Assignment 8

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Date:

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Question 2

$$A = \begin{bmatrix} 1+\mathrm{i} & -1+2\,\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{bmatrix}, B = \begin{bmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{bmatrix}$$

(a)

$$A-2B = \begin{bmatrix} 1+\mathrm{i} & -1+2\,\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{bmatrix} - \begin{bmatrix} 2\mathrm{i} & 6 \\ 4 & -4\mathrm{i} \end{bmatrix} = \begin{bmatrix} 1-\mathrm{i} & -7+2\mathrm{i} \\ -1+2\mathrm{i} & 2+3\mathrm{i} \end{bmatrix}$$

(b)

$$3A+B = \begin{bmatrix} 3+3\mathrm{i} & -3+6\,\mathrm{i} \\ 9+6\mathrm{i} & 6-3\mathrm{i} \end{bmatrix} + \begin{bmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{bmatrix} = \begin{bmatrix} 3+4\mathrm{i} & 6\mathrm{i} \\ 11+6\mathrm{i} & 6-5\mathrm{i} \end{bmatrix}$$

(c)

$$\begin{aligned} A\,B &= \begin{bmatrix} 1+\mathrm{i} & -1+2\,\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{bmatrix} \begin{bmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{i}\,(1+\mathrm{i})+2\,(-1+2\mathrm{i}) & 3\,(1+\mathrm{i})-2\mathrm{i}\,(-1+2\mathrm{i}) \\ \mathrm{i}\,(3+2\mathrm{i})+2\,(2-\mathrm{i}) & 3\,(3+2\mathrm{i})-2\mathrm{i}\,(2-\mathrm{i}) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{i}-1-2+4\mathrm{i} & 3+3\mathrm{i}+2\mathrm{i}+4 \\ 3\mathrm{i}-2+4-2\mathrm{i} & 9+6\mathrm{i}-4\mathrm{i}-2 \end{bmatrix} \\ &= \begin{bmatrix} -3+5\mathrm{i} & 7+5\mathrm{i} \\ 2+\mathrm{i} & 7+2\mathrm{i} \end{bmatrix} \end{aligned}$$

(d)

$$\begin{aligned} B\,A &= \begin{bmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{bmatrix} \begin{bmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{i}\,(1+i)+3\,(3+2\mathrm{i}) & \mathrm{i}\,(-1+2\mathrm{i})+3\,(2-\mathrm{i}) \\ 2\,(1+\mathrm{i})-2\mathrm{i}\,(3+2\mathrm{i}) & 2\,(-1+2\mathrm{i})-2\mathrm{i}\,(2-\mathrm{i}) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{i}-1+9+6\mathrm{i} & -\mathrm{i}-2+6-3\mathrm{i} \\ 2+2\mathrm{i}-6\mathrm{i}+4 & -2+4\mathrm{i}-4\mathrm{i}-2 \end{bmatrix} \\ &= \begin{bmatrix} 8+7\mathrm{i} & 4-4\mathrm{i} \\ 6-4\mathrm{i} & -4 \end{bmatrix} \end{aligned}$$

Question 4

$$A = \begin{bmatrix} 3-2\mathrm{i} & 1+\mathrm{i} \\ 2-\mathrm{i} & -2+3\mathrm{i} \end{bmatrix}$$

(a)

$$A^T = \begin{bmatrix} 3-2\mathrm{i} & 2-\mathrm{i} \\ 1+\mathrm{i} & -2+3\mathrm{i} \end{bmatrix}$$

(b)

$$\bar{A} = \begin{bmatrix} 3+2\mathrm{i} & 1-\mathrm{i} \\ 2+\mathrm{i} & -2-3\mathrm{i} \end{bmatrix}$$

(c)

$$A^* = \bar{A}^T = \begin{bmatrix} 3+2\mathrm{i} & 2+\mathrm{i} \\ 1-\mathrm{i} & -2-3\mathrm{i} \end{bmatrix}$$

<u>Question 12</u>

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Use Gauss-Seidal algorithm to find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \to \cdots$$

$$\cdots \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_3 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right]$$

Change sign of rows 2 and 3 and switch between then to get:

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

<u>Question 14</u>

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{bmatrix}$$

Again, use Gauss-Seidal algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{array}\right]$$

After performing elementary operations that don't affect the determinant of A we've got 2 linearly dependent rows. That means rank $A = 2 < n = 3$, so det $A = 0$ and A is singular.

Question 18

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_4]{R_3 \to R_3 + R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \cdots$$

$$\cdots \xrightarrow{\substack{R_4 \to R_4 + R_3 \\ R_3 \to R_3 + R_2 \\ R_1 \to R_1 + R_2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Question 19

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -4 & 2 \\ 1 & 0 & 1 & 3 \\ -2 & 2 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 + 2R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \cdots$$

$$\cdots \xrightarrow[R_3 \to 4R_3]{R_3 \to R_3 - \frac{1}{4}R_4 - R_2} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -10 & -4 & 4 & -1 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - \frac{2}{5}R_3 \\ R_4 \to R_4 + \frac{1}{5}R_3 \\ R_1 \to R_1 - \frac{1}{2}R_4}} \cdots$$

$$\cdots \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 5 & -10 & -4 & 4 & -1 \\ 0 & 0 & 4 & 0 & 0 & -\frac{4}{5} & \frac{4}{5} & \frac{4}{5} \end{array}\right] \xrightarrow{\substack{R_1 \to R_1 - R_2 \\ R_3 \to \frac{1}{5}R_3 \\ R_4 \to \frac{1}{4}R_4}} \left[\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & -5 & -\frac{11}{5} & \frac{6}{5} & -\frac{4}{5} \\ 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{bmatrix}$$

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Questions: 1,3,9,10,11

Question 1

In matrix form:

$$\overbrace{\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}^{\vec{b}}$$

Solution is given by $\vec{x} = A^{-1}\vec{b}$. Find inverse of A via Gauss-Seidal algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \to \cdots$$

$$\cdots \xrightarrow[]{\substack{R_3 \to R_3 - R_2 \\ R_3 \to -\frac{1}{3}R_3 \\ R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 4R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{7}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{7}{3} & -\frac{1}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{7}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Question 3

$$\overbrace{\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}^{\vec{b}}$$

Note that the second row of A is the sum of the first and third rows, meaning that $\operatorname{rank} A = 2 < n = 3$. This means that there's infinitely many solutions.

We can still solve using the 1st and 3rd row and obtain a formula for the solutions:

$$x_1 + 2x_2 - x_3 = 2 \tag{1}$$

$$2x_1 + x_2 + x_3 = 1 \tag{2}$$

Assuming x_3 is arbitrary: Subtract $2 \cdot \text{eq}\,(1)$ from eq (2).

$$-3x_2 + 3x_3 = -3 \rightarrow x_2 = x_3 + 1$$

$$x_1 + 2\,(x_3 + 1) - x_3 = 2$$

$$x_1 = -x_3$$

The set of solutions is

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

where x_3 is arbitrary.

Question 9

The question is equivalent to finding a set of coefficients $\vec{k} = (k_1, k_2, \ldots, k_n)^T$, n the number of vectors in question, such that

$$k_1\,\vec{x}_1 + k_2\,\vec{x}_2 + \cdots + k_n\,\vec{x}_n = \vec{0}$$

If $k_1, \ldots, k_n = 0$ then the vectors are linearly independent. If not, the relation between the vectors is obtained by finding $k_n \neq 0$ that satisfy the equation

$$A\,\vec{k} = \vec{0}$$

where

$$A = [\ \vec{x}_1\ \ \vec{x}_2\ \ \cdots\ \ \vec{x}_4\ ]$$

We may solve the system of equations via Gaussian elimination. Also, if A is a square matrix, once we can be certain of $\operatorname{rank} A$, if $\operatorname{rank} A = n$ then all vectors are linearly independent.

$$A = [\ \vec{x}_1\ \ \vec{x}_2\ \ \vec{x}_3\ \ \vec{x}_4\ ] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{bmatrix}$$

Try to reduce A to echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 2 & -7 \\ 0 & 3 & 1 & 4 \\ 0 & -1 & 2 & 3 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + 3R_2} \cdots$$

$$\cdots \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 2 & -7 \\ 0 & 0 & 7 & -17 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

We've reduced A to upper-tridiagonal form, which implicates that $\operatorname{rank} A = n$ and all columns are linearly independent.

<u>Question 10</u>

$$A = [\, \vec{x}_1 \;\; \vec{x}_2 \;\; \vec{x}_3 \;\; \vec{x}_4 \,] = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{bmatrix}$$

Reduce to row echelon form:

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 1 & -2 \end{bmatrix} \cdots$$

$$\cdots \xrightarrow{\substack{R_3 \to R_3 + 2R_1 \\ R_3 \to \frac{1}{3}R_3 \\ R_1 \to R_1 - 2R_3}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The rank of matrix A is $3 < n = 4$, so the vectors are linearly dependent

Expand the matrix to obtain

$$k_1 = -k_4, k_2 = -k_4, k_3 = 0$$

Set $k_4 = 1$ and obtain the linear relation between $\vec{x}_1, \ldots, \vec{x}_4$:

$$\vec{x}_1 + \vec{x}_2 - \vec{x}_4 = 0$$

<u>Question 11</u>

Given the vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(m)}$ where each has n components where $n < m$, we shall show that $\vec{x}^{(1)}, \ldots, \vec{x}^{(m)}$ are linearly dependent.

Let A be a matrix containing all the vectors above: $A = [\vec{x}^{(1)}, \ldots, \vec{x}^{(m)}]$. Then A is $n \times m$ and the equation $A\vec{k} = \vec{0}$ corresponds to a system of n equations in m unknowns. If $m > n$, there are more variables than equations, so there must be a free variable. Hence, $A\vec{k} = \vec{0}$ has a non-trivial solution and the columns of A are linearly dependent.

Another approach: The rank of a $n \times m$ matrix always satisfies $\operatorname{rank} A \leq \min(m, n)$. As $n < m$, surely $\operatorname{rank} A \leq n$, so the maximum number of independent rows (or columns) is n. In A the number of colunms is bigger than its rank, so the columns must be linearly dependent.

As $\operatorname{rank} A = \operatorname{rank} A^T$, there are more rows in A^T

There are more rows than independent Therefore, m rows in A are linearly dependent.

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Determine whether the given set of vectors is linearly independent in the vector space of vectors of length 2 with entries of real-valued functions over the real numbers.

(a) $v_1 = \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}, v_2 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}, v_3 = \begin{bmatrix} 3e^{-t} \\ 0 \end{bmatrix}$.

These functions are independent iff for all $t \in \mathbb{R}$:

$$\alpha \cdot \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} + \beta \cdot \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + \gamma \cdot \begin{bmatrix} 3e^{-t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0_f \\ 0_f \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

From the first row:

$$(\alpha + \beta + 3\gamma)e^{-t} = 0$$

$$\alpha + \beta + 3\gamma = 0 \tag{3}$$

From the second row:

$$\alpha\, e^{-2t} + \beta \cdot e^{-t} = 0$$

At $t = 0$

$$\alpha + \beta = 0 \tag{4}$$

Combining (3) and (4) gives $\gamma = 0$.

At $t = \ln 2$ (from second row)

$$\frac{1}{4}\alpha + \frac{1}{2}\beta = 0 \rightarrow \frac{1}{2}\alpha + \beta = 0 \tag{5}$$

Combining (4) and (5) gives

$$\alpha = 0, \beta = 0$$

In conclusion, there is no linear combination of v_1, v_2, v_3 such that $\forall t \in \mathbb{R}$

$$\alpha v_1 + \beta v_2 + \gamma v_3 = \vec{0}_f$$

So v_1, v_2, v_3 are linearly independent.

(b) $v_1 = \left[\begin{smallmatrix} 2\sin t \\ \sin t \end{smallmatrix}\right], v_2 = \left[\begin{smallmatrix} \sin t \\ 2\sin t \end{smallmatrix}\right]$. Find $\alpha, \beta \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$:

$$\begin{aligned} 2\alpha \sin t + \beta \sin t &= 0 \\ \alpha \sin t + 2\beta \sin t &= 0 \end{aligned}$$

at $t = \frac{\pi}{2}$:

$$\begin{aligned} 2\alpha + \beta &= 0 \\ \alpha + 2\beta &= 0 \end{aligned}$$

For both equations to hold, we must have $\alpha = 0, \beta = 0$, so v_1, v_2 are linearly independent.

(c) $v_1 = \left[\begin{smallmatrix} \mathrm{e}^t \\ t\,\mathrm{e}^t \end{smallmatrix}\right], v_2 = \left[\begin{smallmatrix} 1 \\ t \end{smallmatrix}\right]$. Find $\alpha, \beta \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$:

$$\begin{aligned} \alpha\,\mathrm{e}^t + \beta &= 0 \\ \alpha\, t\,\mathrm{e}^t + \beta t &= 0 \end{aligned}$$

Setting $t = 0$ in the first equation gives:

$$\alpha + \beta = 0 \tag{6}$$

Setting $t = \ln 2$ in the first equation gives:

$$2\alpha + \beta = 0 \tag{7}$$

For both equations to hold, we must have $\alpha = 0, \beta = 0$, so v_1, v_2 are linearly independent.

(d) For a real number t_0, the set of vectors $v_1 = [\mathrm{e}^{t_0}, t_0\,\mathrm{e}^{t_0}]$, $v_2 = [1, t_0]$ are linearly dependent if there are α, β (not zero) real scalars such that

$$\alpha v_1 + \beta v_2 = 0$$

Let's try to find α, β that satisfy this condition.

$$\alpha\, v_1 + \beta\, v_2 = 0 \Longleftrightarrow \begin{cases} \alpha\, \mathrm{e}^{t_0} + \beta = 0 \\ \alpha\, t_0\, \mathrm{e}^{t_0} + \beta\, t_0 = 0 \end{cases}$$

From the first equation: $\beta = -\alpha\, \mathrm{e}^{t_0}$. Plug in the second equation to get:

$$\alpha\, t_0\, \mathrm{e}^{t_0} + (-\alpha\, \mathrm{e}^{t_0})\, t_0 = 0 \quad \forall t_0$$

There are indeed α, β real non-zero scalars that satisfy the condition $\alpha\, v_1 + \beta\, v_2 = 0$. Therefore, v_1, v_2 are linearly dependent.