Assignment 9

by Yuval Bernard

Date: TBE

Page 355

Question 24

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \vec{x}$$

$$\vec{x} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} \mathrm{e}^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \mathrm{e}^{2t}$$

Denote $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Assume the solution is of the form

$$\vec{x} = \vec{z}\, \mathrm{e}^{rt}$$

$$\vec{x}' = r\, \vec{z}\, \mathrm{e}^{rt}$$

Plug in the system of ODEs to get

$$r\, \vec{z}\, \mathrm{e}^{rt} = A\, \vec{z}\, \mathrm{e}^{rt}$$

Divide both sides by $\mathrm{e}^{rt} \neq 0$ (for all t) to get

$$A\, \vec{z} = r\, \vec{z}$$

This problem is equivalent to finding the eigenvalues of A. Find λ such that $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} &= (1-\lambda)\,[1 - 2\lambda + \lambda^2 - 1] - 2\,(1 - \lambda + 1) \\ &= 1 - 2\lambda + \lambda^2 - 1 - \lambda + 2\lambda^2 - \lambda^3 + \lambda - 4 + 2\lambda \\ &= -\lambda^3 + 3\lambda^2 - 4 \\ &= (\lambda + 1)\,(-\lambda^2 + 4\lambda - 4) \\ &= -(\lambda + 1)\,(\lambda - 2)^2 = 0 \rightarrow \lambda_1 = -1, \lambda_2 = 2 \end{aligned}$$

Find the eigenvectors that correspond to the calculated eigenvalues.

$\underline{\lambda_1 = -1}$

$$(A - \lambda_1 I)\,\vec{z}_1 = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \vec{z} = 0$$

Apply row operations on the augmented matrix $A|\vec{0}$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

From which we obtain:

$$x_2 = 2x_3,\, x_1 = -\frac{3}{2} x_3$$

Pick $x_3 = -4$ to get:

$$\vec{z}_1 = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix}$$

$\underline{\lambda_2 = 2}$

$$(A - \lambda_2 I)\,\vec{z}_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \vec{z} = 0$$

Obtain

$$x_2 = -x_3,\, x_1 = 0$$

Pick $x_3 = -1$ and get

$$\vec{z}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Finally, the general solution is

$$\vec{x} = c_1\,\vec{z}_1\,\mathrm{e}^{\lambda_1 t} + c_2\,\vec{z}_2\,\mathrm{e}^{\lambda_2 t}$$

If we pick $c_1, c_2 = 1$, we get the unique solution

$$\vec{x} = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} \mathrm{e}^{-t} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \mathrm{e}^{2t}$$

as requested to verify.

Page 367

Question 16

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

Find eigenvalues of A.

$$|\, A - \lambda\, I \,| = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda - 3)\,(1+\lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{2 \pm 4\,\mathrm{i}}{2} = 1 \pm 2\mathrm{i}$$

Find eigenvectors of A.

$\lambda_1 = 1 + 2\mathrm{i}$

Solve $(A - \lambda_1 I)\, \vec{v}_1 = \vec{0}$.

$$A - \lambda_1 I = \begin{bmatrix} 2 - 2\mathrm{i} & -2 \\ 4 & -2 - 2\mathrm{i} \end{bmatrix}$$

Select

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 - \mathrm{i} \end{bmatrix}$$

$\lambda_2 = 1 - 2\mathrm{i}$

Solve $(A - \lambda_2 I)\, \vec{v}_2 = \vec{0}$.

$$A - \lambda_2 I = \begin{bmatrix} 2 + 2\mathrm{i} & -2 \\ 4 & -2 + 2\mathrm{i} \end{bmatrix}$$

Select

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 + \mathrm{i} \end{bmatrix}$$

Question 18

$$A = \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}$$

Find eigenvalues of A.

$$|\, A - \lambda\, I \,| = \begin{vmatrix} 1-\lambda & \mathrm{i} \\ -\mathrm{i} & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - 1 = \lambda\,(\lambda - 2) = 0$$

$$\lambda_{1,2} = 0, 2$$

Find eigenvectors of A.

$\lambda_1 = 0$

$$A - \lambda_1\, I = A = \begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$

$$A - \lambda_2\, I = \begin{bmatrix} -1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}$$

Question 21

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find eigenvalues: (Calculate det $A - \lambda\, I$ along row 1)

$$|\, A - \lambda\, I \,| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\,[\lambda^2 - 2\lambda + 1 + 4] = 0$$

$$(1-\lambda)\,(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda_{1,2,3} = 1, 1 \pm 2\mathrm{i}$$

Find eigenvectors.

$\underline{\lambda_1 = 1}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

$\underline{\lambda_2 = 1 + 2\mathrm{i}}$

$$A - \lambda_2 I = \begin{bmatrix} -2\mathrm{i} & 0 & 0 \\ 2 & -2\mathrm{i} & -2 \\ 3 & 2 & -2\mathrm{i} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ \mathrm{i} \\ 1 \end{bmatrix}$$

$\underline{\lambda_3 = 1 - 2\mathrm{i}}$

$$A - \lambda_3 I = \begin{bmatrix} 2\mathrm{i} & 0 & 0 \\ 2 & 2\mathrm{i} & -2 \\ 3 & 2 & 2\mathrm{i} \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -\mathrm{i} \\ 1 \end{bmatrix}$$

<u>Question 22</u>

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

Find eigenvalues:

$$|\ A - \lambda I\ | = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} \ = \ (3-\lambda)\,[-(4-\lambda)\,(1+\lambda)+4] - 1\,[-2\,(1+\lambda)+8]$$

$$-2\,[2-8+2\lambda] = 0$$

$$(3-\lambda)\,(\lambda^2-3\lambda) - 2(3-\lambda) - 4(\lambda-3) = 0$$

$$(3-\lambda)\,(\lambda^2-3\lambda+2) = 0$$

$$\lambda_{1,2,3} = 3, 1, 2$$

Find eigenvectors:

$\underline{\lambda_1 = 3}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 1}$

$$A - \lambda_2 I = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\underline{\lambda_3 = 2}$

$$A - \lambda_3 I = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

<u>Question 24</u>

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Find eigenvalues:

$$|\ A - \lambda I\ | = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} &= (3-\lambda)\,(\lambda^2 - 3\lambda - 4) - 2\,(6 - 2\lambda - 8) + 4\,(4 + 4\lambda) \\ &= (3-\lambda)\,(\lambda - 4)\,(\lambda + 1) + 4(1+\lambda) + 16\,(1+\lambda) \\ &= -(\lambda+1)\,(\lambda^2 - 7\lambda - 8) = 0 \end{aligned}$$

$$\lambda_{1,2,3} = -1, -1, 8$$

Find eigenvectors:

$\underline{\lambda_{1,2} = -1}$

$$A - \lambda_{1,2} I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

We have one equation in 3 variables. Choose two linearly independent vectors $\vec{v}_{1,2}$ that satisfy $A \vec{v}_{1,2} = \vec{0}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$\underline{\lambda_3 = 8}$

$$A - \lambda_3 I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Two Extra Extremely Exasperating Inquires

Extra Page

Question 1

$$\begin{cases} x_{n+1} = 3x_n - y_n \\ y_{n+1} = 2x_n \end{cases}$$

(a) Denote $\vec{v}_n = \left[\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right]$, Insert coefficients of x_n, y_n from both equations into a matrix A:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

Indeed,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} \equiv \vec{v}_{n+1} = A \vec{v}_n$$

(b) Given initial populations stored in $\vec{v}_0$, we shall prove inductively that

$$\vec{v}_n = A^n \vec{v}_0$$

For $n=0$:

$$\vec{v}_0 = A^0\,\vec{v}_0 = I\,\vec{v}_0 = \vec{v}_0 \checkmark$$

Assume that the statement holds for $n=k$, that is:

$$\vec{v}_k = A^k\,\vec{v}_0$$

Show that it holds for $n=k+1$:

$$\vec{v}_{k+1} = A^{k+1}\,\vec{v}_0 = A\,A^k\,\vec{v}_0 = A\,\vec{v}_k$$

Note here that we've utilized the fact that a matrix commutes with its power. This can be proved inductively, as for $k=0$: $A^0=I$ by definition, and assuming

$$A^{k+1} = A^k\,A$$

indeed:

$$A^{k+1} = A^k\,A = (A\,A^{k-1})\,A = A\,(A^{k-1}\,A) = A\,A^k$$

So $A^k\,A = A\,A^k = A^{k+1}$.

By definition of our problem, $\vec{v}_{k+1} = A\,\vec{v}_k$ (just switch n for k), so the statement

$$\vec{v}_{k+1} = A^{k+1}\,\vec{v}_0$$

holds. In summary, we've proved inductively that

$$\vec{v}_n = A^n\,\vec{v}_0$$

as requested.

(c) Find λ that satisfy:

$$|\ A - \lambda\,I\ | = 0$$

$$|\ A - \lambda\,I\ | = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0$$

$$\lambda_{1,2} = 2, 1$$

Find the eigenvectors $\{\vec{x}_i\}$ that satisfy for each $i \in 1,2$: $(A - \lambda_i\,I)\vec{x}_i = 0$.

$\underline{\lambda_1 = 2}$

$$A - \lambda_1 I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 1}$

$$A - \lambda_2 I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(d) Find explicit formulas for $\vec{v}_n$ in terms of n. Well, we know that

$$\vec{v}_n = A^n \vec{v}_0$$

and we also know that if

$$\begin{cases} A\vec{x}_1 = 2\vec{x}_1 \\ A\vec{x}_2 = \vec{x}_2 \end{cases}$$

Then

$$\begin{cases} A^\ell \vec{x}_1 = 2^\ell \vec{x}_1 \\ A^\ell \vec{x}_2 = 1^\ell \vec{x}_2 = \vec{x}_2 \end{cases}$$

The fact that the i^{th} eigenvalue of A^ℓ is λ_i^ℓ can also be proved by induction:

For $\ell = 0$: (given $\vec{x}_i$ is the eigenvector corresponding to the i^{th} eigenvalue of A)

$$A^0 \vec{x}_i = I\vec{x}_i = \vec{x}_i = \lambda_i^0 \vec{x}_i$$

Assuming $A^\ell \vec{x}_i = \lambda_i^\ell \vec{x}_i$, we show that it holds for $\ell + 1$:

$$A^{\ell+1} \vec{x}_i = A\,A^\ell \vec{x}_i = A\,\lambda_i^\ell \vec{x}_i = \lambda_i^\ell A\,\vec{x}_i = \lambda_i^\ell \lambda_i \vec{x}_i = \lambda_i^{\ell+1} \vec{x}_i$$

thus completing proof by induction.

Now, to compute $A^n \vec{v}_0$, we need to express $\vec{v}_0$ as a linear combination of $\vec{x}_1, \vec{x}_2$. In other words, find scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \vec{x}_1 + \vec{x}_2 = \vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or, in other words,

$$\left[\begin{array}{cc}1 & 1\\ 1 & 2\end{array}\right]\left[\begin{array}{c}c_1\\ c_2\end{array}\right]=\left[\begin{array}{c}x_0\\ y_0\end{array}\right]=\vec{v}_0$$

Denote B as the matrix comprising the eigenvectors of A as its columns:

$$B=\left[\begin{array}{cc}1 & 1\\ 1 & 2\end{array}\right]$$

then

$$\left[\begin{array}{c}c_1\\ c_2\end{array}\right]=B^{-1}\,\vec{v}_0$$

Compute B^{-1} via Gauss-Seidal algorithm:

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{array}\right]\xrightarrow{R_2\to R_2-R_1}\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & 1 & -1 & 1\end{array}\right]\xrightarrow{R_1\to R_1-R_2}\left[\begin{array}{cc|cc}1 & 0 & 2 & -1\\ 0 & 1 & -1 & 1\end{array}\right]$$

Therefore,

$$\left[\begin{array}{c}c_1\\ c_2\end{array}\right]=\left[\begin{array}{cc}2 & -1\\ -1 & 1\end{array}\right]\vec{v}_0$$

In conclusion,

$$\vec{v}_n=A^n\,\vec{v}_0=A^n\,(c_1\,\vec{x}_1+c_2\,\vec{x}_2)=c_1\,A^n\,\vec{x}_1+c_2\,A^n\,\vec{x}_2=c_1\,2^n\,\vec{x}_1+c_2\,\vec{x}_2$$

$$\left[\begin{array}{c}x_n\\ y_n\end{array}\right]=c_1\,2^n\left[\begin{array}{c}1\\ 1\end{array}\right]+c_2\left[\begin{array}{c}1\\ 2\end{array}\right],\qquad c_1,c_2\in\mathbb{R}$$

where c_1, c_2 are given by:

$$\left[\begin{array}{c}c_1\\ c_2\end{array}\right]=\left[\begin{array}{cc}2 & -1\\ -1 & 1\end{array}\right]\vec{v}_0$$

where $\vec{v}_0$ is the vector containing the initial populations. A final, expanded expression:

$$\left[\begin{array}{c}x_n\\ y_n\end{array}\right]=(2\,x_0-y_0)\,2^n\left[\begin{array}{c}1\\ 1\end{array}\right]+(-x_0+y_0)\left[\begin{array}{c}1\\ 2\end{array}\right]$$

(e) If at some year, $\vec{v}_0=\left[\begin{smallmatrix}3\\ 2\end{smallmatrix}\right]$, then 7 years later:

$$\left[\begin{array}{c}x_7\\ y_7\end{array}\right]=(2\cdot3-2)\,2^7\left[\begin{array}{c}1\\ 1\end{array}\right]+(-3+2)\left[\begin{array}{c}1\\ 2\end{array}\right]=512\left[\begin{array}{c}1\\ 1\end{array}\right]-\left[\begin{array}{c}1\\ 2\end{array}\right]=\left[\begin{array}{c}511\\ 510\end{array}\right]$$

Question 2

$a_0 = 0, a_1 = 1$, and for $n > 1$, $a_n = a_{n-1} + a_{n-2}$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{v}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

(a)

$$\vec{v}_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n + a_{n+1} \end{bmatrix}$$

Now calculate $A\,\vec{v}_n$:

$$A\,\vec{v}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n + a_{n+1} \end{bmatrix}$$

We got the same result, thus approving that $\vec{v}_{n+1} = A\,\vec{v}_n$.

(b) the diagonalized form of A is $D = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$, where λ_1, λ_2 are the eigenvalues of A.

$$|\,A - \lambda\,I\,| = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

To find an expression for A^n, we use two facts:

1. For a diagonal matrix $D = \left[\begin{smallmatrix} D_{11} & 0 \\ 0 & D_{22} \end{smallmatrix}\right]$

$$D^n = \begin{bmatrix} D_{11}^n & 0 \\ 0 & D_{22}^n \end{bmatrix}$$

2. Let $A = V D V^{-1}$ where V is a 2×2 invertible matrix and D is a diagonal matrix, then

$$A^n = V D^n V^{-1}$$

The first fact can be proved by induction on n. The base case $n = 1$ is true by definition. Now suppose that

$$D^k = \begin{bmatrix} D_{11}^k & 0 \\ 0 & D_{22}^k \end{bmatrix}$$

then we have (inductive step):

$$D^{k+1} = D\,D^k = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} D_{11}^k & 0 \\ 0 & D_{22}^k \end{bmatrix} = \begin{bmatrix} D_{11}^{k+1} & 0 \\ 0 & D_{22}^{k+1} \end{bmatrix}$$

Hence the inductive step holds, which completes the proof.

The second fact can also be proved by induction on n. The base case $n=1$ is true by definition (the equality $A = VDV^{-1}$ is called *eigendecomposition*, and holds when A is diagonaizable—and it is in this case). For the inductive step, assume that

$$A^k = VD^kV^{-1}$$

then we have

$$A^{k+1} = A\,A^k = (VDV^{-1})\,(VD^kV^{-1}) = VD^{k+1}V^{-1}$$

Hence the inductive step holds, which completes the proof.

The vector V is the eigenvectors of A adjacent to each other in ordering respective to the order of eigenvalues in D.

Calculate the eigenvectors of A:

$\underline{\lambda_1 = \frac{1+\sqrt{5}}{2}}$

$$A - \lambda_1 I = \begin{bmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$\underline{\lambda_2 = \frac{1-\sqrt{5}}{2}}$

$$A - \lambda_2 I = \begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

In conclusion,

$$V = [\ \vec{v}_1 \quad \vec{v}_2\ ] = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

Find V^{-1}. For a 2×2 matrix:

$$V^{-1} = \frac{1}{\det V}\begin{bmatrix} V_{22} & -V_{12} \\ -V_{21} & V_{11} \end{bmatrix} = \frac{1}{-\sqrt{5}}\begin{bmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\frac{1+\sqrt{5}}{2} & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{bmatrix}$$

Thus

$$A^n = VD^nV^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{bmatrix}$$

$$A^n = \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{bmatrix}$$

$$A^n = \frac{1}{\sqrt{5}}\begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^n\left(\frac{1+\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ -\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\left(\frac{1+\sqrt{5}}{2}\right) & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

$$A^n = \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

(c) Given

$$\vec{v}_0 = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_n = A^n\,\vec{v}_0 = \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_n = \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

It is clear that the result obeys the relation

$$\vec{v}_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

which was given to us as a definition. Thus, we can use it to extract the formula for a_n

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$