Assignment 1

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Exercise 1.

(a) Prove De Moivre's theorem by induction for all natural numbers n: if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ then $z^n = r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta))$.

(b) Use part (a) to show the following generalization for all natural numbers n:

If $z^n = r(\cos\theta + \mathrm{i}\sin\theta)$ then $z = r^{1/n}\left(\cos\frac{\theta+2\pi k}{n} + \mathrm{i}\sin\frac{\theta+2\pi k}{n}\right)$ for k a natural number, $0 \leq k \leq n-1$.

Solution.

(a) To prove by induction, we must first prove the statement is true for the lowest natural number. Given

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \tag{1}$$

plugging in $n=1$ into De Moivre's theorem we get the (trivial) equality :

$$z^1 = r^1(\cos(1\theta) + \mathrm{i}\sin(1\theta)) \equiv z = r(\cos\theta + \mathrm{i}\sin\theta) \tag{2}$$

Now we must show that if the theorem is true for n, it must also hold for $n+1$.

Plugging in $n+1$ into De Moivre's theorem we get:

$$\begin{aligned} z^{n+1} &= r^{n+1}(\cos([n+1]\theta) + \mathrm{i}\sin([n+1]\theta)) \\ z \cdot z^n &= r \cdot r^n(\cos([n+1]\theta) + \mathrm{i}\sin([n+1]\theta)) \end{aligned}$$

We may enter $z = r(\cos\theta + \mathrm{i}\sin\theta)$ into the LHS:

$$\cancel{r}(\cos\theta + \mathrm{i}\sin\theta) \cdot z^n = \cancel{r} \cdot r^n(\cos([n+1]\theta) + \mathrm{i}\sin([n+1]\theta)) \tag{3}$$

By the induction hypothesis,

$$z^n = r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta)) \tag{4}$$

plugging (4) into (3) we get:

$$(\cos\theta + \mathrm{i}\sin\theta)\,\cancel{r^n}(\cos(n\theta) + \mathrm{i}\sin(n\theta)) = \cancel{r^n}(\cos([n+1]\theta) + \mathrm{i}\sin([n+1]\theta)) \tag{5}$$

Let's focus on the LHS:

$$(\cos\theta + \mathrm{i}\sin\theta)\,(\cos(n\theta) + \mathrm{i}\sin(n\theta)) = \cos\theta(\cos(n\theta) + \mathrm{i}\sin(n\theta)) + \mathrm{i}\sin\theta\,(\cos(n\theta) + \mathrm{i}\sin(n\theta)) \tag{6}$$

$$\cos\theta(\cos(n\theta) + \mathrm{i}\sin(n\theta)) = \cos\theta\cos(n\theta) + \mathrm{i}\cos\theta\sin(n\theta) \tag{7}$$

$$\mathrm{i}\sin\theta\,(\cos(n\theta) + \mathrm{i}\sin(n\theta)) = \mathrm{i}\sin\theta\cos(n\theta) - \sin\theta\sin(n\theta) \tag{8}$$

Adding (7) and (8) and using trigonometric identities we get:

$$\cos\theta\cos(n\theta) - \sin\theta\sin(n\theta) = \cos(\theta + n\theta) = \cos([n+1]\theta) \tag{9}$$

$$\mathrm{i}\,[\cos\theta\sin(n\theta) + \sin\theta\cos(n\theta)] = \mathrm{i}\,[\sin(\theta + n\theta] = \mathrm{i}\sin([n+1]\theta) \tag{10}$$

Adding (9) and (10) matches between the LHS and RHS of eq. (5), thus concluding that if $z^n = r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta))$ then also $z^{n+1} = r^{n+1}(\cos([n+1]\theta) + \mathrm{i}\sin([n+1]\theta))$, thus proving De Moivre's theorem by induction. $\square$

(b) We start from the relation:

$$z = r^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + \mathrm{i}\sin\frac{\theta + 2\pi k}{n}\right) \tag{11}$$

We may raise both sides of the equation by n, where $n \in \mathbb{N}$.

$$z^n = [r^{1/n}]^n\left(\cos\frac{\theta + 2\pi k}{n} + \mathrm{i}\sin\frac{\theta + 2\pi k}{n}\right)^n \tag{12}$$

$$z^n = r\left(\cos\frac{\theta + 2\pi k}{n} + \mathrm{i}\sin\frac{\theta + 2\pi k}{n}\right)^n$$

We may use the relation $z^{n'} = r^{n'}(\cos(n'\theta) + \mathrm{i}\sin(n'\theta))$, which was proved at part (a):

$$z^n = r\left[\cos\left(n\frac{\theta + 2\pi k}{n}\right) + \mathrm{i}\sin\left(n\frac{\theta + 2\pi k}{n}\right)\right] \tag{13}$$

$$z^n = r\,[\cos(\theta + 2\pi k) + \mathrm{i}\sin(\theta + 2\pi k)] \tag{14}$$

Remember that $\cos(\theta + 2\pi k) = \cos\theta$ and that $\sin(\theta + 2\pi k) = \sin\theta$ if $k \in \mathbb{N}$. This is because adding $2\pi k$ to angle θ essentially rotates the unit vector by whole multiples of a full circle, bringing it back to the starting point determined by θ, rendering the vector defined by $\sin\theta$ and $\cos\theta$ unchanged.

Thus, we can rewrite equation (14) and get:

$$z^n = r(\cos\theta + \mathrm{i}\sin\theta) \tag{15}$$

Which we were instructed to assume is right.□

Exercise 2. Find the three complex cubed roots of $z = 3 - i\sqrt{27}$.

Solution. We may use eq. (11) to calculate the roots of z, where $n = 3$ is the number of roots.

$$z_k = r^{1/3}\left(\cos\frac{\theta + 2\pi k}{3} + \mathrm{i}\sin\frac{\theta + 2\pi k}{3}\right), \quad k = 0, 1, 2$$

First let's calculate θ and r. for a complex number $z' = a + \mathrm{i}\,b$, r and θ are calculated as follows:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} && (16)\\ \theta &= \arctan\frac{b}{a} && (17) \end{aligned}$$

In our case, $r = \sqrt{3^2 + 27} = 6$ and $\theta = \arctan -\frac{\sqrt{27}}{3} = -\frac{\pi}{3}$. Inputting the different values of k we get the three complex roots of z:

$$\begin{aligned} z_{k=0} &= 6^{1/3}\left[\cos\left(\frac{-\frac{\pi}{3} + 0}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 0}{3}\right)\right] = 6^{1/3}\left(\cos\frac{\pi}{9} - \mathrm{i}\sin\frac{\pi}{9}\right)\\ &\approx 1.707 - 0.621\,\mathrm{i}\\ z_{k=1} &= 6^{1/3}\left[\cos\left(\frac{-\frac{\pi}{3} + 2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 2\pi}{3}\right)\right] = 6^{1/3}\left(\cos\frac{5\pi}{9} + \mathrm{i}\sin\frac{5\pi}{9}\right)\\ &\approx -0.315 + 1.789\,\mathrm{i}\\ z_{k=2} &= 6^{1/3}\left[\cos\left(\frac{-\frac{\pi}{3} + 4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 4\pi}{3}\right)\right] = 6^{1/3}\left(\cos\frac{11\pi}{9} + \mathrm{i}\sin\frac{11\pi}{9}\right)\\ &\approx -1.392 - 1.168\,\mathrm{i} \end{aligned}$$

Exercise 3. Calculate the Taylor series for the following functions $f(x)$ at the points a where:

1. $f(x) = 2x^5 - 7x^2 + 4x - 3, \quad a = 0; \quad a = 1.$

2. $f(x) = \sin x + 2\cos x, \quad a = 0.$

3. $f(x) = \sin(x^2), \quad a = 0.$

4. $f(x) = \sqrt{x^2+1}, \quad a = 0$ — only the first 3 terms.

Solution. To calculate the Taylor series we use Taylor's theorem:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{18}$$

1. $f(x) = 2x^5 - 7x^2 + 4x - 3$ has only 5 non-zero derivatives:

   i. $f'(x) = 10x^4 - 14x + 4, \quad f'(0) = 4, \quad f'(1) = 0.$

   ii. $f''(x) = 40x^3 - 14, \quad f''(0) = -14, \quad f''(1) = 26.$

   iii. $f^{(3)}(x) = 120x^2, \quad f^{(3)}(0) = 0, \quad f^{(3)}(1) = 120.$

   iv. $f^{(4)}(x) = 240x, \quad f^{(4)}(0) = 0, \quad f^{(4)}(1) = 240.$

   v. $f^{(5)}(x) = 240 \quad \forall x$

inputting these into (18) gives $[f(0) = -3, \quad f(1) = -4]$ the following Taylor series:

   i. for $a = 0$:

$$\boxed{\begin{aligned} f(x; a=0) &= -3 + \frac{4}{1!}x - \frac{14}{3!}x^3 + \frac{240}{5!}x^5 \\ &= -3 + 4x - \frac{7}{3}x^3 + 2x^5 \end{aligned}}$$

   ii. for $a = 1$:

$$\boxed{\begin{aligned} f(x; a=1) &= -4 + \frac{26}{2!}(x-1)^2 + \frac{120}{3!}(x-1)^3 + \frac{240}{4!}(x-1)^4 + \frac{240}{5!}(x-1)^5 \\ &= -4 + 13(x-1)^2 + 20(x-1)^3 + 10(x-1)^4 + 2(x-1)^5 \end{aligned}}$$

2. $f(x) = \sin x + 2\cos x$. Let's calculate the derivatives at $a = 0$:

$$\begin{aligned} f(0) &= 2 \\ f'(x) &= \cos x - 2\sin x; \quad f'(0) = 1 \\ f''(x) &= -\sin x - 2\cos x; \quad f''(0) = -2 \\ f^{(3)}(x) &= -\cos x + 2\sin x; \quad f^{(3)}(0) = -1 \\ f^{(4)}(x) &= \sin x + 2\cos x; \quad f^{(4)}(0) = 2 \end{aligned}$$

We see that every 4 derivations the original function is retrieved. Therefore:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = 2 + x - 2\frac{x^2}{2!} - \frac{x^3}{3!} + 2\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

This series can actually be written as a sum of two sub-series, one for even n (blue) and one for odd n (red):

$$\boxed{f(x; a=0) = 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

3. $f(x) = \sin(x^2)$. Let's calculate the derivatives at $a = 0$:

$$\begin{aligned}
f(0) &= 0 \\
f'(x) &= 2x\cos(x^2); \quad f'(0) = 0 \\
f''(x) &= 2\cos(x^2) - 4x^2\sin(x^2); \quad f''(0) = 2 \\
f^{(3)}(x) &= -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2); \quad f^{(3)}(0) = 0 \\
f^{(4)}(x) &= -4\sin(x^2) - 8x^2\cos(x^2) - 8\sin(x^2) - 16x^2\cos(x^2) - 24x^2\cos(x^2) + \\
&\quad 16x^4\sin(x^2); \quad f^{(4)}(0) = 0 \\
f^{(5)}(x) &= -8x\cos(x^2) - 16x\cos(x^2) + 16x^3\sin(x^2) - 16x\cos(x^2) - 32x\cos(x^2) + \\
&\quad 32x^3\sin(x^2) - 48x\cos(x^2) + 48x^3\sin(x^2) + 64x^3\sin(x^2) + 32x^5\cos(x^2); \\
&\qquad f^{(5)}(0) = 0 \\
f^{(6)}(x) &= -120\cos(x^2) + \cdots +; \quad f^{(6)}(0) = -120
\end{aligned}$$

We see that every $4n+2$ derivatives a non-zero $f^{(n)}(0)$ is obtained. Also, if we write the Taylor series up to the 7th element we get:

$$\begin{aligned}
f(x) &\approx \frac{2x^2}{2!} - \frac{120x^6}{6!} + \cdots + \\
&= \frac{x^2}{1!} - \frac{x^6}{3!} + \cdots +
\end{aligned}$$

This pattern can be generalized to:

$$\boxed{f(x; a=0) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$$

which is actually what we get if we substitute x for x^2 in the Taylor series we developed for $\sin x$.

4. $f(x) = \sqrt{x^2+1}$ — only the first three terms.

$$\begin{aligned}
f(0) &= 1 \\
f'(x) &= \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}; \quad f'(0) = 0 \\
f''(x) &= (x^2+1)^{-1/2} - x^2(x^2+1)^{-3/2}; \quad f''(0) = 1 \\
f'''(x) &= -3x(x^2+1)^{-3/2} + 3x^3 \cdot (x^2+1)^{-5/2}; \quad f'''(0) = 0 \\
f^{(4)}(x) &= -3(x^2+1)^{-3/2} + 18x^2(x^2+1)^{-5/2} - 15x^4(x^2+1)^{-7/2}; \quad f^{(4)}(0) = -3
\end{aligned}$$

$$\boxed{\begin{aligned}
f(x; a=0) &= 1 + \frac{x^2}{2!} - \frac{3x^4}{4!} + \cdots + \\
&= 1 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots +
\end{aligned}}$$