

# Assignment 12

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## Question 1

Find the Fourier series for  $f(x) = \sin x + \cos x$  on  $[-\pi, \pi]$ .

Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

Calculate Fourier coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin x + \cos x) dx = [-\cos x + \sin x]_{-\pi}^{\pi} = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos(nx) dx + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos(nx) dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sin(x+nx) + \sin(x-nx)] dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(x+nx) + \cos(x-nx)] dx \end{aligned}$$

The integral over the sines is equal to zero because it is an integral of an odd function over a symmetric interval. Calculate the integral of the cosines.

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[ \frac{\sin(x+nx)}{1+n} + \frac{\sin(x-nx)}{1-n} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[ \frac{\sin(\pi+n\pi) - \sin 0}{1+n} + \frac{\sin(\pi-n\pi) - \sin 0}{1-n} \right] = 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin(nx) dx + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin(nx) dx \end{aligned}$$

The second integral is equal to zero because it's an integral of an odd function over a symmetric interval. First integral is:

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{\pi} [\cos(x-nx) - \cos(x+nx)] dx \\ &= \frac{1}{\pi} \left[ \frac{\sin(x-nx)}{1-n} - \frac{\sin(x+nx)}{1+n} \right]_0^{\pi} = 0 \end{aligned}$$

All coefficients are zero! The Fourier series is  $f(x) = 0$ .

## Question 2

Fourier series for

$$f(x) = -x, \quad [-L, L]$$

Calculate Fourier coefficients.

$$a_0 = \int_{-L}^L (-x) dx = [f(x) \text{ odd}] = 0$$

$$a_n = - \int_{-L}^L x \cos \frac{n \pi x}{L} dx$$

Integrate by parts:

$$\int x \cos \frac{n \pi x}{L} dx = \frac{L x}{n \pi} \sin \left( \frac{n \pi x}{L} \right) - \frac{L^2}{n^2 \pi^2} \cos \left( \frac{n \pi x}{L} \right)$$

$$\begin{aligned} a_n &= - \frac{L}{n \pi} \left[ x \sin \left( \frac{n \pi x}{L} \right) - \frac{L}{n \pi} \cos \left( \frac{n \pi x}{L} \right) \right]_{-L}^L \\ &= - \frac{L}{n \pi} \left[ (L \sin(n \pi) + L \sin(-n \pi)) - \frac{L}{n \pi} (\cos(n \pi) - \cos(-n \pi)) \right] = 0 \end{aligned}$$

$$b_n = - \int_{-L}^L x \sin \frac{n \pi x}{L} dx$$

Integrate by parts:

$$\int x \sin \frac{n \pi x}{L} dx = - \frac{L x}{n \pi} \cos \left( \frac{n \pi x}{L} \right) - \frac{L^2}{n^2 \pi^2} \sin \left( \frac{n \pi x}{L} \right)$$

$$\begin{aligned} b_n &= - \frac{L}{n \pi} \left[ x \cos \left( \frac{n \pi x}{L} \right) + \frac{L}{n \pi} \sin \left( \frac{n \pi x}{L} \right) \right]_{-L}^L \\ &= - \frac{L}{n \pi} \left[ (L \cos(n \pi) + L \cos(-n \pi)) + \frac{L}{n \pi} (\sin(n \pi) + \sin(-n \pi)) \right] = - \frac{2L^2 \cos(n \pi)}{n \pi} \end{aligned}$$

$$b_n = - \frac{2L^2}{n \pi} (-1)^n$$

Fourier series for  $f(x)$  on given interval is

$$f(x) = - \sum_{n=1}^{\infty} \frac{2L^2}{n \pi} (-1)^n \sin \frac{n \pi x}{L}$$

### Question 3

Fourier series for

$$f(x) = \begin{cases} x & -\pi \leq x \leq 0 \\ 0 & 0 \leq x < \pi \end{cases}$$

on  $[-\pi, \pi]$ . Calculate Fourier coefficients.

$$a_0 = \int_{-\pi}^0 x dx + 0 = - \frac{\pi^2}{2}$$

$$a_n = \int_{-\pi}^0 x \cos(n x) dx + 0 = - \frac{1}{n} \left[ x \sin(n x) - \frac{1}{n} \cos(n x) \right]_{-\pi}^0 = \frac{1 - \cos(n \pi)}{n^2} = \frac{1 - (-1)^n}{n^2}$$

$$b_n = \int_{-\pi}^0 x \sin(n x) dx + 0 = - \frac{1}{n} \left[ x \cos(n x) + \frac{1}{n} \sin(n x) \right]_{-\pi}^0 = - \frac{\pi}{n} \cos(\pi n) = - \frac{\pi}{n} (-1)^n$$

Fourier series in given interval is

$$f(x) = -\frac{\pi^2}{4} + \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{n^2} \cos(nx) - \frac{\pi}{n} (-1)^n \sin(nx) \right)$$

#### Question 4

Fourier series for

$$f(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ -(1-t) & 1 \leq t < 2 \end{cases}$$

on  $[0, 2]$ . Calculate Fourier coefficients.

$$a_0 = \int_0^1 (1-t) dt - \int_1^2 (1-t) dt = \left[ t - \frac{1}{2} t^2 \right]_0^1 - \left[ t - \frac{1}{2} t^2 \right]_1^2 = 1$$

$$a_n = \int_0^1 (1-t) \cos(n\pi t) - \int_1^2 (1-t) \cos(n\pi t)$$

$$\int (1-t) \cos(n\pi t) = \frac{\sin(n\pi t)}{n\pi} (1-t) - \frac{\cos(n\pi t)}{n^2 \pi^2}$$

$$a_n = -\frac{1}{n\pi} \left[ (1-t) \sin(n\pi t) + \frac{1}{n\pi} \cos(n\pi t) \right]_0^1 + \frac{1}{n\pi} \left[ (1-t) \sin(n\pi t) + \frac{1}{n\pi} \cos(n\pi t) \right]_1^2$$

$$a_n = -\frac{1}{n\pi} \left[ \frac{(-1)^n}{n\pi} - \frac{1}{n\pi} \right] + \frac{1}{n\pi} \left[ \frac{1}{n\pi} - \frac{(-1)^n}{n\pi} \right] = \frac{2}{n^2 \pi^2} [1 - (-1)^n]$$

$$b_n = \int_0^1 (1-t) \sin(n\pi t) - \int_1^2 (1-t) \sin(n\pi t)$$

$$\int (1-t) \sin(n\pi t) = -\frac{\cos(n\pi t)}{n\pi} (1-t) - \frac{\sin(n\pi t)}{n^2 \pi^2}$$

$$b_n = -\frac{1}{n\pi} \left[ (1-t) \cos(n\pi t) + \frac{\sin(n\pi t)}{n\pi} \right]_0^1 + \frac{1}{n\pi} \left[ (1-t) \cos(n\pi t) + \frac{\sin(n\pi t)}{n\pi} \right]_1^2$$

$$b_n = \frac{1}{n\pi} - \frac{1}{n\pi} = 0$$

Fourier series for  $f(t)$  on given interval is

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n\pi t)$$

Another formula for  $f(t)$ :

$$f(t) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{4}{n^2 \pi^2} \cos(n\pi t)$$

#### Question 5

Given the initial value problem:

$$y'' + 4y = f(t), \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

where  $f(t)$  is as in Question 4.

(i) Find a particular solution  $y_p$  of the form

$$y_p = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)]$$

Now we plug in  $y_p$  in the ODE and equate coefficients. By observation of  $f(t)$  and the ODE it is clear that

$$4 \cdot \frac{a_0}{2} = \frac{1}{2} \rightarrow a_0 = \frac{1}{4}$$

Additionally, as there is only a second derivative of  $y_p$ , we can infer that  $b_n = 0$ . (Second derivative of sine and cosine returns scalar multiples of sine and cosine, respectively).

$$y_p'' = \sum_{n=1}^{\infty} (-n^2 \pi^2 a_n) \cos(n\pi t)$$

Plug in the ODE:

$$\begin{aligned} \sum_{n=1}^{\infty} (-n^2 \pi^2 a_n) \cos(n\pi t) + 4 \left( \frac{1}{8} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) \right) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n\pi t) \\ \sum_{n=1}^{\infty} [-n^2 \pi^2 + 4] a_n \cos(n\pi t) &= \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n\pi t) \\ a_n &= \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \end{aligned}$$

The particular solution  $y_p$  is:

$$y_p = \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \cos(n\pi t) \quad \forall t$$

(ii) Find general solution to associated homogeneous equation. Characteristic equation is:

$$\lambda^2 + 4 = 0$$

$$\lambda_{1,2} = \pm 2i$$

General, real solution to the homogeneous equation,  $y_h$ , is:

$$y_h = c_1 \cos(2t) + c_2 \sin(2t) \quad c_{1,2} \in \mathbb{R}, \forall t$$

(iii) General solution to ODE is

$$y = y_h + y_p = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \cos(n\pi t), \quad c_{1,2} \in \mathbb{R}, \forall t$$

Substitute initial conditions:  $y(0) = 1, y'(0) = 0$ .

$$y(0) = c_1 + \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} = 1$$

$$c_1 = \frac{7}{8} - \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)}$$

$$y'(0) = 2c_2 \rightarrow c_2 = 0$$

Unique solution satisfying ICs is:

$$y = \frac{1}{8} + \left( \frac{7}{8} - \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \right) \cos(2t) + \sum_{n=1}^{\infty} \frac{2 [1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \cos(n\pi t), \quad \forall t$$