Assignment 3

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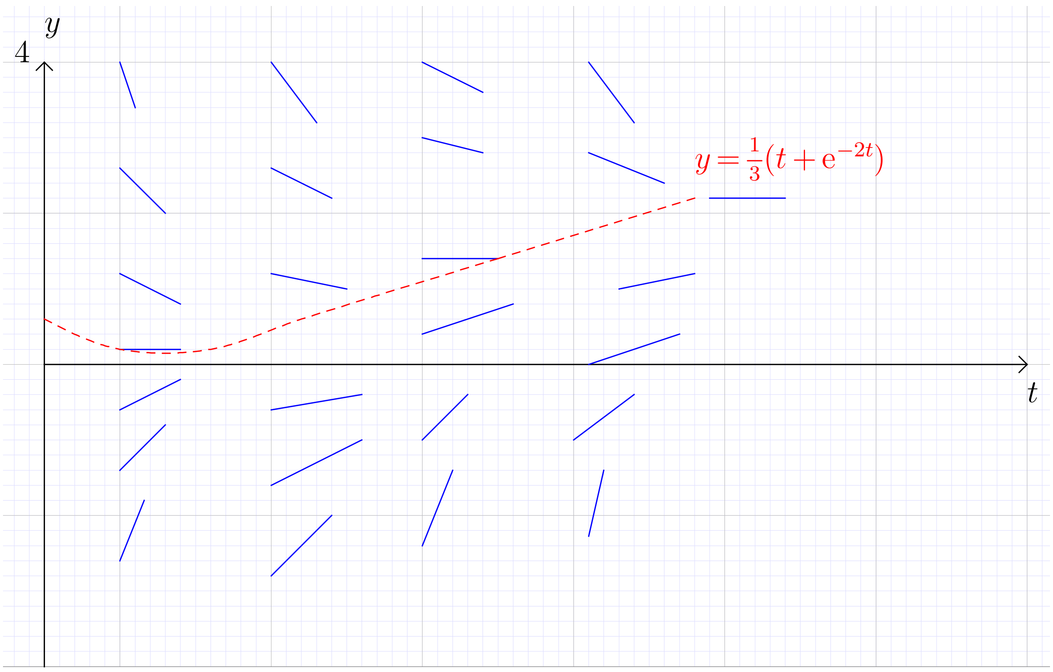
Question 1

(a) $y' + 3y = t + \mathrm{e}^{-2t}$. Rewrite the equation:

$$y' = t + \mathrm{e}^{-2t} - 3y$$

Inspect the expression for y' from different "angles":

- While t is constant, y' becomes increasingly negative for positive y and increasingly positive for negative y.
- While y is constant, y' is decreases a bit in the region $0 < t < \frac{\ln 2}{2}$, where $\partial y' / \partial t < 0$. after which, y' grows almost linearly as the term e^{-2t} effectively diminishes.
- $y' = 0$ on the curve $y = \frac{1}{3}(t + \mathrm{e}^{-2t})$.



(b) Inspection of the direction field suggests that $y \to \infty$ as $t \to \infty$, and grows in a linear fashion.

(c) Solve using integration factor method. For an equation of the form

$$y' + a(t)\, y = b(t)$$

the solution is

$$y = \mathrm{e}^{-\int a(t)\,\mathrm{d}t}\left[\int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}\,\mathrm{d}t + c\right] \tag{1}$$

In our case, $a = 3, b = t + \mathrm{e}^{-2t}$.

$$\begin{aligned} y &= \mathrm{e}^{-3t}\left[\int (t + \mathrm{e}^{-2t})\,\mathrm{e}^{3t}\,\mathrm{d}t + c\right] \\ &= \mathrm{e}^{-3t}\left[\int t\,\mathrm{e}^{3t}\,\mathrm{d}t + \int \mathrm{e}^{t}\,\mathrm{d}t + c\right] \\ &= \left[\mathrm{e}^{-3t}\int t\,\mathrm{e}^{3t}\,\mathrm{d}t\right] + \mathrm{e}^{-2t} + c\,\mathrm{e}^{-3t} \end{aligned}$$

The integral $\int t\,\mathrm{e}^{3t}\,\mathrm{d}t$ is solvable by parts, using the formula:

$$\int u'\,v\,\mathrm{d}t = u\,v - \int u\,v'\,\mathrm{d}t \tag{2}$$

pick $v = t, u' = \mathrm{e}^{3t}$.

$$\begin{aligned} \int t\,\mathrm{e}^{3t}\,\mathrm{d}t &= \frac{1}{3}t\,\mathrm{e}^{3t} - \frac{1}{3}\int \mathrm{e}^{3t}\,\mathrm{d}t \\ &= \frac{1}{3}\mathrm{e}^{3t}\left(t - \frac{1}{3}\right) \end{aligned}$$

Therefore,

$$y = \frac{1}{3}t - \frac{1}{9} + \mathrm{e}^{-2t} + c\,\mathrm{e}^{-3t}$$

Because the exponents approach zero quickly, for large t the solution behaves according to $\frac{t}{3} - \frac{1}{9}$, or in other words, linearly with t.

Question 13

$$y' - y = 2t\,\mathrm{e}^{2t}, \quad y(0) = 1$$

According to eq. (1), $(a = -1, b = 2t\,\mathrm{e}^{2t})$

$$y = \mathrm{e}^{t}\left[\int 2t\,\mathrm{e}^{t}\,\mathrm{d}t + c\right]$$

Solve the integral by parts. Pick $u' = \mathrm{e}^t, v = t$.

$$\int t\,\mathrm{e}^t\,\mathrm{d}t \;=\; t\,\mathrm{e}^t - \int \mathrm{e}^t\,\mathrm{d}t = \mathrm{e}^t(t-1)$$

Therefore,

$$y = 2\mathrm{e}^{2t}(t-1) + c\,\mathrm{e}^t$$

Input the IC to find c:

$$y(0) = 1 = 2(0-1) + c \rightarrow c = 3$$

The unique solution is $y = 2\mathrm{e}^{2t}(t-1) + 3\,\mathrm{e}^t$.

Question 15

$$t\,y' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$$

Given $t \neq 0$ in the requested interval, we can divide by t to get

$$y' + \frac{2}{t}\,y = t - 1 + \frac{1}{t}$$

According to eq. (1), $a = \frac{2}{t}, b = t - 1 + \frac{1}{t}$.

$$\mathrm{e}^{\int a(t)\,\mathrm{d}t} = \mathrm{e}^{2\int \frac{\mathrm{d}t}{t}} = 2\ln|t| = t^2$$

$$\mathrm{e}^{-\int a(t)\,\mathrm{d}t} = \mathrm{e}^{-2\int \frac{\mathrm{d}t}{t}} = -2\ln|t| = \frac{1}{t^2}$$

$$y = \frac{1}{t^2}\left[\int \left(t - 1 + \frac{1}{t}\right) t^2\,\mathrm{d}t + c\right]$$

$$\begin{aligned}\int \left(t - 1 + \frac{1}{t}\right) t^2\,\mathrm{d}t &= \int (t^3 - t^2 + t)\,\mathrm{d}t \\ &= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}\end{aligned}$$

Therefore,

$$y = \frac{1}{t^2}\left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c\right] = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}, \quad t > 0$$

Input IC:

$$y(1) = \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \rightarrow c = \frac{1}{12}$$

The unique solution is $y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}$ in the domain $t > 0$.

Question 18

$$t\,y' + 2y = \sin t, \quad y(\pi/2) = 1$$

In the given IC, $t > 0$. Divide by t assuming $t \neq 0$.

$$y' + \frac{2}{t}\,y = \frac{\sin t}{t}$$

Using eq. (1), $(a = \frac{2}{t}, b = \frac{\sin t}{t})$

$$e^{\int a \mathrm{d}t} = e^{\int \frac{2}{t} \mathrm{d}t} = t^2, \quad e^{-\int a \mathrm{d}t} = \frac{1}{t^2}$$

$$y = \frac{1}{t^2}\left[\int \frac{\sin t}{t} \cdot t^2\,\mathrm{d}t + c\right]$$

Solve the integral using eq. (2). Pick $u' = \sin t, v = t$

$$\begin{aligned}\int \sin t \cdot t\,\mathrm{d}t &= -t\cos t + \int \cos t\,\mathrm{d}t \\ &= -t\cos t + \sin t\end{aligned}$$

The general solution is

$$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c}{t^2}, \quad t \neq 0$$

Input the IC:

$$y(\pi/2) = 1 = 0 + \frac{1}{\pi^2/4} + c\frac{1}{\pi^2/4} \to c = \frac{\pi^2}{4} - 1$$

The unique solution is $y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \left(\frac{\pi^2}{4} - 1\right)\frac{1}{t^2}$ in the domain $t > 0$.

Question 19

$$t^3\,y' + 4t^2\,y = e^{-t}, \quad y(-1) = 0$$

In the given IC, $t < 0$. To find the specific solution we divide by t^3, assuming $t \neq 0$.

$$y' + \frac{4}{t}\,y = \frac{1}{t^3}\,e^{-t}$$

Using eq. (1), $\left(a = \frac{4}{t}, b = \frac{1}{t^3}\,e^{-t}\right)$

$$e^{\int \frac{4}{t} \mathrm{d}t} = t^4, \quad e^{-\int \frac{4}{t} \mathrm{d}t} = \frac{1}{t^4}$$

$$y = \frac{1}{t^4}\left[\int \frac{1}{t^3}\,e^{-t} \cdot t^4\,\mathrm{d}t + c\right]$$

Solve the integral. Pick $u' = \mathrm{e}^{-t}, v = t$

$$\begin{aligned}\int t\,\mathrm{e}^{-t}\,\mathrm{d}t &= -t\,\mathrm{e}^{-t} + \int \mathrm{e}^{-t}\,\mathrm{d}t \\ &= -\mathrm{e}^{-t}(t+1)\end{aligned}$$

Therefore,

$$y = \frac{1}{t^4}[c - \mathrm{e}^{-t}(t+1)], \quad t \neq 0$$

Input the IC:

$$y(-1) = 0 = c$$

The unique solution is $y = -\frac{1}{t^4}\,\mathrm{e}^{-t}(t+1)$ in the domain $t < 0$.

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<u>Question 1</u>

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y}, \quad y \neq 0$$

Integrate:

$$\int y\,\mathrm{d}y = \int x^2\,\mathrm{d}x$$

$$\frac{y^2}{2} = \frac{x^3}{3} + c$$

$$y = \pm\sqrt{\frac{2}{3}x^3 + 2c}, \quad y \neq 0, \quad \left(\frac{x^3}{3} + c\right) > 0$$

<u>Question 5</u>

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos^2 x)(\cos^2 2y)$$

Rewrite and integrate:

$$\int \frac{\mathrm{d}y}{\cos^2 2y} = \int \cos^2 x\,\mathrm{d}x, \quad \cos^2 2y \neq 0$$

Let's focus on the RHS:

$$\begin{aligned}\int \cos^2 x\,\mathrm{d}x &= \frac{1}{2}\int (1+\cos 2x)\mathrm{d}x\\ &= \frac{1}{2}\left(x+\frac{1}{2}\sin 2x+c\right)\end{aligned}$$

Back to the original equation:

$$\int \frac{\mathrm{d}y}{\cos^2(2y)} = \frac{1}{2}\tan 2y = \frac{1}{2}\left(x+\frac{1}{2}\sin 2x+c\right)$$

$$\tan 2y = x+\frac{1}{2}\sin 2x+c, \quad \cos 2y \neq 0$$

$$y = \frac{1}{2}\arctan\left(x+\frac{1}{2}\sin 2x+c\right), \quad \cos 2y \neq 0$$

What if $\cos^2 2y \equiv 0$?

$$\cos 2y = 0 \Longleftrightarrow 2y = \arccos(0) = \frac{\pi}{2}+\pi\cdot n, \quad n\in\mathbb{Z}$$

$$y = \frac{\pi}{4}+\frac{\pi}{2}\cdot n, \quad n\in\mathbb{Z}, \quad \forall x$$

Because for this y, $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \forall x$, this is also a solution.

Question 8

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1+y^2}$$

Integrate.

$$\int (1+y^2)\,\mathrm{d}y = \int x^2\mathrm{d}x$$

$$y+\frac{y^3}{3} = \frac{x^3}{3}+c$$

The solution is given implicitly for all x and y.

Question 23

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2+x\,y^2, \quad y(0)=1$$

Rewrite the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2(2+x)$$

Integrate.

$$\int \frac{\mathrm{d}y}{y^2} = \int (2+x)\,\mathrm{d}x$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} + c, \quad y \neq 0$$

$$y = -\frac{1}{2x + \frac{x^2}{2} + c}, \quad \left(2x + \frac{x^2}{2} + c\right) \neq 0$$

Input the IC:

$$y(0) = 1 = -\frac{1}{c} \rightarrow c = -1$$

The specific solution is

$$y = -\frac{1}{2x + \frac{x^2}{2} - 1}, \quad \left(2x + \frac{x^2}{2} - 1\right) \neq 0$$

y attains its minimum value when $2x + \frac{x^2}{2} - 1$ reaches its maximum value.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2x + \frac{x^2}{2} - 1\right) = 2 + x$$

The expression equals zero when $x = -2$, for which

$$y_{\min} = -\frac{1}{-4+2-1} = \frac{1}{3}$$

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Question 1

Denote $y(t)$ as the dye concentration in the tank. The equation for $y'(t)$ is as follows:

$$y'(t) = -(\text{rate of water flowing in}) \cdot \frac{(\text{initial dye concentration})}{(\text{total transient volume})}\, y(t)$$

Input the numbers:

$$y'(t) = -2\,\mathrm{L}/\mathrm{min} \cdot \frac{1\,\mathrm{g}\,\mathrm{L}^{-1}}{200\,\mathrm{L}} \cdot y(t) = -0.01 y(t), \quad y(0) = 1\,\mathrm{g}\,\mathrm{L}^{-1}$$

The solution to the DE is a decaying exponent:

$$y(t) = y(0)\,\mathrm{e}^{-0.01t} = \mathrm{e}^{-0.01t}$$

We are asked to calculate the time it takes for the dye concentration to reach 1% of its initial value. In other words, find t such that:

$$y(t) = 0.01 = \mathrm{e}^{-0.01t}$$

$$t = -\frac{\ln 0.01}{0.01} \approx 460.5\,\mathrm{min}$$

<u>Question 4</u>

Write the DE for the amount of salt in the tank ($y(t)$):

$y'(t) =$ (rate of salt flowing in)(con. of salt) − (rate of salt flowing out)(con. of salt)

Input the numbers:

$$\begin{aligned} y'(t) &= 1\,\mathrm{lb}/\mathrm{gal}\cdot 3\,\mathrm{gal}/\mathrm{min} - 2\,\mathrm{gal}/\mathrm{min}\cdot\frac{y(t)}{200\,\mathrm{gal} + 1\,\mathrm{gal}/\mathrm{min}\cdot t} \\ y'(t) &= 3 - \frac{2y(t)}{200+t} \end{aligned}$$

Note: the quotient $\frac{2y(t)}{200+t}$ represents the transient concentration at time t. The denominator represents the total volume of liquid in the tank at time t.

Solve the DE:

$$y' + \frac{2}{200+t}\,y(t) = 3$$

Solve according to eq. (1) $\left(a = \frac{2}{200+t}, b = 3\right)$

$$\mathrm{e}^{\int a(t)\,\mathrm{d}t} = \mathrm{e}^{2\int\frac{\mathrm{d}t}{200+t}} = (200+t)^2$$

$$\mathrm{e}^{-\int a(t)\,\mathrm{d}t} = \frac{1}{(200+t)^2}$$

$$y = \frac{1}{(200+t)^2}\left[\int 3\,(200+t)^2\,\mathrm{d}t + c\right]$$

$$\begin{aligned} \int (200+t)^2\,\mathrm{d}t &= \int (t^2 + 400t + 40{,}000)\,\mathrm{d}t \\ &= \frac{t^3}{3} + 200t^2 + 40{,}000t \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{(200+t)^2}\left[\int 3\,(200+t)^2\,\mathrm{d}t + c\right] = \frac{t^3 + 600t^2 + 120{,}000t + c}{(200+t)^2}$$

To find c we use the IC:

$$y(0)=100=\frac{c}{200^2}\to c=4\cdot 10^6$$

$$y(t)=\frac{t^3+600t^2+120{,}000t+4\cdot 10^6}{(200+t)^2}$$

When does the solution begin to overflow? — When the volume reaches 500 gal.

$$V(t)=500=200+t\to t=300\,\mathrm{min}$$

Now let's calculate $y(300)$:

$$y(300)=\frac{300^3+600\cdot 300^2+120{,}000\cdot 300+4\cdot 10^6}{(200+300)^2}=484\,\mathrm{lb}$$

The salt concentration at time of overflowing is $C=\frac{484\,\mathrm{lb}}{500\,\mathrm{gal}}=0.968\,\mathrm{lb\,gal}^{-1}$.

If the tank would have had infinite capacity, the limit concentration is achieved at $t\to\infty$.

$$\lim_{t\to\infty}C=\lim_{t\to\infty}\frac{y(t)}{V(t)}=\lim_{t\to\infty}\left(\frac{t^3+600t^2+120{,}000t+4\cdot 10^6}{(200+t)^2\cdot(200+t)}\right)=1\,\mathrm{lb\,gal}^{-1}$$

The limit approaches the value $C=1$ because the highest order element of both the numerator and denominator is t^3 and in both its coefficient is 1.

Question 9

Given $r=0.1$, the DE for the amount of debt left at time t is:

$$S'(t)=r\,S(t)-k$$

Let's solve.

$$S'(t)-0.1S(t)=-k$$

According to eq. (1), $(a=-0.1, b=-k)$

$$\mathrm{e}^{\int a(t)\,\mathrm{d}t}=\mathrm{e}^{\int -0.1\mathrm{d}t}=\mathrm{e}^{-0.1t}$$

$$\mathrm{e}^{-\int a(t)\,\mathrm{d}t}=\mathrm{e}^{0.1t}$$

$$S(t)=\mathrm{e}^{0.1t}\left[\int -k\,\mathrm{e}^{-0.1t}\,\mathrm{d}t+c\right]$$

$$S(t)=10k+c\,\mathrm{e}^{0.1t}$$

To find c, use IC:

$$S(0) = 8000 = 10k + c$$

$$c = 8000 - 10k$$

$$\Rightarrow S(t) = 10k + (8000 - 10k)\mathrm{e}^{0.1t}$$

$$S(t) = 8000 \cdot \mathrm{e}^{0.1t} + 10k(1 - \mathrm{e}^{0.1t})$$

If the debt is to be re-payed in 3 years, then:

$$S(3) = 0 = 8000 \cdot \mathrm{e}^{0.3} + 10k(1 - \mathrm{e}^{0.3})$$

$$|k| = \frac{8000 \cdot \mathrm{e}^{0.3}}{(1 - \mathrm{e}^{0.3}) \cdot 10} = \$3086.64\,\mathrm{yr}^{-1}$$

The interest paid in 3 years is the amount returned minus the initial loan. In other words, the interest equals $3086.64 \times 3 = \$9259.92$ minus the initial loan of \$8000, for a total of \$1259.92.

Question 14

(a) We are given two points: $Q(t_{1/2}) = \frac{1}{2}Q(0) = 5730\,\mathrm{yrs}$ and $Q(50{,}000) = 0.00236Q(0)$.

For a DE of the form $Q'(t) = -r\,Q(t)$ the solution is $Q(t) = Q(0)\,\mathrm{e}^{-rt}$.

We can determine r using each of the points. For example, using the information of the half-life:

$$\frac{1}{2} = \mathrm{e}^{-r \cdot 5730} \rightarrow r = \frac{\ln 2}{5730} = 1.21 \times 10^{-4}\,\mathrm{yrs}^{-1}$$

(b) The solution is a simple decaying exponent. It may be found via integration factor method: $Q'(t) + r\,Q(t) = 0$. Select $\mu(t) = \mathrm{e}^{rt}$ and get $(Q\,\mathrm{e}^{rt})' = 0$. Integrate to get $Q(t) = c\,\mathrm{e}^{-rt}$, then input $t = 0$ to get $c = Q(0) \equiv Q_0$.

$$Q(t) = Q_0\,\mathrm{e}^{-1.21 \times 10^{-4} \cdot t}$$

(c) We need to find t for which $Q(t) = 0.2\,Q_0$.

$$0.2 = \mathrm{e}^{-rt}$$

$$t = \frac{\ln 5}{r} = 13{,}304.65\,\mathrm{yrs}$$