2022 Exam Solution

Question 1

(a) Solve the equation

$$y' = (5x^3 - 4)\cos^2 y$$

with IC $y(0) = \frac{\pi}{4}$. This is a separable equation.

$$\int \frac{\mathrm{d}y}{\cos^2 y} = \int (5x^3 - 4)\,\mathrm{d}x$$

$$\tan y = \frac{5}{4}x^4 - 4x + c$$

Input IC.

$$1 = c$$

Unique solution is:

$$y = \arctan\left(\frac{5}{4}x^4 - 4x + 1\right)$$

(b) Find the solution to the DE:

$$x^3\,y' + 4x^2\,y = \mathrm{e}^{-x}$$

with IC $y(-1) = 0$.

Assuming $x \neq 0$ on some interval (if $x = 0$ there is no solution), divide by x^3:

$$y' + \frac{4}{x}\,y = \frac{\mathrm{e}^{-x}}{x^3}$$

Solve using integration factors method. For a DE of form $y' + a\,y = b$, the solution is given by:

$$y = \mathrm{e}^{-\int a\,\mathrm{d}x}\left[\int b\,\mathrm{e}^{\int a\,\mathrm{d}x}\,\mathrm{d}x + c\right]$$

In our case, $a = \frac{4}{x}$ and $b = \frac{\mathrm{e}^{-x}}{x^3}$

$$\mathrm{e}^{\int a(x)\,\mathrm{d}x} = \mathrm{e}^{4\int \frac{1}{x}\mathrm{d}x} = \mathrm{e}^{4\ln x} = x^4$$

So:

$$y = \frac{1}{x^4}\left[\int \frac{\mathrm{e}^{-x}}{x^3}\cdot x^4\,\mathrm{d}x + c\right]$$

Calculate the integral by parts:

$$\int x\,\mathrm{e}^{-x}\mathrm{d}x = -x\,\mathrm{e}^{-x} - \int (-1)\cdot \mathrm{e}^{-x}\,\mathrm{d}x = -\mathrm{e}^{-x}\,(x+1)$$

Therefore,

$$y = -\frac{\mathrm{e}^{-x}(x+1)}{x^4} + \frac{c}{x^4}$$

Input IC:

$$y(-1) = 0 = c$$

Unique solution is:

$$y = -\frac{\mathrm{e}^{-x}(x+1)}{x^4}$$

Question 2

Find the solution to the equation

$$y'' + 9y = \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$$

with ICs $y(0) = y'(0) = 0$.

First solve associated homogeneous equation. Characteristic equation is:

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3\mathrm{i}$$

Therefore, general solution to homogeneous equation is:

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

where $c_1, c_2 \in \mathbb{R}$. Now search for a particular solution in a form of a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\,\pi\,x}{L} + b_n \sin\frac{n\,\pi\,x}{L} \right)$$

where $L = \pi$. As the summand in the RHS consists of odd functions only, and because there is only a second derivative in the ODE (which returns sine and cosine given sine and cosine respectively), a_0 and a_n are assumed to be zero. The particular solution is simplified into:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\,x)$$

Second derivative of $f(x)$ is:

$$f''(x) = \sum_{n=1}^{\infty} -n^2 \cdot b_n \sin(n\,x)$$

Input in ODE:

$$\sum_{n=1}^{\infty} -n^2 \cdot b_n \sin(n\,x) + 9\sum_{n=1}^{\infty} b_n \sin(n\,x) = \sum_{n=1}^{\infty} \frac{\sin(n\,x)}{n^2}$$

$$\sum_{n=1}^{\infty} \sin(n\,x)\,[(9-n^2)b_n] = \sum_{n=1}^{\infty} \sin(n\,x) \left[\frac{1}{n^2}\right]$$

Equate coefficients on both sides:

$$(9-n^2)b_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{n^2\,(9-n^2)}$$

Therefore, the general solution to the ODE is:

$$y = c_1\cos(3x) + c_2\sin(3x) + \sum_{n=1}^{\infty} \frac{1}{n^2\,(9-n^2)}\sin(n\,x)$$

Input ICs:

$$\begin{aligned} y(0)=0 &= c_1 \\ y'(0)=0 &= 3c_2 + \sum_{n=1}^{\infty}\frac{1}{n\,(9-n^2)} \to c_2 = \sum_{n=1}^{\infty}\frac{1}{3n\,(n^2-9)} \end{aligned}$$

Unique solution is:

$$y = \sum_{n=1}^{\infty}\frac{1}{3n\,(n^2-9)}\sin(3x) + \sum_{n=1}^{\infty}\frac{1}{n^2\,(9-n^2)}\sin(n\,x)$$

Question 3

A 2-degree of freedom mass system was constructed as follows:

A mass of $m_1 = 2$kg was suspended from a spring with $k_1 = 4$, and a second mass $m_2 = 1$kg was attached by a spring with $k_2 = 2$ to the first one. Let u_1, u_2 be the vertical displacement of masses m_1, m_2.

(a) Construct a system of DEs whose solution gives u_1, u_2. Equations of motion for both masses:

$$\begin{aligned} m_1\,u_1'' &= -k_1\,u_1 + k_2\,(u_2-u_1) \\ m_2\,u_2'' &= -k_2\,(u_2-u_1) \end{aligned}$$

$$\begin{aligned} 2u_1'' &= -4u_1 + 2\,(u_2-u_1) \\ u_2'' &= -2\,(u_2-u_1) \end{aligned}$$

$$\begin{aligned} u_1'' &= -3u_1 + u_2 \\ u_2'' &= 2u_1 - 2u_2 \end{aligned}$$

(b) Derive from it a differential equation of order 4, and solve it for the ICs:

$$u_1(0) = -1, u_1'(0) = 0, u_2(0) = 1, u_2'(0) = -1.$$

From the first equation:

$$u_2 = u_1'' + 3u_1$$

Input in the second equation:

$$u_1^{(4)} + 3u_1'' = 2u_1 - 2\,(u_1'' + 3u_1)$$

$$u_1^{(4)} + 5u_1'' + 4u_1 = 0$$

Characteristic equation is:

$$\lambda^4 + 5\lambda^2 + 4 = 0$$

$$\lambda^2 = -4, -1$$

$$\Rightarrow \lambda = \pm 2\mathrm{i}, \pm \mathrm{i}$$

Solution is:

$$u_1 = c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos t + c_4 \sin t$$

$$u_2 = -c_1 \cos(2t) - c_2 \sin(2t) + 2c_3 \cos t + 2c_4 \sin t$$

Input ICs:

$$u_1(0) = -1 = c_1 + c_3$$

$$u_2(0) = 1 = -c_1 + 2c_3$$

Therefore,

$$c_3 = 0, c_1 = -1$$

$$u_1 = -\cos(2t) + c_2 \sin(2t) + c_4 \sin t$$

$$u_2 = \cos(2t) - c_2 \sin(2t) + 2c_4 \sin t$$

$$u_1'(0) = 0 = 2c_2 + c_4$$

$$u_2'(0) = -1 = -2c_2 + 2c_4$$

$$-1 = 3c_4 \rightarrow c_4 = -\frac{1}{3}$$

$$c_2 = \frac{1}{6}$$

In conclusion, the solution satisfying ICs is:

$$u_1 = -\cos(2t) + \frac{1}{6}\sin(2t) - \frac{1}{3}\sin t$$

$$u_2 = \cos(2t) - \frac{1}{6}\sin(2t) - \frac{2}{3}\sin t$$

Question 4

A closed system of 3 tanks, each with volume 50L, contains a salt solution with pumps maintaining constant flows as follows:

- A flow of 20L/h from tank 1 to tank 2
- A flow of 10L/h from tank 2 to tank 1, and 20L/h from 2 to 3
- A flow of 10L/h from 3 to 1, and 10L/h from 3 to 2

Find the amount of salt in each tank at time t if the initial amount of salt in tank 1 is 100g, in tak 2 200g, and in tank 3 0g.

$$\begin{aligned}
x_1' &= -\frac{2}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3 \\
x_2' &= \frac{2}{5}x_1 - \left(\frac{1}{5} + \frac{2}{5}\right)x_2 + \frac{1}{5}x_3 \\
x_3' &= \frac{2}{5}x_2 - \left(\frac{1}{5} + \frac{1}{5}\right)x_3
\end{aligned}$$

In matrix form:

$$\vec{x}' = A\,\vec{x}$$

$$A = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{1}{5} \\ 0 & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

Note that the sum along each column is zero, so $\lambda = 0$ is an eigenvalue with eigenvector $[1, 1, 1]^T$.

Find the other eigenvalues and eigenvectors.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{2}{5} - \lambda & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} - \lambda & \frac{1}{5} \\ 0 & \frac{2}{5} & -\frac{2}{5} - \lambda \end{vmatrix}$$

$$\det(A - \lambda I) = -\left(\lambda + \frac{2}{5}\right)\left[\left(\lambda + \frac{3}{5}\right)\left(\lambda + \frac{2}{5}\right) - \frac{2}{25}\right] - \frac{2}{5}\left[-\frac{1}{5}\left(\lambda + \frac{2}{5}\right) - \frac{2}{25}\right]$$

$$\det(A - \lambda I) = -\left(\lambda + \frac{2}{5}\right)\left[\lambda^2 + \lambda + \frac{4}{25}\right] + \frac{2}{25}\lambda + \frac{8}{125} = \cdots =$$

$$= -\left[\lambda^3 + \lambda^2 + \frac{4}{25}\lambda + \frac{2}{5}\lambda^2 + \frac{2}{5}\lambda + \frac{8}{125}\right] + \frac{2}{25}\lambda + \frac{8}{125} = -\lambda\left(\lambda^2 + \frac{7}{5}\lambda + \frac{12}{25}\right)$$

Roots are:

$$\lambda_1 = 0, \lambda_{2,3} = \frac{-\frac{7}{5} \pm \frac{1}{5}}{2} = -\frac{3}{5}, -\frac{4}{5}$$

Find eigenvectors. For $\lambda_2 = -\frac{3}{5}$:

$$(A - \lambda_2 I) = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

s.t. $(A-\lambda_2 I)\vec{v}_2=\vec{0}$. For $\lambda_3=-\frac{4}{5}$:

$$A-\lambda_3 I=\begin{bmatrix}\frac{2}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5}\\ 0 & \frac{2}{5} & \frac{2}{5}\end{bmatrix}$$

Pick

$$\vec{v}_3=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

s.t. $(A-\lambda_3 I)\vec{v}_3=\vec{0}$. General solution is:

$$\vec{x}_3=c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\,\mathrm{e}^{-\frac{3}{5}t}\begin{bmatrix}1\\1\\-2\end{bmatrix}+c_3\,\mathrm{e}^{-\frac{4}{5}t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Find $\vec{c}=[c_1,c_2,c_3]^T$ that satisfies ICs. Define

$$B=\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 1 & -2 & -1\end{bmatrix},\vec{b}=\begin{bmatrix}100\\200\\0\end{bmatrix}$$

such that $B^{-1}\vec{b}=\vec{c}$. Find B^{-1}.

$$\det B=[-1+2]-[-1-1]=3$$

$$B^{-1}=\frac{1}{3}\begin{bmatrix}(-1+2) & -(-1-1) & (-2-1)\\ -(-1) & (-1) & -(-2-1)\\ (1) & -(1) & (1-1)\end{bmatrix}^T=\frac{1}{3}\begin{bmatrix}1 & 2 & -3\\ 1 & -1 & 3\\ 1 & -1 & 0\end{bmatrix}^T=\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 3 & 0\end{bmatrix}$$

Therefore:

$$\vec{c}=\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ -3 & 3 & 0\end{bmatrix}\begin{bmatrix}100\\200\\0\end{bmatrix}=\begin{bmatrix}100\\0\\100\end{bmatrix}$$

Question 5

An elastic string of length $L=40\,\mathrm{cm}$ is held down taut at both ends in a frame, and vibrates according to the wave equation:

$$u_{xx}(x,t)=u_{tt}(x,t)$$

where $a=1\,\mathrm{cm}/s$. Assume that the initial position of the string is its equilibrium position, and that it is set in motion with an initial velocity:

$$u_t(x,0)\equiv g(x)=\begin{cases}\frac{x}{10} & x\in[0,10]\\ 1 & x\in(10,30)\\ \frac{40-x}{10} & x\in[30,40]\end{cases}$$

Calculate the series representation of $u(x,t)$ that describes the vibration of the string.

$$u(x,t)=\sum_{n=1}^{\infty}k_n\sin\frac{n\,\pi\,x}{L}\sin\frac{n\,\pi\,a\,t}{L}$$

where

$$k_n = \frac{2}{n\,\pi\,a}\int_0^L g(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Calculate k_n.

$$k_n = \frac{2}{n\pi}\left[\frac{1}{10}\int_0^{10} x\sin\frac{n\,\pi\,x}{40}\,\mathrm{d}x + \int_{10}^{30}\sin\frac{n\,\pi\,x}{40}\,\mathrm{d}x + 4\int_{30}^{40}\sin\frac{n\,\pi\,x}{40}\,\mathrm{d}x - \frac{1}{10}\int_{30}^{40} x\sin\frac{n\,\pi\,x}{40}\,\mathrm{d}x\right]$$

$$\int x\sin\frac{n\,\pi\,x}{40}\,\mathrm{d}x = \frac{40}{n\pi}\left[-x\cos\frac{n\pi\,x}{40} + \frac{40}{n\pi}\sin\left(\frac{n\pi\,x}{40}\right)\right]$$

$$\int \sin\frac{n\pi\,x}{40}\,\mathrm{d}x = -\frac{40}{n\pi}\left[\cos\left(\frac{n\pi\,x}{40}\right)\right]$$

Therefore:

$$\begin{aligned}
k_n &= \frac{2}{n\pi}\left[\frac{1}{10}\cdot\frac{40}{n\pi}\left[-x\cos\frac{n\pi\,x}{40} + \frac{40}{n\pi}\sin\left(\frac{n\pi\,x}{40}\right)\right]_0^{10} - \frac{1}{10}\cdot\frac{40}{n\pi}\left[-x\cos\frac{n\pi\,x}{40} + \frac{40}{n\pi}\sin\left(\frac{n\pi\,x}{40}\right)\right]_{30}^{40}\right] \\
&\quad + \frac{2}{n\pi}\left[-\frac{40}{n\pi}\left[\cos\left(\frac{n\pi\,x}{40}\right)\right]_{10}^{30} - \frac{160}{n\pi}\left[\cos\left(\frac{n\pi\,x}{40}\right)\right]_{30}^{40}\right]
\end{aligned}$$

$$\begin{aligned}
k_n &= \frac{2}{n\pi}\left[\frac{4}{n\pi}\left[-10\cos\left(\frac{n\pi}{4}\right) + \frac{40}{n\pi}\sin\left(\frac{n\pi}{4}\right)\right] - \frac{4}{n\pi}\left[-40\cos(n\pi) + 30\cos\left(\frac{3n\pi}{4}\right) - \frac{40}{n\pi}\sin\left(\frac{3n\pi}{4}\right)\right]\right] \\
&\quad + \frac{2}{n\pi}\left[-\frac{40}{n\pi}\left[\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right)\right] - \frac{160}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{3n\pi}{4}\right)\right]\right]
\end{aligned}$$

$$\begin{aligned}
k_n &= \frac{2}{n\pi}\cdot\frac{4}{n\pi}\left[-10\cos\left(\frac{n\pi}{4}\right) + \frac{40}{n\pi}\sin\left(\frac{n\pi}{4}\right) + 40\cos(n\pi) - 30\cos\left(\frac{3n\pi}{4}\right) + \frac{40}{n\pi}\sin\left(\frac{3n\pi}{4}\right)\right] \\
&\quad - \frac{2}{n\pi}\cdot\frac{40}{n\pi}\left[\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{4}\right) + 4\cos(n\pi) - 4\cos\left(\frac{3n\pi}{4}\right)\right]
\end{aligned}$$

$$\begin{aligned}
k_n &= \frac{2}{n^2\pi^2}\left[\frac{160}{n\pi}\sin\left(\frac{n\pi}{4}\right) - 120\cos\left(\frac{3n\pi}{4}\right) + \frac{160}{n\pi}\sin\left(\frac{3n\pi}{4}\right)\right] \\
&\quad - \frac{2}{n^2\pi^2}\left[40\cos\left(\frac{3n\pi}{4}\right) - 160\cos\left(\frac{3n\pi}{4}\right)\right]
\end{aligned}$$

$$k_n = \frac{2}{n^2\pi^2}\left[\frac{160}{n\pi}\sin\left(\frac{n\pi}{4}\right) + \frac{160}{n\pi}\sin\left(\frac{3n\pi}{4}\right)\right] = -\frac{640}{n^3\pi^3}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right)$$

In conclusion,

$$u(x,t) = -\sum_{n=1}^{\infty}\frac{640}{n^3\pi^3}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right)\sin\frac{n\,\pi\,x}{40}\sin\frac{n\,\pi\,t}{40}$$

Question 6

Given a rod of length $L = 40\text{cm}$ with $\alpha^2 = 0.25$, find $u(x,t)$ if

$$u(x,0) \equiv f(x) = \frac{x(60-x)}{30}, \quad x \in (0,40)$$

and both ends of the rod are insulated.

Solution is given by:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi\, x}{L}\, \mathrm{e}^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi\, x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n = \frac{1}{20}\int_0^{40}\left(2x - \frac{x^2}{30}\right)\cos\frac{n\pi\, x}{L}\,\mathrm{d}x$$

$$\int x\cos\frac{n\pi\, x}{L}\,\mathrm{d}x = \frac{L}{n\pi}\left[x\sin\left(\frac{n\pi\, x}{L}\right) + \frac{L}{n\pi}\cos\left(\frac{n\pi\, x}{L}\right)\right]$$

$$\int x^2\cos\frac{n\pi\, x}{L}\,\mathrm{d}x = \frac{L}{n\pi}\left(x^2\sin\left(\frac{n\pi\, x}{L}\right) - \frac{2L}{n\pi}\left[-x\cos\left(\frac{n\pi\, x}{L}\right) + \frac{L}{n\pi}\sin\left(\frac{n\pi\, x}{L}\right)\right]\right)$$

$$\begin{aligned} c_n &= \frac{1}{20}\cdot 2\cdot\frac{40}{n\pi}\left[x\sin\left(\frac{n\pi\, x}{40}\right) + \frac{40}{n\pi}\cos\left(\frac{n\pi\, x}{40}\right)\right]_0^{40} \\ &\quad + \frac{1}{20}\cdot\left(-\frac{1}{30}\right)\cdot\frac{40}{n\pi}\left(x^2\sin\left(\frac{n\pi\, x}{40}\right) - \frac{40}{n\pi}\left[-x\cos\left(\frac{n\pi\, x}{40}\right) + \frac{40}{n\pi}\sin\left(\frac{n\pi\, x}{40}\right)\right]\right)_0^{40} \end{aligned}$$

$$\begin{aligned} c_n &= \frac{4}{n\pi}\left[\frac{40}{n\pi}\cos(n\pi) - \frac{40}{n\pi}\right] - \frac{1}{15n\pi}\left(-\frac{40}{n\pi}[-40\cos(n\pi)]\right) \\ &= \frac{160}{n^2\pi^2}[\cos(n\pi) - 1] - \frac{320}{3n^2\pi^2}\cdot\cos(n\pi) = \frac{160}{3n^2\pi^2}(-1)^n - \frac{160}{n^2\pi^2} \end{aligned}$$

Calculate c_0.

$$c_0 = \frac{1}{20}\int_0^{40}\left(2x - \frac{x^2}{30}\right)\mathrm{d}x = \frac{1}{20}\left[x^2 - \frac{x^3}{90}\right]_0^{40} = \frac{400}{9}$$

Therefore,

$$c_n = \frac{400}{9} + \sum_{n=1}^{\infty}\left(\frac{160}{3n^2\pi^2}(-1)^n - \frac{160}{n^2\pi^2}\right)\cos\frac{n\pi\, x}{40}\,\mathrm{e}^{-\frac{n^2\pi^2 t}{1600}}$$