Linear Algebra for Chemists — Assignment 9

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Question 1. At the end of year n, #rabbits is x_n and #wolves is y_n.

$$\begin{cases} x_{n+1} = 3\,x_n - y_n \\ y_{n+1} = 2\,x_n \end{cases}.$$

a) Denote $\vec{v}_n = [x_n, y_n]^T$. Consider the matrix A

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}.$$

As we can see,

$$A\,\vec{v}_n = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 3\,x_n - y_n \\ 2\,x_n \end{bmatrix} = \vec{v}_{n+1}\,.$$

b) Denote $\vec{v}_0 = [x_0, y_0]^T$. Prove by induction that $\vec{v}_n = A^n\,\vec{v}_0$.

Proof. For $n = 0$ (base case),

$$\vec{v}_0 = A^0\,\vec{v}_0 = I\,\vec{v}_0 = \vec{v}_0\,.$$

Assume that $\vec{v}_k = A^k\,\vec{v}_0$. For $n = k+1$ (inductive step),

$$\vec{v}_{k+1} = A^{k+1}\,\vec{v}_0 = A\,A^k\,\vec{v}_0 = A\,\vec{v}_k,$$

which is true by the definition of A. $\square$

c) Find eigenvalues for A:

$$|A - \lambda\,I| = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = \lambda - 3\lambda + 2 = (\lambda - 2)\,(\lambda - 1) = 0\,.$$

Find associated eigenvectors: for $\lambda_1 = 2$:

$$[A - 2\,I] = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \rightarrow w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1$:

$$[A - I] = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \rightarrow w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

d) Find an expression for A^n, and express $\vec{v}_n$ as $A^n\,\vec{v}_0$. The matrix of eigenvectors is

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

whose inverse is

$$T^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^n = T\, D^n\, T^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^{n+1} & -2^n \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{n+1}-1 & 1-2^n \\ 2^{n+1}-2 & 2-2^n \end{bmatrix}, \end{aligned}$$

$$\vec{v}_n = A^n\, \vec{v}_0 = \begin{bmatrix} 2^{n+1}-1 & 1-2^n \\ 2^{n+1}-2 & 2-2^n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} (2^{n+1}-1)\, x_0 + (1-2^n)\, y_0 \\ (2^{n+1}-2)\, x_0 + (2-2^n)\, y_0 \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

In a more compact form, we may conclude that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (2\, x_0 - y_0)\, 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (y_0 - x_0) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

e) Given $\vec{v}_0 = [3, 2]^T$,

$$\vec{v}_7 = (2 \times 3 - 2)\, 2^7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2-3) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 511 \\ 510 \end{bmatrix}.$$

Question 2. Find Jordan canonical forms. The number of

a) $P(\lambda) = (\lambda - 1)^2\, (\lambda + 2)^3$.

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

b) $P(\lambda) = (\lambda - 5)^4$.

$$J = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

c) $P(\lambda) = \lambda\, (\lambda + 3)\, (\lambda - 5)^2$.

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Question 3. Find matrix T that transforms the matrix A to Jordan canonical form.

a) $A = \left[\begin{smallmatrix} -12 & 7 \\ -7 & 2 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\, I| = (\lambda + 12)\, (\lambda - 2) + 49 = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0.$$

T consists of two generalized eigenvectors $T = [t_1, t_2]$. $t_1 \in \text{kernel}\,(A + 5\,I)$:

$$[A + 5\,I]\,t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -7 & 7 \\ -7 & 7 \end{bmatrix} t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Pick $t_1 = [-7, -7]^T$. t_2 satisfies $[A + 5\,I]\,t_2 = t_1$.

$$\begin{bmatrix} -7 & 7 \\ -7 & 7 \end{bmatrix} t_2 = \begin{bmatrix} -7 \\ -7 \end{bmatrix}.$$

Pick $t_2 = [1, 0]^T$. The desired matrix T is

$$T = \begin{bmatrix} -7 & 1 \\ -7 & 0 \end{bmatrix}.$$

b) $A = \left[\begin{smallmatrix} 4 & -1 \\ 1 & 2 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\,I| = (\lambda - 4)\,(\lambda - 2) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

T consists of two generalized eigenvectors $T = [t_1, t_2]$. $t_1 \in \text{kernel}(A - 3\,I)$:

$$[A - 3\,I]\,t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Pick $t_1 = [1, 1]^T$. t_2 satisfies $[A - 3\,I]\,t_2 = t_1$.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Pick $t_2 = [1, 0]^T$.

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

c) $A = \left[\begin{smallmatrix} 1 & 3 \\ 4 & -5 \end{smallmatrix}\right]$. Find eigenvalues for A.

$$|A - \lambda\,I| = (\lambda - 1)\,(\lambda + 5) - 12 = \lambda^2 + 4\lambda - 17.$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 + 4 \times 17}}{2} = -2 \pm \sqrt{21}\,.$$

Find associated eigenvectors. For $\lambda_1 = -2 + \sqrt{21}$:

$$\begin{aligned} [A - (\sqrt{21} - 2)\,I] &= \begin{bmatrix} 3 - \sqrt{21} & 3 \\ 4 & -3 - \sqrt{21} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{3 - \sqrt{21}} \\ 4 & -3 - \sqrt{21} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{3}{3 - \sqrt{21}} \\ 0 & -(3 + \sqrt{21}) - \frac{12}{3 - \sqrt{21}} = \frac{-(9 - 21) - 12}{3 - \sqrt{21}} = 0 \end{bmatrix} \end{aligned}$$

$$w_1 = \begin{bmatrix} \frac{3}{\sqrt{21} - 3} \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -2 - \sqrt{21}$:

$$\begin{aligned}[][A + (\sqrt{21} + 2)\, I] &= \begin{bmatrix} 3 + \sqrt{21} & 3 \\ 4 & -3 + \sqrt{21} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{3 + \sqrt{21}} \\ 4 & -3 + \sqrt{21} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & \frac{3}{3 + \sqrt{21}} \\ 0 & -3 + \sqrt{21} - \frac{12}{3 + \sqrt{21}} = \frac{(21 - 9) - 12}{3 + \sqrt{21}} = 0 \end{bmatrix}\end{aligned}$$

$$w_2 = \begin{bmatrix} -\frac{3}{3 + \sqrt{21}} \\ 1 \end{bmatrix}.$$

T is just the matrix of the eigenvectors of A.

$$T = \begin{bmatrix} \frac{3}{\sqrt{21} - 3} & -\frac{3}{3 + \sqrt{21}} \\ 1 & 1 \end{bmatrix}.$$