

Assignment 10

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Date: 16.1.23

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Question 1

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

Denote

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

Solution is of the form $\vec{z} e^{\lambda t}$. \vec{z}, λ are given by calculating the eigenvalues and eigenvectors of A . First find eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4 = \lambda^2 - \lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = 2, -1$$

Find corresponding eigenvectors. For $\lambda = 2$:

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{z}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$:

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \vec{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General solution is a linear combination of $\{z_i e^{\lambda_i t}\}$:

$$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

The solution diverges as $t \rightarrow \infty$.

Question 7

$$\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}, \quad A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

As noted in the book, A has an eigenvalue $\lambda_1 = 0$ (A has linearly dependent rows). Calculate the second eigenvalue:

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & -3 \\ 8 & -6 - \lambda \end{vmatrix} = (\lambda - 4)(\lambda + 6) + 24 = \lambda^2 + 2\lambda = 0$$

$$\lambda_2 = -2$$

Calculate eigenvectors: for $\lambda_1 = 0$:

$$\vec{z}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(As clearly $A\vec{z}_1 = \vec{0}$.)

For $\lambda_2 = -2$:

$$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

Question 11

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \vec{x}$$

Note that the sum along the columns in each row is equal (sum = 4). We can infer that $\vec{z}_1 = [1, 1, 1]^T$ is an eigenvector with an eigenvalue $\lambda_1 = 4$. Find the other eigenvalues:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 & 2 \\ 1 & 2 - \lambda & 1 \\ 2 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)[(2 - \lambda)(1 - \lambda) - 1] - [1 - \lambda - 2] \\ &\quad + 2[1 - 4 + 2\lambda] \\ &= (1 - \lambda)[\lambda^2 - 3\lambda + 1] + (\lambda + 1) + (4\lambda - 6) \\ &= \lambda^2 - 3\lambda + 1 - \lambda^3 + 3\lambda^2 - \lambda + 5\lambda - 5 \\ &= -\lambda^3 + 4\lambda^2 + \lambda - 4 = 0 \end{aligned}$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

Factor out $\lambda = 4$:

$$(\lambda - 4)(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda_{2,3} = 1, -1$$

Find eigenvectors. For $\lambda_2 = 1$:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_3 = -1$:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \vec{z}_3 = \vec{0}$$

$$\vec{z}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad c_{1,2,3} \in \mathbb{R}, \quad \forall t$$

Question 12

$$\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}, \quad A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)[\lambda^2 - 3\lambda - 4] - 2(6 - 2\lambda - 8) + 4(4 + 4\lambda) \\ &= (3-\lambda)(\lambda - 4)(\lambda + 1) + 20(\lambda + 1) \\ &= -(\lambda + 1)(\lambda^2 - 7\lambda - 8) = -(\lambda + 1)^2(\lambda - 8) = 0 \end{aligned}$$

$$\lambda_{1,2,3} = -1, -1, 8$$

Find associated eigenvectors. For $\lambda_{1,2} = -1$:

$n - \text{rank}(A - \lambda_{1,2} I) = 3 - 1 = 2$ linearly independent eigenvectors span the eigenspace:

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \vec{z}_{1,2} = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{z}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 8$:

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \vec{z}_3 = \vec{0}$$

$$\vec{z}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

General solution is:

$$\vec{x} = e^{-t} \left(c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right) + c_3 e^{8t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad c_{1,2,3} \in \mathbb{R}, \quad \forall t$$

Question 15

$$\vec{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find eigenvalues:

$$|A - \lambda I| = (\lambda - 5)(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) = 0$$

$$\lambda_{1,2} = 4, 2$$

Find eigenvectors. For $\lambda_1 = 4$:

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

Find $c_{1,2}$ for given ICs

$$t = 0: \quad c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Rewrite problem in matrix form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{3-1} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -1.5 \end{bmatrix}$$

The unique solution is:

$$\vec{x}(t) = \frac{7}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \forall t$$

As $t \rightarrow \infty$ the solution diverges.

Question 29

Given the 2nd order ODE:

$$a y'' + b y' + c y = 0 \tag{1}$$

and its corresponding characteristic equation:

$$a r^2 + b r + c = 0 \tag{2}$$

(a) Set

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$$

Differentiate:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{b}{a} x_2 - \frac{c}{a} x_1 \end{cases}$$

Write in matrix form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \vec{x}$$

In short:

$$\vec{x}' = A \vec{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$$

(b) The equation that determines the eigenvalues of the coefficient matrix A is given by calculating $\det(A - \lambda I)$.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$

Multiply by a to get:

$$a\lambda^2 + b\lambda + c = 0$$

As requested to show, we've reproduced eq. (2).

Question 30

$$\vec{x}' = \begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} -17 \\ -21 \end{bmatrix}$$

(a) Find eigenvalues and eigenvectors of A (coefficient matrix):

$$\det(A - \lambda I) = \left(\lambda + \frac{1}{10}\right)\left(\lambda + \frac{1}{5}\right) - \frac{3}{400} = \lambda^2 + \frac{3}{10}\lambda + \frac{1}{80} = 0$$

$$\lambda_{1,2} = \frac{-0.3 \pm \sqrt{0.09 - 0.05}}{2} = -0.05, -0.25$$

For $\lambda_1 = -0.05$:

$$\begin{bmatrix} -0.05 & \frac{3}{40} \\ \frac{1}{10} & -0.15 \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -0.25$:

$$\begin{bmatrix} 0.15 & \frac{3}{40} \\ \frac{1}{10} & 0.05 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1 e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad t \geq 0$$

Apply ICs:

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -17 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -17 \\ -21 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 17 \\ 21 \end{bmatrix} = \begin{bmatrix} -\frac{55}{8} \\ \frac{29}{8} \end{bmatrix}$$

Unique solution is:

$$\vec{x} = -\frac{55}{8} e^{-0.05t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{29}{8} e^{-0.25t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t \geq 0$$

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Question 2

$$\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

Find eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Find eigenvector for $\lambda = -1 - 2i$:

$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

Pair of solutions is:

$$\begin{aligned} e^{(-1-2i)t} \begin{bmatrix} -2i \\ 1 \end{bmatrix} &= e^{-t} (\cos 2t - i \sin 2t) \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \dots \\ \dots &= e^{-t} \begin{bmatrix} -2i \cos 2t - 2 \sin 2t \\ \cos 2t - i \sin 2t \end{bmatrix} = e^{-t} \begin{bmatrix} -2 \sin 2t \\ \cos 2t \end{bmatrix} - i e^{-t} \begin{bmatrix} 2 \cos 2t \\ \sin 2t \end{bmatrix} \end{aligned}$$

General solution constituted of real functions is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} -2 \sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \cos 2t \\ \sin 2t \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

As $t \rightarrow \infty$ the solution approaches zero (decaying oscillations).

Question 10

$$\vec{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of the coefficient matrix.

$$\det(A - \lambda I) = \lambda^2 + 4\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Find eigenvector for $\lambda = -2 - i$:

$$\begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Pair of solutions is:

$$\begin{aligned} e^{(-2-i)t} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} &= e^{-2t} (\cos t - i \sin t) \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \dots \\ \dots &= e^{-2t} \begin{bmatrix} \cos t + i \cos t - i \sin t + \sin t \\ \cos t - i \sin t \end{bmatrix} = e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + i e^{-2t} \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix} \end{aligned}$$

General solution is:

$$\vec{x} = c_1 e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

Find $c_{1,2}$ for given ICs.

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

From the second row $c_1 = -2$ and from the first row $c_2 = 3$.

Unique solution is

$$\vec{x} = e^{-2t} \left(-2 \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + 3 \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix} \right) = e^{-2t} \begin{bmatrix} \cos t - 5 \sin t \\ -2 \cos t - 3 \sin t \end{bmatrix}, \quad \forall t$$

As $t \rightarrow \infty$ the solution approaches zero (decaying oscillations).