Linear Algebra for Chemists — Assignment 2

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Question 1. *Show that the span of a set of vectors in a vector space is a subspace.*

Recall the definition of span: let S be a set of vectors in a vector space V. The span of S is the intersection W of all subspaces of V which contain S.

Additionally, recall the theorem that a non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c in the field over which V is defined, the vector $c\,\alpha + \beta$ is again in W.

Proving that the span of S is a subspace can be reduced to proving that the intersection of a collection of subspaces of V is a subspace of V.

Let $\{W_a\}$ be a collection of subspaces of V, and let W be their intersection. Since each W_a is a subspace, each contains the zero vector. Thus, the zero vector is in the intersection W, and W is non-empty. Let v, w be vectors in W and let α be a scalar. By definition of W, both v and w belong to each W_α, and because each W_a is a subspace, the vector $(\alpha\, v + w)$ is in every W_a. Thus $(\alpha\, v + w)$ is again in W. Therefore, W, the span of set S, is a subspace of V.

Question 2. *Check if the following sets of vectors span the subspace W of* $\mathbb{R}^4$*, where W is the set of vectors whose coordinates add up to 0.*

Here we use the theorem: the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

The set S spans W if any $v \in W$ can be expressed as a linear combination of the vectors in S.

a) Given the set S containing $(1,0,0,-1), (1,-1,0,0), (1,-1,1,-1)$, let $v = (x, y, z, w)$ be an arbitrary vector in W such that $x + y + z + w = 0$. We search for a solution to:

$$a\,(1,0,0,-1) + b\,(1,-1,0,0) + c\,(1,-1,1,-1) = (x,y,z,w), \quad \text{s.t.} \quad x + y + z + w = 0.$$

This translates to

$$\begin{aligned} a + b + c &= x \\ -b - c &= y \\ c &= z \\ -a - c &= w \end{aligned}$$

If we sum the set of equations, we see that every choice of a, b, c satisfies $x + y + z + w = 0$. S is thus a spanning set of W.

b) Given the set S containing $(1,2,-4,1), (0,1,1,-2)$, we search for a, b such that

$$a\,(1,2,-4,1) + b\,(0,1,1,-2) = (x,y,z,w), \qquad \text{s.t.} \quad x + y + z + w = 0.$$

The set of equations is

$$\begin{aligned} a &= x \\ 2a+b &= y \\ -4a+b &= z \\ a-2b &= w \end{aligned}$$

The condition $x+y+z+w=0$ holds for every a, b. S is a spanning set.

Question 3. *Which of the following vectors is an element of the subspace W where*

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \text{ in } \mathbb{R}^3: \qquad v = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 1 \\ -18 \end{bmatrix}.$$

A vector u is in the subspace W if and only if there exist scalars c_1, c_2 such that

$$u = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

1. Find c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = v = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

$$\begin{aligned} c_1 &= 4 \\ 2c_1 + c_2 &= 2 \to c_2 = 2 - 2c_1 = -6 \\ -c_1 + c_2 &= 1 \to c_2 = 1 + c_1 = 5 \end{aligned}$$

The system of equations has no solution. v is not in W.

2. Find c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = w = \begin{bmatrix} 3 \\ 1 \\ -18 \end{bmatrix}.$$

$$\begin{aligned} c_1 &= 3 \\ 2c_1 + c_2 &= 1 \to c_2 = 1 - 2c_1 = -5 \\ -c_1 + 3c_2 &= -18 \to c_2 = \frac{c_1 - 18}{3} = -5 \end{aligned}$$

We found $(c_1, c_2) = (3, -5)$. The vector w is thus an element of the subspace W.

Question 4. *Is the function $\sin x$ a linear combination of $\cos x$ and e^x in the space of continuous real functions over the real numbers?*

No. $\sin x$ may only be written as a linear combination of $\cos x$ and e^x in the space of continuous real functions over the field of **complex** numbers, as $\sin x = \mathrm{i} \cos x - \mathrm{i}\, \mathrm{e}^x$. There are no real scalars $\alpha, \beta \in \mathbb{R}$ such that $\sin x = \alpha \cos x + \beta\, \mathrm{e}^x$.