2006 Exam Solution

Question 1

In a 2-degree of freedom mass system, mass of $m_1 = 2\,\text{kg}$ was suspended from a spring with spring constant k_1 and a second mass of $m_2 = 1\,\text{kg}$ was attached by a spring with spring constant $k_2 = 2$ to the first one. Let u_1 be the displacement of the first mass and let u_2 be the verticle displacement of the second mass.

(a) Construct a system of differential equations whose solution gives u_1, u_2.

Write force-balance equations for each mass (after removing $m_i g = k_i \ell_i$ from both equations):

$$m_2\, u_2'' = -k_2\,(u_2 - u_1)$$

$$m_1\, u_1'' = -k_1\, u_1 + k_2\,(u_2 - u_1)$$

Normalize:

$$\begin{aligned} u_1'' &= -\frac{(k_1 + k_2)}{m_1}\, u_1 + \frac{k_2}{m_1}\, u_2 \\ u_2'' &= \frac{k_2}{m_2}\, u_1 - \frac{k_2}{m_2}\, u_2 \end{aligned}$$

(b) Derive from it a differential equation of ordr 4, and solve for some initial conditions (not solved here).

Write u_1 as $f(u_2, u_2'')$:

$$u_1 = \left(u_2'' + \frac{k_2}{m_2}\, u_2\right) \cdot \frac{m_2}{k_2}$$

and insert in the first equation:

$$\frac{m_2}{k_2}\, u_2^{(4)} + u_2'' = -\frac{k_1 + k_2}{m_1} \cdot \left(u_2'' + \frac{k_2}{m_2}\, u_2\right) \cdot \frac{m_2}{k_2} + \frac{k_2}{m_1}\, u_2$$

This can be transformed into a nice 4th order ODE with constant coefficients, which is solved by finding the roots of the characteristic polynomial.

Question 2

Find the general solution to:

$$y^{(3)} - y'' + y' - y = 2\cos x - \mathrm{e}^{3x}$$

First solve the associated homogeneous equation. Characteristic equation is:

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

Note that $\lambda = 1$ solves the equation. Factor out $(\lambda - 1)$:

$$(\lambda - 1)(\lambda^2 + 1) = (\lambda - 1)\,(\lambda + \mathrm{i})\,(\lambda - \mathrm{i}) = 0$$

Roots are $\lambda_1 = 1, \lambda_{2,3} = \pm \mathrm{i}$. Solution to homogeneous equation is:

$$y_h = c_1 + c_2 \cos x + c_3 \sin x, \qquad c_{1,2,3} \in \mathbb{R}$$

Find two particular solutions, each corresponding to a different element in the RHS. For $2\cos x$, guess a solution of form: $y_{p1} \equiv A\cos x + B\sin x$

$$\begin{aligned} y_{p1}' &= -A\sin x + B\cos x \\ y_{p1}'' &= -A\cos x - B\sin x \\ y_{p1}^{(3)} &= A\sin x - B\cos x \end{aligned}$$

Substitute y_{p1} in the ODE:

$$A\sin x - B\cos x + (A\cos x + B\sin x) - A\sin x + B\cos x - A\cos x - B\sin x = 2\cos x$$

$$\sin x\,(A + B - A - B) + \cos x\,(-B + A + B - A) \equiv 0$$

Solution doesn't work. Have to guess solution of form:

$$\begin{aligned} y_{p1} &\equiv x\,(A\cos x + B\sin x) \\ y_{p1}' &= (A\cos x + B\sin x) + x\,(-A\sin x + B\cos x) \\ y_{p1}'' &= 2(-A\sin x + B\cos x) + x\,(-A\cos x - B\sin x) \\ y_{p1}''' &= 3(-A\cos x - B\sin x) + x\,(A\sin x - B\cos x) \end{aligned}$$

Substitute y_{p1} in the ODE:

$$\begin{aligned} 2\cos x &= 3(-A\cos x - B\sin x) + x\,(A\sin x - B\cos x) - 2(-A\sin x + B\cos x) \\ &\quad -x\,(-A\cos x - B\sin x) + (A\cos x + B\sin x) + x\,(-A\sin x + B\cos x) - x\,(A\cos x + B\sin x) \end{aligned}$$

$$\begin{aligned} 2\cos x &= \cos x\,(-3A - Bx - 2B + Ax + A + Bx - Ax) \\ &\quad + \sin x\,(-3B + Ax + 2A + Bx + B - Ax - Bx) \end{aligned}$$

$$2\cos x = \cos x\,(-2A - 2B) + \sin x\,(-2B + 2A)$$

Equate coefficients on both sides:

$$\begin{cases} \sin x\colon & -2B + 2A = 0 \rightarrow A = B \\ \cos x\colon & -2A - 2B = 2 \rightarrow A = -\frac{1}{2} \end{cases}$$

Therefore:

$$y_{p1} = -\frac{1}{2}x(\cos x + \sin x)$$

For the second part of the RHS, guess a solution of form $y_{p2} \equiv A\,\mathrm{e}^{3x}$ and input in ODE:

$$\mathrm{e}^{3x}\,(27A - 9A + 3A - A) = -\mathrm{e}^{3x}$$

$$A = -\frac{1}{20}$$

To summarize, general solution to the ODE is $y = y_h + y_{p1} + y_{p2}$.

$$y = c_1 + c_2\cos x + c_3\sin x - \frac{1}{2}x(\cos x + \sin x) - \frac{1}{20}\mathrm{e}^{3x}$$

Question 3

Find general solution to the system:

$$\vec{x}' = A\,\vec{x}$$

$$A = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$

Find eigenvalues and eigenvectors for A. It is clear already that $\lambda = 0$ is an eigenvalue, as the third row is linearly dependent on the second one ($\operatorname{rank} A^{n\times n} < n$).

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{vmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{vmatrix} 5-\lambda & 5 & 2 \\ 0 & -\lambda & -\lambda \\ 6 & 6 & 5-\lambda \end{vmatrix} = \cdots =$$

$$= (5-\lambda)(\lambda^2 - 5\lambda + 6\lambda) + 6(-5\lambda + 2\lambda) = -(\lambda - 5)\,\lambda\,(\lambda + 1) - 18\lambda = -\lambda\,(\lambda^2 - 4\lambda + 13) = 0$$

$$\lambda_1 = 0, \lambda_{2,3} = \frac{4 \pm 6\mathrm{i}}{2} = 2 \pm 3\mathrm{i}$$

Find eigenvectors. For $\lambda_1 = 0$: find $\vec{v}_1$ such that

$$(A - \lambda_1 I)\,\vec{v}_1 = \vec{0}$$

$$A - \lambda_1 I = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

From $\lambda_2 = 2 - 3\mathrm{i}$ construct two real solutions. Find eigenvector $\vec{v}_2$ such that:

$$(A - \lambda_2 I)\,\vec{v}_2 = \vec{0}$$

$$A - \lambda_2 I = \begin{bmatrix} 3+3\mathrm{i} & 5 & 2 \\ -6 & -8+3\mathrm{i} & -5 \\ 6 & 6 & 3+3\mathrm{i} \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{bmatrix} 3+3\mathrm{i} & 5 & 2 \\ 0 & -2+3\mathrm{i} & -2+3\mathrm{i} \\ 6 & 6 & 3+3\mathrm{i} \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} \frac{3}{3+3\mathrm{i}} \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3\cdot(3-3\mathrm{i})}{18} \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}\mathrm{i} \\ -1 \\ 1 \end{bmatrix}$$

Two real solutions are:

$$\vec{v}_2\,\mathrm{e}^{2x}(\cos(3x) + \mathrm{i}\sin(3x)) = \mathrm{e}^{2x}\left(\begin{bmatrix} \frac{1}{2}\cos 3x + \frac{1}{2}\sin 3x \\ -\cos 3x \\ \cos 3x \end{bmatrix} + \mathrm{i}\begin{bmatrix} \frac{1}{2}\sin 3x - \frac{1}{2}\cos 3x \\ -\sin 3x \\ \sin 3x \end{bmatrix}\right)$$

General solution to system is:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2\, \mathrm{e}^{2x} \begin{bmatrix} \frac{1}{2}\cos 3x + \frac{1}{2}\sin 3x \\ -\cos 3x \\ \cos 3x \end{bmatrix} + c_3\, \mathrm{e}^{2x} \begin{bmatrix} \frac{1}{2}\sin 3x - \frac{1}{2}\cos 3x \\ -\sin 3x \\ \sin 3x \end{bmatrix}$$

Question 4

(a) Solve the equation

$$y'\, y'' - t = 0$$

with ICs $y(1) = 2, y'(1) = 1$.

Set $v = y'$.

$$\frac{\mathrm{d}v}{\mathrm{d}t} \cdot v = t$$

This is a separable equation. It reads

$$\int v\,\mathrm{d}v = \int t\,\mathrm{d}t$$

$$\frac{1}{2} v^2 = \frac{1}{2} t^2 + c_1$$

$$y' = v = \pm\sqrt{t^2 + 2c_1}, \qquad 2c_1 \geq -t^2$$

Integrate again to get

$$y = \pm \int \sqrt{t^2 + 2c_1}\, \mathrm{d}t$$

This integral is hard to calculate... We're asked to find a unique solution. Find c_1 using the IC that $y'(1) = 1$.

$$\frac{1}{2} \cdot y'(1) = \frac{1}{2} \cdot 1^2 + c_1 \rightarrow c_1 = 0$$

This simplifies the solution a bunch, as now we are left with

$$y' = \pm\sqrt{t^2} = \pm t$$

which means

$$y = \int y'\, \mathrm{d}t = \pm \frac{1}{2} t^2 + c_2$$

Insert second IC that $y(1) = 2$:

$$y(1) = 2 = \pm \frac{1}{2} \cdot 1^2 + c_2$$

There are two possibilities: either $c_2 = \frac{3}{2}$ or $c_2 = \frac{5}{2}$. Note though that only the solution

$$y = \frac{1}{2} t^2 + \frac{3}{2}$$

satisfies the IC $y'(1) = 1$. (The second one gives $y'(1) = -1$.)

(b) Solve the equation

$$y' = x\, y^3\, (1 + x^2)^{-1/2}$$

Note that $y \equiv 0$ solves the equation but doesn't satisfy the IC.

with IC $y(0) = 1$. This is a separable equation. Integrate both sides:

$$-\frac{1}{2y^2} = \int \frac{\mathrm{d}y}{y^3} = \int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = \sqrt{x^2+1} + c$$

$$y = \pm\sqrt{-\frac{1}{2\sqrt{x^2+1}+c}}, \quad c \neq -\sqrt{x^2+1}$$

Find solution that satisfies the IC, which constraints solution to only positive values.

$$y(0) = 1 = -\frac{1}{c+2} \to c = -3$$

Unique solution is

$$y = \sqrt{\frac{1}{3-2\sqrt{x^2+1}}}$$

Question 5

Solve the following boundary value problem:

$$25y_{xx} = y_{tt}, \quad x \in (0,3), t > 0$$

With homogeneous BCs: $y(0,t) = y(3,t) = 0$ and ICs: $y(x,0) = \frac{1}{4}\sin(\pi x)$, $y_t(x,0) = 10\sin(2\pi x)$.

This is the wave equation, and it can be solved via D'Alembert method, as the ICs in both cases there is a trivial odd extension of the IC function (in zero initial position and velocity, respectively—we shall treat each case separately) to a $2L$-periodic function on $\mathbb{R}$.

Zero initial velocity case: ICs are $\begin{cases} y(x,0) \equiv f(x) = \frac{1}{4}\sin(\pi x) \\ y_t(x,0) \equiv 0 \end{cases}$

In this case, the solution is

$$u(x,t) = \frac{1}{2}(F(x+a\,t) + F(x-a\,t))$$

where $F(x) = f(x)$ on $x \in \mathbb{R}$, with $a = 5$.

$$u_1(x,t) = \frac{1}{2}\cdot\frac{1}{4}[\sin(\pi x + 5\pi t) + \sin(\pi x - 5\pi t)] = \frac{1}{4}\sin(\pi x)\cos(5\pi t)$$

Zero initial position case: ICs are $\begin{cases} y(x,0) \equiv 0 \\ y_t(x,0) \equiv g(x) = 10\sin(2\pi x) \end{cases}$

Define $H(x)$ as the primitive function of $g(x)$, i.e. $H(x) = \int g(\xi)\,\mathrm{d}\xi$. Then,

$$u(x,t) = \frac{1}{2a}[H(x+a\,t) - H(x-a\,t)]$$

Calculate $H(x)$:

$$H(x) = 10\int \sin(2\pi\,\xi)\,\mathrm{d}\xi = -\frac{10}{2\pi}\cos(2\pi x) = -\frac{5}{\pi}\cos(2\pi x)$$

Therefore,

$$u_2(x,t) = \frac{1}{2\cdot 5}\cdot\left(-\frac{5}{\pi}\right)[\cos(2\pi x + 10\pi t) - \cos(2\pi x - 5t)] = -\frac{1}{\pi}\cos(2\pi x)\cos(10\pi t)$$

Solution to the boundary values problem is the sum of $u_1(x,t)$ and $u_2(x,t)$:

$$u(x,t)=\frac{1}{4}\sin(\pi x)\cos(5\pi t)-\frac{1}{\pi}\cos(2\pi x)\cos(10\pi t)$$

Question 6

Given a rod of length π with thermal diffusivity constant $\alpha^2=3$, find the temperature $u(x,t)$ at point x and time t along the rod if the temperature at time $t=0$ is

$$u(x,0)=4\sin 2x+\frac{10}{\pi}x+15,\quad x\in(0,\pi)$$

and the temperature are the endpoints is held constant so that $u(0,t)=15$ and $u(\pi,t)=25\ \forall t$.

Define $w(x,t)$ and $v(x)$ such that:

$$w(x,t)=u(x,t)-v(x)$$

$$v(x)=\frac{10}{\pi}x+15$$

Basically, $w(x,t)$ satisfies the heat equation with homogeneous BCs.

The solution to such equation is:

$$w(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\mathrm{e}^{-\frac{n^2\pi^2\alpha^2t}{L^2}}$$

where

$$c_n=\frac{2}{L}\int_0^L w(x,0)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n=\frac{2}{\pi}\int_0^{\pi}4\sin 2x\cdot\sin(nx)\,\mathrm{d}x$$

Sines of different frequencies form an orthogonal family, so:

$$c_n=\frac{8}{\pi}\left(\int_0^{\pi}\sin 2x\cdot\sin(nx)\,\mathrm{d}x\right)\cdot\delta(n-2)$$

which is equivalent to

$$c_2=\frac{8}{\pi}\int_0^{\pi}\sin^2 2x\,\mathrm{d}x=\frac{4}{\pi}\left[x-\frac{\sin 4x}{4}\right]_0^{\pi}=4$$

So:

$$w(x,t)=4\sin 2x\cdot\mathrm{e}^{-12t}$$

and:

$$u(x,t)=w(x,t)+v(x)=4\sin 2x\cdot\mathrm{e}^{-12t}+\frac{10}{\pi}x+15$$