Sample Exam Solution

Question 1

(a) A mass of 5 kg is hung on a spring in a viscous medium, and is stretched 1 cm and then released while an external force of $F(t) = 2\cos t$ is applied. When a force of 10 N is applied to the spring, it's stretched by 8 cm. When the velocity of a body in the medium is 1 cm/s it results in a viscous force of 3 N.

Find the position at time t.

Equation of motion is:

$$m\,x'' = F(t) + m\,g - k\,(L + x) - \gamma\,x'$$

Use the fact that $m\,g = k\,L$:

$$x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = \frac{F(t)}{m}$$

Calculate k and γ.

$$k = \frac{10\,\mathrm{N}}{8\,\mathrm{cm}} = 1.25$$

$$\gamma = \frac{3\,\mathrm{N}}{1\,\mathrm{cm\,s^{-1}}} = 3$$

ODE describing the system is:

$$x'' + 0.6x' + 0.25x = 0.4\cos t$$

First solve the associated homogeneous system. Characteristic equation is:

$$\lambda^2 + 0.6\lambda + 0.25 = 0$$

$$\lambda_{1,2} = \frac{-0.6 \pm 0.8\mathrm{i}}{2} = -0.3 \pm 0.4\mathrm{i}$$

Solution to homogeneous system is

$$x_h = c_1\,\mathrm{e}^{-0.3t}\cos 0.4t + c_2\,\mathrm{e}^{-0.3t}\sin 0.4t$$

Now find a particular solution. Guess x_p of the form $A\cos t + B\sin t$. Input in ODE:

$$-A\cos t - B\sin t - 0.6A\sin t + 0.6B\cos t + 0.25A\cos t + 0.25B\sin t = 0.4\cos t$$

$$\cos t\,(-0.75A + 0.6B) + \sin t\,(-0.75B - 0.6A) = 0.4\cos t$$

Equate coefficients on both sides.

$$\begin{cases} \sin t\colon & -\frac{3}{4}B = \frac{6}{10}A \rightarrow A = -\frac{5}{4}B \\ \cos t\colon & -\frac{3}{4}A + \frac{6}{10}B = \frac{4}{10} \rightarrow B = \frac{32}{123},\, A = -\frac{40}{123} \end{cases}$$

General solution to the ODE (i.e. position of the mass at time t) is:

$$x = x_h + x_p = c_1\,\mathrm{e}^{-0.3t}\cos 0.4t + c_2\,\mathrm{e}^{-0.3t}\sin 0.4t - \frac{40}{123}\cos t + \frac{32}{123}\sin t$$

Input ICs: $x(0) = 1\,\mathrm{cm}$, $x'(0) = 0$.

$$x(0) = c_1 - \frac{40}{123} = 1 \rightarrow c_1 = \frac{163}{123}$$

$$x'(0) = 0.4c_2 + \frac{32}{123} = 0 \rightarrow c_2 = \frac{80}{123}$$

$$x = \frac{163}{123}\,\mathrm{e}^{-0.3t}\cos 0.4t + \frac{80}{123}\,\mathrm{e}^{-0.3t}\sin 0.4t - \frac{40}{123}\cos t + \frac{32}{123}\sin t$$

(b) Find the general (implicit) solution to the following equation:

$$y\,y'' - (y')^3 = 0$$

Set $v(x, y) = y'(x)$.

$$y\,v' - v^3 = 0$$

Note that

$$v' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\,v$$

So we get a separable ODE in u, y:

$$y\,\frac{\mathrm{d}v}{\mathrm{d}y}\,v = v^3$$

A possible solution is $v = 0$, equivalent to $y = C \in \mathbb{R}$.

Assuming $v \neq 0$ on some interval, divide by v^3 to get

$$\frac{\mathrm{d}v}{v^2} = \frac{\mathrm{d}y}{y}$$

Integrate:

$$-\frac{1}{v} = \ln|y| + c_1$$

Recall that $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, so we can separate variables again:

$$-\mathrm{d}x = (\ln|y| + c_1)\,\mathrm{d}y$$

Integrate:

$$-x + c_2 = \int \ln|y|\,\mathrm{d}y + c_1\,y$$

Calculate integral via integration by parts. $u' = 1, v = \ln|y|$. (Poor choice of variables)

$$\int \ln|y|\,\mathrm{d}y = y\ln|y| - \int y \cdot \frac{1}{y}\,\mathrm{d}y = y\,(\ln|y| - 1)$$

Solution to the ODE (in implicit form) is: (the term "1" is omitted, as it's not necessary)

$$-x + c_2 = y\,(\ln|y| + c_1)$$

Question 2

Solve the equation

$$y'' - 4y' + 4y = t\,\mathrm{e}^{2t} + 3\sin t$$

with ICs: $y(0) = 2, y'(0) = 1$. First solve associated homogeneous equation. Characteristic equation is:

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda_{1,2} = 2$$

Homogeneous solution is:

$$y_h = c_1\,\mathrm{e}^{2t} + c_2\,t\,\mathrm{e}^{2t} \qquad c_{1,2} \in \mathbb{R}$$

Now find particular solutions corresponding to each element in the RHS. For the sine, guess a solution of the form $y_{p1} = A\cos t + B\sin t$. Input in ODE:

$$-A\cos t - B\sin t - 4(-A\sin t + B\cos t) + 4(A\cos t + B\sin t) = 3\sin t$$

$$\cos t\,(-A - 4B + 4A) + \sin t\,(-B + 4A + 4B) = 3\sin t$$

Equate coefficients on both sides.

$$\begin{cases} \cos t\colon\ \ 3A - 4B = 0 \rightarrow A = \frac{4}{3}B \\ \sin t\colon\ \ 3B + 4A = 3 \rightarrow B = \frac{9}{25}, A = \frac{12}{25} \end{cases}$$

So $y_{p1} = \frac{12}{73}\cos t + \frac{9}{73}\sin t$.

For the second element in RHS, note that scalar multiples of e^{2t} and $t\,\mathrm{e}^{2t}$ solve the homogeneous equation. Try $y_{p2} = A\,t^2\,\mathrm{e}^{2t}$.

$$\begin{aligned} y'_{p2} &= 2A\,t\,\mathrm{e}^{2t} + 2A\,t^2\,\mathrm{e}^{2t} \\ y''_{p2} &= 2A\,\mathrm{e}^{2t} + 8A\,t\,\mathrm{e}^{2t} + 4A\,t^2\,\mathrm{e}^{2t} \end{aligned}$$

Input in ODE: (and divide by $\mathrm{e}^{2t} \neq 0\ \forall t$)

$$(2A + 8A\,t + 4A\,t^2) - 4\,(2A\,t + 2A\,t^2) + 4A\,t^2 = t$$

coefficient of t on LHS vanishes! Guess alternative $y_{p2} = (A\,t^3 + B\,t^2)\mathrm{e}^{2t}$.

$$\begin{aligned} y'_{p2} &= \mathrm{e}^{2t}\,[3A\,t^2 + 2A\,t^3 + 2B\,t + 2B\,t^2] = \mathrm{e}^{2t}\,[2A\,t^3 + (6A + 2B)\,t^2 + 2B\,t] \\ y''_{p2} &= \mathrm{e}^{2t}\,[6A\,t + 12A\,t^2 + 4A\,t^3 + 2B + 8B\,t + 4B\,t^2] \\ &= \mathrm{e}^{2t}\,[4A\,t^3 + (12A + 4B)t^2 + (6A + 8B)t + 2B] \end{aligned}$$

Input in ODE: (and divide by $\mathrm{e}^{2t} \neq 0\ \forall t$)

$$\begin{aligned} t &= [4A\,t^3 + (12A + 4B)t^2 + (6A + 8B)t + 2B] - 4[2A\,t^3 + (6A + 2B)\,t^2 + 2B\,t] \\ &\quad + 4[A\,t^3 + B\,t^2] \end{aligned}$$

$$t = t^2\,(12A + 4B - 24A - 8B + 4B) + t\,(6A + 8B - 8B) + 2B$$

Equate coefficients on both sides to get $A = \frac{1}{6}$, $B = 0$.

General solution to ODE is:

$$y = y_h + y_{p1} + y_{p2} = c_1\,\mathrm{e}^{2t} + c_2\,t\,\mathrm{e}^{2t} + \frac{12}{25}\cos t + \frac{9}{25}\sin t + \frac{1}{6}A\,t^3\,\mathrm{e}^{2t}$$

Input ICs:

$$\begin{aligned} y(0) &= c_1 + \frac{12}{25} = 2 \rightarrow c_1 = \frac{38}{25} \\ y'(0) &= 2c_1 + c_2 + \frac{9}{25} = 1 \rightarrow c_2 = -\frac{12}{5} \end{aligned}$$

Unique solution is:

$$\frac{38}{25}\,\mathrm{e}^{2t} - \frac{12}{5}\,t\,\mathrm{e}^{2t} + \frac{12}{25}\cos t + \frac{9}{25}\sin t + \frac{1}{6}A\,t^3\,\mathrm{e}^{2t}$$

Question 3

Given a closed system of 3 tanks containing a brine solution with constant flow of $r = 10\,\mathrm{L\,min^{-1}}$, find the amount of salt in each tank at time t if the volume of the first tank is $V_1 = 20\,\mathrm{L}$, the second is $V_2 = 50\,\mathrm{L}$ and the third is $V_3 = 20\,\mathrm{L}$.

Write equations for salt concentration in each tank:

$$\begin{aligned} x_1' &= -r\,\frac{x_1}{V_1} + r\,\frac{x_3}{V_3} \\ x_2' &= r\,\frac{x_1}{V_1} - r\,\frac{x_2}{V_2} \\ x_{3'} &= r\,\frac{x_2}{V_2} - r\,\frac{x_3}{V_3} \end{aligned}$$

Convert to matrix form:

$$\vec{x} = A\,\vec{x}$$

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

Find eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{2} - \lambda & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} - \lambda & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} - \lambda \end{vmatrix} = 0$$

$$-\left(\frac{1}{2} + \lambda\right)\left(\frac{1}{5} + \lambda\right)\left(\frac{1}{2} + \lambda\right) - \frac{1}{2}\left[-\frac{1}{10}\right] = 0$$

$$\left(\frac{1}{2} + \lambda\right)\left[\lambda^2 + \frac{7}{10}\lambda + \frac{1}{10}\right] - \frac{1}{20} = 0$$

$$\lambda^3 + \frac{12}{10}\lambda^2 + \frac{9}{20}\lambda = 0$$

$$\lambda\left(\lambda^2 + \frac{12}{10}\lambda + \frac{9}{20}\right) = 0$$

$$\lambda_1 = 0, \lambda_{1,2} = \frac{-\frac{6}{5} \pm \frac{3}{5}\mathrm{i}}{2} = -\frac{3}{5} \pm \frac{3}{10}\mathrm{i}$$

For $\lambda_1 = 0$:

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$$

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \vec{v}_1 = \vec{0}$$

Choose

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -\frac{3}{5} - \frac{3}{10}\mathrm{i}$:

$$(A - \lambda_2 I)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} \frac{1}{10} + \frac{3}{10}\mathrm{i} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{5} + \frac{3}{10}\mathrm{i} & 0 \\ 0 & \frac{1}{5} & \frac{1}{10} + \frac{3}{10}\mathrm{i} \end{bmatrix} \vec{v}_2 = \vec{0}$$

Choose

$$\vec{v}_2 = \begin{bmatrix} -\frac{\frac{1}{2}}{\frac{1}{10} + \frac{3}{10}\mathrm{i}} \\ \frac{1}{2} - \frac{3}{2}\mathrm{i} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{10}{2} \cdot \left(\frac{1}{10} - \frac{3}{10}\mathrm{i}\right) \\ 1 - \frac{3}{2}\mathrm{i} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}\mathrm{i} \\ 1 - \frac{3}{2}\mathrm{i} \\ 1 \end{bmatrix}$$

From $\vec{v}_2$ construct two real solutions:

$$\mathrm{e}^{-\frac{3}{5}t}\left(\cos\left(\frac{3t}{10}\right) + \mathrm{i}\sin\left(\frac{3t}{10}\right)\right) \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}\mathrm{i} \\ 1 - \frac{3}{2}\mathrm{i} \\ 1 \end{bmatrix} = \cdots =$$

$$= \mathrm{e}^{-\frac{3}{5}t} \begin{bmatrix} -\frac{1}{2}\cos\left(\frac{3t}{10}\right) - \frac{3}{2}\sin\left(\frac{3t}{10}\right) \\ \cos\left(\frac{3t}{10}\right) + \frac{3}{2}\sin\left(\frac{3t}{10}\right) \\ \cos\left(\frac{3t}{10}\right) \end{bmatrix} + \mathrm{i}\,\mathrm{e}^{-\frac{3}{5}t} \begin{bmatrix} -\frac{1}{2}\sin\left(\frac{3t}{10}\right) + \frac{3}{2}\cos\left(\frac{3t}{10}\right) \\ \sin\left(\frac{3t}{10}\right) - \frac{3}{2}\left(\frac{3t}{10}\right) \\ \sin\left(\frac{3t}{10}\right) \end{bmatrix}$$

General solution (i.e amount of salt in each tank) is:

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + c_2 e^{-\frac{3}{5}t} \begin{bmatrix} -\frac{1}{2}\cos\left(\frac{3t}{10}\right) - \frac{3}{2}\sin\left(\frac{3t}{10}\right) \\ \cos\left(\frac{3t}{10}\right) + \frac{3}{2}\sin\left(\frac{3t}{10}\right) \\ \cos\left(\frac{3t}{10}\right) \end{bmatrix} + c_3 e^{-\frac{3}{5}t} \begin{bmatrix} -\frac{1}{2}\sin\left(\frac{3t}{10}\right) + \frac{3}{2}\cos\left(\frac{3t}{10}\right) \\ \sin\left(\frac{3t}{10}\right) - \frac{3}{2}\left(\frac{3t}{10}\right) \\ \sin\left(\frac{3t}{10}\right) \end{bmatrix}$$

Question 4

(a) Solve the equation

$$y' + \frac{2}{x}\, y = \frac{\cos x}{x^2}$$

with IC $y(\pi) = 0$. Solve via integration factors method. For an ODE of the form

$$y' + a\, y = b$$

the solution is

$$y = e^{-\int a(x)\,dx} \left[\int b\, e^{\int a(x)\,dx}\, dx + c \right]$$

In our case $a = \frac{2}{x}$ and $b = \frac{\cos x}{x^2}$.

$$e^{-\int a dx} = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$$

$$y = \frac{1}{x^2}\left[\int \frac{\cos x}{x^2} \cdot x^2\, dx + c \right] = \frac{1}{x^2}\left[\sin x + c\right] = \frac{\sin x}{x^2} + \frac{c}{x^2}$$

Find c using IC:

$$y(\pi) = 0 + \frac{c}{\pi^2} = 0 \rightarrow c = 0$$

Unique solution is

$$y = \frac{\sin x}{x^2}$$

(b) Solve the equation

$$y' = \frac{2x}{y(x^2+1)}$$

with IC $y(0) = -2$. Solve via separation of variables:

$$\frac{1}{2}y^2 = \int y\, dy = \int \frac{2x}{x^2+1}\, dx = \ln\left(x^2+1\right) + c$$

$$y = \pm\sqrt{2\ln\left(x^2+1\right) + 2c}$$

Input IC:

$$y(0) = \pm\sqrt{2\cdot 0 + 2c} = -2 \to c = 2$$

Because the initial condition contains a negative y value, the unique solution must be:

$$y = -\sqrt{2\ln(x^2+1)+4}$$

Question 5

Solve the folowing boundary value problem:

$$4y_{xx} = y_{tt}, \qquad x \in (0, \pi), t > 0$$

BCs: $y(0,t) = y(\pi,t) = 0$ and ICs: $y(x,0) = y_t(x,0) = \frac{1}{10}\sin 2x$, $x \in (0,\pi)$.

This is the wave equation with $a = 2$ and $L = \pi$, with homogeneous BCs. Solve the BVP as if once the initial position is zero and once as if the initial velocity is zero.

Zero initial velocity: ICs: $\begin{cases} y^{(1)}(x,0) \equiv f(x) = \frac{1}{10}\sin 2x \\ y_t^{(1)}(x,0) \equiv 0 \end{cases}$

Solution is given by the Fourier series:

$$y^{(1)}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

where c_n is given by:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$\begin{aligned} c_n &= \frac{2}{\pi}\cdot\frac{1}{10}\int_0^{\pi}\sin 2x\cdot\sin(n\,x)\,\mathrm{d}x \\ &= \frac{1}{10\pi}\int_0^{\pi}[\cos[(2-n)x] - \cos[(2+n)x]]\,\mathrm{d}x \\ &= \frac{1}{10\pi}\left[\frac{\sin[(2-n)x]}{2-n} - \frac{\sin(2+n)x}{2+n}\right]_0^{\pi} \\ &= 0 \end{aligned}$$

Here we assumed $n \neq 2$. If $n = 2$ then

$$c_2 = \frac{1}{5\pi}\int_0^{\pi}\sin^2 2x\,\mathrm{d}x = -\frac{1}{10\pi}\int_0^{\pi}(2\cos^2 2x - 2)\,\mathrm{d}x = -\frac{1}{10\pi}\int_0^{\pi}(\cos 4x - 1)\,\mathrm{d}x$$

$$c_2 = -\frac{1}{10\pi}\left[\frac{\sin 4x}{4} - x\right]_0^{\pi} = \frac{1}{10}$$

Therefore, solution is

$$y^{(1)}(x,t) = \frac{1}{10}\sin 2x \cos 4t$$

Zero initial position: ICs: $\begin{cases} y^{(2)}(x,0) \equiv 0 \\ y_t^{(2)}(x,0) \equiv g(x) = \frac{1}{10}\sin 2x \end{cases}$

Solution is given as a Fourier series:

$$y^{(2)}(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \sin\frac{n\pi a t}{L}$$

where

$$k_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

We've already calculated the integral $\int_0^{\pi}\frac{1}{10}\sin 2x \sin\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{10}\delta(n-2)$.

$$\Rightarrow k_2 = \frac{1}{2\pi}\cdot\frac{1}{10} = \frac{1}{20\pi}$$

Therefore,

$$y^{(2)}(x,t) = \frac{1}{20\pi}\sin 2x \sin 4t$$

General solution is $y = y^{(1)} + y^{(2)}$:

$$y(x,t) = \frac{1}{10}\sin 2x \cos 4t + \frac{1}{20\pi}\sin 2x \sin 4t$$

Question 6

Given a rod of length $L = \pi$ with thermal diffusivity constant $\alpha^2 = 9$, find the temperature $u(x,t)$ at point x and time t along the rod if the temperature at time $t = 0$ is 30 for all $x \in [0,\pi]$, and the temperature at the endpoints is held constant so that: $u(0,t) = 20$ and $u(\pi,t) = 40$ for all t.

Transform the BVP into a homogeneous one by defining

$$w(x,t) = u(x,t) - v(x)$$

such that $(0,20),(\pi,40)$

$$v(x) = \frac{20}{\pi}x + 20$$

and

$$w(x,0) = u(x,0) - v(x) = 30 - v(x) = 10 - \frac{20}{\pi}x$$

Solution is given as a Fourier series:

$$w(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\,\pi\,x}{L}\,\mathrm{e}^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$

where

$$c_n=\frac{2}{L}\int_0^L w(x,0)\sin\frac{n\,\pi\,x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n=\frac{2}{\pi}\int_0^{\pi}\left[10\sin(n\,x)\,\mathrm{d}x-\frac{20}{\pi}x\sin(n\,x)\right]\mathrm{d}x$$

Calculate via integration by parts:

$$\int x\sin(n\,x)=-\frac{\cos(n\,x)}{n}x+\frac{\sin(n\,x)}{n^2}$$

Therefore:

$$c_n=-\frac{20}{n\,\pi}\left[\cos(n\,x)\right]_0^{\pi}-\frac{40}{\pi^2}\left[-\frac{\cos(n\,x)}{n}x+\frac{\sin(n\,x)}{n^2}\right]_0^{\pi}$$

$$c_n=\frac{20}{n\,\pi}\left[1-(-1)^n\right]+\frac{40}{n\,\pi^2}\left[(-1)^n\,\pi\right]=\frac{20}{n\,\pi}\left[1+(-1)^n\right]$$

So:

$$u(x,t)=w(x,t)+v(x)$$

$$u(x,t)=\frac{20}{\pi}x+20+\sum_{n=1}^{\infty}\frac{20}{n\,\pi}\left[1+(-1)^n\right]\sin(n\,x)\,\mathrm{e}^{-9n^2t}$$