Assignment 9

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Question 24

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \vec{x}$$

$$\vec{x} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$$

Denote $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Assume the solution is of the form

$$\vec{x} = \vec{z}\, e^{rt}$$

$$\vec{x}' = r\, \vec{z}\, e^{rt}$$

Plug in the system of ODEs to get

$$r\, \vec{z}\, e^{rt} = A\, \vec{z}\, e^{rt}$$

Divide both sides by $e^{rt} \neq 0$ (for all t) to get

$$A\, \vec{z} = r\, \vec{z}$$

This problem is equivalent to finding the eigenvalues of A. Find λ such that $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 & 1 \\ 2 & 1 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} = 0$$

$$
\begin{aligned}
\begin{vmatrix} 1 - \lambda & 1 & 1 \\ 2 & 1 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} &= (1 - \lambda)\left[1 - 2\lambda + \lambda^2 - 1\right] - 2\left(1 - \lambda + 1\right) \\
&= 1 - 2\lambda + \lambda^2 - 1 - \lambda + 2\lambda^2 - \lambda^3 + \lambda - 4 + 2\lambda \\
&= -\lambda^3 + 3\lambda^2 - 4 \\
&= (\lambda + 1)(-\lambda^2 + 4\lambda - 4) \\
&= -(\lambda + 1)(\lambda - 2)^2 = 0 \rightarrow \lambda_1 = -1, \lambda_2 = 2
\end{aligned}
$$

Find the eigenvectors that correspond to the calculated eigenvalues.

$\underline{\lambda_1 = -1}$

$$(A - \lambda_1 I)\,\vec{z}_1 = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \vec{z} = 0$$

Apply row operations on the augmented matrix $A|\vec{0}$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

From which we obtain:

$$x_2 = 2x_3,\ x_1 = -\frac{3}{2}x_3$$

Pick $x_3 = -4$ to get:

$$\vec{z}_1 = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix}$$

$\underline{\lambda_2 = 2}$

$$(A - \lambda_2 I)\,\vec{z}_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \vec{z} = 0$$

Obtain

$$x_2 = -x_3,\ x_1 = 0$$

Pick $x_3 = -1$ and get

$$\vec{z}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Finally, the general solution is

$$\vec{x} = c_1\,\vec{z}_1\,e^{\lambda_1 t} + c_2\,\vec{z}_2\,e^{\lambda_2 t}$$

If we pick $c_1, c_2 = 1$, we get the unique solution

$$\vec{x} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t}$$

as requested to verify.

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Question 16

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

Find eigenvalues of A.

$$| A - \lambda I | = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = (\lambda - 3)(1 + \lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Find eigenvectors of A.

$\underline{\lambda_1 = 1 + 2i}$

Solve $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$.

$$A - \lambda_1 I = \begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix}$$

Select

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$$

$\underline{\lambda_2 = 1 - 2i}$

Solve $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$.

$$A - \lambda_2 I = \begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix}$$

Select

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$$

Question 18

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Find eigenvalues of A.

$$| A - \lambda I | = \begin{vmatrix} 1 - \lambda & i \\ -i & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - 1 = \lambda(\lambda - 2) = 0$$

$$\lambda_{1,2} = 0, 2$$

Find eigenvectors of A.

$\underline{\lambda_1 = 0}$

$$A - \lambda_1 I = A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 2}$

$$A - \lambda_2 I = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Question 21

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find eigenvalues: (Calculate det $A - \lambda I$ along row 1)

$$| A - \lambda I | = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 2 & 1 - \lambda & -2 \\ 3 & 2 & 1 - \lambda \end{vmatrix} = (1 - \lambda)[\lambda^2 - 2\lambda + 1 + 4] = 0$$

$$(1 - \lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda_{1,2,3} = 1, 1 \pm 2i$$

Find eigenvectors.

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$\underline{\lambda_1 = 1}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

$\underline{\lambda_2 = 1 + 2i}$

$$A - \lambda_2 I = \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$\underline{\lambda_3 = 1 - 2i}$

$$A - \lambda_3 I = \begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

Question 22

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

Find eigenvalues:

$$| A - \lambda I | = \begin{vmatrix} 3 - \lambda & 2 & 2 \\ 1 & 4 - \lambda & 1 \\ -2 & -4 & -1 - \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 - \lambda & 2 & 2 \\ 1 & 4 - \lambda & 1 \\ -2 & -4 & -1 - \lambda \end{vmatrix} = (3 - \lambda)\left[-(4 - \lambda)(1 + \lambda) + 4 \right] - 1\left[-2(1 + \lambda) + 8 \right]$$

$$-2\left[2 - 8 + 2\lambda \right] = 0$$

$$(3 - \lambda)(\lambda^2 - 3\lambda) - 2(3 - \lambda) - 4(\lambda - 3) = 0$$

$$(3 - \lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda_{1,2,3} = 3, 1, 2$$

Find eigenvectors:

$\underline{\lambda_1 = 3}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 1}$

$$A - \lambda_2 I = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\underline{\lambda_3 = 2}$

$$A - \lambda_3 I = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\text{Question 24}}$

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Find eigenvalues:

$$| A - \lambda I | = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 3\lambda - 4) - 2(6 - 2\lambda - 8) + 4(4 + 4\lambda)$$

$$= (3-\lambda)(\lambda - 4)(\lambda + 1) + 4(1 + \lambda) + 16(1 + \lambda)$$
$$= -(\lambda + 1)(\lambda^2 - 7\lambda - 8) = 0$$

$$\lambda_{1,2,3} = -1, -1, 8$$

Find eigenvectors:

$\lambda_{1,2} = -1$

$$A - \lambda_{1,2}\,I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

We have one equation in 3 variables. Choose two linearly independent vectors $\vec{v}_{1,2}$ that satisfy $A\,\vec{v}_{1,2} = \vec{0}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$\lambda_3 = 8$

$$A - \lambda_3\,I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Two Extra Extremely Exasperating Inquires

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Question 1

$$\begin{cases} x_{n+1} = 3x_n - y_n \\ y_{n+1} = 2x_n \end{cases}$$

(a) Denote $\vec{v}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$, Insert coefficients of x_n, y_n from both equations into a matrix A:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

Indeed,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} \equiv \vec{v}_{n+1} = A\,\vec{v}_n$$

(b) Given initial populations stored in $\vec{v}_0$, we shall prove inductively that

$$\vec{v}_n = A^n\,\vec{v}_0$$

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For $n = 0$:

$$\vec{v}_0 = A^0 \vec{v}_0 = I \vec{v}_0 = \vec{v}_0 \checkmark$$

Assume that the statement holds for $n = k$, that is:

$$\vec{v}_k = A^k \vec{v}_0$$

Show that it holds for $n = k + 1$:

$$\vec{v}_{k+1} = A^{k+1} \vec{v}_0 = A\, A^k \vec{v}_0 = A\, \vec{v}_k$$

Note here that we've utilized the fact that a matrix commutes with its power. This can be proved inductively, as for $k = 0$: $A^0 = I$ by definition, and assuming

$$A^{k+1} = A^k A$$

indeed:

$$A^{k+1} = A^k A = (A\, A^{k-1})\, A = A\, (A^{k-1}\, A) = A\, A^k$$

So $A^k A = A\, A^k = A^{k+1}$.

By definition of our problem, $\vec{v}_{k+1} = A\, \vec{v}_k$ (just switch n for k), so the statement

$$\vec{v}_{k+1} = A^{k+1} \vec{v}_0$$

holds. In summary, we've proved inductively that

$$\vec{v}_n = A^n \vec{v}_0$$

as requested.

(c) Find λ that satisfy:

$$|\, A - \lambda I \,| = 0$$

$$|\, A - \lambda I \,| = \begin{vmatrix} 3 - \lambda & -1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$$

$$\lambda_{1,2} = 2, 1$$

Find the eigenvectors $\{\vec{x}_i\}$ that satisfy for each $i \in 1, 2$: $(A - \lambda_i I)\vec{x}_i = 0$.

$\underline{\lambda_1 = 2}$

$$A - \lambda_1 I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 1}$

$$A - \lambda_2 I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(d) Find explicit formulas for $\vec{v}_n$ in terms of n. Well, we know that

$$\vec{v}_n = A^n \vec{v}_0$$

and we also know that if

$$\begin{cases} A\vec{x}_1 = 2\,\vec{x}_1 \\ A\vec{x}_2 = \vec{x}_2 \end{cases}$$

Then

$$\begin{cases} A^\ell \vec{x}_1 = 2^\ell\,\vec{x}_1 \\ A^\ell \vec{x}_2 = 1^\ell\,\vec{x}_2 = \vec{x}_2 \end{cases}$$

The fact that the ith eigenvalue of A^ℓ is λ_i^ℓ can also be proved by induction:

For $\ell = 0$: (given $\vec{x}_i$ is the eigenvector corresponding to the ith eigenvector of A)

$$A^0 \vec{x}_i = I\vec{x}_i = \vec{x}_i = \lambda_i^0\,\vec{x}_i$$

Assuming $A^\ell \vec{x}_i = \lambda_i^\ell\,\vec{x}_i$, we show that it holds for $\ell + 1$:

$$A^{\ell+1}\vec{x}_i = A\,A^\ell\vec{x}_i = A\,\lambda_i^\ell\vec{x}_i = \lambda_i^\ell A\,\vec{x}_i = \lambda_i^\ell \lambda_i\,\vec{x}_i = \lambda_i^{\ell+1}\,\vec{x}_i$$

thus proving the inductive step holds.

Now, to compute $A^n\vec{v}_0$, we need to express $\vec{v}_0$ as a linear combination of $\vec{x}_1, \vec{x}_2$. In other words, find scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\vec{x}_1 + \vec{x}_2 = \vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

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or, in other words,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \vec{v}_0$$

Denote B as the matrix comprising the eigenvectors of A as its columns:

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \vec{v}_0$$

Compute B^{-1} via Gauss-Seidal algorithm:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Therefore,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{v}_0$$

In conclusion,

$$\vec{v}_n = A^n \vec{v}_0 = A^n \left(c_1 \vec{x}_1 + c_2 \vec{x}_2 \right) = c_1 A^n \vec{x}_1 + c_2 A^n \vec{x}_2 = c_1 2^n \vec{x}_1 + c_2 \vec{x}_2$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = c_1 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

where c_1, c_2 are given by:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \vec{v}_0$$

where $\vec{v}_0$ is the vector containing the initial populations. A final, expanded expression:

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (2 x_0 - y_0) 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-x_0 + y_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(e) If at some year, $\vec{v}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then 7 years later:

$$\begin{bmatrix} x_7 \\ y_7 \end{bmatrix} = (2 \cdot 3 - 2) 2^7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-3 + 2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 512 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 511 \\ 510 \end{bmatrix}$$

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$a_0 = 0, a_1 = 1$, and for $n > 1$, $a_n = a_{n-1} + a_{n-2}$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{v}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

(a)

$$\vec{v}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n + a_{n+1} \end{bmatrix}$$

Now calculate $A\vec{v}_n$:

$$A\vec{v}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n + a_{n+1} \end{bmatrix}$$

We got the same result, thus approving that $\vec{v}_{n+1} = A\vec{v}_n$.

(b) the diagonalized form of A is $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1, λ_2 are the eigenvalues of A.

$$| A - \lambda I | = \begin{vmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

To find an expression for A^n, we use two facts:

1. For a diagonal matrix $D = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$

$$D^n = \begin{bmatrix} D_{11}^n & 0 \\ 0 & D_{22}^n \end{bmatrix}$$

2. Let $A = VDV^{-1}$ where V is a 2×2 invertible matrix and D is a diagonal matrix, then

$$A^n = VD^nV^{-1}$$

The first fact can be proved by induction on n. The base case $n = 1$ is true by definition. Now suppose that

$$D^k = \begin{bmatrix} D_{11}^k & 0 \\ 0 & D_{22}^k \end{bmatrix}$$

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then we have (inductive step):

$$D^{k+1} = D\,D^k = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} D_{11}^k & 0 \\ 0 & D_{22}^k \end{bmatrix} = \begin{bmatrix} D_{11}^{k+1} & 0 \\ 0 & D_{22}^{k+1} \end{bmatrix}$$

Hence the inductive step holds, which completes the proof.

The second fact can also be proved by induction on n. The base case $n=1$ is true by definition (the equality $A=VDV^{-1}$ is called *eigendecomposition*, and holds when A is diagonaizable—and it is in this case). For the inductive step, assume that

$$A^k = VD^k V^{-1}$$

then we have

$$A^{k+1} = A\,A^k = (VDV^{-1})\,(VD^k V^{-1}) = VD^{k+1} V^{-1}$$

Hence the inductive step holds, which completes the proof.

The vector V is the eigenvectors of A adjacent to each other in ordering respective to the order of eigenvalues in D.

Calculate the eigenvectors of A:

$\lambda_1 = \dfrac{1+\sqrt{5}}{2}$

$$A - \lambda_1 I = \begin{bmatrix} -\dfrac{1+\sqrt{5}}{2} & 1 \\ 1 & 1-\dfrac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \dfrac{1+\sqrt{5}}{2} \end{bmatrix}$$

$\lambda_2 = \dfrac{1-\sqrt{5}}{2}$

$$A - \lambda_2 I = \begin{bmatrix} -\dfrac{1-\sqrt{5}}{2} & 1 \\ 1 & 1-\dfrac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \dfrac{1-\sqrt{5}}{2} \end{bmatrix}$$

In conclusion,

$$V = [\,\vec{v}_1 \ \ \vec{v}_2\,] = \begin{bmatrix} 1 & 1 \\ \dfrac{1+\sqrt{5}}{2} & \dfrac{1-\sqrt{5}}{2} \end{bmatrix}$$

Let's find V^{-1} via Gauss-Seidal method:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{(1+\sqrt{5})}{2}R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -\frac{1+\sqrt{5}}{2} & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_2]{R_2 \to -R_2} \cdots$$

$$V^{-1} = \left[\begin{array}{cc} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{array}\right]$$

Thus,

$$A^n = V D^n V^{-1} = \left[\begin{array}{cc} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{array}\right] \left[\begin{array}{cc} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{array}\right] \left[\begin{array}{cc} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{array}\right]$$

$$A^n = \left[\begin{array}{cc} \left(\frac{1+\sqrt{5}}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{array}\right] \left[\begin{array}{cc} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{array}\right]$$

$$A^n = \left[\begin{array}{cc} \left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\left(\frac{1+\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{array}\right]$$

$$A^n = \left[\begin{array}{cc} -\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ -\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right] & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{array}\right]$$

(c) Given

$$\vec{v}_0 = \left[\begin{array}{c} a_0 \\ a_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

$$\vec{v}_n = A^n \vec{v}_0 = \left[\begin{array}{cc} -\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ -\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right] & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

$$\vec{v}_n = \left[\begin{array}{c} \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{array}\right]$$

It is clear that the result obeys the relation

$$\vec{v}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

which was given to us as a definition. Thus, we can use it to extract the formula for a_n

$$a_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

this is just the subtraction of the eigenvalues of A to the nth power!