Assignment 7

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Question 1

$$\cos 9t - \cos 7t$$

We can use the trigonometric identities

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Note that

$$\begin{aligned}
\sin(\alpha+\beta)\cdot\sin(\alpha-\beta) &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\
&= \sin^2\alpha\cos^2\beta - \sin\alpha\cos\alpha\sin\beta\cos\beta + \sin\alpha\cos\alpha\sin\beta\cos\beta\\
&- \cos^2\alpha\sin^2\beta\\
&= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta\\
&= \frac{1}{4}(1-\cos 2\alpha)(1+\cos 2\beta) - \frac{1}{4}(1+\cos 2\alpha)(1-\cos 2\beta)\\
&= \frac{1}{4}\left(1+\cos 2\beta - \cos 2\alpha - \cos 2\alpha\cos 2\beta\right)\\
&- \frac{1}{4}(1-\cos 2\beta + \cos 2\alpha - \cos 2\alpha\cos 2\beta)\\
&= \frac{1}{2}(\cos 2\beta - \cos 2\alpha)
\end{aligned}$$

If we substitute $\alpha = \frac{a}{2}, \beta = \frac{b}{2}$ then we get

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\cdot\sin\left(\frac{a-b}{2}\right)$$

In our case, $a = 9, b = 7$.

$$\cos 9t - \cos 7t = -2\sin\left(\frac{9+7}{2}t\right)\sin\left(\frac{9-7}{2}t\right) = -2\sin 8t\sin t$$

Question 6

- If a mass of 5 kg stretches a spring 10 cm, then

$$k=\frac{m\,g}{L}=\frac{5\,\mathrm{kg}*9.8\,\mathrm{m\,s^{-2}}}{0.1\,\mathrm{m}}=490\,\mathrm{N\,m^{-1}}$$

- If the viscous medium exerts 2N on the mass when its velocity is 4 cm/s, then

$$\gamma=\frac{F}{v}=\frac{2\,\mathrm{N}}{0.04\,\mathrm{m\,s^{-1}}}=50\,\mathrm{N\,m^{-1}\,s^{-1}}$$

The equation of motion according to Newton's 2nd law is:

$$m\,u''=m\,g-k\,(L+u)-\gamma\,u'+F(t)$$

As $m\,g=k\,L$, the equation can be simplifies and arranged to the form:

$$u''+\frac{\gamma}{m}u'+\frac{k}{m}u=\frac{F}{m}(t)$$

Plug in data:

$$u''+\frac{50}{5}u'+\frac{490}{5}u=2\sin\frac{t}{2}$$

$$u''+10u'+98u=2\sin\frac{t}{2}$$

The given ICs are:

$$\begin{cases} u(0)=0 \\ u'(0)=0.03\,\mathrm{m\,s^{-1}} \end{cases}$$

This is the initial value problem.

Question 8

(a) First solve the associated homogeneous equation. Its characteristic equation is

$$\lambda^2+10\lambda+98=0$$

The roots are

$$\lambda_{1,2}=\frac{-10\pm 2\sqrt{73}\,\mathrm{i}}{2}=-5\pm\sqrt{73}\mathrm{i}$$

The solution of the homogeneous equation is

$$y_h=c_1\,\mathrm{e}^{-5t}\cos\sqrt{73}t+c_2\,\mathrm{e}^{-5t}\sin\sqrt{73}t,\quad c_{1,2}\in\mathbb{R},\quad t>0$$

Now search for a particular solution of the form

$$\begin{aligned}
y_p &= A\cos\frac{t}{2} + B\sin\frac{t}{2} \\
y_p' &= -\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2} \\
y_p'' &= -\frac{A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}
\end{aligned}$$

Input in the ODE:

$$y'' + 10y' + 98y = 2\sin\frac{t}{2}$$

$$-\frac{1}{4}\left(A\cos\frac{t}{2} + B\sin\frac{t}{2}\right) + 5\left(-A\sin\frac{t}{2} + B\cos\frac{t}{2}\right) + 98\left(A\cos\frac{t}{2} + B\sin\frac{t}{2}\right) = 2\sin\frac{t}{2}$$

Equate coefficients on both sides:

$$\begin{cases}
\cos\frac{t}{2}: & -\frac{A}{4} + 5B + 98A = 0 \rightarrow B = -\frac{391}{20}A \\
\sin\frac{t}{2}: & -\frac{B}{4} - 5A + 98B = 2 \rightarrow 5A = 97.75B - 2
\end{cases}$$

$$5A = 97.75 * \left(-\frac{391}{20}\right)A - 2 \rightarrow A = -\frac{160}{153281}, B = \frac{3128}{153281}$$

The general solution of the ODE is given by $y_h + y_p$:

$$y = c_1\,\mathrm{e}^{-5t}\cos\sqrt{73}t + c_2\,\mathrm{e}^{-5t}\sin\sqrt{73}t - \frac{160}{153281}\cos\frac{t}{2} + \frac{3128}{153281}\sin\frac{t}{2}, \quad c_{1,2} \in \mathbb{R}$$

Plug in ICs to find $c_{1,2}$.

$$y(0) = c_1 - \frac{160}{153281} = 0 \rightarrow c_1 = \frac{160}{153281}$$

$$\begin{aligned}
y' &= -5c_1\mathrm{e}^{-5t}\cos\sqrt{73}t - \sqrt{73}c_1\,\mathrm{e}^{-5t}\sin\sqrt{73}t \\
&\quad -5c_2\,\mathrm{e}^{-5t}\sin\sqrt{73}t + \sqrt{73}c_2\,\mathrm{e}^{-5t}\cos\sqrt{73}t \\
&\quad + \frac{80}{153281}\sin\frac{t}{2} + \frac{1564}{153281}\cos\frac{t}{2}
\end{aligned}$$

$$y'(0) = -5c_1 + \sqrt{73}c_2 + \frac{1564}{153281} = 0.03$$

$$c_2 = 2.927 \times 10^{-3}$$

The unique solution to the ODE is:

$$
\begin{aligned}
y = &\frac{160}{153281} \mathrm{e}^{-5t} \cos \sqrt{73} t + 2.927 \times 10^{-3} \mathrm{e}^{-5t} \sin \sqrt{73} t \\
&- \frac{160}{153281} \cos \frac{t}{2} + \frac{3128}{153281} \sin \frac{t}{2}
\end{aligned}
$$

(b) The elements comprising the solution are 2 decaying cosine and sine functions and 2 perpetually oscillating cosine and sine functions. It is clear then that the solution can't approach a steady state, thus concluding that all parts of the solution are transient.

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<u>Question 1</u>

$$u'' + 0.5u' + 2u = 0$$

Denote new variables x_1, x_2 such that

$$\begin{cases} x_1 = u' \\ x_2 = u \end{cases}$$

Their derivatives:

$$\begin{cases} x_1' = u'' \\ x_2' = u' = x_1 \end{cases}$$

Using these substitutions, we can transform the single 2nd order ODE into the following system of 1st order ODEs:

$$\begin{cases} x_1' = -0.5x_1 + 2x_2 \\ x_2' = x_1 \end{cases}$$

<u>Question 4</u>

$$u^{(4)} - u = 0$$

Denote new variables x_1, x_2, x_3, x_4 such that

$$\begin{cases} x_1 = u^{(3)} \\ x_2 = u'' \\ x_3 = u' \\ x_4 = u \end{cases}$$

Their derivatives:

$$\begin{cases} x_1' = u^{(4)} \\ x_2' = u''' = x_1 \\ x_3' = u'' = x_2 \\ x_4' = u' = x_3 \end{cases}$$

Thus, we can transform the original ODE to the following system of 1st order equations:

$$\begin{cases} x_1' = x_4 \\ x_2' = x_1 \\ x_3' = x_2 \\ x_4' = x_3 \end{cases}$$

<u>Question 7</u>

$$\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = x_1 - 2x_2 \end{cases}$$

(a) Solve the equation where x_1 is a function of x_2

$$x_1' + 2x_1 = x_2$$

Via integration factor method:

$$x_1 = \mathrm{e}^{-2x_2}\left[\int x_2\,\mathrm{e}^{2x_2}\,\mathrm{d}x_2 + A\right]$$

Solve the integral via integration by parts:

$$\int x_2\,\mathrm{e}^{2x_2}\,\mathrm{d}x_2 = \frac{1}{2}x_2\,\mathrm{e}^{2x_2} - \frac{1}{2}\mathrm{e}^{2x_2} = \frac{1}{2}\mathrm{e}^{2x_2}(x_2 - 1)$$

$$x_1 = \frac{1}{2}(x_2 - 1) + A\,\mathrm{e}^{-2x_2}, \quad A \in \mathbb{R}$$

Substitute solution into the 2nd equation:

$$x_2' = \frac{1}{2}(x_2 - 1) + A\,\mathrm{e}^{-2x_2} - 2x_2$$

$$x_2' + \frac{3}{2}x_2 = -\frac{1}{2} + A\,\mathrm{e}^{-2x_2}$$

First solve the associated homogeneous equation.

$$\lambda + \frac{3}{2} = 0 \rightarrow \lambda = -\frac{3}{2}$$

The solution to the homogeneous equation is $x_{2,h} = c_1\,\mathrm{e}^{-\frac{3}{2}t}$ where $c_1 \in \mathbb{R}$.

Now find a particular solution that fits the polynomial part of the RHS:

$$x_{2,p1} = \alpha\,t + \beta$$

$$x_{2,p1}' = \alpha$$

Plug in the (polynomial part of the) ODE:

$$\alpha + \frac{3}{2}(\alpha t + \beta) = -\frac{1}{2}$$

Equate coefficients on both sides and get $\alpha = 0$, $\beta = -\frac{1}{3}$. $\Rightarrow x_{2,p1} = -\frac{1}{3}$.

Now find a particular solution that fits the exponent part of the RHS:

$$x_{2,p2} = \gamma\,\mathrm{e}^{-2x_2}$$

$$x_{2,p2}' = -2\gamma\,\mathrm{e}^{-2x_2}$$

....

Question 14

Given the system of equations:

$$\begin{cases} x_1' = a_{11}\,x_1 + a_{12}\,x_2 + g_1(t) \\ x_2' = a_{21}\,x_1 + a_{22}\,x_2 + g_2(t) \end{cases}, \qquad x_{1,2}(0) = x_{1,2}^0$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constants and a_{12}, a_{21} are not both zero, and g_1, g_2 are differentiable.

Differentiate the first (second) equation.

$$x_1'' = a_{11}\,x_1' + a_{12}\,x_2' + g_1'(t)$$

$$x_2'' = a_{21}\,x_1' + a_{22}\,x_2' + g_2'(t)$$

Insert expression for x_2'. (x_1')

$$x_1'' = a_{11}\,x_1' + a_{12}\,(a_{21}\,x_1 + a_{22}\,x_2 + g_2(t)) + g_1'(t)$$

$$x_2'' = a_{21}\,(a_{11}\,x_1 + a_{12}\,x_2 + g_1(t)) + a_{22}\,x_2' + g_2'(t)$$

Note that from the first (second) equation, x_2 (x_1) can be expressed as:

$$x_2 = \frac{x_1' - a_{11}\,x_1 - g_1(t)}{a_{12}}$$

$$x_1 = \frac{x_2' - a_{22}\,x_2 - g_2(t)}{a_{21}}$$

Input back in the above equation:

$$x_1'' = a_{11}\,x_1' + a_{12}\left[a_{21}\,x_1 + \frac{a_{22}}{a_{12}}\,(x_1' - a_{11}\,x_1 - g_1(t)) + g_2(t)\right] + g_1'(t)$$

$$x_2'' = a_{21}\left[\frac{a_{11}}{a_{21}}\,(x_2' - a_{22}\,x_2 - g_2(t)) + a_{12}\,x_2 + g_1(t)\right] + a_{22}\,x_2' + g_2'(t)$$

Further simplification can be made to write the equation as a second order ODE of the form $x'' + p(t)\,x' + q(t) = 0$, however it is sufficiently clear at this point that we've got a second order linear ODE.

Note two things:

1. Two crucial assumptions were made:

   a. $a_{12} \neq 0$

   b. $g_1(t)$ is differentiable.

   Both assumptions are valid given the information we've received at the beginning.

2. We chose rather arbitrarily to differentiate the first equation to obtain a second order ODE in x_1. We could have equally chosen to differentiate the second equation and obtain a similar equation in x_2.

   If we would have done so, the expression would have contained $\frac{a_{11}}{a_{21}}$ and $g_2'(t)$. In order for the would-so obtained equation to hold, a_{22} must be non-zero and $g_2(t)$ must be differentiable. Again, both conditions are satisfied.

Additionally, given the ICs $x_1(0), x_2(0)$, we officially have an initial value problem for a single second order equation.

If $a_{11}, \ldots, a_{22}$ were functions of t, the same procedure could be carried out, with the caveat that $a_{22}(t)\,/\,a_{12}(t)$ and $a_{11}(t)\,/\,a_{21}(t)$ must be continuous (and also differentiable in the interval relevant to the ICs).

Question 16

We're informed that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ solve the following equations:

$$\begin{aligned} x' &= p_{11}(t)\, x + p_{12}(t)\, y + g_1(t) \\ y' &= p_{21}\,(t) x + p_{22}(t)\, y + g_2(t) \end{aligned}$$

These can also be written in matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

Plug in the two solutions:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}' = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}' = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

Subtract the two equations (operations are valid as matrices in $\mathbb{R}$ are a vector subspace).

$$\begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}' = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}$$

Return to equation form and rearrange:

$$(x_1 - x_2)' - p_{11}\,(x_1 - x_2) - p_{12}\,(y_1 - y_2) = 0$$

$$(y_1 - y_2)' - p_{21}\,(x_1 - x_2) - p_{22}\,(y_1 - y_2) = 0$$

We thus proved that $x = x_1 - x_2$, $y = y_1 - y_2$ solve the corresponding homogeneous equation.

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Question 21

(a) Initial conditions are $Q_1(0) = 25\,\mathrm{oz}$, $Q_2(0) = 15\,\mathrm{oz}$. The differential equation for the amount of salt in each tank is given by the amount of salt flowing in minus the amount of salt flowing out. The amount flowing in and out depends on the transient concentraion in each tank.

For tank 1:

$$Q_1'(t) = 1\,\frac{\text{oz}}{\text{gal}} * \frac{3}{2}\,\frac{\text{gal}}{\text{min}} - 3\,\frac{Q_1}{V_1} + \frac{3}{2}\,\frac{Q_2}{V_2}$$

$$Q_1' = -\frac{3}{V_1}\,Q_1 + \frac{3}{2V_2}\,Q_2 + \frac{3}{2}$$

For tank 2:

$$Q_2'(t) = 3\,\frac{\text{oz}}{\text{gal}} * 1\,\frac{\text{gal}}{\text{min}} + 3\,\frac{Q_1}{V_1} - 4\,\frac{Q_2}{V_2}$$

$$Q_2' = \frac{3}{V_1}\,Q_1 - \frac{4}{V_2}\,Q_2 + 3$$

Obtain expressions for V_1, V_2 (in gals):

$$V_1(t) = V_1(0) + 1.5\,\frac{\text{gal}}{\text{min}}t - 3\,\frac{\text{gal}}{\text{min}}\,t + 1.5\,\frac{\text{gal}}{\text{min}}\,t = V_1(0) = 30\,\text{gal}$$

$$V_2(t) = V_2(0) + 1\,\frac{\text{gal}}{\text{min}}\,t + 3\,\frac{\text{gal}}{\text{min}}\,t - 4\,\frac{\text{gal}}{\text{min}}\,t = V_2(0) = 20\,\text{gal}$$

Luckily they remain constant. Plug in the differential equations:

$$\begin{aligned} Q_1' &= -\frac{1}{10}\,Q_1 + \frac{3}{40}\,Q_2 + \frac{3}{2} \\ Q_2' &= \frac{1}{10}Q_1 - \frac{1}{5}Q_2 + 3 \end{aligned}$$

and in matrix form:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}' = \begin{bmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}$$

(b) In equilibrium $\left[\begin{smallmatrix} Q_1 \\ Q_2 \end{smallmatrix}\right]' = 0$. We need to solve:

$$\begin{bmatrix} \frac{1}{10} & -\frac{3}{40} \\ -\frac{1}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} Q_1^E \\ Q_2^E \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}$$

Denote

$$A = \begin{bmatrix} \frac{1}{10} & -\frac{3}{40} \\ -\frac{1}{10} & \frac{1}{5} \end{bmatrix}, \vec{b} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}$$

The solution is given by:

$$\left[\begin{array}{c} Q_1^E \\ Q_2^E \end{array}\right] = A^{-1}\vec{b}$$

The inverse matrix A^{-1} is calculated by taking the adjoint of A and dividing by its determinant.

$$\det A = \frac{1}{10} * \frac{1}{5} - \frac{3}{40} * \frac{1}{10} = \frac{1}{80}$$

$$\left[\begin{array}{cc} \frac{1}{10} & -\frac{3}{40} \\ -\frac{1}{10} & \frac{1}{5} \end{array}\right]$$

$$\operatorname{adj} A = \left[\begin{array}{cc} (-1)^{1+1} \cdot \frac{1}{5} & (-1)^{1+2} \cdot \left(-\frac{1}{10}\right) \\ (-1)^{2+1} \cdot \left(-\frac{3}{40}\right) & (-1)^{2+2} \cdot \frac{1}{10} \end{array}\right]^T = \left[\begin{array}{cc} \frac{1}{5} & \frac{1}{10} \\ \frac{3}{40} & \frac{1}{10} \end{array}\right]^T = \left[\begin{array}{cc} \frac{1}{5} & \frac{3}{40} \\ \frac{1}{10} & \frac{1}{10} \end{array}\right]$$

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} = 80 \cdot \left[\begin{array}{cc} \frac{1}{5} & \frac{3}{40} \\ \frac{1}{10} & \frac{1}{10} \end{array}\right] = \left[\begin{array}{cc} 16 & 6 \\ 8 & 8 \end{array}\right]$$

Finally,

$$\left[\begin{array}{c} Q_1^E \\ Q_2^E \end{array}\right] = A^{-1}\vec{b} = \left[\begin{array}{cc} 16 & 6 \\ 8 & 8 \end{array}\right] \left[\begin{array}{c} \frac{3}{2} \\ 3 \end{array}\right] = \left[\begin{array}{c} 42 \\ 36 \end{array}\right]$$

$Q_1^E = 42\,\text{oz}$ and $Q_2^E = 36\,\text{oz}$.

(c) Note that

$$\begin{aligned} x_1' &= (Q_1(t) + Q_1^E)' = Q_1' \\ x_2' &= (Q_2(t) + Q_2^E) = Q_2' \end{aligned}$$

The differential equatins for x_1, x_2 are the same as for Q_1, Q_2, where Q_1, Q_2 are substituted with $(x_1 + Q_1^E)$ and $(x_2 + Q_2^E)$:

$$\begin{aligned} x_1' &= -\frac{1}{10}(x_1 + 42) + \frac{3}{40}(x_2 + 36) + \frac{3}{2} \\ x_2' &= \frac{1}{10}(x_1 + 42) - \frac{1}{5}(x_2 + 36) + 3 \end{aligned}$$

Simplify:

$$\begin{aligned} x_1' &= -\frac{1}{10}x_1 + \frac{3}{40}x_2 \\ x_2' &= \frac{1}{10}x_1 - \frac{1}{5}x_2 \end{aligned}$$

where

$$x_1(0) = Q_1(0) - Q_1^E = 25 - 42 = -17\,\mathrm{oz}$$

$$x_2(0) = Q_2(0) - Q_2^E = 15 - 36 = -21\,\mathrm{oz}$$

Note that the differential equations are homogeneous, as expected.