Sample Exam Solution

Question 1.

Prove that the union of two subspaces is a subspace if and only if one contains the other as a subset.

Let V be a vector space above $\mathbb{F}$ and let U, W be subspaces of V, then $U \cup W$ is a subspace of V iff $U \subseteq W$ or $W \subseteq U$.

First direction: Given $U \cup W$ is a subspace and prove that $U \subseteq W$ or $W \subseteq U$.

Given $u \in U, w \in W$, we have $u + w \in U \cup W$ as it is closed under addition. By definition of the union of sets, $u + w \in U$ or $u + w \in W$. Additionally, as U, W are subspaces, each contains its respective additive inverse $-u$ and $-w$ and is closed under addition.

If $u + w \in U$ and $-u \in U$ then $(u + w) + (-u) = w \in U$. Any $w \in W$ is in U so $W \subseteq U$.

If $u + w \in W$ and $-w \in W$ then $(u + w) + (-w) = u \in W$. Any $u \in U$ is in W so $U \subseteq W$.

Second direction: Given $U \subseteq W$ or $W \subseteq U$ prove that $U \cup W$ is a subspace.

If $U \subseteq W$ then $U \cup W = W$, and W is a subspace.

If $W \subseteq U$ then $U \cup W = U$, and U is a subspace.

Question 2.

Suppose $T: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation such that:

$$T\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

1. Find bases for the kernel and image of T.
2. Calculate $T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

1. Find $\mathcal{M}(T)$, the matrix representation of T with respect to some basis. According to the fundumental theorem of linear maps, given $T: V \to W$, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

   Linear maps are additive, so

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = T\overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}}^{u_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right) = T\overbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}^{u_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u_1, u_2 are not proportional and therefore linearly independent, and also belong to null T, so $\dim \operatorname{null} T \geq 2$.

Similarly, $v_1 = [3,2,1]^T$ and $v_2 = [1,0,1]^T$ are linearly independent and also belong to range T, so $\dim \operatorname{range} T \geq 2$.

By the theorem above, $\dim \operatorname{range} T = 2$ and $\dim \operatorname{null} T = 2$, so $\{u_1, u_2\}$ is a basis for null T and $\{v_1, v_2\}$ is a basis for range T.

2. Because $T(v+u) = Tv + Tu$ and $T(\alpha v) = \alpha Tv$, we need to find scalars a, b, c, d such that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + d \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \tag{1}$$

then

$$\begin{aligned} T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= a\,T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b\,T \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c\,T \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + d\,T \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \\ &= a \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 + 0. \end{aligned}$$

Eq. (1) is equivalent to solving the system

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the augmented system

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & -2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -7 & 2 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4\left(-\frac{1}{7}\right)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{2}{7} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_4]{R_2 \to R_2 + 3R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 & \frac{1}{7} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{2}{7} \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{2}{7} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 0 & 1 & -\frac{2}{7} \end{array}\right]$$

We can conclude that

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{2}{7} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ -\frac{4}{7} \\ \frac{5}{7} \end{bmatrix}.$$

Question 3.

Let V be the vector space of complex polynomials of degree less or equal to 3. Define two subspaces of V:

$$U = \{p(x) \in V \,|\, p(-1) = p(1) = 0\}, \text{ and } W = \{p(x) \in V \,|\, p(\mathrm{i}) = 0\}.$$

1. Find the dimensions of each of the subspaces $U, W, U \cap W, U + W$.

2. Find a basis for each of the subspaces above and determine whether the sum $U + W$ is direct.

a) $p_u(x) \in U$ is a polynomial of the form

$$p_u(x) = (x-1)(x+1)(a\,x+b), \quad a, b \in \mathbb{C}.$$

Rewrite $p_u(x)$ as

$$\begin{aligned} p_u(x) &= (x^2-1)(a\,x+b) \\ &= a\,x^3 + b\,x^2 - a\,x - b \\ &= a\,(x^3-x) + b\,(x^2-1). \end{aligned}$$

The set $\{x^3-x, x^2-1\}$ is a linearly independent spanning set of U, and therefore a basis for U. Thus, $\dim U = 2$.

b) $p_w(x) \in W$ is a polynomial of the form

$$p_w(x) = (x-\mathrm{i})(a\,x^2+b\,x+c), \quad a, b, c \in \mathbb{C}.$$

Rewrite $p_w(x)$ as

$$\begin{aligned} p_w(x) &= a\,x^3 + b\,x^2 + c\,x - a\,\mathrm{i}\,x^2 - b\,\mathrm{i}\,x - \mathrm{i}\,c \\ &= a\,(x^3 - \mathrm{i}\,x^2) + b\,(x^2 - \mathrm{i}\,x) + c\,(x - \mathrm{i})\,. \end{aligned}$$

The set $\{x^3 - \mathrm{i}\,x^2, x^2 - \mathrm{i}\,x, x - \mathrm{i}\}$ is a linearly independent spanning set of W, and therefore a basis for W. Thus, $\dim W = 3$.

c)

$$U \cap W = \{p(x) \in V \,|\, p(-1) = p(1) = p(\mathrm{i}) = 0\}\,.$$

A polynomial $p(x) \in U \cap W$ is of the form

$$p(x) = (x-1)\,(x+1)\,(x-\mathrm{i})\,a, \quad a \in \mathbb{C}.$$

$$\begin{aligned} p(x) &= a\,(x^2 - 1)\,(x - \mathrm{i}) \\ &= a\,(x^3 - \mathrm{i}\,x^2 - x + \mathrm{i}) \end{aligned}$$

The set $\{x^3 - \mathrm{i}\,x^2 - x + \mathrm{i}\}$ is a linearly independent spanning set of $U \cap W$, and therefore a basis. Thus, $\dim U \cap W = 1$. Since $U \cap W \neq \{0\}$, $U + W$ is **not** a direct sum.

d) There is the theorem that given subspaces U, W: $\dim U + W = \dim U + \dim W - \dim U \cap W$.

In our case, $\dim U + W = 2 + 3 - 1 = 4$. Since $\dim V = 4$, we can conclude that $U + W = V$, so we can just chose the standard basis $\{1, x, x^2, x^3\}$.

Question 4.

Given the following system of equations:

$$\begin{bmatrix} 1 & 1+k & 1 \\ 1+k & 1 & 1 \\ 1 & 1 & 1+k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ 2\,k \end{bmatrix},$$

1. Determine for which values of k, if any, does the system have no solution.
2. Determine for which values of k, if any, does the system have a unique solution, and solve it.
3. Determine for which values of k, if any, does the system have infinitely many solutions, and solve it.

Row reduce the augemented system $A|b$

$$\left[\begin{array}{ccc|c} 1 & 1+k & 1 & 0 \\ 1+k & 1 & 1 & k \\ 1 & 1 & 1+k & 2\,k \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - (1+k)\,R_1} \left[\begin{array}{ccc|c} 1 & 1+k & 1 & 0 \\ 0 & -k\,(k+2) & -k & k \\ 0 & -k & k & 2\,k \end{array}\right] \tag{2}$$

$$\xrightarrow{R_2 \to R_2 - (k+2)\,R_3} \left[\begin{array}{ccc|c} 1 & 1+k & 1 & 0 \\ 0 & 0 & -k\,(k+3) & -k\,(2\,k+3) \\ 0 & -k & k & 2\,k \end{array}\right] \tag{3}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1+k & 1 & 0 \\ 0 & -k & k & 2\,k \\ 0 & 0 & -k\,(k+3) & -k\,(2\,k+3) \end{array}\right] \tag{4}$$

The system is in echelon form. Check special cases for k:

a) $k = 0$. We get

$$A|b = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This corresponds to the equation $x + y + z = 0, \quad x, y, z \in \mathbb{F}$. There are more free variables than equations, so there are infinitely many solutions of the form $(x, y, z) = (x, -x, -x)$.

b) $k = -3$. We get

$$A|b = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -9 \end{array}\right].$$

$\operatorname{rank} A < \operatorname{rank} A|b$ so there is no solution.

c) $k = -\frac{3}{2}$. We get

$$A|b = \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & -3 \\ 0 & 0 & \frac{9}{4} & 0 \end{array}\right].$$

$\operatorname{rank} A = \operatorname{rank} A|b$ so there is a solution. From the third row $z = 0$, so from the second row $y = -2$, and from the first row $x = -1$. $(x, y, z) = (-1, -2, 0)$.

d) Continue solving the augmented system. For $k \neq 0, -3$ we have $\operatorname{rank} A = \operatorname{rank} A|b$ so there is a solution. Copy the augemented system here:

$$\left[\begin{array}{ccc|c} 1 & 1+k & 1 & 0 \\ 0 & -k & k & 2k \\ 0 & 0 & -k(k+3) & -k(2k+3) \end{array}\right].$$

From the third row,

$$z = \frac{2k+3}{k+3},$$

and from the second row $y = (z - 2) = \left(\frac{2k+3-2k-6}{k+3}\right) = -\frac{3}{k+3}$. From the first row, then,

$$x - (1+k)\frac{3}{k+3} + \frac{2k+3}{k+3} = 0$$

$$x + \frac{2k+3-3-3k}{k+3} = 0$$

$$x = \frac{k}{k+3}$$

so the solution is $(x,y,z)=\left(\frac{k}{k+3},-\frac{3}{k+3},\frac{2k+3}{k+3}\right)$ for $k\neq 0,-3$

Question 5.

1. Find eigenvectors and eigenvalues for the matrix $A=\left[\begin{smallmatrix}2 & -3\\ 1 & -2\end{smallmatrix}\right]$

2. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

3. Use the above to calculate A^{99}.

1. Find when $\det(A-\lambda I)=0$.

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda & -3\\ 1 & -(2+\lambda)\end{vmatrix}=-(2-\lambda)(2+\lambda)+3=\lambda^2-1 \quad\Longrightarrow\quad \lambda_{1,2}=\pm 1.$$

find $\operatorname{null}(A-\lambda_{1,2}I)$. For $\lambda_1=1$ we have

$$A-\lambda_1 I=\begin{bmatrix}1 & -3\\ 1 & -3\end{bmatrix} \quad\Rightarrow\quad v_1=\begin{bmatrix}3\\ 1\end{bmatrix}.$$

For $\lambda_2=-1$ we have

$$A-\lambda_2 I=\begin{bmatrix}3 & -3\\ 1 & -1\end{bmatrix} \quad\Rightarrow\quad v_2=\begin{bmatrix}1\\ 1\end{bmatrix}.$$

2. The matrix P is composed of the eigenvectors of A and the diagonal matrix has the corresponding eigenvalues on the diagonal.

$$P=\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix},\quad P^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix},\quad D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

3. We saw in class that

$$A=PDP^{-1},\quad A^k=PD^kP^{-1}=P\begin{bmatrix}d_{11}^k & & 0\\ & \ddots & \\ 0 & & d_{22}^k\end{bmatrix}P^{-1}.$$

Therefore,

$$\begin{aligned}A^{99} &= \begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1^{99} & 0\\ 0 & (-1)^{99}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{3}{2}\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}3 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}4 & -6\\ 2 & -4\end{bmatrix}=\begin{bmatrix}2 & -3\\ 1 & -2\end{bmatrix}.\end{aligned}$$

Question 6.

1. Show there is no solution to the system

$$A\,v = b\colon \quad \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

2. Find a least squares solution to the above system.

1. From the second row $x = 0$ and so from the third row $y = 0$, but that contradicts the first row, which leaves us with $1 \cdot 0 + 1 \cdot 0 = 0 = 1$. There is no solution.

2. Given the subspace $U = C(A)$ (column space of A) and the point b, find the point $u \in U$ such that $\|b - u\|$ is minimal. We've seen in class that the solution is $u = P_U b$, where P_U is the orthogonal projection operator. Given an orthonormal basis $\{e_1, \ldots, e_m\}$ for U,

$$P_U\, b = \langle b, e_1 \rangle\, e_1 + \cdots + \langle b, e_m \rangle\, e_m.$$

Use Gram-Schmidt process to find an orthogonal basis for $C(A)$ given the basis

$$C(A) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\} \equiv \operatorname{span}\{v_1, v_2\}.$$

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle\, e_1}{\|v_2 - \langle v_2, e_1 \rangle\, e_1\|}.$$

$$\begin{aligned}
v_2 - \langle v_2, e_1 \rangle\, e_1 &= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{1}{6} \left\langle \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \frac{1-3}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{8}{3} \end{bmatrix} \\
\|v_2 - \langle v_2, e_1 \rangle\, e_1\| &= \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{64}{9}} = \sqrt{\frac{84}{9}}
\end{aligned}$$

$$e_2 = \frac{1}{\sqrt{84}} \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
P_U b &= \langle b, e_1\rangle\, e_1 + \langle b, e_2\rangle\, e_2 \\
&= \frac{1}{6}\left\langle \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\rangle \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{84}\left\langle \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \right\rangle \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \\
&= -\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 4-1 \\ 2-2 \\ 8+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.
\end{aligned}$$

We found $u = P_U b$ $\|u - b\|$ is minimal, but the question was to find v such that $A\,v = u\ldots$ v is actually the solution of

$$A^T A\, v = A^T b.$$

$$A^T A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}.$$

$$(A^T A)^{-1} = \frac{1}{56}\begin{bmatrix} 10 & 2 \\ 2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}.$$

$$\begin{aligned}
v &= (A^T A)^{-1} A^T b \\
&= \frac{1}{56}\begin{bmatrix} 10 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \frac{1}{56}\begin{bmatrix} 0 \\ 56 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

We can actually verify that

$$A\,v = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = P_U b.$$