Assignment 10

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Question 1

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

Denote

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

Solution is of the form $\vec{z}\, e^{\lambda t}$. $\vec{z}, \lambda$ are given by calculating the eigenvalues and eigenvectors of A. First find eigenvalues:

$$|\, A - \lambda I \,| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda - 3)\,(\lambda + 2) + 4 = \lambda^2 - \lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = 2, -1$$

Find corresponding eigenvectors. For $\lambda = 2$:

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{z}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$:

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \vec{z}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General solution is a linear combination of $\{z_i\, e^{\lambda_i t}\}$:

$$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

The solution diverges as $t \to \infty$.

Question 7

$$\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}, \quad A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

As noted in the book, A has an eigenvalue $\lambda_1 = 0$. Calculate the second eigenvalue:

$$| A - \lambda I | = \begin{vmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{vmatrix} = (\lambda - 4)(\lambda + 6) + 24 = \lambda^2 + 2\lambda = 0$$

$$\lambda_2 = -2$$

Calculate eigenvectors: for $\lambda_1 = 0$:

$$\vec{z}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

For $\lambda_2 = -2$:

$$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2\, e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

Question 11

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Note that the sum of all columns is equal (sum = 4). We can infer that $\vec{z}_1 = [1, 1, 1]^T$ is an eigenvector with an eigenvalue $\lambda_1 = 4$. Find the other eigenvalues:

$$\begin{aligned} | A - \lambda I | = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} &= (1-\lambda)\,[(2-\lambda)(1-\lambda) - 1] - [1 - \lambda - 2] + \\ &\quad 2\,[1 - 4 + 2\lambda] \\ &= (1-\lambda)\,[\lambda^2 - 3\lambda + 1] + (\lambda + 1) + (4\lambda - 6) \\ &= \lambda^2 - 3\lambda + 1 - \lambda^3 + 3\lambda^2 - \lambda + 5\lambda - 5 \\ &= -\lambda^3 + 4\lambda^2 + \lambda - 4 = 0 \end{aligned}$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

Factor out $\lambda = 4$:

$$(\lambda - 4)\,(\lambda - 1)\,(\lambda + 1) = 0$$

$$\lambda_{2,3} = 1, -1$$

Find eigenvectors. For $\lambda_2 = 1$:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_3 = -1$:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \vec{z}_3 = \vec{0}$$

$$\vec{z}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1\,\mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\,\mathrm{e}^{t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3\,\mathrm{e}^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad c_{1,2,3} \in \mathbb{R}, \quad \forall t$$

<u>Question 12</u>

$$\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}, \quad A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} \mid A - \lambda\, I \mid = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} &= (3-\lambda)\,[\lambda^2 - 3\lambda - 4] - 2\,(6 - 2\lambda - 8) + 4\,(4 + 4\lambda) \\ &= (3-\lambda)\,(\lambda - 4)\,(\lambda + 1) + 20(\lambda + 1) \\ &= -(\lambda + 1)\,(\lambda^2 - 7\lambda - 8) = -(\lambda + 1)^2\,(\lambda - 8) = 0 \end{aligned}$$

$$\lambda_{1,2,3} = -1, -1, 8$$

Find associated eigenvectors. For $\lambda_{1,2} = -1$:

$n - \operatorname{rank}(A - \lambda_{1,2}\, I) = 3 - 1 = 2$ linearly independent eigenvectors span the eigen-space:

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \vec{z}_{1,2} = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{z}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_3 = 8$:

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \vec{z}_3 = \vec{0}$$

$$\vec{z}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

General solution is:

$$\vec{x} = \mathrm{e}^{-t} \left(c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right) + c_3\, \mathrm{e}^{8t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad c_{1,2,3} \in \mathbb{R}, \quad \forall t$$

<u>Question 15</u>

$$\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find eigenvalues:

$$|\; A - \lambda\, I \;| = (\lambda - 5)\,(\lambda - 1) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)\,(\lambda - 2) = 0$$

$$\lambda_{1,2} = 4, 2$$

Find eigenvectors. For $\lambda_1 = 4$:

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1\, \mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\, \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad \forall t$$

Find $c_{1,2}$ with given ICs

$$t = 0: \quad c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Rewrite problem in matrix form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{3-1} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -1.5 \end{bmatrix}$$

The unique solution is:

$$\vec{x}(t) = \frac{7}{2} \mathrm{e}^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{2} \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \forall t$$

As $t \to \infty$ the solution diverges.

Question 29

Given the 2nd order ODE:

$$a\, y'' + b\, y' + c\, y = 0 \tag{1}$$

and its corresponding characteristic equation:

$$a\, r^2 + b\, r + c = 0 \tag{2}$$

(a) Set

$$\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$$

Differentiate:

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{b}{a} x_2 - \frac{c}{a} x_1 \end{cases}$$

Write in matrix form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \vec{x}$$

In short:

$$\vec{x}' = A\,\vec{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$$

(b) The equation that determines the eigenvalues of the coefficient matrix A is given by calculating $\det(A - \lambda\, I)$.

$$\det(A - \lambda\, I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$

Multiply by a to get:

$$a\,\lambda^2 + b\,\lambda + c = 0$$

As requested to show, we've reproduced eq. (2).

<u>Question 30</u>

$$\vec{x}' = \begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} -17 \\ -21 \end{bmatrix}$$

(a) Find eigenvalues and eigenvectors of A (coefficient matrix):

$$\det(A - \lambda\, I) = \left(\lambda + \frac{1}{10}\right)\left(\lambda + \frac{1}{5}\right) - \frac{3}{400} = \lambda^2 + \frac{3}{10}\lambda + \frac{1}{80} = 0$$

$$\lambda_{1,2} = \frac{-0.3 \pm \sqrt{0.09 - 0.05}}{2} = -0.05, -0.25$$

For $\lambda_1 = -0.05$:

$$\begin{bmatrix} -0.05 & \frac{3}{40} \\ \frac{1}{10} & -0.15 \end{bmatrix} \vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -0.25$:

$$\begin{bmatrix} 0.15 & \frac{3}{40} \\ \frac{1}{10} & 0.05 \end{bmatrix} \vec{z}_2 = \vec{0}$$

$$\vec{z}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1\,\mathrm{e}^{-0.05t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\,\mathrm{e}^{-0.25t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad c_{1,2} \in \mathbb{R}, \quad t > 0$$

Apply ICs:

$$c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -17 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}^{-1}\begin{bmatrix} -17 \\ -21 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 17 \\ 21 \end{bmatrix} = \begin{bmatrix} -\frac{55}{8} \\ \frac{29}{8} \end{bmatrix}$$

Unique solution is:

$$\vec{x} = -\frac{55}{8}\,\mathrm{e}^{-0.05t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{29}{8}\,\mathrm{e}^{-0.25t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad t > 0$$

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Questions 2,10 (no phase diagrams)

Question 2

$$\vec{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\vec{x}$$

Find eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm 4\mathrm{i}}{2} = -1 \pm 2\mathrm{i}$$

Find eigenvector for $\lambda = -1 - 2\mathrm{i}$:

$$\begin{bmatrix} 2\mathrm{i} & -4 \\ 1 & 2\mathrm{i} \end{bmatrix}\vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} -2\mathrm{i} \\ 1 \end{bmatrix}$$

Pair of solutions is:

$$\mathrm{e}^{(-1-2\mathrm{i})t}\begin{bmatrix} -2\mathrm{i} \\ 1 \end{bmatrix} = \mathrm{e}^{-t}(\cos 2t - \mathrm{i}\sin 2t)\begin{bmatrix} -2\mathrm{i} \\ 1 \end{bmatrix} = \cdots$$

$$\cdots = \mathrm{e}^{-t}\begin{bmatrix} -2\mathrm{i}\cos 2t - 2\sin 2t \\ \cos 2t - \mathrm{i}\sin 2t \end{bmatrix} = \mathrm{e}^{-t}\begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} - \mathrm{i}\,\mathrm{e}^{-t}\begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}$$

General solution is

$$\vec{x} = c_1\,\mathrm{e}^{-t}\begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2\,\mathrm{e}^{-t}\begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}, \quad c_{1,2}\in\mathbb{R}, \quad \forall t$$

As $t\to\infty$ the solution approaches zero.

Question 10

$$\vec{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Find eigenvalues and eigenvectors of the coefficient matrix.

$$\det(A-\lambda I) = \lambda^2 + 4\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm 2\mathrm{i}}{2} = -2 \pm \mathrm{i}$$

Find eigenvector for $\lambda = -2 - \mathrm{i}$:

$$\begin{bmatrix} -1+\mathrm{i} & 2 \\ -1 & 1+\mathrm{i} \end{bmatrix}\vec{z}_1 = \vec{0}$$

$$\vec{z}_1 = \begin{bmatrix} 1+\mathrm{i} \\ 1 \end{bmatrix}$$

Pair of solutions is:

$$\mathrm{e}^{(-2-\mathrm{i})t}\begin{bmatrix} 1+\mathrm{i} \\ 1 \end{bmatrix} = \mathrm{e}^{-2t}(\cos t - \mathrm{i}\sin t)\begin{bmatrix} 1+\mathrm{i} \\ 1 \end{bmatrix} = \cdots$$

$$\cdots = \mathrm{e}^{-2t}\begin{bmatrix} \cos t + \mathrm{i}\cos t - \mathrm{i}\sin t + \sin t \\ \cos t - \mathrm{i}\sin t \end{bmatrix} = \mathrm{e}^{-2t}\begin{bmatrix} \cos t + \sin t \\ \mathrm{co\,st} \end{bmatrix} + \mathrm{i}\,\mathrm{e}^{-2t}\begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix}$$

General solution is:

$$\vec{x} = c_1\,\mathrm{e}^{-2t}\begin{bmatrix} \cos t + \sin t \\ \mathrm{co\,st} \end{bmatrix} + c_2\,\mathrm{e}^{-2t}\begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix}, \quad c_{1,2}\in\mathbb{R}, \quad \forall t$$

As $t\to\infty$ the solution approaches zero.