

Linear Algebra for Chemists — Assignment 5

BY YUVAL BERNARD

ID. 211860754

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Question 1. Write the system of equations in matrix form:

$$A\vec{c}=0 \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Perform Gaussian elimination on $A \in \mathbb{R}^{n \times n}$ to check if $\text{rank}(A) = n$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} R_1 \leftrightarrow R_2 \\ R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 - R_3 \end{matrix}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\text{rank}(A) = 3 < n = 4$, which means that one vector is a linear combination of the others. The system is linearly dependent.

Question 2. In the vector space of vectors of length 2 with entries of real-valued functions,

a) $v_1 = (e^{-t}, 2e^{-t}), v_2 = (e^{-t}, e^{-t}), v_3 = (3e^{-t}, 0).$

Check for linear independence by solving the system

$$e^{-t} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A nontrivial solution (c_1, c_2, c_3) should apply for every $t \in \mathbb{R}$. The system is linearly dependent as the system consists of 3 vectors in \mathbb{R}^2 , whose dimension is 2, and can only have at most two independent vectors.

b) $v_1 = (2 \sin t, \sin t), v_2 = (\sin t, 2 \sin t)$. Perform Gaussian elimination on the system

$$\sin t \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A nontrivial solution (c_1, c_2) should apply for every $t \in \mathbb{R}$. The rank of the coefficient matrix matches the number of columns, so the system is linearly independent.

c) $v_1 = (e^t, te^t), v_2 = (1, t)$. We can see that, for $c_1 = e^t, c_2 = -1$

$$c_1 v_1 + c_2 v_2 = \vec{0}$$

These vectors are not linearly dependent, as there are no *constants* c_1, c_2 such that

$c_1 v_1 + c_2 v_2 = 0$ for all t ; there is a different c_1 for each $t \in \mathbb{R}$.

Question 3. Given $t_0 \in \mathbb{R}$ as scalar and $v_1 = (e^{t_0}, t_0 e^{t_0})$, $v_2 = (1, t_0)$ two vectors in \mathbb{R}^2 , we show that for $c_1 = e^{-t_0}$, $c_2 = -1$, the condition for linear independence,

$$c_1 v_1 + c_2 v_2 = \vec{0},$$

is satisfied.

$$\begin{aligned} c_1 v_1 + c_2 v_2 &= e^{-t_0} (e^{t_0}, t_0 e^{t_0}) - (1, t_0) \\ &= (e^{-t_0} e^{t_0}, t_0 e^{-t_0} e^{t_0}) - (1, t_0) \\ &= (1, t_0) - (1, t_0) = (0, 0). \end{aligned}$$

Question 4. A basis for a space is a sequence of vectors having two properties at once:

1. The vectors are linearly independent.
 2. They span the space.
- a) U is the set of square $n \times n$ real symmetric matrices.

Let $E_{ij} \in \mathbb{R}^{n \times n}$ be a matrix such that its ij^{th} entry is one and all other entries are zero, and let A be a symmetric matrix, such that $a_{ij} = a_{ji}$.

Proposed basis for U is the set $S = \{E_{ii} | i = 1, \dots, n\} \cup \{E_{ij} + E_{ji} | i, j = 1, \dots, n \text{ and } i < j\}$. We first show that the basis spans the set. A can be described completely via the upper triangular entries ($i \leq j$):

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix},$$

because of its symmetry. Note that there are $\frac{n(n+1)}{2}$ distinguishable elements in a symmetric matrix (n on the diagonal and $(n^2 - n)/2$ above the diagonal), which means that the basis dimension shall be $\frac{n(n+1)}{2}$.

A can be decomposed into the following:

$$A = \sum_{i=1}^n a_{ii} E_{ii} + \sum_{\substack{i,j=1 \\ i < j}}^n a_{ij} (E_{ij} + E_{ji}).$$

$A \in U$ is a linear combination of all vectors in S , so S is a spanning set. Because the number of vectors in S matches the basis dimension, S is also linearly independent.

- b) V is the set of square $n \times n$ real matrices whose rows add up to zero.

A general matrix $A^{n \times n} \in V$ has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & -\sum_{j=1}^{n-1} a_{1j} \\ a_{21} & a_{22} & \cdots & -\sum_{j=1}^{n-1} a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & -\sum_{j=1}^{n-1} a_{nj} \end{bmatrix}.$$

We can see that all elements in the n^{th} column are linearly dependent on the elements in their respective rows. The basis dimension is therefore $n^2 - n$ (total number of elements in a square matrix minus number of elements in a row).

Take out the scalars to get a spanning set, and check linear independence to get a basis.

$$\begin{aligned}
A = & a_{11} \begin{bmatrix} 1 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{1(n-1)} \begin{bmatrix} 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
& + a_{21} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + a_{2(n-1)} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
& \vdots \\
& + a_{n1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{bmatrix} + a_{n2} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & -1 \end{bmatrix} + \cdots + a_{n(n-1)} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & -1 \end{bmatrix}
\end{aligned}$$

Let $B_{ij} \in \mathbb{R}^{n \times n}$ be a matrix whose ij^{th} entry is 1 and its in^{th} entry is -1 . A basis for V is

$$S = \{B_{ij} | i = 1, \dots, n \text{ and } j = 1, \dots, n-1\}.$$

As shown above, S is a spanning set: $A = \sum_{i=1}^n \sum_{j=1}^{n-1} B_{ij}$. The set S also contains $n^2 - n$ vectors, so it is linearly independent.

c) W is the set of real polynomial functions.

Every polynomial can be written as the sum:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad a_i \in \mathbb{R}, \quad n \in \mathbb{N},$$

which is spanned by the monomial set:

$$S = \{1, x, x^2, \dots, x^n\}.$$

The set is linearly independent: each monomial can be expressed as a vector of order $n+1$; for example

$$\begin{aligned}
1 & \leftrightarrow [1, 0, 0, \dots, 0] \\
x & \leftrightarrow [0, 1, 0, \dots, 0] \\
x^2 & \leftrightarrow [0, 0, 1, \dots, 0] \\
x^n & \leftrightarrow [0, 0, 0, \dots, 1].
\end{aligned}$$

To check for linear independence, form a matrix from the vectors to get

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The resulting matrix is in canonical form and has a full rank, which means that the set is linearly independent. The set spans the space and is linearly independent. It is therefore a basis.

d) \mathbb{C}^n as a vector space over \mathbb{R} .

Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The set $S = \{e_1, i e_1, e_2, i e_2, \dots, e_n, i e_n\}$ is a basis for the vector space. A generic vector in \mathbb{C}^n is $(a_1 + i b_1, a_2 + i b_2, \dots, a_n + i b_n)$ where $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, n$. The aforementioned vector can be written as a linear combination of the vectors in S :

$$\begin{bmatrix} a_1 + i b_1 \\ a_2 + i b_2 \\ \vdots \\ a_n + i b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 e_1 + a_2 e_2 + \dots + a_n e_n + b_1 i e_1 + b_2 i e_2 + \dots + b_n i e_n.$$

so S is a spanning set. The vectors in S are also linearly independent. Put the vectors in S as row vectors of a matrix and perform Gaussian elimination:

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The matrix is in canonical form so its rows are linearly independent. Furthermore, for each $j = 1, \dots, n$, e_j and $i e_j$ are linearly independent over \mathbb{R} . The set S forms a basis.

Question 5.

a) The subspace of upper triangular 3×3 real matrices.

A generic matrix in the space is

$$A = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}.$$

A can be written as the sum

$$A = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is a spanning set. Additionally, the vectors in the set are independent. Flatten each matrix to a \mathbb{R}^9 row vector, and perform Gaussian elimination on the matrix built from the vectors.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{swapping of rows}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix in row echelon form has no rows of all zeros, so its rows are independent. The set S is a basis, and its dimension is 6.

- b) The subspace of real 2×2 matrices in which the sum of the elements on the main diagonal is zero.

A generic vector in the subspace is

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

A can be written as the sum

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is a spanning set. The vectors in S are also linearly independent. Flatten the matrices to \mathbb{R}^4 row vectors, and perform Gaussian elimination on the matrix built from the vectors.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix is in row echelon form and has no zero rows. Its rows are therefore linearly independent. The set S is a basis, and its dimension is 3.

Question 6. The system can be expressed in matrix form, where $\vec{x} \in \mathbb{R}^5$, $\vec{y} \in \mathbb{R}^3$, and $W \in \mathbb{R}^{3 \times 5}$:

$$W\vec{x} = \vec{y}.$$

- a) The output, \vec{y} , is a linear combination of the columns of W weighted by the coordinates of \vec{x} . This means that \vec{y} is spanned by the columns of W , so \vec{y} resides in the **column space** of W .
- b) We need to find a basis for the column space of W . Perform Gaussian elimination on W .

$$\begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 6 & 4 & 2 & -2 & -10 \end{bmatrix} \xrightarrow{R_3 \rightarrow 7R_3 - 6R_1} \begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 0 & 22 & -22 & -44 & -88 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 22R_2} \begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is in echelon form. We see that the 1st and 2nd column have leading pivots. These are therefore the independent columns of the matrix. A basis for the column space (or output space) is

$$\left\{ \begin{bmatrix} 7 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

- c) The fundamental space of $W\vec{x}=\vec{0}$ is the **nullspace** of W , and its dimension is $n-r$. In this case, $n=5$ and $r=2$ (found from b.) The dimension of the nullspace of W is 3.