# Assignment 6

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## Page 178

Question 1

$$y'' - 2y' - 3y = 3e^{2t}$$

First solve the homogeneous equation. The characteristic polynomial equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_{1,2} = \frac{2 \pm 4}{2} = 3, -1$$

Which gives the solutions $e^{3t}, e^{-t}$. Now find a particular solution. Guess one of the form $y_1 = Ae^{2t}, A \in \mathbb{R}$.

$$\begin{aligned} y_1' &= 2Ae^{2t} \\ y_1'' &= 4Ae^{2t} \end{aligned}$$

Input in the DE.

$$4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t} \rightarrow A = -1$$

The general solution is $y = -e^{2t} + c_1 e^{3t} + c_2 e^{-t}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t.$

Question 6

$$y'' + 2y' + y = 2e^{-t}$$

First solve the homogeneous equation. The characteristic polynomial equation is

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

$\lambda = -1$ is of multiplicity 2. The solutions are $e^{-t}, t\,e^{-t}$. Guess a particular solution of the form $y_1 = A\,t^2\,e^{-t}, \quad A \in \mathbb{R}$.

$$\begin{aligned} y_1' &= 2A\,t\,e^{-t} - A\,t^2\,e^{-t} \\ y_1'' &= 2A\,e^{-t} - 4A\,t\,e^{-t} + A\,t^2\,e^{-t} \end{aligned}$$

Input back in the DE.

$$2A\,\mathrm{e}^{-t} - 4A\,t\,\mathrm{e}^{-t} + A\,t^2\,\mathrm{e}^{-t} + 4A\,t\,\mathrm{e}^{-t} - 2A\,t^2\,\mathrm{e}^{-t} + A\,t^2\,\mathrm{e}^{-t} = 2\mathrm{e}^{-t}$$

$$A = 1$$

The general solution is $y = t^2\,\mathrm{e}^{-t} + c_1\mathrm{e}^{-t} + c_2\,t\,\mathrm{e}^{-t}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t.$

Question 14

$$y'' + 4y = t^2 + 3\mathrm{e}^t, \quad \begin{cases} y(0) = 0 \\ y'(0) = 2 \end{cases}$$

Solve homogeneous counterpart. Characteristic polynomial equation:

$$\lambda^2 + 4 = 0 \rightarrow \lambda_{1,2} = \pm 2\mathrm{i}$$

Break the RHS into parts. To find the solution that suits the polynomial part, pick a particular solution of the form $y_1 = A\,t^2 + B\,t + C, \quad A, B, C \in \mathbb{R}$

$$\begin{aligned} y_1' &= 2A\,t + B \\ y_1'' &= 2A \end{aligned}$$

$$2A + 4A\,t^2 + 4B\,t + 4C = t^2$$

Equate coefficients on both sides. Get $A = \frac{1}{4}, B = 0, C = -\frac{1}{8}$. $y_1 = \frac{1}{4}t^2 - \frac{1}{8}$.

Now find a particular solution of the form $y_2 = \alpha\mathrm{e}^t$ that fits the exponential part of the RHS.

$$\begin{aligned} y_2' &= \alpha\mathrm{e}^t \\ y_2'' &= \alpha\mathrm{e}^t \end{aligned}$$

$$\alpha\mathrm{e}^t + 4\alpha\mathrm{e}^t = 3\mathrm{e}^t \rightarrow \alpha = \frac{3}{5}$$

The general solution is therefore $y = c_1\sin 2t + c_2\cos 2t + \frac{3}{5}\mathrm{e}^t + \frac{1}{4}t^2 - \frac{1}{8}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t.$

Find $c_1, c_2$ via ICs:

$$y(0) = c_2 + \frac{3}{5} - \frac{1}{8} = 0 \rightarrow c_2 = -\frac{19}{40}$$

$$y'(t) = 2c_1\cos 2t - 2c_2\sin 2t + \frac{3}{5}\mathrm{e}^t + \frac{1}{2}t$$

$$y'(0) = 2c_1 + \frac{3}{5} = 2 \rightarrow c_1 = \frac{7}{10}$$

The unique solution is

$$y = \frac{7}{10}\sin 2t - \frac{19}{40}\cos 2t + \frac{3}{5}\mathrm{e}^t + \frac{1}{4}t^2 - \frac{1}{8}, \quad \forall t$$

# Page 197

Question 6

Write equation of motion for the mass:

$$m\,u'' = m\,g - k\,(L + u)$$

Prior to stretching, the mass was at equilibrium and $u$ was equal to zero. At equilibrium,

$$m\,u'' = 0 = m\,g - k\,L \tag{1}$$

That simplifies the previous equation:

$$m\,u'' + k\,u = 0$$

As $m \neq 0$, we can divide by $m$ to obtain the general form of a linear 2nd order homogeneous ODE.

$$u'' + \frac{k}{m}\,u = 0$$

Before we solve, calculate $k$. According to eq. (1), $k = \frac{m\,g}{L}$. Set $g = 9.8\,\frac{m}{s^2}$, and get $k = 19.6\,\frac{N}{m}$. Input in the ODE:

$$u'' + 196u = 0$$

To solve, write the characteristic equation of the ODE:

$$\lambda^2 + 196 = 0$$

$$\lambda_{1,2} = 14\mathrm{i}$$

The general solution is therefore

$$u(t) = c_1\cos(14\,t) + c_2\sin(14\,t) \quad c_{1,2} \in \mathbb{R}, \quad t > 0$$

Use the ICs to find $c_1, c_2$. We are given $u'(0) = 0.1\,\frac{m}{\mathrm{sec}}$ and $u(0) = 0$.

Inserting in $u(t)$ $t = 0$ gives:

$$u(0) = c_1 = 0$$

Differentiate $u(t)$ to utilize the second IC.

$$u'(t) = 14c_2 \cos(14\,t)$$

$$u'(0) = 14c_2 = 0.1m \rightarrow c_2 = \frac{1}{140}$$

The general solution (in SI units) is

$$u(t) = \frac{1}{140} \sin(14\,t) \quad t > 0$$

The mass returns to its equilibrium position when $u(t) = 0$.

$$\frac{1}{140} \sin(14\,t) = 0$$

$$14t = \pi k, k = 1, 2, 3, \ldots$$

The smallest $k$ that satisfies $t > 0$ is $k = 1$. Thus, the mass first returns to equilibrium position after $\frac{\pi}{14}$ seconds.

Question 8

$C = 0.25\mu F$, $L = 1H$, $Q(0) = 1\mu C$, $I(0) = 0$. According to Kirchhoff's's law, The differential equation describing the system is:

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C} Q = 0$$

Or:

$$Q'' + \frac{1}{LC} Q = 0$$

Solve the characteristic polynomial equation:

$$\lambda^2 + \frac{1}{LC} = 0$$

$$\lambda = \pm \mathrm{i}\sqrt{\frac{1}{LC}} = \pm 2000\,\mathrm{i}$$

The general solution is

$$Q(t) = A \cos 2000t + B \sin 2000t, \quad A, B \in \mathbb{R}$$

Use the ICs $Q(0) = 1\mu C$ and $Q'(0) = 0$ to get $A = 10^{-6}$, $B = 0$. The final expression of the charge at time t is

$$Q(t) = 10^{-6} \cos 2000\,t, \quad t > 0$$

Question 11

First find $k$. We are given $F_s = k\,L = 3N$

$$k = \frac{3N}{0.1m} = 30N/m$$

$m = 2\text{kg}$. Second, find $\gamma$. We are given $|F_\gamma| = \gamma v$.

$$\gamma = \frac{F_\gamma}{v} = \frac{3N}{5\frac{m}{s}} = 0.6Ns/m$$

Given ICs: $u(0) = 0.05$ and $v(0) = 0.1$, find $u(t)$. The differential equation of motion (in absence of external force) is:

$$u'' + \frac{\gamma}{m}u' + \frac{k}{m}u = 0$$

$$u'' + 0.3u' + 15u = 0$$

Solve the characteristic equation:

$$\lambda^2 + 0.3\lambda + 15 = 0$$

$$\lambda = \frac{-0.3 \pm \sqrt{0.09 - 60}}{2} = -0.15 \pm 3.87\,\mathrm{i}$$

The general solution is

$$u(t) = A\,\mathrm{e}^{-0.15t}\cos 3.87t + B\mathrm{e}^{-0.15t}\sin 3.87t, \quad A, B \in \mathbb{R}$$

Input ICs:

$$u(0) = A = 0.05$$

$u'(t) = -0.15A\mathrm{e}^{-0.15t}\cos 3.87t - 3.87A\mathrm{e}^{-0.15t}\sin 3.87t - 0.15B\mathrm{e}^{-0.15t}\sin 3.87t + 3.87B\mathrm{e}^{-0.15t}\cos 3.87t$

$$u'(0) = -0.15A + 3.87B = 0.1 \rightarrow B = \frac{1}{36}$$

The unique solution is:

$$u(t) = 0.05\,\mathrm{e}^{-0.15t}\cos 3.87t + \frac{1}{36}\,\mathrm{e}^{-0.15t}\sin 3.87t, \quad t > 0$$

Move to polar coordinates:

$$R = \sqrt{A^2 + B^2} \approx 0.05719$$

$$\delta = \arctan\frac{B}{A} = 0.5071$$

Therefore,

$$u(t) = 0.05719\,\mathrm{e}^{-0.15t}\cos(3.87t - 0.5071), \quad t > 0$$

The quasi frequency $\mu$ is the frequency of the sinusoidal component of the displacement: $\mu \approx 3.87\,\mathrm{rad/sec}$. The ratio between $\mu$ and the natural frequency is:

$$\text{ratio} = \frac{\mu}{\sqrt{\frac{k}{m}}} \approx \frac{3.87}{\sqrt{\frac{30}{2}}} \approx 0.9992$$

Question 18

Critical damping is obtained when the discriminant of the characteristic (quadratic) equation equals zero. The differential equation representing the system is:

$$Q'' + \frac{R}{L}Q' + \frac{1}{L\,C}Q = 0$$

where $L = 0.2H$ and $C = 0.8\mu F$. The characteristic equation is:

$$\lambda^2 + 5R\,\lambda + 6.25 \cdot 10^6 = 0$$

Discriminant is:

$$25R^2 - 4 \cdot 6.25 \cdot 10^6 \stackrel{!}{=} 0$$

$$R = 1000\,\Omega$$

Question 24

Equation of motion is

$$u'' + \frac{2}{3}k\,u = 0$$

With ICs: $u(0) = 2, u'(0) = v$. We are also given $T = \pi\,\mathrm{sec}$, $R = 3m$.

We want to find the polar representation of $u(t)$. Solve the characteristic equation:

$$\lambda^2 + \frac{2k}{3} = 0$$

$$\lambda = \pm\sqrt{\frac{2}{3}k}\,\mathrm{i}$$

Solution is of the form:

$$u(t) = A\cos\sqrt{\frac{2}{3}k}\,t + B\sin\sqrt{\frac{2}{3}k}\,t$$

Input ICs:

$$u(0) = 2 = A$$

$$u'(t) = -\sqrt{\frac{2}{3}k}A\sin\left(\sqrt{\frac{2}{3}kt}\right) + \sqrt{\frac{2}{3}k}B\cos\left(\sqrt{\frac{2}{3}kt}\right)$$

$$u'(0) = v = \sqrt{\frac{2}{3}k}\, B \rightarrow B = \frac{v}{\sqrt{\frac{2}{3}k}}$$

Amplitude of $u(t)$ in polar coordinates:

$$R = \sqrt{A^2 + B^2} = \sqrt{4 + \frac{3v^2}{2k}}$$

Use information given:

$$\text{Amplitude} = R = 3 = \sqrt{4 + \frac{3v^2}{2k}}$$

$$5 = \frac{3v^2}{2k} \rightarrow v = \sqrt{\frac{10k}{3}}$$

$$\text{Period} = T = \frac{2\pi}{\sqrt{\frac{2}{3}k}} = \pi \rightarrow k = 6$$

$$v = \sqrt{\frac{10 * 6}{3}} = \pm 2\sqrt{5}$$