2008 Exam Solution

Question 1 has already been answered in 2006 exam.

Question 2

Find the general solution to the following differential equation

$$2y^{(3)} - 2y'' + 25y' = \sin 2x + x$$

First solve associated homogeneous equation. Characteristic equation:

$$2\lambda^3 - 2y^2 + 25\lambda = 0$$

$$\lambda(2\lambda^2 - 2\lambda + 25) = 0$$

Roots are

$$\lambda_1 = 0, \lambda_{2,3} = \frac{2 \pm 14\mathrm{i}}{4} = \frac{1}{2} \pm \frac{7}{2}\mathrm{i}$$

General solution to homogeneous equation is:

$$c_1 + c_2\,\mathrm{e}^{t/2}\cos\left(\frac{7}{2}t\right) + c_3\,\mathrm{e}^{t/2}\sin\left(\frac{7}{2}t\right)$$

Find particular solution. Split RHS into to elements: (1) $\sin 2x$ and (2) x.

For case (1): Guess solution of form $y_{p1} = A\cos(2x) + B\sin(2x)$

$$\begin{aligned} y'_{p1} &= -2A\sin 2x + 2B\cos 2x \\ y''_{p1} &= -4A\cos 2x - 4B\sin 2x \\ y^{(3)}_{p1} &= 8A\sin 2x - 8B\cos 2x \end{aligned}$$

Substitute in ODE:

$$16A\sin 2x - 16B\cos 2x + 8A\cos 2x + 8B\sin 2x - 50A\sin 2x + 50B\cos 2x = \sin 2x$$

$$\sin 2x\,(16A + 8B - 50A) + \cos 2x\,(-16B + 8A + 50B) = \sin 2x$$

Equate coefficients on both sides:

$$\begin{cases} \cos 2x\text{:} \quad -8A = 34B \rightarrow A = -\frac{17}{4}B \\ \sin 2x\text{:} \quad -34A + 8B = 1 \rightarrow B = \frac{2}{305}, A = -\frac{17}{610} \end{cases}$$

$$y_{p1} = -\frac{17}{610}\cos 2x + \frac{2}{305}\sin 2x$$

For case (2): Guess polynomial of form $y_{p2} = A\,x^3 + B\,x^2 + C\,x$

$$\begin{aligned} y'_{p2} &= 3A\,x^2 + 2B\,x + C \\ y''_{p2} &= 6A\,x + 2B \\ y^{(3)}_{p2} &= 6A \end{aligned}$$

Substitute in ODE:

$$12A - 12A\,x - 4B + 75A\,x^2 + 50B\,x + 25C = x$$

Equate coefficients on both sides to get:

$$\begin{cases} x^2\colon & A = 0 \\ x\colon & B = \frac{1}{50} \\ 1\colon & C = \frac{4}{25}B = \frac{2}{625} \end{cases}$$

$$y_{p2} = \frac{1}{50}x^2 + \frac{2}{625}x$$

In conclusion, general solution to ODE is $y = y_h + y_{p1} + y_{p2}$.

$$y = c_1 + c_2\,\mathrm{e}^{t/2}\cos\left(\frac{7}{2}t\right) + c_3\,\mathrm{e}^{t/2}\sin\left(\frac{7}{2}t\right) + -\frac{17}{610}\cos 2x + \frac{2}{305}\sin 2x + \frac{1}{50}x^2 + \frac{2}{625}x$$

Question 3

Find the solution to the following system of differential equations with ICs $\vec{x}(0) = [1, -3, 5]^T$.

$$\vec{x}' = A\,\vec{x}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

We see that the third row is a scalar multiple of the first row, which means that rank $A^{n\times n} < n$, so $\lambda = 0$ is an eigenvalue. Find all eigenvalues and eigenvectors of A.

$$\det(A - \lambda\,I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = [\text{along third column}] = \cdots =$$

$$= -12 - (\lambda+1) - (\lambda+1)\,[(\lambda+1)\,(\lambda-1) - 12] = \cdots =$$

$$= -\lambda - 13 - (\lambda+1)\,(\lambda^2 - 13) = -[\lambda^3 - 12\lambda + \lambda^2] = -\lambda\,(\lambda^2 + \lambda - 12) = 0$$

$$\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = -4$$

Find associated eigenvectors.

<u>For $\lambda = 0$</u>: Find $\vec{v}_1$ s.t. $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$.

$$(A - \lambda_1 I) = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$$

For $\lambda = 3$:

$$A - \lambda_2 I = \begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

For $\lambda = -4$:

$$A - \lambda_3 I = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

Pick

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

General solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Find unique solution that satisfies ICs.

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} + c_2\, \mathrm{e}^{3t} \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + c_3\, \mathrm{e}^{-4t} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Define

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 1 \\ 6 & 3 & -2 \\ -13 & -2 & -1 \end{bmatrix}$$

s.t.

$$B\,\vec{c} = \vec{b}$$

Therefore,

$$\vec{c} = B^{-1}\,\vec{b}$$

Invert B via Gauss-Seidal algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 6 & 3 & -2 & 0 & 1 & 0 \\ -13 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + 13R_1]{R_2 \to R_2 - 6R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -9 & -8 & -6 & 1 & 0 \\ 0 & 24 & 12 & 13 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + \frac{8}{3}R_2} \cdots$$

$$\cdots \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -9 & -8 & -6 & 1 & 0 \\ 0 & 0 & -\frac{28}{3} & -3 & \frac{8}{3} & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 + \frac{3}{28}R_3]{R_2 \to R_2 - \frac{6}{7}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{19}{28} & \frac{2}{7} & \frac{3}{28} \\ 0 & -9 & 0 & -\frac{24}{7} & -\frac{9}{7} & -\frac{6}{7} \\ 0 & 0 & -\frac{28}{3} & -3 & \frac{8}{3} & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + \frac{2}{9}R_2}$$

$$B^{-1} = \begin{bmatrix} -\frac{1}{12} & 0 & -\frac{1}{12} \\ \frac{8}{21} & \frac{1}{7} & \frac{2}{21} \\ \frac{9}{28} & -\frac{2}{7} & -\frac{3}{28} \end{bmatrix}$$

Coefficients are:

$$\vec{c} = \begin{bmatrix} -\frac{1}{12} & 0 & -\frac{1}{12} \\ \frac{8}{21} & \frac{1}{7} & \frac{2}{21} \\ \frac{9}{28} & -\frac{2}{7} & -\frac{3}{28} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{7} \\ \frac{9}{14} \end{bmatrix}$$

Unique solution to the system of DEs is:

$$\vec{x} = -\frac{1}{2} \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} + \frac{3}{7} \mathrm{e}^{3t} \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + \frac{9}{14} \mathrm{e}^{-4t} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Question 4

(a) Show that the BVP

$$y'' + (\lambda + 1)\, y' + \lambda\, y = 0$$

$y(1) = y'(0) = 0$ has no real eigenvalues.

Characteristic equation:

$$u^2 + (\lambda + 1)u + \lambda = 0$$

$$u_{1,2} = \frac{-(\lambda + 1) \pm \sqrt{\lambda^2 + 2\lambda + 1 - 4\lambda}}{2} = \frac{-(\lambda + 1) \pm (\lambda - 1)}{2} = -1, -\lambda$$

General solution is of the form:

$$y = c_1\, \mathrm{e}^{-t} + c_2\, \mathrm{e}^{-\lambda t}$$

Find $c_{1,}c_2$ that satisfy boundary values.

$$\begin{aligned} y(1) = 0 &= c_1 \cdot \mathrm{e}^{-1} + c_2 \cdot \mathrm{e}^{-\lambda} \rightarrow c_1 = -c_2 \cdot \mathrm{e}^{(1-\lambda)} \\ y'(0) = 0 &= -c_1 - \lambda\, c_2 \rightarrow c_1 = -\lambda\, c_2 \end{aligned}$$

These equations demand that

$$c_2 = 0 \Rightarrow y \equiv 0 \textbf{ or } \lambda = \mathrm{e}^{(1-\lambda)}$$

Only possibility yielding non-trivial solution is $\lambda = 1$. Check if this eigenvalue yields a non-trivial result. Insert $\lambda = 1$:

$$y'' + 2y' + y = 0$$

Characteristic polynomial is

$$\lambda^2 + 2\lambda + 1 = (\lambda^2 + 1)$$

So $\lambda = -1$ is a double root. Solution is then

$$y = c_1\, \mathrm{e}^{-t} + c_2\, t\, \mathrm{e}^{-t}$$

$$\begin{aligned} y(1) = 0 &= c_1 \cdot \mathrm{e}^{-1} + c_2 \cdot \mathrm{e}^{-1} \rightarrow c_1 = -c_2 \\ y'(0) = 0 &= -c_1 + c_2 \rightarrow c_1 = c_2 \end{aligned}$$

So $c_1 = c_2 = 0$, and there is no non-trivial solution for this eigenvalue.

(b) Find the inplicit solution to the equation:

$$\mathrm{e}^x + y'\left(\mathrm{e}^x \frac{\cos y}{\sin y} + \frac{2y}{\sin y}\right) = 0$$

with IC $y(0) = \frac{\pi}{2}$. Open brackets.

$$\mathrm{e}^x + \mathrm{e}^x y' \frac{\cos y}{\sin y} + 2y \frac{y'}{\sin y} = 0$$

Multiply by $\sin y \neq 0$ on some interval: (If $\sin y = 0 \Leftrightarrow y = \pi n, n \in \mathbb{N}$ there is no solution).

$$\mathrm{e}^x \sin y + \mathrm{e}^x y' \cos y + 2y y' = 0$$

Note that

$$(\mathrm{e}^x \sin y + y^2)' = \mathrm{e}^x \sin y + \mathrm{e}^x y' \cos y + 2y y'$$

So, the equation simplifies to

$$(\mathrm{e}^x \sin y + y^2)' = 0$$

$$\mathrm{e}^x \sin y + y^2 = c$$

Find $c \in \mathbb{R}$ s.t IC is satisfied. Input $y(0) = \frac{\pi}{2}$:

$$1 + \frac{\pi^2}{4} = c$$

Question 5

(a) An elastic string of length $L = 30\,\text{cm}$ is held down taut at both ends in a frame, and vibrates according to the wave equation

$$a^2 u_{xx}(x,t) = u_{tt}(x,t)$$

where $a = 5\,\text{cm s}^{-1}$. Assume that the frame is on a truck that crashes into a wall at velocity $u_t = 72\,\text{km hr}^{-1}$ without damaging either the frame or the driver. Calculate the series representation of $u(x,t)$ that describes the vibration of the string (in centimeters).

This is basically the zero position case of the wave equation. Assume homogeneous BCs, i.e. $u(0,t) = u(L,t) = 0$, and ICs: $u(x,0) \equiv 0$, $u_t(x,0) \equiv g(x) = 2000\,\text{cm s}^{-1}$.

Solution is given by:

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}$$

where

$$k_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \,\mathrm{d}x$$

Calculate k_n.

$$k_n = \frac{2}{5\pi n} \int_0^{30} 2000 \cdot \sin \frac{n\pi x}{30} \,\mathrm{d}x = -\frac{800}{\pi n} \cdot \frac{30}{\pi n} \left[\cos \frac{n\pi x}{30}\right]_0^{30} = \frac{24000}{\pi^2 n^2} (1 - (-1)^n)$$

Solution is then:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{24000}{\pi^2 n^2}(1-(-1)^n)\sin\frac{n\,\pi\,x}{30}\cdot\sin\frac{\pi\,n\,t}{6}$$

(b) Solve the DE

$$(1+t^2)\,y''+2t\,y'+\frac{3}{t^2}=0,\quad t>0$$

with ICs $y(1)=2, y'(1)=-1$. Note:

$$\frac{1}{t\,(t^2+1)}=\frac{A}{t}+\frac{B\,t+C}{t^2+1}$$

Set $y'=v$.

$$(1+t^2)\,v'+2t\,v+\frac{3}{t^2}=0$$

Note that:

$$((t^2+1)\,v)'=(t^2+1)\,v'+2t\,v$$

So

$$((t^2+1)\,v)'=-\frac{3}{t^2}$$

Integrate both sides:

$$(t^2+1)\,v=\frac{3}{t}+c_1$$

$$y'=\frac{\frac{3}{t}+c_1}{t^2+1}=\frac{3+c_1\,t}{t\,(t^2+1)}=\frac{A}{t}+\frac{B\,t+C}{t^2+1}$$

Find A,B,C.

$$A\,(t^2+1)+B\,t^2+C\,t=3+c_1\,t$$

Equate coefficients on both sides.

$$\begin{cases} t^2\text{:}\ \ A+B=0 & \\ t\text{:}\ \ \ C=c_1 & \rightarrow B=-3 \\ 1\text{:}\ \ A=3 & \end{cases}$$

Therefore,

$$y'=\frac{3}{t}-\frac{3}{2}\frac{2t}{t^2+1}+\frac{c_1}{t^2+1}$$

Integrate both sides to get:

$$y=3\ln t-\frac{3}{2}\ln\,(t^2+1)+c_1\arctan t+c_2$$

Input ICs.

$$\begin{aligned} y(1)=2 &= -\frac{3}{2}\ln 2+\frac{1}{4}\pi\,c_1+c_2\rightarrow c_2\approx 6.9667 \\ y'(1)=-1 &= 3-\frac{3}{2}+\frac{c_1}{2}\rightarrow c_1=-5 \end{aligned}$$

Solution to DE is

$$y=3\ln t-\frac{3}{2}\ln\,(t^2+1)-5\arctan t+6.9667$$