Assignment 2

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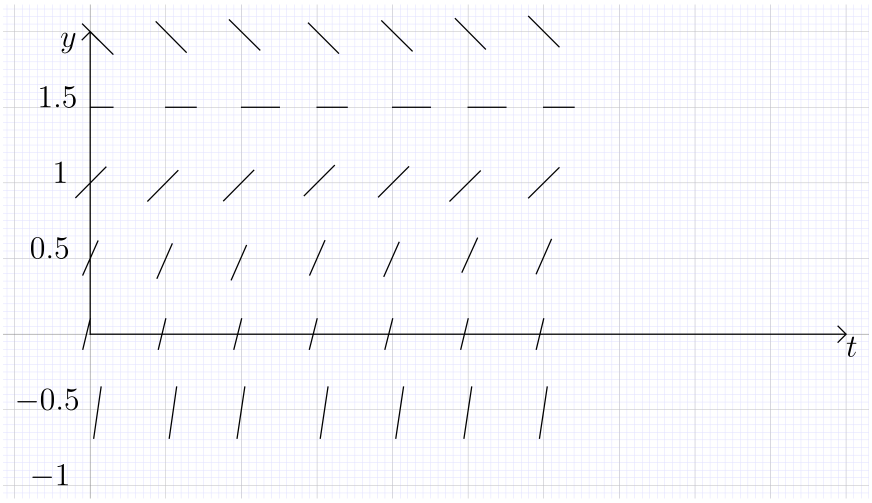
Questions 1,5

(1) $y' = 3 - 2y$.

Notice that y' is only dependent on y, not on t. As a result, across the t-axis replicas of vectors align in a series. Let's calculate some of the values of y' for some y around the equilibrium (for which $y' = 0$): $y(t) = \frac{3}{2}$.

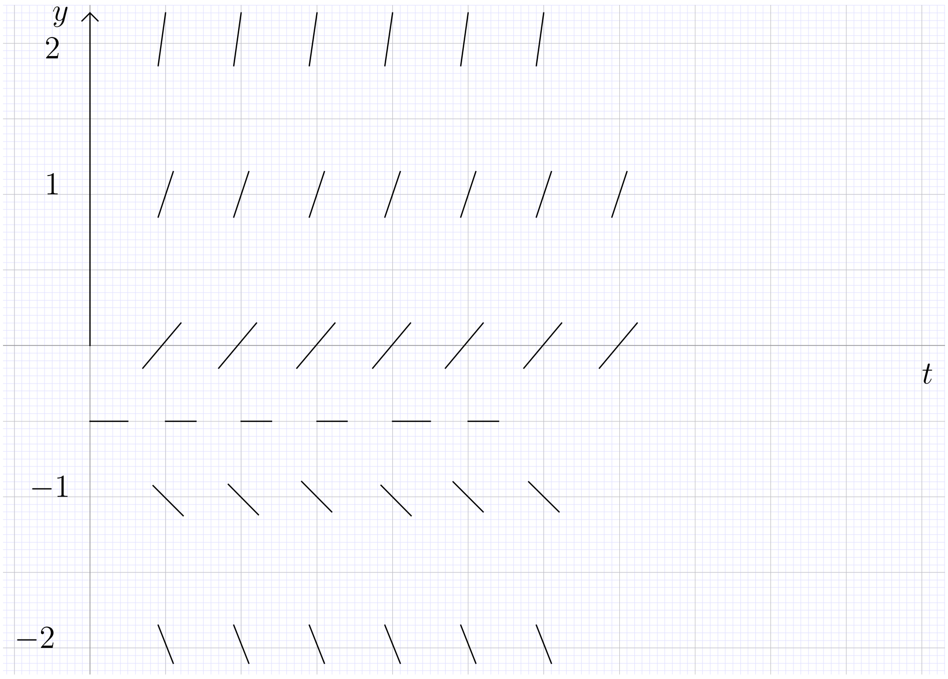
y	y'
-1	5
0	3
0.5	2
1	1
1.5	0
2	-1



Because We can see from the field that the system approaches a stable equilibrium, we can infer that as $\lim_{t\to\infty} y = 1.5$.

(5) $y' = 1 + 2y$. Again, y' is independent of t.

y	y'
-2	-3
-1	-1
-0.5	0
1	3
2	5



It is clear that the equilibrium is unstable. The behavior of $y(t)$ as $t \to \infty$ depends on the initial value of y. There are three cases:

I. $y(0) < -0.5$. In this case the slope will always be increasingly negative, so $\lim_{t\to\infty} y = -\infty$.

II. $y(0) = -0.5$. In this case the slope is always zero, so $y(t)$ won't change. $\lim_{t\to\infty} y = -0.5$.

III. $y(0) > -0.5$. In this case the slope will always be increasingly positive, so $\lim_{t\to\infty} y = \infty$.

Question 17

(a) At any time t, the amount of drug present in the bloodstream ($y(t)$) is described by the following equation:

$$y' = 5 \cdot 100 - 0.4y$$

$$y' + 0.4y = 500$$

(b) We are basically asked to calculate $\lim_{t\to\infty} y(t)$. To calculate the limit we need to solve the differential equation. We shall solve using the integrating factors method. Multiply by a function $\mu(t)$:

$$\mu\, y' + 0.4 y\, \mu = 500\mu$$

We want the LHS to be of the form $\mu\, y' + y\, \mu'$. For it to happen, we need $\mu' = 0.4y$, for which we can choose $\mu(t) = \mathrm{e}^{0.4t}$.

$$(y\, \mathrm{e}^{0.4t})' = 500\, \mathrm{e}^{0.4t}$$

Integrate both sides and divide both sides by $\mathrm{e}^{0.4t}$ to get:

$$\begin{aligned} y &= \mathrm{e}^{-0.4t}\, [1250\, \mathrm{e}^{0.4t} + c] \\ &= 1250 + c\, \mathrm{e}^{-0.4t} \end{aligned}$$

From the general solution we can see that $y \to 1250$ as $t \to \infty$.

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<u>Question 1</u>

(a). $y' + y = 5$, $y(0) = y_0$. Using integration factor method, we've seen in class that the general solution for a linear ODE with $a, b = \text{constant}$ is:

$$\begin{aligned} y &= \mathrm{e}^{-at} \left[\int b\, \mathrm{e}^{at} + c \right] \\ y &= \frac{b}{a} + c\, \mathrm{e}^{-at} \qquad (1) \end{aligned}$$

Here $a = 1, b = 5$.

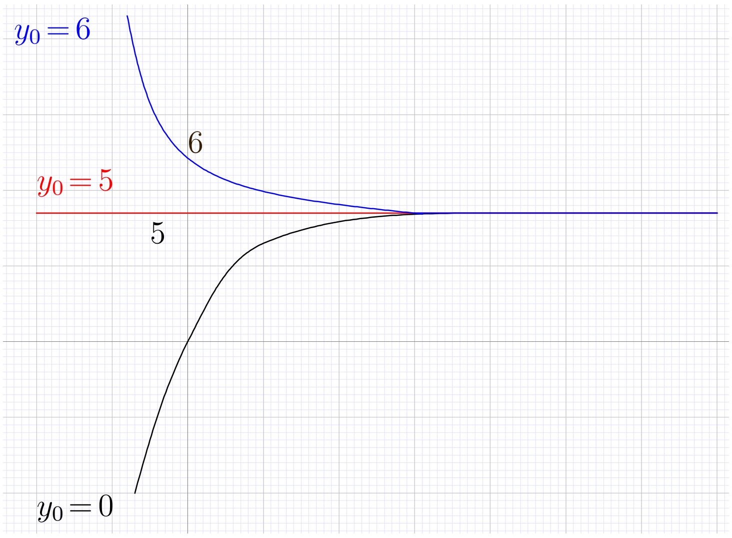
$$y = 5 + c\, \mathrm{e}^{-t}$$

For $y(0) = y_0$:

$$y(0) = y_0 = 5 + c \to c = y_0 - 5$$

$$y = 5 + (y_0 - 5)\, \mathrm{e}^{-t}$$

Let's draw the solution for different values of y_0:



$y_0 = 5$ is the equilibrium solution. For $y_0 > 5$ and $y_0 < 5$ the solutions converge at the equilibrium value, but approach it from above or below, respectively.

(b) $y' + 2y = 5$. According to eq. (1), where $a = 2, b = 5$

$$y \;=\; \frac{5}{2} + c\,\mathrm{e}^{-2t}$$

For $y(0) = y_0$,

$$y_0 = \frac{5}{2} + c \rightarrow c = y_0 - \frac{5}{2}$$

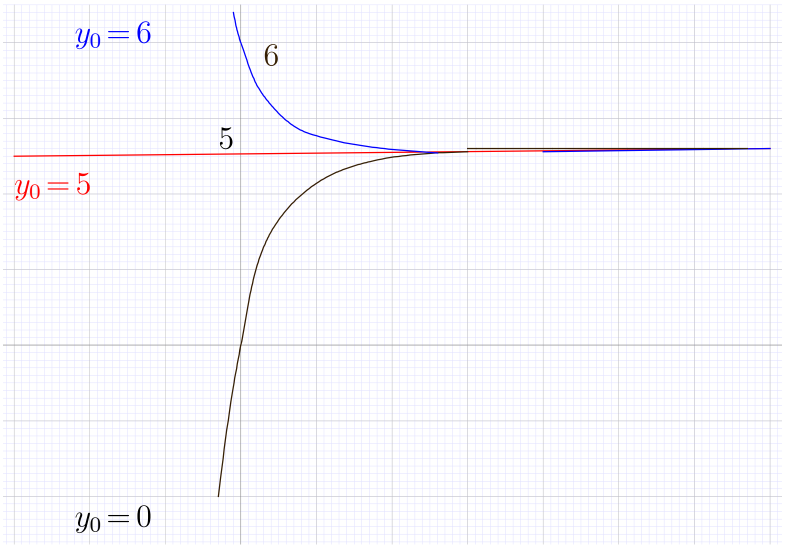
$$y = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right)\mathrm{e}^{-2t}$$

Solutions for different values of y_0:

Result is similar to (a), but the function collapses faster to equilibrium, which is located now at $y = 2.5$.

(c) $y' + 2y = 10$. According to eq. (1), where $a = 2, b = 10$, $c = y_0 - \frac{b}{a}$:

$$y = 5 + (y_0 - 5)\,\mathrm{e}^{-2t}$$



Result is identical to (b) excluding the equilibrium value, which is now $y = 5$.

Question 6

$$p' - 0.5p = -450$$

(a) According to eq. (1), where $a = -0.5, b = -450$, the solution is:

$$p \;=\; 900 + c\,\mathrm{e}^{t/2}$$

given the initial condition $p(0) = 850$,

$$850 = 900 + c \rightarrow c = -50$$

$$p = 900 - 50\,\mathrm{e}^{t/2}$$

The population becomes extinct $(p = 0)$ when:

$$900 = 50\,\mathrm{e}^{t/2}$$

$$t = 2\ln 18 \approx 5.78\,\text{months}$$

(b) For $p = p_0$, where $0 < p_0 < 900$, as we've seen in Question 1, the solution is:

$$p = 900 + (p_0 - 900)\,\mathrm{e}^{t/2}$$

The population becomes extinct when:

$$p = 0 = 900 + (p_0 - 900)\,\mathrm{e}^{t/2}$$

$$t = 2\ln\left(\frac{900}{900 - p_0}\right) \quad \text{months}$$

Indeed, when $0 < p_0 < 900$ we get only positive values for t, which is a physical set of solutions.

(c) Plug in $t = 12\,\text{months}$:

$$0 = 900 + (p_0 - 900)\,\mathrm{e}^{12/2}$$

$$-900\,\mathrm{e}^{-6} = p_0 - 900$$

$$p_0 = 900(1 - \mathrm{e}^{-6}) \approx 897.8$$

Question 8

$$v' + \frac{v}{5} = 9.8, \quad v(0) = 0$$

(a) First solve the equation. Inputting $a = 0.2, b = 9.8$, $c = y(0) - \frac{b}{a}$ in eq. (1):

$$v = 49 + (0 - 49)\,\mathrm{e}^{-t/5} = 49(1 - \mathrm{e}^{-t/5})$$

The limit velocity of the object is $v(t \to \infty) = 49$. At 98%:

$$0.98 = 1 - \mathrm{e}^{-t/5}$$

$$\ln 0.02 = -t/5$$

$$t = 5\ln 50 \approx 19.56\,\mathrm{s}$$

(b) The distance traveled is equal to the integral $\int_0^{t'} v(t)\,\mathrm{d}t$.

$$\begin{aligned}\Delta x &= \int_0^{5\ln 50} 49(1 - \mathrm{e}^{-t/5})\,\mathrm{d}t \\ &= 49 \cdot 5\ln 50 + 49 \cdot 5\,(\mathrm{e}^{-5\ln 50/5} - 1) \\ &\approx 718.34\,\mathrm{m}\end{aligned}$$

Question 11

Rewrite the given equation:

$$Q' + r\,Q = 0$$

Multiply both sides by $\mu(t) = \mathrm{e}^{rt}$:

$$\mathrm{e}^{rt} Q' + r\,\mathrm{e}^{rt} Q = 0$$

$$(\mathrm{e}^{rt} Q)' = 0$$

Integrate both sides and get:

$$Q = c\,\mathrm{e}^{-rt}$$

The half-life time of the radioactive material is defined such that $Q(\tau) = \frac{1}{2} Q(0)$. First find c by applying the initial condition:

$$Q(0) = c\,\mathrm{e}^{0} = c.$$

Now let's get an expression for τ.

$$Q(\tau) = \frac{1}{2} Q(0) = Q(0)\,\mathrm{e}^{-r\tau}$$

Assuming $Q(0) \neq 0$,

$$\frac{1}{2} = \mathrm{e}^{-r\tau} \rightarrow r\,\tau = \ln 2 \tag{2}$$

Question 12

We know that $\tau = 1620\,\text{years}$. Plugging this in the expression for $Q(t)$ we get:

$$Q(1620) = \frac{1}{2} Q(0) = Q(0)\mathrm{e}^{-1620r}$$

$$r = \frac{\ln 2}{1620}$$

For which t do we get $Q(t) = \frac{3}{4} Q(0)$?

$$Q(t) = \frac{3}{4} Q(0) = Q(0)\,\mathrm{e}^{-\frac{\ln 2}{1620} t}$$

$$\ln \frac{4}{3} = \frac{\ln 2}{1620}\, t$$

$$t = 1620 \left(\frac{\ln \frac{4}{3}}{\ln 2} \right) \approx 672.36\,\text{years}$$

Question 14

(a) The rate of change in amount of chemical in the pond, $Q'(t)$, is equal to the rate of chemical entering the pond minus the rate of fraction of water containing chemical leaving the pond. In other words,

$$Q'(t) = 300 \cdot 0.01 - \frac{300}{10^6} Q(t) = 3(1 - 10^{-4} Q), \quad Q(0) = 0$$

Because initially the pond is clean, we set $Q(0) = 0$.

(b) Rewrite the equation:

$$Q' + 3 \cdot 10^{-4} Q = 3$$

The general solution according to eq. (1) is ($a = 3 \cdot 10^{-4}, b = 3, c = -10^4$):

$$Q(t) = 10^4(1 - \mathrm{e}^{-3 \cdot 10^{-4} t})$$

1 year equals 8760 hours (365 days a year multiplied by 24 hours a day).

$$Q(8760) = 10^4(1 - \mathrm{e}^{-3 \cdot 10^{-4} \cdot 8760}) = 9277.77\,\mathrm{g}$$

(c) Now the amount of chemical in the pond just decays at the rate of water leaving the pond.

$$Q'(t) = -3 \cdot 10^{-4} Q(t), \quad Q(0) = 9277.77\,\mathrm{g}$$

(d) The solution to this DE is a simple exponent decaying at a rate of $3 \cdot 10^{-4}$ gal/hr.

$$Q(t) = Q(0) \cdot \mathrm{e}^{-3 \cdot 10^{-4} t}$$

After 1 year = 8760 hours, the amount of chemical in the pond is:

$$Q(8760) = 9277.77 \cdot \mathrm{e}^{-3 \cdot 10^{-4} \cdot 8760} = 670.07\,\mathrm{g}$$

(e) Find for which t: $Q(t) = 10\,\mathrm{g}$.

$$10 = 9277.77 \cdot \mathrm{e}^{-3 \cdot 10^{-4} t}$$

$$t = \frac{\ln 927.78}{3 \cdot 10^{-4}} \approx 2.60\,\mathrm{years}$$

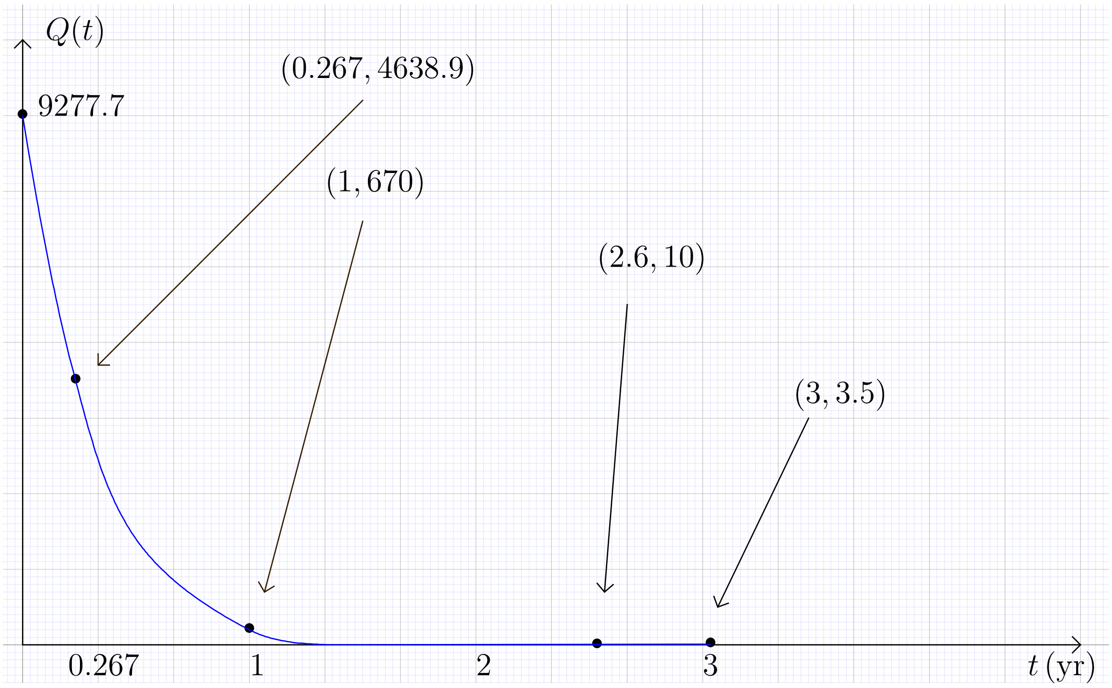
(f) Assuming the question refers to 3 years **after** pure water starts flowing into the pond, let's calculate at least another point to have on the graph. The half-life time of Q is, according to eq. (2),

$$\tau = \frac{\ln 2}{3 \cdot 10^{-4}} \cdot \frac{1\,\mathrm{years}}{8760\,\mathrm{hr}} = 0.264\,\mathrm{years}$$

The amount of chemical after 3 years:

$$Q(3\cdot 8760)=9277.77\cdot \mathrm{e}^{-3\cdot 10^{-4}\cdot 3\cdot 8760}\approx 3.50\,\mathrm{g}$$

Let's draw a graph of $Q(t)$:



If the question refers to 3 years **including** the year that chemical flowed into the pond, the graph is as follows: