Assignment 4

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Date: 05-12-2022

Question 29

$$t\,y'' + y' = 1, \quad t > 0$$

Substitute y' for v.

$$(t\,v)' = 1$$

Integrate and divide by $t \neq 0$ to get:

$$v = 1 + \frac{c_1}{t}$$

Integrate to find y.

$$y = \int v(t)\,\mathrm{d}t = t + c_1 \ln t + c_2, \quad t > 0; \quad c_{1,2} \in \mathbb{R}$$

Note: the integral contains $\ln|t|$ but taking the absolute value is redundant since the given domain is $t > 0$.

Question 30

$$y'' + t\,(y')^2 = 0$$

First, notice that $y \equiv k$, $k \in \mathbb{R}$ is a solution for all t.

Substitute y' for v.

$$v' + t\,v^2 = 0$$

This is a separable equation, which can be rewritten as:

$$\frac{\mathrm{d}v}{v^2} = -t\,\mathrm{d}t, \quad v \neq 0$$

Integrate both sides.

$$\frac{1}{v} = \frac{1}{2}t^2 - c_1$$

Isolate v.

$$v = \frac{2}{t^2 - 2c_1}, \quad t \neq \sqrt{2c_1}; \quad c_1 \in \mathbb{R}$$

There are 3 different kinds of possible solutions, depending whether $c_1 < 0, c_1 = 0$, or $c_1 > 0$.

1. If $c_1 = 0$:

$$v = \frac{2}{t^2}$$

$$y = \int v\,\mathrm{d}t = -\frac{2}{t} + c_2, \quad t \neq 0; \quad c_2 \in \mathbb{R}$$

2. If $c_1 < 0$, substitute $2c_1 = -C^2$, $C > 0$.

$$v = \frac{2}{t^2 + C^2} = \frac{2}{C^2} \cdot \frac{1}{\left(\frac{t}{C}\right)^2 + 1}$$

$$y = \int v \, \mathrm{d}t = \frac{2}{C^2} \cdot \frac{\arctan\left(\frac{t}{C}\right)}{1/C} + c_2$$

$$y = \frac{2}{C} \arctan\left(\frac{t}{C}\right) + c_2, \quad \forall t; \quad c_2 \in \mathbb{R}$$

3. If $c_1 > 0$, substitute $2c_1 = C^2, C > 0$.

$$v = \frac{2}{t^2 - C^2} = 2\frac{1}{(t - C)} \frac{1}{(t + C)}$$

$$y = \int v \, \mathrm{d}t = 2\frac{1}{2C} \int \frac{(t + C) - (t - C)}{(t + C)\,(t - C)} \, \mathrm{d}t = \frac{1}{C} \int \left[\frac{1}{t - C} - \frac{1}{t + C}\right] \mathrm{d}t$$

$$y = \frac{1}{C} \left[\ln\left|\frac{1}{t - C}\right| - \ln\left|\frac{1}{t + C}\right|\right] + c_2$$

$$y = \frac{1}{C} \ln\left|\frac{t + C}{t - C}\right| + c_2, \quad t \neq C; \quad c_2 \in \mathbb{R}$$

Question 32

$$y'' + y' = \mathrm{e}^{-t}$$

Set $v = y'$ and solve using integration factor method.

$$v' + v = \mathrm{e}^{-t}$$

$$v = \mathrm{e}^{-t} \left(\int \mathrm{e}^{-t} \cdot \mathrm{e}^{t} \, \mathrm{d}t + c_1\right) = \mathrm{e}^{-t}(t + c_1)$$

Now integrate v to find y:

$$y = \int v \, \mathrm{d}t = \int (t\,\mathrm{e}^{-t} + c_1 \mathrm{e}^{-t}) \, \mathrm{d}t$$

Via integration by parts,

$$\int t\,\mathrm{e}^{-t} = -t\,\mathrm{e}^{-t} - \mathrm{e}^{-t} = -\mathrm{e}^{-t}\,(t + 1)$$

$$\Rightarrow y = -\mathrm{e}^{-t}\,(t + 1) - c_1\,\mathrm{e}^{-t} + c_2$$

$$y = -t\,\mathrm{e}^{-t} - (c_1 + 1)\,\mathrm{e}^{-t} + c_2, \quad c_{1,2} \in \mathbb{R}$$

Question 36

$$y'' + y\,(y')^3 = 0$$

First observe that $y \equiv k$, $k \in \mathbb{R}$ is a solution.

Set $y' = v$.

$$v' + y v^3 = 0$$

Write v' as $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v$

$$\frac{\mathrm{d}v}{\mathrm{d}y} \cdot v + y \cdot v^3 = 0$$

This is a separable equation. Divide by v assuming $v \neq 0$. (If $v = 0$ then $y \equiv k$ is a solution we've already found.) Rewrite as:

$$\frac{\mathrm{d}v}{v^2} = -y\,\mathrm{d}y$$

Integrate both sides to get

$$-\frac{1}{v} = -\frac{1}{2}y^2 + c_1$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v = \frac{2}{y^2 - 2c_1}$$

We're left with another separable DE. Rewrite as:

$$(y^2 - 2c_1)\,\mathrm{d}y = 2\mathrm{d}t$$

and integrate.

$$\frac{y^3}{3} - 2c_1\,y = 2t + c_2, \quad c_{1,2} \in \mathbb{R}$$

y is given implicitly.

Question 37

$$2y^2\,y'' + 2y\,(y')^2 = 1$$

Set $y' = v$.

$$2y^2\,v' + 2y\,v^2 = 1$$

Divide by $2y^2$ assuming $y \neq 0$.

$$v' + \frac{1}{y}\,v^2 = \frac{1}{2y^2}$$

Input $v' = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v$

$$\frac{\mathrm{d}v}{\mathrm{d}y}\,v + \frac{1}{y}\,v^2 = \frac{1}{2y^2}$$

Set $f(y) = \frac{1}{2}v^2$. $f' = \frac{\mathrm{d}f}{\mathrm{d}y} = v\,\frac{\mathrm{d}v}{\mathrm{d}y}$. Therefore,

$$\frac{\mathrm{d}f}{\mathrm{d}y} + \frac{2}{y}\,f = \frac{1}{2y^2}$$

Solve using integration factor method. $\left(a = \frac{2}{y}, b = \frac{1}{2y^2}; f = \mathrm{e}^{-\int a(y)\,\mathrm{d}y}\left[\int b(y)\,\mathrm{e}^{\int a(y)\,\mathrm{d}y} + c_1\right]\right)$

$$f = \frac{1}{y^2}\left(\int \frac{1}{2y^2} \cdot y^2\,\mathrm{d}y + c_1\right) = \frac{1}{y^2}\left(\frac{1}{2}y + c_1\right)$$

$$f = \frac{1}{2y} + \frac{c_1}{y^2}$$

Input back $f=\frac{v^2}{2}$ and get

$$v^2=\frac{1}{y}+\frac{2c_1}{y^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\pm\left(\frac{1}{y}+\frac{2c_1}{y^2}\right)^{1/2}=\pm\sqrt{\frac{y+2c_1}{y^2}}=\pm\frac{\sqrt{y+2c_1}}{|y|}$$

This is a separable equation

$$\pm\int\frac{|y|}{\sqrt{y+2c_1}}\,\mathrm{d}y=\int\mathrm{d}t=t+c_2$$

Because we already have $\pm$ as a prefix, we don't have to take the absolute value of y.

Let's focus on the LHS: substitute $u=y+2c_1$. $\mathrm{d}u=\mathrm{d}y$.

$$\pm\int\frac{u-2c_1}{\sqrt{u}}\,\mathrm{d}u=\pm\int(u^{1/2}-2c_1\,u^{-1/2})\mathrm{d}u=\pm\left(\frac{2}{3}u^{3/2}-2c_1\cdot2\sqrt{u}\right)$$

Substitute back u and simplify:

$$\pm\left[\frac{2}{3}(y+2c_1)^{3/2}-4c_1\,(y+2c_1)^{1/2}\right]=\pm\left[\frac{2}{3}\sqrt{y+2c_1}\,(y+2c_1-6c_1)\right]=\pm\left[\frac{2}{3}\sqrt{y+2c_1}\,(y-4c_1)\right]$$

To summarize:

$$\pm\left[\frac{2}{3}\sqrt{y+2c_1}\,(y-4c_1)\right]=t+c_2,\quad y\geq-2c_1;\quad c_{1,2}\in\mathbb{R}$$

y is given implicitly.

<u>Question 40</u>

$$y'\,y''=2,\quad y(0)=1, y'(0)=2$$

Set $v=y'$ and divide by $v\neq0$.

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{2}{v},\quad v\neq0$$

Solve separable equation:

$$\int v\,\mathrm{d}v=\int2\,\mathrm{d}t$$

$$\frac{1}{2}v^2=2t+c_1$$

$$v=\pm\sqrt{4t+2c_1}$$

The ICs dictate $v(0)=y'(0)=2>0$. This can simplify the solution. Assume only possitive v.

$$\frac{\mathrm{d}y}{\mathrm{d}t}=v=\sqrt{4t+2c_1}$$

Integrate to find y.

$$y=\frac{1}{6}\,(4t+2c_1)^{3/2}+c_2$$

Find specific solution that satisfies ICs. We have $y'(0)=v(0)=2$. Input in $\frac{1}{2}v^2=2t+c_1$ and get:

$$\frac{1}{2}\cdot2^2=2\cdot0+c_1\rightarrow c_1=2$$

Also:

$$y(0)=1=\frac{1}{6}(4\cdot 0+2\cdot 2)^{3/2}+c_2\to c_2=-\frac{1}{3}$$

The specific solution is

$$y=\frac{1}{6}(4t+4)^{3/2}-\frac{1}{3}$$

$$y=\frac{4}{3}(t+1)^{3/2}-\frac{1}{3}$$

<u>Question 41</u>

$$y''-3y^2=0,\quad y(0)=2, y'(0)=4$$

Set $v=y'$

$$v'-3y^2=0$$

Write v' as $v\frac{\mathrm{d}v}{\mathrm{d}y}$

$$v\frac{\mathrm{d}v}{\mathrm{d}y}=3y^2$$

$$\int v\,\mathrm{d}v=\int 3y^2\,\mathrm{d}y$$

$$\frac{v^2}{2}=y^3+c_1$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=v=\pm\sqrt{2}\sqrt{y^3+c_1}$$

Solve separable DE:

$$\pm\int (y^3+c_1)^{-1/2}\,\mathrm{d}y=\sqrt{2}\int \mathrm{d}t$$

The LHS integral is unsolvable. We can try to find c_1 and may solve the integral if we found that, perhaps, $c_1=0$. We know that $y(0)=2$ and that $y'(0)=v(0)=4$. If we input these in the equation

$$\frac{v^2}{2}=y^3+c_1$$

from above, we get:

$$\frac{4^2}{2}=2^3+c_1\to c_1=0$$

Fortunately, now the integral is solvable! Input $c_1=0$ and get

$$\int (y^3+0)^{-1/2}\,\mathrm{d}y=\int y^{-3/2}\,\mathrm{d}y=-2y^{-1/2}$$

Therefore,

$$\pm(-2y^{-1/2})=\sqrt{2}\,(t+c_2)$$

Raise both sides by -2:

$$y = 2(t + c_2)^{-2}$$

Input the ICs:

$$y(0) = 2 = 2\,(0 + c_2)^{-2} \rightarrow c_2 = 1 \;\text{ or }\; c_2 = -1$$

To find c_2 we use the IC $y'(0) = 4$. If $c_2 = 1$,

$$y' = -4\,(x + 1)^{-3}$$

$$y'(0) = -4\,(0 + 1)^{-3} \neq 4$$

And if $c_2 = -1$,

$$y' = -4\,(x - 1)^{-3}$$

$$y'(0) = -4\,(0 - 1)^{-3} = -4 \quad \checkmark$$

The initial conditions hold only for $c_2 = -1$. To summarize, the unique solution is

$$y = 2(t - 1)^{-2}, \quad t \neq 1$$