Assignment 4

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Question 29

$$t\,y'' + y' = 1, \quad t > 0$$

Substitute y' for v and divide by $t \neq 0$.

$$(t\,v)' = 1$$

Integrate and get

$$v = 1 + \frac{c_1}{t}$$

Integrate to find y.

$$y = \int v(t)\,\mathrm{d}t = t + c_1 \ln t + c_2, \quad t > 0$$

Question 30

$$y'' + t\,(y')^2 = 0$$

First, notice that $y \equiv k$, $k \in \mathbb{R}$ is a solution for all t.

Substitute y' for v.

$$v' + t\,v^2 = 0$$

This is a separable equation, which can be rewritten as:

$$\frac{\mathrm{d}v}{v^2} = -t\,\mathrm{d}t, \quad v \neq 0$$

Integrate both sides.

$$\frac{1}{v} = \frac{1}{2}t^2 - c_1$$

Isolate v.

$$v = \frac{2}{t^2 - 2c_1}$$

There are 3 different kinds of possible solutions, depending whether $c_1 < 0, c_1 = 0$, or $c_1 > 0$.

1. If $c_1 = 0$:

$$v = \frac{2}{t^2}$$

$$y = \int v\,\mathrm{d}t = -\frac{2}{t} + c_2, \quad t \neq 0$$

2. If $c_1 < 0$, substitute $2c_1 = -C^2$, $C > 0$.

$$v = \frac{2}{t^2 + C^2} = \frac{2}{C^2} \cdot \frac{1}{\left(\frac{t}{C}\right)^2 + 1}$$

$$y = \int v \, dt = \frac{2}{C^2} \cdot \frac{\arctan\left(\frac{t}{C}\right)}{1/C} + c_2$$

$$y = \frac{2}{C} \arctan\left(\frac{t}{C}\right) + c_2$$

3. If $c_1 > 0$, substitute $2c_1 = C^2$, $C > 0$.

$$v = = \frac{2}{t^2 - C^2} = 2 \frac{1}{(t - C)} \frac{1}{(t + C)}$$

$$y = \int v \, dt = 2 \frac{1}{2C} \int \frac{(t + C) - (t - C)}{(t + C)(t - C)} \, dt = \frac{1}{C} \int \left[\frac{1}{t - C} - \frac{1}{t + C} \right] dt$$

$$y = \frac{1}{C} \left[\ln \left| \frac{1}{t - C} \right| - \ln \left| \frac{1}{t + C} \right| \right] + c_2$$

$$y = \frac{1}{C} \ln \left| \frac{t + C}{t - C} \right| + c_2, \quad t \neq C$$

Question 32

$$y'' + y' = e^{-t}$$

Set $v = y'$ and solve using integration factor method.

$$v' + v = e^{-t}$$

$$v = e^{-t} \left(\int e^{-t} \cdot e^t \, dt + c_1 \right) = e^{-t}(t + c_1)$$

Now integrate v to find y:

$$y = \int v \, dt = \int (t e^{-t} + c_1 e^{-t}) \, dt$$

Via integration by parts,

$$\int t e^{-t} = -t e^{-t} - e^{-t} = -e^{-t}(t + 1)$$

$$\Rightarrow y = -e^{-t}(t + 1) - c_1 e^{-t} + c_2$$

$$y = -t e^{-t} - (c_1 + 1) e^{-t} + c_2$$

Question 36

$$y'' + y (y')^3 = 0$$

First observe that $y \equiv k$, $k \in \mathbb{R}$ is a solution.

Set $y' = v$.

$$v' + y v^3 = 0$$

2

Write v' as $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}y} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}y} \cdot v$

$$\frac{\mathrm{d}v}{\mathrm{d}y} \cdot v + y \cdot v^3 = 0$$

This is a separable equation. Divide by v assuming $v \neq 0$ (If $v = 0$ then $y \equiv k$ is a solution we've already found.) Rewrite as:

$$\frac{\mathrm{d}v}{v^2} = -y\,\mathrm{d}y$$

Integrate both sides to get

$$-\frac{1}{v} = -\frac{1}{2}y^2 + c_1$$

$$v = \frac{2}{y^2 - 2c_1}$$

Integrate again to find y.

$$y = \int v\,\mathrm{d}x = \frac{2}{y^2 - 2c_1} \cdot t + c_2$$

Notice that to achieve a continuous solution we must set $y \neq \sqrt{2c_1}$. However, $y \equiv \sqrt{2c_1}$ does solve the original DE. Anyway, here y is given implicitly.

Question 37

$$2y^2\,y'' + 2y\,(y')^2 = 1$$

Set $y' = v$.

$$2y^2\,v' + 2y\,v^2 = 1$$

Divide by $2y^2$ assuming $y \neq 0$.

$$v' + \frac{1}{y}\,v^2 = \frac{1}{2y^2}$$

Input $v' = \dfrac{\mathrm{d}v}{\mathrm{d}y} \cdot v$

$$\frac{\mathrm{d}v}{\mathrm{d}y}\,v + \frac{1}{y}\,v^2 = \frac{1}{2y^2}$$

Set $f(y) = \frac{1}{2}v^2$. $f' = \dfrac{\mathrm{d}f}{\mathrm{d}y} = v\,\dfrac{\mathrm{d}v}{\mathrm{d}y}$. Therefore,

$$\frac{\mathrm{d}f}{\mathrm{d}y} + \frac{2}{y}\,f = \frac{1}{2y^2}$$

Solve using integration factor method. $\left(a = \frac{2}{y}, b = \frac{1}{2y^2};\ f = \mathrm{e}^{-\int a(y)\,\mathrm{d}y}\left[\int b(y)\,\mathrm{e}^{\int a(y)\,\mathrm{d}y} + c_1\right] \right)$

$$f = \frac{1}{y^2}\left(\int \frac{1}{2y^2} \cdot y^2\,\mathrm{d}y + c_1\right) = \frac{1}{y^2}\left(\frac{1}{2}y + c_1\right)$$

$$f = \frac{1}{2y} + \frac{c_1}{y^2}$$

Input back $f = \dfrac{v^2}{2}$ and get

$$v^2 = \frac{1}{y} + \frac{2c_1}{y^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\left(\frac{1}{y} + \frac{2c_1}{y^2}\right)^{1/2} = \pm\sqrt{\frac{y + 2c_1}{y^2}} = \pm\frac{\sqrt{y + 2c_1}}{|y|}$$

This is a separable equation

$$\pm \int \frac{|y|}{\sqrt{y + 2c_1}} \, dy = \int dx = x + c_2$$

Because we already have $\pm$ as a prefix, we don't have to take the absolute value of y.

Let's focus on the LHS: substitute $u = y + 2c_1$. $du = dy$.,

$$\pm \int \frac{u - 2c_1}{\sqrt{u}} \, du = \pm \int (\sqrt{u} - 2c_1 \, u^{-1/2}) du = \pm \left(\frac{2}{3} u^{3/2} - 2c_1 \cdot 2\sqrt{u} \right)$$

Substitute back u and simplify:

$$\pm \left[\frac{2}{3} (y + 2c_1)^{3/2} - 4c_1 \, (y + 2c_1)^{1/2} \right] = \pm \left[\frac{2}{3} \sqrt{y + 2c_1} \, (y + 2c_1 - 6c_1) \right] = \pm \left[\frac{2}{3} \sqrt{y + 2c_1} \, (y - 4c_1) \right]$$

Finally,

$$\pm \left[\frac{2}{3} \sqrt{y + 2c_1} \, (y - 4c_1) \right] = x + c_2, \quad y > -2c_1$$

Question 40

$$y' \, y'' = 2, \quad y(0) = 1, y'(0) = 2$$

Set $v = y'$

$$\frac{dv}{dx} = \frac{2}{v}, \quad v \neq 0$$

Solve separable equation:

$$\int v \, dv = \int 2 \, dx$$

$$\frac{1}{2} v^2 = 2x + c_1$$

$$\frac{dy}{dx} = v = \pm\sqrt{4x + 2c_1}$$

From the ICs we have $v(0) = y'(0) = 2 > 0$. This can simplify the solution:

$$\frac{dy}{dx} = v = \sqrt{4x + 2c_1}$$

Integrate again to find y.

$$y = \frac{1}{6} (4x + 2c_1)^{3/2} + c_2$$

Find specific solution to find ICs: $y'(0) = v(0) = 2$. Input in $\frac{1}{2} v^2 = 2x + c_1$ and get:

$$\frac{1}{2} \cdot 2^2 = 2 \cdot 0 + c_1 \rightarrow c_1 = 2$$

$$y(0) = 1 = \frac{1}{6} (4 \cdot 0 + 2 \cdot 2)^{3/2} + c_2 \rightarrow c_2 = -\frac{1}{3}$$

The specific solution is

$$y = \frac{1}{6}\left(4x+4\right)^{3/2} - \frac{1}{3}$$

$$y = \frac{4}{3}\left(x+1\right) - \frac{1}{3}$$

Question 41

$$y'' - 3y^2 = 0, \quad y(0) = 2, \, y'(0) = 4$$

Set $v = y'$

$$v' - 3y^2 = 0$$

Write v' as $v\frac{dv}{dy}$

$$v\frac{dv}{dy} = 3y^2$$

$$\int v\,dv = \int 3y^2\,dy$$

$$\frac{v^2}{2} = y^3 + c_1$$

$$\frac{dy}{dx} = v = \pm\sqrt{2}\,\sqrt{y^3 + c_1}$$

$$\pm\int (y^3 + c_1)^{-1/2}\,dy = \sqrt{2}\int dx$$

The LHS integral is unsolvable. We can try to find c_1 and may solve the integral if we found that, perhaps, $c_1 = 0$. We know that $y(0) = 2$ and that $y'(0) = v(0) = 4$. If we input these in the equation

$$\frac{v^2}{2} = y^3 + c_1$$

from above, we get:

$$\frac{4^2}{2} = 2^3 + c_1 \rightarrow c_1 = 0$$

Fortunately, now the integral is solvable! Input $c_1 = 0$ and get

$$\int (y^3 + 0)^{-1/2}\,dy = \int y^{-3/2}\,dy = -2y^{-1/2}$$

Therefore,

$$\pm(-2y^{-1/2}) = \sqrt{2}\,(x + c_2)$$

Raise both sides by -2:

$$y = 2(x + c_2)^{-2}$$

Input the ICs:

$$y(0) = 2 = 2\,(0 + c_2)^{-2} \rightarrow c_2 = 1$$

The unique solution is

$$y = 2(x+1)^{-2}$$