Differential Equations for Chemists

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Preface: Complex numbers and functions

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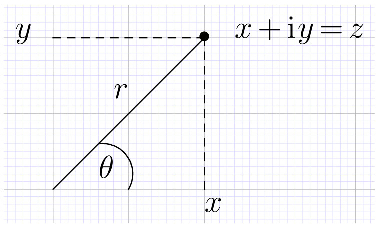
Recollections and definitions

Complex numbers are denoted by $\mathbb{C}=\{x+\mathrm{i}y|x,y\in\mathbb{R}\}$ in Cartesian representation. In Polar representation, we have:



The complex number z can be written using both (x,y) and (r,θ), so that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x}, \qquad \text{if} \quad x>0 \end{aligned}$$

Recall that the *complex conjugate* $\bar{z}$ is defined by: $\bar{z}=x-\mathrm{i}y$ and that the *modulus* squared is

$$z\cdot\bar{z}=\|z\|^2=r^2=x^2+y^2$$

so

$$\|z\|=\sqrt{x^2+y^2} \quad .$$

Another notation:

$$\frac{1}{z}=\frac{\bar{z}}{\|z\|^2}$$

$$\begin{aligned} z^2=(r\cos\theta+\mathrm{i}r\sin\theta)^2 &= r^2(\cos^2\theta-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta) \\ &= r^2(\cos 2\theta+\mathrm{i}\sin 2\theta) \end{aligned}$$

<u>De Moivre's formula:</u>

1. $z^n=r^n(\cos n\theta+\mathrm{i}\sin n\theta), \quad n\in\mathbb{N}$ (integer)

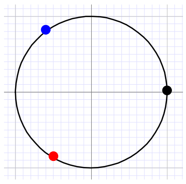
2. if $z^n=r(\cos\theta+\mathrm{i}\sin\theta)$ then:

$$z=r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n}+\mathrm{i}\sin\frac{\theta+2k\pi}{n}\right), \quad 0\le k\le n-1$$

yields n distinct roots.

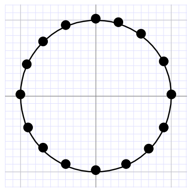
<u>Examples:</u>

1. $z^3 = 1$. $[r = 1 \text{ and } \theta = 0]$



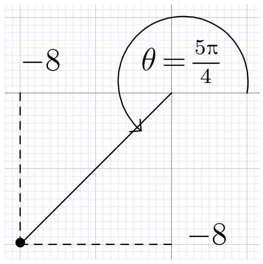
then $z = \cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3}$ where $k = 0, 1, 2$

2. $z^n = 1$. $z = \cos\frac{2\pi k}{n} + \mathrm{i}\sin\frac{2\pi k}{n}$ where $0 \leq k \leq n-1$



n points are distributed equally on the circle.

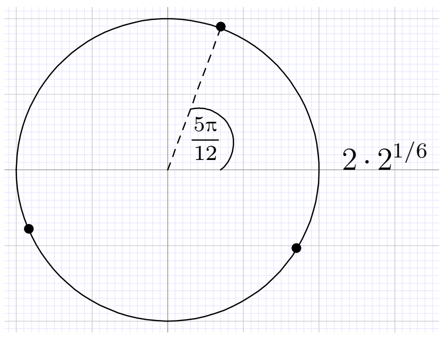
3. $z^3 = -8 - 8\mathrm{i}$. $r = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}$ and $\theta = 5\pi/4$.



so

$$z = (8\sqrt{2})^{1/3} \cdot \left(\cos\frac{\frac{5\pi}{4} + 2k\pi}{3} + \sin\frac{\frac{5\pi}{4} + 2k\pi}{3} \right)$$

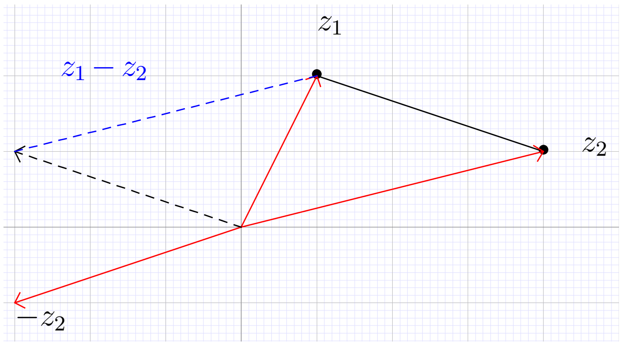
$$z = 2 \cdot 2^{1/6} \left(\cos\frac{5\pi + 8k\pi}{12} + \sin\frac{5\pi + 8k\pi}{12} \right)$$



All points are shifted by $5\pi/12$.

Complex analysis

If we have two points, z_1 and z_2, the distance between them is $z_1 - z_2 = z_1 + (-z_2)$.



In other words, $\|z_1 - z_2\|$ is the distance between z_1 and z_2 in the complex plane, which is a non-negative real number.

Can we use the distance function to define limits and continuity functions etc...?

Limit of a complex sequence

If given the sequence of numbers $\{z_n\}|_{n=1}^{\infty} \subseteq \mathbb{C}$, we say $z_n \rightarrow w$, $w \in \mathbb{C}$, if $\|z_n - w\| \rightarrow 0$ as $n \rightarrow \infty$.

For example, given $z = \frac{1+3\mathrm{i}}{n}$, as $n \rightarrow \infty$ the modulus $\|z_n\| = \frac{\sqrt{10}}{n} \rightarrow 0$ so the sequence goes to zero in the complex plane.

Limit of a series of complex numbers

Given the sequence $\{z_n\}_{n=1}^{\infty}$, the series $\sum_{n=1}^{\infty} z_n$ converges to u if:

$$\left\| \sum_{n=1}^{k} z_n - u \right\| \xrightarrow[k \rightarrow \infty]{} 0$$

For example, given $z_n = z^n$ where $z \in \mathbb{C}$. If $\|z\| < 1$, we have:

$$\sum_{n=0}^{k} z^n = 1 + z + z^2 + \cdots + z^k = \frac{1 - z^{k+1}}{1 - z}$$

$$(1 + z + \cdots + z^k)(1 - z) = 1 - z^{k+1}$$

Note that if $\|z\| < 1$ then $\|z^{k+1}\| \to 0$ as $k \to \infty$. So

$$\left\| \sum_{n=1}^{k} z^n - \frac{1}{1-z} \right\| = \left\| \frac{-z^{k+1}}{1-z} \right\| \xrightarrow[k\to\infty]{} 0$$

Another example: $z = \frac{1}{2} + \frac{1}{2}\mathrm{i}$. The norm is $\|z\| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$. So

$$\sum_{k=1}^{\infty} \left(\frac{1+\mathrm{i}}{2} \right)^k = \frac{1}{1 - \left(\frac{1}{2} + \frac{1}{2}\mathrm{i} \right)} = \frac{1}{\frac{1}{2} - \frac{1}{2}\mathrm{i}} = \frac{\frac{1}{2} + \frac{1}{2}\mathrm{i}}{\frac{1}{2}} = 1 + \mathrm{i}$$

Complex (and analytic) functions

If we have a function f such that $f: \mathbb{C} \to \mathbb{C}$, we say that w is the limit of f at z_0 if $\|f(z) - w\| \to 0$ as $z \to z_0$.

Define *continuity*: f is continuous at the point z_0 if the limit $\lim\limits_{z \to z_0} f(z)$ exists and equals $f(z_0)$.

Definition: $f: \mathbb{C} \to \mathbb{C}$ is *analytic* at z_0 if for the limit:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and $h \in \mathbb{C}$. Then we denote the limit by $f'(z_0) = \frac{\mathrm{d}f}{\mathrm{d}z}|_{z = z_0}$.

If f is analytic at every point we say f is an analytic function.

Turns out that if $f'(z_0)$ exists then so does $f^{(n)}(z)$ $\forall n$ [all higher derivatives of f also exist].

Reminder: Taylor series

A (real) *power series* is a series of the form $\sum_{n=0} a_n x^n$ which converges for some $|x| < r$ ($x \in \mathbb{R}$).

Taylor's theorem:

> If $f: \mathbb{R} \to \mathbb{R}$ is infinitely differentiable in a neighborhood (nbhd) of some point a, then we can represent f *uniquely* as a power series of the form:
>
> $$f(x) = \sum_{n=0}^{\infty} C_n (x - a)^n$$
>
> where
>
> $$C_n = \frac{f^{(n)}(a)}{n!}$$

In particular, if $a=0$ we get the special case

$$f(x)=\frac{\sum f^{(n)}(0)}{n!}x^n$$

(also called the Maclaurin series).

Example:

$f(x)=\sum_{n=0}^{\infty} x^n$ where $x\in\mathbb{R}$ and $|x|<1$. We know that

$$f(x)=\sum_{n=0}^{\infty} x^n=\frac{1}{1-x}$$

So $f(x)=\sum\limits_{n=0}^{\infty} x^n$ must be a Taylor series.

Taylor theorem for analytic functions

If f is analytic (everywhere) then it can be represented as a power series $\sum C_n z^n$ where $C_n=\frac{f^{(n)}(0)}{n!}$.

We can use convergent series to define complex analogues of real functions which have Taylor series.

Examples:

1. $f(x)=\mathrm{e}^x$. Because

   $$\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^x\right)_{x=0}=1 \quad \forall n$$

   we get

   $$C_n=\frac{1}{n!}$$

   Which means its Taylor series is $\sum_{n=0}^{\infty}\frac{x^n}{n!}=\mathrm{e}^x$.

   We can use this to define a complex function:

   $$\mathrm{e}^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

   which in fact does converge for all $z\in\mathbb{C}$.

   In fact e^z "behaves" like an exponential function as we have:

   $$\begin{aligned}\mathrm{e}^{z_1+z_2} &= \mathrm{e}^{z_1}\cdot\mathrm{e}^{z_2} \qquad (1)\\ \mathrm{e}^{z_1\cdot z_2} &= (\mathrm{e}^{z_1})^{z_2}\end{aligned}$$

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2. $f(x) = \sin x$.

$$\sin x = \sum_{n=0}^{\infty} C_n x^n, \qquad C_n = \frac{f^{(n)}(0)}{n!}$$

$$\begin{aligned} \sin' x &= \cos x \\ \cos' x &= -\sin x \\ -\sin' x &= -\cos x \\ -\cos' x &= \sin x \end{aligned}$$

substituting $x = 0$ we get a repeating mini series of $1, 0, -1, 0, \ldots$. If we plug the coefficients we get:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Substituting $z \in \mathbb{C}$ gives a convergent series which we define to be

$$\sin z = \sum_{\text{odd } n} \frac{(-1)^n \cdot x^n}{n!}$$

3. $f(x) = \cos x$.

$$\begin{aligned} \cos' x &= f'(x) = -\sin x \\ f''(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

We get the Taylor series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Substituting $z \in \mathbb{C}$ gives a convergent series which we define to be

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Differentiating our series for $\sin z$ term by term we obtain (just as in the real case) $\sin' z = \cos z$, $\cos' z = -\sin z$. So, if a function is represented by a series that converges *uniformly*, then its derivative can be obtained by differentiating the series term by term.

One of the properties of exponential functions is that using the relations from (1), if we write $z = x + \mathrm{i}y$ $(x, y \in \mathbb{R})$, then $\mathrm{e}^z = \mathrm{e}^x \cdot \mathrm{e}^{\mathrm{i}y}$. And by definition:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}y} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} = 1 + \frac{\mathrm{i}y}{1} - \frac{y^2}{2!} - \frac{\mathrm{i}y^3}{3!} + \frac{y^4}{4!} - \cdots \\
&= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + \mathrm{i}\left(y - \frac{y^3}{3!} + \frac{y^5}{5!}\right) \\
&= \cos y + \mathrm{i}\sin y
\end{aligned}$$

We just got Euler's formula! Additional useful relations:

$$\boxed{\begin{gathered} \mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) \\ z = r(\cos\theta + \mathrm{i}\sin\theta) = r \cdot \mathrm{e}^{\mathrm{i}\theta} \end{gathered}}$$

Notes:

- If f is infinitely differentiable (real or analytic complex) then we have a unique Taylor series at 0, so any power series representation for f will be an alternative form of the Taylor series.

- Taylor series converge uniformly so if f is infinitely differentiable then so is f', therefore it also has a Taylor series. In other words

  $$\text{if } f(x) = \sum C_n x^n \text{ then } f'(x) = \sum_{n=0}^{\infty} n C_n x^{n-1}$$

  is the Taylor series for f'. Similarly, if f is *integrable*, we get

  $$\int f(x)\mathrm{d}x = \sum C_n \frac{x^{n+1}}{n+1}$$

  is the Taylor series for $\int f(x)\mathrm{d}x$.

Example

1. We want to calculate the Taylor series for $\arctan(x)$ at $x = 0$.

$$\arctan x = \sum_{n=0}^{\infty} \frac{\arctan^{(n)}(0)}{n!} x^n$$

Calculating directly we get:

$$\begin{aligned}
\arctan(0) &= 0 \\
\arctan'(x) &= \frac{1}{x^2+1}; \quad \arctan'(0)=1 \\
\arctan''(x) &= \frac{-2x}{(x^2+1)^2}; \quad \arctan''(0)=0
\end{aligned}$$

Further derivatives become more difficult to calculate. Fret naught, there is a shortcut! If we find a power series that fits in some form, we know it is actually *the* Taylor series. If we look at

$$\begin{aligned}
(1-y+y^2-y^3+y^4+\cdots)(1+y) &= \\
=(1-y+y^2-y^3+\cdots+)+(y-y^2+y^3-y^4+\cdots+) &= 1
\end{aligned}$$

So

$$1-y+y^2-y^3+\cdots=\frac{1}{1+y} \quad \text{for} \quad |y|<1$$

So

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8+\cdots \quad \text{for} \quad |x|<1$$

The LHS is the derivative of $\arctan(x)$ so the RHS is its Taylor series at $x=0$.

Integrating the RHS term by term we get:

$$\arctan x=c+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}+\cdots$$

but $c=0$, as can be seen by substituting values.

Note that $\arctan 1=\frac{\pi}{4}$. Plugging in the formula above we get $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots$, which is called Leibniz's series. Actually, we observe convergence at $x\pm 1$. (There is no convergence for $|x|>1$.)

2. Taylor series for $\ln|1+x|$, defined for $x\neq -1$.

$$(\ln|1+x|)'=\frac{1}{1+x}=1-x+x^2-x^3+\cdots+ \quad \text{for} \quad |x|<1$$

Integrating term by term we get:

$$\ln|1+x| \;=\; c+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+$$

Setting $x=0$ we get $c=0$.

$$\ln|1+x|=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+ \quad \text{for} \quad |x|<1$$

Note that the LHS is undefined at $x=-1$ and the RHS equals $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots-$ which diverges! But at $x=1$ it *does* converge, as in fact: $\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+$.

Reminder: Mathematical induction

1. If a claim dependent on a positive integer n is true for $n=1$
2. If it is true for k then it is true for $k+1$

Then the claim is true for all n.

Differential equations

Informally, a differential equation (DE) is a functional equation which involves functions, variables and derivatives of the functions. Note that the solutions to differential equations will always be functions. An example:

$$2y-y'=0, \quad y=y(x)$$

$$y'=2y$$

$$y=\mathrm{e}^{2x} \text{ is a solution.}$$

In fact, $y=c\,e^{2x}$ is a solution for any $c\in\mathbb{R}$.

Example: Radioactive decay

If a function $Q(t)$ is the amount of radioactive material at time t, then the rate of decay is proportional to the amount present (at time t). By these means, let there be $k\in\mathbb{R}$ such that

$$Q'(t)=-k\,Q(t), \quad k>0$$

Note that $Q(t)=c\,\mathrm{e}^{-kt}$ solves the equation for any c. We show these functions are the *only* solutions. Assuming $Q(t)>0$ we can write:

$$\frac{Q'(t)}{Q(t)}=-k$$

The LHS is the logarithmic derivative of $\ln Q(t)$. Therefore, by integration,

$$\ln Q(t)=-k\,t+c$$

and after exponentiation,

$$Q(t) = \mathrm{e}^{-kt+c} = \overbrace{\mathrm{e}^{c}}^{c'} \cdot \mathrm{e}^{-kt}.$$

Removing the assumption that $Q(t) > 0$ we see this solves out equation for any $c \in \mathbb{R}$.

To solve the equation for a *specific* material, *a unique solution*, we need extra information such as the amount of material at $t = 0$, the *initial condition*.

A numerical example

Thorium 234 decays at a rate such that 100 mg decays in a week to 82.04 mg. What is the amount at time t?

$$f'(t) = -k\, f(t)\,.$$

We know $f(t) = c\,\mathrm{e}^{-kt}$. Given $f(0) = 100\,\mathrm{mg}$ and $f(7) = 82.04\,\mathrm{mg}$, we can calculate both c and k.

$$\begin{aligned} 100 &= f(0) = c \\ 82.04 = f(7) &= 100\,\mathrm{e}^{-7k} \\ \ln 0.8204 &= -7k \\ k &\approx 0.02828\,\mathrm{days}. \end{aligned}$$

$$\Rightarrow f(t) = 100\,\mathrm{e}^{-0.2828t}$$

Differential equations can be complicated ... like $2y - y'' \cdot x + y' \cdot x^2 + 7\sin y = 0$.

The *order* of a differential equation is the highest order of the derivative that appears.

There are also *partial differential equations*, like Laplace's equation:

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

Definitions

Ordinary differential equations

Definition: An ordinary differential equation (ODE) is an equation of the form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

where F is some function of $n + 2$ variables.

Partial differential equations

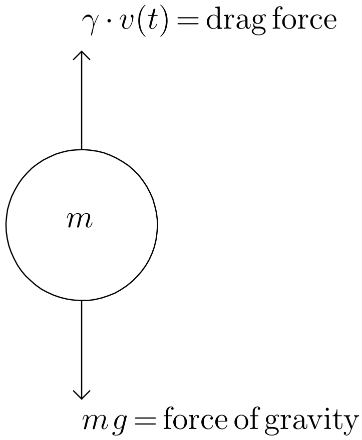
Definition: A partial differential equation (PDE) is one of the form:

$$F\left(x, \ldots, x_k, y, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x_i\, \partial x_j}, \ldots\right) = 0$$

where a solution would be $y = y(x, \ldots, x_k)$, which when substituted with its partial derivatives, solves the equation.

More examples of modeling processes with DEs

1. A falling object:



Define $v(t)$ the velocity of mass m, the drag force is proportionate to $v(t)$.

By Newton's 2nd law, $F = m\,a$.

$$a(t) = v'(t)$$

$$\boxed{m\frac{\mathrm{d}v}{\mathrm{d}t} = m\,g - \gamma\,v}$$

Numerical example: $m = 1\,\mathrm{kg}$, $\gamma = 0.2\,\mathrm{kg\,s^{-1}}$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - \frac{v}{5}$$

Note that if you have some values of the function, you can plot them and get an approximation of the graph of the function and guess what the function (solution to DE) could be.

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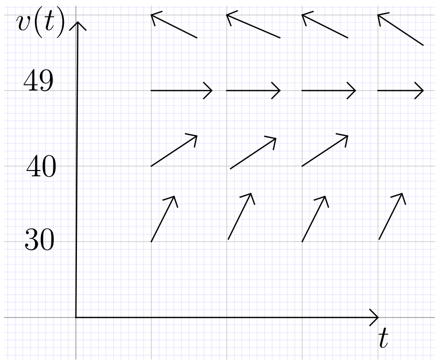
t	$v(t)$	$v'(t)$
0	40	1.8
arb	40	1.8
	50	-0.2
	30	3.8
	49	0



The solutions or *curves* that pass through a given point in the plane are called *integral curves*.

Note that if $v'(t) = 0$ then the function doesn't change, which is a state of equilibrium. Here, the equilibrium is $v(t) \equiv 49 \quad \forall t$. In this case, the drag exactly matches the force of gravity. The velocity that satisfies this condition is called the *limit velocity*.

2. Predator/prey = owls + field mice.

   if no predators assume rate of growth of mouse population, r, $p(t)$ is proportionate to the population level at that time.

   $$\frac{\mathrm{d}p}{\mathrm{d}t} = r \cdot p(t)$$

   Now assume that we have owls who kill a fixed number of mice per month (for example 450) and that $r = 0.5/\text{month}$, then

   $$\frac{\mathrm{d}p}{\mathrm{d}t} = 0.5p - 450$$

   What we get is an *unstable equilibrium*. for $p(0) = 901$, i.e., one mouse survives per month, the population will increase exponentially. for $p(0) = 899$ the mice population goes extinct.

   Equilibrium is unstable if $y(t) \to 0$ for $t \to \infty$ if $y(0) < y_{\text{eq}}$.

Let's solve the ODEs from examples 1 and 2 algebraically. Both are special cases of

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = a\,y + b$$

Assume that $a \neq 0$ and that $y \neq -\frac{b}{a}$ (native equilibrium). Divide by a:

$$\frac{y'}{a} = y + \frac{b}{a}$$

$$\frac{y'}{y + \frac{b}{a}} = a$$

This is the derivative of $\ln\left|y+\frac{b}{a}\right|$. Integrate both sides to get

$$\ln\left|y+\frac{b}{a}\right| = \mathrm{at} + c$$

exponentiate both sides to get

$$\left|y+\frac{b}{a}\right| = \mathrm{e}^{at+c} = \overbrace{\mathrm{e}^{c}}^{C} \cdot \mathrm{e}^{at}$$

an exponent is always positive so we can remove the absolute value by adding a $\pm$ sign.

$$y+\frac{b}{a} = \pm C \cdot \mathrm{e}^{at}, \quad C \in \mathbb{R}$$

our general solution is

$$\boxed{y = k\,\mathrm{e}^{\mathrm{at}} - \frac{b}{a}, \quad k \neq 0, k \in \mathbb{R}} \tag{2}$$

Note that $y \equiv -b/a$ is also a solution — a specific solution.

Note that:

- if $a<0$ then as $t \to \infty$ $y \to \frac{b}{a}$ and we get a stable equilibrium.
- if $a>0$ then:
  - if $k>0$ then $y \to \infty$
  - if $k<0$ then $y \to -\infty$

  and the equilibrium is unstable.

What is the meaning of k? Given initial condition $y(0) = y_0$, then setting $t=0$ in eq. (2) gives: $y_0 = y(0) = k + \frac{b}{a}$. So $k = y_0 - \frac{b}{a}$ is the unique solution satisfying the initial condition (IC): $y = \left(y_0 - \frac{b}{a}\right)\mathrm{e}^{at} + \frac{b}{a}$.

I First order ODEs

First order ODEs are of the form

$$y' = F(t, y)$$

The relationship between t, y can be very complex! Let's review some kinds:

1 Linear ODEs

Linear in y, that is.

$$y' = a(t)\, y + b(t)$$

In the previous examples we solved cases where $a(t)$ and $b(t)$ are constants.

A method for solving linear ODEs:

1.1 Integrating factors method: (due to Leibniz)

Take for example the ODE $y' = -2y + 3$ ($a(t) = -2, b(t) = 3$).

Isolate $f(y)$. rewrite as

$$y' + 2y = 3.$$

Now multiply by a function $\mu(t)$ so that LHS is recognizable as the derivative of something, which is a *product*. Then we can integrate both sides to get the solution. We have:

$$(y \cdot \mu(t))' = \underline{y' \cdot \mu + y \cdot \mu'}$$

If we multiply out equation by $\mu(t)$,

$$\underline{y' \cdot \mu(t) + 2\mu(t) \cdot y} = 3 \cdot \mu(t)$$

All we need is that $\mu'(t) = 2\mu(t)$. We may choose a solution

$$\mu(t) = \mathrm{e}^{2t}$$

and get

$$y' \cdot e^{2t} = 2\mathrm{e}^{2t} \cdot y = 3\mathrm{e}^{2t}$$

Integrate to get

$$y\,\mathrm{e}^{2t} = \int 3\mathrm{e}^{2t}\,\mathrm{d}t = \frac{3}{2}\mathrm{e}^{2t} + c$$

so

$$y = \frac{3}{2} + c \cdot e^{-2t}$$

Note that this method also works for non-linear ODEs.

<u>Another example</u>

$$y' + a\, y = b(t)$$

where a is a constant and $b(t)$ isn't necessarily a constant. Multiply by $\mu(t)$.

$$y'\mu(t) + a\,y\,\mu(t) = b(t)\;\mu(t)$$

we need $\mu(t)$ such that $a\,\mu(t) = \mu'(t)$, so choose $\mu = \mathrm{e}^{at}$ and get

$$(y\cdot\mathrm{e}^{at})' = y'\cdot\mathrm{e}^{at} + a\,y\,\mathrm{e}^{at} = b(t)\,\mathrm{e}^{at}$$

Integrate both sides:

$$y\cdot\mathrm{e}^{at} = \int b(t)\cdot\mathrm{e}^{at}\,\mathrm{d}t + c$$

$$\boxed{y = \mathrm{e}^{-at}\left[\int b(t)\cdot\mathrm{e}^{at}\,\mathrm{d}t + c\right]}$$

Consider the equation

$$y' + 0.5y = 2 + t; \quad y(0) = 2$$

Choose $\mu(t) = \mathrm{e}^{\frac{1}{2}t}$.

$$\left(y\cdot\mathrm{e}^{\frac{1}{2}t}\right) = y'\cdot\mathrm{e}^{\frac{1}{2}t} + \frac{1}{2}\mathrm{e}^{\frac{1}{2}t}\,y = 2\mathrm{e}^{\frac{1}{2}t} + t\,\mathrm{e}^{\frac{1}{2}t}$$

$$y\,\mathrm{e}^{\frac{1}{2}t} = \int 2\mathrm{e}^{\frac{1}{2}t}\,\mathrm{d}t + \int t\,\mathrm{e}^{\frac{1}{2}t}\,\mathrm{d}t$$

The second integral must be calculated via *integration by parts*.

$$\boxed{\begin{aligned}(u\,v)' &= u'\,v + u\,v' \\ u\,v &= \int u'\,v\,\mathrm{d}t + \int u\,v'\,\mathrm{d}t \\ \int u'\,v\,\mathrm{d}t &= u\,v - \int u\,v'\,\mathrm{d}t \\ \int v\,\mathrm{d}u &= u\,v - \int u\,\mathrm{d}v\end{aligned}}$$

take $t = v$, $u' = \mathrm{e}^{\frac{1}{2}t}$, then $v' = 1$, $u = 2\mathrm{e}^{\frac{1}{2}t}$.

$$\int t\,\mathrm{e}^{\frac{1}{2}t}\,\mathrm{d}t = 2t\,\mathrm{e}^{\frac{1}{2}t} - \int 2\mathrm{e}^{\frac{1}{2}t}\,\mathrm{d}t = 2t\mathrm{e}^{\frac{1}{2}t} - 4\mathrm{e}^{t/2} + c$$

So, in total,

$$y\,\mathrm{e}^{\frac{1}{2}t} = 2t\,\mathrm{e}^{t/2} + c$$

$$y = 2t + c\cdot\mathrm{e}^{-t/2}$$

Plug in $y(0)=2$, which gives

$$2=y(0)=2\cdot 0+c\,\mathrm{e}^0=c$$

We get the unique solution $y=2t+2\mathrm{e}^{-t/2}$.

Final example: the general case

For $y'=a(t)\,y+b(t)$, multiply through $\mu(t)$:

$$y'\,\mu+a(t)\,y\,\mu=b(t)\,\mu$$

We want $y'\,\mu+a(t)\,y\,\mu=y\,\mu'$, or equivalently $a(t)\cdot\mu=\mu'$. If $\mu\neq 0$:

$$a(t)=\frac{\mu'}{\mu}=(\ln|\mu(t)|)'\rightarrow\int a(t)\,\mathrm{d}t=\ln|\mu(t)|$$

choose $\mu(t)=\mathrm{e}^{\int a(t)\,\mathrm{d}t}$ as int. factor.

$$\begin{aligned} y'\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}+y\,a(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} &= b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t} \\ &= (y\cdot\mathrm{e}^{\int a(t)\,\mathrm{d}t})' \end{aligned}$$

so that

$$y\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}=\int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}$$

$$\boxed{y=\mathrm{e}^{-\int a(t)\,\mathrm{d}t}\cdot\left[\int b(t)\,\mathrm{e}^{\int a(t)\,\mathrm{d}t}\right]}$$

Example

$$t\,y'+2y=4t^2,\quad t\neq 0,\quad y(1)=2$$

Rewrite as $y'+\frac{2}{t}y=4t$, such that $a(t)=\frac{2}{t}$. Choose $\mu(t)=\mathrm{e}^{\int a(t)\,\mathrm{d}t}=\mathrm{e}^{2\ln|t|}=t^2$.

$$(t^2\,y)'=t^2\,y'+2t\,y=4t^3$$

Integrate to obtain

$$t^2\,y=\int 4t^3\,\mathrm{d}t=t^4+c$$

$$y=t^2+\frac{c}{t^2}$$

Set $y(1)=2$.

$$2=1+\frac{c}{1}\rightarrow c=1$$

The unique solution is $y = t^2 + t^{-2}$.

Question: Is there a solution for $t = 0$? Input $t = 0$ in the original DE:

$$0 \cdot y' + 2y = 0 \rightarrow y(0) = 0$$

First we see that $y(0)$ is *defined*. Then, we can see that if we choose $c = 0$ we get a specific solution $y = t^2$.

2 Separable ODEs

Equations of the form (in Leibniz notation):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x)}{N(y)}$$

for $M(x), N(y)$ functions only of x, y respectively. Sometimes written as:

$$M(x)\,\mathrm{d}x - N(y)\,\mathrm{d}y = 0$$

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1 - y^2}, \quad y \neq \pm 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} \cdot (1 - y^2) = x^2$$

LHS is actually the derivative of $y - \frac{y^3}{3}$ with respect to x! Integrating both sides gives

$$y - \frac{y^3}{3} = \int x^2\,\mathrm{d}x = \frac{x^3}{3} + c$$

So that y is given implicitly.

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Note that if N is a function of y and y is a function of x. Suppose $\int N(y)\,\mathrm{d}y = Q(y)$ so $\frac{\mathrm{d}Q}{\mathrm{d}y} = N(y)$, then, according to the chain rule,

$$\frac{\mathrm{d}Q(y)}{\mathrm{d}x} = \frac{\mathrm{d}Q(y)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = N(y) \cdot y'$$

By taking inverse operation (integrating with respect to x) we find that

$$\int N(y) \cdot y'\,\mathrm{d}x = Q(y) = \int N(y)\,\mathrm{d}y$$

This allows us to write $y'\,\mathrm{d}x = \mathrm{d}y$.

Let's solve the previous example using this notation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1-y^2}$$

$$\int (1-y^2)\,\mathrm{d}y = \int x^2\,\mathrm{d}x$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

When solved before, we had: $y'(1-y^2) = x^2$ and integrated both sides with respect to (wrt) x. What we did here is integrating wrt y on the LHS and wrt x on the RHS. For separable ODEs, non-uniform integration is justified.

Example 2

$$y' = \frac{3x^2+4x+2}{2(y-1)}; \quad y(0) = -1, \quad y \neq 1$$

$$\int 2(y-1)\,\mathrm{d}y = \int (3x^2+4x+2)\,\mathrm{d}x$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Last time we stopped at an implicit expression for y, but here we can go further by applying the initial condition. Set $x=0$: $1+2=c$, so $c=3$. We can write a quadratic expression in y to get a more explicit expression:

$$y^2 - 2y - (x^3 + 2x^2 + 2x + 3) = 0$$

Solve using the quadratic formula:

$$y = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + 3)}}{2}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

On paper, it seems now that applying the initial condition doesn't give us a unique solution, because y has two possible values! However, we shall see that only one quadratic root satisfies the initial condition $y(0) = -1$.

Substituting $x=0$: $y(0)=1\pm\sqrt{4}=\begin{cases}3\\-1\end{cases}$. The unique solution is $y=-1-\sqrt{x^3+2x^2+2x+4}$.

What is its domain of definition? The solution must (a) solve the ODE, (2) satisfy the IC, and (3) be defined in some domain. We need that $\sqrt{x^3+2x^2+2x+4}\geq 0$.

$$x^3+2x^2+2x+4=(x+2)(x^2+2)$$

So $\sqrt{x^3+2x^2+2x+4}\geq 0$ for $x\geq -2$.

But, we defined the DE for $y\neq 1$, so we need $x^3+2x^2+2x+4>0$, so $x>-2$ is the true domain.

Example 3

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2}{\underbrace{y(1+x^3)}_{\neq 0}}$$

$$\frac{y^2}{2}=\int y\,\mathrm{d}y=\int\frac{x^2}{1+x^3}\,\mathrm{d}x\overbrace{=}^{\substack{u=1+x^3\\ \mathrm{d}u=3x^2\mathrm{d}x}}\int\frac{1}{3u}\,\mathrm{d}u=\frac{1}{3}\ln|1+x^3|+c$$

$$y^2=\frac{2}{3}\ln|1+x^3|+c$$

Use the IC: $y(0)=3$.

$$9=\frac{2}{3}\ln 1+c\rightarrow c=9$$

we get a unique solution (positive square root) $y=\sqrt{\frac{2}{3}\ln|1+x^3|+9}$. The domain of the function is which satisfies $y>0$. The domain contains $x=0$, so $x>-1$.

Example 4

Sometimes an equation can be *reduced* to a separable differential equation by a change of variables. For example,

$$y'=x^2+2x\,y+y^2=(x+y)^2$$

We cannot separate the variables here, but we can define $z=x+y$, substitute and get

$$y'=z^2$$

$$\frac{\mathrm{d}z}{\mathrm{d}x}=1+y'=1+z^2$$

the DE for z **is** separable. Get:

$$\int \frac{\mathrm{d}z}{1+z^2} = \int \mathrm{d}x$$

$$\arctan z = x + c$$

Take tangent of both sides:

$$z = x + y = \tan(x+c)$$

$$y = \tan(x+c) - x$$

Cases we need to check to make sure we find **all** possible solutions:

Example:

$$y' + y^2 \sin x = 0$$

We want to divide by y^2 in order to solve, under the assumption for $y \neq 0$. Check some cases first:

1. $y \equiv 0$. Is this a solution? — Yes!

   Out solutions will be differentiable functions and so continuous. So that if $y \not\equiv 0$ there is a point where $y \neq 0$ and around it there's an interval where $y(x) \neq 0$ for all x in that interval. (This is because y is continuous.)

2. $y(x) \neq 0$ for all x in an interval. We can divide by y^2 to get:

   $$\frac{\mathrm{d}y}{y^2} + \sin x \, \mathrm{d}x = 0$$

   $$-\frac{\mathrm{d}y}{y^2} = \sin x \, \mathrm{d}x$$

   Integrate both sides:

   $$-\frac{1}{y} = -\cos x + c$$

   $$y = \frac{1}{c - \cos x}$$

   Note that this solution **never** has the value 0 for **any** x where it is defined.

   If $|c| > 1$ then this holds for all x. Otherwise we need to avoid values where $\cos x = c$.

   What if $y(0) = -\frac{2}{3}$? we get $y(0) = \frac{1}{c-1} \rightarrow c = -\frac{1}{2}$. That's problematic because $\cos x$ can get the value $-\frac{1}{2}$. So our solution holds in the interval $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, which contains $x = 0$.

3. There exists a point x_0 where $y(x_0)=0$ and $y \not\equiv 0$. In that case, we solve as in case 2 for an interval where $y \neq 0$ for all x, as we got in case 2 solution.

   If we had a solution at x_0 where $y(x_0)=0$, and in the interval y is the function in case 2, we cannot get a continuous extension of y to x_0. In other words, this cannot happen!

   We've examined all possible cases, and thus found all the solutions.

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Another example of modeling with first order DEs: Interest compounded continuously

$S(t)=$ amount of money deposited and interest evaluated continuously, then we get that if the rate of change is proportionate to the amount of money.

$$S'(t)=r\cdot S(t)$$

In fact r will be the annual rate of interest so that $S(t)=S(0)\,\mathrm{e}^{rt}$. Why?

If we compute once a year then $S(1)=S(0)+r\,S(0)=S(0)\,(1+r)$.

If you do it twice a year: after 6 months we get $(0.5)=S(0)\left(1+\frac{r}{2}\right)$ and $S(1)=S(0)\left(1+\frac{r}{2}\right)^2$, and in general: $S(t)=S(0)\left(1+\frac{r}{2}\right)^{2t}$.

If we compute n times a year: $S(t)=S(0)\left(1+\frac{r}{n}\right)^{nt}$.

Recall that $\lim_{n\to\infty}\left(1+\dfrac{r}{n}\right)^n=\mathrm{e}^r$. So when $n\to\infty$ and interest is compounded continuously we get $S(t)=S(0)\,\mathrm{e}^{rt}$.

Let's improve our model of money management. In addition to annual rate of interest (r), we also deposit or withdraw k amount of money every year. Thus, the differential equation becomes

$$S'(t)=r\,S(t)+k$$

As before, we get:

$$\begin{aligned} S'(t) &= \left(S_0+\frac{k}{r}\right)\mathrm{e}^{rt}-\frac{k}{r} \\ &= \underbrace{S_0\,\mathrm{e}^{rt}}_{\substack{\text{effect of the}\\ \text{initial investment}\\ \text{at interest rate r}}} + \underbrace{\frac{k}{r}\,(\mathrm{e}^{rt}-1)}_{\substack{\text{result of the}\\ \text{withdraws/deposits}}} \end{aligned}$$

Example

Open a savings plan at age 25 with regular deposits of 2000\$/year with 8% annual rate of interest. What will be the amount saved by age 65?

$$S(40)=S_0\,\mathrm{e}^{rt}+\frac{k}{r}(\mathrm{e}^{rt}-1),\quad S_0=0$$

$$S(40)=\frac{2000}{0.08}\,(\mathrm{e}^{0.08*40}-1)\approx 588{,}313\,\$$$

We invested 80,000$ and made a 508,000$ profit! Note this is a special case called an *autonomous equation*.

2.1 Autonomous ODEs

Definition: An ODE is *autonomous* if the independent variable does not appear explicitly.

For example, $y' = a\,y + b$, In general, this means $y' = F(y)$ so it's separable.

When you plot a direction field for these ODEs you get replicas of vectors along the horizontal axis.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\,y + b}{c\,y + e}, \qquad y \neq -\frac{e}{c}$$

There are 2 cases:

1. $a\,y + b \equiv 0$, so $y \equiv -\frac{b}{a}$. A constant solution.

   (if $a = 0$ then $b = 0$ as well [because we can't divide by zero], and $y = k\ \forall k \in \mathbb{R}$)

2. Assume $a\,y + b \not\equiv 0$, so take $a\,y + b \neq 0$ on some interval. We can now rewrite the equation to get the form: $(a \neq 0)$

   $$\int \frac{c\,y + e}{a\,y + b}\,\mathrm{d}y = \int \mathrm{d}x = x + k$$

   Note that

   $$\begin{aligned} \frac{c\,y + e}{a\,y + b} &= \frac{\frac{c}{a}(a\,y + b) - \frac{c}{a}\,b + e}{a\,y + b} \\ &= \frac{c}{a} + \frac{-\frac{c}{a}\,b + e}{a\,y + b} \\ &= \frac{c}{a} + \frac{a\,e - b\,c}{a\,(a\,y + b)} \end{aligned}$$

   Therefore,

   $$\int \frac{c\,y + e}{a\,y + b}\,\mathrm{d}y = \int \frac{c}{a}\,\mathrm{d}y + \left(\frac{a\,e - b\,c}{a}\right) \int \frac{\mathrm{d}y}{a\,y + b} = x + k$$

   We are left with an implicit expression for $y(x)$:

   $$\frac{c}{a}\,y + \left(\frac{a\,e - b\,c}{a}\right) \ln|a\,y + b| = x + k$$

Note that if $a=0$ we have to solve again from the start. Actually, in this case the equation is easier because we don't have y in the denominator.

2.2 Verhulst's Model (1845)

Similar to the field mice model, we had $y' = r\,y$ where r was a constant rate of population growth and $y(t)$ was the population at time t. We can make the model more realistic by replacing r by a function which depends on y.

$$y' = h(y) \cdot y$$

where $h(y)$ is the rate of population growth at population level y.

Typically, $h(y)$ *decreases* as y *increases*. We approximate this by choosing $h(y)$ to be linearly decreasing in y:

$$h(y) = r - s \cdot y, \quad r, s > 0$$

We get $y' = (r - s\,y)y$, a quadratic autonomous equation! Let's look at the equilibria.

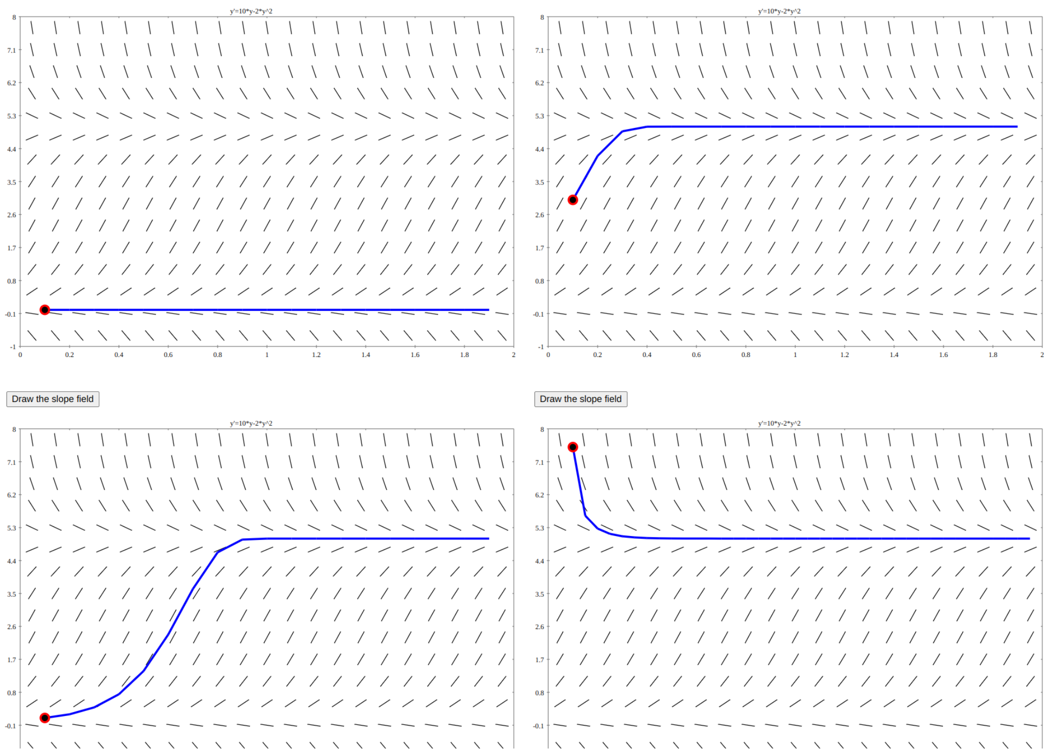
Equilibrium points

$y=0$ or $y = \frac{r}{s} = k$. Rewrite our ODE:

$$y' = r\left(1 - \frac{s}{r}\right) y = r\left(1 - \frac{1}{k}\,y\right) y$$

Before we solve, let's look at the direction field. Take $r=10, k=5$:

$y' = 10(1-5y) = 10y - 2y^2$.



It is clear that $y\equiv 0$ is an unstable equilibrium and that $y\equiv 5$ is a stable equilibrium.

Now let's solve algebraically.

$$y' = r\left(1 - \frac{1}{k}\,y\right)y = r\left(\frac{k-y}{k}\right)y$$

If $y = k$ we get $y\equiv k$ (because $y'(k) = 0$. So now assume $y \neq k$ and separate the variables:

$$\int \frac{k\,\mathrm{d}y}{(k-y)y} = \int r\,\mathrm{d}t = r\,t + c$$

As we have a polynomial at the denominator, we rewrite LHS using partial fractions:

$$\frac{k}{(k-y)y} = \frac{A}{k-y} + \frac{B}{y}$$

where A, B are some constants. Get

$$k = A\,y + B(k-y) = (A-B)y + B\,k$$

Equate coefficients on both sides:

$$\begin{aligned} 0 &= A - B \\ k &= B\,k \end{aligned}$$

We can see that $A = B = 1$.

$$\int \frac{\mathrm{d}y}{k-y} + \int \frac{\mathrm{d}y}{y} = r\,t + c$$

$$\ln\left(\left|\frac{|y|}{|k-y|}\right|\right) = -\ln|k-y| + \ln|y| = r\,t + c$$

Exponentiate both sides:

$$\frac{|y|}{|k-y|} = C\,\mathrm{e}^{rt}$$

Again, there are some cases:

1. $0 < y < k$. Set $y(0) = y_0$.

   $$\frac{y}{k-y} = C\,\mathrm{e}^{rt}$$

   Setting $t = 0$ to get

   $$\frac{y_0}{k-y_0} = C$$

Now we solve for y:

$$\frac{y}{k-y}=\frac{y_0}{k-y}\,\mathrm{e}^{rt}$$

$$y=(k-y)\,\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}$$

$$y\left(1+\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}\right)=\frac{k\,y_0}{k-y_0}\,\mathrm{e}^{rt}$$

$$y=\frac{k\,y_0\,\mathrm{e}^{rt}}{(k-y_0)\left[1+\frac{y_0}{k-y_0}\,\mathrm{e}^{rt}\right]}=\frac{k\,y_0\,\mathrm{e}^{rt}}{k-y_0+y_0\,\mathrm{e}^{rt}}$$

Divide numerator and denominator by e^{rt}.

$$y=\frac{k\,y_0}{(k-y_0)\,\mathrm{e}^{-rt}+y_0}$$

As $t\rightarrow\infty$ we see that $y\rightarrow k$. This means that $y\equiv k$ is a stable equilibrium and that $y\equiv 0$ is an unstable equilibrium.

2. if $y>k$ we get that case (1) solution is still valid ($y=k$ is a stable equilibrium).

The constant $k=\frac{r}{s}$ is called the *saturation level* or *environmental carrying capacity*.

II Second Order ODEs

3 Special 2nd order ODEs

The solution be obtained by *reducing* to a **first** order equation. Two kinds of equations where you can do that:

3.1 2nd order ODE where the dependent variable doesn't appear

An equation of the form:

$$y''=F(x,y')$$

Substitute $v = y'$ and get the first order ODE

$$v' = F(x, v)$$

and then solve

$$y = \int v \, \mathrm{d}x$$

Examples

1.

$$t^2 y'' + 2t y' - 1 = 0, \quad t > 0$$

Set $y' = v$:

$$t^2 v' + 2t v = 1, \quad t > 0$$

Integrate both sides:

$$t^2 v = t + c_1$$

$$y' = v = \frac{1}{t} + \frac{c_1}{t^2}$$

$$y = \ln t - \frac{c_1}{t} + c_2$$

A second order ODE needs 2 initial conditions to get a unique solution. Given $y(0) = b_1$ and $y'(0) = b_2$ we can determine c_1 and c_2.

2.

$$2t^2 y'' + (y')^3 = 2y' t, \quad t > 0$$

Set $y' = v$:

$$2t^2 v' + v^3 = 2v t$$

Examine some cases:

I. $v \equiv 0$. In that case $y \equiv C$. This solves the equation for any $C \in \mathbb{R}$.

II. $v \neq 0$ on some interval. Divide by v^3 and get

$$\frac{2t^2 v'}{v^3} + 1 = \frac{2t}{v^2}$$

$$-\left(\frac{t^2}{v^2}\right)' = \frac{2t^2 v'}{v^3} - \frac{2t}{v^2} = -1$$

Integrate both sides to get

$$\frac{t^2}{v^2} = t + c_1, \quad t + c_1 \neq 0$$

$$(y')^2 = v^2 = \frac{t^2}{t + c_1}$$

$$y' = v = \frac{\pm t}{\sqrt{t + c_1}}, \quad t + c_1 > 0$$

$$y = \pm \int \frac{t\,\mathrm{d}t}{\sqrt{t + c_1}}$$

Integrate by parts. $\int u\,v' = u\,v - \int v\,u'$. Choose $u = t, u' = 1$. $v' = \frac{1}{t + c_1}$ so $v = 2\sqrt{t + c_1}$.

$$\Rightarrow y = \pm \left[2t\,\sqrt{t + c_1} - 2\int \sqrt{t + c_1}\,\mathrm{d}t\right]$$

$$\begin{aligned} y &= \pm\left[2t\,\sqrt{t + c_1} - 2 \cdot \frac{2}{3}(t + c_1)^{3/2} + c_2\right] \\ &= \pm\left[\frac{2}{3}\sqrt{t + c_1}\,(t - 2c_1) + c_2\right] \end{aligned}$$

Take initial conditions: $y(1) = 0, y'(1) = -1$.

$$\begin{aligned} y(1) = 0 &= \frac{2}{3}\sqrt{1 + c_1}\,(1 - 2c_1) + c_2 \\ y'(1) = -1 &= \frac{\pm 1}{\sqrt{1 + c_1}} \rightarrow c_1 = 0 \\ &\Rightarrow\ c_2 = -\frac{2}{3} \end{aligned}$$

$$y = -\frac{2}{3}t^{3/2} + \frac{2}{3}, \quad t > 0$$

3.2 2nd order autonomous ODE

The independent variable doesn't appear.

$$y'' = F(y, y')$$

Set $y' = v$ and get

$$v' = F(y, v)$$

But remember that $v' = \frac{\mathrm{d}v}{\mathrm{d}x}$. We want to express v as a function of y. Using the chain rule,

$$y'' = v' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

Then we get an equation of the form:

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}y} \cdot v = F(y, v)}$$

We got a first order ODE in v as a function of y.

Example

$$y \cdot y'' + (y')^2 = 0$$

Set $v = y'$:

$$y\, v' + v^2 = 0$$

$$v' = \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v$$

Rewrite as an ODE in v as a function of y.

$$y \cdot \frac{\mathrm{d}v}{\mathrm{d}y} \cdot v + v^2 = 0$$

1. $v \equiv 0$ gives $y' \equiv 0$ or $y \equiv C$ is a solution for all C.

2. $v \neq 0$ on some interval. If there is a 2nd derivative it means the 1st derivative is continuous, so if the 2nd derivative is non-zero at a point it is non-zero on the interval.

   $$y \frac{\mathrm{d}v}{\mathrm{d}y} + v = 0$$

   This equation is separable!

   $$y\, \mathrm{d}v + v\, \mathrm{d}y = 0$$

   $$\int \frac{\mathrm{d}v}{v} + \int \frac{\mathrm{d}y}{y} = \int 0\, \mathrm{d}t = C$$

   $$\ln |v\, y| = \ln |v| + \ln |y| = C$$

   $$\pm \mathrm{e}^C = k = v\, y$$

Remember $v = \mathrm{d}y / \mathrm{d}x$ so we get

$$k = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot y$$

$$k \cdot x + c = \int k \, \mathrm{d}x = \int y \, \mathrm{d}y = \frac{y^2}{2}$$

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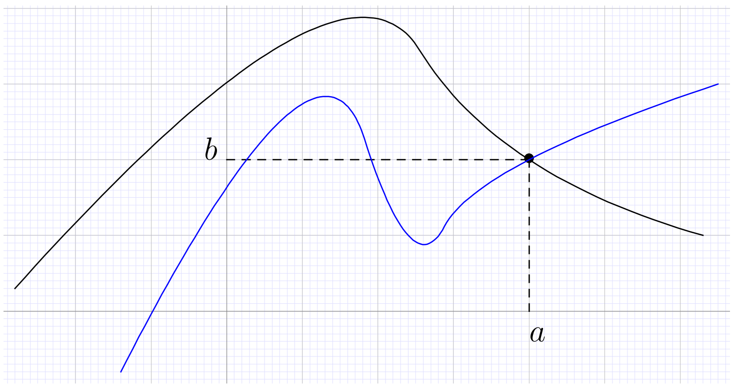
4 Existence and uniqueness theorem for first order ODEs

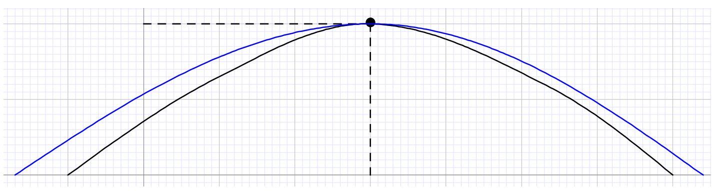
Sometimes knowing that there exists a unique solution helps finding the solution.

Suppose we have the first order ODE $y' = F(x, y)$, $y(a) = b$. If F and $\frac{\partial F}{\partial y}$ are continuous in some open rectangle around (a, b) in the x-y plane, then there exists a unique solution to the ODE satisfying $y(a) = b$.

Note that there is only one curve that passes through this curve and it's defined in some open rectangle. The definition of an open rectangle is a set of points (x, y) such that $\left\{ (x, y) \middle| \begin{array}{l} a_1 < x < a_2 \\ b_1 < y, b_2 \end{array} \right\}$.

Conditions guarantee that we cannot have a solution:





Examples

1. $y' + 2x\, y = x^3\, y^2$

$$y' = x^3\, y^2 - 2x\, y = F(x, y) \quad \text{continuous (polynomial)}$$

$$\frac{\partial F}{\partial y} = 2y\, x^3 - 2x \quad \text{continuous (polynomial)}$$

By the theorem, given any IC there exists a solution.

2. First order linear: $y' = p(x)\, y + q(x) = F(x, y)$.

   F is continuous $\Longleftrightarrow$ $p(x), q(x)$ are continuous.

   $\frac{\partial F}{\partial y} = p(x)$ is continuous $\Longleftrightarrow$ $p(x)$ is continuous.

   Conclusion — we have a unique solution for any IC if p, q are both continuous.

3. $y' = 2\sqrt{y}$. Here $F(x, y)$ is cont. for all x and for all $y > 0$ (and cont. on the right at $y = 0$).

   $\partial F / \partial y = \frac{1}{2\sqrt{y}}$ is **not** cont. at $y = 0$.

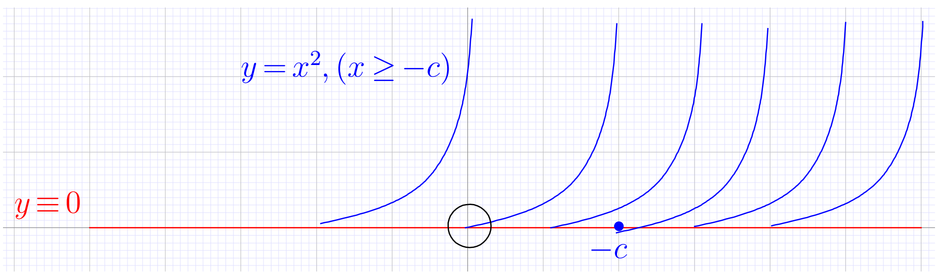
   What happens here when we have the IC: $y(0) = 0$? We solve assuming first that $y \neq 0$ (in some interval) in order to find a non-trivial solution that can be extended continuously to $y(0) = 0$.

$$\sqrt{y} = \int \frac{\mathrm{d}y}{2\sqrt{y}} = \int 1\, \mathrm{d}x = x + c \quad (x + c \geq 0)$$

$$y = (x + c)^2, \quad x \geq -c$$

   Notice that $(x + c) = 0$ is also a solution, therefore we can say that the interval is $x \geq c$ [The function is differentiable from the right].

   Let's draw the solutions. First notice that clearly $y \equiv 0$ is a solution satisfying IC.



   $y = x^2, (x \geq 0)$ is also a solution satisfying IC.

   Actually, we have 2 solutions for every IC such that $y(a) = 0$.

   Why could we extend the interval range to $x = -c$? Have a look at the following example:

   $y' \cdot y = \cos x \cdot y$. Notice that $y \equiv 0$ is a solution. Now assume that $y \not\equiv 0$ and divide by an interval where $y(x) \neq 0$.

$$\begin{aligned} y' &= \cos x \\ y &= \sin x + c \end{aligned}$$

Notice that $y = \sin x$ is a solution for $\sin x \neq 0, x \neq \pi k$.

We then check and see that $y = \sin x$ is a solution *even* for $x = \pi k, k \in \mathbb{N}$.

4. $y^2 + x^2 y' = 0$. Assuming $x, y \neq 0$ we can separate variables.

$$y^2 = -x^2 y'$$

$$-\int \frac{\mathrm{d}x}{x^2} = \int \frac{\mathrm{d}y}{y^2}$$

$$\frac{c\,x+1}{x} = c + \frac{1}{x} = -\frac{1}{y}$$

$$y = -\frac{x}{c\,x+1}, \quad \text{defined for all } x \neq -\frac{1}{c}$$

Note that if $x = 0$ it is defined and then $y = 0$, so all solutions pass through $(0, 0)$.

Given an IC $y(0) = 1$ — there's no solution!

But we do have a unique solution (by the E&U theorem) for every IC $y(a) = b$ where $a \neq 0$, as $y' = F(x, y) = \frac{y^2}{x^2}$ is cont. for $x \neq 0$ and $\frac{\partial F}{\partial y} = \frac{2y}{x}$ also cont. for $x \neq 0$.

Also notice that there are inf. many solutions for $(0, 0)$ according to the solution $y = -\frac{x}{c\,x+1}$.

The E&U theorem fails because $\partial F / \partial y$ is not continuous within the interval.

5. $x\,y' = 2y$. $\Rightarrow$ $y' = \frac{2y}{x}$ is linear. Set $p(x) = \frac{2}{x}$ cont. for $x \neq 0$.

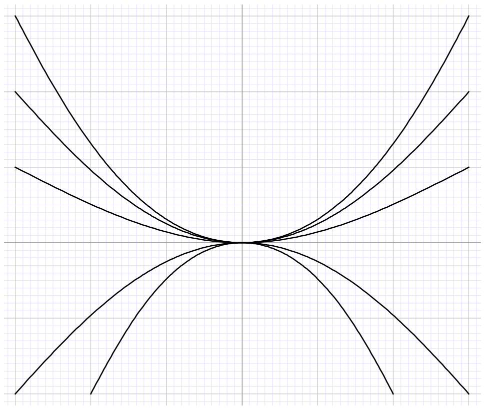
We solve assuming $x, y \neq 0$ and get

$$\frac{1}{2}\ln|y| = \int \frac{\mathrm{d}y}{2y} = \int \frac{\mathrm{d}x}{x} = \ln|x| + c$$

$$\sqrt{y} = k\,|x|$$

$$y = K\,x^2$$

This is a solution for all K, under the restriction $x \neq 0$.



- inf. many solutions satisfy $y(0) = 0$.
- No solution satisfies $y(0) = b$ where $b \neq 0$.
- There exists ($\exists$) a unique (!) solution for $y(a) = b$ for $a \neq 0$, any b.
- Notice that the combination of $y = -K_1 x^2, x < 0$ and $y = K_1 x^2, x \geq 0$ **is** differentiable at point $x = 0$ **and** solves the differential equation!

4.1 Existence and uniqueness theorem for higher order ODEs

Given a linear ODE of order n,

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \cdots + p_0(x)\, y = q(x)$$

and the ICs:

$$\begin{cases} y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$$

If $p_0, p_1, \ldots, p_{n-1}, q$ are continuous for all x in some interval containing a, then the ODE has a unique solution satisfying the ICs for any choice of values $b_0, \ldots, b_{n-1}$.

5 Second order linear ODEs

5.1 Homogeneous ODEs

Homogeneous equations take the general form:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

By E&U thm., for any IC $y(a) = b,\ y'(a) = c$ we have a unique sol. in some neighborhood of a, provided p, q are continuous in this interval.

Reminder:

- The set $V = \{f: \mathbb{R} \to \mathbb{R} | f \text{ is twice differentiable}\}$ is a *vector space* over $\mathbb{R}$ with respect to operations: addition of functions, multiplication by scalars from $\mathbb{R}$.

  Note: V is *closed* under these operations. i.e., if $f, g \in V$ so are $f + g, c \cdot f$.

  Also note that we can define a vector space only in a confined interval $[x_1, x_2]$.

  Note about dimension of V

  $\sin x, \cos x \in V$. V is infinite dimensional over $\mathbb{R}$ as the monomial functions: $1, x, x^2, \ldots,$ are linearly independent over $\mathbb{R}$. (V can contain infinitely many linearly independent functions)

- in any vector space V: the elements $v_1, \ldots, v_k \in V$ are *linearly independent* if $\sum_{i=1}^{k} a_i v_i = 0$ for a_i scalars only if $a_i = 0$ for all i.

  e.g. $\sin x, \cos x$ are linearly independent in our vector space of functions as

  $$a \sin x + b \cos x \equiv 0$$

  i.e.

  $$a \sin x + b \cos x = 0 \quad \forall x$$

  Set $x = 0$ and get $b = 0$, and set $x = \pi/2$ and get $a = 0$.

- A *basis* for V is a set of linearly independent vectors such that every vector in V is a linear combination of these vectors.

Turns out that:

- Every vector space has a basis (assuming axiom of choice).

- Every basis has the same cardinality (number of elements in the set if the set is finite).

- If V has dimension n then any linearly independent set of n elements will be a basis.

Claim: The set of solutions to our ODE

$$y'' + p(x)\, y' + q(x)\, y = 0, \quad p, q \text{ continuous} \tag{3}$$

is a vector space over $\mathbb{R}$ (a subspace of V). In fact, it is a vector space of dimension 2.

Proof: Set of solutions is non-empty as $y \equiv 0$ is a solution.

We need to show that if y_1, y_2 are solutions, then so is $y_1 + y_2$ and also $c \cdot y_1$ for $c \in \mathbb{R}$.

y_1 is a solution so:

$$y_1'' + p(x)\, y_1' + q(x) y_1 = 0, \quad \forall x$$

$$y_2'' + p(x)\, y_2' + q(x) y_2 = 0, \quad \forall x$$

So:

$$\underbrace{(y_1'' + y_2'')}_{(y_1+y_2)''} + p(x)\, \underbrace{(y_1' + y_2')}_{(y_1+y_2)'} + q(x)\, (y_1 + y_2) = 0$$

$$\underbrace{(y_1'' + p(x) y_1' + q(x) y_1)}_{=0} + \underbrace{(y_2'' + p(x) y_2' + q(x) y_2)}_{=0} = 0$$

So $y_1 + y_2$ is a solution and

$$c\, y_1'' + c\, p(x)\, y_1' + c\, q(x)\, y_1 = c\underbrace{(y_1'' + p(x)\, y_1' + q(x) y_1)}_{=0} = 0$$

and so $c\, y_1$ is a solution.

Conclude: Set of all solutions is a subspace of V.

It remains to show dimension of this subspace is 2. Here we use E&U thm. Given a point a we have a solution such that $y(a) = 1$. There is at least one solution, so the dimension must be ≥ 1. [Dimension is zero if only $y \equiv 0$ is a solution]

On the other hand, we have a solution y_1 such that $y_1(a) = 1$ and $y_1'(a) = 0$ and another solution y_2 such that $y_2(a) = 0$ and $y_2'(a) = 1$.

if y_1 and y_2 were linearly dependent, we would have $\alpha \in \mathbb{R}$ such that $y_2 = \alpha \cdot y$. But if we substitute $y_2(a) = \alpha\, y_1(a) = \alpha$, we get $\alpha = 0$. But then $y_2 \equiv 0$, and we know that $y'(a) = 1$ — contradiction!

That means y_1, y_2 are linearly independent and the dimension of the subspace is **at least** 2. Now we only need to show that the dimension is **exactly** 2.

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Let $f(x)$ be an arbitrary solution to (3), such that $f(a) = b$ and $f'(a) = c$. Now look at the following function:

$$g(x) = b\, y_1(x) + c\, y_2(x)$$

$y_1(x), y_2(x)$ are solutions and therefore $g(x)$ also solves (3).

Set $x = a$:

$$g(a) = b\, y_1(a) + c\, y_2(a) = b$$

$$g'(a) = b\, y_1'(a) + c\, y_2'(a) = c$$

So $g(x)$ solves the ODE and satisfies the same ICs as the arbitrary function $f(x)$. But by the E&U thm. there's only one solution satisfying a given set of ICs, so $f(x) = g(x)$ for all x and $f(x)$ is a linear combination of y_1 and y_2.

To conclude, all solutions can be written as linear combinations of y_1, y_2, meaning that the subspace is *spanned* by y_1, y_2, and its dimension is exactly 2.

Example Usage of E&U thm.

Given the ODE

$$y'' + y = 0$$

Note that $\sin x$ and $\cos x$ are solutions. We want the unique sol. such that

$$\begin{cases} y(0) = 3 \\ y'(0) = -2 \end{cases}$$

$\sin x, \cos x$ are linearly independent (as if $\alpha \sin x + \beta \cos x = 0$ for all x, setting $x = 0$ gives: $\alpha \sin 0 + \beta \cos 0 = 0 \rightarrow \beta = 0$, so $\alpha \sin x = 0$ for all x, so $\alpha = 0$.)

$\sin x, \cos x$ are therefore a basis for set of solutions. We want a function $y(x) = a \sin x + b \cos x$ such that the ICs hold. We get

$$3 = y(0) = a \sin 0 + b \cos 0 = b$$

$$-2 = y'(0) = a \cos(0) = a$$

So $y(x) = -2 \sin x + 3 \cos x$ is the unique solution. Knowing that a solution exists and the E&U thm. holds means that $y(x)$ is the **only** solution that satisfies the ICs.

Note

1st order homogeneous linear ODEs are of form: $y' + p(x)\, y = 0$. We solved and found that also solutions were multiples of $\mathrm{e}^{\int p(x)\,\mathrm{d}x}$, i.e. a 1-dimensional space of functions solves the ODE. We expect (and shall see later) that for an ODE of order n, the space of solutions would be n-dimensional.

5.2 Finding a basis for the set of solutions

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If $p(x), q(x)$ are **not** constant functions, this can be difficult.

There are some special situations for which we can use certain tricks to solve.

5.2.1 When one solution is known

Imagine we somehow know one non-zero solution and want to find a second solution which is linearly independent.

Suppose y_1 solves our ODE. We want y_2 which is **not** a multiple of y_1. It means that $\frac{y_2}{y_1}$ is not a constant, but a function, $v(x)$.

In other words, $v(x)$ is non-constant and $y_2(x) = y_1(x) \cdot v(x)$ is a solution. We substitute y_2 in the ODE:

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0$$

$$\begin{aligned} y_2' &= y_1'\, v + y_1\, v' \\ y_2'' &= y_1''\, v + 2 y_1'\, v' + y_1\, v'' \end{aligned}$$

Get

$$(y_1'' v + 2y_1' v' + y_1 v'') + p(x)\,(y_1' v + y_1 v') + q(x)\,(y_1 v) = 0$$

Rewrite as

$$(y_1'' + p(x)\, y_1' + q(x)\, y_1)\, v + (2y_1' + p(x)\, y_1)\, v' + y_1 v'' = 0$$

The red term is equal to zero because y_1 is a solution. Let's look at what's left.

$$(2y_1' + p\, y_1)v' + y_1 v'' = 0$$

This is a 2nd order ODE in v, or a 1st order linear ODE in v'. Therefore we can find y_2 by "reduction of order".

Examples

1. $y + y'' = 0$. Suppose given $y_1 = \sin x$. Set $y_2 = y_1 v = \sin x \cdot v$.

$$\begin{aligned} y_2' &= \cos x \cdot v + \sin x \cdot v' \\ y_2'' &= -\sin x \cdot v + 2\cos x \cdot v' + \sin x \cdot v'' \end{aligned}$$

Substitute and get

$$(-\sin x \cdot v + 2\cos x \cdot v' + \sin x \cdot v'') + \sin x \cdot v = 0$$

$$v'' \cdot \sin x + 2\cos x \cdot v' = 0$$

Set $v' = z$.

$$z' \cdot \sin x + 2\cos x \cdot z = 0$$

This is a separable ODE.

$$-2\cos x \cdot z = \frac{\mathrm{d}z}{\mathrm{d}x} \cdot \sin x$$

$$-2\int \frac{\cos x}{\sin x}\,\mathrm{d}x = \int \frac{\mathrm{d}z}{z}$$

$$-2\ln|\sin x| = \ln|z| + c$$

Take $c = 0$

$$\ln(\sin^{-2} x) = \ln|z|$$

$$v' = z = \frac{1}{\sin^2 x} \Rightarrow v = \cot x \text{ is a solution for } \sin x \neq 0.$$

So out second solution is

$$y_2 = \sin x \cdot v(x) = \sin x \cdot \cot x = \cos x$$

But we can verify $\cos x$ is a solution for all x, not only when $\sin x \neq 0$.

2. $2x^2 y'' + 3xy' - y = 0, \quad x > 0$. By "inspection" we see that $y = \frac{1}{x}$ is a solution.

$$\begin{aligned}\left(\frac{1}{x}\right)' &= -\frac{1}{x^2}\\ \left(\frac{1}{x}\right)'' &= \frac{2}{x^3}\end{aligned}$$

$$\Rightarrow 2x^2 \cdot \frac{2}{x^3} + 3x \cdot \left(-\frac{1}{x^2}\right) - \frac{1}{x} \overset{\checkmark}{=} 0$$

We want $v(x) \equiv C$ such that $y = \frac{v}{x}$ is a solution.

$$\begin{aligned}y' &= \frac{v'}{x} - \frac{v}{x^2}\\ y'' &= \frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\end{aligned}$$

Substitute in the ODE:

$$2x^2\left[\frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\right] + 3x\left[\frac{v'}{x} - \frac{v}{x^2}\right] - \frac{v}{x} = 0$$

$$2x^2 \cdot \frac{v''}{x} - v' = 0$$

Set $v' = z \not\equiv 0$.

$$2x \cdot \frac{\mathrm{d}z}{\mathrm{d}x} = z \longrightarrow \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}x}{2x}$$

$$\ln|z| = \ln\sqrt{x}, \quad x > 0$$

$$v' = z = c \cdot x^{1/2}$$

$$v' = \frac{2}{3} c \cdot x^{3/2}$$

Take $c = \frac{3}{2}$.

$$y = \frac{v}{x} = \frac{x^{3/2}}{x} = \sqrt{x}$$

Out 2nd linearly independent solution will be $\sqrt{x}$ and the general solution to our ODE is $y = a\sqrt{x} + b\frac{1}{x}$.

3. $y'' - 3y' + 2y = 0$. Note that $y = \mathrm{e}^x$ is a solution. We use reduction of order to find a second solution: $y = \mathrm{e}^x \cdot v(x)$.

$$\begin{aligned} y' &= \mathrm{e}^x \cdot v + \mathrm{e}^x \cdot v' = \mathrm{e}^x (v + v') \\ y'' &= \mathrm{e}^x v + 2\mathrm{e}^x v' + \mathrm{e}^x v'' = \mathrm{e}^x (v + 2v' + v'') \end{aligned}$$

Substitute:

$$\mathrm{e}^x (v + 2v' + v'') - 3\mathrm{e}^x (v + v') + 2\mathrm{e}^x v = 0$$

Divide by $\mathrm{e}^x \neq 0 \forall x$, rearrange and get:

$$v'' = v'$$

$v = \mathrm{e}^x$ solves the problem, and $y = \mathrm{e}^{2x}$ is a 2nd solution.

Alternative approach: Suppose we start with e^{2x} and want a 2nd solution of the form $y = v\,\mathrm{e}^{2x}$.

$$\begin{aligned} y' &= 2\mathrm{e}^{2x} v + \mathrm{e}^{2x} v' \\ y'' &= 4\mathrm{e}^{2x} v + 4\mathrm{e}^{2x} v' + \mathrm{e}^{2x} v'' \end{aligned}$$

$$(4\mathrm{e}^{2x} v + 4\mathrm{e}^{2x} v' + \mathrm{e}^{2x} v'') - 3\,(2\mathrm{e}^{2x} v + \mathrm{e}^{2x} v') + 2\mathrm{e}^{2x} v = 0$$

$$\mathrm{e}^{2x} v' + \mathrm{e}^{2x} v'' = 0$$

$$v'' = -v' \Rightarrow v(x) = -c\,\mathrm{e}^{-x}, \quad \text{take } c = -1.$$

$$y = \mathrm{e}^{2x} \cdot \mathrm{e}^{-x} = \mathrm{e}^x$$

We've got the solution from the first approach.

4. $x^2 y'' - 5x\,y + 9y = 0$. Notice $y = x^3$ is a solution. Want a sol. of form $y = x^3 v$.

$$\begin{aligned} y' &= 3x^2 v + x^3 v' \\ y'' &= 6x\,v + 6x^2 v' + x^3 v'' \end{aligned}$$

Get

$$x^2\,(6x\,v + 6x^2 v' + x^3 v'') - 5x\,(3x^2 v + x^3 v') + 9x^3 v = 0$$

$$x^4 v' + x^5 v'' = 0$$

Assuming $x \neq 0$, get

$$v' + x\,v'' = 0$$

Set $z = v'$, solve for z and finally get $v(x) = \ln|x|,\ y = x^3 \cdot \ln|x|$.

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Claim

Given a linear homogeneous ODE of order n,

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = 0$$

the set of solutions is a vector space of functions of dimension n.

Proof

For $n=2$ we've already proved. It is easy to see it is a subspace — showing dim. is n is similar to case we did for $n=2$.

5.2.2 All coefficients are constants

All coeffs are constants: $\forall k$: $p_k(x) \equiv a_k \in \mathbb{R}$.

Rewrite ODE as

$$a_n\,y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_0\,y = 0, \quad a_n \neq 0$$

Example: $y'' - y' = 0$.

Clearly the $y \equiv 1$ and $y = \mathrm{e}^x$ are both solutions and are linearly independent. Therefore, they form a basis for the space of solutions. Given ICs $y(0) = 1, y'(0) = 0$, we can construct the general solution as a linear combination of our basis:

$$\text{General sol.} \quad y = a + b\,\mathrm{e}^x$$

$$\begin{aligned} 1 &= y(0) = a + b \\ 0 &= y'(0) = b \end{aligned}$$

So we get $a = 1, b = 0$ and the unique solution is $y \equiv 1$.

Another example: $y'' - 2y' - 15y = 0$. It makes sense to guess a solution of the form $y = \mathrm{e}^{\lambda x}$. Then:

$$\begin{aligned} y' &= \lambda \mathrm{e}^{\lambda x} \\ y'' &= \lambda^2 \mathrm{e}^{\lambda x} \end{aligned}$$

Substitute and get

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \mathrm{e}^{\lambda x} - 15\mathrm{e}^{\lambda x} = 0$$

$$\mathrm{e}^{\lambda x}\,(\lambda^2 - 2\lambda - 15) = 0$$

$e^{\lambda x}$ is a solution $\iff \lambda^2 - 2\lambda - 15 = 0$. Solve the quadratic equation:

$$\frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2} = \begin{cases} 5 \\ -3 \end{cases}$$

The solutions e^{-3x} and e^{5x} are both linearly independent solutions! So the general solution will be $y = a\, e^{-3x} + b\, e^{5x}$.

In the previous example, we would have gotten $y \equiv 1$ using the method above by finding that $\lambda = 0$.

<u>General case</u>

Given: $a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_0\, y = 0, \quad a_n \neq 0$ we look for solutions of the form $e^{\lambda x}$ and get:

$$y^{(k)}(x) = \lambda^k\, e^{\lambda x} \quad \forall x$$

And when we substitute in the ODE:

$$e^{\lambda x}\, (a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0) = 0$$

Then $e^{\lambda x}$ is a solution if and only if $a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0 = 0$. (This polynomial is also called the *characteristic polynomial of the ODE*.)

If the polynomial has n distinct (real of complex) solutions $\lambda_1, \ldots, \lambda_n$ then $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are n linearly independent solutions and form a basis for space of solutions.

<u>Fundamental thm. of Algebra (Gauss)</u>

Every polynomial equation over $\mathbb{C}$ has n solutions including multiplicities.

<u>Solution of polynomial equations</u>

$$a_n\, \lambda^n + a_{n-1}\, \lambda^{n-1} + \cdots + a_1\, \lambda + a_0 = 0$$

There are **no** formulas that give the roots of polynomials (in general) for $n \geq 5$, in terms of the coefficients.

<u>Example: 3rd order ODE</u>

$$y^{(3)} - 6y'' + 11y' - 6 = 0$$

A trick to solve cubic equations: if you can find one solution λ_0 then you get an equation of the form $(\lambda - \lambda_0)(\text{quadratic equation}) = 0$. So check if $\lambda = 0, \pm 1, \pm 2, \ldots$ are solutions and the problem might be simplified.

In this case we need to solve

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Setting $\lambda = 1$: $1 - 6 + 11 - 6 \overset{\checkmark}{=} 0$.

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)\,(\lambda^2 - 5\lambda + 6) = (\lambda - 1)\,(\lambda - 2)\,(\lambda - 3)$$

The general solution is $y = a\,\mathrm{e}^x + b\,\mathrm{e}^{2x} + c\,\mathrm{e}^{3x}$.

Example: The characteristic polynomial does not have n distinct roots

We still need n linearly independent solutions. What do we do?

$$y'' - 4y' + 4y = 0$$

The characteristic polynomial is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Our method yields only one exponential function: e^{2x}.

We look for a second linearly independent solution. We want a non-constant function $v(x)$ such that $y = \mathrm{e}^{2x}\,v(x)$ solves the ODE.

$$\begin{array}{rcl} y' & = & 2\mathrm{e}^{2x}\,v(x) + \mathrm{e}^{2x}\,v'(x) = \mathrm{e}^{2x}\,(2v + v') \\ y'' & & 2\mathrm{e}^{2x}\,(2v + v') + \mathrm{e}^{2x}\,(2v' + v'') = \mathrm{e}^{2x}\,(4v + 4v' + v'') \end{array}$$

$$\cancel{\mathrm{e}^{2x}}\,(4v + 4v' + v'') - 4\cancel{\mathrm{e}^{2x}}\,(2v + v') + 4\cancel{\mathrm{e}^{2x}}\,v = 0$$

$$v'' = 0 \Rightarrow v \text{ is linear in } x$$

So take $v = x$ and get $\{\mathrm{e}^{2x}, x\,\mathrm{e}^{2x}\}$ is the basis for set of solutions.

> In general
>
> If λ_0 is a root of the characteristic polynomial of multiplicity r, then $\mathrm{e}^{\lambda_0 x}, x\,\mathrm{e}^{\lambda_0 x}, x^2\,\mathrm{e}^{\lambda_0 x}, \ldots, x^{r-1}\,\mathrm{e}^{\lambda_0 x}$ will all solve the ODE and are linearly independent, and will be linearly independent of solutions we obtain from other roots of the characteristic polynomial.

Example: Suppose we know that the char. poly. factors as $\lambda^3\,(\lambda + 3)^2\,(\lambda - 1)^2$: A 7th order linear ODE.

- From $\lambda = 0$ we get the solutions: $y = 1, x, x^2$.
- From $\lambda = 3$ we get the solutions: $y = \mathrm{e}^{-3x}, x\,\mathrm{e}^{-3x}$.

- From $\lambda = 1$ we get the solutions $y = \mathrm{e}^x, x\,\mathrm{e}^x$.

<u>Example: Roots of the characteristic polynomial are not real</u>

$$y'' + y = 0$$

Our method yields the roots of char. poly. $x^2 + 1$: $\pm\mathrm{i}$. Get 2 non-real solutions: $\mathrm{e}^{\mathrm{i}x}$, $\mathrm{e}^{-\mathrm{i}x}$. These span the vector space of all complex solutions.

We note that the span over $\mathbb{C}$ $\{\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}\}$ is a space of dimension 2 of complex functions. We also have 2 real solutions: $\sin x$ and $\cos x$, that are linearly independent. This means that the span can be written as $\{\sin x, \cos x\}$. This meakes sense, as $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, $\mathrm{e}^{-\mathrm{i}x} = \cos x - \mathrm{i}\sin x$, and $\frac{1}{2}(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}) = \cos x, \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}) = \sin x$. $\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}$ are in the span of $\{\sin x, \cos x\}$, and $\cos x, \sin x$ are in the span of $\{\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}\}$. However, $\mathrm{span}_{\mathbb{C}}\{\sin x, \cos x\} \not\supset \mathrm{span}_{\mathbb{R}}\{\sin x, \cos x\}$.

<u>Note</u>: if $z(x)$ is a complex-valued function solving our ODE, then it can be written as:

$$z(x) = u(x) + \mathrm{i}\,v(x), \quad u, v \in \mathbb{R}$$

Then we have

$$(u + \mathrm{i}\,v)^{(n)} + a_{n-1}(u + \mathrm{i}\,v)^{(n-1)} + \cdots + a_1(u + \mathrm{i}\,v)' + a_0(u + \mathrm{i}\,v) = 0$$

$$u^{(n)} + \mathrm{i}\,v^{(n)} + a_{n-1}(u^{(n-1)} + \mathrm{i}\,v^{(n-1)}) + \cdots + a_1(u' + \mathrm{i}\,v') + a_0(u + \mathrm{i}\,v) = 0$$

$$(u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1u' + a_0u) + \mathrm{i}\,(v^{(n)} + a_{n-1}v^{(n-1)} + \cdots + a_1v' + a_0v) = 0$$

This holds for all x if and only if:

$$\begin{aligned} u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1u' + a_0u &= 0 \\ v^{(n)} + a_{n-1}v^{(n-1)} + \cdots + a_1v' + a_0v &= 0 \end{aligned}$$

Therefore, $u(x), v(x)$ are **both** real solutions to the ODE.

Note that if $z(x)$ is a complex solution to our ODE, then so is $\bar{z}(x) = u(x) - \mathrm{i}\,v(x)$, as since we have:

$$z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_1z' + a_0z = 0$$

Since $a_i \in \mathbb{R}$, $\bar{a}_i = a_i$. Because:

$$\overline{z_1 \cdot z_2} = \bar{z}_1\,\bar{z}_2$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

We get

$$\bar{z}^{(n)} + a_{n-1}\bar{z}^{(n-1)} + \cdots + a_1\bar{z}' + a_0\bar{z} = 0$$

In conclusion, **complex roots come in pairs**. So every complex solution $z(x)$ gives rise to another, $\bar{z}(x)$, and these give 2 linearly independent real solutions $u(x), v(x)$.