2021 Exam Solution

Question 1

(a) Find the general solution to the following DE:

$$y'' + 2y' - 8y = 3\sin x - 14\,e^{2x}$$

Solve associated homogeneous equation. Characteristic equation is:

$$\lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2) = 0$$

Solution to homogeneous equation is:

$$y_h = c_1\,e^{-4x} + c_2\,e^{2x}$$

Find particular solution for each part of the RHS. For the sine, guess $y_{p1} = A\cos x + B\sin x$. Input in ODE:

$$-A\cos x - B\sin x - 2A\sin x + 2B\cos x - 8A\cos x - 8B\sin x = 3\sin x$$

$$\cos x\,(-A + 2B - 8A) + \sin x\,(-B - 2A - 8A) = 3\sin x$$

Equate coefficients on both sides:

$$\begin{cases} \cos x\colon & 2B = 9A \to B = \frac{9}{2}A \\ \sin x\colon & -B - 10A = 3 \to A = -\frac{6}{29}, B = -\frac{27}{29} \end{cases}$$

So:

$$y_{p1} = -\frac{6}{29}\cos x - \frac{27}{29}\sin x$$

For the exponent, guess $y_{p2} = A\,x\,e^{2x}$.

$$\begin{aligned} y_{p2}' &= 2A\,x\,e^{2x} + A\,e^{2x} \\ y_{p2}'' &= 4A\,x\,e^{2x} + 4A\,e^{2x} \end{aligned}$$

Input in the ODE:

$$e^{2x}\,[4A\,x + 4A + 4A\,x + 2A - 8A\,x] = -14\,e^{2x}$$

Divide by $e^{2x} \not\equiv 0$

$$x\,(4A + 4A - 8A) + (4A + 2A) = -14$$

Therefore:

$$y_{p2} = -\frac{7}{3}x\,e^{2x}$$

The general solution to the ODE is:

$$y = y_h + y_{p1} + y_{p2} = c_1\,e^{-4x} + c_2\,e^{2x} - \frac{6}{29}\cos x - \frac{27}{29}\sin x - \frac{7}{3}x\,e^{2x}$$

(b) Find all solutions to the differential equation:

$$y \cdot y'' - (y')^3 = 0$$

where y is assumed to be a function of t.

Set $y' = v$, $y'' = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v$.

$$y \cdot v \cdot \frac{\mathrm{d}v}{\mathrm{d}y} = v^3$$

Note that $v \equiv 0$, i.e. $y = c \in \mathbb{R}$, solves the DE. Divide by v on some interval where $v \neq 0$ to obtain a separable DE:

$$\int \frac{\mathrm{d}v}{v^2} = \int \frac{\mathrm{d}y}{y}$$

$$-\frac{1}{v} = \ln|y| + c_1$$

$$y' = -\frac{1}{\ln|y| + c_1}, \quad c_1 \neq -\ln|y|$$

If $c_1 = \ln|y|$ then there is no solution. This is another separable equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{\ln|y| + c_1}$$

$$\int (\ln|y| + c_1)\,\mathrm{d}t = -\int \mathrm{d}t$$

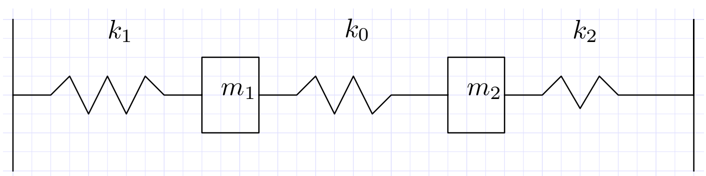
$$y \cdot (\ln|y| + c_1 - 1) = -t + c_2$$

The solution is given in implicit form.

Question 2

In a mass-and-spring system, assume that $m_1 = 1$, $m_2 = 2$, $k_0 = 2$, $k_1 = k_2 = 4$.

Let x_1 be the horizontal displacement of the first mass and let x_2 be the horizontal displacement of the second mass.



(a) Construct a system of DEs whose solution gives x_1 and x_2.

Equation of motion for each mass:

$$\begin{aligned} m_1\, x_1'' &= k_0\,(x_2 - x_1) - k_1\, x_1 \\ m_2\, x_2'' &= k_0\,(x_2 - x_1) - k_2\, x_2 \end{aligned}$$

Insert coefficients:

$$\begin{aligned} x_1'' &= x_1 \cdot (-2-4) + x_2 \cdot 2 \\ 2\,x_2'' &= x_1 \cdot (-2) + x_2 \cdot (2-4) \end{aligned}$$

$$\begin{aligned} x_1'' &= -6x_1 + 2x_2 \\ x_2'' &= -x_1 - x_2 \end{aligned}$$

In matrix form:

$$\vec{x}'' = A\,\vec{x}$$

where:

$$A = \begin{bmatrix} -6 & 2 \\ -1 & -1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) Derive from it a differential equation of order 4, and find the general solution.

From the second equation:

$$x_1 = -x_2'' - x_2$$

Input this expression in the first equation:

$$-x_2^{(4)} - x_2'' = 6x_2'' + 6x_2 + 2x_2$$

$$x_2^{(4)} + 7x_2'' + 8x_2 = 0$$

Solve the system of equations using the matrix form. Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda+6)(\lambda+1) + 2 = \lambda^2 + 7\lambda + 8$$

$$\lambda_{1,2} = \frac{-7 \pm \sqrt{17}}{2} = -\frac{7}{2} \pm \frac{\sqrt{17}}{2}$$

Find eigenvectors. For $\lambda_1 = -\frac{7}{2} + \frac{\sqrt{17}}{2}$, find $\vec{v}_1$ such that

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$A - \lambda_1 I = \begin{bmatrix} -\frac{5}{2} - \frac{\sqrt{17}}{2} & 2 \\ -1 & -\frac{9}{2} - \frac{\sqrt{17}}{2} \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} -\frac{9}{2} - \frac{\sqrt{17}}{2} \\ 1 \end{bmatrix}$$

For $\lambda_2 = -\frac{7}{2} - \frac{\sqrt{17}}{2}$, find $\vec{v}_2$ such that

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0}$$

$$A - \lambda_2 I = \begin{bmatrix} -\frac{5}{2} + \frac{\sqrt{17}}{2} & 2 \\ -1 & -\frac{9}{2} + \frac{\sqrt{17}}{2} \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} -\frac{9}{2} + \frac{\sqrt{17}}{2} \\ 1 \end{bmatrix}$$

General solution to the DE is therefore:

$$\vec{x} = c_1 \, \mathrm{e}^{\left(-\frac{7}{2} + \frac{\sqrt{17}}{2}\right) t} \cdot \begin{bmatrix} -\frac{9}{2} - \frac{\sqrt{17}}{2} \\ 1 \end{bmatrix} + c_2 \, \mathrm{e}^{\left(-\frac{7}{2} + \frac{\sqrt{17}}{2}\right) t} \cdot \begin{bmatrix} -\frac{9}{2} + \frac{\sqrt{17}}{2} \\ 1 \end{bmatrix}$$

where $c_1, c_2 \in \mathbb{R}$, $t > 0$.

Question 3

The diagram shows a 3-tank system, with salt-water in each tank. Fresh water is flowing into the top tank at the rate of 10 L/min, and slat-water is flowing out of tank 1 into tank 2, and also out of tanks 2 and 3, at the rate of 10 L/min.

At time $t = 0$, tank 1 contains 20L of water and 5kg of salt, tank 2 contains 40L of water and 3kg of salt, and tank 3 contains 50L of water and no salt.

Find the amount of salt in each tank at time t.

Because each tanks receives 10L/min and also drops 10L/min, the volume of each tank remains constant. Write DEs representing the change in salt amount in each tank.

$$\begin{aligned} x_1' &= -r \cdot \frac{x_1}{V_1} \\ x_2' &= r \cdot \frac{x_1}{V_1} - r \cdot \frac{x_2}{V_2} \\ x_3' &= r \cdot \frac{x_2}{V_2} - r \cdot \frac{x_3}{V_3} \end{aligned}$$

In matrix form (after inputting coefficients):

$$\vec{x}' = A \, \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

Find eigenvalues and eigenvectors of A.

$$\det (A - \lambda I) = \begin{vmatrix} -\frac{1}{2} - \lambda & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} - \lambda & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} - \lambda \end{vmatrix} = -\left(\lambda + \frac{1}{2}\right)\left[-\left(\lambda + \frac{1}{4}\right) \cdot \left(-\left(\lambda + \frac{1}{5}\right)\right) - 0\right] = 0$$

Roots are:

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = -\frac{1}{4}, \lambda_3 = -\frac{1}{5}$$

Find corresponding eigenvectors. For $\lambda_1 = -\frac{1}{2}$:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$$

s.t. $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$.

For $\lambda_2 = -\frac{1}{4}$:

$$A - \lambda_2 I = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{20} \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$$

s.t. $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$.

For $\lambda_3 = -\frac{1}{5}$:

$$A - \lambda_3 I = \begin{bmatrix} -\frac{3}{10} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

Pick

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

s.t. $(A - \lambda_3 I)\vec{v}_3 = \vec{0}$.

General solution to the system of DEs is:

$$\vec{x} = c_1 \mathrm{e}^{-\frac{1}{2}t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + c_2 \mathrm{e}^{-\frac{1}{4}t} \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + c_3 \cdot \mathrm{e}^{-\frac{1}{5}t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find c_1, c_2, c_3 that satisfy the ICs.

$$\vec{x}(0) = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3c_1 \\ -6c_1 - c_2 \\ 5c_1 + 5c_2 + c_3 \end{bmatrix} \rightarrow c_1 = \frac{5}{3}, c_2 = -13, c_3 = \frac{170}{3}$$

The amount of salt in each tank at time t is given by:

$$\vec{x} = \frac{5}{3} \mathrm{e}^{-\frac{1}{2}t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} - 13\, \mathrm{e}^{-\frac{1}{4}t} \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + \frac{170}{3} \cdot \mathrm{e}^{-\frac{1}{5}t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Question 4

A pandemic is spreading through the population and is passed from those who have been infected with the virus and those who have not. We assume here that anyone who has been infected continues to be infectious indefinitely. Assume the proportion of those who have not had the disease is x, and those who are infected as y. So $x+y=1$. The virus spreads by contacs and the rate of spread $\mathrm{d}y/\mathrm{d}t$ is proportionate to $x\,y$.

(a) Construct a DE in y that models this situation.

$$\frac{\mathrm{d}y}{\mathrm{d}t}=r\,x\,y=r\,(1-y)y, \quad r\in\mathbb{R}$$

(b) Find the equilibrium points for the ODE you found in part (a) and determine for each if it is stable or unstable.

There are two equilibrium points, for which $\frac{\mathrm{d}y}{\mathrm{d}t}=0$: $y=1$ and $y=0$. Using logic, both equilibrium points are stable, because:

1. If $y=0$, i.e. the population is fully infected, then there is no one to infect anymore. Since no one can become uninfected, the infected population becomes stable.

2. If $y=1$. i.e. no one is infected, then the virus can't spread to anyone.

(c) Solve the DE for $y(0)=y_0>0$. Verify that as $t\to\infty$, y tends to the stable equilibrium that you found in part (b).

This is a separable equation:

$$\int\frac{\mathrm{d}y}{y(1-y)}=r\int\mathrm{d}t$$

Simplify:

$$\frac{1}{y(1-y)}=\frac{A}{y}+\frac{B}{1-y}$$

$$A-A\,y+B\,y=1$$

$$A=B=1$$

Therefore,

$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)\mathrm{d}y=r\,t+c$$

$$\ln|y|-\ln|1-y|=r\,t+c$$

$$\ln\left|\frac{y}{1-y}\right|=r\,t+c$$

$$\frac{y}{1-y}=c^\star\,\mathrm{e}^{rt}$$

$$y=c^\star\,\mathrm{e}^{rt}-y\cdot c^\star\,\mathrm{e}^{rt}$$

$$y(1+c^\star\,\mathrm{e}^{rt})=c^\star\,\mathrm{e}^{rt}$$

$$y=\frac{c^\star\,\mathrm{e}^{rt}}{1+c^\star\,\mathrm{e}^{rt}}=\frac{c^\star}{\mathrm{e}^{-rt}+c^\star}$$

Given $y(0)=y_0$,

$$\frac{y_0}{1-y_0}=c^\star$$

Therefore,

$$y=\frac{\frac{y_0}{1-y_0}}{\mathrm{e}^{-rt}+\frac{y_0}{1-y_0}}=\frac{y_0}{(1-y_0)\mathrm{e}^{-rt}+y_0}$$

Indeed, as $t\to\infty$ $y\to 1$.

Question 5

For the following BVP:

$$y''+\lambda\, y=0, \quad y(\pi)=y(-\pi), \quad y'(\pi)=y'(-\pi)$$

(a) Show that the BVP has no negative eigenvalues. Characeteristic equation is:

$$u^2+\lambda=0$$

If λ were negative then if $\mu\equiv-\lambda$ then

$$u=\pm\sqrt{\mu}, \quad \sqrt{\mu}\in\mathbb{R}$$

and the solution is:

$$y=c_1\,\mathrm{e}^{-\sqrt{\mu}t}+c_2\,\mathrm{e}^{\sqrt{\mu}t}$$

Check if boundary values are satisfied.

$$\begin{aligned} y(\pi) &= c_1\,\mathrm{e}^{-\pi\sqrt{\mu}}+c_2\,\mathrm{e}^{\pi\sqrt{\mu}} \\ y(-\pi) &= c_1\,\mathrm{e}^{\pi\sqrt{\mu}}+c_2\,\mathrm{e}^{-\pi\sqrt{\mu}} \end{aligned}$$

i.e.

$$c_1\,(\mathrm{e}^{-\pi\sqrt{\mu}}-\mathrm{e}^{\pi\sqrt{\mu}})=c_2\,(\mathrm{e}^{\pi\sqrt{\mu}}-\mathrm{e}^{-\pi\sqrt{\mu}})$$

i.e. $c_1=-c_2$.

$$\begin{aligned} y'(\pi) &= -\sqrt{\mu}\,c_1\mathrm{e}^{-\pi\sqrt{\mu}}+\sqrt{\mu}\,c_2\,\mathrm{e}^{\pi\sqrt{\mu}} \\ y'(-\pi) &= -\sqrt{\mu}\,c_1\mathrm{e}^{\pi\sqrt{\mu}}+\sqrt{\mu}\,c_2\,\mathrm{e}^{-\pi\sqrt{\mu}} \end{aligned}$$

i.e.

$$-c_1\left(\sqrt{\mu}\,c_1\mathrm{e}^{-\pi\sqrt{\mu}}-\sqrt{\mu}\,c_1\mathrm{e}^{\pi\sqrt{\mu}}\right)=c_2\left(\sqrt{\mu}\,c_2\,\mathrm{e}^{\pi\sqrt{\mu}}-\sqrt{\mu}\,c_2\,\mathrm{e}^{-\pi\sqrt{\mu}}\right)$$

i.e. $c_1=c_2$. These two conditions hold iff $c_1=c_2=0$, which means there is no eigenvalue in this case.

(b) Show that $\lambda=0$ is an eigenvalue and find an eigenfunction for this eigenvalue.

If $\lambda=0$ then the DE becomes

$$y''=0$$

The solution is then:

$$y=c_1\,x+c_2, \quad c_{1,2}\in\mathbb{R}$$

Check if boundary values are satisfied.

$$\begin{aligned} y(\pi) &= \pi c_1 + c_2 \\ y(-\pi) &= -\pi c_1 + c_2 \end{aligned}$$

This holds if $c_1 = 0$,

$$\begin{aligned} y'(\pi) &= c_1 \\ y'(-\pi) &= c_1 \end{aligned}$$

This two conditions leave c_2 to be arbitrary, such that $y = c_2$. The eigenfunction in this case is $f(t) = 1$.

(c) Find all the real positive eigenvalues for the BVP and show that each has two linearly independent eigenfunctions.

Assuming $\lambda > 0$, the solution to the ODE is:

$$y = c_1 \cos(\sqrt{\lambda} t) + c_2 \sin(\sqrt{\lambda} t), \quad c_{1,2} \in \mathbb{R}$$

Check if boundary values are satisfied.

$$\begin{aligned} y(\pi) &= c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\,\pi) \\ y(-\pi) &= c_1 \cos(-\sqrt{\lambda}\,\pi) + c_2 \sin(-\sqrt{\lambda}\,\pi) = c_1 \cos(\sqrt{\lambda}\pi) - c_2 \sin(\sqrt{\lambda}\,\pi) \end{aligned}$$

This implies that $c_2 = 0$ **or**

$$\sin(\sqrt{\lambda}\,\pi) = 0$$

which means that

$$\lambda_n = n^2, \quad n \in \mathbb{N}$$

is an eigenvalue.

Regarding the second boundray value:

$$\begin{aligned} y'(\pi) &= -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda\,\pi}) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\pi) \\ y'(-\pi) &= \sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda\,\pi}) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\pi) \end{aligned}$$

This implies that $c_1 = 0$ **or**

$$\sqrt{\lambda} \sin(\sqrt{\lambda\,\pi}) = 0$$

which yields the same eigenvalue.

In conclusion, two linearly independent eigenfunctions satisfy the BVP: $f(t) = \cos(n\ t)$ and $g(t) = \sin(t)$.

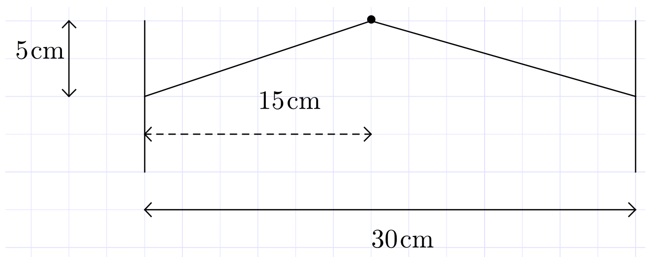
Question 6

An elastic string of length $L = 30$ cm is held down taut at both ends in a frame, and vibrates according to the wave equation:

$$u_{xx}(x,t) = u_{tt}(x,t)$$

where $a = 1$ cm/s. Assume that the string is plucked in the middle of the string, to a height of 5 cm and then released. Find a series representation of the function $u(x,t)$ that describes the vibration of the string.

Well, I wanted to use D'Alembert solution but you insist...



This is a zero velocity case of the homogeneous BCs wave equation with IC:

$$u(x,0) \equiv f(x) = \begin{cases} \frac{x}{3} & x \in (0,15) \\ 10 - \frac{x}{3} & x \in (15,30) \end{cases}$$

Solution in this case is:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\,\pi\,x}{L} \cos\frac{n\,\pi\,a\,t}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n = \frac{1}{15}\int_0^{15}\frac{x}{3}\sin\frac{n\,\pi\,x}{30}\,\mathrm{d}x + \frac{1}{15}\int_{15}^{30}\left(10-\frac{x}{3}\right)\sin\frac{n\,\pi\,x}{30}\,\mathrm{d}x$$

c_n is:

$$\frac{1}{45}\cdot\frac{30}{n\,\pi}\left[-x\cos\frac{n\,\pi\,x}{30}+\frac{30}{n\,\pi}\sin\frac{n\,\pi\,x}{30}\right]_0^{15} - \frac{2}{3}\cdot\frac{30}{n\,\pi}\left[\cos\frac{n\,\pi\,x}{30}\right]_{15}^{30} - \frac{1}{45}\cdot\frac{30}{n\,\pi}\left[-x\cos\frac{n\,\pi\,x}{30}+\frac{30}{n\,\pi}\sin\frac{n\,\pi\,x}{30}\right]_{15}^{30}$$

$$\begin{aligned} c_n &= \frac{2}{3n\pi}\left[-15\cos\left(\frac{n\pi}{2}\right)+\frac{30}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] - \frac{20}{n\pi}\left[\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right] \\ &\quad -\frac{2}{3n\pi}\left[-30\cos(n\pi)+15\cos\left(\frac{n\pi}{2}\right)-\frac{30}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

$$c_n = \frac{2}{3n\pi}\left[-30\cos\left(\frac{n\pi}{2}\right)+\frac{60}{n\pi}\sin\left(\frac{n\pi}{2}\right)+30\cos(n\pi)-30\cos(n\pi)+30\cos\left(\frac{n\pi}{2}\right)\right]$$

$$c_n = \frac{2}{3n\pi}\cdot\frac{60}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\frac{n\,\pi\,x}{30}\cos\frac{n\,\pi\,t}{30}$$