Linear Algebra for Chemists — Assignment 4

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Question 1. Convert the system of equations to matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

Denote A as the coefficient matrix. The solution to the set of equations is given by $A^{-1}b$. Find A^{-1} via Guass-Seidel method.

$$\begin{aligned}
[\,A\,|\,I\,] \quad &= \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \\
&\xrightarrow{R_2 - R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \\
&\xrightarrow{R_1 \to R_1 - R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] = [\,I\,|\,A^1\,].
\end{aligned}$$

The solution to the system of equations is

$$A^{-1}b = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

Question 2. Write the augmented matrix and perform Guassian elimination.

$$\begin{aligned}
[\,A\,|\,I\,] \quad &= \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \\
&\xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to R_2 + \frac{3}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & -3 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] = [\,I\,|\,A^1\,].
\end{aligned}$$

$$A^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -\frac{3}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Question 3. Convert to matrix form.

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}.$$

Row-reduce the augmented matrix $[\,A \mid b\,]$.

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \xrightarrow[R_4 \to R_4 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$$\xrightarrow[R_4 \to \frac{1}{4}R_4]{R_2 \to -R_2,\ R_3 \to \frac{1}{5}R_3} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & \frac{9}{2} & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & \frac{9}{2} & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 0 & \frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 - 2R_2} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{3}R_3]{R_1 \to R_1 + 2R_2} \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus

$$\begin{aligned} x_6 &= \frac{1}{3} \\ x_3 + 2x_4 &= 0 \\ x_1 + 3x_2 + 4x_4 + 2x_5 & \quad 0 \end{aligned}$$

The general solution is given by $\left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3}\right)$ for arbitrary $x_2, x_4, x_5 \in F$.

Question 4.

a) The augmented coefficient matrix is

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & b_1 \\ 1 & 5 & 2 & 0 & b_2 \\ 2 & 9 & 5 & 3 & b_3 \\ 2 & 7 & 4 & 3 & b_4 \end{array}\right].$$

Row reduce the augemented matrix.

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & b_1 \\ 1 & 5 & 2 & 0 & b_2 \\ 2 & 9 & 5 & 3 & b_3 \\ 2 & 7 & 4 & 3 & b_4 \end{array}\right] \xrightarrow[R_4 \to R_4 - 2R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1}} \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & b_1 \\ 0 & 1 & 0 & -1 & b_2 - b_1 \\ 0 & 1 & 1 & 1 & b_3 - 2b_1 \\ 0 & -1 & 0 & 1 & b_4 - 2b_1 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 \to R_2 + R_4 \\ R_3 \to R_3 + R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & -7b_1 + 4b_4 \\ 0 & 0 & 0 & 0 & b_2 - 3b_1 + b_4 \\ 0 & 0 & 1 & 2 & b_3 - 4b_1 + b_4 \\ 0 & -1 & 0 & 1 & b_4 - 2b_1 \end{array}\right].$$

$$\xrightarrow{R_4 \leftrightarrow -R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & -7b_1 + 4b_4 \\ 0 & 1 & 0 & -1 & 2b_1 - b_4 \\ 0 & 0 & 1 & 2 & b_3 - 4b_1 + b_4 \\ 0 & 0 & 0 & 0 & b_2 - 3b_1 + b_4 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & b_1 - 2b_3 + 2b_4 \\ 0 & 1 & 0 & -1 & 2b_1 - b_4 \\ 0 & 0 & 1 & 2 & b_3 - 4b_1 + b_4 \\ 0 & 0 & 0 & 0 & b_2 - 3b_1 + b_4 \end{array}\right]$$

The system has a solution if the rank of the augmented matrix is no greater than the rank of the coefficient matrix. (b_1, b_2, b_3, b_4) must satisfy

$$b_2 - 3b_1 + b_4 = 0,$$

of which the general solution is $(b_1, 3b_1 - b_4, b_3, b_4)$ for arbitrary $b_1, b_3, b_4 \in F$.

b) $(b_1, b_2, b_3, b_4) = (-1, -4, -1, 1)$. Note that the condition from the previous section is satisfied:

$$b_2 - 3b_1 + b_4 = -4 + 3 + 1 \overset{\checkmark}{=\!=} 0.$$

The row echelon form of the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

from which we obtain $(x_1, x_2, x_3, x_4) = (3 - x_4, -3 + x_4, 4 - 2x_4, x_4)$ for $x_4 \in F$.

Question 5.

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & \lambda + 1 & 2 \\ \lambda & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2\lambda \end{bmatrix}.$$

Row-reduce the augmented matrix $[A|b]$.

$$[\,A\,|\,b\,] \quad = \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & \lambda+1 & 2 & 4 \\ \lambda & 1 & 1 & 2\lambda \end{array}\right] \xrightarrow[R_3 \to R_3 - \lambda R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-1 & -2 & 2 \\ 0 & 1-\lambda & 1-2\lambda & \lambda \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-1 & -2 & 2 \\ 0 & 0 & -(2\lambda+1) & \lambda+2 \end{array}\right] \xrightarrow{R_3 \to \frac{-R_3}{2\lambda+1},\quad \lambda \neq -\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-1 & -2 & 2 \\ 0 & 0 & 1 & -\frac{\lambda+2}{2\lambda+1} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{4\lambda+5}{2\lambda+1} \\ 0 & \lambda-1 & 0 & \frac{2(\lambda-1)}{2\lambda+1} \\ 0 & 0 & 1 & -\frac{\lambda+2}{2\lambda+1} \end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{\lambda-1},\quad \lambda \neq 1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{4\lambda+5}{2\lambda+1} \\ 0 & 1 & 0 & \frac{2}{2\lambda+1} \\ 0 & 0 & 1 & -\frac{\lambda+2}{2\lambda+1} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4\lambda+3}{2\lambda+1} \\ 0 & 1 & 0 & \frac{2}{2\lambda+1} \\ 0 & 0 & 1 & -\frac{\lambda+2}{2\lambda+1} \end{array}\right]$$

The system has

- a unique solution for $\lambda \neq -\frac{1}{2}$, $\lambda \neq 1$, as rank of the augmented matrix matches the rank of the matrix and equals the number of columns

- no solution for $\lambda = -\frac{1}{2}$, as the augmented matrix becomes inconsistent (see step 3):

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1.5 & -2 & 2 \\ 0 & 0 & 0 & 1.5 \end{array}\right]$$

- infinitely many solutions for $\lambda = 1$, as in this case we get a row of zeros (see step 5):

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{4\lambda+5}{2\lambda+1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\lambda+2}{2\lambda+1} \end{array}\right]$$

Question 6.

$$Ax = b \quad \text{is} \quad \begin{bmatrix} a & 0 & b \\ a & a & 4 \\ 0 & a & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ b \end{bmatrix}.$$

Row-reduce the augmented matrix $[A|b]$.

$$[\,A\,|\,b\,] \quad = \quad \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right].$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right]$$

The system has

- no solution for $(a,b) = (0,0)$, $(a,b) = (0,4)$, as the augemented matrix becomes inconsistent (we get a row of $[\,0 \;\; 0 \;\; 0 \,|\, 2\,]$).

- infinitely many solutions for $b = 2$ and arbitrary a, as the third row becomes zero and $\operatorname{rank}(A|b) < n$.

- a unique solution for $(a,b) \neq (0,0)$ or $(0,4)$ and for $b \neq 2$, $a, b \in F$, as in these cases $\operatorname{rank}(A|b) = \operatorname{rank}(A) = n$