Differential Equations for Chemists

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Preliminary: Complex numbers and functions

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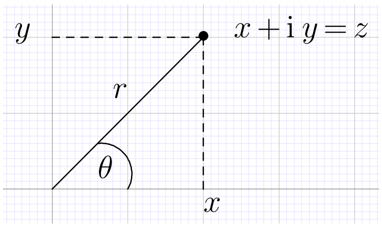
Recollections and definitions

Complex numbers are denoted by $\mathbb{C}=\{x+\mathrm{i}\,y|x,y\in\mathbb{R}\}$ in Cartesian representation. In Polar representation, we have:



The complex number z can be written using both (x,y) and (r,θ), so that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x}, \qquad \text{if} \quad x>0 \end{aligned}$$

Recall that the *complex conjugate* $\bar{z}$ is defined by: $\bar{z}=x-\mathrm{i}\,y$ and that the *modulus* squared is

$$z\cdot\bar{z}=\|z\|^2=r^2=x^2+y^2$$

so

$$\|z\|=\sqrt{x^2+y^2} \quad .$$

Another notation:

$$\frac{1}{z}=\frac{\bar{z}}{\|z\|^2}$$

$$\begin{aligned} z^2=(r\cos\theta+\mathrm{i}\,r\sin\theta)^2 &= r^2(\cos^2\theta-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta) \\ &= r^2(\cos 2\theta+\mathrm{i}\sin 2\theta) \end{aligned}$$

<u>De Moivre's formula:</u>

1. $z^n=r^n(\cos n\theta+\mathrm{i}\sin n\theta), \quad n\in\mathbb{N}$ (integer)

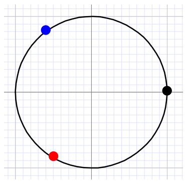
2. if $z^n = r(\cos\theta + \mathrm{i}\sin\theta)$ then:

$$z = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + \mathrm{i}\sin\frac{\theta + 2k\pi}{n}\right), \quad 0 \le k \le n-1$$

yields n distinct roots.

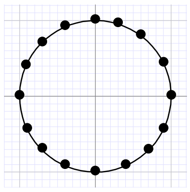
Examples:

1. $z^3 = 1$. $[r = 1$ and $\theta = 0]$



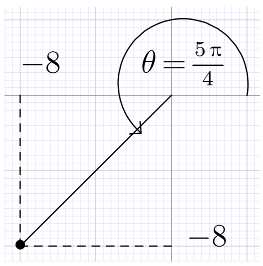
then $z = \cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3}$ where $k = 0, 1, 2$

2. $z^n = 1$. $z = \cos\frac{2\pi k}{n} + \mathrm{i}\sin\frac{2\pi k}{n}$ where $0 \le k \le n-1$



n points are distributed equally on the circle.

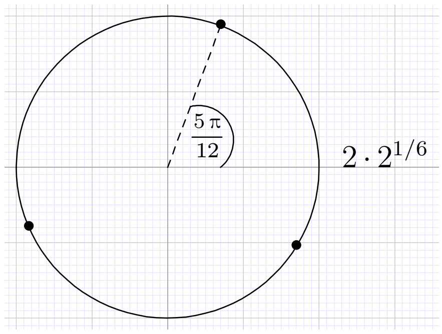
3. $z^3 = -8 - 8\mathrm{i}$. $r = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}$ and $\theta = 5\pi/4$.



so

$$z = (8\sqrt{2})^{1/3}\cdot\left(\cos\frac{\frac{5\pi}{4} + 2k\pi}{3} + \sin\frac{\frac{5\pi}{4} + 2k\pi}{3}\right)$$

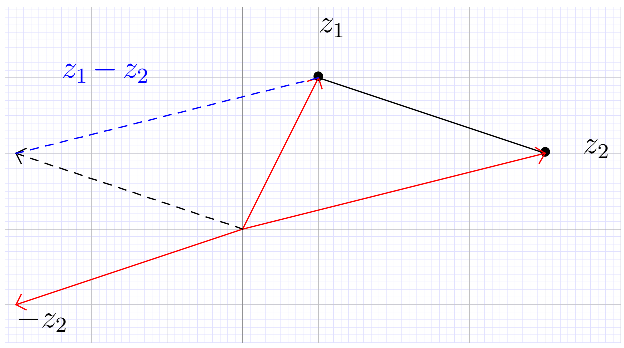
$$z = 2\cdot 2^{1/6}\left(\cos\frac{5\pi + 8k\pi}{12} + \sin\frac{5\pi + 8k\pi}{12}\right)$$



All points are shifted by $5\pi/12$.

Complex analysis

If we have two points, z_1 and z_2, the distance between them is $z_1 - z_2 = z_1 + (-z_2)$.



In other words, $\|z_1 - z_2\|$ is the distance between z_1 and z_2 in the complex plane, which is a non-negative real number.

Can we use the distance function to define limits and continuity functions etc...?

Limit of a complex sequence

If given the sequence of numbers $\{z_n\}|_{n=1}^{\infty} \subseteq \mathbb{C}$, we say $z_n \to w$, $w \in \mathbb{C}$, if $\|z_n - w\| \to 0$ as $n \to \infty$.

For example, given $z = \frac{1+3\,\mathrm{i}}{n}$, as $n \to \infty$ the modulus $\|z_n\| = \frac{\sqrt{10}}{n} \to 0$ so the sequence goes to zero in the complex plane.

Limit of a series of complex numbers

Given the sequence $\{z_n\}_{n=1}^{\infty}$, the series $\sum_{n=1}^{\infty} z_n$ converges to u if:

$$\left\| \sum_{n=1}^{k} z_n - u \right\| \xrightarrow[k \to \infty]{} 0$$

For example, given $z_n = z^n$ where $z \in \mathbb{C}$. If $\|z\| < 1$, we have:

$$\sum_{n=0}^{k} z^n \;=\; 1 + z + z^2 + \cdots + z^k = \frac{1 - z^{k+1}}{1 - z}$$

$$(1 + z + \cdots + z^k)(1 - z) = 1 - z^{k+1}$$

Note that if $\|z\| < 1$ then $\|z^{k+1}\| \to 0$ as $k \to \infty$. So

$$\left\| \sum_{n=1}^{k} z^n - \frac{1}{1 - z} \right\| = \left\| \frac{-z^{k+1}}{1 - z} \right\| \xrightarrow[k \to \infty]{} 0$$

Another example: $z = \frac{1}{2} + \frac{1}{2}\mathrm{i}$. The norm is $\|z\| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$. So

$$\sum_{k=1}^{\infty} \left(\frac{1 + \mathrm{i}}{2}\right)^k = \frac{1}{1 - \left(\frac{1}{2} + \frac{1}{2}\mathrm{i}\right)} = \frac{1}{\frac{1}{2} - \frac{1}{2}\mathrm{i}} = \frac{\frac{1}{2} + \frac{1}{2}\mathrm{i}}{\frac{1}{2}} = 1 + \mathrm{i}$$

Complex (and analytic) functions

If we have a function f such that $f\colon \mathbb{C} \to \mathbb{C}$, we say that w is the limit of f at z_0 if $\|f(z) - w\| \to 0$ as $z \to z_0$.

Define *continuity*: f is continuous at the point z_0 if the limit $\lim\limits_{z \to z_0} f(z)$ exists and equals $f(z_0)$.

Definition

$f\colon \mathbb{C} \to \mathbb{C}$ is *analytic* at z_0 if for the limit:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and $h \in \mathbb{C}$. Then we denote the limit by $f'(z_0) = \frac{\mathrm{d}f}{\mathrm{d}z}|_{z = z_0}$.

If f is analytic at every point we say f is an analytic function.

Turns out that if $f'(z_0)$ exists then so does $f^{(n)}(z)$ $\forall n$ [all higher derivatives of f also exist].

Reminder: Taylor series

A (real) *power series* is a series of the form $\sum_{n=0} a_n x^n$ which converges for some $|x| < r$ $(x \in \mathbb{R})$.

Taylor's theorem

If $f\colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable in a neighborhood (nbhd) of some point a, then we can represent f *uniquely* as a power series of the form:

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

where

$$C_n = \frac{f^{(n)}(a)}{n!}$$

In particular, if $a = 0$ we get the special case

$$f(x) = \frac{\sum f^{(n)}(0)}{n!} x^n$$

(also called the Maclaurin series).

Example:

$f(x) = \sum_{n=0}^{\infty} x^n$ where $x \in \mathbb{R}$ and $|x| < 1$. We know that

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So $f(x) = \sum_{n=0}^{\infty} x^n$ must be a Taylor series.

Taylor theorem for analytic functions

If f is analytic (everywhere) then it can be represented as a power series $\sum C_n z^n$ where $C_n = \frac{f^{(n)}(0)}{n!}$.

We can use convergent series to define complex analogues of real functions which have Taylor series.

Examples:

1. $f(x) = \mathrm{e}^x$. Because

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^x\right)_{x=0} = 1 \quad \forall n$$

we get

$$C_n = \frac{1}{n!}$$

Which means its Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x$.

We can use this to define a complex function:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which in fact does converge for all $z \in \mathbb{C}$.

In fact e^z "behaves" like an exponential function as we have:

$$\begin{aligned} e^{z_1+z_2} &= e^{z_1} \cdot e^{z_2} \\ e^{z_1 \cdot z_2} &= (e^{z_1})^{z_2} \end{aligned} \tag{1}$$

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2. $f(x) = \sin x$.

$$\sin x = \sum_{n=0}^{\infty} C_n x^n, \qquad C_n = \frac{f^{(n)}(0)}{n!}$$

$$\begin{aligned} \sin' x &= \cos x \\ \cos' x &= -\sin x \\ -\sin' x &= -\cos x \\ -\cos' x &= \sin x \end{aligned}$$

substituting $x = 0$ we get a repeating mini series of $1, 0, -1, 0, \ldots$. If we plug the coefficients we get:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Substituting $z \in \mathbb{C}$ gives a convergent series which we define to be

$$\sin z = \sum_{\text{odd} \quad n} \frac{(-1)^n \cdot x^n}{n!}$$

3. $f(x) = \cos x$.

$$\begin{aligned} \cos' x &= f'(x) = -\sin x \\ f''(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

We get the Taylor series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Substituting $z \in \mathbb{C}$ gives a convergent series which we define to be

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Differentiating our series for $\sin z$ term by term we obtain (just as in the real case) $\sin' z = \cos z$, $\cos' z = -\sin z$. So, if a function is represented by a series that converges *uniformly*, then its derivative can be obtained by differentiating the series term by term.

One of the properties of exponential functions is that using the relations from (1), if we write $z = x + \mathrm{i}\, y$ ($x, y \in \mathbb{R}$), then $\mathrm{e}^z = \mathrm{e}^x \cdot \mathrm{e}^{\mathrm{i} y}$. And by definition:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i} y} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\, y)^n}{n!} = 1 + \frac{\mathrm{i}\, y}{1} - \frac{y^2}{2!} - \frac{\mathrm{i}\, y^3}{3!} + \frac{y^4}{4!} - \cdots \\
&= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + \mathrm{i}\left(y - \frac{y^3}{3!} + \frac{y^5}{5!}\right) \\
&= \cos y + \mathrm{i} \sin y
\end{aligned}$$

We just got Euler's formula! Additional useful relations:

$$\boxed{\begin{gathered} \mathrm{e}^z = \mathrm{e}^x (\cos y + \mathrm{i} \sin y) \\ \\ z = r(\cos\theta + \mathrm{i}\sin\theta) = r \cdot \mathrm{e}^{\mathrm{i}\theta} \end{gathered}}$$

Note.

- If f is infinitely differentiable (real or analytic complex) then we have a unique Taylor series at 0, so any power series representation for f will be an alternative form of the Taylor series.

- Taylor series converge uniformly so if f is infinitely differentiable then so is f', therefore it also has a Taylor series. In other words

$$\text{if } f(x) = \sum C_n x^n \text{ then } f'(x) = \sum_{n=0}^{\infty} n C_n x^{n-1}$$

  is the Taylor series for f'. Similarly, if f is *integrable*, we get

$$\int f(x) \mathrm{d}x = \sum C_n \frac{x^{n+1}}{n+1}$$

  is the Taylor series for $\int f(x) \mathrm{d}x$.

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Examples

1. We want to calculate the Taylor series for $\arctan(x)$ at $x=0$.

$$\arctan x = \sum_{n=0}^{\infty} \frac{\arctan^{(n)}(0)}{n!} x^n$$

Calculating directly we get:

$$\begin{aligned}
\arctan(0) &= 0 \\
\arctan'(x) &= \frac{1}{x^2+1}; \quad \arctan'(0) = 1 \\
\arctan''(x) &= \frac{-2x}{(x^2+1)^2}; \quad \arctan''(0) = 0
\end{aligned}$$

Further derivatives become more difficult to calculate. Fret naught, there is a shortcut! If we find a power series that fits in some form, we know it is actually *the* Taylor series. If we look at

$$\begin{aligned}
(1-y+y^2-y^3+y^4+\cdots)(1+y) &= \\
=(1-y+y^2-y^3+\cdots+) + (y-y^2+y^3-y^4+\cdots+) &= 1
\end{aligned}$$

So

$$1-y+y^2-y^3+\cdots = \frac{1}{1+y} \quad \text{for} \quad |y|<1$$

So

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8+\cdots \quad \text{for} \quad |x|<1$$

The LHS is the derivative of $\arctan(x)$ so the RHS is its Taylor series at $x=0$.

Integrating the RHS term by term we get:

$$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$

but $c=0$, as can be seen by substituting values.

Note that $\arctan 1 = \frac{\pi}{4}$. Plugging in the formula above we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$, which is called Leibniz's series. Actually, we observe convergence at $x \pm 1$. (There is no convergence for $|x| > 1$.)

2. Taylor series for $\ln|1+x|$, defined for $x \neq -1$.

$$(\ln|1+x|)' = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \quad \text{for} \quad |x| < 1$$

Integrating term by term we get:

$$\ln|1+x| \;=\; c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots +$$

Setting $x=0$ we get $c=0$.

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \quad \text{for} \quad |x| < 1$$

Note that the LHS is undefined at $x = -1$ and the RHS equals $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots -$ which diverges! But at $x=1$ it *does* converge, as in fact: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots +$.

Reminder: Mathematical induction

1. If a claim dependent on a positive integer n is true for $n=1$

2. If it is true for k then it is true for $k+1$

Then the claim is true for all n.

Differential equations

Informally, a differential equation (DE) is a functional equation which involves functions, variables and derivatives of the functions. Note that the solutions to differential equations will always be functions. An example:

$$2y - y' = 0, \quad y = y(x)$$

$$y' = 2y$$

$$y = \mathrm{e}^{2x} \text{ is a solution.}$$

In fact, $y = c\, e^{2x}$ is a solution for any $c \in \mathbb{R}$.

Example: Radioactive decay

If a function $Q(t)$ is the amount of radioactive material at time t, then the rate of decay is proportional to the amount present (at time t). By these means, let there be $k \in \mathbb{R}$ such that

$$Q'(t) = -k\, Q(t), \quad k > 0$$

Note that $Q(t) = c\, \mathrm{e}^{-kt}$ solves the equation for any c. We show these functions are the *only* solutions. Assuming $Q(t) > 0$ we can write:

$$\frac{Q'(t)}{Q(t)} = -k$$

The LHS is the logarithmic derivative of $\ln Q(t)$. Therefore, by integration,

$$\ln Q(t) = -k\, t + c$$

and after exponentiation,

$$Q(t) = \mathrm{e}^{-kt+c} = \overbrace{\mathrm{e}^{c}}^{c'} \cdot \mathrm{e}^{-kt}.$$

Removing the assumption that $Q(t) > 0$ we see this solves out equation for any $c \in \mathbb{R}$.

To solve the equation for a *specific* material, *a unique solution*, we need extra information such as the amount of material at $t = 0$, the *initial condition*.

A numerical example

Thorium 234 decays at a rate such that 100 mg decays in a week to 82.04 mg. What is the amount at time t?

$$f'(t) = -k\, f(t)\,.$$

We know $f(t) = c\, \mathrm{e}^{-kt}$. Given $f(0) = 100\,\mathrm{mg}$ and $f(7) = 82.04\,\mathrm{mg}$, we can calculate both c and k.

$$\begin{aligned} 100 &= f(0) = c \\ 82.04 = f(7) &= 100\, \mathrm{e}^{-7k} \\ \ln 0.8204 &= -7\, k \\ k &\approx 0.02828\, \mathrm{days}. \end{aligned}$$

$$\Rightarrow f(t) = 100\, \mathrm{e}^{-0.2828 t}$$

Differential equations can be complicated like $2\, y - y'' \cdot x + y' \cdot x^2 + 7 \sin y = 0$.

The *order* of a differential equation is the highest order of the derivative that appears.

There are also *partial differential equations*, like Laplace's equation:

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

Definitions

Ordinary differential equations

Definition

An ordinary differential equation (ODE) is an equation of the form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

where F is some function of $n+2$ variables.

Partial differential equations

Definition

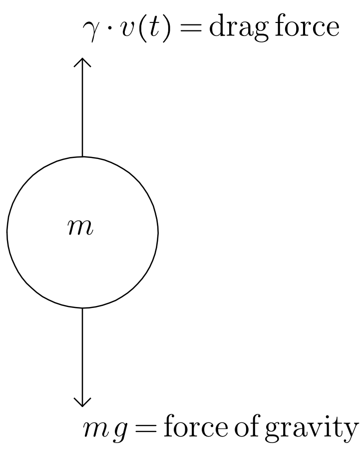
A partial differential equation (PDE) is one of the form:

$$F\left(x, \ldots, x_k, y, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \ldots\right) = 0$$

where a solution would be $y = y(x, \ldots, x_k)$, which when substituted with its partial derivatives, solves the equation.

More examples of modeling processes with DEs

1. A falling object:



Define $v(t)$ the velocity of mass m, the drag force is proportionate to $v(t)$.

By Newton's 2nd law, $F = m\,a$.

$$a(t) = v'(t)$$

$$\boxed{m\,\frac{\mathrm{d}v}{\mathrm{d}t} = m\,g - \gamma\,v}$$

Numerical example: $m = 1\,\mathrm{kg}$, $\gamma = 0.2\,\mathrm{kg\,s^{-1}}$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - \frac{v}{5}$$

Note that if you have some values of the function, you can plot them and get an approximation of the graph of the function and guess what the function (solution to DE) could be.

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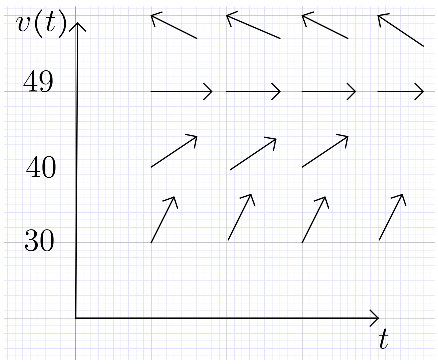
t	$v(t)$	$v'(t)$
0	40	1.8
arb	40	1.8
	50	-0.2
	30	3.8
	49	0



The solutions or *curves* that pass through a given point in the plane are called *integral curves*.

Note that if $v'(t) = 0$ then the function doesn't change, which is a state of equilibrium. Here, the equilibrium is $v(t) \equiv 49 \quad \forall t$. In this case, the drag exactly matches the force of gravity. The velocity that satisfies this condition is called the *limit velocity*.

2. Predator/prey = owls + field mice.

if no predators assume rate of growth of mouse population, r, $p(t)$ is proportionate to the population level at that time.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = r \cdot p(t)$$

Now assume that we have owls who kill a fixed number of mice per month (for example 450) and that $r = 0.5/\text{month}$, then

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0.5\, p - 450$$

What we get is an *unstable equilibrium*. for $p(0) = 901$, i.e., one mouse survives per month, the population will increase exponentially. for $p(0) = 899$ the mice population goes extinct.

Equilibrium is unstable if $y(t) \to 0$ for $t \to \infty$ if $y(0) < y_{\text{eq}}$.

Let's solve the ODEs from examples 1 and 2 algebraically. Both are special cases of

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = a\, y + b$$

Assume that $a \neq 0$ and that $y \neq -\frac{b}{a}$ (native equilibrium). Divide by a:

$$\frac{y'}{a} = y + \frac{b}{a}$$

$$\frac{y'}{y + \frac{b}{a}} = a$$

This is the derivative of $\ln\left|y + \frac{b}{a}\right|$. Integrate both sides to get

$$\ln\left|y + \frac{b}{a}\right| = \text{at} + c$$

exponentiate both sides to get

$$\left|y + \frac{b}{a}\right| = \mathrm{e}^{at+c} = \overbrace{\mathrm{e}^{c}}^{C} \cdot \mathrm{e}^{at}$$

an exponent is always positive so we can remove the absolute value by adding a $\pm$ sign.

$$y + \frac{b}{a} = \pm C \cdot \mathrm{e}^{at}, \quad C \in \mathbb{R}$$

our general solution is

$$\boxed{y = k\,\mathrm{e}^{\mathrm{at}} - \frac{b}{a}, \quad k \neq 0, k \in \mathbb{R}} \tag{2}$$

Note that $y \equiv -b/a$ is also a solution — a specific solution.

Note that:

- if $a < 0$ then as $t \to \infty$ $y \to \frac{b}{a}$ and we get a stable equilibrium.
- if $a > 0$ then:
  - if $k > 0$ then $y \to \infty$
  - if $k < 0$ then $y \to -\infty$

  and the equilibrium is unstable.

What is the meaning of k? Given initial condition $y(0) = y_0$, then setting $t = 0$ in eq. (2) gives: $y_0 = y(0) = k + \frac{b}{a}$. So $k = y_0 - \frac{b}{a}$ is the unique solution satisfying the initial condition (IC): $y = \left(y_0 - \frac{b}{a}\right)\mathrm{e}^{at} + \frac{b}{a}$.

I First order ODEs

First order ODEs are of the form

$$y' = F(t, y)$$

The relationship between t, y can be very complex! Let's review some kinds:

1 Linear ODEs

Linear in y, that is.

$$y' = a(t)\,y + b(t)$$

In the previous examples we solved cases where $a(t)$ and $b(t)$ are constants.

A method for solving linear ODEs:

1.1 Integrating factors method: (due to Leibniz)

Take for example the ODE $y' = -2\,y + 3$ ($a(t) = -2, b(t) = 3$).

Isolate $f(y)$. rewrite as

$$y' + 2\,y = 3.$$

Now multiply by a function $\mu(t)$ so that LHS is recognizable as the derivative of something, which is a *product*. Then we can integrate both sides to get the solution. We have:

$$(y \cdot \mu(t))' = \underline{y' \cdot \mu + y \cdot \mu'}$$

If we multiply out equation by $\mu(t)$,

$$\underline{y' \cdot \mu(t) + 2\,\mu(t) \cdot y} = 3 \cdot \mu(t)$$

All we need is that $\mu'(t) = 2\,\mu(t)$. We may choose a solution

$$\mu(t) = \mathrm{e}^{2t}$$

and get

$$y' \cdot \mathrm{e}^{2t} = 2\,\mathrm{e}^{2t} \cdot y = 3\,\mathrm{e}^{2t}$$

Integrate to get

$$y\,\mathrm{e}^{2t} = \int 3\,\mathrm{e}^{2t}\,\mathrm{d}t = \frac{3}{2}\,\mathrm{e}^{2t} + c$$

so

$$y = \frac{3}{2} + c \cdot e^{-2t}$$

Note that this method also works for non-linear ODEs.

Another example

$$y' + a\,y = b(t)$$

where a is a constant and $b(t)$ isn't necessarily a constant. Multiply by $\mu(t)$.

$$y'\,\mu(t) + a\,y\,\mu(t) = b(t)\;\mu(t)$$

we need $\mu(t)$ such that $a\,\mu(t) = \mu'(t)$, so choose $\mu = \mathrm{e}^{at}$ and get

$$(y \cdot \mathrm{e}^{at})' = y' \cdot \mathrm{e}^{at} + a\,y\,\mathrm{e}^{at} = b(t)\,\mathrm{e}^{at}$$

Integrate both sides:

$$y \cdot \mathrm{e}^{at} = \int b(t) \cdot \mathrm{e}^{at}\,\mathrm{d}t + c$$

$$\boxed{y = e^{-at}\left[\int b(t) \cdot e^{at}\,dt + c\right]}$$

Consider the equation

$$y' + 0.5\,y = 2 + t; \quad y(0) = 2$$

Choose $\mu(t) = e^{\frac{1}{2}t}$.

$$\left(y \cdot e^{\frac{1}{2}t}\right) = y' \cdot e^{\frac{1}{2}t} + \frac{1}{2}\,e^{\frac{1}{2}t}\,y = 2\,e^{\frac{1}{2}t} + t\,e^{\frac{1}{2}t}$$

$$y\,e^{\frac{1}{2}t} = \int 2\,e^{\frac{1}{2}t}\,dt + \int t\,e^{\frac{1}{2}t}\,dt$$

The second integral must be calculated via *integration by parts*.

$$\boxed{\begin{aligned} (u\,v)' &= u'\,v + u\,v' \\ u\,v &= \int u'\,v\,dt + \int u\,v'\,dt \\ \int u'\,v\,dt &= u\,v - \int u\,v'\,dt \\ \int v\,du &= u\,v - \int u\,dv \end{aligned}}$$

take $t = v, u' = e^{\frac{1}{2}t}$, then $v' = 1, u = 2\,e^{\frac{1}{2}t}$.

$$\int t\,e^{\frac{1}{2}t}\,dt = 2\,t\,e^{\frac{1}{2}t} - \int 2\,e^{\frac{1}{2}t}\,dt = 2\,t\,e^{\frac{1}{2}t} - 4\,e^{t/2} + c$$

So, in total,

$$y\,e^{\frac{1}{2}t} = 2\,t\,e^{t/2} + c$$

$$y = 2\,t + c \cdot e^{-t/2}$$

Plug in $y(0) = 2$, which gives

$$2 = y(0) = 2 \cdot 0 + c\,e^0 = c$$

We get the unique solution $y = 2\,t + 2\,e^{-t/2}$.

Final example: the general case

For $y' = a(t)\,y + b(t)$, multiply through $\mu(t)$:

$$y'\,\mu + a(t)\,y\,\mu = b(t)\,\mu$$

We want $y'\,\mu + a(t)\,y\,\mu = y\,\mu'$, or equivalently $a(t)\cdot\mu = \mu'$. If $\mu \neq 0$:

$$a(t) = \frac{\mu'}{\mu} = (\ln|\mu(t)|)' \rightarrow \int a(t)\,\mathrm{d}t = \ln|\mu(t)|$$

choose $\mu(t) = \mathrm{e}^{\int a(t)\,\mathrm{d}t}$ as int. factor.

$$\begin{aligned} y'\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} + y\,a(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} &= b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} \\ &= (y\cdot \mathrm{e}^{\int a(t)\,\mathrm{d}t})' \end{aligned}$$

so that

$$y\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} = \int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}$$

$$\boxed{y = \mathrm{e}^{-\int a(t)\,\mathrm{d}t}\cdot\left[\int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}\right]}$$

<u>Example</u>

$$t\,y' + 2\,y = 4\,t^2, \quad t \neq 0, \quad y(1) = 2$$

Rewrite as $y' + \frac{2}{t}\,y = 4\,t$, such that $a(t) = \frac{2}{t}$. Choose $\mu(t) = \mathrm{e}^{\int a(t)\,\mathrm{d}t} = \mathrm{e}^{2\ln|t|} = t^2$.

$$(t^2\,y)' = t^2\,y' + 2\,t\,y = 4\,t^3$$

Integrate to obtain

$$t^2\,y = \int 4\,t^3\,\mathrm{d}t = t^4 + c$$

$$y = t^2 + \frac{c}{t^2}$$

Set $y(1) = 2$.

$$2 = 1 + \frac{c}{1} \rightarrow c = 1$$

The unique solution is $y = t^2 + t^{-2}$.

Question: Is there a solution for $t = 0$? Input $t = 0$ in the original DE:

$$0\cdot y' + 2\,y = 0 \rightarrow y(0) = 0$$

First we see that $y(0)$ is *defined*. Then, we can see that if we choose $c = 0$ we get a specific solution $y = t^2$.

2 Separable ODEs

Equations of the form (in Leibniz notation):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x)}{N(y)}$$

for $M(x), N(y)$ functions only of x, y respectively. Sometimes written as:

$$M(x)\,\mathrm{d}x - N(y)\,\mathrm{d}y = 0$$

<u>Example</u>

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1-y^2}, \quad y \neq \pm 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} \cdot (1-y^2) = x^2$$

LHS is actually the derivative of $y - \frac{y^3}{3}$ with respect to x! Integrating both sides gives

$$y - \frac{y^3}{3} = \int x^2\,\mathrm{d}x = \frac{x^3}{3} + c$$

So that y is given implicitly.

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Note that if N is a function of y and y is a function of x. Suppose $\int N(y)\,\mathrm{d}y = Q(y)$ so $\frac{\mathrm{d}Q}{\mathrm{d}y} = N(y)$, then, according to the chain rule,

$$\frac{\mathrm{d}Q(y)}{\mathrm{d}x} = \frac{\mathrm{d}Q(y)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = N(y) \cdot y'$$

By taking inverse operation (integrating with respect to x) we find that

$$\int N(y) \cdot y'\,\mathrm{d}x = Q(y) = \int N(y)\,\mathrm{d}y$$

This allows us to write $y'\,\mathrm{d}x = \mathrm{d}y$.

Let's solve the previous example using this notation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1-y^2}$$

$$\int (1-y^2)\,\mathrm{d}y = \int x^2\,\mathrm{d}x$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

When solved before, we had: $y'(1-y^2)=x^2$ and integrated both sides with respect to (wrt) x. What we did here is integrating wrt y on the LHS and wrt x on the RHS. For separable ODEs, non-uniform integration is justified.

Example 2

$$y'=\frac{3\,x^2+4\,x+2}{2(y-1)}; \quad y(0)=-1, \quad y\neq 1$$

$$\int 2(y-1)\,\mathrm{d}y=\int (3\,x^2+4\,x+2)\,\mathrm{d}x$$

$$y^2-2\,y=x^3+2\,x^2+2\,x+c$$

Last time we stopped at an implicit expression for y, but here we can go further by applying the initial condition. Set $x=0$: $1+2=c$, so $c=3$. We can write a quadratic expression in y to get a more explicit expression:

$$y^2-2\,y-(x^3+2\,x^2+2\,x+3)=0$$

Solve using the quadratic formula:

$$y=\frac{2\pm\sqrt{4+4(x^3+2\,x^2+2\,x+3)}}{2}$$

$$y=1\pm\sqrt{x^3+2\,x^2+2\,x+4}$$

On paper, it seems now that applying the initial condition doesn't give us a unique solution, because y has two possible values! However, we shall see that only one quadratic root satisfies the initial condition $y(0)=-1$.

Substituting $x=0$: $y(0)=1\pm\sqrt{4}=\begin{cases}3\\-1\end{cases}$. The unique solution is $y=-1-\sqrt{x^3+2\,x^2+2\,x+4}$.

What is its domain of definition? The solution must (a) solve the ODE, (2) satisfy the IC, and (3) be defined in some domain. We need that $\sqrt{x^3+2\,x^2+2\,x+4}\geq 0$.

$$x^3+2\,x^2+2\,x+4=(x+2)(x^2+2)$$

So $\sqrt{x^3+2\,x^2+2\,x+4}\geq 0$ for $x\geq -2$.

But, we defined the DE for $y\neq 1$, so we need $x^3+2\,x^2+2\,x+4>0$, so $x>-2$ is the true domain.

Example 3

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2}{\underbrace{y(1+x^3)}_{\neq 0}}$$

$$\frac{y^2}{2}=\int y\,\mathrm{d}y=\int\frac{x^2}{1+x^3}\,\mathrm{d}x\overbrace{=}^{\substack{u=1+x^3\\ \mathrm{d}u=3x^2\mathrm{d}x}}\int\frac{1}{3\,u}\,\mathrm{d}u=\frac{1}{3}\ln|1+x^3|+c$$

$$y^2=\frac{2}{3}\ln|1+x^3|+c$$

Use the IC: $y(0)=3$.

$$9=\frac{2}{3}\ln 1+c\rightarrow c=9$$

we get a unique solution (positive square root) $y=\sqrt{\frac{2}{3}\ln|1+x^3|+9}$. The domain of the function is which satisfies $y>0$. The domain contains $x=0$, so $x>-1$.

Example 4

Sometimes an equation can be *reduced* to a separable differential equation by a change of variables. For example,

$$y'=x^2+2\,x\,y+y^2=(x+y)^2$$

We cannot separate the variables here, but we can define $z=x+y$, substitute and get

$$y'=z^2$$

$$\frac{\mathrm{d}z}{\mathrm{d}x}=1+y'=1+z^2$$

the DE for z **is** separable. Get:

$$\int\frac{\mathrm{d}z}{1+z^2}=\int\mathrm{d}x$$

$$\arctan z=x+c$$

Take tangent of both sides:

$$z=x+y=\tan(x+c)$$

$$y=\tan(x+c)-x$$

Cases we need to check to make sure we find **all** possible solutions:

Example:

$$y' + y^2 \sin x = 0$$

We want to divide by y^2 in order to solve, under the assumption for $y \neq 0$. Check some cases first:

1. $y \equiv 0$. Is this a solution? — Yes!

   Out solutions will be differentiable functions and so continuous. So that if $y \not\equiv 0$ there is a point where $y \neq 0$ and around it there's an interval where $y(x) \neq 0$ for all x in that interval. (This is because y is continuous.)

2. $y(x) \neq 0$ for all x in an interval. We can divide by y^2 to get:

$$\frac{\mathrm{d}y}{y^2} + \sin x \, \mathrm{d}x = 0$$

$$-\frac{\mathrm{d}y}{y^2} = \sin x \, \mathrm{d}x$$

   Integrate both sides:

$$-\frac{1}{y} = -\cos x + c$$

$$y = \frac{1}{c - \cos x}$$

   Note that this solution **never** has the value 0 for **any** x where it is defined.

   If $|c| > 1$ then this holds for all x. Otherwise we need to avoid values where $\cos x = c$.

   What if $y(0) = -\frac{2}{3}$? we get $y(0) = \frac{1}{c-1} \to c = -\frac{1}{2}$. That's problematic because $\cos x$ can get the value $-\frac{1}{2}$. So our solution holds in the interval $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, which contains $x = 0$.

3. There exists a point x_0 where $y(x_0) = 0$ and $y \not\equiv 0$. In that case, we solve as in case 2 for an interval where $y \neq 0$ for all x, as we got in case 2 solution.

   If we had a solution at x_0 where $y(x_0) = 0$, and in the interval y is the function in case 2, we cannot get a continuous extension of y to x_0. In other words, this cannot happen!

   We've examined all possible cases, and thus found all the solutions.

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Another example of modeling with first order DEs: Interest compounded continuously

$S(t) =$ amount of money deposited and interest evaluated continuously, then we get that if the rate of change is proportionate to the amount of money.

$$S'(t) = r \cdot S(t)$$

In fact r will be the annual rate of interest so that $S(t) = S(0)\,\mathrm{e}^{rt}$. Why?

If we compute once a year then $S(1) = S(0) + r\,S(0) = S(0)\,(1+r)$.

If you do it twice a year: after 6 months we get $(0.5) = S(0)\,\left(1+\frac{r}{2}\right)$ and $S(1) = S(0)\,(1+\frac{r}{2})^2$, and in general: $S(t) = S(0)\,(1+\frac{r}{2})^{2t}$.

If we compute n times a year: $S(t) = S(0)\,(1+\frac{r}{n})^{nt}$.

Recall that $\lim_{n\to\infty}\left(1+\dfrac{r}{n}\right)^n = \mathrm{e}^r$. So when $n\to\infty$ and interest is compounded continuously we get $S(t) = S(0)\,\mathrm{e}^{rt}$.

Let's improve our model of money management. In addition to annual rate of interest (r), we also deposit or withdraw k amount of money every year. Thus, the differential equation becomes

$$S'(t) = r\,S(t) + k$$

As before, we get:

$$\begin{aligned} S'(t) &= \left(S_0 + \frac{k}{r}\right)\mathrm{e}^{rt} - \frac{k}{r} \\ &= \underbrace{S_0\,\mathrm{e}^{rt}}_{\substack{\text{effect of the}\\ \text{initial investment}\\ \text{at interest rate r}}} + \underbrace{\frac{k}{r}\,(\mathrm{e}^{rt}-1)}_{\substack{\text{result of the}\\ \text{withdraws/deposits}}} \end{aligned}$$

Example

Open a savings plan at age 25 with regular deposits of 2000\$/year with 8% annual rate of interest. What will be the amount saved by age 65?

$$S(40) = S_0\,\mathrm{e}^{rt} + \frac{k}{r}(\mathrm{e}^{rt}-1), \quad S_0 = 0$$

$$S(40) = \frac{2000}{0.08}\,(\mathrm{e}^{0.08*40}-1) \approx 588{,}313\,\$$$

We invested 80,000\$ and made a 508,000\$ profit! Note this is a special case called an *autonomous equation*.

2.1 Autonomous ODEs

Definition

An ODE is *autonomous* if the independent variable does not appear explicitly.

For example, $y' = a\,y + b$, In general, this means $y' = F(y)$ so it's separable.

When you plot a direction field for these ODEs you get replicas of vectors along the horizontal axis.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\,y+b}{c\,y+e}, \qquad y \neq -\frac{e}{c}$$

There are 2 cases:

1. $a\,y+b \equiv 0$, so $y \equiv -\frac{b}{a}$. A constant solution.

   (if $a=0$ then $b=0$ as well [because we can't divide by zero], and $y=k\ \forall k \in \mathbb{R}$)

2. Assume $a\,y+b \not\equiv 0$, so take $a\,y+b \neq 0$ on some interval. We can now rewrite the equation to get the form: $(a \neq 0)$

   $$\int \frac{c\,y+e}{a\,y+b}\,\mathrm{d}y = \int \mathrm{d}x = x+k$$

   Note that

   $$\begin{aligned} \frac{c\,y+e}{a\,y+b} &= \frac{\frac{c}{a}(a\,y+b) - \frac{c}{a}b + e}{a\,y+b} \\ &= \frac{c}{a} + \frac{-\frac{c}{a}b+e}{a\,y+b} \\ &= \frac{c}{a} + \frac{a\,e-b\,c}{a\,(a\,y+b)} \end{aligned}$$

   Therefore,

   $$\int \frac{c\,y+e}{a\,y+b}\,\mathrm{d}y = \int \frac{c}{a}\,\mathrm{d}y + \left(\frac{a\,e-b\,c}{a}\right) \int \frac{\mathrm{d}y}{a\,y+b} = x+k$$

   We are left with an implicit expression for $y(x)$:

   $$\frac{c}{a}\,y + \left(\frac{a\,e-b\,c}{a}\right) \ln|a\,y+b| = x+k$$

   Note that if $a=0$ we have to solve again from the start. Actually, in this case the equation is easier because we don't have y in the denominator.

2.2 Verhulst's Model (1845)

Similar to the field mice model, we had $y' = r\,y$ where r was a constant rate of population growth and $y(t)$ was the population at time t. We can make the model more realistic by replacing r by a function which depends on y.

$$y' = h(y) \cdot y$$

where $h(y)$ is the rate of population growth at population level y.

Typically, $h(y)$ *decreases* as y *increases*. We approximate this by choosing $h(y)$ to be linearly decreasing in y:

$$h(y) = r - s \cdot y, \quad r, s > 0$$

We get $y' = (r - s\,y)\,y$, a quadratic autonomous equation! Let's look at the equilibria.

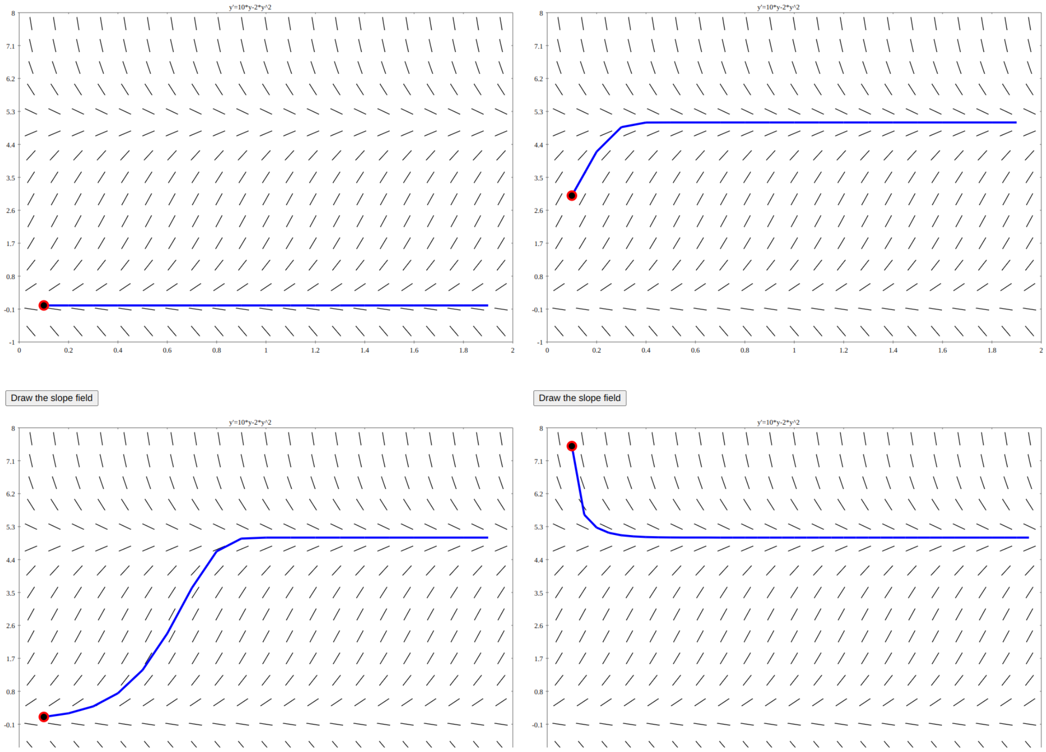
Equilibrium points

$y = 0$ or $y = \frac{r}{s} = k$. Rewrite our ODE:

$$y' = r\left(1 - \frac{s}{r}\right) y = r\left(1 - \frac{1}{k}\,y\right) y$$

Before we solve, let's look at the direction field. Take $r = 10, k = 5$:

$y' = 10(1 - 5\,y) = 10\,y - 2\,y^2$.



It is clear that $y \equiv 0$ is an unstable equilibrium and that $y \equiv 5$ is a stable equilibrium.

Now let's solve algebraically.

$$y' = r\left(1 - \frac{1}{k}\,y\right) y = r\left(\frac{k - y}{k}\right) y$$

If $y = k$ we get $y \equiv k$ (because $y'(k) = 0$. So now assume $y \neq k$ and separate the variables:

$$\int \frac{k\,\mathrm{d}y}{(k - y)\,y} = \int r\,\mathrm{d}t = r\,t + c$$

As we have a polynomial at the denominator, we rewrite LHS using partial fractions:

$$\frac{k}{(k-y)\,y}=\frac{A}{k-y}+\frac{B}{y}$$

where A, B are some constants. Get

$$k=A\,y+B(k-y)=(A-B)\,y+B\,k$$

Equate coefficients on both sides:

$$\begin{aligned} 0 &= A-B \\ k &= B\,k \end{aligned}$$

We can see that $A=B=1$.

$$\int\frac{\mathrm{d}y}{k-y}+\int\frac{\mathrm{d}y}{y}=r\,t+c$$

$$\ln\left(\frac{|y|}{|k-y|}\right)=-\ln|k-y|+\ln|y|=r\,t+c$$

Exponentiate both sides:

$$\frac{|y|}{|k-y|}=C\,\mathrm{e}^{rt}$$

Again, there are some cases:

1. $0<y<k$. Set $y(0)=y_0$.

$$\frac{y}{k-y}=C\,\mathrm{e}^{rt}$$

Setting $t=0$ to get

$$\frac{y_0}{k-y_0}=C$$

Now we solve for y:

$$\frac{y}{k-y}=\frac{y_0}{k-y}\,\mathrm{e}^{rt}$$

$$y=(k-y)\,\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}$$

$$y\left(1+\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}\right)=\frac{k\,y_0}{k-y_0}\,\mathrm{e}^{rt}$$

$$y=\frac{k\,y_0\,\mathrm{e}^{rt}}{(k-y_0)\left[1+\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}\right]}=\frac{k\,y_0\,\mathrm{e}^{rt}}{k-y_0+y_0\,\mathrm{e}^{rt}}$$

Divide numerator and denominator by e^{rt}.

$$y = \frac{k\, y_0}{(k - y_0)\, \mathrm{e}^{-rt} + y_0}$$

As $t \to \infty$ we see that $y \to k$. This means that $y \equiv k$ is a stable equilibrium and that $y \equiv 0$ is an unstable equilibrium.

2. if $y > k$ we get that case (1) solution is still valid ($y = k$ is a stable equilibrium).

The constant $k = \frac{r}{s}$ is called the *saturation level* or *environmental carrying capacity*.

II Second Order ODEs

3 Special 2nd order ODEs

The solution be obtained by *reducing* to a **first** order equation. Two kinds of equations where you can do that:

3.1 2nd order ODE where the dependent variable doesn't appear

An equation of the form:

$$y'' = F(x, y')$$

Substitute $v = y'$ and get the first order ODE

$$v' = F(x, v)$$

and then solve

$$y = \int v\, \mathrm{d}x$$

<u>Examples</u>

1.

$$t^2\, y'' + 2\, t\, y' - 1 = 0, \quad t > 0$$

Set $y' = v$:

$$t^2\, v' + 2\, t\, v = 1, \quad t > 0$$

Integrate both sides:

$$t^2 v = t + c_1$$

$$y' = v = \frac{1}{t} + \frac{c_1}{t^2}$$

$$y = \ln t - \frac{c_1}{t} + c_2$$

A second order ODE needs 2 initial conditions to get a unique solution. Given $y(0) = b_1$ and $y'(0) = b_2$ we can determine c_1 and c_2.

2.

$$2\, t^2\, y'' + (y')^3 = 2\, y'\, t, \quad t > 0$$

Set $y' = v$:

$$2\, t^2\, v' + v^3 = 2\, v\, t$$

Examine some cases:

I. $v \equiv 0$. In that case $y \equiv C$. This solves the equation for any $C \in \mathbb{R}$.

II. $v \neq 0$ on some interval. Divide by v^3 and get

$$\frac{2\, t^2\, v'}{v^3} + 1 = \frac{2\, t}{v^2}$$

$$-\left(\frac{t^2}{v^2}\right)' = \frac{2\, t^2\, v'}{v^3} - \frac{2\, t}{v^2} = -1$$

Integrate both sides to get

$$\frac{t^2}{v^2} = t + c_1, \quad t + c_1 \neq 0$$

$$(y')^2 = v^2 = \frac{t^2}{t + c_1}$$

$$y' = v = \frac{\pm t}{\sqrt{t + c_1}}, \quad t + c_1 > 0$$

$$y = \pm \int \frac{t\, \mathrm{d}t}{\sqrt{t + c_1}}$$

Integrate by parts. $\int u\,v' = u\,v - \int v\,u'$. Choose $u = t, u' = 1$.

$v' = \frac{1}{t + c_1}$ so $v = 2\sqrt{t + c_1}$.

$$\Rightarrow y = \pm\left[2\,t\,\sqrt{t + c_1} - 2\int\sqrt{t + c_1}\,\mathrm{d}t\right]$$

$$\begin{aligned} y &= \pm\left[2\,t\,\sqrt{t + c_1} - 2 \cdot \frac{2}{3}(t + c_1)^{3/2} + c_2\right] \\ &= \pm\left[\frac{2}{3}\sqrt{t + c_1}\,(t - 2\,c_1) + c_2\right] \end{aligned}$$

Take initial conditions: $y(1) = 0, y'(1) = -1$.

$$\begin{aligned} y(1) = 0 &= \frac{2}{3}\sqrt{1 + c_1}\,(1 - 2\,c_1) + c_2 \\ y'(1) = -1 &= \frac{\pm 1}{\sqrt{1 + c_1}} \rightarrow c_1 = 0 \\ &\Rightarrow \; c_2 = -\frac{2}{3} \end{aligned}$$

$$y = -\frac{2}{3}t^{3/2} + \frac{2}{3}, \quad t > 0$$

3.2 2nd order autonomous ODE

The independent variable doesn't appear.

$$y'' = F(y, y')$$

Set $y' = v$ and get

$$v' = F(y, v)$$

But remember that $v' = \frac{\mathrm{d}v}{\mathrm{d}x}$. We want to express v as a function of y. Using the chain rule,

$$y'' = v' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

Then we get an equation of the form:

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}y} \cdot v = F(y, v)}$$

We got a first order ODE in v as a function of y.

Example

$$y \cdot y'' + (y')^2 = 0$$

Set $v = y'$:

$$y\,v' + v^2 = 0$$

$$v' = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v$$

Rewrite as an ODE in v as a function of y.

$$y \cdot \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v + v^2 = 0$$

1. $v \equiv 0$ gives $y' \equiv 0$ or $y \equiv C$ is a solution for all C.

2. $v \neq 0$ on some interval. If there is a 2nd derivative it means the 1st derivative is continuous, so if the 2nd derivative is non-zero at a point it is non-zero on the interval.

$$y\frac{\mathrm{d}v}{\mathrm{d}y} + v = 0$$

   This equation is separable!

$$y\,\mathrm{d}v + v\,\mathrm{d}y = 0$$

$$\int \frac{\mathrm{d}v}{v} + \int \frac{\mathrm{d}y}{y} = \int 0\,\mathrm{d}t = C$$

$$\ln|v\,y| = \ln|v| + \ln|y| = C$$

$$\pm \mathrm{e}^C = k = v\,y$$

   Remember $v = \mathrm{d}y/\mathrm{d}x$ so we get

$$k = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot y$$

$$k \cdot x + c = \int k\,\mathrm{d}x = \int y\,\mathrm{d}y = \frac{y^2}{2}$$

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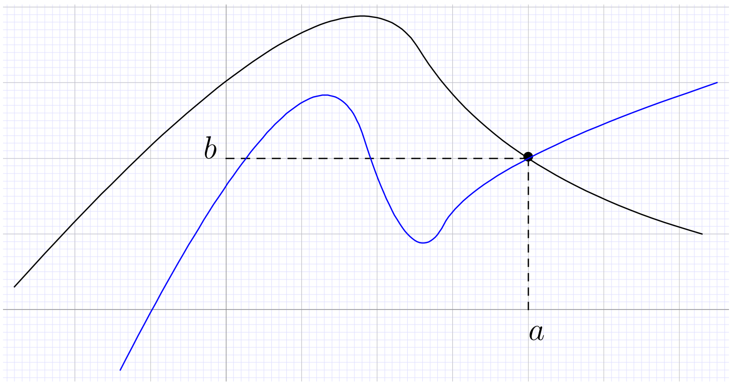
4 Existence and uniqueness theorem for first order ODEs

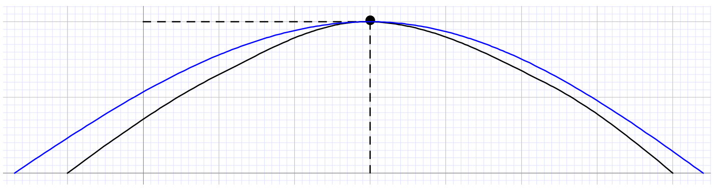
Sometimes knowing that there exists a unique solution helps finding the solution.

Suppose we have the first order ODE $y'=F(x,y),\ y(a)=b$. If F and $\frac{\partial F}{\partial y}$ are continuous in some open rectangle around (a,b) in the x-y plane, then there exists a unique solution to the ODE satisfying $y(a)=b$.

Note that there is only one curve that passes through this curve and it's defined in some open rectangle. The definition of an open rectangle is a set of points (x,y) such that $\left\{(x,y)\middle|\begin{array}{l} a_1<x<a_2\\ b_1<y,b_2\end{array}\right\}$.

Conditions guarantee that we cannot have a solution:





Examples

1. $y'+2\,x\,y=x^3\,y^2$

$$y'=x^3\,y^2-2\,x\,y=F(x,y)\quad \text{continuous (polynomial)}$$

$$\frac{\partial F}{\partial y}=2\,y\,x^3-2\,x\quad \text{continuous (polynomial)}$$

By the theorem, given any IC there exists a solution.

2. First order linear: $y'=p(x)\,y+q(x)=F(x,y)$.

F is continuous $\iff p(x), q(x)$ are continuous.

$\frac{\partial F}{\partial y} = p(x)$ is continuous $\iff p(x)$ is continuous.

Conclusion — we have a unique solution for any IC if p, q are both continuous.

3. $y' = 2\sqrt{y}$. Here $F(x,y)$ is cont. for all x and for all $y > 0$ (and cont. on the right at $y = 0$).

   $\partial F / \partial y = \frac{1}{2\sqrt{y}}$ is **not** cont. at $y = 0$.

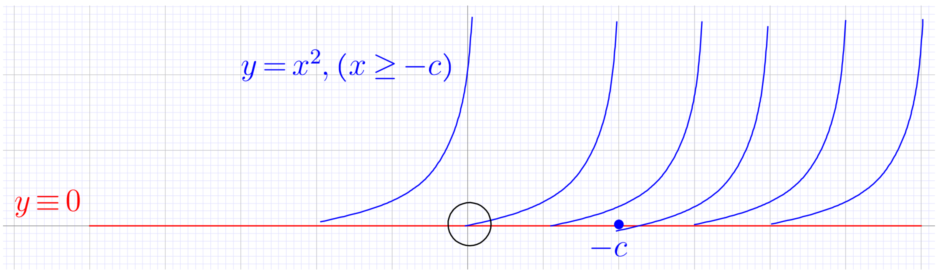
   What happens here when we have the IC: $y(0) = 0$? We solve assuming first that $y \neq 0$ (in some interval) in order to find a non-trivial solution that can be extended continuously to $y(0) = 0$.

$$\sqrt{y} = \int \frac{\mathrm{d}y}{2\sqrt{y}} = \int 1\,\mathrm{d}x = x + c \quad (x + c \geq 0)$$

$$y = (x+c)^2, \quad x \geq -c$$

   Notice that $(x+c) = 0$ is also a solution, therefore we can say that the interval is $x \geq c$ [The function is differentiable from the right].

   Let's draw the solutions. First notice that clearly $y \equiv 0$ is a solution satisfying IC.



   $y = x^2, (x \geq 0)$ is also a solution satisfying IC.

   Actually, we have 2 solutions for every IC such that $y(a) = 0$.

   Why could we extend the interval range to $x = -c$? Have a look at the following example:

   $y' \cdot y = \cos x \cdot y$. Notice that $y \equiv 0$ is a solution. Now assume that $y \not\equiv 0$ and divide by an interval where $y(x) \neq 0$.

$$\begin{aligned} y' &= \cos x \\ y &= \sin x + c \end{aligned}$$

Notice that $y=\sin x$ is a solution for $\sin x\neq 0, x\neq \pi k$.

We then check and see that $y=\sin x$ is a solution *even* for $x=\pi k, k\in\mathbb{N}$.

4. $y^2+x^2 y'=0$. Assuming $x,y\neq 0$ we can separate variables.

$$y^2=-x^2 y'$$

$$-\int \frac{\mathrm{d}x}{x^2}=\int \frac{\mathrm{d}y}{y^2}$$

$$\frac{c\,x+1}{x}=c+\frac{1}{x}=-\frac{1}{y}$$

$$y=-\frac{x}{c\,x+1}, \quad \text{defined for all } x\neq -\frac{1}{c}$$

Note that if $x=0$ it is defined and then $y=0$, so all solutions pass through $(0,0)$.

Given an IC $y(0)=1$ — there's no solution!

But we do have a unique solution (by the E&U theorem) for every IC $y(a)=b$ where $a\neq 0$, as $y'=F(x,y)=\frac{y^2}{x^2}$ is cont. for $x\neq 0$ and $\frac{\partial F}{\partial y}=\frac{2\,y}{x}$ also cont. for $x\neq 0$.

Also notice that there are inf. many solutions for $(0,0)$ according to the solution $y=-\frac{x}{c\,x+1}$.

The E&U theorem fails because $\boldsymbol{\partial F/\partial y}$ is not continuous within the interval.

5. $x\,y'=2\,y$. $\Rightarrow$ $y'=\frac{2\,y}{x}$ is linear. Set $p(x)=\frac{2}{x}$ cont. for $x\neq 0$.

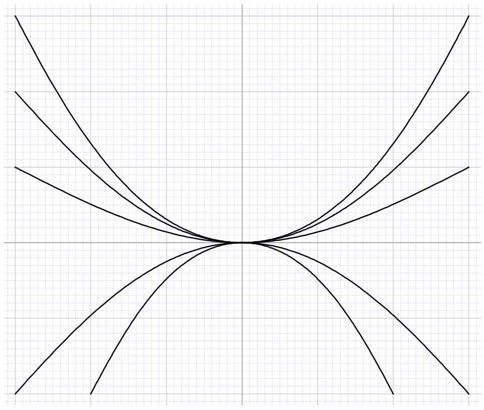
We solve assuming $x,y\neq 0$ and get

$$\frac{1}{2}\ln|y|=\int \frac{\mathrm{d}y}{2\,y}=\int \frac{\mathrm{d}x}{x}=\ln|x|+c$$

$$\sqrt{y}=k|x|$$

$$y=K\,x^2$$

This is a solution for all K, under the restriction $x\neq 0$.



- inf. many solutions satisfy $y(0)=0$.
- No solution satisfies $y(0)=b$ where $b\neq 0$.
- There exists ($\exists$) a unique (!) solution for $y(a)=b$ for $a\neq 0$, any b.
- Notice that the combination of $y=-K_1\,x^2, x<0$ and $y=K_1\,x^2, x\geq 0$ **is** differentiable at point $x=0$ **and** solves the differential equation!

4.1 Existence and uniqueness theorem for higher order ODEs

Theorem

Given a linear ODE of order n,

$$y^{(n)}+p_{n-1}(x)\,y^{(n-1)}+p_{n-2}(x)\,y^{(n-2)}+\cdots+p_0(x)\,y=q(x)$$

and the ICs:

$$\begin{cases} y(a)=b_0 \\ y'(a)=b_1 \\ \vdots \\ y^{(n-1)}(a)=b_{n-1} \end{cases}$$

If $p_0, p_1, \ldots, p_{n-1}, q$ are continuous for all x in some interval containing a, then the ODE has a unique solution satisfying the ICs for any choice of values $b_0, \ldots, b_{n-1}$.

5 Second order linear ODEs

5.1 Homogeneous ODEs

Homogeneous equations take the general form:

$$y''+p(x)\,y'+q(x)\,y=0$$

By E&U thm., for any IC $y(a)=b$, $y'(a)=c$ we have a unique sol. in some neighborhood of a, provided p, q are continuous in this interval.

Reminder:

- The set $V = \{f: \mathbb{R} \to \mathbb{R} | f \text{ is twice differentiable}\}$ is a *vector space* over $\mathbb{R}$ with respect to operations: addition of functions, multiplication by scalars from $\mathbb{R}$.

  Note: V is *closed* under these operations. i.e., if $f, g \in V$ so are $f + g, c \cdot f$.

  Also note that we can define a vector space only in a confined interval $[x_1, x_2]$.

  Note about dimension of V

  $\sin x, \cos x \in V$. V is infinite dimensional over $\mathbb{R}$ as the monomial functions: $1, x, x^2, \ldots$,are linearly independent over $\mathbb{R}$. (V can contain infinitely many linearly independent functions)

- in any vector space V: the elements $v_1, \ldots, v_k \in V$ are *linearly independent* if $\sum_{i=1}^{k} a_i v_i = 0$ for a_i scalars only if $a_i = 0$ for all i.

  e.g. $\sin x, \cos x$ are linearly independent in our vector space of functions as

$$a \sin x + b \cos x \equiv 0$$

  i.e.

$$a \sin x + b \cos x = 0 \quad \forall x$$

  Set $x = 0$ and get $b = 0$, and set $x = \pi/2$ and get $a = 0$.

- A *basis* for V is a set of linearly independent vectors such that every vector in V is a linear combination of these vectors.

Turns out that:

- Every vector space has a basis (assuming axiom of choice).

- Every basis has the same cardinality (number of elements in the set if the set is finite).

- If V has dimension n then any linearly independent set of n elements will be a basis.

Claim: The set of solutions to our ODE

$$y'' + p(x)\, y' + q(x)\, y = 0, \quad p, q \text{ continuous} \tag{3}$$

is a vector space over $\mathbb{R}$ (a subspace of V). In fact, it is a vector space of dimension 2.

Proof: Set of solutions is non-empty as $y \equiv 0$ is a solution.

We need to show that if y_1, y_2 are solutions, then so is $y_1 + y_2$ and also $c \cdot y_1$ for $c \in \mathbb{R}$. y_1 is a solution so:

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 = 0, \quad \forall x$$

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0, \quad \forall x$$

So:

$$\underbrace{(y_1'' + y_2'')}_{(y_1+y_2)''} + p(x)\, \underbrace{(y_1' + y_2')}_{(y_1+y_2)'} + q(x)\, (y_1 + y_2) = 0$$

$$\underbrace{(y_1'' + p(x)\, y_1' + q(x)\, y_1)}_{=0} + \underbrace{(y_2'' + p(x)\, y_2' + q(x)\, y_2)}_{=0} = 0$$

So $y_1 + y_2$ is a solution and

$$c\, y_1'' + c\, p(x)\, y_1' + c\, q(x)\, y_1 = c\underbrace{(y_1'' + p(x)\, y_1' + q(x)\, y_1)}_{=0} = 0$$

and so $c\, y_1$ is a solution.

Conclude: Set of all solutions is a subspace of V.

It remains to show dimension of this subspace is 2. Here we use E&U thm. Given a point a we have a solution such that $y(a) = 1$. There is at least one solution, so the dimension must be ≥ 1. [Dimension is zero if only $y \equiv 0$ is a solution]

On the other hand, we have a solution y_1 such that $y_1(a) = 1$ and $y_1'(a) = 0$ and another solution y_2 such that $y_2(a) = 0$ and $y_2'(a) = 1$.

if y_1 and y_2 were linearly dependent, we would have $\alpha \in \mathbb{R}$ such that $y_2 = \alpha \cdot y$. But if we substitute $y_2(a) = \alpha\, y_1(a) = \alpha$, we get $\alpha = 0$. But then $y_2 \equiv 0$, and we know that $y'(a) = 1$ — contradiction!

That means y_1, y_2 are linearly independent and the dimension of the subspace is **at least** 2. Now we only need to show that the dimension is **exactly** 2.

lec 7 27.11.22 Let $f(x)$ be an arbitrary solution to (3), such that $f(a) = b$ and $f'(a) = c$. Now look at the following function:

$$g(x) = b\, y_1(x) + c\, y_2(x)$$

$y_1(x), y_2(x)$ are solutions and therefore $g(x)$ also solves (3).

Set $x = a$:

$$g(a) = b\, y_1(a) + c\, y_2(a) = b$$

$$g'(a) = b\, y_1'(a) + c\, y_2'(a) = c$$

So $g(x)$ solves the ODE and satisfies the same ICs as the arbitrary function $f(x)$. But by the E&U thm. there's only one solution satisfying a given set of ICs, so $f(x) = g(x)$ for all x and $f(x)$ is a linear combination of y_1 and y_2.

To conclude, all solutions can be written as linear combinations of y_1, y_2, meaning that the subspace is *spanned* by y_1, y_2, and its dimension is exactly 2.

Example Usage of E&U thm.

Given the ODE

$$y'' + y = 0$$

Note that $\sin x$ and $\cos x$ are solutions. We want the unique sol. such that

$$\begin{cases} y(0) = 3 \\ y'(0) = -2 \end{cases}$$

$\sin x, \cos x$ are linearly independent (as if $\alpha \sin x + \beta \cos x = 0$ for all x, setting $x = 0$ gives: $\alpha \sin 0 + \beta \cos 0 = 0 \rightarrow \beta = 0$, so $\alpha \sin x = 0$ for all x, so $\alpha = 0$.)

$\sin x, \cos x$ are therefore a basis for set of solutions. We want a function $y(x) = a \sin x + b \cos x$ such that the ICs hold. We get

$$3 = y(0) = a \sin 0 + b \cos 0 = b$$

$$-2 = y'(0) = a \cos(0) = a$$

So $y(x) = -2 \sin x + 3 \cos x$ is the unique solution. Knowing that a solution exists and the E&U thm. holds means that $y(x)$ is the **only** solution that satisfies the ICs.

Note. 1st order homogeneous linear ODEs are of form: $y' + p(x)\, y = 0$. We solved and found that also solutions were multiples of $e^{\int p(x)\,\mathrm{d}x}$, i.e. a 1-dimensional space of functions solves the ODE. We expect (and shall see later) that for an ODE of order n, the space of solutions would be n-dimensional.

5.2 Finding a basis for the set of solutions

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If $p(x), q(x)$ are **not** constant functions, this can be difficult.

There are some special situations for which we can use certain tricks to solve.

5.2.1 When one solution is known

Imagine we somehow know one non-zero solution and want to find a second solution which is linearly independent.

Suppose y_1 solves our ODE. We want y_2 which is **not** a multiple of y_1. It means that $\frac{y_2}{y_1}$ is not a constant, but a function, $v(x)$.

In other words, $v(x)$ is non-constant and $y_2(x) = y_1(x) \cdot v(x)$ is a solution. We substitute y_2 in the ODE:

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0$$

$$\begin{aligned} y_2' &= y_1'\, v + y_1\, v' \\ y_2'' &= y_1''\, v + 2\, y_1'\, v' + y_1\, v'' \end{aligned}$$

Get

$$(y_1'' v + 2\, y_1' v' + y_1 v'') + p(x)\,(y_1' v + y_1 v') + q(x)\,(y_1 v) = 0$$

Rewrite as

$$(y_1'' + p(x)\, y_1' + q(x)\, y_1)\, v + (2\, y_1' + p(x)\, y_1)\, v' + y_1 v'' = 0$$

The red term is equal to zero because y_1 is a solution. Let's look at what's left.

$$(2\, y_1' + p\, y_1)\, v' + y_1 v'' = 0$$

This is a 2nd order ODE in v, or a 1st order linear ODE in v'. Therefore we can find y_2 by "reduction of order".

Examples

1. $y + y'' = 0$. Suppose given $y_1 = \sin x$. Set $y_2 = y_1 v = \sin x \cdot v$.

$$\begin{aligned} y_2' &= \cos x \cdot v + \sin x \cdot v' \\ y_2'' &= -\sin x \cdot v + 2 \cos x \cdot v' + \sin x \cdot v'' \end{aligned}$$

   Substitute and get

$$(-\sin x \cdot v + 2 \cos x \cdot v' + \sin x \cdot v'') + \sin x \cdot v = 0$$

$$v'' \cdot \sin x + 2 \cos x \cdot v' = 0$$

   Set $v' = z$.

$$z' \cdot \sin x + 2 \cos x \cdot z = 0$$

   This is a separable ODE.

$$-2 \cos x \cdot z = \frac{\mathrm{d}z}{\mathrm{d}x} \cdot \sin x$$

$$-2 \int \frac{\cos x}{\sin x}\, \mathrm{d}x = \int \frac{\mathrm{d}z}{z}$$

$$-2 \ln |\sin x| = \ln |z| + c$$

Take $c=0$

$$\ln(\sin^{-2} x)=\ln|z|$$

$$v'=z=\frac{1}{\sin^2 x}\Rightarrow v=\cot x \text{is a solution for } \sin x\neq 0.$$

So out second solution is

$$y_2=\sin x\cdot v(x)=\sin x\cdot\cot x=\cos x$$

But we can verify $\cos x$ is a solution for all x, not only when $\sin x\neq 0$.

2. $2x^2y''+3xy'-y=0,\quad x>0$. By "inspection" we see that $y=\frac{1}{x}$ is a solution.

$$\begin{aligned}\left(\frac{1}{x}\right)' &= -\frac{1}{x^2}\\ \left(\frac{1}{x}\right)'' &= \frac{2}{x^3}\end{aligned}$$

$$\Rightarrow 2x^2\cdot\frac{2}{x^3}+3x\cdot\left(-\frac{1}{x^2}\right)-\frac{1}{x}\overset{\checkmark}{=}0$$

We want $v(x)\equiv C$ such that $y=\frac{v}{x}$ is a solution.

$$\begin{aligned}y' &= \frac{v'}{x}-\frac{v}{x^2}\\ y'' &= \frac{v''}{x}-\frac{2v'}{x^2}+\frac{2v}{x^3}\end{aligned}$$

Substitute in the ODE:

$$2x^2\left[\frac{v''}{x}-\frac{2v'}{x^2}+\frac{2v}{x^3}\right]+3x\left[\frac{v'}{x}-\frac{v}{x^2}\right]-\frac{v}{x}=0$$

$$2x^2\cdot\frac{v''}{x}-v'=0$$

Set $v'=z\not\equiv 0$.

$$2x\cdot\frac{\mathrm{d}z}{\mathrm{d}x}=z\longrightarrow\frac{\mathrm{d}z}{z}=\frac{\mathrm{d}x}{2x}$$

$$\ln|z|=\ln\sqrt{x},\quad x>0$$

$$v'=z=c\cdot x^{1/2}$$

$$v'=\frac{2}{3}c\cdot x^{3/2}$$

Take $c=\frac{3}{2}$.

$$y=\frac{v}{x}=\frac{x^{3/2}}{x}=\sqrt{x}$$

Out 2nd linearly independent solution will be $\sqrt{x}$ and the general solution to our ODE is $y = a\sqrt{x} + b\frac{1}{x}$.

3. $y'' - 3y' + 2y = 0$. Note that $y = e^x$ is a solution. We use reduction of order to find a second solution: $y = e^x \cdot v(x)$.

$$\begin{aligned} y' &= e^x \cdot v + e^x \cdot v' = e^x(v + v') \\ y'' &= e^x v + 2e^x v' + e^x v'' = e^x(v + 2v' + v'') \end{aligned}$$

Substitute:

$$e^x(v + 2v' + v'') - 3e^x(v + v') + 2e^x v = 0$$

Divide by $e^x \neq 0 \forall x$, rearrange and get:

$$v'' = v'$$

$v = e^x$ solves the problem, and $y = e^{2x}$ is a 2nd solution.

Alternative approach: Suppose we start with e^{2x} and want a 2nd solution of the form $y = v\,e^{2x}$.

$$\begin{aligned} y' &= 2e^{2x} v + e^{2x} v' \\ y'' &= 4e^{2x} v + 4e^{2x} v' + e^{2x} v'' \end{aligned}$$

$$(4e^{2x} v + 4e^{2x} v' + e^{2x} v'') - 3(2e^{2x} v + e^{2x} v') + 2e^{2x} v = 0$$

$$e^{2x} v' + e^{2x} v'' = 0$$

$$v'' = -v' \Rightarrow v(x) = -c\,e^{-x}, \quad \text{take } c = -1.$$

$$y = e^{2x} \cdot e^{-x} = e^x$$

We've got the solution from the first approach.

4. $x^2 y'' - 5xy + 9y = 0$. Notice $y = x^3$ is a solution. Want a sol. of form $y = x^3 v$.

$$\begin{aligned} y' &= 3x^2 v + x^3 v' \\ y'' &= 6xv + 6x^2 v' + x^3 v'' \end{aligned}$$

Get

$$x^2(6xv + 6x^2 v' + x^3 v'') - 5x(3x^2 v + x^3 v') + 9x^3 v = 0$$

$$x^4 v' + x^5 v'' = 0$$

Assuming $x \neq 0$, get

$$v' + x v'' = 0$$

Set $z = v'$, solve for z and finally get $v(x) = \ln|x|$, $y = x^3 \cdot \ln|x|$.

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Proposition

Given a linear homogeneous ODE of order n,

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = 0$$

the set of solutions is a vector space of functions of dimension n.

Proof

For $n = 2$ we've already proved. It is easy to see it is a subspace — showing dim. is n is similar to case we did for $n = 2$.

5.2.2 All coefficients are constants

All coeffs are constants: $\forall k$: $p_k(x) \equiv a_k \in \mathbb{R}$.

Rewrite ODE as

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_0\, y = 0, \quad a_n \neq 0$$

Example: $y'' - y' = 0$.

Clearly the $y \equiv 1$ and $y = \mathrm{e}^x$ are both solutions and are linearly independent. Therefore, they form a basis for the space of solutions. Given ICs $y(0) = 1$, $y'(0) = 0$, we can construct the general solution as a linear combination of our basis:

$$\text{General sol.} \quad y = a + b\,\mathrm{e}^x$$

$$\begin{aligned} 1 &= y(0) = a + b \\ 0 &= y'(0) = b \end{aligned}$$

So we get $a = 1, b = 0$ and the unique solution is $y \equiv 1$.

Another example: $y'' - 2\,y' - 15\,y = 0$. It makes sense to guess a solution of the form $y = \mathrm{e}^{\lambda x}$. Then:

$$\begin{aligned} y' &= \lambda\,\mathrm{e}^{\lambda x} \\ y'' &= \lambda^2\,\mathrm{e}^{\lambda x} \end{aligned}$$

Substitute and get

$$\lambda^2\,\mathrm{e}^{\lambda x} - 2\,\lambda\,\mathrm{e}^{\lambda x} - 15\,\mathrm{e}^{\lambda x} = 0$$

$$\mathrm{e}^{\lambda x}\,(\lambda^2 - 2\,\lambda - 15) = 0$$

$e^{\lambda x}$ is a solution $\Longleftrightarrow \lambda^2 - 2\lambda - 15 = 0$. Solve the quadratic equation:

$$\frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2} = \begin{cases} 5 \\ -3 \end{cases}$$

The solutions e^{-3x} and e^{5x} are both linearly independent solutions! So the general solution will be $y = a\, e^{-3x} + b\, e^{5x}$.

In the previous example, we would have gotten $y \equiv 1$ using the method above by finding that $\lambda = 0$.

General case

Given: $a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_0\, y = 0, \quad a_n \neq 0$ we look for solutions of the form $e^{\lambda x}$ and get:

$$y^{(k)}(x) = \lambda^k\, e^{\lambda x} \quad \forall x$$

And when we substitute in the ODE:

$$e^{\lambda x}\,(a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0) = 0$$

Then $e^{\lambda x}$ is a solution if and only if $a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0 = 0$. (This polynomial is also called the *characteristic polynomial of the ODE*.)

If the polynomial has n distinct (real of complex) solutions $\lambda_1, \ldots, \lambda_n$ then $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are n linearly independent solutions and form a basis for space of solutions.

Fundamental theorem of Algebra (Gauss)

Every polynomial equation over $\mathbb{C}$ has n solutions including multiplicities.

Solution of polynomial equations

$$a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0 = 0$$

There are **no** formulas that give the roots of polynomials (in general) for $n \geq 5$, in terms of the coefficients.

Example: 3rd order ODE

$$y^{(3)} - 6\, y'' + 11\, y' - 6 = 0$$

A trick to solve cubic equations: if you can find one solution λ_0 then you get an equation of the form $(\lambda - \lambda_0)(\text{quadratic equation}) = 0$. So check if $\lambda = 0, \pm 1, \pm 2, \ldots$ are solutions and the problem might be simplified.

In this case we need to solve

$$\lambda^3 - 6\, \lambda^2 + 11\, \lambda - 6 = 0$$

Setting $\lambda = 1$: $1 - 6 + 11 - 6 \stackrel{\checkmark}{=} 0$.

$$\lambda^3 - 6\,\lambda^2 + 11\,\lambda - 6 = (\lambda - 1)\,(\lambda^2 - 5\,\lambda + 6) = (\lambda - 1)\,(\lambda - 2)\,(\lambda - 3)$$

The general solution is $y = a\,\mathrm{e}^x + b\,\mathrm{e}^{2x} + c\,\mathrm{e}^{3x}$.

<u>Example: The characteristic polynomial does not have n distinct roots</u>

We still need n linearly independent solutions. What do we do?

$$y'' - 4\,y' + 4\,y = 0$$

The characteristic polynomial is $\lambda^2 - 4\,\lambda + 4 = (\lambda - 2)^2$. Our method yields only one exponential function: e^{2x}.

We look for a second linearly independent solution. We want a non-constant function $v(x)$ such that $y = \mathrm{e}^{2x}\,v(x)$ solves the ODE.

$$\begin{array}{lcl} y' & = & 2\,\mathrm{e}^{2x}\,v(x) + \mathrm{e}^{2x}\,v'(x) = \mathrm{e}^{2x}\,(2\,v + v') \\ y'' & & 2\,\mathrm{e}^{2x}\,(2\,v + v') + \mathrm{e}^{2x}\,(2\,v' + v'') = \mathrm{e}^{2x}\,(4\,v + 4\,v' + v'') \end{array}$$

$$\cancel{\mathrm{e}^{2x}}\,(4\,v + 4\,v' + v'') - 4\cancel{\mathrm{e}^{2x}}\,(2\,v + v') + 4\cancel{\mathrm{e}^{2x}}\,v = 0$$

$$v'' = 0 \Rightarrow v \ \text{ is linear in } x$$

So take $v = x$ and get $\{\mathrm{e}^{2x}, x\,\mathrm{e}^{2x}\}$ is the basis for set of solutions.

<u>In general</u>

If λ_0 is a root of the characteristic polynomial of multiplicity r, then $\mathrm{e}^{\lambda_0 x}, x\,\mathrm{e}^{\lambda_0 x}, x^2\,\mathrm{e}^{\lambda_0 x}, \ldots, x^{r-1}\,\mathrm{e}^{\lambda_0 x}$ will all solve the ODE and are linearly independent, and will be linearly independent of solutions we obtain from other roots of the characteristic polynomial.

<u>Example</u>: Suppose we know that the char. poly. factors as $\lambda^3\,(\lambda + 3)^2\,(\lambda - 1)^2$: A 7th order linear ODE.

- From $\lambda = 0$ we get the solutions: $y = 1, x, x^2$.
- From $\lambda = 3$ we get the solutions: $y = \mathrm{e}^{-3x}, x\,\mathrm{e}^{-3x}$.
- From $\lambda = 1$ we get the solutions $y = \mathrm{e}^x, x\,\mathrm{e}^x$.

<u>Example: Roots of the characteristic polynomial are not real</u>

$$y'' + y = 0$$

Our method yields the roots of char. poly. x^2+1: $\pm\mathrm{i}$. Get 2 non-real solutions: $\mathrm{e}^{\mathrm{i}x}$, $\mathrm{e}^{-\mathrm{i}x}$. These span the vector space of all complex solutions.

We note that the span over $\mathbb{C}$ $\{\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}\}$ is a space of dimension 2 of complex functions. We also have 2 real solutions: $\sin x$ and $\cos x$, that are linearly independent. This means that the span can be written as $\{\sin x, \cos x\}$. This makes sense, as $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, $\mathrm{e}^{-\mathrm{i}x} = \cos x - \mathrm{i}\sin x$, and $\frac{1}{2}(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}) = \cos x$, $\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}) = \sin x$.

$\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}$ are in the span of $\{\sin x, \cos x\}$, and $\cos x, \sin x$ are in the span of $\{\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}\}$. However, $\mathrm{span}_{\mathbb{C}}\{\sin x, \cos x\} \not\supset \mathrm{span}_{\mathbb{R}}\{\sin x, \cos x\}$.

Note: if $z(x)$ is a complex-valued function solving our ODE, then it can be written as:

$$z(x) = u(x) + \mathrm{i}\, v(x), \quad u, v \in \mathbb{R}$$

Then we have

$$(u+\mathrm{i}\,v)^{(n)} + a_{n-1}(u+\mathrm{i}\,v)^{(n-1)} + \cdots + a_1(u+\mathrm{i}\,v)' + a_0(u+\mathrm{i}\,v) = 0$$

$$u^{(n)} + \mathrm{i}\,v^{(n)} + a_{n-1}(u^{(n-1)} + \mathrm{i}\,v^{(n-1)}) + \cdots + a_1(u' + \mathrm{i}\,v') + a_0(u+\mathrm{i}\,v) = 0$$

$$(u^{(n)} + a_{n-1}\,u^{(n-1)} + \cdots + a_1\,u' + a_0\,u) + \mathrm{i}\,(v^{(n)} + a_{n-1}\,v^{(n-1)} + \cdots + a_1\,v' + a_0\,v) = 0$$

This holds for all x if and only if:

$$\begin{cases} u^{(n)} + a_{n-1}\,u^{(n-1)} + \cdots + a_1\,u' + a_0\,u = 0 \\ v^{(n)} + a_{n-1}\,v^{(n-1)} + \cdots + a_1\,v' + a_0\,v = 0 \end{cases}$$

Therefore, $u(x), v(x)$ are **both** real solutions to the ODE.

Note that if $z(x)$ is a complex solution to our ODE, then so is $\bar{z}(x) = u(x) - \mathrm{i}\,v(x)$, as since we have:

$$z^{(n)} + a_{n-1}\,z^{(n-1)} + \cdots + a_1\,z' + a_0\,z = 0$$

Since $a_i \in \mathbb{R}$, $\bar{a}_i = a_i$. Because:

$$\overline{z_1 \cdot z_2} = \bar{z}_1\,\bar{z}_2$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

We get

$$\bar{z}^{(n)} + a_{n-1}\,\bar{z}^{(n-1)} + \cdots + a_1\,\bar{z}' + a_0\,\bar{z} = 0$$

In conclusion, **complex roots come in pairs**. So every complex solution $z(x)$ gives rise to another, $\bar{z}(x)$, and these give 2 linearly independent real solutions $u(x), v(x)$.

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Example: Roots are complex, where the real part is non-zero

$$y''+y'+y=0$$

Characteristic polynomial is $\lambda^2+\lambda+1$,

$$\lambda^2+\lambda+1=0 \Longleftrightarrow \lambda=\frac{-1\pm\sqrt{1-4}}{3}=\frac{-1\pm\mathrm{i}\sqrt{3}}{2}$$

We get 2 complex solutions to the ODE: $\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)x}, \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)x}$. Note that

$$\mathrm{e}^{-\frac{1}{2}x}\cdot \mathrm{e}^{\frac{\sqrt{3}}{2}\mathrm{i}x}=\mathrm{e}^{-\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x+\mathrm{i}\sin\frac{\sqrt{3}}{2}x\right)$$

So we get

$$\begin{cases} u(x)=\mathrm{e}^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x \\ v(x)=\mathrm{e}^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x \end{cases}$$

are 2 real, linearly independent solutions.

Conclude: if z is a non-real solution to $a_n\lambda^n+\cdots+a_1\lambda+a_0=0$ then so is $\bar{z}$, and if we write

$$z=\alpha+\mathrm{i}\,\beta$$

we get 2 linearly independent real solutions to the ODE:

$$\begin{cases} \mathrm{e}^{\alpha x}\cos\beta \\ \mathrm{e}^{\alpha x}\sin\beta \end{cases}$$

Final case: the characteristic polynomial has non-real roots which are multiple roots. Then we multiply the solutions we obtained by powers of x.

For example,

$$y^{(4)}+2\,y''+y=0$$

Characteristic equation is

$$\lambda^4+2\,\lambda^2+1=0$$

$$(\lambda^2+1)^2=0$$

roots are $\pm\mathrm{i}$ both of multiplicity 2. Get 4 linearly independent real solutions: $\cos x, \sin x, x\cos x, x\sin x$.

$$y=a\cos x+b\sin x+c\,x\cos x+d\,x\sin x$$

Let's show these 4 are linearly independent. Assume linear combinations $=0$. Set $x=0$ and get $a=0$. Set $x=\pi$ and get $c=0$. Set $x=\frac{\pi}{2}$ and get $b+\frac{\pi}{2}d=0$. Set $x=-\frac{\pi}{2}$ and get $-b+\frac{\pi}{2}d=0$. Therefore $b,d=0$.

<u>Summarize</u>

Suppose the roots of the characteristic polynomial are:

<u>real:</u> $\lambda_1,\ldots,\lambda_k$ of respective multiplicities $r_1,\ldots,r_k$.

<u>non-real:</u>

$$\begin{cases} \mu_1, \bar{\mu}_1 \text{ of multiplicity } s_1 \\ \mu_2, \bar{\mu}_2 \text{ of multiplicity } s_2 \\ \vdots \\ \mu_\ell, \bar{\mu}_\ell \text{ of multiplicity } s_\ell \end{cases}$$

Then we can obtain $r_1+\cdots+r_k$ linearly independent real solutions from $\lambda_1,\ldots,\lambda_k$ and $2(s_1+\cdots+s_\ell)$ non-real solutions from $\mu_1,\bar{\mu}_1,\ldots,\mu_\ell,\bar{\mu}_\ell$.

<u>Example</u>

Suppose out ODE has a characteristic polynomial which factors as follows:

$$\underbrace{(\lambda-1)^3}_{1}\underbrace{(\lambda-2)^4}_{2}\underbrace{(\lambda^2+\lambda+4)^3}_{\frac{1\pm\mathrm{i}\sqrt{15}}{2}}$$

The degree of the polynomial is 13. We want 13 linearly independent solutions to the ODE.

- $\lambda=1$: $\mathrm{e}^x, x\,\mathrm{e}^x, x^2\,\mathrm{e}^x$.
- $\lambda=2$: $\mathrm{e}^{2x}, x\,\mathrm{e}^{2x}, x^2\,\mathrm{e}^{2x}, x^3\,\mathrm{e}^{2x}$
- $\lambda=\frac{-1+\mathrm{i}\sqrt{15}}{2}$: $\mathrm{e}^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{15}\,x}{2}\right)$,$\mathrm{e}^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{15}\,x}{2}\right)$, $\mathrm{e}^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{15}\,x}{2}\right)$, $x\,\mathrm{e}^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{15}\,x}{2}\right)$, $x^2\,\mathrm{e}^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{15}\,x}{2}\right)$, $x^2\,\mathrm{e}^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{15}\,x}{2}\right)$.

<u>Example</u>

$$y^{(3)}-y''+2\,y'-8\,y=0$$

Characteristic equation is

$$\lambda^3-\lambda^2+2\,\lambda-8=0$$

Look for an integer root. $\lambda=2$ is a root. Factor out $\lambda-2$.

$$\lambda^3-\lambda^2+2\,\lambda-8=(\lambda-2)(\lambda^2+\lambda+4)$$

5.3 Non-homogeneous linear ODEs

Given a second order ODE of form:

$$y'' + p(x)\, y' + q(x)\, y = g(x), \quad g(x) \not\equiv 0$$

Note that sums and multiples of solutions will **not** solve our equation!

So the set of all solutions is **not** a vector space.

Claim: If y_1, y_2 both solve the ODE then $y_1 - y_2$ solves the associated homogeneous equation.

Proof: We know that:

$$\begin{aligned} y_1'' + p(x)\, y_1' + q(x)\, y_1 &= g(x) \\ y_2'' + p(x)\, y_2' + q(x)\, y_2 &= g(x) \end{aligned}$$

Subtract to get:

$$(y_1'' - y_2'') + p(x)\,(y_1' - y_2') + q(x)\,(y_1 - y_2) = 0$$

So $y_1 - y_2$ solves

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Claim: Let y_1 be a solution of the ODE. Then *every* solution will be of the form:

$$y = y_1 + y_0,$$

where y_0 solves the associated homogeneous equation. Moreover, every solution of this form solves the non-homogeneous equation. Therefore, if we find a y_1, we can find the general solution by finding the set of solutions to the homogeneous equation and add them up.

Proof: Suppose y_0 solves the homogeneous equation. We substitute $y_1 + y_0$ in the ODE:

$$(y_1 + y_0)'' + p(x)\,(y_1 + y_0)' + q(x)\,(y_1 + y_0)$$

$$\underbrace{(y_1'' + p(x)\, y_1' + q(x)\, y_1)}_{g(x)} + \underbrace{(y_0'' + p(x)\, y_0' + q(x)\, y_0)}_{0} = g(x) + 0 = g(x)$$

Now take any solution y_2 of the ODE. We shall show it is of the form $y_0 + y_1$.

Write:

$$y_2 = y_1 + (y_2 - y_1)$$

$(y_2 - y_1)$ is the difference of 2 solutions to the non-homogeneous ODE and so solves the homogeneous equations. So y_2 is of the required form.

Conclusion

To find the general solution to a non-homogeneous linear ODE we need to find one particular solution and a basis for solution space of the homogeneous equation.

Question: How do we find a particular solution to the non-homogeneous ODE?

5.3.1 Method of undetermined coefficients

Applies for the special case where our ODE has constant coefficients. The RHS can be a non-constant function, though.

$$a_2\, y'' + a_1\, y' + a_0\, y = g(x), \quad g(x) \not\equiv 0$$

This works if $g(x)$ belongs to one of several families of "nice" functions: combination of exponentials, trigonometric functions (sine and cosine), and polynomials.

Note: multiplication of these "nice" functions also counts.

Idea: We guess a solution which has "same" form as $g(x)$. If $g(x)$ is a poly, then we guess a poly. If $g(x)$ is an exponential, guess an exponential, etc.

Note: This method does **not** work for rational functions on RHS! (Poly divided by poly.)

Examples: Which function to guess?

1. $y'' - 3\, y' - 4\, y = 3\, \mathrm{e}^{2x}$. Guess a solution of form $y(x) = A\, \mathrm{e}^{2x}$. We substitute in the equation and get:

$$4\, A\, \mathrm{e}^{2x} - 6\, A\, \mathrm{e}^{2x} - 4\, A\, \mathrm{e}^{2x} = 3\, \mathrm{e}^{2x}$$

Divide by $\mathrm{e}^{2x} \neq 0$.get

$$4\, A - 6\, A - 4\, A = 3 \rightarrow A = -\frac{1}{2}$$

We found a particular solution $y = -\frac{1}{2}\, \mathrm{e}^{2x}$.

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Now find the basis of the associated homogeneous equation.

$$y'' - 3\, y' - 4\, y = 0$$

$$\lambda^2 - 3\, \lambda - 4 = 0 \rightarrow \lambda = 4, -1$$

So $\mathrm{e}^{4x}, \mathrm{e}^{-x}$ is a basis for solutions space of the homo. equation. The general solution to the ODE is

$$y = -\frac{1}{2}\, \mathrm{e}^{2x} + c_1\, \mathrm{e}^{4x} + c_2\, \mathrm{e}^{-x}, \quad c_1, c_2 \in \mathbb{R}$$

2. $y'' - 3\,y' - 4\,t = 2\sin x$.

   Wrong guess: $y = A\sin x$. What happens if we substitute?

$$-A\sin x - 3\,A\cos x - 4\,A\sin x = 2\sin x$$

$$-5\,A\sin x - 3\,A\cos x = 2\sin x \quad \forall x$$

   This doesn't hold! Set $x = 0$ and find a contradiction.

   Right guess: $y = A\sin x + B\cos x$.

$$y' = A\cos x - B\sin x$$

$$y'' = -A\sin x - B\cos x$$

$$(-A\sin x - B\cos x) - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$$

$$(-5\,A + 3\,B)\sin x + (-5\,B - 3\,A)\cos x = 2\sin x$$

   Equate coefficients of $\sin x$ and $\cos x$ on both sides.

$$\begin{cases} \sin x\colon -5\,A + 3\,B = 2 \\ \cos x\colon -5\,B - 3\,A = 0 \end{cases} \Rightarrow A = -\frac{5}{17}, B = \frac{3}{17}$$

   So $y_2 = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$ is a particular solution. We've already solved the homogeneous equation (example 1.) The general solution is

$$-\frac{5}{17}\sin x + \frac{3}{17}\cos x + c_1\,\mathrm{e}^{4x} + c_2\,\mathrm{e}^{-x}, \quad c_1, c_2 \in \mathbb{R}$$

3. $y'' - 3\,y' - 4\,y = 4\,x^2 - 1$. Guess a polynomial of the same degree as the RHS.

   Guess $y = A\,x^2 + B\,x + C$.

$$\begin{aligned} y' &= 2\,A\,x + B \\ y'' &= 2\,A \end{aligned}$$

   Substitute:

$$2\,A - 3(2\,A\,x + B) - 4(A\,x^2 + B\,x + C) = 4\,x^2 - 1$$

   Equate coefficients on both sides.

$$\begin{cases} x^2\colon -4\,A = 4 \\ x\colon -6\,A - 4\,B = 0 \\ \text{const}\colon 2\,A - 3\,B - 4\,C = -1 \end{cases} \Rightarrow A = -1, B = \frac{3}{2}, C = -\frac{11}{8}$$

$y = -x^2 + \frac{3}{2}x - \frac{11}{8}$ is a particular solution.

4. $y'' - 3y' - 4y = -8e^x \cos x$. Guess $y = Ae^x \cos x + Be^x \sin x$ for a particular solution.

5. $y'' - 3y' - 4y = 3e^{2x} + 2\sin x$.

   We found $y_1 = -\frac{1}{2}e^{2x}$ solves $y'' - 3y' - 4y = 3e^{2x}$ and $y_2 = -\frac{5}{17}\sin x + \frac{3}{17}\cos x$ solves $y'' - 3y' - 4y = 2\sin x$.

   So if we substitute $y_1 + y_2$ in LHS we get: $3e^{2x} + 2\sin x$.

   This is how you deal with a complex RHS. Don't try to solve it all in one shot. Divide the RHS into easily solvable groups and add them up later.

6. $y'' + 9y = 2\sin 3x$.

   Here, as before, we try $y = A\sin 3x + B\cos 3x$.

$$\begin{aligned} y' &= 3A\cos 3x - 3B\sin 3x \\ y'' &= -9(A\sin 3x + B\cos 3x) \end{aligned}$$

   When we substitute we get

$$\underbrace{-9(A\sin 3x + B\cos 3x) + 9(A\sin 3x + B\cos 3x)}_{=0} = 2\sin 3x$$

$$0 = 2\sin 3x \Rightarrow \text{no solution?!}$$

   What went wrong? If you look at the associated homogeneous equation, it turns out that $y = A\sin 3x + B\cos 3x$ solves it.

$$y'' + 9y = 0$$

$$\lambda^2 + 9 = 0 \rightarrow \lambda = \pm 3\,\mathrm{i}$$

   We get a basis for solution space: $\{\sin 3x, \cos 3x\}$.

   We have to guess something else. What works here — Multiply by x.

   Guess instead $y = x(A\cos 3x + B\sin 3x)$

$$\begin{aligned} y' &= A\cos 3x + B\sin 3x + x(-3A\sin 3x + 3B\cos 3x) \\ y'' &= -3A\sin 3x + 3B\cos 3x - 3A\sin 3x + 3B\cos 3x \\ &+ x(-9A\cos 3x - 9B\sin 3x) \\ y'' &= -6A\sin 3x + 6B\cos 3x - 9x(A\cos 3x + B\sin 3x) \end{aligned}$$

   Substitute:

$$y'' + 9y = -6A\sin 3x + 6B\cos 3x = 2\sin 3x$$

Equate coefficients on both sides and get $A = -\frac{1}{3}$, $B = 0$. The particular solution is $y = -\frac{1}{3}\, x \cos 3\, x$.

7. $y^{(4)} - y' = x^3 + 2\, x^2 + 3$. Find solution to homogeneous version. Note that we can't guess a polynomial of degree 3 because the LHS has a characteristic polynomial of a higher degree.

$$\lambda(\lambda - 1)(\lambda^2 + 2\,\lambda + 1) = \lambda(\lambda^3 - 1) = 0$$

Roots are $\lambda = 0, \lambda = 1, \lambda = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2}$.

Get 4 linearly independent solutions: $1, \mathrm{e}^x, \mathrm{e}^{-\frac{1}{2}x} \cos \frac{\sqrt{3}\, x}{2}, \mathrm{e}^{-\frac{1}{2}x} \sin \frac{\sqrt{3}\, x}{2}$.

It is clear that polynomials of degree 0 solve the homogeneous equation, however it wouldn't benefit to finding a particular solution.

Guess a 4th degree polynomial, which would give us a third degree polynomial at both RHS and LHS. $y = x(A\, x^3 + B\, x^2 + C\, x + D$.

$$\begin{aligned} y' &= 4\, A\, x^3 + 3\, B\, x^2 + 2\, C\, x + D \\ y'' &= 12\, A\, x^2 + 6\, B\, x + 2\, C \\ y''' &= 24\, A\, x + 6\, B \\ y^{(4)} &= 24\, A \end{aligned}$$

Substitute and get:

$$24\, A - (4\, A\, x^3 + 3\, B\, x^2 + 2\, C\, x + D) = x^3 + 2\, x^2 + 3$$

Equate coefficients of equal powers of x on both sides:

$$\begin{cases} x^3\!: & -4\, A = 1 \rightarrow A = -\frac{1}{4} \\ x^2\!: & -3\, B = 2 \rightarrow B = -\frac{2}{3} \\ x\!: & 2\, C = 0 \rightarrow C = 0 \\ 1\!: & 24\, A - D = 3 \rightarrow D = -9 \end{cases}$$

The general solution is: $y = -\frac{1}{4}\, x^4 - \frac{2}{3}\, x^2 - 9\, x + c_1 + c_2\, \mathrm{e}^x + \mathrm{e}^{-\frac{1}{2}x} \left(c_3 \cos \frac{\sqrt{3}}{2}\, x + c_4 \sin \frac{\sqrt{3}}{2}\, x \right)$.

<u>Note</u>: If the ODE were $y^{(4)} - y'' = x^3 + 2\, x^2 + 3$ then we would have chosen a 5th degree polynomial as a particular solution.

<u>Note</u>: If the homogeneous equation would have had repeated roots and RHS involved a solution to the homogeneous equation, we would have multiplied by higher powers of x.

8. $y'' + 3\, y' = 2\, x^4 + x^2\, \mathrm{e}^{-3x} + \sin 3\, x$. Solve the homogeneous equation.

$$\lambda(\lambda + 3) = \lambda^2 + 3\, \lambda = 0$$

$\lambda = 0, -3$, giving the basis: $1, \mathrm{e}^{-3x}$.

It won't be enough to guess either a poly. of degree 4 of an exponent multiplied by x^2. Divide the problem into parts.

I. Find a particular solution when RHS is $2\,x^4$. Guess a 5th degree poly. with constant zero (as it's redundant).

$$y_1 = x\,(A_4\,x^4 + A_3\,x^3 + A_2\,x^2 + A_1\,x + A_0)$$

II. Now find a particular solution when RHS is $x^2\,\mathrm{e}^{-3x}$.

If we guess (quadratic)e^{-3x}, i.e. $y_2 = (B_2\,x^2 + B_1\,x)\,\mathrm{e}^{-3x}$, then

$$\begin{aligned} y_2' &= (2\,B_2\,x + B_1)\,\mathrm{e}^{-3x} - 3\,\mathrm{e}^{-3x}(B_2\,x^2 + B_1\,x) \\ y_2'' &= 2\,B_2\,\mathrm{e}^{-3x} + B_1\,\mathrm{e}^{-3x} - 3\,\mathrm{e}^{-3x}(2\,B_2\,x + B_1) \\ &+\; -3\,\mathrm{e}^{-3x}(2\,B_2\,x + B_1) + 9\,\mathrm{e}^{-3x}(B_2\,x^2 + B_1\,x) \end{aligned}$$

Substitute on LHS:

$$(9\,\mathrm{e}^{-3x}\,B_2\,x^2 + \cdots) + 3(-3\,\mathrm{e}^{-3x}\,B_2\,x^2 + \cdots +) = (\text{lin. poly.})\,\mathrm{e}^{-3x}$$

We don't have a high enough order poly. multiplied by e^{-3x}. Guess instead $y_2 = x((B_2\,x^2 + B_1\,x + B_0)\,\mathrm{e}^{-3x})$.

III. Now find particular solution when RHS is $\sin 3\,x$. Guess $y_3 = C_1 \sin 3\,x + D_2 \cos 3\,x$.

The general solution will be $y_1 + y_2 + y_3 + c_1 + c_2\,\mathrm{e}^{-3x}$.

Conclude

- First, solve the associated homogeneous equation
- If the RHS is a solution to the homogeneous equation, guess a particular solution of the form:

  $$(\text{power of } x) \cdot (\text{RHS form})$$

  Such that the form of LHS matches the form of RHS.

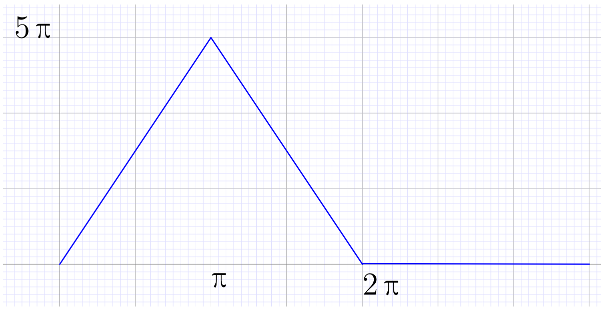
Cases where the RHS is split (not differentiable)

For example,

$$u'' + u = f(t), \quad \begin{cases} u(0) = 0 \\ u'(0) = 0 \end{cases}$$

$$f(t) = \begin{cases} 5\,t, \quad 0 \leq t \leq \pi \\ 5\,(2\,\pi - t), \quad \pi \leq t \leq 2\,\pi \\ 0, \quad t > 2\,\pi \end{cases}$$

$f(t)$ looks like this:



Solve associated homogeneous equation:

$$x^2 + 1 = 0 \Longleftrightarrow x = \pm \mathrm{i}$$

General solution is a linear combination of $\sin t, \cos t$.

Particular solution on $[0, \pi]$: In this interval $f(t)$ is a polynomial. Guess $u = A\,t + B$, substitute and get

$$A\,t + B = 5\,t \rightarrow A = 5, B = 0$$

General solution on $[0, \pi]$: $x(t) = a_1 \cos t + a_2 \sin t + 5\,t$. Inserting $t = 0$ gives a_1, a_2.

Particular solution on $[\pi, 2\,\pi]$: Guess again $u = A\,t + B$, substitute and get

$$A\,t + B = 10\,\pi - 5\,t \rightarrow A = -5, B = 10\,\pi$$

General solution on $[\pi, 2\,\pi]$: $y(t) = b_1 \cos t + b_2 \sin t - 5\,t + 10\,\pi$.

General solution on $[2\,\pi, \infty)$: $z(t) = c_1 \cos t + c_2 \sin t$.

We want a unique solution satisfying the ICs, which is twice differentiable (equivalent to order of the ODE). We need to determine the coefficients $a_{1,2}, b_{1,2}, c_{1,2}$ so that this condition holds.

Setting $t=0$:

$$x(0)=a_1=0$$

$$x'(0)=a_2+5=0\rightarrow a_2=-5$$

$$x(t)=-5\sin t+5\,t$$

Now we need to make sure that $x(\pi)=y(\pi), x'(\pi)=y'(\pi)$.

$$x(\pi)=5\,\pi, x'(\pi)=10$$

Set “initial conditions” for $y(t)$: $y(\pi)=5\,\pi$, $y'(\pi)=10$.

$$y(\pi)=b_1\cos\pi+b_2\sin\pi-5\,\pi+10\,\pi=-b_1+5\,\pi\rightarrow b_1=0$$

$$y'(\pi)=-b_2-5\rightarrow b_2=-15$$

$$y(t)=-15\sin t-5\,t+10\,\pi$$

Make sure $y(2\,\pi)=z(2\,\pi), y'(2\,\pi)=z'(2\,\pi)$. $y(2\,\pi)=0, y'(2\,\pi)=-20$.

$$z(2\,\pi)=c_1\cos 2\,\pi+c_2\sin 2\,\pi\rightarrow c_1=0$$

$$z'(2\,\pi)=-c_1\sin 2\,\pi+c_2\cos 2\,\pi\rightarrow c_2=-20$$

$$z(t)=-20\sin t$$

In summary, we have a unique solution:

$$u(t)=\begin{cases}-5\sin t+5\,t, & 0\leq t\leq\pi\\ -15\sin t-5\,t+10\,\pi, & \pi<t\leq 2\,\pi\\ -20\sin t, & t>2\,\pi\end{cases}$$

We constructed $u(t)$ so that it is continuous everywhere. What about the second derivative?

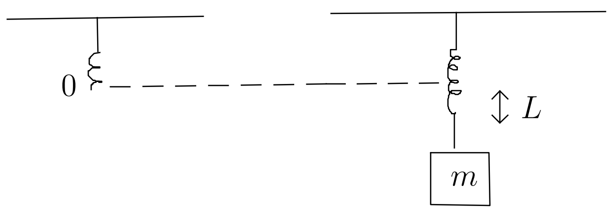
$$u''(t)=\begin{cases}5\sin t, & 0\leq t\leq\pi\\ 15\sin t, & \pi<t\leq 2\,\pi\\ 20\sin t, & t>2\,\pi\end{cases}$$

The second derivative is continuous (which is what we expected). Note though that at π and $2\,\pi$, $u'''(t)$ does not exist!

Our method of indeterminate coefficients works also for a split RHS case, so long that the solution is differentiable and continuous within the given intervals.

5.3.2 Mechanical and electrical vibrations

Start with the mechanical case: mass on a spring.



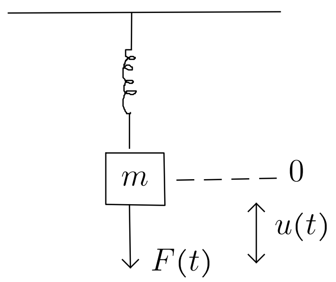
When we hang a mass on a spring it lengthens by L.

When in a state of equilibrium, by Newton's 2nd law: $\sum F = 0$.

$$\begin{aligned} \text{Force downward} &= m\,g \\ \text{Force upward exerted by spring} &= F_s = k\,L \end{aligned}$$

So $m\,g = F_s = k\,L$, where k is the spring's constant. (The stronger the spring, the smaller L would be.)

We act on the spring with a new force $F(t)$ and it displaces the mass by $u(t)$ at time t.



Forces acting on mass:

- $F(t)$, external force (down)
- $m\,g$, gravity (down).
- Force exerted by the spring (up)
- Damping force (drag) (up), which is proportionate to the velocity. $\gamma u'(t)$.

Sum of forces is $m\,a(t)$.

$$m\,u''(t) = F(t) + m\,g - k(L + u(t)) - \gamma u'(t)$$

$$m\,u'' = F(t) - k\,u - \gamma u'$$

lec 11 11.12.22 Rearrange:

$$m\,u'' + \gamma u' + k\,u = F(t)$$

This is a 2nd order linear ODE with constant coefficients. Initial conditions: $u(0)=u_0$, $u'(0)=v_0$.

Example: A mass weighing 4N stretches a spring by 10 cm. We stretch another 20 cm down and let go. The mass is in a sticky medium that exerts a resistance of 6N when mass has velocity of $1\,\mathrm{m\,s^{-1}}$. At $t=0$ $F=0$.

$$m\,u''+\gamma\,u'+k\,u=0$$

$$m\,g=4\rightarrow m\approx 0.4\,\mathrm{kg}$$

$$m\,g=k\,L\rightarrow k=\frac{0.4\,\mathrm{N}}{0.1\,\mathrm{m}}=40\,\mathrm{kg\,s^{-2}}$$

$$\gamma\,u'=6\ \text{ when } u'=1\,\mathrm{m\,s^{-1}}\rightarrow\gamma=6\,\mathrm{kg\,s^{-1}}$$

Initial conditions are $u(0)=0.2\,\mathrm{m}$ and $u'(0)=0$. Get:

$$\frac{4}{10}\,u''+6\,u'+40\,u=0$$

$$u''+15\,u'+100\,u=0$$

Solve characteristic equation:

$$\lambda^2+15\,\lambda+100=0$$

$$\lambda_{1,2}=\frac{-15\pm\sqrt{225-400}}{2}=\frac{-15\pm 5\sqrt{7}\,\mathrm{i}}{2}$$

We actually expect non-real roots, because that means the mass oscillates!

General solution:

$$u(t)=c_1\,\mathrm{e}^{-\frac{15}{2}t}\cos\frac{5\sqrt{7}}{2}\,t+c_2\,\mathrm{e}^{-\frac{15}{2}t}\sin\frac{5\sqrt{7}}{2}\,t$$

Set ICs:

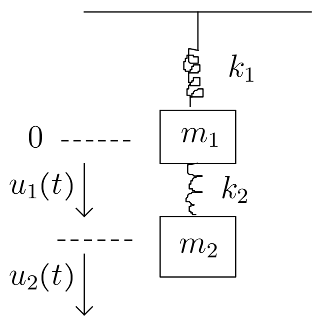
$$u(0)\ =\ c_1=\frac{1}{5}$$

$$u'(t)\ =\ -\frac{15}{2}\left(c_1\,\cos\,\frac{5\sqrt{7}}{2}\,t\,+\,c_2\,\sin\,\frac{5\sqrt{7}}{2}\,t\right)\,+\,\mathrm{e}^{-\frac{15}{2}t}\left(-c_1\,\frac{5\sqrt{7}}{2}\,\sin\,\frac{5\sqrt{7}}{2}\,t\,+\,c_2\,\frac{5\sqrt{7}}{2}\cos\frac{5\sqrt{7}}{2}\,t\right)$$

$$u'(0)=-\frac{15}{2}\,c_1+\frac{5\sqrt{7}}{2}\,c_2=0\rightarrow c_2=\frac{3}{5\sqrt{7}}$$

The unique solution is: $u(t)=\mathrm{e}^{-\frac{15}{2}t}\left(\frac{1}{5}\cos\frac{5\sqrt{7}}{2}t+\frac{3}{5\sqrt{7}}\sin\frac{5\sqrt{7}}{2}t\right)$

Another example: 2-degree of freedom spring-mass system.



$u_i(t)$ is the position of mass i at time t.

We assume there is no damping and no external force: $F(t)=0, \gamma=0$.

$$\begin{aligned} m_1 u_1'' &= -k_1 u_1+(u_2-u_1)\, k_2 \\ m_2 u_2'' &= -k_2(u_2-u_1) \end{aligned}$$

Solve by reducing to one 4th order ODE. Say $m_1=m_2=1, k_1=3, k_2=2$.

$$\begin{aligned} u_1'' &= -3\, u_1+(u_2-u_1)\cdot 2 \\ u_2'' &= -2(u_2-u_1) \end{aligned}$$

$$\begin{aligned} u_1'' &= -5\, u_1+2\, u_2 \rightarrow u_2=\frac{u_1''+5\, u_1}{2} \\ u_2'' &= 2\, u_1-2\, u_2 \end{aligned}$$

Substitute u_2 and get

$$\left(\frac{u_1''+5\, u_1}{2}\right)''=2\, u_1-(u_1''+5\, u_1)$$

$$u_1^{(4)}+7\, u_1''+6\, u_1=0$$

Solve characteristic equation: $\lambda^4+7\,\lambda^2+6=0$

$$\lambda=\pm\mathrm{i}, \pm\mathrm{i}\sqrt{6}$$

General solution:

$$u_1=c_1\cos t+c_2\sin t+c_3\cos\sqrt{6}\, t+c_4\sin\sqrt{6}\, t$$

Taking IC:

$$\begin{cases} u_1(0) = 1 \\ u_1'(0) = 0 \\ u_2(0) = 2 \\ u_2'(0) = 0 \end{cases}$$

yields:

$$u_1 = \cos t$$

$$u_2 = \frac{u_1'' + 5\,u_1}{2} = \frac{-\cos t + 5\cos 5}{2} = 2\cos t$$

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Special cases of $m\,u'' + \gamma\,u' + k\,u = F(t)$:

Undamped free vibrations

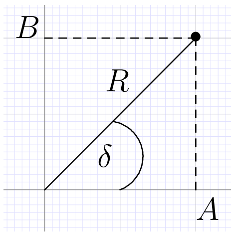
Here $F(t) = 0$ and $\gamma = 0$. Equation is now $m\,u'' + k\,u = 0$.

$$u'' + \frac{k}{m}\,u = 0$$

We need to solve $\lambda^2 + \frac{k}{m} = 0$. $\lambda_{1,2} = \pm\sqrt{\frac{k}{m}}$. Denote $\sqrt{\frac{k}{m}} = \omega_0$. General solution is

$$u(t) = A\cos\omega_0\,t + B\sin\omega_0\,t$$

Rewrite using polar coordinates:



$$R = \sqrt{A^2 + B^2}, \delta = \arctan\frac{B}{A}$$

$$A = R\cos\delta, B = R\sin\delta$$

So

$$\begin{aligned} u(t) &= R(\cos\delta\cos\omega_0\,t + \sin\delta\sin\omega_0\,t) \\ &= R\cos\,(\omega_0\,t - \delta) \end{aligned}$$

We get oscillations with period $T = \frac{2\,\pi}{\omega_0} = 2\,\pi\sqrt{\frac{m}{k}}$.

ω_0 is called the "natural frequency" of the vibration, R is the "amplitude", and δ is the "phase angle".

Damped free vibrations

$F(t) = 0$, so we have

$$m\,u'' + \gamma\,y' + k\,u = 0$$

Roots of the char. poly. are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\,m\,k}}{2\,m} = \frac{\gamma}{2\,m}\left[-1 \pm \sqrt{1 - \frac{4\,m\,k}{\gamma^2}}\right]$$

You can see that the square roots may or may not be real, but their real part is **always** negative.

3 cases:

1. $\gamma^2 - 4\,m\,k > 0$.

   r_1, r_2 are real. Since $\gamma > \sqrt{\gamma^2 - 4\,m\,k}$, roots are both negative.

   The general solution is

   $$u(t) = A\,\mathrm{e}^{r_1 t} + B\,\mathrm{e}^{r_2 t}$$

   As $t \to \infty$, $u(t) \to 0$, no vibrations! This results in *over-damped motion*.

2. $\gamma^2 - 4\,m\,k = 0$.

   $$u(t) = \mathrm{e}^{-\frac{\gamma}{2m}t}\,(A + B\,t)$$

   Still no vibrations! This is called *critical damping*.

3. $\gamma^2 - 4\,m\,k < 0$.

   $$r_{1,2} = \frac{-\gamma}{2\,m} \pm \mathrm{i}\,\frac{\sqrt{4\,k\,m - \gamma^2}}{2\,m} \equiv -\frac{\gamma}{2\,m} \pm \mathrm{i}\,\mu$$

   General solution:

   $$u(t) = \mathrm{e}^{-\frac{\gamma}{2m}t}\,(A\cos(\mu\,t) + B\sin(\mu\,t))$$

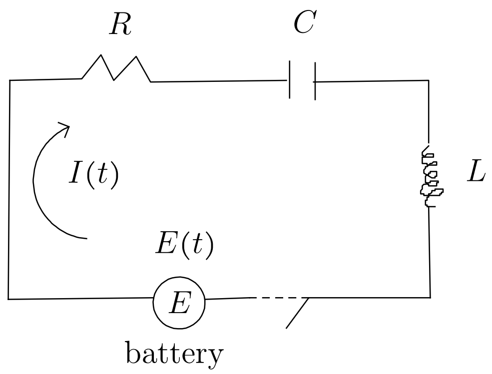
   Here we have vibrations but as $t \to \infty$ they die away ($u(t) \to 0$).

   As before we can rewrite this is

   $$u(t) = R\,\mathrm{e}^{-\frac{\gamma}{2m}t}\cos(\mu\,t - \delta)$$

   This is called *damped vibration*, or *small damping*.

Electrical vibrations



$Q(t)$ is the charge within the circuit and $I(t)$ is the current. $I(t)=\frac{\mathrm{d}Q}{\mathrm{d}t}$.

The voltage drops on each unit:

- On inductor: $L\cdot\frac{\mathrm{d}I}{\mathrm{d}t},\quad L>0$
- On resistor: $R\cdot I,\quad R>0$
- On capacitor: $\frac{Q}{C},\quad C>0$

Kirchhoff's's law says that the sum of voltage that drops on circuit is equal to the applied voltage $E(t)$.

$$L\frac{\mathrm{d}I}{\mathrm{d}t}+R\,I+\frac{1}{C}\,Q=E(t)$$

Substitute $Q=\frac{\mathrm{d}I}{\mathrm{d}t}$

$$L\,Q''+R\,Q'+\frac{1}{C}\,Q=E(t)$$

You may also differentiate the first equation and get

$$L\,I''+R\,I'+\frac{1}{C}\,I=E'(t)$$

Get the "mechanical-electrical analogy"

$$\begin{cases}\gamma\leftrightarrow R\\ m\leftrightarrow L\\ k\leftrightarrow\frac{1}{C}\end{cases}$$

Note that ICs for Q can yield ICs for I, as if we're given

$$\begin{cases}Q(t_0)=Q_0\\ Q'(t_0)=I_0\end{cases}$$

Then we can derive $I'(t_0)$ to form the equation:

$$L\,I' + R\,I + \frac{1}{C}\,Q = E(t)$$

$$I'(t_0) = \frac{E(t_0) - \frac{1}{C}\,Q_0 - R\,I_0}{L}$$

Let's treat the mechanical version again, but with force applied:

Forced mechanical vibrations

$$m\,u'' + \gamma\,u' + k\,u = F(t)$$

$F(t) \neq 0$ and is periodic. For example, $F(t) = F_0 \cos\omega\,t$.

1. No damping. $\gamma = 0$

$$m\,u'' + k\,u = F_0 \cos\omega\,t$$

   Recall that solution to the homogeneous equation was $A \sin\omega_0\,t + B \cos\omega_0\,t$, $\omega_0 = \sqrt{\frac{k}{m}}$.

   I. $\omega_0 \neq \omega$. Guess particular solution of form

$$\begin{aligned} u &= c_1 \cos\omega\,t + c_2 \sin\omega\,t \\ u'' &= -c_1\,\omega^2 \cos\omega\,t - c_2\,\omega^2 \sin\omega\,t \end{aligned}$$

$$-m\,\omega^2\,(c_1 \cos\omega\,t + c_2 \sin\omega\,t) + k\,(c_1 \cos\omega\,t + c_2 \sin\omega\,t) = F_0 \cos\omega\,t$$

   Equate coefficient on both sides.

$$\begin{aligned} 0 &= -m\,c_2\,\omega^2 + k\,c_2 = c_2\,(k - m\,\omega^2) \\ F_0 &= -m\,c_1\,\omega^2 + k\,c_1 \end{aligned}$$

   We assumed $\omega_0 \neq \omega$, so $k \neq m\,\omega^2$ and we get $c_2 = 0$.

$$c_1 = \frac{F_0}{k - m\,\omega^2}$$

General solution in this case:

$$u(t) = \frac{F_0}{k - m\,\omega^2}\cos\omega\,t + A\sin\omega_0\,t + B\cos\omega_0\,t$$

The behavior of solution depends on the relation between ω_0 and ω. $k - m\,\omega^2 = m\,(\omega_0 - \omega^2)$. If very close, coefficient of $\cos\omega\,t$ is very large. In other words, the oscillation amplitude will increase as the force oscillation frequency approaches the natural frequency of the spring.

Taking ICs, $u(0) = 0, u'(0) = 0$, we get

$$\begin{cases} B = -\frac{F_0}{m\,(\omega_0^2 - \omega^2)} \\ A = 0 \end{cases}$$

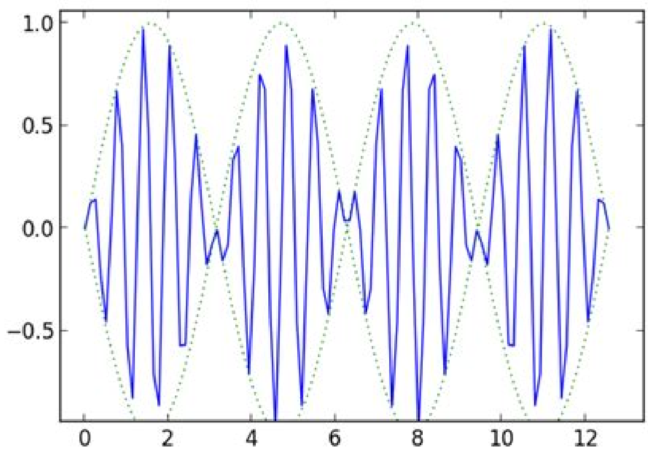
Our solution is then

$$u(t) = \frac{F_0}{m\,(\omega_0^2 - \omega^2)}\,(\cos\omega\,t - \cos\omega_0\,t)$$

Using trig. identities we can rewrite solution as a product of sine functions:

$$u(t) = \frac{2\,F_0}{m\,(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega}{2}\,t\right)\sin\left(\frac{\omega_0 + \omega}{2}\,t\right)$$

We get "beats".



Note: the $\sin\left(\frac{\omega_0 - \omega}{2}\,t\right)$ term is responsible to the outer oscillations and the $\sin\left(\frac{\omega_0 + \omega}{2}\,t\right)$ term is responsible to the inner oscillations.

II. $\omega = \omega_0$. External force is applied at resonance.

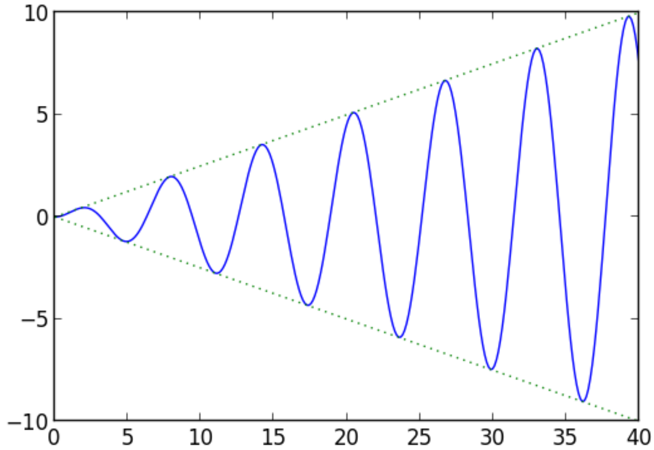
$$m\,u'' + k\,u = F_0 \cos \omega_0\, t$$

Guess a particular solution: $c_1\, t \cos \omega_0\, t + c_2\, t \sin \omega_0\, t$.

General solution:

$$u(t) = A \cos \omega_0\, t + B \sin \omega_0\, t + \frac{F_0}{2\, m\, \omega_0}\, t \sin \omega_0\, t$$

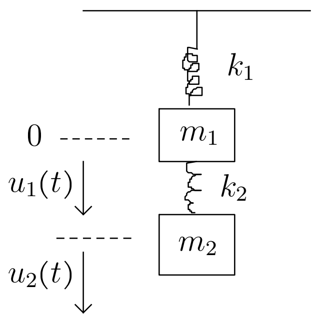
The solution diverges!



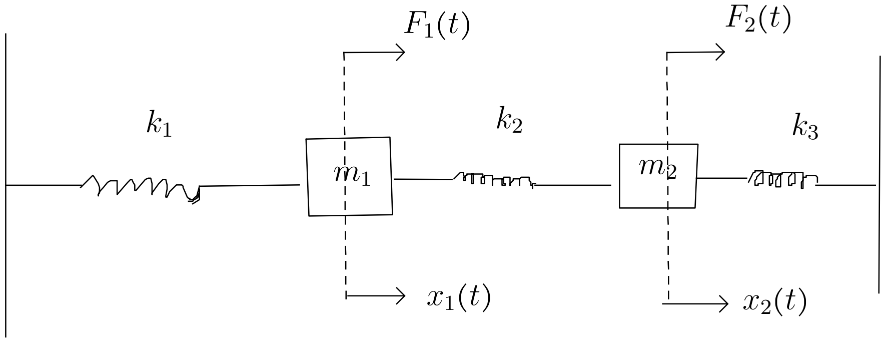
III System of ODES

6 Examples of problems represented by a system of ODEs

1. 2nd order system in u_1, u_2.



2. 2-degree of freedom spring-mass system with dash-pot.



System of motion equations (Assume no damping):

$$\begin{cases} m_1 x_1'' = F_1(t) - k_1 x_1 + k_2 (x_2 - x_1) \\ m_2 x_2'' = F_2(t) - k_2 (x_2 - x_1) - k_3 x_2 \end{cases}$$

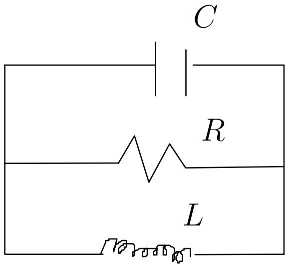
or:

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2) x_1 + k_2 x_2 + F_1(t) \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3) x_2 + F_2(t) \end{aligned}$$

This is a linear 2nd order system of ODEs. In matrix form:

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{F_1(t)}{m_1} \\ \frac{F_2(t)}{m_2} \end{pmatrix}$$

3. Parallel electrical circuit.



$$\begin{cases} V(t) = \text{ voltage at time } t \\ I(t) = \text{ current at time } t \end{cases}$$

Get:

$$\begin{cases} \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \frac{V(t)}{L} \\ \frac{\mathrm{d}V(t)}{\mathrm{d}t} = -\frac{I(t)}{C} - \frac{V(t)}{RC} \end{cases}$$

This is a linear first order system.

$$\begin{bmatrix} I(t) \\ V(t) \end{bmatrix}' = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{\mathrm{RC}} \end{bmatrix} \begin{bmatrix} I(t) \\ V(t) \end{bmatrix}$$

4. Predator-prey situation, model of Lotka & Volterra (1925).

   - population of food fish $= x(t)$
   - population of sharks $= y(t)$

   Principles:

   a. If no predators, prey pop. increases at a constant rate $a > 0$.

   $$\frac{\mathrm{d}x}{\mathrm{d}t} = a\,x$$

   b. If no prey, predator pop. decreases at constant rate $c > 0$.

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = -c\,y$$

   c. If both prey and predators are present, number of encounters is proportionate to $x\,y$.

   Result is net decline of $b\,x\,y$ in fish and increase of $r\,x\,y$ in sharks. $(r, b > 0)$

   $$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a\,x - b\,x\,y = x(a - b\,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -c\,x + r\,x\,y = y(-c + r\,x) \end{cases}$$

   This is a first order non-linear system.

   Inspect equilibrium of populations: $x = 0 = y$ **or** $y = \frac{a}{b}$ & $x = \frac{c}{r}$.

   Note: The system:

   $$\begin{cases} y_1' = y_1 \\ y_2' = y_2 \end{cases}$$

   Clearly cannot express y_2 in terms of y_1 and reduce to one equation.

In general:

A first order system of ODEs is of the form:

$$\begin{aligned} y_1'(x) &= f_1(x, y, \ldots, y_n) \\ y_2'(x) &= f_2(x, y, \ldots, y_n) \\ &\vdots \\ y_n'(x) &= f_n(x, y, \ldots, y_n) \end{aligned}$$

We call it *linear* if $f_0, \ldots, f_n$ are linear in $y_1, \ldots, y_n$.

So $f_i(x, y_1, \ldots, y_n) = p_{i1}(x)\, y_1 + p_{i_2}(x)\, y_2 + \cdots + p_{i_n}(x)\, y_n + g_i(x)$ where $p_{i_1}, \ldots, p_{i_n}$ are functions of x.

Linear systems can be written in matrix form:

$$\begin{bmatrix} y_1'(x) \\ \vdots \\ y_n'(x) \end{bmatrix} = \begin{bmatrix} p_{1_1}(x) & \ldots & p_{1_n}(x) \\ \vdots & \ddots & \\ p_{n_1}(x) & & p_{n_n}(x) \end{bmatrix} \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix} + \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

Or

$$\vec{Y}' = A\vec{Y} + \vec{b}$$

Example

$$\begin{cases} y_1' = x^2\, y_1 + e^x\, y_2 + \sin x\, y_3 + x^2 \\ y_2' = x\, y_2 + y_3 + e^{2x} \\ y_3' = \sqrt{x}\, y_1 + 2\, y_3 + \ln x \end{cases}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} x^2 & e^x & \sin x \\ 0 & x & 1 \\ \sqrt{x} & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} x^2 \\ e^{2x} \\ \ln x \end{bmatrix}$$

A solution is a vector of functions $[\, y_1 \;\ldots\; y_n \,]^T$ satisfying the system.

ICs: $y_1(a) = b_1, y_2(a) = b_2, \ldots, y_n(a) = b_n$.

7 Existence & Uniqueness theorem for first order systems

Given

$$\begin{aligned} y_1'(x) &= f_1(x, y, \ldots, y_n) \\ y_2'(x) &= f_2(x, y, \ldots, y_n) \\ &\vdots \\ y_n'(x) &= f_n(x, y, \ldots, y_n) \end{aligned}$$

and ICs

$$y_1(a) = b_1, y_2(a) = b_2, \ldots, y_n(a) = b_n$$

such that $f_1,\ldots, f_n$ are continuous in $x, y_1,\ldots, y_n$ and $\partial f_i / \partial y_j$ exist and are continuous in some neighborhood of $(a, b_1, \ldots, b_n)$ in $\mathbb{R}^{n+1}$, there exists a unique solution satisfying ICs defined for some interval around a. This holds for linear systems if and only if $g_i, p_{i,j}$ are continuous.

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Theorem

The cardinality (number of elements in a group) of a basis is determined uniquely. This cardinality is the dimension of the space.

Theorem

If the dimension of a space w is k, then any linearly independent set of k elements will be a basis. Also, any spanning set with k elements will be a basis.

Note.

- The dimension of the vector space of all real functions in infinite. $(2^{\aleph_0})$
- The dimension of vector space of polynomial functions is $\aleph_0$ are $1, x, x^2, \ldots$ is a basis

8 Reducing a higher order ODE to a system of first order ODEs

Any higher order linear ODE can be reduced to a first order linear system.

Example: $y'' + y = 0$.

We can define a 2×2 system which is equivalent:

$$\begin{cases} y_1 = y \\ y_2 = y' \end{cases}$$

Then:

$$\begin{cases} y_1' = y_2 \\ y_2' = y'' = -y_1 \end{cases}$$

Write in matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can see that $y_1 = \sin x, y_2 = \cos x$ solve this system.

Another example: $y^{(3)} - 2\,y'' + 3\,y' + 4\,y = \sin x$

This is equivalent to the following 3×3 system:

$$\begin{cases} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{cases}$$

Get:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -4\,y_1 + -3\,y_2 + 2\,y_3 + \sin x \end{cases}$$

In matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin x \end{bmatrix}$$

Another example

$$y''' + 7\,x\,y'' - \sin x\,y' + 4\,y = 0$$

We define

$$\begin{cases} y_1 = y \\ y_2 = y' \\ y_3 = y'' \end{cases}$$

and obtain the following system:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y''' = -4\,y_1 + \sin x\,y_2 - 7\,x\,y_3 \end{cases}$$

If we solve the system then $y = y_1$ will solve the ODE.

9 System of algebraic equations

A system of equations of the form:

$$\begin{cases} a_{11}\,x_1 + \cdots + a_{1n}\,x_n = b_1 \\ \vdots \\ a_{m1}\,x_1 + \cdots + a_{m,n}\,x_n = b_n \end{cases}$$

or in matrix form:

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

or

$$A\,\vec{x} = \vec{b}$$

To solve, use Gaussian elimination on the *extended matrix* $(A|\vec{b})$ to get an equivalent system (with the same set of solutions) in (upper) echelon form. Then, solve by back substitution.

A matrix is in *row echelon form* if all rows consisting of only zeros are at the bottom, and a leading non-zero coefficient ("pivot") of a row is always strictly to the right of any leading coefficient of the row above. For example:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian elimination uses elementary row operations. Three types of operations:

1. Permute rows. $(R_i \leftrightarrow R_j)$
2. Replace row i by a non-zero multiple of itself. $(\alpha R_i \rightarrow R_i, \quad \alpha \neq 0)$
3. Add a multiple of row i to row j. $(\alpha R_i + R_j \rightarrow R_j$, any $\alpha)$

Note. r is the rank of a matrix, and is equal to the number of non-zero rows in a matrix *after* bringing it to upper echelon form.

Definition

For an $m \times n$ matrix A, we define fundamental subspaces:

1. Set of solutions to $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$ (= nullspace of A).

   Set of solutions to $A\vec{x} = \vec{b}$ $(\vec{b} \neq \vec{0})$ is not a subspace
2. Span of rows of A = row-space of A is a subspace of $\mathbb{R}^n$
3. Span of columns of A = column-space of A, is a subspace of $\mathbb{R}^m$.

Theorems

- Rank of $A = r$ is the dimension of row-space and dimension of the column-space.
- Dimension of the nullspace is $n - r$.

Example

$$\begin{bmatrix} 0 & 1 & 2 & -1 & 0 & 4 \\ 0 & 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

Dependent variables are the variables which correspond to the **pivot** columns, in our case x_2, x_4, x_5.

Independent variables are all the rest, in our case x_1, x_3, x_6.

We want to express the solution in terms of the independent variables.

$$\begin{aligned}
-x_5+4\,x_6 &= -3 \rightarrow x_5=4\,x_6+3\\
3\,x_4+x_5+2\,x_6 &= 1 \rightarrow x_4=\frac{1}{3}(1-x_5-2\,x_6)=\frac{1}{3}(1-(4\,x_6+3)-2\,x_6)=-2\,x_6-\frac{2}{3}\\
x_2+2\,x_3-x_4+4\,x_6 &= 0 \rightarrow x_2=-2\,x_3+x_4-4\,x_6=\cdots=-2\,x_3-6\,x_6-\frac{2}{3}
\end{aligned}$$

General solution:

$$\begin{bmatrix} x_1 \\ -2\,x_3-\frac{2}{3}-6\,x_6 \\ x_3 \\ -2\,x_6-\frac{2}{3} \\ 4\,x_6+3 \\ x_6 \end{bmatrix} = x_1 \begin{bmatrix} 1\\0\\0\\0\\0\\0 \end{bmatrix} + x_3 \begin{bmatrix} 0\\-2\\1\\0\\0\\0 \end{bmatrix} + x_6 \begin{bmatrix} 0\\-6\\0\\-2\\4\\1 \end{bmatrix} + \begin{bmatrix} 0\\-\frac{2}{3}\\0\\-\frac{2}{3}\\3\\0 \end{bmatrix}, \quad x_1, x_3, x_6 \in \mathbb{R}$$

Note that the last vector solves the inhomogeneous equation, and the first three vectors solve the associated homogeneous system. Also, they are linearly independent and therefore span the basis of the nullspace.

Another approach: First solving the associated homogeneous system:

$$\begin{bmatrix} 0 & 1 & 2 & -1 & 0 & 4 \\ 0 & 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
-x_5+4\,x_6 &= 0 \rightarrow x_5=4\,x_6\\
3\,x_4+x_5+2\,x_6 &= 0 \rightarrow x_4=-2\,x_y\\
x_2+2\,x_3-x_4+4\,x_6 &= 0 \rightarrow x_2=-2\,x_3-6\,x_3
\end{aligned}$$

The set of solutions (nullspace of the matrix) is a subspace of $\mathbb{R}^6$.

The number of free variables is the dimension: #columns – rank(coeff. matrix) $=3$

We can construct a basis for this by taking in turn on free variable $=1$ and the rest 0.

1. $x_1=1, x_3=x_6=0.$ $\begin{bmatrix} 1\\0\\0\\0\\0\\0 \end{bmatrix}$

2. $x_3=1, x_1=x_6=0.$ $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3. $x_6=1, x_1=x_3=0.$ $\begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}$

These vectors are linearly independent and span the space of solutions.

General solution (of the associated homogeneous system) is:

$$x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \cdot \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix}$$

To solve the inhomogeneous system, we just need to find a particular solution. We had the general solution:

$$\begin{bmatrix} x_1 \\ -2\,x_3 - \frac{2}{3} - 6\,x_6 \\ x_3 \\ -2\,x_6 - \frac{2}{3} \\ 4\,x_6 + 3 \\ x_6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -6 \\ 0 \\ -2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 0 \\ -\frac{2}{3} \\ 3 \\ 0 \end{bmatrix}, \qquad x_1, x_3, x_6 \in \mathbb{R}$$

It is clean then that $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 0 \\ -\frac{2}{3} \\ 3 \\ 0 \end{bmatrix}$ is a particular solution.

In general, given A $m \times n$ matrix such that $A\vec{x} = \vec{b}$ and a particular solution $v \in \mathbb{R}^n$, the set of all solutions is the set $\{v + v_0 |\ v_0$ is in nullspace of $A\}$.

Theorem

For an $m \times n$ matrix A, $n = \dim N(A) + \dim$ (row space) such that $n - r = \dim$ (nullspace). [dim of row space is equal to the dim of column space of A.]

2 more examples:

1.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ -1 & 1 & -2 & 1 \\ 2 & -1 & 3 & -4 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Can solve. Dimension of nullspace is 1 ($n=3, r=2$).

2.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ -1 & 1 & -2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$r > n$, No solution!

Note: There exist solutions to $A\vec{x} = \vec{b}$ if and only if $\operatorname{rank}(A) = \operatorname{rank}(A|B)$.

9.1 Algorithm to invert (square) matrices: Gauss-Seidel algorithm

Works for *square matrices* ($n \times n$).

Definition

An elementary matrix is obtained from I (identity matrix) by performing one elementary operation on its rows. e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \xleftarrow{-2R_1 + R_3 \to R_3} I$$

Corollary

Multiplying a matrix A on the left by an elementary matrix E performs the same operation defining E on rows of A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{11} + a_{31} & -2a_{12} + a_{32} & -2a_{13} + a_{33} \end{bmatrix}$$

Idea: If we perform elementary operation on rows of A until we reach I, it is as though we multiplied A by a sequence of elementary matrices.

$$\underbrace{E_n \cdots E_2 E_1}_{A^{-1}} A = I$$

Algorithm

Algorithm

Take $(A|I)$ $n \times 2\,n$ matrix and perform elementary operations until $(I|A^{-1})$ is obtained.

If the algorithm "gets stuck" (don't have enough pivots) then $\operatorname{rank}(A) < n$ and A is not invertible.

Note. The method works iff $\operatorname{rank} A = n \Longleftrightarrow A$ is invertible.

Example

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & -1 & 1 & 0 \\ 0 & 4 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_2\to R_2} \cdots$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 4 & -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow{-4R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{4}{3} & 1 \end{array}\right]$$

Notice that now we know that $r = 3$. Eventually, after some operations, we get:

$$A^{-1} = \begin{bmatrix} -\frac{7}{2} & -3 & -\frac{5}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{5}{2} & 2 & -\frac{3}{2} \end{bmatrix}$$

9.2 Review of determinants

Assume throughout this subsection that all matrices are square.

Note: If A is an $n \times n$ real matrix, $\det A$ is a real number.

We define def $A = |A|$ recursively:

if $n = 1$: $A = [\,a\,]$ then $\det A = a$.

For an $n \times n$ matrix $A = [\,a_{ij}\,]$, denote by A_{ij} = det of matrix obtained from A by omitting row i and column j (ij^{th} minor of A).

Define:

Definition

$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij}$ for any fixed i ("along row i").

<u>if $n=2$</u>:

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Along row 1:

$$\det A=\sum_{j=1}^{2} a_{1j}\,A_{1j}\cdot(-1)^{1+j}=a_{ij}\,A_{11}-a_{12}\,A_{12}=a_{11}\,a_{22}-a_{12}\,a_{21}$$

Along row 2:

$$\det A=\sum_{j=1}^{2} a_{2j}\,A_{2j}\cdot(-1)^{2+j}=-a_{21}\,A_{21}+a_{22}\,A_{22}=-a_{21}\,a_{12}+a_{22}\,a_{11}$$

For 2×2 matrices you can also cross-multiply diagonals and subtract main from sub-diagonal.

<u>if $n=3$</u>:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}\det A &= \sum_{j=1}^{3} (-1)^{1+j}\,a_{1j}\,A_{1j}=a_{11}\,A_{11}-a_{12}\,A_{12}+a_{13}\,A_{13} \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}\,(a_{22}\,a_{33}-a_{23}\,a_{32})-a_{12}\,(a_{21}\,a_{33}-a_{23}\,a_{31})+a_{13}\,(a_{21}\,a_{32}-a_{22}\,a_{31}) \\ &= (a_{11}\,a_{22}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{32})-(a_{11}\,a_{23}\,a_{32}+a_{12}\,a_{21}\,a_{33}+a_{13}\,a_{22}\,a_{31})\end{aligned}$$

Theorem

We can express a det also using columns: $\det A=\sum_{i=1}^{n}(-1)^{i+j}\,a_{ij}\,A_{ij}$ for a fixed j ("down column j").

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<u>if $n=4$</u>:

Along row i,

$$\det A=\sum_{j=1}^{4}(-1)^{i+j}\,a_{ij}\,A_{ij} = (-1)^{i+1}a_{i1}A_{i1}+(-1)^{i+2}a_{i2}A_{i2}+(-1)^{i+3}a_{i3}A_{i3}+\cdots+ +(-1)^{i+4}\,a_{i4}\,A_{i4}$$

In fact: $\det A$ = sum of all choices of products of elements chosen, one from each row and one from each column, multiplied by a power of (-1). In total, $n!$ summands.

9.3 Cramer's rule

A strategy to find solutions of a system based on determinant:

In a 2×2 case, where

$$\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{cases}$$

Operate $R_2 - R_1 \rightarrow R_2$

$$\begin{cases} a_{21} a_{11} x_1 + a_{21} a_{12} x_2 = a_{21} b_1 \\ a_{11} a_{21} x_1 + a_{11} a_{22} x_2 = a_{11} b_2 \end{cases}$$

Subtract

$$(a_{11} a_{22} - a_{21} a_{12})\, x_2 = a_{11} b_2 - a_{21} b_1$$

If $\det A \neq 0$ we can solve uniquely:

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A}$$

It turns out that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\det A}$$

This is called Cramer's rule for $n \times n$ systems. This method is practical only for small matrices, as it makes us calculate $n+1$ determinants to find the solutions.

9.4 Properties of determinants

1. If A has a row/column of zeros then $\det A = 0$.
2. $\det I = 1$. (I is the identity matrix)
3. $\det (c \cdot A) = c^n \det A$, where $c \in \mathbb{C}$.
4. $\det A = \det A^T$.
5. $\det (A \cdot B) = \det A \cdot \det B$.
6. $\det (A + B) \neq \det A + \det B$ (proof by splitting I to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$).

Note that from 2. and 5. $\det A^{-1} = \frac{1}{\det A}$.

How do elementary row/column operations affect $\det A$?

1. If B is obtained from A by one row/column switch, then $\det B = -\det A$.

2. If B is obtained from A by multiplying a row/column by a scalar c, then $\det B = c \cdot \det A$.

3. If B is obtained from A by $\alpha R_i + R_j \to R_j$, then $\det B = \det A$.

4. If A is upper-tridiagonal $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$ (elements bellow the main diagonal are only zeros), then $\det A = \prod_{i=1}^{n} a_{ii}$.

Conclude:

If $\det A \neq 0$ and we use Gaussian elimination on the rows of A to get it in upper echelon form, U, then $\det U \neq 0$. So U will have no rows of zeros, so $\operatorname{rank} U = \operatorname{rank} A = n$.

That means that all rows/columns of A are linearly independent!

Note also that if we had $\det A = 0$, then $\det U = 0$ and $\operatorname{rank} U = \operatorname{rank} A < n$.

Theorem

Rows/columns of A are linearly independent iff $\det A \neq 0$, iff $\operatorname{rank} A = n$ iff A is invertible (non-singular).

If $\det A \neq 0$ we can write:

$$ij^{\text{th}}\,\text{element of}\; A^{-1} = \frac{1}{\det A} \underbrace{((-1)^{i+j} A_{ji})}_{\text{adjoint of } A}$$

$(-1)^{i+j} A_{ji}$ represents the ij^{th} element of $\operatorname{Adj} A$.

For $n = 2$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11} a_{22} - a_{21} a_{12}} \begin{bmatrix} A_{11} & -A_{21} \\ -A_{12} & A_{22} \end{bmatrix} = \frac{1}{a_{11} a_{22} - a_{21} a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

This is used in Cramer's rule: suppose you have a system of equations $A\vec{x} = \vec{b}$, then if $\det A \neq 0$ we have a unique solution $\vec{x} = A^{-1}\vec{b}$. Each element of $\vec{x}$ is obtained by:

$$x_i = \frac{\det\, A \text{ where column } i \text{ is replaced by } \vec{b}}{\det A}$$

Different ways of calculating determinant of a 4×4 matrix A:

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 \end{bmatrix}$$

Calculate down column 2.

$$\det A = \sum_{i=1}^{4} (-1)^{i+2} a_{i2} A_{i2} \;=\; (-1)^{1+2} a_{12} A_{12} + (-1)^{2+2} a_{22} A_{22} + (-1)^{3+2} a_{32} A_{32} + (-1)^{4+2} a_{42} A_{42}$$

Notice that $a_{12}, a_{42} = 0$

$$\begin{aligned} \det A &= 3 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \\ &= 3\,((-2-3+2)-(12+1+1)) + 1 \cdot (1-1+8-2+2-2) = -45 \end{aligned}$$

Another approach: use elementary operations of type 3 to eliminate below first pivot (this does not change the determinant)

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 \end{bmatrix} \to \cdots \to \det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & 4 & -7 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & 4 & -7 \end{bmatrix} =$$

$$= 3 \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & -6 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -5 \end{bmatrix} = 3 \cdot 1 \cdot 1 \cdot 3 \cdot (-5) = -45$$

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9.5 Review of eigenvalues and eigenvectors

Assume A is square $n \times n$. The map that sends a vector in $\mathbb{R}^n$ $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ to $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a *linear operator*.

Special case

$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Clearly $A \cdot \vec{0} = \vec{0}$. We're interested in a non-zero $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$ for some scalar λ.

Example:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Definition

If $\vec{x}$ is a non-zero vector such that $A\,\vec{x} = \lambda\,\vec{x}$ for some scalar λ, we say it is an eigenvector for A and λ is its associated eigenvalue.

To find an eigenvector we need to solve:

$$A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

This is a non-linear system in $n+1$ unknowns: $x_1, \ldots, x_n, \lambda$.

Suppose $\vec{x}$ solves $A\,\vec{x} = \lambda\,\vec{x}$, then: $(A - \lambda \cdot I)\,\vec{x} = \vec{0}$. So $\vec{x}$ is a solution to the homogeneous linear system. In other words, $\vec{x}$ is a non-trivial element in the nullspace of $A - \lambda\,I$.

nullspace of $A - \lambda\,I$ is called the *eigenspace*.

Recall that $\dim \mathrm{nullspace} = n - r$, so $n - 1 \geq 1$. In other words, we have non-trivial solutions to $(A - \lambda\,I)\,\vec{x} = \vec{0}$ iff $n > r$ iff $\det(A - \lambda\,I) = 0$.

Conclude: λ is an eigenvalue iff $|A - \lambda\,I| = 0$.

Example: Find eigenvectors and eigenvalues of matrix A:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

1. Solve $|A - \lambda\,I| = 0$.

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\,(4-\lambda) - 2 = 0$$

$$\lambda^2 - 7\,\lambda + 10 = 0$$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-40}}{2} = 5, 2$$

2. Solve $(A - 5\,I)\,\vec{x} = \vec{0}$ and $(A - 2\,I)\,\vec{x} = \vec{0}$.

   a. $\lambda = 5$:

$$\begin{bmatrix} 3-5 & 2 \\ 1 & 4-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note this is a matrix of rank 1, as expected. Solve $x-y=0$ and get set of solutions $\left[\begin{smallmatrix} x \\ -x \end{smallmatrix}\right]$ for any $x\neq 0$ will be an eigenvector. e.g. $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

b. $\lambda=2$:

$$\begin{bmatrix} 3-2 & 2 \\ 1 & 4-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

choose an eigenvector $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$.

<u>More examples:</u>

1.

$$A=\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$$

Solve $|A-\lambda I|=0$

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & -6-\lambda \end{vmatrix}=-(1-\lambda)(6+\lambda)+6=\lambda^2+5\lambda=0$$

The eigenvalues are $\lambda=0,-5$. We got $\lambda=0$ because rank $A<2$. $\lambda=0$ is an eigenvalue iff $\det A=0$.

<u>$\lambda=0$</u>: Find eigenvectors:

$$\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x+2y=0$. Get $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$ basis for eigenspace.

<u>$\lambda=-5$</u>:

$$\begin{bmatrix} 6 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x+y=0$. Get $\left[\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right]$ basis for eigenspace.

2. What if every non-zero vector is an eigenvector?

$$A=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

Solve $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0 \Longleftrightarrow \lambda = 3$$

Note that setting $\lambda=3$ means when we solve: $[A-\lambda I]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ we get $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and every vector is a solution. Here eigenspace has dim 2.

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3.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\det(A-\lambda I) = \lambda^2+1$. There are no real eigenvalues. Over $\mathbb{C}$ there are $\lambda = \pm \mathrm{i}$ and we can find the eigenvectors over $\mathbb{C}^2$.

<u>Set $\lambda = \mathrm{i}$</u>

Find nullspace of $\left[\begin{smallmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. $-\mathrm{i}\,x - y = 0 \to \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ -\mathrm{i} \end{smallmatrix}\right]$.

<u>Set $\lambda = -\mathrm{i}$</u>

Find nullspace of $\left[\begin{smallmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. $\mathrm{i}\,x - y = 0 \to \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ \mathrm{i} \end{smallmatrix}\right]$.

Notice that the eigenvectors than correspond to conjugate eigenvalues are conjugates of each other.

4.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$\det(A-\lambda I) = (2-\lambda)^2$. $\lambda = 2$. Find nullspace of $A-\lambda I$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ is a basis for the eigenspace.

Theorem

If A is $n \times n$ than $\det(A-\lambda I)$ is a polynomial of degree n.

Theorem

If r_0 is the multiplicity of an eigenvalue λ_0 in the characteristic polynomial of A, then the dimension of the corresponding eigenspace is less or equal to r_0.

Also, dimension of eigenspace is called *geometric multiplicity*, and multiplicity of eigenvalue in characteristic polynomial is called the *algebraic multiplicity*. So $1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity.

Example: Suppose $(\lambda-3)\,(\lambda-2)^5\,(\lambda-1)^2$ is the characteristic polynomial for a matrix.

- $\lambda=3$: eigenspace has dim 1
- $\lambda=2$: eigenspace has $1\leq \dim \leq 5$.
- $\lambda=1$: eigenspace has dim 1 or 2.

5.

$$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Calculate character polynomial

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

Clearly setting $\lambda=-1$ gives a matrix of ones, with rank 1, so $\lambda=-1$ will be an eigenvalue. (Then $\det(A-\lambda I)=0$.)

$$-\lambda^3+1+1+\lambda+\lambda+\lambda=-\lambda^3+3\lambda+2$$

Factor our $\lambda+1$:

$$-\lambda^3+3\lambda+2=(\lambda+1)\,(-\lambda^2+\lambda+2)=0$$

$$\lambda_{1,2,3}=2,-1,-1$$

Calculate eigenvectors. For $\lambda=2$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenspace is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The nullspace is of dimension 2. Basis for eigenspace can be $\left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right]$.

Given the three eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we form a matrix $P = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$. Note that rank $P = 3$, so P is invertible. Calculate $A\,P$.

$$A\,P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = [\ A\,\vec{v}_1 \quad A\,\vec{v}_2 \quad A\,\vec{v}_3\ ] = [\ 2\,\vec{v}_1 \quad -\vec{v}_2 \quad -\vec{v}_3\ ]$$

Now multiply P by a diagonal matrix whose diagonal contains the eigenvalues of A (in same order that the eigenvectors are aligned in P):

$$P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = A\,P$$

Conclusion:

$$P^{-1}\,A\,P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

9.6 Diagonalizability, eigendecomposition and power of matrices

Theorem

If $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors and we have $\lambda_1, \ldots, \lambda_n$ corresponding eigenvalues (not necessarily distinct), then if P is the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_n$, then P is invertible and we have:

$$A\,P = P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$P^{-1}\,A\,P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Definition

An $n \times n$ matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1} A P$ is the diagonal matrix.

Theorem

If A has n linearly independent eigenvectors, then it is diagonalizable.

Theorem

If A is diagonalizable, then it has n linearly independent eigenvectors.

Note that in the situation that we do have $P^{-1} A P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ then column i of P is an eigenvector for λ_i.

Theorem

If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ then $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Corollary *(Sufficient condition [not necessary] for diagonalizability)*

If A has n distinct eigenvalues then A is diagonalizable.

Show linear independence of eigenvectors corresponding to distinct eigenvalues for $k = 3$: Suppose

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}, \quad c_i \in \mathbb{R}$$

We need to show $c_1 = c_2 = c_3 = 0$. So:

$$\begin{aligned} A\,(c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3) &= A\,\vec{0} = \vec{0} \\ &= c_1 A \vec{v}_1 + c_2 A \vec{v}_2 + c_3 A \vec{v}_3 \\ &= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3 \end{aligned}$$

Multiplying again by A gives

$$c_1 \lambda_1^2 \vec{v}_1 + c_2 \lambda_2^2 \vec{v}_2 + c_3 \lambda_3^2 \vec{v}_3 = \vec{0}_f.$$

The matrix of coefficients here is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vandermonde matrix has a non-zero det!

Note.

1. There can be less than n distinct eigenvalues, but still has n linearly independent eigenvectors and be diagonalizable.

2. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues, so it **is not** diagonalizable over $\mathbb{R}$, but **is** diagonalizable over $\mathbb{C}$.

3. Calculations can be much easier if A is diagonalizable. e.g. finding A^k for high values of k. For a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$ it is easy to show (by induction) that

$$D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_k^k \end{bmatrix}$$

4. If $P^{-1} A P = D$ then $A = P D P^{-1}$.

$$A^2 = (P D P^{-1})(P D P^{-1}) = P D^2 P^{-1}$$

$$\boxed{A^k = P D^k P^{-1}}$$

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Method *Calculate the product $A^k \vec{v}_0$ without inversion.*

Given a basis $\vec{v}_1, \ldots, \vec{v}_n$ of eigenvectors for A (corresponding to $\lambda_1, \ldots, \lambda_n$ eigenvalues), as $\vec{v}_0 \in \mathbb{R}^n$ we can find $c_1, \ldots, c_n$ such that $\vec{v}_0 = \sum c_i \vec{v}_i$ and get

$$A^k \vec{v}_0 = A^k \left(\sum_{i=1}^n c_i \vec{v}_i \right) = \sum c_i (A^k \vec{v}_i) = \sum c_i \lambda_i^k \vec{v}_i$$

This method is good for a fixed vector $\vec{v}_0$, when A is hard to diagonalize.

Theorem

If A is a complex matrix and it is Hermitian (i.e. $\bar{A}^T = A$) then:

1. All its eigenvalues are real.
2. It has n linearly independent eigenvectors (so it is diagonalizable).
3. Eigenvectors corresponding to distinct eigenvalues are *orthogonal* (i.e. the dot product is zero.)

Note. If A is real symmetric then it is Hermitian.

Definition

In $\mathbb{C}^n$, for $\vec{v} = [x_1, \ldots, x_n]^T, \vec{w} = [y_1, \ldots, y_n]^T$ we define the inner product of $\vec{v}$ and $\vec{w}$ as:

$$(\vec{v}, \vec{w}) = \sum_{j=1}^{n} x_j \, \bar{y}_j$$

Note. If $\vec{v}, \vec{w}$ are real this is simply their scalar product. $\vec{v}, \vec{w}$ are orthogonal if $(\vec{v}, \vec{w}) = 0$. In fact, orthogonality implies linear independence!

Note. A real matrix is symmetric iff it is Hermitian.

10 Systems of First-order linear ODEs

Suppose we have the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = A(t) \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

In the special case that $A(t)$ is a matrix of constants and $\vec{g}$(t) is a vector of constants, then

$$\vec{x}'(t) = A \, \vec{x}(t) + \vec{g}$$

Theorem

Given a non-homogeneous system

$$\vec{x}' = A(t)\,\vec{x} + \vec{g}(t)$$

and two solutions $\vec{u}_1(t), \vec{u}_2(t)$, then $\vec{u}_1 - \vec{u}_2(t)$ solves the associated homogeneous system.

Moreover, every solution to the non-homogeneous system is of the form $\vec{u}_1(t) + \vec{u}_0(t)$, where $\vec{u}_1$ is a fixed particular solution and $\vec{u}_0$ solves the homogeneous system.

10.1 Homogeneous system

Suppose we have a homogeneous system:

$$\vec{x}'(t) = A(t)\,\vec{x}(t)$$

$A(t) = [a_{ij}(t)]$, $\vec{x} = [x_1(t), \ldots, x_n(t)]^T$.

Theorem

The set of solutions is a vector space of dimension n

Note. This means that we only need to search for n solutions.

Proof. It's easy to show it is a vector space, as linear combination of solutions is also a solution. Additionally, $\vec{0}_f$ is a solution.

Regarding dimensionality, since we have a solution for every IC by the E&U theorem, we have n solutions satisfying different ICs:

$$\begin{cases} x_i(t_0) = 1 & \\ x_j(t_0) = 0 & i \neq j \end{cases} \qquad \forall i \leq j \leq n$$

These will be linearly independent and span the solution space.

As if $\vec{x}(t)$ is a solution we write $\vec{x}(0) = [b_1, \ldots, b_n]^T$ and individual solutions satisfying ICs are $\vec{y}_1, \ldots, \vec{y}_n$, then $b_1\,\vec{y}_1 + \cdots + b_n\,\vec{y}_n$ is a solution satisfying the same ICs as $x(t)$, so by uniqueness we must have $\vec{x}(t) = \sum_{i=1}^{n} b_i\,\vec{y}_i(t)$. □

Assume throughout this subsection that A is an $n \times n$ matrix of constants.

Example: $n = 1$

$$x'(t) = a\,x(t), \quad a \in \mathbb{R}$$

We saw that set of solutions is a 1-dim space spanned by e^{at}.

For $n>1$ we shall have an analogous situation. We look for solutions of the form:

$$x_i(t)=c_i\,\mathrm{e}^{\lambda t}$$

or in vector form:

$$\vec{x}=\mathrm{e}^{\lambda t}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\equiv \mathrm{e}^{\lambda t}\,\vec{v}$$

We substitute $\vec{x}=\mathrm{e}^{\lambda t}\,\vec{v}$ in our system of equations:

$$\vec{x}'=\lambda\,\mathrm{e}^{\lambda t}\,\vec{v}$$

So get

$$\lambda\,\mathrm{e}^{\lambda t}\,\vec{v}=\vec{x}'=A\,\vec{x}=A\,\mathrm{e}^{\lambda t}\,\vec{v},\quad \forall t$$

$$\Updownarrow \mathrm{e}^{\lambda t}\neq 0\quad \forall t$$

$$\lambda\,\vec{v}=A\,\vec{v}$$

Conclude: if λ eigenvalue for A with eigenvector $\vec{v}$ then $\mathrm{e}^{\lambda t}\,\vec{v}$ solves the system.

Example

$$\begin{cases} x_1'=x_1+x_2 \\ x_2'=4\,x_1+x_2 \end{cases},\qquad A=\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Find eigenvalues and eigenvectors for A:

$$|\,A-\lambda\,I\,|=\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}=\lambda^2-2\,\lambda-3=0$$

$$\lambda_{1,2}=3,-1$$

For $\lambda=3$: Solve

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

get eigenvector $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. For $\lambda=-1$:

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

get eigenvector $\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]$. We have 2 solutions to the system of equations:

$$e^{3t}\begin{bmatrix}1\\2\end{bmatrix},\quad e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$$

These are linearly independent as if: $(\forall t)$

$$a\,e^{3t}\begin{bmatrix}1\\2\end{bmatrix}+b\,e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

then (set $t=0$):

$$a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

but $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]$ are linearly independent so $a=b=0$.

Given ICs, say: $x_1(0)=1, x_2(0)=3$. Want a, b that satisfy ICs.

$$\begin{bmatrix}1\\3\end{bmatrix}=a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}1\\-2\end{bmatrix}$$

$$b=1-a\rightarrow 2\,a-2(1-a)=3\rightarrow a=\frac{5}{4}, b=-\frac{1}{4}$$

Theorem

If $\lambda_1,\ldots,\lambda_k$ are distinct eigenvalues with corresponding eigenvectors $\vec{v}_1,\ldots,\vec{v}_k$, then the vectors of functions $e^{\lambda_1 t}\,\vec{v}_1, e^{\lambda_2 t}\,\vec{v}_2,\ldots,e^{\lambda_k t}\,\vec{v}_k$ are linearly independent. (Particularly in space of vectors of functions.)

Theorem

If A has n linearly independent eigenvectors $\vec{v}_1,\ldots,\vec{v}_n$ with corresponding eigenvalues $\lambda_1,\ldots,\lambda_n$ (not necessarily distinct) then again the n vectors of functions: $e^{\lambda_1 t}\,\vec{v}_1, e^{\lambda_2 t}\,\vec{v}_2,\ldots,e^{\lambda_k t}\,\vec{v}_n$ will be linearly independent. *(Useful if A is symmetric.)*

Example

$$\begin{cases} x_1'=x_2+x_3\\ x_2'=x_1+x_3\\ x_3'=x_1+x_2\end{cases}$$

Matrix of coefficients is $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Recall: there are 2 linearly independent eigenvectors for $\lambda=-1$: $[1,0,-1]^T,[0,1,-1]^T$ and 1 eigenvector for $\lambda=2$: $[1,1,1]^T$.

General solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=c_1\,\mathrm{e}^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+c_2\,\mathrm{e}^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+c_3\,\mathrm{e}^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Note. The set of equations will *always* have a unique solution, as the matrix of coefficients has rank n.

10.2 Real coefficient matrix with non-real eigenvalues

What happens if A has non-real eigenvalues? Suppose A is real and we have

$$\vec{x}'=A\,\vec{x}$$

and $\lambda\in\mathbb{C}$ is a non-real eigenvalue. If $\vec{v}\in\mathbb{C}^n$ is an eigenvector, then $\mathrm{e}^{\lambda t}\,\vec{v}$ is a (complex) solution. We want to find corresponding real solutions.

Note. In this situation, $\bar{\lambda}$ is also an eigenvalue with eigenvector $\bar{\vec{v}}$.

Find a pair of real solutions that correspond to the pair of complex solutions $\vec{x}=\{\mathrm{e}^{\lambda t}\,\vec{v},\mathrm{e}^{\bar{\lambda} t}\,\bar{\vec{v}}\}$. We can write:

$$\vec{v}=\vec{a}+\vec{b}\,\mathrm{i},\quad \vec{a},\vec{b}\in\mathbb{R}^n$$

$$\lambda=\alpha+\beta\,\mathrm{i},\quad \alpha,\beta\in\mathbb{R}$$

Get:

$$\vec{x}=\mathrm{e}^{\lambda t}\,\vec{v}=\mathrm{e}^{\alpha t}\,(\cos\beta t+\mathrm{i}\sin\beta t)\;(\vec{a}+\vec{b}\,\mathrm{i})$$

$$\vec{x}=\mathrm{e}^{\alpha t}\,[\cos(\beta t)\,\vec{a}-\sin(\beta t)\,\vec{b}]+\mathrm{i}\,\mathrm{e}^{\alpha t}[\sin(\beta t)\,\vec{a}+\cos(\beta t)\,\vec{b}]\equiv\vec{u}(t)+\mathrm{i}\,\vec{w}(t)$$

Proposition

$\vec{u},\vec{w}$ are real solutions to our system of ODEs and are linearly independent.

We have

$$\vec{x}'=A\,\vec{x}$$

or

$$(\vec{u}+\mathrm{i}\,\vec{w})' = A\,(\vec{u}+\mathrm{i}\,\vec{w})$$

$$\vec{u}'+\mathrm{i}\,\vec{w}' = A\,\vec{u}+\mathrm{i}\,A\,\vec{w}$$

Equate real and imaginary parts on both sides:

$$\begin{aligned}\vec{u}' &= A\,\vec{u}\\ \vec{w}' &= A\,\vec{w}\end{aligned}$$

<u>Example</u>:

$$\begin{cases} x_1' = 4\,x_1 - 3\,x_2 \\ x_2' = 3\,x_1 + 4\,x_2 \end{cases}, \qquad A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$$

Find eigenvectors and eigenvalues for A:

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = \lambda^2 - 8\,\lambda + 25 = 0$$

$$\lambda_{1,2} = 4 \pm 3\,\mathrm{i}$$

Take $\lambda = 4 + 3\,\mathrm{i}$ and find eigenvector:

$$\begin{bmatrix} 4-(4+3\,\mathrm{i}) & -3 \\ 3 & 4-(4+3\,\mathrm{i}) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3\,\mathrm{i} & -3 \\ 3 & -3\,\mathrm{i} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

take $\left[\begin{smallmatrix} 1 \\ -\mathrm{i} \end{smallmatrix}\right]$. Construct complex solutions:

$$\mathrm{e}^{(4+3\mathrm{i})t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

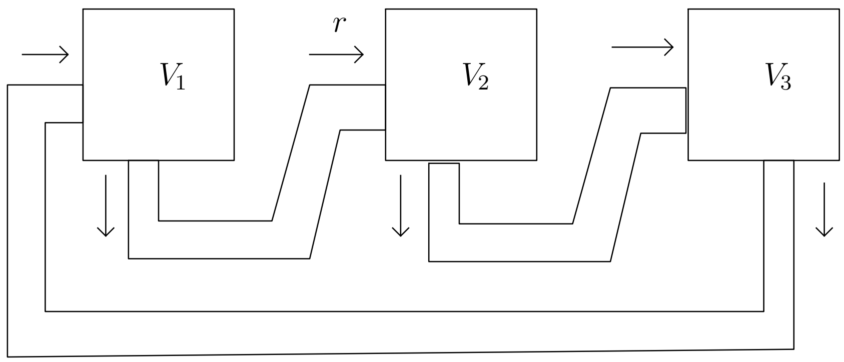
and break them up:

$$\mathrm{e}^{4t}\,(\cos 3\,t + \mathrm{i}\sin 3\,t)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \mathrm{e}^{4t}\begin{bmatrix} \cos 3\,t + \mathrm{i}\sin 3\,t \\ -\mathrm{i}\cos 3\,t + \sin 3\,t \end{bmatrix} = \mathrm{e}^{4t}\begin{bmatrix} \cos 3\,t \\ \sin 3\,t \end{bmatrix} + \mathrm{i}\,\mathrm{e}^{4t}\begin{bmatrix} \sin 3\,t \\ -\cos 3\,t \end{bmatrix}$$

General real solution will be:

$$\vec{x}(t) = c_1\,\mathrm{e}^{4t}\begin{bmatrix} \cos 3\,t \\ \sin 3\,t \end{bmatrix} + c_2\,\mathrm{e}^{4t}\begin{bmatrix} \sin 3\,t \\ -\cos 3\,t \end{bmatrix}$$

<u>Example</u>: Closed system of 3 tanks of salt solution.



- V_i is the volume of solution in tank i.
- r is the rate of flow (L/min).
- $x_i(t)$ is the amount of salt in tank i.

We get

$$\begin{aligned} x_1'(t) &= -r\,\frac{x_1}{V_1} + r\,\frac{x_3}{V_3} \\ x_2'(t) &= r\,\frac{x_1}{V_1} - r\,\frac{x_2}{V_2} \\ x_3'(t) &= r\,\frac{x_2}{V_2} - r\,\frac{x_3}{V_3} \end{aligned}$$

In matrix notation:

$$\vec{x}' = \begin{bmatrix} -\frac{r}{V_1} & 0 & \frac{r}{V_3} \\ \frac{r}{V_1} & -\frac{r}{V_2} & 0 \\ 0 & \frac{r}{V_2} & -\frac{r}{V_3} \end{bmatrix} \vec{x}$$

Take $r = 10, V_1 = 50, V_2 = 25, V_3 = 50$. Get

$$\vec{x}' = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \vec{x}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{5} - \lambda & 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} - \lambda & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5} - \lambda \end{vmatrix} = \left(\frac{2}{5} + \lambda\right)\left(\frac{1}{5} + \lambda\right)^2 + \frac{2}{125} = \cdots$$

$$\cdots = -\lambda^3 - \frac{4}{5}\lambda^2 - \frac{1}{5}\lambda = -\lambda\left(\lambda^2 + \frac{4}{5}\lambda + \frac{1}{5}\right) = 0$$

$$\lambda_{1,2,3} = 0, -\frac{2}{5} \pm \frac{1}{5}\,\mathrm{i}$$

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Interlude:

10.3 Coefficient matrix is not diagonalizable & generalized eigenvectors

Interlude: what happens if a matrix isn't diagonalizable?

Suppose we have a matrix of the following type:

$$A = \begin{bmatrix} \alpha & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & & 1 \\ 0 & & 0 & \alpha \end{bmatrix}$$

Matrix is called a Jordan cell.

Calculate eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} \alpha - \lambda & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & & 1 \\ 0 & & 0 & \alpha - \lambda \end{vmatrix} = (\alpha - \lambda)^n$$

$\lambda = \alpha$ is the only eigenvalue, with multiplicity n. Find eigenvectors:

$$(A - \alpha I)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

$$A - \alpha I = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & & 1 \\ 0 & & 0 & 0 \end{bmatrix}$$

rank $(A - \lambda I) = n - 1$, so nullspace is of dim. 1 and $[1, 0, 0, \ldots, 0]^T$ spans the eigenspace.

Note. If we square the matrix $A - \alpha I$ we get:

$$(A - \alpha I)^2 = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & & 1 \\ 0 & & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The ones diagonal shifts up one row! Also, rank $(A-\alpha I)^2 = n-2$.

What's the nullspace? Basis for the nullspace is $[1,0,\ldots,0]^T,\ldots,[0,1,0,\ldots,0]^T$.

In general, if $\vec{v}$ is a vector such that for and eigenvalue λ of a matrix A we have

$$(A-\lambda I)^k \vec{v} = \vec{0}$$

for some k, then $\vec{v}$ is called a *generalized eigenvector*.

Turns out that there are *always* n linearly independent generalized eigenvectors for a matrix A. It's basically the next best thing to diagonalize if A a is non-diagonalizable.

Now back to the tank question. We have 3 eigenvalues where 2 are complex conjugates and the third is zero. For $\lambda = 0$ the eigenvector is $[2,1,2]^T$.

For the other eigenvectors, we want 2 linearly independent real eigenvectors. Work, e.g. with $\lambda = -\frac{2}{5} - \frac{1}{5}\mathrm{i}$.

$$\begin{aligned} A-\lambda I &= \begin{bmatrix} -\frac{1}{5}-\left(-\frac{2}{5}-\frac{1}{5}\mathrm{i}\right) & 0 & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5}-\left(-\frac{2}{5}-\frac{1}{5}\mathrm{i}\right) & 0 \\ 0 & \frac{2}{5} & -\frac{1}{5}-\left(-\frac{2}{5}-\frac{1}{5}\mathrm{i}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5}+\frac{\mathrm{i}}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{\mathrm{i}}{5} & 0 \\ 0 & \frac{2}{5} & \frac{1}{5}+\frac{\mathrm{i}}{5} \end{bmatrix} \end{aligned}$$

Nullspace of this is our eigenspace. The vector $[1,\mathrm{i},-1-\mathrm{i}]^T$ spans the eigenspace.

Get complex solution:

$$\mathrm{e}^{\left(-\frac{2}{5}-\frac{\mathrm{i}}{5}\right)t}\begin{bmatrix} 1 \\ \mathrm{i} \\ -1-\mathrm{i} \end{bmatrix} = \cdots = \underbrace{\mathrm{e}^{-\frac{2}{5}t}\begin{bmatrix} \cos\frac{t}{5} \\ \sin\frac{t}{5} \\ -\cos\frac{t}{5}-\sin\frac{t}{5} \end{bmatrix}}_{\vec{u}} + \mathrm{i}\,\underbrace{\mathrm{e}^{-\frac{2}{5}t}\begin{bmatrix} -\sin\frac{t}{5} \\ \cos\frac{t}{5} \\ -\cos\frac{t}{5}+\sin\frac{t}{5} \end{bmatrix}}_{\vec{w}}$$

$\vec{u},\vec{w}$ are both real linearly independent solutions.

General solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_2\,\mathrm{e}^{-\frac{2}{5}t}\begin{bmatrix} \cos\frac{t}{5} \\ \sin\frac{t}{5} \\ -\cos\frac{t}{5}-\sin\frac{t}{5} \end{bmatrix} + c_3\,\mathrm{e}^{-\frac{2}{5}t}\begin{bmatrix} -\sin\frac{t}{5} \\ \cos\frac{t}{5} \\ -\cos\frac{t}{5}+\sin\frac{t}{5} \end{bmatrix}$$

Note. If we add x_1, x_2, x_3 we get $5\,c_1$. This makes sense, as this is a closed system, which means the salt amount does not change.

Additionally, as $t \to \infty$ the solution approaches the steady state $\begin{bmatrix} 2\,c_1 \\ c_1 \\ 2\,c_1 \end{bmatrix}$. Take into account the volumes of the tanks and reach to the conclusion that at steady state the salt concentration is ubiquitous in the tanks.

Only case we haven't touched yet is where A does not have n linearly independent eigenvectors, i.e. for some eigenvalue λ the eigenspace has dimension **smaller** than the multiplicity of λ in the characteristic polynomial. In this case, generalized eigenvectors come in.

10.4 Coefficient matrix does not have n linearly independent eigenvectors

Example:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \qquad \vec{x}' = A\,\vec{x}$$

$$\det\,(A - \lambda\,I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda - 2)^2$$

$\lambda = 2$ only eigenvalue.

$$A - 2\,I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Nullspace has dimension 1, so there is only one linearly independent eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which yields the solution $\mathrm{e}^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We need to find another linearly independent solution. Try to find a solution of the form $\vec{x} = \vec{v}_1\,t\,\mathrm{e}^{2t} + \vec{v}_2\,\mathrm{e}^{2t}$. Substitute in $\vec{x}' = A\,\vec{x}$.

$$\vec{x}' = 2\,\vec{v}_1\,t\,\mathrm{e}^{2t} + \vec{v}_1\,\mathrm{e}^{2t} + 2\,\vec{v}_2\,\mathrm{e}^{2t} = A\,(\vec{v}_1\,t\,\mathrm{e}^{2t} + \vec{v}_2\,\mathrm{e}^{2t})$$

This has to hold for all t. Set $t = 0$:

$$\vec{v}_1 + 2\,\vec{v}_2 = A\,\vec{v}_2 \tag{4}$$

$$(A - 2\,I)\,\vec{v}_2 = \vec{v}_1 \tag{5}$$

Factoring e^{2t} from eq. (4) and dividing we get:

$$2\,\vec{v}_1\,t + \vec{v}_1 + 2\,\vec{v}_2 = A\,\vec{v}_1\,t + A\,\vec{v}_2$$

$$2\,\vec{v}_1\,t + \vec{v}_1 - A\,\vec{v}_1\,t = A\,\vec{v}_2 - 2\,\vec{v}_2$$

Use eq. (5) to simplify:

$$2\,\vec{v}_1\,t + \vec{v}_1 - A\,\vec{v}_1\,t = \vec{v}_1$$

Now we have an equation in $\vec{v}_1$ only. Set $t = 1$:

$$2\,\vec{v}_1 = A\,\vec{v}_1$$

$\vec{v}_1$ is an eigenvector for $\lambda = 2$. Take $\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. We now look at eq. (5):

$$(A - 2\,I)\,\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and solve.

Note. We can utilize the fact that $(A - 2\,I)\,\vec{v}_1 = 0$ and multiply both sides (from the left) by $(A - 2\,I)$. Then:

$$(A - 2\,I)^2\,\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $\vec{v}_2$ is a generalized eigenvector.

Back to finding $\vec{v}_2$.

$$(A - 2\,I)\,\vec{v}_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Select $\vec{v}_2 = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$. Final solution:

$$\vec{x} = t\,\mathrm{e}^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \mathrm{e}^{2t} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

is a linear combination of an eigenvector and a generalized eigenvector.

11 Two-point boundary value problems (BVPs)

Given a 2nd order linear ODE:

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

we had initial conditions $y(t_0) = b_1$, $y'(t_0) = b_2$. By the E&U theorem we knew that if p, q, g were continuous and differentiable we would have a unique solution.

In many physical situations, we have a time variable t. But sometimes, our variable might be a space-variable (dependent on distance, e.g. x). So instead of ICs we might be given 2 values of y at different points x_1, x_2: $y(x_1) = b_1$, $y(x_2) = b_2$.

Typically (but not always), x_1, x_2 will be endpoints, or *boundary values* (or boundary conditions).

We now assume we have:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \quad \text{on} \quad [x_1, x_2]$$

and boundary values

$$\begin{cases} y(x_1) = b_1 \\ y(x_2) = b_2 \end{cases}$$

We call a BVP homogeneous if $b_1, b_2 = 0$ and $g(x) \equiv 0$.

Note. The E&U theorem does **not** hold for **all** BVPs. There can be no solutions, unique solutions, or infinitely many solutions, just as in linear algebraic systems of equations for non-homogeneous systems.

Note. Homogeneous BVPs *always* have a solution $y \equiv 0$.

Examples

1. $y'' + 2\,y = 0$, $\begin{cases} y(0) = 1 \\ y(\pi) = 0 \end{cases}$. This is a non-homogeneous BVP.

   Solve characteristic equation

   $$\lambda^2 + 2 = 0 \rightarrow \lambda = \pm\sqrt{2}\,\mathrm{i}$$

   General solution is

   $$y = c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)$$

   Substitute the BVs:

   $$\begin{aligned} y(0) = 1 &= c_1 \\ y(\pi) = 0 &= \cos(\sqrt{2}\,\pi) + c_2 \sin(\sqrt{2}\,\pi) \rightarrow c_2 = -\cot(\sqrt{2}\,\pi) \end{aligned}$$

   Get a unique solution

   $$y = \cos(\sqrt{2}\,x) - \cot(\sqrt{2}\,\pi) \sin(\sqrt{2}\,x)$$

2. $y'' + 2\,y = 0$, $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$. This is a homogeneous BVP.

General solution is

$$y = c_1 \cos(\sqrt{2}\, x) + c_2 \sin(\sqrt{2}\, x)$$

Set

$$\begin{aligned} y(0) &= 0 \rightarrow c_1 = 0 \\ y(\pi) &= c_2 \sin(\sqrt{2}\, \pi) = 0 \rightarrow c_2 = 0 \end{aligned}$$

Unique solution is $y \equiv 0$.

3. $y'' + y = 0, \quad \begin{cases} y(0) = 1 \\ y(\pi) = 7 \end{cases}.$

   General solution is

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$$\begin{aligned} y(0) = 1 &\rightarrow c_1 = 1 \\ y(\pi) = 7 &\rightarrow 7 = -1 \end{aligned}$$

   Oopsie no solution.

4. $y'' + y = 0, \quad \begin{cases} y(0) = 1 \\ y(\pi) = -1 \end{cases}.$

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$$\begin{aligned} y(0) = 1 &\rightarrow c_1 = 1 \\ y(\pi) = -1 &\rightarrow -1 = -1 \end{aligned}$$

   There are infinitely many solutions, as $c_2 \in \mathbb{R}$ is arbitrary.

$$y = \cos x + c_2 \sin x, \quad c_2 \in \mathbb{R}$$

5. $y'' + y = 0, \quad \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}.$

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 = 0 \\ y(\pi) = 0 &\rightarrow 0 = 0 \end{aligned}$$

   Again there are infinitely many solutions.

$$y = c_2 \sin x, \quad c_2 \in \mathbb{R}$$

Both equations we looked at were of the form $y'' + \lambda y = 0$. For which λ do we have non-trivial solutions for the homogeneous BVP?

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11.1 Eigenvalue problems

Given homogeneous BVPs with a parameter λ:

$$\text{either:} \begin{cases} y'' + \lambda\, p(x)\, y' + q(x)\, y = 0 \\ y'' + p(x)\, y' + \lambda\, q(x)\, y = 0 \end{cases}$$

with boundary values $y(a) = y(b) = 0$, for what (real) values of λ do these have a non-trivial solution? This problem is called an *eigenvalue problem*. The values λ giving non-trivial solutions are called *eigenvalues* and the non-trivial solutions are called *eigenfunctions*.

Note. The set of eigenfunctions, together with 0_f, is a vector space.

Proof. If f, g are eigenfunctions for λ

$$f'' + p(x)\, f' + \lambda\, q(x)\, f = 0$$

$$g'' + p(x)\, g' + \lambda\, q(x)\, g = 0$$

so

$$f'' + g'' + p\,(f' + g') + \lambda\, q\,(f + g) = 0$$

$$\begin{cases} f(a) = 0 = f(b) \\ g(a) = 0 = g(b) \end{cases}$$

so

$$(f + g)(a) = 0 = (f + g)(b)$$

If $f + g \neq 0$ then it will be an eigenfunction. Additionally, if $c \neq 0$ then $c\, f$ is an eigenfunction and $c\, f(a) = 0 = c\, f(b)$. $\square$

Special case:

$$y'' + \lambda\, y = 0, \quad y(0) = y(L) = 0, \quad L > 0$$

Characteristic equation is

$$u^2 + \lambda = 0$$

There are 3 possible cases:

1. Two real roots. $\lambda < 0$

2. Two non-real roots. $\lambda > 0$

3. One double (real) root. $\lambda = 0$.

Case 1: $\lambda < 0$

General solution can be written as follows (denoting $\mu = \sqrt{-\lambda}$)

$$y = c_1 \, \mathrm{e}^{\mu x} + c_2 \, \mathrm{e}^{-\mu x}$$

Plug in BVs

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 + c_2 = 0 \\ y(L) = 0 &\rightarrow c_1 \, \mathrm{e}^{\mu L} + c_2 \, \mathrm{e}^{-\mu L} = 0 \end{aligned}$$

Get

$$c_1 \, \mathrm{e}^{\mu L} - c_1 \, \mathrm{e}^{-\mu L} = 0$$

If $c_1 \neq 0$ then get $\mathrm{e}^{\mu L} = \mathrm{e}^{-\mu L}$ or $\mathrm{e}^{2\mu L} = 1$. As $\mu, L > 0$ problem is unsolvable unless $c_1, c_2 = 0$. Only solution is the trivial solution, and $\lambda < 0$ are **not** eigenvalues.

Case 2: $\lambda > 0$

Characteristic polynomial has complex roots $\pm\sqrt{\lambda}\,\mathrm{i}$.

General solution is (denoting $\mu = \sqrt{\lambda}$)

$$y = c_1 \cos(\mu \, x) + c_2 \sin(\mu \, x)$$

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 = 0 \\ y(L) = 0 &\rightarrow c_2 \sin(\mu \, L) = 0 \end{aligned}$$

$$\mu \, L = \pi \, n, \quad n \in \mathbb{N}$$

So $\lambda = \dfrac{n^2 \, \pi^2}{L^2}$. Then we can take as eigenfunction $y_n(x) = \sin \dfrac{n \, \pi}{L} x$, and all eigenfunctions are scalar multiples of these.

Note. If $L = \pi$ then the eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are $\sin(n \, x)$.

Case 3: $\lambda = 0$

ODE is $y'' = 0$ and general solution is $y = c_1 \, x + c_2$. Set BVs:

$$\begin{aligned} y(0) = 0 &\rightarrow c_2 = 0 \\ y(L) = 0 &\rightarrow c_1 \, L = 0 \rightarrow c_1 = 0 \end{aligned}$$

Only solution is the trivial solution.

Conclude: Only eigenvalues are $\lambda_n = \frac{n^2 \pi^2}{L^2}$ with corresponding eigenfunctions $\sin \frac{n \pi x}{L}$ (and non-zero scalar multiples of these).

Note. Boundary values can be given for the derivative of y, instead of y itself, but an different points.

Example of such BVP

$$y'' + y = 0, \quad \begin{cases} y(0) = 0 \\ y'(\pi) = 1 \end{cases}$$

General solution is

$$y = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 = 0 \\ y'(\pi) = 1 &\rightarrow c_2 \cos \pi = 1 \rightarrow c_2 = -1 \end{aligned}$$

$$y = -\sin x$$

Note though that this is **not** a homogeneous BVP, because of the boundary values are not all zero.

Another eigenvalue problem

$$y'' + \lambda y' + 2 y = 0, \quad \begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases}$$

Characteristic equation is

$$u^2 + \lambda u + 2 = 0$$

$$u_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 8}}{2}$$

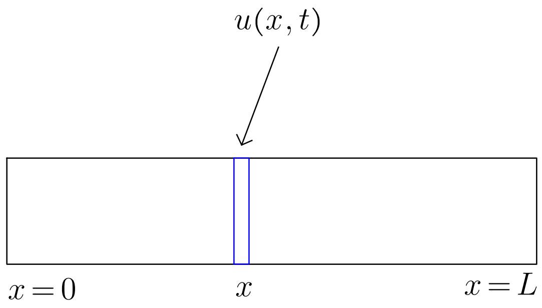
Get a double root if $\lambda^2 = 8$, get two real roots if $\lambda^2 > 8$ and two non-real roots if $\lambda^2 < 8$. Each case must be dealt with separately.

IV Partial differential equations

In this section we review some examples of PDVs.

12 Heat conduction in a rod (Heat equation)

Heat conduction in a finite rod. Solution was given by Fourier (1768–1830).



Assume the rod has a uniform cross-section and the material is homogeneous. Sides are perfectly insulated.

Denote $u(x,y)$ the temperature at a point x and time t.

The heat equation that expresses this is:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

α^2 is a constant representing the thermal diffusivity of the bar, in units of $(\text{length})^2/\text{time}$.

Different notation:

$$\alpha^2 u_{xx} = u_t$$

Given the initial temperature of the rod at every point & assuming that the temperature at the endpoints is kept constant, we want to find a unique solution $u(x,t)$.

Initial conditions: $u(x,0) = f(x)$ in $x \in [0,L]$.

Boundary conditions: $u(0,t) = u(L,t) = 0$ for all t. (Homogeneous BCs.)

[Boundary conditions could also have been u' at $x = 0, L$.]

Note. If $f(x) \equiv 0$ then $u(x,t) \equiv 0$ $\forall t$ is a solution of the homogeneous BVP.

Note. Set of solutions to heat equation satisfying the homogeneous BCs is a vector space, which is infinitely dimensional.

How do we find a basis for an infinitely dimensional vector space? We do the best we can... We find an infinite set of linearly independent solutions and form a *series* of functions with unknown coefficients, which can be determined so as to satisfy the ICs. We usually approximate the solution as a finite series.

In formal nomenclature: Find a set of non-trivial solutions to the PDE and BCs $\{u_n\}$ and form a series of these: $\sum_{n=1}^{\infty} c_n u_n$, so that it satisfies the ICs as well.

12.1 Solution by separation of variables

Method

Step 1: Try to find an infinite set of solutions of a special form:

$$u(x,t) = X(x)\,T(t)$$

Then we can convert the problem to two ODEs, one in x and one in t.

Step 2: Use the fundamental solutions to find a unique solution that also satisfies the ICs.

Step 1: Substitute in the heat equation

$$\alpha^2 u_{xx} = u_t$$

and get

$$\alpha^2 X'' T = X T'$$

Assume $X, T \not\equiv 0$ on some interval. Separate variables:

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} \quad \forall x, t$$

Note. Suppose we have

$$F(x) = G(t) \quad \forall x, t$$

Differentiate wrt x:

$$F'(x) = 0 \quad \forall x$$

So $F(x)$ is a constant function, say $F(x) \equiv c$.

Now differentiate wrt t:

$$G'(t) = 0 \quad \forall t$$

So $G(t)$ is a constant function, say $G(t) \equiv c^{\star}$. But $F(x) = G(t)$ so $c = c^{\star}$ and we have

$$F(x) = G(t) = c$$

By our note, both sides are equal to one constant we denote as $-\lambda$.

$$\frac{X''}{X} = -\lambda = \frac{1}{\alpha^2}\frac{T'}{T}$$

We get two ODEs, which together solve the heat equation:

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases}$$

We want $X(x)\,T(t)$ to satisfy the BCs. Say $u(0,t) = u(L,t) = 0$ for all $t \geq 0$. Set $u = X(x)\,T(t)$:

$$X(0)\,T(t) = 0 \quad \forall t$$

$T(t) \not\equiv 0$ so this gives $X(0) = 0$. Similarly $X(L) = 0$.
For $X(x)$ we need to find non-zero solutions to

$$X'' + \lambda X = 0, \quad \begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$$

That's an eigenvalue problem! The only non-trivial solutions we found (Section 6.6.1) were

$$\lambda_n = \frac{\pi^2 n^2}{L^2}, \quad n \geq 1$$

with functions which are multiples of $\sin\left(\sqrt{\lambda_n}\, x\right)$. Define

$$X_n(x) = \sin \frac{\pi n}{L}$$

and substitute λ_n in our ODE for $T(t)$:

$$T' + \alpha^2 \lambda_n T = 0$$

So $T(t)$ = constant multiple of $\mathrm{e}^{-\alpha^2 \lambda_n t}$. Define

$$T_n(t) = \exp\left(-\frac{\alpha^2 \pi^2 n^2}{L^2}\, t\right)$$

Indeed,

$$u_n(x,t) = X_n(x)\,T_n(t) = \exp\left(-\frac{\alpha^2 \pi^2 n^2}{L^2}\, t\right) \sin\left(\frac{\pi n}{L}\, x\right), \quad n \geq 1, t \geq 0$$

solves the heat equation and satisfies the BCs.

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Note. linear combinations of $\{u_n(x,t)\}$ will also satisfy the PDE and BCs.

After finding the fundamental solutions satisfying the PDE and BCs, try to form an infinite series.

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, u_n(x,t)$$

That will satisfy the PDE and BCs and **also** the ICs. Generally, though, if a finite series satisfies the PDE it doesn't necessarily mean that the infinite series will satisfy it, because it might not be differentiable. However, if the series converges uniformly, then it will still satisfy the required conditions. We essentially differentiate the series term by term:

$$\left(\sum_{i=1}^{\infty} f_i\right)' = \sum_{i=1}^{\infty} f_i'$$

So we need to find $\{c_i\}$ such that $u(x,0) = f(x)$, where

$$f(x) = \sum c_n \, \mathrm{e}^{-\frac{\alpha^2 \pi^2 n^2}{L^2} \cdot 0} \sin \frac{n\pi x}{L} = \sum c_n \sin\left(\frac{n\pi}{L} x\right)$$

The coefficients $\{c_n\}$ are called *Fourier coefficients*.

12.1.1 Interlude: Fourier series

Suppose we want to express a function which is periodic using a series of trigonometric functions.

Definition

$f(x)$ is periodic if there exists a $p > 0$ such that for all x

$$f(x) = f(x+p)$$

p is called the *period* of $f(x)$.

Note that periodic functions don't have to be continuous.

Definition

A periodic function of period $2L$ is piece-wise smooth if f, f' are continuous except for at most a finite number of points on $[-L, L]$, and at the points of discontinuity a we have only jump discontinuities, i.e. $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and are finite.

Fourier Convergence Theorem

If f is periodic of period $2L$ and piece-wise smooth, then it can be represented as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

where a_n, b_n are determined *uniquely*.

The series converges "uniformly" to:

- $f(x)$ for all x if f is continuous
- $\frac{1}{2}[\lim_{x\to a^-} f(x) + \lim_{x\to a^+} f(x)]$ if f has a discontinuity at a.

Note. Periodic functions which are piece-wise smooth of period $2L$ form a vector space. In fact, if f, g are in this vector space then also $f \cdot g$ will be as well.

If f_n are in this vector space and $\sum f_n$ converges to a function f then so is f in the vector space.

12.1.2 Inner products on function spaces & orthogonality

On $\mathbb{C}^n$ we had $\langle \vec{v}, \vec{w} \rangle$ inner product for $\vec{v}, \vec{w} \in \mathbb{C}^n$:

$$\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{w} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}: \quad \langle \vec{v}, \vec{w} \rangle = \sum_{j=1}^{n} x_j \bar{y}_j$$

By analogy, for u, v 2 integrable real functions on $[a, b]$ we define:

$$\langle u, v \rangle = \int_a^b u(t)\, v(t)\, \mathrm{d}t$$

Properties

1. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Longleftrightarrow u \equiv 0$ on $[a, b]$.
2. Linearity. $\langle c_1 u_1 + c_2 u_2, v \rangle = c_1 \langle u_1, v \rangle + c_2 \langle u_2, v \rangle$ (Originates from linearity of integrals)
3. Symmetry. $\langle u, v \rangle = \langle v, u \rangle$

Definition

u, v are orthogonal with respect to inner product if $\langle u, v \rangle = 0$.

We say $\{u_n\}_{n=1}^{\infty}$ are an orthogonal family of functions if $\langle u_i, u_j \rangle = 0$ for all $i \neq j$ and if $u_n \not\equiv 0$ for all n.

Corollary

If $\{u_n\}_{n=1}^{\infty}$ are an orthogonal family of functions then they are linearly independent (converse not necessarily true).

Proof. Suppose we have $\sum_{i=1}^{k} c_i u_i \equiv 0_f$ and $\{u_i\}$ are orthogonal. Then:

$$0 = \langle 0, u_j \rangle = \Big\langle \sum_{i=1}^{k} c_i u_i, u_j \Big\rangle = \sum_{i=1}^{k} c_i \langle u_i, u_j \rangle = c_j \langle u_j, u_j \rangle \neq 0$$

so $c_j = 0$ for all j and $\{u_i\}$ are linearly independent. $\square$

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Corollary

The family of functions $\{1\} \cup \{\cos \frac{n\pi x}{L}\}_{n=1}^{\infty} \cup \{\sin \frac{n\pi x}{L}\}_{n=1}^{\infty}$ are an orthogonal family wrt the inner product on $[-L, L]$ $(L > 0)$

Note. So these are linearly independent and periodic (of period $2L$) but are **not** a basis for the space of periodic functions of period $2L$.

Proof. Show that the following are equal to zero:

- $\langle 1, \cos \frac{n\pi x}{L}\rangle, \langle 1, \sin \frac{n\pi x}{L}\rangle$ for $n \geq 1$
- $\left\langle \cos \frac{n\pi x}{L}, \sin \frac{k\pi x}{L}\right\rangle$ for all n, k
- $\left\langle \cos \frac{n\pi x}{L}, \cos \frac{k\pi x}{L}\right\rangle, \left\langle \sin \frac{n\pi x}{L}, \sin \frac{k\pi x}{L}\right\rangle$ for all $n \neq k$

We show for example:

$$\left\langle \cos \frac{n\pi x}{L}, \cos \frac{k\pi x}{L}\right\rangle = 0 \quad \forall n \neq k$$

We want to calculate

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right) \mathrm{d}x$$

Use the trigonometric identities:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

to get:

$$\frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)] = \cos\alpha \cos\beta$$

So,

$$\int_{-L}^{L} \frac{1}{2}\left[\cos \frac{(n+k)\pi x}{L} + \cos \frac{(n-k)\pi x}{L}\right] \mathrm{d}x = \cdots$$

$$\cdots = \frac{1}{2}\left[\frac{L}{(n+k)\pi} \sin \frac{(n+k)\pi x}{L} + \frac{L}{(n-k)\pi} \sin \frac{(n-k)\pi x}{L}\right]_{-L}^{L} = 0$$

Remember that because $n \neq k$ we can divide by $n \pm k$ and inserting $x = \pm L$ gives us integer multiples of π inside each sine, which always results in zero.

$\square$

Example. Look at the following inner product:

$$\begin{aligned}
\left\langle \cos\frac{n\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle &= \int_{-L}^{L} \cos^2\frac{n\,\pi\,x}{L}\,\mathrm{d}x = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{2\,n\,\pi\,x}{L}+1\right)\mathrm{d}x \\
&= \frac{1}{2}\left(\frac{L}{2\,n\,\pi}\sin\frac{2\,n\,\pi\,x}{L}+x\right)_{-L}^{L} = \frac{1}{2}\,(L-(-L))
\end{aligned}$$

$$\Rightarrow \left\langle \cos\frac{n\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle = L$$

Similarly one may obtain $\langle \sin\frac{n\,\pi\,x}{L}, \sin\frac{n\,\pi\,x}{L}\rangle = L$.

Note. If a series of functions converges uniformly $\sum_{n=1}^{\infty} f_n = f$ then we have infinite additivity in inner product, i.e.

$$\left\langle \sum_{n=1}^{\infty} f_n, g\right\rangle = \sum_{n=1}^{\infty} \langle f_n, g\rangle$$

We shall use this fact in Fourier series calculations.

For a fixed $n \geq 1$, use Fourier series for $f(x)$:

$$\left\langle f(x), \cos\frac{n\,\pi\,x}{L}\right\rangle = \left\langle \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\,\pi\,x}{L} + \sum_{k=1}^{\infty} b_k \sin\frac{k\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle$$

Using “infinite linearity”:

$$\begin{aligned}
\left\langle f(x), \cos\frac{n\,\pi\,x}{L}\right\rangle &= \left\langle \frac{a_0}{2}, \cos\frac{n\,\pi\,x}{L}\right\rangle + \sum_{k=1}^{\infty} \left\langle a_k \cos\frac{k\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle \\
&\quad + \sum_{k=1}^{\infty} \left\langle b_k \sin\frac{k\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle \\
&= \frac{a_0}{2}\left\langle 1, \cos\frac{n\,\pi\,x}{L}\right\rangle + \sum_{k=1}^{\infty} a_k \left\langle \cos\frac{k\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle \\
&\quad + \sum_{k=1}^{\infty} b_k \left\langle \sin\frac{k\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle
\end{aligned}$$

Use the claim that all functions involved in the Fourier series are an orthogonal family, and get

$$\left\langle f(x), \cos\frac{n\,\pi\,x}{L}\right\rangle = a_n \left\langle \cos\frac{n\,\pi\,x}{L}, \cos\frac{n\,\pi\,x}{L}\right\rangle = a_n\,L$$

So a_n is calculated by taking the inner product

$$\boxed{a_n = \frac{1}{L}\left\langle f(x), \cos\frac{n\pi x}{L}\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x}$$

Similarly,

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x}$$

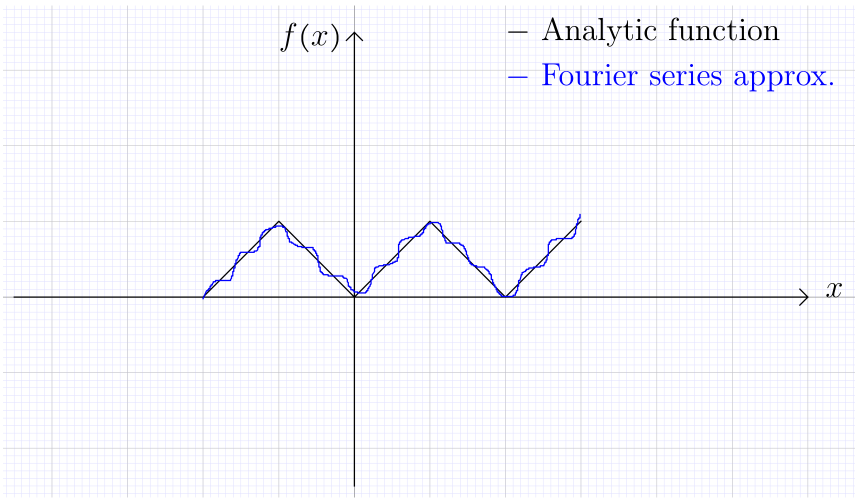
We now calculate

$$\begin{aligned}\langle f(x), 1\rangle &= \frac{a_0}{2}(1,1) + \underbrace{\sum_{n=1}^{\infty} a_n\left\langle\cos\frac{n\pi x}{L}, 1\right\rangle}_{=0} + \underbrace{\sum_{n=1}^{\infty} b_n\left\langle\sin\frac{n\pi x}{L}, 1\right\rangle}_{=0} \\ &= \frac{a_0}{2}\int_{-L}^{L} 1\cdot 1\,\mathrm{d}x = a_0 L\end{aligned}$$

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x}$$

12.1.3 Popular examples of Fourier series

1. Triangular wave function.



$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x, & 0 \leq x \leq 2 \end{cases} \qquad f(x+4) = f(x)\ \forall x$$

Want a Fourier series for $f(x)$ on $[-2, 2]$.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-2}^{0}(-x)\,\mathrm{d}x + \int_{0}^{2} x\,\mathrm{d}x\right] = \frac{1}{2}\cdot 4 = 2$$

For $n > 0$:

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\,\pi\,x}{2}\,\mathrm{d}x = \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\,\pi\,x}{2}\,\mathrm{d}x + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\,\pi\,x}{2}\,\mathrm{d}x$$

Integrate by parts:

$$\int x\cos\frac{n\,\pi\,x}{2}\,\mathrm{d}x = \frac{2\,x}{\pi\,n}\sin\frac{n\,\pi\,x}{2} + \frac{4}{n^2\,\pi^2}\cos\frac{n\,\pi\,x}{2}$$

Insert back and get

$$\begin{aligned} a_n &= \frac{1}{2}\left[-\frac{2\,x}{n\,\pi}\sin\frac{n\,\pi\,x}{2} - \frac{4}{n^2\,\pi^2}\cos\frac{n\,\pi\,x}{2}\right]_{-2}^{0} \\ &\quad + \frac{1}{2}\left[\frac{2\,x}{n\,\pi}\sin\frac{n\,\pi\,x}{2} + \frac{4}{n^2\,\pi^2}\cos\frac{n\,\pi\,x}{2}\right]_{0}^{2} \end{aligned}$$

Get

$$a_n = \frac{4}{n^2\,\pi^2}(\cos(n\,\pi) - 1) = \begin{cases} \frac{-8}{n^2\,\pi^2} & n\,\text{odd} \\ 0 & n\,\text{even} \end{cases}$$

In a similar fashion,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\,\pi\,x}{2}\,\mathrm{d}x = 0$$

Finally,

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{\text{odd } n}\frac{\cos\frac{n\,\pi\,x}{2}}{n^2} = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi\,x}{2} + \frac{\cos\frac{3\,\pi\,x}{2}}{9} + \frac{\cos\frac{5\,\pi\,x}{2}}{25} + \cdots\right)$$

Another formula:

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{(2\,k-1)\,\pi\,x}{2}\right)}{(2\,k-1)^2}$$

Note. Setting $x = 0$:

$$0 = f(0) = 1 - \frac{8}{\pi^2}\sum_{\text{odd } n}\frac{1}{n^2}$$

So

$$\sum_{\text{odd } n}\frac{1}{n^2} = \frac{\pi^2}{8}$$

Note. If $L = \pi$ get the Fourier series

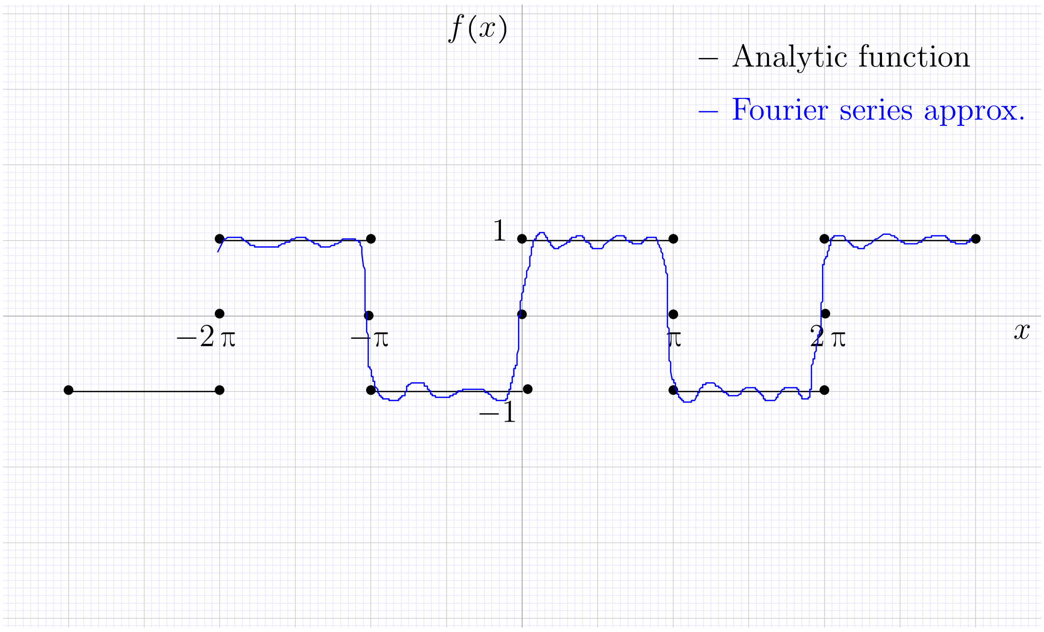
$$\frac{a_0}{2} + \sum\,[a_n\cos(n\,x) + b_n\sin(n\,x)]$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n\,x)\, \mathrm{d}x$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n\,x)\, \mathrm{d}x$$

2. Square wave function.



Definition on interval $[-\pi, \pi]$

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0, \pm\pi \\ 1, & x < \pi \end{cases} \qquad f(x + 2\,\pi\,k) = f(x), \quad k \in \mathbb{Z}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\cos n\,x)\, \mathrm{d}x + \int_{0}^{\pi} (\cos n\,x)\, \mathrm{d}x \right] \\ &= \frac{1}{\pi} \left[\left(-\frac{1}{n} \sin(n\,x) \right)_{-\pi}^{0} + \left(\frac{1}{n} \sin(n\,x) \right)_{0}^{\pi} \right] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \left[\int_{-\pi}^{0} (-\sin n\,x)\, \mathrm{d}x + \int_{0}^{\pi} (\sin n\,x)\, \mathrm{d}x \right] \\ &= \frac{1}{\pi} \left[\left(\frac{1}{n} \cos(n\,x) \right)_{-\pi}^{0} + \left(-\frac{1}{n} \cos(n\,x) \right)_{0}^{\pi} \right] \\ &= \frac{1}{\pi} \left[\left(\frac{1}{n} - \frac{1}{n} \cos(-n\,\pi) \right) + \left(-\frac{1}{n} \cos(n\,\pi) + \frac{1}{n} \right) \right] \end{aligned}$$

Get

$$b_n = \frac{2}{\pi n}[1 - \cos(n\pi)] = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

The Fourier series is

$$f(x) = \frac{4}{\pi} \sum_{\text{odd } n} \sin n x = \frac{4}{\pi}\left(\sin x + \frac{\sin(3x)}{3} + \cdots\right)$$

Setting $x = \frac{\pi}{2}$ gives

$$1 = f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \cdots\right) = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

Get Leibniz formula: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$

3. $f(x) = \sin(\pi x)$ on $[-1, 1]$.

$$a_0 = \int_{-1}^{1} \sin(\pi x)\, \mathrm{d}x = 0$$

$$a_n = \int_{-1}^{1} \sin(\pi x)\cos(n\pi x)\, \mathrm{d}x = 0 \quad \forall n \geq 1$$

$$b_n = \int_{-1}^{1} \sin(\pi x)\sin(n\pi x)\, \mathrm{d}x = \begin{cases} 0 & n > 1 \\ 1 & n = 1 \end{cases}$$

In other words, $\sin(\pi x)$ is its own Fourier series (Duh).

Note. This is just like any polynomial is its own Taylor series.

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12.1.4 Utilizing properties of even and odd functions

Definition

f is *even* if $f(-x) = f(x)$

f is *odd* if $f(-x) = -f(x)$

More properties:

- If f is even then

$$\int_{-L}^{L} f(x)\, \mathrm{d}x = 2\int_0^L f(x)\, \mathrm{d}x$$

If f is odd then

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = 0$$

- If f is even then f' is odd and $\int f(x)\,\mathrm{d}x$ is odd.

  If f is odd then f' is even and $\int f(x)\,\mathrm{d}x$ is even.

How can we apply this to Fourier series?

- If f is even $2L$-periodic and piece-wise smooth then: $f(x)\cos\frac{n\pi x}{L}$ is even and $f(x)\sin\frac{n\pi x}{L}$ is odd. Therefore, the Fourier sine coefficients $b_n = 0\ \forall n$ and the Fourier cosine coefficients $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$ and the Fourier series of f will be

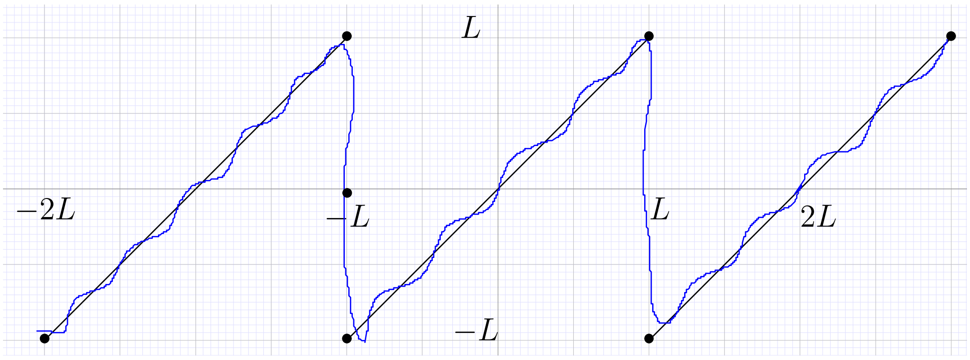
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

- Similarly, if f is odd $2L$-periodic piece-wise smooth then $f(x)\sin\frac{n\pi x}{L}$ is even and $f(x)\cos\frac{n\pi x}{L}$ is odd. So $a_n = 0\ \forall n \geq 0$ and $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$, and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Example: Sawtooth wave.

$$f(x) = \begin{cases} x & -L < x < L \\ 0 & x = \pm L \end{cases}, f(x) = f(x+2L)$$



f is odd so we get a sine Fourier series.

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{2}{L}\left[-\frac{Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L \end{aligned}$$

$$f(x) = \frac{2L}{\pi}\sum \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

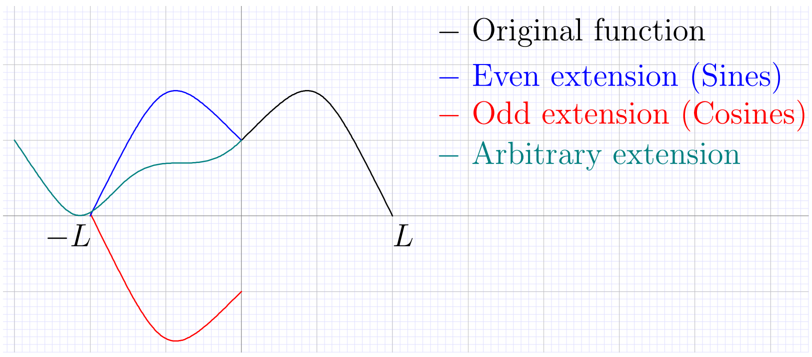
Example: $2L$-periodic triangular wave. A triangular wave is even and coincides with with the sawtooth wave on $[0, L]$. Remember that for a triangular wave:

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{(2k-1)\pi x}{2}\right)}{(2k-1)^2}$$

which is a sum of cosines. How can a two functions coincide if one is a sum of sines and the other is a sum of cosines? Well, the Fourier convergence theorem holds for the **full** interval, so on **partial** intervals two series can converge to the same function $f(x) = x$, $x \in [0, L]$.

In fact, we could extend arbitrarily to $[-L, 0]$ and extend periodically with period $2L$. There are infinitely many Fourier series that converge to some function on a partial section of the period.

In general, given **any** function defined on $[0, L]$ can be extended to $[-L, 0]$ in any way we please, and then extend periodically. Each option gives a Fourier series and all converge to the same function on $[0, L]$.



12.2 Back to heat equation

Summary so far: PDV is:

$$\alpha^2 u_{xx} = u_t$$

ICs: $u(x, 0) = f(x), x \in [0, L]$.

BCs: $u(0, t) = u(L, t) = 0$, $\forall t \geq 0$.

We had constructed fundamental solutions:

$$u_n(x, t) = \exp\left(-\frac{\alpha^2 \pi^2 n^2}{L^2} t\right) \sin\left(\frac{\pi n}{L} x\right)$$

satisfying the heat equation and the BCs. We then formed a series

$$u(x,t)=\sum_{n=1}^{\infty} c_n\, u_n(x,t)$$

satisfying these conditions, and we demand that $u(x,t)$ satisfies the ICs.

$$f(x)=u(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\,\pi\,x}{L}$$

This is the sine Fourier series for $f(x)$, such that

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}$$

Take for example $L=50\,\text{cm}$, $\alpha^2=1$, $u(0,t)=u(L,t)=0$ $\forall t\geq 0$ and that the initial temperature is 20°C on all bar: $u(x,0)=20, x\in[0,L]$. Solutions is

$$u(x,t)=\sum c_n\, \mathrm{e}^{-\frac{n^2\pi^2 t}{2500}}\sin\frac{n\,\pi\,x}{50}$$

$$c_n=\frac{2}{50}\int_0^{50} 20\cdot\sin\frac{n\,\pi\,x}{50}\,\mathrm{d}x=\frac{4}{5}\cdot\frac{50}{n\,\pi}\left[-\cos\frac{n\,\pi\,x}{50}\right]_0^{50}=\frac{40}{n\,\pi}[(-1)^n+1]=\begin{cases}0 & n \text{ even}\\ \frac{80}{n\,\pi} & n \text{ odd}\end{cases}$$

$$u(x,t)=\frac{80}{n\,\pi}\sum_{n\text{ odd}}\frac{1}{n}\mathrm{e}^{-\frac{n^2\pi^2 t}{2500}}\sin\frac{n\,\pi\,x}{50}$$

The exponent has n^2 in the numerator so the series converges fast.

Let's say we want to estimate when the entire bar's temperature is smaller or equal to 1°C using first term of series. We find t such that

$$1=u(25,t)\approx\frac{80}{\pi}\mathrm{e}^{-\frac{n^2\pi^2 t}{2500}}\underbrace{\sin\frac{\pi\cdot 25}{50}}_{=1}=\frac{80}{\pi}\mathrm{e}^{-\frac{n^2\pi^2 t}{2500}}$$

$$\ln\frac{80}{\pi}=\frac{\pi^2\,t}{2500}\Rightarrow t=\frac{2500}{\pi^2}\ln\frac{80}{\pi}\sim 820\,\text{sec}$$

12.2.1 Non-homogeneous boundary conditions

A case where $u(0,t)=T_1$ and $u(L,t)=T_2$.

Method

Solve by reducing to the homogeneous case. Subtract some linear function $v(x)$ from $u(x,t)$ to get the homogeneous BCs.

$$\begin{aligned} v(0) &= T_1 \\ v(L) &= T_2 \end{aligned}$$

As $v(x)$ is linear,

$$v(x) = (T_2 - T_1)\,\frac{x}{L} + T_1$$

Define

$$w(x,t) \equiv u(x,t) - v(x)$$

such that $w(0,t) = w(L,t) = 0$. What else w satisfies?

$$\alpha^2\, w_{xx} = \alpha^2\,(u_{xx} - v''(x)) = \alpha^2\, u_{xx}$$

$$w_t = u_t - \frac{\mathrm{d}v}{\mathrm{d}t} = u_t$$

$w(x,t)$ satisfies the heat equation and the BCs. What about the ICs?

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$$

So we solve for w with these ICs and substitute

$$u(x,t) = w(x,t) + v(x)$$

In fact, we get

$$u(x,t) = \sum c_n\,\mathrm{e}^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin\frac{n\,\pi\,x}{L} + (T_2 - T_1)\,\frac{x}{L} + T_1$$

where

$$c_n = \frac{2}{L}\int_0^L \left[f(x) - (T_2 - T_1)\,\frac{x}{L} - T_1\right] \sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Example: Rod of length 30, $\alpha^2 = 1$, with ICs $u(x,0) = 60 - 2x, x \in [0,30]$ and BCs $u(0,t) = 20, u(30,t) = 50$.

$$v(x) = 20 + x$$

Solve $\alpha^2 w_{xx} = w_t$.

$$w(x,0) = 60 - 2x - (20 + x) = 49 - 3x$$

$$w(0,t) = 0 = w(L,t)$$

and then add $20 + x$ to our solution $w(x,t)$ to get $u(x,t)$.

12.3 Bar with insulated ends

$$\alpha^2 u_{xx} = u_t, x \in [0, L]$$

ICs: $u(x,0) = f(x)$. Now assume that no heat passes through the ends in the x direction, so that $u_x(0,t) = u_x(L,t) = 0$.

Solve by finding fundamental solutions of the form $X(x)\,T(t)$ and get as before:

$$a^2 X'' T = X T'$$

Separate variables to get:

$$\frac{X''}{X} = \lambda = \frac{T'}{\alpha^2 T}$$

and

$$\begin{aligned} X'' + \lambda X &= 0 \\ T' + \lambda \alpha^2 T &= 0 \end{aligned}$$

The difference is in the BCs:

$$X'(0) = X'(L) = 0$$

We need to solve this eigenvalue problem for $X(x)$. Different cases:

1. Characteristic polynomial has 2 real roots: $\lambda < 0$, denote $\mu = \sqrt{-\lambda} > 0$

$$y^2 + \lambda = 0$$

$$X'' - \mu^2 X = 0$$

general solution is

$$X(x) = c_1 \mathrm{e}^{\mu x} + c_2 \mathrm{e}^{-\mu x}$$

Set BCs:

$$\begin{aligned} X'(0) = 0 &\;\rightarrow\; \mu\,(c_1 - c_2) = 0 \rightarrow c_1 = c_2 \\ X'(L) = 0 &\;\rightarrow\; c_1\,\mathrm{e}^{\mu L} - \mu\,c_2\,\mathrm{e}^{-\mu L} = 0 \rightarrow c_1 = c_2 = 0 \end{aligned}$$

No non-zero solutions in this case.

2. Characteristic polynomial has one double root: $\lambda = 0$.

$$X(x) = c_1\,x + c_2$$

$$X'(0) = 0 \;\rightarrow\; c_1 = 0$$

c_2 is arbitrary. $X(x) = c_2 = 1$ is an eigenfunction.

3. Characteristic polynomial has 2 non-real roots: $\lambda > 0$

$$X'' + \lambda\,X = 0$$

Write $\mu = \sqrt{\lambda} > 0$ and get general solution

$$X(x) = c_1 \cos(\mu\,x) + c_2 \sin(\mu\,x)$$

$$X'(x) = -c_1\,\mu \sin(\mu\,x) + c_2 \cos(\mu\,x)$$

Set BCs:

$$\begin{aligned} X'(0) = 0 &\;\rightarrow\; c_2\,\mu = 0 \rightarrow c_2 = 0 \\ X'(L) = 0 &\;\rightarrow\; -c_1\,\mu \sin(\mu\,L) = 0 \end{aligned}$$

If $c_1 \neq 0$ then $\sin(\mu\,L) = 0$.

$$\sin(\mu\,L) = 0 \Longleftrightarrow \mu\,L = n\,\pi, \quad n \geq 1 \text{ integer}$$

$$\lambda_n = \frac{n^2\,\pi^2}{L^2}$$

Eigenfunctions are scalar multiples of

$$\boxed{X_n(x) = \cos\frac{n\,\pi\,x}{L}, \quad n \geq 0}$$

Now solve for T:

$$T' + \alpha^2\,\lambda\,T = 0$$

$$\begin{cases} \lambda = 0 & T_0 \equiv 1 \\ \lambda_n = \frac{n^2\,\pi^2}{L^2} & T = \mathrm{e}^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \end{cases}$$

Fundamental solutions are:

$$\begin{cases} \lambda = 0 & u_0(x,t) = X_0(x)\, T_0(t) \equiv 1 \\ \lambda_n = \frac{n^2 \pi^2}{L^2} & u_n(x,t) = X_n(x)\, T_n(t) = \cos \frac{n \pi x}{L} \cdot \mathrm{e}^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \end{cases}$$

Form an infinite series:

$$\begin{aligned} u(x,t) &= \frac{c_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} c_n\, u_n(x,t) \\ &= \frac{c_0}{2} + \sum c_n \cos \frac{n \pi x}{L} \mathrm{e}^{-\frac{n^2 \pi^2 \alpha^2}{L^2} t} \end{aligned}$$

which satisfies the PDE & BCs and set $t = 0$ to get ICs:

$$f(x) = u(x,0) = \frac{c_0}{2} + \sum c_n \cos \frac{n \pi x}{L}, \quad x \in [0, L]$$

c_n is the Fourier cosine series coefficients for $f(x)$.

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} \, \mathrm{d}x, \quad n \geq 0$$

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Example of another eigenvalue problem

$$y'' + y' + 2\lambda\, y = 0$$

BVs: $y(0) = y(\pi) = 0$. Characteristic equation is

$$x^2 + x + 2\lambda = 0$$

$$\text{roots} = \frac{-1 \pm \sqrt{1 - 8\lambda}}{2}$$

Cases depend on sign of discriminant.

1. $1 - 8\lambda > 0$, so $\lambda < \frac{1}{8}$ and 2 real roots.

   General solution is

   $$y = c_1 \mathrm{e}^{\frac{-1 + \sqrt{1 - 8\lambda}}{2} t} + c_2 \mathrm{e}^{\frac{-1 - \sqrt{1 - 8\lambda}}{2} t}$$

2. $1 - 8\lambda = 0$, so $\lambda = \frac{1}{8}$ and one double root.

   General solution is

   $$y = c_1 \mathrm{e}^{-\frac{1}{2} t} + c_2\, t\, \mathrm{e}^{-\frac{1}{2} t}$$

3. $1-8\lambda<0$, so $\lambda>\frac{1}{8}$ and 2 complex roots.

   General solution is

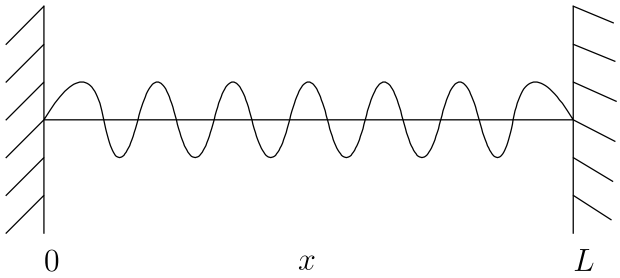
$$y=c_1\,\mathrm{e}^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{8\lambda-1}}{2}\,t\right)+c_2\,\mathrm{e}^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{8\lambda-1}}{2}\,t\right)$$

From the third case we get the eigenvalues and eigenfunctions (homogeneous BVs):

$$\lambda_n=\frac{4n^2+1}{8},\, y_n=\mathrm{e}^{-\frac{1}{2}t}\sin\left(n\,t\right)$$

13 The wave equation

Represent e.g. vibrations of an elastic string.



$u(x,t)$ is the vertical displacement of a string at point x and time t.

Assume that the string vibrates in the vertical plane, with no damping.

The string has the properties ρ, the *linear density* of the string (mass/length) and T, the tension force. The wave equation is:

$$\left(\frac{T}{\rho}\right)u_{xx}=u_{tt}$$

The velocity, a, satisfies $a^2=T\,/\,\rho$.

BCs are $u(0,t)=u(L,t)=0$.

ICs are $u(x,0)=f(t), u_t(x,0)=g(x)$. (Initial position and initial velocity).

Now we examine different ICs.

13.1 Zero initial velocity

Initial velocity $g(x)\equiv0$.

We have

$$a^2\,u_{xx} = u_{tt}$$

BVs: $u(0,t) = u(L,t) = 0$. ICs: $u(x,0) = f(x), u_t(x,0) = 0$.

We find "fundamental solutions" satisfying the PDE, the homogeneous BVs and the homogeneous IC: $u_t = 0$.

We look for solutions of the form $X(x)\,T(t)$. We get

$$a^2\,X''\,T = X\,T''$$

Separate variables:

$$\frac{X''}{X} = \frac{T''}{a^2\,T} = -\lambda$$

We get:

$$\begin{aligned} X'' + \lambda\,X &= 0 \\ T'' + a^2\,\lambda\,T &= 0 \end{aligned}$$

BVs:

$$X(0)\,T(t) = X(L)\,T(t) \quad \forall t$$

As $T(t) \not\equiv 0$ we get

$$X(0) = X(L) = 0$$

This is an eigenvalue problem we've already solved for the heat equation. It has non-zero solutions only for

$$\lambda_n = \frac{n^2\,\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\,\pi\,x}{L}$$

Substitute these in the equation for $T(t)$.

$$T'' + \frac{a^2\,n^2\,\pi^2}{L^2}\,T = 0$$

General solution is

$$T(t) = c_1\cos\frac{n\,a\,\pi\,t}{L} + c_2\sin\frac{n\,a\,\pi\,t}{L}$$

We have $u_t(x,0) \equiv 0\ \forall x$ in this case.

$$u_t(x,0) = X(x)\,T'(0) = 0$$

As $X(x) \not\equiv 0$, $T'(0) = 0$.

$$T'(0) = c_2 \frac{n\,a\,\pi}{L} = 0 \rightarrow c_2 = 0$$

That's the only condition we have so c_1 is arbitrary. Choose a basis

$$T_n(t) = \cos\left(\frac{n\,a\,\pi}{L}\,t\right)$$

and get fundamental solutions

$$\boxed{u_n(x,t) = \sin\left(\frac{n\,\pi}{L}\,x\right)\cos\left(\frac{n\,a\,\pi}{L}\,t\right)}$$

u_n satisfies the following:

$$a^2\,u_{xx} = -a^2\,\frac{n^2\,\pi^2}{L^2}\sin\frac{n\,\pi\,x}{L}\cos\frac{n\,\pi\,a\,t}{L}$$

$$u_{tt} = -\sin\frac{n\,\pi\,x}{L}\cdot\frac{n^2\,\pi^2\,a^2}{L^2}\cdot\cos\frac{n\,\pi\,a\,t}{L}$$

Indeed the PDE is satisfied. Regarding BVs:

$$u_n(0,t) = u_n(L,t) = 0$$

$$u_t = \sin\frac{n\,\pi\,x}{L}\left(-\frac{n\,\pi\,a}{L}\right)\sin\frac{n\,\pi\,a\cdot 0}{L} = 0$$

The only thing u_n doesn't satisfy is $u(x,0) = f(x)$. We form a series:

$$u(x,t) = \sum c_n\,u_n(x,t)$$

and demand that $u(x,0) = f(x)$.

$$u(x,0) = \sum c_n \sin\frac{n\,\pi\,x}{L}$$

So c_n must be Fourier sine coefficients of $f(x)$.

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

One problem though: with the heat equation it can be shown that you can take the series and differentiate it, and the series converges. There, we had fundamental solutions that involved decaying exponents. Here, have cosines instead of exponents.

Note. The series $\sum c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L}$ **cannot** be differentiated twice (wrt either x or t) to get a convergent series.

We need to find a different representation of the solution in order to show that the wave equation holds. (Take care of that later.)

13.2 Zero initial position

$u(x,0) = f(x) \equiv 0 \ \forall x$ and $u_t(x,0) = g(x)$.

Again we look for fundamental solutions of the form $X(x)\,T(t)$ and get

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

$$T'' + a^2 \lambda T = 0$$

with $X(x)\,T(t) \equiv 0$, which implies $T(0) = 0$. The eigenvalue problem for $X(x)$ yields eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

and eigenfunctions

$$X_n(x) = \sin \frac{n\pi x}{L}$$

We solve for $T(t)$:

$$\begin{aligned} T(t) &= c_1 \cos \frac{n\pi a t}{L} + c_2 \sin \frac{n\pi a t}{L} \\ T(0) = 0 &\rightarrow c_1 = 0 \end{aligned}$$

Get basis for solutions

$$T_n(t) = \sin \frac{n\pi a t}{L}$$

and fundamental solutions:

$$u_n(x,t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}$$

u_n satisfies the PDE, the BCs and $u_n(x,0) \equiv 0$.

Form the series

$$u(x,t) = \sum k_n \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}$$

and demand

$$u_t(x,0) = g(x)$$

Get:

$$\sum k_n \sin\frac{n\,\pi\,x}{L}\cdot\left(\frac{n\,\pi\,a}{L}\right)\underbrace{\cos\frac{n\,\pi\,a\cdot 0}{L}}_{=1}=g(x)$$

So

$$\sum \left(k_n\cdot\frac{n\,\pi\,a}{L}\right)\sin\frac{n\,\pi\,x}{L}=g(x)$$

In other words, $k_n\cdot\frac{n\,\pi\,a}{L}$ is the sine Fourier coefficient of $g(x)$.

$$k_n\frac{n\,\pi\,a}{L}=\frac{2}{L}\int_0^L g(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

so

$$\boxed{k_n=\frac{2}{n\,\pi\,a}\int_0^L g(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x}$$

13.3 General case: arbitrary initial position and velocity

The general case is with ICs:

$$\begin{cases} u(x,0)=f(x) \\ u_t(x,0)=g(x)\end{cases},\quad x\in[0,L]$$

We use the previous cases: suppose

$$\begin{cases} v(x,t) \text{ solves case in section 13.1} \\ w(x,t) \text{ solves case in section 13.2}\end{cases}$$

Set $u(x,t)=v(x,t)+w(x,t)$. Then we have

$$a^2\,u_{xx}=a^2\,v_{xx}+a^2\,w_{xx}=v_{tt}+w_{tt}=u_{tt}$$

Wave equation is satisfied.

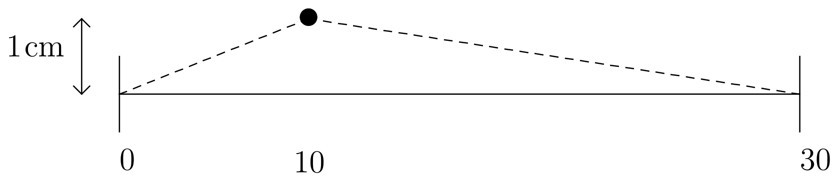
$$\begin{cases} u(0,t)=v(0,t)+w(0,t)=0 \\ u(L,t)=v(L,t)+w(L,t)=0\end{cases}$$

BCs are satisfied.

Regarding ICs:

$$\begin{cases} u(x,0)=v(x,0)+w(x,0)=f(x) \\ u_t(x,0)=v_t(x,0)+w_t(x,0)=g(x)\end{cases}$$

Example: Vibrating string of length 30 cm, $a=2$. We pluck at the point 10 cm from the left to a height of 1 cm and let go.



ICs and BCs:

$$u(x,0)=\begin{cases}\frac{x}{10} & x\in[0,10]\\ \frac{30-x}{20} & x\in(10,30]\end{cases}$$

$$\begin{cases}u_t(x,0)=0\\ u(0,t)=u(30,t)=0\end{cases}$$

The PDE is

$$4\,u_{xx}=u_{tt}$$

1. Initial zero velocity.

$$u(x,t)=\sum c_n\sin\frac{n\pi x}{30}\cos\frac{2n\pi t}{30}$$

lec 25 29.01.23 Calculate c_n

$$c_n=\frac{2}{30}\int_0^{30}f(x)\sin\frac{n\pi x}{30}\,\mathrm{d}x$$

$$A(x)\equiv\int x\sin\frac{n\pi x}{30}\,\mathrm{d}x=\frac{-30\,x\cos\frac{n\pi x}{30}}{n\pi}+\frac{900}{n^2\pi^2}\sin\frac{n\pi x}{30}$$

$$\begin{aligned}
c_n &= \frac{2}{30}\left[\frac{1}{10}A(x)|_0^{10}+\frac{1}{20}\int_{10}^{30}30\sin\frac{n\pi x}{30}\,\mathrm{d}x-\frac{1}{20}A(x)|_{10}^{30}\right]\\
&= \frac{2}{300}\left[A(10)+\frac{30}{2}\left(-\cos\frac{n\pi x}{30}\cdot\frac{30}{n\pi}\right)_{10}^{30}-\frac{1}{2}A(30)+\frac{1}{2}A(10)\right]\\
&= \frac{2}{300}\left(\left[\frac{3}{2}\,A(10)\,+\,\frac{900}{2\pi n}\,\left(-\cos\,(n\,\pi)\,+\,\cos\,\frac{n\,\pi}{3}\right)\,-\,\frac{1}{2}\,\left[-\frac{900}{n\,\pi}\,\cos\,(n\,\pi)\,+\right.\right.\right.\\
&\qquad \left.\left.\left.\frac{900}{n^2\pi^2}\sin\,(n\,\pi)\right]\right]\right)\\
&= \frac{2}{300}\left[\frac{3}{2}\left(-\frac{300}{n\pi}\cos\frac{n\pi}{3}+\frac{900}{n^2\pi^2}\sin\frac{n\pi}{3}\right)+\frac{900}{2\pi n}\cos\frac{n\pi}{3}\right]\\
&= \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}
\end{aligned}$$

So,

$$u(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\,\pi}{3}\right)\sin\frac{n\,\pi\,x}{30}\cos\frac{2\,n\,\pi\,t}{30}$$

Note. Differentiating the series on the right twice by x or by t, we get a series whose coefficients never approach zero as $n \to \infty$. In other words, the series' second derivative doesn't converge, and so can't be equal either u_{xx} or u_{tt}. We need another way to write the solution so we can differentiate.

Example: $a = 5$, string of length 3 cm, with BCs: $u(0,t) = u(3,t) = 0$ an ICs

$$\begin{cases} u(x,0) = \frac{1}{4}\sin(\pi\,x) \\ u_t(x,0) = 10\sin(2\,\pi\,x) \end{cases}$$

1. Solve for ICs:

$$\begin{cases} u(x,0) = \frac{1}{4}\sin(\pi\,x) \\ u_t(x,0) \equiv 0 \end{cases}$$

$$u_1(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\,\pi\,x}{3}\cos\frac{5\,n\,\pi\,t}{3}$$

$$A_n = \frac{2}{3}\int_0^3 \frac{1}{4}\sin(\pi\,x)\sin\frac{n\,\pi\,x}{3} = [\text{orthogonality}] = \begin{cases} 0, & n \neq 3 \\ \frac{1}{4}, & n = 3 \end{cases}$$

$$u_1(x,t) = \frac{1}{4}\sin(\pi\,x)\cos(5\,\pi\,t)$$

2. Solve for ICs:

$$\begin{cases} u(x,0) \equiv 0 \\ u_t(x,0) = 10\sin(2\,\pi\,x) \end{cases}$$

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\,\pi\,x}{3}\sin\frac{5\,n\,\pi\,t}{3}$$

$$B_n = \frac{2}{5\,n\,\pi}\int_0^3 10\sin(2\,\pi\,x)\cdot\sin\frac{n\,\pi\,x}{3} = \begin{cases} 0, & n \neq 6 \\ \frac{1}{\pi}, & n = 6 \end{cases}$$

So

$$u_2(x,t) = \frac{1}{\pi}\sin(2\,\pi\,x)\sin(10\,\pi\,t)$$

Solution to original problem is $u(x,t) = u_1 + u_2$.

$$u(x,t) = \frac{1}{4}\sin(5\pi t)\sin(\pi t) + \frac{1}{\pi}\sin(10\pi t)\sin(2\pi x)$$

13.4 D'Alembert solution to the wave equation

D'Alembert method preceded Fourier's method (1717–1783).

13.4.1 Zero initial velocity

The ICs are:

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) \equiv 0 \end{cases}, \quad x \in [0, L]$$

Theorem

Let $F(x)$ denote the odd extension of $f(x)$ to a periodic function of period $2L$ on the entire real line. Then

$$u(x,t) = \frac{1}{2}(F(x + a\,t) + F(x - a\,t))$$

is the solution to the wave equation $u_{tt} = a^2 u_{xx}$ with homogeneous BCs and ICs above.

- Show that the solution satisfies homogeneous BCs.

$$u(0,t) = \frac{1}{2}[F(a\,t) - F(-a\,t)] = [F \text{ odd}] = \frac{1}{2}[F(a\,t) - F(a\,t)] = 0$$

$$u(L,t) = \frac{1}{2}[F(L + a\,t) + F(L - a\,t)] = \frac{1}{2}[F(L + a\,t) - F(a\,t - L)]$$

  Using the fact that F is $2L$-periodic,

$$u(L,t) = \frac{1}{2}[F(L + a\,t) - F(2\,L + a\,t - L)] = \frac{1}{2}[F(L + a\,t) - F(L + a\,t)] = 0$$

- Show that the solution solves the wave equation.

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}[F'(x + a\,t)\cdot a + F'(x - a\,t)\cdot(-a)] \\ u_{tt}(x,t) &= \frac{1}{2}[F''(x + a\,t)\,a^2 + F''(x - a\,t)\,a^2] = \frac{a^2}{2}[F''(x + a\,t) + F''(x - a\,t)] = a^2 u_{xx} \end{aligned}$$

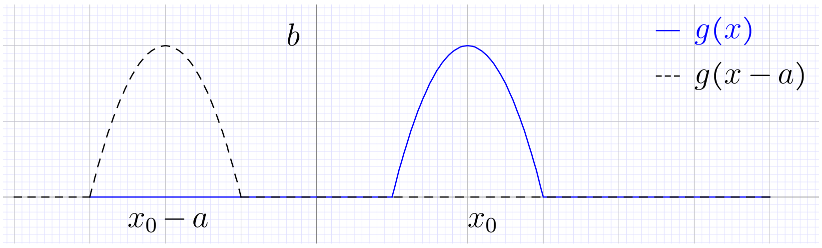
- Check ICs:

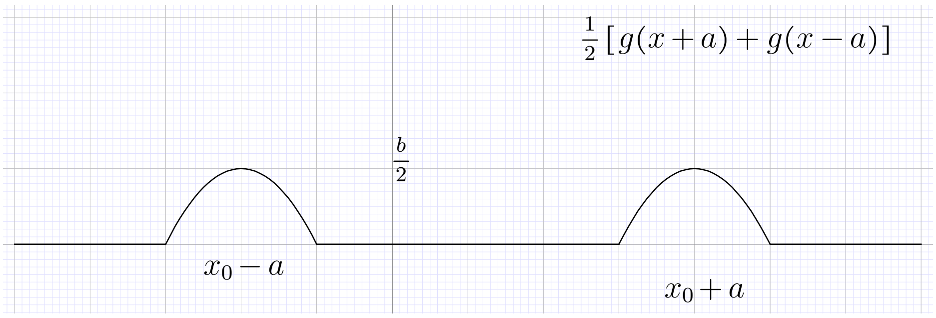
$$u(x,0)=\frac{1}{2}[F(x)+F(x)]=F(x)=f(x), x\in[0,L]$$

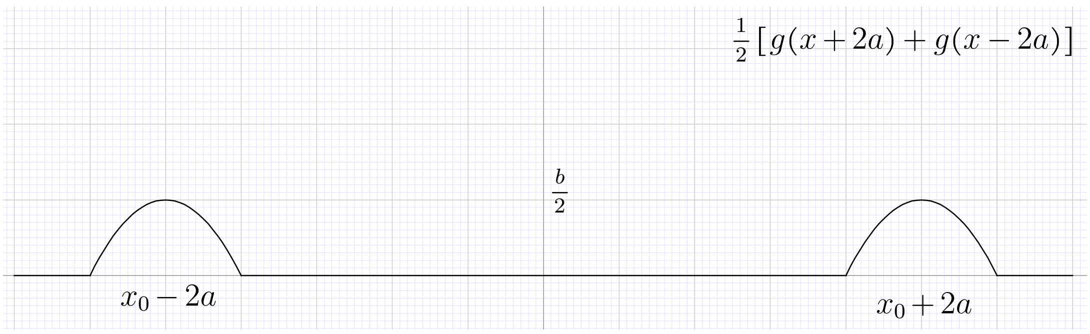
$$u_t(x,t)=\frac{a}{2}[F'(x+a\,t)-F(x'-a\,t)]$$

$$u_t(x,0)=\frac{a}{2}[F'(x)-F'(x)]=0$$

Why use D'alembert's version? Suppose $g(x)$ has a graph:







As we can see $u(x,t)$ is just a sum of waves that get further and further apart. (Similar to the trajectory of a pebble thrown flatly above water.)

Recall example

$$25\,u_{xx}=u_{tt}$$

$$u(0,t)=u(3,t)=0$$

$$\begin{cases} u(x,0)=\frac{1}{4}\sin(\pi\,x) \\ u_t(x,0)\equiv 0 \end{cases}$$

Solution we had was $u(x,t)=\frac{1}{4}\sin(\pi x)\cos(5\pi t)$.

In this case the odd periodic extension of $f(x)=\frac{1}{4}\sin(\pi x)$ on $[0,3]$ is $F(x)=\frac{1}{4}\sin(\pi x)$ on $\mathbb{R}$. The D'Alembert solution gives:

$$\begin{aligned}
u(x,t) &= =\frac{1}{2}[F(x+a\,t)+F(x-a\,t)] \\
&= \frac{1}{2}\left[\frac{1}{4}\sin(\pi(x+a\,t))+\frac{1}{4}\sin(\pi(x-a\,t))\right] \\
&= \frac{1}{8}[\sin(\pi x)\cos(\pi a\,t)+\sin(a\,t)\cos(\pi x)] \\
&\quad +\frac{1}{8}[\sin(\pi x)\cos(\pi a\,t)-\sin(a\,t)\cos(\pi x)] \\
&= \frac{1}{4}\sin(\pi x)\cos(\pi a\,t)
\end{aligned}$$

Fourier series solution and D'Alembert solution give the same result.

Why is the D'Alembert solution equal to the Fourier series solution in general? (If so, then the Fourier series solution **is** the solution to the wave equation, because we verified that D'Alembert solution works.)

Still assuming $u_t\equiv 0$ and $u(x,0)=f(x)$ on $[0,L]$. The Fourier series solution:

$$w(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a\,t}{L}$$

c_n = Fourier sine coefficients for f so that $f(x)=\sum c_n\sin\frac{n\pi x}{L}$ on $[0,L]$ and RHS is a function defined for all $x\in\mathbb{R}$ and is odd! So RHS = $F(x)$ and is $2L$-periodic.

Note. $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$, so that

$$w(x,t)=\sum c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi a\,t}{L}$$

Denote $\alpha=\frac{n\pi x}{L}$ and $\beta=\frac{n\pi a\,t}{L}$. Then,

$$\begin{aligned}
w(x,t) &= \frac{1}{2}\left[\sum c_n\left(\sin\left(\frac{n\pi x}{L}+\frac{n\pi a\,t}{L}\right)+\sin\left(\frac{n\pi x}{L}-\frac{n\pi a\,t}{L}\right)\right)\right] \\
&= \frac{1}{2}\sum c_n\left(\sin\frac{n\pi}{L}(x+a\,t)+\sin\frac{n\pi}{L}(x-a\,t)\right) \\
&= \frac{1}{2}[F(x+a\,t)+F(x-a\,t)]
\end{aligned}$$

which is D'Alembert solution.

13.4.2 Zero initial position

$$a^2u_{xx}=u_{tt}$$

BCs: $u(0,t)=u(L,t)=0$ and ICs:

$$\begin{cases} u(x,0)\equiv 0 \\ u_t(x,0)=g(x) \end{cases}, x\in[0,L]$$

Let $G(x)$ be the odd period-$2L$ extension of $g(x)$ and let $H(x)$ be the *primitive function* of $G(x)$, i.e

$$H(x)=\int_0^x G(\xi)\,d\xi$$

so $H'(x)=G(x)$. The solution to the above wave equation for this case is given by:

$$u(x,t)=\frac{1}{2a}\left[H(x+a\,t)-H(x-a\,t)\right]$$

It can be shown that

$$u_{tt}(x,t)=\frac{a^2}{2a}\left[H''(x+a\,t)-H''(x-a\,t)\right]=a^2\,u_{xx}$$

Check ICs:

$$u(x,0)=\frac{1}{2a}\left[H(x)-H(x)\right]=0$$

$$\begin{aligned} u_t(x,t) &= \frac{1}{2a}\cdot a\left[H'(x+a\,t)+H'(x-a\,t)\right] \\ &= \frac{1}{2}\left[H'(x+a\,t)+H'(x-a\,t)\right] \end{aligned}$$

Remember that $H'(x)=G(x)$.

$$u_t(x,0)=\frac{1}{2}\left[G(x)+G(x)\right]=G(x)=g(x)\text{ on }[0,L]$$

Check BCs.

$$u(0,t) = \frac{1}{2a}\left[H(a\,t)-H(-a\,t)\right]$$

Because G is odd H is even. So

$$u(0,t)=\frac{1}{2a}\left[H(a\,t)-H(a\,t)\right]=0$$

Similarly, using periodicity, $u(L,t)=0$.

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14 Green's function

14.1 Introduction—Heat conduction on an infinite rod

Green's function can be used to solve many PDEs. Look at the heat equation

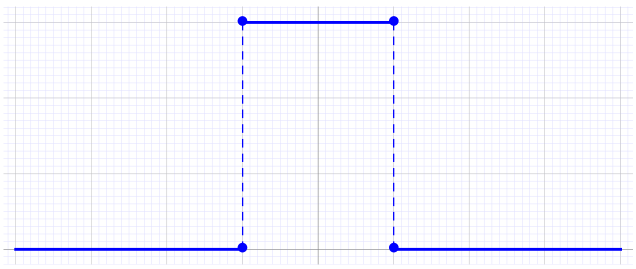
$$\alpha^2 u_{xx} = u_t$$

on an infinite rod, i.e. $x \in (-\infty, \infty)$. BCs are: $\lim_{x\to\infty} u(x,t) = \lim_{x\to-\infty} u(x,t) = 0$. ICs are: $u(x,0) = f(x)$, $x \in \mathbb{R}$.

Our previous method using the orthogonal family $\{\sin\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\}$ in space of periodic functions will not work. The method of Green's function uses one special function that can be used to compute a solution for a given IC.

We first need to introduce a generalized function (or distribution) called *the Dirac delta function*, also called the Impulse function: $\delta(x)$. We define $\delta(x)$ as the limit of a sequence of functions $\delta_n(x)$ such that

$$\delta_n(x) = \begin{cases} n, & x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right] \\ 0 & \text{otherwise} \end{cases}$$



Properties of $\delta_n(x)$:

1. $\delta_n(x) \geq 0 \quad \forall x, n$
2. $\lim_{n\to\infty} \delta_n(x) = 0 \quad \forall x \neq 0$
3. $\lim_{n\to\infty} \delta_n(0) = \infty$
4. $\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{d}x = 1 \quad \forall n$

We define $\delta(x)$ as the limit $\lim_{n\to\infty} \delta_n(x)$, meaning infinite impulse at $x = 0$.

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

We can shift $\delta(x)$ so that the spike is at a point ξ by taking

$$\delta(x-\xi) = \lim_{n\to\infty} \delta_n(x-\xi)$$

Definition

The Green function (also called propagator) for the heat equation with the given boundary conditions, $G(x,\xi,t)$, is the solution to the heat equation & BCs, where the IC is: $u(x,0)=\delta(x-\xi)$.

Essentially, the IC means infinite heat at $x=\xi$ and 0 elsewhere.

$$G(x,\xi,t)=\frac{1}{2\alpha\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4\alpha^2 t}}$$

Note. We have t in the denominator of G, thus G is not at $t=0$. However, it can be extended continuously to $t=0$.

We need to verify 3 things: G satisfies the heat equation, $\lim_{t\to 0}G(x,\xi,t)=\delta(x,\xi)$ and that it satisfies the BCs.

- Check that $\alpha^2 G_{xx}=G_t$

$$\begin{aligned}\frac{\partial G}{\partial t} &= -\frac{1}{2t\sqrt{t}}\cdot\frac{\mathrm{e}^{-\frac{(x-\xi)^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi}}+\frac{1}{2\alpha\sqrt{\pi t}}\cdot\frac{-(x-\xi)^2}{4\alpha^2(-t^2)}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4\alpha^2 t}}\\ &= \frac{1}{2t}\cdot G+\frac{(x-\xi)^2}{4\alpha^2 t^2}\cdot G\end{aligned}$$

.

$$\begin{aligned}\frac{\partial G}{\partial x} &= \frac{-2(x-\xi)}{4\alpha^2 t}\cdot G\\ \frac{\partial^2 G}{\partial x^2} &= -\frac{1}{2\alpha^2 t}\cdot G+\frac{(x-\xi)^2}{4\alpha^4 t^2}\cdot G\end{aligned}$$

Indeed, $G(x,\xi,t)$ solves the heat equation.

- Check BCs. As $\exp\left(-\frac{(x-\xi)^2}{4\alpha^2 t}\right)\to 0$ as $x\to\pm\infty$, we have:

$$\lim_{x\to\infty} G(x,\xi,t)=0=\lim_{x\to-\infty} G(x,\xi,t)$$

- It remains to show that $G(x,\xi,0)=\delta(x-\xi)$. (ICs.) And for that we need to show that $G(x,\xi,t)$ has a limit as $t\to 0$ and that this limit equals $\delta(x,\xi)$.

$$\begin{aligned}\lim_{t\to 0}\frac{1}{2\alpha^2\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(x-\xi)^2}{4\alpha^2 t}} &\overset{t\to 1/s}{=} \lim_{s\to\infty}\frac{\sqrt{s}}{2\alpha^2\sqrt{\pi}}\,\mathrm{e}^{-\frac{(x-\xi)^2 s}{4\alpha^2}}\\ &= \lim_{s\to\infty}\frac{1}{2\alpha^2\sqrt{\pi}}\cdot\frac{\sqrt{s}}{\mathrm{e}^{\frac{(x-\xi)^2 s}{4\alpha^2}}}\end{aligned}$$

By L'hopital rule we get

$$\lim_{s\to\infty} \frac{\sqrt{s}}{\mathrm{e}^s} = 0$$

So

$$\lim_{s\to\infty} \frac{1}{2\alpha^2\sqrt{\pi}} \cdot \frac{\sqrt{s}}{\mathrm{e}^{\frac{(x-\xi)^2 s}{4\alpha^2}}} = 0, \quad x \neq \xi$$

When $x = \xi$ we get

$$\lim_{t\to 0} G(\xi, \xi, t) = \infty$$

so that

$$\lim_{t\to 0} G(x, \xi, t) = \begin{cases} 0, & x \neq \xi \\ \infty, & x = \xi \end{cases} = \delta(x - \xi)$$

How do we use Green's functions in the general problem? (With "true" ICs.)

14.2 Solving the general case for the heat equation

Theorem

Given Our BVP with IC $u(x, 0) = f(x)$ for $f(x)$ continuous, the solution is given by:

$$u(x, t) = \int_{-\infty}^{\infty} G(x, \xi, t)\, f(\xi)\, \mathrm{d}\xi$$

Note. In most cases this will be a difficult integral, but it can be approximated numerically.

Verify that $u(x, t)$ is indeed the solution.

- Show that $u(x, t)$ satisfies the heat equation.

$$\alpha^2 u_{xx} = \alpha^2 \cdot \frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{\infty} G(x, \xi, t)\, f(\xi)\, \mathrm{d}\xi\right]$$

Use the fact that $\frac{\partial^2}{\partial x^2}\int (\cdot) = \int \frac{\partial^2}{\partial x^2}(\cdot)$.

$$\begin{aligned} \alpha^2 u_{xx} &= \alpha^2 \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}(G(x, \xi, t))\, f(\xi)\, \mathrm{d}\xi = \alpha^2 \int_{-\infty}^{\infty} G_{xx}\, f(\xi)\, \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} G_t\, f(\xi)\, \mathrm{d}\xi = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} G\, f(\xi)\, \mathrm{d}\xi = u_t \end{aligned}$$

- Show that BCs are satisfied.

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \int_{-\infty}^{\infty} G(x,\xi,t)\, f(\xi)\, \mathrm{d}\xi = \int_{-\infty}^{\infty} \Big(\lim_{x\to\infty} G(x,\xi,t)\Big)\, f(\xi)\, \mathrm{d}\xi$$

We already calculated the limit and got $\lim_{x\to\infty} G(x,\xi,t) = 0$, so

$$\lim_{x\to\infty} u(x,t) = 0$$

- Show that ICs are satisfied. To show that $u(x,0) = f(x)$ we need to show a property of the delta function: The *sifting* property of $\delta(x)$.

> The *sifting* property of the Dirac delta function. For all x and continuous $f(x)$:
>
> $$\int_{-\infty}^{\infty} \delta(x-\xi)\, f(\xi)\, \mathrm{d}\xi = f(x)$$
>
> The integral is equivalent to the convolution of δ and f, denoted $\delta * f$.

Proof. We calculate

$$\int_{-\infty}^{\infty} \delta_n(x-\xi)\, f(\xi)\, \mathrm{d}\xi = \int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} n\cdot f(\xi)\, \mathrm{d}\xi = n\cdot \int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} f(\xi)\, \mathrm{d}\xi$$

By the *integral intermediate value* theorem (for continuous functions), there exists a point $c\in[a,b]$ such that

$$\int_a^b f(\xi)\, \mathrm{d}\xi = f(c)\cdot(b-a)$$

So we have $x_n \in \left[x - \frac{1}{2n}, x + \frac{1}{2n}\right]$ such that

$$\int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} f(\xi)\, \mathrm{d}\xi = f(x_n)\cdot\frac{1}{n}$$

So:

$$\int_{-\infty}^{\infty} \delta_n(x-\xi)\, f(\xi)\, \mathrm{d}\xi = f(x_n) \xrightarrow{n\to\infty} f(x)$$

Conclude that

$$\int_{-\infty}^{\infty} \delta(x-\xi)\, f(\xi)\, \mathrm{d}\xi = f(x)$$

□

We now use this to show that $u(x,0)=f(x)$:

$$u(x,0)=\int_{-\infty}^{\infty} G(x,\xi,0)\,f(\xi)\,\mathrm{d}\xi=\int_{-\infty}^{\infty}\delta(x-\xi)\,f(\xi)\,\mathrm{d}\xi=f(x)$$

This method can be used for many PDEs!

The Fourier series method can also be used for many PDEs, where we have finite conditions, e.g. 2-dimensional heat equation, Laplace's equation etc.

Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

on a rectangle. Equation can be solved by separation of variables, fundamental solutions method etc.

Note. There are a class of PDEs of the form: $u(x,t)=u$

$$r(x)\,u_t=(p(x)\,u_x)_x-q(x)\,u$$

If p,r are positive contstant functions and $q\equiv 0$ this is $r\,u_t=p\,u_{xx}$, which is the heat equation!

There are also BVPs of the form:

$$a_1\,u(0,t)+a_2\,u_x(0,t)=0$$

$$b_1\,u(L,t)+b_2\,u_x(L,t)=0$$

This is called a *Strum-Liouville problem*. Can be solved by separation of variables by finding solutions of the form $u(x,t)=X(x)\cdot T(t)$.