symmetry group if we introduce sources for such operators.

We stress again that the validity of the symmetry relations for the Green's functions relies on the existence of the regularization and renormalization procedures which preserve this symmetry. In particular renormalization of Green's functions involving products of composite operators, in addition to multiplicative renormalization, may also require subtractions in the form of polynomials in momenta in momentum space or of δ-functions and their derivatives in position space.

Finally we observe that in the special cases of no massless Goldstone bosons coupled to the current j^α_μ, Ward identities like (10.11) can be integrated over space-time and the surface terms neglected. Thus in the case of exact symmetry which is not spontaneously broken one gets

$$\delta^a \langle 0|T \prod_i \Phi_i(y_i)|0\rangle = 0 \tag{10.16}$$

which simply means an exact invariance of the Green's functions under the symmetry transformations. When the symmetry is explicitly broken by perturbation $\mathcal{L}_1$ we have

$$\begin{aligned} \delta^a \langle 0|T \prod_i \Phi_i(y_i)|0\rangle &= -\mathrm{i} \int \mathrm{d}^4 x \, \langle 0|T \delta^a \mathcal{L} \prod_i \Phi_i(y_i)|0\rangle \\ &\equiv -\mathrm{i}\langle 0|T \delta^a S \prod_i \Phi_i(y_i)|0\rangle \end{aligned} \tag{10.17}$$

Note that in the latter case the symmetry may also be spontaneously broken since in the presence of perturbation $\mathcal{L}_1$ the massless Goldstone bosons acquire masses. This can, for instance, already be seen at the level of the tree approximation, i.e. by analysing the classical lagrangian.

Comparison with the operator language

The equal-time bare current commutation relations following from the equal-time canonical commutation relations for the bare field operators are

$$[j_0^a(\mathbf{x}, t), j_0^b(\mathbf{y}, t)] = \mathrm{i}c^{abc} j_0^c(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{y}) \tag{10.18}$$

Relations (10.18) are true in the symmetric theory and also with the symmetry breaking term $\mathcal{L}_1$ included as long as such terms do not contain derivatives of the fields. However, in the presence of the symmetry breaking term $\mathcal{L}_1$ the currents are no longer conserved and the corresponding charges are time-dependent. Defining

$$Q^a(t) = \int \mathrm{d}^3x\, j_0^a(\mathbf{x}, t)$$

we get from (10.18)

$$[Q^a(t), Q^b(t)] = \mathrm{i}c^{abc} Q^c(t) \tag{10.19}$$

In accordance with the transformation properties (10.1) we have furthermore

$$[Q^a(t), \Phi_i(\mathbf{x}, t)] = -T_{ij}^a \Phi_j(\mathbf{x}, t) \tag{10.20}$$

or

$$[j_0^a(\mathbf{x}, t), \Phi_i(\mathbf{y}, t)] = -T_{ij}^a \Phi_j(\mathbf{y}, t)\delta(\mathbf{x} - \mathbf{y}) \tag{10.21}$$

If the symmetry breaking is soft (10.18)–(10.21) hold for the renormalized quantities as well because wave-function renormalization constants can remain invariant (for properly chosen renormalization schemes) and as will be seen shortly the currents are not subject to renormalization. The time dependence of the charges is given by

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = \int \mathrm{d}^3x \frac{\mathrm{d}}{\mathrm{d}t} j_0^a(\mathbf{x}, t) = \mathrm{i} \int \mathrm{d}^3x\, [\mathcal{H}'(\mathbf{x}, t), Q^a(t)] \tag{10.22}$$

where $\mathcal{H}'$ is the symmetry breaking part of the hamiltonian, and therefore

$$\partial^\mu j_\mu^a(\mathbf{x}, t) = \mathrm{i}[\mathcal{H}'(\mathbf{x}, t), Q^a(t)] \tag{10.23}$$

(the surface terms at infinity can be neglected because due to perturbation $\mathcal{L}_1$ the would-be Goldstone bosons acquire masses).

The Ward identity (10.11) is the same as one derived for the time-ordered product of operators defined as

$$Tj_\mu^a(x)\Phi(y) = j_\mu^a(x)\Phi(y)\Theta(x^0 - y^0) + \Phi(y)j_\mu^a(x)\Theta(y^0 - x^0) \qquad (10.24)$$

(with a straightforward generalization to products of more than two operators) by direct differentiation of the Green's function and use of the commutation relations (10.21). For instance

$$(\partial/\partial x_\mu)\langle 0|Tj_\mu^a(x)\Phi(y)|0\rangle = \langle 0|T\partial^\mu j_\mu^a(x)\Phi(y)|0\rangle + \delta(x^0 - y^0)\langle 0|[j_0^a(x), \Phi(y)]|0\rangle \qquad (10.25)$$

and using (10.21) together with (10.5) for the operator $\partial^\mu j_\mu^a$ we recover the Ward identity (10.11). Similarly, one can derive Ward identities for Green's functions involving products of currents and they are the same as the relations following from (10.13) or (10.15) if we neglect in the current commutators the presence of the derivatives of the δ-functions (the so called Schwinger terms). However, in many cases these terms certainly do not vanish (see Jackiw (1972) for a systematic discussion) and one may wonder about the compatibility of the two methods. The answer is that the Green's functions defined by the path integral do not always coincide with the vacuum expectation values of the T-products defined by (10.24). The former are always understood to be regularized or renormalized covariant quantities, whereas the latter are not and they may differ by local terms, the so-called seagull terms, which cancel the Schwinger terms. The path integral approach tells us that it is always possible to arrange such cancellation by a suitable renormalization prescription if there exists one which preserves the symmetry of the classical lagrangian. Otherwise anomalies appear. We shall come back to these problems in some detail in the following.

Ward identities and short-distance singularities of the operator products

Consider, for instance, the Ward identity (10.25) with $\Phi(x)$ replaced by any operator $O(x)$. This relation is obtained by differentiating the T-product defined by (10.24). However, generally speaking, the meaning of time-ordering as given by (10.24) is clear everywhere except at coinciding points, i.e. at $x_0 = y_0$. Therefore there is an apparent ambiguity

$$TA(x)B(0) \to TA(x)B(0) + c_1\delta(x) + c_2\partial_x\delta(x) + \cdots \qquad (10.26)$$

(Lorentz and other symmetry group indices must, of course, be properly taken into account). The question thus arises of the role of such terms which in momentum space are polynomials in momentum.

To discuss this point we recall first that equations like (10.25) are for distributions, i.e. they hold in the sense that both sides are integrated over $\int d^4x\, f(x)$ with $f(x)$ being any 'smooth' test function vanishing at $x_\mu \to \pm\infty$. In particular, the derivative $\partial_\mu T(x)$ of the distribution $T(x)$ is defined as follows:

$$\int d^4x\, f(x)\partial_\mu T(x) \underset{\text{df}}{\equiv} \int d^4x\, [-\partial_\mu f(x)]T(x) \tag{10.27}$$

We also define the Fourier transform of a distribution

$$(2\pi)^{-4}\int d^4q\, \tilde{f}(-q)\tilde{T}(q) = \int d^4x\, f(x)T(x) \tag{10.28}$$

where $\tilde{f}(q)$ is the Fourier transform of $f(x)$ and use the notation

$$\tilde{T}(q) = \int d^4x \exp(iqx)T(x)$$

The Fourier transform $(\widetilde{\partial_\mu T})$ of $\partial_\mu T(x)$ is

$$\begin{aligned}(2\pi)^{-4}\int d^4q\, \tilde{f}(-q)(\widetilde{\partial_\mu T}) &= \int d^4x\, f(x)\partial_\mu T(x)\\ &= -\int d^4x\, \partial_\mu f(x)T(x)\end{aligned} \tag{10.29}$$

i.e.

$$(\widetilde{\partial_\mu T}) = -iq_\mu \tilde{T}(q) \tag{10.30}$$

In the other notation

$$\int d^4x \exp(iqx)\partial_\mu T(x) = -iq_\mu \int d^4x \exp(iqx)T(x) \tag{10.31}$$

We expect the meaningfulness of the Ward identity (10.25) to be related to the short-distance singularity structure of the distributions on both sides of (10.25).

To see this clearly, let us rederive (10.25) paying explicit attention to the singularities at $x \sim 0$. Using the definition (10.27) the l.h.s. of (10.25) can be written as

$$-\lim_{\varepsilon\to 0}\int d^4x\, \partial_\mu f(x)\langle 0|Tj_\mu(x)O(0)|0\rangle\Theta(|x_0| - \varepsilon) \tag{10.32}$$

Performing Wick's rotation we can work in the Euclidean space, so that the light-cone singularities of the operator product collapse to the origin. Then the integrand in (10.32) is finite and unambiguous for $\varepsilon > 0$. Integrating by parts we get

$$
\begin{aligned}
-\lim_{\varepsilon\to 0}&\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\mathrm{d}x_0\int \mathrm{d}^3x\,\partial_\mu f(x)\langle 0|Tj^\mu O|0\rangle\\
&= -\lim_{\varepsilon\to 0}\int \mathrm{d}^3x\, f(x)\langle 0|O(0)j_0(\mathbf{x},-\varepsilon) - j_0(\mathbf{x},\varepsilon)O(0)|0\rangle\\
&\quad + \lim_{\varepsilon\to 0}\int \mathrm{d}^4x\, f(x)\langle 0|T\partial^\mu j_\mu(x)O(0)|0\rangle\Theta(|x_0|-\varepsilon)
\end{aligned}
\tag{10.33}
$$

In the last term the operator ∂_μ has been interchanged with T because, after excluding $x = 0$, $Tj_\mu(x)O(0)$ is regular. We have also assumed the absence of massless particles; then the surface terms at $\pm\infty$ do not contribute.

Similarly, we may consider the Fourier transform

$$
\begin{aligned}
\int &\mathrm{d}^4x\exp(\mathrm{i}px)\partial^\mu\langle 0|Tj_\mu(x)O(0)|0\rangle\\
&= \lim_{\varepsilon\to 0}\int \mathrm{d}^4x\,[-\partial^\mu\exp(\mathrm{i}px)]\langle 0|Tj_\mu O|0\rangle\Theta(|x_0|-\varepsilon)\\
&= -\lim_{\varepsilon\to 0}\int \mathrm{d}^4x\exp(\mathrm{i}px)\langle 0|Tj_0(x)O(0)|0\rangle[\delta(x_0+\varepsilon)-\delta(x_0-\varepsilon)]\\
&\quad + \lim_{\varepsilon\to 0}\int \mathrm{d}^4x\exp(\mathrm{i}px)\langle 0|T\partial_\mu j^\mu(x)O(0)|0\rangle\Theta(|x_0|-\varepsilon)
\end{aligned}
\tag{10.34}
$$

If the limit $\varepsilon \to 0$ exists, we have

$$
-\mathrm{i}p_\mu\tilde{T}(p) = \Delta(p) + \int \mathrm{d}^3x\exp(\mathrm{i}\mathbf{px})\langle 0|[j_0(\mathbf{x},0), O(0)]|0\rangle \tag{10.35}
$$

where $\Delta(p)$ is the Fourier transform of $\langle 0|T\partial^\mu j_\mu(x)O(0)|0\rangle$.

Let us discuss the limit $\varepsilon \to 0$ in (10.33) and (10.34) using the OPE. One can immediately convince oneself that the limit exists if

$$
Tj_\mu(x)O(0) \sim C^1_\mu(x)O_1(0) + \text{less singular terms} \tag{10.36}
$$

where $C^1_\mu(x) \sim O(1/x^3)$, and similarly for $\partial^\mu j_\mu(x)$. Indeed, with $O(1/x^3)$ singularity the integrals in (10.33) and (10.34) are convergent. From (10.36) we also get (see Section 7.4)

$$
[j_\mu(\mathbf{x},0), O(0)] = a^1 O_1(\mathbf{x},0)\delta(\mathbf{x}) \tag{10.37}
$$

(no Schwinger terms, a^1 finite). Thus, with no additional subtractions the Ward identity (10.33) or (10.35) with the equal-time commutator given by (10.37) is a well-defined relation.

Often $Tj_\mu(x)O(0)$ is more singular than $O(1/x^3)$ (for example $Tj^{\text{em}}_\mu(x)j^{\text{em}}_\nu(0) \sim (1/x^6)\cdot \mathbb{1}$). Stronger than $O(1/x^3)$ singularity in the OPE of the T-product of operators translates into terms additional to the $\delta(\mathbf{x})$ term with a finite coefficient in their equal-time commutators (see Section 7.4). These are the so-called Schwinger terms. However, at the same time subtractions at $x = 0$ are necessary to have a well-defined limit when $\varepsilon \to 0$ in the integrals in (10.33) and (10.34). Depending on the nature of the singularity, without subtractions these integrals may depend on the order of integration over x_0 and $\mathbf{x}$ (and be then non-covariant) or may be divergent. Subtractions are of the form

$$c_1\delta(x) + c_2\partial_{(x)}\delta(x) + \cdots$$

possibly with divergent c_i. For instance, let

$$TA(x)B(0) \sim \left(\frac{1}{x^2 - \mathrm{i}\varepsilon}\right)^2 \mathbb{1} + \cdots$$

Performing Wick's rotation and integrating in Euclidean space we conclude that for a well-defined $\varepsilon \to 0$ limit in the l.h.s. integral in (10.33) a subtraction of the form

$$\mathrm{i}\pi^2 \ln \varepsilon\delta(0) + c\delta(0)$$

is necessary. The finite constant c is to be fixed by some renormalization conditions. This subtractive renormalization of the Green's functions comes in addition to the multiplicative renormalization of operators. The path integral approach teaches us that when regularization and renormalization schemes exist which preserve the classical symmetry of the lagrangian it is possible to define the renormalized T-products so that they satisfy Ward identities in a form like (10.11), i.e. as if the Schwinger terms were absent. In other words by suitable renormalization the Schwinger terms can be cancelled by subtraction terms. Our last remark is that in applications of Ward identities to physical problems one should pay attention to the fact that part of their content may be subtraction dependent.

Renormalization of currents

As the first application of (10.11) we discuss the renormalization properties of currents and their divergences. Let us consider, for instance, the following specific case of (10.11):

$$\begin{aligned}\partial^\mu_{(x)}\langle 0|Tj^a_\mu(x)\Psi^b(y)\bar{\Psi}^c(z)|0\rangle = {}& \langle 0|T\partial^\mu j^a_\mu(x)\Psi^b(y)\bar{\Psi}^c(z)|0\rangle \\ & - \delta(x-y)T^a_{bd}\langle 0|T\Psi^d(y)\bar{\Psi}^c(z)|0\rangle \\ & + \delta(x-z)T^a_{dc}\langle 0|T\Psi^b(y)\bar{\Psi}^d(z)|0\rangle \qquad (10.38)\end{aligned}$$

We take (10.38) to be for bare quantities. In the symmetry limit

$$\partial^\mu j_\mu^a = 0 \tag{10.39}$$

$$\Psi_{\mathrm{B}}^a = Z_2^{1/2}\Psi_{\mathrm{R}}^a \tag{10.40}$$

where Z_2 is invariant under symmetry transformations. Expressing the bare fermion fields in (10.38) by the renormalized fields and dividing both sides by Z_2 we immediately conclude from the finiteness of the r.h.s. of (10.38) that

$$j_{\mu\mathrm{B}}^a(x) = j_{\mu\mathrm{R}}^a(x)$$

up to a finite multiplicative constant which is fixed as one by the current commutation relations. Thus the conserved currents are not subject to renormalization, for example,

$$j_{\mu\mathrm{B}}^a = \bar{\Psi}_{\mathrm{B}} T^a \gamma_\mu \Psi_{\mathrm{B}} = Z_2 \bar{\Psi}_{\mathrm{R}} T^a \gamma_\mu \Psi_{\mathrm{R}} = j_{\mu\mathrm{R}}^a \tag{10.41}$$

In the case of explicit symmetry breaking by soft ($d < 4$) terms one has $\partial^\mu j_\mu(x) \neq 0$ but, as discussed before, (10.40) can still be assumed. Writing in general $j_{\mu\mathrm{B}}^a = Z_j j_{\mu\mathrm{R}}^a$ and $\partial^\mu j_{\mu\mathrm{B}}^a = Z_{\mathrm{D}}\partial^\mu j_{\mu\mathrm{R}}^a$ we first conclude, arguing as before, that $Z_j = Z_{\mathrm{D}}$. Next, Fourier transforming (10.38) and taking the limit $p = 0$ (p is conjugate to x) we get $Z_{\mathrm{D}} = 1$ since in the limit $p = 0$ the Fourier transform of the l.h.s. vanishes (explicit symmetry breaking accompanies the spontaneous symmetry breaking and therefore there are no massless bosons in the spectrum coupled to j_μ^a currents). In conclusion, even if the symmetry is broken explicitly but only softly the currents and their divergences are not subject to renormalization. For instance, for chiral symmetry broken by a mass term

$$\mathcal{L}_1 = -m\bar{\Psi}\Psi \qquad (d = 3) \tag{10.42}$$

for the divergence of the axial currents $j_{5\mu}^a$ we have (from the equations of motion)

$$\partial^\mu j_{5\mu}^a = 2m\mathrm{i}\bar{\Psi}T^a\gamma_5\Psi \tag{10.43}$$

and the non-renormalization theorem implies

$$Z_m = Z_{\bar{\Psi}\gamma_5\Psi}^{-1} \tag{10.44}$$

where

$$(\bar{\Psi}\gamma_5 T^a \Psi)_{\mathrm{B}} = Z_{\bar{\Psi}\gamma_5\Psi}(\bar{\Psi}\gamma_5 T^a\Psi)_{\mathrm{R}} \tag{10.45}$$

Absence of wave-function renormalization for conserved or partially conserved currents does not, of course, exclude subtractive renormalizations for time-ordered products of such currents, as discussed earlier.

10.2 Quark masses and chiral perturbation theory

Simple approach

Using the Ward identity for the Green's function involving the axial current $\mathcal{A}_\mu^a(x)$ and the pseudoscalar density $\Pi^a(x)$ we have proved in Section 9.5 that in chirally symmetric QCD with chiral symmetry spontaneously broken there exist massless Goldstone particles in the physical spectrum. The same Ward identity gave us the relation (9.101) between the parameters f_π and $\langle\bar{\Psi}\Psi\rangle$ of spontaneous breaking of chiral symmetry

$$Zf_\pi = -\langle\bar{\Psi}\Psi\rangle \tag{9.101}$$

where the constants f_π and Z are defined by (9.97) and (9.98)

$$\langle 0|\Pi^a(x)|\pi^b(p)\rangle = Z\delta^{ab}\exp(-\mathrm{i}px)$$

and

$$\langle 0|\mathcal{A}_\mu^a(x)|\pi^b(p)\rangle = \mathrm{i}f_\pi\delta^{ab}p_\mu\exp(-\mathrm{i}px)$$

respectively. (In view of our discussion in the previous section check, using the OPE for $T\mathcal{A}_\mu^a(x)\Pi^b(y)$, that the Ward identity (9.91) does not need subtractions and therefore in the chiral limit the relation (9.101) is free of any subtraction-dependent parameters.)

Imagine now a realistic case in which the chiral symmetry is also broken explicitly by the mass term (10.42). Then the proof of Goldstone's theorem breaks down because

$$\partial^\mu\mathcal{A}_\mu^a(x) = 2m\mathrm{i}\bar{\Psi}T^a\gamma_5\Psi \neq 0 \tag{10.43a}$$

Instead, one can now calculate masses of the would-be (pseudo-) Goldstone bosons in terms of the explicit symmetry breaking parameter: quark mass m. Such calculation is based on perturbative treatment of the symmetry breaking interaction (chiral perturbation theory) but takes account of all orders of the interactions described by the chirally symmetric part $\mathcal{L}_0$ of the lagrangian.

It is worth mentioning at this point that the subject of chiral perturbation theory is not limited to the problem of the pseudoscalar meson masses but covers the whole low energy physics of pions and kaons. Several techniques have been developed to expand the bound state masses, Green's functions and any other quantity in the low energy pseudoscalar meson sector (which in the limit of exact chiral symmetry depend on only two parameters f_π and $\langle\bar{\Psi}\Psi\rangle$) in powers of the light quark masses m_u, m_d and m_s (Dashen 1969, Dashen & Weinstein 1969, Pagels 1975). The most systematic approach proposed in a very interesting series of papers (Leutwyler 1984, Gasser & Leutwyler 1984, 1985*a*, *b*) relies on the effective lagrangian technique (see Chapter 14).

Here we shall only be concerned with the pseudoscalar meson masses calculated in the first order in the quark mass m. The answer is almost immediate. Eq. (10.43a) sandwiched between the vacuum and the one-pion state together with the definitions (9.97) and (9.98) imply the exact relation

$$f_\pi m_\pi^2 = 2mZ \tag{10.46}$$

Working to the first order in m we can now use (9.101), valid in the chiral limit, to get

$$m_\pi^2 = -(1/f_\pi^2)2m\langle 0|\bar{\Psi}\Psi|0\rangle \tag{10.47}$$

which is the desired result valid to the lowest order in chiral perturbation theory. It is the pion mass, for example, in $SU(2) \times SU(2)$ symmetric QCD with $\Psi = (u, d)$ and $m_u = m_d = m$. The pion mass has been expressed in terms of the parameters of chiral symmetry breaking: spontaneous $\langle 0|\bar{\Psi}\Psi|0\rangle$ and explicit m.

Our discussion is always in terms of renormalized quantities. We recall that interactions described by the chirally invariant part $\mathcal{L}_0$ of the lagrangian are included to all orders in our discussion. Therefore $m = m(\mu)$, where μ is some renormalization point, and operators $\bar{\Psi}\gamma_5 T^a\Psi$ and $\bar{\Psi}\Psi$ are also renormalized at the same point. However, as discussed in Section 10.1 the products $m\bar{\Psi}\gamma_5 T^a\Psi$, $m\bar{\Psi}\Psi$ are renormalization-scheme-independent.

The result (10.47) can be extended to the case of three flavours with $m_u \neq m_d \neq m_s$ (for heavy flavours the breaking of chiral symmetry by the mass terms is too strong to be reliably considered as a perturbation). We get the following:

$$\left.\begin{aligned}
&m_{\pi^0}^2 = m_{\pi^\pm}^2 = (m_u + m_d)C + O(m_q^2)\\
&m_{K^+}^2 = (m_u + m_s)C + \cdots\\
&m_{K^0}^2 = m_{\bar{K}^0}^2 = (m_d + m_s)C + \cdots\\
&C = -(2/f_\pi^2)\langle 0|\bar{u}u|0\rangle,\ \langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle
\end{aligned}\right\} \tag{10.48}$$

To extract reliable values for the quark mass ratios we need, however, to include the electromagnetic mass corrections which will be considered in Section 10.4.

Finally we would like to stress that the quark masses we talk about are the renormalized quark mass parameters of the lagrangian (the so-called current quark masses). For a mass-independent renormalization procedure ratios of the renormalized masses do not depend on μ.

Approach based on use of the Ward identity

The result (10.47) can also be derived using a Ward identity directly for the realistic case with the chiral symmetry broken explicitly. This approach can be used in other problems too and its generalizations are used to calculate higher order corrections in the chiral perturbation theory.

The starting point is the Ward identity for the Green's function $\langle 0|T\mathcal{A}_\mu^a(x)\partial^\nu\mathcal{A}_\nu^b(0)|0\rangle$, where $\mathcal{A}_\mu^a(x)$ is the axial current and $\partial^\nu\mathcal{A}_\nu^b$ stands for $2m\mathrm{i}\bar{\Psi}\gamma_5 T^b\Psi$; similar considerations hold for any other current corresponding to a generator broken both spontaneously and explicitly when according to (10.23)

$$\partial^\mu j_\mu^a(\mathbf{x}, t) = \mathrm{i}[\mathcal{H}'(\mathbf{x}, t), Q^a(t)] \tag{10.23}$$

The Ward identity reads

$$\partial^\mu_{(x)}\langle 0|T\mathcal{A}_\mu^a(x)\partial^\nu\mathcal{A}_\nu^b(0)|0\rangle = D_5^{ab}(x) + \delta(x_0)\langle 0|[\mathcal{A}_0^a(x), \partial^\nu\mathcal{A}_\nu^b(0)]|0\rangle \tag{10.49}$$

where

$$D_5^{ab}(x) = \langle 0|T\partial^\mu\mathcal{A}_\mu^a(x)\partial^\nu\mathcal{A}_\nu^b(0)|0\rangle$$

The divergence $\partial^\mu\mathcal{A}_\mu$ is soft: the identity (10.49) can then be regarded as an identity for the renormalized Green's functions.

Defining the Fourier transform $\Delta_5^{ab}(q)$ of the $D_5^{ab}(x)$

$$\Delta_5^{ab}(q) = \mathrm{i}\int \mathrm{d}^4x\, \exp(\mathrm{i}qx) D_5^{ab}(x) \tag{10.50}$$

Fourier transforming both sides of (10.49) and taking the limit $q \to 0$ one gets the following relation:

$$0 = \Delta_5^{ab}(q=0) + \mathrm{i}\langle 0|[Q_5^a(t=0), \partial^\nu\mathcal{A}_\nu^b(0)]|0\rangle \tag{10.51}$$

which in a general case, using (10.23), can also be written as

$$\Delta_5^{ab}(q=0) = -\langle 0|[Q_5^a(t=0), [Q_5^b(t=0), \mathcal{H}'(0)]]|0\rangle \tag{10.52}$$

For us $\mathcal{H}' = m\bar{\Psi}\Psi$. In the limit $q \to 0$ the l.h.s. of (10.49) is zero because in the present case (explicitly broken symmetry) there are no massless bosons in the physical spectrum and therefore no poles at $q^2 = 0$. The relation (10.52) can be used to calculate the would-be Goldstone boson (pion) masses in the first order in chiral perturbation theory if we approximate the Fourier transform $\Delta_5^{ab}(q)$ by the pion pole. In general $\Delta_5^{ab}(q)$ is given by the pole and the continuum contribution starting at $s = 9m_\pi^2$

$$\Delta_5^{ab}(q^2) = \frac{f_\pi^2 m_\pi^4 \delta^{ab}}{m_\pi^2 - q^2} + \int_{(3m_\pi)^2}^\infty \mathrm{d}s\, \frac{\rho(s, m_\pi^2)}{s - q^2} \tag{10.53}$$

We have used $\langle 0|\partial^\mu\mathcal{A}_\mu^a(0)|\pi^b\rangle = f_\pi m_\pi^2\delta^{ab}$ and have written (10.50) as the spectral integral using the standard technique (see, for example, Bjorken & Drell (1965)). For $q^2 = 0$ one gets

$$\Delta_5^{ab}(q^2 = 0) = f_\pi^2 m_\pi^2 \delta^{ab} + O(m_\pi^4) \tag{10.54}$$

The leading term in m_π^2 is given by the pion pole because this is the only term where in the chiral symmetry limit the zero of the numerator in (10.53) partially cancels with the denominator. The pion mass measures the departure from chiral symmetry induced by perturbation $\mathcal{H}'(x)$, so expansion in m_π^2 is the chiral perturbation theory. In the order $O(m_\pi^2)$ we get the so-called Dashen's relation

$$\delta^{ab} m_\pi^2 = -\frac{1}{f_\pi^2} \langle 0 | [Q_5^a(t=0), [Q_5^b(t=0), \mathcal{H}'(0)]] | 0 \rangle \tag{10.55}$$

where f_π and $|0\rangle$ should be taken, for consistency, as in the chiral limit. In our specific problem, from commutation relations (9.90), we easily get (10.47).

It is also worth observing that the Green's function $D_5^{ab}(x)$ or equivalently the integral in the spectral representation (10.53) does require subtractions. For instance, it can be seen from the OPE (see Chapter 7) that the product $T \partial^\mu \mathcal{A}_\mu^a(x) \partial^\nu \mathcal{A}_\nu^b(0)$ has a singularity $O(m^2 x^{-6})$. The subtraction ambiguity is $O(m^2)$ or according to our discussion following (10.54) it is $O(m_\pi^4)$ and does not affect the first order results. However, in higher orders the subtraction constants appear as additional free parameters.

10.3 Dashen's theorems

Formulation of Dashen's theorems

We have learned in Section 9.4 that if a global symmetry G of a theory is spontaneously broken by the vacuum state $|0\rangle$ to a subgroup H then there is a set of degenerate vacua corresponding to different orientations of H in G. This set of vacua is generated by the action of the G transformations on the state $|0\rangle$

$$|\Omega(g)\rangle = U(g)|0\rangle \tag{10.56}$$

where $U(g)$ is a unitary operator corresponding to a group element g. If the vacuum $|0\rangle$ is left invariant by H then $|\Omega(g)\rangle$ is left invariant by the equivalent G-rotated subgroup gHg^{-1}. The set of degenerate vacua is isomorphic to the coset G/H space. If we introduce generators X^a and Y^a which are, respectively, broken and unbroken by the $|0\rangle$ then the transformation (10.56) can be written as

$$|\Omega(\Theta)\rangle = \exp(-\mathrm{i}\Theta^a X^a)|0\rangle \tag{10.57}$$

Transformations in G/H space correspond to variations of the vacuum expectation values for which the effective action is level at its minimum. Zero mass particles are quantized excitations – one for each orthogonal direction in G/H (see (9.51) and Chapter 14).

A small symmetry breaking perturbation $\mathcal{H}'(x)$ may change the above picture in a qualitative way: it may lift the degeneracy of the vacuum. The criteria for identifying the correct vacuum in the presence of a small perturbation $\mathcal{H}'(x)$ are called Dashen's theorems, and generally speaking, ensure the vacuum stability. To the leading order in the perturbation $\mathcal{H}'(x)$ the energy density in each of the degenerate vacua is given by

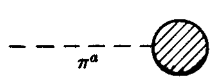
$$\Delta E(\Theta) = \langle\Omega(\Theta)|\mathcal{H}'(0)|\Omega(\Theta)\rangle = \langle 0|\exp(\mathrm{i}\Theta^a X^a)\mathcal{H}'(0)\exp(-\mathrm{i}\Theta^a X^a)|0\rangle \tag{10.58}$$

To identify $|0\rangle$ as the true vacuum in the presence of the perturbation $\mathcal{H}'(x)$, $\Delta E(\Theta)$ should have a minimum (at least a local one) for $\Theta = 0$. We therefore get the following two conditions:

$$\frac{\partial}{\partial\Theta^a}\Delta E(\Theta)|_{\Theta=0} = \mathrm{i}\langle 0|[X^a, \mathcal{H}'(0)]|0\rangle = 0 \tag{10.59}$$

$$\frac{\partial^2}{\partial\Theta^a\partial\Theta^b}\Delta E(\Theta)|_{\Theta=0} = -\langle 0|[X^b, [X^a, \mathcal{H}'(0)]]|0\rangle \geq 0 \tag{10.60}$$

In view of the relation $\partial^\mu j^a_\mu(x) = \mathrm{i}[\mathcal{H}'(x), Q^a]$, condition (10.59) also follows from the requirement of the translational invariance of the vacuum: the vacuum value of a divergence then vanishes. Since charges X^a which do not annihilate the vacuum couple to Goldstone bosons $\langle 0|X^a|\pi^a\rangle \neq 0$, (10.59) says that $\langle\pi^a|\mathcal{H}'|0\rangle = 0$, i.e. the Goldstone boson tadpoles



vanish. Eq. (10.60) has a straightforward interpretation too: up to the renormalization factor the second derivative of the potential with respect to Θ is the Goldstone boson mass matrix (see also Chapter 14) and (10.60) ensures that in the first order perturbation in $\mathcal{H}'(x)$ the pseudo-Goldstone boson masses m^2_{ab} are positive. So we rederive Dashen's formula (10.55) referring, however, to the previous derivation for the fixing of the normalization factor.

Let us first see what the conditions (10.59) and (10.60) give for the σ-model with chiral symmetry explicitly broken by the $\mathcal{H}'(x) = +\varepsilon\sigma'$ term. From the commutation relations (9.30) we get

$$\langle 0|\pi|0\rangle = 0 \qquad -\varepsilon\langle 0|\sigma'|0\rangle \geq 0 \tag{10.61}$$

for the correct vacuum, in agreement with (9.36). Using the normalization given in (10.55) we also rederive the result (9.39) for the pseudo-Goldstone boson mass.

For the $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2)$ symmetric QCD lagrangian with chiral symmetry broken spontaneously to $SU(2)$ by a condensate $U_{\mathrm{R}}\langle\Phi\rangle U_{\mathrm{L}}^{+}$, where $\langle\Phi\rangle = v\mathbb{1}$ as in (9.87) and explicitly by the mass term $\mathcal{H}' = m\bar{\Psi}\Psi$ one gets

$$\langle 0|\bar{\Psi}\gamma_5\Psi|0\rangle = 0 \qquad -\langle 0|\bar{\Psi}\Psi|0\rangle > 0 \tag{10.62}$$

This follows from the commutation relation (9.88) and from the relations $\bar{\Psi}\Psi = \mathrm{Tr}[\Phi + \Phi^{\dagger}]$ and $\bar{\Psi}\gamma_5\Psi = \mathrm{Tr}[\Phi - \Phi^{\dagger}]$, where Φ is defined by (9.84). In particular we recover (9.86) which we have taken as the definitions of the vacuum in the chiral limit and in a specified basis for the chiral fields. Explicit symmetry breaking lifts the degeneracy and leaves (10.61) and (10.62) as the only possibilities.

Dashen's conditions find very interesting applications in models of the dynamical breaking of the gauge symmetry of weak interactions (technicolour and extended technicolour models). Some of these ideas will be discussed in Section 11.3. Here we give a general technical introduction to such problems which will also be useful in the next section where we calculate the $\pi^{+} - \pi^{0}$ mass difference.

Dashen's conditions and global symmetry broken by weak gauge interactions

A physically very interesting case of an explicit global symmetry breaking by a small perturbation is the following one: consider a strongly interacting theory (a gauge theory) with some exact global symmetry G, for example, the chiral symmetry which is spontaneously broken. Imagine then that some of the G-symmetry currents couple to some of the gauge bosons of the group G_{W} of weak interactions (in the rest of this section all sums over the group indices are explicitly written down)

$$\mathcal{L}_1 = -\sum_{\alpha} g^{\alpha} A_{\mu}^{\alpha}(x) j_{\alpha}^{\mu}(x) \tag{10.63}$$

where $j_{\alpha}^{\mu}(x)$ are linear combinations of some of the G currents; operators corresponding to quantum numbers commuting with G are also possible but irrelevant for our discussion. In the first non-vanishing order of perturbation theory in g^{α} the G-symmetry breaking term of such a theory is given by the following effective hamiltonian:

$$\mathcal{H}'(0) = -\sum_{\alpha,\beta} \tfrac{1}{2} g^{\alpha} g^{\beta} \int \mathrm{d}^4x\, \Delta_{\alpha\beta}^{\mu\nu}(x) T j_{\mu}^{\alpha}(x) j_{\nu}^{\beta}(0) \tag{10.64}$$

where $\Delta_{\alpha\beta}^{\mu\nu}(x)$ is the G_{W} gauge boson propagator

$$\Delta_{\alpha\beta}^{\mu\nu}(x) = \mathrm{i}\langle 0|T A_{\alpha}^{\mu}(x) A_{\beta}^{\nu}(0)|0\rangle \equiv \delta^{\alpha\beta}\Delta^{\mu\nu}(x) \tag{10.65}$$

The question is whether our considerations in Section 10.1 based on the assumed softness of $\mathcal{H}'$ apply to this case when the symmetry breaking term $\mathcal{H}'(0)$ is not soft ($d = 6$), and at first glance it appears that it could affect the wave-function renormalization of fields. It turns out, however, that this does not necessarily happen. The problem can be studied by means of the operator expansion of the product of the two currents in (10.64). First of all the counterterms affected by $\mathcal{H}'(0)$ are easily enumerated (by assumption $\mathcal{H}'(0)$ is used only in the first order perturbation): they involve only those operators of the expansion which are accompanied by coefficients that produce non-integrable singularities. It is clear from (10.64) and (10.65) (remember that $\Delta^{\mu\nu}(x) \sim 1/x^2$ for $x \to 0$) that these are operators of dimension ≤ 4. The higher dimension operators in the expansion of $j^\alpha_\mu j^\beta_\nu$ do not cause a non-integrable singularity at $x = 0$, do not require renormalization and are irrelevant for our discussion; notice the difference in the case if $\mathcal{H}'(0)$ was considered in any order of perturbation theory. Among the operators of dimension 4 there may be singlets under G which are again irrelevant. The non-singlets with $d = 4$ would affect our discussion but in several interesting cases they do not appear (see, for example, Section 10.4 for the breaking of the chiral group by $\mathcal{H}'$ given by the product of two electromagnetic currents). We assume here that the expression (10.64) converges without subtractions which are non-singlets under the group G.

With perturbation $\mathcal{H}'(0)$ given by (10.64), Dashen's conditions (10.59) and (10.60) can be rewritten in a more useful form if we express the vacuum energy density $\Delta E(\Theta)$ given by (10.58) in terms of the G currents (Peskin 1981, Preskill 1981). We recall that (10.58) and (10.64) together give the following expression

$$\Delta E(\Theta) = -\tfrac{1}{2} \int \mathrm{d}^4x\, \Delta^{\mu\nu}(x) \sum_\alpha g^2_\alpha \langle 0|T U^\dagger j^\alpha_\mu(x) U U^\dagger j^\alpha_\nu(0) U|0\rangle \tag{10.66}$$

where U is any unitary operator representing the group G. The G_W currents are linear combinations of the G currents

$$g_\alpha j^\mu_\alpha(x) = \sum_a g^a_\alpha j^\mu_a(x) \tag{10.67}$$

(α is the G_W index, a is the G index and there is no sum over α on the l.h.s.) and consequently

$$g_\alpha U^\dagger j^\mu_\alpha(x) U = \sum_a g^a_\alpha U^\dagger j^\mu_a(x) U = \sum_{a,c} g^a_\alpha R^{ac}(\Theta) j^\mu_c(x) \tag{10.68}$$

where $R(\Theta)$ is the adjoint representation of G. Because the vacuum is H-invariant ($G \to H$ spontaneously) only the H-invariant part of the product $j^c_\mu j^d_\nu$ contributes to (10.66) expressed in terms of (10.68). For any given vacuum from the degenerate set the generators of G split into two groups: the generators Y^a such that $Y^a|0\rangle = 0$

and X^a where $X^a|0\rangle \neq 0$. Correspondingly, each G current carries index Y or X. Since Y^a span a single real representation of H (the adjoint representation), by Schur's lemma the only H-invariant term in the product ${}^Yj^c_\mu {}^Yj^d_\nu$ is proportional to δ^{cd}:

$$\langle 0|T\,{}^Yj^c_\mu(x)\,{}^Yj^d_\nu(0)|0\rangle = \delta^{cd}\langle 0|T\,{}^Yj_\mu\,{}^Yj_\nu|0\rangle = \mathrm{Tr}[Y^cY^d]\langle 0|T\,{}^Yj_\mu\,{}^Yj_\nu|0\rangle \tag{10.69}$$

(here we normalize the broken and unbroken generators to $\mathrm{Tr}[Y^aY^b] = \mathrm{Tr}[X^aX^b] = \delta^{ab}$, $\mathrm{Tr}[Y^aX^b] = 0$), where ${}^Yj_\mu$ denotes any single current ${}^Yj^a_\mu$; there is no summation in $T\,{}^Yj_\mu\,{}^Yj_\nu$. We will limit our discussion to cases in which in the product X^aX^b also the only H-invariant term is δ^{ab}. It can be shown (Peskin 1981) that this restriction implies the existence of a parity operation P which preserves the Lie algebra of G, such that

$$PY^aP^{-1} = Y^a, \qquad PX^aP^{-1} = -X^a \tag{10.70}$$

For chiral symmetry (10.70) are, of course, true since $PQ^a_{\mathrm{R,L}}P^{-1} = Q^a_{\mathrm{L,R}}$. Eqs. (10.70) also imply that the joint expectation values of one ${}^Yj^a_\mu$ and one ${}^Xj^b_\nu$ are zero by parity conservation. The decomposition in (10.68) can be split into the Y part and the X part

$$\begin{aligned} U^\dagger g_\alpha j^\mu_\alpha(x)U &= U^\dagger g_\alpha j^\mu_\alpha(x)U|_Y + U^\dagger g_\alpha j^\mu_\alpha(x)U|_X \\ &\equiv {}^Yj^\mu_{\alpha U}(x) + {}^Xj^\mu_{\alpha U}(x) \end{aligned} \tag{10.71}$$

(the last line defines our notation) such that only $Y(X)$ currents from the sum $\sum_{a,c} g^a_\alpha R^{ac} j^c_\mu$ contribute to the first (second) component. Taking all this into account we obtain for $\Delta E(\Theta)$ the following:

$$\begin{aligned} \Delta E(\Theta) = -\tfrac{1}{2}\int \mathrm{d}^4x\, \Delta^{\mu\nu}(x) \sum_\alpha \{ &\mathrm{Tr}[{}^Y\hat{J}_{\alpha U}\,{}^Y\hat{J}_{\alpha U}]\langle 0|T\,{}^Yj_\mu(x)\,{}^Yj_\nu(0)|0\rangle \\ &+ \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,{}^X\hat{J}_{\alpha U}]\langle 0|T\,{}^Xj_\mu(x)\,{}^Xj_\nu(0)|0\rangle\} \end{aligned} \tag{10.72}$$

where $\hat{J}_{\alpha U}$ is the linear combination of the G generators corresponding to the current $U^\dagger g_\alpha j^\alpha_\mu(x)U$ with g_α included. From the orthogonality of X^a and Y^b ($\mathrm{Tr}[X^aY^b] = 0$)

$$\left.\begin{aligned} \mathrm{Tr}[{}^Y\hat{J}_{\alpha U}\,{}^Y\hat{J}_{\alpha U}] &= \mathrm{Tr}[{}^Y\hat{J}_{\alpha U}\,\hat{J}_{\alpha U}] \\ \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,{}^X\hat{J}_{\alpha U}] &= \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,\hat{J}_{\alpha U}] \end{aligned}\right\} \tag{10.73}$$

Using in addition an obvious relation

$$\mathrm{Tr}[{}^Y\hat{J}_{\alpha U}\,\hat{J}_{\alpha U}] = \mathrm{Tr}[\hat{J}_{\alpha U}\,\hat{J}_{\alpha U}] - \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,\hat{J}_{\alpha U}] \tag{10.74}$$

we finally get

$$\Delta E(\Theta) = -\tfrac{1}{2}\int \mathrm{d}^4x\ \Delta^{\mu\nu}(x)\sum_\alpha\{\mathrm{Tr}[\hat{J}_{\alpha U}\hat{J}_{\alpha U}]\langle 0|T\,{}^Yj_\mu(x)\,{}^Yj_\nu(0)|0\rangle\}$$
$$-\tfrac{1}{2}\left\{\int \mathrm{d}^4x\ \Delta^{\mu\nu}(x)\langle 0|T[{}^Xj_\mu(x)\,{}^Xj_\nu(0) - {}^Yj_\mu(x)\,{}^Yj_\nu(0)]|0\rangle\right\}$$
$$\times\sum_\alpha \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,{}^X\hat{J}_{\alpha U}] \tag{10.75}$$

The first term in the sum is Θ-independent (without subtraction it may even be infinite) and in the second term Θ-dependence has been factored out from the current matrix elements. Assuming that the coefficient of the last trace in (10.75) is negative as can be argued in some cases, one gets the preferred vacuum simply by minimizing the quantity

$$\sum_\alpha \mathrm{Tr}[{}^X\hat{J}_{\alpha U}\,{}^X\hat{J}_{\alpha U}]$$

i.e. by minimizing the trace for projections of currents $\hat{J}^\alpha$ on the subspace of possible equivalent representations of the X^a and Y^a or, regarding the vacuum $|0\rangle$ as fixed so that the rotation is applied to the perturbation $\mathcal{H}'$, for projections of transformed currents on the subspace of the original broken generators. One sees that the preferred vacuum orientation corresponds to the minimal number of broken generators in the decomposition of $\hat{J}^\alpha$ into the generators of G.

We can rewrite Dashen's conditions (10.59) and (10.60) in the present notation. Writing $U = \exp(-\mathrm{i}\Theta^a X^a)$ one has

$$\frac{\partial}{\partial\Theta^a}\sum_\alpha \mathrm{Tr}({}^X\hat{J}_{\alpha U})^2|_{U=1} = 2\mathrm{i}\sum_\alpha \mathrm{Tr}[{}^X[X^a,\hat{J}_\alpha]\,{}^X\hat{J}_\alpha] = 2\mathrm{i}\sum_\alpha \mathrm{Tr}[X^a[{}^Y\hat{J}_\alpha, {}^X\hat{J}_\alpha]] \tag{10.76}$$

where the last result follows from the commutation relations $[X, X] = \mathrm{i}Y$ following from (10.70), orthogonality of Y and X and the trace property $\mathrm{Tr}[[A, B]C] = \mathrm{Tr}[A[B, C]]$. Thus, the vacuum orientation is a stationary point of ΔE if $[{}^Y\hat{J}_\alpha, {}^X\hat{J}_\alpha] = 0$ for all αs.

The second Dashen's condition reads

$$m^2_{ab} = \frac{1}{2}\frac{\partial^2}{\partial\Theta^a\partial\Theta^b}\sum_\alpha \mathrm{Tr}({}^X\hat{J}_{\alpha U})^2|_{U=1} \times M^2 \geq 0 \tag{10.77}$$

where, from (10.75)

$$M^2 = (1/f_\pi^2)\int \mathrm{d}^4x\ \Delta^{\mu\nu}(x)\langle 0|T[{}^Yj_\mu(x)\,{}^Yj_\nu(0) - {}^Xj_\mu(x)\,{}^Xj_\nu(0)]|0\rangle \tag{10.78}$$

and ${}^Yj_\mu$ (${}^Xj_\mu$) denotes, as before, any single current corresponding to an unbroken

(broken) generator of G. The Θ-dependence is only in the trace which can be expressed as follows

$$\begin{aligned}
&\frac{\partial^2}{\partial\Theta^a\partial\Theta^b}\sum_\alpha Tr({}^X\hat{J}_{\alpha U})^2|_{U=1} \\
&\quad = -2\sum_\alpha\{\mathrm{Tr}[{}^X[X^a, [X^b, \hat{J}_\alpha]]{}^X\hat{J}_\alpha] + \mathrm{Tr}[{}^X[X^a, \hat{J}_\alpha]{}^X[X^b, \hat{J}_\alpha]]\} \\
&\quad = 2\sum_\alpha \mathrm{Tr}[[{}^Y\hat{J}_\alpha, [{}^Y\hat{J}_\alpha, X^a]]X^b - [{}^X\hat{J}_\alpha, [{}^X\hat{J}_\alpha, X^\alpha]]X^b]
\end{aligned} \tag{10.79}$$

Dashen's condition (10.60) or equivalently the positivity of the pseudo-Goldstone bosons mass matrix depends on the signs of the trace (10.79) and of the integral (10.78). The latter can often be studied by means of the spectral function sum rules. Several applications of the results of this section will be discussed in the next section and in Section 11.3.

10.4 Electromagnetic π^+–π^0 mass difference and spectral function sum rules

Electromagnetic π^+–π^0 mass difference from Dashen's formula

In Section 10.2 we have studied an explicit chiral symmetry breaking in QCD caused by non-zero current quark masses. Another source of such a breaking is the electromagnetic interaction of quarks which couples the T^3 generator of the isospin group to the electromagnetic gauge field: $Q = T^3 + Y$. The effective hamiltonian is given by (10.64) with $j_\mu(x)$ being the electromagnetic current and $\Delta_{\mu\nu}$ the photon propagator. Following our discussion in Section 10.3 we notice that the expansion of the operator product $Tj_\mu(x)j^\mu(x)$ reads

$$\begin{aligned}
Tj_\mu(x)j^\mu(x) = C_1(x)\mathbb{1} + \sum_q C_q(x)m_q\bar{q}q + C_G(x)G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha \\
+ \text{operators with } d > 4 + \text{non-scalar operators}
\end{aligned} \tag{10.80}$$

where $G^\alpha_{\mu\nu}$ is the gluon field strength. Only scalar operators contribute to the integral in (10.64). Thus, the only $d = 4$ operator ($G^\alpha_{\mu\nu}G^{\mu\nu}_\alpha$) is a singlet under the chiral group and our considerations of Sections 10.1 and 10.3 apply. We also see that in the chiral limit $m_q = 0$ all the divergent terms in $\mathcal{H}_{\mathrm{em}}$ are singlets under the chiral group and may be disregarded: they induce a universal energy shift for all states, including the vacuum.

This section is devoted to a study of the electromagnetic contribution to the pion masses. The quark masses will be neglected in the calculation of the electromagnetic masses in the spirit of the first order perturbation in any explicit symmetry breaking term. As mentioned, in this case the electromagnetic pion mass difference is finite without renormalization. The calculation can be performed by means of Dashen's formula.

Using (10.77), (10.79) and the fact that $\hat{J}_Y^\alpha = eT^3$, $\hat{J}_X^\alpha = 0$ for the electromagnetic current (as in Section 9.6 we define the vacuum of the $SU_\mathrm{L}(2) \times SU_\mathrm{R}(2)$ symmetric QCD so that the $SU_\mathrm{L+R}(2)$ remains unbroken; then $\hat{J}_X^\alpha = 0$ and Dashen's condition for the stationary vacuum is satisfied in the presence of the electromagnetic interactions) we immediately get the following result for the one-photon exchange contribution to the pion mass:

$$(m_{\pi^+}^2)^\gamma = e^2 M^2, \quad (m_{\pi^0}^2)^\gamma = 0 \tag{10.81}$$

where M^2 can be explicitly rewritten in terms of the vector and axial currents

$$M^2 = (1/f_\pi^2) \int \mathrm{d}^4x\, \Delta^{\mu\nu}(x) \langle 0|T[\mathcal{V}_\mu^3(x)\mathcal{V}_\nu^3(0) - \mathcal{A}_\mu^3(x)\mathcal{A}_\nu^3(0)]|0\rangle \tag{10.82}$$

We recall again that all, possibly infinite, singlet contributions to the electromagnetic masses have been disregarded in (10.81) and (10.82) (see the derivation of (10.77) in the previous section). We also remember that both equations are valid only in the first order in the symmetry breaking, i.e. for $m_q = 0$. Non-zero quark masses introduce second order corrections $O(\alpha m_q)$ which need renormalization and are therefore renormalization-scale-dependent. Formula (10.82) was derived by Das *et al.* (1967). To calculate the integral (10.82) one can use the spectral function sum rules.

Spectral function sum rules

Spectral function sum rules, originally derived by Weinberg (1967*a*) for the $SU(2) \times SU(2)$ chiral symmetry, can be discussed in a general way by invoking the OPE (Bernard, Duncan, Lo Secco & Weinberg 1975). Let us consider the Källen–Lehmann spectral representation (Bjorken & Drell 1965) for the time-ordered product of two currents

$$\langle 0|Tj_\mu^a(x)j_\nu^b(0)|0\rangle = \int_0^\infty \mathrm{d}\mu^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \exp(-\mathrm{i}kx) \frac{\mathrm{i}}{k^2 - \mu^2 + \mathrm{i}\varepsilon} \rho_{\mu\nu}^{ab}(k, k^2 = \mu^2) \tag{10.83}$$

where the spectral function ρ can be decomposed into the spin-one and spin-zero parts:

$$(2\pi)^3 \sum_{\substack{n\\ \text{spin 1}}} \langle 0|j^a_\mu(0)|n\rangle\langle n|j^b_\nu(0)|0\rangle\delta(k-k_n) = -(g_{\mu\nu} - k_\mu k_\nu/\mu^2)\rho^{(1)ab}(\mu^2) \tag{10.84}$$

$$(2\pi)^3 \sum_{\substack{n\\ \text{spin 0}}} \langle 0|j^a_\mu(0)|n\rangle\langle n|j^b_\nu(0)|0\rangle\delta(k-k_n) = k_\mu k_\nu \rho^{(0)ab}(\mu^2) \tag{10.85}$$

Taking the Fourier transform of (10.83)

$$\text{F.T.} = \int \mathrm{d}^4x \exp(\mathrm{i}kx)\langle 0|Tj^a_\mu(x)j^b_\nu(0)|0\rangle = \mathrm{i}\int_0^\infty \mathrm{d}\mu^2\, \frac{\rho^{ab}_{\mu\nu}(k, k^2=\mu^2)}{k^2-\mu^2} \tag{10.86}$$

and expanding it formally in powers of $1/k^2$ we get

$$\begin{aligned}\text{F.T.} = \mathrm{i}\frac{k_\mu k_\nu}{k^2}\int_0^\infty \mathrm{d}\mu^2\left[\frac{\rho^{(1)}_{ab}(\mu^2)}{\mu^2} + \rho^{(0)}_{ab}(\mu^2)\right] - \mathrm{i}\frac{g_{\mu\nu}}{k^2}\int_0^\infty \mathrm{d}\mu^2\,\rho^{(1)}_{ab}(\mu^2)\\ + \mathrm{i}\frac{k_\mu k_\nu}{(k^2)^2}\int_0^\infty \mathrm{d}\mu^2\,[\rho^{(1)}_{ab}(\mu^2) + \mu^2\rho^{(0)}_{ab}(\mu^2)] + \cdots\end{aligned} \tag{10.87}$$

This formal expansion can be meaningful only when the Fourier transform $\int \mathrm{d}^4x \exp(\mathrm{i}kx)\langle 0|Tj^a_\mu(x)j^b_\nu(0)|0\rangle$ behaves for large k^2 as $O(1)$ or softer. Otherwise the coefficients of the expansion must be divergent. If one constructs a linear combination of currents such that the Fourier transform (10.86) is for large k^2 softer than $O(1)$, then the first few terms in the expansion (10.87) must vanish, for example, for softer than $(1/k^2)$ behaviour

$$\left.\begin{aligned}\int_0^\infty \mathrm{d}s\,[\rho^{(1)}_{ab}(s)/s + \rho^{(0)}_{ab}(s)] = 0\\ \int_0^\infty \mathrm{d}s\,\rho^{(1)}_{ab}(s) = 0\\ \int_0^\infty \mathrm{d}s\,s\rho^{(0)}_{ab}(s) = 0\end{aligned}\right\} \tag{10.88}$$

and these are the spectral function sum rules.

The OPE is useful in finding appropriate combinations of current products with soft high energy behaviour. In looking for such products one can use the global symmetry of the lagrangian which is respected by the OPE coefficient functions even if the symmetry is spontaneously broken. In our problem we are interested in the group $G = SU_\mathrm{L}(2) \times SU_\mathrm{R}(2)$ broken to the $H = SU_{\mathrm{L+R}}(2)$. Spectral function sum rules can be derived by studying the high momentum limit of

$$G^{ab}_{\mu\nu}(k) = \int \mathrm{d}^4x \exp(\mathrm{i}kx)\langle 0|T j^a_{\mu\mathrm{L}}(x) j^b_{\nu\mathrm{R}}(0)|0\rangle$$

which transforms as the adjoint representation (3, 3) under $SU(2) \times SU(2)$. The asymptotic behaviour of $G^{ab}_{\mu\nu}(k)$ is determined by the lowest-dimension operator in the expansion of $j^a_{\mathrm{L}\mu} j^b_{\mathrm{R}\nu}$ which has a non-zero vacuum expectation value. This operator must be Lorentz-invariant, gauge-invariant, H-invariant, i.e. a singlet under $SU_{\mathrm{L+R}}(2)$, and, because the expansion coefficient functions respect the G symmetry, it must transform as the adjoint representation (3, 3) under $SU(2) \times SU(2)$. The lowest-dimension operator satisfying these requirements is a four-fermion operator of dimension (mass)^6 ($\bar{\Psi}\Psi$ transforms as $(2, \bar{2}) \oplus (\bar{2}, 2)$). Therefore, $G^{ab}_{\mu\nu} \sim (k^2)^{-2}$ up to logarithms and the sum rules (10.88) hold for the spectral functions ρ^{ab}_{LR}.

According to (10.82) we are interested in the combination $\mathcal{VV} - \mathcal{AA} = 4 j_\mathrm{L} j_\mathrm{R}$. Taking into account that

$$\langle 0|\mathcal{A}^a_\mu|\pi^b\rangle = \mathrm{i}k_\mu f_\pi \delta^{ab}$$

and therefore

$$\rho^{(0)}_{\mathcal{A}ab}(\mu^2) = \delta_{ab} f^2_\pi \delta(\mu^2) + \cdots$$

and neglecting any continuum contribution to $\rho^{(0)}_{\mathcal{A}}$ which should be small we get from (10.88) the following sum rules:

$$\left.\begin{aligned} &\int_0^\infty (\mathrm{d}s/s)[\rho^{(1)}_{\mathcal{V}ab}(s) - \rho^{(1)}_{\mathcal{A}ab}(s)] = f^2_\pi \delta_{ab} \\ &\int_0^\infty \mathrm{d}s\, [\rho^{(1)}_{\mathcal{V}ab}(s) - \rho^{(1)}_{\mathcal{A}ab}(s)] = 0 \end{aligned}\right\} \tag{10.89}$$

which are valid in the chirally symmetric theory. If the chiral symmetry is broken by non-zero quark masses the OPE for the difference $\mathcal{VV} - \mathcal{AA}$ contains a term proportional to $m\bar{q}q$. This can be seen by a spurion analysis (Wilson 1969*a*): the current product belongs to (3, 3) of the $SU(2) \times SU(2)$, whereas the operator $\bar{q}q$ belongs to $(2, \bar{2}) \oplus (\bar{2}, 2)$. However, the symmetry is broken by the mass term $m\bar{q}q$. Representing the symmetry breaking interaction by a spurion which must also belong to $(2, \bar{2}) \oplus (\bar{2}, 2)$ we see that combining one spurion with the $\bar{q}q$ operator one can produce the (3, 3) representation. Thus, in the expansion of the two-current product we need one power of the symmetry breaking parameter m for the coefficient function in front of the operator $\bar{q}q$ to be non-zero. The second sum rule is then no longer valid: the coefficient of this $m\bar{q}q$ term scales as x^{-2}, so its

Fourier transform behaves as k^{-2}. The corrections are, however, expected to be small because the masses m_u and m_d are very small in comparison to the scale of QCD.

Results

We are now ready to calculate M^2, (10.82). Introducing explicitly

$$\Delta^{\mu\nu}(x) = g_{\mu\nu} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\exp(-\mathrm{i}k'x)}{k'^2 + \mathrm{i}\varepsilon}$$

we get

$$M^2 = -\frac{\mathrm{i}}{f_\pi^2} \int_0^\infty \mathrm{d}\mu^2\, 3[\rho_{\mathcal{V}}^{(1)}(\mu^2) - \rho_{\mathcal{A}}^{(1)}(\mu^2)] \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2(k^2 - \mu^2)} \tag{10.90}$$

The momentum integral is logarithmically divergent and gives, using dimensional regularization,

$$\frac{1}{(4\pi)^2} \frac{1}{\varepsilon} + \frac{1}{(4\pi)^2} \ln \frac{\mu_0^2}{\mu^2} + \text{const.} \tag{10.91}$$

However, it follows from the second sum rule (10.89) that the coefficient of the divergent term vanishes. This just reflects the already discussed fact that the first order electromagnetic contribution is finite after the singlet piece under $SU(2) \times SU(2)$ has been disregarded. The remaining finite term gives

$$(m_{\pi^+}^2)^\gamma = \frac{3\alpha}{4\pi (f_\pi)^2} \int_0^\infty \mathrm{d}s \left(\ln \frac{\mu_0^2}{s} \right) [\rho_{\mathcal{V}}^{(1)}(s) - \rho_{\mathcal{A}}^{(1)}(s)] \tag{10.92}$$

and on account of the second sum rule (10.89) it is renormalization point (μ_0^2) independent. Finally we can calculate (10.92) assuming that it is reasonable to saturate the integral (10.92) with the two lowest-lying narrow resonances, one (ρ) coupled to the vector current and the other (A_1) to the axial current. Taking

$$\rho_{\mathcal{V}}(s) = f_\rho^2 s \delta(s - m_\rho^2)$$
$$\rho_{\mathcal{A}}(s) = f_{A_1}^2 s \delta(s - m_{A_1}^2)$$

and using both sum rules (10.89) one gets

$$f_\rho^2 = f_{A_1}^2 + f_\pi^2$$

and

$$m_\rho^2 f_\rho^2 = f_{A_1}^2 m_{A_1}^2$$

Therefore we can eliminate the A_1 parameters to get the final answer in terms of the ρ parameters:

$$(m_{\pi^+}^2)^\gamma = \frac{3\alpha}{4\pi} m_\rho^2 \left(\frac{f_\rho}{f_\pi}\right)^2 \ln \frac{f_\rho^2}{f_\rho^2 - f_\pi^2} \tag{10.93}$$

To calculate the π^+–π^0 mass difference we have to combine the results of Section 10.2 and of this section. In the first order in both symmetry breaking perturbations, quark masses and electromagnetic interactions, we have, using the first equation (10.48),

$$m_{\pi^+}^2 = m_{\pi^0}^2 + (m_{\pi^+}^2)^\gamma \tag{10.94}$$

where

$$m_{\pi^0}^2 = C(m_u + m_d), \qquad C = -(2/f_\pi^2)\langle 0|\bar{u}u|0\rangle$$

and therefore

$$m_{\pi^+} - m_{\pi^0} \approx (m_{\pi^+}^2)^\gamma / 2m_{\pi^0}$$

Using $f_\pi = 93$ MeV and $f_\rho = (145 \pm 8)$ MeV, one gets $m_{\pi^+} - m_{\pi^0} = (4.9 \pm 0.2)$ MeV. Although the error due to the resonance saturation is hard to estimate accurately this result should be regarded as at least in qualitative agreement with the experimental value 4.6 MeV.

Corrections to this result due to the π^0–η mixing caused by $m_u \neq m_d$ can be estimated to be negligibly small (Gasser & Leutwyler 1982). Finally, as already mentioned, corrections of order αm_q require renormalization but are also expected to be small (Gasser & Leutwyler 1982).

Extending our considerations to the $SU(3) \times SU(3)$ case we get $(m_{\mathrm{K}^+}^2)^\gamma = (m_{\pi^+}^2)^\gamma$ and $(m_{\mathrm{K}^0}^2)^\gamma = 0$. Therefore, again using (10.48),

$$\left.\begin{aligned} m_{\mathrm{K}^+}^2 &= C(m_u + m_s) + (m_{\pi^+}^2)^\gamma \\ m_{\mathrm{K}^0}^2 &= C(m_d + m_s) \end{aligned}\right\} \tag{10.95}$$

and the K^+–K^0 mass difference calculated in the first order of the explicit chiral symmetry breaking is not purely electromagnetic but it also reflects a possible m_u–m_d difference. Eqs. (10.94) and (10.95) taken together can be used to estimate the quark mass ratios (Weinberg 1977). Completing our discussion given at the end of

Section 10.2 we get

$$\frac{m_d}{m_u} = \frac{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}}{2m^2_{\pi^0} + m^2_{K^+} - m^2_{K^0} - m^2_{\pi^+}} = 1.8$$

$$\frac{m_s}{m_d} = \frac{m^2_{K^0} + m^2_{K^+} - m^2_{\pi^+}}{m^2_{K^0} - m^2_{K^+} + m^2_{\pi^+}} = 20.1$$

We stress that these results are valid in the first order of the chiral symmetry breaking.

11

Higgs mechanism in gauge theories

11.1 Higgs mechanism

The proof of Goldstone's theorem given in Chapter 9 for theories with spontaneously broken global symmetry requires Lorentz invariance and Hilbert space with positive-definite scalar products. Gauge theories do not obey both requirements simultaneously. In a covariant gauge, the theory contains states of negative norm. In a gauge in which the theory has only states of positive norm it is not manifestly covariant. In consequence, Goldstone's theorem does not hold and the so-called Higgs mechanism operates (see, for instance, Abers & Lee (1973) and references therein).

We shall consider the simplest example of the $U(1)$ gauge theory. This example illustrates all the essential points. The Higgs mechanism requires the presence of a scalar field coupled to the gauge field. For a $U(1)$ symmetric theory it has to be a complex scalar field.

The lagrangian for a single self-interacting complex field Φ'

$$\mathcal{L} = \partial_\mu \Phi'^* \partial^\mu \Phi' - \lambda(\Phi'^* \Phi' + m^2/2\lambda)^2 \tag{11.1}$$

is invariant under global $U(1)$ transformation: ${}^{\mathrm{U}}\Phi' = \exp(-\mathrm{i}\Theta)\Phi'$. The $U(1)$ symmetry is spontaneously broken (see Section 1.2) for $m^2 < 0$. For the real fields $\Phi' = (\varphi' + \mathrm{i}\chi')/\sqrt{2}$ we can choose, for instance,

$$\langle 0|\varphi'|0\rangle = v = (-m^2/\lambda)^{1/2}, \qquad \langle 0|\chi'|0\rangle = 0$$

and defining the physical fields $\varphi' = \varphi - v$, $\chi = \chi'$ we get

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi + \partial_\mu\chi\partial^\mu\chi) - v^2\lambda\varphi^2 - \tfrac{1}{4}\lambda(\varphi^2+\chi^2)^2 - \lambda\varphi v(\varphi^2+\chi^2) \tag{11.2}$$

This lagrangian describes the interaction of the massive particle φ and the massless Goldstone boson χ.

It is also useful to introduce 'angular' variables and parametrize the field Φ' as follows (see (9.78))

$$\Phi'(x) = \rho'(x)\exp[-\mathrm{i}\eta'(x)T]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \rho' \sin\eta' \\ \rho' \cos\eta' \end{pmatrix}; \qquad T = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{11.3}$$

Using the freedom in our choice of the vacuum expectation value among the $U(1)$ degenerate set (Section 1.2) we can write

$$\Phi'(x) - \langle 0|\Phi'(x)|0\rangle = [\rho'(x) - v]\exp[-\mathrm{i}\eta'(x)T]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11.4}$$

and in terms of the fields $\rho = \rho' - v$ and $\eta = v\eta'$ the lagrangian (11.1) reads

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\rho\partial^\mu\rho + \partial_\mu\eta\partial^\mu\eta) - v^2\lambda\rho^2 + \text{cubic and higher order terms} \tag{11.5}$$

Thus, the particle interpretation of the fields ρ and η is the same as that of φ and χ.

Now let us introduce a gauge field A_μ; a lagrangian invariant under local gauge transformation is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\Phi')^*(D_\mu\Phi') - \lambda(\Phi'^*\Phi' + m^2/2\lambda)^2 \tag{11.6}$$

where

$$D_\mu = \partial_\mu + \mathrm{i}gA_\mu$$

The gauge freedom has profound implications for the mechanism of spontaneous symmetry breaking. This is easiest to see working with the angular variables. Choosing the gauge function to be $-\eta'(x) = -\eta(x)/v$ and using (11.4) we get (in the real basis)

$$\left.\begin{aligned} &{}^{\mathrm{U}}\Phi'(x) - \langle 0|\Phi'(x)|0\rangle^{\mathrm{U}} = [\rho'(x) - v]\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &A_\mu^{\mathrm{U}}(x) = A_\mu(x) + (1/gv)\partial_\mu\eta(x) \end{aligned}\right\} \tag{11.7}$$

Since the lagrangian (11.6) is gauge-invariant we can write it in terms of fields ${}^{\mathrm{U}}\Phi$ and A_μ^{U} and then

$$\mathcal{L} = -\tfrac{1}{4}F^{\mathrm{U}\mu\nu}F^{\mathrm{U}}_{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho + \tfrac{1}{2}g^2(\rho+v)^2A^{\mathrm{U}}_\mu A^{\mathrm{U}\mu} - \tfrac{1}{4}\lambda(\rho^2 + 2\rho v)^2 \tag{11.8}$$

where

$$\rho = \rho' - v$$

In this gauge, called the unitary gauge (introduced in Section 1.3), the particle spectrum is evident. There is a scalar meson ρ with the bare mass $(2\lambda v^2)^{1/2}$ and a massive vector meson with mass gv and no particle corresponding to the field η which has disappeared from the lagrangian. The η degree of freedom has been traded for the longitudinal component of the vector field, now massive. Thus, the gauge symmetry is spontaneously broken and no massless Goldstone boson remains in the physical spectrum. The Higgs mechanism is one of the fundamental ideas for condensed matter physics and it is also the crucial part of unified electroweak theory (see Chapter 12).

Quantization in the unitary gauge gives a theory which is manifestly unitary order-by-order in perturbation theory but not manifestly renormalizable. Because of the massive vector meson propagator which for large k grows like $k_\mu k_\nu/m^2k^2$ the Green's functions contain divergences that cannot be removed by the renormalization counterterms. Such divergences vanish in the S-matrix elements but this we know from the proof of renormalizability of a spontaneously broken gauge theory given in covariant gauges (Abers & Lee (1973) and references therein). Hence it is useful to extend our discussion to covariant gauges.

Let us first take a gauge fixed by the standard gauge-fixing term $-(\partial_\mu A^\mu)^2/2a$. Returning to the variables φ and χ we get from (11.6) the following lagrangian:

$$
\begin{aligned}
\mathcal{L} = {} & -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi + \partial_\mu\chi\partial^\mu\chi) - \lambda v^2\varphi^2 - (1/2a)(\partial_\mu A^\mu)^2 && (S_0)\\
& + \tfrac{1}{2}g^2v^2A_\mu A^\mu + gvA_\mu\partial^\mu\chi && (\tilde{S}_0)\\
& \left.\begin{aligned} & - gA_\mu\partial^\mu\varphi\chi + gA_\mu\partial^\mu\chi\varphi + \tfrac{1}{2}g^2A_\mu A^\mu(\varphi^2 + 2\varphi v)\\ & + \tfrac{1}{2}g^2A_\mu A^\mu\chi^2 - \tfrac{1}{4}\lambda(\varphi^2+\chi^2)^2 - \lambda(\varphi^2+\chi^2)\varphi v \end{aligned}\right\} && (S_\mathrm{I})
\end{aligned}
\tag{11.9}
$$

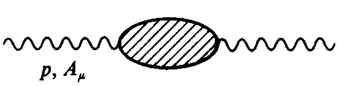
One can now proceed in several different ways. One possibility is to take S_0 as the free lagrangian of the theory with the corresponding definitions of the free particle states (which may not be physical states). With the standard methods of Chapters 2 and 3 we get then the following free particle propagators (for example, in the Landau gauge $a \to 0$):

$$
\left.\begin{array}{ll}
A_\mu(\text{massless}): & -\mathrm{i}(g_{\mu\nu} - p_\mu p_\nu/p^2)/(p^2+\mathrm{i}\varepsilon) \equiv -\mathrm{i}D_{\mu\nu}\\
\varphi(\text{massive}): & \mathrm{i}/(p^2 - 2\lambda v^2 + \mathrm{i}\varepsilon)\\
\chi(\text{massless}): & \mathrm{i}/(p^2+\mathrm{i}\varepsilon)
\end{array}\right\}
\tag{11.10}
$$

The full gauge boson propagator

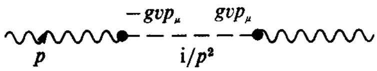
$$\Delta_{\mu\nu}(p) = -\mathrm{i}D_{\mu\nu}\frac{1}{1-\Pi(p^2)} \tag{11.11}$$

where $\Pi(p^2)$ is defined by the gauge-invariant vacuum polarization tensor,

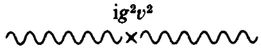


$$\equiv \mathrm{i}\Pi_{\mu\nu}(p^2) = \mathrm{i}(g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi(p^2) \tag{11.12}$$

gets a contribution from the remaining terms in the lagrangian and in particular from $\tilde{S}_0$ which, although quadratic in fields too, is counted as an interaction term. In the lowest order the $\tilde{S}_0$ contribution is



$$\equiv -\mathrm{i}(p_\mu p_\nu/p^2)g^2v^2 \tag{11.13a}$$



$$\equiv +\mathrm{i}g^2v^2g_{\mu\nu} \tag{11.13b}$$

Hence

$$\Pi(p^2) = g^2v^2/p^2 \tag{11.14}$$

and consequently

$$\Delta_{\mu\nu}(p) = -\mathrm{i}(g_{\mu\nu} - p_\mu p_\nu/p^2)/(p^2 - g^2v^2) \tag{11.15}$$

which can be rewritten as a combination of the standard propagator for a massive vector boson and of a term corresponding to a scalar negative norm boson coupled to the source of the vector meson gradiently

$$\Delta_{\mu\nu}(p) = -\mathrm{i}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{g^2v^2}\right)\frac{1}{p^2 - g^2v^2 + \mathrm{i}\varepsilon} - \mathrm{i}\frac{p_\mu p_\nu}{g^2v^2}\frac{1}{p^2 + \mathrm{i}\varepsilon} \tag{11.16}$$

Note also that the χ propagator remains massless in the Landau gauge.

One sometimes says that the coupling (11.13a) of a gauge boson to a massless Goldstone boson is the origin of the gauge boson mass and of the spontaneous breaking of gauge invariance. In this context one should observe that the massless Goldstone boson contributes only to the $p_\mu p_\nu$ term in the $\Pi_{\mu\nu}$ but by gauge invariance the $g_{\mu\nu}$ term must then be present. This remark will be particularly relevant in Section 11.3 where other mechanisms for spontaneous gauge symmetry breaking are discussed. Secondly, such an interpretation is certainly procedure- and gauge-dependent. Indeed, we can as well take $S_0 + \tilde{S}_0$ as our free particle lagrangian. The presence of the mixing term $gvA_\mu\partial^\mu\chi$ leads to a system of coupled differential equations for A_μ and χ propagators and in the limit $a \to 0$ we get (11.15) as the *free* boson propagator. The coupling  disappears from the theory as well as its previous interpretation.

Finally we can get rid of the coupling $gvA_\mu\partial^\mu\chi$ in the lagrangian by a clever choice of gauge. This so-called R_ξ gauge is in wide use. Take the gauge-fixing term as

$$f(A_\mu, \chi) = -\frac{1}{2\xi}(\partial^\mu A_\mu - \xi gv\chi)^2 \tag{11.17}$$

Then, integrating by parts when necessary,

$$\tilde{S}_0 + f(A_\mu, \chi) = \tfrac{1}{2}g^2v^2A_\mu A^\mu - \tfrac{1}{2}\xi g^2v^2\chi^2 - (1/2\xi)(\partial_\mu A^\mu)^2 \tag{11.18}$$

and the free propagators are

$$
\begin{aligned}
A_\mu:\quad & \frac{-\mathrm{i}}{p^2 - M^2 + \mathrm{i}\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi M^2}\right] \\
& = -\mathrm{i}\frac{g_{\mu\nu} - p_\mu p_\nu/M^2}{p^2 - M^2 + \mathrm{i}\varepsilon} - \mathrm{i}\frac{p_\mu p_\nu/M^2}{p^2 - \xi M^2 + \mathrm{i}\varepsilon} \\
\chi:\quad & \mathrm{i}/(p^2 - \xi M^2 + \mathrm{i}\varepsilon) \\
\varphi:\quad & \mathrm{i}/(p^2 - 2\lambda v^2 + \mathrm{i}\varepsilon)
\end{aligned}
$$

where $M = gv$. In the limit $\xi = 0$ we get the Landau gauge and for $\xi = 1$ the so-called 't Hooft–Feynman gauge. The unitary gauge is recovered in the limit $\xi \to \infty$.

For any finite value of ξ there are unphysical poles at $p^2 = \xi M^2$ in the gauge boson propagator and in the χ propagator. They cancel, however, in the S-matrix elements which are obtained from the Green's functions by removing external lines, setting external momenta on the mass-shell and contracting tensor indices with appropriate physical polarization vectors. This can be shown, for instance, by proving that the renormalized S-matrix is independent of ξ (Abers & Lee 1973).

11.2 Spontaneous gauge symmetry breaking by radiative corrections

In the model of the previous section gauge symmetry has already been spontaneously broken at the tree level. Here we discuss spontaneous gauge symmetry breakdown by radiative corrections (Coleman & Weinberg 1973). This can occur when elementary scalar fields of zero mass are present in the theory (Georgi & Pais 1977). In the tree approximation there is then no spontaneous symmetry breaking because in this approximation a non-zero vacuum expectation value of the scalar fields can appear only as a result of interplay between the Φ^4 interaction terms and the scalar mass terms. Spontaneous symmetry breaking is, however, produced by non-zero vacuum expectation values induced by the higher order corrections. This is also an example of a dimensional transmutation: a dimensionful parameter $\langle 0|\Phi|0\rangle$ is generated in a quantum field theory whose classical limit is described by a scale-invariant lagrangian.

We consider first $\lambda\Phi^4$ theory with a massless meson field and discuss spontaneous breaking of global reflection symmetry $\Phi \to -\Phi$. The lagrangian density for this theory written in terms of the renormalized quantities

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - (\lambda/4!)\Phi^4 + \tfrac{1}{2}A(\partial_\mu\Phi)(\partial^\mu\Phi) - \tfrac{1}{2}B\Phi^2 - 1/4!C\Phi^4 \quad (11.19)$$

contains the usual wave-function and coupling-constant renormalization counterterms. Also a mass renormalization counterterm is present, even though we are studying the massless theory, because the theory possesses no symmetry which would guarantee vanishing bare mass in the limit of vanishing renormalized mass (the scale invariance is broken by anomalies). To study spontaneous symmetry breaking in such a theory we will use the effective potential formalism developed in Section 2.6. To the one-loop approximation the effective potential is given by (2.165) and (2.166) (with $m^2 = 0$ and with the contributions from the mass and coupling-constant counterterms included)

$$V = \frac{\lambda}{4!}\varphi^4 + \tfrac{1}{2}B\varphi^2 + \frac{1}{4!}C\varphi^4 + \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \ln\left(1 + \frac{\lambda\varphi^2}{2k^2}\right) \tag{11.20}$$

In this expression we have performed the usual Wick rotation so the integral is in Euclidean space: $\mathrm{d}^4k = \frac{1}{2}k^2\,\mathrm{d}k^2\,\mathrm{d}\Omega_4$, $\Omega_4 = 2\pi^2$. Observe also that, even though for $m^2 = 0$ the series in (2.166) is disastrously IR divergent, its sum exhibits only a logarithmic singularity at $\varphi = 0$. Cutting off the integral in (11.20) at $k^2 = \Lambda^2$ (exceptionally we use the UV cut-off rather than the dimensional regularization method), performing the elementary integration and throwing away terms which vanish as Λ^2 goes to infinity we obtain

$$V = \frac{\lambda}{4!}\varphi^4 + \tfrac{1}{2}B\varphi^2 + \frac{1}{4!}C\varphi^4 + \frac{\lambda\Lambda^2}{64\pi^2}\varphi^2 + \frac{\lambda^2\varphi^4}{256\pi^2}\left(\ln\frac{\lambda\varphi^2}{2\Lambda^2} - \frac{1}{2}\right) \tag{11.21}$$

Imposing now the definitions of the renormalized mass and coupling constant we determine the value of the counterterms. Defining the renormalized squared mass of the field $\Phi(x)$ as the value of the inverse propagator at zero momentum and recalling the interpretation following (2.164) of the derivatives of the effective potential we get

$$\mathrm{d}^2V/\mathrm{d}\varphi^2|_{\varphi=0} = 0 \tag{11.22}$$

as the mass renormalization condition. We take the derivative at $\varphi = 0$ because we are interested in the inverse propagator of the original (not shifted) field $\Phi(x)$; note also that the perturbative calculation of $V(\varphi)$ makes use of the Feynman rules for the lagrangian written in terms of the field $\Phi(x)$ and therefore it is based on (2.164) even though the theory may be spontaneously broken. From (11.21) and (11.22) we get

$$B = -\lambda\Lambda^2/32\pi^2$$

A possible definition of the renormalized coupling constant would be to identify it with the value of the 1PI four-point function in the original fields $\Phi(x)$ at zero external momenta

$$\mathrm{d}^4V/\mathrm{d}\varphi^4|_{\varphi=0} = \lambda$$

Unfortunately, for a massless field we encounter the logarithmic IR singularities.

In the standard renormalization procedure one defines in such a case the coupling constant at some off-mass-shell position in momentum space. Using the effective potential formalism it is much more convenient to adopt an alternative definition, namely

$$\mathrm{d}^4 V/\mathrm{d}\varphi^4|_{\varphi=M} = \lambda(M) \tag{11.23}$$

where M is some arbitrary number with the dimension of mass. It is clear from our discussion in Section 2.6 (see (2.156)) that the fourth derivative of $V(\varphi)$ at $\varphi = M$ is the 1PI Green's function, at zero external momenta, for the shifted fields $\Phi'(x) = \Phi(x) - M$. In particular if we take $M = \langle 0|\Phi|0\rangle$ it is the four-point function taken at zero external momenta for the physical massive field $\bar{\Phi}(x)$: $\langle 0|\bar{\Phi}(x)|0\rangle = 0$.

Imposing condition (11.23) on the effective potential $V(\varphi)$ given by (11.21) we finally get

$$V = \frac{\lambda(M)}{4!}\varphi^4 + \frac{\lambda^2(M)\varphi^4}{256\pi^2}\left(\ln\frac{\varphi^2}{M^2} - \frac{25}{6}\right) \tag{11.24}$$

We see that the one-loop radiative corrections have turned the minimum at the origin $\varphi = 0$ into a maximum and caused a new minimum of V to appear at

$$\lambda(M)\ln(\langle 0|\Phi|0\rangle/M) = -\tfrac{32}{3}\pi^2 + O(\lambda) \tag{11.25}$$

Thus, the one-loop corrections have generated spontaneous breaking of reflection symmetry. However, in our example, the new minimum lies outside the range of validity of perturbation theory as higher orders will give higher powers of $\lambda\ln(\varphi^2/M^2)$ which according to (11.25) is large for $\varphi = \langle 0|\Phi|0\rangle$.

Physically interesting theories in which spontaneous symmetry breaking can be studied in perturbation theory are gauge theories with elementary massless scalar fields and possibly fermion fields coupled to the gauge fields. Let us consider the case of a single gauge field coupled to a charged scalar to see qualitatively what happens. In the one-loop approximation the effective potential of the scalar sector gets a contribution from the scalar loops which, aside from irrelevant numerical factors (there are now two real scalar fields in the theory: $\Phi = (\Phi_1 + \mathrm{i}\Phi_2)/\sqrt{2}$; the scalar self-coupling is $\lambda(\Phi\Phi^*)^2$; the effective potential can only depend on $\varphi^2 = (\varphi_1^2 + \varphi_2^2)$ because the theory is $U(1)$-invariant), have the same structure as before, and from the gauge field loops in Fig. 11.1 arising from the minimal coupling $|(\partial_\mu + \mathrm{i}gA_\mu)\Phi|^2$. The trilinear terms lead to diagrams such as

which vanish if we work in the Landau gauge, where the gauge boson propagator

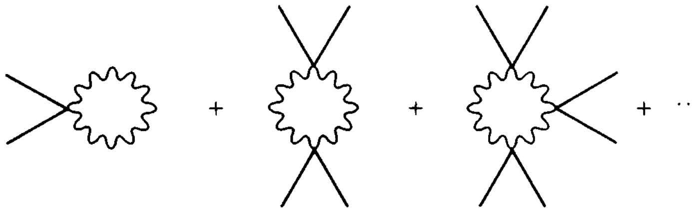


Fig. 11.1.

is

$$D_{\mu\nu} = -\mathrm{i}\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 + \mathrm{i}\varepsilon}$$

This is because we calculate diagrams with zero external momenta, and therefore the momentum of the internal scalar meson is the same as that of the internal gauge boson.

It is obvious now that the gauge boson loop contribution to the effective potential has the same structure as the scalar loop contribution, so we get

$$V = \lambda(M)\varphi^4 + [(1/16\pi^2)\lambda^2(M) + bg^4(M)]\varphi^4[\ln(\varphi^2/M^2) - \tfrac{25}{6}] \tag{11.26}$$

where b is some numerical factor. If it happens that $\lambda(M) \ll g^2(M) \ll 1$ we can neglect the $O(\lambda^2)$ term and then $\mathrm{d}V/\mathrm{d}\varphi = 0$ for

$$|\langle\varphi\rangle| = (\langle 0|\Phi_1|0\rangle^2 + \langle 0|\Phi_2|0\rangle^2)^{1/2} = M\exp\left[\frac{11}{6} - \frac{1}{2b}\frac{\lambda(M)}{g^4(M)}\right] \tag{11.27}$$

Thus, in the one-loop approximation we find in our theory spontaneous breaking of the $U(1)$ symmetry and, this time, the new minimum is in the range of validity of our approximation: due to our choice of the renormalization point such that $\lambda(M) \ll g^2(M)$, (11.27) constrains only the value of $\ln(\langle\varphi\rangle^2/M^2)$ and for small λ and g the terms $\lambda\ln(\langle\varphi\rangle^2/M^2)$ and $g^2\ln(\langle\varphi\rangle^2/M^2)$ are small. A non-vanishing field vacuum expectation value generated by radiative corrections is an example of the dimensional transmutation phenomenon. This can be seen more clearly if we choose $M = \langle\varphi\rangle$. Then, from (11.27)

$$\lambda(\langle\varphi\rangle) = \tfrac{11}{3}bg^4(\langle\varphi\rangle) \tag{11.28}$$

and we recall that $M = \langle\varphi\rangle$ corresponds to on-shell renormalization for the shifted fields. We can take dimensionless λ and g or g and dimensionful $\langle\varphi\rangle$ as independent parameters of our theory.

Spontaneous gauge symmetry breaking leads to the physical consequences discussed in Section 11.1: the would-be Goldstone boson combines with the gauge boson to make a massive vector meson $m^2(V) = g^2\langle\varphi\rangle^2$ and the remaining scalar field, after shifting, corresponds to a scalar meson of mass $m^2(S) = V''(\langle\varphi\rangle) \sim g^4\langle\varphi\rangle^2$.

Physics cannot depend on the arbitrary parameter M but one choice of M can be more convenient than the others, particularly since we are working in perturbation theory and have to control higher order corrections. We have already learned that the perturbative calculation of $\langle\varphi\rangle$ is reliable when the renormalization point M is such that $\lambda(M) \ll g^2(M) \ll 1$. It is also worth noting that from (11.27)

$$\langle\varphi\rangle = M_0 e^{11/6} \tag{11.29}$$

where $\lambda(M_0) = 0$. Usually, we suppose that $\lambda(M)$ and $g(M)$ are given at some scale $M = \tilde{M}$, for example, at the grand unification scale, and we can then use renormalization group arguments to look for a renormalization scale suitable for our calculation (for example, M_0). In our renormalization procedure the scalar quartic coupling is defined in terms of the fourth derivative of V. Thus, to study $\lambda(M)$ one should derive the RGE for this fourth derivative (see Coleman & Weinberg (1973) and Gildener (1976)). We know, however, that the function $\beta(\lambda)$ in the one-loop approximation does not depend on the renormalization conventions. Therefore to the lowest order we can use the results of calculations which adopt momentum space renormalization conventions. The one-loop RGEs for λ and g then read

$$\left.\begin{aligned} 16\pi^2 M(\mathrm{d}\lambda/\mathrm{d}M) &= A\lambda^2(M) + B'\lambda(M)g^2(M) + Cg^4(M) \\ 16\pi^2 M(\mathrm{d}g^2/\mathrm{d}M) &= -bg^4(M) \end{aligned}\right\} \tag{11.30}$$

where the dimensionless coefficients A, B' and C are given, respectively, by the diagrams shown in Fig. 11.2. To solve (11.30) we introduce $\alpha = \lambda/g^2$. Then we get ($t = \ln M$)

$$\begin{aligned} \frac{16\pi^2}{g^2(t)}\frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -bg^2(t)\frac{\mathrm{d}\alpha}{\mathrm{d}g^2} = A\alpha^2(t) + B\alpha(t) + C \\ &= A(\alpha - \alpha_1)(\alpha - \alpha_2), \quad \alpha_1 < \alpha_2 \end{aligned} \tag{11.31}$$

where $B = B' + b$. For real roots there is a UV stable fixed point at $\alpha = \alpha_1 < \alpha_2$ and an IR stable fixed point at $\alpha = \alpha_2$. Thus, if $0 < \alpha(\tilde{M}) = \lambda(\tilde{M})/g^2(\tilde{M}) < \alpha_1$ we expect the RGEs (11.31) to have a solution $\alpha(M) = 0$ for $M \neq 0$. A direct integration gives

$$\exp\left(-b\int_0^{\alpha(\tilde{M})} \frac{\mathrm{d}\alpha}{A\alpha^2 + B\alpha + C}\right) = \frac{g^2(\tilde{M})}{g^2(M)} = 1 + \frac{1}{16\pi^2}g^2(\tilde{M})b\ln\frac{M}{\tilde{M}} \tag{11.32}$$

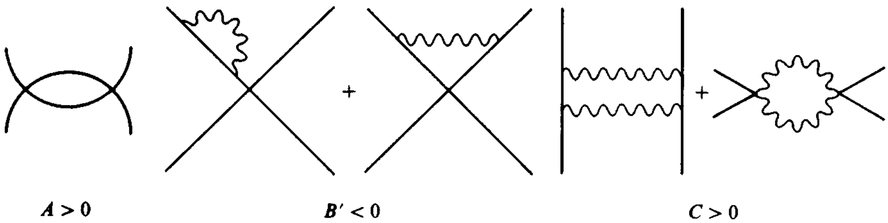


Fig. 11.2.

or, more explicitly, $\alpha(M) = 0$ for

$$M = \tilde{M} \exp\left[-\frac{16\pi^2}{g^2(\tilde{M})} F(\alpha(\tilde{M}))\right] \tag{11.33}$$

where

$$F(\alpha(\tilde{M})) = \frac{1}{b}\left[1 - \exp\left(-b\int_0^{\alpha(\tilde{M})} \frac{d\alpha}{A\alpha^2 + B\alpha + C}\right)\right] \tag{11.34}$$

If $\alpha(\tilde{M})$ is of the order $O(1)$ (i.e. if $\lambda(\tilde{M}) \sim g^2(\tilde{M})$) then $F(\alpha(\tilde{M}))$ is also of order unity and

$$\frac{M}{\tilde{M}} = \exp\left[-\frac{O(1)}{g^2(\tilde{M})}\right] \tag{11.35}$$

Thus, using (11.29) we conclude that for a reasonably small value of $g^2(\tilde{M})$, the dimensional transmutation can generate a scale M of spontaneous symmetry breaking which is enormously smaller than the scale $\tilde{M}$ at which as we suppose $\alpha(\tilde{M})$ is given. This small ratio of mass scales is not put into the theory by hand but arises automatically from the assumption of a vanishing scalar mass. Thus in order for this to happen some massless scalars which escape getting masses of order $\tilde{M}$ must be available in our theory. For instance, if $\tilde{M}$ is the grand unification scale they must remain massless after the spontaneous breakdown of the grand unification group. Such a situation can perhaps arise naturally from supersymmetries which are unbroken at $\tilde{M}$. The mechanism described here very much resembles the generation of the scale Λ at which QCD coupling becomes large (see (7.53a)). It can also be generalized to the case of several quartic scalar couplings present in the theory (Gildener & Weinberg 1976).

The generation of a very small ratio of mass scales by radiative corrections and the dimensional transmutation phenomenon is to be contrasted with making $M/\tilde{M}$ very small in the tree approximation. The latter is highly 'unnatural'. To be more specific, we have in mind the following: consider a gauge group G which we want to be spontaneously broken to G_1 at some scale $\tilde{M}$, and G_1 to be broken to G_2 at M with $M/\tilde{M}$ very small. It turns out that one can obtain such a pattern of spontaneous symmetry breaking in the tree approximation by a proper choice of the scalar potential, but only at the expense of tuning the parameters of the potential to an accuracy much higher than the neglected one-loop corrections. This is called the gauge hierarchy problem.

The formalism presented in this section can be extended to non-abelian gauge theories with scalars and also fermions (see Coleman & Weinberg (1973) and Problem 11.3).

11.3 Dynamical breaking of gauge symmetries and vacuum alignment

Dynamical breaking of gauge symmetry

We consider now the possibility of spontaneous gauge symmetry breaking in the absence of elementary scalar fields. The mechanism presented here is often termed dynamical gauge symmetry breaking. Our discussion in this section is closely connected with that in Section 10.3.

Imagine a spontaneously broken chiral *global* symmetry G of some strong gauge interactions with gauge group G_{H} (we shall call it the hypercolour group) also to be broken explicitly by weak interactions with a gauge symmetry group G_{W}; some of the chiral currents of G couple to gauge bosons of G_{W}. By assumption, it is a good approximation to treat G_{W} interaction to the lowest order (see (10.64)). In Section 10.3 we have learned how to find the preferred vacuum which minimizes the vacuum energy in the presence of the perturbation G_{W}. Assuming the positivity condition for certain current matrix elements the correct vacuum corresponds to the minimal value of the quantity (see (10.75))

$$\sum_\alpha \mathrm{Tr}[{}^X\hat{J}^\alpha_U {}^X\hat{J}^\alpha_U] \tag{11.36}$$

built of the transformed currents projected onto the subspace of the broken generators in the original vacuum; we recall that $\hat{J}^\alpha$ is the linear combination of the G generators corresponding to the G_{W} current $g_\alpha j^\alpha_\mu(x)$ and ${}^X\hat{J}^\alpha$ is that part of $\hat{J}^\alpha$ which contains only broken generators X^α. Equivalently, the correct vacuum must satisfy Dashen's conditions (10.59) and (10.60). Assume that the correct vacuum has been found: it gives a specific orientation of the spontaneously unbroken subgroup H in the G, a set of broken generators X^α and the corresponding Goldstone bosons, some of which get masses induced by the perturbation G_{W}. In general one might also single out another subgroup of G: the maximal subgroup S of elements of G which commute with all the generators of G_{W}. Then the coupling of gauge bosons to the currents of G breaks G explicitly to $G_{\mathrm{W}} \times S$ and we get

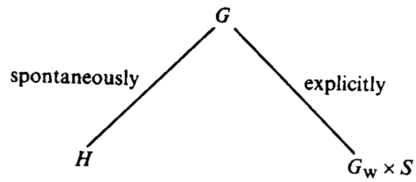


Fig. 11.3.

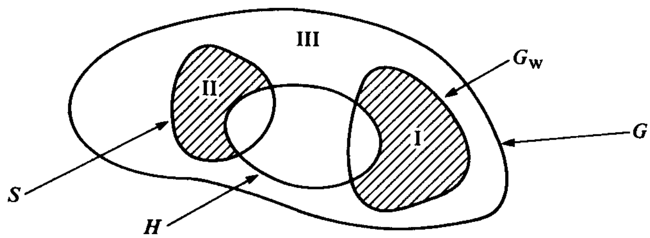


Fig. 11.4.

the picture of the G symmetry breaking shown in Fig. 11.3 where the orientation of different subgroups is illustrated in Fig. 11.4.

In the zeroth order in the G_W couplings there are Goldstone bosons associated with each spontaneously broken generator (regions I + II + III). In presence of the G_W interactions the Goldstone bosons split into three different groups which are easy to classify. The $G_W \times S$ symmetry remains the exact symmetry of the lagrangian and therefore the Goldstone bosons in regions I and II remain massless in any order in the G_W interactions, as can be proved following the arguments of Section 9.6. The generators in region III do not correspond to exact symmetries of the full lagrangian and the corresponding Goldstone bosons acquire, in general, masses which in the first order in G_W are given by (10.55) or (10.77) and (10.78). From the calculation of the π^+–π^0 mass difference we can see that the masses are of the order $g\Lambda(m_\rho \sim \Lambda)$, where g is the coupling constant of the weak interactions and Λ is the confinement scale of the hypercolour interactions. The Goldstone bosons in regions I and II still split into two groups: clearly, those in region II remain in the physical spectrum; however, those in region I are absorbed by the G_W gauge bosons which get masses through the Higgs mechanism. One then talks about the dynamical breaking of the G_W gauge symmetry. Let us discuss the mass generation for the G_W gauge bosons somewhat more explicitly. The Goldstone bosons in region I couple to the G_W currents j^α_μ

$$\langle 0|j^\alpha_\mu(0)|\pi^a\rangle = \mathrm{i}p_\mu f^{\alpha a} \tag{11.37}$$

and therefore the gauge-invariant vacuum polarization tensor $\Pi^{\alpha\beta}_{\mu\nu}$ defined by

$$\begin{aligned} \mathrm{i}\Pi^{\alpha\beta}_{\mu\nu}(p^2) &= g^\alpha g^\beta \int \mathrm{d}^4x \exp(\mathrm{i}px)\langle 0|T j^\alpha_\mu(x) j^\beta_\nu(0)|0\rangle \\ &= \mathrm{i}(g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi^{\alpha\beta}(p^2) \end{aligned} \tag{11.38}$$

has a pole at $p^2 = 0$ (see also Section 11.1)

$$\lim_{p^2\to 0} \Pi^{\alpha\beta}_{\mu\nu}(p^2) = -(g_{\mu\nu}p^2 - p_\mu p_\nu)\sum_a g^\alpha g^\beta f^{\alpha a} f^{\beta a}/p^2 \tag{11.39}$$

It is important to remember the role of gauge invariance in writing (11.39): the Goldstone bosons contribute only to the $p_\mu p_\nu$ term of the polarization tensor; the $g_{\mu\nu}$ term must be present by gauge invariance and we do not need to specify its origin more explicitly. The full gauge boson propagator $\Delta^{\alpha\beta}_{\mu\nu}$ reads†

$$\Delta^{\alpha\beta}_{\mu\nu} = -\mathrm{i}D_{\mu\nu}[\delta^{\alpha\beta} - \Pi^{\alpha\beta}(p^2)] = -\mathrm{i}D_{\mu\nu}(1+\Pi)^{-1\alpha\beta} + \text{higher order terms} \tag{11.40}$$

We are working in the Landau gauge and $-\mathrm{i}D_{\mu\nu}\delta^{\alpha\beta}$ is the free propagator in this gauge. Thus, the residue (11.39) gives the vector boson mass matrix

$$m^2_{\alpha\beta} = \sum_a g_\alpha g_\beta f_{\alpha a} f_{\beta a}$$

The Goldstone boson decay constants $f^{\alpha a}$ can be expressed in terms of the appropriate traces. Under assumption (10.70) we have

$$\langle 0|j^a_\mu(0)|\pi^b\rangle = \mathrm{i}p_\mu f_\pi \delta^{ab} = \mathrm{i}p_\mu f_\pi \,\mathrm{Tr}[\mathrm{X}^a\mathrm{X}^b] \tag{11.41}$$

where j^a_μ is a G current and consequently

$$m^2_{\alpha\beta} = f^2_\pi \sum_a \mathrm{Tr}[\hat{J}^\alpha X^a]\,\mathrm{Tr}[\hat{J}^\beta X^a] = f^2_\pi\,\mathrm{Tr}[{}^X\hat{J}^{\alpha\,X}\hat{J}^\beta] \tag{11.42}$$

(where no sum has been taken over α and β). This result has been obtained for the vacuum state $|0\rangle$. For any other vacuum state related by the unitary transformation $U|0\rangle$ we find

$$m^2_{\alpha\beta} = f^2_\pi\,\mathrm{Tr}[{}^X\hat{J}^{\alpha\,X}_U\hat{J}^\beta_U] \tag{11.43}$$

Comparing with (10.75) we see that

$$\Delta E(\Theta) \sim \mathrm{Tr}\, m^2 \tag{11.44}$$

The conclusion is that the energetically preferred vacuum alignment and the relative orientation of the groups G_W and H in the presence of the perturbation

† $\Delta^{\alpha\beta}_{\mu\nu} = (-\mathrm{i}D_{\mu\nu}\delta^{\alpha\beta}) + (-\mathrm{i}D_{\mu\lambda}\delta^{\alpha\beta'}(-\mathrm{i}\Pi^{\lambda\rho,\beta'\gamma'})(-\mathrm{i}D_{\rho\nu}\delta^{\gamma'\beta}))$.

G_W are such that the trace of the vector boson mass matrix is the minimal one (the gauge symmetry is broken as little as possible). The weak interactions determine their own pattern of symmetry breaking by the choice of the vacuum orientation. We note also that the vector boson mass matrix obeys, in general, certain relations which are consequences of H invariance.

The above ideas may be relevant for the gauge symmetry breaking in the Glashow–Salam–Weinberg theory of the weak interactions (see, for instance, Fahri & Susskind (1981)), however, no convincing model has been constructed yet. We will illustrate them with two simple examples.

Examples

Consider a strong hypercolour gauge group G_H with $2N$ left-handed fermions and $2N$ right-handed fermions transforming as the same representation of G_H. The theory has $G = SU_L(2N) \times SU_R(2N)$ chiral symmetry which, we assume, breaks down spontaneously to $H = SU(2N)$. The embedding of H in G is characterized by a fermion condensate

$$\langle 0|\bar{\Psi}_{Li}\Psi_{Rj}|0\rangle = \Phi_{ij} \tag{11.45}$$

As in Section 9.6, assuming $\Phi_{ij} = v\delta_{ij}$, the chiral group is broken to $SU_{L+R}(2N)$. Any equivalent orientation of the unbroken $SU(2N)$ corresponds to a condensate obtained from (11.45) by a unitary transformation

$$U_R \Phi U_L^\dagger = v U_R U_L^\dagger \tag{11.46}$$

(remember that $U^\dagger \Phi U = U_R \Phi U_L^\dagger$; U is an unitary operator, $U_{L(R)}$ are unitary matrices), i.e. it is specified by a unimodular unitary matrix $U_R U_L^\dagger$. We take the weak gauge group to be $G_W = SU_L(2) \times U(1)$ with fermions in N left-handed G_W doublets and $2N$ right-handed singlets, and with the $U(1)$ charge assignments as follows (see Chapter 12):

$$\begin{array}{cccc} & \begin{pmatrix} \mathcal{U}_i \\ \mathcal{D}_i \end{pmatrix}_L & \mathcal{U}_{iR} & \mathcal{D}_{iR} \\ Y = & q & q + \frac{1}{2} & q - \frac{1}{2} \end{array}$$

Consider first the case $N = 1$. Denoting the gauge bosons of the Weinberg–Salam model by W_μ^α, B_μ the weak coupling is

$$\Delta\mathcal{L} = -\sum_\alpha g W_\mu^\alpha \bar{\Psi}_L \gamma^\mu T^\alpha \Psi_L - g' B_\mu (\bar{\Psi}_R \gamma^\mu T^3 \Psi_R + q\bar{\Psi}\gamma^\mu \Psi) \tag{11.47}$$

where T^α is an $SU(2)$ generator normalized to $\mathrm{Tr}[T^\alpha T^\beta] = \frac{1}{2}\delta^{\alpha\beta}$ and q is the mean electric charge of the doublet. Assuming the condensate (11.45) and expressing the

G_W currents j^α_μ in terms of the G currents corresponding to the unbroken subgroup (vector currents) and to the broken generators (axial currents) $j_L = \frac{1}{2}(\mathcal{V}-\mathcal{A})$, $j_R = \frac{1}{2}(\mathcal{V}+\mathcal{A})$ one gets in the (Ψ_L, Ψ_R) basis the following expression:

$$\begin{aligned}\hat{\jmath}^\alpha &= g\left(\begin{array}{c|c} T^\alpha & \\ \hline & 0\end{array}\right) + g'\left(\begin{array}{c|c} q & \\ \hline & q+T^3\end{array}\right) = \left[\tfrac{1}{2}g\left(\begin{array}{c|c} T^\alpha & \\ \hline & T^\alpha\end{array}\right) + \tfrac{1}{2}g'\left(\begin{array}{c|c} 2q+T^3 & \\ \hline & 2q+T^3\end{array}\right)\right]\\ &\quad + \left[\tfrac{1}{2}g\left(\begin{array}{c|c} T^\alpha & \\ \hline & -T^\alpha\end{array}\right) + \tfrac{1}{2}g'\left(\begin{array}{c|c} -T^3 & \\ \hline & T^3\end{array}\right)\right] \qquad (11.48)\end{aligned}$$

Inserting (11.48) into (11.42) gives the familiar vector boson matrix

	W_1	W_2	W_3	B
W_1	$\frac{1}{4}g^2 f_\pi^2$			
W_2		$\frac{1}{4}g^2 f_\pi^2$		
W_3			$\frac{1}{4}g^2 f_\pi^2$	$-\frac{1}{4}gg' f_\pi^2$
B			$-\frac{1}{4}gg' f_\pi^2$	$\frac{1}{4}g'^2 f_\pi^2$

After diagonalization one gets

$$\left.\begin{array}{l} m_{W_{1,2}} = \frac{1}{2}gf_\pi, \quad m_Z = \frac{1}{2}f_\pi(g'^2+g^2)^{1/2} = m_W/\cos\Theta_W \\ m_\gamma = 0 \end{array}\right\} \qquad (11.49)$$

where Θ_W, the Weinberg angle, is given by

$$\cos\Theta_W = \frac{g}{(g'^2+g^2)^{1/2}}$$

and

$$\begin{aligned}\gamma &= \cos\Theta_W B + \sin\Theta_W W_3\\ Z &= -\sin\Theta_W B + \cos\Theta_W W_3\end{aligned}$$

In this model all the Goldstone bosons have been used to give masses to the weak gauge bosons. There is no problem of vacuum alignment since all the equivalent vacua (11.46) can be transformed into (11.45) by the gauge transformations in the weak sector.

For $N > 1$ left-handed doublets with the same $U(1)$ charge assignment the situation is more complex. If we introduce the $4N$-plet $\Psi = (\mathcal{U}_{1L}\mathcal{D}_{1L}\ldots\mathcal{D}_{NL}\mathcal{D}_{NL}\mathcal{U}_{1R}\mathcal{D}_{1R}\ldots\mathcal{U}_{NR}\mathcal{D}_{NR})$ then the Salam–Weinberg coupling is

$$\Delta\mathcal{L} = -gW^\alpha_\mu\bar{\Psi}\gamma^\mu\left(\begin{array}{c|c} T^\alpha\otimes 1\!\!1 & \\ \hline & 0\end{array}\right)\Psi - g'B_\mu\bar{\Psi}\gamma^\mu\left(\begin{array}{c|c} q & \\ \hline & q+T^3\otimes 1\!\!1\end{array}\right)\Psi \qquad (11.50)$$

where the $2N\times 2N$ block $T^\alpha\otimes 1\!\!1$ is a direct product of the 2×2 weak isospin and the $N\times N$ matrix on the space of N doublets. The weak coupling with the gauge symmetry $G_{\mathrm{W}} = SU_{\mathrm{L}}(2) \times U(1)$ breaks the original chiral symmetry $G = SU_{\mathrm{L}}(2N) \times SU_{\mathrm{R}}(2N)$ of the strong interaction sector into $G \to G_{\mathrm{W}} \times S$ where $S = SU_{\mathrm{L}}(N) \times SU_{\mathrm{R}}(N) \times SU_{\mathrm{R}}(N)$ with the first $SU(N)$ corresponding to horizontal transformations among N left-handed doublets, and the second (the third) $SU(N)$ corresponding to transformations among N right-handed singlets $\mathcal{U}_{i\mathrm{R}}(\mathcal{D}_{i\mathrm{R}})$; $\mathcal{U}_{i\mathrm{R}}$ and $\mathcal{D}_{i\mathrm{R}}$ have different charges and therefore the original $SU_{\mathrm{R}}(2N)$ is broken into $SU_{\mathrm{R}}(N) \times SU_{\mathrm{R}}(N)$. The spontaneous breakdown of G into $SU_{\mathrm{L+R}}(2N)$ induced by (11.46) leaves the subgroup $SU_{\mathrm{L+R}}(N)$ of S unbroken. The spontaneous breakdown of the original $SU(2N) \times SU(2N)$ chiral symmetry produces $4N^2 - 1$ Goldstone bosons. Three of them are used to give masses to the gauge bosons. Again, there is no vacuum alignment problem. The breaking of S into $SU(N)$ which remains the exact symmetry of the full theory leaves $2(N^2 - 1)$ exactly massless Goldstone bosons. They are electrically neutral because they correspond to generators of transformations among fermions with the same charge. There remain $2(N^2 - 1)$ charged Goldstone bosons which do not correspond to generators of exact symmetries of the full lagrangian and they should, therefore, receive masses from the G_{W} interactions. Splitting the weak currents into the unbroken ${}^{Y}\hat{J}^{\alpha}$ and the broken parts ${}^{X}\hat{J}^{\alpha}$ as in (11.48)

$$\left.\begin{aligned} {}^{Y}\hat{J}^{\alpha} &\equiv \tfrac{1}{2} g \left(\begin{array}{c|c} T^{\alpha} \otimes \mathbb{1} & \\ \hline & T^{\alpha} \otimes \mathbb{1} \end{array}\right) + \tfrac{1}{2} g' \left(\begin{array}{c|c} 2q + T^{3} \otimes \mathbb{1} & \\ \hline & 2q + T^{3} \otimes \mathbb{1} \end{array}\right) \\ {}^{X}\hat{J}^{\alpha} &\equiv -\tfrac{1}{2} g \left(\begin{array}{c|c} -T^{\alpha} \otimes \mathbb{1} & \\ \hline & T^{\alpha} \otimes \mathbb{1} \end{array}\right) + \tfrac{1}{2} g' \left(\begin{array}{c|c} -T^{3} \otimes \mathbb{1} & \\ \hline & T^{3} \otimes \mathbb{1} \end{array}\right) \end{aligned}\right\} \tag{11.51}$$

it is evident from (10.79) that the Goldstone bosons receive no mass in the first order in G_{W}. The broken generators X_a are of the block-diagonal form

$$\left(\begin{array}{c|c} -A & \\ \hline & A \end{array}\right)$$

where the As are $4N^2 - 1$ matrices which can be easily constructed for any N, and therefore for ${}^{Y}\hat{J}$ and ${}^{X}\hat{J}$ given by (11.51) the two contributions to the trace in (10.79) cancel each other. Actually, one can see that if G_{W} commutes with one of the chiral groups $SU_{\mathrm{L}}(2N)$ or $SU_{\mathrm{R}}(2N)$ then there will be pseudo-Goldstone bosons which remain massless to the lowest order in G_{W}. Indeed, one sees directly from (10.66), using H invariance as in (10.69), that the Θ-dependent (U-dependent) part of the vacuum energy $\Delta E(\Theta)$ relevant for the pseudo-Goldstone boson masses (10.79) comes from the term

$$\Delta E(\Theta) = -\tfrac{1}{2} \int \mathrm{d}^4 x\, \Delta^{\mu\nu}(x) \sum_{\alpha} g_{\alpha}^{2} \langle 0 | T U_{\mathrm{L}}^{\dagger} j_{\mu\mathrm{L}}^{\alpha}(x) U_{\mathrm{L}} U_{\mathrm{R}}^{\dagger} j_{\nu\mathrm{R}}^{\alpha}(0) U_{\mathrm{R}} | 0 \rangle \tag{11.52}$$

whereas $j_{\mu\mathrm{L}}^{\alpha} j_{\nu\mathrm{L}}^{\alpha}$ and $j_{\mu\mathrm{R}}^{\alpha} j_{\nu\mathrm{R}}^{\alpha}$ terms contribute only the Θ-independent constants.

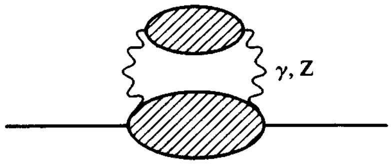


Fig. 11.5.

In our example the $U(1)$ current also does not contribute to $\Delta E(\Theta)$ (although it has both the left-handed and the right-handed parts) because the left-handed $U(1)$ current is an $SU_{\mathrm{L}}(2N)$ singlet. The $U(1)$ contribution is therefore $SU_{\mathrm{L}}(2N)$-invariant and also $SU_{\mathrm{L+R}}(2N)$-invariant because the vacuum is, so it is $SU_{\mathrm{L}}(2N) \times SU_{\mathrm{R}}(2N)$-invariant and hence Θ-independent. The charged pseudo-Goldstone bosons receive masses in the next order in G_{W} due to the mass splitting of the γ and Z which has to be taken into account in that order, for example, as in Fig. 11.5.

In the model discussed above there remain massless neutral Goldstone bosons in the physical spectrum because the $SU(N)$ group of transformations between doublets remains the exact symmetry of the theory. This can be avoided by giving the left-handed doublets different $U(1)$ charge. Such a model is very instructive because then the $U(1)$ boson exchange contributes the Θ-dependent term to the $\Delta E(\Theta)$ and the pattern of the G_{W} breakdown depends on the $U(1)$ charge assignments.

To be specific we consider a model with four left-handed and four right-handed fermions with the G_{W} representation content given by

$$\begin{array}{cccccc} \begin{pmatrix} \mathcal{U}_1 \\ \mathcal{D}_1 \end{pmatrix}_{\mathrm{L}} & \begin{pmatrix} \mathcal{U}_2 \\ \mathcal{D}_2 \end{pmatrix}_{\mathrm{L}} & \mathcal{U}_{1\mathrm{R}} & \mathcal{D}_{1\mathrm{R}} & \mathcal{U}_{2\mathrm{R}} & \mathcal{D}_{2\mathrm{R}} \\ Y = \Delta, & -\Delta, & (\Delta + \frac{1}{2}), & (\Delta - \frac{1}{2}), & (-\Delta + \frac{1}{2}), & (-\Delta - \frac{1}{2}) \end{array} \tag{11.53}$$

Similarly to (11.50) the G_{W} coupling in the $\Psi_{\mathrm{L}}, \Psi_{\mathrm{R}}$ basis ($\Psi = (\mathcal{U}_{1\mathrm{L}}\mathcal{D}_{1\mathrm{L}}\mathcal{U}_{2\mathrm{L}}\mathcal{D}_{2\mathrm{L}}\mathcal{U}_{1\mathrm{R}}\mathcal{D}_{1\mathrm{R}}\mathcal{U}_{2\mathrm{R}}\mathcal{D}_{2\mathrm{R}})$) is

$$\begin{aligned} \Delta\mathcal{L} = &-gW_\mu^\alpha \bar{\Psi}\gamma^\mu \left(\begin{array}{c|c} T^\alpha \otimes 1\!\!1 & \\ \hline & 0 \end{array}\right)\Psi - g' B_\mu \bar{\Psi}\gamma^\mu \left[\left(\begin{array}{c|c} 0 & \\ \hline & T^3 \otimes 1\!\!1 \end{array}\right) \right. \\ &\left. + \Delta \left(\begin{array}{c|c} 1\!\!1 \otimes T^3 & \\ \hline & 1\!\!1 \otimes T^3 \end{array}\right) \right]\Psi \end{aligned} \tag{11.54}$$

where in the product $A \otimes B$ the first matrix is in the 2×2 weak space and the second one is on the space of doublets. Let us first assume that the $SU(4) \times SU(4)$ symmetry of the G_{H} lagrangian is broken by the

$$\langle 0|\bar{\mathcal{U}}_{\mathrm{L}i}\mathcal{U}_{\mathrm{R}j}|0\rangle = \langle 0|\bar{\mathcal{D}}_{\mathrm{L}i}\mathcal{D}_{\mathrm{R}j}|0\rangle = v\delta_{ij} \tag{11.55}$$

condensates. This defines the vacuum and the breaking $SU_{\mathrm{L}}(4) \times SU_{\mathrm{R}}(4) \to SU_{\mathrm{L+R}}(4)$ in the basis (11.53). We can now split the G_{W} currents into unbroken and broken parts in the vacuum (11.55). It is clear from our previous discussion that only the $U(1)$ current is interesting to us; as before the contribution of the $SU(2)$ current to the two terms under the trace in (10.79) cancels. For the $U(1)$ current we have

$$\hat{J} = \tfrac{1}{2}g' \left(\begin{array}{c|c} T^3 \otimes \mathbb{1} + 2\Delta \mathbb{1} \otimes T^3 & \\ \hline & T^3 \otimes \mathbb{1} + 2\Delta \mathbb{1} \otimes T^3 \end{array}\right) + \tfrac{1}{2}g' \left(\begin{array}{c|c} -T^3 \otimes \mathbb{1} & \\ \hline & T^3 \otimes \mathbb{1} \end{array}\right) \tag{11.56}$$

There are 15 Goldstone bosons corresponding to the broken generators

$$X = \frac{1}{2}\left(\begin{array}{c|c} -A & \\ \hline & A \end{array}\right)$$

where A represents $T^a \otimes \mathbb{1}$, $2T^a \otimes T^3$, $\mathbb{1} \otimes T^3$ and eight matrices of the form, for example,

$$\frac{1}{\sqrt{2}}\left(\begin{array}{c|c} 0 & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} & 0 \end{array}\right) \quad \text{or} \quad \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} 0 & \begin{matrix} -\mathrm{i} & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} \mathrm{i} & 0 \\ 0 & 0 \end{matrix} & 0 \end{array}\right) \tag{11.57}$$

linking one of $(\mathcal{U}_1, \mathcal{D}_1)$ with one of $(\mathcal{U}_2, \mathcal{D}_2)$. A look at (10.79) and (11.56) tells us that Goldstone bosons corresponding to broken generators $T^a \otimes \mathbb{1}$, $T^a \otimes T^3$ and $\mathbb{1} \otimes T^3$ receive no mass in the first order in G_{W}. An explicit calculation based on (10.79), (11.56) and (11.57) gives the following masses for the remaining pseudo-Goldstone bosons which are denoted by the flavours linked by the corresponding generators: non-diagonal terms of the mass matrix vanish and bosons corresponding to the two generators like (11.57) are degenerate in mass and

$$\left.\begin{aligned} m^2_{\mathcal{U}_1\mathcal{U}_2} = m^2_{\mathcal{D}_1\mathcal{D}_2} &\sim g'^2\Delta^2 M^2 \\ m^2_{\mathcal{U}_1\mathcal{D}_2} &\sim g'^2\Delta(\Delta+1)M^2 \\ m^2_{\mathcal{D}_1\mathcal{U}_2} &\sim g'^2\Delta(\Delta-1)M^2 \end{aligned}\right\} \tag{11.58}$$

where M^2 is the integral given by (10.78). Assuming that the integral M^2 is dominated by the lowest-lying vector mesons of the correct quantum numbers we find, as for the π^+–π^0 mass difference in Section 10.4, $M^2 > 0$. Therefore, we find an interesting result: for $|\Delta| > 1$ Dashen's condition is satisfied by masses (11.58) and the vacuum (11.55) is the preferred vacuum in the presence of perturbation G_{W}. However, for the $U(1)$ charge assignments for doublets such that $|\Delta| < 1$ the vacuum (11.55) turns out to be unstable. We must, therefore, find the correct vacuum in the set of vacua obtained from (11.55) by all unitary transformations U of $SU(4) \times SU(4)$. According to (11.36) the right transformation is the one which minimizes the $\mathrm{Tr}\,{}^X\hat{J}_U^2$ or maximizes the $\mathrm{Tr}\,{}^Y\hat{J}_U^2$ of the projection of the transformed current on the subspace of the original broken (X^a) and unbroken (Y^a) generators, respectively. Before we find this transformation let us notice that in the original basis (11.54) the term in the lagrangian proportional to Δ is $SU_{\mathrm{L+R}}(4)$-invariant. It is then clear that for $|\Delta| > 1$ that basis is the right one: the $\Delta\mathcal{L}$ given by (11.54) is 'as far as possible' $SU_{\mathrm{L+R}}(4)$-invariant. However, for $|\Delta| < 1$ one can imagine that a transformation $U^\dagger \hat{J} U$, which breaks the $SU_{\mathrm{L+R}}(4)$-invariance of the Δ term, simultaneously transforms the other term of the $U(1)$ current to such a form that the full transformed current has 'better' $SU_{\mathrm{L+R}}(4)$ symmetry than the original one. The right transformation can be found by noting that the transformed $U(1)$ current, split into unbroken and broken parts, is as follows:

$$U^\dagger \left(\begin{array}{cccc|cccc} \Delta &&&&&&& \\ & \Delta &&&&&& \\ && -\Delta &&&&& \\ &&& -\Delta &&&& \\ \hline &&&& 1+\Delta &&& \\ &&&&& -1+\Delta && \\ &&&&&& 1-\Delta & \\ &&&&&&& -1-\Delta \end{array}\right) U$$

$$= \left(\begin{array}{c|c} A & \\ \hline & A \end{array}\right) + \left(\begin{array}{c|c} B & \\ \hline & -B \end{array}\right)$$

where the matrix on the l.h.s. is the $U(1)$ current in the original basis (11.54). Since the trace of B^2 should be the minimal one, the transformed current should have both 4×4 blocks as similar as possible ($A + B \approx A - B \approx A$). Hence, for $|\Delta| < 1$ the right transformation is the one which gives

$$U^\dagger \hat{J} U = \left(\begin{array}{cccc|cccc} \Delta &&&&&&& \\ & \Delta &&&&&& \\ && -\Delta &&&&& \\ &&& -\Delta &&&& \\ \hline &&&& 1+\Delta &&& \\ &&&&& 1-\Delta && \\ &&&&&& -1+\Delta & \\ &&&&&&& -1-\Delta \end{array}\right)$$

From (11.54) one sees that this is the transformation $\mathcal{D}_{1R} \leftrightarrow \mathcal{U}_{2R}$. The new vacuum is then defined by the condensate

$$U_R \Phi U_L^\dagger = v \begin{array}{c|c} \begin{array}{cccc} \bar{\mathcal{U}}_{1L} & \bar{\mathcal{D}}_{1L} & \bar{\mathcal{U}}_{2L} & \bar{\mathcal{D}}_{2L} \end{array} & \\ \hline \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \mathcal{U}_{1R} \\ \mathcal{D}_{1R} \\ \mathcal{U}_{2R} \\ \mathcal{D}_{2R} \end{array} \end{array}$$

so the electrically charged composite field $\bar{\mathcal{U}}_2\mathcal{D}_1$ acquires a non-zero vacuum expectation value. Thus, the $U(1)_{\rm em}$ subgroup of the gauge group $SU(2) \times U(1)$ is broken and the photon has a mass when $|\Delta| < 1$. Of course, one can easily find a set of broken and unbroken generators corresponding to the new vacuum as well as all the gauge boson masses and pseudo-Goldstone boson masses.

An interesting conclusion of the above exercise is that the coupling of a weak gauge group to the chiral currents of a spontaneously broken chiral symmetry of strong interactions may, for dynamical reasons, lead to different theories depending on the specific assignment of the chiral multiplets to the weak group representations.

Problems

11.1 Derive the remaining Feynman rules for the $U(1)$ theory of Section 11.1 in the R_ξ gauge.

11.2 Check by explicit calculation that at the tree level the unphysical ξ-dependent poles cancel in the S-matrix for the process $\varphi_1 + \varphi_2 \to A_1 + A_2$ where φ and A are the physical scalar particle and the massive vector meson of the $U(1)$ theory in Section 11.1, respectively.

11.3 Extend the discussion of Section 11.2 to non-abelian gauge theories including fermions. Calculate the Higgs particle (physical scalar) mass in terms of the W and Z masses in the $SU(2) \times U(1)$ electroweak theory with one massless scalar complex doublet and taking fermions as massless.

12

Standard electroweak theory

The basic concepts of the electroweak theory (Glashow 1961, Weinberg 1967*b*, Salam 1968) have been outlined in the Introduction. The underlying symmetry group $SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ emerges from study of the charged weak currents, supplemented by the idea of minimal unification with electromagnetic forces. The symmetry is assumed to be local gauge symmetry, in accordance with general remarks in the Introduction and as required by unification with electromagnetism. Thus, we have the following gauge vector fields:† W_μ^a for $SU_{\mathrm{L}}(2)$ and B_μ for $U_{\mathrm{Y}}(1)$. The matter fields consist of quarks and leptons forming three generations. In terms of the left-handed Weyl fields each generation consists of:

$$\left.\begin{array}{ll} l_A & \left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right) \\ e_A^c & (\mathbf{1}, \mathbf{1}, +1) \\ q_A & \left(\mathbf{3}, \mathbf{2}, +\frac{1}{6}\right) \\ d_A^c & \left(\mathbf{3}^\star, \mathbf{1}, +\frac{1}{3}\right) \\ u_A^c & \left(\mathbf{3}^\star, \mathbf{1}, -\frac{2}{3}\right) \end{array}\right\} \tag{12.1}$$

where $A = 1, 2, 3$ is the generation index. (Any reader unfamiliar with the two-component spinor notation should consult Chapter 1 and Appendix A.) The first and second entries show the transformation properties under the colour $SU(3)$ and the weak $SU_{\mathrm{L}}(2)$ groups, respectively. The last entry shows the hypercharge of each field. We recall that the hypercharge is defined as $Y = Q - T^3$, where Q is the electric charge of the field, and T^3 is the third component of its weak isospin $SU_{\mathrm{L}}(2)$. At this stage the superscript 'c' does not have the same meaning as in Chapter 1. It is just used to denote the left-handed Weyl fields which are singlets of $SU_{\mathrm{L}}(2)$. It will acquire the proper meaning of charge conjugation (as in Chapter 1) after the electroweak symmetry breaking to the electromagnetic $U(1)$. The hermitean conjugates of the fields carrying it will then combine with appropriate

† The $SU(3)$ gauge group of QCD was extensively discussed in Chapter 8. The complete $SU_{\mathrm{c}}(3) \times SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ theory is called the Standard Model of elementary interactions.

components of the doublets to form Dirac fields, with charge conjugation acting only on their electric and colour (i.e. conserved) charges. Writing explicitly the $SU_{\mathrm{L}}(2)$ doublet components and using the conventional terminology we have

$$l_1 \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \qquad l_2 \equiv \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \qquad l_3 \equiv \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$
$$e_1^c \equiv e^c, \qquad e_2^c \equiv \mu^c, \qquad e_3^c \equiv \tau^c$$

for leptons and

$$q_1 \equiv \begin{pmatrix} u \\ d \end{pmatrix}, \qquad q_2 \equiv \begin{pmatrix} c \\ s \end{pmatrix}, \qquad q_3 \equiv \begin{pmatrix} t \\ b \end{pmatrix}$$
$$u_1^c \equiv u^c, \qquad u_2^c \equiv c^c, \qquad u_3^c \equiv t^c$$
$$d_1^c \equiv d^c, \qquad d_2^c \equiv s^c, \qquad d_3^c \equiv b^c$$

for quarks.

We can also introduce the four-component chiral fermion fields, as in (1.72):

$$\left.\begin{array}{l} (\Psi^i_{l_A})_{\mathrm{L}} \equiv \begin{pmatrix} l^i_A \\ 0 \end{pmatrix}, \qquad (\Psi_{e_A})_{\mathrm{R}} \equiv \begin{pmatrix} 0 \\ \bar{e}^c_A \end{pmatrix}, \\ (\Psi^i_{q_A})_{\mathrm{L}} \equiv \begin{pmatrix} q^i_A \\ 0 \end{pmatrix}, \qquad (\Psi_{d_A})_{\mathrm{R}} \equiv \begin{pmatrix} 0 \\ \bar{d}^c_A \end{pmatrix}, \qquad (\Psi_{u_A})_{\mathrm{R}} \equiv \begin{pmatrix} 0 \\ \bar{u}^c_A \end{pmatrix} \end{array}\right\} \tag{12.2}$$

where $i = 1, 2$. We recall that the unbarred (barred) two-component spinors carry lower (upper) Lorentz indices. The left-(right-)handed chiral fields $\Psi_{\mathrm{L(R)}} \equiv \frac{1}{2}(1 \mp \gamma_5)\Psi_{\mathrm{L(R)}}$ are doublets (singlets) under $SU_{\mathrm{L}}(2)$ and the electric and colour charges of the left- and right-handed fields are pairwise the same. We shall also use the notation $(\Psi^1_{l_A})_{\mathrm{L}} \equiv (\Psi_{\nu_A})_{\mathrm{L}}$.

The gauge symmetry $SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ must be spontaneously broken to the electromagnetic $U_{\mathrm{EM}}(1)$ or, more precisely, the so-called Higgs mechanism (discussed in Section 11.1) must be invoked. To this end one introduces a doublet of scalar fields H, called the Higgs doublet, with the quantum numbers $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ with respect to the full $SU(3) \times SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ gauge group. One Higgs doublet is the minimal set of scalar fields which allows for the required symmetry breaking. Since the presence of more doublets (for constraints on the presence of other representations of scalars see Problem 12.1) has no particular theoretical motivation (apart from the supersymmetric models which require an even number of Higgs doublets), the one-doublet theory is what is called the Standard Model. The Higgs mechanism is the origin of the masses of the $W^\pm$ and Z^0 vector bosons (which are linear combinations of the original W^a_μ and B_μ fields) and of the masses of elementary fermions, which are strictly massless in the symmetry limit. It does not destroy the renormalizability of the theory ('t Hooft 1971a, b).

Moreover, interestingly enough, Higgs doublets and only Higgs doublets can have renormalizable Yukawa couplings to the left- and right-handed chiral matter fields. Therefore, after symmetry breaking, they can generate the Dirac mass terms for those particles.

As discussed in Chapter 9, the pattern $SU(2) \times U(1) \to U'(1)$ is unique if the Higgs mechanism is generated by vacuum expectation values of the components of a complex scalar $SU(2)$ doublet field $H(x)$. By proper redefinition of the group generator basis the remaining $U'(1)$ can always be interpreted as the electromagnetic $U_{\mathrm{EM}}(1)$. With the basis fixed by the charge assignment (12.1) the desired pattern is obtained with

$$H(x) = \begin{pmatrix} H^+(x) \\ H^0(x) \end{pmatrix} \tag{12.3}$$

where

$$\langle 0|H(x)|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{12.4}$$

and the components H^+ and H^0 are complex fields. It may be useful to stress that, according to (12.4), the vacuum carries quantum numbers of the component H^0. This is different from quantum vacuum fluctuations which are singlets with respect to all symmetries, and are eliminated by normal ordering (a redefinition of the vacuum energy) which is understood also to be present in (12.4). As discussed in Chapters 1 and 9, spontaneously broken symmetries can be intuitively depicted as the property of the vacuum which is a big reservoir of the scalars H^0.

The electroweak theory has been confirmed experimentally with the precision of one per mille, mainly by the LEP experiments, as the correct theory of elementary interactions for the energy range up to $\mathcal{O}(100\ \mathrm{GeV})$. In particular, the quantum one- and partly two-loop corrections to the tree level predictions have been tested (see later). However, the Higgs scalar has not yet (1999) been discovered, and is extensively searched for in present and planned experiments. From the point of view of the electroweak theory considered in the range up to $\mathcal{O}(100\ \mathrm{GeV})$, the Higgs boson mass range can extend up to $\mathcal{O}(1\ \mathrm{TeV})$. It can be elementary or composite or even only the effective description of a strongly interacting (at the scale $\mathcal{O}(1\ \mathrm{TeV})$) gauge boson sector (Section 11.3). Thus, full experimental clarification of the Higgs sector is expected to provide an important bridge to physics beyond the Standard Model.

12.1 The lagrangian

With the ideas and the formalism introduced so far in this book we are well equipped to formulate the electroweak theory (Glashow 1961, Weinberg 1967*b*, Salam 1968, 't Hooft 1971a, b) as a gauge theory. The full classical gauge-invariant lagrangian consists of several parts:

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{Higgs}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{g.fix}} + \mathcal{L}_{\mathrm{ghosts}} \tag{12.5}$$

The gauge boson part has the standard form:

$$\mathcal{L}_{\mathrm{gauge}} = -\tfrac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{12.6}$$

where $W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\varepsilon^{abc} W^b_\mu W^c_\nu$ and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the field strengths of the W^a_μ and B_μ gauge fields and ε^{abc} are the $SU_{\mathrm{L}}(2)$ group structure constants. Its kinetic and interaction parts are, respectively,

$$\mathcal{L}^{\mathrm{kin}}_{\mathrm{gauge}} = \tfrac{1}{2} W^a_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) W^a_\nu + \tfrac{1}{2} B_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) B_\nu$$

$$\mathcal{L}^{\mathrm{int}}_{\mathrm{gauge}} = g\varepsilon^{abc}\left(\partial^\mu W^{a\nu}\right) W^b_\mu W^c_\nu - \tfrac{1}{4} g^2 \left(W^{a\mu} W^{b\nu} W^a_\mu W^b_\nu - W^{a\mu} W^{a\nu} W^b_\mu W^b_\nu\right)$$

The matter field lagrangian in the two-component spinor notation reads

$$\begin{aligned}
\mathcal{L}_{\mathrm{matter}} = {} & \mathrm{i}\bar{l}_A \bar{\sigma}^\mu \left(\partial_\mu + \mathrm{i}g W^a_\mu T^a - \frac{\mathrm{i}}{2} g' B_\mu\right) l_A \\
& + \mathrm{i}\bar{e}^c_A \bar{\sigma}^\mu \left(\partial_\mu + \mathrm{i}g' B_\mu\right) e^c_A \\
& + \mathrm{i}\bar{q}_A \bar{\sigma}^\mu \left(\partial_\mu + \mathrm{i}g W^a_\mu T^a + \frac{\mathrm{i}}{6} g' B_\mu\right) q_A \\
& + \mathrm{i}\bar{d}^c_A \bar{\sigma}^\mu \left(\partial_\mu + \frac{\mathrm{i}}{3} g' B_\mu\right) d^c_A \\
& + \mathrm{i}\bar{u}^c_A \bar{\sigma}^\mu \left(\partial_\mu - \mathrm{i}\frac{2}{3} g' B_\mu\right) u^c_A
\end{aligned} \tag{12.7}$$

where T^a are the $SU_{\mathrm{L}}(2)$ generators in the fundamental representation and g, g' are the two coupling constants associated with $SU_{\mathrm{L}}(2)$ and $U_{\mathrm{Y}}(1)$ groups, respectively. The generation index A is summed from 1 to 3. In terms of four-component chiral fermions defined in (12.2) the lagrangian (12.7) can be rewritten in the more familiar form

$$\begin{aligned}
\mathcal{L}_{\mathrm{matter}} = {} & \mathrm{i}\overline{(\Psi_{l_A})_{\mathrm{L}}} \gamma^\mu \left(\partial_\mu + \mathrm{i}g W^a_\mu T^a - \frac{\mathrm{i}}{2} g' B_\mu\right) (\Psi_{l_A})_{\mathrm{L}} \\
& + \mathrm{i}\overline{(\Psi_{e_A})_{\mathrm{R}}} \gamma^\mu \left(\partial_\mu - \mathrm{i}g' B_\mu\right) (\Psi_{e_A})_{\mathrm{R}} \\
& + \mathrm{i}\overline{(\Psi_{q_A})_{\mathrm{L}}} \gamma^\mu \left(\partial_\mu + \mathrm{i}g W^a_\mu T^a + \frac{\mathrm{i}}{6} g' B_\mu\right) (\Psi_{q_A})_{\mathrm{L}} \\
& + \mathrm{i}\overline{(\Psi_{d_A})_{\mathrm{R}}} \gamma^\mu \left(\partial_\mu - \frac{\mathrm{i}}{3} g' B_\mu\right) (\Psi_{d_A})_{\mathrm{R}} \\
& + \mathrm{i}\overline{(\Psi_{u_A})_{\mathrm{R}}} \gamma^\mu \left(\partial_\mu + \mathrm{i}\frac{2}{3} g' B_\mu\right) (\Psi_{u_A})_{\mathrm{R}}
\end{aligned} \tag{12.8}$$

(Note the change of signs of the $U(1)$ charges of the $SU_{\mathrm{L}}(2)$ singlets in (12.8) compared to (12.7).)

The lagrangian for the Higgs field has the form

$$\mathcal{L}_{\text{Higgs}} = H^{\dagger}\left(\overleftarrow{\partial}_{\mu} - \mathrm{i}gT^{a}W_{\mu}^{a} - \frac{\mathrm{i}}{2}g'B_{\mu}\right)\left(\overrightarrow{\partial}_{\mu} + \mathrm{i}gT^{a}W_{\mu}^{a} + \frac{\mathrm{i}}{2}g'B_{\mu}\right)H \\ - m^{2}H^{\dagger}H - \frac{\lambda}{2}\left(H^{\dagger}H\right)^{2} \tag{12.9}$$

Interactions of the Higgs doublet with the matter fields are described by the Yukawa part of the full lagrangian (12.5):

$$\mathcal{L}_{\text{Yukawa}} = -Y_{l}^{BA}H_{i}^{\star}l_{iA}e_{B}^{c} - Y_{d}^{BA}H_{i}^{\star}q_{iA}d_{B}^{c} - Y_{u}^{BA}\varepsilon_{ij}H_{i}q_{jA}u_{B}^{c} + \text{h.c.} \tag{12.10}$$

where $\varepsilon_{12} = -\varepsilon_{21} = -1$, Y^{AB} are 3×3 complex Yukawa coupling matrices and summation over the $SU_{\mathrm{L}}(2)$ indices i, j and over the fermion generation indices A and B is understood. We have also used the fact that the two-dimensional representation of $SU(2)$ is real (see Appendix E) and $\mathrm{i}\tau_2 H$ transforms as $H^{\star}$, i.e. as $2^{\star}$ of $SU_{\mathrm{L}}(2)$. Therefore, $(\mathrm{i}\tau_2 H)_j q_j = \varepsilon_{ij}H_i q_j$ is also an invariant of $SU_{\mathrm{L}}(2)$.† Using the four-component chiral fields this can be rewritten as

$$\mathcal{L}_{\text{Yukawa}} = -Y_{l}^{BA}H_{i}^{\star}\overline{(\Psi_{e_B})_{\mathrm{R}}}(\Psi_{l_A}^{i})_{\mathrm{L}} - Y_{d}^{BA}H_{i}^{\star}\overline{(\Psi_{d_B})_{\mathrm{R}}}(\Psi_{q_A}^{i})_{\mathrm{L}} \\ - Y_{u}^{BA}\varepsilon_{ij}H_{i}\overline{(\Psi_{u_B})_{\mathrm{R}}}(\Psi_{q_A}^{j})_{\mathrm{L}} + \text{h.c.} \tag{12.11}$$

When $m^2 < 0$ is chosen in (12.9) the Higgs doublet acquires the vacuum expectation value. Indeed, the minimum of the potential is for

$$\langle H^{\dagger}H\rangle = -\frac{m^2}{\lambda} \equiv \frac{v^2}{2} \tag{12.12}$$

By $SU_{\mathrm{L}}(2)$ rotation we can always redefine the vacuum so that only the vacuum expectation value of the lower component of the Higgs doublet is non-vanishing. The $SU_{\mathrm{L}}(2)\times U_{\mathrm{Y}}(1)$ symmetry is then broken down to $U'(1)$ (which is identified with the electromagnetic $U(1)$ with $Q = T^3 + Y$) because

$$T^{a}\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \neq 0, \quad \text{for } a = 1,2,3, \qquad (T^{3}+Y)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = 0 \tag{12.13}$$

where Y is the hypercharge generator.

† In the lagrangian (12.10) we have not included the Yukawa couplings for neutrinos since our list (12.2) of physical fields does not contain the right-handed neutrinos. Note, however, that the right-handed neutrinos, if they exist, are sterile ($T_3 = Y = 0$) with respect to the electroweak interactions and their inclusion would be a no-cost extension of the theory. The neutrino Yukawa terms and explicit Majorana mass terms for the right-handed neutrinos can then be added to the electroweak lagrangian without destroying the renormalizability of the theory. Such an extension may be necessary in view of the accumulating evidence for neutrino masses.

The gauge-fixing term and the Fadeev–Popov ghost term of the lagrangian (12.5) will be discussed later. First, we shall discuss the predictions of the theory which follow from the underlying gauge symmetry and the Higgs mechanism, i.e. from the tree level structure of the lagrangian.

12.2 Electroweak currents and physical gauge boson fields

The gauge boson couplings to matter follow from $\mathcal{L}_{\text{matter}}$ and have the form

$$\mathcal{L}_{\text{matter}} \supset -g J^{a\mu} W^a_\mu - g' J^\mu_{\text{Y}} B_\mu \tag{12.14}$$

where (see (12.8))

$$J^{a\mu} = \sum_f \bar{\Psi}^f_{\text{L}} \gamma^\mu T^a \Psi^f_{\text{L}} \tag{12.15}$$

$$J^\mu_{\text{Y}} = \sum_f \bar{\Psi}^f_{\text{L}} \gamma^\mu Y_f \Psi^f_{\text{L}} + \sum_f \bar{\Psi}^f_{\text{R}} \gamma^\mu Y_f \Psi^f_{\text{R}} \tag{12.16}$$

The $SU_{\text{L}}(2)$ current has, of course, explicit V–A structure. Note that the $U(1)$ current is chiral, too. Both left- and right-handed fermions contribute to it, but with different $U(1)$ charges. This is a potential source of chiral anomalies (see Chapter 13).

It follows from the unification hypothesis that the photon field A_μ is a combination of the Yang–Mills fields W^3_μ and B_μ. This combination can be easily found either from the requirement that the photon does not couple to the neutral component of the Higgs field, or from the requirement that its coupling to fermions is vector-like with the strength given by their electric charges. We find

$$A_\mu = \frac{1}{(g^2 + g'^2)^{1/2}} (g B_\mu + g' W^3_\mu) \equiv \cos\theta_{\text{W}} B_\mu + \sin\theta_{\text{W}} W^3_\mu \tag{12.17}$$

where the Weinberg angle is defined as

$$\sin^2\theta_{\text{W}} \equiv \frac{g'^2}{g^2 + g'^2}, \qquad \tan\theta_{\text{W}} \equiv \frac{g'}{g} \tag{12.18}$$

and the electric charge is ($e > 0$)

$$e \equiv \frac{g' g}{(g^2 + g'^2)^{1/2}} = g \sin\theta_{\text{W}} \tag{12.19}$$

The orthogonal combination defines another electrically neutral field of the theory:

$$Z^0_\mu = \frac{1}{(g^2 + g'^2)^{1/2}} (g W^3_\mu - g' B_\mu) \equiv \cos\theta_{\text{W}} W^3_\mu - \sin\theta_{\text{W}} B_\mu \tag{12.20}$$

Its couplings to fermions then follow from (12.14):

$$\frac{g}{\cos\theta_{\mathrm{W}}} J_Z^{\mu} Z_{\mu}^0 = \frac{g}{\cos\theta_{\mathrm{W}}} (J^{3\mu} - \sin^2\theta_{\mathrm{W}} J_{\mathrm{EM}}^{\mu}) Z_{\mu}^0 \tag{12.21}$$

with

$$J_{\mathrm{EM}}^{\mu} = \sum_f \bar{\Psi}_{\mathrm{L}}^f \gamma^{\mu} Q_f \Psi_{\mathrm{L}}^f + \sum_f \bar{\Psi}_{\mathrm{R}}^f \gamma^{\mu} Q_f \Psi_{\mathrm{R}}^f \tag{12.22}$$

The mixing angle θ_{W} measures the departure of the weak neutral current J_Z^{μ} from pure V–A structure! It is measured experimentally, for example, in atomic parity violation experiments (for other measurements of $\sin^2\theta_{\mathrm{W}}$ – see Section 12.4), and together with the electric charge can be used as independent input parameter fixing the couplings g and g'. It is worth stressing that the departure in the neutral current from pure V–A Lorentz structure is the basic prediction of the unified theory. Moreover, the value of the Weinberg angle has *a priori* nothing to do with the vector boson masses, but is just a measure of this departure. Such a relation is, however, obtained after the $W^{\pm}$ and Z^0 bosons are given masses by the Higgs mechanism, and it depends crucially on the $SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ representation content of the Higgs fields.

The physical gauge and Higgs boson spectrum of the electroweak theory is most easily seen in the so-called unitary gauge introduced in Chapter 11. Using the 'angular' variables for the Higgs field

$$H(x) = \frac{1}{\sqrt{2}} \exp\left[\mathrm{i}\frac{\eta^a(x)}{v} T^a\right] \begin{pmatrix} 0 \\ v + h^0(x) \end{pmatrix} \tag{12.23}$$

and choosing the gauge rotations $\Theta_{\mathrm{L}}^a(x) = -\eta^a(x)/v$, we again see from (12.9) that the minimum of the scalar potential occurs for $v^2 = -2m^2/\lambda$. Furthermore, three of the four degrees of freedom present in the Higgs doublet $H(x)$ are exchanged for the longitudinal components of the gauge bosons $W_{\mu}^{\pm} \equiv \frac{1}{\sqrt{2}}(W_{\mu}^1 \mp \mathrm{i}W_{\mu}^2)$ and Z^0, which obtain the masses

$$M_W^2 = \tfrac{1}{4} g^2 v^2, \qquad M_Z^2 = \tfrac{1}{4}(g^2 + g'^2) v^2 \tag{12.24}$$

The photon remains massless, in accord with the fact that the $U(1)_{\mathrm{EM}}$ gauge symmetry remains exact. The $W^{\pm}$ and Z^0 boson masses are related by

$$\frac{M_W}{M_Z} = \cos\theta_{\mathrm{W}} \tag{12.25}$$

(actually the same result holds with any number of Higgs doublets; the reader is invited to check this by explicit calculation). Also from (12.9) we see that the fourth degree of freedom originally present in the field $H(x)$ remains in the physical spectrum as an electrically neutral massive Higgs scalar with the mass $M_h^2 = \lambda v^2$.

Since in the unitary gauge the theory is not manifestly renormalizable, in the following we choose to work with the covariant gauge-fixing condition. In this class of gauges, the unphysical Higgs fields (the three Goldstone bosons 'eaten up' in the unitary gauge by the longitudinal components of the gauge boson fields) remain in the lagrangian (and in the Feynman rules). However, as discussed in Section 11.1, the unphysical poles of the Green's functions do not, of course, contribute to the S-matrix elements. We recall that S-matrix elements are obtained from the Green's functions by removing external lines, setting external momenta on the mass-shell, and contracting tensor indices with appropriate *physical* polarization vectors.

We complete this section by discussing some aspects of the lagrangian in covariant (renormalizable) gauges. Before fixing the gauge we write† in general

$$H = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h^0 + \mathrm{i}G^0) \end{pmatrix} \tag{12.26}$$

where h^0 is assumed to have a vanishing vacuum expectation value. Inserting (12.26) in the scalar potential in (12.9) one finds (with $v^2 = -2m^2/\lambda$ fixed as usual in its minimum) that the fields G^0 and $G^\pm$ remain in the lagrangian (12.9) and mix with the gauge bosons (v can always be taken as a real parameter):

$$\mathcal{L}_{\text{Higgs}} = \cdots + \frac{\mathrm{i}}{2} g v (W_\mu^+ \partial^\mu G^- - W_\mu^- \partial^\mu G^+) - \tfrac{1}{2}(g^2 + g'^2)^{1/2} v Z_\mu^0 \partial^\mu G^0 \tag{12.27}$$

In the so-called R_ξ gauge these terms are, however, cancelled by the appropriate choice of the $\mathcal{L}_{\text{g.fix}}$ part of the lagrangian. In lagrangian (12.9) we also have the canonical kinetic terms for h^0, G^0 and G^+ fields as well as their self-interactions and interactions with the gauge fields A_μ, Z_μ^0 and $W_\mu^\pm$. These terms are straightforward to obtain and their derivation is left as an exercise for the reader. Finally, one finds the mass terms for the gauge bosons

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= \cdots + \tfrac{1}{8} g^2 v^2 (W^{1\mu} W_\mu^1 + W^{2\mu} W_\mu^2) + \tfrac{1}{8} v^2 (g W^{3\mu} - g' B^\mu)(g W_\mu^3 - g' B_\mu) \\ &\equiv M_W^2 W^{+\mu} W_\mu^- + \tfrac{1}{2} M_Z^2 Z^{0\mu} Z_\mu^0 \end{aligned} \tag{12.28}$$

The fields G^0 and $G^\pm$ are massless at this stage but they will be given masses by the gauge-fixing term.

The terms (12.27) which mix $W^\pm$ and Z^0 vector bosons with the Goldstone

† Compared to (12.3) we have renamed the Goldstone boson fields which in the unitary gauge are 'eaten up' by the longitudinal components of the physical massive gauge boson fields. Also note that the neutral field is split into two real fields.

fields are cancelled by the following gauge-fixing lagrangian†

$$\mathcal{L}_{\text{g.fix}} = -\frac{1}{2\xi}\sum_{a=0}^{3}\mathcal{F}^a\mathcal{F}^a \tag{12.29}$$

with

$$\left.\begin{aligned}\mathcal{F}^0 &\equiv \partial^\mu B_\mu - \mathrm{i}\xi g'\left(H^\dagger Y\langle H\rangle - \langle H^\dagger\rangle Y H\right)\\ \mathcal{F}^a &\equiv \partial^\mu W^a_\mu - \mathrm{i}\xi g\left(H^\dagger T^a\langle H\rangle - \langle H^\dagger\rangle T^a H\right) \quad (a = 1, 2, 3)\end{aligned}\right\} \tag{12.30}$$

or, more explicitly,

$$\begin{aligned}\mathcal{F}^0 &\equiv \partial^\mu B_\mu - \frac{1}{2}\xi g' v G^0\\ \mathcal{F}^3 &\equiv \partial^\mu W^3_\mu + \frac{1}{2}\xi g v G^0\\ \mathcal{F}^1 &\equiv \partial^\mu W^1_\mu - \frac{\mathrm{i}}{2\sqrt{2}}\xi g v(G^- - G^+)\\ \mathcal{F}^2 &\equiv \partial^\mu W^2_\mu - \frac{1}{2\sqrt{2}}\xi g v(G^- + G^+)\end{aligned}$$

Integrating these terms by parts, one gets

$$\begin{aligned}\mathcal{L}_{\text{g.fix}} = {}&\frac{1}{2\xi}Z^0_\mu\partial^\mu\partial^\nu Z^0_\nu + \frac{1}{2\xi}A_\mu\partial^\mu\partial^\nu A_\nu + \frac{1}{\xi}W^+_\mu\partial^\mu\partial^\nu W^-_\nu\\ &- \frac{1}{2}\xi M_Z^2 G^0G^0 - \xi M_W^2 G^+G^-\\ &+ \frac{1}{2}(g^2 + g'^2)^{1/2} v Z^0_\mu\partial^\mu G^0 - \frac{\mathrm{i}}{2} g v\left(W^+_\mu\partial^\mu G^- - W^-_\mu\partial^\mu G^+\right)\end{aligned} \tag{12.31}$$

The last two terms indeed cancel the terms in (12.27). The first three terms are gauge-dependent corrections to the kinetic terms for the gauge bosons, and the remaining two terms are mass terms for the unphysical Goldstone bosons.

Finally, the ghost lagrangian can be obtained using the general procedure described in Chapter 3. Performing infinitesimal gauge transformations on the fields in $\mathcal{F}^a$ in (12.30):

$$\left.\begin{aligned}&\delta_{\mathrm{Y}} H = -\mathrm{i}g'\delta\Theta_{\mathrm{Y}} Y H, \qquad \delta_{\mathrm{L}} H = -\mathrm{i}g\delta\Theta^a_{\mathrm{L}} T^a H\\ &\delta_{\mathrm{Y}} H^\dagger = \mathrm{i}g'\delta\Theta_{\mathrm{Y}} H^\dagger Y, \qquad \delta_{\mathrm{L}} H^\dagger = \mathrm{i}g H^\dagger\delta\Theta^a_{\mathrm{L}} T^a\\ &\delta_{\mathrm{Y}} B_\mu = \partial_\mu\delta\Theta_{\mathrm{Y}}, \qquad \delta_{\mathrm{Y}} W^a_\mu = 0\\ &\delta_{\mathrm{L}} B_\mu = 0, \qquad \delta_{\mathrm{L}} W^a_\mu = -g\varepsilon^{abc}W^b_\mu\delta\Theta^c_{\mathrm{L}} + \partial_\mu\delta\Theta^a_{\mathrm{L}}\end{aligned}\right\} \tag{12.32}$$

† In general, one can introduce different gauge-fixing parameters ξ for different components of the W^a field and for the B field. We follow here the simplest approach and introduce only one such parameter.

we compute the corresponding Fadeev–Popov determinants (the variations are taken always at $\Theta = 0$):

$$\left.\begin{aligned}
M_{\mathrm{YY}}(x, y) &\equiv \frac{\delta\mathcal{F}^0(x)}{\delta\Theta_{\mathrm{Y}}(y)} = \left[\partial^2 + \xi g'^2\big(H^\dagger YY\langle H\rangle + \langle H^\dagger\rangle YYH\big)\right]\delta^{(4)}(x-y)\\
M_{\mathrm{LL}}^{ac}(x, y) &\equiv \frac{\delta\mathcal{F}^a(x)}{\delta\Theta_{\mathrm{L}}^c(y)} = \Big[\partial^2 + gc^{abc}\overleftarrow{\partial}^\mu W_\mu^b + \xi g^2\left(H^\dagger T^c T^a\langle H\rangle\right.\\
&\qquad\qquad\left. + \langle H^\dagger\rangle T^a T^c H\right)\Big]\delta^{(4)}(x-y)\\
M_{\mathrm{LY}}^{a}(x, y) &\equiv \frac{\delta\mathcal{F}^a(x)}{\delta\Theta_{\mathrm{Y}}(y)} = \xi g g'\left(H^\dagger Y T^a\langle H\rangle + \langle H^\dagger\rangle T^a Y H\right)\delta^{(4)}(x-y)\\
M_{\mathrm{YL}}^{a}(x, y) &\equiv \frac{\delta\mathcal{F}^0(x)}{\delta\Theta_{\mathrm{L}}^a(y)} = \xi g g'\left(H^\dagger T^a Y\langle H\rangle + \langle H^\dagger\rangle Y T^a H\right)\delta^{(4)}(x-y)
\end{aligned}\right\} \tag{12.33}$$

(the symbol $\overleftarrow{\partial}^\mu$ means that the derivative acts on the object standing to its left in the full expression for the ghost action). The ghost action then reads

$$\begin{aligned}
S_{\mathrm{ghost}} = \int \mathrm{d}x \int \mathrm{d}y\, \big[&\bar{\eta}_{\mathrm{Y}}(x) M_{\mathrm{YY}}(x, y)\eta_{\mathrm{Y}}(y) + \bar{\eta}_{\mathrm{L}}^a(x) M_{\mathrm{LL}}^{ab}(x, y)\eta_{\mathrm{L}}^a(y)\\
&+ \bar{\eta}_{\mathrm{Y}}(x) M_{\mathrm{YL}}^a(x, y)\eta_{\mathrm{L}}^a(y) + \bar{\eta}_{\mathrm{L}}^a(x) M_{\mathrm{LY}}^a(x, y)\eta_{\mathrm{Y}}(y)\big]
\end{aligned} \tag{12.34}$$

Since the variations of $\mathcal{F}^a$ contain factors $\delta^{(4)}(x-y)$, this can be written as an integral of a local lagrangian density

$$S_{\mathrm{ghost}} = \int \mathrm{d}x\, \mathcal{L}_{\mathrm{ghost}}(x) \tag{12.35}$$

Finally, defining

$$\left.\begin{aligned}
\eta_Z &= \cos\theta_{\mathrm{W}}\eta_{\mathrm{L}}^3 - \sin\theta_{\mathrm{W}}\eta_{\mathrm{Y}},\\
\eta_A &= \sin\theta_{\mathrm{W}}\eta_{\mathrm{L}}^3 + \cos\theta_{\mathrm{W}}\eta_{\mathrm{Y}},\\
\eta_\pm &= \tfrac{1}{\sqrt{2}}\left(\eta_{\mathrm{L}}^1 \mp \mathrm{i}\eta_{\mathrm{L}}^2\right),
\end{aligned}\right\} \tag{12.36}$$

and expressing the functional derivatives (12.33) through the physical fields $W_\mu^\pm$, Z_μ^0, A_μ and h^0, one arrives at $\mathcal{L}_{\mathrm{ghost}}$ generating the Feynman rules collected in Appendix C. We shall work in the so-called 't Hooft–Feynman gauge with $\xi = 1$.

12.3 Fermion masses and mixing

The Yukawa terms (12.10) in the electroweak lagrangian couple the left-handed Weyl fermion doublets of $SU_{\mathrm{L}}(2)$ with the left-handed Weyl fermion singlets and the Higgs boson. Clearly, all those fields have well-defined symmetry properties under $SU_{\mathrm{L}}(2) \times U_{\mathrm{Y}}(1)$ and we call them to be in an 'electroweak basis'. With three families of fermions, this basis is defined up to rotations in three-dimensional flavour space, independently for each category of fields q_A, l_A, e^c_A, u^c_A, d^c_A ($A = 1, 2, 3$). Yukawa couplings are free parameters of the present theory. One may hope that the future, more fundamental theory will fix those parameters in some electroweak basis.

A non-zero vacuum expectation value of the Higgs field (12.26) generates fermion mass terms:

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}} Y_l^{BA} e_A e^c_B - \frac{v}{\sqrt{2}} Y_d^{BA} d_A d^c_B - \frac{v}{\sqrt{2}} Y_u^{BA} u_A u^c_B + \text{h.c.} \tag{12.37}$$

The physical interpretation of these terms becomes more transparent when we rewrite the lagrangian (12.37) in terms of the four-component chiral fields, defined in (12.2):

$$\begin{aligned}\mathcal{L}_{\text{mass}} = &-\frac{v}{\sqrt{2}} Y_l^{BA} \overline{(\Psi_{e_B})_{\text{R}}} (\Psi_{e_A})_{\text{L}} - \frac{v}{\sqrt{2}} Y_d^{BA} \overline{(\Psi_{d_B})_{\text{R}}} (\Psi_{d_A})_{\text{L}} \\ &- \frac{v}{\sqrt{2}} Y_u^{BA} \overline{(\Psi_{u_B})_{\text{R}}} (\Psi_{u_A})_{\text{L}} + \text{h.c.}\end{aligned} \tag{12.38}$$

The mass matrices are:

$$[m_e] = \frac{v}{\sqrt{2}} [Y_e], \quad [m_d] = \frac{v}{\sqrt{2}} [Y_d], \quad [m_u] = \frac{v}{\sqrt{2}} [Y_u] \tag{12.39}$$

In general the Yukawa matrices can be the most general complex 3×3 matrices. They can be diagonalized by two unitary transformations (check it): $V^\dagger [m] U \rightarrow [m]_{\text{diagonal}}$ with real diagonal entries; the latter follows from the fact that the transformations V and U can include phase rotations independently for the left- and right-handed fields. Thus, to diagonalize the mass matrices we perform unitary transformations on the right- and left-handed fields, independently for up and down quarks and charged leptons:

$$\left.\begin{aligned}(\Psi_{u_A})_{\text{L}} &= (\mathcal{U}_{\text{L}})_{AB} (\Psi'_{u_B})_{\text{L}}, \qquad (\Psi_{u_A})_{\text{R}} = (\mathcal{U}_{\text{R}})_{AB} (\Psi'_{u_B})_{\text{R}} \\ (\Psi_{d_A})_{\text{L}} &= (\mathcal{D}_{\text{L}})_{AB} (\Psi'_{d_B})_{\text{L}}, \qquad (\Psi_{d_A})_{\text{R}} = (\mathcal{D}_{\text{R}})_{AB} (\Psi'_{d_B})_{\text{R}}\end{aligned}\right\} \tag{12.40}$$

and analogously for the down quarks and charged leptons e_A (i.e. e, μ, τ). The primed fields describe physical particles: after diagonalization we can combine chiral fields into Dirac fields $\Psi = \Psi_{\text{L}} + \Psi_{\text{R}}$, which are mass eigenstates.

As mentioned in Section 12.1 one consequence of the lagrangian (12.10) is that neutrinos remain massless. The accumulating experimental evidence for neutrino masses and oscillations (for a review see, for example, Fisher, Kayser & McFarland (1999)) may call for some extension of the theory. An attractive and straightforward extension is the inclusion of the right-handed neutrinos or, strictly speaking, of a set of new left-handed Weyl fields ν^c_{L} in the list (12.1) with four-component $(\Psi_{\nu_A})_{\mathrm{R}}$ given by the relation analagous to (12.2). New renormalizable terms can then be added to the electroweak lagrangian, namely, neutrino Yukawa terms $Y_\nu^{BA}\epsilon_{ij}H_i l_j^A \nu_{\mathrm{L}}^{cB}$ and the right-handed neutrino Majorana mass matrix $M^{BA}\nu_{\mathrm{L}}^{cA}\nu_{\mathrm{L}}^{cB}$ + h.c. Assuming that the entries of the matrix $\hat{M}$ are much larger than the electroweak scale one can integrate out the heavy right-handed neutrino fields. One then obtains the standard electroweak theory supplemented by the effective dimension five operator

$$\hat{Y}_\nu \hat{M}^{-1}\hat{Y}_\nu^{\mathrm{T}}(\epsilon_{ij}H_i l_j)(\epsilon_{km}H_k l_m) \tag{12.41}$$

(from tree diagrams with ν_{L}^c exchange) which, after the Higgs field aquires a non-zero vacuum expectation value, gives masses to the almost pure Majorana left-handed neutrinos. This see-saw mechanism (Gell-Mann, Ramond & Slansky 1979) can naturally explain the neutrino masses in the experimentally suggested range provided the Majorana mass M is of the order of the Grand Unification scale.

We can now rewrite the matter couplings (12.14) to the gauge bosons in terms of the physical fermion fields:

$$\mathcal{L}_{\mathrm{matter}} \supset -\frac{g}{\sqrt{2}}(J_\mu^- W^{+\mu} + J_\mu^+ W^{-\mu}) - \frac{g}{\cos\theta_{\mathrm{W}}} J_Z^\mu Z_\mu^0 - eJ_{\mathrm{EM}}^\mu A_\mu \tag{12.42}$$

where (returning first to the Weyl spinor notation – rewrite (12.40) in terms of the left-handed Weyl fields)

$$\begin{aligned} J_\mu^- &= \sum_A \left(\bar{u}_A\bar{\sigma}_\mu d_A + \bar{\nu}_A\bar{\sigma}_\mu e_A\right) \\ &= \sum_{AB} \bar{u}'_A\bar{\sigma}_\mu (V_{\mathrm{CKM}})_{AB} d'_B + \sum_A \bar{\nu}'_A\bar{\sigma}_\mu e'_A \\ &\equiv \sum_{AB} \bar{\Psi}'_{u_A}\gamma_\mu P_{\mathrm{L}}(V_{\mathrm{CKM}})_{AB}\Psi'_{d_B} + \sum_A \bar{\Psi}'_{\nu_A}\gamma_\mu P_{\mathrm{L}}\Psi'_{e_A}, \end{aligned} \tag{12.43}$$

and $V_{\mathrm{CKM}} \equiv \mathcal{U}_{\mathrm{L}}^\dagger \mathcal{D}_{\mathrm{L}}$ is called the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix. In (12.37) neutrinos are massless and, therefore, the neutrino fields can always be rotated in such a way that there is no lepton mixing in (12.43). It is also important that the matrices $\mathcal{U}_{\mathrm{R}}$ and $\mathcal{D}_{\mathrm{R}}$ drop out completely from the final lagrangian, because the right-handed fields only have diagonal couplings to the B_μ gauge field. The neutral weak current and the electromagnetic current have the same form in the mass eigenstate and in the electroweak basis. Note, in particular, that due to the doublet structure of all left-handed quark multiplets, the neutral weak current remains diagonal in flavour in the mass eigenstate basis. Finally, the Yukawa couplings of the physical scalar h^0 and of the unphysical pseudoscalar Goldstone boson G^0 (present in the R_ξ gauge) become diagonal in flavour in the mass eigenstate basis. Indeed, they are determined by the same Yukawa coupling matrices which give the masses, and are diagonalized simultaneously with the latter.† The coupling of the unphysical charged Goldstone boson G^+ to physical quarks involves the CKM matrix as for the charged weak currents. This completes the discussion of the electroweak lagrangian in the R_ξ gauge. The Feynman rules for the full lagrangian can be derived by the standard methods of Chapter 2, and are summarized in Appendix C.

We end this section with a few more remarks. The Cabibbo–Kobayashi–Maskawa quark mixing matrix is a 3×3 unitary matrix. In the general case of n quark doublets, the $n \times n$ unitary mixing matrix has $2n^2 - n^2 = n^2$ real parameters, but not all of them are observable. Indeed, $2n - 1$ phases can be removed by phase redefinitions of the physical quark mass eigenstates. Notice that the phase transformations on the left- and right-handed quarks have to be the same to preserve real diagonal mass terms. Moreover, the matrix V_{CKM} remains invariant when we change all quarks by the same phase. So only $2n - 1$ phases can be removed. The number of real parameters is therefore $n^2 - 2n + 1$, of which $\frac{1}{2}n(n-1)$ are angles of $n \times n$ orthogonal matrix. Therefore, $n^2 - 2n + 1 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$ is the number of phases. For $n = 2$ no physical phase in the CKM matrix is present. For $n = 3$ there is one physical phase. In the world with three generations of quarks the CP symmetry is explicitly broken if this phase is different from zero. The two most often used parametrizations of V_{CKM} read

$$V_{\mathrm{CKM}} = \begin{pmatrix} c_\theta c_\beta & s_\theta c_\beta & s_\beta \exp(-\mathrm{i}\delta) \\ -s_\theta c_\gamma - c_\theta s_\beta s_\gamma \exp(\mathrm{i}\delta) & c_\theta c_\gamma - s_\theta s_\beta s_\gamma \exp(\mathrm{i}\delta) & c_\beta s_\gamma \\ s_\theta s_\gamma - c_\theta s_\beta c_\gamma \exp(\mathrm{i}\delta) & -c_\theta s_\gamma - s_\theta s_\beta c_\gamma \exp(\mathrm{i}\delta) & c_\beta c_\gamma \end{pmatrix} \tag{12.44}$$

with the angles θ, β, γ and the phase δ fixed by experiment, and

$$V_{\mathrm{CKM}} \approx \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda - \mathrm{i}A\lambda^5\eta & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - \mathrm{i}\eta) & -A\lambda^2 - \mathrm{i}A\lambda^4\eta & 1 \end{pmatrix} \tag{12.45}$$

The second one (Wolfenstein 1983) is only approximate (in this parametrization V_{CKM} is unitary only up to order $\mathcal{O}(\lambda^4)$), but is sufficient for the present level of accuracy of the measurements. We know from experiment that $\lambda \approx 0.23$ (the Cabibbo angle) and $A \approx 1$, and from (12.45) we clearly see the order of magnitude of the elements of the V_{CKM} matrix.

† This is no longer true if the Higgs sector consists of several Higgs doublets, unless for some symmetry reasons only one doublet couples to the up-type quarks and one to the down-type. Thus, with several Higgs doublets one generically has phenomenologically dangerous flavour changing neutral-current transitions mediated by Higgs boson exchange. This is another reason (apart from economy) to consider one-doublet theory.

12.4 Phenomenology of the tree level lagrangian

The electroweak theory discussed in this chapter correctly describes all known phenomena induced by the electromagnetic and/or weak forces up to the energy scale $\mathcal{O}(100 \text{ GeV})$. For processes induced by the weak forces there are two regions of energy scale which have been studied experimentally particularly thoroughly. These are: (*a*) very small momentum, $q \approx 0$, processes (mainly weak decays of leptons and hadrons); (*b*) the region of the Z^0 resonance at $s^{1/2} \approx 91$ GeV studied experimentally with very high precision in e^+e^- collisions at LEP1.† We are going to concentrate on these two regions and to provide sample calculations in the standard electroweak theory. This will illustrate the basic logic of such calculations. One should also remember that weak decays of elementary particles played a crucial role in the development of the electroweak theory, with non-renormalizable four-fermion lagrangian introduced on purely phenomenological grounds in 1933 (Fermi theory).

The parameters of the electroweak gauge sector of the Standard Model lagrangian are the gauge couplings g and g' and the vacuum expectation value v of the Higgs field. Of course, they can be replaced by some combinations of them, and a convenient choice is, for example,

$$\sin^2\theta_W = \frac{g'^2}{g^2 + g'^2}, \qquad \alpha = \frac{g^2 \sin^2\theta_W}{4\pi}, \qquad M_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2 \tag{12.46}$$

Three independent measurements are thus necessary to fix these input parameters of the theory (beyond tree level they are renormalization-scale-dependent). Any further measurement is already a test of its predictions. At present the best measured electroweak quantities are α_{EM}, G_μ and $\hat{M}_Z$, where $\alpha_{EM} = 1/137.0359895(61)$ is the fine structure constant, $G_\mu = 1.16639(2) \times 10^{-5}\ \text{GeV}^{-2}$ is the so-called Fermi constant directly related to the muon decay width (see (12.50)) and $\hat{M}_Z = 91.1867 \pm 0.002$ GeV is the physical Z^0-boson mass (we distinguish it by the hat from the mass introduced as the parameter of the lagrangian; this distinction becomes important beyond the tree level). We shall use these observables as the three measurements fixing the parameters of the lagrangian order by order in perturbation theory, i.e. we need to calculate them to the required accuracy in the electroweak theory parametrized by the set (12.46). It is clear from Section 12.1 that at the tree level we get $\alpha = \alpha_{EM}$, $M_Z = \hat{M}_Z$. Moreover, as we shall see, $\sin^2\theta_W$ is fixed in terms of G_μ, α_{EM} and $\hat{M}_Z$.

The complete set of the electroweak parameters also includes fermion masses, CKM angles, the Higgs boson mass and the strong coupling constant. The fermion masses are known with good (or very good) accuracy for leptons and less precisely

† At present (1999) the second phase of the LEP accelerator, the so-called LEP2 is operating, which will reach the c.m. energy of 200 GeV, in a search for some departures from the Standard Model at those high energies.

for quarks. One should also remember that the predictions for the electroweak processes in the hadronic sector generically have in addition uncertainties coming from our limited knowledge of the quark current matrix elements taken between the hadronic states (i.e. from the lack of precise theory of hadrons as bound states of quarks and gluons).

Effective four-fermion interactions

We proceed with the discussion of the tree level predictions. As advocated above, first we have to calculate the μ^- decay widths, and then we shall be able to discuss the predictions of the theory.

All known elementary fermions, except for the top quark, are much lighter than the $W^\pm$ and Z^0 bosons. Charge current processes with four external light fermions, and with their momenta $q \ll M_{W,Z}$, can be described in the tree level approximation by the effective four-fermion lagrangian

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(x) = -2\sqrt{2}\, G_{\text{F}} J_\mu^-(x) J^{\mu+}(x) \tag{12.47}$$

where

$$\sqrt{2}\, G_{\text{F}} = \frac{\pi\alpha}{s^2 c^2 M_Z^2} \tag{12.48}$$

We have abbreviated our notation, $s \equiv \sin\theta_{\text{W}}$, $c \equiv \cos\theta_{\text{W}}$, and used relation (12.25) (this relation is actually strictly valid for the parameters of the lagrangian in any order of perturbation theory in the $\overline{\text{MS}}$ scheme). The lagrangian (12.47) accounts for the amplitude that, for example, for the decay $\mu^- \rightarrow e\nu_\mu\bar{\nu}_e$, corresponds to the diagram in Fig. 12.1. The currents are the same as in (12.43). The V–A current is always built of a pair of fermions which form an $SU_{\text{L}}(2)$ doublet, and includes the appropriate CKM matrix element. It is important to stress that we introduce the effective four-fermion lagrangian as the four-point part of the integrand in the effective action (2.132), which is approximately local for the considered range of phase space. The origin of this approximate locality is in the decoupling of the heavy particles ($W^\pm$, Z^0 and Higgs bosons and the top quark). This is a convenient way to organize certain calculations in the complete electroweak theory, since we can discuss the whole class of processes at once. The concept of the effective low energy theory, in its full sense introduced in Chapter 7, will be discussed in Section 12.6.

At the tree level, the coefficient G_{F} is universal for all four-fermion operators present in the lagrangian (12.47). It can be determined in terms of the muon decay widths. Calculating the muon decay ($\mu \rightarrow e\nu_\mu\bar{\nu}_e$) width in the tree approximation

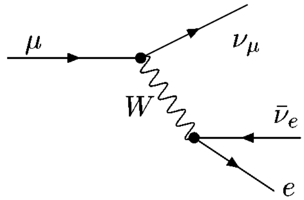


Fig. 12.1. Tree level diagram for the charged currents interactions in the Standard Model.

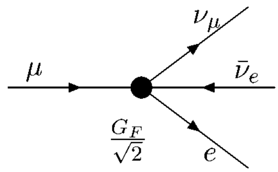


Fig. 12.2. Tree level diagram for muon decay in the Fermi theory.

we get

$$1/\tau_\mu = \Gamma_\mu = G_{\mathrm{F}}^2 m_\mu^5 / 192\pi^3 \tag{12.49}$$

where τ_μ and Γ_μ are the muon life time and its full decay width, respectively.

Traditionally (but not quite consistently for a tree level calculation), the G_{F} in (12.48) is expressed in terms of the muon lifetime after correcting the theoretical prediction for $\mathcal{O}(\alpha_{\mathrm{EM}})$ QED radiative effects. Historically, this calculation has been performed in the Fermi theory, in a four-dimensional regularization scheme and with the on-shell renormalization (Berman 1958, Kinoshita & Sirlin 1959). Computing the full set of diagrams shown in Figs. 12.2 and 12.3 and integrating over the whole phase space, i.e. taking into account muon decays with one photon in the final state, one gets the following result:†

$$1/\tau_\mu = \Gamma_\mu = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \times f\left(\frac{m_e^2}{m_\mu^2}\right)\left[1 + \frac{\alpha_{\mathrm{EM}}}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right] \tag{12.50}$$

where α_{EM} is the fine structure constant, $f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x$ and G_{F} has been renamed G_μ. Actually, the standard attitude is that (12.50) defines a new fundamental constant, the so-called Fermi constant, which will be denoted by G_μ. From this formula and the measured muon lifetime $\tau_\mu = (2.19703 \pm 0.00004) \times 10^{-6}$ s the value $G_\mu = 1.16639(2) \times 10^{-5}\ \mathrm{GeV}^{-2}$ is extracted. Once G_{F} is determined by its identification with G_μ, its comparison with

† This result is finite. In four-dimensional regularization schemes the counterterms shown in Fig. 12.3 cancel out. The absence of the IR and mass singularities follows from the Lee–Nauenberg–Kinoshita theorem (Chapter 5).

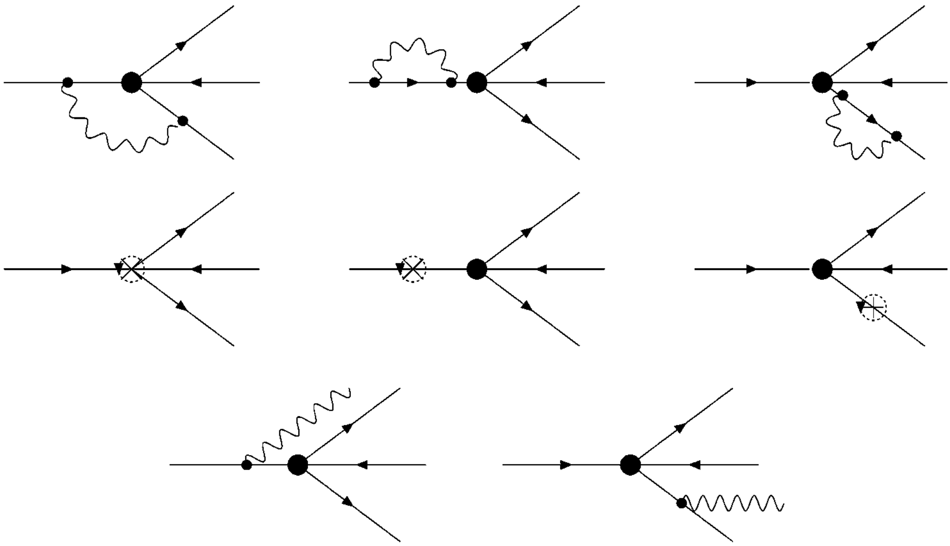


Fig. 12.3. QED corrections to the muon decay in the Fermi theory. The crossed circles denote counterterms.

any other experimentally measured decay constant $G_{ll'}$ (defined by the analogue of (12.50) for beta decay of lepton l into lepton l') is a tree level test of the electroweak theory (with the $\mathcal{O}(\alpha_{\mathrm{EM}})$ QED corrections included). As said earlier, at the tree level the parameters α and M_Z of the lagrangian have to be identified with the fine structure constant α_{EM} and with the physical Z-boson mass $\hat{M}_Z$. The relation (12.48), with $G_{\mathrm{F}} = G_\mu$, can then be used to exchange the third parameter $\sin^2\theta_{\mathrm{W}}$ for the accurately measured G_μ. With the three input parameters taken from experiment we can predict at the tree level the values of all other electroweak observables.

For instance, for the $W^{\pm}$-boson mass one gets (using again (12.25))

$$\hat{M}_W^{(0)} = \hat{M}_Z\left\{\tfrac{1}{2}\left[1 + (1-4A)^{1/2}\right]\right\}^{1/2} \tag{12.51}$$

where $A \equiv \pi\alpha_{\mathrm{EM}}/\sqrt{2}\,G_\mu\hat{M}_Z^2$. Numerically, we get $\hat{M}_W^{(0)} = 80.94$ GeV (the superscript (0) means the tree level prediction for the physical W-mass). The recent (1997) combination of different measurements at LEP and Tevatron colliders gives $\hat{M}_W = 80.39 \pm 0.08$. The discrepancy signals the need of including radiative corrections (see next section).

The neutral-current transitions (like, for example, $\mathrm{e}^-\nu_\mu \to \mathrm{e}^-\nu_\mu$) in the limit $q \approx 0$, for vanishing fermion masses and in the tree approximation, can be described by the effective four-fermion lagrangian

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{NC}} = -\rho_{ff'}2\sqrt{2}\,G_\mu J_Z^\mu J_{Z\mu} \tag{12.52}$$

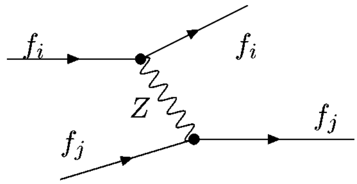


Fig. 12.4. Tree level diagram for neutral-current interactions in the Standard Model.

where the neutral current is

$$\begin{aligned} J_Z^\mu &= \sum_f \bar{\Psi}_f \gamma^\mu P_\mathrm{L} T_f^3 \Psi_f - \sum_f Q_f \sin^2\theta_{f(\mathrm{low})}^\mathrm{eff} \bar{\Psi}_f \gamma^\mu \Psi_f \\ &\equiv \sum_f \bar{\Psi}_f \gamma^\mu (g_\mathrm{L}^f P_\mathrm{L} + g_\mathrm{R}^f P_\mathrm{R}) \Psi_f \end{aligned} \tag{12.53}$$

Calculating the Feynman amplitude corresponding to Fig. 12.4 and using (12.21) and (12.25) we see that at the tree level

$$\sin^2\theta_{f(\mathrm{low})}^\mathrm{eff} = \sin^2\theta_\mathrm{W} \tag{12.54}$$

and, moreover, we obtain the remarkable tree level prediction of the SM†

$$\rho_{ff'} \equiv \rho = 1 \tag{12.55}$$

This follows from the relation (12.25) (used in the derivation of (12.48)), which in turn is the consequence of using only Higgs doublets to break $SU_\mathrm{L}(2) \times U_\mathrm{Y}(1)$ symmetry. In this case the Higgs sector has the so-called custodial $SU_\mathrm{V}(2)$ symmetry (see Problem 12.3). These predictions can be compared with experimental determination of the parameters ρ (strictly speaking $\rho_{ff'}$) and $\sin^2\theta_{f(\mathrm{low})}^\mathrm{eff}$ defined by the lagrangian (12.52), interpreted as a low energy phenomenological lagrangian. For instance, the parameter ρ measured in the leptonic sector of the neutral-current transitions is $\rho_\mathrm{EXP} = 1.006 \pm 0.034$ (Vilain *et al.* 1994).

Z^0 *couplings*

For the on-shell Z^0 boson coupled to leptons and quarks we have (by analogy with (12.52); the differences are easy to understand)

$$\mathcal{L}_{Zff} = -(\rho_f \sqrt{2}\, G_\mu)^{1/2} 2\hat{M}_Z J_Z^\mu Z_\mu^0 \tag{12.56}$$

where, as before, J_Z^μ is given by (12.53) with

$$g_\mathrm{L}^f = T_f^3 - Q_f \sin^2\theta_f^\mathrm{eff}, \qquad g_\mathrm{R}^f = -Q_f \sin^2\theta_f^\mathrm{eff} \tag{12.57}$$

† Note the factor of 2 in the denominator in the effective coupling $2\sqrt{2}\, G_\mu \rho = g^2/2M_Z^2 \cos^2\theta_\mathrm{W}$ obtained from the tree level diagram. It follows from the identity of the two currents.

(one also often defines vector and axial couplings $g_{\mathrm{V}} = g_{\mathrm{L}} + g_{\mathrm{R}},\ \ g_{\mathrm{A}} = g_{\mathrm{L}} - g_{\mathrm{R}}$).

At the tree level one gets

$$\sin^2\theta_f^{\mathrm{eff}} = \sin^2\theta_{f\,(\mathrm{low})}^{\mathrm{eff}} = \sin^2\theta_{\mathrm{W}} \tag{12.58}$$

Also, then $\rho_f = \rho = 1$.

The decay width of Z^0 into a fermion–antifermion pair is

$$\Gamma(Z^0 \to \bar{f}f) = \frac{N_c\rho_f\sqrt{2}\,G_\mu\hat{M}_Z^3}{12\pi}\left(1 - \frac{4m_f^2}{\hat{M}_Z^2}\right)^{1/2} \times\left[(g_{\mathrm{L}}^{f2} + g_{\mathrm{R}}^{f2})\left(1 - \frac{m_f^2}{\hat{M}_Z^2}\right) + 6g_{\mathrm{L}}^f g_{\mathrm{R}}^f \frac{m_f^2}{\hat{M}_Z^2}\right] \tag{12.59}$$

where $N_c = 1(3)$ for leptons (quarks). Thus, at the tree level we get $\Gamma(Z^0 \to \mathrm{e}^+\mathrm{e}^-) = 84.82$ MeV, which should be compared with the LEP measurement 83.91 ± 0.10 MeV. The experimental values for the parameters ρ_f and $\sin^2\theta_f^{\mathrm{eff}}$ can be obtained by measuring the widths and various asymmetries in the $Z^0 \to \bar{f}f$ decays. The importance of higher order effects can be judged from the comparison of the experimental world average $\sin^2\theta_e^{\mathrm{eff}} = 0.23151 \pm 0.00022$ with the tree level prediction $\sin^2\theta_f^{\mathrm{eff}} = \sin^2\theta_{\mathrm{W}} = \frac{1}{2}[1 - (1 - 4A)^{1/2}] = 0.21215$, where A is defined below (12.51).

12.5 Beyond tree level

Renormalization and counterterms

The electroweak theory is a renormalizable field theory. A formal proof of this fact can be found in 't Hooft (1971*a*, *b*). Crucial for renormalization is spontaneous (and not explicit) breaking of the gauge symmetry and also the matter content. The latter ensures the absence of chiral anomalies (Chapter 13). The interplay of quarks and leptons in obtaining the anomaly-free theory is a big puzzle of the Standard Model, which suggests the existence of a still deeper (more unified?) theory of elementary interactions. In this book we shall be mainly concerned with practical aspects of the renormalization programme to provide a basis for systematic higher order calculations. A vast literature exists on precision calculations and precision tests of the electroweak theory which include all one-loop effects and also, partially, higher order contributions. The complexity of the theory together with arbitrariness in the choice of the 'best' renormalization scheme often make it difficult to follow and to compare different approaches. Strictly speaking, part of the renormalization scheme is the choice of the input observables that are used to fix the parameters of the lagrangian. At present, after the experimental results of LEP1, there is a consensus at least with regard to that latter point. The commonly used input parameters are $\alpha_{\rm EM}$, G_μ and the mass $\hat{M}_Z$ of the Z^0 boson. The remaining freedom in the renormalization scheme is the choice of the renormalization conditions that fix the counterterms in the intermediate steps of the calculation. As in the rest of the book, we choose to work in the Minimal Subtraction scheme (actually in the $\overline{\rm MS}$ scheme). Complete fixing of the renormalization scheme then requires only a choice of the renormalization scale μ. The Minimal Subtraction scheme is a convenient reference frame for comparison of different approaches. Any other renormalization scheme (for example, the often used on-shell OS scheme) is obtained by finite renormalization. Once we know the theory is renormalizable, we are assured that calculation of any physical observable in terms of the input physical parameters (such as $\alpha_{\rm EM}$, G_μ, $\hat{M}_Z$) must give a finite answer. Therefore, in the one-loop calculations considered in this book, the most pragmatic attitude to the renormalization programme (useful in a theory as complex as the electroweak theory) is to use lagrangian (12.5) as it stands simply without explicitly listing the counterterms. Since we know the infinities cancel out, we can as well subtract them in the dimensionally regularized result, without keeping track of the necessary counterterms. Thus, in one-loop calculations we can proceed as follows:

(1) Choose the set of the input observables in terms of which the others will be calculated. We use the set $a_i = (\alpha_{\rm EM}, G_\mu, \hat{M}_Z)$.

(2) Compute all input observables a_i, dimensionally regularized, in terms of the parameters of lagrangian (12.5) interpreted as the renormalization scale-dependent renormalized parameters in the $\overline{\rm MS}$ (we shall use the set $\alpha(\mu)$, $s^2(\mu)$ and $M_Z(\mu)$ defined in (12.46)); keep only the terms which are finite in the limit $\varepsilon \to 0$. We get $\alpha_{\rm EM}^{(1)} = \alpha_{\rm EM}^{(0)} + \delta\alpha_{\rm EM}$, $\hat{M}_Z^{2(1)} = \hat{M}_Z^{2(0)} + \delta \hat{M}_Z^2$, $G_\mu^{(1)} = G_\mu^{(0)} + \delta G_\mu$, in terms of the same set of the renormalized lagrangian parameters. The quantities $\alpha_{\rm EM}^{(1)}$, $G_\mu^{(1)}$ and $\hat{M}_Z^{2(1)}$ are then identified with experimentally measured values $\alpha_{\rm EM}$, G_μ and $\hat{M}_Z^2$, and the equations can then be solved for $\alpha(\mu)$, $s^2(\mu)$ and $M_Z(\mu)$. Of course, at the tree level we have

$$\alpha_{\rm EM}^{(0)} = \alpha, \qquad \hat{M}_Z^{(0)} = M_Z, \qquad G_\mu^{(0)} = \frac{\pi\alpha}{\sqrt{2}\, s^2(1-s^2)M_Z^2} \tag{12.60}$$

and in this order one gets

$$c^2 = 1 - s^2 = \frac{1}{2}\left[1 + \left(1 - \frac{4\pi\alpha_{\rm EM}}{\sqrt{2}G_\mu \hat{M}_Z^2}\right)^{1/2}\right] \equiv c_0^2 \tag{12.61}$$

(3) Compute in the one-loop approximation any other observable quantity of interest in terms of the same set of the renormalized parameters of the lagrangian; keep only the terms which are finite in the limit $\varepsilon \to 0$.

(4) Obtain finite predictions for the computed observables in terms of the chosen physical input observables $\alpha_{\rm EM}$, G_μ and $\hat{M}_Z$ by inserting into the result of step (3) the expressions for the parameters $\alpha(\mu)$, $s^2(\mu)$ and $M_Z^2(\mu)$ obtained in step (2).

The above prescriptions define the one-loop perturbative calculation up to some terms of the second order which may appear depending on the particular way of realizing this programme. One way of organizing such calculations is as follows (Barbieri 1993): calculate a renormalized matrix element (or any observable quantity)

$$\mathcal{M}^{(1)} = \mathcal{M}^{(0)}(\alpha(\mu), M_Z^2(\mu), s^2(\mu)) + \delta\mathcal{M}(\alpha(\mu), M_Z^2(\mu), s^2(\mu)) \quad (12.62)$$

Using the relations $M_Z^2 \equiv \hat{M}_Z^{2(0)} = \hat{M}_Z^2 - \delta\hat{M}_Z^2$, where $\hat{M}_Z$ is the measured physical mass, $\alpha(\mu) \equiv \alpha_{\text{EM}}^{(0)} = \alpha_{\text{EM}} - \delta\alpha_{\text{EM}}$ and similarly replacing $s^2(\mu)$ by the Fermi constant, we can rewrite the last equation as follows:

$$\begin{aligned}\mathcal{M}^{(1)}(\alpha_{\text{EM}}, G_\mu, \hat{M}_Z) = {} & \mathcal{M}^{(0)}(\alpha_{\text{EM}}, G_\mu, \hat{M}_Z) + \delta\mathcal{M}(\alpha_{\text{EM}}, G_\mu, \hat{M}_Z) \\ & - \frac{\partial\mathcal{M}^{(0)}}{\partial\alpha_{\text{EM}}}\delta\alpha_{\text{EM}} - \frac{\partial\mathcal{M}^{(0)}}{\partial G_\mu}\delta G_\mu - \frac{\partial\mathcal{M}^{(0)}}{\partial M_Z^2}\delta\hat{M}_Z^2 \quad (12.63)\end{aligned}$$

For higher order calculations it is necessary to go beyond the 'short cut' approach described above and to introduce the counterterms explicitly (see the end of Section 4.3). This is also helpful in getting a more profound flavour of the renormalization programme (for example, Ward identities) and is necessary in studying the renormalization group equations for the parameters of the theory (Problem 12.4). The gauge invariance of the theory is easily taken care of if we introduce the renormalization constants at the level of the original lagrangian (12.5). We summarize here the renormalization constants of the complete Standard Model. For the gauge fields and couplings we follow the definitions introduced in Chapter 8 for QCD but we change the notation in the QCD sector for a more convenient one in the complete Standard Model framework (g_s is the QCD coupling constant, G_μ^a denote the gluon fields, the counterterms Z_3 and Z_{1YM} are renamed as Z_G and Z_S, respectively; the superscript $^{\text{B}}$ means 'bare'):

$$\left.\begin{aligned} G_\mu^{a\text{B}} &= Z_G^{1/2} G_\mu^a \\ W_\mu^{a\text{B}} &= Z_W^{1/2} W_\mu^a \\ B_\mu^{\text{B}} &= Z_B^{1/2} B_\mu \\ H^{\text{B}} &= Z_{\text{H}}^{1/2} H \\ g_s^{\text{B}} &= Z_G^{-3/2} Z_s g_s \\ g'^{\text{B}} &= Z_B^{-3/2} Z_{g'} g' \\ g^{\text{B}} &= Z_W^{-3/2} Z_g g \end{aligned}\right\} \quad (12.64)$$

The renormalization constants for the non-abelian couplings g and g_s are defined as in Chapter 8. It is convenient to define the abelian $Z_{g'}$ in the same way. It is helpful at this point to recall the notation for QED in Chapter 5: there $e^{\text{B}} =$

$(Z_1/Z_2)Z_3^{-1/2}e \equiv Z_e e$, where Z_1, Z_2, Z_3 are the vertex, electron and photon self-energy renormalization constants, respectively. The $U(1)$ Ward identity ensures $Z_1 = Z_2$ and $Z_e = Z_3^{-1/2}$. If for QED we redefine as in (12.64), $e^{\mathrm{B}} = Z_3^{-3/2}\tilde{Z}_e e$, we get $\tilde{Z}_e = Z_3$. In the present case the same argument applies separately to the left- and right-handed chiral parts of the $U_{\mathrm{Y}}(1)$ lagrangian, and it follows that the relation $Z_B = Z_{g'}$ can be imposed.

For fermion fields we use the chiral notation of (12.2):

$$\left.\begin{aligned}
\left(\Psi_{l_A}\right)_{\mathrm{L}}^{\mathrm{B}} &= (Z_l^{1/2})^A \left(\Psi_{l_A}\right)_{\mathrm{L}}\\
\left(\Psi_{e_A}\right)_{\mathrm{R}}^{\mathrm{B}} &= (Z_e^{1/2})^A \left(\Psi_{e_A}\right)_{\mathrm{R}}\\
\left(\Psi_{q_A}\right)_{\mathrm{L}}^{\mathrm{B}} &= (Z_q^{1/2})^{AC} \left(\Psi_{q_C}\right)_{\mathrm{L}}\\
\left(\Psi_{u_A}\right)_{\mathrm{R}}^{\mathrm{B}} &= (Z_u^{1/2})^{AC} \left(\Psi_{u_C}\right)_{\mathrm{R}}\\
\left(\Psi_{d_A}\right)_{\mathrm{R}}^{\mathrm{B}} &= (Z_d^{1/2})^{AC} \left(\Psi_{d_C}\right)_{\mathrm{R}}\\
\left(Y_l^A\right)^{\mathrm{B}} &= Z_{\mathrm{H}}^{-1/2}(Z_e^{-1/2})^A(Z_{Y_l}^A Y_l^A)(Z_l^{-1/2})^A\\
\left(Y_u^{AC}\right)^{\mathrm{B}} &= Z_{\mathrm{H}}^{-1/2}(Z_u^{-1/2})^{AX}(Z_{Yu}Y)^{XY}(Z_q^{-1/2})^{YC}\\
\left(Y_d^{AC}\right)^{\mathrm{B}} &= Z_{\mathrm{H}}^{-1/2}(Z_d^{-1/2})^{AX}(Z_{Yd}Y)^{XY}(Z_q^{-1/2})^{YC}
\end{aligned}\right\} \qquad (12.65)$$

Since there is no flavour mixing in the lepton sector, the Yukawa matrices Y_l^A are diagonal.† The quark wave-function renormalization constants are hermitean. The Yukawa coupling renormalization is determined by the vertex counterterms Z_Ys defined by the matrix equations, for example, $Y_u^{\mathrm{B}}\bar{\Psi}_{\mathrm{R}}^{\mathrm{B}}\Psi_{\mathrm{L}}^{\mathrm{B}}H^{\mathrm{B}} = Z_{Y_u}Y_u\bar{\Psi}_{\mathrm{R}}\Psi_{\mathrm{L}}H$, and it is convenient to define the renormalization constant matrices δYs, for example, $Z_{Y_u}Y_u = Y_u + \delta Y_u$. In Appendix C the vertex counterterms are then expressed in terms of δYs.

Finally, for the Higgs boson mass parameter m^2, its self-coupling λ and the vacuum expectation value v we write‡

$$\left.\begin{aligned}
v^{\mathrm{B}} &= Z_{\mathrm{H}}^{1/2}(v - \delta v)\\
\lambda^{\mathrm{B}} &= Z_{\mathrm{H}}^{-2}(\lambda + \delta\lambda)\\
(m^2)^{\mathrm{B}} &= Z_{\mathrm{H}}^{-1}(m^2 + \delta m^2)
\end{aligned}\right\} \qquad (12.66)$$

The additive notation is again useful, in particular since λ is additively renormalized (for example, by fermionic loops proportional to the Yukawa couplings). Replacing the unrenormalized fields and parameters in the original lagrangian (12.5) with the help of the above relations and splitting all Zs: $Z_i = 1 + \delta Z_i$,

† With massive neutrinos it is still convenient to work in the basis with diagonal matrix Y_l^A and non-diagonal effective neutrino mass operator $m_{AB}(\epsilon_{ij}l_A^i H^j)(\epsilon_{ij}l_B^i H^j)$.

‡ In the R_ξ gauge it is convenient but not necessary to introduce the counterterm δv for the Higgs vacuum expectation value.

we can generate the necessary counterterms. (We do not explicitly write down the renormalization constants for the ghost fields but they are included in the Feynman rules given in Appendix C.) We are already familiar with the fact that the counterterms which have been introduced at the level of the original lagrangian (12.5) are also sufficient to renormalize this theory after spontaneous breaking of the gauge symmetry. The Feynman rules derived from the lagrangian including the counterterms, and in the four-component Dirac notation for fermions (appropriate after spontaneous gauge symmetry breaking) are presented in Appendix C.

It is worth stressing once again that, going beyond the tree level, we must carefully distinguish parameters of the lagrangian which depend on the renormalization scale (and may also be gauge-dependent) and the physical quantities. For instance, $\hat{M}_W$ and $\hat{M}_Z$ must be distinguished from the μ-dependent lagrangian parameters M_W and M_Z (the same remark applies to the other masses too) which are given by (12.24). We recall that in the $\overline{\mathrm{MS}}$ scheme, in any order of perturbation theory, $c^2(\mu) = M_W^2(\mu)/M_Z^2(\mu)$. Likewise, the lagrangian parameter $s^2(\mu) \equiv \sin^2\theta_{\mathrm{W}}$ defined by (12.18) should be distinguished from several quantities directly related to experimental measurements and often also called in the literature the Weinberg angles. Important examples are various $\sin^2\theta^{\mathrm{eff}}$ introduced in Section 12.4 and the ratio of the physical gauge boson masses $1 - \hat{M}_W^2/\hat{M}_Z^2$. To avoid any confusion, in this book we shall never call the latter a sine of the Weinberg angle. Finally, we recall that, if the same physical quantity is calculated in terms of the same input physical observables to the same order in perturbation theory, the result does not depend on the remaining freedom in the choice of the renormalization scheme. If, however, a given observable is calculated in terms of two different sets of input observables, finite order results can numerically depend on the choice of the input parameters. Although this dependence is, in principle, of higher order, to estimate the uncertainty one would have to perform full next order calculation.

In the following we discuss certain building blocks of the one-loop quantum corrections in the electroweak theory, and then consider a sample of the most relevant physical applications. In particular, as we shall see, it will be convenient also to recast one-loop corrections to the selected processes in the form of corrections to the effective lagrangians introduced in Section 12.4.

Corrections to gauge boson propagators

In general, radiative corrections to the processes described in Section 12.4 can be conveniently divided into two classes: universal corrections to the gauge boson propagators (called sometimes the ‘oblique’ corrections) which in most cases give the dominant contribution, and process-dependent corrections consisting of corrections to the external fermion lines, corrections to the vertices and corrections

Fig. 12.5. Contributions to gauge boson self-energy.

generated by box diagrams. Process-dependent corrections are usually (but not always) in the commonly used gauges, like, for example, Feynman ($\xi = 1$) or Landau ($\xi = 0$) gauge, much smaller than the 'oblique' corrections, and are therefore frequently neglected. One should, however, keep in mind, that the splitting into universal and process-dependent parts is gauge-dependent and to get physical, gauge-independent results one should include both types of correction. Also, in general, the corrections to the gauge boson propagators are not separately renormalization scheme-independent (in a given order) since they share the counterterms, for example, with the vertex corrections. Thus, in the calculation in the $\overline{\text{MS}}$ scheme, in which only $1/\varepsilon$ terms are subtracted, the dependence on the renormalization scale μ of the loop corrections to gauge boson propagators alone does not disappear. Moreover, in some schemes (like the $\overline{\text{MS}}$ scheme) inclusion of at least part of the process-dependent corrections may be necessary to avoid spurious singularities in the amplitudes in some kinematical limits (see below).

We begin with the process-independent corrections to the gauge boson propagators. The full vector-boson self-energy amplitude, which consists in general of two parts corresponding to diagrams shown in Fig. 12.5, will be denoted by†

$$\mathrm{i}\Pi^{\mu\nu}_{V_1V_2}(p) = \mathrm{i}P^{\mu\nu}\Pi^{\mathrm{T}}_{V_1V_2}(p^2) + \mathrm{i}L^{\mu\nu}\Pi^{\mathrm{L}}_{V_1V_2}(p^2) \tag{12.67}$$

where $P_{\mu\nu}$ and $L_{\mu\nu}$ are transverse and longitudinal projection operators. The tadpole diagrams must be included if one wants to verify gauge invariance of the mass counterterms. They cancel out, however, in physical amplitudes and will be omitted in the following. Moreover, the longitudinal terms $\Pi^{\mathrm{L}}_{V_1V_2}$ of the gauge boson self-energies do not contribute to the physical matrix elements. This is immediately seen for processes with external vector bosons ($\varepsilon_\mu k^\mu = 0$) and with external fermions in the limit $m_f^2 \to 0$ (conserved currents). In the general case these terms cancel (order by order in perturbation theory) Goldstone boson self-energy amplitudes as a consequence of the Slavnov–Taylor identities.‡

† Note the sign convention which is different from the one used for fermion self-energies. This is convenient since the free fermion and boson propagators have opposite signs and in both cases, after summing up one-particle reducible diagrams, the self-energy amplitudes at $k^2 = M_V^2$ are corrections to the physical masses.

‡ The gauge dependence of Π^{T}s is cancelled by the vertex corrections. Note also that, for on-shell amplitudes, the gauge-dependent parts of the tree level propagators cancel with the gauge-dependent Goldstone boson propagators. The net result is equivalent to using the propagators in the unitary gauge.

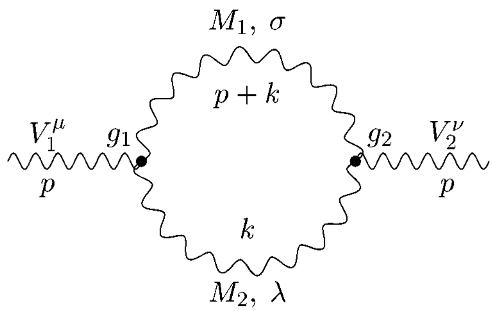


Fig. 12.6. Vector boson contribution to the vector boson self-energy.

As an illustration, we explicitly compute in the 't Hooft–Feynman gauge the contribution of the gauge boson loop to the gauge boson self-energies. The formulae for other contributions (from fermion, Higgs boson and mixed Higgs–vector boson loops) are given in Appendix D. The diagram to be evaluated is shown in Fig. 12.6. Using the trilinear vector boson couplings given in Appendix C we get then for Fig. 12.6 (in $n = 4 - \varepsilon$ dimensions, with the dimensionless couplings $g \to g\mu^{\varepsilon/2}$)

$$
\begin{aligned}
(\mathrm{i}g_1)(\mathrm{i}g_2)\mu^\varepsilon \int \frac{\mathrm{d}^n k}{(2\pi)^n} & \frac{-\mathrm{i}g^{\sigma\sigma'}}{(k+p)^2 - M_1^2} \frac{-\mathrm{i}g^{\lambda\lambda'}}{k^2 - M_2^2} \\
& \times \left[g^{\mu\sigma}(k+2p)^\lambda + g^{\mu\lambda}(k-p)^\sigma - g^{\sigma\lambda}(2k+p)^\mu \right] \\
& \times \left[g^{\nu\sigma'}(k+2p)^{\lambda'} + g^{\nu\lambda'}(k-p)^{\sigma'} - g^{\sigma'\lambda'}(2k+p)^\nu \right]
\end{aligned}
$$

where the couplings are, in principle, renormalization scale-dependent but to this order of calculation they can be replaced by their tree level values. With some algebra (and remembering that $g^{\mu\nu}g_{\nu\mu} = n$) we find

$$
\begin{aligned}
-\mathrm{i}g_1 g_2 \mu^\varepsilon \int \frac{\mathrm{d}^n k}{(2\pi)^n} & \frac{\mathrm{i}}{[(k+p)^2 - M_1^2][k^2 - M_2^2]} \left[g^{\mu\nu}(2k^2 + 2kp + 5p^2) \right. \\
& \left. + (4n-6)k^\mu k^\nu + (n-6)p^\mu p^\nu + (2n-3)(k^\mu p^\nu + p^\mu k^\nu) \right]
\end{aligned}
$$

$\Pi^{\mathrm{T}}_{V_1V_2}(p^2)$ and $\Pi^{\mathrm{L}}_{V_1V_2}(p^2)$ can now be expressed in terms of the functions a, b_0, A and B given by (D.1)–(D.11). For instance,

$$
\begin{aligned}
\Pi^{\mathrm{T}}_{V_1V_2}(p^2) = -\frac{g_1 g_2}{(4\pi)^2} & \left[10A(p^2, M_1, M_2) + (4p^2 + M_1^2 + M_2^2) b_0(p^2, M_1, M_2) \right. \\
& \left. + a(M_1) + a(M_2) + 2M_1^2 + 2M_2^2 - \tfrac{2}{3}p^2 \right]
\end{aligned}
$$

With some labour the other contributions to the vector boson self-energies can be obtained in a fairly straightforward manner, using our experience from the chapters on the scalar field theory and on QED. We recall in particular that, due to $U_{\mathrm{EM}}(1)$

gauge invariance, only the transverse part of the photon propagator is renormalized and, moreover, the photon self-energy can be written as (see Section 5.2)

$$\mathrm{i}\Pi^{\mu\nu}_{\gamma\gamma}(p) = \mathrm{i}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) p^2 \tilde{\Pi}_{\gamma\gamma}(p^2) \tag{12.68}$$

i.e. $\Pi^{\mathrm{L}}_{\gamma\gamma}(p^2) \equiv 0$ and $\Pi^{\mathrm{T}}_{\gamma\gamma}(0) = 0$. Using the results of Section 5.2 we can include the fermion loop corrections to the photon propagator and, for instance, in the limit $p^2 = 0$, which is useful for us one gets the following result:

$$\tilde{\Pi}_{\gamma\gamma}(0) = \frac{\alpha}{3\pi}\left(-\frac{23}{4}\eta + \frac{1}{2} - \frac{5}{2}\ln\frac{M_W^2}{\mu^2} + \frac{1}{4}\ln\frac{M_W^2}{\mu^2} + \sum_f N_{c_f} Q_f^2 \ln\frac{m_f^2}{\mu^2}\right) \tag{12.69}$$

Here N_{c_f} and Q_f are the colour factor (1 for leptons, 3 for quarks) and charge of the fermion, respectively; the two M_W-dependent terms are the $W^\pm$ and the Goldstone boson contributions, μ is the $\overline{\mathrm{MS}}$ renormalization scale and the divergence $\eta \equiv 2/(4-n) + \ln(4\pi) - \gamma_E$ may be removed by adding appropriate counterterms. Of course, $M_W^2 = c^2 M_Z^2$ and all the parameters are renormalized at the scale μ, but to this order of calculation they can be replaced by the physical parameters α_{EM}, $\hat{M}_Z^2$ and the appropriate function of G_μ.

Another self-energy which requires a comment is the γ–Z^0 mixing amplitude. At one loop it is given by the same set of Feynman diagrams (Figs D.1 and D.2) as the corrections to the photon self-energy with appropriate changes in one of the vertices (see Appendix C). It will prove useful to decompose it as follows

$$\Pi^{\mathrm{T}}_{\gamma Z}(p^2) = \Pi^{\mathrm{T}}_{\gamma Z}(0) + p^2 \tilde{\Pi}_{\gamma Z}(p^2) \tag{12.70}$$

In the case of the photon self-energy, the contributions of the diagrams shown in Figs D.2(*a*)–(*c*) and D.1(*a*)–(*c*) plus D.2(*d*) separately vanish at $p^2 = 0$. For the γ–Z^0 mixing amplitude diagram D.2(*d*) (with W^+G^- and W^-G^+ circulating in the loop) does not cancel the contribution of D.1(*a*)–(*c*) at $p^2 = 0$. Comparing with the $\Pi^{\mathrm{T}}_{\gamma\gamma}$ case and isolating the combination of the two point functions which does vanish at $p^2 = 0$, one easily finds

$$\Pi^{\mathrm{T}}_{\gamma Z}(0) = -2\frac{\alpha}{4\pi}\frac{c}{s}M_Z^2\left(-\eta + \ln\frac{M_W^2}{\mu^2}\right) \tag{12.71}$$

It is important that $\Pi^{\mathrm{T}}_{\gamma Z}(0)$ does not contain a fermionic contribution.

Eight self-energies: $\Pi^{\mathrm{T(L)}}_{WW}(p^2)$, $\Pi^{\mathrm{T(L)}}_{ZZ}(p^2)$, $\Pi^{\mathrm{T(L)}}_{\gamma Z}(p^2)$, $\Pi^{\mathrm{T(L)}}_{\gamma\gamma}(p^2)$ are made finite by including counterterms shown in Appendix C, which depend on five independent renormalization constants δZ_{W}, δZ_B, δZ_g, δZ_{H} and δv. This is a non-trivial and practical test of the computation.

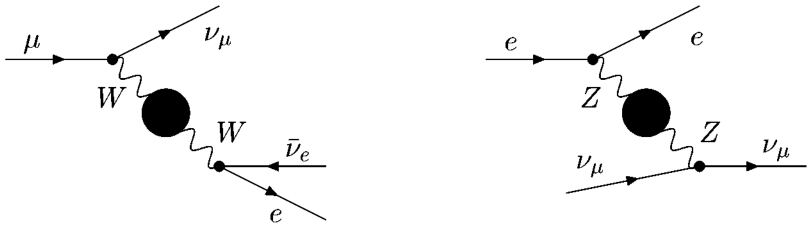


Fig. 12.7. 'Oblique' correction to muon decay and elastic neutrino scattering.

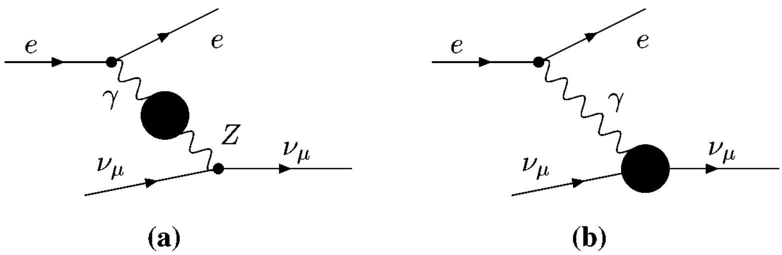


Fig. 12.8. Corrections to elastic neutrino scattering which are singular at $p^2 = 0$.

In the limit $p^2 \approx 0$ and neglecting terms $\mathcal{O}(m_f^2/M_W^2)$, the one-loop corrections to the gauge boson propagators, inserted into four-fermion Green's functions (Fig. 12.7), can be summarized in the form of corrections to the effective four-fermion lagrangians:

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(\text{oblique}) = \left(1 - \frac{\Pi_{WW}^{\text{T}}(0)}{M_W^2}\right) \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \tag{12.72}$$

$$\mathcal{L}_{\text{eff}}^{\text{NC}}(\text{oblique}) = \left(1 - \frac{\Pi_{ZZ}^{\text{T}}(0)}{M_Z^2}\right) \times \mathcal{L}_{\text{eff}}^{\text{NC}}(\text{Born}) \tag{12.73}$$

where, as it is clear from (12.47) and (12.48), for example,

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \equiv -\frac{2\pi\alpha}{s^2 c^2 M_Z^2} J_\mu^- J^{\mu +}.$$

It is important to remark that an additional contribution which should in principle be included is the one shown in Fig. 12.8(a). Being singular at $p^2 = 0$, it cannot be incorporated as it stands into the neutral-current effective lagrangian. However, the singularity at $p^2 = 0$ of the diagram shown in Fig. 12.8(a) cancels against the similar singularity contained in the vertex correction shown in Fig. 12.8(b). The sum of these two graphs is finite at $p^2 = 0$, numerically small and can be neglected.

The first interesting physical result which we can already get at this point is that, with one-loop accuracy and neglecting corrections other than the ones to the $W^{\pm}$ and Z^0 gauge boson propagators, we have

$$\Delta\rho_f \equiv \rho_f - 1 \approx \frac{\Pi^{\mathrm{T}}_{WW}(0)}{M_W^2} - \frac{\Pi^{\mathrm{T}}_{ZZ}(0)}{M_Z^2} \tag{12.74}$$

Of course, $\Delta\rho_f$, as any other physical observable, is calculable (finite) in terms of α_{EM}, $\hat{M}_Z$, G_μ and the Higgs boson mass, i.e. a systematic application of the renormalization programme gives a finite (and, therefore, renormalization scale-independent) result for $\Delta\rho_f$ in each order of perturbation theory. However, as mentioned earlier, as a generic situation in the case of some truncations in this programme, the above combination of the self-energy amplitudes is not separately renormalization-scale-independent. Still, the dominant contribution to (12.74) is finite by itself. It comes from the third generation fermion loops. Indeed, we have already noted that $\Delta\rho_f = 0$ in the limit of exact custodial $SU_{\mathrm{V}}(2)$ symmetry. This symmetry is explicitly broken by the g' coupling, the Higgs sector and the fermion mass splittings within each generation. The latter contribution is finite by itself since it vanishes when the splitting vanishes. The largest violation of the custodial $SU_{\mathrm{V}}(2)$ originates from the large top quark–bottom quark mass splitting. Thus, the dominant contribution to $\Delta\rho$ is readily calculable.

With the help of the formulae for the fermionic contribution to the gauge boson self-energies collected in Appendix D one finds

$$\left.\begin{aligned}
\Pi^{\mathrm{T}(t,b)}_{WW}(0) &= \frac{N_c}{4\pi}\frac{\alpha}{s^2}\left[\frac{1}{12}(m_t^2+m_b^2) - \frac{1}{6}\frac{m_t^2 m_b^2}{m_t^2-m_b^2}\ln\frac{m_t^2}{m_b^2}\right.\\
&\left. -\frac{1}{3}a(m_t) - \frac{1}{3}a(m_b) - \frac{1}{6}(m_t^2+m_b^2)b_0(0,m_t,m_b) - \frac{1}{3}m_t^2 - \frac{1}{3}m_b^2\right]\\
\Pi^{\mathrm{T}(t,t)}_{ZZ}(0) &= \frac{N_c}{4\pi}\frac{\alpha}{2s^2c^2}\left\{\left(1-\frac{8}{3}s^2+\frac{32}{9}s^4\right)\left[-\frac{2}{3}a(m_t) - \frac{2}{3}m_t^2\right.\right.\\
&\left.\left. -\frac{1}{3}m_t^2 b_0(0,m_t,m_t)\right] + \left(-\frac{8}{3}s^2+\frac{32}{9}s^4\right)m_t^2 b_0(0,m_t,m_t)\right\}\\
\Pi^{\mathrm{T}(b,b)}_{ZZ}(0) &= \frac{N_c}{4\pi}\frac{\alpha}{2s^2c^2}\left\{\left(1-\frac{4}{3}s^2+\frac{8}{9}s^4\right)\left[-\frac{2}{3}a(m_b) - \frac{2}{3}m_b^2\right.\right.\\
&\left.\left. -\frac{1}{3}m_b^2 b_0(0,m_b,m_b)\right] + \left(-\frac{4}{3}s^2+\frac{8}{9}s^4\right)m_b^2 b_0(0,m_b,m_b)\right\}
\end{aligned}\right\} \tag{12.75}$$

where $N_c = 3$ is the colour factor. It is easy to check that

$$\frac{\Pi_{WW}^{\mathrm{T}(t,b)}(0)}{\hat{M}_W^2} - \frac{\Pi_{ZZ}^{\mathrm{T}(t,t)}(0)}{\hat{M}_Z^2} - \frac{\Pi_{ZZ}^{\mathrm{T}(b,b)}(0)}{\hat{M}_Z^2}$$
$$= \frac{N_c}{4\pi}\frac{\alpha}{4s^2c^2\hat{M}_Z^2}\left(m_t^2 + m_b^2 - \frac{2m_t^2m_b^2}{m_t^2 - m_b^2}\ln\frac{m_t^2}{m_b^2}\right) \tag{12.76}$$

This result is indeed renormalization-scale-independent (as for a finite contribution), and since $m_t \gg m_b$ (experimentally $m_t = 175.6 \pm 5.5$ GeV whereas $m_b \approx 5$ GeV) it can be approximated as

$$\Delta\rho_f \approx N_c \frac{\alpha_{\mathrm{EM}}}{16\pi s^2c^2}\frac{m_t^2}{\hat{M}_Z^2} \tag{12.77}$$

Expressing the Weinberg angle in terms of G_μ (in a one-loop calculation the tree level formula can be used), we obtain in our approximation the first one-loop prediction of the electroweak theory: $\Delta\rho_f = 0.0096$.

So far we have discussed the self-energies of the vector bosons in a systematic perturbation theory. At each order the physical mass of the $W^\pm$ boson is given in terms of the running mass $M_W^2(\mu)$ by the requirement that the real part of the two-point non-local vertex of the effective action (also called 1PI two-point Green's function), introduced in Section 2.6,

$$\Gamma_{WW}^{\mathrm{T}}(k^2) = k^2 - M_W^2 - \Pi_{WW}^{\mathrm{T}}(k^2) \tag{12.78}$$

vanishes at $p^2 = \hat{M}_W^2$.† This condition applies to the transverse part of Γ_{WW}. There are, however, important physical situations such as the phenomena at energies in the vicinity of the physical masses of unstable particles, for example, the $\mathrm{e}^+\mathrm{e}^-$ scattering close to M_Z (in the resonance region), where a systematic perturbation theory result is insufficient, and should be replaced by the Breit–Wigner form of the full propagator. Since the transverse part of the full $W^\pm$ propagator is given by (2.152) (or, equivalently, can be obtained by summing all 1PI diagrams), we see that it indeed has the Breit–Wigner form with the decay width of $W^\pm$ given by $\mathrm{Im}\,\Pi_{WW}^{\mathrm{T}}(k^2 = M_W^2)$ (we ignore here the subtleties related to the definition of the mass of unstable particles and to the gauge-independent definition of their width). The imaginary part of $\Pi_{WW}^{\mathrm{T}}(k^2)$ can be obtained by applying the Landau–Cutkosky rule to the Feynman integrals or by taking the imaginary part of the final result (see Appendix D).

† We have not yet given all the calculations necessary for solving this equation with one-loop accuracy in terms of α_{EM}, G_μ and $\hat{M}_Z^2$. For that, we need the one-loop relation between $M_W^2(\mu)$ and the above three measured parameters, which will be discussed later in this section.

For the longitudinal part of the $W^\pm$ propagator the situation is more complicated as it receives a contribution from the $W^\pm$–Goldstone boson mixing. The two-point vertex function Γ is now a 2×2 matrix and the longitudinal part of the full propagator is obtained by inverting this matrix. Of course, the physical mass and the width of W obtained from this procedure are the same (due to Ward identities) as those defined for the transverse part.

The same procedure is necessary for the Z^0 boson propagator because of the Z^0–γ mixing. We begin with the 2×2 matrix for the inverse propagator

$$\Gamma^{\mathrm{T}}_{Z\gamma}(k^2) = \begin{pmatrix} k^2 - \Pi^{\mathrm{T}}_{\gamma\gamma}(k^2) & -\Pi^{\mathrm{T}}_{\gamma Z}(k^2) \\ -\Pi^{\mathrm{T}}_{\gamma Z}(k^2) & k^2 - M_Z^2 - \Pi^{\mathrm{T}}_{ZZ}(k^2) \end{pmatrix} \tag{12.79}$$

Using 2×2 matrix inversion we find for the transverse parts of the Z^0–γ propagators

$$\left.\begin{aligned} G^{\mathrm{T}}_{ZZ}(k^2) &= \frac{1}{k^2 - M_Z^2 - \Pi^{\mathrm{T}}_{ZZ}(k^2) - \dfrac{[\Pi^{\mathrm{T}}_{\gamma Z}(k^2)]^2}{k^2 - \Pi_{\gamma\gamma}(k^2)}} \sim \frac{1}{k^2 - M_Z^2 - \Pi^{\mathrm{T}}_{ZZ}(k^2)} \\ G^{\mathrm{T}}_{\gamma\gamma}(k^2) &= \frac{1}{k^2 - \Pi^{\mathrm{T}}_{\gamma\gamma}(k^2) - \dfrac{[\Pi^{\mathrm{T}}_{\gamma Z}(k^2)]^2}{k^2 - M_Z^2 - \Pi^{\mathrm{T}}_{ZZ}(k^2)}} \sim \frac{1}{k^2 - \Pi^{\mathrm{T}}_{\gamma\gamma}(k^2)} \\ G^{\mathrm{T}}_{\gamma Z} &= \frac{\Pi_{\gamma Z}(k^2)}{[k^2 - \Pi_{\gamma\gamma}(k^2)][k^2 - M_Z^2 - \Pi^{\mathrm{T}}_{ZZ}(k^2)] - [\Pi^{\mathrm{T}}_{\gamma Z}(k^2)]^2} \sim \frac{\Pi_{\gamma Z}(k^2)}{k^2(k^2 - M_Z^2)} \end{aligned}\right\} \tag{12.80}$$

Fermion self-energies

We compute also the gauge boson and scalar contributions to the fermion self-energy. Denoting the amplitude corresponding to the general self-energy diagrams shown in Fig. 12.9 as

$$-\mathrm{i}\Sigma \equiv -\mathrm{i}\,(\Sigma_{\mathrm{L}} \not{p} P_{\mathrm{L}} + \Sigma_{\mathrm{R}} \not{p} P_{\mathrm{R}} + \Sigma_m) \equiv -\mathrm{i}(\Sigma_{\mathrm{V}} \not{p} + \Sigma_{\mathrm{A}} \gamma_5 \not{p} + \Sigma_m) \tag{12.81}$$

we have for Fig. 12.9(a):

$$\begin{aligned} -\mathrm{i}\Sigma = (\mathrm{i})^2 \mu^\varepsilon \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\mathrm{i}}{[(k+p)^2 - m^2]} \frac{-\mathrm{i} g_{\mu\nu}}{[k^2 - M^2]} \\ \times \gamma^\mu (c_{\mathrm{V}}^\star - c_{\mathrm{A}}^\star \gamma^5)\,(\not{k} + \not{p} + m)\,\gamma^\nu (c_{\mathrm{V}} - c_{\mathrm{A}} \gamma^5) \end{aligned} \tag{12.82}$$

Performing the Dirac algebra and using the formulae (D.1) and (D.8) we obtain the

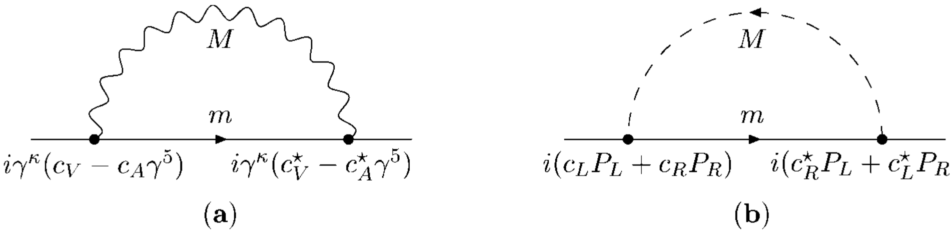


Fig. 12.9. General vector boson and scalar contributions to a fermion self-energy.

result

$$(4\pi)^2\Sigma = \frac{n-2}{2p^2}\left[a(M) - a(m) + (p^2 + m^2 - M^2)b_0(p^2, m, M)\right] \times \not{p}\left(|c_V + c_A|^2 P_L + |c_V - c_A|^2 P_R\right) - m(|c_V|^2 - |c_A|^2)\, n\, b_0(p^2, m, M) \tag{12.83}$$

Similarly, for a scalar contribution (Fig. 12.9(*b*)) we find

$$(4\pi)^2\Sigma = \frac{1}{2p^2}\left[a(M) - a(m) + (p^2 + m^2 - M^2)b_0(p^2, m, M)\right] \times \not{p}\left(c_L c_L^\star P_L + c_R c_R^\star P_R\right) + m\left(c_L c_R^\star P_L + c_R c_L^\star P_R\right) b_0(p^2, m, M) \tag{12.84}$$

For example, applying the general formula (12.83) to the $W^\pm$ and Z^0 corrections to the electron self-energy we get in the limit $m_e \ll M_{W,Z}$:

$$\left.\begin{aligned}\Sigma_{eL}(0) &= \frac{\alpha}{(4\pi)}\left[\frac{1}{2s^2}\left(\frac{1}{2} + \ln\frac{M_W^2}{\mu^2}\right) + \frac{(1-2s^2)^2}{4s^2c^2}\left(\frac{1}{2} + \ln\frac{M_Z^2}{\mu^2}\right)\right] \\ \Sigma_{eR}(0) &= \frac{\alpha}{(4\pi)}\frac{s^2}{c^2}\left(\frac{1}{2} + \ln\frac{M_W^2}{\mu^2}\right)\end{aligned}\right\} \tag{12.85}$$

All the parameters are, as usual, the running parameters.

Running $\alpha(\mu)$ in the electroweak theory

Following our general strategy, we now have to calculate loop corrections to the input observables to relate the renormalization scale-dependent lagrangian parameters to the measured α_{EM}, $\hat{M}_Z$ and G_F. These corrections enter into the calculation of any other (predicted) observable if accuracy beyond the tree level is needed. It is also important to stress that the renormalization-scale-dependent coupling $\alpha(\mu)$ in the complete electroweak theory is different from the analogous coupling in QED, which is the low energy effective theory obtained after integrating out the weak

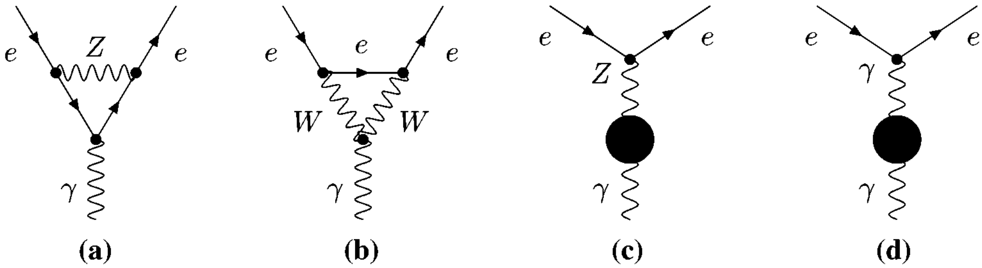


Fig. 12.10. Corrections to the photon–electron interaction. The external photon is real with the four-momentum $k \approx 0$.

sector. This point will be discussed in detail in Section 12.6. When necessary, we shall keep the distinct notation $\alpha_{\rm SM}(\mu)$ and $\alpha_{\rm EM}(\mu)$. Here we want to find the relation between $\alpha_{\rm SM}(\mu)$ and the fine structure constant $\alpha_{\rm EM} = 1/137$, and the subscript 'SM' will be omitted.

The fine structure constant can be defined by the classical cross section for Thomson scattering (see, for example, Bjorken & Drell 1964)

$$\sigma = \frac{8\pi}{3}\frac{\alpha_{\rm EM}^2}{m_e^2} \tag{12.86}$$

In any order of perturbation theory the lagrangian parameter $\alpha(\mu)$ can be expressed in terms of $\alpha_{\rm EM}$ and $\hat{M}_Z^2$ by calculating the Thomson scattering in the limit of $k \to 0$ (k is the photon four-momentum) in the complete electroweak theory renormalized at some scale μ. At the one-loop level one has to compute the diagrams shown in Fig. 12.10 (the black blobs denote one-loop diagrams) and one-loop $W^\pm$ and Z^0 contributions to the electron self-energies.† The diagram similar to Fig. 12.10(a) but with the photon replacing Z^0, at $k = 0$ and with the electrons on-shell is exactly cancelled by the photon contribution to the external line renormalization (as demonstrated in Chapter 5). It is convenient to summarize the result of this calculation, performed in the $\overline{\rm MS}$ scheme, in a form such that it will then be easy to compare it with (12.86). By explicit calculation one finds that the result obtained in the complete electroweak theory, in the limit $k = 0$, is reproduced by the following tree level lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm eff}^{\rm low} = &-\tfrac{1}{4}(1+\delta z_\gamma)F_{\rho\kappa}F^{\rho\kappa} + (1+\delta z_2^{\rm L})\bar{\Psi}_e \mathrm{i}\not{\partial} P_{\rm L}\Psi_e + (1+\delta z_2^{\rm R})\bar{\Psi}_e \mathrm{i}\not{\partial} P_{\rm R}\Psi_e \\ &- e(\mu)(1+\delta z_1^{\rm L})\bar{\Psi}_e \not{A} P_{\rm L}\Psi_e - (\delta m + m\delta z_{\rm L})\bar{\Psi}_e P_{\rm L}\Psi_e \\ &- e(\mu)(1+\delta z_1^{\rm R})\bar{\Psi}_e \not{A} P_{\rm R}\Psi_e - (\delta m + m\delta z_{\rm R})\bar{\Psi}_e P_{\rm R}\Psi_e\end{aligned} \tag{12.87}$$

† Box diagrams with a gauge boson (γ or Z^0) attached to the external electron lines vanish in the limit $k = 0$.

All the fields and parameters are, of course, those of the complete theory. In the $\overline{\text{MS}}$ scheme and for $k = 0$ we get

$$\begin{aligned}\delta z_\gamma &= -\tilde{\Pi}_{\gamma\gamma}(0)\\ e(\mu)\delta z_1^{\mathrm{L}} &= -\frac{e(\mu)(1-2s^2)}{2sc}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{M_Z^2} - \frac{e^3(\mu)}{(4\pi)^2}\left[\frac{(1-2s^2)^2}{4s^2c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right)\right.\\ &\quad\left. + \frac{3}{2s^2}\left(\frac{1}{6}+\ln\frac{M_W^2}{\mu^2}\right)\right] \end{aligned} \tag{12.88}$$

$$e(\mu)\delta z_1^{\mathrm{R}} = e(\mu)\frac{s}{c}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{M_Z^2} - \frac{e^3(\mu)}{(4\pi)^2}\frac{s^2}{c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right) \tag{12.89}$$

where $\tilde{\Pi}_{\gamma\gamma}(0)$ is given by (12.69) and $\Pi_{\gamma Z}$ by (12.71). One-loop $W^\pm$ and Z^0 contributions to the electron self-energies have already been computed and are given by (12.85). They give

$$\left.\begin{aligned}\delta z_{\mathrm{L}} &= -\frac{e^2(\mu)}{(4\pi)^2}\left[\frac{1}{2s^2}\left(\frac{1}{2}+\ln\frac{M_W^2}{\mu^2}\right) + \frac{(1-2s^2)^2}{4s^2c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right)\right]\\ \delta z_{\mathrm{R}} &= -\frac{e^2(\mu)}{(4\pi)^2}\frac{s^2}{c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right)\end{aligned}\right\} \tag{12.90}$$

Comparison with (12.86) is immediate if we rescale the fields in the lagrangian (12.87) to get the canonical form of the kinetic terms. The effective charge

$$e(\mu)\left(1+\delta z_1^{\mathrm{L}} - \delta z_2^{\mathrm{L}} - \tfrac{1}{2}\delta z_\gamma\right) \equiv e(\mu)\left(1+\delta z_1^{\mathrm{R}} - \delta z_2^{\mathrm{R}} - \tfrac{1}{2}\delta z_\gamma\right) \tag{12.91}$$

can be identified† with α_{EM}, and we get the one-loop relation

$$\begin{aligned}\alpha_{\mathrm{EM}} &= \alpha(\mu)\left(1+\tilde{\Pi}_{\gamma\gamma}(0) + 2\frac{s}{c}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{M_Z^2}\right)\\ &= \alpha(\mu)\left[1+\frac{\alpha}{4\pi}\left(\frac{2}{3} - 7\ln\frac{M_W^2}{\mu^2} + \frac{4}{3}\sum_f N_{c_f}Q_f^2\ln\frac{m_f^2}{\mu^2}\right)\right]\end{aligned} \tag{12.92}$$

We remind the reader once again that $\alpha(\mu) \equiv e^2(\mu)/4\pi \equiv \alpha_{\mathrm{SM}}(\mu)$.

One complication in using (12.92) is that $\tilde{\Pi}_{\gamma\gamma}(0)$ contains light quark contribution, and the result obtained for it in perturbation theory is not reliable. In this regime strong interaction corrections become large and non-perturbative. To deal

† Two facts about (12.91) are worth noting: it is an equivalence relation (as we checked by explicit calculation) due to the unbroken $U_{\mathrm{EM}}(1)$ symmetry and, moreover, $\delta z_1^{\mathrm{L}} - \delta z_2^{\mathrm{L}} = \delta z_1^{\mathrm{R}} - \delta z_2^{\mathrm{R}} \neq 0$ because of the weak contributions.

with this problem one writes

$$\begin{aligned}\tilde{\Pi}_{\gamma\gamma}(0) &= \tilde{\Pi}^{\text{had}}_{\gamma\gamma}(0) + \tilde{\Pi}^{\text{remainder}}_{\gamma\gamma}(0) \\ &= -\hat{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) + \tilde{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) + \tilde{\Pi}^{\text{remainder}}_{\gamma\gamma}(0)\end{aligned}$$

where $\tilde{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2)$ can already be computed in perturbation theory, and the finite quantity

$$\hat{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) = \tilde{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) - \tilde{\Pi}^{\text{had}}_{\gamma\gamma}(0) \tag{12.93}$$

is the photon self-energy scalar function in the on-shell scheme. It can be computed via the dispersion relation with one subtraction at $p^2 = 0$ (since $\hat{\Pi}(0) = 0$). The dispersion relation reads

$$\hat{\Pi}^{\text{had}}_{\gamma\gamma}(s) = \frac{1}{\pi} s \int_{4m_\pi^2}^{\infty} \mathrm{d}s' \frac{\operatorname{Im}\hat{\Pi}(s')}{s'(s'-s)} \tag{12.94}$$

and $\operatorname{Im}\hat{\Pi}(p^2)$ can be expressed in terms of the measured ratio $R(s) = \sigma(\mathrm{e}^+\mathrm{e}^- \to \text{hadrons})/\sigma(\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-)$ (where s is the centre of mass energy of the process): $\operatorname{Im}\hat{\Pi}(s) = \frac{1}{3}\alpha_{\text{EM}} R(s)$. Evaluation of the integral gives the value $\hat{\Pi}^{\text{had}}_{\gamma\gamma}(M_Z^2) = 0.0280 \pm 0.0007$ (Eidelman & Jegerlehner 1995).

Muon decay in the one-loop approximation

The most straightforward approach to using the muon decay width for fixing the parameters of the electroweak theory at the one-loop level is simply to express them in terms of the width itself. It is, however, much more elegant to consider the constant G_μ defined by (12.50) as a fundamental constant, to write down the result of the one-loop calculation in the electroweak theory in the same form and to express the lagrangian parameters in terms of the constant G_μ. This can be achieved if we succeed in splitting the full result into QED corrections to the tree level effective lagrangian and the genuine one-loop electroweak corrections to the coupling G_{F}. Thus, in practice, we have to subtract from the full one-loop Green's function the Green's function obtained by calculating the diagrams in Figs. 12.2 and 12.3. As we shall see, in the limit $q \sim 0$ and neglecting fermion masses, the remaining piece can indeed be summarized in the form of the effective four-fermion lagrangian (in fact this is true in any order of perturbation theory). This is true for all charged current processes with four external fermions but, now, the effective four-fermion lagrangian has to be written as a sum of flavour-dependent pieces:

$$\mathcal{L}_{\text{eff}}^{\text{CC}} = - \sum_{A,B,C,D} 2\sqrt{2}\, G_f \left(\bar{\Psi}_{u_A} \gamma^\mu P_{\text{L}} (V_{\text{CKM}})_{AB} \Psi_{d_B}\right)\left(\bar{\Psi}_{d_C} \gamma_\mu P_{\text{L}} (V_{\text{CKM}}^\dagger)_{CD} \Psi_{u_D}\right) \tag{12.95}$$

with similar terms (with obvious simplifications) for the leptonic and semi-leptonic processes. The subscript f of the couplings G denotes the dependence on all the fermion flavours of a given four-fermion operator. Due to the process-dependent corrections, they are no longer universal and become fermion quantum number dependent. Thus, for example, the lepton universality is no longer strictly true and each decay: $\mu \to e\bar{\nu}_e \nu_\mu$, $\tau \to \mu \bar{\nu}_\mu \nu_\tau$ and $\tau \to e\bar{\nu}_e \nu_\tau$ has its own Fermi constant. In the following we shall use G_μ (best measured experimentally) as the input observable and the others will be the predictions of the theory at each order of perturbation. These predictions can, for instance, be expressed as the values of the parameters κ_f defined by the equations: $G_f = \kappa_f G_\mu$.

For muon decay, we have to complete the calculation of one-loop corrections to the relevant piece of the effective lagrangian (12.95) or, more precisely, to the coefficient of the four-fermion operator $\left(\bar{\Psi}_e \gamma_\mu P_{\text{L}} \Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu} \gamma^\mu P_{\text{L}} \Psi_\mu\right)$, which describes muon decay. The subtle point of our subtraction procedure is that the result (12.50) has been obtained with a four-dimensional regularization and in the on-shell renormalization scheme, whereas now we work with the dimensional regularization and in the $\overline{\text{MS}}$ scheme. One obvious consequence is that our final result (suppose it has the form (12.50)) will depend on the renormalization scale μ: α_{EM} in (12.50) will be replaced by $\alpha(\mu)$. This difference is a higher order effect and can be neglected at one-loop accuracy. Another consequence is more delicate.

As mentioned earlier, with four-dimensional regularization prescriptions, the vertex counterterm and the wave-function counterterms in Fig. 12.3 cancel out. This is, in general, no longer true in dimensional regularization. By explicit calculation one can check that the vertex correction to the lagrangian (12.47) has the structure

$$\begin{aligned}
&\frac{\alpha}{\pi}\frac{G_{\text{F}}}{\sqrt{2}}\left(\frac{1}{\varepsilon} + \text{const.}\right)\left(\bar{\Psi}_e \gamma_\mu \gamma_\nu \gamma_\rho P_{\text{L}} \Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu} \gamma_\rho \gamma_\nu \gamma_\mu P_{\text{L}} \Psi_\mu\right) \\
&\qquad \equiv \frac{\alpha}{\pi}\frac{G_{\text{F}}}{\sqrt{2}}\left(\frac{1}{\varepsilon} + \text{const.}\right)\mathcal{O}_1 = \frac{\alpha}{\pi}\frac{G_{\text{F}}}{\sqrt{2}}\left(\frac{1}{\varepsilon} + \text{const.}\right)[4\mathcal{O}_2 + (\mathcal{O}_1 - 4\mathcal{O}_2)]
\end{aligned}$$

where $\mathcal{O}_2 \equiv \left(\bar{\Psi}_e \gamma_\lambda P_{\text{L}} \Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu} \gamma^\lambda P_{\text{L}} \Psi_\mu\right)$. The diagrams with the fermion self-energy correction give

$$-4\frac{\alpha}{\pi}\frac{G_{\text{F}}}{\sqrt{2}}\left(\frac{1}{\varepsilon} + \text{const.}'\right)\mathcal{O}_2$$

In four dimensions there is the identity†

$$\left(\bar{\Psi}_e\gamma^\lambda\gamma^\kappa\gamma^\rho P_{\mathrm{L}}\Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu}\gamma^\rho\gamma^\kappa\gamma^\lambda P_{\mathrm{L}}\Psi_\mu\right) = 4\left(\bar{\Psi}_e\gamma^\kappa P_{\mathrm{L}}\Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu}\gamma^\kappa P_{\mathrm{L}}\Psi_\mu\right) \tag{12.96}$$

and the two divergent contributions would cancel each other. The finite result would then be a finite correction to (12.47). However, the identity (12.96) is not valid in $4-\varepsilon$ dimensions and instead for the sum of the vertex and the self-energy corrections we get an additional piece

$$\left(\frac{1}{\varepsilon} + \text{const.}\right)(\mathcal{O}_1 - 4\mathcal{O}_2) \tag{12.97}$$

Our calculation requires, therefore, an additional counterterm in the form of the operator $(\mathcal{O}_1 - 4\mathcal{O}_2)$ and an additional renormalization condition. Thus, the original $\overline{\text{MS}}$ scheme must be supplemented by this renormalization condition, and the finite part in (12.97) depends on the choice of it. It is possible to chose the subtraction scheme so that one subtracts all the $1/\varepsilon$ poles and, in addition, these finite pieces (Dugan & Grinstein 1991). There is then no need to calculate matrix elements of the (so-called evanescent) operator $(\mathcal{O}_1 - 4\mathcal{O}_2)$. The finite result is the same as if (12.96) was used in n dimensions. Another way of reaching the same result is to perform the one-loop QED calculations starting from the outset with the tree level effective lagrangian written in the 'charge retention' form (12.98), i.e. after the Fierz rearrangement

$$\left(\bar{\Psi}_e\gamma^\lambda P_{\mathrm{L}}\Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu}\gamma_\lambda P_{\mathrm{L}}\Psi_\mu\right) = \left(\bar{\Psi}_e\gamma^\lambda P_{\mathrm{L}}\Psi_\mu\right)\left(\bar{\Psi}_{\nu_\mu}\gamma_\lambda P_{\mathrm{L}}\Psi_{\nu_e}\right) \tag{12.98}$$

which again follows easily from (A.41).

With the above background we can proceed with the subtraction procedure. The finite parts of the last four graphs shown in Fig. 12.3 are equal to their obvious analogue in the full amplitude and the latter should be simply omitted. The vertex graph in Fig. 12.3 is in the *SM* replaced by the box diagram shown in Fig. 12.11.

† In the Weyl spinor notation it has the form

$$\left(\bar{e}\bar{\sigma}^\lambda\sigma^\kappa\bar{\sigma}^\rho\nu_e\right)\left(\bar{\nu}_\mu\bar{\sigma}^\rho\sigma^\kappa\bar{\sigma}^\lambda\mu\right) = 4\left(\bar{e}\bar{\sigma}^\kappa\nu_e\right)\left(\bar{\nu}_\mu\bar{\sigma}^\kappa\mu\right)$$

and follows directly from the completeness relation for the σ matrices summarized in Appendix A (A.41). In the four-component notation it follows from the identity

$$\gamma_\mu\gamma_\rho\gamma_\nu = \gamma_\mu g_{\rho\nu} + \gamma_\nu g_{\mu\rho} - \gamma_\rho g_{\mu\nu} - \mathrm{i}\varepsilon_{\mu\rho\nu\sigma}\gamma^\sigma\gamma_5$$
$$\varepsilon_{\alpha\beta\gamma\mu}\varepsilon^{\alpha\beta\gamma}{}_\nu = -6g_{\mu\nu},$$

which imply

$$(\gamma_\mu\gamma_\rho\gamma_\nu P_{\mathrm{L}})_{ij}(\gamma^\nu\gamma^\rho\gamma^\mu P_{\mathrm{L}})_{kl} = 4(\gamma_\mu P_{\mathrm{L}})_{ij}(\gamma^\mu P_{\mathrm{L}})_{kl}$$

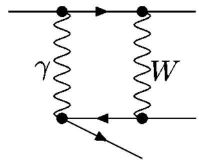


Fig. 12.11. W–γ box in the Standard Model.

Hence, the result for that diagram should be subtracted from the one for the box diagram.

We calculate the contribution to the effective lagrangian from the box diagram shown in Fig. 12.11 in the limit of zero external momenta. The corresponding amplitude is finite and is reproduced by the following effective lagrangian:

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(\gamma W) = \mathrm{i}\frac{e^4}{2s^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\left[\bar{\Psi}_{\nu_\mu}\gamma^\lambda P_{\text{L}}(\not{k} + m_\mu)\gamma^\rho \Psi_\mu\right]\left[\bar{\Psi}_e \gamma_\rho(\not{k} + m_e)\gamma_\lambda P_{\text{L}}\Psi_e\right]}{[k^2 - m_\mu^2][k^2 - m_e^2][k^2 - M_W^2]k^2}$$

In the limit of zero fermion masses and introducing the photon mass λ_γ to regularize the integral in the infrared one gets

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(\gamma W) = \frac{e^4}{2s^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}k^\mu k^\nu}{k^4[k^2 - M_W^2][k^2 - \lambda_\gamma^2]} \times \left(\bar{\Psi}_{\nu_\mu}\gamma^\lambda\gamma_\mu\gamma^\rho P_{\text{L}}\Psi_\mu\right)\left(\bar{\Psi}_e\gamma_\rho\gamma_\nu\gamma_\lambda P_{\text{L}}\Psi_e\right)$$

Replacing $k^\mu k^\nu \to (1/n)k^2 g^{\mu\nu} \to \frac{1}{4}k^2 g^{\mu\nu}$ under the integral and using the identity (12.96) we get (see (D.15) and (D.18))

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{CC}}(\gamma W) &= \frac{e^4}{2(4\pi)^2 s^2} F(0, M_W, \lambda_\gamma)\left(\bar{\Psi}_{\nu_\mu}\gamma^\lambda P_{\text{L}}\Psi_\mu\right)\left(\bar{\Psi}_e\gamma_\lambda P_{\text{L}}\Psi_e\right) \\ &= \frac{\alpha}{4\pi}\ln\frac{M_W^2}{\lambda_\gamma^2} \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \end{aligned} \tag{12.99}$$

where $\mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born})$ now denotes only that part of the tree level effective lagrangian (12.47) which is relevant for the muon decay (we have retained only terms singular in the limit $\lambda_\gamma \to 0$).

Computing in the same limit (again introducing the photon mass) the effective lagrangian generated by the first diagram shown in Fig. 12.3 one gets

$$-\frac{\alpha}{4\pi}\left(\frac{1}{2} + \ln\frac{\lambda_\gamma^2}{\mu^2}\right) \times \mathcal{L}_{\text{eff}}^{\text{CC}} \tag{12.100}$$

where μ is the renormalization scale. As explained earlier, this result is obtained either by starting with the charge retention form of the four-fermion effective

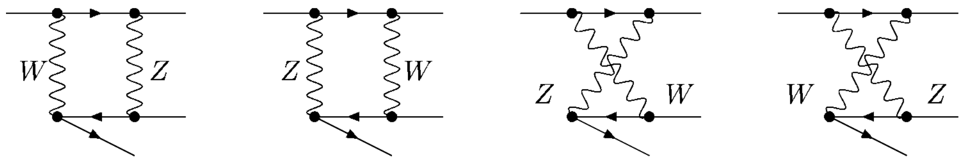


Fig. 12.12. Remaining box diagrams for the muon decay in the Standard Model.

lagrangian, or by imposing the particular renormalization condition for the new counterterm which is necessary in n dimensions if we work with the lagrangian (12.47). Subtracting (12.100) from (12.99) (and identifying to this order $\mathcal{L}_{\text{eff}}^{\text{CC}}$ with $\mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born})$ in (12.100)) we finally get

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{CC}}(\gamma W) &= \frac{\alpha}{4\pi}\left(\frac{1}{2} + \ln\frac{M_W^2}{\mu^2}\right) \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \\ &\equiv B_{\gamma W} \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \qquad (12.101)\end{aligned}$$

Note that the dependence on the IR cut-off has disappeared. The remaining box diagrams which have to be evaluated are shown in Fig. 12.12. The computation of the first two makes use of identity (12.96). The result is again the Born effective lagrangian multiplied by the coefficients $-(e^2/16\pi^2)(1/4s^4)\ln(M_W^2/M_Z^2)$ and $-(e^2/16\pi^2)[(1-2s^2)^2/4s^4]\ln(M_W^2/M_Z^2)$, respectively. The other two diagrams give amplitudes which are equal to each other and, after using the identity

$$\left(\bar{\Psi}_e\gamma^\lambda\gamma^\kappa\gamma^\rho P_{\text{L}}\Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu}\gamma^\lambda\gamma^\kappa\gamma^\rho P_{\text{L}}\Psi_\mu\right) = 16\left(\bar{\Psi}_e\gamma^\kappa P_{\text{L}}\Psi_{\nu_e}\right)\left(\bar{\Psi}_{\nu_\mu}\gamma^\kappa P_{\text{L}}\Psi_\mu\right) \qquad (12.102)$$

each give $-(e^2/16\pi^2)(2/s^4)(1-2s^2)\ln(M_W^2/M_Z^2)$ as the coefficient of the effective Born lagrangian. Together, these four box diagrams generate the effective lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{\text{CC}}(WZ\text{ boxes}) &= -\frac{\alpha}{4\pi}\left(1 - \frac{5}{s^2} + \frac{5}{2s^4}\right)\ln\frac{M_W^2}{M_Z^2} \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \\ &\equiv B_{WZ} \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \qquad (12.103)\end{aligned}$$

Vertex corrections to be computed are shown in Fig. 12.13. In the limit of vanishing external momenta and neglecting the fermion masses, when inserted in the full diagram for the muon decay, these corrections give

$$\mathcal{L}_{\text{eff}}^{\text{CC}}(\text{vertex}) = \Lambda \times \mathcal{L}_{\text{eff}}^{\text{CC}}(\text{Born}) \qquad (12.104)$$

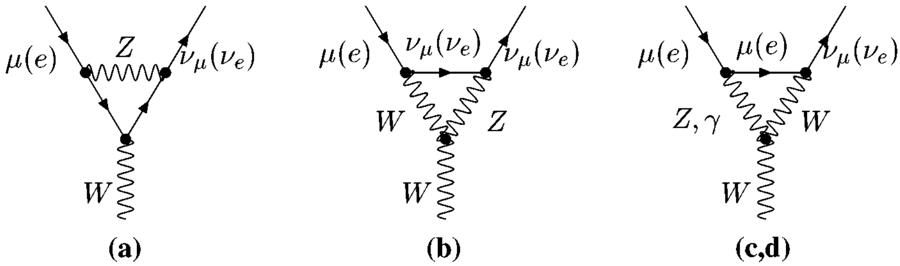


Fig. 12.13. Vertex corrections.

where $\Lambda \equiv \Lambda^{(a)} + \Lambda^{(b)} + \Lambda^{(c)} + \Lambda^{(d)}$ and

$$\left.\begin{aligned}
\Lambda^{(a)} &= \frac{\alpha}{4\pi}\frac{1-2s^2}{4s^2c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right)\\
\Lambda^{(b)} &= -\frac{\alpha}{4\pi}\frac{3}{2s^2}\left(-\frac{5}{6}+\ln\frac{M_Z^2}{\mu^2}-\frac{c^2}{s^2}\ln\frac{M_W^2}{M_Z^2}\right)\\
\Lambda^{(c)} &= -\frac{\alpha}{4\pi}\frac{3-6s^2}{2s^2}\left(-\frac{5}{6}+\ln\frac{M_Z^2}{\mu^2}-\frac{c^2}{s^2}\ln\frac{M_W^2}{M_Z^2}\right)\\
\Lambda^{(d)} &= -\frac{3\alpha}{4\pi}\left(-\frac{5}{6}+\ln\frac{M_Z^2}{\mu^2}+\ln\frac{M_W^2}{M_Z^2}\right)
\end{aligned}\right\} \tag{12.105}$$

Diagrams renormalizing external fermion lines are shown in Fig. 12.9. The $W^\pm$ and Z^0 gauge boson contributions to the muon and electron self-energies are given by (12.85). For the neutrino self-energy, using the general formula (12.83), one gets

$$\Sigma_{\nu_\mu(\nu_e)\mathrm{L}}(0) = \frac{\alpha}{4\pi}\left[\frac{1}{2s^2}\left(\frac{1}{2}+\ln\frac{M_W^2}{\mu^2}\right)+\frac{1}{4s^2c^2}\left(\frac{1}{2}+\ln\frac{M_Z^2}{\mu^2}\right)\right] \tag{12.106}$$

where we have again neglected the charged lepton masses. In the calculation of the box, vertex and external wave-function renormalization corrections we also neglect the contributions of the Higgs and Goldstone bosons whose couplings are proportional to the masses of fermions to which they couple. Due to the presence of the projections P_L in the $\mu W \nu_\mu$ and $eW\nu_e$ vertices only Σ_Ls contribute to the muon decay amplitude. We conclude that, up to terms $\mathcal{O}(m_f^2/M_W^2)$, the contribution of the external line renormalization to the effective lagrangian for muon decay reads

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{CC}}(\text{ext lines}) = \tfrac{1}{2}\left(\Sigma_{\mu\mathrm{L}}+\Sigma_{\nu_\mu\mathrm{L}}+\Sigma_{e\mathrm{L}}+\Sigma_{\nu_e\mathrm{L}}\right)\times\mathcal{L}_{\mathrm{eff}}^{\mathrm{CC}}(\text{Born}) \tag{12.107}$$

Collecting together all the corrections we get (note that corrections to the electron and muon 'sides' are equal in the limit $m_{\mu,e} \to 0$)

$$B_{\gamma W} + B_{WZ} + 2 \times \Lambda + 2 \times \tfrac{1}{2}\left(\Sigma_{\mu\mathrm{L}} + \Sigma_{\nu_\mu \mathrm{L}}\right) = \frac{\alpha}{4\pi s^2}\left(-4\ln\frac{M_Z^2}{\mu^2} + 6 + \frac{7-12s^2}{2s^2}\ln\frac{M_W^2}{M_Z^2}\right) \tag{12.108}$$

Finally, adding the correction (12.72) to the $W^{\pm}$ boson propagator computed earlier we get

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha(\mu)}{2s^2c^2M_Z^2(\mu)}\left[1 - \frac{\Pi_{WW}^{\mathrm{T}}(0)}{c^2M_Z^2(\mu)} + B_{\gamma W} + B_{W,Z} + 2\Lambda + \Sigma_{\mu\mathrm{L}} + \Sigma_{\nu_\mu\mathrm{L}}\right] \tag{12.109}$$

After (12.92), this is the second equation which relates the measured quantities to the running parameters. The third one is the one-loop correction to the physical mass of the Z^0 boson. From (12.79) and (12.80) we have

$$\hat{M}_Z^2 = M_Z^2(\mu) + \Pi_{ZZ}^{\mathrm{T}}(M_Z^2) \tag{12.110}$$

The three equations can be solved for the parameters $\alpha(\mu)$, $s^2(\mu)$ and $M_Z^2(\mu)$ in terms of α_{EM}, $\hat{M}_Z^2$ and G_μ. This task is simplified if we notice that, to the order we are working, we can replace the running parameters by the measured ones in all one-loop corrections. Eqs. (12.92) and (12.110) can immediately be inverted. Inserting the obtained $\alpha(\mu)$ and $M_Z^2(\mu)$ into (12.109) we get

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha_{\mathrm{EM}}}{2s^2c^2\hat{M}_Z^2}(1+\Delta) \tag{12.111}$$

where with one-loop precision

$$\Delta(\mu) = -\frac{\Pi_{WW}^{\mathrm{T}}(0)}{c_0^2\hat{M}_Z^2} + \frac{\Pi_{ZZ}^{\mathrm{T}}(\hat{M}_Z^2)}{\hat{M}_Z^2} - 2\frac{s_0}{c_0}\frac{\Pi_{\gamma Z}^{\mathrm{T}}(0)}{\hat{M}_Z^2} - \tilde{\Pi}_{\gamma\gamma}(0) + B_{\gamma W} + B_{WZ} + 2\Lambda + \Sigma_{\mu\mathrm{L}} + \Sigma_{\nu_\mu\mathrm{L}} \tag{12.112}$$

and c_0^2 and s_0^2 are given by (12.61). Eq. (12.111) can be used to express $s^2(\mu)$ in terms of G_μ. We get

$$c^2(\mu) = 1 - s^2(\mu) = \frac{1}{2}\left[1 + (1-4A)^{1/2} - \frac{2A\Delta}{(1-4A)^{1/2}}\right] = c_0^2 - \frac{A\Delta}{c_0^2 - s_0^2} \tag{12.113}$$

and, as in (12.51), $A \equiv \pi\alpha_{\mathrm{EM}}/\sqrt{2}\,G_\mu\hat{M}_Z^2 = s_0^2c_0^2$.

With these results we can predict any other electroweak observable. For instance, we can obtain the one-loop prediction for the $W^{\pm}$ mass. Using the equation

$$\hat{M}_W^2 = M_W^2(\mu) + \Pi_{WW}^{\mathrm{T}}(M_W^2) = c^2\hat{M}_Z^2 + \Pi_{WW}^{\mathrm{T}}(c_0^2\hat{M}_Z^2) - c_0^2\Pi_{ZZ}^{\mathrm{T}}(\hat{M}_Z^2) \quad (12.114)$$

and expressing $c^2(\mu)$ in terms of G_μ, (12.113), one gets the final one-loop result:

$$\begin{aligned}\hat{M}_W^2 &= c_0^2\hat{M}_Z^2\left[1 - \frac{\Pi_{ZZ}^{\mathrm{T}}(\hat{M}_Z^2)}{\hat{M}_Z^2} + \frac{\Pi_{WW}^{\mathrm{T}}(c_0^2\hat{M}_Z^2)}{c_0^2\hat{M}_Z^2} - \frac{s_0^2}{c_0^2 - s_0^2}\Delta\right] \\ &\equiv c_0^2\hat{M}_Z^2\left[1 - \frac{s_0^2}{c_0^2 - s_0^2}\Delta r\right] \qquad (12.115)\end{aligned}$$

This result is, of course, renormalization-scale-independent, as it should be when a physical observable is expressed in a given order of perturbation theory in terms of other physical observables. It should be remembered that the important step was the use of the strictly one-loop relations (12.92) and (12.69) expressing $\alpha_{\mathrm{SM}}(\mu)$ in terms of α_{EM}. Those relations contain logarithms, and there is no choice of the scale μ (which is arbitrary from the point of the final result (12.115)) which can make all of them simultaneously small. Thus, the result (12.115) suffers from considerable inaccuracy introduced when expressing $\alpha_{\mathrm{SM}}(\mu)$ in terms of α_{EM}. We shall return to this point in Section 12.6. We shall show there that it is possible to calculate $\alpha_{\mathrm{SM}}(\mu \approx M_Z)$ in terms of α_{EM} with all large logarithms summed up to all orders. Then using that $\alpha_{\mathrm{SM}}(M_Z)$ in (12.109) is an important improvement. Although it introduces only partial summation of higher order effects and, as such, μ dependence in (12.115), the numerical accuracy is improved since these are the largest corrections to M_W. In the last step of (12.115) we have defined the correction Δr, which is the common notation in the literature. From the previous equations we get

$$\begin{aligned}\Delta r = &-\frac{\Pi_{WW}^{\mathrm{T}}(0)}{c_0^2\hat{M}_Z^2} + \frac{\Pi_{WW}^{\mathrm{T}}(c_0^2\hat{M}_Z^2)}{c_0^2\hat{M}_Z^2} - \frac{c_0^2}{s_0^2}\left[\frac{\Pi_{WW}^{\mathrm{T}}(c_0^2\hat{M}_Z^2)}{c_0^2\hat{M}_Z^2} - \frac{\Pi_{ZZ}^{\mathrm{T}}(\hat{M}_Z^2)}{\hat{M}_Z^2}\right] \\ &- \tilde{\Pi}_{\gamma\gamma}(0) - 2\frac{s_0}{c_0}\frac{\Pi_{\gamma Z}^{\mathrm{T}}(0)}{\hat{M}_Z^2} + B_{\gamma W} + B_{WZ} + 2\Lambda + \Sigma_{\mu\mathrm{L}} + \Sigma_{\nu_\mu\mathrm{L}} \quad (12.116)\end{aligned}$$

Numerically, the most important contributions to Δr are contained in factors $\tilde{\Pi}_{\gamma\gamma}(0)$ and $[\Pi_{WW}^{\mathrm{T}}(M_W^2)/M_W^2] - [\Pi_{ZZ}^{\mathrm{T}}(M_Z^2)/M_Z^2]$. The second term in (12.116) can be approximated as

$$\Delta r = \cdots - \frac{c_0^2}{s_0^2}\Delta\rho + \cdots \qquad (12.117)$$

because

$$\left(\frac{\Pi^{\mathrm{T}}_{WW}(M_W^2)}{M_W^2} - \frac{\Pi^{\mathrm{T}}_{ZZ}(M_Z^2)}{M_Z^2}\right) \approx \left(\frac{\Pi^{\mathrm{T}}_{WW}(0)}{M_W^2} - \frac{\Pi^{\mathrm{T}}_{ZZ}(0)}{M_Z^2}\right) \approx \Delta\rho \quad (12.118)$$

With a bit more work, using the formulae for the b_0 function, it is possible to show that the leading m_t dependence of Δr is:

$$\Delta r \approx -\frac{c_0^2}{s_0^2}\frac{3\alpha_{\mathrm{EM}}}{16\pi s_0^2}\frac{m_t^2}{c_0^2\hat{M}_Z^2} + \frac{\alpha_{\mathrm{EM}}}{12\pi s_0^2}\ln\frac{m_t}{c_0\hat{M}_Z} \quad (12.119)$$

For the dominant contribution of the Higgs boson to Δr one finds

$$\Delta r \approx \frac{\alpha_{\mathrm{EM}}}{16\pi s_0^2}\frac{11}{3}\ln\frac{M_h^2}{c_0^2\hat{M}_Z^2} \quad (12.120)$$

The terms $\sim M_h^2/M_W^2$ cancel out. This is an illustration of the important screening of heavy Higgs boson effects (Veltman 1994).

In the limit of $q \sim 0$ and vanishing fermion masses, loop corrections to flavour conserving neutral-current processes can be summarized in terms of the parameters $\rho_f G_\mu$ and $\sin^2\theta^{\mathrm{eff}}_{f(\mathrm{low})}$ already introduced in the lagrangian (12.52). They become flavour-dependent, i.e. different for different flavour pieces of the lagrangian (12.52) and relations (12.54) and (12.55) are modified. The custodial $SU_{\mathrm{V}}(2)$ symmetry is broken by the Yukawa couplings and by the $U_{\mathrm{Y}}(1)$ current.

Corrections to the Z^0 partial decay widths

A very precise test of the electroweak theory comes from the measurements of the Z^0 boson properties at CERN and SLAC e^+e^- colliders.† It measures the partial decay widths of the Z^0 boson and various asymmetries of the decay products. All these observables are calculable from the effective lagrangian for the on-shell Z^0 coupled to a pair of fermions, introduced in (12.56). Indeed, higher order corrections can also be summarized in terms of that effective lagrangian. Two-body Z^0 decays can be described by a local effective lagrangian since they depend on kinematical invariants whose values are fixed in terms of the external particle masses. Higher order effects can be parametrized by the effective parameters $\rho_f G_\mu$ and $\sin^2\theta^{\mathrm{eff}}_f$, which are different for different fermions and different from the analogous parameters present in the low energy neutral-current lagrangian. Below we will compute radiative corrections to the effective $Z^0 f\bar{f}$ lagrangian.

† For several years CERN LEP and linear SLAC electron–positron colliders operated at the centre of mass energy $\sqrt{s} = M_Z$ producing millions of Z^0s at rest and allowing for detailed study of its couplings to light fermions.

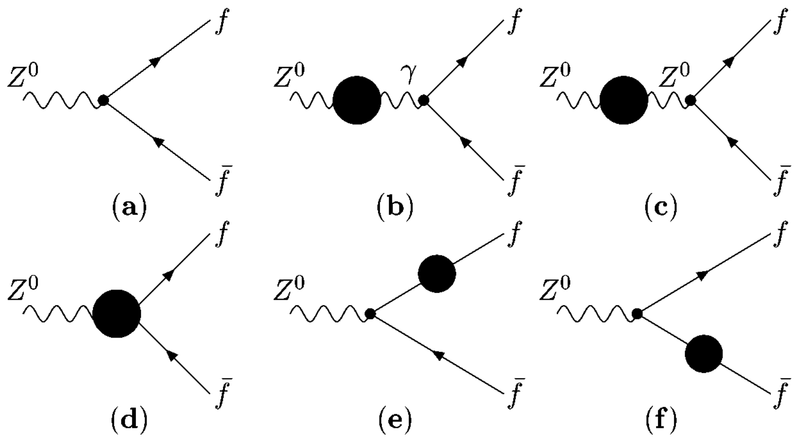


Fig. 12.14. Diagrams for Z^0 boson decay.

The relevant diagrams are shown in Fig. 12.14. Denoting the momentum of the Z^0 boson by q and the momenta of the outgoing fermion and antifermion by p_1 and p_2, respectively, we have for these diagrams the following amplitudes (the factor $T_f^3 - Q_f s^2$ in the couplings can be written as $\pm\frac{1}{2}(1-2|Q_f|s^2)$ with the sign $+$ $(-)$ for up (down) type fermions)

$$
\begin{aligned}
(a) &= \mp\frac{ie}{2sc}\bar{u}(p_1)\gamma^\lambda\left[(1-2|Q_f|s^2)P_\mathrm{L} + (-2|Q_f|s^2)P_\mathrm{R}\right]v(p_2)\varepsilon_\lambda \\
(b) &= \mp\frac{ie}{2sc}\bar{u}(p_1)\gamma^\lambda\left[2sc|Q_f|\frac{\Pi^\mathrm{T}_{\gamma Z}(M_Z^2)}{M_Z^2}\right]v(p_2)\varepsilon_\lambda \\
(c) &= \mp\frac{ie}{2sc}\bar{u}(p_1)\gamma^\lambda\left[(1-2|Q_f|s^2)\frac{1}{2}\Pi^{\mathrm{T}\prime}_{ZZ}(M_Z^2)P_\mathrm{L}\right. \\
&\qquad\left. + (-2|Q_f|s^2)\frac{1}{2}\Pi^{\mathrm{T}\prime}_{ZZ}(M_Z^2)P_\mathrm{R}\right]v(p_2)\varepsilon_\lambda \\
(d) &= \mp\frac{ie}{2sc}\bar{u}(p_1)\gamma^\lambda\left[F_\mathrm{L}\,P_\mathrm{L} + F_\mathrm{R}\,P_\mathrm{R}\right]v(p_2)\varepsilon_\lambda \\
(e)+(f) &= \mp\frac{ie}{2sc}\bar{u}(p_1)\gamma^\lambda\left[(1-2|Q_f|s^2)\delta\mathcal{Z}_\mathrm{L}P_\mathrm{L} + (-2|Q_f|s^2)\delta\mathcal{Z}_\mathrm{R}P_\mathrm{R}\right]v(p_2)\varepsilon_\lambda
\end{aligned}
$$

The factors F_L and F_R denote one-loop vertex amplitudes (note that $e/2sc$ has been factored out), $\varepsilon_\lambda(q)$ is the polarization vector of the Z^0 boson, $\Pi^{\mathrm{T}\prime}_{ZZ}(M_Z^2)$ is the derivative at $q^2 = M_Z^2$ of its transverse self-energy, and

$$
\delta\mathcal{Z}_\mathrm{L,R} = \Sigma_\mathrm{L,R}(m_f^2) + 2m_f^2\Sigma'_V(m_f^2) + 2m_f\Sigma'_m(m_f^2) \tag{12.121}
$$

are the external fermion wave-function renormalization factors (see (12.81) and compare with (5.42); each external wave-function renormalization factor is $\frac{1}{2}\delta\mathcal{Z}_{\mathrm{L,R}}$). Collecting all the factors together, expressing the $\overline{\mathrm{MS}}$ parameters s, c and e through the measurable quantities (as we did in the case of Δr) and replacing the amplitude by the corresponding contribution to the effective action, the final one-loop result can be written in the form (12.56) with

$$\rho_f = 1 - \Delta + \Pi^{\mathrm{T}\prime}_{ZZ}(\hat{M}_Z^2) + 2\frac{c_0}{s_0}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{\hat{M}_Z^2} + 2(\hat{F}_{\mathrm{L}} - \hat{F}_{\mathrm{R}}) \tag{12.122}$$

Here

$$\left.\begin{aligned} \hat{F}_{\mathrm{L}} &\equiv F_{\mathrm{L}} + (1 - 2|Q_f|s_0^2)\delta\mathcal{Z}_{\mathrm{L}} - 2\frac{c_0}{s_0}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{\hat{M}_Z^2} \\ \hat{F}_{\mathrm{R}} &\equiv F_{\mathrm{R}} - 2|Q_f|s_0^2\delta\mathcal{Z}_{\mathrm{R}} \end{aligned}\right\} \tag{12.123}$$

are the 'renormalized' (scale-independent) vertex corrections,† and Δ is given by (12.112). Its presence originates from expressing with one-loop accuracy the factor $e(\mu)/s(\mu)c(\mu)$ through measurable quantities. For the effective mixing angle we find

$$\sin^2\theta_f^{\mathrm{eff}} = s^2\left[1 - \frac{c_0}{s_0}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{c_0}{s_0}\frac{\Pi^{\mathrm{T}}_{\gamma Z}(0)}{M_Z^2} - \hat{F}_{\mathrm{L}} + \left(1 - \frac{1}{2|Q_f|s_0^2}\right)\hat{F}_{\mathrm{R}}\right] \tag{12.124}$$

where

$$s^2 = s_0^2\left(1 + \frac{c_0^2}{c_0^2 - s_0^2}\Delta\right) \tag{12.125}$$

The final result for the partial decay width is still subject to QED and (for $f = q$) QCD radiative corrections. With $\mathcal{O}(\alpha_{\mathrm{EM}}\alpha_s)$ accuracy (including virtual and real emissions) we get

$$\Gamma(Z^0 \to f\bar{f}) = \Gamma_0(Z^0 \to f\bar{f})\left(1 + \frac{3\alpha_{\mathrm{EM}}}{4\pi}Q_f^2\right)\left(1 - \frac{\alpha_s}{\pi}\right), \tag{12.126}$$

with Γ_0 given by (12.59). For all fermions f except for the b quark‡ the vertex corrections are small. The corrections to ρ_f and $\sin^2\theta_f^{\mathrm{eff}}$ are then dominated by the factor $\Delta\rho$ present in Δ – see (12.74) and (12.118) – which contains a term proportional to m_t^2/M_W^2 (12.77), and which originates solely from expressing the parameters s and c through s_0 and c_0. Since the effective coefficient of $\Delta\rho$ is positive, $\Gamma(Z^0 \to f\bar{f})$ increases with increasing top-quark mass.

† Indeed, in the popular on-shell renormalization scheme, where the counterterms also contain finite parts (fixed by some physical conditions), one finds precisely the combinations (12.123) for the renormalized vertex corrections.

‡ For kinematical reasons Z^0 cannot decay into a $t\bar{t}$ quark pair.

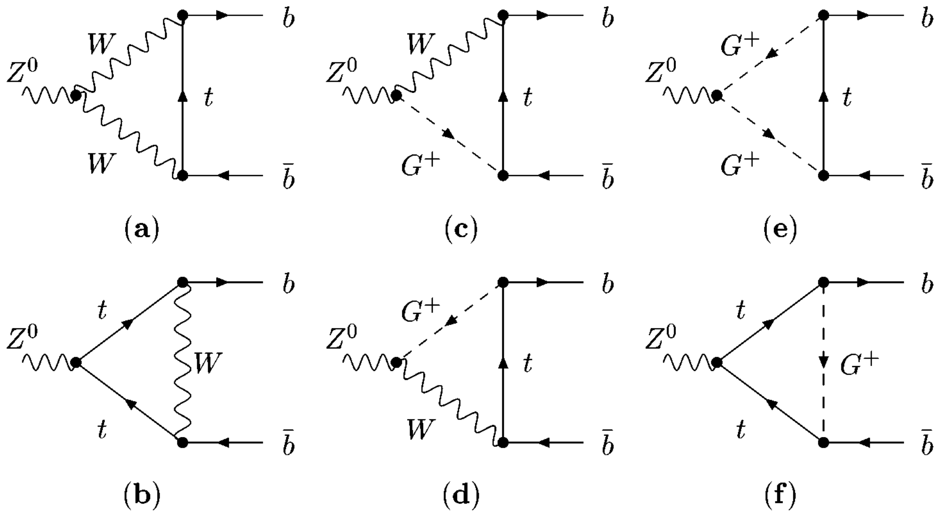


Fig. 12.15. m_t-dependent vertex corrections to the $Z^0 b\bar{b}$ decay.

$$+\mathrm{i}\frac{e}{\sqrt{2}s}\frac{m_t}{M_W}\left(P_\mathrm{R}-\frac{m_b}{m_t}P_\mathrm{L}\right) \qquad +\mathrm{i}\frac{e}{\sqrt{2}s}\frac{m_t}{M_W}\left(P_\mathrm{L}-\frac{m_b}{m_t}P_\mathrm{R}\right)$$

Fig. 12.16. Charged Goldstone boson couplings to the (t, b) quark pair.

For the $b\bar{b}$ pair in the final state there is an additional contribution of order m_t^2/M_W^2 to the parameters ρ_b and $\sin^2\theta_b^{\mathrm{eff}}$. It comes from $\hat{F}_\mathrm{L}$ (the vertex corrections) and from the renormalization of the external b-quark line (diagrams (d)–(f) in Fig. 12.14). This is an interesting example of when the Appelquist–Carrazzone decoupling theorem does not work. The reason is that the top-quark mass is generated by spontaneous breaking of the gauge symmetry and, therefore, larger mass requires larger Yukawa coupling. We will compute the leading term. The relevant vertex corrections are shown in Fig. 12.15. Inspection of the Feynman rules in Fig. 12.16 and Appendix C reveals that only diagrams (e) and (f) contain the factor m_t^2/M_W^2, coming from the Yukawa coupling G^+tb of the Goldstone field G^+. Diagrams (c) and (d) are finite. The loop integral goes as $1/m_t$, compensating factors m_t/M_W coming from the G^+tb vertices. Finally, diagrams (a) and (b) do not have any m_t/M_W factors in their vertices, and the logarithmically divergent loop integrals can at most give $\ln m_t$.

In the limit $M_{W,Z} \ll m_t$, i.e. for vanishing external momenta, and neglecting the b quark mass Fig. 12.15(e) gives†

$$\mathrm{i}\frac{e}{2sc}F_{\mathrm{L}}^{(e)} = \frac{e^3}{2s^2}\frac{1-2s^2}{2sc}\frac{m_t^2}{M_W^2}\int\frac{\mathrm{d}^n k}{(2\pi)^n}P_{\mathrm{R}}\frac{1}{\not{k}-m_t}P_{\mathrm{L}}\frac{2k^\lambda}{(k^2-M_W^2)^2}$$
$$= -\mathrm{i}\frac{e^3}{(4\pi)^2}\frac{1-2s^2}{8s^3c}\frac{m_t^2}{M_W^2}\left[-\eta-\frac{3}{2}+\ln\frac{m_t^2}{\mu^2}+\mathcal{O}\left(\frac{1}{m_t}\right)\right]\gamma^\lambda P_{\mathrm{L}}$$

Similarly, for the Fig. 12.15(f) we find

$$\mathrm{i}\frac{e}{2sc}F_{\mathrm{L}}^{(f)} = \frac{e}{2sc}\frac{e^2}{2s^2}\frac{m_t^2}{M_W^2}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{1}{k^2-M_W^2}P_{\mathrm{R}}\frac{1}{\not{k}-m_t}\gamma^\lambda$$
$$\times\left[\left(1-\frac{4}{3}s^2\right)P_{\mathrm{L}}-\frac{4}{3}s^2P_{\mathrm{R}}\right]\frac{1}{\not{k}-m_t}P_{\mathrm{L}}$$
$$= \frac{e^3}{4s^3c}\frac{m_t^2}{M_W^2}\left[-\frac{4}{3}s^2\frac{2-n}{n}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{k^2}{(k^2-M_W^2)(k^2-m_t^2)^2}\right.$$
$$\left.+\left(1-\frac{4}{3}s^2\right)\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{m_t^2}{(k^2-M_W^2)(k^2-m_t^2)^2}\right]\gamma^\lambda P_{\mathrm{L}}$$
$$= \frac{e^3}{4s^3c}\frac{m_t^2}{M_W^2}\left[-\frac{4}{3}s^2\frac{2-n}{n}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{1}{(k^2-M_W^2)(k^2-m_t^2)}\right.$$
$$\left.+\left(1-\frac{2}{3}s^2\right)\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{m_t^2}{(k^2-M_W^2)(k^2-m_t^2)^2}\right]\gamma^\lambda P_{\mathrm{L}}$$
$$= -\mathrm{i}\frac{e}{(4\pi)^2}\frac{e^2}{8s^3c}\frac{m_t^2}{M_W^2}\left[\frac{4}{3}s^2\left(-\eta-\frac{1}{2}+\ln\frac{m_t^2}{\mu^2}\right)+2\left(1-\frac{2}{3}s^2\right)\right]\gamma^\lambda P_{\mathrm{L}}$$

Finally, the full contribution of order m_t^2/M_W^2 to the renormalization factor $\delta\mathcal{Z}_{\mathrm{L}} \approx \Sigma_{\mathrm{L}}$ can be obtained by calculating the contribution of $G^\pm$ to the self-energy of the b quark. One obtains

$$\Sigma_{\mathrm{L}}^{G^\pm}(0) = \frac{1}{(4\pi)^2}\frac{e^2}{4s^2}\frac{m_t^2}{M_W^2}\left[\left(-\eta-\frac{3}{2}+\ln\frac{m_t^2}{Q^2}\right)+\mathcal{O}\left(\frac{1}{m_t}\right)\right] \tag{12.127}$$

Collecting all contributions together (and replacing s, c by s_0, c_0) we get

$$2\hat{F}_{\mathrm{L}} = 2\left[F_{\mathrm{L}}+\left(1-\frac{2}{3}s_0^2\right)\Sigma_{\mathrm{L}}^{G^\pm}+\cdots\right] = -\frac{\alpha}{4\pi s_0^2c_0^2}\frac{m_t^2}{\hat{M}_Z^2} \tag{12.128}$$

There is no similar contribution of order m_t^2/M_W^2 to $\hat{F}_{\mathrm{R}}$. Comparing with $\Delta\rho$, (12.77), we find that for b quarks the new negative contribution (12.128) of order

† We also approximate the CKM matrix element V_{tb} by 1.

m_t^2/M_W^2 to the parameter ρ_b overcompensates the positive universal contribution of the same order. As a result, $\Gamma(Z^0 \to \bar{b}b)$ decreases as m_t grows. The same correction (12.128) is also responsible for the slightly slower, compared to effective Weinberg angles for other fermions, decrease of $\sin^2\theta_b^{\text{eff}}$ with m_t.

12.6 Effective low energy theory for electroweak processes

In Sections 12.4 and 12.5 we have presented a sample of systematic tree level and one-loop calculations in the standard electroweak theory. We have used the language of effective lagrangians, but that was merely as a local approximation to the effective action for the considered range of phase space. It is also very interesting to look at the previous results from another perspective, namely, to introduce the effective low energy theory in its full sense defined in Section 7.7. In a renormalizable theory with several mass scales, the subsequent decoupling of heavy particles gives field theories which are valid in a smaller and smaller energy range. In general, they are described by lagrangians which consist of dimension four or less (renormalizable part) and higher dimension operators. An effective theory for light particles can be obtained by calculating the Green's functions with only light field external legs at small momenta (compared to heavy masses) and with all heavy and mixed light–heavy particle loops. The UV divergences arising in these graphs must be renormalized by appropriate counterterms (for example, in the $\overline{\text{MS}}$ scheme). In the final step one redefines the light fields and parameters so that the whole dependence on the heavy fields is absorbed into those new, effective, parameters. They can be considered as given in terms of the parameters of the original theory (matching conditions) or as free, renormalizable parameters of the local effective theory. The main virtue of such a separation of mass scales is the possibility of easy resummation to all orders of large logarithms, $\ln(E/M)$, which one encounters when calculating low energy, $E \ll M$, processes in the full theory containing the large mass scale M.

It is interesting to discuss this technique in the electroweak theory, with the muon decay amplitude as an example, and to compare it with the systematic perturbative calculation performed in the previous section. Of course, summing up the terms $\mathcal{O}(\alpha\ln(M_W/m_f))$ is less important than the terms $\mathcal{O}(\alpha_s\ln(M_W/m_f))$ (QCD corrections) discussed in the next section, but it is interesting as an illustration of the method.

Suppose we calculate at one-loop level the muon decay amplitude in the complete theory. This result has not been explicitly written down, but by inspection of (12.109) and (12.50) one can convince oneself that it does not contain large logarithms, $\ln(M_W^2/m_\mu^2)$, apart from those present in the running coupling $\alpha(\mu)$.

This is pathological from the point of view of perturbative calculations in the full theory, which generically give such logarithms of two different mass scales, and these are the logarithms that one would like to resum to all orders. Putting aside (for a moment) the pathology of our example, let us proceed with the construction of the effective theory, describing muon decay, by decoupling heavy particles: vector and Goldstone bosons, Higgs bosons and the top quark. Clearly, at the tree level, such a procedure gives the effective theory described by the lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{QED}} - 2\sqrt{2}\, G_{\mathrm{F}} J^{+\mu} J^{-}_{\mu} \tag{12.129}$$

where G_{F} is given by (12.49), and has now the interpretation of the Wilson coefficient.

The local, dimension six, operators present in lagrangian (12.129) are the lowest dimension operators describing weak transitions, obtained for four-fermion processes in the limit $q \to 0$, i.e. after decoupling heavy particles. The Wilson coefficients of the effective four-fermion operators can be calculated in each order of perturbation by comparing the leading term in the expansion in powers of $(1/M_W^2)^n$ of the amplitudes calculated in the complete electroweak theory with the amplitude generated by (12.129), with QED corrections included.† Thus, for instance, to be able to identify the Wilson coefficient at one-loop accuracy, we have to require equality of the full SM one-loop amplitude and the amplitude generated by lagrangian (12.129), with one-loop QED corrections included. In fact, we know already the one-loop corrections to the Wilson coefficient G_{F}: they are given by (12.109)! And here comes an important observation: the correction to the Wilson coefficient depends on the renormalization scale in the effective theory (remember that in the calculation of the previous section the scale μ is the renormalization scale for QED corrections to the four-fermion operator) and involves a logarithm which can be large. It is minimized for $\mu \sim M_W$.

In consequence, from the point of view of minimizing higher order corrections to the Wilson coefficient $G_{\mathrm{F}}(\mu)$, the tree level effective lagrangian must be interpreted as the lagrangian at the decoupling scale $\mu \sim M_W$, and (12.48) is the equation for the Wilson coefficient $G_{\mathrm{F}}(\mu)$ at $\mu \sim M_W$. The basic idea is then to apply the RGE in the effective theory to evolve $G_{\mathrm{F}}(\mu)$ down to the scale $\mu \sim m_\mu$, and to calculate the muon-decay matrix element at that scale. In the present, pathological, example $G_{\mathrm{F}}(\mu)$ is renormalization-scale-independent at the one-loop level (a fact which is directly connected to the absence of logarithms of two separate scales in complete one-loop calculation), since the counterterm diagrams in Fig 12.3 cancel out. With

† A systematic expansion in inverse powers of M_W^2 would also give higher dimension operators, for example, $m_f^2 J_\mu^- J^{\mu+}$, as well as terms with different chirality structure ($V+A$ currents and scalar currents). Compared to (12.129), they are suppressed by $\mathcal{O}(m_f^2/M_W^2)$, and for the light fermions are negligible compared to the corrections $\mathcal{O}(\alpha G_\mu)$ to the Wilson coefficients G_f.

$G_F(M_W) = G_F(m_\mu)$, we can finally calculate the muon-decay matrix element at the tree level (to match the overall accuracy). Again, at the tree level that matrix element does not depend on the renormalization scale but it should be understood as taken at the scale $\mu \sim m_\mu$, to minimize higher order corrections to it. (Here we encounter the same pathology: one-loop corrections to the matrix element, of course, also do not contain any logarithms of the renormalization scale.) The main difference of this approach compared to the tree level calculation in the full theory is that α in (12.48), and in consequence in (12.51), is now $\alpha(M_Z)$. This is an important improvement for calculating at the tree level electroweak observables at the $W^\pm$ mass scale, for example, the $W^\pm$ mass, in terms of the low energy observable G_μ.

In the next order of perturbation theory, we include one-loop corrections to G_F, (12.109). Choosing $\mu \sim M_Z$, we expect to minimize two-loop corrections. Next, for consistency, we should evolve $G_F(M_W)$ to $G_F(m_\mu)$ with two-loop RGE in the effective theory. In this way we would resum all terms $\alpha(\alpha \ln(M_W/m_f))^n)$ of the full theory, if they had been present. In our pathological example G_F is renormalization-scale-independent at any order. One can see it by the following argument. Suppose we calculate higher order QED corrections to the four-fermion operator written in the charge retention form $\left(\bar{\Psi}_e \gamma^\lambda P_L \Psi_\mu\right)\left(\bar{\Psi}_{\nu_\mu} \gamma^\lambda P_L \Psi_{\nu_e}\right)$, obtained by the Fierz rearrangement, (12.98). This is a product of the neutrino current (which does not interact with photons) and the current which, in the limit of vanishing fermion masses, is conserved (and, therefore, does not undergo renormalization). Thus, all infinities generated by this four-fermion vertex can be renormalized by the field wave-function renormalization constants and the Wilson coefficient is renormalization-scale-independent (see Section 7.7). Finally we calculate the muon decay matrix element by calculating finite parts of the one-loop QED corrections in the effective theory. Although those corrections do not contain logarithms of the renormalization scale, the latter should, in principle, be taken as $\mu \sim m_\mu$ to minimize two-loop corrections to the matrix element (remember, however, that our example is pathological). Eventually we can express $G_F(M_W)$ in terms of G_μ with the next-to-leading logarithm accuracy.

The perturbation theory based on lagrangian (12.129) would include higher order contributions generated by perturbation in the four-fermion operator itself. Such contributions are effective dimension > 6 operators, with new dimensionful coupling constants $\leq \mathcal{O}(G_f^2)$, and of course each new coupling requires new renormalization constant (the theory is not renormalizable with a finite number of counterterms). The power of the effective theory approach resides in the truncation of this series, and in the present case we use $\mathcal{L}_{\text{eff}}^{\text{CC}}$ only in the Born approximation, consistent with the fact that operators with dimension > 6 have already been neglected in the construction of the effective theory.

QED as the effective low energy theory

It is now worthwhile to discuss in more detail the connection between the complete electroweak theory and the renormalizable part of the effective theory defined by lagrangian (12.129), namely the QED of Chapter 5. Actually, even at the level of pure QED we can discuss several steps of 'effectiveness': in the top-down approach, starting with the decoupling of the weak sector and the even heavier top quark we obtain QED describing the interaction of photons, three lepton flavours (e, μ, τ) and five quark flavours (u, d, c, s, b). This effective QED is valid for the energy range $0 < E < \hat{M}_Z$. However, it still contains hugely different mass scales (fermion masses). Thus, the main virtue of effective theories (which is easy resummation of large logarithms) is achieved if this theory is used only in the energy range $m_b < E < \hat{M}_Z$. In the next step, we can integrate out subsequent quark and lepton flavours (in the order of their heaviness), obtaining each time a new renormalizable effective theory valid in a smaller and smaller energy range, and finally arriving at the QED of photons and electrons valid in the range $0 < E < m_\mu$. Each effective QED is described by its proper scale-dependent lagrangian parameters and, in particular, by its own coupling constant $\alpha_{\text{EM}}^{(n_e,n_d,n_u)}(\mu)$, where n_e, n_d, n_u are, respectively, the numbers of the charged leptons, down- and up-type quarks, present in the effective theory.

In the energy range of interest, the corresponding 'minimal' effective renormalizable theory is not only sufficient but also best suited to describe physics in that range (up to corrections $\mathcal{O}(q^2/m^2)$, where m^2 is the next mass threshold). Eventually, our goal is to use each effective QED only in the energy range $m_n < E < m_{n+1}$. Since the theory is renormalizable, its structure does not contain any information about the energy range of its applicability. This is the usual bottom-up attitude in which we study physics in a certain energy range without even knowing the more complete theory. In the present case we are fortunate to know both the complete theory and the effective theory, and we are able to prove by the decoupling procedure that the usual bottom-up attitude is correct (Appelquist & Carrazzone 1975).

The first point to stress is that the renormalized (in the $\overline{MS}$ scheme) coupling constant for each effective QED can be fixed in terms of α_{EM}. In the one-loop approximation we get:

$$\alpha_{\text{EM}} = \alpha_{\text{EM}}^{(n_e,n_d,n_u)}(\mu)\left(1 + \frac{\alpha}{3\pi}\sum_{n_e,n_d,n_u} N_{c_f} Q_f^2 \ln\frac{m_f^2}{\mu^2}\right) + \mathcal{O}\left(\frac{m_n^2}{m_{(n+1)}^2}\right) \quad (12.130)$$

These equations are obtained by calculating in each effective QED the $k \to 0$ limit of the Thomson scattering (i.e. by repeating in these simpler cases the procedure used for calculating α_{EM} in the complete electroweak theory). The result is given by the corrections to the photon propagator. Of course, for fixed n_e, n_d, n_u the renormalized coupling is defined only in the range $0 < \mu < m_h$, where m_h is the mass of the lightest decoupled fermion, and the μ scale in (12.130) can take any value in this range. We also remark that (12.130) is valid up to non-perturbative corrections to the photon propagator discussed in Section 12.5. Eq. (12.130) reflects quite clearly the problem of large logarithm resummation. When we calculate the low energy observable, α_{EM}, in an effective QED with several flavours, (12.130) contains large logarithms for any value of μ, and in this theory improved accuracy can be achieved only by calculating higher order corrections to this equation.

However, we can proceed in a different way. It is clear from (12.130) that

$$\alpha_{\text{EM}} = \alpha_{\text{EM}}^{(1,0,0)}(\mu \approx m_e) + \mathcal{O}(\alpha^2), \tag{12.131}$$

the result already known from Chapters 5 and 6. The accuracy of the determination of $\alpha_{\text{EM}}^{(1,0,0)}(\mu^2 \approx m_e^2)$ in terms of α_{EM} is $\mathcal{O}(\alpha^2)$ since no large logarithm appears in this case in the one-loop (12.130). (In fact, they are also absent in higher order corrections to this relation used in the effective QED of only photons and electrons.)

Once $\alpha_{\text{EM}}^{(1,0,0)}(\mu)$ is fixed by (12.131) at some $\mu_0 \approx m_e$, its μ dependence in the $\overline{\text{MS}}$ is determined by the RGE. Summing up leading logarithms, i.e. in the one-loop approximation for the β-functions, we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \alpha_{\text{EM}}^{(1,0,0)}(\mu) = \frac{1}{2\pi} b_1(1,0,0) (\alpha_{\text{EM}}^{(1,0,0)})^2 \tag{12.132}$$

where (in general)

$$b_1(n_e, n_d, n_u) = \tfrac{4}{3}(n_e + \tfrac{1}{3} n_d + \tfrac{4}{3} n_u) \tag{12.133}$$

is the first coefficient of the β-function $\beta(n_e, n_d, n_u)$ with n_e, n_d, n_u 'active' fermions. For any μ in the range of the validity of the effective theory, $0 < \mu < m_\mu$, the solution reads:

$$\frac{1}{\alpha_{\text{EM}}^{(1,0,0)}(\mu)} = \frac{1}{\alpha_{\text{EM}}^{(1,0,0)}(\mu_0)} - \frac{1}{4\pi} b_1(1,0,0) \ln \frac{\mu^2}{\mu_0^2} \tag{12.134}$$

Thus, we obtain $\alpha_{\text{EM}}^{(1,0,0)}$ at the scale $\mu = m_\mu$, with all terms $[\alpha \ln(m_\mu/m_e)]^n$ summed up. We can then switch to the next effective QED parametrized by the running $\alpha_{\text{EM}}^{(2,0,0)}(\mu)$ (with $\beta(2,0,0)$), up to the next threshold etc., until we reach $\alpha_{\text{EM}}^{(3,3,2)}(\mu)$.

In order to accomplish this programme, we have to know the relations between the couplings of two different effective QEDs in the range of μ where both are valid, for example, between $\alpha_{\text{EM}}^{(1,0,0)}(\mu)$ and $\alpha_{\text{EM}}^{(2,0,0)}(\mu)$ for $\mu \approx m_\mu^2$ (for the sake of our examples we assume that all quarks are heavier than leptons). One way to get these relations is to calculate the same physical quantity in both theories. Taking it as α_{EM}, with one-loop accuracy we get from (12.130):

$$\alpha_{\mathrm{EM}}^{(1,0,0)}(\mu) = \alpha_{\mathrm{EM}}^{(2,0,0)}(\mu)\left(1 + \frac{\alpha}{3\pi}\ln\frac{m_\mu^2}{\mu^2}\right) + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right), \tag{12.135}$$

for $\mu \approx m_\mu$. These are the so-called matching conditions. We see that in this simple example, for $\mu \approx m_\mu$, the two couplings are equal up to corrections $\mathcal{O}(\alpha^2)$. Such corrections can be neglected as long as we work with one-loop RGE.† (Another way to get (12.135) is to begin with two-lepton theory and to follow the decoupling procedure. Then, $\alpha_{\mathrm{EM}}^{1,0,0}(\mu)$ is the effective coupling constant of the 'effective' one-lepton theory obtained from the full two-lepton theory, and (12.135) illustrates the general feature of decoupling in the $\overline{\mathrm{MS}}$ scheme: the dependence on the heavy states is absorbed into effective parameters of the lower energy theory, which are free, renormalizable parameters from the point of view of the latter theory.) It is very easy to see that our procedure finally gives

$$\frac{1}{\alpha_{\mathrm{EM}}^{(3,3,2)}(\mu)} = \frac{1}{\alpha_{\mathrm{EM}}^{(1,0,0)}(m_e^2)} + \frac{1}{3\pi}\sum_{f\neq t} N_{c_f} Q_f^2 \ln\frac{m_f^2}{\mu^2} \tag{12.136}$$

i.e. the leading logarithms are summed up to all orders (we remember that $\alpha_{\mathrm{EM}}^{(1,0,0)}(m_e^2) = \alpha_{\mathrm{EM}}$).

The procedure which we have outlined here is important in the following context. Suppose that, as in Section 12.5, we want to fix $\alpha_{\mathrm{SM}}(\mu)$ in terms of α_{EM}. The one-loop approximation, (12.92), contains large logarithms (there is no scale μ for which all of them can be small). It is, therefore, very desirable to go beyond the accuracy of (12.92) and to express $\alpha_{\mathrm{SM}}(\mu)$ in terms of α_{EM} with all the large logarithms summed up to all orders. This is indeed achieved by using our effective theory approach.

The final step is to use the matching condition

$$\begin{aligned}\alpha_{\mathrm{SM}}(\mu) &= \alpha_{\mathrm{EM}}^{(3,3,2)}(\mu)\left[1 - \frac{\alpha}{4\pi}\left(\frac{2}{3} - 7\ln\frac{M_W^2}{\mu^2} + \frac{16}{9}\ln\frac{m_t^2}{\mu^2}\right)\right] \\ &= \alpha_{\mathrm{EM}}^{(3,3,2)}(\mu) + \mathcal{O}(\alpha^2),\end{aligned} \tag{12.137}$$

for $\mu \approx M_Z$. Since at $\mu \approx M_Z$ the logarithms in (12.137) are small we can rewrite

† Of course, for $\mu = m_\mu$ we get $\alpha_{\mathrm{EM}}^{(1,0,0)} = \alpha_{\mathrm{EM}}^{(2,0,0)}$, but this is not a necessary choice of μ. Eq. (12.135) can be used for any μ such that $(\alpha/3\pi)\ln(m_\mu^2/\mu^2) \ll 1$.

our result in the compact form:

$$\frac{1}{\alpha_{\text{SM}}(M_Z)} = \frac{1}{\alpha_{\text{EM}}} + \frac{1}{(4\pi)}\left(\frac{2}{3} - 7\ln\frac{M_W^2}{M_Z^2} + \frac{4}{3}\sum_f N_{c_f} Q_f^2 \ln\frac{m_f^2}{M_Z^2}\right) \tag{12.138}$$

where the sum extends to all fermions including the top quark. This is the desired improvement over the one-loop formula (12.92).

In some very interesting physical questions, for example, in theories of grand unification, $\alpha_{\text{SM}}(M_Z)$ is used as input parameter in the renormalization group evolution up to scales $\mu \approx 10^{16}$ GeV; see for example (Chankowski, Płuciennik & Pokorski 1995). Then, $\ln(\mu/M_Z) \gg \ln(m_f/M_Z)$ and the accuracy of (12.138) requires for consistency the two-loop renormalization group evolution in that large μ range.

Our final comment is that (12.137) illustrates two generic points. One is that often several mass scales must be decoupled at the same time. If both logarithms are small enough the whole procedure is self-consistent. Another general fact is the presence of some constant factors in the one-loop matching conditions. In such cases the two-loop evolution of coupling constants is discontinuous. If one wishes, one can avoid the constant $(\frac{2}{3})$ in (12.137) (the so-called finite threshold correction) by using Dimensional Reduction ($\overline{\text{DR}}$) scheme instead of the $\overline{\text{MS}}$.

12.7 Flavour changing neutral-current processes

We have learned already in Section 12.3 that the structure of the electroweak theory is such that it ensures the absence of the tree level flavour changing neutral currents. This follows from the $SU_{\text{L}}(2) \times U_{\text{Y}}(1)$ gauge invariance, renormalizability and the particle content of the theory. Both neutral gauge boson and Higgs boson couplings are diagonal in the flavour mass eigenstate basis. Moreover, the lepton number conservation (separately for each lepton flavour) is exact. It reflects the $U_e(1) \times U_\mu(1) \times U_\tau(1)$ global symmetry of the lagrangian (12.5). For quarks, only the global baryon number is conserved, while the quark mixing explicitly breaks the quark flavour $U(1)$s. Thus, the flavour changing neutral-current processes involving quarks are generated in higher orders in the electroweak interactions. Since they are strongly suppressed in Nature, it is interesting to discuss the predictions for them in the electroweak theory. Such processes are, of course, calculable without any new counterterms.

At the quark level, the generic examples of flavour changing neutral-current transitions are the reactions $d\bar{s} \to \bar{d}s$ ($\Delta S = 2$), $b\bar{d} \to \bar{b}d$ ($\Delta B = 2$) and $s \to d\gamma$ ($\Delta S = 1$), $b \to s\gamma$ ($\Delta B = 1$), with a real or virtual photon. Such transitions are responsible for physical processes like K^0–$\bar{\text{K}}^0$ and B^0–$\bar{\text{B}}^0$ mixing (and, in particular, for CP violation in the K^0–$\bar{\text{K}}^0$ system), for radiative flavour changing

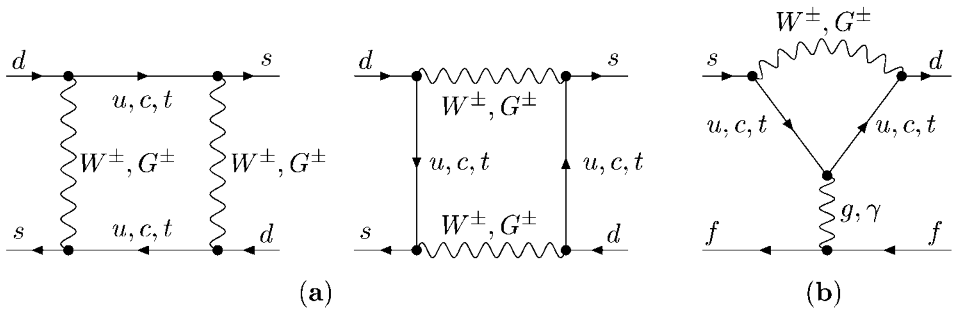


Fig. 12.17. *(a)* Leading Standard Model contribution to K^0–$\bar{K}^0$ mixing; *(b)* penguin diagram contribution to a $\Delta S = 1$ process.

decays of strange and bottom mesons (like, for example, the celebrated $b \to s\gamma$ decay) and for decays like $K \to \pi e^+e^-$ or $B \to K^\star e^+e^-$. Similar processes occur in the up-quark sector, too, but are less interesting phenomenologically.

Predictions for physical processes are obtained by convoluting the quark off-shell amplitudes with the hadron wave-functions or (in practice – as we shall see) by calculating the hadronic matrix elements of operators present in the effective lagrangian. For the time being, we remain at the quark level and estimate qualitatively the expected order of magnitude for the $\Delta S(B) = 2$ and $\Delta S(B) = 1$ amplitudes.

For the $\Delta S(B) = 2$ transitions, for example, for the $\bar{s}d \to s\bar{d}$ transition, we have the contribution from the diagrams shown in Fig. 12.17(a) ($G^\pm$ denote the charged unphysical Goldstone bosons). The two sets of diagrams enter with a relative minus sign, as can be checked by general techniques developed in Chapter 2 for obtaining the Feynman rules from a lagrangian, due to Fermi statistics of the quark fields. We shall be interested in the off-shell 1PI Green's functions which build the effective action, (2.132). The corresponding contribution to the effective action is finite and has dimension $[mass]^{-2}$. In the limit of small (compared to the gauge boson mass) external momenta which interests us, it is approximately local and can be interpreted as an effective lagrangian. For the sum of the two diagrams 12.17(a) with the W^+W^- exchange, in the 't Hooft–Feynman gauge, we have

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(\frac{e}{\sqrt{2}s}\right)^4 \sum_{i,j=u,c,t} V_{is}^\star V_{id} V_{js}^\star V_{jd}$$
$$\times \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{[\bar{\Psi}_s \gamma_\mu P_\mathrm{L}(\not{q} + m_{q_i})\gamma_\nu P_\mathrm{L}\Psi_d][\bar{\Psi}_s \gamma^\nu P_\mathrm{L}(\not{q} + m_{q_j})\gamma^\mu P_\mathrm{L}\Psi_d]}{(q^2 - m_{q_i}^2)(q^2 - m_{q_j}^2)(q^2 - M_W^2)^2} \tag{12.139}$$

The colour indices are contracted within square brackets. Suppose we want to recover the contribution of the effective vertex (12.139) to the Green's function $\langle 0|T\left(\Psi_s\bar{\Psi}_d\Psi_s\bar{\Psi}_d\right)|0\rangle$. We insert the effective lagrangian (12.139) into (2.202). There are four possible ways of contracting the fields in $\mathcal{L}_{\text{eff}}$ (12.139) with the external fields that contribute to the 1PI four point Green's function. Two contractions give the two sets of Fig. 12.17(a), with the relative minus sign. The other two contractions give the same result. Therefore, to recover the result of the calculation in the original theory, we need an extra combinatorial factor of $\frac{1}{2}$ in front of (12.139).

The contribution from the top quark exchange in (12.139) is strongly suppressed by its very small mixing with the first two generations of quarks. (As we shall see, this suppression is just the right one for accounting for the measurements). As far as the u and c exchange is concerned, we can take the limit $m_{q_i} \ll M_W$, and make use of unitarity of the Cabibbo–Kobayashi–Maskawa (CKM) matrix. On dimensional grounds, we then get the following estimate for the $\bar{s}d \to s\bar{d}$ transition amplitude with double W-boson, u- and/or c-quark exchange:

$$
\begin{aligned}
A_{uc} &\sim \left(\frac{e}{\sqrt{2}s}\right)^4 \frac{1}{M_W^2} \sum_{i,j=u,c} V_{is}^\star V_{id} V_{js}^\star V_{jd} \left[1 + \mathcal{O}\left(\frac{m_{q_i}^2}{M_W^2}, \frac{m_{q_j}^2}{M_W^2}\right)\right] \\
&\sim \alpha G_{\text{F}} \left\{\left(V_{ts}^\star V_{td}\right)^2 + \mathcal{O}\left(\sum_{i,j=u,c} V_{is}^\star V_{id} V_{js}^\star V_{jd} \frac{m_q^2}{M_W^2}\right)\right\}
\end{aligned}
\tag{12.140}
$$

In the last line, unitarity of the CKM matrix has been used: $\sum_{i,j=u,c} V_{is}^\star V_{id} = -V_{ts}^\star V_{td}$. Consequently, the leading αG_{F} term is suppressed by the same very small CKM angles as the double top quark exchange contribution. The remaining terms, which are proportional to larger CKM angles, are in turn suppressed by light quark masses, i.e. they are $\mathcal{O}\left(\alpha G_{\text{F}} m_q^2/M_W^2\right)$.† The same refers to contributions from $G^\pm$ exchange, because couplings of $G^\pm$ to light quarks are suppressed by m_q/M_W.

Such a mechanism of suppression of the flavour changing neutral-current amplitudes is known as the Glashow–Iliopoulos–Maiani (GIM) mechanism. We see that the strong suppression of the flavour changing neutral-current transitions is indeed a prediction of the Standard Model. However, this prediction follows not only from the structure of the theory but also depends on the empirical pattern of the quark masses and mixing angles. Thus, from the point of view of the Standard Model, the successful predictions for the flavour changing neutral-current processes should be considered as fairly accidental and, in fact, it is probably the (unknown) physics beyond the Standard Model which is responsible for it (namely, the physics which determines the quark Yukawa couplings).

† The dimensionless coefficient of the term $1/M_W^2$ could also include the term $\ln(m_q/M_W)$. For the box diagram there is no logarithmic term. We shall return to this point shortly.

It is also interesting to comment on the $\Delta S(B) = 1$ transitions at the qualitative level. At one-loop, they receive contributions from box diagrams and also from the so-called penguin diagrams like, for example, the one shown in Fig. 12.17(b). The corresponding amplitude behaves as

$$A_{uc} \sim \alpha G_{\mathrm{F}} \sum_{i,j=u,c} V_{id}^{\star} V_{is} \ln \frac{m_{q_i}^2}{M_W^2} + \mathcal{O}(V_{td}^{\star} V_{ts}) = \alpha G_{\mathrm{F}} V_{ud}^{\star} V_{us} \ln \frac{m_u^2}{m_c^2} + \mathcal{O}(V_{td}^{\star} V_{ts}). \tag{12.141}$$

This time, the dimensionless coefficient of the αG_{F} term† contains logarithms of light quark masses. We often say that the GIM mechanism is power-like in the case of box diagrams, but only logarithmic in the case of certain penguin diagrams. (In practice, since the masses of the up and charm quarks are quite different, no additional suppression except for the usual αG_{F} factor occurs in this case, unlike the previously considered box diagrams.)

This qualitative difference between penguin and box diagrams can be seen either by explicit calculation or by the following heuristic argument. The M_W dependence of the amplitude can be traced back to the behaviour of the effective field theory diagrams which are obtained by contracting the heavy internal lines in the box and penguin diagrams, respectively. In the first case we get either a tree level (UV-finite) diagram or a one-loop diagram with two four-quark vertices. On dimensional grounds, the latter diagram is proportional to $1/M_W^4$, which means that the power-like GIM mechanism occurs. In the penguin case, we obtain a UV-divergent one-loop diagram with a single four-quark interaction. This signals the possibility of only logarithmic GIM mechanism.

In processes where the GIM mechanism is only logarithmic, perturbative calculations can be trustworthy only if all the quark masses under logarithms are larger than the QCD confinement scale. Thus (12.141) could be used in a quantitative calculation only if the up-quark mass was larger than $\sim$1 GeV. Since this is not the case in Nature, (12.141) can be considered only as an indication that non-perturbative strong interactions play an important role in the quark-level $s \to d\mathrm{e}^+\mathrm{e}^-$ transition amplitude.

Since the flavour changing neutral-current transitions involve strongly interacting quarks, one of the most interesting field theoretical aspects are the QCD corrections to them. They are often important phenomenologically, and their calculation strongly relies on the effective theory approach. In the final subsection of this chapter we describe these techniques using the concrete example of CP violation in the neutral kaon system.

† The leading term behaves like $1/M_W^2$. The $1/k^2$ from the photon propagator is cancelled by the k^2/M_W^2 factor coming from the vertex function. The logarithmic part of the vertex function is independent of quark masses and vanishes due to unitarity of the CKM matrix.

QCD corrections to CP violation in the neutral kaon system

We are interested in CP violation in the kaon–antikaon transition amplitude (for the general discussion of CP violation in the kaon system consult Section 1.5). At the quark level, this amplitude is generated by the $\bar{s}d \rightarrow s\bar{d}$ amplitude which receives the leading-order contribution from the box diagrams presented in Fig. 12.17(a). In the following, we shall first discuss the calculation of these diagrams and QCD corrections to them. Next, we shall turn to the explicit form of their relation to the CP violation parameter $\varepsilon_{\rm K}$.

The complete result for the $\mathcal{L}_{\rm eff}$ given by (12.139) is easy to obtain. Using the formulae from Appendix D and the identity (12.96), it can be written in the following form:

$$\mathcal{L}_{\rm eff}(WW) = -\frac{G_{\rm F}^2 M_W^2 \lambda_i \lambda_j}{4\pi^2} A(x_i, x_j) \left(\bar{\Psi}_s \gamma_\mu P_{\rm L} \Psi_d\right) \left(\bar{\Psi}_s \gamma^\mu P_{\rm L} \Psi_d\right) \tag{12.142}$$

with

$$A(x_i, x_j) = \frac{1}{x_i - x_j}\left[\frac{x_i^2 \ln x_i}{(1-x_i)^2} - (i \rightarrow j)\right] + \frac{1}{(1-x_i)(1-x_j)}. \tag{12.143}$$

To this order, $G_{\rm F}$ is given by (12.48) and $x_i = m_{q_i}^2/M_W^2$. Summation over indices $i, j = u, c, t$ is understood in (12.142). The products of the CKM matrix elements have been denoted shortly by $\lambda_i = V_{is}^* V_{id}$ and colour indices are contracted within brackets.

The remaining box diagrams (remember that (12.139) accounts only for the W–W exchange diagrams) with either two unphysical Goldstone scalars in the loop or one such scalar and one $W^\pm$-boson are calculated in an analogous way. The sum of these diagrams is reproduced by an effective lagrangian which has the same form as $\mathcal{L}_{\rm eff}(WW)$ with $A(x_i, x_j)$ replaced by

$$\tfrac{1}{4} x_i x_j [A(x_i, x_j) - 8B(x_i, x_j)] \tag{12.144}$$

where

$$B(x_i, x_j) = \frac{1}{x_i - x_j}\left[\frac{x_i \ln x_i}{(1-x_i)^2} - (i \leftrightarrow j)\right] + \frac{1}{(1-x_i)(1-x_j)} \tag{12.145}$$

Thus, up to $\mathcal{O}$(external momenta$/M_W$), the diagrams shown in Fig. 12.17(a) can be reproduced by the following effective four-quark interaction lagrangian

$$\mathcal{L}_{\rm eff}(\text{box}) = -\frac{G_{\rm F}^2 M_W^2}{4\pi^2} \sum_{i,j=u,c,t} \lambda_i \lambda_j \tilde{S}(x_i, x_j) \left(\bar{\Psi}_s \gamma_\mu P_{\rm L} \Psi_d\right) \left(\bar{\Psi}_s \gamma^\mu P_{\rm L} \Psi_d\right) \tag{12.146}$$

where

$$\tilde{S}(x_i, x_j) = \left(1 + \tfrac{1}{4} x_i x_j\right) A(x_i, x_j) - 2 x_i x_j B(x_i, x_j) \tag{12.147}$$

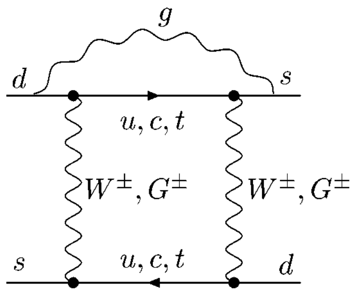


Fig. 12.18. Sample QCD correction to K^0–$\bar{\mathrm{K}}^0$ mixing.

The $\bar{s}d \rightarrow s\bar{d}$ amplitude presented in Fig. 12.17(a) is subject to QCD corrections. They are described by diagrams obtainable from those shown in Fig. 12.17(a) by connecting the quark lines by an arbitrary number of gluon lines, for example, like that shown in Fig. 12.18.

An important property of these corrections is that their ordinary perturbative expansion is actually an expansion in powers of $\alpha_s \ln(M_W^2/Q_{\mathrm{low}}^2)$ and $\alpha_s \ln(m_t^2/Q_{\mathrm{low}}^2)$ rather then in powers of α_s itself. Here, Q_{low}^2 stands for all possible Lorentz invariants made out of momenta and masses of external particles in the considered amplitude, i.e. it is at most of order of the neutral kaon mass $m_{\mathrm{K}} \simeq 0.5$ GeV squared. Since $M_W \simeq 80$ GeV and $m_t \simeq 175$ GeV, the logarithms are large, and the ordinary expansion in α_s breaks down. We need to resum these large logarithms from all orders of the perturbation series. It is here where the main virtue of using the effective theory is, since we can then easily apply the (perturbative) renormalization group methods for resummation of large logarithms. This procedure will be described in some detail below.

The kaon–antikaon transition amplitude we look for is not perturbatively calculable even when large QCD logarithms are resummed. In order to find it, one needs to use non-perturbative methods like, for instance, lattice calculations. One could then ask why we intend to perform the perturbative logarithm resummation instead of applying non-perturbative methods immediately. The answer to this question is that the accessible non-perturbative methods are not suited for calculating quantities which receive significant contributions from rapidly varying field configurations. In other words, gluons and quarks with momenta close to M_W or m_t (i.e. very much larger than Λ_{QCD}) cannot be accounted for by non-perturbative methods in the case when external states are light hadrons at rest. As we shall see, the purely perturbative resummation of large logarithms in the $\bar{s}d \rightarrow s\bar{d}$ amplitude will automatically provide us with quantities suitable for applying non-perturbative methods.

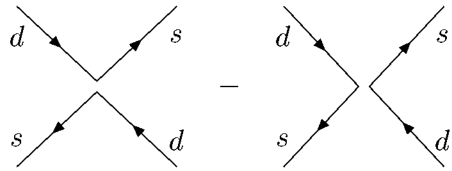


Fig. 12.19. Tree level $d\bar{s} \to \bar{s}d$ amplitude generated by $\mathcal{L}_{\text{eff}}$. Each of the two diagrams has a combinatoric factor of 2.

For simplicity, we shall discuss the procedure of large logarithm resummation in detail only for the top quark contribution, i.e. for the amplitude generated by the diagrams in Fig. 12.17(a) with only top quarks in the electroweak loop, and by all the QCD corrections to this diagram. The fact of basic importance here is that this amplitude can be reproduced to all orders in QCD by the following effective lagrangian (see Section 7.7):

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{QCD}}(\text{gluons and light quarks})\\ &\quad - \frac{G_{\text{F}}^2 M_W^2}{4\pi^2}\lambda_t^2\, C(\mu)\left(\bar{\Psi}_s\gamma_\mu P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right) \end{aligned} \tag{12.148}$$

up to non-leading corrections in $Q_{\text{low}}^2/(M_W^2$ or $m_t^2)$ and higher order electroweak corrections.

The Wilson coefficient $C(\mu)$ depends on the scale μ at which the effective theory fields and parameters are renormalized (see Section 7.7). The numerical value of $C(\mu)$ can be found perturbatively in QCD by requiring equality of the effective theory and full Standard Model amplitudes up to $\mathcal{O}(\text{external momenta}/M_W)$. At the leading order in QCD, this means requiring equality of the box diagrams in Fig. 12.17(a) (with only top quarks in the loop) and the tree level $d\bar{s} \to s\bar{d}$ amplitude generated by $\mathcal{L}_{\text{eff}}$ (see Fig. 12.19). At the next-to leading order in QCD, we would need to take into account diagrams like the one shown in Fig. 12.18 on the Standard Model side. On the effective theory side we would have to include one-loop QCD corrections to the vertex generated by $\mathcal{L}_{\text{eff}}$ (see Fig. 12.20).

At the leading order in QCD, the result we have already obtained in (12.146) immediately implies that

$$C(\mu_0) = \tilde{S}(x_t, x_t) + \mathcal{O}(\alpha_s) \tag{12.149}$$

Here, μ_0 denotes the renormalization scale used in the effective theory when the comparison of the full and the effective theory amplitudes is performed. At the first glance, it may seem strange that we specify this renormalization scale even though the tree level diagram in Fig. 12.19 obviously requires no renormalization. However, one needs to keep in mind that we intend to effectively resum large

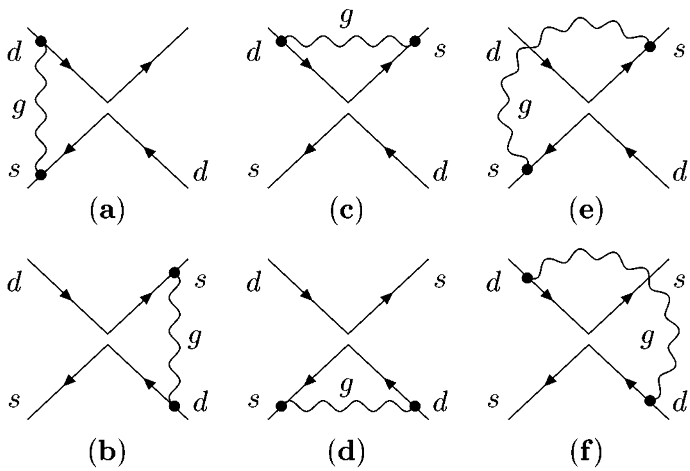


Fig. 12.20. One-loop QCD corrections to the $d\bar{s} \to \bar{s}d$ amplitude generated by the effective lagrangian. Each of the diagrams shown has its partner in which fermion lines in the four-fermion vertex are differently contracted, and which enters with a relative minus sign due to the Fermi statistics of quark fields, as in Fig. 12.19.

logarithms from all orders of the QCD perturbation series for the full Standard Model amplitudes. Thus, even if we restrict ourselves to the leading terms in the evaluation of $C(\mu_0)$, we must make sure that no large logarithms reside in the neglected higher order QCD corrections to this coefficient coming from comparing amplitudes like the one in Fig. 12.18 with those in Fig. 12.20. It turns out that such large logarithms are absent so long as the renormalization scale μ_0 is chosen to be of the same order as the heavy particle masses M_W and m_t in the full Standard Model diagrams.

Let us have a closer look at how the size of μ_0 is being specified. The QCD corrections to the tree level diagram, shown in Fig. 12.20, are UV-divergent. After renormalization, they contain logarithms $\ln(\mu_0^2/Q_{\rm low}^2)$. On the other hand, the box diagram in Fig. 12.18 contains logarithms $\ln(M_W^2/Q_{\rm low}^2)$ and $\ln(m_t^2/Q_{\rm low}^2)$, i.e. the very logarithms we would like to resum. The dependence on $\ln Q_{\rm low}^2$ cancels in the difference of the full and effective theory amplitudes to all orders in QCD, up to non-leading corrections in $Q_{\rm low}^2/(M_W^2$ or $m_t^2)$. (This is a general fact: Wilson coefficients are insensitive to low energy behaviour of the considered theory). Consequently, the QCD corrections to the leading order result for the coefficient $C(\mu_0)$ depend on $\alpha_s \ln(M_W^2/\mu_0^2)$ and $\alpha_s \ln(m_t^2/\mu_0^2)$. In order to make the latter two logarithms small (i.e. not much larger than unity), we choose μ_0 of the same order as M_W and m_t.

Once the Wilson coefficient $C(\mu)$ is found, we need to evaluate the $\mathrm{K}^0 \to \bar{\mathrm{K}}^0$ transition amplitude generated by $\mathcal{L}_{\text{eff}}$. This requires finding the matrix element of the four-quark operator present in $\mathcal{L}_{\text{eff}}$ between the kaon and antikaon states at rest. This matrix element is usually parametrized as follows (for states normalized to unity):

$$\int \mathrm{d}^3\mathbf{x}\, \langle \bar{\mathrm{K}}^0 | (\bar{s}_{\mathrm{L}}\gamma_\mu d_{\mathrm{L}})(\bar{s}_{\mathrm{L}}\gamma^\mu d_{\mathrm{L}}) | \mathrm{K}^0 \rangle = \tfrac{1}{3} B_{\mathrm{K}}(\mu) m_{\mathrm{K}} f_{\mathrm{K}}^2 \tag{12.150}$$

where m_{K} is the kaon mass, and $f_{\mathrm{K}} \simeq 161$ MeV is its decay constant. The parameter $B_{\mathrm{K}}(\mu)$ cannot be calculated perturbatively because it depends on QCD interactions at energy scales close to Λ_{QCD}.† Nevertheless, it is still a function of the renormalization scale μ used for UV renormalization of amplitudes generated by the considered four-quark interaction. Non-perturbative methods used for evaluation of $B_{\mathrm{K}}(\mu)$ are not applicable when μ is arbitrarily large.‡ Consequently, renormalization group evolution of the Wilson coefficient from $\mu_0 \sim M_W$ to $\mu_{\text{low}} = (\text{a few}) \times \Lambda_{\text{QCD}}$ must be performed before calculation of the final low energy amplitude. Performing renormalization group evolution of the Wilson coefficient would be necessary even if the matrix element was perturbatively calculable. We would then argue that the presence of large renormalization scale μ_0 worsens the perturbative expansion of the matrix element due to large logarithms $\ln(\mu_0^2/Q_{\text{low}}^2)$. Thus, the argument for using low μ in calculation of the matrix element would be identical to the one we have used to fix the size of μ_0.

The RGE for the Wilson coefficient $C(\mu)$ is discussed in Section 7.7. In order to render the amplitudes generated by $\mathcal{L}_{\text{eff}}$ finite, we need to perform the standard renormalization of the $\mathcal{L}_{\text{QCD}}$ part of it, as well as renormalize couplings and fields in the effective part:

$$C(\mu) \to Z_c C(\mu) \tag{12.151}$$

and

$$\left(\bar{\Psi}_s \gamma_\mu P_{\mathrm{L}} \Psi_d\right)\left(\bar{\Psi}_s \gamma^\mu P_{\mathrm{L}} \Psi_d\right) \to Z_2^2 \left(\bar{\Psi}_s \gamma_\mu P_{\mathrm{L}} \Psi_d\right)\left(\bar{\Psi}_s \gamma^\mu P_{\mathrm{L}} \Psi_d\right) \tag{12.152}$$

The constant Z_2 in the above equation is the usual QCD renormalization constant of the quark wave-function (see (8.4)). We recall the RGE for $C(\mu)$, (7.138), (7.139):

$$\left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} - \gamma_c\right) C(\mu) = 0 \tag{12.153}$$

† The factor $B_{\mathrm{K}}(\mu)$ is defined so that $B_{\mathrm{K}}(\mu) = 1$ when only the vacuum state is introduced as an intermediate state between the two currents in the matrix elements in (12.150), i.e. in the so-called vacuum saturation approximation.

‡ For instance, maximal energy scales available in lattice calculations are much smaller than $\Lambda_{\text{QCD}} \times$ (size of the lattice)/(lattice spacing).

where

$$\gamma_c = -\mu \frac{dZ_c}{d\mu} \frac{1}{Z_c}. \tag{12.154}$$

We are going to use this equation for evolving $C(\mu)$ from μ_0 down to μ_{low} at the leading order in QCD (i.e. in the so-called leading logarithmic approximation). Thus, we need to find the explicit $\mathcal{O}(\alpha_s)$ contribution to the renormalization constant Z_c. At the leading order in QCD, the renormalization constant Z_c in the $\overline{\text{MS}}$ scheme is found from pole parts of one-loop diagrams shown in Fig. 12.20. Up to $\mathcal{O}(Q^2_{\text{low}}/M_W^2)$, the contribution to the effective lagrangian of Fig. 12.20(a), (b) and the 'negative partners' of Fig. 12.20(c), (d) reads

$$\begin{aligned}\mathcal{L}_1 &= 2 \times (-X)(-\mathrm{i}g_s)^2(-\mathrm{i})\mathrm{i}^2\mu^\varepsilon \\ &\times \int \frac{d^n q}{(2\pi)^n} \frac{[\bar{\Psi}_s \gamma_\mu P_{\text{L}}(\not{q} + m_d)\gamma_\nu T^a \Psi_d][\bar{\Psi}_s \gamma^\nu T^a(\not{q} + m_s)\gamma^\mu P_{\text{L}}\Psi_d]}{(q^2 - m_d^2)(q^2 - m_s^2)q^2}\end{aligned} \tag{12.155}$$

where $X = G_{\text{F}}^2 M_W^2 \lambda_t^2 C(\mu)/4\pi^2$ abbreviates the factor standing in front of the four-quark interaction in $\mathcal{L}_{\text{eff}}$. The $SU(3)$ generators and gauge coupling are denoted by T^a and g_s, respectively. As usually, colour indices are contracted within brackets. The factor of 2 is the combinatorial factor for the diagrams summarized by $\mathcal{L}_1$. It can be traced back to the possibility of two different (but indistinguishable) contractions of the fields in the four-fermion operator in (12.148) with the external fields after its insertion into (2.202) to build the Green's function $\langle 0|T\left(\Psi_s \bar{\Psi}_d \Psi_s \bar{\Psi}_d\right)|0\rangle$ (observe that for $\mathcal{L}_1$ all contributions are distinguishable and see Fig. 12.19).

By power counting, we can verify that only the double $\not{q}$ term in the numerator can contribute to UV divergence. Thus, the only integral we need is

$$\int \frac{d^n q}{(2\pi)^n} \frac{q^\rho q^\sigma}{(q^2 - m_d^2)(q^2 - m_s^2)q^2} = \frac{\mathrm{i}g^{\rho\sigma}}{32\pi^2\varepsilon} + \text{finite terms} \tag{12.156}$$

Substituting this result into the considered amplitude, one finds the pole part of $\mathcal{L}_1$

$$\mathcal{L}_1^{\text{pole}} = -\frac{Xg_s^2}{16\pi^2\varepsilon}\left(\bar{\Psi}_s \gamma_\mu P_{\text{L}} \gamma_\rho \gamma_\nu T^a \Psi_d\right)\left(\bar{\Psi}_s \gamma^\nu \gamma^\rho \gamma^\mu P_{\text{L}} T^a \Psi_d\right) \tag{12.157}$$

The pole parts of Fig. 12.20(c), (d) and the 'negative partners' of Fig. 12.20(a), (b) are found in an analogous manner and yield the contribution to the effective lagrangian:

$$\mathcal{L}_2^{\text{pole}} = -\frac{Xg_s^2}{16\pi^2\varepsilon}\left(\bar{\Psi}_s \gamma_\nu T^a \gamma_\rho \gamma_\mu P_{\text{L}} \gamma^\rho \gamma^\nu T^a \Psi_d\right)\left(\bar{\Psi}_s \gamma^\mu P_{\text{L}} \Psi_d\right)$$

Pole parts of Fig. 12.20(e), (f) (and their 'negative' companions) are summarized by

$$\mathcal{L}_3^{\text{pole}} = +\frac{Xg_s^2}{32\pi^2\varepsilon}\left(\bar{\Psi}_s\gamma_\nu T^a\gamma_\rho\gamma_\mu P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\nu T^a\gamma^\rho\gamma^\mu P_{\text{L}}\Psi_d\right)$$
$$\mathcal{L}_4^{\text{pole}} = +\frac{Xg_s^2}{32\pi^2\varepsilon}\left(\bar{\Psi}_s\gamma_\mu P_{\text{L}}\gamma_\rho\gamma_\nu T^a\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\gamma^\rho\gamma^\nu T^a\Psi_d\right)$$

Compared with $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ have no combinatorial factors of 2 since two indistinguishable contractions appear when $\mathcal{L}_3$ ($\mathcal{L}_4$) is used in (2.202) (we recall once again that for $\mathcal{L}_1$ and $\mathcal{L}_2$ all contractions are distinguishable and the factor of 2 is necessary to match the combinatorics of the tree level diagrams described by the operator in (12.148), which gives pairs of indistinguishable contractions). Summing up all contributions one finds for the pole part

$$\begin{aligned}\sum_{k=1}^{4}\mathcal{L}_k^{\text{pole}} = -\frac{Xg_s^2}{16\pi^2\varepsilon}&\left[\left(\bar{\Psi}_s\gamma_\mu\gamma_\rho\gamma_\nu P_{\text{L}}T^a\Psi_d\right)\left(\bar{\Psi}_s\gamma^\nu\gamma^\rho\gamma^\mu P_{\text{L}}T^a\Psi_d\right)\right.\\ &+\left(\bar{\Psi}_s\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu P_{\text{L}}T^aT^a\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right)\\ &\left.-\left(\bar{\Psi}_s\gamma_\mu\gamma_\rho\gamma_\nu P_{\text{L}}T^a\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu\gamma^\rho\gamma^\nu P_{\text{L}}T^a\Psi_d\right)\right]\end{aligned} \tag{12.158}$$

In order to simplify the Dirac algebra in the first expression, we apply the identity (12.96). It holds in four dimensions only, but we are allowed to use four-dimensional Dirac algebra so long as we are interested only in calculating one-loop renormalization constants in the $\overline{\text{MS}}$ scheme (see also the discussion in Section 12.5 on muon decay). The second term in (12.158) is simplified by using $\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu = 4\gamma_\mu + \mathcal{O}(\varepsilon)$ and $(T^aT^a)_{\alpha\beta} = \frac{4}{3}\delta_{\alpha\beta}$. After all that we find

$$\begin{aligned}\sum_{k=1}^{4}\mathcal{L}_k^{\text{pole}} = \frac{Xg_s^2}{16\pi^2\varepsilon}&\left[12\left(\bar{\Psi}_s\gamma^\mu T^a P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu T^a P_{\text{L}}\Psi_d\right)\right.\\ &\left.-\frac{16}{3}\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right)\right]\end{aligned} \tag{12.159}$$

Finally, we can get rid of the $SU(3)$ generators T^a by applying the identity

$$(T^a)_{\alpha\beta}(T^a)_{\gamma\delta} = \tfrac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta} - \tfrac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{12.160}$$

together with the Fierz rearrangement formula (12.98) for anticommuting fields. After all these simplifications, we obtain the final result

$$\sum_{k=1}^{4}\mathcal{L}_k^{\text{pole}} = -\frac{Xg_s^2}{12\pi^2\varepsilon}\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right) \tag{12.161}$$

with colour indices contracted within brackets. Now, we require that the divergent part of the one-loop diagrams we have calculated is cancelled by the counterterm

$$\Delta\mathcal{L}_{\text{eff}} = -X(Z_c Z_2^2 - 1)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right)\left(\bar{\Psi}_s\gamma^\mu P_{\text{L}}\Psi_d\right) \tag{12.162}$$

which implies that

$$Z_c Z_2^2 - 1 = -\frac{g_s^2}{12\pi^2\varepsilon} + \mathcal{O}(g_s^4) \tag{12.163}$$

Since the quark wave-function renormalization constant is known to be (see Chapter 8)

$$Z_2 = 1 - \frac{g_s^2}{6\pi^2\varepsilon} + \mathcal{O}(g_s^4) \tag{12.164}$$

the renormalization constant Z_c must be equal to

$$Z_c = 1 + \frac{g_s^2}{4\pi^2\varepsilon} + \mathcal{O}(g_s^4) = 1 + \frac{\alpha_s}{\pi\varepsilon} + \mathcal{O}(\alpha_s^2). \tag{12.165}$$

Consequently,

$$\gamma_c(\mu) = \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2). \tag{12.166}$$

The solution to the RGE (12.153) is easily found to be

$$C(\mu) = \exp\left\{\int_{\ln\mu_0}^{\ln\mu} \mathrm{d}(\ln\mu')\ \gamma_c(\mu')\right\} C(\mu_0). \tag{12.167}$$

Thus, we obtain (up to higher orders in α_s)

$$\begin{aligned} C(\mu) &= \exp\left\{\frac{1}{\pi}\int_{\ln\mu_0}^{\ln\mu} \mathrm{d}(\ln\mu')\alpha_s(\mu')\right\} C(\mu_0) \\ &= \exp\left\{\frac{1}{b_1}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\mathrm{d}\alpha_s'}{\alpha_s'}\right\} C(\mu_0) = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{-1/b_1} C(\mu_0) \end{aligned} \tag{12.168}$$

where b_1 is the coefficient in one-loop RGE for the QCD coupling constant (see Chapters 6, 8 and (6.45)):

$$\mu\frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = \alpha_s\left[b_1\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right] \tag{12.169}$$

The value of $b_1 = -\frac{1}{2}(11 - \frac{2}{3}N_f)$ depends on the number N_f of quark flavours active in the considered effective theory. In the full Standard Model, we have $N_f = 6$. In the effective theory introduced at the scale μ_0, the top quark is already absent. Consequently, we substitute $N_f = 5$ to b_1 when running the Wilson coefficient from $\mu = \mu_0$ to $\mu = \mu_b \sim m_b$. At this latter scale, the b quark should be decoupled. Below μ_b, we substitute $N_f = 4$ to b_1. Finally, the charm quark can be

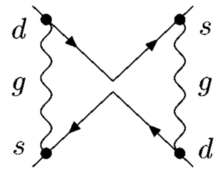


Fig. 12.21. Sample two-loop QCD corrections to the effective vertex.

decoupled at $\mu = \mu_c \sim m_c$. The ultimate expression for the Wilson coefficient at scales $\mu_{\text{low}} < \mu_c$ is thus

$$C(\mu_{\text{low}}) = \left[\frac{\alpha_s(\mu_c)}{\alpha_s(\mu_{\text{low}})}\right]^{6/27} \left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu_c)}\right]^{6/25} \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu_b)}\right]^{6/23} \tilde{S}(x_t, x_t) \; + \; \mathcal{O}(\alpha_s). \tag{12.170}$$

The expression we have obtained for $C(\mu_{\text{low}})$ is perturbative in α_s but exact in $\alpha_s \ln(\mu_0/\mu_b)$, $\alpha_s \ln(\mu_b/\mu_c)$ and $\alpha_s \ln(\mu_c/\mu_{\text{low}})$. In other words, we have resummed powers of $\alpha_s \times$(large logarithm) from all orders of the usual perturbation series. Achieving this resummation has been the main reason for introducing the effective lagrangian (12.148).

Resummation of large logarithms can be systematically performed at higher orders in QCD, too. At the next-to-leading order, it requires calculating the $\mathcal{O}(\alpha_s)$ part of $C(\mu_{\text{low}})$ exactly in $\alpha_s \ln(\mu_0/\mu_b)$, $\alpha_s \ln(\mu_b/\mu_c)$ and $\alpha_s \ln(\mu_c/\mu_{\text{low}})$, i.e. summing all terms $\alpha_s(\alpha_s \ln(\mu_0/\mu_b))^n$ etc. In order to achieve this, one needs to include $\mathcal{O}(\alpha_s)$ contributions to the matching between the full and effective theories at the scale μ_0, and to the running of the Wilson coefficient down to the scale μ_{low}. As mentioned earlier, next-to-leading contributions to the matching on the full theory side are given by two-loop diagrams like the one shown in Fig. 12.18. On the effective theory side, one has to evaluate finite parts of the diagrams presented in Fig. 12.20. Equating the two amplitudes renormalized at the scale $\mu_0 \sim M_W$, one finds $C(\mu_0)$ including the $\mathcal{O}(\alpha_s)$ part of it. Running the Wilson coefficient down to the low energy scale μ_{low} at the next-to-leading order in QCD requires knowing the two-loop contribution to the anomalous dimension γ_c. In the MS-scheme, it is given by pole parts of two-loop corrections to the effective four-quark vertex. An example of such a correction is shown in Fig. 12.21. Details of the next-to-leading QCD calculation are presented in Buchalla, Buras & Harlander (1990).

Including the $\mathcal{O}(\alpha_s)$ part of $C(\mu_{\text{low}})$ in (12.170) automatically reduces its dependence on the arbitrariness in the choice of scales μ_0, μ_b and μ_c. These scales have to be of the same order of magnitude as M_W, m_b and m_c, respectively. However, changing them by, for example, a factor of 2 is still allowed. For instance, when μ_0 in (12.170) is replaced by $2\mu_0$, the leading order (LO) contribution to the coefficient $C(\mu_{\mathrm{low}})$ changes by

$$\begin{aligned}\Delta C^{\mathrm{LO}}(\mu_{\mathrm{low}}) &= \left(\left[\frac{\alpha_s(2\mu_0)}{\alpha_s(\mu_0)}\right]^{6/23} - 1\right) C^{\mathrm{LO}}(\mu_{\mathrm{low}}) \\ &= -\frac{\alpha_s(\mu_0)}{\pi}(\ln 2)\cdot C(\mu_{\mathrm{low}}) + \mathcal{O}(\alpha_s^2)\end{aligned} \tag{12.171}$$

Before the $\mathcal{O}(\alpha_s)$ term in (12.170) is found, the above dependence on μ_0 can serve as an estimate of the error we make by neglecting this very term. On the other hand, when the $\mathcal{O}(\alpha_s)$ term is calculated explicitly, one finds that its μ_0 dependence is precisely such that it cancels the μ_0 dependence of the leading term. Consequently, the μ_0 dependence of $C(\mu_{\mathrm{low}})$ becomes a $\mathcal{O}(\alpha_s^2)$ effect (see, for example, Buras, Misiak, Münz & Pokorski (1995) for an extensive discussion of this point in the case of the decay $b \to s\gamma$). The same phenomenon can be proven to happen at any order in perturbation theory. The renormalization-scale-dependence is always formally a higher order effect. Thus, what matters in practical calculations are only orders of magnitude of renormalization scales, not their precise values. Choosing the proper order of magnitude for the renormalization scale is important for (asymptotic) convergence of the perturbation series. On the other hand, the effect of changing the scale by a factor of 2 can always be absorbed into unknown higher order corrections.

In the remainder of this subsection, we shall briefly describe how the evaluated Wilson coefficient is used to find the CP-violating observable ε_{K} (defined in Chapter 1). We recall that (1.273)

$$\varepsilon_{\mathrm{K}} \simeq \frac{\mathrm{Im}(M_{12}A_0^2)}{\sqrt{2}\Delta m|A_0|^2}\exp(\mathrm{i}\pi/4). \tag{12.172}$$

where M_{12} is the hermitean part of the weak hamiltonian, Δm is introduced in (1.273) and A_0 is defined in (1.270). In order to simplify further discussion, we shall assume that we use the parametrization of the CKM matrix as in (12.44). When this parametrization is used, the amplitude A_0 is approximately real. This is because the dominant contribution to A_0 originates from the tree level $s \to u\bar{u}d$ weak transition. Therefore, A_0 is to a good approximation proportional to $V_{12}V_{11}^*$ which is real in the standard parametrization (12.44) of the CKM matrix. Note that if A_0 was exactly real, then the parameters ε_{K} in (1.270) and $\bar{\varepsilon}$ in (1.265) would be equal. Our approximate expression for ε_{K} now becomes proportional to $\mathrm{Im}\,M_{12}$

$$\varepsilon_{\mathrm{K}} \simeq \frac{\mathrm{Im}\,M_{12}}{\sqrt{2}\Delta m}\exp(\mathrm{i}\pi/4) \tag{12.173}$$

As follows from (1.261), the imaginary part of M_{12} is equal to

$$\mathrm{Im}\,M_{12} = \mathrm{Im}\langle \mathrm{K}^0|H_{\mathrm{W}}|\bar{\mathrm{K}}^0\rangle + \mathcal{O}(H_{\mathrm{W}}^2) = -\,\mathrm{Im}\langle \bar{\mathrm{K}}^0|H_{\mathrm{W}}|\mathrm{K}^0\rangle + \mathcal{O}(H_{\mathrm{W}}^2) \tag{12.174}$$

The matrix element in the r.h.s. of (12.174) is generated by the box diagram in Fig. 12.17 and by QCD corrections to it. Similarly to (12.146), we can write it as a sum of contributions proportional to different products of CKM matrix elements

$$\mathrm{Im}\, M_{12} = \mathrm{Im} \sum_{i,j=u,c,t} \lambda_i \lambda_j \; M_{12}^{(ij)} \tag{12.175}$$

The part proportional to $\mathrm{Im}\,\lambda_t^2$ in the above expression is given by the matrix element of the effective four-quark interaction present in (12.148) between the kaon and antikaon states. Using (12.150) and (12.174), we find

$$\begin{aligned}\mathrm{Im}\, M_{12} = &-\frac{G_{\mathrm{F}}^2 M_W^2}{12\pi^2}\, \mathrm{Im}(\lambda_t^2) C(\mu_{\mathrm{low}}) B_{\mathrm{K}}(\mu_{\mathrm{low}}) m_{\mathrm{K}} f_{\mathrm{K}}^2 \\ &+ (\text{contributions proportional to other products } \lambda_i \lambda_j)\end{aligned} \tag{12.176}$$

with $C(\mu_{\mathrm{low}})$ given by (12.170).

Although the double top-quark contribution is really the largest in $\mathrm{Im}\, M_{12}$, the other contributions cannot be neglected, because $|\lambda_c| \gg |\lambda_t|$. When the other contributions are included and the unitarity of the CKM matrix is used, one obtains a result which is usually written in the following form

$$\begin{aligned}\mathrm{Im}\, M_{12} = &-\frac{G_{\mathrm{F}}^2 M_W^2}{12\pi^2} B_{\mathrm{K}} m_{\mathrm{K}} f_{\mathrm{K}}^2 \left\{\mathrm{Im}(\lambda_c^2) S(x_c)\eta_1 \right. \\ &\left. + \mathrm{Im}(\lambda_t^2) S(x_t)\eta_2 + 2\,\mathrm{Im}(\lambda_t \lambda_c) S(x_c, x_t)\eta_3\right\}\end{aligned} \tag{12.177}$$

where

$$\left.\begin{aligned} S(x) &= \tilde{S}(x,x) + \tilde{S}(0,0) - 2\tilde{S}(x,0) = \frac{-3x^3 \ln x}{2(1-x)^3} + \frac{x^3 - 11x^2 + 4x}{4(1-x)^2} \\ S(x_c, x_t) &= \tilde{S}(x_c, x_t) + \tilde{S}(0,0) - \tilde{S}(x_t, 0) - \tilde{S}(x_c, 0) \\ &= x_c \left[-\ln x_c + \frac{x_t^2 - 8x_t + 4}{4(1-x_t)^2} \ln x_t - \frac{3x_t}{4(1-x_t)} \right] + \mathcal{O}(x_c^2) \end{aligned}\right\} \tag{12.178}$$

and the μ-independent parameter B_{K} is defined by

$$B_{\mathrm{K}} = B_{\mathrm{K}}(\mu)[\alpha_s(\mu)]^{-2/9}[1 + \mathcal{O}(\alpha_s)] \tag{12.179}$$

The quantity η_2 which parametrizes the short-distance QCD effects in the double top quark contribution is directly related to the calculated expression for the coefficient $C(\mu)$. Namely

$$\eta_2 = [\alpha_s(\mu_c)]^{\frac{2}{9}} \left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu_c)}\right]^{6/25} \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu_b)}\right]^{6/23} + \mathcal{O}(\alpha_s) \tag{12.180}$$

The calculation of η_1 and η_3 is performed with the use of the same technique as η_2, i.e. with help of an effective theory and renormalization group evolution of Wilson coefficients. However, it is technically much more involved.

Problems

12.1 Show that with N Higgs multiplets with weak isospin $T_n^2 = t_n(t_n+1)$, $n = 1, \ldots, N$, with vacuum expectation values v_n, one gets the tree level relation:

$$\rho \equiv \frac{M_W^2}{\cos^2\theta_W M_Z^2} = \frac{\sum_n \left[t_n(t_n+1) - t_{3n}^2\right] v_n^2}{2\sum_n t_{3n}^2 v_n^2} \tag{12.181}$$

12.2 Show that a convenient measure of the CP violation in the K^0–$\bar{K}^0$ mixing is the parameter $J = \mathrm{Im}(V_{ub}V_{cd}V_{ud}^\star V_{cb}^\star)$. Show that it can be expressed in terms of four moduli of the elements of the CKM matrix:

$$\begin{aligned} 4J^2 &= 4|V_{ub}|^2|V_{cb}|^2|V_{ud}|^2|V_{cd}|^2 \\ &\quad - (1 - |V_{ud}|^2 - |V_{cd}|^2 - |V_{cb}|^2 - |V_{ub}|^2 + |V_{ud}|^2|V_{cb}|^2 + |V_{ub}|^2|V_{cd}|^2)^2 \end{aligned} \tag{12.182}$$

12.3 Check that the relation $\rho = 1$ follows from the $SU_V(2)$ symmetry of the Higgs potential after spontaneous symmetry breaking (like in the σ model).

12.4 Calculate the β-functions for the renormalization group evolution of the gauge couplings in the Standard Model.

12.5 Derive RGE for Yukawa couplings in the Standard Model.

12.6 Calculate the partial decay width for $h^0 \to W^+W^-$. Show that in the limit $M_h \gg M_W$ the same result is obtained by calculating the decay width for $h^0 \to G^+G^-$ where $G^\pm$ are the Goldstone bosons in the 't Hooft–Feynman gauge. This is an example of the 'equivalence theorem': $A(W_L^\pm, Z_L^0, \ldots) = A(G^\pm, G^0, \ldots) + \mathcal{O}(M_W/E)$. Show that the equivalence theorem also holds for decays $h^0 \to W^-W^+ \to 4f$ and $h^0 \to G^+G^- \to 4f$ (for $M_W < M_h < 2M_W$), even in the limit of zero fermion masses (i.e. for vanishing $G^\pm ff$ coupling), due to a singularity in the phase space integration.

12.7 Show that $\Gamma(h^0 \to \gamma\gamma) \sim (g^2/M_W^2)\alpha^2 M_h^3$. Construct the effective lagrangian for this process.

12.8 Calculate the forward–backward and left–right asymmetries for the process $e^+e^- \to Z^0 \to \bar{f}f$ for $s^{1/2} \approx M_Z$.

12.9 Calculate the box diagram for K^0–$\bar{K}^0$ mixing in the unitary gauge.

13

Chiral anomalies

13.1 Triangle diagram and different renormalization conditions

Introduction

Anomalies have already been mentioned in this book on several occasions. In this chapter we systematically discuss the fermion anomaly in $(3+1)$-dimensional quantum field theory (Adler 1969, Bell & Jackiw 1969). Its existence can be traced back to the short-distance singularity structure of products of local operators.

To be specific let us consider QCD. As discussed in Chapter 9, apart from being invariant under gauge transformations in the colour space its lagrangian is also invariant under the global $SU(N) \times SU(N) \times U(1) \times U_{\mathrm{A}}(1)$ chiral group of transformations acting in the flavour space.† Fermions belong to a vector-like, i.e. real, representation of the gauge group and for $N > 2$ to a complex representation of the chiral group (see Appendix E). We know from Chapters 9 and 10 that the Noether currents corresponding to the global flavour symmetry, although external with respect to the strong interaction gauge group, acquire important dynamical sense. The axial non-abelian currents couple to Goldstone bosons (pseudoscalar mesons) and the left-handed chiral currents couple to the intermediate vector bosons, i.e. they are gauge currents of the weak interaction gauge group. Moreover, the conservation of the $U(1)$ current corresponds to baryon number conservation whereas the conservation of the $U_{\mathrm{A}}(1)$ current is a problem (the so-called $U_{\mathrm{A}}(1)$ problem): it can be seen that the spontaneous breakdown of the $SU(N) \times SU(N)$ implies the same for the $U_{\mathrm{A}}(1)$ but there is no good candidate for the corresponding Goldstone boson in the particle spectrum.

It is clear from the preceding discussion that Green's functions involving chiral or axial currents are of direct physical interest. In the framework of the renormalization programme in QCD the finiteness of the matrix elements of

† We take quarks as massless. For massive quarks the chiral group is softly broken but, as we shall see, this is irrelevant for the discussion of anomalies.

the T-ordered products of such local composite operators as chiral currents is an additional requirement which, in general, must be separately verified. They may require an additional renormalization. Anomalies reflect the absence of a renormalization procedure such that Green's functions respect all the symmetries of the lagrangian. In other words an anomaly means a deviation of the renormalized theory from the canonical behaviour.

Historically, the main tool for studying anomalies is the Ward identities in perturbation theory. Relations obtained in this way between various renormalized Green's functions involving currents can then be summarized in operator language: certain currents have anomalous divergences. Ward identities, being a generalization of Gauss' theorem, also provide a beautiful manifestation of the UV–IR interrelation in the theory: short-distance singularities manifest themselves in low energy theorems. In this way anomalies can be seen experimentally. However, if gauge fields are coupled to linear combinations of vector and axial-vector flavour currents, as in the standard $SU(2) \times U(1)$ model of weak interactions, these combinations must be anomaly-free. This is an important constraint on any theory. Otherwise, the anomaly spoils the conservation of the currents coupled to gauge bosons and gauge theories with gauge fields coupled to non-conserved currents are inconsistent. In the next section we will see explicitly that the anomaly constitutes a breakdown of gauge invariance.

Anomalies originate in the short-distance singularities of the free fermion propagator which generate renormalization ambiguities in the Green's functions containing chiral currents. It turns out that in four space-time dimensions the Green's function which cannot be renormalized preserving all the classical symmetries of the lagrangian is the one containing two vector and one axial currents or three chiral currents. The basic structures to study are the triangle diagrams depicted in Fig. 13.1. As we know, in perturbation theory, the requirement of unitarity in each channel makes any one-loop amplitude well-defined up to a polynomial in the external momenta. This freedom is used in the renormalization procedure when we get rid of all infinite pieces. The triangle diagram is anomalous because it is impossible to add a polynomial so that all the classical symmetries are simultaneously preserved.

In the following we calculate explicitly the triangle amplitude and study its Ward identities. All internal quantum numbers of fermions are suppressed in this calculation. We consider the $U(1)$ and $U_{\mathrm{A}}(1)$ currents of one-fermion free-field theory. All the necessary generalizations can be easily introduced at the end.

To complete the introductory remarks it is worth mentioning that anomalies are also of basic importance for most theories going beyond the Standard Model. In particular the problem of chiral anomalies in more than four space-time dimensions and of gravitational anomalies as well as new techniques for their calculation

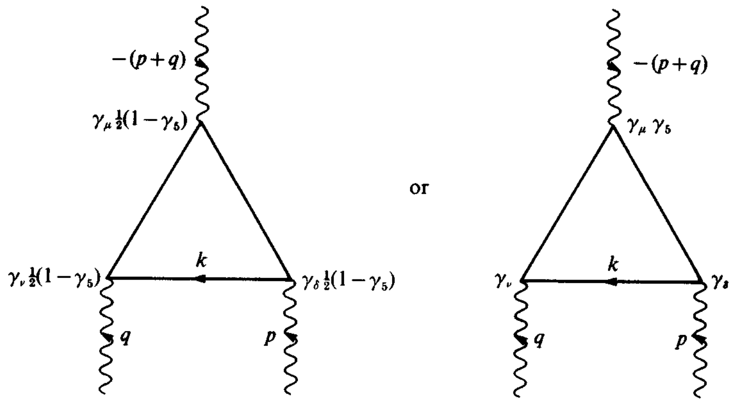


Fig. 13.1.

have been vigorously studied (for example, Frampton & Kephart (1983), Alvarez-Gaumé & Witten (1983), Zumino, Wu & Zee (1984)). We shall come back to some of these subjects later on.

Calculation of the triangle amplitude

We consider a vacuum matrix element of three chiral currents $j^{\mathrm{L}}_{\mu}(x)$, say, or of one axial vector current $j^5_{\mu}(x)$, and two vector currents $j_{\mu}(x)$. For the time being we treat them as abelian currents of the one-fermion free-field theory which, at the classical level, are conserved. If we allow for a Dirac mass term in the lagrangian then the axial-vector current is partially conserved

$$\partial^{\mu} j^5_{\mu}(x) = -2\partial^{\mu} j^{\mathrm{L}}_{\mu}(x) = 2m\mathrm{i}\bar{\Psi}\gamma_5\Psi$$

Using the techniques of Section 10.1 and the fact that the canonical equal-time commutators $[j_0(\mathbf{x},t), j_{\mu}(\mathbf{y},t)]$, $[j^5_0(\mathbf{x},t), j_{\mu}(\mathbf{y},t)]$ and $[j^5_0(\mathbf{x},t), j^5_{\mu}, (\mathbf{y},t)$ vanish one can derive the standard Ward identities for the matrix elements $\langle 0|Tj^{\mathrm{L}}_{\mu}(z)j^{\mathrm{L}}_{\nu}(y)j^{\mathrm{L}}_{\delta}(x)|0\rangle$ and $\langle 0|Tj^5_{\mu}(z)j_{\nu}(y)j_{\delta}(x)|0\rangle$. In momentum space, for example, they read

$$-(p+q)^{\mu}\Gamma^5_{\mu\nu\delta}(p,q) = p^{\delta}\Gamma^5_{\mu\nu\delta}(p,q) = q^{\nu}\Gamma^5_{\mu\nu\delta}(p,q) = 0 \tag{13.1}$$

for $m = 0$ or, for $m \neq 0$

$$-(p+q)^{\mu}\Gamma^5_{\mu\nu\delta}(p,q) = 2m\Gamma^5_{\nu\delta}(p,q) \tag{13.2}$$

and

$$p^{\delta}\Gamma^5_{\mu\nu\delta}(p,q) = q^{\nu}\Gamma^5_{\mu\nu\delta}(p,q) = 0 \tag{13.3}$$

where p and q are momenta conjugate to x and y, respectively, $\Gamma^5_{\mu\nu\delta}(p,q)$ is the Fourier transform of $\langle 0|Tj^5_\mu(z)j_\nu(y)j_\delta(x)|0\rangle$

$$\begin{aligned}\Gamma^5_{\mu\nu\delta}&(p,q)(2\pi)^4\delta(r+p+q)\\ &=\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\exp[-\mathrm{i}(px+qy+rz)]\langle 0|Tj^5_\mu(z)j_\nu(y)j_\delta(x)|0\rangle\end{aligned}\tag{13.4}$$

and $\Gamma^5_{\nu\delta}(p,q)$ is the Fourier transform of $\langle 0|TP(z)j_\nu(y)j_\delta(x)|0\rangle$ with $P=\bar\Psi\gamma_5\Psi$.

In a quantized theory, the current matrix elements get, at the one-loop level, a contribution from the triangle diagrams shown in Fig. 13.1 and from crossed diagrams with the external lines interchanged, $p\leftrightarrow q$ and $\delta\leftrightarrow\nu$. As we shall see, these contributions destroy the naive Ward identities (13.1)–(13.3).

We shall first consider the diagram on the r.h.s. of Fig. 13.1. The amplitude corresponding to this diagram reads

$$F_{\mu\nu\delta}(p,q)=(-1)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{\mathrm{i}}{\not k+\not q-m}\gamma_\nu\frac{\mathrm{i}}{\not k-m}\gamma_\delta\frac{\mathrm{i}}{\not k-\not p-m}\gamma_\mu\gamma_5\right]\tag{13.5}$$

and

$$\Gamma^5_{\mu\nu\delta}(p,q)=F_{\mu\nu\delta}(p,q)+F_{\mu\delta\nu}(q,p)\tag{13.6}$$

We take here a non-zero fermion mass but, as we shall see, the anomaly is mass-independent. Thus the result we will obtain is valid for the $m=0$ case, too. We also keep in mind that, if currents are coupled to gauge fields with some coupling constant g, an extra factor $(-\mathrm{i}g)$ appears in each such vertex.

The first observation worth making is the symmetry property

$$F_{\mu\nu\delta}(p,q)=F_{\mu\delta\nu}(q,p)\tag{13.7}$$

This follows from (13.5) by using the change of variables $k'=-k$, anticommuting $\gamma_\mu\gamma_5=-\gamma_5\gamma_\mu$ and reversing the order of factors

$$\mathrm{Tr}[\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\delta\gamma_\gamma\gamma_\mu\gamma_5]=\mathrm{Tr}[\gamma_\mu\gamma_\gamma\gamma_\delta\gamma_\beta\gamma_\nu\gamma_\alpha\gamma_5]\tag{13.8}$$

We also notice that the sign of the mass factor is irrelevant because the only non-vanishing mass-dependent terms in the trace are proportional to m^2.

By power counting the amplitude $F_{\mu\nu\delta}(p,q)$ is superficially linearly divergent. It is impossible to renormalize this amplitude preserving simultaneously all three identities (13.1) or (13.2) and (13.3). It depends on our choice of the renormalization constraints, dictated by the physics of the problem, where the anomaly is actually placed.

For clarity of the following discussion it is useful to first write down the most general form of the amplitude $\Gamma^5_{\mu\nu\delta}(p,q)=2F_{\mu\nu\delta}(p,q)$ consistent with the requirements of parity and Lorentz invariance. It reads

$$\begin{aligned}\Gamma^5_{\mu\nu\delta}(p,q) = {} & A_1(p,q)\varepsilon_{\mu\nu\delta\alpha}p^\alpha + A_2(p,q)\varepsilon_{\mu\nu\delta\alpha}q^\alpha + B_1(p,q)p_\nu\varepsilon_{\mu\delta\alpha\beta}p^\alpha q^\beta \\ & + B_2(p,q)q_\nu\varepsilon_{\mu\delta\alpha\beta}p^\alpha q^\beta + B_3(p,q)p_\delta\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \\ & + B_4(p,q)q_\delta\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \end{aligned} \tag{13.9}$$

where As and Bs are scalar functions. The terms $p_\nu\varepsilon_{\nu\delta\alpha\beta}p^\alpha q^\beta$ and $q_\mu\varepsilon_{\nu\delta\alpha\beta}p^\alpha q^\beta$ are not linearly independent. The requirement of Bose symmetry under interchange of the two vector currents $\Gamma^5_{\mu\nu\delta}(p,q) = \Gamma^5_{\mu\delta\nu}(q,p)$ implies that $A_1(p,q) = -A_2(q,p)$, $B_1(p,q) = -B_4(q,p)$, $B_2(p,q) = -B_3(q,p)$. Using again power counting arguments it is clear that only the functions A_1 and A_2 are divergent, in fact only logarithmically divergent, whereas the functions B_i are finite. Subtractions are, in principle, necessary for the functions A_1 and A_2. However, these functions can be expressed in terms of Bs when definite renormalization conditions are imposed. Imagine, for instance, that we insist on the vector current conservation

$$q^\nu\Gamma^5_{\mu\nu\delta}(p,q) = p^\delta\Gamma^5_{\mu\nu\delta}(p,q) = 0 \tag{13.10}$$

for the renormalized vertex $\Gamma^5_{\mu\nu\delta}$. Then we must have

$$\left.\begin{aligned} A_1 + p\cdot qB_1 + q^2B_2 = 0 \\ A_2 + p^2B_3 + p\cdot qB_4 = 0 \end{aligned}\right\} \tag{13.11}$$

and indeed the conditions (13.11) determine the divergent amplitudes A_i in terms of finite B_is. From Bose symmetry $A_1(p,q) = -A_2(q,p)$ follows so that only one of the equations (13.11) is independent.

There are many ways to calculate anomalies. One is to regularize the amplitude (13.5) by some regularization prescription and to calculate Ward identities for the regularized $\Gamma^5_{\mu\nu\delta}$. If the regularization procedure satisfies the constraints which we want to impose as our renormalization constraints then in the limit of no regulator we obtain the anomalous Ward identity we are looking for. Given this identity, by additional finite subtractions one can always transform it into another anomalous Ward identity, satisfying different renormalization constraints. Let us follow this approach choosing to work with the Pauli–Villars regularization (Itzykson & Zuber 1980) of the amplitude (13.5). Dimensional regularization in this case has some problems with ambiguities in the extension of γ_5 to n dimensions (for a review of this matter see, for instance, Ovrut (1983) and references therein). Let us choose as our renormalization conditions the relations (13.10). Thus we insist on vector current conservation for the renormalized vertex $\Gamma^5_{\mu\nu\delta}$

The Pauli–Villars regularization of the amplitude $\Gamma^5_{\mu\nu\delta}(p,q)$ consists of considering it to be a function of the mass of the fermion circulating in the loop:

$\Gamma^5_{\mu\nu\delta}(p,q) = \Gamma^5_{\mu\nu\delta}(p,q,m)$. A regulated amplitude is defined as the difference between the given amplitude and the same amplitude taken at some other value M of the mass (M is the regulator)

$$\Gamma^5_{\mu\nu\delta}(p,q,m,M) = \Gamma^5_{\mu\nu\delta}(p,q,m) - \Gamma^5_{\mu\nu\delta}(p,q,M) \tag{13.12}$$

The regularized amplitude $\Gamma^5_{\mu\nu\delta}(p,q,m,M)$ is finite. This can easily be checked by expanding the integrand in the integrals over d^4k in powers of $1/k^2$: the linear divergence of $\Gamma^5_{\mu\nu\delta}(p,q,m)$ is cancelled by $\Gamma^5_{\mu\nu\delta}(p,q,M)$. At the end one lets the mass M go to infinity and the final answer is finite with no additional subtractions, because it satisfies our renormalization conditions fixing A_i in terms of B_i

First we show that the regularized amplitude satisfies the 'normal' Ward identities

$$-(p+q)^\mu\Gamma^5_{\mu\nu\delta}(p,q,m,M) = 2m\Gamma^5_{\nu\delta}(p,q,m) - 2M\Gamma^5_{\nu\delta}(p,q,M) \tag{13.13}$$

$$p^\delta\Gamma^5_{\mu\nu\delta}(p,q,m,M) = q^\nu\Gamma^5_{\mu\nu\delta}(p,q,m,M) = 0 \tag{13.14}$$

Indeed, using the relation $\not p + \not q = (\not q + \not k - m) - (\not k + \not p - m)$ we can write down the following equation:

$$\begin{aligned}
-(p+q)^\mu&\Gamma^5_{\mu\nu\delta}(p,q,m,M)\\
&= 2\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not k+\not q-m}\gamma_\nu\frac{1}{\not k-m}\gamma_\delta\gamma_5 - (m\to M)\right]\\
&\quad + 2\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\nu\frac{1}{\not k-m}\gamma_\delta\frac{1}{\not k-\not p-m}\gamma_5 - (m\to M)\right]\\
&\quad + 2\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not k+\not q-m}\gamma_\nu\frac{1}{\not k-m}\gamma_\delta\frac{1}{\not k-\not p-m}2m\gamma_5 - (m\to M)\right]
\end{aligned} \tag{13.15}$$

All the integrals are finite. The first two terms are second rank pseudotensors depending on only one momentum variable and hence vanish. The last term gives just the Ward identity (13.13). Consider now $p^\delta\Gamma^5_{\mu\nu\delta}(p,q,m,M)$. Using $\not p = (\not k - m) - (\not k - \not p - m)$ this becomes

$$\begin{aligned}
p^\delta&\Gamma^5_{\mu\nu\delta}(p,q,m,M)\\
&= 2\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not k+\not q-m}\gamma_\nu\frac{1}{\not k-\not p-m}\gamma_\mu\gamma_5 - (m\to M)\right]\\
&\quad - 2\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{1}{\not k+\not q-m}\gamma_\nu\frac{1}{\not k-m}\gamma_\mu\gamma_5 - (m\to M)\right]
\end{aligned} \tag{13.16}$$

The last term vanishes for the same reason as above. The first term also vanishes because we can shift the momentum integration variable to $k' = k - p$ and again apply the same argument. This shift of the integration variable is legitimate because the integral is finite. If it were linearly divergent the translation of the integration variable might involve a non-vanishing surface term which would spoil the argument† (see Problem 13.1). In a quite analogous way we can show that

$$q^\nu \Gamma^5_{\mu\nu\delta}(p, q, m, M) = 0$$

Thus, our regularization procedure is consistent with our renormalization requirements (13.10). Our last step is to take the limit $M \to \infty$ in the Ward identity (13.13). The anomaly results from the non-vanishing of the regulator term in (13.13) in that limit. The $\Gamma^5_{\nu\delta}(p, q, m)$ vertex can be calculated explicitly since the Feynman integral is convergent:

$$\begin{aligned}\Gamma^{5\nu\delta}(p, q, m) &= 2\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \operatorname{Tr}\left[\frac{1}{\not k + \not q - m}\gamma^\nu \frac{1}{\not k - m}\gamma^\delta \frac{1}{\not k - \not p - m}\gamma_5\right] \\ &= 2 \times 4m\mathrm{i}\varepsilon^{\nu\delta\beta\alpha} p_\beta q_\alpha F(p, q, m) \qquad (13.17)\end{aligned}$$

where

$$F(p, q, m) = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k+q)^2 - m^2} \frac{1}{k^2 - m^2} \frac{1}{(p-k)^2 - m^2} \qquad (13.18)$$

and the factor 2 in front of the integral in (13.17) takes account of the crossed diagram.

Using the Feynman parametrization (B.16) and integrating over $\mathrm{d}^4 k$ one gets

$$F(p, q, m) = -\frac{1}{16\pi^2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, [q^2 y(1-y) + p^2(1-x)x + 2pqxy - m^2]^{-1} \qquad (13.19)$$

It is now straightforward to calculate the limit

$$\lim_{M\to\infty} 2M\Gamma^{5\nu\delta}(p, q, M) = +(\mathrm{i}/2\pi^2)\varepsilon^{\nu\delta\beta\alpha} p_\beta q_\alpha \qquad (13.20)$$

and the final form of the axial Ward identity consistent with our renormalization constraints reads

$$\left.\begin{aligned} &-(p+q)^\mu \Gamma^5_{\mu\nu\delta}(p, q, m) = 2m\Gamma^5_{\nu\delta}(p, q, m) - (\mathrm{i}/2\pi^2)\varepsilon_{\nu\delta\beta\alpha} p^\beta q^\alpha \\ &p^\delta \Gamma^5_{\mu\nu\delta} = q^\nu \Gamma^5_{\mu\nu\delta} = 0 \end{aligned}\right\} \qquad (13.21)$$

The anomalous term is m-independent.

Another way to obtain the anomalous Ward identity (13.21) is to calculate explicitly the finite functions B_i starting with the unregularized amplitude (13.5) and then to use renormalization constraints (13.11). This we leave as an exercise for the reader.

† Note the difference between a shift of the integration variable by a constant and a change of variables $k \to -k$ used in proving (13.7).

Different renormalization constraints for the triangle amplitude

The renormalization conditions (13.10) lead to the result (13.21). Whether we should impose them or not depends on the physics of the problem. This will be illustrated in the next section. It may happen that we must, instead, insist on the relation

$$-(p+q)^{\mu}\Gamma^5_{\mu\nu\delta}(p,q,m) = 2m\Gamma^5_{\nu\delta}(p,q,m) \tag{13.22}$$

as our renormalization constraint. The divergent amplitudes A_1 and A_2 are again uniquely specified by the condition (13.22). Indeed, with an additional finite subtraction for the A_1 and A_2

$$a_1\varepsilon_{\mu\nu\delta\alpha}p^{\alpha} + a_2\varepsilon_{\mu\nu\delta\alpha}q^{\alpha} \tag{13.23}$$

we see, by comparing (13.21) and (13.22), that the latter is satisfied for

$$a_1 - a_2 = \mathrm{i}/2\pi^2$$

Since $a_1(p,q) = -a_2(q,p)$ from the Bose symmetry under interchange of the two vector currents discussed earlier, we conclude that

$$a_1 = -a_2 = \mathrm{i}/4\pi^2 \tag{13.24}$$

Thus, with (13.22) as the renormalization condition one gets

$$\left.\begin{aligned} p^{\delta}\Gamma^5_{\mu\nu\delta}(p,q,m) &= -(\mathrm{i}/4\pi^2)\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}q^{\beta} \\ q^{\nu}\Gamma^5_{\mu\nu\delta}(p,q,m) &= (\mathrm{i}/4\pi^2)\varepsilon_{\mu\delta\alpha\beta}p^{\alpha}q^{\beta} \end{aligned}\right\} \tag{13.25}$$

One cannot satisfy simultaneously all three Ward identities (13.2) and (13.3).

Finally let us consider the triangle on the l.h.s. of Fig. 13.1 with the left-handed currents at each vertex and in the limit $m = 0$. At the level of the divergent Feynman integral the bare amplitude $\Gamma^{\mathrm{L}}_{\mu\nu\delta}(p,q)$ formally satisfies the following relation

$$\Gamma^{\mathrm{L}}_{\mu\nu\delta}(p,q) = \tfrac{1}{2}\Gamma_{\mu\nu\delta}(p,q) - \tfrac{1}{2}\Gamma^5_{\mu\nu\delta}(p,q) \tag{13.26}$$

where $\Gamma_{\mu\nu\delta}(p,q)$ is the three-vector current amplitude which has no anomaly. Thus formally the anomaly of $\Gamma^{\mathrm{L}}_{\mu\nu\delta}(p,q)$ can also be calculated from the triangle on the r.h.s. of Fig. 13.1. However, our previous renormalization conditions are inappropriate since now for the Green's function of three identical currents we must require Bose symmetry of the renormalized $\Gamma^5_{\mu\nu\delta}(p,q)$ (which we denote by $\tilde{\Gamma}^5_{\mu\nu\delta}(p,q)$ to distinguish it from the AVV amplitude considered earlier) under interchange of any pair of vertices. We can get the right answer by applying a finite renormalization to the previous result (13.21). Adding a finite polynomial term, (13.23), to the amplitudes A_1 and A_2 we get (for $m = 0$)

$$\left.\begin{aligned} -(p+q)^\mu \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= (-\mathrm{i}/2\pi^2 + a_1 - a_2)\varepsilon_{\nu\delta\alpha\beta}p^\alpha q^\beta \\ p^\delta \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= a_2 \varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \\ q^\nu \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= a_1 \varepsilon_{\mu\delta\alpha\beta}p^\alpha q^\beta \end{aligned}\right\} \tag{13.27}$$

Insisting on symmetry under the interchange $-(p+q) \leftrightarrow p, \mu \leftrightarrow \delta$ and $p \leftrightarrow q, \delta \leftrightarrow \nu$ one gets the following equations

$$\left.\begin{aligned} -\mathrm{i}/2\pi^2 + a_1 - a_2 &= a_2 \\ a_1 &= -a_2 \end{aligned}\right\} \tag{13.28}$$

Therefore

$$a_1 = -a_2 = \mathrm{i}/6\pi^2 \tag{13.29}$$

and

$$\left.\begin{aligned} -(p+q)^\mu \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= -(\mathrm{i}/6\pi^2)\varepsilon_{\nu\delta\alpha\beta}p^\alpha q^\beta \\ p^\delta \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= -(\mathrm{i}/6\pi^2)\varepsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \\ q^\nu \tilde{\Gamma}^5_{\mu\nu\delta}(p,q) &= +(\mathrm{i}/6\pi^2)\varepsilon_{\mu\delta\alpha\beta}p^\alpha q^\beta \end{aligned}\right\} \tag{13.30}$$

Thus, the same triangle amplitude leads to three different sets of anomalous Ward identities depending on the imposed renormalization conditions. The latter must be chosen according to the physical requirements of the problem under consideration. We recall that our present interest is in the amplitude $\Gamma^{\mathrm{L}}_{\mu\nu\delta}(p,q)$ so we combine (13.30) with (13.26).

Important comments

In the next section we will discuss in more detail the anomaly problem in the context of specific physical theories. Here, to illustrate the meaning of arbitrariness in renormalization conditions for the triangle diagram, we consider two models given by the lagrangians

$$\mathcal{L} = Z_2 \bar{\Psi} \mathrm{i} \not{D} \Psi - \tfrac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} \tag{13.31}$$

and

$$\mathcal{L} = Z_2^{\mathrm{R}} \bar{\Psi}_{\mathrm{R}} \mathrm{i} \not{\partial} \Psi_{\mathrm{R}} + Z_2^{\mathrm{L}} \bar{\Psi}_{\mathrm{L}} \mathrm{i} \not{D} \Psi_{\mathrm{L}} - \tfrac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} \tag{13.32}$$

respectively, where

$$D_\mu = \partial_\mu + \mathrm{i} g A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\otimes \equiv (\mathrm{i}/16\pi^2) g^2 \epsilon^{\mu\nu\rho\delta} F_{\mu\nu} F_{\rho\delta} \equiv (\mathrm{i}/16\pi^2) 2g^2 F_{\mu\nu} \tilde{F}^{\mu\nu}$$

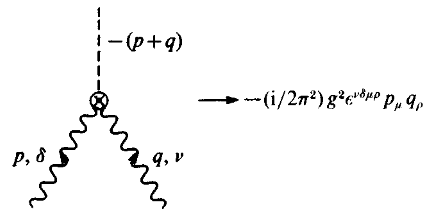


Fig. 13.2.

and the lagrangians are written in terms of renormalized quantities. We define the bare currents, for instance vector and axial-vector currents in the theory (13.31), which in terms of the renormalized fields are as follows:

$$j^5_\mu(x) = Z_2 \bar{\Psi}(x) \gamma_\mu \gamma_5 \Psi(x) \tag{13.33}$$

and

$$j_\mu(x) = Z_2 \bar{\Psi}(x) \gamma_\mu \Psi(x) \tag{13.34}$$

A comment is in order here on the renormalization of currents in presence of the axial anomaly. In Section 10.1 we have proved that a conserved or partially conserved current satisfying normal Ward identities is not renormalized. This conclusion is no longer valid in the presence of the anomaly. The current is still multiplicatively renormalized ${}^{\mathrm{R}}j^5_\mu = Z_5 j^5_\mu$, but Z_5 is not finite (we will come back to this point shortly). However, as we know from the previous subsection, at the one-loop level the current matrix elements are finite after imposing certain renormalization conditions, without additional counterterms. Thus, at this level $Z_5 = 1$.

The first theory is just QED of a single fermion field. The vector current is coupled to the electromagnetic field and the axial-vector current, conserved in the Noether sense, is not coupled to any gauge field. Gauge invariance of the theory requires the vector current conservation. Thus for the triangle diagram involving j^5_μ and two vector currents we must impose the renormalization conditions (13.10) which lead to the anomalous Ward identity (13.21). In operator language this corresponds to an anomaly in the divergence of the axial-vector current

$$\partial^\mu j^5_\mu(x) = (g^2/16\pi^2) \varepsilon^{\mu\nu\rho\delta} F_{\mu\nu}(x) F_{\rho\delta}(x) \tag{13.35}$$

Eq. (13.35) can be verified using the Feynman rules for the operator insertion

$$\varepsilon^{\mu\nu\rho\delta} F_{\mu\nu}(x) F_{\rho\delta}(x) = 4\varepsilon^{\mu\nu\rho\delta} \partial_\mu A_\nu(x) \partial_\rho A_\delta(x) \tag{13.36}$$

(see Section 7.6). Considering, for example, the Green's function $\langle 0|T \partial_\mu A_\nu(z)$

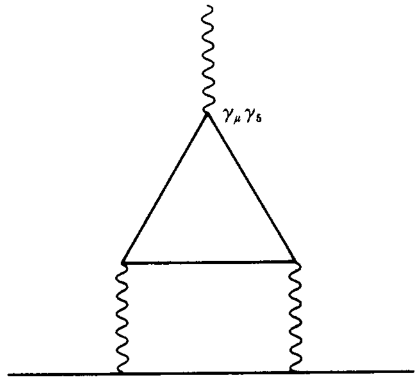


Fig. 13.3.

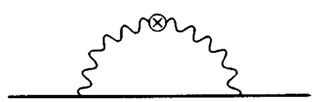


Fig. 13.4.

$\partial_\rho A_\delta(z) A_\alpha(y) A_\beta(x)|0\rangle$ in momentum space one gets the Feynman diagram shown in Fig. 13.2 (propagators are, as usual, removed from the external lines). Therefore, in order g^2 the Fourier transform (13.21)† (with $m = 0$) of the divergence $\partial^\mu \langle 0|T j^5_\mu j_\nu j_\delta|0\rangle$ is given by the Feynman diagram in Fig. 13.2 and (13.35) follows if we recall the general form (10.11) of Ward identities. Using (13.35) we can calculate perturbatively other anomalous Ward identities involving axial-vector current $j^5_\mu(x)$. For instance, the derivative of the Green's function $\Gamma^{5\mu}_{\bar{\Psi}\Psi} = \langle 0|T\Psi(x) j^{5\mu}(0)\bar{\Psi}(y)|0\rangle$ is anomalous because the latter gets a contribution from the diagram in Fig. 13.3 and in the lowest non-vanishing order in g^2 the anomaly is given by the diagram in Fig. 13.4. The anomalous Ward identity for $\Gamma^5_{\bar{\Psi}\Psi}$ (derive it using the path integral technique of Sections 10.2 and 13.3) can be used to study the renormalization properties of the axial-vector current. Closely following the procedure of Section 10.1 we conclude that Z_5 is not finite as the operator $\tilde{F}_{\mu\nu} F^{\mu\nu}$ does not have finite matrix elements: e.g. the diagram in Fig. 13.4 is logarithmically divergent.

Our second example, the lagrangian (13.32), is a theory in which only the left-handed chiral current couples to the gauge fields. The triangle diagram with gauge source currents at each vertex should be evaluated imposing symmetry under interchange of any pair of indices. Using relations (13.26) and (13.30) we now get

† Now with $(-ig)$ at each vector vertex included.

the anomaly in the chiral current

$$\partial^\mu j_\mu^{\mathrm{L}}(x) = -(g^2/48\pi^2) F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{13.37}$$

Because of the anomaly the gauge source current is no longer conserved and the theory is not gauge invariant.

Writing the operator equation (13.35) for the bare current we have used the result obtained in the lowest order in g^2 in perturbation theory. Equivalently one can say that the gauge field has been treated as a classical background field. One believes that the anomalies are not modified by higher order corrections, i.e. (13.35) is exact to any order in g^2 for the bare current in the regularized theory or for the renormalized operators (see also the next section) with g and $F_{\mu\nu}$ being the renormalized quantities defined by the lagrangian (13.31). For the vector abelian theory considered here this has been proved to all orders in perturbation theory (Adler & Bardeen 1969). The proof is technical but its main point can be summarized as follows: the necessity of renormalization of the higher order corrections in g^2, i.e. of photon insertions in the triangle diagram, does not interfere with chiral symmetry, so the chiral anomaly should arise only from the lowest order fermion loops. (On the contrary, the scale invariance anomaly discussed in Chapter 7 does obtain corrections because renormalization violates scale invariance.) The Adler–Bardeen theorem also holds in non-abelian gauge theory but it may be regularization-scheme-dependent (Bardeen 1972), particularly in supersymmetric theories. In gauge theories with chiral gauge symmetry the anomalies must be absent for a consistent theory. According to the Adler–Bardeen theorem generalized to the non-abelian case a theory is completely free of anomalies if, and only if, all the triangle diagram anomalies are absent.

Our last remark in this section is concerned with generalizing the triangle diagram calculation to the case in which several species of fermion fields which have some internal degrees of freedom are present. Let the currents in the vertices have coupling matrices $\lambda_1^a, \lambda_2^b, \lambda_3^c$. The currents may be coupled to gauge fields or not and the matrices $\lambda_1^a, \lambda_2^b, \lambda_3^c$ may transform non-trivially under different internal symmetry groups and be block-diagonal in the remaining internal symmetry spaces. For the triangle diagram we get then an extra factor $\frac{1}{2}\,\mathrm{Tr}[\lambda_1^a\{\lambda_2^b, \lambda_3^c\}]$, where the trace corresponds to summing over different fermions in the loop and the anticommutator is due to the symmetrization over the two diagrams in Fig. 13.5 The necessary modifications of operator equations, like (13.35), will be discussed later.

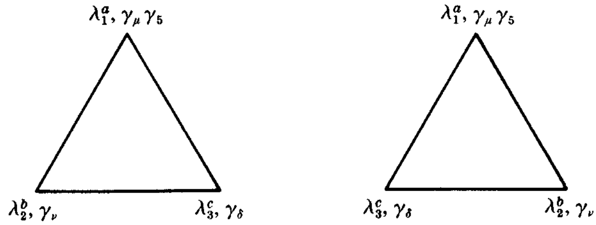


Fig. 13.5.

13.2 Some physical consequences of the chiral anomalies

Chiral invariance in spinor electrodynamics

The lagrangian (13.31) of spinor electrodynamics is invariant both under $U(1)$ and axial $U_A(1)$ transformations. However, in quantum theory if we want to preserve vector gauge invariance an anomaly appears in the divergence of the axial-vector current. If we introduce a cut-off which preserves photon gauge invariance then to any order in g^2 the divergence of the unrenormalized axial-vector current is given by (13.35). As discussed in the previous section, the triangle diagram amplitude turns out to be finite even in the limit of no cut-off but the higher order corrections induce infinities in the axial-current matrix elements. Not all of these can be absorbed in the renormalization constants present in the lagrangian (13.31). Therefore due to the anomaly the axial-vector current is renormalized:

$$ {}^{\mathrm{R}}j_\mu^5 = Z_5 j_\mu^5 $$

and we must re-express (13.35) in terms of the renormalized operators. Defining

$$ (g^2/8\pi^2)[F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)]^{\mathrm{R}} = (g^2/8\pi^2)F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x) + (Z_5 - 1)\partial^\mu j_\mu^5(x) \tag{13.38} $$

we may write (13.35) in terms of the renormalized operators

$$ \partial^\mu\, {}^{\mathrm{R}}j_\mu^5(x) = (g^2/8\pi^2)[F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)]^{\mathrm{R}} \tag{13.39} $$

However, this renormalized axial-vector current does not generate chiral transformations. The axial charge is not conserved, it does not commute with the photon field and does not have the correct equal-time commutation relations with the renormalized spinor field, due to the presence of a dimension four operator $(F_{\mu\nu}\tilde{F}^{\mu\nu})^{\mathrm{R}}$ in the divergence of the current (see Section 10.1).

Nevertheless, it turns out that chiral invariance remains a symmetry of the quantum theory based on the lagrangian (13.31) and it is possible to construct the corresponding charge Q^5. Following Adler (1970) we first define a new bare current: symmetry current

$$j^{\mathrm{S}}_{5\mu}(x) = j^5_\mu(x) - (g^2/4\pi^2)A^\nu(x)\tilde{F}_{\mu\nu}(x) \tag{13.40}$$

which, from (13.35), is conserved

$$\partial^\mu j^{\mathrm{S}}_{5\mu} = 0 \tag{13.41}$$

Its existence is directly related to our discussion of different renormalization conditions of the triangle diagram; compare (13.21) and (13.22). This current is a finite operator (Bardeen 1974) and does not need renormalization. The current $j^{\mathrm{S}}_{5\mu}$ is conserved but it is explicitly gauge-dependent due to the presence of the gauge field in its definition and therefore is not an observable operator. But the associate charge

$$Q_5(t) = \int \mathrm{d}^3x\, j^{\mathrm{S}}_{50}(\mathbf{x}, t) \tag{13.42}$$

is from (13.41) time-independent, its commutator with $\Psi(x)$, calculated by use of the canonical commutation relations, is

$$[Q_5(t), \Psi(\mathbf{x}, t)] = -\gamma_5\Psi(x) \tag{13.43}$$

it commutes with the photon field and it is gauge-invariant. Let us check the last property. Under the gauge transformation

$$\left.\begin{aligned} \delta\Psi(x) &= -\mathrm{i}g\Theta(x)\Psi(x) \\ \delta A_\mu(x) &= \partial_\mu\Theta(x) \end{aligned}\right\} \tag{13.44}$$

the symmetry current transforms as follows:

$$\delta j^{\mathrm{S}}_{5\mu} = (g^2/4\pi^2)\partial^\nu\Theta(x)\tilde{F}_{\mu\nu}(x) = (g^2/4\pi^2)\partial^\nu(\Theta(x)\tilde{F}_{\mu\nu}(x)) \tag{13.45}$$

Therefore

$$\delta Q_5(t) = \int \mathrm{d}^3x\, \delta j^{\mathrm{S}}_{50}(x) \sim \int \mathrm{d}^3x\, \partial^\nu(\Theta(x)\tilde{F}_{0\nu}(x)) = \int \mathrm{d}^3x\, \partial^k(\Theta(x)\tilde{F}_{0k}(x)) = 0 \tag{13.46}$$

Gauge invariance of Q_5 in spinor electrodynamics follows from the vanishing of $\Theta(x)\tilde{F}_{0k}(x)$ at spatial infinity. The situation is different in non-abelian gauge theories (see Section 8.3) where not all finite gauge transformations can be reached by iterating infinitesimal transformations. We conclude that in spite of the presence of the axial anomaly chiral symmetry remains a good symmetry in massless spinor electrodynamics.

$\pi^0 \to 2\gamma$

The relevance of the chiral anomaly to the decay $\pi^0 \to 2\gamma$ relies on our identification of pions as the Goldstone bosons of the spontaneously broken chiral symmetry of QCD (see Chapter 9). Thus, the axial-vector flavour currents of the $SU(2) \times SU(2)$ symmetric QCD lagrangian (9.1) acquire dynamical sense through their coupling to pions, see (9.65)

$$\langle 0|j_\mu^{5a}(x)|\pi^b\rangle = \mathrm{i} f_\pi p_\mu \delta^{ab} \exp(-\mathrm{i}px)$$

where p_μ is the pion four-momentum. To calculate the $\pi^0 \to 2\gamma$ decay rate we consider the amplitude ($j_\mu^{5a}(x)$ is the renormalized axial isospin current)

$$\begin{aligned}\int \mathrm{d}^4x \exp(-\mathrm{i}qx)\langle\gamma_1(k_1)\gamma_2(k_2)|TSj_\mu^{53}(x)|0\rangle &= \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z \exp(+\mathrm{i}k_1y) \\ \times \exp(+\mathrm{i}k_2z)\exp(-\mathrm{i}qx)\langle 0|Tj_\mu^{53}(x)j_\nu(y)j_\delta(z)|0\rangle &\varepsilon_1^\nu(k_1)\varepsilon_2^\delta(k_2) \qquad (13.47)\end{aligned}$$

In (13.47) we have used (2.188), (2.68) and the equations of motion for the gauge fields $A_\mu(x)$. Next, we use the PCAC (partial conservation of the axial-vector current) approximation relating physical quantities to the results obtained in the exact symmetry limit $p^2 = m_\pi^2 = 0$. In this limit there is a pion pole contribution to the l.h.s. of (13.47) (see, for example, (2.194), (2.195) and (2.196))

$$\begin{aligned}\lim_{q^2\to p^2}\int \mathrm{d}^4x \exp(-\mathrm{i}qx)\langle\gamma_1(k_1)\gamma_2(k_2)|TSj_\mu^{53}(x)|0\rangle &= f_\pi(p_\mu/p^2)\langle\gamma_1\gamma_2|S|\pi^0\rangle \\ + O(k_1,k_2) & \qquad (13.48)\end{aligned}$$

where $O(k_1, k_2)$ has no pole in p^2.

We now take the divergence of (13.47). The divergence of the r.h.s. of (13.47) is given by the results of the previous section with the triangle diagram specified as shown in Fig. 13.6. The fermion lines are quark lines, the Pauli matrix σ^3 is the neutral axial-vector coupling matrix and Q is the diagonal quark electric charge matrix. In this case, for $\Gamma^5_{\mu\nu\delta}$ defined by (13.4) we get the following:

$$p^\mu\Gamma^5_{\mu\nu\delta}(k_1,k_2) = (\mathrm{i}/2\pi^2)\,\mathrm{Tr}[\tfrac{1}{2}\sigma^3Q^2]\varepsilon_{\nu\delta\beta\alpha}k_1^\beta k_2^\alpha \qquad (13.49)$$

plus terms of at least third order in k_i from the second diagram: the factor k_1k_2 comes from the gauge-invariant coupling (through the field strength tensor $F_{\mu\nu}$) of external photons and an extra factor (k_1+k_2) comes from the anomalous divergence term. The effect of the last diagram is only the coupling-constant renormalization. With u and d quarks of three colours we get

$$\mathrm{Tr}[\tfrac{1}{2}\sigma^3Q^2] = 3\times\tfrac{1}{2}(\tfrac{4}{9}-\tfrac{1}{9})e^2 = \tfrac{1}{2}e^2 \qquad (13.50)$$

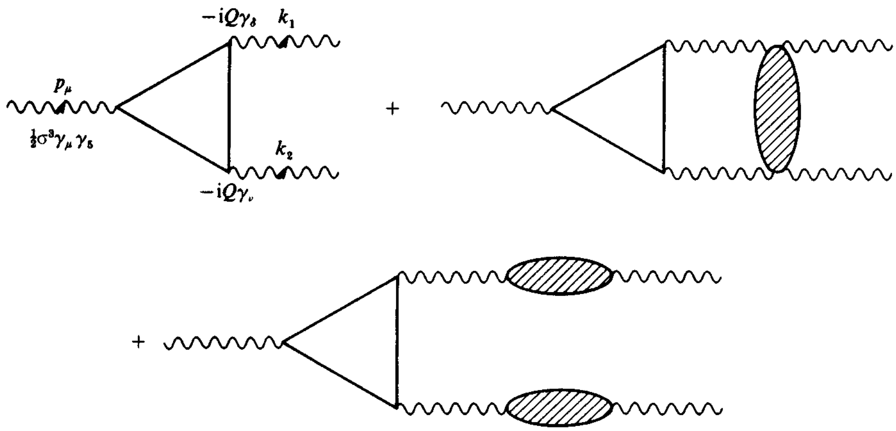


Fig. 13.6.

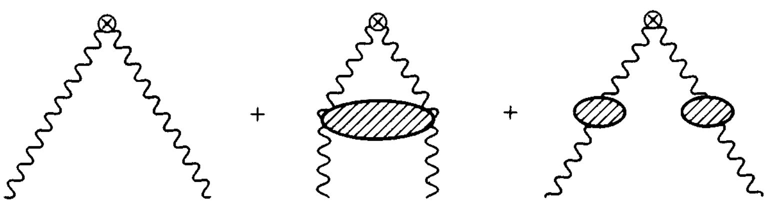


Fig. 13.7.

Therefore, from (13.47) and (13.48)

$$\langle\gamma_1\gamma_2|S|\pi^0\rangle = \frac{1}{f_\pi}\frac{e^2}{4\pi^2}\mathrm{i}\varepsilon_{\nu\delta\beta\alpha}k_1^\beta k_2^\alpha\varepsilon_1^\nu\varepsilon_2^\delta(2\pi)^4\partial(p-k_1-k_2)+O(k_1,k_2,p) \tag{13.51}$$

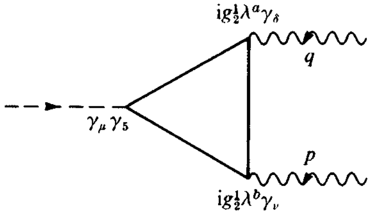
Our calculation of the divergence of (13.47) is, of course, equivalent to calculating perturbatively the matrix element $\langle\gamma_1\gamma_2|TS\partial^\mu j_\mu^{5a}(x)|0\rangle$ with the divergence given by

$$\partial^\mu j_\mu^{5a}(x) = (e^2/8\pi^2)\tfrac{1}{2}\,\mathrm{Tr}[\tfrac{1}{2}\sigma^a\{Q,Q\}]F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{13.52}$$

The contributing diagrams are those shown in Fig. 13.7. Terms $O(k_1,k_2)$ in (13.48) and $O(k_1k_2,p)$ in (13.51) are at least second and third order in momenta, respectively.

From Lorentz and parity invariance

$$\langle\gamma_1\gamma_2|S|\pi^0\rangle = f(p^2)\mathrm{i}\varepsilon_{\nu\delta\beta\alpha}k_1^\beta k_2^\alpha\varepsilon_1^\nu\varepsilon_2^\delta(2\pi)^4\delta(p-k_1-k_2) \tag{13.53}$$



$\lambda^a \equiv$ colour $SU(3)$ Gell–Mann matrices

Fig. 13.8.

Comparing (13.51) and the last equation we conclude that

$$f(0) = \frac{1}{f_\pi}\left(\frac{e^2}{4\pi^2}\right) = \frac{\alpha}{f_\pi \pi} \tag{13.54}$$

and the rate for $\pi^0 \to 2\gamma$ is

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}\sum_{\varepsilon_1,\varepsilon_2}\int \frac{\mathrm{d}^3k_1\,\mathrm{d}^3k_2}{2\pi^6 4k_{10}k_{20}}|f(0)\varepsilon_{\nu\delta\beta\alpha}k_1^\beta k_2^\alpha \varepsilon_1^\nu \varepsilon_2^\delta|^2(2\pi)^4\delta(p-k_1-k_2)\\ &= \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} = 7.63\ \mathrm{eV}\end{aligned} \tag{13.55}$$

(Bose statistics implies integration over a 2π solid angle only). The experimental value is $\Gamma^{\mathrm{Exp}} = (7.7 \pm 0.5\ \mathrm{eV})$. We make two important observations. The result (13.55) is entirely given by the axial anomaly. In the absence of the anomaly one would get $f(0) = 0$ and PCAC could not be reconciled with the $\pi^0 \to 2\gamma$ decay rate. Secondly, the agreement with the experimental value relies on the existence of three colour states of quarks.

Chiral anomaly for the axial $U(1)$ current in QCD; $U_\mathrm{A}(1)$ problem

Global symmetries of the QCD lagrangian with massless quarks are (see Section 9.1): chiral $SU(n) \times SU(n)$ invariance and invariance under abelian vector $U(1)$ and axial-vector $U_\mathrm{A}(1)$ transformation. We study now the anomaly of the $U_\mathrm{A}(1)$ current. The $U_\mathrm{A}(1)$ current is a singlet under the colour group and the relevant triangle diagram is shown in Fig. 13.8. We must have $D^\mu j_\mu^a = 0$ to maintain gauge invariance. A derivation of the equation $D^\mu j_\mu^a = 0$ in a *quantized* gauge-invariant theory is given in the next section. For the triangle diagram we effectively get the condition $\partial^\nu\langle 0|Tj_\mu^5 j_\nu^a j_\delta^b|0\rangle = 0$ and therefore we must impose the renormalization constraints (13.10). Thus, by analogy with (13.21), the divergence $-(p+q)\Gamma^5_{\mu\nu\delta}(p,q)$ of the Fourier transform of the matrix element

$\langle 0|Tj^5_\mu j^a_\nu j^b_\delta|0\rangle$ reads

$$-(p+q)^\mu \Gamma^5_{\mu\nu\delta}(p,q) = (\mathrm{i}g^2/2\pi^2)n\,\mathrm{Tr}[\tfrac{1}{2}\lambda^a\tfrac{1}{2}\lambda^b]\varepsilon_{\nu\delta\beta\alpha}p^\alpha q^\beta \tag{13.56}$$

where n is the number of quark flavours. The operator equation (13.35) can also be easily generalized to the present case if we remember that $U_\mathrm{A}(1)$ current is a colour singlet. The only colour-invariant operator which for the triangle diagram gives the result (13.56) is

$$\partial^\mu j^5_\mu(x) = -(n/8\pi^2)\,\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] \tag{13.57}$$

where Tr is in the internal quantum number space and $G_{\mu\nu} = \mathrm{i}gG^a_{\mu\nu}T^a$, $T^a = \frac{1}{2}\lambda^a$. A new feature of (13.57) is that, due to the self-coupling of the non-abelian gauge fields, see (1.111), the anomalous divergence of the $U_\mathrm{A}(1)$ current involves not only terms quadratic in gauge fields but also terms of the third order in gauge fields

$$\begin{aligned}\partial^\mu j^5_\mu(x) &= -(n/16\pi^2)\,\mathrm{Tr}[4\partial_\mu A_\nu\partial_\rho A_\sigma + 4\partial_\mu A_\nu[A_\rho, A_\sigma]]\varepsilon^{\mu\nu\rho\sigma}\\ &= -(n/4\pi^2)\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\,\mathrm{Tr}[A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma]\end{aligned} \tag{13.58}$$

Thus, square diagrams also contribute to the anomaly for the divergence of the axial current. However, this contribution is not independent of the triangle anomaly: given the latter, it is determined by the gauge invariance of the axial-vector current. All the anomalous Ward identities follow from the operator equation (13.57). For an explicit discussion of the square diagrams in the spirit of the previous section see, for instance, Aviv & Zee (1972).

The anomaly (13.57) of the axial-vector current is crucial for resolving the so-called $U_\mathrm{A}(1)$ problem in QCD. As we know from Chapter 9 the massless QCD lagrangian has the chiral $(U(1) \mp U_\mathrm{A}(1))$ symmetry. As with the $SU(n)\times SU(n)$ it does not seem to be the symmetry of the hadron spectra so $U_\mathrm{A}(1)$ must be broken ($U(1)$ manifests itself in baryon number conservation). However, unlike the case of the spontaneously broken $SU(n)\times SU(n)$, there is no good candidate for the $U_\mathrm{A}(1)$ Goldstone boson in the particle spectrum. The resolution of the problem is in the existence of the anomaly (13.57) and of field configurations with a non-zero topological charge (see Section 8.3). These properties offer a mechanism of breaking of the $U_\mathrm{A}(1)$ which does not imply the existence of the massless Goldstone boson coupled to the gauge-invariant $U_\mathrm{A}(1)$ current.

As in the first subsection of this section we observe that the very existence of the anomaly does not destroy the $U_\mathrm{A}(1)$ symmetry since the conserved current

$$j^\mathrm{S}_{5\mu} = j_{5\mu} - 2nK_\mu, \quad \partial^\mu K_\mu = -(1/16\pi^2)\,\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}]$$

and the conserved charge

$$Q_5 = \int \mathrm{d}^3x\, j^\mathrm{S}_{50}$$

can be defined. Now, however, Q_5 is not gauge-invariant: it is not gauge-invariant under large gauge transformations $\mathcal{G}_N$ with the winding number N. We have (see Section 8.3 and in particular (8.65) and (8.67) and Problem 8.2)

$$\mathcal{G}_N Q_5 \mathcal{G}_N^{-1} = Q_5 - 2nN$$

From Section 8.3 we also remember that

$$\mathcal{G}_N|\Theta\rangle = \exp(-\mathrm{i}N\Theta)|\Theta\rangle \tag{8.58}$$

Thus we get

$$\mathcal{G}_N \exp(\mathrm{i}\Theta' Q_5)|\Theta\rangle = \exp[-\mathrm{i}N(\Theta + 2n\Theta')] \exp(\mathrm{i}\Theta' Q_5)|\Theta\rangle$$

or

$$\exp(\mathrm{i}\Theta' Q_5)|\Theta\rangle = |\Theta + 2n\Theta'\rangle$$

The conclusion is that the $U_\mathrm{A}(1)$ rotation by an angle Θ' changes the $|\Theta\rangle$ vacuum into the $|\Theta + 2n\Theta'\rangle$ vacuum. $U_\mathrm{A}(1)$ symmetry is spontaneously broken because the vacuum is not $U_\mathrm{A}(1)$-invariant. Different Θ-vacua are degenerate states since for massless fermions H commutes with Q_5. However, it can be seen from the chiral Ward identities that the spontaneous breakdown occurring because of the axial-vector anomaly and the existence of field configurations with non-zero topological charge (hence the set of the Θ-vacua) is consistent with the assumption of no Goldstone boson coupled to the gauge-invariant current j^5_μ.

Anomaly cancellation in the $SU(2) \times U(1)$ *electroweak theory*

We discuss this problem at the level of the triangle diagrams relegating to the next section the construction of the full anomalous divergence of a non-abelian current coupled to gauge fields. This does not limit the generality of our discussion: a theory is anomaly-free if, and only if, the triangle diagram anomalies are absent. For consistency of a quantum gauge theory the gauge source currents must be covariantly conserved, i.e. anomaly-free (Gross & Jackiw 1972; see also Jackiw 1984 and our discussion in the next section).

In the electroweak theory the $SU(2)$ gauge bosons couple only to the left-handed currents and the $U(1)$ gauge boson couples to both the left- and right-handed currents but with different $U(1)$ charges Y. It will also be convenient to remember that by construction the fermion electric charge operator is given by $Q = T^3 + Y$. Thus, $Y_\mathrm{R} = Q$ and $Y_\mathrm{L} = (Q - T^3)$. We can now systematically list possible anomalous triangle diagram contributions to the divergences of the gauge source currents. Take first the $SU(2)$ current $j^a_{\mathrm{L}\mu}$: we can have the triangle diagrams shown in Fig. 13.9 with the $j^a_{\mathrm{L}\mu}$ in one of the vertices and the corresponding

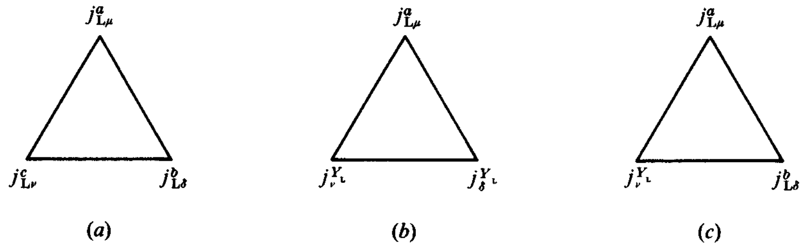


Fig. 13.9.

symmetrized diagrams ($j^{Y_L}_\mu$ is the left-handed part of the $U(1)$ current). It is clear from our discussion in the previous section that the eventual anomalous contribution of these diagrams after summing over all fermions in the loops is proportional to

$\frac{1}{2}\,\mathrm{Tr}[T^a\{T^b, T^c\}]$ for Fig. 13.9(a)
$\frac{1}{2}\,\mathrm{Tr}[T^a\{Y, Y\}]$ for Fig. 13.9(b)
$\frac{1}{2}\,\mathrm{Tr}[T^a\{Y, T^b\}]$ for Fig. 13.9(c)

($T^a = \sigma^a$ are $SU(2)$ generators). We immediately see that Fig. 13.9(b) does not contribute because $[Y, T^a] = 0$ and $\mathrm{Tr}\,T^a = 0$. If we express Y in terms of Q the remaining contributions are

$$\frac{1}{2}\,\mathrm{Tr}[T^a\{Q, T^b\}] = \frac{1}{2}\,\mathrm{Tr}[Q\{T^b, T^a\}] = \delta^{ab}\,\mathrm{Tr}[Q(T^a)^2] = \frac{1}{2}\delta^{ab}\sum_i Q_i \quad (13.59)$$

where the sum is taken over the charges of all fermions in the loop, and

$$\mathrm{Tr}[T^a\{T^b, T^c\}] = 0$$

for the group $SU(2)$.

The anomalous contribution to the divergence of the $U(1)$ current $j^Y_\mu = \sum_i(\bar{\Psi}^i_L\gamma_\mu Y\Psi^i_L + \bar{\Psi}^i_R\gamma_\mu Y\Psi^i_R)$, where the sum is over all fermions in the theory, can be studied in a similar way. We recall that the $U(1)$ coupling is not vector-like because of different Y quantum number assignments to the left- and right-handed fields. In this case all possible triangle diagrams are proportional to one of the following factors:

$$\frac{1}{2}\,\mathrm{Tr}[Y\{Y, T^a\}], \quad \frac{1}{2}\,\mathrm{Tr}[Y\{T^a, T^b\}] \quad \text{and} \quad \frac{1}{2}\,\mathrm{Tr}[Y\{Y, Y\}]$$

The first two traces have already been calculated. The last one appears in two types of loop, with left- and right-handed fermions, which contribute with opposite signs because of the $(1 \pm \gamma_5)$ factors (see (13.26)). We calculate

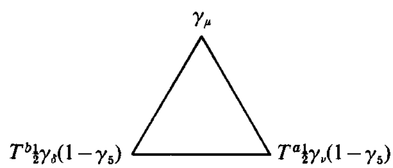
$$\frac{1}{2}\,\mathrm{Tr}[Y\{Y, Y\}] = \mathrm{Tr}\,Y^3 = \mathrm{Tr}(Q - T_3)^3 = \mathrm{Tr}[Q^3 - 3Q^2T_3 + 3Q(T_3)^2 - (T_3)^3] \quad (13.60)$$

Since Q is the same for fermions with either of the chiralities, the Q^3 term is cancelled separately for each fermion when the sum over the two types of loop is taken. The remaining terms give again

$$\sum_i Q_i = 0 \tag{13.61}$$

since $\mathrm{Tr}[Q^2T^3] = \mathrm{Tr}[(T_3+Y)^2T_3] = \mathrm{Tr}[(T_3)^3+2Y(T_3)^2+Y^2T_3] = \mathrm{Tr}\,Y = \mathrm{Tr}\,Q$. Thus (13.61) is the only condition for the anomaly cancellation. We conclude that the $SU(2)\times U(1)$ theory is anomaly-free if the sum of charges of all fermions in the theory vanishes. The quarks and leptons discovered so far strikingly satisfy this condition: to each $SU(2)$ doublet of leptons corresponds a doublet of quarks, fractionally charged but coming in three colours.

An interesting point to mention in the context of the $SU(2)\times U(1)$ theory is that once we insist on conservation of the chiral source currents for the electroweak gauge fields (one needs suitable sets of fermion multiplets to ensure this property) the baryon number current, $U(1)$ vector current, acquires an anomaly. This is clear if we consider the triangle diagram with the baryon current and two gauge source currents, for instance,



Insisting on chiral gauge current conservation we are formally back to the renormalization conditions (13.10) and the anomaly is placed in the vector current. Notice that the condition of vanishing divergence of two currents can always be imposed on triangle diagrams. Restrictions on the fermion representations appear when we consider triangle diagrams with three gauge currents.

Since $U(1)$ current is an $SU(2)$ singlet, in the operator notation we have, analogously to (13.35),

$$\partial^\mu j_\mu = -(1/16\pi^2)\,\mathrm{Tr}[\varepsilon^{\mu\nu\rho\sigma}G_{\mu\nu}G_{\rho\sigma}] \tag{13.62}$$

where $G_{\mu\nu} = \mathrm{i}gG^a_{\mu\nu}T^a$ is the $SU(2)$ field strength tensor. An interesting possibility is that the vacuum expectation value of the r.h.s. of (13.62) does not vanish due to some non-perturbative effects like tunnelling, with instantons as the dominant field configurations, or monopoles. One would then expect 'topological' baryon number non-conservation.

Anomaly-free models

We conclude with more general remarks. As we know the triangle diagram anomaly is given by a multiple of

$$D_{abc} = \frac{1}{2}\,\mathrm{Tr}[T^a\{T^b, T^c\}] \tag{13.63}$$

where T^a, T^b and T^c are generators of the group of transformations of currents attached to the three vertices. We take all three currents to be gauge source currents. One can distinguish three cases of anomaly-free models (Georgi & Glashow 1972). The right-handed fermion loop and the left-handed fermion loop anomalies may cancel. This happens when fermions with either of the chiralities transform according to the same representation of the gauge group, i.e. the theory is vector-like (Appendix E). If this is not the case, for an anomaly-free theory the quantity D_{abc} must vanish. Two cases may be distinguished here. It may be that, for all representations of the group, $D_{abc} = 0$. These are called 'safe' groups. They include $SU(2)$, all orthogonal groups except $SO(6) \approx SU(4)$ and all symplectic groups. The other case is when the group is not safe, like $SU(N)$ with $N > 2$ or $SU(2) \times U(1)$, but there exist some 'safe' representations for which D_{abc} vanishes. This then gives a limitation on the allowed fermion representations.

13.3 Anomalies and the path integral

Introduction

For a systematic discussion of chiral anomalies it is convenient to use the generating functional and the path integral technique. We consider a theory of massless fermions coupled to non-abelian gauge fields. For the purpose of the present discussion it is convenient to treat the gauge fields as classical background fields. Depending on the physical problem at hand we may need to use vector and axial-vector couplings or left- and right-handed couplings. In the first case the gauge-invariant lagrangian density reads†

$$\mathcal{L} = \bar{\Psi}\mathrm{i}\gamma^\mu(\partial_\mu + A_\mu^{\mathrm{V}} + \gamma_5 A_\mu^{\mathrm{A}})\Psi = \bar{\Psi}\mathrm{i}\gamma^\mu\Psi - j_{\mathrm{V}}^{\alpha\mu} A_{\alpha\mu}^{\mathrm{V}} - j_{\mathrm{A}}^{\alpha\mu} A_{\alpha\mu}^{\mathrm{A}} \tag{13.64}$$

where

$$A_\mu^{\mathrm{V,A}} = \mathrm{i}A_{\alpha\mu}^{\mathrm{V,A}} T^\alpha$$

and the T^αs are the generators of the gauge group in fermion space. The transformation properties of the fermion and the gauge fields under infinitesimal

† Since in this section fields A_μ are treated as background fields, the coupling constant g is included in the definition of A_μ.

vector and axial-vector gauge transformations are

$$\left.\begin{aligned}
&\Psi' = (1 - \Theta_{\rm V})\Psi \\
&\delta_{\Theta_{\rm V}} A^{\rm V}_{\mu} = \partial_{\mu}\Theta_{\rm V} - [\Theta_{\rm V}, A^{\rm V}_{\mu}] = D_{\mu}\Theta_{\rm V} \\
&\delta_{\Theta_{\rm V}} A^{\rm A}_{\mu} = -[\Theta_{\rm V}, A^{\rm A}_{\mu}]
\end{aligned}\right\} \tag{13.65}$$

$$\left.\begin{aligned}
&\Psi' = (1 - \gamma_5\Theta_{\rm A})\Psi \\
&\delta_{\Theta_{\rm A}} A^{\rm V}_{\mu} = -[\Theta_{\rm A}, A^{\rm A}_{\mu}] \\
&\delta_{\Theta_{\rm A}} A^{\rm A}_{\mu} = \partial_{\mu}\Theta_{\rm A} - [\Theta_{\rm A}, A^{\rm V}_{\mu}]
\end{aligned}\right\} \tag{13.66}$$

respectively, where as usual $\Theta = {\rm i}\Theta^{\alpha}T^{\alpha}$. We see, in particular, that in a non-abelian theory a pure axial-vector interaction with Dirac fermions would not be gauge-invariant. The lagrangian (13.64) is also invariant under global $U(1)$ and $U_{\rm A}(1)$ transformations. However, no gauge bosons are coupled to the corresponding abelian currents.

It is often more convenient to use the left- and right-handed fields and then

$$\left.\begin{aligned}
\mathcal{L} &= \bar{\Psi}_{\rm L}{\rm i}\gamma^{\mu}(\partial_{\mu} + {\rm i}A^{\alpha}_{{\rm L}\mu}T^{\alpha})\Psi_{\rm L} + \bar{\Psi}_{\rm R}{\rm i}\gamma^{\mu}(\partial_{\mu} + {\rm i}A^{\alpha}_{{\rm R}\mu}T^{\alpha})\Psi_{\rm R} \\
&= \sum_{\rm L,R} \bar{\Psi}_{\rm L,R}{\rm i}\gamma^{\mu}\partial_{\mu}\Psi_{\rm L,R} - j^{{\rm L}\alpha}_{\mu}A^{\mu}_{{\rm L}\alpha} - j^{{\rm R}\alpha}_{\mu}A^{\mu}_{{\rm R}\alpha}
\end{aligned}\right\} \tag{13.67}$$

where

$$j^{\alpha}_{{\rm L,R}\mu} = \bar{\Psi}_{\rm L,R}\gamma_{\mu}T^{\alpha}_{\rm L,R}\Psi_{\rm L,R}$$

Invariance under gauge transformations on the chiral fields requires standard transformation properties for the gauge fields $A^{\mu}_{\rm L,R}$ (see Section 1.3). At the classical level the $U(1)$ currents are conserved: $\partial^{\mu}j^{\rm V}_{\mu} = \partial^{\mu}j^{\rm A}_{\mu} = 0$ and the non-abelian currents are covariantly conserved: $D_{\mu}j^{\alpha\mu}_{\rm L,R} = 0$ (see (1.125)).

Depending on the physical problem considered, the gauge fields in the lagrangian (13.64) or (13.67) can be regarded as dynamical gauge fields or as auxiliary background fields, introduced to write down the generating functional for the anomalous Ward identities involving currents which in the framework of the considered theory are not coupled to any dynamical gauge fields. This is, for instance, the case with the flavour symmetry currents in QCD. Of course, the physical interpretation of the non-abelian structure in the lagrangians (13.64) and (13.67) also depends on the problem. In addition the same fermions may couple to other gauge fields, too, corresponding to a gauge group acting in a different space.

Since, by assumption, the gauge fields are classical background fields, only two types of Feynman diagram appear in our theory: diagrams in which the external fields couple to a free spinor line and diagrams in which the external fields couple to a spinor loop. This, however, has no impact on the generality of the discussion of chiral anomalies due to the Adler–Bardeen non-renormalization theorem quoted in Section 13.1.

Abelian anomaly

Let us first consider the generating functional for Ward identities involving the $U(1)$ axial-vector current. We use the lagrangian (13.64) and specify $A_\mu^{\mathrm{A}} = 0$. Thus we consider the axial abelian anomaly in a QCD-like theory. Applying the general formalism developed in Section 10.1 to derive Ward identities we take the functional

$$\exp\{\mathrm{i}Z[A_\mu, \eta, \bar{\eta}]\} = \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi} \exp\left[\mathrm{i}\int \mathrm{d}^4x\, (\mathcal{L}_{\mathrm{kin}} - j_\alpha^\mu A_\mu^\alpha + \bar{\eta}\Psi + \bar{\Psi}\eta)\right] \tag{13.68}$$

and use its independence of the choice of the integration variables changed according to the transformation

$$\Psi(x) \to \exp[-\mathrm{i}\gamma_5\Theta(x)]\Psi(x), \quad \bar{\Psi}(x) \to \bar{\Psi}(x)\exp[-\mathrm{i}\gamma_5\Theta(x)] \tag{13.69}$$

(the gauge fields A_μ^α, are not altered). Naively we then get (10.9) which generates Ward identities corresponding to $\partial^\mu j_\mu^5 = 0$ and is wrong in view of our perturbative calculations and of (13.57) in particular. Using the latter we can anticipate the correct modification of (10.9). It reads

$$\begin{aligned}
0 &= \delta Z[A, \eta, \bar{\eta}]/\delta\Theta|_{\Theta=0} \\
&= \frac{1}{\exp\{\mathrm{i}Z[A, \eta, \bar{\eta}]\}} \int \mathrm{d}^4z \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi} \\
&\quad \times \exp\left[\mathrm{i}\int \mathrm{d}^4x\, (\bar{\Psi}\mathrm{i}\not{\partial}\Psi - j_\alpha^\mu A_\mu^\alpha + \bar{\eta}\Psi + \bar{\Psi}\eta)\right] \\
&\quad \times \{-(n/8\pi^2)\,\mathrm{Tr}[G_{\mu\nu}(z)\tilde{G}^{\mu\nu}(z)] - \partial^\mu j_\mu^5(z) + \bar{\eta}(-\mathrm{i}\gamma_5\Psi) + (-\mathrm{i}\bar{\Psi}\gamma_5)\eta\}
\end{aligned} \tag{13.70}$$

One may ask where the anomalous term comes from in the path integral formalism. The answer has been given by Fujikawa (1980) who noticed that the anomaly is due to the non-invariance of the fermionic path integral measure under the transformation (13.69). We shall come back to this point in Section 13.4.

Eq. (13.70) generates the desired Ward identities when we differentiate it functionally with respect to the A_μ^α, η and $\bar{\eta}$ and set $\bar{\eta} = \eta = 0$. For instance the anomalous Ward identity for the matrix element $\langle 0|Tj_\mu^5(z)j_\nu^\alpha(y)j_\delta^\beta(x)|0\rangle$ is obtained from (13.70) by taking the derivative

$$\frac{\partial}{\partial A_\nu^\alpha(y)}\frac{\partial}{\partial A_\delta^\beta(x)}$$

and setting $\bar{\eta} = \eta = 0$. One gets

$$\partial^\mu \langle 0|T j^5_\mu(z) j^\alpha_\nu(y) j^\beta_\delta(x)|0\rangle = -(n/8\pi^2)\langle 0|T\, \mathrm{Tr}[G_{\mu\rho}\tilde{G}^{\mu\rho}] j^\alpha_\nu j^\beta_\delta|0\rangle$$

and in momentum space this is just equivalent to (13.56). Differentiating with respect to $\bar{\eta}$ and η and setting $\bar{\eta} = \eta = 0$ one gets the anomalous Ward identity for the matrix element $\langle 0|T\Psi j^5_\mu \bar{\Psi}|0\rangle$.

Non-abelian anomaly and gauge invariance

Our next task is to study the generating functional for the anomalous Ward identities involving non-abelian currents. We again consider the functional

$$\exp\{\mathrm{i}Z[A_\mu]\} = \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi} \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \bar{\Psi}\mathrm{i}\not{D}\Psi\right) \tag{13.71}$$

corresponding to the lagrangian (13.64) or (13.67). The Ψ stands for Dirac or chiral fermion fields and A_μ for vector and axial-vector gauge fields or left- and right-handed gauge fields, respectively; other fields are suppressed. First, we study the gauge dependence of the functional $Z[A_\mu]$. The gauge transformation of the fermion fields is again just a change of integration variables but under the accompanying transformation of the gauge fields $A_\mu(x) \to A_\mu(x) + \delta A_\mu(x)$ the functional $Z[A_\mu]$ changes as follows

$$Z[A_\mu] \to Z[A_\mu] - 2\int \mathrm{d}^4x\, \mathrm{Tr}\left[\frac{\delta Z[A_\mu]}{\delta A_\mu(x)}\delta A_\mu(x)\right] \tag{13.72}$$

where

$$\frac{\delta}{\delta A_\mu} = \frac{\delta}{\delta A^\alpha_\mu}(\mathrm{i}T^\alpha)$$

and

$$\frac{\delta Z[A_\mu]}{\delta A_\mu(x)} = \frac{1}{\exp\{\mathrm{i}Z[A_\mu]\}}\int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\, j^\mu(x) \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \bar{\Psi}\mathrm{i}\not{D}\Psi\right) \equiv \hat{J}^\mu \tag{13.73}$$

and $j_\mu(x)$ is the current coupled to the field $A_\mu(x)$ whereas $\hat{J}^\mu(x)$ is defined by (13.73). The explicit formulae (13.71)–(13.73) and that for δA_μ depend, of course, on whether we use vector and axial-vector fields or chiral fields. If we work with vector and axial-vector gauge fields, then $A_\mu(x)$ in (13.72) includes both A^{V}_μ and A^{A}_μ which transform under vector or axial-vector gauge transformation according to (13.65) and (13.66). On the other hand, if we use chiral fields, the gauge transformations on the left- and right-handed fields separately form a group and we have

$$\delta_\Theta A^{\mathrm{L,R}}_\mu = \partial_\mu\Theta^{\mathrm{L,R}} - [\Theta^{\mathrm{L,R}}, A^{\mathrm{L,R}}_\mu] = D_\mu\Theta^{\mathrm{L,R}} \tag{13.74}$$

In each case we may rewrite (13.72) and (13.73) using the following relation

$$\int \mathrm{d}^4x \, \mathrm{Tr}[\hat{J}^\mu(x) D_\mu \Theta(x)] = -\int \mathrm{d}^4x \, \mathrm{Tr}[D^\mu \hat{J}_\mu(x)\Theta(x)] \tag{13.75}$$

which is easy to verify. With the help of (13.75) we get

$$\delta_\Theta Z = 2\int \mathrm{d}^4x \, \mathrm{Tr}[X(x)Z[A_\mu]\Theta(x)] \tag{13.76}$$

or

$$\delta_\Theta Z[A_\mu]/\delta\Theta_\alpha(x)|_{\Theta(x)=0} = -X_\alpha(x)Z[A_\mu] \tag{13.77}$$

where

$$X_\mathrm{V}(x) = \partial_\mu \frac{\delta}{\delta A_\mu^\mathrm{V}(x)} + \left[A_\mu^\mathrm{V}(x), \frac{\delta}{\delta A_\mu^\mathrm{V}(x)}\right] + \left[A_\mu^\mathrm{A}(x), \frac{\delta}{\delta A_\mu^\mathrm{A}(x)}\right] \tag{13.78}$$

(where $X = \mathrm{i}X^\alpha T^\alpha$) for the vector gauge transformations on the vector and axial-vector fields,

$$X_\mathrm{A}(x) = \partial_\mu \frac{\delta}{\delta A_\mu^\mathrm{A}(x)} + \left[A_\mu^\mathrm{V}(x), \frac{\delta}{\delta A_\mu^\mathrm{A}(x)}\right] + \left[A_\mu^\mathrm{A}(x), \frac{\delta}{\delta A_\mu^\mathrm{V}(x)}\right] \tag{13.79}$$

for the axial-vector gauge transformations on the vector and axial-vector fields, and

$$X_\mathrm{L,R}(x) = \partial_\mu \frac{\delta}{\delta A_\mu^\mathrm{L,R}(x)} + \left[A_\mu^\mathrm{L,R}(x), \frac{\delta}{\delta A_\mu^\mathrm{L,R}(x)}\right] = D_\mu \frac{\delta}{\delta A_\mu^\mathrm{L,R}(x)} \tag{13.80}$$

for chiral gauge transformations on the chiral gauge fields (all commutators are for matrices T^α and not for fields which are c-numbers). By explicit calculation† one verifies that the Xs satisfy the gauge group commutation relations

$$\left.\begin{aligned}
[X_\mathrm{V}^\alpha(x), X_\mathrm{V}^\beta(y)] &= c^{\alpha\beta\gamma} X_\mathrm{V}^\gamma \delta(x-y) \\
[X_\mathrm{V}^\alpha(x), X_\mathrm{A}^\beta(y)] &= c^{\alpha\beta\gamma} X_\mathrm{A}^\gamma \delta(x-y) \\
[X_\mathrm{A}^\alpha(x), X_\mathrm{A}^\beta(Y)] &= c^{\alpha\beta\gamma} X_\mathrm{V}^\gamma \delta(x-y) \\
[X_\mathrm{L,R}^\alpha(x), X_\mathrm{L,R}^\beta(Y)] &= c^{\alpha\beta\gamma} X_\mathrm{L,R}^\gamma \delta(x-y)
\end{aligned}\right\} \tag{13.81}$$

which, applied to $Z[A_\mu]$, are equivalent to the relation

$$\delta_{\Theta_1}\delta_{\Theta_2} Z[A_\mu] - \delta_{\Theta_2}\delta_{\Theta_1} Z[A_\mu] = \delta_{[\Theta_1,\Theta_2]} Z[A_\mu] \tag{13.82}$$

expressing the group property of the gauge transformations.

† It is convenient to use a test functional.

Eqs. (13.76), (13.77) and (13.81) or (13.82) have several implications. Firstly, for invariance of our quantum theory under one of the previously defined gauge transformations we must have

$$\delta_\Theta Z[A_\mu] = 0 \quad \Rightarrow \quad X(x)Z[A_\mu] = 0 \tag{13.83}$$

where the X is one of the Xs given by (13.78)–(13.80) depending on the considered gauge transformation. In a theory with only vector or only chiral gauge fields, using (13.73) we recover the equation

$$D^\mu j_\mu^{\mathrm{V}} = 0 \quad \text{or} \quad D^\mu j_\mu^{\mathrm{L,R}} = 0 \tag{13.84}$$

obtained in Section 1.3 on the classical level. The non-abelian source currents for gauge fields must be covariantly conserved at any order of perturbation theory for gauge invariance to be maintained in quantum theory. With dynamical gauge fields the generating functional $Z[A_\mu]$ differs from the $Z[J, \eta, \bar{\eta}]$ of the complete theory, where $J, \eta, \bar{\eta}$ are the gauge and fermion field sources, by the absence of the functional integration over $\mathcal{D}A_\mu$ and of the gauge field kinetic terms which are gauge-invariant. To study gauge invariance of the complete quantum theory it is sufficient to study the gauge invariance of the $Z[A_\mu]$.

The anomaly means that the variation of $Z[A_\mu]$ under gauge transformation does not vanish

$$D^\mu j_\mu^{\mathrm{L,R}}(x) = G^{\mathrm{L,R}}(x) \tag{13.85}$$

or equivalently

$$-X^\alpha_{\mathrm{L,R}}(x)Z[A_\mu] = G^\alpha_{\mathrm{L,R}}(x) \tag{13.86}$$

Similarly, if we work with vector and axial-vector currents and insist on the vector current conservation $D^\mu j_\mu^{\mathrm{V}}(x) = 0$ then the divergence of the axial current is anomalous and

$$D^\mu j_\mu^{\mathrm{A}} = \partial^\mu j_\mu^{\mathrm{A}} + [A_{\mathrm{V}}^\mu, j_\mu^{\mathrm{A}}] = G^{\mathrm{A}}(x) \tag{13.87}$$

or

$$-X^\alpha_{\mathrm{A}}(x)Z[A_\mu] = G^\alpha_{\mathrm{A}}(x) \tag{13.88}$$

Indeed, from the triangle diagram calculation in Section 13.1 we know partial contributions to the $G_\alpha^{\mathrm{L,R}}(x)$ in (13.86) and the $G_\alpha^{\mathrm{A}}(x)$ in (13.88) which are (at the triangle diagram level $D^\mu = \partial^\mu$)

$$\pm(1/12\pi^2)\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[T^\alpha\partial_\mu A_\nu^{\mathrm{L,R}}\partial_\rho A_\sigma^{\mathrm{L,R}}] \tag{13.89}$$

and

$$-(1/2\pi^2)\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[T^\alpha(\partial_\mu A_\nu^{\mathrm{V}}\partial_\rho A_\sigma^{\mathrm{V}} + \tfrac{1}{3}\partial_\mu A_\nu^{\mathrm{A}}\partial_\rho A_\sigma^{\mathrm{A}})] \tag{13.90}$$

respectively. In the second expression the first term comes from the AVV diagram and the second one from the AAA diagram. Different numerical coefficients reflect different renormalization conditions for the two diagrams: conservation of the vector currents for the first one and symmetry between all three vertices for the second one. Observe that as long as for the anomaly the minimal form depending only upon the gauge fields and not other external fields is taken, (13.86) and (13.88) follow from (13.85) and (13.87), respectively, since the $G^\alpha(A)$s factorize out of the path integral which cancels with the $\exp\{-\mathrm{i}Z[A]\}$.

One way to get the full anomalous terms $G^\alpha(A)$ is to use the so-called Wess–Zumino consistency condition (Wess & Zumino 1971). Using (13.81) or (13.82) and (13.85) we get for the anomaly $G_\alpha^{\mathrm{L,R}}(x)$ the equation

$$X^\beta_{\mathrm{L,R}}(x)G^\alpha_{\mathrm{L,R}}(y) - X^\alpha_{\mathrm{L,R}}(y)G^\beta_{\mathrm{L,R}}(x) = -c^{\alpha\beta\gamma}G^\gamma_{\mathrm{L,R}}(x)\delta(x-y) \tag{13.91}$$

The importance of this condition follows from the fact that the operator X is non-linear in the gauge potential A_μ. Therefore the Wess–Zumino condition completely determines $G^\alpha[A_\mu]$ once the term (13.89) is given. After some calculation we can get

$$G^\alpha_{\mathrm{L,R}}(x) = \pm(1/12\pi^2)\,\mathrm{Tr}[T^\alpha\partial^\mu\varepsilon_{\mu\nu\rho\sigma}(A^\nu_{\mathrm{L,R}}\partial^\rho A^\sigma_{\mathrm{L,R}} + \tfrac{1}{2}A^\nu_{\mathrm{L,R}}A^\rho_{\mathrm{L,R}}A^\sigma_{\mathrm{L,R}})] \tag{13.92}$$

The expression for the axial anomaly must satisfy the following consistency conditions (again, see (13.81))

$$\left.\begin{aligned} X^{\mathrm{V}}_\alpha G^{\mathrm{A}}_\beta(y) &= c_{\alpha\beta\gamma}G^{\mathrm{A}}_\gamma\delta(x-y)\\ X^{\mathrm{A}}_\alpha G^{\mathrm{A}}_\beta(y) - X^{\mathrm{A}}_\beta G^{\mathrm{A}}_\alpha(y) &= 0 \end{aligned}\right\} \tag{13.93}$$

and it is much more complicated (Bardeen 1969). All the anomalous Ward identities follow from the relations (13.86) or (13.88) (we may need to introduce additional sources to the functional Z).

Consistent and covariant anomaly

Finally we study the transformation properties of the anomalous current. Naively, this current would be expected to transform covariantly under gauge transformations. To determine the effect of the anomalies, in addition to the gauge variation

$$\left.\begin{aligned} \delta_\Theta A_\mu &= D_\mu\Theta\\ \delta_\Theta Z[A] &= -2\int \mathrm{d}^4x\,\mathrm{Tr}[(\delta Z/\delta A_\mu)\delta_\Theta A_\mu(x)] = 2\int \mathrm{d}^4x\,\mathrm{Tr}[D_\mu\hat{J}^\mu\Theta(x)]\\ &= -\int \mathrm{d}^4x\,\Theta^\alpha(x)G^\alpha(A) \end{aligned}\right\} \tag{13.94}$$

we introduce the variation which defines the current. By definition

$$\tilde{\delta}_B Z[A] = -2 \int \mathrm{d}^4 x\, \mathrm{Tr}[(\delta Z/\delta A_\mu)\delta A_\mu] = -2 \int \mathrm{d}^4 x\, \mathrm{Tr}[\hat{J}^\mu B_\mu(x)] \tag{13.95}$$

and

$$\tilde{\delta}_B A_\mu(x) = \delta A_\mu \equiv B_\mu(x)$$

The following operator equation holds

$$\tilde{\delta}_B \delta_\Theta - \delta_\Theta \tilde{\delta}_B = \tilde{\delta}_{[B,\Theta]} \tag{13.96}$$

Applying this operator to the functional $Z[A]$ we obtain

$$\left.\begin{aligned}(\tilde{\delta}_B \delta_\Theta - \delta_\Theta \tilde{\delta}_B) Z[A] &= \int \mathrm{d}^4 x\, \{[\tilde{\delta}_B G_\alpha(A)]\Theta^\alpha(x) - (\delta_\Theta \hat{J}^\mu_\alpha) B^\alpha_\mu(x)\} \\ &= \int \mathrm{d}^4 x\, \hat{J}^\mu_\alpha \{[B_\mu(x), \Theta(x)]\}^\alpha\end{aligned}\right\} \tag{13.97}$$

The gauge transformation properties of the non-abelian current follow from (13.97)

$$\int \mathrm{d}^4 x\, (\delta_\Theta \hat{J}^\mu_\alpha) B^\alpha_\mu(x) = -\int \mathrm{d}^4 x\, \{[\Theta, \hat{J}^\mu]_\alpha B^\alpha_\mu - [\tilde{\delta}_B G_\alpha(A)]\Theta^\alpha\} \tag{13.98}$$

The first term on the r.h.s. gives the usual transformation property of the current; the second term is due to the anomaly. One can show (Bardeen & Zumino 1984) that in the case of the anomaly (13.92) there exists the covariant non-abelian current

$$\tilde{J}^{\mathrm{L,R}\mu}_\alpha = \hat{J}^{\mathrm{L,R}\mu}_\alpha + P^{\mathrm{L,R}\mu}_\alpha(A) \tag{13.99}$$

Its explicit form can be found by searching for a polynomial $P^{\mathrm{L,R}\mu}_\alpha$ in gauge fields with an anomalous gauge transformation rule opposite to that of the current $\hat{J}^\mu_\alpha$. Bardeen & Zumino (1984) show that

$$P^{\mathrm{L,R}\mu}_\alpha = \mp(1/24\pi^2)\varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}[T_\alpha(A_\nu G_{\rho\sigma} + G_{\rho\sigma} A_\nu - A_\nu A_\rho A_\sigma)] \tag{13.100}$$

The current $\tilde{J}^\alpha_\mu$ is also anomalous

$$(D_\mu \tilde{J}^{\mathrm{L,R}\mu})_\alpha = \tilde{G}_\alpha(A) = \pm(1/16\pi^2)\varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}[T_\alpha G_{\mu\nu} G_{\rho\sigma}] \tag{13.101}$$

The anomalies (13.92) and (13.101) are called consistent and covariant anomalies, respectively. Each can be used to study anomaly cancellation. The physical sense of the original non-abelian current has been discussed earlier. The covariant current may have some physical significance when coupled to other external non-gauged fields.

13.4 Anomalies from the path integral in Euclidean space

Introduction

In this section we discuss the techniques of anomaly calculation using the path integrals in Euclidean space (Fujikawa 1980). First let us summarize the rules of continuation from Minkowski to Euclidean space. For the spatial coordinates we have (see Section 2.3)

$$\left.\begin{array}{ll} x_0 = -\mathrm{i}\hat{x}_4 & \hat{x}_i = (-x, -y, -z) \\ x_i = \hat{x}_i & \hat{x}^\mu = -\hat{x}_\mu \end{array}\right\} \tag{13.102}$$

where $\mu = 1, 2, 3, 4$ and correspondingly

$$\left.\begin{array}{ll} p_0 = -\mathrm{i}\hat{p}_4 & \hat{p}_i = (-p_x, -p_y, -p_z) \\ p_i = \hat{p}_i & \hat{p}_\mu = \mathrm{i}(\partial/\partial\hat{x}^\mu) \end{array}\right\} \tag{13.103}$$

where again $\mu = 1, 2, 3, 4$. We define the Euclidean vector potential $\hat{A}_\mu$ as

$$A_0 = \mathrm{i}\hat{A}_4, \quad A_i = -\hat{A}_i \tag{13.104}$$

so that $\hat{A}_\mu$ form a Euclidean four-vector. Thus for the covariant derivative $D_\mu = \partial_\mu + A_\mu = \partial_\mu + \mathrm{i}gA^a_\mu T^a$ we get ($\partial_\mu = (\partial/\partial x_0, -\partial/\partial x_i)$)

$$D_0 = \mathrm{i}\hat{D}_4, \quad D_i = -\hat{D}_i, \quad \hat{D}_\mu = -\partial/\partial\hat{x}^\mu + \mathrm{i}g\hat{A}^a_\mu T^a \tag{13.105}$$

The field strength tensor is

$$G^a_{ij} = \hat{G}^a_{ij} \quad (i, j = 1, 2, 3), \quad G^a_{0i} = -\mathrm{i}\hat{G}^a_{4i} \tag{13.106}$$

where

$$\hat{G}^a_{\mu\nu} = -\frac{\partial}{\partial\hat{x}^\mu}\hat{A}^a_\nu + \frac{\partial}{\partial\hat{x}^\nu}\hat{A}^a_\mu + gc^{abc}\hat{A}^b_\mu\hat{A}^c_\nu$$

We define the Euclidean γ-matrices to be hermitean matrices $\hat{\gamma}_n$ obeying

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\delta_{\mu\nu} \qquad \mu, \nu = 1, \ldots, 4 \tag{13.107}$$

and explicitly

$$\hat{\gamma}_4 = \gamma_0, \quad \hat{\gamma}_i = -\mathrm{i}\gamma_i, \quad i = 1, 2, 3 \tag{13.108}$$

The matrix $\hat{\gamma}_5$ is taken as

$$\hat{\gamma}_5 = -\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\hat{\gamma}_4 \tag{13.109}$$

and it is also hermitean. The Dirac operator $\mathrm{i}\hat{\not{D}}$ is hermitean. The fermion fields Ψ and $\bar{\Psi}$ are independent variables in the path integral and in Euclidean space we define

$$\hat{\Psi} = \Psi, \quad \hat{\bar{\Psi}} = \mathrm{i}\bar{\Psi} \tag{13.110}$$

Under infinitesimal rotations in Euclidean space $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ transform as follows:

$$\delta\hat{\Psi} = \tfrac{1}{8}(\hat{\gamma}_\mu\hat{\gamma}_\nu - \hat{\gamma}_\nu\hat{\gamma}_\mu)\hat{\omega}_{\mu\nu}\hat{\Psi} \tag{13.111}$$

where

$$\omega_{ij} = \hat{\omega}_{ij}, \quad \omega_{0i} = \mathrm{i}\hat{\omega}_{4i}$$

and

$$\delta\hat{\Psi}^\dagger = -\tfrac{1}{8}\hat{\Psi}^\dagger(\hat{\gamma}_\mu\hat{\gamma}_\nu - \hat{\gamma}_\nu\hat{\gamma}_\mu)\hat{\omega}_{\mu\nu} \tag{13.112}$$

so that $\hat{\Psi}_1^\dagger\hat{\Psi}_2$ is a scalar and we identify the transformation properties of $\hat{\bar{\Psi}}$ as those of $\hat{\Psi}^\dagger$ (in Minkowski space $\Psi_1^\dagger\gamma_0\Psi_2$ is a scalar). The Euclidean action is

$$\mathrm{i}S = -\hat{S} \tag{13.113}$$

where

$$\hat{S} = \int \mathrm{d}^4\hat{x}\left[\tfrac{1}{4}\hat{G}^a_{\mu\nu}\hat{G}^a_{\mu\nu} + \hat{\bar{\Psi}}(\mathrm{i}\hat{\not{D}} + \mathrm{i}M)\hat{\Psi}\right]$$

To complete this introduction let us discuss the eigenvalue problem for the hermitean Dirac operator $\mathrm{i}\hat{\not{D}}$ in Euclidean space. For simplicity we assume that $\mathrm{i}\hat{\not{D}}$ has a purely discrete spectrum. This corresponds to stereographically projecting Euclidean four-space onto a four-sphere and changes the determinant but only by a factor which is independent of the gauge field. Since $\mathrm{i}\hat{\not{D}}$ is hermitean it has real eigenvalues λ_n

$$\mathrm{i}\hat{\not{D}}\hat{\Psi}_n = \lambda_n\hat{\Psi}_n \tag{13.114}$$

and because

$$\{\hat{\gamma}_5, \mathrm{i}\hat{\not{D}}\} = 0 \tag{13.115}$$

we get

$$\mathrm{i}\hat{\not{D}}\hat{\gamma}_5\hat{\Psi}_n = -\lambda_n\hat{\gamma}_5\hat{\Psi}_n \tag{13.116}$$

Thus non-vanishing eigenvalues always occur in pairs of opposite sign. We observe also that for $\lambda_n \neq 0$ chiral fields are not eigenvectors of $\mathrm{i}\hat{\not{D}}$

$$\mathrm{i}\hat{\not{D}}(1 \pm \hat{\gamma}_5)\hat{\Psi}_n = \lambda_n(1 \mp \hat{\gamma}_5)\hat{\Psi}_n \tag{13.117}$$

On the other hand eigenfunctions of the vanishing eigenvalue can always be chosen to be eigenfunctions of γ_5 and thus to be chiral fields. Since $\hat{\gamma}_5^2 = 1$ its eigenvalues are ± 1 and

$$\hat{\gamma}_5(1 \pm \hat{\gamma}_5)\hat{\Psi}_n = \pm(1 \pm \hat{\gamma}_5)\hat{\Psi}_n \tag{13.118}$$

In the rest of this section we work in Euclidean space and the caret is omitted.

Abelian anomaly with Dirac fermions

As mentioned in the previous section the anomaly of the ungauged axial $U(1)$ current $j^5_\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi$ can be understood as a non-invariance of the path integral measure under local transformations on fermion fields

$$\left.\begin{aligned}\Psi' &= \exp[\mathrm{i}\Theta(x)\gamma_5]\Psi \approx (1+\mathrm{i}\Theta(x)\gamma_5)\Psi\\ \bar{\Psi}' &= \bar{\Psi}\exp[\mathrm{i}\Theta(x)\gamma_5] \approx \bar{\Psi}(1+\mathrm{i}\Theta(x)\gamma_5)\end{aligned}\right\} \tag{13.119}$$

The generating functional in the Euclidean space

$$\exp\{-Z[A]\} = \int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\,\exp\Bigl(-\int \mathrm{d}^4x\,\bar{\Psi}\mathrm{i}\not{D}\Psi\Bigr) \tag{13.120}$$

where Ψ and $\bar{\Psi}$ are two independent variables describing Dirac fermions coupled to gauge potentials transforming under a vector-like gauge group, can be defined precisely by expanding Ψ and $\bar{\Psi}$ in terms of the eigenfunctions φ_n of $\mathrm{i}\not{D}$

$$\left.\begin{aligned}&\Psi(x) = \sum_n a_n\varphi_n(x), && \bar{\Psi}(x) = \sum_n \varphi^\dagger_n(x)\bar{b}_n\\ &\mathrm{i}\not{D}\varphi_n(x) = \lambda_n\varphi_n(x), && \int \mathrm{d}^4x\,\varphi^\dagger_n(x)\varphi_m(x) = \delta_{mn}\end{aligned}\right\} \tag{13.121}$$

The coefficients a_n and $\bar{b}_n$ are Grassmann variables. The measure becomes

$$\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi} = \prod_n \mathrm{d}a_n \prod_m \mathrm{d}\bar{b}_m \tag{13.122}$$

up to an irrelevant arbitrary normalization factor. Under local infinitesimal axial transformations (13.119) the lagrangian is transformed into

$$\mathcal{L}'(x) = \mathcal{L}(x) + \partial_\mu\Theta j^\mu_5(x)$$

and therefore

$$\int \mathrm{d}^4x\,\bar{\Psi}\mathrm{i}\not{D}\Psi \to \int \mathrm{d}^4x\,\bar{\Psi}\mathrm{i}\not{D}\Psi - \int \mathrm{d}^4x\,\Theta(x)\partial_\mu j^\mu_5(x) \tag{13.123}$$

If the measure were invariant under axial transformations then using the invariance of the path integral under a change of variables one could derive (10.11) (with $\delta^a\mathcal{L}(x) = 0$) implying the conservation of axial current at the quantum level. However, the measure (13.122) is not invariant and following Fujikawa we can explicitly evaluate the Jacobian of the change of variables. We get

$$\Psi'(x) = \sum_n a'_n\varphi_n(x) = \sum_n a_n \exp[\mathrm{i}\Theta(x)\gamma_5]\varphi_n(x) \tag{13.124}$$

or

$$a'_m = \sum_n \int \mathrm{d}^4x\,\varphi^\dagger_m(x)\exp[\mathrm{i}\Theta(x)\gamma_5]\varphi_n(x)a_n = \sum_n C_{mn}a_n \tag{13.125}$$

Thus (see (2.116))

$$\prod_m \mathrm{d}a'_m = \det{}^{-1} C \prod_n \mathrm{d}a_n \tag{13.126}$$

and the Jacobian for $\mathcal{D}\bar{\Psi}$ gives the identical factor. Furthermore, we have

$$C = \exp[\mathrm{i}\Theta(x)\gamma_5]$$

and

$$\det C = \exp\{\mathrm{i}\,\mathrm{Tr}[\Theta(x)\gamma_5]\} = \exp\left[\mathrm{i}\sum_n \int \mathrm{d}^4x\,\Theta(x)\varphi_n^\dagger(x)\gamma_5\varphi_n(x)\right] \tag{13.127}$$

where the trace is taken over the whole Hilbert space. Hence the measure is multiplied by

$$\exp\left[-2\mathrm{i}\int \mathrm{d}^4x\,\Theta(x)\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\right] \tag{13.128}$$

which is ill defined as it stands and must be regularized. In our problem we want to choose a regularization procedure which preserves the invariance under the vector-like gauge group with gauge potentials A_μ. Actually, with this fact in mind we have been working in the basis (13.121) of the eigenfunctions of $\mathrm{i}\not{D} = \mathrm{i}(\not{\partial} + \not{A})$ which are gauge-invariant, i.e. gauge transformed eigenfunctions are eigenfunctions of the gauge transformed operator. Hence we may simply regularize the expression (13.128) with a gaussian cut-off for large eigenvalues λ_n

$$\begin{aligned}
&\int \mathrm{d}^4x\,\Theta(x)\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\\
&\quad\equiv \lim_{M\to\infty}\int \mathrm{d}^4x\,\Theta(x)\sum_n \varphi_n^\dagger(x)\gamma_5\varphi_n(x)\exp(-\lambda_n^2/M^2)\\
&\quad= \lim_{M\to\infty}\int \mathrm{d}^4x\,\Theta(x)\sum_n \widetilde{\mathrm{Tr}}[\gamma_5\exp[-(\mathrm{i}\not{D})^2/M^2]\varphi_n(x)\varphi_n^\dagger(x)]\\
&\quad= \lim_{M\to\infty}\int \mathrm{d}^4x\,\Theta(x)\lim_{y\to x}\{\gamma_5\exp[-(\mathrm{i}\not{D}_x)^2/M^2]\}^{ab}_{kl}\sum_n[\varphi_n(x)]^b_l[(\varphi_n^*(y)]^a_k
\end{aligned} \tag{13.129}$$

where $\widetilde{\mathrm{Tr}}$ means the trace over the group and the Dirac indices. Going over to a plane wave basis and using the completeness relation for the $\varphi_n(x)$ we get

$$\begin{aligned}
&\sum_n [\varphi_n(x)]^b_l [\varphi^*_n(y)]^a_k \\
&\qquad = \delta^{ab}\delta^{kl} \sum_n \langle x|n\rangle\langle n|y\rangle = \delta^{ab}\delta^{kl}\langle x|y\rangle \\
&\qquad = \delta^{ab}\delta^{kl} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \langle x|k\rangle\langle k|y\rangle \\
&\qquad = \delta^{ab}\delta^{kl} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp(-\mathrm{i}kx)\exp(+\mathrm{i}ky)
\end{aligned} \tag{13.130}$$

and therefore†

$$\begin{aligned}
&\int \mathrm{d}^4 x\, \Theta(x) \sum_n \varphi^\dagger_n(x)\gamma_5\varphi_n(x) \\
&= \lim_{M\to\infty} \int \mathrm{d}^4 x\, \Theta(x) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \widetilde{\mathrm{Tr}}[\gamma_5 \exp(+\mathrm{i}kx)\exp[-(\mathrm{i}\not{D})^2/M^2]\exp(-\mathrm{i}kx)] \\
&= \lim_{M\to\infty} \int \mathrm{d}^4 x\, \Theta(x)\, \widetilde{\mathrm{Tr}}[\gamma_5([\gamma^\mu,\gamma^\nu]G_{\mu\nu})^2\left(\frac{1}{4M^2}\right)^2 \frac{1}{2!} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp(+k^\mu k_\mu/M^2) \\
&= \frac{1}{16\pi^2} \int \mathrm{d}^4 x\, \Theta(x)\, \widetilde{\mathrm{Tr}}[G_{\mu\nu}\tilde{G}^{\mu\nu}]
\end{aligned} \tag{13.131}$$

where the last trace is over the group indices only. Using (13.123), (13.128) and invariance of the path integral under the change of variables we finally get

$$\partial^\mu j^5_\mu(x) = (\mathrm{i}/8\pi^2)\,\mathrm{Tr}[G_{\mu\nu}\tilde{G}^{\mu\nu}] \tag{13.132}$$

which corresponds to the previous result (13.57) obtained in Minkowski space by the triangle diagram calculation.

As a by-product of this derivation of the abelian anomaly we can easily understand the content of the so-called index theorem for the Dirac operator $\mathrm{i}\not{D}$. The theorem relates the number of positive chirality minus the number of negative chirality zero modes of the operator $\mathrm{i}\not{D}$ to the topological charge (Pontryagin index) of the background gauge field A_μ. Using the result (13.131) for $\Theta(x)$ independent of x, observing that

$$\int \mathrm{d}^4 x \sum_n \varphi^\dagger_n(x)\gamma_5\varphi_n(x) \tag{13.133}$$

vanishes for eigenfunctions with eigenvalue $\lambda_n \neq 0$ since $\gamma_5\varphi_n$ has eigenvalue $-\lambda_n$ and is thus orthogonal to φ_n, and writing

$$\varphi^\dagger_n(x)\gamma_5\varphi_n(x) = \tfrac{1}{4}\varphi^\dagger_n(1+\gamma_5)^2\varphi_n - \tfrac{1}{4}\varphi^\dagger_n(1-\gamma_5)^2\varphi_n$$

† In Euclidean space we define $\varepsilon^{1234} = -1$ and then $\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\delta] = 4\varepsilon^{\mu\nu\rho\sigma}$.

for the eigenfunctions with eigenvalue $\lambda_n = 0$, we get

$$n_+ - n_- = Q \tag{13.134}$$

where n_+ (n_-) is the number of positive chirality (negative chirality) zero modes of the operator $\mathrm{i}\not{D}$ and Q is the topological charge defined by (8.62). As in Section 8.3 we assume here that the background gauge fields rapidly approach the pure gauge configuration of $|x_\mu| \to \infty$ so that the Euclidean action is finite and the topological charge Q is well defined. The topological charge density is connected to the anomaly density.

Non-abelian anomaly and chiral fermions

We consider now the structure of the non-abelian anomaly in theories of chiral fermions coupled to an external dynamical or auxiliary gauge field A_μ and described by the effective action

$$\exp\{-Z[A]\} = \int \mathcal{D}\Psi_\mathrm{L}\mathcal{D}\bar{\Psi}_\mathrm{L} \exp\left(-\int \mathrm{d}^4x\, \bar{\Psi}_\mathrm{L}\mathrm{i}\not{D}\Psi_\mathrm{L}\right) \tag{13.135}$$

The left-handed fermions Ψ_L and the right-handed fermions $\bar{\Psi}_\mathrm{L}$ are independent variables. In Section 13.3 we have understood the non-abelian anomaly as a breakdown of the gauge invariance of the functional $Z[A]$. This suggests a possible approach to its calculation using the path integral formalism: calculate

$$Z[A] = -\ln\det \mathrm{i}\not{D}(A) \tag{13.136}$$

and its change under gauge transformation. However, as we know from the introduction to this section the operator $\mathrm{i}\not{D}(A)$ maps positive chirality spinors to negative chirality spinors

$$\mathrm{i}\not{D}(A)\Psi_\mathrm{L} \to \Psi_\mathrm{R}$$

and consequently it does not have a well-defined eigenvalue problem and a well-defined determinant on the space of chiral fields. The only exception is when chiral fermions transform as a real representation of the gauge group, i.e. left- and right-handed chiralities transform under the same representation. Then we can formulate our theory in terms of the Dirac fermions $\Psi = \Psi_\mathrm{L} + \Psi_\mathrm{R}$ and in Euclidean space we are back to the hermitean eigenvalue problem $\mathrm{i}\not{D}\Psi_n = \lambda_n\Psi_n$ with real λ_n. In this case $\det \mathrm{i}\not{D}$ is real and, as we know from the previous discussion, it can always be regularized in a gauge-invariant anomaly-free manner (in other words positive and negative chirality anomalies cancel each other).

Now consider chiral fermions in a complex representation R. We can always imagine an extended theory with additional fermions in R^* so that

$$Z_{R+R^*}[A] = Z_R[A] + Z_{R^*}[A] = Z_R[A] + Z_R^*[A] = 2\,\mathrm{Re}\,Z_R[A] \qquad (13.137)$$

But $R + R^*$ is a real representation and consequently $Z_{R+R^*}[A]$ is gauge-invariant. We arrive at an interesting observation (Alvarez-Gaumé & Witten 1983): in Euclidean space only the imaginary part of the functional $Z_R[A]$ may suffer from the anomalies which give its anomalous variation under gauge transformation. Equivalently, if $\det \mathrm{i}\not{D}$ was defined in the space of chiral fermions, we could say that only the phase of $\mathrm{i}\not{D}$ may be gauge-non-invariant. One can circumvent the problem of defining $\det \mathrm{i}\not{D}$ observing that it suffices to calculate the change of $\ln\det \mathrm{i}\not{D}$ under a gauge transformation. And

$$\ln\det \mathrm{i}\not{D}(A^g) - \ln\det \mathrm{i}\not{D}(A) = \ln\det[-\mathrm{i}\not{D}^{-1}(A)\mathrm{i}\not{D}(A^g)] \qquad (13.138)$$

is meaningful since $\not{D}^{-1}\not{D}$ maps a space of fermion fields with given chirality into itself. The overall phase of the determinant is irrelevant for us and so is the dependence on A; we can put, for example, $A = 0$. Thus we see that the calculation of the non-abelian anomaly from the path integral is a well-defined problem. The main difficulty is to invent an appropriate regularization procedure. For actual calculation, and different regularization methods we refer the reader to the original literature (Fujikawa 1984, Balachandran, Marmo, Nair & Trahern 1982, Alvarez-Gaumé & Ginsparg 1984, Leutwyler 1984). Of course, the result agrees with (13.93) which in Euclidean space reads

$$G^\alpha_\mathrm{L} = D_\mu \frac{\partial Z}{\partial A^\alpha_{\mu\mathrm{L}}} = -\frac{\mathrm{i}}{12\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[T^\alpha \partial^\mu (A_\nu \partial_\rho A_\sigma - \tfrac{1}{2}\mathrm{i}A_\nu A_\rho A_\sigma)] \qquad (13.139)$$

Problems

13.1 Derive the anomalous Ward identity (13.21) starting with the unregularized integral representation (13.5), performing a shift of the integration variable such that (13.3) is satisfied for this integral representation and studying the surface term induced by this change of the integration variable in the linearly divergent integral.

13.2 Show that Ward identities for the Green's function of the three-vector currents are anomaly-free. Discuss this point also in Euclidean space using the material of Section 13.4.

13.3 In Fig. 13.6 replace photons by gluons and check that in QCD there is no anomaly for the axial isospin current.

13.4 Check that

$$G_A^\alpha = -(1/4\pi^2)\varepsilon^{\mu\nu\sigma\tau}\,\mathrm{Tr}[\lambda^\alpha[\tfrac{1}{4}F_{\mu\nu}^{\mathrm{V}}F_{\sigma\tau}^{\mathrm{V}} + \tfrac{1}{12}F_{\mu\nu}^{\mathrm{A}}F_{\sigma\tau}^{\mathrm{A}} + \tfrac{2}{3}(A_\mu A_\nu F_{\sigma\tau}^{\mathrm{V}} - A_\mu F_{\nu\sigma}^{\mathrm{V}}A_\tau + F_{\mu\nu}^{\mathrm{V}}A_\sigma A_\tau) - \tfrac{8}{3}A_\mu A_\nu A_\sigma A_\tau]]$$

where

$$V_\mu = +\mathrm{i}\lambda^i V_\mu^i, \quad A_\mu = +\mathrm{i}\lambda^i A_\mu^i$$
$$F_{\mathrm{V}}^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu] + [A^\mu, A^\nu]$$
$$F_{\mathrm{A}}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, V^\nu] + [A^\mu, A^\nu]$$

satisfies the consistency conditions (13.93).

13.5 The determinant of an operator A with real positive discrete and increasing without bound eigenvalues λ_n: $Af_n = \lambda_n f_n$ can be defined as

$$\det A = \prod_n \lambda_n = \exp[-(\mathrm{d}\zeta_A/\mathrm{d}t)(0)]$$

where the so-called ζ-function, defined as

$$\zeta_A(t) = \sum_n 1/\lambda_n^t$$

converges for $\mathrm{Re}\, t > 2$ and can be analytically extended to a meromorphic function of t with poles only at $t = 1$ and $t = 2$ and is regular at $t = 0$. Using the integral representation for the Γ-function

$$\Gamma(t) = \int_0^\infty x^{t-1}\exp(-x)\,\mathrm{d}x$$

check that the ζ_A-function can be calculated as

$$\zeta_A(t) = \frac{1}{\Gamma(t)}\int_0^\infty \mathrm{d}\tau\,\tau^{t-1}\int \mathrm{d}x\,h(x, x, \tau)$$

where

$$h(x, y, \tau) = \sum_n \exp(-\lambda_n\tau)f_n(x)f_n^*(y) = \langle x|\exp(-A\tau)|y\rangle$$

and $h(x, y, \tau)$ obeys the so-called 'heat equation'

$$A_x h(x, y, \tau) = -(\partial/\partial\tau)h(x, y, \tau), \quad h(x, y, \tau = 0) = \delta(x - y)$$

Note also that

$$\tilde{\zeta}_A(t, x, y) \equiv \sum_n \frac{f_n(x)f_n^*(y)}{\lambda_n^t} = \frac{1}{\Gamma(t)}\int_0^\infty \mathrm{d}\tau\,\tau^{t-1}h(x, y, \tau)$$

The last equation can be used to regularize quantities such as

$$\sum_n \varphi_n^+(x)\gamma_5\varphi_n(x) = \lim_{\substack{t\to 0\\ y\to x}} \mathrm{Tr}[\gamma_5\tilde{\zeta}_A(x, y, t)]$$

with $A = (\not{D})^2$. The heat kernel $h(x, y, \tau)$ then has the asymptotic expansion (Nielsen, Grisaru, Römer & van Nieuwenhuizen 1978)

$$h(x, y, \tau) = \frac{1}{16\pi^2\tau^2} \exp\left[-\frac{(x-y)^2}{4\tau}\right] \sum_{n=0}^{\infty} a_n(x, y)\tau^n$$

for small τ. Check that

$$\tilde{\zeta}_{\hat{\not{D}}^2}(x, x, 0) = (1/16\pi^2) a_2(x, x)$$

13.6 Determine the abelian and non-abelian chiral anomalies in $2n$-dimensional space-time (Frampton & Kephart 1983, Zumino, Wu & Zee 1984, Leutwyler 1984).

13.7 Prove (13.134) starting with the anomalous divergence in Euclidean space

$$\partial^\mu j^5_\mu = 2m\bar{\Psi}\gamma_5\Psi - 2\mathrm{i}Q(x)$$

(Integrate over the Euclidean volume, take the vacuum expectation value and then take the limit $m \to 0$.)

14

Effective lagrangians

14.1 Non-linear realization of the symmetry group

Non-linear σ-model

To become familiar with the effective lagrangian technique it is useful to follow the original approach and to begin with the discussion of the $SU(2) \times SU(2)$-invariant σ-model in which the chiral symmetry has been spontaneously broken: $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2) \rightarrow SU_{\mathrm{V}}(2)$. We consider the σ-model with bosons only. The quadruplet of real fields $\Phi_0 = (\pi^1 \pi^2 \pi^3 \sigma_0)^{\mathrm{T}}$ transforming as $(2, 2)$ under the chiral group splits then into massless Goldstone bosons and a massive $\sigma = \sigma_0 - v$ meson which transform under the unbroken subgroup $SU(2)$ as a triplet and a singlet, respectively. The σ mass is proportional to the vacuum expectation value v and can be made arbitrarily heavy. Imagine we are interested in the physics at low energy, $E \ll m_\sigma$, only. It is natural to expect that the low energy region can be effectively described in terms of the Goldstone boson fields, with the σ meson decoupled. However, an inspection of the lagrangian (9.38) shows that this does not happen: the $\pi\pi\sigma$ coupling is proportional to v and therefore, for example, the σ-exchange contribution to the low energy $\pi\pi \rightarrow \pi\pi$ scattering cannot be neglected when $m_\sigma \rightarrow \infty$. The reason is that a naive decoupling of the σ field by its elimination from the lagrangian (9.38) explicitly breaks the $SU(2) \times SU(2)$ invariance of the lagrangian. Both the $\pi\pi\pi\pi$ and the $\pi\pi\sigma$ couplings originate from the $SU(2) \times SU(2)$ invariant term $(\sigma_0^2 + \boldsymbol{\pi}^2)^2$ in the original lagrangian. Equivalently, decoupling of the σ means setting $\sigma_0 = v$ and this breaks the invariance since $(\boldsymbol{\pi}, \sigma_0)$ does no longer transform as a quadruplet under $SU(2) \times SU(2)$. Can we find an effective low energy description in terms of the Goldstone bosons only which would be $SU(2) \times SU(2)$-invariant?

Our goal is to implement the $SU(2) \times SU(2)$ symmetry in such a way that the two irreducible multiplets, the triplet and the singlet of the unbroken $SU(2)$ which are contained in the quadruplet Φ_0 do not mix under general $SU(2) \times SU(2)$

transformations. This can be achieved if we use the freedom of choice of the broken symmetry vacuum (Section 9.5) among the degenerate set $|\alpha\rangle$

$$|\alpha\rangle = \exp(+\mathrm{i}\alpha_a X_a)|0\rangle \tag{14.1}$$

The corresponding transformation of the field operators reads

$$\Phi' = \exp(\mathrm{i}\alpha_a X_a)\Phi \exp(-\mathrm{i}\alpha_a X_a) \tag{14.2}$$

or in the matrix form

$$\Phi' = \exp(-\mathrm{i}\alpha_a X_a)\Phi \tag{14.3}$$

where the X_as are broken generators in (14.1) and (14.2) and their matrix representation in (14.3). Let us take some $\Phi_0 = (\boldsymbol{\pi}(x), \sigma_0(x))^{\mathrm{T}}$ in the originally defined vacuum. First of all we observe that at a given point x the Goldstone boson fields can always be set to zero by a suitable redefinition of the vacuum. In fact using the real representation (E.38) for the generators we can write

$$\Phi_0(x) = \begin{pmatrix} \pi^1(x) \\ \pi^2(x) \\ \pi^3(x) \\ \sigma_0(x) \end{pmatrix} = \exp(-\mathrm{i}\xi_a X_a) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma(x) \end{pmatrix} = \begin{pmatrix} -\sigma(\xi_1/\xi)\sin\xi \\ -\sigma(\xi_2/\xi)\sin\xi \\ -\sigma(\xi_3/\xi)\sin\xi \\ \sigma\cos\xi \end{pmatrix} \tag{14.4}$$

where

$$\xi = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2}$$

and

$$\sigma^2(x) = \boldsymbol{\pi}^2(x) + \sigma_0^2(x)$$

and therefore due to (14.3) a rotation of the vacuum by the angle $\boldsymbol{\xi}(x)$ sets the new Goldstone boson fields to zero locally at x. Any chiral transformation on the vector $\Phi_0(x)$ can then be written as follows:

$$\Phi_0'(x) = \exp(-\mathrm{i}\alpha_a G_a)\Phi_0(x) = \begin{pmatrix} \pi_1'(x) \\ \pi_2'(x) \\ \pi_3'(x) \\ \sigma_0'(x) \end{pmatrix} = \exp[-\mathrm{i}\xi_a'(\xi, \alpha)X_a] \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma(x) \end{pmatrix} \tag{14.5}$$

where

$$G_a \equiv (X_a, Y_a), \quad \mathrm{Tr}[G_j G_i] = \delta_{ji}$$

and X_a and Y_a are broken and unbroken generators of $SU(2)\times SU(2)$, respectively, with the same $\sigma(x)$ as in (14.4) because $\sigma^2(x) = \boldsymbol{\pi}^2(x) + \sigma_0^2(x)$ is invariant under chiral rotations. Thus, a general chiral transformation on $\Phi_0(x)$ can be represented by the new angles $\xi'_a(\boldsymbol{\xi}, \boldsymbol{\alpha})$ specifying the vacuum orientation which sets the transformed Goldstone boson fields locally to zero. The vector $(0\,0\,0\,\sigma(x))^{\mathrm{T}}$ refers now to the new, rotated, vacuum.

Since $\sigma(x)$ is an $SU_{\mathrm{V}}(2)$ singlet we suspect that in general a chiral transformation can be factorized into an unbroken-$SU(2)$-subgroup transformation (which in our case acts on the $SU(2)$ singlet $\sigma(x)$) and a rotation of the local vacuum orientation. However, before we generalize our discussion, for the purpose of further considerations let us check the transformation properties of the $\xi_a(x)$s under the unbroken $SU(2)$ isospin group

$$\begin{aligned}\exp(-\mathrm{i}u_a Y_a)\Phi_0 &= \exp(-\mathrm{i}u_a Y_a)\exp(-\mathrm{i}\xi_a X_a)\exp(\mathrm{i}u_a Y_a)\exp(-\mathrm{i}u_a Y_a)\begin{pmatrix}0\\0\\0\\\sigma\end{pmatrix}\\ &= \exp(-\mathrm{i}\xi_a R_{ab} X_b)\begin{pmatrix}0\\0\\0\\\sigma\end{pmatrix} = \exp(-\mathrm{i}\xi'_b X_b)\begin{pmatrix}0\\0\\0\\\sigma\end{pmatrix}\end{aligned} \tag{14.6}$$

Hence

$$\xi'_b = R_{ab}\xi_a$$

R is the matrix representation of the unbroken $SU(2)$ under which the matrices X_a transform

$$\exp(-\mathrm{i}u_c Y_c) X_a \exp(\mathrm{i}u_c Y_c) = R_{ab} X_b$$

i.e. $\xi_a(x)$s indeed transform linearly under the $SU_{\mathrm{V}}(2)$, belong to the same representation as X_as and transform according to R^{T}. These results can be generalized as follows (Coleman, Wess & Zumino 1969, Callan, Coleman, Wess & Zumino 1969). We consider a theory with some symmetry group G which is spontaneously broken to a subgroup H. Let G be a compact, connected, semi-simple Lie group. From the properties of the exponentials it follows that, in some neighbourhood of the identity of G, every group element can be decomposed uniquely into a product of the form

$$g = \exp(\mathrm{i}\xi_a X_a)\exp(\mathrm{i}u_a Y_a) \tag{14.7}$$

where X_as and Y_as are broken and unbroken generators of the group G, respectively.† If Φ_0 is a field which transforms according to a linear representation of G, we can recast it as

$$\Phi_0(x) = \exp[-\mathrm{i}\xi_a(x)X_a]\Phi(x) \equiv U\Phi \tag{14.8}$$

(X_a is the matrix representation of the broken generator X_a appropriate for $\Phi_0(x)$) where under a general G transformation $g(\alpha)$

$$\left.\begin{aligned} \boldsymbol{\xi}' &= \boldsymbol{\xi}'(\boldsymbol{\xi}, \boldsymbol{\alpha}) \\ \Phi' &= \exp[-\mathrm{i}u_i(\boldsymbol{\xi}, \boldsymbol{\alpha})Y_i]\Phi \end{aligned}\right\} \tag{14.9}$$

and $\boldsymbol{\xi}'(\boldsymbol{\xi}, \boldsymbol{\alpha})$ and $\mathbf{u}(\boldsymbol{\xi}, \boldsymbol{\alpha})$ are non-linear functions of $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}$. Indeed, using (14.7)

$$\begin{aligned} \Phi_0'(x) &= \exp(-\mathrm{i}\alpha_a G_a)\Phi_0(x) = \exp(-\mathrm{i}\alpha_a G_a)\exp[-\mathrm{i}\xi_a(x)X_a]\Phi(x) \\ &= \exp[-\mathrm{i}\xi_a'(\boldsymbol{\xi}, \boldsymbol{\alpha})X_a]\exp[-\mathrm{i}u_a(\boldsymbol{\xi}, \boldsymbol{\alpha})Y_a]\Phi(x) \end{aligned} \tag{14.10}$$

since $\exp(-\mathrm{i}\alpha\cdot G)\exp(-\mathrm{i}\zeta\cdot X)$ belongs to G and can be decomposed as in (14.7).‡ Thus, a general global transformation belonging to the group G is realized as a non-linear transformation on the fields $\xi_a(x)$ and as a gauge (local) transformation, belonging to the unbroken H, on the multiplet Φ. We say that the fields (ξ, Φ) provide a non-linear realization of the group G. Under transformations belonging to the unbroken subgroup H the fields $\xi_i(x)$ transform linearly, according to the generalized (14.6).

Let us go back to the σ-model. Given the transformation rule (14.5) we observe that the field $\sigma(x)$ can be decoupled at $E \ll m_\sigma$ in a chirally invariant manner by setting $\sigma(x) = v$ or equivalently by imposing on the field $\Phi_0(x)$ a chirally invariant constraint $\Phi_0^{\mathrm{T}}\Phi_0 = v^2$. After rescaling $\Phi_0/v \to \Phi_0$ we obtain the so-called non-linear σ-model described by the lagrangian

$$\mathcal{L} = \tfrac{1}{2}f_\pi^2\partial_\mu\Phi_0^{\mathrm{T}}\partial^\mu\Phi_0, \quad \Phi_0^{\mathrm{T}}\Phi_0 = 1, \quad f_\pi^2 = v^2 \tag{14.11}$$

Imagine that using the lagrangian (14.11) we want to calculate matrix elements involving Goldstone bosons. We can work either with Green's functions of currents or with the on-mass-shell matrix elements. The first method is more general (see, for example, Gasser & Leutwyler (1984)) and consists in gauging the full chiral group, introducing the external gauge fields by changing the ordinary derivatives acting on the linearly transforming fields Φ_0 into covariant derivatives and expanding the generating functional, given by the vacuum-to-vacuum amplitude in the

† The scalar fields $\xi(x)$ are coordinates of the manifold of left cosets G/H at each point of space-time. The set of group elements $l(\xi) = \exp(\mathrm{i}\xi_a X_a)$ parametrizes this manifold. Eq. (14.7) means that once we have a parametrization $l(\xi)$ each group element g can be uniquely decomposed into a product $g = l \cdot h$, where $h \in H$. The l is the representative member of the coset to which g belongs and h connects l to g within the coset.

‡ A product of g with an arbitrary group element and in particular with some $l(\xi)$ defines another $l(\xi')$ and an h according to $g \cdot l(\xi) = l(\xi') \cdot h$, where $\xi' = \xi'(\xi, g)$ and $h = h(\xi, g)$.

presence of external gauge fields, in powers of external fields. If we are interested in the on-mass-shell matrix elements only, it is best to interpret the transforming non-linearly $\xi_a(x)$s, rather than the $\pi_a(x)$s, as the pion fields (Weinberg 1968, Coleman, Wess & Zumino 1969). Such a redefinition of the pion field does not change the S-matrix elements if the redefinition is local and leaves the origin of field space invariant and as long as the new object has the right quantum numbers. Different definitions lead to different matrix elements off the mass-shell but they all give the same results on the mass-shell. We shall briefly discuss this second method. Irrespective of the method, however, one finds that beyond the tree level our theory has UV divergences which require counterterms of higher and higher order in derivatives of fields reflecting the non-renormalizability of the non-linear σ-model. This leads us to the following important comment on the physical significance of the effective lagrangian.

We have arrived at the lagrangian (14.11) by freezing the σ degree of freedom in the linear σ-model. However, one can convince oneself that this lagrangian is the most general one in order p^2 which is Lorentz-, parity- and chiral-invariant and can be constructed in terms of the real $O(4)$ vector Φ_0 of unit length $\Phi_0^{\mathrm{T}}\Phi_0 = 1$.† Thus we can think about Φ_0 as representing the degrees of freedom of composite Goldstone bosons in the $SU(2) \times SU(2)$-invariant QCD as well. The real virtue of the effective lagrangians is that they provide the most general systematic low energy expansion in the light particle sector, invariant under full symmetry G of the underlying theory. In this context the non-renormalizability of (14.11) is not a problem. Working to order p^4 or higher we anyway have to supplement the lagrangian (14.11) with additional terms with free parameters. Let us again consider QCD. Its chiral symmetry is spontaneously broken into the isovector subgroup H. There are Goldstone bosons, composites of quark fields which form a multiplet under H, and possibly other states also with well-defined transformation properties under H. Due to spontaneous symmetry breaking, chiral multiplets split into H-multiplets with quite different masses in general. Imagine we are interested in the low energy description of the strong interactions. We are not able yet to get it directly from QCD in terms of its fundamental parameter Λ (and quark masses, if explicit breaking of chiral symmetry is taken into account). There is also no reason to expect that the linear σ-model is a good approximation to the low energy QCD. On the other hand the non-linear effective lagrangian is a tool for exploiting QCD as much as possible without solving the confinement problem, namely for exploiting its underlying chiral symmetry. This happens at the expense of a set of new unknown constants at every level of the low energy expansion.

† $\Phi_0^{\mathrm{T}}\partial_\mu\Phi_0$ vanishes and $\Phi_0^{\mathrm{T}}\partial^\mu\partial_\mu\Phi_0 = -\partial_\mu\Phi_0^{\mathrm{T}}\partial^\mu\Phi_0$.

Effective lagrangian in the $\xi_a(x)$ *basis*

We return to the problem mentioned earlier of calculating the on-mass- shell matrix elements from effective lagrangians. Because of the constraint $\Phi_0^{\mathrm{T}}\Phi_0 = 1$ the form (14.11) is not convenient for constructing the Feynman rules. They can be derived easily if we work in the $\xi_a(x)$ basis. We can introduce the covariant derivative of the field $\xi(x)$, defined by its transformation properties under general transformation $G(\alpha)$ as follows

$$[(D_\mu\xi)_a(x)]' \equiv (D_\mu\xi)_b R_{ba}(\exp[-\mathrm{i}u_c(\boldsymbol{\xi},\boldsymbol{\alpha})Y_c]) \tag{14.12}$$

where the matrix R_{ba} is defined by the relation

$$\exp[-\mathrm{i}u_a(\boldsymbol{\xi},\boldsymbol{\alpha})Y_a]X_b\exp[\mathrm{i}u_a(\boldsymbol{\xi},\boldsymbol{\alpha})Y_a] = R_{bc}(\exp[-\mathrm{i}u_a(\boldsymbol{\xi},\boldsymbol{\alpha})Y_a])X_c \tag{14.13}$$

and Y_a and X_a are representation matrices of unbroken and broken generators of G, respectively, in the representation to which the Φ_0 belongs. The functions $u_a(\xi,\alpha)$ are defined by (14.9). The covariant derivative can be constructed explicitly by considering the derivative $\partial_\mu\Phi_0$ which transforms under G transformations in the same way as Φ_0 does. Using the non-linear notation (14.8) for a vector Φ_0 we can, on general grounds, expand as follows:

$$\partial_\mu(U\Phi) = \exp[-\mathrm{i}\xi_a(x)X_a]\{\mathrm{i}\partial_\mu\xi_a(x)D_{ab}(\xi)X_b\Phi + [\mathrm{i}\partial_\mu\xi_a(x)E_{ab}(\xi)Y_b + \partial_\mu]\Phi\} \tag{14.14}$$

and correspondingly for $\partial_\mu(U\Phi)'$ with $\xi_a \to \xi_a'$, $D_{ab} \to D_{ab}'$, $E_{ab} \to E_{ab}'$, $\Phi \to \Phi'$, where the functions $D(\xi)$ and $E(\xi)$ are defined by†

$$\exp(\mathrm{i}\xi_a X_a)\partial_\mu\exp(-\mathrm{i}\xi_a X_a) = \mathrm{i}\partial_\mu\xi_a[D_{ab}(\xi)X_b + E_{ab}(\xi)Y_b] \tag{14.15}$$

The transformation properties under G of both sides in (14.14) are the same as those of the Φ_0, so the object in the braces must be a multiplet of H and have the local gauge transformation law (14.9). Thus

$$\begin{aligned}
&[\mathrm{i}\partial_\mu\xi_a'(\boldsymbol{\xi},\boldsymbol{\alpha})D_{ab}'(\xi')X_b + (\mathrm{i}\partial_\mu\xi_a'E_{ab}'(\xi')Y_b + \partial_\mu)]\Phi' \\
&\quad = \exp(-\mathrm{i}u_cY_c)[\mathrm{i}\partial_\mu\xi_aD_{ab}X_b + (\mathrm{i}\partial_\mu\xi_aE_{ab}Y_b + \partial_\mu)]\underbrace{\exp[\mathrm{i}u_c(\boldsymbol{\xi},\boldsymbol{\alpha})Y_c]\Phi'}_{\Phi}
\end{aligned}$$

Identifying

$$(D_\mu\xi)_b = \mathrm{i}\partial_\mu\xi_aD_{ab}(\xi) \equiv \mathrm{Tr}[U^\dagger\partial_\mu UX_b] \tag{14.16}$$

$$D_\mu\Phi = [\partial_\mu + \mathrm{i}\partial_\mu\xi_aE_{ab}(\xi)Y_b]\Phi \equiv \{\partial_\mu + \mathrm{Tr}[U^\dagger\partial_\mu UY_b]Y_b\}\Phi \tag{14.17}$$

† Those who are familiar with differential geometry will recognize that in the expansion (14.15) $D_{ab}(\xi)$ represents the *vielbein* in G/H and $E_{ab}(\xi)$ is an H-connection.

and equating coefficients of X_a we recover for the object $(D_\mu\xi)_b$ the transformation law (14.12). Similarly we also get

$$(D_\mu\Phi)' = \exp[-\mathrm{i}u_a(\boldsymbol{\xi},\boldsymbol{\alpha})Y_a]D_\mu\Phi \tag{14.18}$$

Thus, under general transformation $G(\alpha)$ functions Φ, $D_\mu\Phi$ and $(D_\mu\xi)_b$ all transform according to some finite matrix representations of the group H. An important conclusion following from these transformation rules is that a function of Φ, $D_\mu\Phi$ and $(D_\mu\xi)_b$ which is invariant under H transformations is also automatically invariant under G transformations.

The explicit form of the covariant derivative can be obtained from (14.15). First, expanding the l.h.s. (14.15) in power series one can check the identity

$$\begin{aligned}&\exp(\mathrm{i}\xi_a X_a)\partial_\mu \exp(-\mathrm{i}\xi_a X_a)\\ &\quad= \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^{n+1}}{(n+1)!}[\ldots[[\partial_\mu\xi_a\cdot X_a, \underset{1}{\xi_b\cdot X_b}], \underset{2}{\xi_d\cdot X_d}]\underset{\ldots}{\ldots}\underset{n}{\xi_w\cdot X_w}]\end{aligned} \tag{14.19}$$

The commutators give

$$\partial_\mu\xi_\alpha\cdot \underset{1}{\mathrm{i}c_{abc}\xi_b}\cdot \underset{2}{\mathrm{i}c_{cde}\xi_d}\underset{\ldots}{\ldots}\underset{n}{\mathrm{i}c_{uwz}\xi_w}G_z$$

where

$$[G_a, G_b] = \mathrm{i}c_{abc}G_c$$

is the G group algebra and G_z is a broken generator X for n even, and unbroken Y for n odd. Denoting

$$\mathrm{i}c_{abc}\xi_b = (c\cdot\xi)_{ac} \tag{14.20}$$

one gets

$$\exp(\mathrm{i}\xi_a\cdot X_a)\partial_\mu\exp(-\mathrm{i}\xi_a\cdot X_a) = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^{n+1}}{(n+1)!}\partial_\mu\xi_a(c\cdot\xi)^n_{az}G_z \tag{14.21}$$

and summing the terms even in n

$$D_{ab}(\xi) = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)!}(c\cdot\xi)^{2m}_{ab} = -\left(\frac{\sin c\cdot\xi}{c\cdot\xi}\right)_{ab} \tag{14.22}$$

Similarly

$$E_{ab}(\xi) = \sum_{m=0}^{\infty}\frac{\mathrm{i}(-1)^m}{(2m+2)!}(c\cdot\xi)^{2m+1}_{ab} = \mathrm{i}\left(\frac{1-\cos c\cdot\xi}{c\cdot\xi}\right)_{ab} \tag{14.23}$$

Using the transformation properties of the covariant derivative $(D_\mu\xi)_a$ the effective lagrangian of the non-linear σ-model can be written down in the $\xi(x)$ basis. It reads†

$$\mathcal{L} = \tfrac{1}{2} f_\pi^2 (D_\mu\xi)_a (D^\mu\xi)_a \tag{14.24}$$

As expected, there is no dependence on the ξ field other than that present in the covariant derivative. Any such dependence would break G invariance because one may always set $\xi(x)$ to zero at any given x by rotating the vacuum. Using the explicit form of the covariant derivative we can now easily read off the Feynman rules.

We can extend the effective lagrangian description to include the interaction of Goldstone bosons with other fields which may be relevant for the effective low energy theory. According to the previously described general approach a multiplet Ψ_0 transforming linearly under G is replaced by the fields $(\xi(x), \Psi)$, where

$$\Psi_0 = \exp[-\mathrm{i}\xi_a(x)X_a]\Psi$$

providing a non-linear realization of the group G which is linear on the unbroken subgroup H. Any H-invariant form built of $\Psi(x)$, $D_\mu\Psi(x)$ and $(D_\mu\xi(x))_a$ is automatically G-invariant. For definiteness let us take Ψ to be a fermion field and G to be the chiral symmetry spontaneously broken to the isospin subgroup H. The leading low energy terms are

$$\bar{\Psi}\Psi, \quad \bar{\Psi}\gamma_\mu\gamma_5 Y^a \Psi (D^\mu\xi)^a, \quad \bar{\Psi}\gamma_\mu D^\mu \Psi, \ldots$$

The presence of the first term reflects the fact that the chiral symmetry of the lagrangian tells us nothing about the fermion masses as the symmetry may be spontaneously broken. The second term contains the fermion–fermion–pion vertex and the effective gradient coupling is known as the Adler zero for the soft pion emission. If we want to use the effective lagrangian beyond the tree approximation, terms of higher order in derivatives must be systematically included.

Matrix representation for Goldstone boson fields

In QCD-like theories with chiral $U(N)\times U(N)$ symmetry spontaneously broken to $SU_\mathrm{V}(N)\times U_\mathrm{V}(1)$‡ it is very convenient to collect the Goldstone boson fields into a unitary matrix U transforming linearly under chiral $U(N)\times U(N)$ transformations. Let us consider a general matrix

$$\tilde{\Sigma} = \bar{\Psi}_{\mathrm{L}j}\Psi_{\mathrm{R}i} \tag{14.25}$$

† Or $\mathcal{L} = \frac{1}{2} f_\pi^2 g_{ab}\partial_\mu\xi^a\partial^\mu\xi^b$, where g_{ab} is given in terms of the *vielbein* $D_{ab}(\xi)$ and is a G-invariant metric on G/H.

‡ See Section 13.2 for the discussion of the spontaneous breaking of $U_\mathrm{A}(1)$.

where $\Psi^i_{\mathrm{L,R}}$ are chiral fermion fields with flavour i. Under chiral transformations $\tilde{\Sigma}$ transforms linearly and according to (9.85)

$$\tilde{\Sigma}' = U_{\mathrm{R}} \tilde{\Sigma} U^{\dagger}_{\mathrm{L}} \tag{14.26}$$

Matrices U_{R} and U_{L} act in the space of fermions Ψ_{R} and Ψ_{L}, respectively, and have the form

$$U_{\mathrm{L,R}} = \exp(-\mathrm{i}\alpha^a_{\mathrm{L,R}} T^a) \tag{14.27}$$

where $a = 0, 1, \ldots, N^2 - 1$ and $T^0 = \mathbb{1}$ and T^a, $a = 1, 2, 3, \ldots, N^2 - 1$, are the $SU(N)$ generators in the fermion representation with $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$. The transformations under the unbroken vector group $U_{\mathrm{V}}(N)$ are represented by $\alpha_{\mathrm{L}} = \alpha_{\mathrm{R}}$ and axial transformations by $\alpha_{\mathrm{L}} = -\alpha_{\mathrm{R}}$. Observe that now we always work with $SU(N)$ generators T^a and do not need to introduce explicitly the broken and unbroken generators of the full chiral $SU(N) \times SU(N)$. Thus a change of the vacuum orientation among the degenerate set corresponds to $\tilde{\Sigma} \to \tilde{\Sigma}'$, where

$$\tilde{\Sigma}' = \exp(\mathrm{i}\alpha_a T_a)\tilde{\Sigma}\exp(\mathrm{i}\alpha_a T_a) \tag{14.28}$$

where $a = 0, 1, \ldots, N^2 - 1$. Imagine now that we take $\tilde{\Sigma} = \mathbb{1}$ which according to (9.87) is an ansatz for the spontaneous breaking $SU(N) \times SU(N) \to SU_{\mathrm{V}}(N)$. The matrix

$$\Sigma = \exp(2\mathrm{i}\alpha_a T_a)\mathbb{1} \equiv U \cdot \mathbb{1} \tag{14.29}$$

corresponds to a rotated vacuum and represents the degrees of freedom necessary to describe the Goldstone boson sector. We check that any matrix $\tilde{\Sigma}$ written as a product of a unitary and hermitean matrix $\Sigma = U \cdot h$ transforms under general chiral transformations as follows:

$$\Sigma' = U_{\mathrm{R}} U h U^{\dagger}_{\mathrm{L}} = U_{\mathrm{R}} U U^{\dagger}_{\mathrm{L}} U_{\mathrm{L}} h U^{\dagger}_{\mathrm{L}} = U' h' \tag{14.30}$$

so U also transforms linearly and h undergoes only vector transformations. Thus once Σ is written in this product form any full $U_{\mathrm{V}}(N)$ multiplet can be decoupled from the matrix h in the chirally invariant way. Since (14.29) is in the product form it provides us with the desired chirally invariant decoupling of all but Goldstone degrees of freedom. Actually, the unitary matrix $U(x)$ contains N^2 fields

$$U(x) = \exp[\tfrac{1}{3}\mathrm{i}\alpha_0(x)]\exp[2\mathrm{i}\alpha_a(x) T_a] \tag{14.31}$$

where $a = 1, \ldots, N^2 - 1$ and T^a are hermitean and traceless. For instance, in the case of $U(3) \times U(3)$ it includes the η' degree of freedom which, due to the abelian anomaly, is not a Goldstone boson (see Section 13.2). The single component field $\alpha_0(x)$ is related to the determinant of U

$$\det U = \exp(\mathrm{i}\alpha_0) \tag{14.32}$$

Under a $U_A(1)$ axial transformation

$$U' = \exp(2\mathrm{i}\beta)U \tag{14.33}$$

Under an arbitrary chiral $SU(N) \times SU(N)$ transformation

$$\begin{aligned}\exp(-\tfrac{1}{3}\mathrm{i}\alpha_0)U' &= \exp(-\mathrm{i}\beta_R^a T^a)\exp(2\mathrm{i}\alpha^c T^c)\exp(\mathrm{i}\beta_L^a T^a)\\ &= \exp[2\mathrm{i}\alpha_c'(\beta_R, \beta_L, \alpha)T_c]\end{aligned} \tag{14.34}$$

where $c = 1, \ldots, N^2 - 1$ so the fields $(\boldsymbol{\alpha}, h)$ provide a non-linear realization of this chiral group. One can also easily check that α_as transform linearly under vector transformations $U_R = U_L$.

Because U is an unitary matrix $UU^\dagger = 1$, it is impossible to write a $U(3) \times U(3)$-invariant interaction without derivatives. In order p^2 there are two $U(3) \times U(3)$ invariants

$$\begin{aligned}I_1 &= \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]\\ I_2 &= -\partial_\mu\alpha_0\partial^\mu\alpha_0 = \{\mathrm{Tr}[U^\dagger\partial_\mu U]\}^2\end{aligned}$$

(from (14.32) $\alpha_0 = -\mathrm{i}\ln\det U = -\mathrm{i}\,\mathrm{Tr}\ln U$). On the basis of arguments which are not discussed in this book such as the Okubo–Zweig–Iizuka rule and the large N_c (number of colours) limit of QCD (Veneziano 1979, Witten 1979, Di Vecchia 1980) one expects the I_1 term to be the dominant one. Since

$$U^{-1}\partial_\mu U \equiv \mathrm{i}\partial_\mu\alpha_a D_{ab}(\alpha_c)T_b \equiv (D_\mu\alpha)_b T_b \tag{14.35}$$

where T_b form a group, the effective lagrangian can now be written in one of the following equivalent ways

$$\left.\begin{aligned}\mathcal{L} &= \tfrac{1}{4}f_\pi^2\,\mathrm{Tr}[\partial_\mu U\partial^\mu U^\dagger]\\ &= \tfrac{1}{4}f_\pi^2\,\mathrm{Tr}[U^{-1}\partial_\mu U(U^{-1}\partial^\mu U)^\dagger]\\ &= \tfrac{1}{2}f_\pi^2\,\mathrm{Tr}[U^{-1}\partial_\mu U T_a]\,\mathrm{Tr}[(U^{-1}\partial^\mu U)^\dagger T_a]\\ &= \tfrac{1}{8}f_\pi^2(D_\mu\alpha)_b(D\ \alpha)_b^\dagger\end{aligned}\right\} \tag{14.36}$$

Using the definition (14.35) and (14.30) one can easily check that for general chiral transformations the 'covariant derivative' $(D_\mu\alpha)_a$ transforms only under *global* rotations of the unbroken $SU_V(N)$. Thus if we want to include in the effective lagrangian other multiplets of the unbroken H we can work with their ordinary derivatives.

14.2 Effective lagrangians and anomalies

In QCD and in models of composite quarks and leptons one starts from a fundamental lagrangian describing a non-abelian gauge theory with elementary fermions. However, usually we are unable to solve the original theory and find the spectrum and the interaction of the composite states. The virtue of the effective lagrangian approach is that it provides a systematic method of isolating the composite states that are relevant at low energy and of studying their interactions in terms of a finite number of free effective parameters. The basic requirement in the construction of an effective lagrangian is that it possesses the same symmetries as the original theory. If this theory has anomalies the effective lagrangian must also satisfy the same anomalous transformation laws. Then the anomalous Ward identities will be satisfied at least to the tree order.

Abelian anomaly

In QCD the axial current has the $U_{\mathrm{A}}(1)$ anomaly (Section 13.2)

$$\partial_\mu j_5^\mu = -2N(1/32\pi^2)\varepsilon^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma} \equiv 2Nq(x) \tag{13.57}$$

where N is the number of flavours, determined by the response of the theory to the axial $U_{\mathrm{A}}(1)$ transformations on the fermion fields. The effective lagrangian (14.36) describing the interaction of Goldstone bosons is, however, invariant under the $U_{\mathrm{A}}(1)$ transformations. We want to generalize our effective theory so that (13.57) is satisfied by virtue of the equations of motion. Then the anomalous Ward identities will be satisfied to the tree order. The problem can be solved (Rosenzweig, Schechter & Trahern 1980, Di Vecchia & Veneziano 1980, Witten 1980) by introducing the pseudoscalar 'glueball' field $q(x)$ into the effective lagrangian which is modified to

$$\mathcal{L} = \tfrac{1}{4} f_\pi^2 \operatorname{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \tfrac{1}{2}\mathrm{i} q(x) \operatorname{Tr}[\ln U - \ln U^\dagger] + c q^2(x) \tag{14.37}$$

Every term is invariant under $SU(N) \times SU(N)$ transformation. Under $U_{\mathrm{A}}(1)$ transformation $U' = \exp(\mathrm{i}\Theta)U$ the action transforms as follows

$$S' = S - \Theta N \int \mathrm{d}^4x\, q(x) \tag{14.38}$$

The $U_{\mathrm{A}}(1)$ matter current found by Noether's theorem is

$$j_\mu^5 = \tfrac{1}{4}\mathrm{i} f_\pi^2 \operatorname{Tr}[\partial_\mu U^\dagger U - U^\dagger \partial_\mu U] \tag{14.39}$$

and therefore

$$\partial^\mu j_\mu^5 = \tfrac{1}{4}\mathrm{i} f_\pi^2 \operatorname{Tr}[U \Box U^\dagger - U^\dagger \Box U] \tag{14.40}$$

On the other hand from the equations of motion we get

$$\tfrac{1}{4} f_\pi^2 (\Box U - U \Box U^\dagger U) + \mathrm{i} q(x) U = 0 \tag{14.41}$$

Multiplying (14.41) on the left by $U^\dagger$ and subtracting the complex conjugate equation gives

$$\partial^\mu j^5_\mu = 2Nq(x)$$

Once the effective field $q(x)$ has been introduced into the lagrangian the presence of the chirally invariant term $q^2(x)$ is crucial for the consistency of this approach† and for the successful phenomenology. The additional free parameter c is needed for the phenomenological solution to the $U_\mathrm{A}(1)$ problem, i.e. for the correct description of the masses of the pseudoscalar nonet (including η') in the framework of the effective lagrangian approach (see Problem 14.1). Of course, to this end one must also introduce a term which accounts for the explicit breaking of the chiral symmetry by the quark masses. The quark mass terms transform under $SU(3) \times SU(3)$ as $(3, \bar{3}) \oplus (\bar{3}, 3)$ so we must construct, from U, terms which transform under $SU(3) \times SU(3)$ in the same way. Since the components of U transform as $(3, \bar{3})$, to lowest order in the number of derivatives the necessary correction is

$$\Delta\mathcal{L} = \tfrac{1}{4} f_\pi^2 \{\mathrm{Tr}[MU] + \mathrm{Tr}[M^\dagger U^\dagger]\}$$

where M is some matrix which must be a multiple of the quark mass matrix.

It is convenient to eliminate the field $q(x)$ from the lagrangian (14.37) by using its equation of motion. We finally get

$$\mathcal{L} = \tfrac{1}{4} f_\pi^2\, \mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger] + \tfrac{1}{4} f_\pi^2 \{\mathrm{Tr}[MU] + \mathrm{Tr}[M^\dagger U^\dagger]\} + \frac{1}{8c}\{\mathrm{Tr}[\ln U - \ln U^\dagger]\}^2 \tag{14.42}$$

Effects connected to the existence of a non-vanishing vacuum angle Θ can also be studied by adding to the lagrangian (14.42) a term $\Theta q(x)$.

The Wess–Zumino term

In Chapter 13 we related the presence of the non-abelian anomaly in theories with chiral fermions to the gauge invariance breaking of the functional $Z[A]$, see (13.76):

$$\delta_\Theta Z[A] = \int \mathrm{d}^4x\, [-\Theta_a(x) X_a(x)] Z[A]$$

where A denotes dynamical or auxiliary gauge fields coupled to the fermionic currents and $Z[A]$ is the effective action of the theory with fermions integrated

† In general arbitrary chiral-invariant functions of $q(x)$ are possible. The term $q^2(x)$ is the only one which survives in the limit $N_\mathrm{c} \to \infty$, where N_c is the number of colours (see, for instance, Di Vecchia (1980)).

out

$$\exp\{iZ[A]\} = \int \mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\,\exp\left(i\int d^4x\,\mathcal{L}\right)$$

In Section 13.3 we have also found certain consistency conditions (integrability conditions) which must be satisfied in the presence of anomalies. Our question now is how to take account of the non-abelian anomaly using the effective lagrangian for a QCD-like theory with $SU(N) \times SU(N)$ symmetry spontaneously broken to $SU_V(N)$. The Goldstone boson fields are collected in the matrix $U = \exp[2i\alpha^a(x)T^a]$, $a = 1, 2, \ldots, N^2 - 1$. The field $\alpha_0(x)$ in (14.31) which is a singlet under $SU(N) \times SU(N)$ is irrelevant for the present discussion. The term (14.36) constructed in the previous section is chirally invariant and does not satisfy the Wess–Zumino consistency condition. To find the correct generalization of the effective lagrangian we imagine an addition to the original fermionic lagrangian of the effective Goldstone fields coupled to fermions. When fermions are integrated out one now gets the effective action $Z[\alpha^a, A]$ (rather than $Z[A]$) which should satisfy the Wess–Zumino condition.

Let us consider a finite axial gauge transformation

$$Z[\alpha', A'] = \exp\left\{\int d^4x\,\Theta_a(x)[-X_a^A(x) + V_a(x)]\right\}Z[\alpha, A] \tag{14.43}$$

where $X_a^A(x)$ is given in (13.79) and $V_a(x)$ denotes an infinitesimal axial gauge transformation operating on the Goldstone boson fields α_a. Using the notation

$$\int d^4x\,\Theta_a(-X_a^A + V_a) = \Theta\cdot(-X^A + V) \tag{14.44}$$

we observe that

$$[\Theta\cdot(-X^A+V)]^n Z[\alpha, A] = [\Theta\cdot(-X^A+V)]^{n-1}(\Theta\cdot G_A) = (-\Theta\cdot X^A)^{n-1}(\Theta\cdot G_A) \tag{14.45}$$

where G_A^a is the axial anomaly, see (13.87), depending only on the gauge fields. Therefore the solution to (14.43) reads

$$Z[\alpha', A'] = Z[\alpha, A] - \frac{\exp(-\Theta\cdot X^A) - 1}{\Theta\cdot X^A}\Theta\cdot G_A \tag{14.46}$$

We know that for each $\boldsymbol{\alpha}(x)$ there exists an axial rotation such that $\boldsymbol{\alpha}'(x) = 0$. Setting $\Theta^a(x) = -\alpha^a(x)$ we get

$$Z[0, A'] = Z[\alpha, A] - \frac{\exp(\alpha\cdot X^A) - 1}{\alpha\cdot X^A}\alpha\cdot G_A \tag{14.47}$$

Since $Z[0, A']$ does not involve the fields $\alpha^a(x)$ we can assume $Z[0, A'] = 0$ and then

$$Z[\alpha, A] = -\frac{1 - \exp(\alpha \cdot X^{\mathrm{A}})}{\alpha \cdot X^{\mathrm{A}}} \alpha \cdot G_{\mathrm{A}} = \int_0^1 \mathrm{d}t \exp(t\alpha \cdot X^{\mathrm{A}}) \alpha \cdot G_{\mathrm{A}} \qquad (14.48)$$

The explicit solution for $Z[\alpha, A]$ can be used for reading off the vertices of the effective lagrangian which describe the consequence of the anomalies in the Ward identities. Expanding the exponential and using the explicit expression for X^{A}, see (13.79), and G_{A} (see Problem 13.4) one can work out the effective vertices order-by-order in $\alpha^a(x) = \pi^a(x)/f_\pi$. An interesting point is that due to the derivatives with respect to the gauge fields A present in the X^{A} the effective action does not vanish for $A = 0$. Thus the anomalous effective lagrangian contains vertices describing interactions between Goldstone bosons. Using (13.79) and Problem 13.4 one finds, after a short calculation, that the lowest order term is

$$\frac{1}{6\pi^2 f_\pi^5} \varepsilon^{\mu\nu\sigma\rho} \operatorname{Tr}[\pi\, \partial_\mu \pi\, \partial_\nu \pi\, \partial_\sigma \pi\, \partial_\rho \pi] \qquad (14.49)$$

where $\pi = \pi^a T^a$. In a similar way one can also work out vertices involving pseudoscalars and one vector abelian gauge field describing the photon coupled to the neutral component of the $SU_{\mathrm{V}}(N)$ current (see Wess & Zumino (1971) and Problem 14.2).

Further attention has been paid to the construction and analysis of the Wess–Zumino terms in the effective action (see, for instance, Witten (1983), Adkins, Nappi & Witten (1983), Alvarez-Gaumé & Ginsparg (1984)).

Problems

14.1 In terms of the free parameters M and c obtain the mass spectrum of the pseudoscalar mesons from the quadratic in the fields $\alpha^a(x)$ part of the lagrangian (14.42); $U = \exp(\mathrm{i}\alpha^i T^i)$, $i = 0, 1, \ldots, 8$ (Veneziano 1979).

14.2 Using (14.48) and the axial anomaly $G_{\mathrm{A}}(A)$ given in Problem 13.4 find the effective vertices describing the processes $\pi^0 \to 2\gamma$, $\gamma \to \pi^+\pi^-\pi^0$ and $\gamma + \gamma \to 3\pi$ (consider the two-flavour case).

15

Introduction to supersymmetry

15.1 Introduction

This last chapter is intended as a brief introduction to supersymmetric field theories. In recent years they have become a topic of intense research. The motivation for this research is so far purely theoretical. The Standard Model $SU(3)\times SU(2)\times U(1)$ describes very accurately the physics at presently accessible energies. Present experimental results tell us that if supersymmetry is relevant in Nature it has to be broken in the energy range explored so far. Nevertheless, there are several theoretical reasons to expect that the Standard Model is not a complete theory. The main motivation for considering supersymmetry in the search for such a theory is the improved convergence properties in the UV of supersymmetric field theories. It is this feature which we eventually discuss in this chapter, after introducing the formalism of superfields.

UV divergences, which are softer than in non-supersymmetric theories, may be crucial for solving the problem of the 'naturalness' of the Higgs sector, considered as one of the most serious defects of standard electroweak theory. Higgs particles are scalar particles and their masses are subject to quadratic divergences in perturbation theory. For example, one-loop radiative corrections to elementary scalar masses are

$$\delta m^2 = g^2 \int^{\Lambda} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m} = O\left(\frac{\alpha}{\pi}\right)\Lambda^2$$

Since the theory is renormalizable we can take the limit $\Lambda \to \infty$ and as long as we treat these masses as free parameters there is no reason to pay attention to the magnitude of δm^2: bare parameters and corrections to them are not observable quantities. However, ultimately we want to understand the magnitude of these masses and then we must require

$$\delta m^2 \lesssim O(m^2)$$

Thus the quadratic divergence should be physically cut off at mass scale Λ of the order of the scalar particle mass and originating from new physics beyond the Standard Model. The Standard Model requires the mass of the scalars to be $O(100\ \text{GeV})$, since otherwise the couplings in the Higgs sector would be too large for perturbation theory to be valid. So we need

$$\Lambda \lesssim 1\ \text{TeV}$$

On the other hand if the Standard Model is a complete theory of everything but gravity the only cut-off which remains at our disposal is due to the fact that all particles participate in the gravitational interactions. The energy at which gravity and quantum effects become of comparable strength can be estimated from the only expression with the dimension of energy that can be formed from the fundamental constants c, $\hbar$ and G_{Newton}. We have then

$$\Lambda \sim M_{\text{Planck}} = c^2(\hbar c/G_{\text{Newton}})^{1/2} \approx 10^{19}\ \text{GeV}$$

Two strategies have been proposed to resolve the above dilemma. One is to postulate that the scalar Higgs particle is composite with the compositeness scale $\Lambda \lesssim 1$ TeV which provides a natural cut-off in the quadratically divergent loops. This 'technicolour' scenario has been discussed in Chapter 11. Interesting as it is it cannot easily be reconciled with small fermion masses and with the magnitude of the flavour-changing neutral interactions.

The other strategy is to cancel the loops noticing that the boson and fermion loops have opposite signs. This occurs in supersymmetric field theories and is part of the non-renormalization theorem to be discussed at the end of this chapter. Of course, this cancellation cannot be exact since supersymmetry, if at all relevant, must be a broken symmetry. What we need, however, is

$$\delta m^2 = +O(\alpha/\pi)\Lambda^2|_{\text{bosons}} - O(\alpha/\pi)\Lambda^2|_{\text{fermions}} \lesssim 1\ \text{TeV}^2$$

or effectively

$$|m_{\text{B}}^2 - m_{\text{F}}^2| \lesssim 1\ \text{TeV}^2$$

Thus, this strategy leads to the expectation that the supersymmetric partners of known particles have masses lighter than $O(1\ \text{TeV})$.

The reader interested in studying supersymmetry at a level going beyond our introductory discussion is advised to consult, for example, Gates *et al.* (1983), Sohnius (1985), Wess & Bagger (1983), Nilles (1984) and references therein.

15.2 The supersymmetry algebra

The continuous global symmetries of quantum field theory are generated by infinitesimal transformations

$$\delta_a \varphi = \mathrm{i}\Theta[Q_a, \varphi] \tag{15.1}$$

where Θ is an infinitesimal parameter. The conserved charges Q_a are expressed in terms of fields in the way given by Noether's procedure. The group of the Poincaré transformations with the Lorentz generators $M^{\mu\nu}$ and the energy–momentum operators P^μ satisfying the Poincaré algebra (see Problem 7.1) and various groups of internal symmetry transformations with generators transforming under the Lorentz group as spin zero operators and forming a Lie algebra

$$[Q_a, Q_b] = \mathrm{i}c_{abc} Q_c \tag{15.2}$$

are well known to be important symmetries of physical theories. Actually, Coleman & Mandula (1967) have proved that all generators, other than the Poincaré group generators, of symmetry transformations which form Lie groups with real parameters must have spin zero. Thus, any symmetry group of the S-matrix whose generators obey well-defined *commutation relations* must commute with the Poincaré group. Since the Casimir operators of the Poincaré group are the mass square operator $P^2 = P_\mu P^\mu$ and the generalized spin operator $W^2 = W_\mu W^\mu$ where W^μ is the Pauli–Lubanski vector

$$W^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$

one can conclude that all members of an irreducible multiplet of the internal symmetry group must have the same mass and the same spin.

The Coleman–Mandula theorem, however, leaves open the possibility of including symmetry operations whose generators obey anticommutation relations. Such a generalization of a Lie algebra is called a graded algebra or a superalgebra. The generators of a graded Lie algebra consist of even elements A_i and odd elements B_α and have commutation rules of the form

$$\left.\begin{aligned} [A_i, A_j] &= c_{ijk} A_k \\ [A_i, B_\alpha] &= g_{i\alpha\beta} B_\beta \\ \{B_\alpha, B_\beta\} &= h_{\alpha\beta i} A_i \end{aligned}\right\} \tag{15.3}$$

We are interested in an extension of the Poincaré algebra into a graded Lie algebra which contains the Poincaré algebra as its subalgebra. This is easily accomplished if we introduce spinorial charge Q which is a Majorana spinor or equivalently a left-handed Weyl spinor Q_α ($\alpha = 1, 2$), where

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}, \qquad \bar{Q} = \left(Q^\alpha, \bar{Q}_{\dot\alpha}\right) \tag{15.4}$$

where $\bar{Q} = Q^\dagger \gamma^0$ and $Q_\alpha^\dagger = \bar{Q}_{\dot\alpha}$ (at this point we advise the reader to consult Appendix A for the two-component notation). Thus we have

$$\left.\begin{aligned} [M_{\mu\nu}, Q_\alpha] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \\ [M_{\mu\nu}, \bar{Q}_{\dot\alpha}] &= -\tfrac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\bar{Q}_{\dot\beta} \end{aligned}\right\} \tag{15.5}$$

The anticommutator $\{Q_\alpha, Q_\beta^\dagger\}$ transforms under the Lorentz transformations as $(\frac{1}{2}, \frac{1}{2})$ and, since P_μ is the only generator of the Poincaré algebra in such a representation, we must have

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \tag{15.6}$$

The factor 2 is the normalization convention and the sign is determined by the requirement that the energy $E = P_0$ should be a semi-positive definite operator

$$\sum_{\alpha=1}^{2}\{Q_\alpha, Q_\alpha^\dagger\} = 2\,\mathrm{Tr}[\sigma^\mu P_\mu] = 4P_0 \tag{15.7}$$

The supersymmetry generators must commute with the momenta

$$[Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \tag{15.8}$$

Indeed, the commutator of the Q with a P_μ could contain the Lorentz representations $(1, \frac{1}{2})$ and $(0, \frac{1}{2})$. Since there are no $(1, \frac{1}{2})$ generators present we get, as the most general possibility,

$$[Q_\alpha, P_\mu] = C(\sigma_\mu)_{\alpha\dot\beta}\bar{Q}^{\dot\beta}$$

and the adjoint equation

$$[\bar{Q}^{\dot\beta}, P_\mu] = C^*(\bar\sigma_\mu)^{\dot\beta\alpha} Q_\alpha$$

Therefore

$$[[Q_\alpha, P_\mu], P_\nu] = CC^*(\sigma_\mu\bar\sigma_\nu)_\alpha{}^\beta Q_\beta$$

and from the Jacobi identity and from $[P_\mu, P_\nu] = 0$ we get $C = 0$ (a graded Lie algebra is defined by (15.3) and by the requirement that the graded Jacobi identities are fulfilled; the graded cyclic sum is defined just as the cyclic sum, except that there is an additional minus sign if two fermionic operators are interchanged). Finally, the anticommutator of Q_α with Q_β, by Lorentz covariance, must be a linear combination of Poincaré generators in the representations $(0, 0)$ and $(1, 0)$

of the Lorentz group. In addition, due to (15.8) and the Jacobi identity they must commute with P_μ. Such operators do not exist and we are left with (15.5), (15.6), (15.8) and

$$\{Q_\alpha, Q_\beta\} = 0 \tag{15.9}$$

which together with the Poincaré algebra form the simplest supersymmetric algebra which has the Poincaré group as space-time symmetry. It is called $N = 1$ supersymmetry, meaning that there is one two-spinor supercharge Q_α. Haag, Łopuszański & Sohnius (1975) have shown that only two-spinor supercharges are acceptable as generators of graded Lie algebras of symmetries of the S-matrix which are consistent with relativistic quantum field theory. Thus the only possible extension of the $N = 1$ supersymmetry is to have $N > 1$ two-spinor supercharges $Q_{\alpha i}$, $i = 1, \ldots, N$, which carry some representation of the internal symmetry†

$$[Q_{\alpha i}, T_j] = c_{ijk} Q_{\alpha k} \tag{15.10}$$

We shall limit our discussion to $N = 1$ supersymmetry which has been the most extensively studied one. In this case the full symmetry group of physical theory can be a direct product of $N = 1$ supersymmetry and of some internal symmetry group.

To conclude these considerations we note the four-component form of (15.6) (remember that in four-component notation Q is a Majorana spinor)

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu \tag{15.11}$$

15.3 Simple consequences of the supersymmetry algebra

We notice that

$$\left.\begin{aligned} &[Q_\alpha, W^2] \neq 0 \\ &[Q_\alpha, P^2] = 0 \end{aligned}\right\} \tag{15.12}$$

i.e. supersymmetry multiplets contain different spins but are always degenerate in mass. Supersymmetry transformations relate fermions to bosons as follows immediately from the fact that the Q_αs have spin one-half. Since in Nature we do not observe mass degeneracy among particles of different spin, supersymmetry must be broken either explicitly or spontaneously if it is relevant to physics. Spontaneous breaking is theoretically more appealing.

† In the $N = 1$ case the only non-trivially acting internal symmetry is a single $U(1)$, known under the name of R-symmetry: $[Q_\alpha, R] = Q_\alpha$, $[\bar{Q}_\alpha, R] = -\bar{Q}_\alpha$. Since under parity $Q_\alpha \leftrightarrow \bar{Q}^\alpha$ we must have $R \to -R$, i.e. the $U(1)$ symmetry group is chiral. In the four-spinor notation $[Q, R] = \mathrm{i}\gamma_5 Q$.

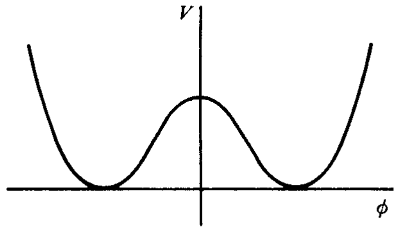


Fig. 15.1.

The vacuum structure of supersymmetric theories is of a special character. It follows immediately from (15.7) that if there exists a supersymmetrically invariant state

$$Q_\alpha|0\rangle = 0 \qquad \text{and} \qquad \bar{Q}_{\dot\alpha}|0\rangle = 0 \tag{15.13}$$

then obviously

$$E_{\text{vac}} = 0 \tag{15.14}$$

Since from (15.7) the spectrum of H is semi-positive definite, $E_{\text{vac}} = 0$ implies that the supersymmetrically invariant state is always at the absolute minimum of the potential, i.e. it is the true vacuum state. Thus, if the supersymmetric state exists, it is the ground state and supersymmetry is not spontaneously broken. In this respect supersymmetry is different from ordinary symmetries with which a symmetric state may exist without being the ground state. Only if a state invariant under supersymmetry does not exist is supersymmetry spontaneously broken and from

$$Q_\alpha|0\rangle \neq 0 \qquad \text{and/or} \qquad \bar{Q}_\alpha|0\rangle \neq 0 \tag{15.15}$$

and from (15.7). It follows that

$$E_{\text{vac}} > 0$$

Thus supersymmetry is unbroken if, and only if, $E_{\text{vac}} = 0$. It is spontaneously broken if, and only if, the energy of the vacuum is greater than zero. An important fact is that a non-zero vacuum expectation value of a scalar field does not necessarily mean that supersymmetry is spontaneously broken. This is illustrated in Fig. 15.1 which shows the case in which the expectation value of a scalar field breaks an internal symmetry but does not break supersymmetry because the vacuum energy is zero.

In the case of spontaneously broken supersymmetry one can prove an analogue of Goldstone's theorem. Consider the state obtained by acting on the vacuum with Q_α (or $\bar{Q}_{\dot{\alpha}}$)

$$|\Psi\rangle = Q_\alpha|0\rangle \quad (\text{or } |\Psi\rangle = \bar{Q}_{\dot{\alpha}}|0\rangle) \tag{15.16}$$

By the definition of spontaneously broken symmetry, $|\Psi\rangle$ is not zero. It is a state of odd fermion number and, because $[Q_\alpha, H] = 0$, it is degenerate with the vacuum. A state with such a property is a state of a single massless fermion of momentum $\mathbf{p}$, in the limit $\mathbf{p} \to 0$. So spontaneously broken supersymmetry requires the existence of a massless fermion, the so-called Goldstone fermion or goldstino.

Finally let us construct a hermitean operator

$$Q = (1/\sqrt{2})(Q_\alpha + Q_\alpha^\dagger) \tag{15.17}$$

for $\alpha = 1$ or 2 and restrict our attention to the subspace of states with zero three-momentum $\mathbf{p} = 0$.† In the space of states with $\mathbf{p} = 0$ we have

$$Q^2 = H \tag{15.18}$$

and all states of non-zero energy are paired by the action of Q. Given any boson state $|b\rangle$ of non-zero energy E one gets a fermion state $|f\rangle$ acting with Q on $|b\rangle$

$$Q|b\rangle = E^{1/2}|f\rangle$$

and vice versa

$$Q|f\rangle = E^{1/2}|b\rangle$$

However, the zero-energy states are annihilated by Q and therefore they are not paired in this way. Any bosonic or fermionic state of zero energy (vacuum state) forms a one-dimensional supersymmetry multiplet.

15.4 Superspace and superfields for $N = 1$ supersymmetry

Superspace

To formulate a supersymmetric field theory we must work out representations of supersymmetry algebra on field operators. The supersymmetry transformations must act linearly on field multiplets and leave the action invariant. An elegant formalism in which $N = 1$ supersymmetry is manifest is the formalism of superfields in superspace. To construct the superfield multiplets we first represent the supersymmetry algebra in terms of differential operators acting in superspace.

† For a detailed discussion of the supersymmetric spectrum, including massless states, see, for example, Sohnius (1985).

To do so we can imitate the construction of differential operators representing the Poincaré generators in Minkowski space which can proceed as follows.

Take a quantum field $\Phi(x)$ which depends on the four coordinates x^μ. We can consider $\Phi(x)$ to have been translated from $x^\mu = 0$

$$\Phi(x) = \exp(\mathrm{i}x \cdot P)\Phi(0)\exp(-\mathrm{i}x \cdot P) \tag{15.19}$$

or, differentially,

$$\mathrm{i}[P_\mu, \Phi] = \partial_\mu \Phi \tag{15.20}$$

The important point is that this transformation is compatible with the multiplication law

$$\exp(\mathrm{i}x \cdot P)\exp(\mathrm{i}y \cdot P) = \exp[\mathrm{i}(x+y) \cdot P] \tag{15.21}$$

which holds because the operators P_μ commute with each other.

The differential version of a Lorentz transformation can be derived writing

$$\begin{aligned}\Phi'(x) &= \exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M)\,\Phi(x)\exp(\tfrac{1}{2}\mathrm{i}\omega \cdot M) \\ &= \exp(\mathrm{i}x' \cdot P)\exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M)\,\Phi(0)\exp(\tfrac{1}{2}\mathrm{i}\omega \cdot M)\exp(-\mathrm{i}x' \cdot P)\end{aligned} \tag{15.22}$$

where $\omega \cdot M = \omega^{\mu\nu} M_{\mu\nu}$ and x' is determined by the equation

$$\begin{aligned}\exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M)\exp(\mathrm{i}x \cdot P) &= \exp(\mathrm{i}x' \cdot P)\exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M) \\ &= \exp[\mathrm{i}x \cdot (P - \tfrac{1}{2}\mathrm{i}\omega \cdot [M, P])]\exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M) + O(\omega^2)\end{aligned} \tag{15.23}$$

or explicitly, for infinitesimal $\omega^{\mu\nu}$ and using (see Problem 7.1)

$$[M^{\mu\nu}, P^\lambda] = -\mathrm{i}(g^{\mu\lambda}P^\nu - g^{\nu\lambda}P^\mu)$$

we get

$$x'^\lambda = x^\lambda + \omega^{\lambda\nu}x_\nu \tag{15.24}$$

Finally, writing

$$\exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot M)\Phi(0)\exp(\tfrac{1}{2}\mathrm{i}\omega \cdot M) = \exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot \Sigma)\Phi(0) \tag{15.25}$$

where the Σ is some matrix representation of the algebra of the $M_{\mu\nu}$ which depends on the spin of Φ, and using (15.22) one has

$$\Phi'(x) = \exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot \Sigma)\,\Phi(x') \tag{15.26}$$

In the differential form we get

$$\mathrm{i}[M_{\mu\nu}, \Phi(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu + \mathrm{i}\Sigma_{\mu\nu})\Phi(x) \tag{15.27}$$

This result follows essentially from the multiplication law (15.23).

We can now repeat a similar construction for the case of the $N = 1$ supersymmetry algebra. The first observation is that the presence of the anticommutator in the supersymmetry algebra calls for an extension of the Minkowski space into a superspace including some Grassmann parameters, in which differential operators will act representing the supersymmetry algebra. This stems from the fact that the algebra must be exponentiated into a group, i.e. the product of two group elements like

$$\exp(\mathrm{i}\Theta Q_\alpha)\exp(\mathrm{i}\bar{Q}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}) \tag{15.28}$$

where Θ^α and $\bar{\Theta}^{\dot{\alpha}}$ are for the time being arbitrary parameters, must be a group element. This is indeed ensured if we take Θ^α and $\bar{\Theta}^{\dot{\beta}}$ to be anticommuting components of two-spinors

$$\{\Theta^\alpha, \Theta^\beta\} = \{\bar{\Theta}^{\dot{\alpha}}, \bar{\Theta}^{\dot{\beta}}\} = \{\Theta^\alpha, \bar{\Theta}^{\dot{\beta}}\} = 0 \tag{15.29}$$

Since

$$\left.\begin{aligned} &[\Theta^\alpha Q_\alpha, \bar{Q}_{\dot{\beta}}\bar{\Theta}^{\dot{\beta}}] = 2\Theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\Theta}^{\dot{\beta}}P_\mu \\ &[\Theta^\alpha Q_\alpha, \Theta^\beta Q_\beta] = [\bar{Q}_{\dot{\beta}}\bar{\Theta}^{\dot{\beta}}, \bar{Q}_{\dot{\alpha}}\Theta^{\dot{\alpha}}] = 0 \end{aligned}\right\} \tag{15.30}$$

we can write the product (15.28) as a group element using the Baker–Campbell–Hausdorf formula (see Problem 2.9) in which the multiple commutators vanish for generators of the supersymmetry algebra

$$\exp(\mathrm{i}\Theta^\alpha Q_\alpha)\exp(\mathrm{i}\bar{Q}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}) = \exp[\mathrm{i}(\Theta^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}} + \mathrm{i}\Theta^\alpha\sigma^\mu{}_{\alpha\dot{\beta}}\bar{\Theta}^{\dot{\beta}}P_\mu)] \tag{15.31}$$

The commonly used choice for Θ and $\bar{\Theta}$ is the 'real' superspace $z = (x^\mu, \Theta^\alpha, \bar{\Theta}^{\dot{\alpha}})$ with $\bar{\Theta}^{\dot{\alpha}} = (\Theta^\alpha)^*$ which in the four-component notation corresponds to choosing $\Theta_i,\ i = 1, \ldots, 4$, as a Majorana spinor.

As for the Poincaré group we can discuss separately the Lorentz transformations L which form a subgroup of the super-Poincaré group and supertranslations which form a coset manifold with respect to L. Any element of the super-Poincaré group can be uniquely decomposed into a product

$$S(x, \Theta, \bar{\Theta})\, L(\omega) \tag{15.32}$$

where a finite supertranslation $S(x, \Theta, \bar{\Theta})$ is parametrized as follows

$$S(x, \Theta, \bar{\Theta}) = \exp(\mathrm{i}x \cdot P + \mathrm{i}\Theta^\alpha Q_\alpha + \mathrm{i}\bar{Q}_{\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}) \tag{15.33}$$

The multiplication law for two successive supertranslations, analogous to (15.21), can be evaluated with the help of the Baker–Campbell–Hausdorf formula and (15.30):

$$S(y, \eta, \bar{\eta})\, S(x, \Theta, \bar{\Theta}) = S(x', \Theta', \bar{\Theta}') \tag{15.34}$$

where

$$\left.\begin{aligned} x'^{\mu} &= x^{\mu} + y^{\mu} - \mathrm{i}\Theta^{\alpha}\sigma^{\mu}{}_{\alpha\dot\beta}\bar\eta^{\dot\beta} + \mathrm{i}\eta^{\alpha}\sigma^{\mu}{}_{\alpha\dot\beta}\bar\Theta^{\dot\beta} \\ \Theta' &= \eta + \Theta \\ \bar\Theta' &= \bar\eta + \bar\Theta \end{aligned}\right\} \tag{15.35}$$

In complete analogy with (15.23), multiplication with the Lorentz group gives

$$L(\omega)\, S(x, \Theta, \bar\Theta) = S(x', \Theta', \bar\Theta')\, L(\omega) \tag{15.36}$$

where, for infinitesimal transformations, using (15.5) we get

$$\left.\begin{aligned} x'^{\lambda} &= x^{\lambda} + \omega^{\lambda\nu}x_{\nu} \\ \Theta'^{\alpha} &= \Theta^{\alpha} - \tfrac{1}{4}\mathrm{i}\omega^{\mu\nu}(\sigma_{\mu\nu})_{\beta}{}^{\alpha}\Theta^{\beta} \\ \bar\Theta'^{\dot\alpha} &= \bar\Theta^{\dot\alpha} + \tfrac{1}{4}\mathrm{i}\omega^{\mu\nu}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\Theta^{\dot\beta} \end{aligned}\right\} \tag{15.37}$$

A superfield $\Phi(x, \Theta, \bar\Theta)$ is now defined as a function of the parameters Θ and $\bar\Theta$ in addition to x_μ such that (see (15.19))

$$\Phi(x, \Theta, \bar\Theta) = S(x, \Theta, \bar\Theta)\Phi(0, 0, 0)S^{-1}(x, \Theta, \bar\Theta) \tag{15.38}$$

with the coordinate transformations given by (15.35), and

$$L(\omega)\Phi(x, \Theta, \bar\Theta)L^{-1}(\omega) = \exp(-\tfrac{1}{2}\mathrm{i}\omega \cdot \Sigma)\Phi(x', \Theta', \bar\Theta') \tag{15.39}$$

where for infinitesimal ω the parameters x', Θ', $\bar\Theta'$ are given by (15.37). Expanding

$$\begin{aligned} \Phi(x', \Theta', \bar\Theta') &= \Phi(x, \Theta, \bar\Theta) + (y - \mathrm{i}\Theta\sigma\bar\eta + \mathrm{i}\eta\sigma\bar\Theta)(\partial\Phi/\partial x) \\ &\quad + \eta(\partial\Phi/\partial\Theta) + \bar\eta(\partial\Phi/\partial\bar\Theta) + \cdots \\ &= \Phi(x, \Theta, \bar\Theta) + \mathrm{i}y[P, \Phi] + \mathrm{i}[\eta Q, \Phi] + \mathrm{i}[\bar Q\bar\eta, \Phi] + \cdots \end{aligned} \tag{15.40}$$

we get the differential form of the supersymmetry generators acting on a superfield $\Phi(x, \Theta, \bar\Theta)$

$$\left.\begin{aligned} \mathrm{i}[P_\mu, \Phi] &= \partial_\mu\Phi \\ \mathrm{i}[\eta^{\alpha}Q_{\alpha}, \Phi] &= \eta^{\alpha}(\partial/\partial\Theta^{\alpha} + \mathrm{i}\sigma^{\mu}{}_{\alpha\dot\beta}\bar\Theta^{\dot\beta}\partial_\mu)\Phi \\ \mathrm{i}[\bar Q_{\dot\alpha}\bar\eta^{\dot\alpha}, \Phi] &= \bar\eta^{\dot\alpha}(\partial/\partial\bar\Theta^{\dot\alpha} + \mathrm{i}\Theta^{\beta}\sigma^{\mu}{}_{\beta\dot\alpha}\partial_\mu)\Phi \end{aligned}\right\} \tag{15.41}$$

Using (15.37) and (15.39) we also get

$$\mathrm{i}[M_{\mu\nu}, \Phi] = x_\mu\partial_\nu - x_\nu\partial_\mu + \tfrac{1}{2}\mathrm{i}\Theta^{\alpha}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}(\partial/\partial\Theta^{\beta}) - \tfrac{1}{2}\mathrm{i}\bar\Theta^{\dot\beta}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\partial/\partial\bar\Theta^{\dot\alpha}) + \mathrm{i}\Sigma_{\mu\nu} \tag{15.42}$$

If Φ does not have overall spinor indices we can factorize out η in the last two equations of (15.41), for example,

$$\mathrm{i}[Q_\alpha, \Phi] = (\partial/\partial\Theta^{\alpha} + \mathrm{i}\sigma^{\mu}{}_{\alpha\dot\beta}\bar\Theta^{\dot\beta}\partial_\mu)\Phi \tag{15.43}$$

The important notion in supersymmetry is that of covariant derivatives. As usual one wants to generalize

$$\frac{\partial}{\partial\Theta^\alpha}\Phi \to D_\alpha\Phi, \qquad \frac{\partial}{\partial\bar{\Theta}^{\dot{\alpha}}}\Phi \to \bar{D}_{\dot{\alpha}}\Phi$$

so that the quantities $D_\alpha\Phi$, and $\bar{D}_{\dot{\alpha}}\Phi$ transform under supertranslations as the fields themselves. (Note that this requirement is satisfied by $\partial_\mu\Phi$.) Thus we require

$$\left.\begin{aligned}\{D_\beta, Q_\alpha\} = 0\\ \{D_\beta, \bar{Q}_{\dot{\alpha}}\} = 0\end{aligned}\right\} \tag{15.44}$$

and similarly for $\bar{D}_{\dot{\beta}}$. This gives

$$\left.\begin{aligned}D_\beta &= \partial/\partial\Theta^\beta - \mathrm{i}\sigma^\mu{}_{\beta\dot{\beta}}\bar{\Theta}^{\dot{\beta}}\partial_\mu\\ \bar{D}_{\dot{\beta}} &= -\partial/\partial\bar{\Theta}^{\dot{\beta}} + \mathrm{i}\Theta^\beta\sigma^\mu{}_{\beta\dot{\beta}}\partial_\mu\end{aligned}\right\} \tag{15.45}$$

One can check that

$$\left.\begin{aligned}\{D_\alpha, D_\beta\} &= \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0\\ \{D_\alpha, \bar{D}_{\dot{\beta}}\} &= 2\mathrm{i}\sigma^\mu{}_{\alpha\dot{\beta}}\partial_\mu\end{aligned}\right\} \tag{15.46}$$

In the four-component notation we have

$$D = \begin{pmatrix} D_\alpha \\ \bar{D}^{\dot{\alpha}}\end{pmatrix}, \qquad \Theta = \begin{pmatrix}\Theta_\alpha \\ \bar{\Theta}^{\dot{\alpha}}\end{pmatrix}, \qquad \bar{\Theta} = (\Theta^\alpha, \bar{\Theta}_{\dot{\alpha}})$$

where

$$\bar{D}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\beta}} = \partial/\partial\bar{\Theta}_{\dot{\alpha}} - \mathrm{i}\bar{\sigma}^{\mu\dot{\alpha}\beta}\Theta_\beta\partial_\mu \tag{15.47}$$

(since $\partial/\partial\bar{\Theta}_{\dot{\beta}} = -\varepsilon^{\dot{\beta}\dot{\alpha}}\,\partial/\partial\bar{\Theta}^{\dot{\alpha}}$) and consequently

$$D = \partial/\partial\bar{\Theta} - \mathrm{i}\gamma^\mu\Theta\partial_\mu \tag{15.48}$$

Superfields

We are now prepared to construct superfields as power series expansions in Θ^α and $\bar{\Theta}^{\dot{\alpha}}$. The most general superfield in the two-component notation reads (we denote it by $F(x, \Theta, \bar{\Theta})$ reserving the Φ for the chiral superfields)

$$\begin{aligned}F(x, \Theta, \bar{\Theta}) = f(x) + \Theta\chi(x) + \bar{\Theta}\bar{\chi}(x) + \Theta\Theta m(x) + \bar{\Theta}\bar{\Theta}n(x) + \Theta\sigma^\mu\bar{\Theta}v_\mu(x)\\ + \Theta\Theta\bar{\Theta}\bar{\lambda}(x) + \bar{\Theta}\bar{\Theta}\Theta\lambda(x) + \Theta\Theta\bar{\Theta}\bar{\Theta}d(x)\end{aligned} \tag{15.49}$$

This is the most general superfield due to the vanishing of the square of each component of Θ^α and $\bar{\Theta}^{\dot{\alpha}}$. Remember that

$$\Theta\Theta = \Theta^\alpha\Theta_\alpha = \varepsilon_{\alpha\beta}\Theta^\alpha\Theta^\beta, \qquad \bar{\Theta}\bar{\Theta} = -\bar{\Theta}^{\dot{\alpha}}\bar{\Theta}_{\dot{\alpha}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\Theta}^{\dot{\alpha}}\bar{\Theta}^{\dot{\beta}}$$

or in other words

$$\left.\begin{aligned}\Theta^\alpha\Theta^\beta &= -\tfrac{1}{2}\varepsilon^{\alpha\beta}\Theta\Theta\\ \bar\Theta^{\dot\alpha}\bar\Theta^{\dot\beta} &= \tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\Theta\bar\Theta\end{aligned}\right\} \tag{15.50}$$

Assuming that the superfield has no overall Lorentz indices, this superfield contains four complex scalar fields $f(x)$, $m(x)$, $n(x)$ and $d(x)$, two spinors $\chi(x)$ and $\lambda(x)$ in the $(\frac{1}{2}, 0)$ representation and two unrelated spinors $\bar\chi(x)$ and $\bar\lambda(x)$ in the $(0, \frac{1}{2})$ representation of the Lorentz group, and one complex vector field $v_\mu(x)$. Thus we have altogether 16 bosonic and 16 fermionic degrees of freedom (a degree of freedom is an unconstrained single real field).

The transformation laws under supersymmetric transformations for the components of $F(x, \Theta, \bar\Theta)$ can be obtained by comparing powers of Θ and $\bar\Theta$ in the equation

$$\delta F = \delta_\eta F + \delta_{\bar\eta} F = \delta f(x) + \Theta\delta\chi(x) + \bar\Theta\delta\bar\chi(x) + \Theta\Theta\delta m(x) + \cdots \tag{15.51}$$

where the l.h.s. of (15.51) is given by (15.41). Note that it follows from (15.41) that the variation of the highest component of the superfield must always be a space-time derivative.

One can see that the general superfield yields the component multiplet $(f, \chi, \bar\chi, m, n, v_\mu, \lambda, \bar\lambda, d)$ which is reducible under supertranslations. To get irreducible component multiplets one must impose some conditions on the superfields. These conditions must be covariant with respect to supersymmetric transformations but otherwise there is no general rule concerning their choice. One widely used set of such conditions is that which defines a so-called chiral (left-handed) superfield $\Phi(x, \Theta, \bar\Theta)$ with

$$\bar D_{\dot\alpha}\Phi = 0 \tag{15.52}$$

and an antichiral (right-handed) superfield $\bar\Phi(x, \Theta, \bar\Theta)$ with

$$D_\alpha\bar\Phi = 0 \tag{15.53}$$

Using (15.45) and solving the first order partial differential equations we get

$$\Phi(x, \Theta, \bar\Theta) = \exp(-\mathrm{i}\Theta^\beta\sigma^\mu{}_{\beta\dot\alpha}\bar\Theta^{\dot\alpha}\partial_\mu)\hat\Phi(x, \Theta) \tag{15.54}$$

and

$$\bar\Phi(x, \Theta, \bar\Theta) = \exp(\mathrm{i}\Theta^\beta\sigma^\mu{}_{\beta\dot\alpha}\bar\Theta^{\dot\alpha}\partial_\mu)\hat{\bar\Phi}(x, \bar\Theta) \tag{15.55}$$

The fields $\hat\Phi(x, \Theta)$ and $\hat{\bar\Phi}(x, \bar\Theta)$ have the expansions

$$\hat\Phi(x, \Theta) = A(x) + 2\Theta\Psi(x) - \Theta\Theta F(x) \tag{15.56}$$

$$\hat{\bar\Phi}(x, \bar\Theta) = \bar A(x) + 2\bar\Psi(x)\bar\Theta - \bar\Theta\bar\Theta\bar F(x) \tag{15.57}$$

and each contains Weyl spinors of only one chirality. The chiral superfield $\Phi(x, \Theta, \bar{\Theta})$ is formally obtained from $\hat{\Phi}(x, \Theta)$ by the translation (15.54)

$$\begin{aligned}\Phi(x, \Theta, \bar{\Theta}) = \hat{\Phi}(x^{\mu} - \mathrm{i}\Theta^{\beta}\sigma^{\mu}{}_{\beta\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}, \Theta) &= A(x) - \mathrm{i}\Theta\sigma^{\mu}\bar{\Theta}\partial_{\mu}A(x) \\ &- \tfrac{1}{4}\Theta\Theta\bar{\Theta}\bar{\Theta}\partial^{2}A(x) + 2\Theta\Psi(x) + \mathrm{i}\Theta\Theta\partial_{\mu}\Psi(x)\sigma^{\mu}\bar{\Theta} - \Theta\Theta F(x)\end{aligned} \tag{15.58}$$

and the analogous expression also holds for $\hat{\Phi}(x, \Theta, \bar{\Theta})$. In deriving (15.58) we have used the relation

$$(\Theta\sigma^{\mu}\bar{\Theta})(\Theta\sigma^{\nu}\bar{\Theta}) = \tfrac{1}{2}g^{\mu\nu}(\Theta\Theta)(\bar{\Theta}\bar{\Theta}) \tag{15.59}$$

The transformation rules under supertranslation of the component fields A, Ψ and F follow from the general formula (15.51) and read

$$\left.\begin{aligned}\delta A &= \delta_{\eta}A + \delta_{\bar{\eta}}A = 2\eta^{\alpha}\Psi_{\alpha} \\ \delta\Psi_{\alpha} &= -\eta_{\alpha}F - \mathrm{i}(\sigma^{\mu}\bar{\eta})_{\alpha}\partial_{\mu}A \\ \delta F &= -2\mathrm{i}\partial_{\mu}\Psi\sigma^{\mu}\bar{\eta}\end{aligned}\right\} \tag{15.60}$$

We see explicitly that the component multiplet of the chiral superfield is irreducible under supertranslations. Let us also mention that the chiral superfield is not the only one used in the construction of supersymmetric theories. In particular, supersymmetric gauge theories require vector superfields defined by the condition $V(x, \Theta, \bar{\Theta}) = V^{\dagger}(x, \Theta, \bar{\Theta})$ that contain spin one bosons.

15.5 Supersymmetric lagrangian; Wess–Zumino model

Let us begin our considerations with the observation that the product of two superfields is again a superfield

$$F_1(x, \Theta, \bar{\Theta})F_2(x, \Theta, \bar{\Theta}) = F_3(x, \Theta, \bar{\Theta}) \tag{15.61}$$

This follows from the representation (15.41) of the superalgebra which is of first order in differential operators. From two superfields Φ_1 and Φ_2 of the same chirality one can construct another chiral superfield

$$\Phi_3 = \Phi_1\Phi_2 \tag{15.62}$$

and from two superfields of different chirality – two vector superfields

$$\left.\begin{aligned}V_{\mathrm{S}} &= \tfrac{1}{2}(\Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2) \\ V_{\mathrm{A}} &= \tfrac{1}{2}\mathrm{i}(\Phi_1\bar{\Phi}_2 - \bar{\Phi}_1\Phi_2)\end{aligned}\right\} \tag{15.63}$$

corresponding to the symmetrized and antisymmetrized product, respectively. The Φ_3 in (15.62) is chiral due to the fact that $\bar{D}$ is a first-order differential operator which obeys the Leibnitz rule

$$\bar{D}(\Phi_1\Phi_2) = (\bar{D}\Phi_1)\Phi_2 + \Phi_1\bar{D}\Phi_2 \tag{15.64}$$

A very important chiral superfield is the one which, as we shall see, can be identified with the kinetic part of the supersymmetric lagrangian. It reads

$$\Phi_{\mathrm{K}} = \tfrac{1}{4}\Phi\bar{D}^2\bar{\Phi} \tag{15.65}$$

where $\bar{\Phi} \equiv \Phi^\dagger$ and it is a left-handed chiral superfield since $\bar{D}^2\bar{\Phi}$ is a left-handed chiral superfield (see (15.46))

$$\bar{D}\bar{D}^2\bar{\Phi} = 0 \tag{15.66}$$

and the product of two left-handed superfields is again a left-handed superfield.

We will need the notion of integration over the Grassmann variables Θ and $\bar{\Theta}$. This has been introduced in Section 2.5 and we just add the following definitions:

$$\int \mathrm{d}^2\Theta = \int \mathrm{d}\Theta^2\,\mathrm{d}\Theta^1, \qquad \int \mathrm{d}^2\bar{\Theta} = \int \mathrm{d}\bar{\Theta}^1\,\mathrm{d}\bar{\Theta}^2 \tag{15.67}$$

Therefore

$$\int \mathrm{d}^2\Theta\Theta\Theta = \int \mathrm{d}^2\bar{\Theta}\bar{\Theta}\bar{\Theta} = -2 \tag{15.68}$$

We also note that the Dirac δ-functions are

$$\left.\begin{array}{ll} \delta(\Theta_\alpha) = \Theta_\alpha, & \delta(-\Theta_\alpha) = -\delta(\Theta_\alpha) \\ \delta^{(2)}(\Theta) = -\frac{1}{2}\Theta\Theta, & \delta^{(2)}(\bar{\Theta}) = -\frac{1}{2}\bar{\Theta}\bar{\Theta} \end{array}\right\} \tag{15.69}$$

The crucial observation for constructing supersymmetric lagrangians is the fact that for the general scalar superfield (15.49) the integral

$$S_D = \tfrac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,F(x,\Theta,\bar{\Theta}) \tag{15.70}$$

and for the chiral superfield the integral

$$S_F = \tfrac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^2\Theta\,\Phi(x,\Theta,\bar{\Theta}) + \mathrm{h.c.} \tag{15.71}$$

are invariant under supertranslations. Indeed we see from (15.49) that

$$\int \mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,F(x,\Theta,\bar{\Theta}) = 4d(x) \tag{15.72}$$

and we remember that the variation of $d(x)$ under supertranslation is a four-derivative. Thus S_D is an invariant. For chiral fields S_D vanishes but the $\int \mathrm{d}^4x\,\mathrm{d}^2\Theta$

alone gives an invariant result. Under $\int \mathrm{d}^4x$ we can use the expansion (15.56) since the transformation (15.54) in its infinitesimal form gives $\delta\Phi$ which is a four-derivative. Thus

$$S_F = \frac{1}{2}\int \mathrm{d}^4x\, F(x) + \text{h.c.} \tag{15.73}$$

where $F(x)$ is the highest component of the chiral superfield, and S_F is an invariant of supertranslations since $\delta F(x)$ is also a four-derivative. (We always assume that the fields fall off fast enough at infinity.) Of course the superfields in S_D and S_F must in general arise as products of 'single-particle' superfields since the lagrangian has to contain terms at least bilinear in these elementary fields.

As the last step towards constructing the supersymmetric lagrangian let us do some dimensional analysis. Clearly, from (15.7) and (15.28) each Θ and $\bar{\Theta}$ carries a mass dimension $-\frac{1}{2}$. Thus taking the canonical dimensions 1 and $\frac{3}{2}$ for the boson and fermion fields we conclude that the whole 'single-particle' superfield has dimension 1. The derivatives D_α and $\bar{D}_{\dot{\alpha}}$ have dimension $\frac{1}{2}$ each and from (15.68) the dimension of the measure $\mathrm{d}^4x\,\mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta} \equiv \mathrm{d}z$ is -2 and the dimension of the chiral measure $\mathrm{d}^4x\,\mathrm{d}^2\Theta$ is -3. In order to have a renormalizable theory we have to construct a dimensionless action built of single-particle superfields so that no negative-dimension coupling constants are present.

We are now ready to write down the supersymmetric lagrangian for a chiral superfield Φ with complex field components A, Ψ^α and F describing four bosonic and four fermionic degrees of freedom. In four dimensions this is the smallest possible number of degrees of freedom which can be contained in a superfield since any multiplet must contain a spinor which has at least two complex (Weyl) or four real (Majorana) components. It is clear from our considerations that the most general action for Φ (the so-called Wess–Zumino action) with the superkinetic term quadratic in the field reads

$$S_{\mathrm{WZ}} = \frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^2\Theta\left(\tfrac{1}{8}\Phi\bar{D}^2\bar{\Phi} - \tfrac{1}{2}m\Phi^2 - \tfrac{1}{3}g\Phi^3\right) + \text{h.c.} \tag{15.74}$$

(the term $\lambda\Phi$ can be eliminated by a redefinition of fields). The first term in (15.74) can also be written as $\mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\;\Phi\bar{\Phi}$.

In terms of the field components denoted as follows

$$\mathrm{Re}\,A \equiv S, \quad \mathrm{Im}\,A \equiv P, \quad \mathrm{Re}\,F \equiv F_S, \quad \mathrm{Im}\,F \equiv F_P$$

and

$$\Psi = \begin{pmatrix} \Psi_\alpha \\ \bar{\Psi}^{\dot{\alpha}} \end{pmatrix}$$

where Ψ is the four-component Majorana spinor, one can derive from (15.74) the following lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\big(\partial_\mu S \partial^\mu S + \partial_\mu P \partial^\mu P + \mathrm{i}\bar{\Psi}\not{\partial}\Psi + F_S^2 + F_P^2\big) - m(SF_S + PF_P + \tfrac{1}{2}\bar{\Psi}\Psi)$$
$$- g\big[(S^2 - P^2)F_S + 2SPF_P + \bar{\Psi}(S - \mathrm{i}\gamma_5 P)\Psi\big] + 4\text{-div} \qquad (15.75)$$

This is the lagrangian of the original Wess–Zumino supersymmetric model. It gives the following equations of motion

$$\left.\begin{aligned}
\partial^2 S &= -mF_S - 2g(SF_S + PF_P + \tfrac{1}{2}\bar{\Psi}\Psi)\\
\partial^2 P &= -mF_P - 2g(SF_p - PF_s - \tfrac{1}{2}\bar{\Psi}\mathrm{i}\gamma_5\Psi)\\
\mathrm{i}\not{\partial}\Psi &= m\Psi + 2g(S - \mathrm{i}\gamma_5 P)\Psi\\
F_S &= mS + g(S^2 - P^2)\\
F_P &= mP + 2gSP
\end{aligned}\right\} \qquad (15.76)$$

An important observation is that the equations for the fields F_S and F_P are purely algebraic and consequently the F_S and F_P can be eliminated from the lagrangian and from the equations of motion. We then get

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu S\partial^\mu S - m^2 S^2) + \tfrac{1}{2}(\partial_\mu P\partial^\mu P - m^2 P^2) + \tfrac{1}{2}\bar{\Psi}(\mathrm{i}\not{\partial} - m)\Psi$$
$$- mgS(S^2 + P^2) - g\bar{\Psi}(S - \mathrm{i}\gamma_5 P)\Psi - \tfrac{1}{2}g^2(S^2 + P^2)^2 \qquad (15.77)$$

and correspondingly

$$\left.\begin{aligned}
(\partial^2 + m^2)S &= -mg(3S^2 + P^2) - 2g^2 S(S^2 + P^2) - g\bar{\Psi}\Psi\\
(\partial^2 + m^2)P &= -2mgSP - 2g^2 P(S^2 + P^2) + g\bar{\Psi}\mathrm{i}\gamma_5\Psi\\
(\mathrm{i}\not{\partial} - m)\Psi &= 2g(S - \mathrm{i}\gamma_5 P)\Psi
\end{aligned}\right\} \qquad (15.78)$$

The elimination of the auxiliary fields F_S and F_P gives, however, a theory which is supersymmetric only 'on-shell'. The lagrangian (15.77) is invariant under transformations obtained from the supersymmetry transformations by elimination of the fields F_S and F_P, using their own equations of motion, and the on-shell algebra closes only if the equations of motion of the dynamical fields hold. This is understandable by counting the off-shell and on-shell number of degrees of freedom for scalar and fermion fields.

15.6 Supersymmetry breaking

Supersymmetry has many appealing theoretical features, but cannot be an exact symmetry of Nature: it predicts that fermions and scalars appear in supersymmetric multiplets with all particles in the multiplet having the same mass, in obvious contradiction to experiment. The only possible explanation is that, if supersymmetric partners of known particles exist, they must be significantly heavier (they cannot be weakly coupled and therefore escape detection because the structure of the couplings follows strictly from the constructions of the supersymmetric lagrangian). Therefore, one has to propose mechanisms of supersymmetry breaking, which would, however, preserve the main virtue of supersymmetry which is the absence of quadratic divergences. The most natural and elegant choice is spontaneous supersymmetry breaking, analogous to that described in Chapter 9: supersymmetry is broken by a non-symmetric ground state (the vacuum).

Before discussing specific examples, we consider the most general renormalizable lagrangian describing interaction of n chiral fields Φ_i, $i = 1, \ldots, n$:

$$\mathcal{L} = \tfrac{1}{8}\Phi_i \bar{D}^2 \bar{\Phi}_i - W(\Phi_i) + \text{h.c.} \tag{15.79}$$

where $W(x_1, \ldots, x_n)$ is an arbitrary polynomial of degree 3 of n variables

$$W(\Phi_1, \ldots, \Phi_n) = \lambda_i \Phi_i + \tfrac{1}{2} m_{ij} \Phi_i \Phi_j + \tfrac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \tag{15.80}$$

Auxiliary fields F_i may be eliminated by their (purely algebraic) equations of motion:

$$F_i = -\frac{\partial W^*}{\partial A_i^*} \tag{15.81}$$

(in this and the following equations we introduce the function $W = W(A_1, \ldots, A_n)$ which has the same dependence on the scalar fields as the superpotential does on the superfields).

In terms of the physical component fields A_i, ψ_i lagrangian (15.79) may be read (after dropping all total four-derivatives):

$$\begin{aligned}\mathcal{L} = {} & \mathrm{i}\partial_\mu \bar{\psi}_i \bar{\sigma}^\mu \psi_i + \partial^\mu A_i^* \partial^\mu A_i \\ & - \frac{1}{2}\left[\frac{\partial^2 W}{\partial A_i \partial A_j}\psi_i \psi_j + \text{h.c.}\right] - V(A_i, A_j^*)\end{aligned} \tag{15.82}$$

where the scalar potential V is

$$V = F_i^* F_i = \left|\frac{\partial W}{\partial A_i}\right|^2 \tag{15.83}$$

In the simplest case, lagrangian (15.79) contains only one superfield Φ (compare with (15.74)–(15.78) for the Wess–Zumino lagrangian). The condition $V = 0$ for a supersymmetric minimum of the potential has the form (we keep the linear term in the superpotential):

$$\lambda + mA + gA^2 = 0 \tag{15.84}$$

Denoting

$$\operatorname{Re} A \equiv S \qquad \operatorname{Im} A \equiv P \tag{15.85}$$

(15.84) gives (assuming that λ, m and g are real):

$$\left.\begin{aligned} &\lambda + mS + g(S^2 - P^2) = 0 \\ &mP + 2gSP = 0 \end{aligned}\right\} \tag{15.86}$$

Eqs. (15.86) can always be satisfied. The solution depends on the sign of the expression $m^2 - 4g\lambda$:

$$\left.\begin{aligned} &\langle S\rangle = -\frac{1}{2g}[m \pm (m^2 - 4g\lambda)^{1/2}] \\ &\langle P\rangle = 0 \end{aligned}\right\} \quad \text{for } m^2 - 4g\lambda \geq 0 \tag{15.87}$$

$$\left.\begin{aligned} &\langle S\rangle = -\frac{1}{2g}m \\ &\langle P\rangle = \pm\frac{1}{2g}(4g\lambda - m^2)^{1/2} \end{aligned}\right\} \quad \text{for } m^2 - 4g\lambda \leq 0 \tag{15.88}$$

For each of these solutions $V_{\min} = 0$, i.e. in the simplest version of the Wess–Zumino model supersymmetry remains unbroken. Moreover, shifting the fields $S \to S - \langle S\rangle$, $P \to P - \langle P\rangle$ one can eliminate the term $\lambda\Phi$ from the lagrangian and shift one of the minima to $\langle S\rangle = \langle P\rangle = 0$. It is worth noting that for λ, $g \neq 0$ the considered lagrangian does not possess any global symmetry, so such a shift does not lead to any global symmetry breaking and no massless Goldstone boson appears in the spectrum.

Spontaneous supersymmetry breaking is possible and, in some special cases, even unavoidable in models with more chiral superfields. It was shown (O'Raifeartaigh 1975) that at least three scalar superfields are necessary to break supersymmetry spontaneously in the absence of gauge fields. Consider a model with three chiral superfields with the following superpotential:

$$W = m\Phi_1\Phi_2 + \Phi_3(\lambda + g\Phi_1^2) \tag{15.89}$$

Therefore the scalar potential V:

$$V = \sum_{i=1}^{3} F_i^* F_i = |mA_2 + 2gA_1A_3|^2 + m^2|A_1|^2 + |\lambda + gA_1^2|^2 \tag{15.90}$$

The conditions for a supersymmetric minimum of the potential, $F_i = 0$, read

$$mA_2 + 2gA_1A_3 = 0 \tag{15.91}$$

$$mA_1 = 0 \tag{15.92}$$

$$\lambda + gA_1^2 = 0 \tag{15.93}$$

One sees that for $\lambda \neq 0$ (15.91)–(15.93) cannot be simultaneously fulfilled.

The potential (15.90) is always positive and supersymmetry in this model is spontaneously broken.

It is very instructive to analyse the mass spectrum of the O'Raifeartaigh model. For simplicity we assume $m^2 > 2g\lambda$, the opposite case may be easily analysed in a similar way. For $m^2 > 2g\lambda$ the minimum of the potential,

$$V_{\min} = \lambda^2 \tag{15.94}$$

is achieved for

$$\langle A_1 \rangle = \langle A_2 \rangle = 0 \tag{15.95}$$

and with arbitrary $\langle A_3 \rangle$. The potential possesses the so-called flat direction. Introduce a new parameter μ defined by

$$\langle A_3 \rangle = \frac{\mu}{2g} \tag{15.96}$$

(μ can always be chosen real by proper field redefinition). With this choice, the term in the lagrangian

$$-\tfrac{1}{2} m_{ij} \psi_i \psi_j - g_{ijk} \psi_i \psi_j \langle A_k \rangle + \text{h.c.} \tag{15.97}$$

gives for the Weyl fermion fields the following mass matrix:

$$M_\psi = \begin{pmatrix} \mu & m & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{15.98}$$

with the mass eigenvalues

$$\left.\begin{aligned} m_{\psi_{1,2}} &= (m^2 + \mu^2/4)^{1/2} \pm \mu/2 \\ m_{\psi_3} &= 0 \end{aligned}\right\} \tag{15.99}$$

For scalar S_i, P_i fields the potential (15.90) gives the following mass matrices:

$$\left.\begin{aligned} M_S^2 &= \begin{pmatrix} \mu^2 + m^2 + 2g\lambda & m\mu & 0 \\ m\mu & m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ M_P^2 &= \begin{pmatrix} \mu^2 + m^2 - 2g\lambda & m\mu & 0 \\ m\mu & m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}\right\} \tag{15.100}$$

The physical masses are

$$\left.\begin{aligned} m^2_{\tilde{S}_{1,2}} &= \mu^2/2 + m^2 + g\lambda \pm [(\mu^2/2 + g\lambda)^2 + m^2\mu^2]^{1/2} \\ m^2_{\tilde{P}_{1,2}} &= \mu^2/2 + m^2 - g\lambda \pm [(\mu^2/2 - g\lambda)^2 + m^2\mu^2]^{1/2} \\ m^2_{\tilde{S}_3} &= m^2_{\tilde{P}_3} = 0 \end{aligned}\right\} \tag{15.101}$$

As can be seen, mass matrices (15.98), (15.100) obtained in the presence of supersymmetry breaking fulfil the following condition:

$$M_S^2 + M_P^2 = 2M_\psi^2 \tag{15.102}$$

instead of being simply equal $M_S^2 = M_P^2 = M_\psi^2$, like in the unbroken phase. Also, they do not commute and cannot be simultaneously diagonalized. Hence, the massive physical fermion and scalar fields are not degenerate and they are different mixtures of the initial supermultiplets. Taking the trace of (15.102) one finds the very important relation connecting the masses of the physical states:

$$\sum_{\text{fermion states}} m_i^2 = \sum_{\text{boson states}} m_i^2 \tag{15.103}$$

Since the number of physical fields with spin s is $2s + 1$, we can rewrite (15.103) in the form:

$$S\,\mathrm{Tr}\,\mathcal{M}^2 \equiv \sum_s (-1)^s (2s+1) m_s^2 = 0 \tag{15.104}$$

In the case of unbroken supersymmetry this condition is trivially fulfilled, as the masses of all particles inside each supersymmetric multiplet are equal and (15.104) reduces to

$$\sum_{\text{(supermultiplets)}} \; \sum_{\text{(states in supermultiplet)}} (-1)^s (2s+1) = 0 \tag{15.105}$$

which reflects the basic rule that the number of the fermionic and bosonic states in the supermultiplets are the same.

We have derived (15.104) for the special case of the O'Raifeartaigh lagrangian, but similar equations hold generally for all models with spontaneously broken supersymmetry, and also those including the gauge interactions. We give the proof here only for a model defined by the lagrangian (15.79) generalized to the case of completely arbitrary (not necessarily polynomial) superpotential $W(\Phi_1, \ldots, \Phi_n)$ (Ferrara, Girardello & Palumbo 1979). The scalar potential given by (15.83) is semi-positive definite as a consequence of supersymmetry. As discussed in Section 15.3, supersymmetry is spontaneously broken if, and only if, $V_{\min} > 0$.

The minimum of the potential is given by the conditions

$$\left.\begin{aligned} \frac{\partial V}{\partial A_i} &= \frac{\partial^2 W}{\partial A_i \partial A_j}\frac{\partial W^*}{\partial A_j^*} = 0 \\ \frac{\partial V}{\partial A_i^*} &= \frac{\partial W}{\partial A_j}\frac{\partial^2 W^*}{\partial A_i^* \partial A_j^*} = 0 \end{aligned}\right\} \tag{15.106}$$

The scalar mass matrix can be obtained by expanding the potential (15.83) around the minimum:

$$\begin{aligned} V = V_{\min} &+ \frac{\partial^2 V}{\partial A_i \partial A_j^*}(A_i - \langle A_i \rangle)(A_j^* - \langle A_j \rangle^*) \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial A_i \partial A_j}(A_i - \langle A_i \rangle)(A_j - \langle A_j \rangle) \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial A_i^* \partial A_j^*}(A_i^* - \langle A_i \rangle^*)(A_j^* - \langle A_j \rangle^*) \end{aligned} \tag{15.107}$$

where $\langle A_i \rangle$ are solutions to (15.106). Denoting the shifted scalar fields as

$$A_i - \langle A_i \rangle = S_i + \mathrm{i}P_i \tag{15.108}$$

one can rewrite (15.107) as

$$\begin{aligned} V = V_{\min} &+ \frac{\partial^2 V}{\partial A_i \partial A_j^*}[S_i S_j + P_i P_j - \mathrm{i}(S_i S_j - S_j P_i)] \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial A_i \partial A_j}[S_i S_j - P_i P_j + \mathrm{i}(S_i S_j + S_j P_i)] \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial A_i^* \partial A_j^*}[S_i S_j - P_i P_j - \mathrm{i}(S_i S_j + S_j P_i)] \end{aligned} \tag{15.109}$$

Hence the diagonal mass matrix elements for scalars S_i and pseudoscalars P_i are

$$\left.\begin{aligned} (\mathcal{M}_S^2)_{ii} &= \frac{\partial^2 V}{\partial A_i \partial A_i^*} + \frac{1}{2}\left(\frac{\partial^2 V}{\partial A_i \partial A_i^*} + \frac{\partial^2 V}{\partial A_i^* \partial A_i^*}\right) \\ (\mathcal{M}_P^2)_{ii} &= \frac{\partial^2 V}{\partial A_i \partial A_i^*} - \frac{1}{2}\left(\frac{\partial^2 V}{\partial A_i \partial A_i^*} + \frac{\partial^2 V}{\partial A_i^* \partial A_i^*}\right) \end{aligned}\right\} \tag{15.110}$$

Therefore, the trace over the scalar mass matrices is equal to

$$\mathrm{Tr}(\mathcal{M}_S^2 + \mathcal{M}_P^2) = 2\frac{\partial^2 V}{\partial A_i \partial A_i^*} = 2\frac{\partial^2 W}{\partial A_i \partial A_j}\frac{\partial^2 W^*}{\partial A_i^* \partial A_j^*} \tag{15.111}$$

The trace (15.111) is identical, up to the overall factor 2, to the trace of the square

$\mathcal{M}_F^\dagger \mathcal{M}_F$ of the fermion mass term which can be read off from (15.82)

$$(\mathcal{M}_F)_{ij} = \frac{\partial^2 W}{\partial A_i \partial A_j} \tag{15.112}$$

Therefore, we have proven the general version of (15.102) and (15.104).

It is worth noting that the existence of $F_i \neq 0$ solutions to (15.106) implies (by (15.112)) that also the equation

$$\mathcal{M}_F \psi_0 = 0 \tag{15.113}$$

has a non-zero solution. Therefore, the existence of the SUSY breaking minima of the scalar potential implies the existence of at least one massless fermion in the theory, the so-called goldstino. In this way we also proved the supersymmetric generalization of the Goldstone theorem (Goldstone 1961).

Eq. (15.104) and its generalizations mean that models with spontaneous supersymmetry breaking, although very appealing theoretically (a limited number of free parameters, a well-defined mass spectrum), are difficult to accept from the phenomenological point of view. Eq. (15.104) predicts that at least some of the scalar masses in the theory should be smaller than the maximal fermion mass. Therefore a realistic model should be able to explain why we do not observe such light scalar particles. An interesting hypothesis is that supersymmetry is broken spontaneously in a 'hidden' sector which communicates with the observable sector (described by the supersymmetric extension of the Standard Model) only via gravitational interactions. Such models are invariant under *local* supersymmetry transformations and also include gravitational interactions (the so-called supergravity models).† The only effect of spontaneous supersymmetry breaking in the hidden sector on the visible sector is the presence of some additional, explicitly non-supersymmetric, terms in the matter lagrangian. The structure of such terms is very special: although they explicitly break supersymmetry, they preserve its most important virtue: the absence of the quadratic divergences in the loop diagrams. We call this a soft explicit breaking.

The dynamics of supergravity theories is very complicated and still remains mostly unknown. A simplified approach is often employed: the supersymmetric lagrangian is directly extended by adding all possible terms which do not introduce quadratic divergences to the theory (Girardello & Grisaru 1982). Any two- and trilinear scalar couplings are allowed and, after including gauge interactions, also mass terms for the gauginos (supersymmetric partners of the gauge bosons). Supergravity gives us the theoretical motivation for this procedure, even if at

† Operators of supersymmetry transformations anticommute with Lorentz transformations; local supersymmetry transformations anticommute with local Lorentz transformations, i.e. local Minkowski space transformations. Hence it is natural that local supersymmetry must include (super)gravitational interactions.

present we are unable to calculate the magnitude of the additional terms and we may only try to estimate (restrict) their size on the basis of the existing experimental data. Such effective low energy models with soft supersymmetry breaking appear to be fully acceptable phenomenologically. An important example is the so-called Minimal Supersymmetric Standard Model, the minimal realistic supersymmetric theory containing the Standard Model.

15.7 Supergraphs and the non-renormalization theorem

Perturbation theory in superspace may be developed as an extension of ordinary perturbation theory. Our aim is to calculate superfield Green's functions ($z \equiv (x, \Theta, \bar{\Theta})$)

$$\langle 0|T\Phi(z_1)\ldots\Phi(z_k)\ldots\bar{\Phi}(z_{k+1})\ldots\bar{\Phi}(z_n)|0\rangle \tag{15.114}$$

from which one can obtain the component-field Green's functions by power series expansion in $\Theta_1, \bar{\Theta}_1 \ldots \Theta_n, \bar{\Theta}_n$. We use the generating functional technique with generating functionals defined in Section 2.6. For discussing the necessary counterterms in particular, it is very convenient (see Section 4.2) to calculate the effective action

$$\begin{aligned}\Gamma[\Phi\ldots\bar{\Phi}\ldots] &= \sum_n \int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\,\mathrm{d}^2\Theta_1\ldots\mathrm{d}^2\bar{\Theta}_n\,\Gamma^{(n)}(z_1,\ldots,z_n)\Phi(z_1)\ldots\bar{\Phi}(z_n)\\ &= \sum_n \int \prod_{i=1}^n \left[\frac{\mathrm{d}^4p_i}{(2\pi)^4}\right]\mathrm{d}^2\Theta_1\ldots\mathrm{d}^2\bar{\Theta}_n(2\pi)^4\delta\left(\sum_{i=1}^n p_i\right)\\ &\quad\times\tilde{\Gamma}^{(n)}(p_1,\ldots,p_n;\Theta_1,\ldots,\bar{\Theta}_n)\Phi(p_1,\Theta_1)\ldots\Phi(p_n,\bar{\Theta}_n)\end{aligned} \tag{15.115}$$

where the $\Gamma^{(n)}$s are the 1PI Green's functions.

Let us begin with several auxiliary remarks. The difference between $\bar{D}_{\dot{\alpha}}$ and $-\partial/\partial\bar{\Theta}^{\dot{\alpha}}$ is a derivative term which does not contribute under the d^4x integral. Thus, using the equivalence of integration over and differentiation with respect to Grassmann variables and our conventions (15.50) and (15.67) we can convert

$$\bar{D}^2 \to 2\mathrm{d}^2\bar{\Theta} \tag{15.116}$$

under the d^4x integral. Similarly, we have

$$D^2 \to 2\mathrm{d}^2\Theta \tag{15.117}$$

For the operators $\bar{D}^2D^2$ and $D^2\bar{D}^2$ acting on chiral and antichiral fields, respectively, we can write

$$\bar{D}^2D^2\Phi = [\bar{D}^2, D^2]\Phi = (-16\partial^2 - 8\mathrm{i}D\sigma^\mu\partial_\mu\bar{D})\Phi = -16\partial^2\Phi \tag{15.118}$$

and similarly

$$D^2\bar{D}^2\bar{\Phi} = -16\partial^2\bar{\Phi} \tag{15.119}$$

Thus, for example

$$\int \mathrm{d}^2\Theta\,\Phi = \int \mathrm{d}^2\Theta\,(-\bar{D}^2D^2/16\partial^2)\Phi = \int \mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,(-D^2/8\partial^2)\Phi \tag{15.120}$$

and

$$\begin{aligned}\int \mathrm{d}^2\Theta\,\Phi^2 &= \int \mathrm{d}^2\Theta\,\Phi(-\bar{D}^2D^2/16\partial^2)\Phi = \int \mathrm{d}^2\Theta\,(\bar{D}^2/2)[\Phi(-D^2/8\partial^2)\Phi]\\ &= \int \mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,\Phi(-D^2/8\partial^2)\Phi\end{aligned} \tag{15.121}$$

where we have used the fact that $\bar{D}\Phi = 0$.

The free-field part of the action (15.74) can now be written as follows

$$\begin{aligned}S_0 &= \tfrac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^2\Theta\,(\tfrac{1}{8}\Phi\bar{D}^2\bar{\Phi} - \tfrac{1}{2}m\Phi^2) + \text{h.c.}\\ &= \int \mathrm{d}^4x\,\mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,[\tfrac{1}{8}\bar{\Phi}\Phi - \tfrac{1}{8}m\Phi(-D^2/8\partial^2)\Phi - \tfrac{1}{8}m\bar{\Phi}(-\bar{D}^2/8\partial^2)\bar{\Phi}]\\ &= \int \mathrm{d}^4x\,\mathrm{d}^2\Theta\,\mathrm{d}^2\bar{\Theta}\,\tfrac{1}{16}(\Phi,\bar{\Phi})A\begin{pmatrix}\Phi\\ \bar{\Phi}\end{pmatrix}\end{aligned} \tag{15.122}$$

where

$$A = \begin{pmatrix} mD^2/4\partial^2 & 1\\ 1 & m\bar{D}^2/4\partial^2\end{pmatrix} \tag{15.123}$$

To derive the Feynman rules we shall use the generating functional technique. We define the functional derivative for the chiral fields by the equation

$$\frac{\delta}{\delta\Phi(x,\Theta,\bar{\Theta})}\int \mathrm{d}^4x'\,\mathrm{d}^2\Theta'\,\mathcal{F}[\Phi(x',\Theta',\bar{\Theta}')] = \mathcal{F}'[\Phi(x,\Theta,\bar{\Theta})] \tag{15.124}$$

This definition implies

$$\frac{\delta\Phi(x',\Theta',\bar{\Theta}')}{\delta\Phi(x,\Theta,\bar{\Theta})} = \tfrac{1}{2}\bar{D}^2[\delta^{(2)}(\Theta'-\Theta)\delta^{(2)}(\bar{\Theta}'-\bar{\Theta})\delta^{(4)}(x'-x)] \tag{15.125}$$

(to get (15.124) integrate by parts using (15.125) and analogously for $\bar{\Phi}$.

The generating functional of the theory is

$$W[J,\bar{J}] = \exp(\mathrm{i}S_{\text{int}}[\delta/\mathrm{i}\delta J, \delta/\mathrm{i}\delta\bar{J}])W_0[J,\bar{J}] \tag{15.126}$$

where $W_0[J, \bar{J}]$ is the generating functional for the superfield Green's functions:

$$W_0[J, \bar{J}] = \left\langle 0 \left| T \exp\left[\mathrm{i} \int \mathrm{d}^4x \, \mathrm{d}^2\Theta \, \mathrm{d}^2\bar{\Theta} \, \tfrac{1}{2}(\Phi, \bar{\Phi}) \begin{pmatrix} -(D^2/4\partial^2)J \\ -(\bar{D}^2/4\partial^2)\bar{J} \end{pmatrix} \right] \right| 0 \right\rangle \tag{15.126a}$$

and J and $\bar{J}$ are chiral sources. Thus

$$G^{(n),0}_{\Phi\ldots\bar{\Phi}}(z_1, \ldots, z_n) = (-\mathrm{i})^n \frac{\delta}{\delta J(z_1)} \cdots \frac{\delta}{\delta \bar{J}(z_n)} W_0[J, \bar{J}]|_{J=\bar{J}=0} \tag{15.127}$$

Using the path integral representation for the $W_0[J, \bar{J}]$

$$W_0[J, \bar{J}] = \int \mathcal{D}\Phi \, \mathcal{D}\bar{\Phi} \exp\left\{ \mathrm{i} \int \mathrm{d}z \left[\tfrac{1}{16}(\Phi, \bar{\Phi}) A \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix} + \tfrac{1}{2}(\Phi, \bar{\Phi}) B \right] \right\} \tag{15.128}$$

where $\mathrm{d}z = \mathrm{d}^4x \, \mathrm{d}^2\Theta \, \mathrm{d}^2\bar{\Theta}$ and

$$B = \begin{pmatrix} -(D^2/4\partial^2)J \\ -(\bar{D}^2/4\partial^2)\bar{J} \end{pmatrix}$$

and performing the Gaussian integration over Φ and $\bar{\Phi}$ one gets (see (2.81))

$$W_0[J, \bar{J}] = \exp\left(-\mathrm{i} \int \mathrm{d}z \, B^{\mathrm{T}} A^{-1} B \right) \tag{15.129}$$

where

$$A^{-1} = \begin{pmatrix} -\dfrac{m\bar{D}^2}{4(\partial^2 + m^2)} & 1 + \dfrac{m\bar{D}^2 D^2}{16\partial^2(\partial^2 + m^2)} \\ 1 + \dfrac{m D^2 \bar{D}^2}{16\partial^2(\partial^2 + m^2)} & -\dfrac{m D^2}{4(\partial^2 + m^2)} \end{pmatrix} \tag{15.130}$$

Finally we have

$$\begin{aligned} W_0[J, \bar{J}] = \exp\biggl[-2\mathrm{i} \int \mathrm{d}z \biggl(&-\bar{J} \frac{1}{\partial^2 + m^2} J + \tfrac{1}{2} J \frac{\frac{1}{4} m D^2}{\partial^2(\partial^2 + m^2)} J \\ &+ \tfrac{1}{2} \bar{J} \frac{\frac{1}{4} m \bar{D}^2}{\partial^2(\partial^2 + m^2)} \bar{J} \biggr) \biggr] \end{aligned} \tag{15.131}$$

Using (15.125), (15.127), (15.131) and integrating by parts when necessary we obtain the free superpropagators. For instance

$$
\begin{aligned}
G^{(2),0}_{\Phi\bar{\Phi}}(z_1, z_2) &= -2\mathrm{i}(\partial/\partial J(z_1))(\partial/\partial \bar{J}(z_2)) \\
&\qquad \times \int \mathrm{d}z_3\, \mathrm{d}z_4\, \bar{J}(z_3)\delta^{(4)}(\Theta_{34})G(x_3 - x_4)J(z_4) \\
&= -2\mathrm{i} \int \mathrm{d}z_3\, \mathrm{d}z_4\, [\tfrac{1}{2}\bar{D}_1^2\delta(z_{14})]\delta^{(4)}(\Theta_{34})G(x_3 - x_4)[\tfrac{1}{2}D_2^2\delta(z_{23})] \\
&= -2\mathrm{i}\tfrac{1}{2}\bar{D}_1^2\tfrac{1}{2}D_2^2[\delta^{(4)}(\Theta_{12})G(x_1 - x_2)]
\end{aligned}
\tag{15.132}
$$

where

$$
\delta(z_{ij}) \equiv \delta^{(4)}(\Theta_i - \Theta_j)\delta^{(4)}(x_i - x_j) \equiv \delta^{(4)}(\Theta_{ij})\delta^{(4)}(x_i - x_j)
$$

and

$$
(\partial_1^2 + m^2)G(x_1 - x_2) = \delta^{(4)}(x_1 - x_2)
$$

and

$$
\left.
\begin{aligned}
D_1^2\delta^{(2)}(\Theta_1 - \Theta_2) &= -D_1^2\tfrac{1}{2}(\Theta_1 - \Theta_2)^2 = 2\exp[\mathrm{i}(\Theta_1 - \Theta_2)\sigma^\mu\bar{\Theta}_1\partial_\mu^{(1)}] \\
\bar{D}_1^2\delta^{(2)}(\bar{\Theta}_1 - \bar{\Theta}_2) &= -\bar{D}_1^2\tfrac{1}{2}(\bar{\Theta}_1 - \bar{\Theta}_2)^2 = 2\exp[-\mathrm{i}\Theta_1\sigma^\mu(\bar{\Theta}_1 - \bar{\Theta}_2)\partial_\mu^{(1)}]
\end{aligned}
\right\}
\tag{15.133}
$$

(remember that $\partial/\partial\Theta^\alpha = -\varepsilon_{\alpha\beta}\partial/\partial\Theta_\beta$). To obtain the component-field propagators we can use (15.133), (15.54) and (15.55) to write

$$
\hat{G}^{(2),0}_{\Phi\bar{\Phi}}(z_1, z_2) = \langle 0|T\hat{\Phi}(x_1, \Theta_1)\hat{\bar{\Phi}}(x_2, \bar{\Theta}_2)|0\rangle = -2\mathrm{i}\exp(\mathrm{i}2\Theta_1\sigma^\mu\bar{\Theta}_2\partial_\mu^{(1)})G(x_1 - x_2)
\tag{15.134}
$$

and then use (15.56) and (15.57) to expand in Θ_1 and $\bar{\Theta}_2$. Analogously, we get†

$$
\hat{G}^{(2),0}_{\Phi\Phi}(z_1, z_2) = 2\mathrm{i}m\frac{\bar{D}_2^2 D_2^2}{16\partial^2}\bar{D}_1^2[G(x_1 - x_2)\delta^{(4)}(\Theta_{12})]
\tag{15.135}
$$

$$
= -2\mathrm{i}m\bar{D}_1^2[\delta^{(4)}(\Theta_{12})G(x_1 - x_2)]
\tag{15.135a}
$$

† $\bar{D}_1^2[\delta^{(4)}(\Theta_{12})f(x_1 - x_2)] = \bar{D}_2^2[\delta^{(4)}(\Theta_{12})f(x_1 - x_2)]$.

In momentum space we thus get, for example,

$$G(x_1 - x_2) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4 p \, \frac{\exp[-\mathrm{i}p(x_1 - x_2)]}{p^2 - m^2}$$

$$\hat{G}^{(2),0}_{\Phi\bar{\Phi}} = \frac{2\mathrm{i}}{p^2 - m^2}\exp[(2\Theta_1\bar{\Theta}_2 - \Theta_1\bar{\Theta}_1 + \Theta_2\bar{\Theta}_2)^{\alpha\dot{\beta}}\sigma^{\mu}_{\alpha\dot{\beta}}p_\mu] \tag{15.136}$$

It is, however, more convenient to take as free superpropagators the following:†

$$\left.\begin{aligned} G^{(2),0}_{\Phi\bar{\Phi}} &= \frac{2\mathrm{i}}{p^2 - m^2}\delta^{(4)}(\Theta_{12}) \\ G^{(2),0}_{\Phi\Phi} &= \frac{2\mathrm{i}}{p^2(p^2 - m^2)}\tfrac{1}{4}mD^2\delta^{(4)}(\Theta_{12}) \\ G^{(2),0}_{\bar{\Phi}\bar{\Phi}} &= \frac{2\mathrm{i}}{p^2(p^2 - m^2)}\tfrac{1}{4}m\bar{D}^2\delta^{(4)}(\Theta_{12}) \end{aligned}\right\} \tag{15.137}$$

where

$$D^2 = D^2(p, \Theta_1)$$

$$D_\alpha(p, \Theta) = \partial/\partial\Theta^\alpha - (\sigma^\mu\bar{\Theta})_\alpha p_\mu$$

Using this convention we associate the factors $\frac{1}{2}\bar{D}_1^2$ and $\frac{1}{2}\bar{D}_2^2$ present in (15.132) and $\frac{1}{2}\bar{D}_2^2$ and $\frac{1}{2}\bar{D}_1^2$ present in (15.135) with the chiral vertices rather than with the propagators. Notice that these factors cancel the p^{-2} factors in the $\Phi\Phi$ and $\bar{\Phi}\bar{\Phi}$ propagators (15.137)

The vertices can be obtained from the general formula (15.126) where

$$S_{\mathrm{int}}\left[\frac{\delta}{\mathrm{i}\delta J}, \frac{\delta}{\mathrm{i}\delta \bar{J}}\right] = -\tfrac{1}{12}g\int \mathrm{d}^4x\left[\mathrm{d}^2\Theta\left(\frac{1}{\mathrm{i}\delta J}\right)^3 + \mathrm{d}^2\bar{\Theta}\left(\frac{1}{\mathrm{i}\delta\bar{J}}\right)^3\right] \tag{15.138}$$

Let us consider, for instance, the three-point Green's function $G^{(3)}_{\Phi\Phi\Phi}(z_1, z_2, z_3)$ in the first order in g. The first term in (15.138) gives, up to the overall combinatorial factor,

$$\begin{aligned} G^{(3)}_{\Phi\Phi\Phi}(z_1, z_2, z_3) &= -S_{\mathrm{int}}\left[\frac{\delta}{\mathrm{i}\delta J}\right]\frac{\delta}{\delta J(z_1)}\frac{\delta}{\delta J(z_2)}\frac{\delta}{\delta J(z_3)}W_0[J, \bar{J}] \\ &\sim -\mathrm{i}g\int \mathrm{d}^4x_4\,\mathrm{d}^2\Theta_4\, G^{(2),0}_{\Phi\Phi}(z_1, z_4)G^{(2),0}_{\Phi\Phi}(z_2, z_4)G^{(2),0}_{\Phi\Phi}(z_3, z_4) \end{aligned} \tag{15.139}$$

where $G^{(2),0}_{\Phi\Phi}$ are given by (15.135). It is now convenient to regroup the factors $\frac{1}{2}\bar{D}^2$ present in each $G^{(2),0}_{\Phi\Phi}$ (due to the functional differentiation over δJ) as follows: use one $\frac{1}{2}\bar{D}^2$ to replace $\int \mathrm{d}^2\Theta_4$ by $\mathrm{d}^4\Theta_4$, associate the other two $\frac{1}{2}\bar{D}^2$s with the

† Factors 2 in the propagators correspond to our normalization (15.74) appropriate for the real component fields, see (15.75).

Fig. 15.2.

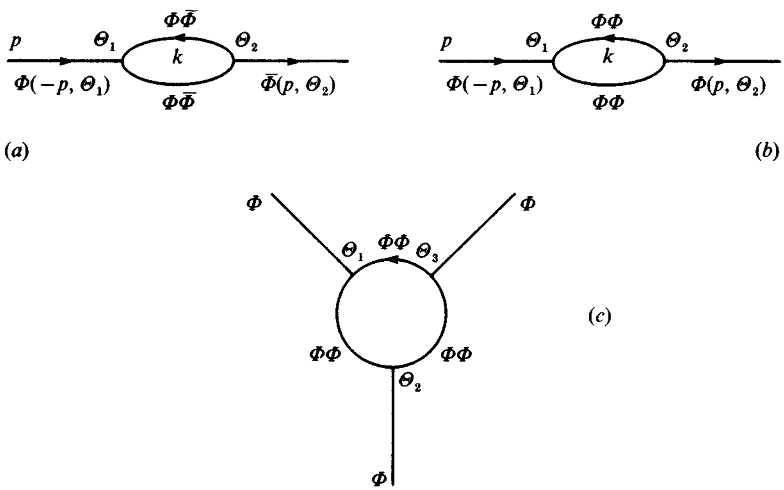


Fig. 15.3.

Φ^3 vertex and leave the remaining $\bar{D}^2$s for use in the adjacent vertices. Thus a Φ^3 vertex has two factors $\frac{1}{2}\bar{D}^2$ acting on two of three propagators entering it. We then get the following rules:

(i) free propagators are given by (15.137)

(ii) vertices are read from S_{int} with an extra $\frac{1}{2}\bar{D}^2$ (or $\frac{1}{2}D^2$) for each chiral (or antichiral) superfield, but omitting one $\frac{1}{2}\bar{D}^2$ (or $\frac{1}{2}D^2$) for converting $d^2\Theta$ (or $d^2\bar{\Theta}$) into $d^4\Theta$

(iii) for each vertex there is an integration over $d^4\Theta$ and for each loop the usual integration over $d^4k/(2\pi)^4$

(iv) to obtain the effective action we compute amputated 1PI diagrams i.e. we replace the external propagators by the appropriate superfield with a $\frac{1}{2}\bar{D}^2$ (or $\frac{1}{2}D^2$) omitted at a vertex for each external chiral (or antichiral) superfield

The counting of the $\frac{1}{2}\bar{D}^2$ factors becomes even more evident when we consider the one-loop and two-loop diagrams in Fig. 15.2. In the first case after amputating the external lines we are left with two propagators, i.e. with four factors $\frac{1}{2}\bar{D}^2$ or $\frac{1}{2}D^2$. Two of them are used to convert two $d^2\Theta$ integrals into $d^4\Theta$ integrals and two are left as vertex factors, in agreement with our general rules. In the second case the internal vertices have two $\frac{1}{2}\bar{D}^2$ or $\frac{1}{2}D^2$ factors each.

Let us use these rules to calculate the amplitudes corresponding to the 1PI

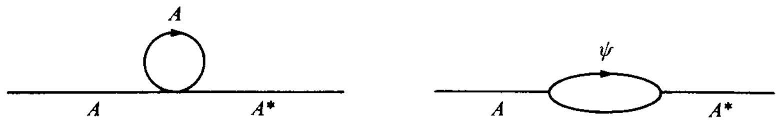


Fig. 15.4.

diagrams given in Fig. 15.3. Diagram (*a*), after integrating by parts, gives

$$\Gamma(\Phi, \bar{\Phi}) \sim -g^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(p+k^2)} \int d^4\Theta_1 \, d^4\Theta_2 \, \Phi(-p, \Theta_1) \times \bar{\Phi}(p, \Theta_2) \delta^{(4)}(\Theta_{12}) \bar{D}_1^2 D_2^2 \delta^{(4)}(\Theta_{12})$$
$$\sim g^2 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(p+k^2)} \int d^4\Theta \, \Phi(-p, \Theta) \bar{\Phi}(p, \Theta) \tag{15.140}$$

We have used the fact that (see (15.133))

$$\bar{D}_1^2 D_2^2 \delta^{(4)}(\Theta_{12})_{\Theta_1 = \Theta_2} = 4$$

This diagram is logarithmically divergent and contributes to the wave-function renormalization constant for the superfield ($\int d^4\Theta \, \Phi\bar{\Phi}$ counterterm).

Writing down the amplitudes for Fig. 15.3(*b*) and (*c*) we immediately see that they vanish. Indeed we end up with integrals $\int d^4\Theta \, \Phi\Phi = 0$ because the integrand is chiral. Thus, we see that at the one-loop level there is no mass and coupling-constant renormalization other than that induced by the wave-function renormalization, neither infinite nor finite. Expressed in terms of the component fields this result means that the contributions to the mass renormalization constant of, for example, the diagrams in Fig. 15.4 generated by the lagrangian (15.77) (we now use the complex field notation $A \equiv S + iP$) cancel each other. Actually there is no *infinite* renormalization of the mass and the coupling-constant to any order of perturbation theory. This is the non-renormalization theorem for chiral superfields. (In general, there is finite renormalization of these parameters in higher orders. It vanishes when renormalizing at vanishing external momenta.)

The above-mentioned validity of the non-renormalization theorem to any order of perturbation theory follows from the fact that, by integrating by parts the factors $\frac{1}{2}\bar{D}^2$ or $\frac{1}{2}D^2$ one-by-one, the contribution from any Feynman diagram to the effective action can be written in the form

$$\int \prod_{i=1}^{n} d^4 p_i \, d^4\Theta \, f(p_1, \ldots, p_n) \Phi(p_1, \Theta_1) \ldots \bar{\Phi}(p_n, \Theta_n) \tag{15.141}$$

i.e. it can be expressed as an integral over a single $d^4\Theta$ and with the function

$f(p_1, \ldots, p_n)$ translationally invariant. Of course, our result (15.140) is the simplest example of the general rule (15.141). From (15.141) it follows in particular that all vacuum-to-vacuum diagrams vanish because without any superfields the expression is annihilated by the $d^4\Theta$ integration. Thus, the normalization factor in (15.131) is one.

Appendix A

Spinors and their properties

Our conventions are:

$$\hbar = c = 1, \quad x^\mu = (t, \mathbf{x}), \quad p^\mu = (E, \mathbf{p}), \quad g^{\mu\nu} = (1, -1, -1, -1),$$
$$x^2 = g_{\mu\nu} x^\mu x^\nu = t^2 - \mathbf{x}^2, \qquad \mathbf{p} = -\mathrm{i}\partial/\partial\mathbf{x}, \qquad p^\mu = \mathrm{i}\partial/\partial x_\mu$$

A plane wave solution to the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\mathbf{p}\rangle = E|\mathbf{p}\rangle \tag{A.1}$$

is $\langle\mathbf{x}|\mathbf{p}, t\rangle = \exp(-\mathrm{i}p \cdot x) = \Phi(t, \mathbf{x}) = \Phi(x)$. Space-time translation (change of the origin of the reference frame) $x'_\mu = x_\mu + \varepsilon_\mu$ gives $\Phi'(x') = \langle\mathbf{x}'|\mathbf{p}\rangle = \Phi(x) = \exp(-\mathrm{i}p \cdot x)$. Hence,

$$\Phi'(x) = \Phi(x - \varepsilon) = \exp[-\mathrm{i}p \cdot (x - \varepsilon)] \approx (1 + \mathrm{i}p^\mu \varepsilon_\mu)\Phi(x) \tag{A.2}$$

Thus, the translation of the state $|\mathbf{p}\rangle$ is described by the operator

$$|\mathbf{p}'\rangle = \exp(\mathrm{i}\varepsilon_\mu p^\mu)|\mathbf{p}\rangle \tag{A.3}$$

Indeed $\Phi'(x) = \langle\mathbf{x}|\mathbf{p}'\rangle \approx (1 + \mathrm{i}p^\mu \varepsilon_\mu)\Phi(x)$.

Lorentz transformations and two-dimensional representations of the group $SL(2, C)$

The Lorentz group is the group of transformations which leave invariant the square $x^2 \equiv x^\mu x_\mu \equiv g_{\mu\nu} x^\mu x^\nu$ of the four-vector x^μ. The infinitesimal form of these transformations reads

$$x^{\mu\prime} \approx x^\mu + \omega^\mu{}_\nu x^\nu \tag{A.4}$$

where $\omega_{\mu\nu} \equiv g_{\mu\kappa}\omega^{\kappa}{}_{\nu} = -\omega_{\nu\mu}$. Identifying $\omega_{\mu\nu}$ with the transformation parameters, (A.4) can be rewritten in the form

$$x^{\mu\prime} \approx x^{\mu} - \frac{\mathrm{i}}{2}\omega_{\kappa\rho}\,(M^{\kappa\rho})^{\mu}{}_{\nu}x^{\nu} \tag{A.5}$$

with the Lorentz group generators in the vector representation given by

$$(M^{\kappa\rho})^{\mu}{}_{\nu} = \mathrm{i}(g^{\kappa\mu}g^{\rho}{}_{\nu} - g^{\rho\mu}g^{\kappa}{}_{\nu}) \tag{A.6}$$

The generators $M^{\mu\nu}$ satisfy the following commutation relations

$$\left[M^{\kappa\rho}, M^{\mu\nu}\right] = \mathrm{i}(g^{\kappa\nu}M^{\rho\mu} - g^{\kappa\mu}M^{\rho\nu} - g^{\rho\nu}M^{\kappa\mu} + g^{\rho\mu}M^{\kappa\nu}) \tag{A.7}$$

In this abstract (representation-independent) form the commutation relations (A.7) define the Lie algebra of the Lorentz group. In any representation, the finite transformations $U(\omega)$ can be then written as

$$U(\omega) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\kappa\rho}M^{\kappa\rho}\right) \tag{A.8}$$

with $M^{\kappa\rho}$ in the appropriate representation.

The connection of the group $SL(2, C)$ (the group of 2×2 complex matrices M of determinant 1) to the Lorentz group is analogous to the connection of the group $SU(2)$ of the two-dimensional unitary unimodular matrices to the rotation group in three dimensions (for example, Werle (1966)). It can be established through the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.9}$$

(note that $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$, $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$) in the following way. To every space-time point x^{μ} we can assign a matrix

$$\sigma \cdot x \equiv \sum_{\mu=0}^{3} \sigma^{\mu}x_{\mu} = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \tag{A.10}$$

where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.11}$$

such that

$$\det(\sigma \cdot x) = x^2 \tag{A.12}$$

The transformation $x \to x'$ is defined by the equation

$$\sigma \cdot x' = M(\sigma \cdot x)M^{\dagger} \qquad \det M = 1 \tag{A.13}$$

Since

$$\det(\sigma \cdot x') = x'^2 = \det(\sigma \cdot x) = x^2 \tag{A.14}$$

the transformation

$$x'^\mu = \Lambda^\mu{}_\nu(M)\, x^\nu \tag{A.15}$$

is a Lorentz transformation. The group $SL(2, C)$ is homomorphic to the restricted Lorentz group (i.e. its proper orthochronous subgroup) $\det \Lambda = +1$, $\Lambda^0{}_0 \geq 1$: $\mathrm{L}_+^\uparrow = SL(2, C)/Z_2$ (because M and $-M$ generate the same Lorentz transformation).

The group $SL(2, C)$ has two inequivalent two-dimensional (fundamental) representations

$$\lambda'_\alpha(x') = M_\alpha{}^\beta\, \lambda_\beta(x) \tag{A.16}$$

$$\bar{\chi}'^{\dot{\alpha}}(x') = (M^{\dagger -1})^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\chi}^{\dot{\beta}}(x) \tag{A.17}$$

where λ and χ are two-component complex vectors (dots are introduced to remind us that the indices α and $\dot{\alpha}$ are the indices of the two different representations and cannot be contracted) and x' is given by (A.15). Two other possible two-dimensional representations,

$$\lambda'^\alpha = (M^{T-1})^\alpha{}_\beta\, \lambda^\beta \equiv \lambda^\beta (M^{-1})_\beta{}^\alpha \tag{A.18}$$

and

$$\bar{\chi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}\, \bar{\chi}_{\dot{\beta}} \equiv \bar{\chi}_{\dot{\beta}} (M^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}} \tag{A.19}$$

are, as can be easily checked, equivalent to the representations (A.16) and (A.17), respectively, via the unitary transformations

$$\left.\begin{aligned} \lambda^\alpha &= \varepsilon^{\alpha\beta}\lambda_\beta \quad & \bar{\chi}^{\dot{\alpha}} &= \bar{\chi}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} \\ \lambda_\alpha &= \varepsilon_{\alpha\beta}\lambda^\beta \quad & \bar{\chi}_{\dot{\alpha}} &= \bar{\chi}^{\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\alpha}} \end{aligned}\right\} \tag{A.20}$$

where epsilons are defined by $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = (\mathrm{i}\sigma_2)^{\alpha\beta}$, i.e.

$$\left.\begin{aligned} \varepsilon^{12} &= -\varepsilon^{21} = -\varepsilon_{12} = \varepsilon_{21} = 1 \\ \varepsilon^{\dot{1}\dot{2}} &= -\varepsilon^{\dot{2}\dot{1}} = -\varepsilon_{\dot{1}\dot{2}} = \varepsilon_{\dot{2}\dot{1}} = 1 \end{aligned}\right\} \tag{A.21}$$

and satisfy

$$\left.\begin{aligned} \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} &= \delta_\alpha^\gamma \\ \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\gamma}} &= \delta_{\dot{\gamma}}^{\dot{\alpha}} \end{aligned}\right\} \tag{A.22}$$

Since matrices M are unimodular, the εs are $SL(2, C)$ invariant tensors:

$$\left.\begin{array}{ll}\varepsilon_{\alpha\beta} = M_\alpha{}^\gamma M_\beta{}^\delta \varepsilon_{\gamma\delta} & \varepsilon^{\alpha\beta} = \varepsilon^{\gamma\delta} M_\gamma{}^\alpha M_\delta{}^\beta \\ \varepsilon_{\dot\alpha\dot\beta} = \varepsilon_{\dot\gamma\dot\delta}(M^{\dagger-1})^{\dot\gamma}{}_{\dot\alpha}(M^{\dagger-1})^{\dot\delta}{}_{\dot\beta} & \varepsilon^{\dot\alpha\dot\beta} = (M^{\dagger-1})^{\dot\alpha}{}_{\dot\gamma}(M^{\dagger-1})^{\dot\beta}{}_{\dot\delta}\varepsilon^{\dot\gamma\dot\delta}\end{array}\right\} \tag{A.23}$$

From (A.16), (A.18) and (A.17), (A.19), using anticommutativity of ϕ, λ, it follows that the combinations

$$\left.\begin{array}{l}\lambda\cdot\phi \equiv \lambda^\alpha\phi_\alpha = \phi^\alpha\lambda_\alpha = -\lambda_\alpha\phi^\alpha = \phi\cdot\lambda \\ \bar\eta\cdot\bar\chi \equiv \bar\eta_{\dot\beta}\bar\chi^{\dot\beta} = \bar\chi_{\dot\beta}\bar\eta^{\dot\beta} = -\bar\eta^{\dot\beta}\bar\chi_{\dot\beta} = \bar\chi\cdot\bar\psi\end{array}\right\} \tag{A.24}$$

are Lorentz-invariant.

To make the connection between the $SL(2, C)$ representations ((A.16), (A.17)) and the Lorentz group more explicit recall that the Lorentz transformations (A.8) can also be parametrized in terms of

$$N^i_\pm = \tfrac{1}{2}(J^i \pm \mathrm{i}K^i) \tag{A.25}$$

where

$$J^i = \tfrac{1}{2}\varepsilon^{ijk}M^{jk}, \quad K^i = M^{0i} \tag{A.26}$$

satisfy

$$\left.\begin{array}{l}\left[J^i, J^j\right] = \mathrm{i}\varepsilon^{ijk}J^k \\ \left[K^i, K^j\right] = -\mathrm{i}\varepsilon^{ijk}J^k \\ \left[J^i, K^j\right] = \mathrm{i}\varepsilon^{ijk}K^k\end{array}\right\} \tag{A.27}$$

and

$$\left.\begin{array}{l}\left[N^i_\pm, N^j_\pm\right] = \mathrm{i}\varepsilon^{ijk}N^k_\pm \\ \left[N^i_+, N^j_-\right] = 0\end{array}\right\} \tag{A.28}$$

We see that the Lie algebra of the Lorentz group can locally be represented as a direct product of two (complexified) $SU(2)$ algebras. However, the resulting two $SL(2, C)$ groups† are not independent. This follows from the fact that their group parameters must be complex conjugate to each other:

$$\exp\left(-\mathrm{i}\omega_{0i}M^{0i} - \frac{\mathrm{i}}{2}\omega_{ij}M^{ij}\right) \equiv \exp\left(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i\right) = \exp\left(-\mathrm{i}\zeta_i N^i_+ - \mathrm{i}\zeta_i^* N^i_-\right) \tag{A.29}$$

with $\zeta_i \equiv \eta_i - \mathrm{i}\xi_i$. The representations can be built by exploiting this decomposition. Two obvious ways in which the commutation relations (A.27) can be satisfied

† A complexified $SU(2)$ is just the $SL(2, C)$ group.

are

$$r_1(N_+^i) = \tfrac{1}{2}\sigma^i, \quad r_1(N_-^j) = 0 \tag{A.30}$$

$$r_2(N_+^i) = 0, \quad r_2(N_-^j) = \tfrac{1}{2}\sigma^j \tag{A.31}$$

It is clear that the finite-dimensional (non-unitary) representations of the Lorentz group can be labelled by the pair (m, n), where $m(m+1)$ is the eigenvalue of the $N_+^i N_+^i$ operator and $n(n+1)$ is the eigenvalue of the $N_-^i N_-^i$ operator. Thus, the representations (A.30) and (A.31) correspond to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. Since $J^i = N_+^i + N_-^i$, we can identify the spin of the representation with $m+n$. Using (A.25), (A.30) and (A.31) we have

$$r_1\big(\exp(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i)\big) = \exp\left(-\frac{\mathrm{i}}{2}\sigma^i(\eta_i - \mathrm{i}\xi_i)\right) \tag{A.32}$$

$$r_2\big(\exp(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i)\big) = \exp\left(-\frac{\mathrm{i}}{2}\sigma^i(\eta_i + \mathrm{i}\xi_i)\right) \tag{A.33}$$

Identifying (A.32) with $M_\alpha{}^\beta$ (acting on spinors λ_β) we must identify (A.33) with $(M^{\dagger -1})^{\dot\alpha}{}_{\dot\beta}$ which acts on spinors $\bar\chi^{\dot\beta}$. Introducing two sets of matrices $(\sigma^\mu)_{\alpha\dot\beta}$ and $(\bar\sigma^\mu)^{\dot\alpha\beta}$ defined by

$$\sigma^\mu = (I, \sigma^i), \quad \bar\sigma^\mu = (I, -\sigma^i) \tag{A.34}$$

and satisfying

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2g^{\mu\nu} \tag{A.35}$$

the Lorentz group generators in representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ can be written compactly as

$$\left.\begin{aligned} r_1(M^{\mu\nu}) &= \tfrac{1}{2}\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) \\ r_2(M^{\mu\nu}) &= \tfrac{1}{2}\bar\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right) \end{aligned}\right\} \tag{A.36}$$

with the obvious assignment of spinor indices: $(\sigma^{\mu\nu})_\alpha{}^\beta$ and $(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$. Here $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are two-dimensional matrices. We thus have

$$\lambda'_\alpha(x') = \{\exp[-(\mathrm{i}/4)\omega_{\mu\nu}\sigma^{\mu\nu}]\}_\alpha{}^\beta\lambda_\beta(x) \tag{A.37}$$

$$\bar\chi'^{\dot\alpha}(x') = \{\exp[-(\mathrm{i}/4)\omega_{\mu\nu}\bar\sigma^{\mu\nu}]\}^{\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta}(x) \tag{A.38}$$

As can be verified using (A.35), the matrices σ^μ and $\bar\sigma^\mu$ defined in (A.34) satisfy the following identities

$$\left.\begin{aligned} \tfrac{1}{2}\sigma^{\kappa\rho}\sigma^\mu - \tfrac{1}{2}\sigma^\mu\bar\sigma^{\kappa\rho} - \mathrm{i}(\sigma^\kappa g^{\rho\mu} - \sigma^\rho g^{\kappa\mu}) = 0 \\ \tfrac{1}{2}\bar\sigma^{\kappa\rho}\bar\sigma^\mu - \tfrac{1}{2}\bar\sigma^\mu\sigma^{\kappa\rho} - \mathrm{i}(\bar\sigma^\kappa g^{\mu\rho} - \bar\sigma^\rho g^{\kappa\mu}) = 0 \end{aligned}\right\} \tag{A.39}$$

Taking into account (A.16)–(A.19), (A.37), (A.38) and the explicit form of the Lorentz group generators in the vector representation (A.6) the identities (A.39) turn out to be the infinitesimal forms of the relations

$$\left.\begin{aligned} \Lambda^\nu{}_\mu M\sigma^\mu M^\dagger &= \sigma^\nu \\ \Lambda^\nu{}_\mu M^{\dagger-1}\bar{\sigma}^\mu M^{-1} &= \bar{\sigma}^\nu \end{aligned}\right\} \tag{A.40}$$

Thus, σ^μ and $\bar{\sigma}^\mu$ are numerically invariant tensors of the Lorentz group provided the index μ transforms according to the vector representation of $O(1,3)$. They are therefore the Clebsch–Gordan coefficients which relate the representation $(\frac{1}{2},\frac{1}{2})$ of $SL(2,C)$ to the vector representation of $O(1,3)$.

From the completeness relation

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}_\mu)^{\dot{\kappa}\rho} = 2\delta_\alpha^\rho\delta_{\dot{\beta}}^{\dot{\kappa}} \tag{A.41}$$

one derives the Fierz transformation for anticommuting (Grassmann) Weyl spinors†

$$(\lambda\psi)(\bar{\chi}\bar{\eta}) = \tfrac{1}{2}(\lambda\sigma^\mu\bar{\chi})(\psi\sigma_\mu\bar{\eta}) = -\tfrac{1}{2}(\lambda\sigma^\mu\bar{\chi})(\bar{\eta}\bar{\sigma}_\mu\psi) \tag{A.42}$$

Other useful relations are

$$\left.\begin{aligned} \sigma^\mu\bar{\sigma}^\nu &= g^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu} \\ \bar{\sigma}^\mu\sigma^\nu &= g^{\mu\nu} - \mathrm{i}\bar{\sigma}^{\mu\nu} \end{aligned}\right\} \tag{A.43}$$

and

$$\left.\begin{aligned} \varepsilon_{\dot{\kappa}\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\varepsilon_{\beta\delta} &= (\sigma^\mu)_{\delta\dot{\kappa}} \\ \varepsilon^{\delta\beta}(\sigma^\mu)_{\beta\dot{\alpha}}\varepsilon^{\dot{\alpha}\dot{\kappa}} &= (\bar{\sigma}^\mu)^{\dot{\kappa}\delta} \end{aligned}\right\} \tag{A.44}$$

from which (for anticommuting spinors λ, $\bar{\chi}$) follows the identity

$$(\bar{\chi}\bar{\sigma}^\mu\lambda) = -(\lambda\sigma^\mu\bar{\chi}) \tag{A.45}$$

Finally, from the hermiticity of the σ^μs we have:

$$(\sigma^\mu_{\alpha\dot{\beta}})^* = \sigma^\mu_{\beta\dot{\alpha}} \qquad ((\bar{\sigma}^\mu)^{\dot{\alpha}\beta})^* = (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \tag{A.46}$$

Four-component Dirac spinors are built from two Weyl spinors transforming as $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of $SL(2,C)$ (and carrying the same charges under all other transformations):

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\Psi} = \left(\chi^\alpha, \bar{\lambda}_{\dot{\alpha}}\right) = \Psi^\dagger\gamma^0 \tag{A.47}$$

† For commuting (c-numbers) spinors one gets + sign on the r.h.s. of (A.42).

We have, for instance, $\lambda\cdot\chi+\bar\lambda\cdot\bar\chi=\bar\Psi\Psi$, $\bar\lambda\cdot\bar\chi-\lambda\cdot\chi=\bar\Psi\gamma^5\Psi$, $\bar\lambda\bar\sigma^\mu\lambda+\chi\sigma^\mu\bar\chi=\bar\Psi\gamma^\mu\Psi$. For $\chi\equiv\lambda$ one obtains in this way the Majorana spinors. Dirac matrices have in this representation the following form:

$$\gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} \tag{A.48}$$

This is the so-called chiral (or Weyl) representation. The matrix γ^5, defined as

$$\gamma^5\equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.49}$$

has in this representation the form

$$\gamma^5=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix} \tag{A.50}$$

The standard Dirac representation

$$\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} \tag{A.51}$$

is obtained with the help of the unitary transformation:

$$\gamma^\mu_{\text{Dirac}}=U^\dagger\gamma^\mu_{\text{Weyl}}U,\quad \Psi_{\text{Dirac}}=U^\dagger\Psi_{\text{Weyl}},\quad \text{with}\quad U=\frac{1}{\sqrt2}\begin{pmatrix}I & -I\\ I & I\end{pmatrix} \tag{A.52}$$

It is also useful to introduce chiral Dirac spinors in the chiral representation

$$\Psi_{\mathrm{L}}=\begin{pmatrix}\lambda_\alpha\\ 0\end{pmatrix},\quad \Psi_{\mathrm{R}}=\begin{pmatrix}0\\ \bar\chi^{\dot\alpha}\end{pmatrix} \tag{A.53}$$

which satisfy the relations

$$P_{\mathrm{L,R}}\Psi_{\mathrm{L,R}}\equiv\frac{1\mp\gamma^5}{2}\Psi_{\mathrm{L,R}}=\Psi_{\mathrm{L,R}},\quad P_{\mathrm{L,R}}\Psi_{\mathrm{R,L}}=0 \tag{A.54}$$

These can be introduced with no restriction on the internal charges of λ and χ. In addition, every Dirac spinor can be decomposed as $\Psi=\Psi_{\mathrm{L}}+\Psi_{\mathrm{R}}$.

Under the Lorentz transformation both, Ψ_{Dirac} and $\Psi_{\mathrm{L,R}}$ transform according to

$$\Psi'(\Lambda x)=\exp\left(-\frac{\mathrm{i}}{4}\omega_{\rho\kappa}\sigma^{\rho\kappa}\right)\Psi(x) \tag{A.55}$$

with

$$\sigma^{\rho\kappa}=\frac{\mathrm{i}}{2}\left[\gamma^\rho,\gamma^\kappa\right] \tag{A.56}$$

which are four-dimensional matrices. However, the reducibility of the four-dimensional representation is manifest only in the Weyl form where

$$\sigma^{\rho\kappa}_{4\times 4} = \begin{pmatrix} \sigma^{\rho\kappa} & 0 \\ 0 & \bar{\sigma}^{\rho\kappa} \end{pmatrix} \tag{A.57}$$

Note that we use the same symbol for four- and two-dimensional matrices $\sigma^{\mu\nu}$.

Solutions of the free Weyl and Dirac equations and their properties

We begin by recalling the properties of the free particle solutions of the massless Weyl equation

$$\mathrm{i}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\partial_{\mu}\lambda_{\beta}(x) = 0 \tag{A.58}$$

Writing the positive and negative frequency solutions as

$$\left.\begin{aligned} \lambda_{\text{pos}}(x) &= \exp(-\mathrm{i}k\cdot x)\, a(\mathbf{k}) \\ \lambda_{\text{neg}}(x) &= \exp(+\mathrm{i}k\cdot x)\, b(\mathbf{k}) \end{aligned}\right\} \tag{A.59}$$

(lower case indices are understood) multiplying (A.58) by σ^0 from the left and recalling the definitions (A.34) we get (remember $E \equiv k^0 = |\mathbf{k}|$)

$$\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}a(\mathbf{k}) = -a(\mathbf{k}) \tag{A.60}$$

and the same equation for $b(\mathbf{k})$. For $\mathbf{k}/|\mathbf{k}| = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ the well-known solutions to these equations read

$$\left.\begin{aligned} a(\mathbf{k}) &= \mathcal{N}_a \exp(\mathrm{i}\varphi_a) a_-(\mathbf{k}) \\ b(\mathbf{k}) &= \mathcal{N}_b \exp(\mathrm{i}\varphi_b) a_-(\mathbf{k}) \end{aligned}\right\} \tag{A.61}$$

where $\mathcal{N}_{a,b}$ are normalization factors, and

$$a_-(\mathbf{k}) = \begin{pmatrix} -\sin(\theta/2)\exp(-\mathrm{i}\phi) \\ \cos(\theta/2) \end{pmatrix} \tag{A.62}$$

is normalized to unity.

Thus, the solution $\lambda_{\text{pos}}(x)$ describes a fermion state with the spin projection onto the direction of its momentum ($+\mathbf{k}$) equal to $-1/2$, i.e. with helicity $-1/2$. The solution $\lambda_{\text{neg}}(x)$, being an eigenstate of the three-momentum operator $\hat{\mathbf{p}} \equiv -\mathrm{i}\partial/\partial\mathbf{x}$ with the eigenvalue $-\mathbf{k}$, consequently describes a negative energy fermion with helicity $+1/2$. In the Dirac sea interpretation it represents therefore the absence of a negative energy particle with momentum $-\mathbf{k}$ and helicity $+1/2$, i.e. the presence of a positive energy antiparticle with momentum $+\mathbf{k}$ and helicity again $+1/2$ (because the absence of the spin antiparallel to $-\mathbf{k}$ is equivalent to the presence of the spin antiparallel to $\mathbf{k}$). This interpretation becomes more obvious in the second quantization language.

Similarly, for the solutions of the equation

$$\mathrm{i}(\sigma^{\mu})_{\alpha\dot{\beta}}\partial_{\mu}\bar{\chi}^{\dot{\beta}}(x) = 0 \tag{A.63}$$

we get

$$\left.\begin{aligned}\bar{\chi}_{\text{pos}}(x) &= \exp(-\mathrm{i}k\cdot x)a'(\mathbf{k}) = \mathcal{N}_{a'}\exp(-\mathrm{i}k\cdot x)\exp(\mathrm{i}\varphi'_a)a_+(\mathbf{k})\\ \bar{\chi}_{\text{neg}}(x) &= \exp(+\mathrm{i}k\cdot x)b'(\mathbf{k}) = \mathcal{N}_{b'}\exp(+\mathrm{i}k\cdot x)\exp(\mathrm{i}\varphi'_b)a_+(\mathbf{k})\end{aligned}\right\} \tag{A.64}$$

(upper case dotted indices are understood), where

$$a_+(\mathbf{k}) = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\exp(\mathrm{i}\phi)\end{pmatrix} \tag{A.65}$$

is normalized to unity and such that we have

$$\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}a'(\mathbf{k}) = a'(\mathbf{k}) \tag{A.66}$$

and the same relation for $b'(\mathbf{k})$. Therefore, $\bar{\chi}_{\text{pos}}(x)$ represents a $+1/2$ helicity particle, whereas $\bar{\chi}_{\text{neg}}(x)$ describes a $-1/2$ helicity antiparticle. Normalization factors $\mathcal{N}$ are determined by requiring that there are $2E$ particles per unit volume, i.e. that

$$\int_{\text{unit volume}} \mathrm{d}^3x\,\bar{\lambda}\bar{\sigma}^0\lambda = \int_{\text{unit volume}} \mathrm{d}^3x\,\chi\sigma^0\bar{\chi} = 2E \tag{A.67}$$

leading to $\mathcal{N} = (2E)^{1/2}$ for all $\mathcal{N}$. With this normalization we get

$$\left.\begin{aligned}a(\mathbf{k})_{\alpha}a^*(\mathbf{k})_{\dot{\beta}} &= b(\mathbf{k})_{\alpha}b^*(\mathbf{k})_{\dot{\beta}} = k_{\mu}\sigma^{\mu}_{\alpha\dot{\beta}}\\ a'(\mathbf{k})^{\dot{\alpha}}a'^*(\mathbf{k})^{\beta} &= b'(\mathbf{k})^{\dot{\alpha}}b'^*(\mathbf{k})^{\beta} = k_{\mu}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\end{aligned}\right\} \tag{A.68}$$

which are the analogues of (A.80) for four-component spinors.

In the four-component spinor language ((A.47)–(A.54)) (A.58) and (A.63) both take the form

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\Psi_{\mathrm{L,R}} = 0 \tag{A.69}$$

supplemented with different conditions $\gamma^5\Psi_{\mathrm{L}} = -\Psi_{\mathrm{L}}$ and $\gamma^5\Psi_{\mathrm{R}} = \Psi_{\mathrm{R}}$, respectively which in the Weyl representation are trivially satisfied with

$$\Psi_{\mathrm{L}} = \begin{pmatrix}\lambda_{\alpha}\\ 0\end{pmatrix}, \quad \Psi_{\mathrm{R}} = \begin{pmatrix}0\\ \bar{\chi}_{\dot{\alpha}}\end{pmatrix} \tag{A.70}$$

Thus, for massless fermions chirality eigenstates are at the same time helicity eigenstates. Chiral spinors in the Dirac representation are obtained by (A.52).

The Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \tag{A.71}$$

has four independent free-particle solutions (two of which correspond to positive and two other to negative energies) given by

$$\left.\begin{aligned} \Psi(x)_{\text{pos}} &= \exp(-\mathrm{i}p\cdot x)\, u(\mathbf{p}, s), \quad s = 1, 2 \\ \Psi(x)_{\text{neg}} &= \exp(+\mathrm{i}p\cdot x)\, v(\mathbf{p}, s), \quad s = 1, 2 \end{aligned}\right\} \tag{A.72}$$

where $p^\mu = (E, \mathbf{p})$, $E \equiv p^0 = (\mathbf{p}^2 + m^2)^{1/2} > 0$ is the energy–momentum of the particle and the s numbers its polarizations in a chosen basis (see below). From (A.71) we get

$$(\not{p} - m)u(\mathbf{p}, s) = 0, \quad (\not{p} + m)v(\mathbf{p}, s) = 0 \tag{A.73}$$

The solutions of (A.73), valid in any representation of the Dirac matrices, read:

$$\left.\begin{aligned} u(\mathbf{p}, s) &= \frac{\not{p} + m}{[2m(m+E)]^{1/2}} u(\mathbf{O}, s) \\ v(\mathbf{p}, s) &= \frac{-\not{p} + m}{[2m(m+E)]^{1/2}} v(\mathbf{O}, s) \end{aligned}\right\} \tag{A.74}$$

In (A.74) $u(\mathbf{O}, s)$ and $v(\mathbf{O}, s)$ are four linearly independent solutions in the rest frame of the particle in which $p^\mu_{\text{rest}} = (m, \mathbf{O})$. Obviously, $p^\mu = \Lambda^\mu{}_\nu p^\nu_{\text{rest}}$, with $\Lambda^\mu{}_\nu$ being the appropriate Lorentz boost. In the rest frame $s = 1$ (2) numbers the solutions with spin projection onto a chosen unit spin vector $\mathbf{s} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ equal $+\frac{1}{2}$ ($-\frac{1}{2}$). In the *Dirac representation* (A.51) of the γ^μ matrices, the two linearly independent (and orthogonal) solutions with positive energy read

$$u(\mathbf{O}, 1) = \mathcal{N}\begin{pmatrix} a_+(\mathbf{s}) \\ 0 \\ 0 \end{pmatrix}, \quad u(\mathbf{O}, 2) = \mathcal{N}\begin{pmatrix} a_-(\mathbf{s}) \\ 0 \\ 0 \end{pmatrix} \tag{A.75}$$

where $\mathcal{N}$ are normalization factors and two-component spinors $a_\pm$ are defined in (A.65) and (A.62). For negative energy solutions we have

$$v(\mathbf{O}, 1) = \mathcal{N}\begin{pmatrix} 0 \\ 0 \\ -a_-(\mathbf{s}) \end{pmatrix}, \quad v(\mathbf{O}, 2) = \mathcal{N}\begin{pmatrix} 0 \\ 0 \\ a_+(\mathbf{s}) \end{pmatrix} \tag{A.76}$$

(to match the Dirac sea interpretation, the solutions with $+\frac{1}{2}$ and $-\frac{1}{2}$ spin projections are numbered in reversed order). Eqs. (A.74)–(A.76) are sufficient to explicitly construct four-component spinors for massive fermions in the Dirac representation (see, for example, Haber (1993)). Using the transformation (A.52) one can also construct them in the chiral representation. The helicity eigenstates

$$\frac{\Sigma p}{|p|}u(p,h) = \pm\frac{1}{2}u(p,h) \tag{A.77}$$

can be constructed by specifying the directions of s and p to be parallel: $\sigma p a^{+}(s) = a^{+}(s)$.

The spinors (A.74) are normalized as follows ($\bar{u} \equiv u^{\dagger}\gamma^{0}$; cf. (A.47) and (A.48)):

$$\left.\begin{aligned} \bar{u}(\mathbf{p},s)u(\mathbf{p},s') &= \mathcal{N}^2\delta_{ss'}, \quad \bar{u}(\mathbf{p},s)v(\mathbf{p},s') = 0, \\ \bar{v}(\mathbf{p},s)v(\mathbf{p},s') &= -\mathcal{N}^2\delta_{ss'}, \quad \bar{v}(\mathbf{p},s)u(\mathbf{p},s') = 0 \end{aligned}\right\} \tag{A.78}$$

or, equivalently,

$$u^{\dagger}(\mathbf{p},s)u(\mathbf{p},s') = \mathcal{N}^2\frac{E}{m}\delta_{ss'}, \quad v^{\dagger}(\mathbf{p},s)v(\mathbf{p},s') = \mathcal{N}^2\frac{E}{m}\delta_{ss'} \tag{A.79}$$

As for the massless chiral spinors we choose $\mathcal{N} = (2m)^{1/2}$ corresponding to $2E$ particles in a unit volume. By explicit calculation we then find

$$\left.\begin{aligned} \sum_{s=1,2} u(\mathbf{p},s)\bar{u}(\mathbf{p},s) &= \not{p} + m \\ \sum_{s=1,2} v(\mathbf{p},s)\bar{v}(\mathbf{p},s) &= \not{p} - m \end{aligned}\right\} \tag{A.80}$$

One can also define projection operators onto the states with positive and negative energy

$$\Lambda_{\pm}(p) \equiv \frac{\pm\not{p} + m}{2m} \tag{A.81}$$

with properties $\Lambda_{\pm}^2(p) = \Lambda_{\pm}(p)$, $\Lambda_{\pm}(p)\Lambda_{\mp}(p) = 0$, $\Lambda_{+}(p) + \Lambda_{-}(p) = 1$ and satisfying

$$\left.\begin{aligned} \Lambda_{+}(p)\Psi(x)_{\text{pos}} &= \Psi(x)_{\text{pos}}, \quad \Lambda_{-}(p)\Psi(x)_{\text{pos}} = 0, \\ \Lambda_{-}(p)\Psi(x)_{\text{neg}} &= \Psi(x)_{\text{neg}}, \quad \Lambda_{+}(p)\Psi(x)_{\text{neg}} = 0 \end{aligned}\right\} \tag{A.82}$$

Projection operators on the state which in the rest frame has spin parallel (antiparallel) to the direction **s** for positive (negative) energy solution reads

$$P(s) = \frac{1 + \gamma^5\not{s}}{2} \tag{A.83}$$

where the four-vector s^{μ} ($s^2 = -1$, $p \cdot s = 0$) is related to **s** by the appropriate Lorentz boost: $s^{\mu} = \Lambda^{\mu}{}_{\nu}s^{\nu}_{\text{rest}}$, $s^{\nu}_{\text{rest}} = (0, \mathbf{s})$. Because of the reverse assignment (A.76) $P(s)$ projects the positive energy solutions onto the states with spin in the rest frame parallel to **s** and negative energy solutions onto the states with spin antiparallel to $\mathbf{s}$ (which in the Dirac sea interpretation represent the antiparticle with spin parallel to $\mathbf{s}$).

A useful basis for the spin states is obtained by choosing the four-vector s^μ in the form:

$$s^\mu = \left(\frac{|\mathbf{p}|}{m}, \frac{E}{m}\frac{\mathbf{p}}{|\mathbf{p}|}\right) \tag{A.84}$$

which is obtained from the unit vector $\mathbf{s} = \mathbf{p}/|\mathbf{p}|$ by the boost relating the rest frame of the particle to the frame where it has a momentum $\mathbf{p}$. In this case $P(s)$ projects the positive energy solutions onto the states with spin parallel to $\mathbf{s}$, i.e. to the momentum of the state and, for negative energy solutions, onto the states with spin antiparallel to $\mathbf{s}$. Consequently, $P(s)$ projects onto the positive helicity states for both signs of energy since for negative energy solutions the momentum of the state (in the sense of the eigenvalue of the operator $-\mathrm{i}\partial/\partial\mathbf{x}$) is $-\mathbf{p}$, i.e. parallel to $-\mathbf{s}$. This is confirmed by noting that for s^μ given by (A.84)

$$P(s)\Lambda_\pm(p) = \frac{1}{2}\left(1 \pm \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\Lambda_\pm(p) \tag{A.85}$$

where

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{A.86}$$

is the spin operator. Thus, for the negative energy solutions the operator (A.85) projects onto the states which in the Dirac sea interpretation represent the states of the antiparticle with physical momentum $+\mathbf{p}$ and the spin parallel to $\mathbf{p}$, i.e. onto the positive helicity states of the antiparticle.

For the matrix elements of processes involving massive Majorana particles external wave-functions are provided by the solutions of the free Dirac equation.†

Parity

Transformation of the space reflection is the change of the description of the physical system from the one given in the right-handed coordinate frame to the description in the left-handed frame. Thus, $x^\mu \to x^{\mu\prime}$ with $x^{0\prime} = x^0$, $\mathbf{x}' = -\mathbf{x}$.

A positive (negative) energy solution (A.59) of the Weyl equation (A.58) for left-handed fields upon substituting $\mathbf{x} = -\mathbf{x}'$ becomes the positive (negative) energy solution of (A.63) (in the primed variables) for right-handed fields with $\mathbf{k}' = -\mathbf{k}$. Similarly, the solutions of the equation for right-handed fields become

† Massless Majorana fermions are indistinguishable from massless Weyl fermions (there are just two states connected by CPT).

in the reflected frame the solutions of the equation for the left-handed fields with reversed momentum. This follows from the the following relations:

$$\left.\begin{aligned} (\sigma^0)_{\alpha\dot{\beta}}a'(-\mathbf{k})^{\dot{\beta}} &= \mathrm{i}\exp[\mathrm{i}(\varphi'_a - \varphi_a)]a(\mathbf{k})_\alpha \\ (\bar{\sigma}^0)^{\dot{\alpha}\beta}a(-\mathbf{k})_\beta &= \mathrm{i}\exp[\mathrm{i}(\varphi_a - \varphi'_a)]a'(\mathbf{k})^{\dot{\alpha}} \end{aligned}\right\} \tag{A.87}$$

and similar relations for $b'(-\mathbf{k})$ and $b(-\mathbf{k})$.

For the four-component Dirac spinors, the positive (negative) energy solutions (A.72) of the Dirac equation (A.71) with momentum $\mathbf{p}$ and spin projection label s upon substituting $\mathbf{x} = -\mathbf{x}'$ and multiplying by γ^0 become the positive (negative) energy solutions of the Dirac equation in the reflected frame with momentum $-\mathbf{p}$ and the same spin label s. This follows from the relations

$$u(-\mathbf{p}, s) = \gamma^0 u(\mathbf{p}, s), \quad v(-\mathbf{p}, s) = -\gamma^0 v(\mathbf{p}, s) \tag{A.88}$$

For helicity spinors, from (A.62) and (A.65), we get instead

$$\left.\begin{aligned} u(-p, -h) &= (-1)^{\frac{1}{2}-h}\exp(-2\mathrm{i}h\phi)\gamma^0 u(p, h) \\ v(-p, -h) &= (-1)^{\frac{1}{2}+h}\exp(2\mathrm{i}h\phi)\gamma^0 v(p, h) \end{aligned}\right\} \tag{A.89}$$

Time reversal

Transformation of the time reversal can be viewed as a change of the direction of the time axis of the reference frame. Thus, $t' = -t$, $\mathbf{x}' = \mathbf{x}$. A state with momentum $\mathbf{k}$ and helicity $-1/2$ $(+1/2)$ described originally by a solution ((A.59), (A.64)) of the Weyl equation ((A.58), (A.63)) in the new frame should be described by a solution (with the same sign of the energy) of the Weyl equation ((A.58), (A.63)) (in the primed variables) with momentum $\mathbf{k}' = -\mathbf{k}$. The solutions in the new frame $\lambda_{T\alpha}(x')$ and $\bar{\chi}_T^{\dot{\alpha}}(x')$ are obtained from the original ones by the operations

$$\left.\begin{aligned} \lambda_{T\alpha}(x') &\equiv \exp(\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar{\sigma}^3)_\alpha{}^\beta(\lambda^\star(x))_\beta \\ \bar{\chi}_T^{\dot{\alpha}}(x') &\equiv \exp(\mathrm{i}\bar{\gamma})(\mathrm{i}\bar{\sigma}^1\sigma^3)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{\chi}^\star(x))^{\dot{\beta}} \end{aligned}\right\} \tag{A.90}$$

Substituting $t = -t'$ and taking the complex conjugate brings back the exponential factors to the right form $\exp[-\mathrm{i}(Et' - \mathbf{k}' \cdot \mathbf{x}')]$ $(\exp[+\mathrm{i}(Et' - \mathbf{k}' \cdot \mathbf{x}')])$ for positive (negative) energy solutions. The correct behaviour of the spinor factors follow from the properties

$$(\mathrm{i}\sigma^1\bar{\sigma}^3)a(-\mathbf{k}) = \exp(2\mathrm{i}\varphi_a)a^\star(\mathbf{k}), \quad (\mathrm{i}\bar{\sigma}^1\sigma^3)a'(-\mathbf{k}) = -\exp(2\mathrm{i}\varphi'_a)a'^\star(\mathbf{k}) \tag{A.91}$$

$$\mathrm{i}\sigma^1\bar{\sigma}^3 = \mathrm{i}\bar{\sigma}^1\sigma^3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{A.92}$$

and similar relation for $b(\mathbf{k})$ and $b'(\mathbf{k})$.

For the four-component Dirac spinors, the positive (negative) energy solutions (A.72) of the Dirac equation (A.71) with momentum $\mathbf{p}$ and spin projection label s upon substituting $t = -t'$, complex conjugating and multiplying by $\mathrm{i}\gamma^1\gamma^3$ become the positive (negative) energy solutions of the Dirac equation in the reflected frame with momentum $-\mathbf{p}$ and opposite spin $s' = -s$. This follows from the relations

$$\left.\begin{aligned} \mathrm{i}\gamma^1\gamma^3 u(-\mathbf{p}, s) &= \mathrm{i}(-1)^s u^\star(\mathbf{p}, 3-s) \\ \mathrm{i}\gamma^1\gamma^3 v(-\mathbf{p}, s) &= -\mathrm{i}(-1)^s v^\star(\mathbf{p}, 3-s) \end{aligned}\right\} \tag{A.93}$$

Charge conjugation

Consider a charged field Φ transforming as $\Phi \rightarrow \exp(-\mathrm{i}q\Theta)\Phi$ under the $U(1)$ group of gauge transformations interacting with a classical electromagnetic potential $A_\mu(x)$. The positive energy solution† $\Phi_{\mathrm{pos}}(x)$ of the equation

$$[(\partial + \mathrm{i}qeA)^2 + m^2]\Phi_{\mathrm{pos}}(x) = 0 \tag{A.94}$$

describes the behaviour of a particle carrying charge q in the vector potential $A_\mu(x)$. The negative energy solution $\Phi_{\mathrm{neg}}(x)$ cannot by itself have such an interpretation. However, the function $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x) = \exp(\mathrm{i}\gamma)(\Phi_{\mathrm{neg}}(x))^\star$ having positive energy satisfies the equation

$$[(\partial - \mathrm{i}qeA)^2 + m^2]\Phi^{\mathrm{c}}_{\mathrm{pos}}(x) = 0 \tag{A.95}$$

and describes the behaviour of another particle (called the antiparticle) carrying charge $-q$ in the same potential. It transforms as $\Phi^{\mathrm{c}}_{\mathrm{pos}} \rightarrow \exp(+\mathrm{i}q\Theta)\Phi^{\mathrm{c}}_{\mathrm{pos}}$ under the $U(1)$ group. In the potential $A^{\mathrm{c}}_\mu(x) \equiv -A_\mu(x)$, the solution $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x)$ has obviously (up to a constant phase factor) an identical space-time form as the solution $\Phi_{\mathrm{pos}}(x)$ in the original potential $A_\mu(x)$. Thus, the simultaneous change of a particle for its antiparticle and of the sign of the potential is a symmetry of the theory because the resulting physical system behaves like the original one (their wave-functions are identical). The same applies to a set of fields Φ_i transforming as a representation R of a non-abelian gauge group G and interacting with the non-abelian potential $T^a A^a_\mu(x)$ (T^a are the generators of the group) provided we substitute $T^a A^a_\mu(x) \rightarrow T^a A^{\mathrm{c}a}_\mu(x) = -T^{a\star} A^a_\mu(x)$. $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x)$ transforms as $R^\star$ under the action of G.

For left-handed Weyl fields λ^i, transforming as a representation R of the gauge group, the positive energy solution of the equation

$$\mathrm{i}\bar{\sigma} \cdot (\partial + \mathrm{i}gA^a T^a) \cdot \lambda_{\mathrm{pos}}(x) = 0 \tag{A.96}$$

describes a helicity $h = -1/2$ particle interacting with the external potential

† Strictly speaking, for energy of the solution to be well defined, the potential A_μ should not depend on time.

$T^a A^a_\mu(x)$. The negative energy solution of this equation, $\lambda^i_{\text{neg}}(x)$, upon complex conjugation (and raising the spinor index) becomes the positive energy solution of the equation

$$\mathrm{i}\sigma \cdot (\partial - \mathrm{i}g A^a T^{a\star}) \cdot \bar{\lambda}_{\text{pos}}(x) = 0 \tag{A.97}$$

and describes a helicity $h = +1/2$ antiparticle ($\bar{\lambda}_{\text{pos}}(x)$ transforms as $R^\star$ under the action of the gauge group). This confirms the interpretation given to the negative energy solutions of (A.58) and (A.63). Clearly, there is no choice of $T^a A^{ca}_\mu(x)$ in which this antiparticle would behave like the helicity $h = -1/2$ particle in the original potential. The wave-function of a helicity $h = -1/2$ antiparticle denoted as λ^{c} transforming as $R^\star$ under the gauge transformations is a positive energy solution of the equation

$$\mathrm{i}\bar{\sigma} \cdot (\partial - \mathrm{i}g A^a T^{a\star}) \cdot \lambda^{\mathrm{c}}_{\text{pos}}(x) = 0 \tag{A.98}$$

Obviously, in the potential $T^a A^{ca}_\mu(x) = -T^{a\star} A^a_\mu(x)$ the wave-function of the helicity $h = -1/2$ antiparticle is identical (up to a constant phase factor) to the one of the helicity $h = -1/2$ particle in the original potential $T^a A^a_\mu(x)$. Thus, if the physical system consists of both sets of fields, λ_i and λ^{c}_i, the charge conjugation is its symmetry.

Positive energy solutions of the equation

$$\left(\mathrm{i}\not{\partial} - g \not{A}^a T^a - m\right) \Psi_{\text{pos}}(x) = 0 \tag{A.99}$$

for the Dirac field describe two polarization states of a massive, spin $1/2$ particle and transform as representation R under the action of the gauge group. The positive energy wave-function

$$\Psi^{\mathrm{c}}_{\text{pos}}(x) = \exp(\mathrm{i}\gamma) C \gamma^0 \Psi^\star_{\text{neg}}(x) = \exp(\mathrm{i}\gamma) C \bar{\Psi}^{\mathrm{T}}_{\text{neg}}(x) \tag{A.100}$$

where $\Psi_{\text{neg}}(x)$ is a negative energy solution of (A.99) and the matrix C is chosen so that

$$(C\gamma^0)\gamma^{\mu\star}(C\gamma^0)^{-1} = -\gamma^\mu \tag{A.101}$$

satisfies the equation†

$$\left(\mathrm{i}\not{\partial} + g \not{A}^a T^{a\star} - m\right) \Psi^{\mathrm{c}}_{\text{pos}}(x) = 0 \tag{A.102}$$

and (having positive energy) can be interpreted as a wave-function of the corresponding antiparticle in the same potential $A^a T^a$. Obviously, $\Psi^{\mathrm{c}}_{\text{pos}}(x)$ transforms as $R^\star$. In the potential $T^a A^{ca}_\mu(x) = -T^{a\star} A^a_\mu(x)$ the solution $\Psi^{\mathrm{c}}_{\text{pos}}(x)$ has the same space-time form as the solution $\Psi_{\text{pos}}(x)$ in the original potential and, therefore, charge conjugation is a symmetry of the physical system.

† Of course $\Psi^{\mathrm{c}}_{\text{neg}}(x) = \exp(\mathrm{i}\gamma) C \bar{\Psi}^{\mathrm{T}}_{\text{pos}}(x)$ satisfies (A.102) with negative energy.

The matrix C defined by (A.99) can be taken as

$$C = \mathrm{i}\gamma^0\gamma^2 = \begin{pmatrix} 0 & -\mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix} \tag{A.103}$$

which, for example, in the Weyl representation is given by (cf. (A.20) and (A.47))

$$C = \begin{pmatrix} \{\varepsilon_{\alpha\beta}\} & 0 \\ 0 & \{\varepsilon^{\dot\alpha\dot\beta}\} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.104}$$

The matrix C satisfies the following relations:

$$C^{-1} = C^\dagger = C^\mathrm{T} = -C, \quad C^2 = -I, \quad \left[C, \gamma^5\right] = 0 \tag{A.105}$$

Under the operation (A.100) a solution of (A.99) characterized by the momentum and spin four-vectors p^μ and s^μ, i.e. satisfying

$$\Psi(x) = \left(\frac{\varepsilon \not p + m}{2m}\right)\left(\frac{1+\gamma^5 \not s}{2}\right)\Psi(x) \tag{A.106}$$

transforms into

$$\begin{aligned}\Psi^\mathrm{c}(x) &\equiv \exp(\mathrm{i}\gamma) C \bar\Psi^\mathrm{T}(x) = \exp(\mathrm{i}\gamma)(C\gamma^0)\left(\frac{\varepsilon \not p + m}{2m}\right)^\star \left(\frac{1+\gamma^5 \not s}{2}\right)^\star \Psi^\star(x) \\ &= \left(\frac{-\varepsilon \not p + m}{2m}\right)\left(\frac{1+\gamma^5 \not s}{2}\right)\Psi^\mathrm{c}(x)\end{aligned} \tag{A.107}$$

where $\varepsilon = \pm 1$, i.e. the solution with negative energy, momentum $-\mathbf{p}$ and spin antiparallel to $\mathbf{s}$ is transformed into the solution with positive energy, momentum $\mathbf{p}$ and spin parallel to $\mathbf{s}$ etc. Therefore

$$\left.\begin{aligned} C\bar u^\mathrm{T}(\mathbf{p}, s) &= v(\mathbf{p}, s) \\ C\bar v^\mathrm{T}(\mathbf{p}, s) &= u(\mathbf{p}, s) \end{aligned}\right\} \tag{A.108}$$

The same relations are valid for the helicity eigenstates. For a field with definite chirality

$$\Psi(x) = \left(\frac{\varepsilon \not p + m}{2m}\right)\left(\frac{1 \pm \gamma^5}{2}\right)\Psi(x) \tag{A.109}$$

we get

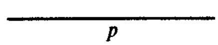
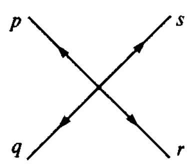
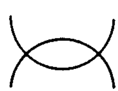
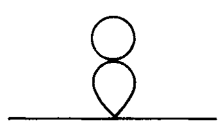
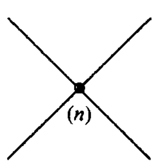
$$\begin{aligned}\Psi^\mathrm{c}(x) &= \exp(\mathrm{i}\gamma)(C\gamma^0)\left(\frac{\varepsilon \not p + m}{2m}\right)^\star \left(\frac{1 \pm \gamma^5}{2}\right)^\star \Psi^\star(x) \\ &= \left(\frac{-\varepsilon \not p + m}{2m}\right)\left(\frac{1 \mp \gamma^5}{2}\right)\Psi^\mathrm{c}(x)\end{aligned} \tag{A.110}$$

Appendix B

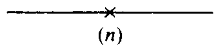
Feynman rules for QED and QCD and Feynman integrals

In this Appendix we collect Feynman rules used in Chapters 4, 5 and 8 for $\lambda\phi^4$ theory, QED and QCD, respectively.

Feynman rules for the $\lambda\Phi^4$ theory:

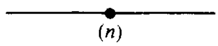
propagator		$\dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$
loop integration		$\displaystyle\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$
vertex		$-\mathrm{i}\lambda, \quad p + q + r + s = 0$
symmetry factors		$S = \frac{1}{2}$
		$S = \frac{1}{6}$
		$S = \frac{1}{2}$
		$S = \frac{1}{4}$
vertex counterterm		$-\mathrm{i}\lambda(Z_1 - 1)^{(n)}$

mass counterterm



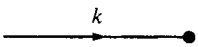
$-\mathrm{i}(Z_0 - 1)^{(n)} m^2$

wave-function renormalization counterterm

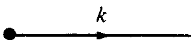


$+\mathrm{i}(Z_3 - 1)^{(n)} p^2$

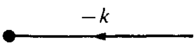
Feynman rules for QED



incoming electron with momentum k: $u(k, s)$



outgoing electron with momentum k: $\bar{u}(k, s)$



outgoing positron with momentum k: $v(k, s)$

incoming positron with momentum k: $\bar{v}(k, s)$

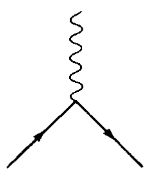
fermion propagator

$$\frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon}$$

photon propagator

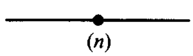
$$\frac{-\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[g_{\mu\nu} - (1 - a)\frac{k_\mu k_\nu}{k^2}\right]$$

fermion–fermion–photon vertex



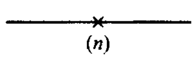
$+\mathrm{i}e\gamma_\mu$

fermion wave-function renormalization counterterm



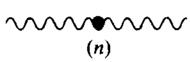
$\mathrm{i}(Z_2 - 1)^{(n)} \not{p}$

fermion mass counterterm



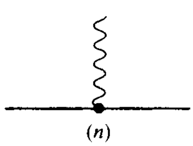
$-\mathrm{i}m(Z_0 - 1)^{(n)} \equiv \mathrm{i}\delta m^{(n)}$

photon wave-function renormalization counterterm



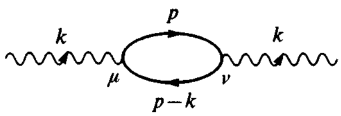
$-\mathrm{i}(Z_3 - 1)^{(n)}$
$(k^2 g_{\mu\nu} - k_\mu k_\nu)$

vertex counterterm



$+\mathrm{i}e(Z_1 - 1)^{(n)} \gamma_\mu$

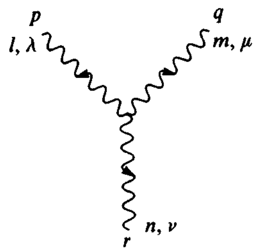
Fermion loop parametrization



$$= (-1)\,\mathrm{Tr}\left[\frac{\mathrm{i}}{\not p - m}\gamma_\mu \frac{\mathrm{i}}{\not p - \not k - m}\gamma_\nu\right](-e^2)$$

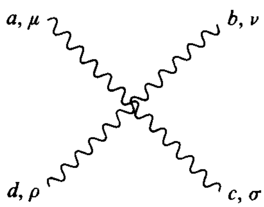
Feynman rules for QCD

three-gluon coupling



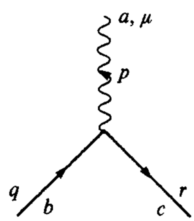
$$-gc_{lmn}[(r-p)_\mu g_{\lambda\nu} + (r-q)_\lambda g_{\mu\nu} + (q-p)_\nu g_{\lambda\mu}]$$

four-gluon coupling



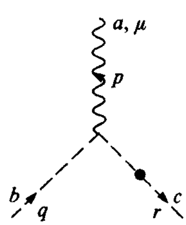
$$(-\mathrm{i}g^2)[c_{abe}c_{cde}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + c_{ace}c_{bde}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) + c_{ade}c_{cbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma})]$$

quark–quark–gluon vertex



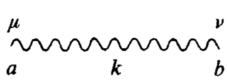
$$-\mathrm{i}g\gamma_\mu (T^a)_{bc}$$
$$\mathrm{Tr}[T^a T^b] = \tfrac{1}{2}\delta^{ab}$$

ghost–ghost–gluon vertex



$$+gc_{abc}r_\mu$$

gluon propagator



covariant gauge:

$$-\frac{1}{2a}(\partial_\mu A_a^\mu)^2$$

$$\frac{-\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\delta_{ab}\left[g_{\mu\nu} - (1-a)\frac{k_\mu k_\nu}{k^2}\right]$$

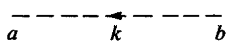
Coulomb gauge: $\partial_\mu A_a^\mu - (n_\mu \partial^\mu)(n_\mu A_a^\mu) = 0, \quad n_\mu = (1, 0, 0, 0)$

$$\frac{-\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\delta_{ab}\left[g_{\mu\nu} - \frac{k \cdot n(k_\mu n_\nu + k_\nu n_\mu) - k_\mu k_\nu}{(k \cdot n)^2 - k^2}\right]$$

axial gauge: $n_\mu A_a^\mu = 0, \quad n^2 = 0$ or $n^2 < 0$

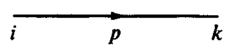
$$\frac{-\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\delta_{ab}\left[g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} + \frac{n^2}{(n \cdot k)^2}k_\mu k_\nu\right]$$

ghost propagator



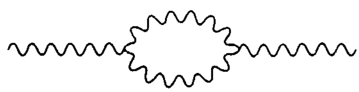
$$-\frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\delta_{ab}$$

quark propagator

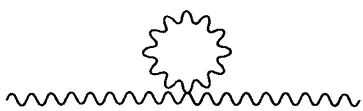


$$\delta_{ik}\frac{\mathrm{i}}{\not{p} - m + \mathrm{i}\varepsilon}$$

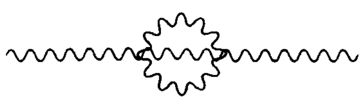
symmetry factors



$$S = \frac{1}{2!}$$



$$S = \frac{1}{2!}$$



$$S = \frac{1}{3!}$$

For each fermion and ghost loop there is a minus sign. As for $\lambda\Phi^4$ and for QED, counterterms can be explicitly introduced into the Feynman rules. The complete list is given in Appendix C.

Dirac algebra in n dimensions

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{B.1}$$

$$g^\mu{}_\mu = n \tag{B.2}$$

$$\gamma^\mu \gamma_\mu = n \tag{B.3}$$

$$\gamma_\mu \not{a} \gamma^\mu = (2 - n)\not{a} \tag{B.4}$$

$$\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a \cdot b + (n - 4)\not{a}\not{b} \tag{B.5}$$

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} - (n - 4)\not{a}\not{b}\not{c} \tag{B.6}$$

$$\mathrm{Tr}\, \mathbb{1} = 4 \quad \mathrm{Tr}(\text{odd } \gamma) = 0 \quad \mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \tag{B.7}$$

In four dimensions ($\varepsilon_{0123} = -\varepsilon^{0123} = 1$)

$$\gamma^5 = \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \mathrm{i}\varepsilon_{\alpha\beta\gamma\rho}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\rho/4! \tag{B.8}$$

$$\mathrm{Tr}[\gamma^5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\rho] = +\mathrm{i}\varepsilon^{\alpha\beta\gamma\rho}\,\mathrm{Tr}\,\mathbb{1} \tag{B.9}$$

$$\gamma_\mu\gamma_\rho\gamma_\nu = (S_{\mu\rho\nu\sigma} + \mathrm{i}\varepsilon_{\mu\rho\nu\sigma}\gamma_5)\gamma^\sigma \tag{B.10}$$

$$S_{\mu\rho\nu\sigma} = g_{\mu\rho}g_{\nu\sigma} + g_{\rho\nu}g_{\rho\sigma} - g_{\mu\nu}g_{\rho\sigma} \tag{B.11}$$

The Dirac and chiral representations for γs are given in Appendix A.

$$(\gamma^5)^\dagger = \gamma^5, \quad (\gamma^\mu)^\dagger = \gamma_\mu = \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^\mu & \mu \geq 1 \end{cases} \tag{B.12}$$

Feynman parameters

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x\, \frac{1}{[(1-x)b + xa]^2} \tag{B.13}$$

$$\frac{1}{a^n b} = n\int_0^1 \mathrm{d}x\, \frac{x^{n-1}}{[(1-x)b + xa]^{n+1}} \tag{B.14}$$

$$\begin{aligned}\frac{1}{abc} &= 2\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \frac{x}{[axy + bx(1-y) + c(1-x)]^3} \\ &= 2\int_0^1 \mathrm{d}u \int_0^{1-u} \mathrm{d}w\, \frac{1}{[aw + b(1-u-w) + cu]^3}\end{aligned} \tag{B.15}$$

$$\begin{aligned}\frac{1}{a_1 a_2 \ldots a_n} &= (n-1)! \int_0^1 \mathrm{d}x_n\, \mathrm{d}x_{n-1} \ldots \mathrm{d}x_2 \\ &\times \frac{x_n^{n-2} x_{n-1}^{n-3} \ldots x_3^1 x_2^0}{[(1-x_n)a_n + x_n[(1-x_{n-1})a_{n-1} + x_{n-1}[\cdots + x_3[(1-x_2)a_2 + x_2 a_1]] \ldots]^n}\end{aligned} \tag{B.16}$$

Feynman integrals in n dimensions

$$\begin{aligned}I_0 &= \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{1}{(p^2 + 2k\cdot p + M^2 + \mathrm{i}\varepsilon)^\alpha} \\ &= \frac{\mathrm{i}(-\pi)^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - \frac{1}{2}n)}{\Gamma(\alpha)} \frac{1}{(M^2 - k^2 + \mathrm{i}\varepsilon)^{\alpha - n/2}}\end{aligned} \tag{B.17}$$

$$I_\mu = \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{p_\mu}{(p^2 + 2k\cdot p + M^2 + \mathrm{i}\varepsilon)^\alpha} = -k_\mu I_0 \tag{B.18}$$

$$
\begin{aligned}
I_{\mu\nu} &= \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{p_\mu p_\nu}{(p^2 + 2k \cdot p + M^2 + \mathrm{i}\varepsilon)^\alpha} \\
&= I_0 \left[k_\mu k_\nu + \tfrac{1}{2} g_{\mu\nu} (M^2 - k^2) \frac{1}{\alpha - \frac{1}{2}n - 1} \right]
\end{aligned} \tag{B.19}
$$

$$
\begin{aligned}
I_{\mu\nu\rho} &= \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{p_\mu p_\nu p_\rho}{(p^2 + 2k \cdot p + M^2 + \mathrm{i}\varepsilon)^2} \\
&= -I_0 \left[k_\mu k_\nu k_\rho + \tfrac{1}{2} (g_{\mu\nu} k_\rho + g_{\mu\rho} k_\nu + g_{\nu\rho} k_\mu)(M^2 - k^2) \frac{1}{\alpha - \frac{1}{2}n - 1} \right]
\end{aligned} \tag{B.20}
$$

Also, by convention

$$
\int \mathrm{d}^n p \, (p^2)^{-1} = 0
$$

α-representation

$$
\frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{d}\alpha \exp[\mathrm{i}\alpha (p^2 - m^2 + \mathrm{i}\varepsilon)] \tag{B.21}
$$

Gaussian integrals

$$
\int \mathrm{d}^4 k \exp[\mathrm{i}(ak^2 + 2b \cdot k)] = \frac{1}{\mathrm{i}} \left(\frac{\pi}{a} \right)^2 \exp\left(-\mathrm{i} \frac{b^2}{a} \right) \tag{B.22}
$$

$$
\int \mathrm{d}^4 k \, k_\mu \exp[\mathrm{i}(ak^2 + 2b \cdot k)] = \frac{1}{\mathrm{i}} \left(\frac{\pi}{a} \right)^2 \exp\left(-\mathrm{i} \frac{b^2}{a} \right) \left(-\frac{b_\mu}{a} \right) \tag{B.23}
$$

$$
\int \mathrm{d}^4 k \, k_\mu k_\nu \exp[\mathrm{i}(ak^2 + 2b \cdot k)] = \frac{1}{\mathrm{i}} \left(\frac{\pi}{a} \right)^2 \exp\left(-\mathrm{i} \frac{b^2}{a} \right) \left(\frac{\mathrm{i}a g_{\mu\nu} + 2b_\mu b_\nu}{2a^2} \right) \tag{B.24}
$$

λ-parameter integrals

$$
\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} \exp[\mathrm{i}(A + \mathrm{i}\varepsilon)\lambda] = -\ln(A + \mathrm{i}\varepsilon) + \infty \tag{B.25}
$$

$$
\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} [\exp(\mathrm{i}A\lambda) - \exp(\mathrm{i}B\lambda)] \exp(-\varepsilon\lambda) = \ln \frac{B + \mathrm{i}\varepsilon}{A + \mathrm{i}\varepsilon} \tag{B.26}
$$

$$
\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^2} \exp(-\varepsilon\lambda) f(\lambda) \underset{\varepsilon \to 0}{=} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} \exp(-\varepsilon\lambda) \frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda} \tag{B.27}
$$

Feynman integrals in light-like gauge $n \cdot A = 0$, $n^2 = 0$

Using

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \, \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} \tag{B.28}$$

one can show that

$$\begin{aligned} I(\alpha) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2-2p\cdot k+M^2)^\alpha (k\cdot n)^\beta} \\ &= \frac{1}{(p\cdot n)^\beta} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{(k^2-2p\cdot k+M^2)^\alpha} \end{aligned} \tag{B.29}$$

$$\begin{aligned} I_\mu(\alpha) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k_\mu}{(k^2-2p\cdot k+M^2)^\alpha (k\cdot n)^\beta} \\ &= I(\alpha)\left[p_\mu - \frac{\beta}{2(\alpha-\frac{1}{2}n-1)}(M^2-p^2)\frac{n_\mu}{p\cdot n}\right] \end{aligned} \tag{B.30}$$

$$\begin{aligned} I_{\mu\nu}(\alpha) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2-2p\cdot k+M^2)^\alpha (k\cdot n)^\beta} \\ &= I(\alpha)\left[\tfrac{1}{2}(M^2-p^2)\frac{g_{\mu\nu}}{\alpha-1-\frac{1}{2}n} \right. \\ &\quad + p_\mu p_\nu - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}(M^2-p^2)\frac{\beta}{2(\alpha-1-\frac{1}{2}n)} \\ &\quad \left. + \frac{n_\mu n_\nu}{(p\cdot n)^2}(M^2-p^2)^2 \frac{\beta(\beta+1)}{4(\alpha-1-\frac{1}{2}n)(\alpha-2-\frac{1}{2}n)} \right] \end{aligned} \tag{B.31}$$

Convention for the logarithm

The logarithm has a cut along the negative real axis ($\ln z = \ln|z| + \mathrm{i}\arg z$, $-\pi < \arg z < \pi$). With this convention the rule for the logarithm of a product is

$$\ln(ab) = \ln a + \ln b + \eta(a,b) \tag{B.32}$$

where

$$\begin{aligned} \eta(a,b) = 2\pi\mathrm{i}[&\Theta(-\operatorname{Im} a)\Theta(-\operatorname{Im} b)\Theta(+\operatorname{Im} ab) \\ &- \Theta(\operatorname{Im} a)\Theta(\operatorname{Im} b)\Theta(-\operatorname{Im} ab)] \end{aligned} \tag{B.33}$$

Important consequences are:

(i) $\ln(ab) = \ln a + \ln b$ if $\operatorname{Im} a$ and $\operatorname{Im} b$ have different signs;
(ii) $\ln(a/b) = \ln a - \ln b$ if $\operatorname{Im} a$ and $\operatorname{Im} b$ have the same sign;
(iii) if a and b are real then

$$\ln(ab + \mathrm{i}\varepsilon) = \ln(a + \mathrm{i}(\varepsilon/b)) + \ln(b + \mathrm{i}(\varepsilon/a))$$

The integration $\int_0^1 [\mathrm{d}x/(ax+b)]$ for arbitrary complex a and b requires some care. Naively we have $a^{-1}\ln(ax+b)$ but if for $0 < x < 1$ the argument can be real and negative, one must actually split the integral into two intervals. It is easier to first divide out a:

$$\frac{1}{a}\int_0^1 \frac{\mathrm{d}x}{x+b/a} = \frac{1}{a}\ln\left(x+\frac{b}{a}\right)\bigg|_0^1 \tag{B.34}$$

No matter what the sign is of $\mathrm{Im}\, b/a$ the argument of ln never crosses the cut for real x. Thus the answer is $(1/a)\ln[(a+b)/b]$ since the imaginary parts of $(1+b/a)$ and b/a have the same sign.

Spence functions:

$$F(\xi) = \int_0^\xi \frac{\ln(1+x)}{x}\mathrm{d}x \tag{B.35}$$

$$F(\xi) + F(1/\xi) = \tfrac{1}{6}\pi^2 + \tfrac{1}{2}\ln^2\xi \tag{B.36}$$

$$F(-\xi) + F(\xi - 1) = -\tfrac{1}{6}\pi^2 + \ln\xi\,\ln(1-\xi) \tag{B.37}$$

$$F(1) = \tfrac{1}{12}\pi^2, \quad F(-1) = -\tfrac{1}{6}\pi^2 \tag{B.38}$$

$$F(\xi) = \xi - \tfrac{1}{4}\xi^2 + \tfrac{1}{9}\xi^3 + \cdots \tag{B.39}$$

$$L(x) = \int_0^x \frac{\ln(1-u)}{u}\mathrm{d}u = F(-x) \tag{B.40}$$

$$L(x) = \int_0^x \frac{\ln|1-u|}{u}\mathrm{d}u \pm \mathrm{i}\pi\ln x \tag{B.41}$$

Indefinite integrals

$$\int(\mathrm{d}z/z)\ln(z-z_0) = \tfrac{1}{2}\ln^2 z - L(z_0/z) \tag{B.42}$$

$$\int(\mathrm{d}z/z)\ln(z+z_0) = \ln z\ln z_0 + F(z/z_0) \tag{B.43}$$

Appendix C

Feynman rules for the Standard Model

We use the notation introduced in Chapter 12. Moreover, y_f^A ($f = l, u, d$) stand for eigenvalues of the corresponding renormalized Yukawa coupling matrices Y_f^{AC}. Obviously, we have $y_f^A \equiv (e/\sqrt{2}sc)(m_{f_A}/M_Z)$ where quantities on the r.h.s. are the renormalized lagrangian parameters. Since there is no flavour mixing in the lepton sector we can also write $Y_l^{AC} = Y_l^A \delta^{AC} = y_l^A \delta^{AC}$, $\delta Y_l^{AC} = \delta Y_l^A \delta^{AC} = \delta y_l^A \delta^{AC}$.

The QCD renormalization constants Z_S, Z_G, $\tilde{Z}_{\eta_G}$ are, respectively, $Z_{1\mathrm{YM}}$, Z_3, $\tilde{Z}_2$ of Chapter 8 (we use the scheme in which $g = \tilde{g} = \tilde{\tilde{g}}$; see the comment following (8.3)). The $Z_{\eta_{\mathrm{L}}}$ and $Z_{\eta_{\mathrm{Y}}}$ denote renormalization of ghost fields associated with the $SU_{\mathrm{L}}(2)$ and $U_{\mathrm{Y}}(1)$ gauge fields, respectively. The gauge fixing parameter in QCD is a and in the electroweak theory it is denoted by ξ. Wherever more convenient we use the explicit splitting of the renormalization constants Z_i into $Z = 1 + \delta Z_i$. For one-loop calculations all counterterms can be approximated by expressions linear in δZ_is and δY_is.

Propagators of fermions

Leptons

$\Psi_{\nu_B} \xrightarrow{\;p\;} \Psi_{\nu_A}$

$$\frac{\mathrm{i}}{\not p} P_{\mathrm{L}}\, \delta^{AB}$$

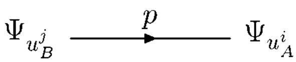
$$\mathrm{i}\delta Z_l^A \not p P_{\mathrm{L}} \delta^{AB}$$

$\Psi_{e_B} \xrightarrow{\;p\;} \Psi_{e_A}$

$$\frac{\mathrm{i}}{\not p - m_{e_A}} \delta^{AB}$$

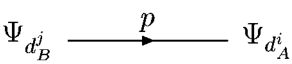
$$\mathrm{i}\left(\delta Z_l^A \not p P_{\mathrm{L}} + \delta Z_e^A \not p P_{\mathrm{R}}\right) - \frac{\mathrm{i}}{\sqrt{2}} \left[(v - \delta v)(y_l^A + \delta y_l^A) - y_l^A v\right] \delta^{AB}$$

Quarks



$$\frac{\mathrm{i}}{\not{p} - m_{u_A}}\delta^{AB}\delta^{ij}$$

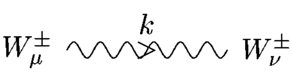
$$\begin{aligned}&\mathrm{i}\left[\delta Z_q^{AB}\not{p}P_{\mathrm{L}} + \delta Z_u^{AB}\not{p}P_{\mathrm{R}}\right]\delta^{ij}\\ &-\frac{\mathrm{i}}{\sqrt{2}}\big[(v-\delta v)(y_u^A\delta^{AB} + \delta Y_u^{AB})P_{\mathrm{L}}\\ &\quad + (v-\delta v)(y_u^A\delta^{AB} + (\delta Y_u^{\dagger})^{AB})P_{\mathrm{R}}\\ &- v y_u^A\delta^{AB}\big]\delta^{ij}\end{aligned}$$



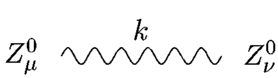
$$\frac{\mathrm{i}}{\not{p} - m_{d_A}}\delta^{AB}\delta^{ij}$$

$$\begin{aligned}&\mathrm{i}\left[(V^{\dagger}\delta Z_q V)^{AB}\not{p}P_{\mathrm{L}} + i\delta Z_d^{AB}\not{p}P_{\mathrm{R}}\right]\delta^{ij}\\ &-\frac{\mathrm{i}}{\sqrt{2}}\big[(v-\delta v)(y_d^A\delta^{AB} + \delta Y_d^{AB})P_{\mathrm{L}}\\ &+ (v-\delta v)(y_d^A\delta^{AB} + (\delta Y_d^{\dagger})^{AB})P_{\mathrm{R}}\\ &- v y_d^A\delta^{AB}\big]\delta^{ij}\end{aligned}$$

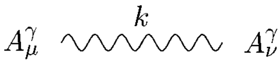
Propagators of the gauge bosons



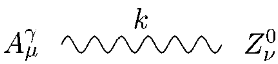
$$\begin{aligned}&\frac{-\mathrm{i}}{k^2 - M_W^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_W^2}\right]\\ &- \mathrm{i}\delta Z_W\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)\\ &+ \mathrm{i}M_W^2\left[Z_{\mathrm{H}}\left(1 - \frac{\delta v}{v}\right)^2 Z_g^2 Z_W^{-2} - 1\right]g^{\mu\nu}\end{aligned}$$



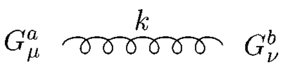
$$\begin{aligned}&\frac{-\mathrm{i}}{k^2 - M_Z^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_Z^2}\right]\\ &-\mathrm{i}(c^2\delta Z_W + s^2\delta Z_B)\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)\\ &+\mathrm{i}M_Z^2\bigg[Z_{\mathrm{H}}\left(1 - \frac{\delta v}{v}\right)^2\\ &\times\left(c^4 Z_g^2 Z_W^{-2} + s^4 + 2c^2 s^2 Z_g Z_W^{-1}\right) - 1\bigg]g^{\mu\nu}\end{aligned}$$



$$\frac{-\mathrm{i}}{k^2}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$
$$-\mathrm{i}(c^2\delta Z_B+s^2\delta Z_W)\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)$$
$$+\mathrm{i}s^2c^2M_Z^2Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)^2\left(1-Z_gZ_W^{-1}\right)^2g^{\mu\nu}$$

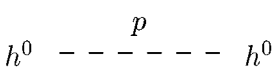


$$-\mathrm{i}sc(\delta Z_W-\delta Z_B)\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)$$
$$+\mathrm{i}scM_Z^2Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)^2\left(s^2+c^2Z_gZ_W^{-1}\right)$$
$$\times\left(Z_gZ_W^{-1}-1\right)g^{\mu\nu}$$

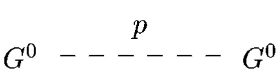


$$\frac{-\mathrm{i}}{k^2}\left[g^{\mu\nu}-(1-a)\frac{k^\mu k^\nu}{k^2}\right]\delta^{ab}$$
$$-\mathrm{i}\delta Z_G\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\delta^{ab}$$

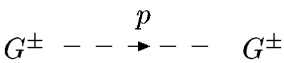
Propagators of the Higgs and Goldstone bosons



$$\frac{\mathrm{i}}{p^2-M_h^2}$$
$$\mathrm{i}\delta Z_\mathrm{H}p^2-\mathrm{i}\Big\{\delta m^2$$
$$+\frac{3}{2}\left[(\lambda+\delta\lambda)\left(1-\frac{\delta v}{v}\right)^2-\lambda\right]v^2\Big\}$$

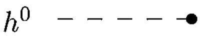


$$\frac{\mathrm{i}}{p^2-\xi M_Z^2}$$
$$\mathrm{i}\delta Z_\mathrm{H}p^2-\mathrm{i}\Big\{\delta m^2$$
$$+\frac{1}{2}\left[(\lambda+\delta\lambda)\left(1-\frac{\delta v}{v}\right)^2-\lambda\right]v^2\Big\}$$



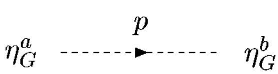
$$\frac{\mathrm{i}}{p^2 - \xi M_W^2}$$

$$\mathrm{i}\delta Z_{\mathrm{H}} p^2 - \mathrm{i}\left\{\delta m^2 + \frac{1}{2}\left[(\lambda + \delta\lambda)\left(1 - \frac{\delta v}{v}\right)^2 - \lambda\right]v^2\right\}$$



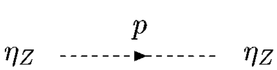
$$-\mathrm{i}\left\{v\delta m^2 + \frac{1}{2}\left[(\lambda + \delta\lambda)\left(1 - \frac{\delta v}{v}\right)^3 - \lambda\right]v^3\right\}$$

Propagators of the ghost fields



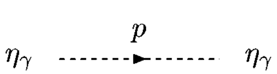
$$\frac{-\mathrm{i}\delta^{ab}}{p^2}$$

$$-\mathrm{i}\delta Z_{\eta_G}\delta^{ab} p^2$$



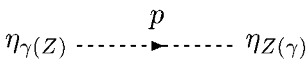
$$\frac{-\mathrm{i}}{p^2 - \xi M_Z^2}$$

$$-\mathrm{i}(c^2\delta Z_{\eta_{\mathrm{L}}} + s^2\delta Z_{\eta_Y})p^2$$
$$+\mathrm{i}\xi M_Z^2\left\{Z_{\mathrm{H}}\left(1 - \frac{\delta v}{v}\right)^2\left[s^4 Z_{\eta_{\mathrm{Y}}} + c^4 Z_{\eta_{\mathrm{L}}} Z_g^2 Z_W^{-2} + s^2c^2 Z_{\eta_{\mathrm{Y}}}^{1/2} Z_{\eta_{\mathrm{L}}}^{1/2} Z_g (Z_W^{-3/2} Z_B^{1/2} + Z_W^{-1/2} Z_B^{-1/2})\right] - 1\right\}$$

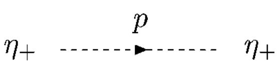


$$\frac{-\mathrm{i}}{p^2}$$

$$-\mathrm{i}(s^2\delta Z_{\eta_{\mathrm{L}}} + c^2\delta Z_{\eta_Y})p^2$$
$$+\mathrm{i}s^2c^2 M_Z^2 Z_{\mathrm{H}}\left(1 - \frac{\delta v}{v}\right)^2\left[Z_{\eta_{\mathrm{Y}}} + Z_{\eta_{\mathrm{L}}} Z_g^2 Z_W^{-2} - Z_{\eta_{\mathrm{Y}}}^{1/2} Z_{\eta_{\mathrm{L}}}^{1/2} Z_g (Z_W^{-3/2} Z_B^{1/2} + Z_W^{-1/2} Z_B^{-1/2})\right]$$

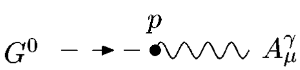


$$
\begin{aligned}
&-\mathrm{i}cs(\delta Z_{\eta_\mathrm{L}} - \delta Z_{\eta_Y})p^2 \\
&+\mathrm{i}cs\xi M_Z^2 Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)^2 \\
&\times\left[s^2\left(Z_{\eta_\mathrm{Y}}^{1/2}Z_{\eta_\mathrm{L}}^{1/2}Z_g Z_W^{-1/2}Z_B^{-1/2} - Z_{\eta_\mathrm{Y}}\right)\right. \\
&\left.+\,c^2 Z_g Z_W^{-1}\left(Z_{\eta_\mathrm{L}}Z_g Z_W^{-1} - Z_{\eta_\mathrm{Y}}^{1/2}Z_{\eta_\mathrm{L}}^{1/2}Z_W^{-1/2}Z_B^{1/2}\right)\right]
\end{aligned}
$$

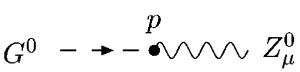


$$
\begin{aligned}
&\frac{-\mathrm{i}}{p^2-\xi M_W^2} \\
&-\mathrm{i}\delta Z_{\eta_\mathrm{L}}p^2 + \mathrm{i}\xi M_W^2\left[Z_\mathrm{H}Z_{\eta_\mathrm{L}}Z_g^2 Z_W^{-2}\left(1-\frac{\delta v}{v}\right)^2 - 1\right]
\end{aligned}
$$

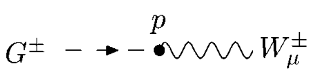
Mixed propagators (only counterterms exist)



$$-scM_Z Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)\left(Z_g Z_W^{-1}-1\right)p_\mu$$



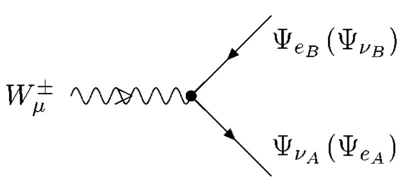
$$-M_Z\left[Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)\left(s^2+c^2 Z_g Z_W^{-1}\right)-1\right]p_\mu$$



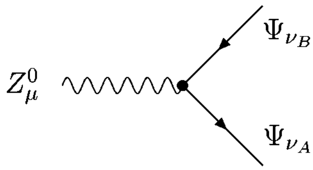
$$\mp\mathrm{i}M_W\left[Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)Z_g Z_W^{-1}-1\right]p_\mu$$

Gauge interactions of fermions

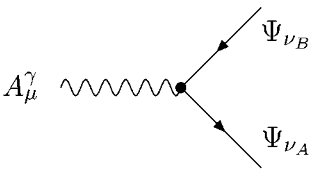
Leptons



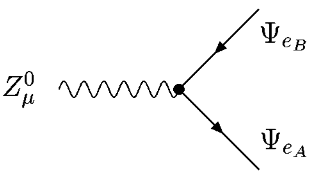
$$
\begin{aligned}
&-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu\delta^{AB}P_\mathrm{L} \\
&-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu(Z_l^A Z_g Z_W^{-1}-1)\delta^{AB}P_\mathrm{L}
\end{aligned}
$$



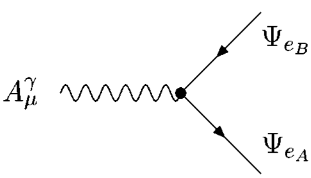
$$-\mathrm{i}\frac{e}{2sc}\gamma_\mu \delta^{AB} P_\mathrm{L}$$
$$-\mathrm{i}\frac{e}{2sc}\gamma_\mu \left[Z_l^A(c^2 Z_g Z_W^{-1} + s^2) - 1\right]\delta^{AB} P_\mathrm{L}$$



$$-\mathrm{i}\frac{e}{2}\gamma_\mu Z_l^A(Z_g Z_W^{-1} - 1)\delta^{AB} P_\mathrm{L}$$

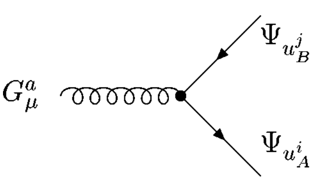


$$+\mathrm{i}\frac{e}{2sc}\gamma_\mu \delta^{AB}\left[(1-2s^2)P_\mathrm{L} - 2s^2 P_\mathrm{R}\right]$$
$$+\mathrm{i}\frac{e}{2sc}\gamma_\mu \delta^{AB}\Big\{\big[Z_l^A(c^2 Z_g Z_W^{-1} - s^2) - (1-2s^2)\big]P_\mathrm{L} - 2s^2\delta Z_e^A P_\mathrm{R}\Big\}$$

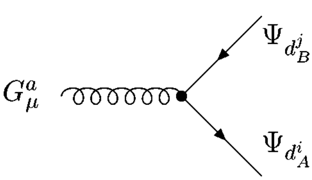


$$+\mathrm{i}e\gamma_\mu\delta^{AB}$$
$$+\mathrm{i}e\gamma_\mu\delta^{AB}\Big\{\big[\tfrac{1}{2}Z_l^A(1 + Z_g Z_W^{-1}) - 1\big]P_\mathrm{L} + \delta Z_e^A P_\mathrm{R}\Big\}$$

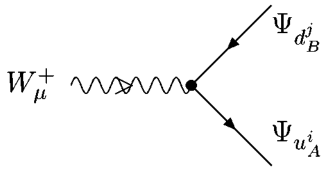
Quarks



$$-\mathrm{i}g_s\gamma_\mu T^a_{ij}\delta^{AB}$$
$$-\mathrm{i}g_s\gamma_\mu T^a_{ij}\Big[\big(Z_q^{AB} Z_S Z_G^{-1} - \delta^{AB}\big)P_\mathrm{L} + \big(Z_u^{AB} Z_S Z_G^{-1} - \delta^{AB}\big)P_\mathrm{R}\Big]$$

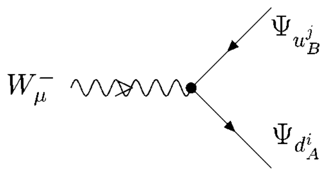


$$-\mathrm{i}g_s\gamma_\mu T^a_{ij}\delta^{AB}$$
$$-\mathrm{i}g_s\gamma_\mu T^a_{ij}\Big\{\big[(V^\dagger Z_q V)^{AB} Z_S Z_G^{-1} - \delta^{AB}\big]P_\mathrm{L} + \big(Z_d^{AB} Z_S Z_G^{-1} - \delta^{AB}\big)P_\mathrm{R}\Big\}$$



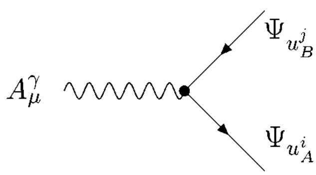
$$-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu V^{AB}\delta^{ij}P_\mathrm{L}$$

$$-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu\left[(Z_q V)^{AB}Z_S Z_G^{-1} - V^{AB}\right]\delta^{ij}P_\mathrm{L}$$



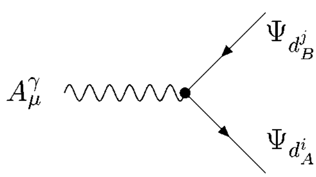
$$-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu (V^\dagger)^{AB}\delta^{ij}P_\mathrm{L}$$

$$-\mathrm{i}\frac{e}{\sqrt{2}s}\gamma_\mu\left[(V^\dagger Z_q)^{AB}Z_S Z_G^{-1} - (V^\dagger)^{AB}\right]\delta^{ij}P_\mathrm{L}$$



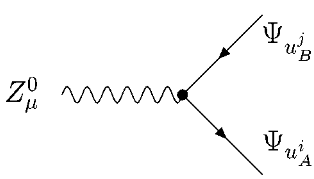
$$-\mathrm{i}\tfrac{2}{3}e\gamma_\mu\delta^{AB}\delta^{ij}$$

$$-\mathrm{i}\tfrac{2}{3}e\gamma_\mu\Big\{\Big[\tfrac{1}{4}Z_q^{AB}(3Z_g Z_W^{-1}+1) - \delta^{AB}\Big]P_\mathrm{L} + \delta Z_u^{AB}P_\mathrm{R}\Big\}\delta^{ij}$$



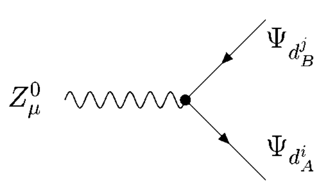
$$+\mathrm{i}\tfrac{1}{3}e\gamma_\mu\delta^{AB}\delta^{ij}$$

$$+\mathrm{i}\tfrac{1}{3}e\gamma_\mu\Big\{\Big[\tfrac{1}{2}(V^\dagger Z_q V)^{AB}(3Z_g Z_W^{-1}-1) - \delta^{AB}\Big]P_\mathrm{L} + \delta Z_d^{AB}P_\mathrm{R}\Big\}\delta^{ij}$$



$$-\mathrm{i}\frac{e}{2sc}\gamma_\mu\left[(1-\tfrac{4}{3}s^2)P_\mathrm{L} - \tfrac{4}{3}s^2 P_\mathrm{R}\right]\delta^{AB}\delta^{ij}$$

$$-\mathrm{i}\frac{e}{2sc}\gamma_\mu\Big\{\Big[Z_q^{AB}(c^2 Z_g Z_W^{-1} - \tfrac{1}{3}s^2) - (1-\tfrac{4}{3}s^2)\delta^{AB}\Big]P_\mathrm{L} - \tfrac{4}{3}s^2\delta Z_u^{AB}P_\mathrm{R}\Big\}\delta^{ij}$$

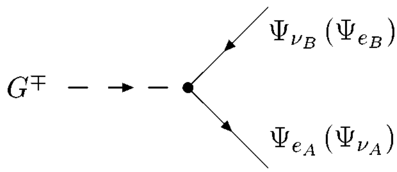


$$+\mathrm{i}\frac{e}{2sc}\gamma_\mu\left[(1-\tfrac{2}{3}s^2)P_\mathrm{L} - \tfrac{2}{3}s_W^2 P_\mathrm{R}\right]\delta^{AB}\delta^{ij}$$

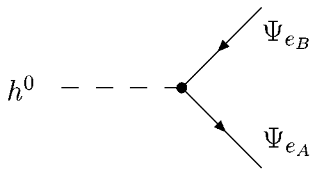
$$+\mathrm{i}\frac{e}{2sc}\gamma_\mu\Big\{\Big[(V^\dagger Z_q V)^{AB}(c^2 Z_g Z_W^{-1} + \tfrac{1}{3}) - (1-\tfrac{2}{3}s^2)\delta^{AB}\Big]P_\mathrm{L} - \tfrac{2}{3}s^2\delta Z_d^{AB}P_\mathrm{R}\Big\}\delta^{ij}$$

Yukawa interactions of fermions

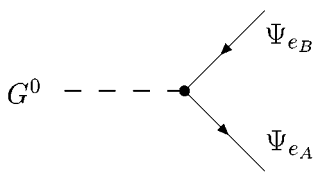
Leptons



$$-\mathrm{i}y_l^A\delta^{AB}P_{\mathrm{L(R)}}$$
$$-\mathrm{i}\delta y_l^A\delta^{AB}P_{\mathrm{L(R)}}$$

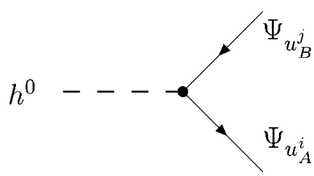


$$-\frac{\mathrm{i}}{\sqrt{2}}y_l^A\delta^{AB}$$
$$-\frac{\mathrm{i}}{\sqrt{2}}\delta y_l^A\delta^{AB}$$

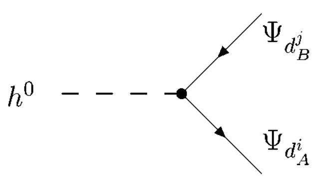


$$+\frac{1}{\sqrt{2}}y_l^A\gamma^5\delta^{AB}$$
$$+\frac{1}{\sqrt{2}}\delta y_l^A\gamma^5\delta^{AB}$$

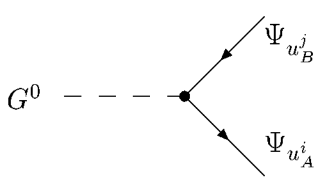
Quarks



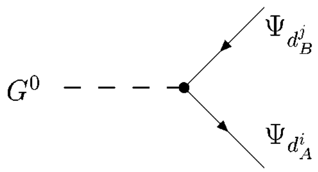
$$-\frac{\mathrm{i}}{\sqrt{2}}y_u^A\delta^{AB}\delta^{ij}$$
$$-\frac{\mathrm{i}}{\sqrt{2}}\left(\delta Y_u^{AB}P_{\mathrm{L}}+(\delta Y_u^{\dagger})^{AB}P_{\mathrm{R}}\right)\delta^{ij}$$



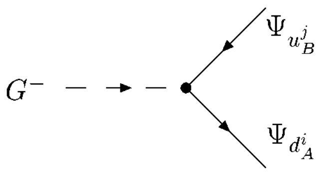
$$-\frac{\mathrm{i}}{\sqrt{2}}y_d^A\delta^{AB}\delta^{ij}$$
$$-\frac{\mathrm{i}}{\sqrt{2}}\left(\delta Y_d^{AB}P_{\mathrm{L}}+(\delta Y_d^{\dagger})^{AB}P_{\mathrm{R}}\right)\delta^{ij}$$



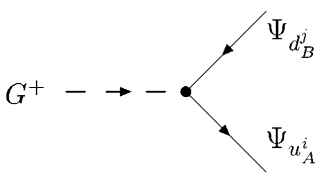
$$-\frac{1}{\sqrt{2}}y_u^A\gamma^5\delta^{AB}\delta^{ij}$$
$$+\frac{1}{\sqrt{2}}\left(\delta Y_u^{AB}P_{\mathrm{L}}-(\delta Y_u^{\dagger})^{AB}P_{\mathrm{R}}\right)\delta^{ij}$$



$$+\frac{1}{\sqrt{2}} y_d^A \gamma^5 \delta^{AB} \delta^{ij}$$
$$-\frac{1}{\sqrt{2}} \left(\delta Y_d^{AB} P_{\mathrm{L}} - (\delta Y_d^\dagger)^{AB} P_{\mathrm{R}}\right) \delta^{ij}$$



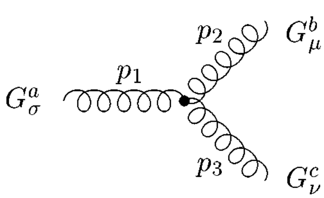
$$-\mathrm{i} \left(y_d^A P_{\mathrm{L}} - y_u^B P_{\mathrm{R}}\right) (V^\dagger)^{AB} \delta^{ij}$$
$$-\mathrm{i} \left[(\delta Y_d V^\dagger)^{AB} P_{\mathrm{L}} - (V^\dagger \delta Y_u^\dagger)^{AB} P_{\mathrm{R}}\right] \delta^{ij}$$



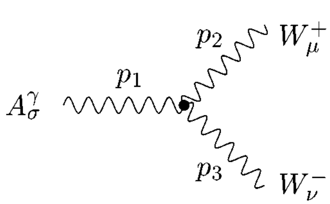
$$+\mathrm{i} \left(y_u^A P_{\mathrm{L}} - y_d^B P_{\mathrm{R}}\right) V^{AB} \delta^{ij}$$
$$+\mathrm{i}\left[(\delta Y_u V)^{AB} P_{\mathrm{L}} - (V \delta Y_d^\dagger)^{AB} P_{\mathrm{R}}\right] \delta^{ij}$$

Gauge interactions of the gauge bosons

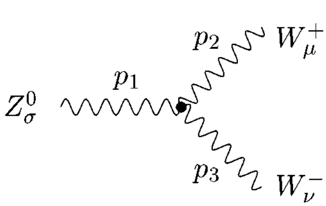
All momenta are outgoing.



$$+g_s c^{abc}\left[g^{\sigma\mu}(p_1 - p_2)^\nu + g^{\mu\nu}(p_2 - p_3)^\sigma + g^{\nu\sigma}(p_3 - p_1)^\mu\right]$$
$$+g_s \delta Z_S c^{abc}\left[g^{\sigma\mu}(p_1 - p_2)^\nu + g^{\mu\nu}(p_2 - p_3)^\sigma + g^{\nu\sigma}(p_3 - p_1)^\mu\right]$$

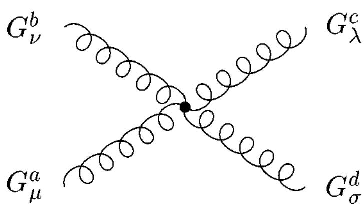


$$-\mathrm{i}e\left[g^{\sigma\mu}(p_1 - p_2)^\nu + g^{\mu\nu}(p_2 - p_3)^\sigma + g^{\nu\sigma}(p_3 - p_1)^\mu\right]$$
$$-\mathrm{i}e\delta Z_g\left[g^{\sigma\mu}(p_1 - p_2)^\nu + g^{\mu\nu}(p_2 - p_3)^\sigma + g^{\nu\sigma}(p_3 - p_1)^\mu\right]$$



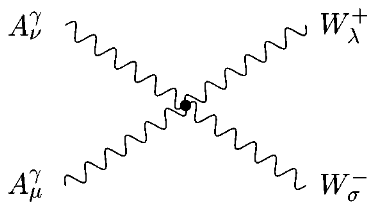
$$-\mathrm{i}e\frac{c}{s}\big[g^{\sigma\mu}(p_1-p_2)^\nu+g^{\mu\nu}(p_2-p_3)^\sigma + g^{\nu\sigma}(p_3-p_1)^\mu\big]$$

$$-\mathrm{i}e\delta Z_g\frac{c}{s}\big[g^{\sigma\mu}(p_1-p_2)^\nu + g^{\mu\nu}(p_2-p_3)^\sigma+g^{\nu\sigma}(p_3-p_1)^\mu\big]$$



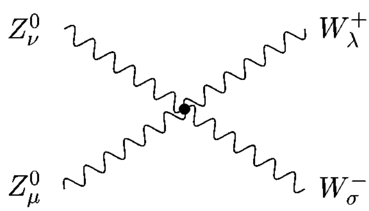
$$-\mathrm{i}g_s^2[c^{abe}c^{ecd}(g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\sigma}) + c^{ace}c^{edb}(g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma}) + c^{ade}c^{ebc}(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\sigma}g^{\nu\lambda})]$$

$$-\mathrm{i}g_s^2(Z_S^2Z_G^{-1}-1)\big[c^{abe}c^{ecd}(g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\sigma}) + c^{ace}c^{edb}(g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma}) + c^{ade}c^{ebc}(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\sigma}g^{\nu\lambda})\big]$$



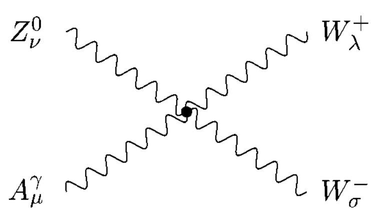
$$-\mathrm{i}e^2\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$

$$-\mathrm{i}e^2\left(Z_g^2Z_W^{-1}-1\right)\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$



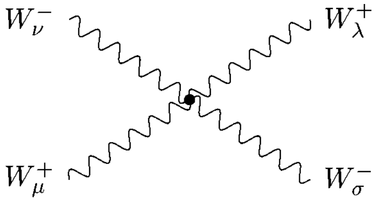
$$-\mathrm{i}e^2\frac{c^2}{s^2}\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$

$$-\mathrm{i}e^2\frac{c^2}{s^2}\left(Z_g^2Z_W^{-1}-1\right)\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$



$$-\mathrm{i}e^2\frac{c}{s}\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$

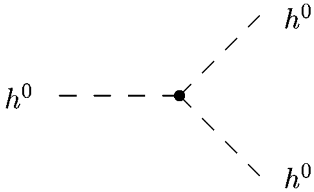
$$-\mathrm{i}e^2\frac{c}{s}\left(Z_g^2Z_W^{-1}-1\right)\left[2g^{\mu\nu}g^{\sigma\lambda}-g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\lambda}\right]$$



$$+\mathrm{i}\frac{e^2}{s^2}\left[2g^{\mu\lambda}g^{\sigma\nu} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right]$$

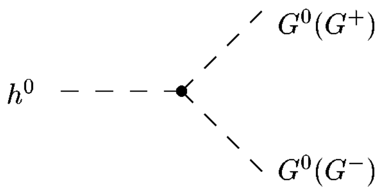
$$+\mathrm{i}\frac{e^2}{s^2}\left(Z_g^2 Z_W^{-1} - 1\right)\left[2g^{\mu\lambda}g^{\sigma\nu} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right]$$

Self-interactions of the Higgs and Goldstone bosons



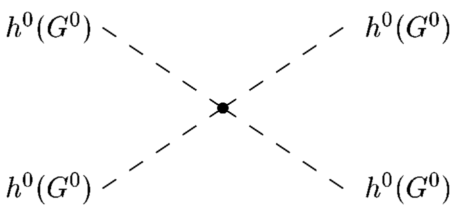
$$-\mathrm{i}3v\lambda$$

$$-\mathrm{i}3\left[(\lambda + \delta\lambda)(v - \delta v) - \lambda v\right]$$



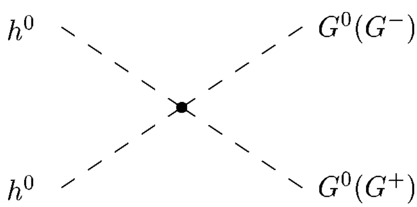
$$-\mathrm{i}v\lambda$$

$$-\mathrm{i}\left[(\lambda + \delta\lambda)(v - \delta v) - \lambda v\right]$$



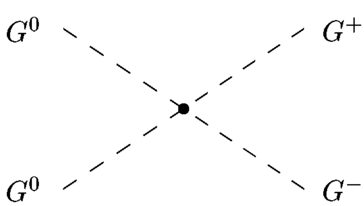
$$-\mathrm{i}3\lambda$$

$$-\mathrm{i}3\delta\lambda$$



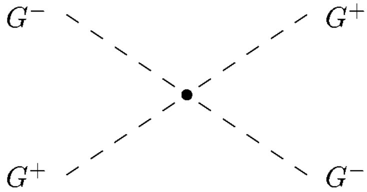
$$-\mathrm{i}\lambda$$

$$-\mathrm{i}\delta\lambda$$



$$-\mathrm{i}\lambda$$

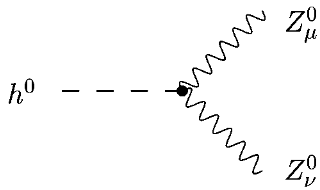
$$-\mathrm{i}\delta\lambda$$



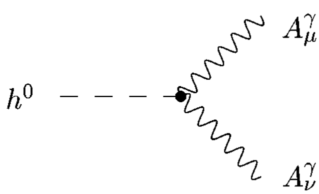
$-\mathrm{i}2\lambda$

$-\mathrm{i}2\delta\lambda$

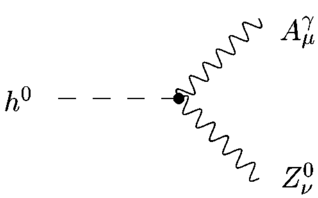
Gauge interactions of the Higgs and Goldstone bosons



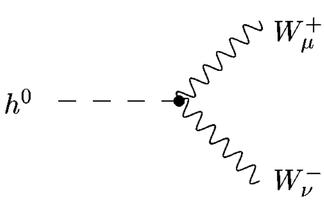
$$+\mathrm{i}\frac{e}{sc}M_Z g_{\mu\nu}$$
$$+\mathrm{i}\frac{e}{sc}M_Z\Big[Z_\mathrm{H}\Big(1-\frac{\delta v}{v}\Big)\big(c^4 Z_g^2 Z_W^{-2} + 2s^2c^2 Z_g Z_W^{-1} + s^4\big) - 1\Big]g_{\mu\nu}$$



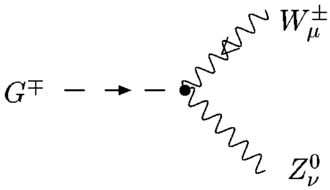
$$+\mathrm{i}escM_Z Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)\left(Z_g Z_W^{-1} - 1\right)^2 g_{\mu\nu}$$



$$+\mathrm{i}eM_Z Z_\mathrm{H}\left(1-\frac{\delta v}{v}\right)\left(Z_g Z_W^{-1} - 1\right)$$
$$\times\left(s^2 + c^2 Z_g Z_W^{-1}\right) g_{\mu\nu}$$

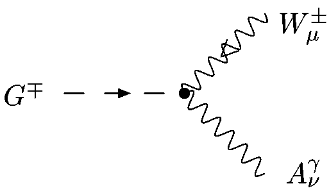


$$+\mathrm{i}\frac{e}{s}M_W g_{\mu\nu}$$
$$+\mathrm{i}\frac{e}{s}M_W\left[Z_\mathrm{H} Z_g^2 Z_W^{-2}\left(1-\frac{\delta v}{v}\right) - 1\right] g_{\mu\nu}$$



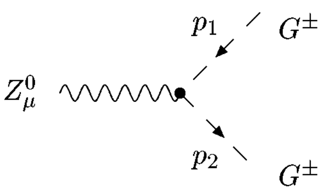
$$-\mathrm{i}esM_Z g_{\mu\nu}$$

$$-\mathrm{i}esM_Z \left[Z_{\mathrm{H}} Z_g Z_W^{-1} \left(1 - \frac{\delta v}{v}\right) - 1 \right] g_{\mu\nu}$$



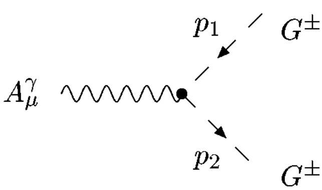
$$+\mathrm{i}ecM_Z g_{\mu\nu}$$

$$+\mathrm{i}ecM_Z \left[Z_{\mathrm{H}} Z_g Z_W^{-1} \left(1 - \frac{\delta v}{v}\right) - 1 \right] g_{\mu\nu}$$



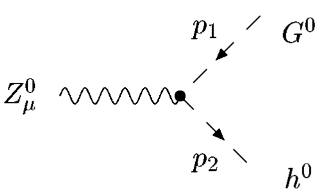
$$\mp\mathrm{i}\frac{e}{2sc}(1 - 2s^2)(p_1 + p_2)_\mu$$

$$\mp\mathrm{i}\frac{e}{2sc}\left[Z_{\mathrm{H}}(c^2 Z_g Z_W^{-1} - s^2) - (1 - 2s^2) \right] \times (p_1 + p_2)_\mu$$



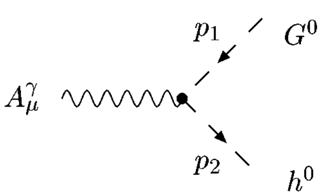
$$\mp\mathrm{i}e(p_1 + p_2)_\mu$$

$$\mp\mathrm{i}\frac{e}{2}\left[Z_{\mathrm{H}}(1 + Z_g Z_W^{-1}) - 2 \right](p_1 + p_2)_\mu$$

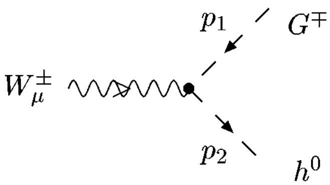


$$-\frac{e}{2sc}(p_1 + p_2)_\mu$$

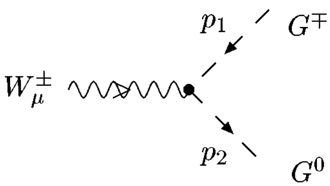
$$-\frac{e}{2sc}\left[Z_{\mathrm{H}}(c^2 Z_g Z_W^{-1} + s^2) - 1 \right](p_1 + p_2)_\mu$$



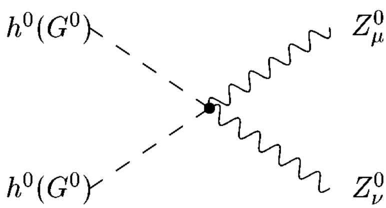
$$-\frac{e}{2} Z_{\mathrm{H}}(Z_g Z_W^{-1} - 1)(p_1 + p_2)_\mu$$



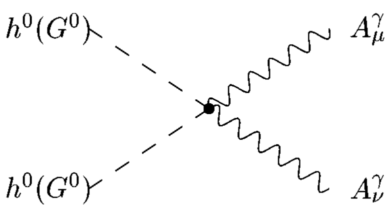
$$\pm \mathrm{i}\frac{e}{2s}(p_1+p_2)_\mu$$
$$\pm \mathrm{i}\frac{e}{2s}\left(Z_\mathrm{H} Z_g Z_W^{-1}-1\right)(p_1+p_2)_\mu$$



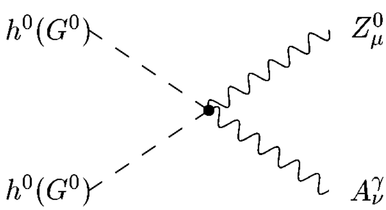
$$-\frac{e}{2s}(p_1+p_2)_\mu$$
$$-\frac{e}{2s}\left(Z_\mathrm{H} Z_g Z_W^{-1}-1\right)(p_1+p_2)_\mu$$



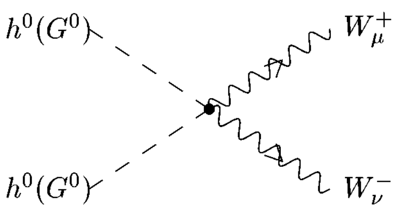
$$+\mathrm{i}\frac{e^2}{2s^2c^2}g_{\mu\nu}$$
$$+\mathrm{i}\frac{e^2}{2s^2c^2}\big[Z_\mathrm{H}(c^4 Z_g^2 Z_W^{-2} + 2s^2c^2 Z_g Z_W^{-1} + s^4) - 1\big]g_{\mu\nu}$$



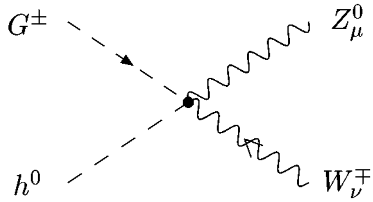
$$+\mathrm{i}\frac{e^2}{2}Z_\mathrm{H}\left(Z_g Z_W^{-1}-1\right)^2 g_{\mu\nu}$$



$$+\mathrm{i}\frac{e^2}{2sc}Z_\mathrm{H}\left(Z_g Z_W^{-1}-1\right)\left(c^2 Z_g Z_W^{-1}+s^2\right)g_{\mu\nu}$$

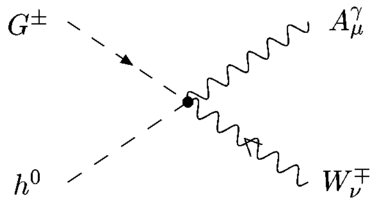


$$+\mathrm{i}\frac{e^2}{2s^2}g_{\mu\nu}$$
$$+\mathrm{i}\frac{e^2}{2s^2}\left(Z_\mathrm{H} Z_g^2 Z_W^{-2}-1\right)g_{\mu\nu}$$



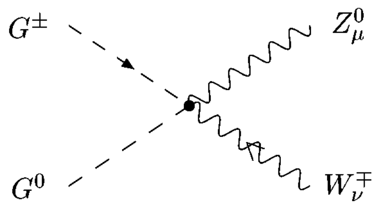
$$-\mathrm{i}\frac{e^2}{2c}g_{\mu\nu}$$

$$-\mathrm{i}\frac{e^2}{2c}\left(Z_{\mathrm{H}}Z_g Z_W^{-1}-1\right)g_{\mu\nu}$$



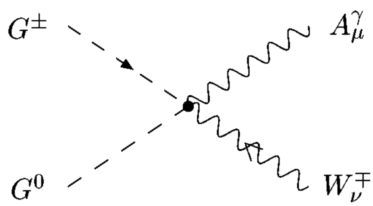
$$+\mathrm{i}\frac{e^2}{2s}g_{\mu\nu}$$

$$+\mathrm{i}\frac{e^2}{2s}\left(Z_{\mathrm{H}}Z_g Z_W^{-1}-1\right)g_{\mu\nu}$$



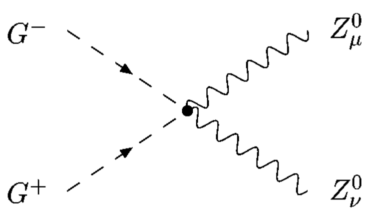
$$\mp\frac{e^2}{2c}g_{\mu\nu}$$

$$\mp\frac{e^2}{2c}\left(Z_{\mathrm{H}}Z_g Z_W^{-1}-1\right)g_{\mu\nu}$$



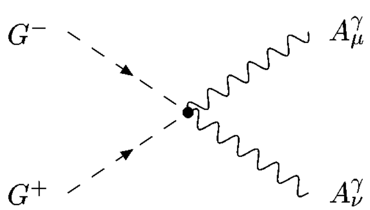
$$\pm\frac{e^2}{2s}g_{\mu\nu}$$

$$\pm\frac{e^2}{2s}\left(Z_{\mathrm{H}}Z_g Z_W^{-1}-1\right)g_{\mu\nu}$$



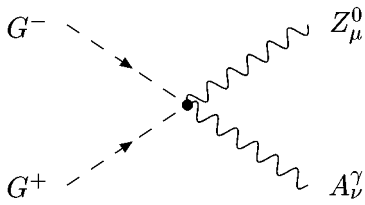
$$+\mathrm{i}\frac{e^2}{2s^2c^2}(1-2s^2)^2 g_{\mu\nu}$$

$$+\mathrm{i}\frac{e^2}{2s^2c^2}\big[Z_{\mathrm{H}}(c^4Z_g^2Z_W^{-2}+s^4$$
$$-2s^2c^2Z_gZ_W^{-1})-(1-2s^2)^2\big]g_{\mu\nu}$$



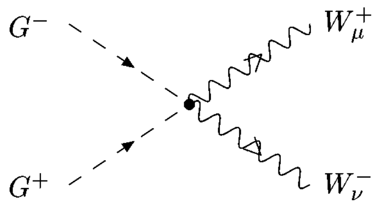
$$+\mathrm{i}2e^2 g_{\mu\nu}$$

$$+\mathrm{i}\frac{e^2}{2}\left[Z_{\mathrm{H}}\left(1+Z_gZ_W^{-1}\right)^2-4\right]g_{\mu\nu}$$



$$+\mathrm{i}\frac{e^2}{sc}(1-2s^2)g_{\mu\nu}$$

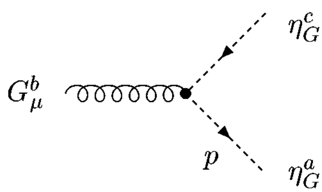
$$+\mathrm{i}\frac{e^2}{2sc}\left[Z_\mathrm{H}(1+Z_g Z_W^{-1})(c^2 Z_g Z_W^{-1}-s^2) - 2(1-2s^2)\right]g_{\mu\nu}$$



$$+\mathrm{i}\frac{e^2}{2s^2}g_{\mu\nu}$$

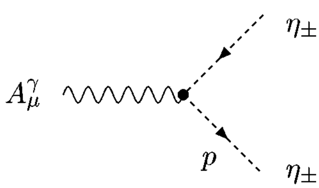
$$+\mathrm{i}\frac{e^2}{2s^2}\left(Z_\mathrm{H}Z_g^2 Z_W^{-2}-1\right)g_{\mu\nu}$$

Gauge interactions of the ghost fields



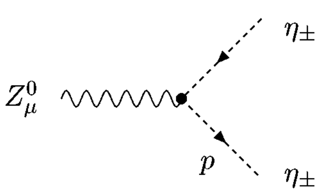
$$-g_s c^{abc} p_\mu$$

$$-g_s(Z_{\eta_G} Z_S Z_G^{-1}-1)c^{abc}p_\mu$$



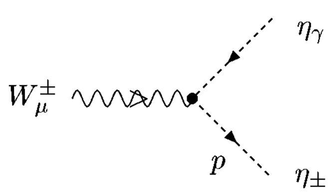
$$\pm\mathrm{i}ep_\mu$$

$$\pm\mathrm{i}e(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1}-1)p_\mu$$



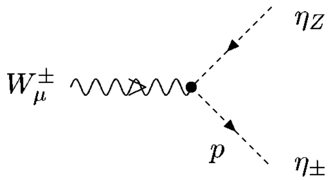
$$\pm\mathrm{i}e\frac{c}{s}p_\mu$$

$$\pm\mathrm{i}e\frac{c}{s}(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1}-1)p_\mu$$



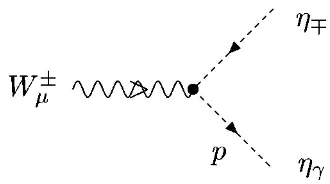
$$\mp\mathrm{i}ep_\mu$$

$$\mp\mathrm{i}e(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1}-1)p_\mu$$



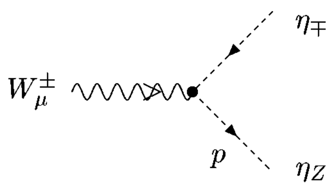
$$\mp \mathrm{i} e \frac{c}{s} p_\mu$$

$$\mp \mathrm{i} e \frac{c}{s} (Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - 1) p_\mu$$



$$\pm \mathrm{i} e p_\mu$$

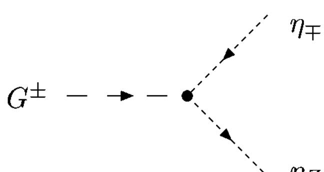
$$\pm \mathrm{i} e (Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - 1) p_\mu$$



$$\pm \mathrm{i} e \frac{c}{s} p_\mu$$

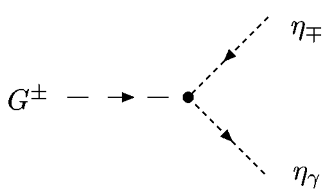
$$\pm \mathrm{i} e \frac{c}{s} (Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - 1) p_\mu$$

Interactions of ghosts with Higgs and Goldstone bosons



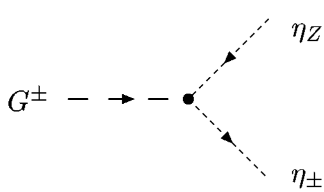
$$-\mathrm{i} \frac{e}{2s} \xi M_Z$$

$$-\mathrm{i} \frac{e}{2s} \xi M_Z \left[Z_\mathrm{H} \left(1 - \frac{\delta v}{v} \right) Z_g Z_W^{-1} \big(c^2 Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} + s^2 Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_W^{-1/2} Z_B^{1/2} \big) - 1 \right]$$



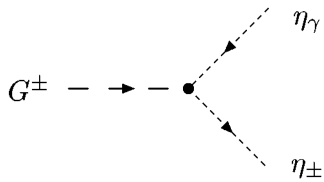
$$-\mathrm{i} \xi M_W$$

$$-\mathrm{i} \frac{e}{2} \xi M_W Z_\mathrm{H} \left(1 - \frac{\delta v}{v} \right) Z_g Z_W^{-1} \big(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_W^{-1/2} Z_B^{1/2} \big)$$

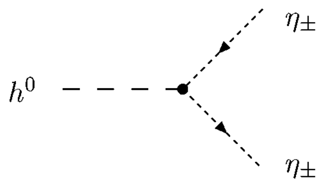


$$+\mathrm{i} \frac{e}{2s}$$

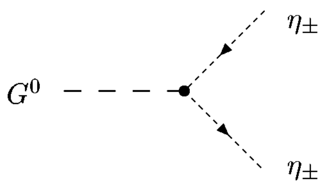
$$-\mathrm{i} \frac{e}{2s} \xi M_Z \left[Z_\mathrm{H} \left(1 - \frac{\delta v}{v} \right) Z_g Z_W^{-1} \big(c^2 Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - s^2 Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_W^{1/2} Z_B^{-1/2} \big) - (1 - 2s^2) \right]$$



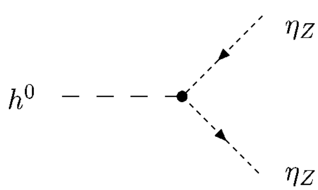
$$
\begin{aligned}
&+\mathrm{i}e\xi M_W \\
&+\mathrm{i}\frac{e}{2}\xi M_W\Big[Z_\mathrm{H}\Big(1-\frac{\delta v}{v}\Big)Z_g Z_W^{-1}\big(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} \\
&+ Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_W^{1/2} Z_B^{-1/2}\big) - 2\Big]
\end{aligned}
$$



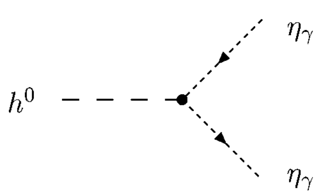
$$
\begin{aligned}
&+\mathrm{i}\frac{e}{2s}\xi M_W \\
&+\mathrm{i}\frac{e}{2s}\xi M_W\left[Z_\mathrm{H} Z_{\eta_\mathrm{L}} Z_g^2 Z_W^{-2}\left(1-\frac{\delta v}{v}\right)-1\right]
\end{aligned}
$$



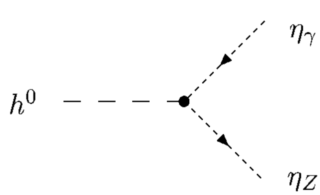
$$
\begin{aligned}
&\mp\frac{e}{2s}\xi M_W \\
&\mp\frac{e}{2s}\xi M_W\left[Z_\mathrm{H} Z_{\eta_\mathrm{L}} Z_g^2 Z_W^{-2}\left(1-\frac{\delta v}{v}\right)-1\right]
\end{aligned}
$$



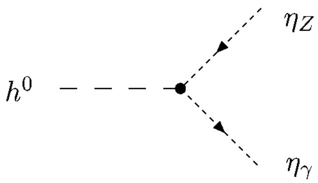
$$
\begin{aligned}
&\mathrm{i}\frac{e}{2sc}\xi M_Z \\
&+\mathrm{i}\frac{e}{2sc}\xi M_Z\Big[Z_\mathrm{H}\Big(1-\frac{\delta v}{v}\Big)\big(s^4 Z_{\eta_\mathrm{Y}} + c^4 Z_{\eta_\mathrm{L}} Z_g^2 Z_W^{-2} \\
&+ s^2c^2 Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_g (Z_W^{-3/2} Z_B^{1/2} + Z_W^{-1/2} Z_B^{-1/2})\big) - 1\Big]
\end{aligned}
$$



$$
\begin{aligned}
&+\mathrm{i}\frac{e}{2}\xi sc M_Z Z_\mathrm{H}\Big(1-\frac{\delta v}{v}\Big) \\
&\times\big[Z_{\eta_\mathrm{Y}} + Z_{\eta_\mathrm{L}} Z_g^2 Z_W^{-2} \\
&- Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_g (Z_W^{-3/2} Z_B^{1/2} + Z_W^{-1/2} Z_B^{-1/2})\big]
\end{aligned}
$$



$$
\begin{aligned}
&+\frac{\mathrm{i}}{2}e\xi M_Z Z_\mathrm{H}\Big(1-\frac{\delta v}{v}\Big) \\
&\times\big[s^2(Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_g Z_W^{-1/2} Z_B^{-1/2} - Z_{\eta_\mathrm{Y}}) \\
&+ c^2 Z_g Z_W^{-1}(Z_{\eta_\mathrm{L}} Z_g Z_W^{-1} - Z_{\eta_\mathrm{Y}}^{1/2} Z_{\eta_\mathrm{L}}^{1/2} Z_W^{-1/2} Z_B^{1/2})\big]
\end{aligned}
$$



$$+\frac{\mathrm{i}}{2}e\xi M_Z Z_{\mathrm{H}}\left(1-\frac{\delta v}{v}\right)$$
$$\times\left[s^2(Z_{\eta_{\mathrm{Y}}}^{1/2}Z_{\eta_{\mathrm{L}}}^{1/2}Z_g Z_W^{-3/2}Z_B^{1/2}-Z_{\eta_{\mathrm{Y}}})\right.$$
$$\left.+\,c^2 Z_g Z_W^{-1}(Z_{\eta_{\mathrm{L}}}Z_g Z_W^{-1}-Z_{\eta_{\mathrm{Y}}}^{1/2}Z_{\eta_{\mathrm{L}}}^{1/2}Z_W^{1/2}Z_B^{-1/2})\right]$$

We end this appendix with a few more remarks. We have chosen our conventions to classify physical objects as particles and antiparticles described by charge conjugate fields (for example, the electron is usually called a particle and described by the field ψ and the positron is called an antiparticle and described by the charge conjugate field ψ^{c}). A spin $\frac{1}{2}$ particle state with momentum k and spin projection s is created by the operator $b^+(k,s)$ and the antiparticle state by $b^{\mathrm{c}+}(k,s) = d^+(k,s)$. Using for instance, the rules of canonical quantization introduced in Sections 1.4 and 2.7 it is easy to see that a general fermion–boson vertex $\mathrm{i}\Gamma$, under interchange of incoming and outgoing particles (or antiparticles) is transformed into $\mathrm{i}\gamma_0\Gamma^+\gamma_0$. This follows by taking the matrix element of the interaction lagrangian

$$h_{ijk}\phi_i\bar{\psi}'_j\Gamma\psi'_k + h^*_{ijk}\phi_i^+\bar{\psi}_k\gamma_0\Gamma^+\gamma_0\psi_j$$

between the appropriate one-particle states built of creation and annihilation operators. Changing in the vertex particles (antiparticles) into antiparticles (particles) gives the same transformation for the vertex. The vertex remains, of course, unchanged under both transformations performed simultaneously. We also recall once more that each incoming (outgoing) fermion with momentum k and spin projection s is described by the spinor $u(k,s)$ ($\bar{u}(k,s)$) and each incoming (outgoing) antifermion with momentum k and spin projection s by $\bar{v}(k,s)$ ($v(k,s)$).

In some theories, for example, in the Minimal Supersymmetric Standard Model (MSSM), some fields appear in the lagrangian simultaneously with their charge conjugate partners. For instance, in the MSSM, defining the positively charged chargino χ^+ as a particle, we have the couplings

$$\overline{\chi^+}\Gamma u\tilde{d}^* + \overline{(\chi^+)^{\mathrm{c}}}\tilde{\Gamma}d\tilde{u}^* + \mathrm{h.c.}$$

where u and d are quark fields and $\tilde{u}$ and $\tilde{d}$ are scalar fields of the respective supersymmetric partners of the quarks (called squarks). Both couplings may be simultaneously relevant in calculations of some amplitudes (for example, $Z \to \chi^+\chi^-$ at the one-loop level), so that the redefinition $\chi^+ = (\chi^-)^{\mathrm{c}}$ does not eliminate the charge conjugate field from the Wick contractions. If χ^{c} appears to be an internal line then, after Wick contractions, we encounter the propagator

$\langle 0|T\chi^{\mathrm{c}}\overline{\chi^{\mathrm{c}}}|0\rangle = C\langle 0|T\chi\bar{\chi}|0\rangle^{\mathrm{T}}C^{-1}$. In momentum space, $CS^{\mathrm{T}}(p)C^{-1} = 1/(-\not{p} - m) = S(-p)$, where the field χ carries momentum p. When the field χ and χ^{c} describe external particles like in $Z \to \chi^+\chi^-$ it is more convenient to rewrite one of the interaction terms, for example,

$$\overline{\chi^{\mathrm{c}}}\tilde{\Gamma}d\tilde{u}^* = \overline{d^{\mathrm{c}}}\tilde{\Gamma}'\chi\tilde{u}^{\mathrm{c}*}$$

where $\tilde{\Gamma}' = C\tilde{\Gamma}^{\mathrm{T}}C^{-1}$. Then, the charged conjugate fermion appears as an internal line and we can apply the simple propagator rule. A similar trick is useful in deriving Feynman rules for Majorana fermions (Denner 1992).

Appendix D
One-loop Feynman integrals

In this appendix we collect some useful formulae for integrals appearing in the computations of one-loop Feynman diagrams and general expressions for Feynman diagrams contributing to the gauge boson self-energies.

Two-point functions

The two basic integrals are†

$$\left.\begin{aligned} \mu^{\varepsilon}\int\frac{\mathrm{d}^n k}{(4\pi)^n}\frac{\mathrm{i}}{k^2-m^2} &\equiv \frac{1}{(4\pi)^2}a(m) \\ \mu^{\varepsilon}\int\frac{\mathrm{d}^n k}{(4\pi)^n}\frac{\mathrm{i}}{[k^2-m_1^2][(k+p)^2-m_2^2]} &\equiv \frac{1}{(4\pi)^2}b_0(p^2,m_1,m_2) \end{aligned}\right\} \tag{D.1}$$

where

$$a(m) = m^2\left(-\eta - 1 + \ln\frac{m^2}{\mu^2}\right) \tag{D.2}$$

and

$$b_0(p^2,m_1,m_2) = -\eta + \int_0^1 \mathrm{d}x \ln\frac{x(x-1)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2}. \tag{D.3}$$

Here μ is the renormalization scale and $\eta = 2/(4-n) + \ln(4\pi) - \gamma_E$. The b_0-function is symmetric in m_1 and m_2. Useful limiting cases for the b_0-function are

† The factor $-\mathrm{i}\varepsilon$ in all masses squared is understood.

listed below:

$$
\begin{aligned}
b_0(0, m_1, m_2) &= -\eta - 1 + \frac{m_1^2}{m_1^2 - m_2^2} \ln \frac{m_1^2}{\mu^2} + \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{\mu^2} \\
&= -\eta - 1 + \ln \frac{m_1^2}{\mu^2} + \frac{m_2^2}{m_2^2 - m_1^2} \ln \frac{m_2^2}{m_1^2} \\
&= \frac{1}{m_1^2 - m_2^2} \left(a(m_1) - a(m_2)\right) \qquad \text{(D.4)}
\end{aligned}
$$

$$
b_0(0, 0, m) = -\eta - 1 + \ln \frac{m^2}{\mu^2} \qquad \text{(D.5)}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}p^2} b_0(0, m_1, m_2) = -\frac{1}{2} \frac{m_1^2 + m_2^2}{(m_1^2 - m_2^2)^2} + \frac{m_1^2 m_2^2}{(m_1^2 - m_2^2)^3} \ln \frac{m_1^2}{m_2^2} \qquad \text{(D.6)}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}p^2} b_0(m^2, m, \lambda) = -\frac{1}{m^2} - \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} + \mathcal{O}(\lambda^2) \qquad \text{(D.7)}
$$

Any other integral appearing in the computation of two-point Feynman diagrams can be reduced to these. For example,

$$
\begin{aligned}
&\mu^\varepsilon \int \frac{\mathrm{d}^n k}{(4\pi)^n} \frac{\mathrm{i}k^\mu}{[k^2 - m_1^2][(k+p)^2 - m_2^2]} \\
&= \frac{1}{(4\pi)^2} \frac{p^\mu}{2p^2} \left[a(m_1) - a(m_2) - (p^2 + m_1^2 - m_2^2) b_0(p^2, m_1, m_2)\right] \qquad \text{(D.8)}
\end{aligned}
$$

$$
\begin{aligned}
&\mu^\varepsilon \int \frac{\mathrm{d}^n k}{(4\pi)^n} \frac{\mathrm{i}k^\mu k^\nu}{[k^2 - m_1^2][(k+p)^2 - m_2^2]} \\
&= \frac{1}{(4\pi)^2} \left[g^{\mu\nu} A(p^2, m_1, m_2) + \frac{p^\mu p^\nu}{p^2} B(p^2, m_1, m_2)\right] \qquad \text{(D.9)}
\end{aligned}
$$

where

$$
\begin{aligned}
A(p^2, m_1, m_2) &= \frac{1}{12} a(m_1) + \frac{1}{12} a(m_2) + \frac{1}{6}\left(m_1^2 + m_2^2 - \frac{p^2}{2}\right) b_0(p^2, m_1, m_2) \\
&\quad + \frac{m_1^2 - m_2^2}{12p^2} \left[a(m_1) - a(m_2) - (m_1^2 - m_2^2) b_0(p^2, m_1, m_2)\right] \\
&\quad - \frac{1}{6}\left(m_1^2 + m_2^2 - \frac{p^2}{3}\right) \qquad \text{(D.10)}
\end{aligned}
$$

$$B(p^2, m_1, m_2) = -\frac{1}{3}a(m_1) + \frac{2}{3}a(m_2) - \frac{1}{3}m_1^2 b_0(p^2, m_1, m_2)$$
$$+ \frac{(p^2 + m_1^2 - m_2^2)^2}{3p^2} b_0(p^2, m_1, m_2)$$
$$- \frac{m_1^2 - m_2^2}{3p^2}\,[a(m_1) - a(m_2)] + \frac{1}{6}\left(m_1^2 + m_2^2 - \frac{p^2}{3}\right) \tag{D.11}$$

Using the derivative (D.7) one easily finds

$$A(0, m_1, m_2) = \frac{1}{12}a(m_1) + \frac{1}{12}a(m_2) + \frac{1}{6}(m_1^2 + m_2^2)\,[-1 + b_0(0, m_1, m_2)]$$
$$+ \frac{1}{24}\left[m_1^2 + m_2^2 - \frac{2m_1^2 m_2^2}{(m_1^2 - m_2^2)} \ln \frac{m_1^2}{m_2^2}\right] \tag{D.12}$$

In the limit $m_1 \to m_2$ the last square bracket vanishes.

The imaginary part of any two-point amplitude can easily be calculated knowing that

$$\operatorname{Im} b_0(p^2, m_1, m_2) = -\pi \lambda^{1/2}(p^2, m_1^2, m_2^2)\theta\left[1 - \frac{(m_1 + m_2)^2}{p^2}\right] \tag{D.13}$$

where $\lambda(p^2, m_1^2, m_2^2)$ is the standard phase space factor

$$\lambda(p^2, m_1^2, m_2^2) = 1 - \frac{2(m_1^2 + m_2^2)}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^4} \tag{D.14}$$

Three- and four-point functions

Three- and four-point functions for general non-vanishing external momenta and all masses different are very complicated. A compact expression exists only for the scalar three-point function and involves twelve Spence functions ('t Hooft & Veltman 1979). Here we restrict ourselves to the case of vanishing external momenta. In this case the two basic integrals are

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{[k^2 - m_1^2][k^2 - m_2^2][k^2 - m_3^2]} \equiv \frac{1}{(4\pi)^2} F(m_1, m_2, m_3) \tag{D.15}$$

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{[k^2 - m_1^2][k^2 - m_2^2][k^2 - m_3^2][k^2 - m_4^2]} \equiv \frac{1}{(4\pi)^2} H(m_1, m_2, m_3, m_4) \tag{D.16}$$

Other integrals can be reduced to these two. Note that these are finite functions. They can be calculated by using the decomposition of the denominators into simple fractions and then by using $a(m)$- or $b_0(0, m_1, m_2)$-functions. For (D.15) we find

$$F(m_1, m_2, m_3) = \frac{1}{m_1^2 - m_2^2} \left[b_0(0, m_1, m_3) - b_0(0, m_2, m_3) \right] \tag{D.17}$$

In this form, the limit $m_1^2 \to m_2^2$ is easily calculable by noting that (D.17) becomes in this limit just the definition of the derivative of $b_0(0, m_1, m_3)$ with respect to m_1^2. Using the explicit form (D.4) of the b_0 function we find

$$\begin{aligned} F(m_1, m_2, m_3) & \\ &= \frac{1}{m_1^2 - m_2^2} \left(\frac{m_1^2}{m_1^2 - m_3^2} \ln \frac{m_1^2}{m_3^2} - \frac{m_2^2}{m_2^2 - m_3^2} \ln \frac{m_2^2}{m_3^2} \right) & \text{(D.18)} \\ &= -\frac{m_1^2 m_2^2 \ln(m_1^2/m_2^2) + m_2^2 m_3^2 \ln(m_2^2/m_3^2) + m_3^2 m_1^2 \ln(m_3^2/m_1^2)}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)} & \text{(D.19)} \end{aligned}$$

Knowing the function $F(m_1, m_2, m_3)$, the function $H(m_1, m_2, m_3, m_4)$ can be represented as

$$H(m_1, m_2, m_3, m_4) = \frac{F(m_1, m_3, m_4) - F(m_2, m_3, m_4)}{m_1^2 - m_2^2} \tag{D.20}$$

Again, the limit $m_1^2 \to m_2^2$ can be found by applying the trick with the derivative. As an application we compute the integral appearing in (12.139). After performing the algebra of the Dirac matrices we are left with

$$\begin{aligned} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} & \frac{\mathrm{i} k^2}{[k^2 - m_i^2][k^2 - m_j^2][k^2 - M_W^2]^2} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{\mathrm{i}}{[k^2 - m_i^2][k^2 - m_j^2][k^2 - M_W^2]} \right. \\ &\quad \left. + \frac{\mathrm{i} M_W^2}{[k^2 - m_i^2][k^2 - m_j^2][k^2 - M_W^2]^2} \right] \\ &= \frac{1}{(4\pi)^2} \left[F(m_i, m_j, M_W) + M_W^2 \frac{\mathrm{d}}{\mathrm{d} M_W^2} F(m_i, m_j, M_W) \right] \\ &= \frac{1}{(4\pi)^2 M_W^2} A(x_i, x_j) \end{aligned}$$

where $A(x_i, x_j)$ in which $x_i \equiv m_i^2/M_W^2$ defined in (12.143) is obtained by using (D.18) and $x_i \equiv m_i^2/M_W^2$.

General expressions for the one-loop vector boson self-energies

The general self-energy of a vector boson shown in Fig. 12.5(a) is decomposed into the transverse and longitudinal parts according to (12.67). We define here

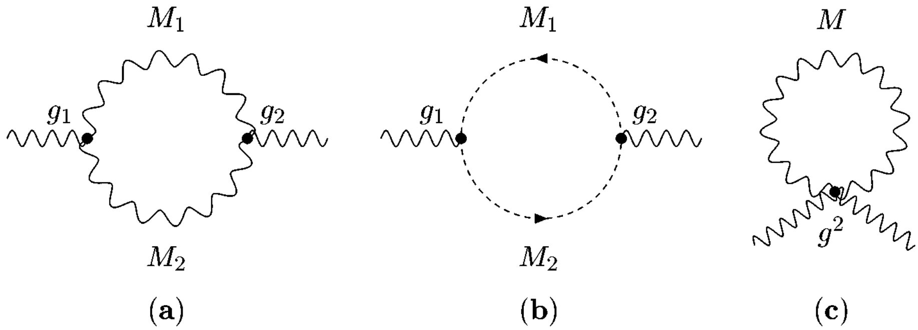


Fig. D.1. Gauge boson sector contributions to $\Pi^{\mu\nu}_{V_1,V_2}(p^2)$.

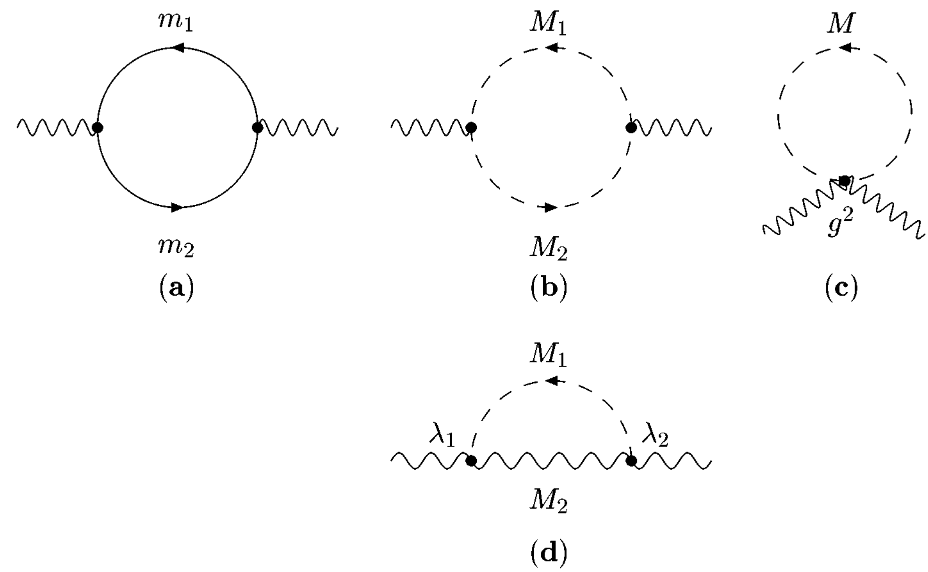


Fig. D.2. Matter sector contributions to $\Pi^{\mu\nu}_{V_1,V_2}(p^2)$.

another decomposition

$$\mathrm{i}\Pi^{\mu\nu}_{V_1,V_2}(p^2) = \mathrm{i}g^{\mu\nu}\Pi^{\mathrm{T}}_{V_1,V_2}(p^2) + \mathrm{i}\frac{p^\mu p^\nu}{p^2}\tilde{\Pi}^{\mathrm{L}}_{V_1,V_2}(p^2) \tag{D.21}$$

so that $\tilde{\Pi}^{\mathrm{L}}_{V_1,V_2} \equiv \Pi^{\mathrm{L}}_{V_1,V_2} - \Pi^{\mathrm{T}}_{V_1,V_2}$. This decomposition is more convenient for a concise presentation of the results of the calculation. Typical one-loop contributions to $\Pi^{\mu\nu}_{V_1,V_2}(p^2)$ are shown in Figs. D.1 and D.2. Evaluating diagrams shown in Fig. D.1 we find

$$\Pi^{\mathrm{T}}_{(a)}(p^2) = -\frac{g_1 g_2}{(4\pi)^2}\big[10A(p^2, M_1, M_2) + (4p^2 + M_1^2 + M_2^2)b_0(p^2, M_1, M_2) + a(M_1) + a(M_2) + 2M_1^2 + 2M_2^2 - \tfrac{2}{3}p^2\big] \tag{D.22}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(a)}(p^2) = -\frac{g_1 g_2}{(4\pi)^2}\big[10B(p^2, M_1, M_2) - 2p^2 b_0(p^2, M_1, M_2) - 5\big(a(M_2) - a(M_1) - (p^2 - M_1^2 + M_2^2)b_0(p^2, M_1, M_2)\big) + \tfrac{2}{3}p^2\big] \tag{D.23}$$

$$\Pi^{\mathrm{T}}_{(b)}(p^2) = \frac{g_1 g_2}{(4\pi)^2} A(p^2, M_1, M_2) \tag{D.24}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(b)}(p^2) = \frac{g_1 g_2}{(4\pi)^2}\big[B(p^2, M_1, M_2) + \tfrac{1}{2}\big(a(M_2) - a(M_1) - (p^2 - M_1^2 + M_2^2)b_0(p^2, M_1, M_2)\big)\big] \tag{D.25}$$

$$\Pi^{\mathrm{T}}_{(c)}(p^2) = \frac{g^2}{(4\pi)^2}\left[6a(M) + 4M^2\right] \tag{D.26}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(c)}(p^2) = 0 \tag{D.27}$$

For the diagrams shown in Fig. D.2 we get (the coupling of fermions to the vector boson is $\mathrm{i}\gamma^\kappa(c_V - c_A\gamma^5)$)

$$\Pi^{\mathrm{T}}_{(a)}(p^2) = \frac{4}{(4\pi)^2}(c_V^2 + c_A^2)\big[2A(p^2, m_1, m_2) - \tfrac{1}{2}a(m_1) - \tfrac{1}{2}a(m_2) + \tfrac{1}{2}(p^2 - m_1^2 - m_2^2)b_0(p^2, m_1, m_2)\big] + \frac{4}{(4\pi)^2}(c_V^2 - c_A^2)m_1 m_2 b_0(p^2, m_1, m_2) \tag{D.28}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(a)}(p^2) = \frac{4}{(4\pi)^2}(c_V^2 + c_A^2)\big[2B(p^2, m_1, m_2) + \tfrac{1}{2}a(m_2) - \tfrac{1}{2}a(m_1) - (p^2 - m_1^2 + m_2^2)b_0(p^2, m_1, m_2)\big] \tag{D.29}$$

$$\Pi^{\mathrm{T}}_{(b)}(p^2) = -\frac{4}{(4\pi)^2} g_1 g_2 A(p^2, M_1, M_2) \tag{D.30}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(b)}(p^2) = -\frac{g_1 g_2}{(4\pi)^2}\big[4B(p^2, M_1, M_2) + b_0(p^2, M_1, M_2) + 2\big(a(M_2) - a(M_1) - (p^2 - M_1^2 + M_2^2)b_0(p^2, M_1, M_2)\big)\big] \tag{D.31}$$

$$\Pi^{\mathrm{T}}_{(c)}(p^2) = \frac{1}{(4\pi)^2} g^2 a(M) \tag{D.32}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(c)}(p^2) = 0 \tag{D.33}$$

$$\Pi^{\mathrm{T}}_{(d)}(p^2) = \frac{1}{(4\pi)^2}\lambda_1\lambda_2 b_0(p^2, M_1, M_2) \tag{D.34}$$

$$\tilde{\Pi}^{\mathrm{L}}_{(d)}(p^2) = 0 \tag{D.35}$$

The functions $a(m)$, $b_0(q^2, m_1, m_2)$ and $A(q^2, m_1, m_2)$ are defined in (D.9).

In the Standard Model, using the Feynman rules and taking proper account of the combinatoric factors we find for $\Pi^{\mathrm{T}}_{WW}(p^2)$ (in units $e^2/(4\pi)^2 s^2$)

$$\begin{aligned}
&\tfrac{1}{2}\sum_{k=1}^{3}\left[4A(p^2, m_{e_k}, 0) - a(m_{e_k}) + (p^2 - m_{e_k}^2)b_0(p^2, m_{e_k}, 0)\right] \\
&\quad + \tfrac{3}{2}\sum_{k,l=1}^{3}|V_{\mathrm{CKM}}^{kl}|^2\big[4A(p^2, m_{u_k}, m_{d_l}) - a(m_{u_k}) - a(m_{d_l}) \\
&\quad + (p^2 - m_{u_k}^2 - m_{d_l}^2)b_0(p^2, m_{u_k}, m_{d_l})\big] \\
&\quad - A(p^2, M_W, M_H) - A(p^2, M_W, M_H) + \tfrac{1}{2}a(M_W) + \tfrac{1}{4}a(M_Z) + \tfrac{1}{4}a(M_H) \\
&\quad + s^4 M_Z^2 b_0(p^2, M_W, M_Z) + s^2 M_W^2 b_0(p^2, M_W, 0) + M_W^2 b_0(p^2, M_W, M_H) \\
&\quad - s^2\left[8A(p^2, M_W, 0) + (4p^2 + M_W^2)b_0(p^2, M_W, 0) + 2a(M_W) + \tfrac{2}{3}p^2\right] \\
&\quad - c^2\big[8A(p^2, M_W, M_Z) + (4p^2 + M_W^2 + M_Z^2)b_0(p^2, M_W, M_Z) \\
&\quad - 2a(M_W) - 2a(M_Z) + \tfrac{2}{3}p^2\big]
\end{aligned}$$

For $\Pi^{\mathrm{T}}_{ZZ}(p^2)$ (in units $e^2/(4\pi)^2 s^2 c^2$) we get:

$$\begin{aligned}
&\tfrac{3}{4}\left[A(p^2, 0, 0) + p^2 b_0(p^2, 0, 0)\right] \\
&\quad + \tfrac{1}{2}\sum_{f} N_{c_f} a_+^f\left[2A(p^2, m_f, m_f) - a(m_f) + (\tfrac{1}{2}p^2 - m_f^2)b_0(p^2, m_f, m_f)\right] \\
&\quad + \tfrac{1}{2}\sum_{f} N_{c_f} a_-^f m_f^2 b_0(p^2, m_f, m_f) \\
&\quad - (c^2 - s^2)\left[A(p^2, M_W, M_W) - \tfrac{1}{2}a(M_W)\right] \\
&\quad - A(p^2, M_Z, M_H) - \tfrac{1}{4}a(M_Z) - \tfrac{1}{4}a(M_H) \\
&\quad + 2s^4 M_W^2 b_0(p^2, M_W, M_W) + M_Z^2 b_0(p^2, M_Z, M_H) \\
&\quad - c^4\left[8A(p^2, M_W, M_W) + (4p^2 + 2M_W^2)b_0(p^2, M_W, M_W) - 4a(M_W) - \tfrac{2}{3}p^2\right]
\end{aligned}$$

where the sum in the second line extends to all fermions except neutrinos, $a_+^f = 1 - 4|Q_f|s^2 + 8|Q_f|^2 s^4$, $a_-^f = -4|Q_f|s^2 + 8|Q_f|^2 s^4$ and Q_f, N_{c_f} are the electric charge and colour factor (1 for leptons, 3 for quarks) of the particle, respectively.

The expression for $\Pi^{\mathrm{T}}_{\gamma\gamma}(p^2)$ in units $e^2/(4\pi)^2$ reads

$$\begin{aligned}
&2\sum_f N_{c_f} Q_f^2 \left[4A(p^2, m_f, m_f) - 2a(m_f) + p^2 b_0(p^2, m_f, m_f)\right] \\
&\quad - 4A(p^2, M_W, M_W) - 2a(M_W) + 2M_W^2 b_0(p^2, M_W, M_W) \\
&\quad - \left[8A(p^2, M_W, M_W) + (4p^2 + 2M_W^2) b_0(p^2, M_W, M_W) - 4a(M_W) - \tfrac{2}{3}p^2\right]
\end{aligned}$$

Finally, $\Pi^{\mathrm{T}}_{\gamma Z}(p^2)$ in units $e^2/(4\pi)^2 sc$ is given by

$$\begin{aligned}
&2\sum_f N_{c_f} |Q_f|(1 - 4|Q_f|s^2) \left[2A(p^2, m_f, m_f) - a(m_f) + p^2 b_0(p^2, m_f, m_f)\right] \\
&\quad - (c^2 - s^2)\left[2A(p^2, M_W, M_W) - a(M_W)\right] - 2s^2 M_W^2 b_0(p^2, M_W, M_W) \\
&\quad - c^2\left[8A(p^2, M_W, M_W) + (4p^2 + 2M_W^2) b_0(p^2, M_W, M_W) - 4a(M_W) - \tfrac{2}{3}p^2\right]
\end{aligned}$$

Appendix E
Elements of group theory

For a concise introduction to group theory and its application in particle physics see, for instance, Werle (1966). Here we recall only those definitions and properties most useful in reading this book, with no attempt for a complete and self-consistent presentation.

Definitions

Let H be a proper (i.e. different from the group G itself and from the unit element alone) subgroup of G, and g an arbitrary element of G. The sets gH and Hg with fixed $g \in G$, are called left and right cosets of H. Two left cosets $g_i H$ and $g_k H$ (or two right cosets Hg_i, and Hg_k) of the same subgroup H contain exactly the same elements of G or have no common elements at all. Taking all different left (or right) cosets of H we can decompose the group G into the sum

$$G = H + g_1 H + \cdots + g_{\nu-1} H \tag{E.1}$$

where $g_1 \in G$, $g_1 \notin H$, $g_2 \in G$, $g_2 \notin H$, $g_2 \notin g_1 H$ etc. The number ν is called the index of H in G. We can define equivalence classes of elements of G; two elements g and g' are considered equivalent if they belong to the same coset with respect to H. The set of all cosets is a manifold denoted by G/H.

Two elements $g, h \in G$ are said to be mutually conjugate if there exists such an element $a \in G$ that

$$h = aga^{-1}$$

One can divide all the group elements into separate classes of conjugate elements.

If a set H with elements h is a subgroup of G then the set $H' = gHg^{-1}$ is another subgroup of G which is called conjugate to H. If for arbitrary $g \in G$ all the conjugate subgroups gHg^{-1} are identical (i.e. contain the same elements) $gHg^{-1} = H$, we call H a normal subgroup. A normal subgroup H of G consists of whole undivided classes of G. Groups which contain no proper normal subgroups are said to be *simple*. Groups which contain no proper normal abelian subgroups are said to be *semi-simple*.

Consider the products of an element of the coset gH by an element of the coset fH, where H is a normal subgroup of G. All such products belong to the coset gfH. Hence we can define a new type of multiplication of cosets

$$(gH)(fH) = (gfH) \tag{E.2}$$

The set of all different cosets of a normal subgroup H is a group with respect to this multiplication law. This so-called quotient group F is not a subgroup of G as its elements are whole sets (cosets). We write symbolically $F = G/H$.

A function φ is said to be defined on a group G if to each element $g \in G$ a unique complex number $\varphi(g)$ is assigned. In the case of a continuous k-parameter group $G_{(k)}$ a function $\varphi(g)$ can be regarded as an ordinary function $\varphi(\alpha_1, \ldots, \alpha_k)$ of k real variables that are necessary to specify the group elements. A group is called *compact* if the group manifold M is compact, i.e. if every infinite subset of M contains a sequence converging to a limit in M, or equivalently if any function $\varphi(\alpha_i)$ which is continuous in all points of M is also bounded. For compact groups we can define the average of a group function by the expression

$$\mathrm{Av}[\varphi] = (1/V) \int \varphi(\alpha)\rho(\alpha)\,\mathrm{d}\alpha \tag{E.3}$$

where

$$V = \int \rho(\alpha)\,\mathrm{d}\alpha \tag{E.4}$$

and $\rho(\alpha)$ is a properly chosen weight function such that

(i) $\mathrm{Av}[\varphi(g)] = c$ if $\varphi(g) = \text{const}$ for all $g \in G$
(ii) $\mathrm{Av}[\varphi(g)] \geq 0$ if $\varphi(g) \geq 0$
(iii) $\mathrm{Av}[\varphi(hg)] = \mathrm{Av}[\varphi(gh)] = \mathrm{Av}[\varphi(g)]$ for any fixed $h \in G$.

All representations of a semi-simple group are fully reducible. Unitary representations of any group are fully reducible. Every representation of a finite group or of a compact Lie group is equivalent to a unitary representation. Lie groups which are not compact may have representations which are not equivalent to any unitary representation. Any finite group has a finite number and any compact Lie group has a countable infinite number of inequivalent, unitary, irreducible representations which are all finite-dimensional. Non-compact Lie groups have uncountable numbers of inequivalent, unitary irreducible representations which are all infinite dimensional.

Transformation of operators

Consider the set of all possible linear operators $\hat{O}$ which transform the vectors $|v\rangle$ of some Hilbert space $\mathcal{H}$ into vectors $|w\rangle$ of the same space $|w\rangle = \hat{O}|v\rangle$. Suppose that under a group of transformations G the vectors of $\mathcal{H}$ are transformed as follows

$$|w_g\rangle = U(g)|w\rangle, \qquad |v_g\rangle = U(g)|v\rangle \tag{E.5}$$

where $U(g) = \exp(\mathrm{i}\alpha^a Q^a)$ are unitary operators and Q^a are generators of G. The operator $\hat{O}_g$ which transforms $|v_g\rangle$ into $|w_g\rangle$ is given by

$$\hat{O}_g = U(g)\hat{O}U^{-1}(g) \tag{E.6}$$

Let us define the linear transformation $T(g)$ which transforms $\hat{O} \to \hat{O}_g$

$$\hat{O}_g = T(g)\hat{O} = U(g)\hat{O}U^{-1}(g) \tag{E.7}$$

The transformation $T(g) = \exp(-\mathrm{i}\alpha^a T^a)$ forms a unitary representation of G. For infinitesimal transformations we have

$$\hat{O}_g \cong (1 - \mathrm{i}\delta\alpha^a T^a)\hat{O} = \hat{O} + \mathrm{i}\delta\alpha^a[Q^a, \hat{O}] \tag{E.8}$$

Complex and real representations

Let T_i, $i = 1, \ldots, N$ form a representation of generators of a group G

$$[T_i, T_j] = \mathrm{i}c_{ijk}T^k \tag{E.9}$$

The structure constants c_{ijk} are real if T_i are hermitean. Taking the complex conjugate of these commutation relations we see that $(-T_i^*)$ also form a representation of the group

$$[(-T_i^*), (-T_j^*)] = \mathrm{i}c_{ijk}(-T^{k*}) \tag{E.10}$$

If T_i and $(-T_i^*)$ are equivalent, i.e.

$$-T_i^* = UT_iU^\dagger \tag{E.11}$$

where U is a unitary transformation, the representation is said to be real (the term real is motivated by an alternative convention in which $\mathrm{i}T_i$ is the representation matrix). For instance for 2 and 2^* of $SU(2)$ we have

$$-T_i^* = -T_i^{\mathrm{T}} = -\tfrac{1}{2}\sigma_i^{\mathrm{T}} \tag{E.12}$$

and

$$\mathrm{i}\sigma_2(-\tfrac{1}{2}\sigma_i^{\mathrm{T}})(\mathrm{i}\sigma_2)^\dagger = \tfrac{1}{2}\sigma_i \tag{E.13}$$

But this is not true for $SU(N)$, $N > 2$.

Take a set of n fields $\Phi_a(x)$ which transform according to $n \times n$-dimensional representation matrices T_i:

$$[Q^i, \Phi_a] = -(T^i)_{ab}\Phi_b \tag{E.14}$$

Then the fields $\Phi_a^\dagger$ transform according to the complex conjugate representation (take the hermitean conjugate of (E.14)):

$$[Q^i, \Phi_a^\dagger(x)] = +T_{ab}^{i*}\Phi_b^\dagger(x) = -(-T_{ab}^{i*})\Phi_b^\dagger(x) \tag{E.15}$$

Let all the left-handed fermion fields Ψ_{L} and $(\Psi^{\mathrm{C}})_{\mathrm{L}}$ (C means charge conjugation, see Appendix A) in a theory form the representation f_{L} under some symmetry group. From (A.110)

$$\Psi_{\mathrm{R}} = C(\overline{\Psi^{\mathrm{C}}})_{\mathrm{L}}{}^{\mathrm{T}} \quad \text{and} \quad (\Psi^{\mathrm{C}})_{\mathrm{R}} = (\Psi_{\mathrm{L}})^{\mathrm{C}} = C\bar{\Psi}_{\mathrm{L}}{}^{\mathrm{T}} \tag{E.16}$$

so Ψ_{R} and $(\Psi^{\mathrm{C}})_{\mathrm{R}}$ transform under the complex conjugate representation f_{L}^*. If f_{L} is real then the left-handed and the right-handed fermions transform according to the same representation of the symmetry group. The theory is said to be vector-like. If f_{L} is complex ($f_{\mathrm{R}} = f_{\mathrm{L}}^* \neq f_{\mathrm{L}}$) the theory is said to be chiral.

QCD is vector-like because

$$\begin{aligned} f_{\mathrm{L}} &= 3 + 3^* \\ &\quad (u_{\mathrm{L}})(u^{\mathrm{C}})_{\mathrm{L}} \end{aligned} \tag{E.17}$$

while

$$f_{\mathrm{R}} = f_{\mathrm{L}}^* = 3^* + 3 = f_{\mathrm{L}} \tag{E.18}$$

The standard model $SU(3) \times SU(2) \times U(1)$ is chiral. For one family we have:

$$\begin{array}{llllll} f_{\mathrm{L}} = & (3, 2, \frac{1}{6}) + & (3^*, 1, -\frac{2}{3}) + & (3^*, 1, \frac{1}{3}) + & (1, 2, -\frac{1}{2}) + & (1, 1, 1) \\ & \begin{pmatrix} u \\ d \end{pmatrix}_{\mathrm{L}} & (u^{\mathrm{C}})_{\mathrm{L}} & (d^{\mathrm{C}})_{\mathrm{L}} & \begin{pmatrix} \nu \\ \mathrm{e}^- \end{pmatrix}_{\mathrm{L}} & (\mathrm{e}^+)_{\mathrm{L}} \end{array} \tag{E.19}$$

and

$$\begin{array}{llllll} f_{\mathrm{R}} = f_{\mathrm{L}}^* = & (3^*, 2, -\frac{1}{6}) + & (3, 1, \frac{2}{3}) + & (3, 1, -\frac{1}{3}) + & (1, 2, \frac{1}{2}) & + (1, 1, -1) \neq f_{\mathrm{L}} \\ & \begin{pmatrix} d^{\mathrm{C}} \\ -u^{\mathrm{C}} \end{pmatrix}_{\mathrm{R}} & u_{\mathrm{R}} & d_{\mathrm{R}} & \begin{pmatrix} \mathrm{e}^+ \\ -\nu^{\mathrm{C}} \end{pmatrix}_{\mathrm{R}} & \mathrm{e}_{\mathrm{R}}^- \end{array} \tag{E.20}$$

(We have used the equivalence $2^* = 2$ and the transformation rule (E.13)).

Traces

Let

$$[T^a, T^b] = \mathrm{i}c_{abc}T^c, \; c_{abc} \equiv c^{abc} \tag{E.21}$$

Then

$$\mathbf{T}^2 = \sum_{a,k} T^a_{ik} T^a_{kj} = \delta_{ij} C_R \tag{E.22}$$

where R means representation R.

$$\mathrm{Tr}[T^a T^b] = T_R \delta^{ab} \tag{E.23}$$

$$T_R d(G) = C_R d(R) \tag{E.24}$$

where $d(G)$ is the dimension of the group and $d(R)$ is the dimension of the representation R.

For $SU(N)$:

$$\begin{aligned} d(R) &= N && \text{for the fundamental representation } (F) \\ d(R) &= N^2 - 1 = d(G) && \text{for the adjoint representation } (A) \end{aligned}$$

$$\begin{aligned} &T_{\mathrm{F}} = \tfrac{1}{2}, \quad C_{\mathrm{F}} = \frac{N^2-1}{2N} \\ &T_A = C_A = N \end{aligned}$$

$\sum_{i,k} c_{aik} c_{bik} = \delta_{ab} C_R = N \delta_{ab}$ because in the adjoint representation $(T^a)_{ik} = -\mathrm{i} c_{aik}$.

For $SU(3)$ in the fundamental representation $T^a = \frac{1}{2}\lambda^a$ with

$$\left.\begin{aligned} &\lambda^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3; \\ &\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} & & -\mathrm{i} \\ & 0 & \\ \mathrm{i} & & \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^1 \end{pmatrix} \\ &\lambda^7 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \end{aligned}\right\} \tag{E.25}$$

$$\{T^a, T^b\} = \tfrac{1}{3}\delta^{ab} \mathbb{1} + d^{abc} T^c, d^{abc} = d_{abc} \tag{E.26}$$

is totally symmetric. One has

$$\begin{aligned} 1 &= c_{123} = 2c_{147} = 2c_{246} = 2c_{257} = 2c_{345} = -2c_{156} \\ &= -2c_{367} = 2c_{458}/\sqrt{3} = 2c_{678}/\sqrt{3} \end{aligned}$$

$$\begin{aligned}
1/\sqrt{3} &= d_{118} = d_{228} = d_{338} = -d_{888} \\
-1/2\sqrt{3} &= d_{448} = d_{558} = d_{668} = d_{778} \\
\tfrac{1}{2} &= d_{146} = d_{157} = d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} \\
T^a T^b &= \tfrac{1}{2}(\mathrm{i}c^{abn} + d^{abn})T^n + \tfrac{1}{6}\delta^{ab}, \quad \mathrm{Tr}[T^a T^b] = \tfrac{1}{2}\delta^{ab} && \text{(E.27)} \\
T^a T^b T^c &= \tfrac{1}{2}(d^{abn} + \mathrm{i}c^{abn})T^n T^c + \tfrac{1}{6}\delta^{ab} T^c, \\
&\mathrm{Tr}[T^a T^b T^c] = \tfrac{1}{4}(d^{abc} + \mathrm{i}c^{abc}) && \text{(E.28)} \\
\mathrm{Tr}[T^a T^b T^c T^d] &= \tfrac{1}{8}(d^{abn} + \mathrm{i}c^{abn})(d^{cdn} + \mathrm{i}c^{cdn}) + \tfrac{1}{12}\delta^{ab}\delta^{cd} && \text{(E.29)} \\
\mathrm{Tr}[T^a T^b T^a T^b] &= \tfrac{1}{8}(d^{abn}d_{abn} - c^{abn}c_{abn}) + \tfrac{8}{12} = \tfrac{1}{8}(d^2 - f^2) + \tfrac{8}{12}, \\
&d^2 = \tfrac{40}{3}, \quad f^2 = 24 && \text{(E.30)}
\end{aligned}$$

σ-model

We derive the transformation properties of the π and σ fields in the σ-model. We assume the nucleon doublets N_R and N_L transforming under the $SU_R(2) \times SU_L(2)$ as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively. Therefore

under $SU_R(2)$

$$\begin{aligned} N_R &\to N_R - \mathrm{i}\delta\boldsymbol{\alpha}_R \cdot (\tfrac{1}{2}\boldsymbol{\sigma})N_R \\ N_L &\to N_L \end{aligned} \tag{E.31}$$

under $SU_L(2)$

$$\begin{aligned} N_L &\to N_L - \mathrm{i}\delta\boldsymbol{\alpha}_L \cdot (\tfrac{1}{2}\boldsymbol{\sigma})N_L \\ N_R &\to N_R \end{aligned} \tag{E.32}$$

(remember that $\sigma^a_R N_R = \sigma^a \otimes P_R N_R = \sigma^a \otimes (P_R + P_L) N_R = \sigma^a N_R$). For invariance of the lagrangian (9.27) under each of these groups of transformations one needs that the change of $\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}$ compensates the change of the fermion fields:

$$\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma} \to \sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma} - \mathrm{i}\delta(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) \tag{E.33}$$

where for $SU_L(2)$

$$\begin{aligned} -\mathrm{i}\delta(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) &= -\mathrm{i}\delta\boldsymbol{\alpha}_L(\tfrac{1}{2}\boldsymbol{\sigma})(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) \\ &= (-\mathrm{i}\tfrac{1}{2}\boldsymbol{\sigma})[\sigma'\delta\boldsymbol{\alpha}_L + (\boldsymbol{\pi} \times \delta\boldsymbol{\alpha}_L)] + \tfrac{1}{2}\boldsymbol{\pi} \cdot \delta\boldsymbol{\alpha}_L \end{aligned} \tag{E.34}$$

$$(\boldsymbol{\sigma} \cdot \mathbf{a}\boldsymbol{\sigma} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathrm{i}\boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b})$$

and for $SU_R(2)$

$$\begin{aligned} -\mathrm{i}\delta(\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma}) &= (\sigma' + \mathrm{i}\boldsymbol{\pi} \cdot \boldsymbol{\sigma})(\mathrm{i}\tfrac{1}{2}\boldsymbol{\sigma})\delta\boldsymbol{\alpha}_R \\ &= (-\mathrm{i}\tfrac{1}{2}\boldsymbol{\sigma})[-\sigma'\delta\boldsymbol{\alpha}_R + (\boldsymbol{\pi} \times \delta\boldsymbol{\alpha}_R)] - \tfrac{1}{2}\boldsymbol{\pi} \cdot \delta\boldsymbol{\alpha}_R \end{aligned} \tag{E.35}$$

Therefore we get the following transformation of fields:

$SU_{\mathrm{L}}(2)$

$$\left.\begin{aligned} \sigma' &\to \sigma' + \tfrac{1}{2}\boldsymbol{\pi}\cdot\delta\boldsymbol{\alpha}_{\mathrm{L}} \\ \boldsymbol{\pi} &\to \boldsymbol{\pi} - \tfrac{1}{2}\boldsymbol{\pi}\times\delta\boldsymbol{\alpha}_{\mathrm{L}} - \tfrac{1}{2}\sigma'\delta\boldsymbol{\alpha}_{\mathrm{L}} \end{aligned}\right\} \tag{E.36}$$

$SU_{\mathrm{R}}(2)$

$$\left.\begin{aligned} \sigma' &\to \sigma' - \tfrac{1}{2}\boldsymbol{\pi}\cdot\delta\boldsymbol{\alpha}_{\mathrm{R}} \\ \boldsymbol{\pi} &\to \boldsymbol{\pi} - \tfrac{1}{2}\boldsymbol{\pi}\times\delta\boldsymbol{\alpha}_{\mathrm{R}} + \tfrac{1}{2}\sigma'\delta\boldsymbol{\alpha}_{\mathrm{R}} \end{aligned}\right\} \tag{E.37}$$

The above results correspond to the commutation relations (9.30) (as can be immediately seen using (9.32)). By taking appropriate linear combinations we also get the relations (9.28).

Sometimes it is convenient to work in the real four-dimensional representation $\Phi \equiv (\pi^1, \pi^2, \pi^3, \sigma')^{\mathrm{T}}$. It is easy to check that

$$[Q^a, \Phi_b] = -(T^a)_{bi}\Phi_i$$

where

$$\begin{gathered} T^1 = \left(\begin{array}{cc|cc} & & & \\ & & -\mathrm{i} & \\ \hline & \mathrm{i} & & \\ & & & \end{array}\right) \quad T^2 = \left(\begin{array}{cc|cc} & & \mathrm{i} & \\ & & & \\ \hline -\mathrm{i} & & & \\ & & & \end{array}\right) \quad T^3 = \left(\begin{array}{cc|cc} & -\mathrm{i} & & \\ \mathrm{i} & & & \\ \hline & & & \\ & & & \end{array}\right) \\ {}_5T^1 = \left(\begin{array}{cc|cc} & & & \mathrm{i} \\ & & & \\ \hline & & & \\ -\mathrm{i} & & & \end{array}\right) \quad {}_5T^2 = \left(\begin{array}{cc|cc} & & & \\ & & & \mathrm{i} \\ \hline & & & \\ & -\mathrm{i} & & \end{array}\right) \quad {}_5T^3 = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & & \mathrm{i} \\ & & -\mathrm{i} & \end{array}\right) \end{gathered} \tag{E.38}$$

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