Assignment 12

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Question 1

Find the Fourier series for $f(x) = \sin x + \cos x$ on $[-\pi, \pi]$.

Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \right)$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L}\, \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n \pi x}{L}\, \mathrm{d}x$$

Calculate Fourier coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin x + \cos x)\, \mathrm{d}x = [-\cos x + \sin x]_{-\pi}^{\pi} = 0$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos (n\, x)\, \mathrm{d}x + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \cos (n\, x)\, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sin (x + n\, x) + \sin (x - n\, x)]\, \mathrm{d}x + \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos (x + n\, x) + \cos (x - n\, x)]\, \mathrm{d}x
\end{aligned}$$

The integral over the sines is equal to zero because it is an integral of odd functions over a symmetric interval. Calculate the integral of the cosines.

$$\begin{aligned}
a_n &= \frac{1}{\pi} \left[\frac{\sin (x + n\, x)}{1 + n} + \frac{\sin (x - n\, x)}{1 - n} \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[\frac{\sin (\pi + n\, \pi) - \sin 0}{1 + n} + \frac{\sin (\pi - n\, \pi) - \sin 0}{1 - n} \right] = 0
\end{aligned}$$

Here we assumed $n \neq 1$. If $n = 1$:

$$a_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos (2\, x) + \cos (0)]\, \mathrm{d}x = 1$$

Calculate the other coefficient.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin (n\, x)\, \mathrm{d}x + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos x \sin (n\, x)\, \mathrm{d}x$$

The second integral is equal to zero because it's an integral of an odd function over a symmetric interval. First integral is:

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{\pi} [\cos (x - n\, x) - \cos (x + n\, x)]\, \mathrm{d}x \\
&= \frac{1}{\pi} \left[\frac{\sin (x - n\, x)}{1 - n} - \frac{\sin (x + n\, x)}{1 + n} \right]_0^{\pi} = 0
\end{aligned}$$

Here we also assumed $n \neq 1$. If we set $n = 1$:

$$b_1 = \frac{1}{\pi} \int_0^{\pi} [\cos(0) - \cos(2x)] \, \mathrm{d}x = 1$$

In conclusion, a_n, b_n are zero for all n except $n = 1$, for which $a_1, b_1 = 1$. If we plug the coefficients in the formula for the series we get:

$$f(x) = a_1 \cos \frac{1 \pi x}{\pi} + b_1 \sin \frac{1 \pi x}{\pi} = \cos x + \sin x$$

As expected, $\sin x + \cos x$ is its own Fourier sreies.

Question 2

Fourier series for

$$f(x) = -x, \quad [-L, L]$$

Calculate Fourier coefficients.

$$a_0 = \frac{1}{L} \int_{-L}^{L} (-x) \, \mathrm{d}x = [f(x) \text{ odd}] = 0$$

$$a_n = -\frac{1}{L} \int_{-L}^{L} x \cos \frac{n \pi x}{L} \, \mathrm{d}x$$

Integrate by parts:

$$\int x \cos \frac{n \pi x}{L} \, \mathrm{d}x = \frac{L x}{n \pi} \sin\left(\frac{n \pi x}{L}\right) - \frac{L^2}{n^2 \pi^2} \cos\left(\frac{n \pi x}{L}\right)$$

$$\begin{aligned} a_n &= -\frac{1}{n \pi}\left[x \sin\left(\frac{n \pi x}{L}\right) - \frac{L}{n \pi} \cos\left(\frac{n \pi x}{L}\right)\right]_{-L}^{L} \\ &= -\frac{1}{n \pi}\left[(L \sin(n \pi) + L \sin(-n \pi)) - \frac{L}{n \pi}(\cos(n \pi) - \cos(-n \pi))\right] = 0 \end{aligned}$$

$$b_n = -\frac{1}{L} \int_{-L}^{L} x \sin \frac{n \pi x}{L} \, \mathrm{d}x$$

Integrate by parts:

$$\int x \sin \frac{n \pi x}{L} \, \mathrm{d}x = -\frac{L x}{n \pi} \cos\left(\frac{n \pi x}{L}\right) + \frac{L^2}{n^2 \pi^2} \sin\left(\frac{n \pi x}{L}\right)$$

$$\begin{aligned} b_n &= \frac{1}{n \pi}\left[x \cos\left(\frac{n \pi x}{L}\right) - \frac{L}{n \pi} \sin\left(\frac{n \pi x}{L}\right)\right]_{-L}^{L} \\ &= \frac{1}{n \pi}\left[(L \cos(n \pi) + L \cos(-n \pi)) - \frac{L}{n \pi}(\sin(n \pi) - \sin(-n \pi))\right] = \frac{2L \cos(n \pi)}{n \pi} \end{aligned}$$

$$b_n = \frac{2L}{n \pi} (-1)^n$$

Fourier series for $f(x)$ on given interval is

$$f(x) = \sum_{n=1}^{\infty} \frac{2L}{n \pi} (-1)^n \sin \frac{n \pi x}{L}$$

Question 3

Fourier series for

$$f(x) = \begin{cases} x & -\pi \le x \le 0 \\ 0 & 0 \le x < \pi \end{cases}$$

on $[-\pi, \pi]$. Calculate Fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,\mathrm{d}x + 0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(n\,x)\,\mathrm{d}x + 0 = \frac{1}{n\,\pi}\left[x\sin(n\,x) + \frac{1}{n}\cos(n\,x)\right]_{-\pi}^{0} = \frac{1}{n\,\pi}\left[\frac{1}{n} - \frac{1}{n}\cos(n\,\pi)\right] = \frac{1-(-1)^n}{n^2\,\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(n\,x)\,\mathrm{d}x + 0 = \frac{1}{\pi\,n}\left[-x\cos(n\,x) + \frac{1}{n}\sin(n\,x)\right]_{-\pi}^{0} = -\frac{\cos(n\,\pi)}{n} = -\frac{(-1)^n}{n}$$

Fourier series in given interval is

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\,\pi}\cos(n\,x) - \frac{(-1)^n}{n}\sin(n\,x)\right)$$

Question 4

Fourier series for

$$f(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ -(1-t) & 1 \le t < 2 \end{cases}$$

on $[0, 2]$. Calculate Fourier coefficients.

$$a_0 = \int_0^1 (1-t)\,\mathrm{d}t - \int_1^2 (1-t)\,\mathrm{d}t = \left[t - \frac{1}{2}t^2\right]_0^1 - \left[t - \frac{1}{2}t^2\right]_1^2 = 1$$

$$a_n = \int_0^1 (1-t)\cos(n\,\pi\,t) - \int_1^2 (1-t)\cos(n\,\pi\,t)$$

$$\int (1-t)\cos(n\,\pi\,t) = \frac{\sin(n\,\pi\,t)}{n\,\pi}(1-t) - \frac{\cos(n\,\pi\,t)}{n^2\,\pi^2}$$

$$a_n = -\frac{1}{n\,\pi}\left[(1-t)\sin(n\,\pi\,t) + \frac{1}{n\,\pi}\cos(n\,\pi\,t)\right]_0^1 + \frac{1}{n\,\pi}\left[(1-t)\sin(n\,\pi\,t) + \frac{1}{n\,\pi}\cos(n\,\pi\,t)\right]_1^2$$

$$a_n = -\frac{1}{n\,\pi}\left[\frac{(-1)^n}{n\,\pi} - \frac{1}{n\,\pi}\right] + \frac{1}{n\,\pi}\left[\frac{1}{n\,\pi} - \frac{(-1)^n}{n\,\pi}\right] = \frac{2}{n^2\,\pi^2}[1-(-1)^n]$$

$$b_n = \int_0^1 (1-t)\sin(n\,\pi\,t) - \int_1^2 (1-t)\sin(n\,\pi\,t)$$

$$\int (1-t)\sin(n\,\pi\,t) = -\frac{\cos(n\,\pi\,t)}{n\,\pi}(1-t) - \frac{\sin(n\,\pi\,t)}{n^2\,\pi^2}$$

$$b_n = -\frac{1}{n\,\pi}\left[(1-t)\cos(n\,\pi\,t) + \frac{\sin(n\,\pi\,t)}{n\,\pi}\right]_0^1 + \frac{1}{n\,\pi}\left[(1-t)\cos(n\,\pi\,t) + \frac{\sin(n\,\pi\,t)}{n\,\pi}\right]_1^2$$

$$b_n = \frac{1}{n\,\pi} - \frac{1}{n\,\pi} = 0$$

Fourier series for $f(t)$ on given interval is

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n \pi t)$$

Another formula for $f(t)$:

$$f(t) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{4}{n^2 \pi^2} \cos(n \pi t)$$

Question 5

Given the initial value problem:

$$y'' + 4y = f(t), \quad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

where $f(t)$ is as in Question 4.

(i) Find a particular solution y_p of the form

$$y_p = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n \pi t) + b_n \sin(n \pi t)]$$

Now we plug in y_p in the ODE and equate coefficients. By observation of $f(t)$ and the ODE it is clear that

$$4 \cdot \frac{a_0}{2} = \frac{1}{2} \to a_0 = \frac{1}{4}$$

Additionaly, as there is only a second derivative of y_p, we can infer that $b_n = 0$. (Second derivative of sine and cosine returns scalar multiples of sine and cosine, respectively).

$$y_p'' = \sum_{n=1}^{\infty} (-n^2 \pi^2 a_n) \cos(n \pi t)$$

Plug in the ODE:

$$\sum_{n=1}^{\infty} (-n^2 \pi^2 a_n) \cos(n \pi t) + 4\left(\frac{1}{8} + \sum_{n=1}^{\infty} a_n \cos(n \pi t)\right) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n \pi t)$$

$$\sum_{n=1}^{\infty} [-n^2 \pi^2 + 4] a_n \cos(n \pi t) = \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [1 - (-1)^n] \cos(n \pi t)$$

$$a_n = \frac{2[1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)}$$

The particular solution y_p is:

$$y_p = \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n^2 \pi^2 (4 - n^2 \pi^2)} \cos(n \pi t) \quad \forall t$$

(ii) Find general solution to associated homogeneous equation. Characteristic equation is:

$$\lambda^2 + 4 = 0$$

$$\lambda_{1,2} = \pm 2\mathrm{i}$$

General, real solution to the homogeneous equation, y_h, is:

$$y_h = c_1 \cos(2\,t) + c_2 \sin(2\,t) \quad c_{1,2} \in \mathbb{R}, \forall t$$

(iii) General solution to ODE is

$$y = y_h + y_p = c_1 \cos(2\,t) + c_2 \sin(2\,t) + \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)} \cos(n\,\pi\,t), \quad c_{1,2} \in \mathbb{R}, \forall t$$

Substitute initial conditions: $y(0)=1$, $y'(0)=0$.

$$y(0) = c_1 + \frac{1}{8} + \sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)} = 1$$

$$c_1 = \frac{7}{8} - \sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)}$$

$$y'(0) = 2\,c_2 \rightarrow c_2 = 0$$

Unique solution satisfying ICs is:

$$y = \frac{1}{8} + \left(\frac{7}{8} - \sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)}\right) \cos(2\,t) + \sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)} \cos(n\,\pi\,t), \quad \forall t$$

Can further simplify:

$$y = \frac{1}{8} + \frac{7}{8}\cos(2t) - 2\sum_{n=1}^{\infty} \frac{2\,[1-(-1)^n]}{n^2\,\pi^2\,(4-n^2\,\pi^2)} \sin\left(\left(n\,\frac{\pi}{2}+1\right)t\right) \sin\left(\left(n\,\frac{\pi}{2}-1\right)t\right), \qquad \forall t$$