Assignment 11

by Yuval Bernard

Date: TBE

Page 547

Question 2

$$y'' + 2\,y = 0, \qquad \begin{cases} y'(0) = 1 \\ y'(\pi) = 0 \end{cases}$$

Solve characteristic equation.

$$x^2 + 2 = 0$$

$$x = \pm\sqrt{2}\,\mathrm{i}$$

General solution is:

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + c_2 \sin\left(\sqrt{2}\,x\right), \quad c_{1,2} \in \mathbb{R}, \quad \forall x$$

Find $c_{1,2}$ that satisfy the boundary values.

$$y' = -\sqrt{2}\,c_1 \sin\left(\sqrt{2}\,x\right) + \sqrt{2}\,c_2 \cos\left(\sqrt{2}\,x\right)$$

Insert BVs.

$$y'(0) = 1 \quad \rightarrow \quad c_2 = \frac{\sqrt{2}}{2}$$

$$y'(\pi) = 0 \quad \rightarrow \quad -\sqrt{2}\,c_1 \sin\left(\sqrt{2}\,\pi\right) + \cos\left(\sqrt{2}\,\pi\right) \rightarrow c_1 = \frac{\sqrt{2}}{2} \cot\left(\sqrt{2}\,\pi\right)$$

Unique solution to the BVP is

$$y = \frac{\sqrt{2}}{2} \cot\left(\sqrt{2}\,\pi\right) \cos\left(\sqrt{2}\,x\right) + \frac{\sqrt{2}}{2} \sin\left(\sqrt{2}\,x\right), \quad x \in [0, \pi]$$

Question 5

$$y'' + y = x, \qquad \begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

First solve associated homogeneous equation, then find a particular solution. Characteristic equation:

$$x^2 + 1 = 0 \rightarrow x = \pm\mathrm{i}$$

General solution of homogeneous equation is

$$y_h = c_1 \cos x + c_2 \sin x, \quad c_{1,2} \in \mathbb{R}, \quad \forall x$$

Guess a particular solution of the form $y_p = A\,x^2 + B\,x + C$

$$\begin{aligned} y_p' &= 2\,A\,x + B \\ y_p'' &= 2\,A \end{aligned}$$

Subsitute back in BVP

$$2\,A + A\,x^2 + B\,x + C = x$$

Equate coefficients on both sides:

$$\begin{cases} x^2\colon \quad A = 0 \\ x\colon \quad B = 1 \\ 1\colon \quad 2\,A + C = 0 \to C = 0 \end{cases}$$

Unique solution is:

$$y = c_1 \cos x + c_2 \sin x + x$$

Find $c_{1,2}$ that satisfies the BVs.

$$\begin{aligned} y(0) = 0 &\;\to\; c_1 = 0 \\ y(\pi) = 0 &\;\to\; -c_1 + \pi = 0 \to c_1 = \pi \end{aligned}$$

Contradiction! The boundary value problem has no solution.

Question 9

$$y'' + 4\,y = \cos x, \quad \begin{cases} y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$$

Find solution to associated homogeneous equation. Characterisitc equation is:

$$x^2 + 4 = 0 \to x = \pm 2\,\mathrm{i}$$

$$y_h = c_1 \cos(2\,x) + c_2 \sin(2\,x), \quad c_{1,2} \in \mathbb{R}, \quad \forall x$$

Guess particular solution of the form $y_p = A \cos x + B \sin x$

$$y_p'' = -A \cos x - B \sin x$$

Substitute back and get:

$$-A\cos x - B\sin x + 4\,(A\cos x + B\sin x) = \cos x$$

$$A = \frac{1}{3}, B = 0$$

Unique solution is

$$y = y_h + y_p = c_1\cos(2\,x) + c_2\sin(2\,x) + \frac{1}{3}\cos x, \quad c_{1,2} \in \mathbb{R}$$

Find $c_{1,2}$ that satisfy BVs.

$$y' = -2\,c_1\sin(2\,x) + 2\,c_2\cos(2\,x) - \frac{1}{3}\sin x$$

$$\begin{aligned} y'(0) = 0 &\;\rightarrow\; c_2 = 0 \\ y'(\pi) = 0 &\;\rightarrow\; c_2 = 0 \end{aligned}$$

We've don't have sufficient information to find a specific c_1, meaning that it can be arbitrary. The BVP has infinitely many solutions:

$$y = c_1\cos(2\,x) + \frac{1}{3}\cos x, \quad c_1 \in \mathbb{R}, \quad x \in [0, \pi]$$

<u>Question 11</u>

$$y'' + \lambda\,y = 0, \quad \begin{cases} y(0) = 0 \\ y'(\pi) = 0 \end{cases}$$

Characteristic polynomial is:

$$u^2 + \lambda = 0$$

There are different cases for when $\lambda = 0, \lambda < 0$ or $\lambda > 0$. We shall treat each one separately.

1. $\lambda = 0$. $u = 0$ is a double root and the solution to the ODE is:

   $$y = c_1\,x + c_2, \quad c_{1,2} \in \mathbb{R}$$

   Find $c_{1,2}$ that satisfy the boundary conditions.

   $$\begin{aligned} y(0) = 0 &\;\rightarrow\; c_2 = 0 \\ y'(\pi) = 0 &\;\rightarrow\; c_1 = 0 \end{aligned}$$

Only solution is trivian solution, therefore there are no eigenvalues associated with it.

2. $\lambda < 0$. Denote $\mu = \sqrt{\lambda}$. General solution is

$$y = c_1\, e^{\mu x} + c_2\, e^{-\mu x}, \quad c_{1,2} \in \mathbb{R}$$

Find $c_{1,2}$ that satisfy the BCs.

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 = -c_2 \\ y'(\pi) = 0 &\rightarrow c_1\, \mu\, e^{\mu\pi} + c_1\, \mu\, e^{-\mu\pi} = 0 \end{aligned}$$

Once again $c_1, c_2 = 0$. No non-trivial solution.

3. $\lambda > 0$. Denote $\mu = \sqrt{\lambda}$. General solution is:

$$y = c_1 \cos(\mu\, x) + c_2 \sin(\mu\, x)$$

Find $c_{1,2}$ that satisfy the BCs.

$$\begin{aligned} y(0) = 0 &\rightarrow c_1 = 0 \\ y'(\pi) = 0 &\rightarrow \mu\, c_2 \cos(\mu\, \pi) = 0 \end{aligned}$$

Non-trivial solution is obtained only when $\cos(\mu\, \pi) = 0$

$$\mu\, \pi = \pi\, n - \frac{\pi}{2}, \quad n \in \mathbb{N}$$

$$\sqrt{\lambda} = \mu = \frac{2\, n - 1}{2}, \quad n \in \mathbb{N}$$

$$\lambda_n = \frac{4\, n^2 - 4\, n + 1}{4}, \quad n \in \mathbb{N}$$

$\{\lambda_n\}$ are the eigenvalues of the BVP and the eigenfunctions $\{f_n\}$ are all the scalar multiples of:

$$y_n = \sin\left(\frac{2\, n - 1}{2}\, x\right)$$

Question 13

Same question as before, with different BCs:

$$\begin{cases} y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$$

Re-examine the three cases:

1. $\lambda = 0$. General solution is:

$$y = c_1 x + c_2, \quad c_{1,2} \in \mathbb{R}$$

   Find $c_{1,2}$ that satisfy the boundary conditions.

$$\begin{aligned} y'(0) = 0 &\rightarrow c_1 = 0 \\ y'(\pi) = 0 &\rightarrow c_1 = 0 \end{aligned}$$

   There is not enough information to determine c_2, so it's arbitrary. There are infinitely many solutions for the eigenvalue $\lambda = 0$, which is all $y = c_2 \in \mathbb{R}$. Eigenfunctions for this eigenvalue are all real scalars.

2. $\lambda < 0$. Denote $\mu = \sqrt{\lambda}$. General solution is

$$y = c_1 \mathrm{e}^{\mu x} + c_2 \mathrm{e}^{-\mu x}, \quad c_{1,2} \in \mathbb{R}$$

   Find $c_{1,2}$ that satisfy the BCs.

$$\begin{aligned} y'(0) = 0 &\rightarrow \mu c_1 - \mu c_2 = 0 \rightarrow c_1 = c_2 \\ y'(\pi) = 0 &\rightarrow c_1 \mu \mathrm{e}^{\mu\pi} - c_1 \mu \mathrm{e}^{-\mu\pi} = 0 \end{aligned}$$

   $c_1, c_2 = 0$. No non-trivial solution and no eigenvalues associated with this condition.

3. $\lambda > 0$. Denote $\mu = \sqrt{\lambda}$. General solution is:

$$y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

   Find $c_{1,2}$ that satisfy the BCs.

$$\begin{aligned} y'(0) = 0 &\rightarrow c_2 = 0 \\ y'(\pi) = 0 &\rightarrow -\mu c_1 \sin(\mu \pi) = 0 \end{aligned}$$

   Only non-trivial solution is when $\sin(\mu \pi) = 0$.

$$\mu \pi = \pi n, \quad n \in \mathbb{N}$$

$$\sqrt{\lambda} = \mu = n, \quad n \in \mathbb{N}$$

   Eigenvalues are $\lambda_n = n^2$, and their associated eigenfunctions are all scalar multiples of

$$y_n = \cos(n x), \quad n \in \mathbb{N}$$