Killing vector & Isometry

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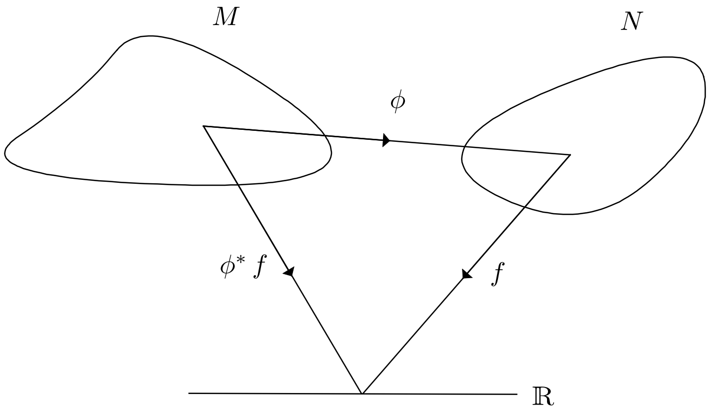
Might be easier to work with active viewpoint?

1 Pull back & push forward

For the coordinate transformation $x^\mu \rightarrow x^\mu + \xi^\mu$, the passive viewpoint suggests that the points on the manifold did not move, while the coordinate frame moved, therefore the components of the points changed . From the active viewpoint, it is simply the diffeomorphism $\phi: M \rightarrow M$, which means we map the old points to new points.



In general case, if a function f is defined on N, one can naturally define a function on M by $\phi^* f \equiv f \circ \phi$, which is called pull back. Since vectors eat functions, if a vector field $\boldsymbol{v}$ is defined on M, which means it eats $\phi^* f$, one can naturally define a vector field on N by $(\phi_* \boldsymbol{v})(f) \equiv \boldsymbol{v}(\phi^* f)$, which is called push forward. In other words, when $\boldsymbol{v}$ eats $\phi^* f$, it implicitly eats f, so there must be a thing eating f (vector field on N), which we call $\phi_* \boldsymbol{v}$. Similarly, dual vectors eat vectors, we can therefore pull back dual vectors based on the same arguement.

The above means that the diffeomorphism ϕ not only moves the points, but also moves the tensors associated with the points, i.e. before performing the diffeomorphism, the metric tensor at point p is given by $g_{\mu\nu}|_p$, while after the diffeomorphism it becomes $\phi^* g_{\mu\nu}|_p$. This pull back/push forward map allows us to see how much the tensor field changes via Lie derivative.

2 One-parameter group of diffeomorphism

This concept is needed to define Lie derivative. Consider the simple coordinate transformation $x \mapsto x + a$, $y \mapsto y$, and $z \mapsto z$, or equivalently a diffeomorphism $\phi_a: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ controled by the displacement parameter a. In fact, the collection of all such diffeomorphisms forms a group since one can define the group multiplication

$$\phi_a \phi_b \equiv \phi_{a+b}$$

with identity ϕ_0 and inverse $\phi_a^{-1} \equiv \phi_{-a}$, thus we call it a one-paramter group of diffeomorphisms. We can deduce that $\phi: \mathbb{R} \times M \rightarrow M$, since the parameter must be specified to define the map ϕ_a: $M \rightarrow M$. Or, we can sepecify the point p on M so that now $\phi_p: \mathbb{R} \rightarrow M$ is a curve $\phi_p(a)$ with parameter a on M passing through point p, which ultimately gives a tangent vector $\boldsymbol{v}$ (to the curve) at p.

The above means that the one-parameter group of diffeomorphisms gives a vector field $\boldsymbol{v}$, and the vector field $\boldsymbol{v}$ gives a one-parameter group of diffeomorphisms on M.

3 Lie derivative

Using the fact that a (smooth) vector field $\boldsymbol{v}$ gives a one-parameter group of diffeomorphisms ϕ, with the diffeomorphism ϕ_t controled by the paramter t, we can define the Lie derivative of tensor field T along vector field $\boldsymbol{v}$ by

$$\mathfrak{L}_{\boldsymbol{v}} T \equiv \lim_{t \rightarrow 0} \frac{(\phi_t^* T - T)}{t}$$

which measures how much the tensor field T changes under the diffeomorphism ϕ given by the vector field $\boldsymbol{v}$.

4 Killing vector field

Among many possible diffeomorphisms $\phi_t: M \rightarrow M$, there is one special type of them which preserves the metric tensor $g_{\mu\nu}$, called isometry. As mentioned in section 2, each group of diffeomorphisms is given by a vector field $\boldsymbol{v}$, so the vector field corresponding to the group of isometry is called the Killing vector field $\boldsymbol{\xi}$. Formally speaking, isometry means $\phi_t^* g_{\mu\nu} = g_{\mu\nu}$, which is also equivalent to $\mathfrak{L}_{\boldsymbol{\xi}} g_{\mu\nu} = 0$ (simply saying that the metric doesn't change), leading to the so called Killing equation

$$\begin{aligned} \mathfrak{L}_{\boldsymbol{\xi}} g_{ab} &= \xi^c \nabla_c g_{ab} + g_{cb} \nabla_a \xi^c + g_{ac} \nabla_b \xi^c \\ &= \nabla_a \xi^b + \nabla_b \xi^a \\ &= 0 \end{aligned}$$

where the first line follows from the definition of the Lie derivative of $(0, 2)$ tensor, and the first term vanishes due to the compatability condition of the covariant derivative, i.e. $\nabla_c g_{ab} = 0$ is required to properly define parallel tranportation.