<u>Linear Algebra for Chemists</u>

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Assignments

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Assignments are due on Mondays. They should be sent in PDF files to Jan Kadlec before 09:00.

Please write the name of the course, and the assignment number in the title of the email, and include your own name in the filename.

Late assignments will not be accepted unless there are special circumstances, and permission has been given in advance.

Assignment 1

1. Determine whether the given set is a vector space in the following cases. If it is not, list the axioms that do not hold.

   a. The set of real polynomials of degree less or equal to n with positive constant term.
   b. The set of real polynomials of degree less or equal to n.
   c. The set of points in $\mathbb{R}^3$ lying on a plane passing through the origin.
   d. Show that the set of positive real numbers forms a vector space over the real numbers under the operations addition $\oplus$ and multiplication by scalars $\odot$ defined:

$$x \oplus y = xy, \quad \alpha \odot x = x^{\alpha},$$

   - where x, y are positive real numbers and α is real.

2. Check whether W is a subspace of V for the following cases:

   a. $V = \mathbb{R}^n$, W = set of vectors in V whose coordinates are nonnegative.
   b. $V = \mathbb{R}^n$, W = set of vectors in V whose coordinates add up to 1.
   c. V = space of square $n \times n$ matrices,
      W = set of symmetric matrices in V.
   d. V = space of square $n \times n$ matrices,
      W = set of matrices in V whose rows add up to zero.
   e. V = set of one-variabled real functions,
      W = set of polynomial functions of degree 2 in V.
   f. $V = \mathbb{C}^n$, as a vector space over $\mathbb{C}$, $W = \mathbb{R}^n$.
   g. $V = \mathbb{C}^n$, as a vector space over $\mathbb{R}$,$W = \mathbb{R}^n$.
   h. V = set of one-variabled real continuous functions,
      W = set of real differentiable functions.

3. Show that the zero element in a vector space is unique.

4. Show that the additive inverse of an element in a vector space is unique.

Assignment 2

1. Show that the span of a set of vectors in a vector space is a subspace.

2. Check if the following sets of vectors spans the subspace W of $\mathbb{R}^4$ where W is the set of vectors whose coordinates add up to 0:
   a. The set: (1, 0, 0, -1), (1, -1, 0, 0), (1, -1, 1, -1).
   b. The set: (1, 2, -4, 1), (0, 1, 1, -2)

3. Which of the following vectors is an element of the subspace W where
$$W = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\right\} \text{ in } \mathbb{R}^3: \qquad v = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 3 \\ 1 \\ -18 \end{pmatrix} \ ?$$

4. Is the function $\mathrm{Sin}x$ a linear combination of the functions $\mathrm{Cos}x$ and e^x in the space of continuous real functions over the real numbers? Prove or disprove.

Assignment 3

1. A linear neuron calculates $y = \vec{w} \cdot \vec{x}$, where $\vec{w}$ = (-3,4,7,1,2,-9) and x is a vector of electronic inputs. Calculate the output of the neuron on the following vectors of inputs:
   a. (9,21,3,6,-4,15) b. (2,0,0,-13,-18,6)

2. We define the following vectors and matrices:
3.

$$\vec{v} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix},\ \vec{u} = \begin{pmatrix} 9 \\ 3 \\ -4 \\ -2 \end{pmatrix}, A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 2 & 7 & -1 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate ("by hand") the following:

(i) Av	(ii) v^Tv	(iii) vv^T	(iv) $v^T Av$	(v) A^TA	(vi) $2AB$
(vii) AD	(viii) DA	(ix) $B^T A$	(x) $v^T Bu$	(xi) . $v^T Dv$	(xii) D^{-1}

3. **Definition**: A is symmetric if $A = A^T$
   Prove that for square matrices A and B:
   a. $(AB)^T = B^T A^T$

   and use it to prove the following:
   b. AA^T and $A + A^T$ are symmetric matrices.
   c. If A is invertible and symmetric then so is its inverse.
   d. If A and B are symmetric matrices then:
   AB is symmetric if and only if $AB = BA$.

4. Show that if A and B are square invertible matrices of the same order, then so is their product and we have: $(AB)^{-1} = B^{-1}A^{-1}$.

5. Use Gaussian elimination to determine the ranks of the following matrices:

$$\begin{pmatrix} 1 & -1 & 0 \\ -3 & 4 & 6 \\ 4 & -5 & -6 \end{pmatrix},\ \begin{pmatrix} 1 & 2 & 5 \\ -1 & 6 & 6 \\ 4 & -5 & -6 \end{pmatrix},\ \begin{pmatrix} 1 & 4 & -3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 2 & -2 & 7 & 0 \\ 0 & 1 & 3 & 5 \end{pmatrix}$$

Assignment 4

1. Solve the following system of linear equations:

$$\begin{cases} u+v+w=2 \\ u+3v+3w=0 \\ u+3v+5w=2 \end{cases}$$

2. Invert the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ using the Gauss-Seidel method.

3. Solve the following system of linear equations:

$$\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$$

4. a. Find all quadruples of real numbers (b_1,b_2, b_3, b_4) such that the following system of equations has a solution:

$$\begin{cases} x_1+4x_2+2x_3+x_4=b_1 \\ x_1+5x_2+2x_3=b_2 \\ 2x_1+9x_2+5x_3+3x_4=b_3 \\ 2x_1+7x_2+4x_3+3x_4=b_4 \end{cases}$$

   b. Solve the system for the case where (b_1,b_2, b_3, b_4) = (-1,-4,-1,1).

5. For what values of λ does the following system of equations have:
   a. a unique solution
   b. no solution.
   c. an infinite number of solutions.

$$\begin{aligned} &x+y+2z=1 \\ &2x+(\lambda+1)y+2z=4 \\ &\lambda x+y+z=2\lambda \end{aligned}$$

6. For what values of a and b does the following system of equations have:
   a. a unique solution
   b. no solution
   c. an infinite number of solutions.

$$\begin{aligned} &ax+bz=2 \\ &ax+ay+4z=4 \\ &ay+2z=b \end{aligned}$$

Assignment 5

1. Check if the following set of vectors is linearly dependent in $\mathbb{R}^4$ directly by trying to solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = 0$:

$$\vec{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \vec{v}_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix},\ \vec{v}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix},\ \vec{v}_4 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

2. Determine whether the given set of vectors is linearly independent in the vector space of vectors of length 2 with entries of real-valued functions:

   a. $v_1 = (e^{-t}, 2e^{-t}),\quad v_2 = (e^{-t}, e^{-t}),\quad v_3 = (3e^{-t}, 0)$
   b. $v_1 = (2\,sin\,t\,, sin\,t),\quad v_2 = (sin\,t\,, 2\,sin\,t)$
   c. $v_1 = (e^{t}, te^{t}),\quad v_2 = (1, t)$

3. Show that if t_0 is a real number the set of vectors:
   $v_1 = (e^{t_0}, t_0e^{t_0}),\quad v_2 = (1, t_0)$ is linearly dependent in $\mathbb{R}^2$.

4. Find a basis for each of the following subspaces, and justify in each case why it is a basis:
   a. U = set of square $n \times n$ real symmetric matrices
   b. V = set of square $n \times n$ real matrices whose rows add up to zero.
   c. W = set of real polynomial functions.
   d. $\mathbb{C}^n$ as a vector space over $\mathbb{R}$.

5. Find the dimensions and give a basis for the following spaces:
   a. the subspace of upper triangular 3×3 real matrices.
   b. the subspace of real 2×2 matrices in which the sum of the elements on the main diagonal is zero.

6. In a neural network we are given 5 neuron inputs $x_1, x_2, \dots, x_5$, and 3 neuron outputs y_1, y_2, y_3. The connection between the inputs outputs is expressed by $y_i = \sum_{j=1}^{5} w_{ij}x_j \;\; i = 1,2,3,$ for numerical constants w_{ij}.
   a. Find which of the fundamental spaces of the matrix (w_{ij}) equals the space of outputs.
   b. Find a basis for the space of outputs if the matrix of connections is:

$$W = \begin{bmatrix} 7 & 1 & 6 & 5 & 3 \\ 0 & -1 & 1 & 2 & 4 \\ 6 & 4 & 2 & -2 & -10 \end{bmatrix}$$

   c. Find the dimension of the space of inputs such that the network in (b) gives the output (0,0,0).

Assignment 6

1. Find bases for the following spaces where A is the matrix:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

   a. the column space of A.
   b. the row space of A.
   c. the null space of A.
   d. the null space of A^T.

2. For the set of vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$:

   a. Find a matrix whose row space equals the span of the vectors.
   b. Find a matrix whose null space equals the span of the vectors.
   ((b) is optional – not included in the grade!)

3. Find a basis for the solution space for the following systems of equations:

   a. $\begin{cases} x - 3y + z = 0 \\ -2x + 2y - 3z = 0 \\ 4x - 8y + 5z = 0 \end{cases}$

   b. $\begin{cases} x - y - z = 0 \\ 2x - y + z = 0 \end{cases}$

4. Solve the following systems of equations, expressing the solution as a particular solution added to the set of solutions of the associated homogeneous system, by finding a basis for the nullspace of the matrix of coefficients:

   a. $\begin{cases} u - y + z = 7 \\ 2x + 2y - 3z = -1 \\ 4u - x - y = 0 \\ -2u + x + 4z = 2 \end{cases}$

   b. $\begin{cases} 2x + 3y - z = 5 \\ -x + 2y + 3z = 0 \\ 4x - y + z = -1 \end{cases}$

Assignment 7

1. Calculate the following determinants:

a. $\begin{bmatrix} -9 & 14 \\ -2 & 7 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 14 & 27 & 35 \\ 22 & 11 & 19 \\ -4 & 0 & 0 \end{bmatrix}$ d. $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$

e. $\begin{bmatrix} 24 & 35 & 9 & 17 \\ 0 & 1 & 100 & 38 \\ 0 & 0 & 2 & 97 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ f. $\begin{bmatrix} t+2 & 0 & 1 \\ t+2 & t-2 & 1 \\ 0 & t-2 & t+4 \end{bmatrix}$ g. $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$

2. Determine whether the following matrices are linearly independent in the space of 2×2 real matrices. **Hint:** Use determinants.

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 \\ 4 & 9 \end{pmatrix}$$

3. For a square matrix A, prove the following claims using theorems stated in class:

a. If A is invertible and both A and A^{-1} have integer entries then $\det(A) = \pm 1$.

b. If A is invertible and has integer entries, and $\det(A) = \pm 1$ then A^{-1} has integer entries.
(**Hint:** Use the determinant formula for inverse of A.)

c. If $AA^T = I$ then $det(A) = \pm 1$.

d. If $A^2 = A$ and $A \neq I$ then $\det(A) = 0$.

e. If A is upper triangular then det(A) is the product of the elements on the main diagonal. (Stated but not proved in class!)

f. If the sum of the entries in every row of A is 0 then $\det(A) = 0$.

Assignment 8

1. Guided exercise for calculating an explicit formula for the Fibonacci sequence.

   We define the Fibonacci sequence as follows: $a_0 = 0, \quad a_1 = 1$ and for $n > 1, \quad a_n = a_{n-1} + a_{n-2}$.

   Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\vec{v}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$.

   a. Verify that $\vec{v}_{n+1} = A\vec{v}_n$.
   b. Diagonalize *A* to get an expression for A^n.
   c. Calculate $\vec{v}_n = A^n\vec{v}_0$ and use it to get an explicit formula for a_n.

2. Calculate the real eigenvalues and eigenvectors for each of the following matrices and determine if it is diagonalizable over the real numbers,. If so, find an invertible matrix *P* and diagonal matrix *D* such that $P^{-1}AP = D$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$$

3. a. Show that if $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$.
   b. Use part (a) to calculate A^8 for $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

4. Prove the following claims:
   a. If λ is an eigenvalue for A then λ^k is an eigenvalue for A^k.
   b. If $A^2 = A$ and if λ is an eigenvalue for A then $\lambda = 0$ or $\lambda = 1$.

Assignment 9

1. In a forest there live a number of rabbits and wolves. At the end of year number *n*, the number of rabbits is x_n, and the number of wolves is y_n. The relation between the populations from year to year is given by the following system of equations:

$$\begin{cases} x_{n+1} = 3x_n - y_n \\ y_{n+1} = 2x_n \end{cases}$$

   a. Show that if we denote $\vec{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, find a matrix *A* such that we can represent the relation in the form $\vec{v}_{n+1} = A\vec{v}_n$.
   b. Show that if the initial populations are given by the vector $\vec{v}_0$, then $\vec{v}_n = A^n\vec{v}_0$.
   c. Find eigenvalues and eigenvectors for *A*.
   d. Use the previous parts of the question and question 3(a) to find explicit formulas for $x_n$and y_n in terms of *n*.
   e. At the end of a given year there were 3 rabbits and 2 wolves in the forest. Use part (d) to calculate the populations 7 years later.

2. Find all possible Jordan canonical forms (up to permutation of blocks) for matrices whose characteristic polynomials are given below:
   a. $(\lambda - 1)^2(\lambda + 2)^3$
   b. $(\lambda - 5)^4$
   c. $\lambda(\lambda + 3)(\lambda - 5)(\lambda - 5)$

3. Find an invertible matrix that transforms the following matrices to its Jordan canonical form:

$$\begin{pmatrix} -12 & 7 \\ -7 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 \\ 4 & -5 \end{pmatrix},$$

.

Assignment 10

1. a. Which of the following functions are linear transformations?

$T: \mathbb{R}^3 \to \mathbb{R}^3,\quad T(x,y,z) = (x+y-2z, x+y, 2z)$

$T: \mathbb{R}^2 \to \mathbb{R}^2,\quad T(x,y) = (x+y+1, y)$

$T: \mathbb{R}^n \to \mathbb{R}^n,\quad T(x_1, x_2, \dots, x_n) = (0, x_1, x_2, \dots, x_{n-1})$

$T: \mathbb{R}^3 \to \mathbb{R},\quad T(x,y,z) = 5x - 2z.$

$T: \mathbb{R}^3 \to \text{vector space of real} - \text{valued functions } f(t),$

$T(x,y,z) = xe^t + (y-z)\sin t.$

$T: \mathbb{R}^3 \to \mathbb{R}^2,\quad T(x,y,z) = (x+y, xy).$

b. Find bases for the kernels and images of each of the linear transformations you found in part (a).

2. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\2\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\\0\end{bmatrix} \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\-4\\0\end{bmatrix}$$

a. Calculate the matrix of T and find its rank.

b. Find a basis for the kernel and image of T.

3. Calculate the matrix of the linear transformation T on $\mathbb{R}^3$ that rotates each vector anti-clockwise around the z-axis at an angle of 45 degrees.

4. Calculate the matrix of the linear transformation S on $\mathbb{R}^3$ that rotates each vector anti-clockwise around the x-axis at an angle of 90 degrees.

5. Use the matrices from the last two questions to show that the operators do not commute.

Assignment 11

1. In $\mathbb{R}^2$, and let $v = (x_1, x_2), w = (y_1, y_2)$.
   Show that $\langle v, w\rangle = x_1y_1 + 3x_2y_2$ defines an inner product on $\mathbb{R}^2$.

2. Show that the following expression defines an inner product on the space of real integrable functions defined on [0,1]:

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

3. With respect to the inner product defined in the last question, find an orthonormal basis for the subspace of real polynomial functions of degree less or equal to 2. Use Gram-Schmidt starting with the standard basis for this subspace.

4. a. Find an orthogonal complement in $\mathbb{R}^3$ for the subspace:

$$U = \text{span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}.$$

   b. Write a decomposition of the vector $\begin{pmatrix}-1\\3\\0\end{pmatrix}$ as a sum of a vector in U and a vector in $U^{\perp}$.

Assignment 12

1. Show that in $\mathbb{C}^n$ we have $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

2. Use the last question to show that the eigenvalues of Hermitian matrices are real.

3. Show that eigenvectors of a hermitian matrix corresponding to distinct eigenvalues are orthogonal.

An $\boldsymbol{n \times n}$ complex matrix $\boldsymbol{A}$ is called unitary if $\boldsymbol{A^* = A^{-1}}$.

4. Show that A is unitary if and only if its columns form an orthonormal basis for $\mathbb{C}^n$.

5. Show that if A is unitary then $|\det A| = 1$.