Assignment 4

Question 1

Solve the system: $\begin{cases} u+v+w=2 \\ u+3v+3w=0 \\ u+3v+5w=2 \end{cases}$

We will get the extended coefficient matrix, use elementry row operations to get an equivalent system of upper echilon form and get the solution from there

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 3 & 0 \\ 1 & 3 & 5 & 2 \end{array}\right] \xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R_3-R_2\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{2}R_2\to R_2 \\ \frac{1}{2}R_3\to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We got the system:

$$\begin{cases} u+v+w=2 \\ \quad v+w=-1 \\ \qquad w=1 \end{cases} \quad v=-2 \quad u=3$$

The solution to the system: $u=3$, $v=-2$, $w=1$

Question 2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3-R_1\to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_2+3R_3\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & -3 \\ 0 & 1 & 0 & -\frac{3}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

The inverse is: $\begin{bmatrix} 4 & -2 & -3 \\ -\frac{3}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

Question 3

$$\begin{cases} x_1+3x_2-2x_3+2x_5=0 \\ 2x_1+6x_2-5x_3-2x_4+4x_5-3x_6=-1 \\ 5x_3+10x_4+15x_6=5 \\ 2x_1+6x_2+8x_4+4x_5+18x_6=6 \end{cases}$$

We will solve similarly to question 1.

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} R_2-2R_1\to R_2 \\ R_4-2R_1\to R_4 \\ \frac{1}{5}R_3\to R_3 \end{array} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} R_3+R_2\to R_3 \\ R_4+4R_2\to R_4 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} -R_2\to R_2 \\ R_3\leftrightarrow R_4 \\ \frac{1}{6}R_3\to R_3 \end{array} \begin{array}{c} \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \end{array} \quad R_2-3R_3\to R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We got $x_6=\frac{1}{3}$, x_2, x_4, x_5 are free variables and x_1, x_3 dependent

Solving while expressing the dependent vars in terms of the free vars:

$x_3+2x_4=0 \Rightarrow x_3=-2x_4$

$x_1+3x_2-2x_3+2x_5=0 \Rightarrow x_1=-3x_2+2x_3-2x_5=-3x_2-4x_4-2x_5$

The solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3x_2-4x_4-2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ \frac{1}{3} \end{pmatrix} = x_2\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

$x_2, x_4, x_5 \in \mathbb{R}$.

Question 4

(a) $\begin{cases} x_1+4x_2+2x_3+x_4=b_1 \\ x_1+5x_2+2x_3=b_2 \\ 2x_1+9x_2+5x_3+3x_4=b_3 \\ 2x_1+7x_2+4x_3+3x_4=b_4 \end{cases}$, Assuming $b_i \in \mathbb{R}$

Using the coefficient matrix, we will find a condition for b_i to have a solution:

$$\left[\begin{array}{cccc|c} 1&4&2&1&b_1 \\ 1&5&2&0&b_2 \\ 2&9&5&3&b_3 \\ 2&7&4&3&b_4 \end{array}\right] \begin{array}{l} R_2-R_1\to R_2 \\ R_3-2R_1\to R_3 \\ R_4-2R_1\to R_4 \end{array} \left[\begin{array}{cccc|c} 1&4&2&1&b_1 \\ 0&1&0&-1&b_2-b_1 \\ 0&1&1&1&b_3-2b_1 \\ 0&-1&0&1&b_4-2b_1 \end{array}\right] \begin{array}{l} R_3-R_2\to R_3 \\ R_4+R_2\to R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1&4&2&1&b_1 \\ 0&1&0&-1&b_2-b_1 \\ 0&0&1&2&b_3-b_2-b_1 \\ 0&0&0&0&b_4+b_2-3b_1 \end{array}\right]$$

The condition for the system to have a solution: $b_2+b_4-3b_1=0$

$b_4=3b_1-b_2$

$$\begin{pmatrix} b_1\\b_2\\b_3\\b_4 \end{pmatrix} = b_1\begin{pmatrix}1\\0\\0\\3\end{pmatrix} + b_2\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} + b_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad b_1,b_2,b_3 \in \mathbb{R}$$

(b) First, check the condition: $b_2+b_4-3b_1 = -4+1+3=0$.

x_4 is a free variable and we will solve in terms of it.

$x_3+2x_4=b_3-b_2-b_1 = -1+4+1=4 \Rightarrow x_3=4-2x_4$

$x_2-x_4=b_2-b_1=-4+1=-3 \Rightarrow x_2=x_4-3$

$x_1+4x_2+2x_3+x_4=-1 \Rightarrow x_1=-1-4(x_4-3)-2(4-2x_4)-x_4=$

$= -1-\cancel{4x_4}+12-8+\cancel{4x_4}-x_4=3-x_4$

The solution:

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4 \end{pmatrix} = \begin{pmatrix} 3-x_4\\x_4-3\\4-2x_4\\x_4 \end{pmatrix} = x_4\begin{pmatrix}-1\\1\\-2\\1\end{pmatrix} + \begin{pmatrix}3\\-3\\4\\0\end{pmatrix}, \quad x_4 \in \mathbb{R}$$

Question 5

$$\begin{cases} x+y+2z=1 \\ 2x+(1+\lambda)y+2z=4 \\ \lambda x+y+z=2\lambda \end{cases}$$ Assuming $\lambda \in R$.

The coefficient matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1+\lambda & 2 & 4 \\ \lambda & 1 & 1 & 2\lambda \end{array}\right] \begin{array}{l} R_2-2R_1 \to R_2 \\ R_3-\lambda R_1 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-1 & -2 & 2 \\ 0 & 1-\lambda & 1-2\lambda & \lambda \end{array}\right] R_3+R_2 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda-1 & -2 & 2 \\ 0 & 0 & -2\lambda-1 & \lambda+2 \end{array}\right]$$

(b) No Solution: for $\lambda=-\frac{1}{2}$, the last row is $[0\ 0\ 0 \mid 1.5]$ and there is no solution.

(c) An infinite number of solutions: for $\lambda=1$, the matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -3 & 3 \end{array}\right] \begin{array}{l} R_3-\frac{3}{2}R_2 \to R_3 \\ -\frac{1}{2}R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ We get y as a free var and so inf. many sols.

(a) A unique solution: We get a unique sol for any $\lambda \neq 1, -\frac{1}{2} \in R$. This way we have 3 pivots = rank of matrix.

Question 6

$$\begin{cases} ax + bz = 2 \\ ax + ay + 4z = 4 \\ ay + 2z = b \end{cases}$$, assuming $a, b \in \mathbb{R}$

Similarly to question 5:

$$\left[\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right] \quad R_2 - R_1 \to R_2 \quad \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right] \quad R_3 - R_2 \to R_3 \quad \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right]$$

(c) An infinite number of solutions occur at $b=2$ for any $a \in \mathbb{R}$.
In this case we have z as a free variable $(a \neq 0)$ or x, y free $(a=0)$.

(b) No solution: This will occur for $a=0$ and $b=0$ or $a=0$ and $b=4$
(first and second row zeros, accordingly).

(a) A unique solution will happen for $(a,b) \neq (0,0), (0,4)$ and $b \neq 2$, $a, b \in \mathbb{R}$