Linear Algebra for Chemists — Assignment 9

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Question 1. At the end of year n, #rabbits is x_n and #wolves is y_n.

$$\begin{cases} x_{n+1} = 3\,x_n - y_n \\ y_{n+1} = 2\,x_n \end{cases}.$$

a) Denote $\vec{v}_n = [x_n, y_n]^T$. Consider the matrix A

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}.$$

As we can see,

$$A\,\vec{v}_n = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 3\,x_n - y_n \\ 2\,x_n \end{bmatrix} = \vec{v}_{n+1}\,.$$

b) Denote $\vec{v}_0 = [x_0, y_0]^T$. Prove by induction that $\vec{v}_n = A^n\,\vec{v}_0$.

Proof. For $n = 0$ (base case),

$$\vec{v}_0 = A^0\,\vec{v}_0 = I\,\vec{v}_0 = \vec{v}_0\,.$$

Assume that $\vec{v}_k = A^k\,\vec{v}_0$. For $n = k+1$ (inductive step),

$$\vec{v}_{k+1} = A^{k+1}\,\vec{v}_0 = A\,A^k\,\vec{v}_0 = A\,\vec{v}_k,$$

which is true by the definition of A. $\square$

c) Find eigenvalues for A:

$$|A - \lambda\,I| = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = \lambda - 3\lambda + 2 = (\lambda - 2)\,(\lambda - 1) = 0\,.$$

Find associated eigenvectors: for $\lambda_1 = 2$:

$$[A - 2\,I] = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \rightarrow w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1$:

$$[A - I] = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \rightarrow w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

d) Find an expression for A^n, and express $\vec{v}_n$ as $A^n\,\vec{v}_0$. The matrix of eigenvectors is

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

whose inverse is

$$T^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^n = T\,D^n\,T^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 1^n \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2^{n+1} & -2^n \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{n+1}-1 & 1-2^n \\ 2^{n+1}-2 & 2-2^n \end{bmatrix}, \\ \vec{v}_n = A^n\,\vec{v}_0 &= \begin{bmatrix} 2^{n+1}-1 & 1-2^n \\ 2^{n+1}-2 & 2-2^n \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} (2^{n+1}-1)\,x_0 + (1-2^n)\,y_0 \\ (2^{n+1}-2)\,x_0 + (2-2^n)\,y_0 \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \end{aligned}$$

In a more compact form, we may conclude that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (2\,x_0 - y_0)\,2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (y_0 - x_0)\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

e) Given $\vec{v}_0 = [3, 2]^T$,

$$\vec{v}_7 = (2 \times 3 - 2)\,2^7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2-3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 511 \\ 510 \end{bmatrix}.$$

Question 2. Find Jordan canonical forms. For each eigenvalue, the number of unique Jordan blocks (up to permuations) equals its algebraic multiplicity. The number of possible Jordan forms equals the product of the numbers of unique Jordan blocks (of each eigenvalue).

a) $P(\lambda) = (\lambda-1)^2\,(\lambda+2)^3$.

$$J_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, J_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, J_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$J_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, J_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

b) $P(\lambda) = (\lambda-5)^4$.

$$J_1 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, J_2 = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, J_3 = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, J_4 = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

c) $P(\lambda) = \lambda\,(\lambda+3)\,(\lambda-5)^2$.

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Question 3. Find matrix T that transforms the matrix A to Jordan canonical form.

a) $A = \left[\begin{smallmatrix} -12 & 7 \\ -7 & 2 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\,I| = (\lambda+12)\,(\lambda-2) + 49 = \lambda^2 + 10\lambda + 25 = (\lambda+5)^2 = 0.$$

T consists of two generalized eigenvectors $T = [t_1, t_2]$. $t_1 \in \text{kernel}\,(A + 5\,I)$:

$$[A + 5\,I]\,t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -7 & 7 \\ -7 & 7 \end{bmatrix} t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Pick $t_1 = [-7, -7]^T$. t_2 satisfies $[A + 5\,I]\,t_2 = t_1$.

$$\begin{bmatrix} -7 & 7 \\ -7 & 7 \end{bmatrix} t_2 = \begin{bmatrix} -7 \\ -7 \end{bmatrix}.$$

Pick $t_2 = [1, 0]^T$. The desired matrix T is

$$T = \begin{bmatrix} -7 & 1 \\ -7 & 0 \end{bmatrix}.$$

b) $A = \left[\begin{smallmatrix} 4 & -1 \\ 1 & 2 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\,I| = (\lambda-4)\,(\lambda-2) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2 = 0.$$

T consists of two generalized eigenvectors $T = [t_1, t_2]$. $t_1 \in \text{kernel}(A - 3\,I)$:

$$[A - 3\,I]\,t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Pick $t_1 = [1, 1]^T$. t_2 satisfies $[A - 3\,I]\,t_2 = t_1$.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Pick $t_2 = [1, 0]^T$.

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

c) $A = \left[\begin{smallmatrix} 1 & 3 \\ 4 & -5 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\,I| = (\lambda-1)\,(\lambda+5) - 12 = \lambda^2 + 4\lambda - 17.$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 + 4 \times 17}}{2} = -2 \pm \sqrt{21}\,.$$

Find associated eigenvectors. For $\lambda_1 = -2 + \sqrt{21}$:

$$\begin{aligned}[A - (\sqrt{21} - 2)\, I] &= \begin{bmatrix} 3 - \sqrt{21} & 3 \\ 4 & -3 - \sqrt{21} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{3 - \sqrt{21}} \\ 4 & -3 - \sqrt{21} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{3}{3 - \sqrt{21}} \\ 0 & -(3 + \sqrt{21}) - \frac{12}{3 - \sqrt{21}} = \frac{-(9 - 21) - 12}{3 - \sqrt{21}} = 0 \end{bmatrix}\end{aligned}$$

$$w_1 = \begin{bmatrix} \frac{3}{\sqrt{21} - 3} \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -2 - \sqrt{21}$:

$$\begin{aligned}[A + (\sqrt{21} + 2)\, I] &= \begin{bmatrix} 3 + \sqrt{21} & 3 \\ 4 & -3 + \sqrt{21} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{3 + \sqrt{21}} \\ 4 & -3 + \sqrt{21} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{3}{3 + \sqrt{21}} \\ 0 & -3 + \sqrt{21} - \frac{12}{3 + \sqrt{21}} = \frac{(21 - 9) - 12}{3 + \sqrt{21}} = 0 \end{bmatrix}\end{aligned}$$

$$w_2 = \begin{bmatrix} -\frac{3}{3 + \sqrt{21}} \\ 1 \end{bmatrix}.$$

T is just the matrix of the eigenvectors of A.

$$T = \begin{bmatrix} \frac{3}{\sqrt{21} - 3} & -\frac{3}{3 + \sqrt{21}} \\ 1 & 1 \end{bmatrix}.$$