Linear Algebra for Chemists — Assignment 6

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Question 1. Find bases where A is

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

a) The column space of A. Notice that the first two columns are independent and that the 3rd one is identical to the first. A basis for the column space is therefore

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

b) The row space of A. Reduce A to echelon form

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[]{\substack{R_4 \to R_4 - 3R_3 + R_2 \\ R_1 \to R_1 - 2R_3 - 2R_2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rowspace of A is spanned by the following independent vectors:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the dimension of the row space is $r = 2$. As the spanning set contains two vectors, the set is a basis.

c) The nullspace of A. Let $x = (x_1, x_2, x_3)$ be a vector such that $A\,x = 0$. From the reduced form of A, we see that

$$\begin{aligned} x_2 &= 0 \\ x_3 &= -x_1 \,. \end{aligned}$$

The solution to $A\,x = 0$ is a vector of the form $x = (x_1, 0, -x_1) = x_1\,(1, 0, -1)$, $x_1 \in \mathbb{F}$.

A basis for the nullspace of A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

d) The nullspace of A^T.

$$A^T = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 2 & 0 & 1 & 1 \end{bmatrix}.$$

Row reduce A^T.

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 2 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $x = (x_1, x_2, x_3, x_4)$ such that $A^T x = 0$. We have

$$\begin{aligned} 2x_1 + x_3 + x_4 &= 0 \\ x_2 - x_3 - 2x_4 &= 0, \end{aligned}$$

where $x_3, x_4 \in \mathbb{F}$. A general solution to $A^T x = 0$ is $x = \left(-\frac{x_3 + x_4}{2}, x_3 + 2x_4, x_3, x_4\right),\ x_3, x_4 \in \mathbb{F}$. A basis for the nullspace of A^T is therefore

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the dimension of the nullspace of A^T is $m - r = 2$. As we found a spannig set containing two vectors, the set is a basis.

Question 2. Given the set of vectors

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

a) A matrix whose row space equals the span of the vectors.

The row space of a matrix is the space which is spanned by its rows. Therefore, the matrix whose rows are the vectors in S has its row space spanned by S.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix}.$$

b) A matirx whose nullspace equals the span of the vectors.

First find a basis for S. Row reduce the spanning set of S

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}.$$

Finding a homogeneous system of equations that S its is solution space corresponds to finding the conditions to lie in S. Write a general in $\mathbb{F}^3$ and write it as the last row of a matrix with the basis vectors of S and perform elimination.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ x & y & z \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & y-x & z \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & z+3y-3x \end{bmatrix}.$$

The last row becomes full of zeros if and only if

$$-3x+3y+z=0.$$

An example of a matrix whose nullspace equals the span of the vectors is

$$A=\begin{bmatrix} -3 & 3 & 1 \\ 1 & -1 & -\frac{1}{3} \\ -6 & 6 & 2 \end{bmatrix}.$$

Question 3. Find a basis for the solution space of

a)

$$\begin{cases} x-3y+z=0 \\ -2x+2y-3z=0 \\ 4x-8y+5z=0 \end{cases}$$

Row reduce the coefficient matrix.

$$\begin{bmatrix} 1 & -3 & 1 \\ -2 & 2 & -3 \\ 4 & -8 & 5 \end{bmatrix} \xrightarrow[R_3\to R_3-4R_1]{R_2\to R_2+2R_1} \begin{bmatrix} 1 & -3 & 1 \\ 0 & -4 & -1 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1\to R_1+R_2]{R_3\to R_3+R_2} \begin{bmatrix} 1 & -7 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2\to -R_2} \begin{bmatrix} 1 & -7 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution space is given by $(7a, a, -4a)$, $a\in\mathbb{F}$. A basis for the solution space is

$$\left\{\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix}\right\}.$$

b)

$$\begin{cases} x-y-z=0 \\ 2x-y+z=0 \end{cases}$$

Row reduce the coefficient matrix.

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow{R_2\to R_2-2R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1\to R_1+R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

The solution space is given by $(-2a, -3a, a)$, $a \in \mathbb{F}$. A basis for the solution space is

$$\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Question 4. Find a basis for the nullspace of the matrix of coefficients of the following, and find a particular solution.

a) Solving

$$\begin{bmatrix} 0 & 1 & -1 & 1 \\ 2 & 0 & 2 & -3 \\ -1 & 4 & -1 & 0 \\ 1 & -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ u \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

Row reduce the augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 7 \\ 2 & 0 & 2 & -3 & -1 \\ -1 & 4 & -1 & 0 & 0 \\ 1 & -2 & 0 & 4 & 2 \end{array}\right] \xrightarrow[R_2 \to R_2 + 2R_3]{R_4 \to R_4 + R_3} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 7 \\ 0 & 8 & 0 & -3 & -1 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & 2 & -1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow -R_3} \left[\begin{array}{cccc|c} 1 & -4 & 1 & 0 & 0 \\ 0 & 8 & 0 & -3 & -1 \\ 0 & 1 & -1 & 1 & 7 \\ 0 & 2 & -1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_4} \left[\begin{array}{cccc|c} 1 & -4 & 1 & 0 & 0 \\ 0 & 8 & 0 & -3 & -1 \\ 0 & -1 & 0 & -3 & 5 \\ 0 & 2 & -1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{cccc|c} 1 & -4 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 & -6 \\ 0 & -1 & 0 & -3 & 5 \\ 0 & 2 & -1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2/9} \left[\begin{array}{cccc|c} 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & -1 & 0 & -3 & 5 \\ 0 & 2 & -1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow[\substack{R_4 \to R_4 - 2R_2 \\ R_1 \to R_1 + 4R_2}]{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -\frac{8}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & -3 & \frac{13}{3} \\ 0 & 0 & -1 & 4 & \frac{10}{3} \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -\frac{8}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{13}{9} \\ 0 & 0 & -1 & 4 & \frac{10}{3} \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 4R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -\frac{8}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{13}{9} \\ 0 & 0 & -1 & 0 & \frac{82}{9} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{58}{9} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{13}{9} \\ 0 & 0 & -1 & 0 & \frac{82}{9} \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow -R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{58}{9} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & -\frac{82}{9} \\ 0 & 0 & 0 & 1 & -\frac{13}{9} \end{array}\right]$$

The nullspace of the coefficient matrix is empty, and a particular solution to the system is $(x,u,y,z) = \left(\frac{58}{9}, -\frac{2}{3}, -\frac{82}{9}, -\frac{13}{9}\right)$.

b) Solving

$$\begin{bmatrix} 2 & 3 & -1 \\ -1 & 2 & 3 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}.$$

Row reduce the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ -1 & 2 & 3 & 0 \\ 4 & -1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 + 4R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 0 & 7 & 5 & 5 \\ -1 & 2 & 3 & 0 \\ 0 & 7 & 13 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 0 & 7 & 5 & 5 \\ -1 & 2 & 3 & 0 \\ 0 & 0 & 8 & -6 \end{array}\right]$$

$$\xrightarrow[R_2 \to -R_2]{R_3 \to R_3/8} \left[\begin{array}{ccc|c} 0 & 7 & 5 & 5 \\ 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 5R_3]{R_2 \to R_2 + 3R_3} \left[\begin{array}{ccc|c} 0 & 7 & 0 & \frac{35}{4} \\ 1 & -2 & 0 & -\frac{9}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1/7} \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{5}{4} \\ 1 & -2 & 0 & -\frac{9}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{5}{4} \\ 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right].$$

Again, the nullspace of the coefficient matrix is empty. A particular solution to the system of equations is $(x, y, z) = \left(\frac{1}{4}, \frac{5}{4}, -\frac{3}{4}\right)$