Linear Algebra for Chemists — Assignment 5

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Question 1. Write the system of equations in matrix form:

$$A\vec{c}=0 \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Perform Gaussian elimination on $A\in\mathbb{R}^{n\times n}$ to check if $\operatorname{rank}(A)=n$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1\to R_1-R_2} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_4\to R_4-R_3]{\substack{R_1\leftrightarrow R_2 \\ R_3\to R_3-R_2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\operatorname{rank}(A)=3<n=4$, which means that one vector is a linear combination of the others. The system is linearly dependent.

Question 2. In the vector space of vectors of length 2 with entries of real-valued functions,

a) $v_1=(\mathrm{e}^{-t},2\,\mathrm{e}^{-t}), v_2=(\mathrm{e}^{-t},\mathrm{e}^{-t}), v_3=(3\mathrm{e}^{-t},0)$.

Perform Gaussian elimination on the system

$$\begin{bmatrix} \mathrm{e}^{-t} & 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} & \mathrm{e}^{-t} \\ 3\,\mathrm{e}^{-t} & 0 \end{bmatrix} \xrightarrow{\substack{R_2-R_2-R_1 \\ R_3\to R_3-3R_1}} \begin{bmatrix} \mathrm{e}^{-t} & 2\,\mathrm{e}^{-t} \\ 0 & -\mathrm{e}^{-t} \\ 0 & -6\,\mathrm{e}^{-t} \end{bmatrix} \xrightarrow{R_3-6R_2} \begin{bmatrix} \mathrm{e}^{-t} & 2\,\mathrm{e}^{-t} \\ 0 & -\mathrm{e}^{-t} \\ 0 & 0 \end{bmatrix}$$

The system is linearly dependent. (We could have also claimed this based on the fact that the dimension of $\mathbb{F}^n$ is n.)

b) $v_1=(2\sin t,\sin t), v_2=(\sin t,2\sin t)$. Perform Gaussian elimination on the system

$$\begin{bmatrix} 2\sin t & \sin t \\ \sin t & 2\sin t \end{bmatrix} \xrightarrow{R_2\to R_2-\frac{1}{2}R_2} \begin{bmatrix} 2\sin t & \sin t \\ 0 & \frac{3}{2}\sin t \end{bmatrix}.$$

The rank of the coefficient matrix equals its number of columns.

The system is linearly independent, because there are no c_1,c_2 such that $c_1\,v_1(t)+c_2\,v_2(t)=\vec{0}$ for all t.

c) $v_1=(\mathrm{e}^t,t\,\mathrm{e}^t), v_2=(1,t)$. We can see that, for $c_1=\mathrm{e}^t, c_2=-1$

$$c_1\,v_1+c_2\,v_2=\vec{0}$$

These vectors are not linearly dependent, as there are no *constants* c_1,c_2 such that

$c_1\,v_1 + c_2\,v_2 = 0$ *for all* t; there is a different c_1 for each t.

Question 3. Given $t_0 \in \mathbb{R}$ as scalar and $v_1 = (\mathrm{e}^{t_0}, t_0\,\mathrm{e}^{t_0})$, $v_2 = (1, t_0)$ two vectors in $\mathbb{R}^2$, we show that for $c_1 = \mathrm{e}^{-t_0}$, $c_2 = -1$, the condition for linear independence,

$$c_1\,v_1 + c_2\,v_2 = \vec{0},$$

is satisfied.

$$\begin{aligned} c_1\,v_1 + c_2\,v_2 &= \mathrm{e}^{-t_0}\,(\mathrm{e}^{t_0}, t_0\,\mathrm{e}^{t_0}) - (1, t_0) \\ &= (\mathrm{e}^{-t_0}\,\mathrm{e}^{t_0}, t_0\,\mathrm{e}^{-t_0}\,\mathrm{e}^{t_0}) - (1, t_0) \\ &= (1, t_0) - (1, t_0) = (0, 0)\,. \end{aligned}$$

Question 4. A basis for a space is a sequence of vectors having two properties at once:

1. The vectors are linearly independent.
2. They span the space.

a) U is the set of square $n \times n$ real symmetric matrices.

Let $E_{ij} \in \mathbb{R}^{n\times n}$ be a matrix such that its ij^{th} entry is one and all other entries are zero, and let A be a symmetric matrix, such that $a_{ij} = a_{ji}$.

Proposed basis for U is the set $S = \{E_{ii} | i = 1, \ldots, n\} \cup \{E_{ij} + E_{ji} | i, j = 1, \ldots, n \text{ and } i < j\}$. We first show that the basis spans the set. A can be described completely via the upper triangular entries ($i \leq j$):

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix},$$

because of its symmetry. Note that there are $\frac{n\,(n+1)}{2}$ distinct elements in a symmetric matrix (n from the diagonal and $(n^2 - n)/2$ above the diagonal), which means that the basis dimension shall be $\frac{n\,(n+1)}{2}$.

A can be decomposed into the following:

$$A = \sum_{i=1}^{n} a_{ii}\,E_{ii} + \sum_{\substack{i,j=1 \\ i<j}}^{n} a_{ij}\,(E_{ij} + E_{ji}).$$

$A \in U$ is a linear combination of all vectors in S, so it is a spanning set. Because the number of vectors in S matches the basis dimension, S is also linearly independent.

b) V is the set of square $n \times n$ real matrices whose rows add up to zero.

A general matrix $A^{n\times n} \in V$ has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & -\sum_{j=1}^{n-1} a_{1j} \\ a_{21} & a_{22} & \cdots & -\sum_{j=1}^{n-1} a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & -\sum_{j=1}^{n-1} a_{nj} \end{bmatrix}.$$

We can see that all elements in the n^{th} column are linearly dependent on the elements in their respective rows. The basis dimension is therefore n^2-n (total number of elements in a square matrix minus number of elements in a row).

Take out the scalars to get a spanning set, and check linear independence to get a basis.

$$\begin{aligned}
A &= a_{11}\begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{1(n-1)}\begin{bmatrix} 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&+ a_{21}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{2(n-1)}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&\vdots \\
&+ a_{n1}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{bmatrix} + a_{n2}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & -1 \end{bmatrix} + \cdots + a_{n(n-1)}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -1 \end{bmatrix}
\end{aligned}$$

Let $B_{ij} \in \mathbb{R}^{n\times n}$ be a matrix shose ij^{th} entry is 1 and its in^{th} entry is -1. A basis for V is

$$S = \{B_{ij} | i = 1, \ldots, n \text{ and } j = 1, \ldots, n-1\}.$$

As shown above, S is a spanning set: $A = \sum_{i=1}^{n} \sum_{j=1}^{n-1} B_{ij}$. The set S also contains n^2-n vectors, so it is linearly independent.

c) W is the set of real polynomial functions.

Every polynomial can be written as the sum:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \qquad a_i \in \mathbb{R}, \quad n \in \mathbb{N},$$

which is spanned by the monomial set:

$$S = \{1, x, x^2, \ldots, x^n\},$$

whose dimension is $n+1$. The set is linearly independent: each monomial can be expressed as a vector of order $n+1$; for example

$$\begin{aligned} 1 &\leftrightarrow [1, 0, 0, \ldots, 0] \\ x &\leftrightarrow [0, 1, 0, \ldots, 0] \\ x^2 &\leftrightarrow [0, 0, 1, \ldots, 0] \\ x^n &\leftrightarrow [0, 0, 0, \ldots, 1]. \end{aligned}$$

To check for linear independence, form a matrix from the vectors to get

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The resulting matrix is in canonical form and has a full rank, which means that the set is linearly independent. The set spans the space and is linearly independent. It is therefore a basis.

d) $\mathbb{C}^n$ as a vector space over $\mathbb{R}$.

Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Thet set $S = \{e_1, \mathrm{i}\, e_1, e_2, \mathrm{i}\, e_2, \dots, e_n, \mathrm{i}\, e_n\}$ is a basis for the vector space. A generic vector in $\mathbb{C}^n$ is $(a_1 + \mathrm{i}\, b_1, a_2 + \mathrm{i}\, b_2, \dots, a_n + \mathrm{i}\, b_n)$ where $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, n$. The aforementioned vector can be written as a linear combination of the vectors in S:

$$\begin{bmatrix} a_1 + \mathrm{i}\, b_1 \\ a_2 + \mathrm{i}\, b_2 \\ \vdots \\ a_n + \mathrm{i}\, b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \mathrm{i} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1\, e_1 + a_2\, e_2 + \cdots + a_n\, e_n + b_1\, \mathrm{i}\, e_1 + b_2\, \mathrm{i}\, e_2 + \cdots + b_n\, \mathrm{i}\, e_n\,.$$

so S is a spanning set. The vectors in S are also linearly independent. Put the vectors in S as row vectors of a matrix and perform Gaussian elimination:

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The matrix is in canonical form so its rows are linearly independent. Furthermore, for each $j = 1, \dots, n$, e_j and $\mathrm{i}\, e_j$ are linearly independent over $\mathbb{R}$. The set S forms a basis.

Question 5.

a) The subspace of upper triangualar 3×3 real matrices.

A generic matrix in the space is

$$A = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}.$$

A can be written as the sum

$$A = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is a spanning set. Additionally, the vectors in the set are independent. Flatten each matrix to a $\mathbb{R}^9$ row vector, and perform Gaussian elimination on the matrix built from the vectors.

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \end{bmatrix} \xrightarrow{\text{swapping of rows}} \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

The matrix in row echelon form has no rows of all zeros, so its rows are independent. The set S is a basis, and its dimension is 6.

b) The subspace of real 2×2 matrices in which the sum of the elements on the main diagonal is zero.

A generic vector in the subspace is

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

A can be written as the sum

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is a spanning set. The vectors in S are also linearly independent. Flatten the matrices to $\mathbb{R}^4$ row vectors, and perform Gaussian elimination on the matrix built from the vectors.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix is in row echelon form and has no zero rows. Its rows are therefore linearly independent. The set S is a basis, and its dimension is 3.

Question 6. The system can be expressed in matrix form, where $\vec{x} \in \mathbb{R}^5$, $\vec{y} \in \mathbb{R}^3$, and $W \in R^{3\times 5}$:

$$W\vec{x} = \vec{y}\,.$$

a) The output, $\vec{y}$, is a linear combination of the columns of W weighted by the coordinates of $\vec{x}$. This means that $\vec{y}$ is spanned by the columns of W, so $\vec{y}$ resides in the **column space** of W.

b) We need to find a basis for the column space of W. Perform Gaussian elimination on W.

$$\begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 6 & 4 & 2 & -2 & -10 \end{bmatrix} \xrightarrow{R_3 \to 7R_3 - 6R_1} \begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 0 & 22 & -22 & -44 & -88 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 22R_2} \begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is in echelon form. We see that the 1st and 2nd column have leading pivots. These are therefore the independent columns of the matrix. A basis for the column space (or output space) is

$$\left\{ \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

c) The fundumental space of $W\vec{x} = \vec{0}$ is the **nullspace** of W, and its dimension is $n - r$. In this case, $n = 5$ and $r = 2$ (found from b.) The dimension of the nullspace of W is 3.