What is it really? I mean $d\,s^2 = g_{ab}\,d\,x^a\,d\,x^b \ldots$

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Please do let me know if you see anything that doesn't look right

1 What causes the confusion or you didn't notice that at all

It seems that this equation is totally fine, it is nothing more than the infinitesimal change of length when you change the variable by a little bit, where "a little bit" is referring to $d\,x^a$. However, this $d\,x^a$,which we are all very familiar with since high school time, is causing serious problems in understanding the meaning of $d\,s^2$!

1.1 better to use Δx instead of $d\,x$

In fact, the symbol $d\,x$ is considered as a dual vector/differential form/1-form in differential geometry, which is a tensor but not a scalar. What we are actually looking for is the thing called Δx, a infinitesimal scalar/a small real number, i.e. $\Delta x = 0.000001$ means our x-coordinate has changed by 0.000001 units. Hence, the below expression should make sense

$$(\Delta\,s)^2 = g_{ij}\,\Delta\,x^i\,\Delta\,x^j$$

where the LHS is clearly a scalar/number, so is the RHS. By far, I guess you might agree with the above discussion, but you may not get the point of doing so, I promise the point will be clear to you very shortly.

1.2 what's wrong with replacing Δx by $d\,x$

We all learnt something like takling the limit of $\frac{\Delta\,x}{\Delta\,t}$, then it becomes a derivative like this $\frac{d\,x}{d\,t}$, it is nothing but using another symbol. But, please do try to distinguish the difference between the scalar $\Delta\,x$ and the dual vector/1-fomr $d\,x$ from now! Let's now consider a function f/physicists call it a scalar field defined on a manifold M, and we will come back to $d\,x$ later coz they are the same thing/functions. Formally speaking, the dual vector $d\,f$ is a map from vector space to real numbers,

$$d\,f\colon V \longrightarrow \mathbb{R}$$

where V is the vector space attached to the manifold at every single point, remember that we call it tangent space. First, you will agree with me that this definition make perfect sense, because $d\,f$ is kind of like the change in f when we move from one point on the manifold to another point. While, I mean there are as many different directions as you want in order to move to a new place, and each direction/vector corresponds to a particular number/the change in our function f in that direction. Hence, you provide a direction/vector and get a number, which is exactly how $d\,f$ is defined. In case you are interested in what number you get after specifying the direction of movement, here you go

$$d\,f|_p(v) := v(f), \qquad \forall v \in V_p$$

This says that at a point p on M, if we move along v which is a vector at p, we will observe that the function f changes by $v(f)=v^i\frac{\partial}{\partial x^i}(f)$. Again, I should say that v^i is called a vector sometimes in our lecture notes, which is not true, please bear in mind that $v^i\frac{\partial}{\partial x^i}$ is the actually vector in this case! Well... I think it is good to define vector as well, of course not by how it transforms but by what it is

$$\frac{\partial}{\partial x^i}\colon \text{function on } M \longrightarrow \text{function on } M$$

forgive me for the informal language used here, a bit lazy to type it out explicitly. So, in differential geometry, vectors are in fact maps, i.e. you feed a function to a vector, then you get a new function. By far, I believe you might get a feel of what actually $d f$ and $\frac{\partial}{\partial x}$ are, they are maps, that's it. Therefore, come back to $d x$, there is nothing more I should say about $d x$ because the coordinates of a point on M are simply functions which are called x^i. Hence, it makes no sense to say $d x = 0.00001$, it is a dual vector/a map from vector to real number, what does that mean if I tell you the map $d x$ is equal to 0.00001, it is non-sense.

1.3 Now you can see the problem with line element

With the above discussion in mind, the expression for $d s^2$ should be more confusing to you now, but again will be clear later

$$d s^2 = g_{ij}\, d x^i\, d x^j$$

The LHS again is a scalar, the RHS is a coefficien g_{ij} multiplied by two dual vectors $d x^i$ and $d x^j$. So the equation does not make sense at all since we put different objects on both sides, one side with real number, another side with dual vectors. That is why I think it is better to write

$$(\Delta s)^2 = g_{ij}\, \Delta x^i\, \Delta x^j$$

to remind us that we are working with real numbers Δx instead of dual vectors $d x$. So, I believe the expression in terms of Δ is what we really mean I guess...

2 interestingly $d s^2 = g$ if we do maths rigorous enough!

It turns out that the line element $d s^2$ is just another name of our metric tensor g from a differnetial geometric point of view. The following section will explain why it is so. Altough I mentioned that Δx and $d x$ should be considered different, the $d x$ still means something similar and should make certain sense.

2.1 what happens if we replace Δx by $d x/2.0$ version

Now, lets start with a vector $\boldsymbol{x} = \nu^i\frac{\partial}{\partial x^i}$, where ν^i is the expansion coeficient and $\frac{\partial}{\partial x^i}$ is the basis vector. The idea is that, we could first find out the infinitesimal displacement vector $d\boldsymbol{x}$, and then use the inner product $\langle , \rangle$ defined by metric tensor to measure the lenght squred of $d\boldsymbol{x}$, which is also known as $d s^2$. Originally, our displacement vector is a column vector as below

$$d\boldsymbol{x} = \begin{pmatrix} d x^1 \\ d x^2 \\ \vdots \\ d x^i \\ \vdots \end{pmatrix}$$

what we then did is to treat the entries of the vector as numbers resulting in a non-sense express. Let now do it carefully, we could still consider $d\,x^i$ as sort of "coefficients" and write out the basis expansion for $d\,\boldsymbol{x}$

$$d\,\boldsymbol{x} = \frac{\partial}{\partial x^i} \otimes d\,x^i$$

In our language, this guy $d\,\boldsymbol{x}$ is a mixed tensor/a vector-valued 1-form. This indeed makes sense, if we choose the diretion of movement to be $\boldsymbol{v} = v^j\,\frac{\partial}{\partial x^j}$, then the change in the vector $\boldsymbol{x}$ along $\boldsymbol{v}$ is given by

$$\begin{aligned}
d\,\boldsymbol{x}(\boldsymbol{v}) &= \frac{\partial}{\partial x^i} \otimes d\,x^i(\boldsymbol{v}) \\
&= v^j\,\frac{\partial}{\partial x^i} \otimes d\,x^i\!\left(\frac{\partial}{\partial x^j}\right) \\
&= v^j\,\frac{\partial}{\partial x^i}\,\frac{\partial x^i}{\partial x^j} \\
&= v^j\,\frac{\partial}{\partial x^i}\,\delta^i_j \\
&= v^j\,\frac{\partial}{\partial x^j} \\
&= \boldsymbol{v}
\end{aligned}$$

surely, if we start with a vector $\boldsymbol{x}$ then move along $\boldsymbol{v}$, the change in $\boldsymbol{x}$ is just $\boldsymbol{v}$, because it is what we've done. Right, we could now use inner product defined by g to calculate the length squred

$$\begin{aligned}
d\,s^2 &= \langle d\,\boldsymbol{x}, d\,\boldsymbol{x} \rangle \\
&= \left\langle \frac{\partial}{\partial x^i} \otimes d\,x^i, \frac{\partial}{\partial x^j} \otimes d\,x^j \right\rangle \\
&= g_{ij}\,d\,x^i \otimes d\,x^j
\end{aligned}$$

the last step follows from the fact that the metric tensor/(0,2) tensor will only contract with the vector part of the $d\,\boldsymbol{x}$, and also by definition

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

Therefore, what we end up with is a tensor equation

$$\begin{aligned}
d\,s^2 &= g_{ij}\,d\,x^i \otimes d\,x^j \\
d\,s \otimes d\,s &= g_{ij}\,d\,x^i \otimes d\,x^j
\end{aligned}$$

where both sides are $(0,2)$ tensors and g_{ij} is the component of metric tensor.

2.2 now you can see $d\,s^2 = g$

In face, the above equation is nothing but the basis expansion of the metric tensor g, the set of tensors $\{d\,x^i \otimes d\,x^j\}$ forms a basis for $(0,2)$ tensor product space, which means any $(0,2)$ tensor can be represented by the linear combination of such basis

$$g = g_{ij}\,d\,x^i \otimes d\,x^j$$

therefore, we identify that $d\,s^2$ is just another name for metric tensor in this language.