HW3

Question 1. $\vec{w} = (-3, 4, 7, 1, 2, -9)$.

1. $\vec{x} = (9, 21, 3, 6, -4, 15)$.

$$\begin{aligned} \vec{w} \cdot \vec{x} &= (-3, 4, 7, 1, 2, -9) \cdot (9, 21, 3, 6, -4, 15) \\ &= -27 + 84 + 21 + 6 - 8 - 135 \\ &= -59. \end{aligned}$$

2. $\vec{x} = (2, 0, 0, -13, -18, 6)$.

$$\begin{aligned} \vec{w} \cdot \vec{x} &= (-3, 4, 7, 1, 2, -9) \cdot (2, 0, 0, -13, -18, 6) \\ &= -6 - 13 - 36 - 54 \\ &= -109. \end{aligned}$$

Question 2.

1.

$$A v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 + 2 \\ -9 + 4 \\ 36 + 6 + 12 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 54 \end{bmatrix}$$

2.

$$v^T v = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = 16 + 9 + 4 = 29$$

3.

$$v v^T = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -12 & 8 \\ -12 & 9 & -6 \\ 8 & -6 & 4 \end{bmatrix}$$

4.

$$v^T A v = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \\ 54 \end{bmatrix} = 115$$

5.

$$\begin{aligned} A^T A &= \begin{bmatrix} -1 & 0 & 9 \\ 0 & 3 & -2 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (1 + 81) & (-18) & (-1 + 54) \\ (-18) & (9 + 4) & (6 - 12) \\ (-1 + 54) & (6 - 12) & (1 + 4 + 36) \end{bmatrix} = \begin{bmatrix} 82 & -18 & 53 \\ -18 & 13 & -6 \\ 53 & -6 & 41 \end{bmatrix} \end{aligned}$$

6.

$$\begin{aligned} 2AB &= 2\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= 2\begin{bmatrix} (-2) & (-7) & (1) & (-4+1) \\ (0) & (9) & (6) & (12+2) \\ (18) & (63-6) & (-9-4) & (36-8+6) \end{bmatrix} = \begin{bmatrix} -4 & -14 & 2 & -6 \\ 0 & 18 & 12 & 28 \\ 36 & 114 & -26 & 68 \end{bmatrix} \end{aligned}$$

7.

$$AD = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ 0 & -9 & 4 \\ 36 & 6 & 12 \end{bmatrix}$$

8.

$$DA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 4 \\ 0 & -9 & -6 \\ 18 & -4 & 12 \end{bmatrix}$$

9.

$$B^T A = \begin{bmatrix} 2 & 0 & 0 \\ 7 & 3 & 0 \\ -1 & 2 & 0 \\ 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & 2 \\ 9 & -2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -7 & 9 & (7+6) \\ 1 & 6 & (-1+4) \\ (-4+9) & (12-2) & (4+8+6) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -7 & 9 & 13 \\ 5 & 10 & 18 \end{bmatrix}$$

10.

$$\begin{aligned} v^T B u &= \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 7 & -1 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 3 \\ -4 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 18+21+4-8 \\ 9-8-8 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 35 \\ -7 \\ -2 \end{bmatrix} \\ &= 140+21-4 = 157 \end{aligned}$$

11.

$$v^T D v = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 16 \\ 9 \\ 4 \end{bmatrix} = 64-27+8 = 45$$

12. Calculate via Gaussian eliminatoin.

$$[D \mid I] = \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] = [I \mid D^{-1}].$$

$$D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Question 3.

1. By definition, for square $n \times n$ matrices A, B, we have

$$(A\,B)^T_{ji} = (A\,B)_{ij} = \sum_{k=1}^{n} A_{ik}\,B_{kj}\,.$$

On the other hand,

$$(B^T A^T)_{ji} = \sum_{k=1}^{n} B^T_{jk}\,A^T_{ki},$$

and using $B^T_{jk} = B_{kj}$ and $A^T_{ki} = A_{ik}$,

$$\sum_{k=1}^{n} B^T_{jk}\,A^T_{ki}, = \sum_{k=1}^{n} B_{kj}\,A_{ik} = \sum_{k=1}^{n} A_{ik}\,B_{kj} = (A\,B)_{ij}\,.$$

By transitivity, we get that each element in $(A\,B)^T$ equals each element in $(B^T A^T)$. We can conclude then that $(A\,B)^T = (B^T A^T)$.

2. A matrix A is symmetric if $A = A^T$. Let $B \equiv A\,A^T$.

$$B^T = (A\,A^T)^T = (A^T A)\,.$$

Since $A = A^T$, we can write

$$B^T = (A\,A^T) \equiv B\,.$$

B is symmetric. Let $C \equiv (A + A^T)$.

$$C^T = (A + A^T)^T = (A^T + A)\,.$$

Because matrix addition is commutative, we have

$$C^T = (A + A^T) \equiv C\,.$$

C is symmetric.

3. A is invertible. Therefore, there exists A^{-1} such that

$$A\,A^{-1} = I, \qquad A^{-1} A = I\,.$$

Transpose both sides of the expression:

$$(A\,A^{-1})^T = I^T = I = (A^{-1})^T A^T\,.$$

Since A is symmetric,

$$(A^{-1})^T A^T = (A^{-1})^T A = I\,.$$

Multiply on the right by A^{-1} to get

$$(A^{-1})^T A\,A^{-1} = (A^{-1})^T = A^{-1}\,.$$

We thus proved that if A is symmetric then also its inverse is symmetric.

4. Let A, B be symmetric matrices, i.e. $A^T = A$, $B^T = B$. We have:

$$(A\,B)^T = B^T A^T = B\,A,$$

$$(B\,A)^T = A^T B^T = A\,B.$$

Because $(A\,B)^T = (B\,A)$, if $(A\,B)$ is symmetric then $(B\,A)$ must also be symmetric. If $(B\,A)$ is symmetric then

$$(B\,A)^T = B\,A,$$

and

$$(A\,B)^T = B\,A = (B\,A)^T = A\,B.$$

In conclusion, $(A\,B)$ is symmetric if and only if $A\,B = B\,A$.

Question 4. Let there be two matrices A, B such that

$$A\,A^{-1} = I, \quad A^{-1}A = I, \quad B\,B^{-1} = I, \quad B^{-1}B = I$$

The product $A\,B$ is invertible if and only if there is a matrix C such that

$$(A\,B)\,C = C\,(A\,B) = I.$$

We show that C must be $B^{-1}A^{-1}$.

$$A\,B\,B^{-1}A^{-1} = A\,I\,A^{-1} = A\,A^{-1} = I,$$

and

$$B^{-1}A^{-1}A\,B = B^{-1}I\,B = B^{-1}B = I.$$

We thus proved that $A\,B$ is invertible, and the matrix C satisfies: $C = (A\,B)^{-1} = B^{-1}A^{-1}$.

Question 5.

1.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 4 & 6 \\ 4 & -5 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 6 \\ 0 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$, as it's row-echelon form has 2 non-zero rows.

2.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 6 & 6 \\ 4 & -5 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 8 & 11 \\ 0 & -13 & -26 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 8 & 11 \\ 0 & 0 & -\frac{65}{8} \end{bmatrix}$$

rank$(A) = 3$, as it's row-echelon form has 3 non-zero rows.

3.

$$A = \begin{bmatrix} 1 & 4 & -3 & 2 \\ 1 & 2 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 3 & -1 & -1 \\ 0 & 5 & -3 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 8 & -4 & 0 \\ 0 & 0 & -18 & -24 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

rank$(A) = 4$, as it's row-echelon form has 4 non-zero rows.