# Linear Algebra for Chemists — Assignment 1

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**Question 1.** Assuming the vector space is over $\mathbb{R}$:

a) *The set of real polynomials of degree less or equal to $n$ with positive constant term.*

This is not a vector space, as the set is not closed under multiplication by a negative scalar. There is also no identity element, because the zero constant term is not within the subset. Axioms that do not hold:

- Inverse elements of vector addition.
- Identity element of vector addition.

b) *The set of real polynomials of degree less or equal to $n$.*

This is a vector space. Let $f(x)$, $g(x)$ be both real polynomials of degree $n$,

- $f(x) + g(x)$ is still a real polynomial of degree $n$.
- $\alpha\, f(x)$ where $\alpha$ is a (real) scalar is a real polynomial of degree $n$.

The set is closed under addition and multiplication. The set is a vector space.

c) *The set of points in $\mathbb{R}^3$ lying on a plane passing through the origin.*

This is a vector space. A plane that passes through the origin satisfies $\vec{n} \cdot \vec{P} = 0$, where $\vec{n}$ is the normal to the plane and $\vec{P}$ is any point on the plane. Let $\vec{P}_1, \vec{P}_2$ be vectors that represent points on the plane and let $a$ be a scalar. We have:

$$\vec{n} \cdot (\vec{P}_1 + \vec{P}_2) = \vec{n} \cdot \vec{P}_1 + \vec{n} \cdot \vec{P}_2 = 0 + 0 = 0,$$

$$\vec{n} \cdot (a\, \vec{P}_1) = a\, \vec{n} \cdot \vec{P}_1 = a \cdot 0 = 0.$$

The set is closed under addition and multiplication. The set is a vector space.

d) *The set of positive real numbers under addition $\oplus$ and multiplication by scalar $\odot$ defined:*

$$x \oplus y = x\, y, \qquad \alpha \odot x = x^{\alpha}.$$

The set is a vector space. Let $x, y \in \mathbb{R}^+$ and $a \in \mathbb{R}$.

- The result of vector addition, $x\, y$, is also a positive real number, as it is a product of positive real numbers.
- The result of vector multiplication by scalar, $x^a$, is also a positive real number, as any power of a positive real number is also a positive real number.

The set is closed under addition and multiplication. The set is a vector space.

**Question 2.**

a) *$V = \mathbb{R}^n$, $W$ is the set of vectors in $V$ whose coordinates are nonnegative.*

$W$ is not a subspace. Let $\vec{w}_1, \vec{w}_2 \in W$ s.t. $\vec{w}_1 = 2\,\vec{w}_2$. The linear combination $\vec{w}_1 - 2\,\vec{w}_2 = \vec{0} \notin W$.

b) *$V = \mathbb{R}^n$, $W$ is the set of vectors in $V$ whose coordinates add up to 1.*

$W$ is not a subspace. The definition of each vector in $\vec{w} \in W$ is

$$\sum_i w_i = 1\,.$$

Let $\vec{w}_1, \vec{w}_2 \in W$ s.t. $\vec{w}_1 = 2\,\vec{w}_2$. The linear combination $\vec{w}_1 + \vec{w}_2 = 3\,\vec{w}_2$ is not in $W$, as

$$\sum_i (\vec{w}_1 + \vec{w}_2) = 3 \sum_i \vec{w}_2 = 3 > 1\,.$$

c) *$V$ is the space of square $n \times n$ matrices and $W$ is the set of symmetric matrices in $V$.*

$W$ is a subspace. The matrix $A$ is symmetric if for every $i, j$ we have $a_{ji} = a_{ij}$. Let $A, B$ be symmetric matrices in $W$, i.e. $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$.

The matrix $C = A + B$ satisfies $c_{ji} = a_{ji} + b_{ji} = c_{ij} = a_{ij} + b_{ij}$. Additionally, $C = \alpha \cdot A$ satisfies $c_{ji} = \alpha\, a_{ji} = c_{ij} = \alpha\, a_{ij}$. The subset in closed under addition and multiplication by scalar.

d) *$V$ is the space of square $n \times n$ matrices and $W$ is the set of matrices in $V$ whose rows add up to zero.*

$W$ is a subspace. Each matrix $A^{n \times n} \in W$ satisfies:

$$\sum_{j=1}^{n} a_{ij} = 0, \quad \forall i \in n$$

Let $A, B$ both be matrices in $W$. The sum $A + B$ satisfies:

$$\sum_{j=1}^{n} (a_{ij} + b_{ij}) = \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} b_{ij} = 0 + 0 = 0,$$

and the multiplication of $A$ by a scalar $\alpha$ satisfies:

$$\sum_{j=1}^{n} \alpha\, a_{ij} = \alpha \sum_{j=1}^{n} a_{ij} = \alpha \cdot 0 = 0\,.$$

The subset in closed under addition and multiplication by scalar.

e) *$V$ is the set of one-variabled real functions, and $W$ is the set of polynomial functions of degree 2 in $V$.*

The subset is not closed under addition. Let $f(x) = x^2 + x$ and let $g(x) = -x^2 + x$. $f(x) + g(x) = 2\,x$ is a polynomial of degree 1, and is thus not in $W$.

f) *$V = \mathbb{C}^n$, as a vector space over $\mathbb{C}$, $W = \mathbb{R}^n$.*

$W$ is not a subspace of $V$, as it is not closed under multiplication by a scalar $\alpha \in \mathbb{C}$.

g) *$V = \mathbb{C}^n$, as a vector space over $\mathbb{R}$, $W = \mathbb{R}^n$.*

$W$ is a subspace of $V$. Let $A, B \in \mathbb{R}^n$. These matrices satisfy:

$$A + B \in \mathbb{R}^n, \qquad \alpha A \in \mathbb{R}^n \quad \forall \alpha \in \mathbb{R}.$$

h) *$V$ is the set of one-variabled real continuous functions, and $W$ is the set of real differentiable functions.*

   $W$ is a subspace of $V$. All functions that are differentiable are also continuous (in the domain where they are differentiable). Additionally, the set of all real differentiable functions are closed under linear combinations.

**Question 3.**

The zero element $\mathbf{0}$ in a vector space $V$ is defined such that for all $\boldsymbol{v} \in V$

$$\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}.$$

Assume there are two distinct zero elements of the same vector space, $\mathbf{0}$ and $\mathbf{0}'$. By definition,

$$\boldsymbol{v} + \mathbf{0} = \boldsymbol{v},$$

$$\boldsymbol{v} + \mathbf{0}' = \boldsymbol{v}.$$

By transitivity, we get

$$\boldsymbol{v} + \mathbf{0} = \boldsymbol{v} + \mathbf{0}'.$$

Subtract $\boldsymbol{v}$ from both sides of the equation to get

$$\mathbf{0} = \mathbf{0}'.$$

Contradiction! The zero element in a vector space is unique.

**Question 4.**

The additive inverse of an element $-\boldsymbol{v}$ in a vector space $V$ satisfies, that, for all $\boldsymbol{v} \in V$ there exists $-\boldsymbol{v}$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$.

Assume there are two distinct inverse elements $-\boldsymbol{v}, -\boldsymbol{v}'$ s.t.

$$\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0},$$

$$\boldsymbol{v} + (-\boldsymbol{v}') = \mathbf{0}.$$

By transitivity, we get

$$\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{v} + (-\boldsymbol{v}').$$

Subtract $\boldsymbol{v}$ from both sides of the equation to get

$$(-\boldsymbol{v}) = (-\boldsymbol{v}').$$

Contradition! The inverse additive element of a vector space is unique.