PROJECT

M1 CCA

Polynomial interpolation in two variables

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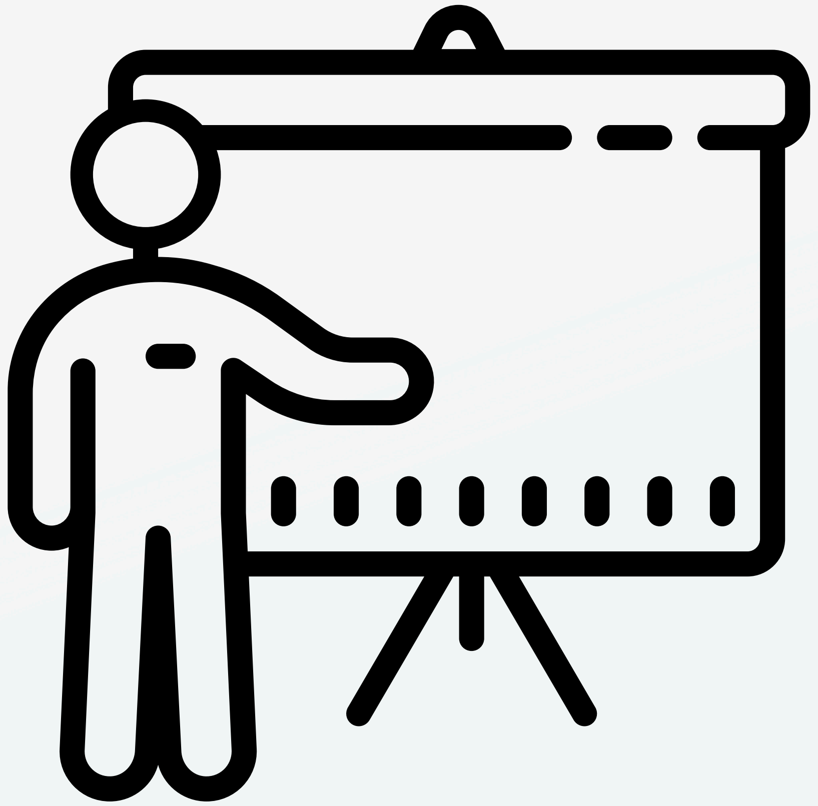
CONTENT

INTRODUCTION

Our project is dedicated to studying Polynomial interpolation in two variables:

- Studying the polynomials of vanish on a set in the distinct and no-distinct cases.
- Finding polynomials of low total degree under general conditions.
- Extension to N>= 3 variable.



Example:

$$g_0 = \prod_{i=0}^{n-1}(x - x_i) \quad \text{and} \quad g_1 = y - h(x)$$

$$E = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), (x_n, y_n)\}$$

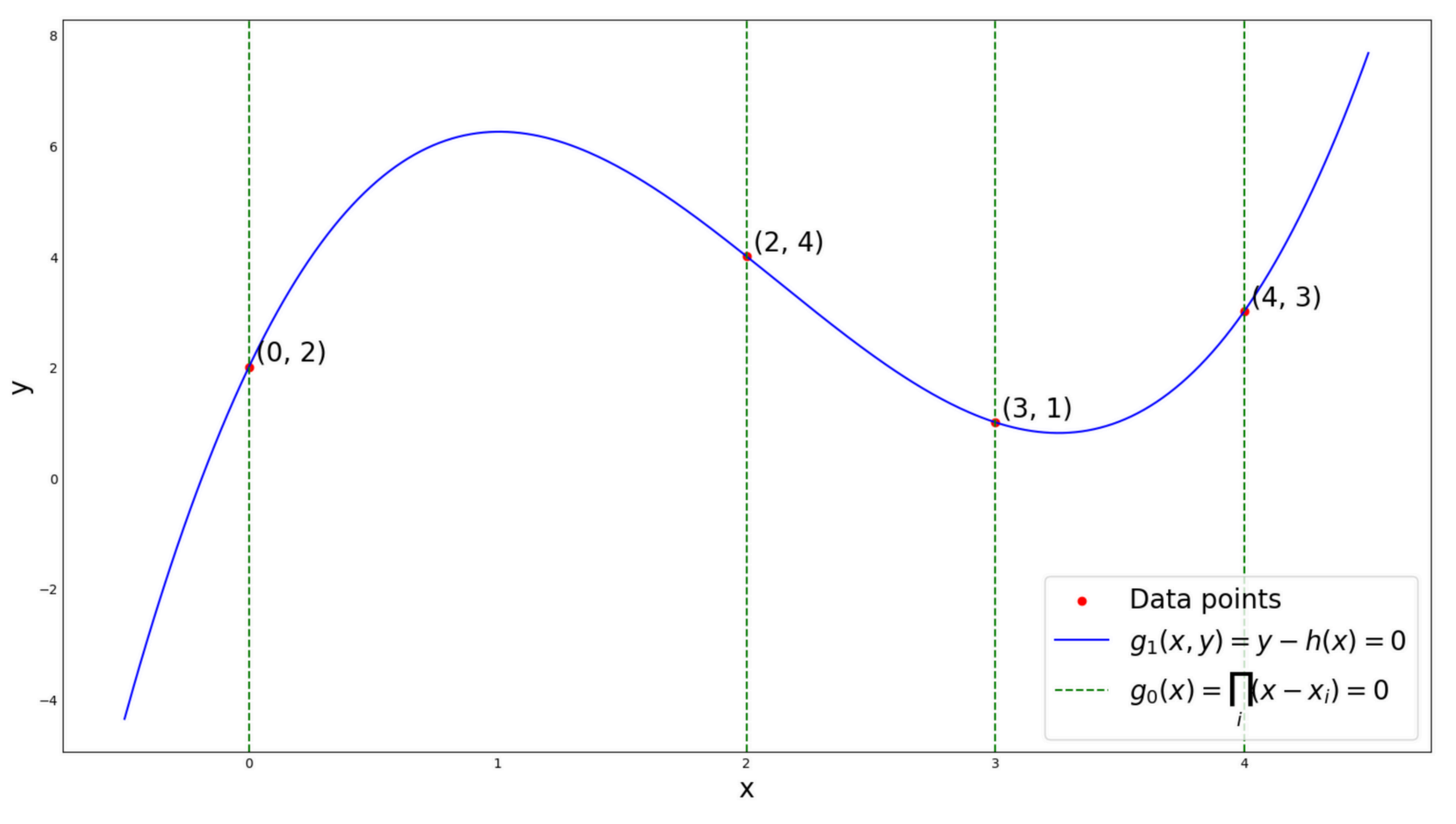
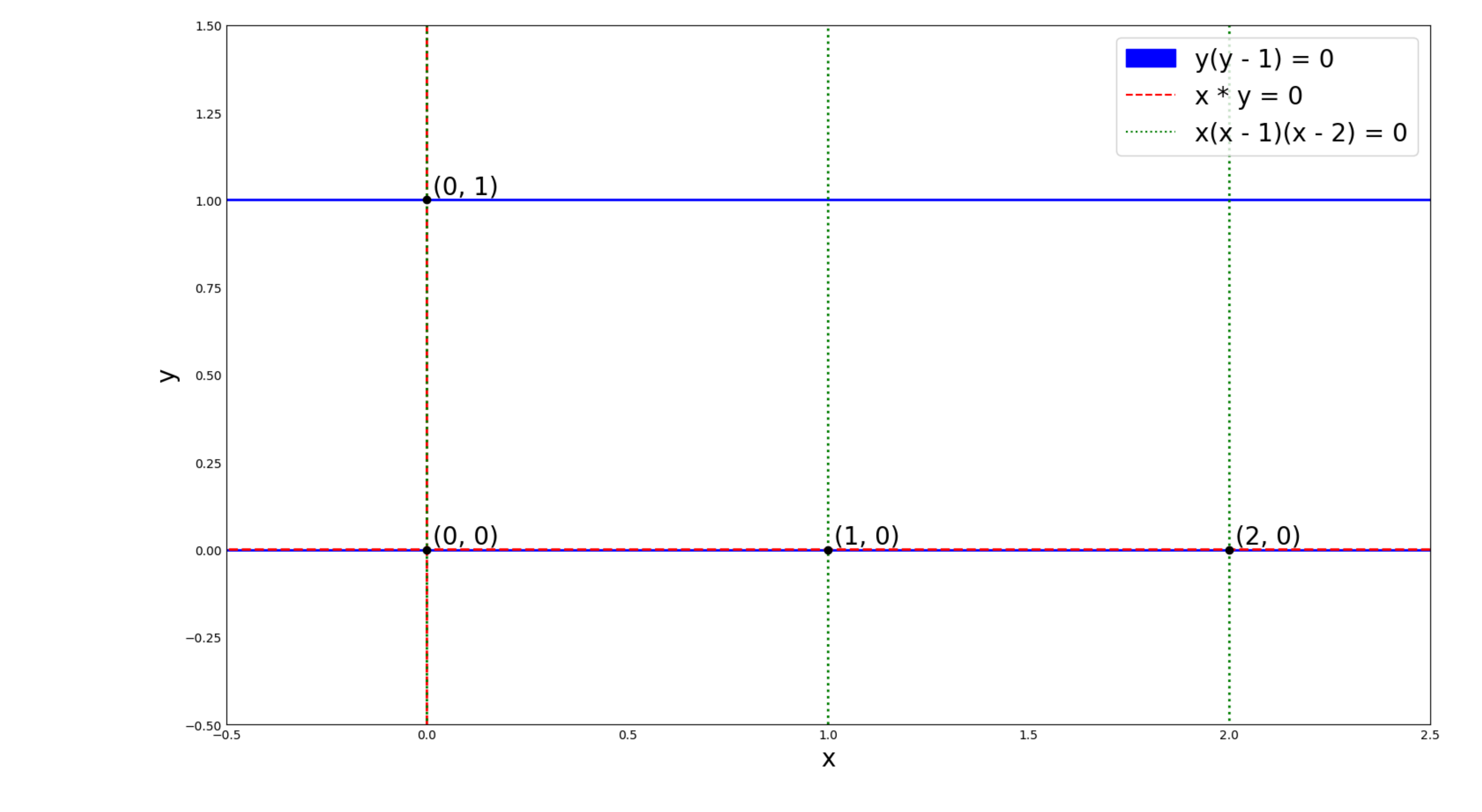


Illustration of g_0 and g_1



Points E and their vanishing ideal

VANDERMONDE METHOD

Two variable (x,y) (under general conditons)

$$p(x,y) = g(x)q(x)+(y-h(x))(x-x_n)q_1(x,y)+(y-h_1(x))(y-h_2(x))q_2(x,y).$$

$$[W(x,y) = \begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_0^n & y_0\,x_0^0 & y_0\,x_0^1 & \cdots & y_0\,x_0^n & \cdots & y_0^n\,x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n & y_1\,x_1^0 & y_1\,x_1^1 & \cdots & y_1\,x_1^n & \cdots & y_1^n\,x_1^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n & y_n\,x_n^0 & y_n\,x_n^1 & \cdots & y_n\,x_n^n & \cdots & y_n^n\,x_n^n \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)^2}]$$

Example (3 points) :

$$(x_0,y_0) = (1,0), \quad (x_1,y_1) = (2,0), \quad (x_2,y_2) = (0,1).$$

$$W = \begin{pmatrix} x_0^0 & x_0^1 & x_0^2 & y_0x_0^0 & y_0x_0^1 & y_0x_0^2 & y_0^2x_0^0 & y_0^2x_0^1 & y_0^2x_0^2 \\ x_1^0 & x_1^1 & x_1^2 & y_1x_1^0 & y_1x_1^1 & y_1x_1^2 & y_1^2x_1^0 & y_1^2x_1^1 & y_1^2x_1^2 \\ x_2^0 & x_2^1 & x_2^2 & y_2x_2^0 & y_2x_2^1 & y_2x_2^2 & y_2^2x_2^0 & y_2^2x_2^1 & y_2^2x_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

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1. Let:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{pmatrix}.$$

2. Then:

$$W\mathbf{c} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Expand into individual equations:

$$\begin{cases} c_1 + c_2 + c_3 = 0, \\ c_1 + 2c_2 + 4c_3 = 0, \\ c_1 + c_4 + c_7 = 0. \end{cases}$$

4. find the polynomials:

The vanishing ideal of p(x,y) is:

$$\{\, y^2 - y,\ xy,\ x^2 - 3x - 2y + 2\}$$

OPTIMIZATIONS

Example:

$$x^3 + a_{20}x^2 + a_{10}x + a_{00} = 0$$
$$xy + b_{01}y + b_{20}x^2 + b_{10}x + b_{00} = 0$$
$$y^4 + c_{03}y^3 + c_{02}y^2 + c_{01}y + c_{20}x^2 + c_{10}x + c_{00} = 0$$

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cancel{x_0^3} \\ 1 & x_1 & x_1^2 & \cancel{x_1^3} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cancel{x_n^3} \end{pmatrix}}_{\text{(block 1)}} \quad \underbrace{\begin{pmatrix} y_0 & \cancel{y_0 x_0} & \cancel{y_0 x_0^2} \\ y_1 & \cancel{y_1 x_1} & \cancel{y_1 x_1^2} \\ \vdots & \vdots & \vdots \\ y_n & \cancel{y_n x_n} & \cancel{y_n x_n^2} \end{pmatrix}}_{\text{(block 2)}} \quad \underbrace{\begin{pmatrix} y_0^2 & \cdots & y_0^3 & \cdots & \cancel{y_0^4} \\ y_1^2 & \cdots & y_1^3 & \cdots & \cancel{y_1^4} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ y_n^2 & \cdots & y_n^3 & \cdots & \cancel{y_n^4} \end{pmatrix}}_{\text{(block 3)}}$$

Algorithm:

- **Global record**
  - $L \leftarrow \emptyset$: the set of leading monomials we have found.
    Use L to skip monomials that are already covered.

- **Monomial sets**
  - $M = \{x^i y^j \mid i + j \leq d\}$: all monomials of total degree at most d.
  - $M_{\text{std}} = \{\, m \in M \mid \nexists \ell \in L : \ \ell \mid m\}$: Removing from M any monomial divisible by L.

In the optimization process, if an elemen like x^2y can be replaced by a previously known expression, then we can remove that element.

OPTIMIZATIONS EXAMPLE:

$$E = \{(0,0),\ (1,0),\ (2,0),\ (0,1)\}, \qquad \text{variables } x, y.$$

Degree 0

$$M_0 = \{1\}.$$

$$M_{\text{std}} = M_0,$$

- **Evaluation matrix**

$$V_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Degree 1

$$M_1 = \{1,\ x,\ y\}.$$

$$M_{\text{std}} = \{1,\ x,\ y\},$$

- **Evaluation matrix** (4×3)

$$V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Degree 2

$$M_2 = \{1,\ x,\ y,\ x^2,\ xy,\ y^2\}.$$

$$M_{\text{std}} = M_2,$$

- **Evaluation matrix** (4×6; columns ordered $1, x, y, x^2, xy, y^2$):

A basis of $\ker V_2$ gives two polynomials

$$y - y^2, \qquad xy.$$

- **Leading terms** (lex order $x > y$): $\ell_1 = xy,\ \ell_2 = y^2$. Record

$$L = \{xy,\ y^2\}.$$

OPTIMIZATIONS EXAMPLE:

Degree 3

- **All monomials**

$$M_3 = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$$

- **Standard monomials** Remove any monomial divisible by xy or y^2:

$$M_{\mathrm{std}} = \{1,\ x,\ y,\ x^2,\ x^3\}.$$

- **Evaluation matrix** $(4 \times 5$; columns $1, x, y, x^2, x^3)$:

$$V_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 4 & 8 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Taking $c_4 = 1$ yields the (unique) polynomial

$$x^3 - 3x^2 + 2x = x(x-1)(x-2).$$

- **Leading term** $\ell_3 = x^3$. Update

$$L = \{xy,\ y^2,\ x^3\}.$$

Degree 4

- **All monomials** number 15. After removing those divisible by xy, y^2 or x^3:

$$M_{\mathrm{std}} = \{1,\ x,\ y,\ x^2\}.$$

- **Evaluation matrix** $(4 \times 4$; columns $1, x, y, x^2)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 4 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

- The determinant is non-zero, so $\ker = 0$.

Final set of generators

$$\boxed{\{\, xy,\ y - y^2,\ x^3 - 3x^2 + 2x \,\}} = \{\, xy,\ y(1-y),\ x(x-1)(x-2) \}.$$

EXTENSION TO N>=3 VARIABLES

Three variable (x,y,z) (no-distinct under general conditons)

Given nonnegative integer d, consider all monomials:

$$\{\, x^i y^j z^k : i, j, k \geq 0,\ i + j + k \leq d\}$$

$$N(d) \ = \ \binom{d+3}{3} \ = \ \frac{(d+1)(d+2)(d+3)}{6}$$

Hence we have the matrix:

$$W(x,y,z) \ = \ \begin{pmatrix} x_0^0 y_0^0 z_0^0 & \cdots & x_0^n y_0^0 z_0^0 & \cdots & x_0^0 y_0^n z_0^0 & \cdots & x_0^0 y_0^0 z_0^n & \cdots & x_0^n y_0^n z_0^n \\ x_1^0 y_1^0 z_1^0 & \cdots & x_1^n y_1^0 z_1^0 & \cdots & x_1^0 y_1^n z_1^0 & \cdots & x_1^0 y_1^0 z_1^n & \cdots & x_1^n y_1^n z_1^n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ x_n^0 y_n^0 z_n^0 & \cdots & x_n^n y_n^0 z_n^0 & \cdots & x_n^0 y_n^n z_n^0 & \cdots & x_n^0 y_n^0 z_n^n & \cdots & x_n^n y_n^n z_n^n \end{pmatrix} \ \in \ \mathbb{R}^{(n+1)\times\binom{n+3}{3}}.$$

Similarly, we can extend it to n variables . . .

COMPLEXITY

Method	Evaluation matrix size	Total time complexity
Naïve enumeration	$n \times \binom{D+2}{2} \left(\approx n \cdot \frac{D^2}{2}\right)$	(set $D = n$) $\mathbf{O}(n^5)$
Filtered “standard monomial” method (2-D)	$\leq n \times n$	$\mathbf{O}(n^3)$
Filtered “standard monomial” method (3-D, `xyz` script)	$\leq n \times n$	$\mathbf{O}(n^3)$

CONCLUSION

- Theoretical Contributions
- Algorithm & Implementation
- Extensions & Verification
- Limitations & Future Work

DEMO

THANK'S FOR WATCHING