Assignment 13

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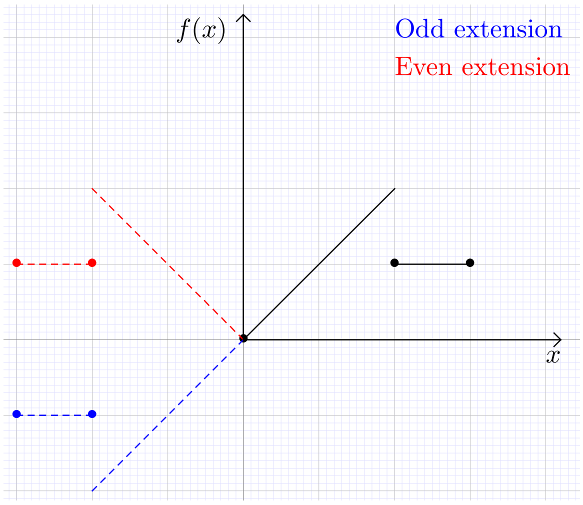
Question 7

$$f(x) = \begin{cases} x, & x \in [0, 2) \\ 1, & x \in [2, 3) \end{cases}$$

For a function $f(x), x \in [0, L]$, its odd extension is $f(-x) = -f(x)$ and its even extension is $f(-x) = f(x)$.



Question 17

Cosine series of period 2π.

$$f(x) = 1 \quad x \in [0, \pi]$$

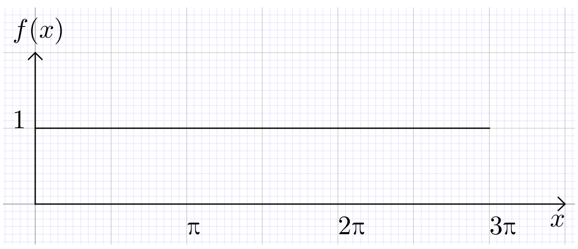
For a cosine series b_n need not be calculated.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \mathrm{d}x = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \cos(n\,x)\,\mathrm{d}x = 2\left[\frac{\sin(n\,x)}{n\,\pi}\right]_0^{\pi} = 0$$

Fourier series is just

$$f(x) = \frac{a_0}{2} = 1 \quad \forall x$$



Question 18

Sine series of period 2π.

$$f(x) = 1 \qquad x \in (0, \pi)$$

For a sine series only b_n need be calculated.

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(n\,x)\,\mathrm{d}x = -\frac{2}{n\,\pi}\,[\cos(n\,x)]_0^{\pi} = -\frac{2\,[(-1)^n - 1]}{n\,\pi}$$

Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2\,[1 - (-1)^n]}{n\,\pi}\sin(n\,x)$$

Every $x = k\,\pi, k \in \mathbb{N}$, $f(x)$ passes through zero, thus changing sign. The series looks (roughly) like this:

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Question 7

The following heat conduction problem is given:

$$100\,u_{xx} = u_t \quad x \in (0,1), \;\; t > 0$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0$$

$$u(x,0) = f(x) = \sin(2\,\pi\,x) - \sin(5\,\pi\,x) \qquad x \in [0,1]$$

We've seen in class that the solution satisfying homogeneous BCs is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\!\left(\frac{-\alpha^2\,\pi^2\,n^2}{L^2}\,t\right)\sin\left(\frac{\pi\,n}{L}\,x\right)$$

Here $\alpha = 10, L = 1$.

where c_n is given by:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

In our case $L=1$. Calculate c_n:

$$
\begin{aligned}
c_n &= 2\int_0^1 [\sin(2\pi x) - \sin(5\pi x)]\sin(n\pi x)\,\mathrm{d}x \\
&= 2\int_0^1 \sin(2\pi x)\sin(n\pi x)\,\mathrm{d}x - 2\int_0^1 \sin(5\pi x)\sin(n\pi x)\,\mathrm{d}x \\
&= \int_0^1 \cos((2-n)\pi x) - \cos((2+n)\pi x) - \int_0^1 \cos((5-n)\pi x) - \cos((5+n)\pi x) \\
&= \left[\frac{\sin((2-n)\pi x)}{(2-n)\pi} - \frac{\sin((2+n)\pi x)}{(2+n)\pi}\right]_0^1 - \left[\frac{\sin((5-n)\pi x)}{(5-n)\pi} - \frac{\sin((5+n)\pi x)}{(5+n)\pi}\right]_0^1
\end{aligned}
$$

$$
c_n = \begin{cases} 0, & n \neq 2, 5 \\ 1, & n = 2, 5 \end{cases}
$$

Therefore,

$$
u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(\frac{-\alpha^2\pi^2 n^2}{L^2}t\right)\sin\left(\frac{\pi n}{L}x\right)
$$

$$
u(x,t) = 2\,\mathrm{e}^{-400\pi^2 t}\sin(2\pi x) + 5\,\mathrm{e}^{-2500\pi^2 t}\sin(5\pi x)
$$

Question 10

Given $\alpha = 1, L = 40$,

$$
u(x,0) = f(x) = \begin{cases} x, & x \in [0,20] \\ 40 - x, & x \in [20,40] \end{cases}
$$

We are asked to solve the heat equation. $u(x,t)$ is given by:

$$
u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(\frac{-\alpha^2\pi^2 n^2}{L^2}t\right)\sin\left(\frac{\pi n}{L}x\right)
$$

Calculate c_n.

$$
c_n = \frac{2}{40}\int_0^{20} x\sin\frac{n\pi x}{40}\,\mathrm{d}x + \frac{2}{40}\int_{20}^{40}(40-x)\sin\frac{n\pi x}{40}\,\mathrm{d}x
$$

Calculate integrals by parts.

$$
\int x\sin\frac{n\pi x}{40} = -\frac{40}{n\pi}x\cos\frac{n\pi x}{40} + \frac{1600}{n^2\pi^2}\sin\frac{n\pi x}{40}
$$

$$
\begin{aligned}
c_n = {} & \frac{1}{20}\frac{40}{n\pi}\left[-x\cos\frac{n\pi x}{40} + \frac{40}{n\pi}\sin\frac{n\pi x}{40}\right]_0^{20} - \frac{1}{20}\frac{40}{n\pi}\left[-x\cos\frac{n\pi x}{40} + \frac{40}{n\pi}\sin\frac{n\pi x}{40}\right]_{20}^{40} \\
& -\left[\frac{40}{20}\frac{40}{n\pi}\cos\frac{n\pi x}{40}\right]_{20}^{40}
\end{aligned}
$$

$$
\begin{aligned}
c_n = {} & \frac{2}{n\pi}\left[-20\cos\left(\frac{n\pi}{2}\right) + \frac{40}{n\pi}\sin\left(\frac{n\pi}{2}\right) + 40\cos(n\pi) - 20\cos\left(\frac{n\pi}{2}\right) + \frac{40}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] \\
& + \frac{2}{n\pi}\left[-40\cos(n\pi) + 40\cos\left(\frac{n\pi}{2}\right)\right]
\end{aligned}
$$

$$
c_n = \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
$$

Therefore,

$$u(x,t)=\sum_{n=1}^{\infty} c_n \exp\left(\frac{-\alpha^2\,\pi^2\,n^2}{L^2}\,t\right)\sin\left(\frac{\pi\,n}{L}\,x\right)$$

$$u(x,t)=\sum_{n\text{ odd}} \frac{160}{n^2\,\pi^2}\sin\left(\frac{n\,\pi}{2}\right)\exp\left(\frac{-\pi^2\,n^2}{1600}\,t\right)\sin\left(\frac{\pi\,n\,x}{40}\right)$$

Question 18(a)

Given $\alpha^2 = 1.71$ and $L = 20$. Also, $u(x,0) = 100$ and the shenanigans about the plunging the rod ends to a cold bath are just a preposterous excuse to claim: $u(0,t) = u(L,t) = 0$, i.e. homogeneous BCs.

We are asked to calculate $u(10,30)$.

$$u(x,t)=\sum_{n=1}^{\infty} c_n \exp\left(\frac{-\alpha^2\,\pi^2\,n^2}{L^2}\,t\right)\sin\left(\frac{\pi\,n}{L}\,x\right)$$

Calculate c_n:

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x=\frac{1}{10}\int_0^{20} 100\sin\frac{n\,\pi\,x}{20}\,\mathrm{d}x=\frac{200}{n\,\pi}\left[-\cos\frac{n\,\pi\,x}{20}\right]_0^{20}=\frac{200}{n\,\pi}(1-(-1)^n)$$

In other words,

$$c_n=\begin{cases}0, & n\text{ even}\\ \frac{400}{n\,\pi}, & n\text{ odd}\end{cases}$$

So,

$$u(x,t)=\sum_{n\text{ odd}} \frac{400}{n\,\pi}\exp\left(\frac{-1.71\,\pi^2\,n^2}{400}\,t\right)\sin\left(\frac{\pi\,n\,x}{20}\right)$$

Josie instructed to take only the first term ($n=1$) as an estimate of $u(x,t)$.

$$u(10,30)\approx\frac{400}{\pi}\exp\left(\frac{-1.71\,\pi^2}{400}\cdot 30\right)\sin\left(\frac{3\,\pi}{2}\right)\approxeq -35.695$$

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Question 14(a)

The system is described as follows: $L = 30, \alpha^2 = 1,\ u_x(0,t) = u_x(L,t) = 0$,

$$u(x,0)=f(x)=\begin{cases}25, & x\in(5,10)\\ 0, & \text{otherwise}\end{cases},\ x\in[0,30]$$

We are asked to find an expression for $u(x,t)$. The formula given in class for a bar with insulated ends is:

$$u(x,t)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n \exp\left(\frac{-\alpha^2\,\pi^2\,n^2}{L^2}\,t\right)\cos\left(\frac{n\,\pi\,x}{L}\right)$$

where the ICs satisfy:

$$f(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty} c_n\cos\left(\frac{n\,\pi\,x}{L}\right)$$

and

$$c_0 = \frac{2}{L} \int_0^L f(x)\, \mathrm{d}x$$

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\,\pi\,x}{L}\, \mathrm{d}x$$

Calculate c_n and c_0.

$$c_n = \frac{1}{15} \int_5^{10} 25 \cos \frac{n\,\pi\,x}{30}\, \mathrm{d}x = \frac{50}{n\,\pi} \left[\sin\left(\frac{n\,\pi\,x}{30}\right) \right]_5^{10} = \frac{50}{n\,\pi} \left[\sin\left(\frac{n\,\pi}{3}\right) - \sin\left(\frac{n\,\pi}{6}\right) \right]$$

We may simplify c_n by using the identity

$$\cos(a) \sin(b) = \frac{1}{2} [\sin(a+b) - \sin(a-b)]$$

$$\begin{aligned} a + b &= \frac{n\,\pi}{3} \\ a - b &= \frac{n\,\pi}{6} \end{aligned}$$

$$a = \frac{n\,\pi}{4}, b = \frac{n\,\pi}{12}$$

$$\Rightarrow c_n = \frac{25}{n\,\pi} \cos\left(\frac{n\,\pi}{4}\right) \sin\left(\frac{n\,\pi}{12}\right)$$

$$c_0 = \frac{1}{15} \int_5^{10} 25\, \mathrm{d}x = \frac{25 \cdot 5}{15} = \frac{25}{3}$$

Therefore,

$$u(x,t) = \frac{25}{6} + \sum_{n=1}^{\infty} \frac{25}{n\,\pi} \cos\left(\frac{n\,\pi}{4}\right) \sin\left(\frac{n\,\pi}{12}\right) \exp\left(\frac{-\pi^2\, n^2}{900}\, t\right) \cos\left(\frac{n\,\pi\,x}{30}\right)$$

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Question 1(a)

As we've seen in class, the fundamental solution satisfying the wave equation and homogeneous BCs in case of zero initial velocity is

$$u_n(x,t) = \sin\left(\frac{n\,\pi\,x}{L}\right) \cos\left(\frac{n\,a\,\pi\,t}{L}\right), \quad n = 1, 2, 3, \ldots$$

The solution $u(x,t)$ is given as a spectral expansion of u_n:

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, u_n(x,t)$$

where c_n satisfyies

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\,\pi\,x}{L}\right) \mathrm{d}x$$

$f(x) = u(x,0)$. In our case $L = 10, a = 1$, and

$$f(x) = \begin{cases} \frac{x}{5}, & x \in [0,5] \\ \frac{10-x}{5}, & x \in [5,10] \end{cases}$$

We're asked to find an expression for $u(x,t)$. Calculate c_n:

$$c_n = \frac{1}{5}\int_0^5 \frac{1}{5} x \sin\left(\frac{n\,\pi\,x}{10}\right) \mathrm{d}x + \frac{1}{5}\int_5^{10} \frac{1}{5}(10-x)\sin\left(\frac{n\,\pi\,x}{10}\right)\mathrm{d}x$$

It might prove useful to calculate the integral more generaly:

$$\int x \sin\left(\frac{n\,\pi\,x}{L}\right)\mathrm{d}x = [\text{integration by parts}] = -\frac{L}{n\,\pi}\left[-\cos\left(\frac{n\,\pi\,x}{L}\right)x + \frac{L}{n\,\pi}\sin\left(\frac{n\,\pi\,x}{L}\right)\right]$$

Therefore,

$$\begin{aligned} c_n &= \frac{1}{25}\left(\frac{10}{n\,\pi}\left[-\cos\left(\frac{n\,\pi\,x}{10}\right)x + \frac{10}{n\,\pi}\sin\left(\frac{n\,\pi\,x}{10}\right)\right]_0^5 - \left[-\cos\left(\frac{n\,\pi\,x}{10}\right)x + \frac{10}{n\,\pi}\sin\left(\frac{n\,\pi\,x}{10}\right)\right]_5^{10}\right) \\ &\quad -\frac{10}{25}\left[\frac{10}{n\,\pi}\cos\left(\frac{n\,\pi\,x}{10}\right)\right]_5^{10} \end{aligned}$$

$$\begin{aligned} c_n &= \frac{2}{5\,n\,\pi}\left[-5\cos\left(\frac{n\,\pi}{2}\right) + \frac{10}{n\,\pi}\sin\left(\frac{n\,\pi}{2}\right) + 10\cos(n\,\pi) - 5\cos\left(\frac{n\,\pi}{2}\right) + \frac{10}{n\,\pi}\sin\left(\frac{n\,\pi}{2}\right)\right] \\ &\quad + \frac{2}{5\,n\,\pi}\left[-10\cos(n\,\pi) + 10\cos\left(\frac{n\,\pi}{2}\right)\right] \end{aligned}$$

$$c_n = \frac{8}{n^2\,\pi^2}\sin\left(\frac{n\,\pi}{2}\right)$$

Input value of c_n into formula for $u(x,t)$ to get:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\,\pi^2}\sin\left(\frac{n\,\pi}{2}\right)\sin\left(\frac{n\,\pi\,x}{10}\right)\cos\left(\frac{n\,\pi\,t}{10}\right)$$

Question 8(a)

Given $L = 10, a = 1$ and ICs

$$u_t(x,0) = g(x) = \begin{cases} 1, & x \in [4,6] \\ 0, & \text{otherwise} \end{cases}$$

This is a case of zero initial position, The formula developed in class for $u(x,t)$ in case of homogeneous BCs is:

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\,\pi\,x}{L}\right)\sin\left(\frac{n\,\pi\,a\,t}{L}\right)$$

where k_n satisfies

$$k_n = \frac{2}{n\,\pi\,a}\int_0^L g(x)\sin\left(\frac{n\,\pi\,x}{L}\right)\mathrm{d}x$$

In our case,

$$k_n = \frac{2}{n\,\pi}\int_4^6 \sin\left(\frac{n\,\pi\,x}{10}\right)\mathrm{d}x = \frac{20}{n^2\,\pi^2}\left[\cos\left(\frac{2\,n\,\pi}{5}\right) - \cos\left(\frac{3\,n\,\pi}{5}\right)\right]$$

We can further simpify the expression for k_n using the identity

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

Here,

$$\begin{aligned} a-b &= \frac{2\,n\,\pi}{5} \\ a+b &= \frac{3\,n\,\pi}{5} \end{aligned}$$

$$\Rightarrow a = \frac{\pi\,n}{2}, b = \frac{\pi\,n}{10}$$

So,

$$k_n = \frac{10}{n^2\,\pi^2}\sin\left(\frac{\pi\,n}{2}\right)\sin\left(\frac{\pi\,n}{10}\right)$$

Input expression for k_n into formula for $u(x,t)$ to get:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{10}{n^2\,\pi^2}\sin\left(\frac{\pi\,n}{2}\right)\sin\left(\frac{\pi\,n}{10}\right)\sin\left(\frac{n\,\pi\,x}{10}\right)\sin\left(\frac{n\,\pi\,t}{10}\right)$$