

Assignment 3

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Page 38

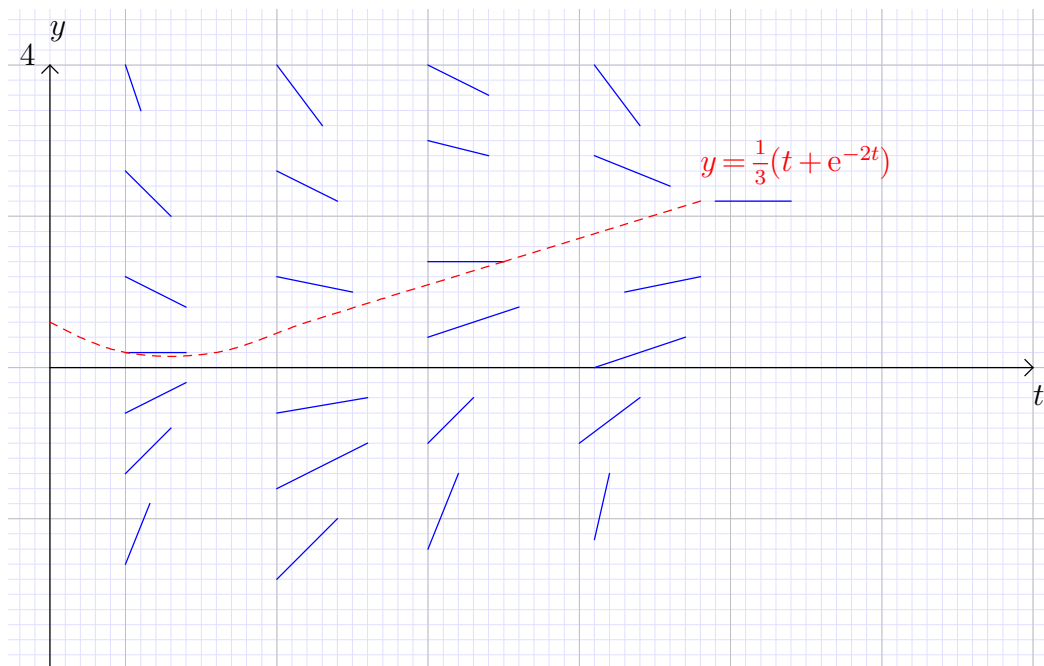
Question 1

(a) $y' + 3y = t + e^{-2t}$. Rewrite the equation:

$$y' = t + e^{-2t} - 3y$$

Inspect the expression for y' from different “angles”:

- While t is constant, y' becomes increasingly negative for positive y and increasingly positive for negative y .
- While y is constant, y' decreases a bit in the region $0 < t < \frac{\ln 2}{2}$, where $\partial y' / \partial t < 0$. after which, y' grows almost linearly as the term e^{-2t} effectively diminishes.
- $y' = 0$ on the curve $y = \frac{1}{3}(t + e^{-2t})$.



(b) Inspecting the direction field suggests that $y \rightarrow \infty$ as $t \rightarrow \infty$, and grows in a linear fashion.

(c) Solve using integration factor method. For an equation of the form

$$y' + a(t)y = b(t)$$

the solution is

$$y = e^{-\int a(t) dt} \left[\int b(t) e^{\int a(t) dt} dt + c \right] \quad (1)$$

In our case, $a = 3, b = t + e^{-2t}$.

$$\begin{aligned} y &= e^{-3t} \left[\int (t + e^{-2t}) e^{3t} dt + c \right] \\ &= e^{-3t} \left[\int t e^{3t} dt + \int e^t dt + c \right] \\ &= \left[e^{-3t} \int t e^{3t} dt \right] + e^{-2t} + c e^{-3t} \end{aligned}$$

The integral $\int t e^{3t} dt$ is solvable by parts, using the formula:

$$\int u' v dt = u v - \int u v' dt \quad (2)$$

pick $v = t, u' = e^{3t}$.

$$\begin{aligned} \int t e^{3t} dt &= \frac{1}{3} t e^{3t} - \frac{1}{3} \int e^{3t} dt \\ &= \frac{1}{3} e^{3t} \left(t - \frac{1}{3} \right) \end{aligned}$$

Therefore,

$$y = \frac{1}{3} t - \frac{1}{9} + e^{-2t} + c e^{-3t}$$

Because the exponents approach zero quickly, for large t the solution behaves according to $\frac{t}{3} - \frac{1}{9}$, or in other words, linearly with t .

Question 13

$$y' - y = 2t e^{2t}, \quad y(0) = 1$$

According to eq. (1), ($a = -1, b = 2t e^{2t}$)

$$y = e^t \left[\int 2t e^t dt + c \right]$$

Solve the integral by parts. Pick $u' = e^t, v = t$.

$$\int t e^t dt = t e^t - \int e^t dt = e^t(t - 1)$$

Therefore,

$$y = 2e^{2t}(t - 1) + c e^t$$

Input the IC to find c :

$$y(0) = 1 = 2(0 - 1) + c \rightarrow c = 3$$

The unique solution is $y = 2e^{2t}(t - 1) + 3e^t$.

Question 15

$$t y' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$$

Given $t \neq 0$ in the requested interval, we can divide by t to get

$$y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$$

According to eq. (1), $a = \frac{2}{t}, b = t - 1 + \frac{1}{t}$.

$$e^{\int a(t) dt} = e^{2 \int \frac{dt}{t}} = 2 \ln |t| = t^2$$

$$e^{-\int a(t) dt} = e^{-2 \int \frac{dt}{t}} = -2 \ln |t| = \frac{1}{t^2}$$

$$y = \frac{1}{t^2} \left[\int \left(t - 1 + \frac{1}{t} \right) t^2 dt + c \right]$$

$$\begin{aligned} \int \left(t - 1 + \frac{1}{t} \right) t^2 dt &= \int (t^3 - t^2 + t) dt \\ &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} \end{aligned}$$

Therefore,

$$y = \frac{1}{t^2} \left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c \right] = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}, \quad t > 0$$

Input IC:

$$y(1) = \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \rightarrow c = \frac{1}{12}$$

The unique solution is $y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$, $t > 0$.

Question 18

$$t y' + 2y = \sin t, \quad y(\pi/2) = 1$$

In the given IC, $t > 0$. Divide by t assuming $t \neq 0$.

$$y' + \frac{2}{t} y = \frac{\sin t}{t}$$

Using eq. (1), ($a = \frac{2}{t}, b = \frac{\sin t}{t}$)

$$e^{\int a dt} = e^{\int \frac{2}{t} dt} = t^2, \quad e^{-\int a dt} = \frac{1}{t^2}$$

$$y = \frac{1}{t^2} \left[\int \frac{\sin t}{t} \cdot t^2 dt + c \right]$$

Solve the integral using eq. (2). Pick $u' = \sin t, v = t$

$$\begin{aligned} \int \sin t \cdot t dt &= -t \cos t + \int \cos t dt \\ &= -t \cos t + \sin t \end{aligned}$$

The general solution is

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}, \quad t > 0$$

Input the IC:

$$y(\pi/2) = 1 = 0 + \frac{1}{\pi^2/4} + c \frac{1}{\pi^2/4} \rightarrow c = \frac{\pi^2}{4} - 1$$

The unique solution is $y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \left(\frac{\pi^2}{4} - 1 \right) \frac{1}{t^2}$, $t > 0$.

Question 19

$$t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0$$

For the given IC, $t < 0$. To find the specific solution we divide by t^3 , assuming $t \neq 0$.

$$y' + \frac{4}{t} y = \frac{1}{t^3} e^{-t}$$

Using eq. (1), $(a = \frac{4}{t}, b = \frac{1}{t^3} e^{-t})$

$$e^{\int \frac{4}{t} dt} = t^4, \quad e^{-\int \frac{1}{t^3} dt} = \frac{1}{t^4}$$

$$y = \frac{1}{t^4} \left[\int \frac{1}{t^3} e^{-t} \cdot t^4 dt + c \right]$$

Solve the integral. Pick $u' = e^{-t}, v = t$

$$\begin{aligned} \int t e^{-t} dt &= -t e^{-t} + \int e^{-t} dt \\ &= -e^{-t}(t+1) \end{aligned}$$

Therefore,

$$y = \frac{1}{t^4} [c - e^{-t}(t+1)], \quad t \neq 0$$

Input the IC:

$$y(-1) = 0 = c$$

The unique solution is $y = -\frac{1}{t^4} e^{-t}(t+1)$ in the domain $t < 0$.

Page 45

Question 1

$$\frac{dy}{dx} = \frac{x^2}{y}, \quad y \neq 0$$

Integrate:

$$\int y dy = \int x^2 dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c$$

$$y = \pm \sqrt{\frac{2}{3}x^3 + 2c}, \quad y \neq 0, \quad \left(\frac{x^3}{3} + c \right) > 0$$

Question 5

$$\frac{dy}{dx} = (\cos^2 x)(\cos^2 2y)$$

Rewrite and integrate:

$$\int \frac{dy}{\cos^2 2y} = \int \cos^2 x \, dx, \quad \cos^2 2y \neq 0$$

Let's focus on the RHS:

$$\begin{aligned} \int \cos^2 x \, dx &= \frac{1}{2} \int (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x + c \right) \end{aligned}$$

Back to the original equation:

$$\begin{aligned} \int \frac{dy}{\cos^2 2y} &= \frac{1}{2} \tan 2x = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x + c \right) \\ \tan 2y &= x + \frac{1}{2} \sin 2x + c, \quad \cos 2y \neq 0 \end{aligned}$$

The solution is given implicitly.

What if $\cos^2 2y \equiv 0$?

$$\cos 2y = 0 \iff 2y = \arccos(0) = \frac{\pi}{2} + \pi \cdot n, \quad n \in \mathbb{Z}$$

$$y = \frac{\pi}{4} + \frac{\pi}{2} \cdot n, \quad n \in \mathbb{Z}, \quad \forall x$$

This is also a solution.

Question 8

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

Integrate.

$$\int (1+y^2) \, dy = \int x^2 \, dx$$

$$y + \frac{y^3}{3} = \frac{x^3}{3} + c$$

The solution is given implicitly for all x and y .

Question 23

$$\frac{dy}{dx} = 2y^2 + x y^2, \quad y(0) = 1$$

Rewrite the equation:

$$\frac{dy}{dx} = y^2(2+x)$$

Integrate.

$$\int \frac{dy}{y^2} = \int (2+x) dx$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} + c, \quad y \neq 0$$

$$y = -\frac{1}{2x + \frac{x^2}{2} + c}, \quad \left(2x + \frac{x^2}{2} + c\right) \neq 0$$

Input the IC:

$$y(0) = 1 = -\frac{1}{c} \rightarrow c = -1$$

The specific solution is

$$y = -\frac{1}{2x + \frac{x^2}{2} - 1}, \quad \left(2x + \frac{x^2}{2} - 1\right) \neq 0$$

y attains its minimum value when $2x + \frac{x^2}{2} - 1$ reaches its maximum value.

$$\frac{d}{dx} \left(2x + \frac{x^2}{2} - 1 \right) = 2 + x$$

The expression equals zero when $x = -2$, for which

$$y_{\min} = -\frac{1}{-4 + 2 - 1} = \frac{1}{3}$$

Page 57

Question 1

Denote $y(t)$ the dye concentration in the tank. The equation for $y'(t)$ is as follows:

$$y'(t) = -2 \text{ L/min} \cdot \frac{1 \text{ g L}^{-1}}{200 \text{ L}} \cdot y(t) = -0.01 y(t), \quad y(0) = 1 \text{ g L}^{-1}$$

The solution to the DE is a decaying exponent:

$$y(t) = y(0) e^{-0.01t} = e^{-0.01t}$$

We are asked to calculate the time it takes for the dye concentration to reach 1% of its initial value. In other words, find t such that:

$$y(t) = 0.01 = e^{-0.01t}$$

$$t = -\frac{\ln 0.01}{0.01} \approx 460.5 \text{ min}$$

Question 4

Write the DE for the amount of salt in the tank ($y(t)$):

$$\begin{aligned} y'(t) &= 1 \text{ lb/gal} \cdot 3 \text{ gal/min} - 2 \text{ gal/min} \cdot \frac{y(t)}{200 \text{ gal} + 1 \text{ gal/min} \cdot t} \\ y'(t) &= 3 - \frac{2y(t)}{200+t} \end{aligned}$$

Note: the quotient $\frac{2y(t)}{200+t}$ represents the transient concentration at time t . The denominator represents the total volume of liquid in the tank at time t .

Solve the DE:

$$y' + \frac{2}{200+t} y(t) = 3$$

Solve according to eq. (1) ($a = \frac{2}{200+t}$, $b = 3$)

$$e^{\int a(t) dt} = e^{2 \int \frac{dt}{200+t}} = (200+t)^2$$

$$e^{-\int a(t) dt} = \frac{1}{(200+t)^2}$$

$$y = \frac{1}{(200+t)^2} \left[\int 3(200+t)^2 dt + c \right]$$

$$\begin{aligned} \int (200+t)^2 dt &= \int (t^2 + 400t + 40,000) dt \\ &= \frac{t^3}{3} + 200t^2 + 40,000t \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{(200+t)^2} \left[\int 3(200+t)^2 dt + c \right] = \frac{t^3 + 600t^2 + 120,000t + c}{(200+t)^2}$$

To find c we use the IC:

$$y(0) = 100 = \frac{c}{200^2} \rightarrow c = 4 \cdot 10^6$$

$$y(t) = \frac{t^3 + 600t^2 + 120,000t + 4 \cdot 10^6}{(200 + t)^2}$$

When does the solution begin to overflow? — When the volume reaches 500 gal.

$$V(t) = 500 = 200 + t \rightarrow t = 300 \text{ min}$$

Now let's calculate $y(300)$:

$$y(300) = \frac{300^3 + 600 \cdot 300^2 + 120,000 \cdot 300 + 4 \cdot 10^6}{(200 + 300)^2} = 484 \text{ lb}$$

The salt concentration at time of overflowing is $C = \frac{484 \text{ lb}}{500 \text{ gal}} = 0.968 \text{ lb gal}^{-1}$.

If the tank would have had infinite capacity, the limit concentration is achieved at $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} C = \lim_{t \rightarrow \infty} \frac{y(t)}{V(t)} = \lim_{t \rightarrow \infty} \left(\frac{t^3 + 600t^2 + 120,000t + 4 \cdot 10^6}{(200 + t)^2 \cdot (200 + t)} \right) = 1 \text{ lb gal}^{-1}$$

The limit approaches the value $C = 1$ because the highest order element of both the numerator and denominator is t^3 and in both its coefficient is 1.

Question 9

Given $r = 0.1$, the DE for the amount of debt left at time t is:

$$S'(t) = r S(t) - k$$

Let's solve.

$$S'(t) - 0.1S(t) = -k$$

According to eq. (1), ($a = -0.1, b = -k$)

$$e^{\int a(t) dt} = e^{\int -0.1 dt} = e^{-0.1t}$$

$$e^{-\int a(t) dt} = e^{0.1t}$$

$$S(t) = e^{0.1t} \left[\int -k e^{-0.1t} dt + c \right]$$

$$S(t) = 10k + c e^{0.1t}$$

To find c , use IC:

$$\begin{aligned} S(0) &= 8000 = 10k + c \\ c &= 8000 - 10k \\ \Rightarrow S(t) &= 10k + (8000 - 10k)e^{0.1t} \\ S(t) &= 8000 \cdot e^{0.1t} + 10k(1 - e^{0.1t}) \end{aligned}$$

If the debt is to be re-paid in 3 years, then:

$$\begin{aligned} S(3) &= 0 = 8000 \cdot e^{0.3} + 10k(1 - e^{0.3}) \\ |k| &= \frac{8000 \cdot e^{0.3}}{(1 - e^{0.3}) \cdot 10} = \$3086.64 \text{ yr}^{-1} \end{aligned}$$

The interest paid in 3 years is the amount returned minus the initial loan. In other words, the interest equals $3086.64 \times 3 = \$9259.92$ minus the initial loan of \$8000, for a total of \$1259.92.

Question 14

(a) We are given two points: $Q(t_{1/2}) = \frac{1}{2}Q(0) = 5730 \text{ yrs}$ and $Q(50,000) = 0.00236Q(0)$.

For a DE of the form $Q'(t) = -rQ(t)$ the solution is $Q(t) = Q(0)e^{-rt}$.

We can determine r using each of the points. For example, using the information of the half-life:

$$\frac{1}{2} = e^{-r \cdot 5730} \rightarrow r = \frac{\ln 2}{5730} = 1.21 \times 10^{-4} \text{ yrs}^{-1}$$

(b) The solution is a simple decaying exponent. It may be found via integration factor method: $Q'(t) + rQ(t) = 0$. Select $\mu(t) = e^{rt}$ and get $(Qe^{rt})' = 0$. Integrate to get $Q(t) = ce^{-rt}$, then input $t = 0$ to get $c = Q(0) \equiv Q_0$.

$$Q(t) = Q_0 e^{-1.21 \times 10^{-4} \cdot t}$$

(c) We need to find t for which $Q(t) = 0.2Q_0$.

$$\begin{aligned} 0.2 &= e^{-rt} \\ t &= \frac{\ln 5}{r} = 13,304.65 \text{ yrs} \end{aligned}$$