2009 Exam Solution

Question 1

A closed system of 3 tanks, each with volume $V = 50\,\mathrm{L}$, contains a salt solution with pumps maintatning constant flows as follows:

- A flow of $20\,\mathrm{L\,hr^{-1}}$ from tank 1 to tank 2.
- A flow of $10\,\mathrm{L\,hr^{-1}}$ from tank 2 to tank 1, and a flow of $20\,\mathrm{L\,hr^{-1}}$ from tank 2 to tank 3.
- A flow of $10\,\mathrm{L\,hr^{-1}}$ from tank 3 to tank 1, and a flow of $10\,\mathrm{L\,hr^{-1}}$ from tank 3 to tank 2.

Find the amount of salt in each tank at time t if the initial amount of salt in tank 1 is $100\,\mathrm{g}$, in tank 2 $200\,\mathrm{g}$, and in tank 3 $0\,\mathrm{g}$.

System of equations is:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\frac{2}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{2}{5}x_1 - \frac{1}{5}x_2 - \frac{2}{5}x_2 + \frac{1}{5}x_3\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= \frac{2}{5}x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_3
\end{aligned}$$

And in matrix form:

$$\vec{x}' = A\,\vec{x}$$

$$A = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{2}{5} & -\frac{3}{5} & \frac{1}{5}\\ 0 & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

To solve system of equations, find eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{2}{5} - \lambda & \frac{1}{5} & \frac{1}{5}\\ \frac{2}{5} & -\frac{3}{5} - \lambda & \frac{1}{5}\\ 0 & \frac{2}{5} & -\frac{2}{5} - \lambda \end{vmatrix} = \cdots =$$

$$= -\left(\lambda + \frac{2}{5}\right)\left[-\left(\lambda + \frac{3}{5}\right)\cdot\left(-\left(\lambda + \frac{2}{5}\right)\right) - \frac{2}{25}\right] - \frac{2}{5}\left[-\frac{1}{5}\left(\lambda + \frac{2}{5}\right) - \frac{2}{25}\right] = \cdots =$$

$$= -\left(\lambda + \frac{2}{5}\right)\left(\lambda^2 + \lambda + \frac{4}{25}\right) + \frac{2}{5}\left(\frac{1}{5}\lambda + \frac{4}{25}\right) = -\left(\lambda^3 + \lambda^2 + \frac{4}{25}\lambda + \frac{2}{5}\lambda^2 + \frac{2}{5}\lambda + \frac{8}{125}\right) + \frac{2}{25}\lambda + \frac{8}{125}$$

$$= -\left(\lambda^3 + \frac{7}{5}\lambda^2 + \frac{12}{25}\lambda\right) = -\lambda\left(\lambda^2 + \frac{7}{5}\lambda + \frac{12}{25}\right)$$

Roots are:

$$\lambda_1 = 0, \lambda_2 = -\frac{4}{5}, \lambda_3 = -\frac{3}{5}$$

Find eigenvectors. For λ_1, find $\vec{v}_1$ s.t. $(A-\lambda_1 I)\vec{v}_1=\vec{0}$.

$$A-\lambda_1 I=\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{1}{5} \\ 0 & \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$

Pick

$$\vec{v}_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For λ_2:

$$A-\lambda_2 I=\begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

Pick

$$\vec{v}_2=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

For λ_3:

$$A-\lambda_3 I=\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{1}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Pick

$$\vec{v}_3=\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

General solution is

$$\vec{x}=c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+c_2\,\mathrm{e}^{-\frac{4}{5}t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}+c_3\,\mathrm{e}^{-\frac{3}{5}t}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Find c_1, c_2, c_3 that satisfy ICs. Define

$$\vec{c}=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \vec{b}=\begin{bmatrix} 100 \\ 200 \\ 0 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

such that

$$\vec{c}=B^{-1}\vec{b}$$

Find B^{-1} via Gauss-Seidal algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3\to R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3\to R_3/(-3)]{R_1\to R_1+\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

Therefore,

$$\vec{c} = B^{-1}\vec{b} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 0 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 0 \end{bmatrix}$$

In conclusion, the amount of salt in each tank at time t is:

$$\vec{x} = 100 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 100\,\mathrm{e}^{-\frac{4}{5}t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Question 2

Find the general solution to the following DE:

$$y'' - 2y' - 15y = \sin t + 3\,\mathrm{e}^{5t}$$

with ICs: $y(0) = \frac{33}{65}$, $y'(1) = 0$. First solve associated homogeneous equation.

$$\lambda^2 - 2\lambda - 15 = 0$$

$$\lambda_{1,2} = \frac{2 \pm 8}{2} = 5, -3$$

$$y_h = c_1\,\mathrm{e}^{5t} + c_2\,\mathrm{e}^{-3t}$$

Find particular solution of form $y_{p1} = A\cos t + B\sin t$.

$$-A\cos t - B\sin t + 2A\sin t - 2B\cos t - 15A\cos t - 15B\sin t = \sin t$$

$$\sin t\,(-B + 2A - 15B) + \cos t\,(-A - 2B - 15A) = \sin t$$

Equate coefficients on both sides.

$$\begin{cases} \cos t\colon & -16A = 2B \to B = -8A \\ \sin t\colon & -16B + 2A = 1 \to A = \frac{1}{130}, B = -\frac{4}{65} \end{cases}$$

$$y_{p1} = \frac{1}{130}\cos t - \frac{4}{65}\sin t$$

Find another particular solution of form $y_{p2} = A\,t\,\mathrm{e}^{5t}$

$$\begin{aligned} y'_{p2} &= A\,\mathrm{e}^{5t} + 5A\,t\,\mathrm{e}^{5t} \\ y''_{p2} &= 10A\,\mathrm{e}^{5t} + 25A\,t\,\mathrm{e}^{5t} \end{aligned}$$

$$\mathrm{e}^{5t}\,[10A + 25A\,t - 2A - 10A\,t - 15A\,t] = 3\,\mathrm{e}^{5t}$$

Equate coefficients on both sides:

$$\begin{cases} t\colon & 25A - 10A - 25A = 0 \\ 1\colon & 10A - 2A = 3 \to A = \frac{3}{8} \end{cases}$$

General solution to ODE is:

$$y = c_1\,\mathrm{e}^{5t} + c_2\,\mathrm{e}^{-3t} + \frac{1}{130}\cos t - \frac{4}{65}\sin t + \frac{3}{8}\,t\,\mathrm{e}^{5t}$$

Find c_1, c_2 that satisfy ICs.

$$\begin{aligned}
y'(1) = 0 &= 5c_1\,\mathrm{e}^5 - 3c_2\,\mathrm{e}^{-3} + \frac{1}{130}\cos(1) - \frac{4}{65}\sin(1) + \frac{3}{8}\mathrm{e}^5 + \frac{15}{8}\mathrm{e}^5 \\
&= \mathrm{e}^5\left(5c_1 + \frac{18}{8}\right) - 3c_2\,\mathrm{e}^{-3}\frac{1}{130}\cos(1) - \frac{4}{65}\sin(1) \\
y(0) = \frac{33}{65} &= c_1 + c_2 + \frac{1}{130} \rightarrow c_1 = \frac{1}{2} - c_2
\end{aligned}$$

$$\mathrm{e}^5\left(\frac{5}{2} - 5c_2 + \frac{18}{8}\right) - 3c_2\,\mathrm{e}^{-3}\frac{1}{130}\cos(1) - \frac{4}{65}\sin(1) = 0$$

c_2 and c_1 are indeed numbers…

Question 3

(a) Solve the following BVP:

$$25\,y_{xx} = y_{tt}, \quad x \in (0,3), t > 0$$

with homogeneous BCs and ICs: $y(x,0) = \frac{1}{4}\sin(\pi x)$, $y_t(x,0) = 10\sin(2\pi x)$.

Solve separately in case of zero initial position and zero initial velocity. Note that in both cases, $y(x,0), y_t(x,0)$ are odd functions over the real line.

Zero initial velocity, i.e. $y(x,0) \equiv f(x) = \frac{1}{4}\sin(\pi x)$, $y_t(x,0) \equiv 0$.

Solution is ofform

$$y_1(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\,\pi\,x}{L}\cos\frac{n\,\pi\,a\,t}{L}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n = \frac{2}{3}\cdot\frac{1}{4}\int_0^3 \sin(\pi\,x)\cdot\sin\frac{n\,\pi\,x}{3}\,\mathrm{d}x$$

As sines of different frequencies are orthogonal, the integral isn't zero only for $n = 3$.

$$c_3 = \frac{1}{6}\int_0^3 \sin^2(\pi\,x)\,\mathrm{d}x = \frac{1}{6\pi}\left[\frac{\pi x}{2} - \frac{\sin(2\pi\,x)}{4}\right]_0^3 = \frac{1}{4}$$

Therefore,

$$y_1(x,t) = \frac{1}{4}\sin(\pi\,x)\cos(5\pi\,x)$$

Zero initial position, i.e. $y(x,0) \equiv 0, y_t(x,0) \equiv g(x) = 10\sin(2\pi\,x)$

Solve via D'alembert method. Let $G(x) = g(x)$ be the odd extension of $g(x)$ over $\mathbb{R}$. Let $H(x)$ be the promitive function of $g(x)$:

$$H(x) = \int g(\xi)\,\mathrm{d}\xi = -\frac{5}{\pi}\cos(2\pi x)$$

The solution to the wave equation in this case is:

$$y_2(x,t) = \frac{1}{10}\left[H(x+5t) - H(x-5t)\right]$$

$$y_2(x,t) = \frac{1}{10}\cdot\left(-\frac{5}{\pi}\right)\left[\cos(2\pi x + 10\pi t) - \cos(2\pi x - 10\pi t)\right] = -\frac{1}{\pi}\cos(2\pi x)\cos(10\pi t)$$

General solution to the wave equation is thus:

$$y(x,t) = y_1 + y_2 = \frac{1}{4}\sin(\pi x)\cos(5\pi x) - \frac{1}{\pi}\cos(2\pi x)\cos(10\pi t)$$

(b) Find the general real solution to the following system of differential equations:

$$\vec{x}' = A\,\vec{x}, \qquad A = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}$$

Find eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 7 & 1-\lambda \end{vmatrix} = (\lambda-3)(\lambda-1) + 7 = \lambda^2 - 4\lambda + 10$$

Roots are

$$\lambda_{1,2} = \frac{4 \pm 2\sqrt{6}\mathrm{i}}{2} = 2 \pm \sqrt{6}\mathrm{i}$$

Construct two real solutions from the eigenvector of $\lambda_1 = 2 - \sqrt{6}\mathrm{i}$. Find $\vec{v}$ such that $(A - \lambda_1 I)\vec{v} = \vec{0}$.

$$A - \lambda_1 I = \begin{bmatrix} 1+\sqrt{6}\mathrm{i} & -1 \\ 7 & -1+\sqrt{6}\mathrm{i} \end{bmatrix}$$

Pick

$$\vec{v} = \begin{bmatrix} 1 \\ 1+\sqrt{6}\mathrm{i} \end{bmatrix}$$

$$\mathrm{e}^{\lambda_1 t}\,\vec{v} = \mathrm{e}^{2t}\left(\cos(\sqrt{6}t) + \mathrm{i}\sin(\sqrt{6}t)\right)\begin{bmatrix} 1 \\ 1+\sqrt{6}\mathrm{i} \end{bmatrix} = \cdots =$$

$$= \mathrm{e}^{2t}\left(\begin{bmatrix} \cos\sqrt{6}t \\ \cos\sqrt{6}t - \sqrt{6}\sin\sqrt{6}t \end{bmatrix} + \mathrm{i}\begin{bmatrix} \sin\sqrt{6}t \\ \sin\sqrt{6}t + \sqrt{6}\cos\sqrt{6}t \end{bmatrix}\right)$$

General solution to the DE is:

$$\vec{x} = c_1\,\mathrm{e}^{2t}\begin{bmatrix} \cos\sqrt{6}t \\ \cos\sqrt{6}t - \sqrt{6}\sin\sqrt{6}t \end{bmatrix} + c_2\,\mathrm{e}^{2t}\begin{bmatrix} \sin\sqrt{6}t \\ \sin\sqrt{6}t + \sqrt{6}\cos\sqrt{6}t \end{bmatrix}$$

Question 4

(a) Solve the equation

$$y' \cdot y'' - t = 0$$

with ICs $y(1) = 1, y'(1) = 2$. Set $v = y'$.

$$v' \cdot v = t$$

This is a separable DE.

$$\int v \, \mathrm{d}v = \int t \, \mathrm{d}t$$

$$v^2 = t^2 + c_1$$

$$v = \pm\sqrt{t^2 + c_1}$$

Find c_1 using IC.

$$y'(1) = 2 \rightarrow 4 = 1 + c_1 \rightarrow c_1 = 3$$

As a side note, the IC implies that

$$y' = +\sqrt{t^2 + 3}$$

Integrate again:

$$y = \int \sqrt{t^2 + 3} \, \mathrm{d}t = \cdots$$

(b) Solve the equation

$$y' = x\, y^3\, (1 + x^2)^{1/2}$$

with IC $y(0) = 2$. This is a separable equation.

$$\int \frac{\mathrm{d}y}{y^3} = \int \frac{x}{\sqrt{1 + x^2}} \, \mathrm{d}x$$

$$-\frac{1}{2} \frac{1}{y^2} = \sqrt{1 + x^2} + c$$

Insert IC.

$$y(0) = 2 \rightarrow -\frac{1}{2} \cdot \frac{1}{4} = 1 + c \rightarrow c = -\frac{9}{8}$$

$$y = \sqrt{-\frac{1}{2\sqrt{x^2 + 1} - \frac{9}{4}}}$$

Question 5

Given a rod of length $L = \pi$ with $\alpha^2 = 25$, find the temperature $u(x, t)$ along the rod if $u(x, 0) = 25$ and $u(0, t) = 10, u(\pi, t) = 30$ for all t.

Define

$$w(x, t) = u(x, t) - v(x)$$

such that

$$v(x) = 10 + \frac{20}{\pi}\, x$$

$$w(x,0) = 15 - \frac{20}{\pi}\, x$$

$w(x,t)$ solves the heat equation with homogeneous BCs. The solution in this case is given by:

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\,\pi\,x}{L} \exp\!\left(\frac{-n^2\pi^2\alpha^2\,t^2}{L^2}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L w(x,0)\,\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n = \frac{2}{\pi}\int_0^{\pi}\left(15 - \frac{20}{\pi}\,x\right)\sin\left(n\,x\right)\mathrm{d}x = \frac{2}{\pi}\left[\frac{15}{n}\cos\left(n\,x\right)\right]_0^{\pi} - \frac{40}{\pi^2}\int x\sin\left(n\,x\right)\mathrm{d}x$$

$$\int x\sin\left(n\,x\right)\mathrm{d}x = -x\,\frac{1}{n}\cos\left(n\,x\right) + \frac{1}{n^2}\sin\left(n\,x\right)$$

$$\Rightarrow c_n = \frac{2}{\pi}\left[\frac{15}{n}\cos\left(n\,x\right)\right]_0^{\pi} - \frac{40}{\pi^2}\,\frac{1}{n}\left[-\cos\left(n\,x\right) + \frac{\sin\left(n\,x\right)}{n}\right]_0^{\pi}$$

$$c_n = \frac{30}{\pi\,n}\left[(-1)^n - 1\right] - \frac{40}{\pi^2\,n}\left[1 - (-1)^n\right]$$

Therefore:

$$u(x,t) = 15 - \frac{20}{\pi}\,x + \sum_{n=1}^{\infty}\left(\frac{30}{\pi\,n}\left[(-1)^n - 1\right] - \frac{40}{\pi^2\,n}\left[1 - (-1)^n\right]\right)\sin\left(n\,x\right)\mathrm{e}^{-25n^2t^2}$$