Linear Algebra for Chemists — Assignment 11

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Question 1. Let $v=(x_1,x_2), w=(y_1,y_2)$ and define $\langle v,w\rangle \equiv x_1\,y_1+3\,x_2\,y_2$. Show that the following 5 properties hold:

1. $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ for all $u,v,w\in\mathbb{R}^2$. (Denote $u=(z_1,z_2)$).

$$\begin{aligned}\langle u+v,w\rangle &= \langle (x_1+z_1,x_2+z_2),(y_1,y_2)\rangle\\ &= (x_1+z_1)\,y_1+3\,(x_2+z_2)\,y_2\\ &= x_1\,y_1+3\,x_2\,y_2+z_1\,y_1+3\,z_2\,y_2\\ &= \langle v,w\rangle+\langle u,w\rangle\,.\end{aligned}$$

2. $\langle \alpha\,v,w\rangle=\alpha\,\langle v,w\rangle$ for all $v,w\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$.

$$\begin{aligned}\langle \alpha\,v,w\rangle &= \langle (\alpha\,x_1,\alpha\,x_2),(y_1,y_2)\rangle\\ &= \alpha\,x_1\,y_1+3\,\alpha\,x_2\,y_2\\ &= \alpha\,(x_1\,y_1+3\,x_2\,y_2)\\ &= \alpha\,\langle v,w\rangle\,.\end{aligned}$$

3. $\langle w,v\rangle=\overline{\langle v,w\rangle}$ for $v,w\in\mathbb{R}^2$.

   For all $\alpha\in\mathbb{R}$, $\alpha=\bar{\alpha}$.

$$\begin{aligned}\overline{\langle v,w\rangle} &= \overline{x_1\,y_1+3\,x_2\,y_2}\\ &= \overline{x_1\,y_1}+3\,\overline{x_2\,y_2}\\ &= x_1\,y_1+3\,x_2\,y_2\\ &= \langle v,w\rangle\,.\end{aligned}$$

$$\begin{aligned}\langle w,v\rangle &= y_1\,x_1+3\,y_2\,x_2\\ &= x_1\,y_1+3\,x_2\,y_2\\ &= \langle v,w\rangle\,.\end{aligned}$$

   By transitivity, $\langle w,v\rangle=\overline{\langle v,w\rangle}$.

4. $\langle v,v\rangle\geq 0$ for all $v\in\mathbb{R}^2$

$$\begin{aligned}\langle v,v\rangle &= x_1\,x_1+3\,x_2\,x_2\\ &= x_1^2+3\,x_2^2\,.\end{aligned}$$

   For any $\alpha\in\mathbb{R}$, $\alpha^2\in\mathbb{R}$ is non-negative, and the sum of two non-negative (real) numbers is non-negative. Therefore $\langle v,v\rangle\geq 0$.

5. $\langle v,v\rangle=0$ iff $v=(0,0)$.

The sum of two non-negative numbers is zero iff they are both zero, so x_1, x_2 must be zero.

Question 2. Given $f(x), g(x)$ two real integrable functions defined on $[0, 1]$,

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, \mathrm{d}x.$$

Show that the 5 properties hold. Again, show that the 5 properties of a vector space hold.

1.

$$\begin{aligned} \langle f + h, g \rangle &= \int_0^1 (f(x) + h(x))\, g(x)\, \mathrm{d}x \\ &= \int_0^1 (f(x)\, g(x) + h(x)\, g(x))\, \mathrm{d}x \\ &= \int_0^1 f(x)\, g(x)\, \mathrm{d}x + \int_0^1 h(x)\, g(x)\, \mathrm{d}x \\ &= \langle f, g \rangle + \langle h, g \rangle \quad . \end{aligned}$$

2.

$$\begin{aligned} \langle \alpha f, g \rangle &= \int_0^1 \alpha f(x)\, g(x)\, \mathrm{d}x \\ &= \alpha \int_0^1 f(x)\, g(x)\, \mathrm{d}x \\ &= \alpha \langle f, g \rangle. \end{aligned}$$

3. Note that for real functions $\bar{f}(x) = f(x)$.

$$\begin{aligned} \overline{\langle f, g \rangle} &= \int_0^1 \bar{f}(x)\, \bar{g}(x)\, \mathrm{d}x = \int_0^1 f(x)\, g(x)\, \mathrm{d}x \\ &= \int_0^1 g(x)\, f(x)\, \mathrm{d}x \\ &= \langle g, f \rangle. \end{aligned}$$

4. $\langle f, f \rangle = \int_0^1 f(x)\, f(x)\, \mathrm{d}x = \int_0^1 f^2(x)\, \mathrm{d}x$. The area between the x-axis and a squared function is always non-negative.

5. For the integral to be zero, the area between the x-axis and the function must also be zero. This is possible iff the function completely lies on the x-axis, that is, $f(x) = 0$.

Question 3. Given an inner product space V over $\mathbb{R}$ and $v_1, \ldots, v_n$ a basis for V, then an orthonormal basis $u_1, \ldots, u_n$ for V exists, such that $\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$ for all $k = 1, \ldots, n$, where $u_1, \ldots, u_k$ are given as follows:

$$\text{Define } \psi_1 = v_1, \quad \text{then} \quad u_1 = \frac{\psi_1}{\|\psi_1\|}.$$

$$\psi_k = v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle\, u_j, \quad \text{then} \quad u_k = \frac{\psi_k}{\|\psi_k\|}.$$

For $V = \mathbb{R}_2[x]$ with the inner product space defined on $[0,1]$, a basis for V is $v_1 = 1, v_2 = x, v_3 = x^2$. Perform the Gram-Schmidt process to get an orthonormal basis for V.

$$\begin{aligned}
\psi_1 &= v_1 = 1 \\
\|\psi_1\| &= \sqrt{\int_0^1 1^2 \,\mathrm{d}x} = \sqrt{[x]_0^1} = 1 \\
u_1 &= \frac{\psi_1}{\|\psi_1\|} = 1.
\end{aligned}$$

$$\begin{aligned}
\psi_2 &= v_2 - \langle v_2, u_1 \rangle u_1 \\
&= x - \left(\int_0^1 x \cdot 1 \,\mathrm{d}x \right) \cdot 1 \\
&= x - \left[\frac{1}{2} x^2 \right]_0^1 \\
&= x - \frac{1}{2} \\
\|\psi_2\| &= \sqrt{\int_0^1 \left(x - \frac{1}{2} \right)^2 \mathrm{d}x} = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{4} \right) \mathrm{d}x} \\
&= \sqrt{\left[\frac{1}{3} x^3 - \frac{1}{2} x^2 + \frac{1}{4} x \right]_0^1} = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \frac{1}{\sqrt{12}} \\
u_2 &= \frac{\psi_2}{\|\psi_2\|} = \sqrt{12} \left(x - \frac{1}{2} \right).
\end{aligned}$$

$$\begin{aligned}
\psi_3 &= v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2 \\
&= x^2 - \int_0^1 x^2 \,\mathrm{d}x - \left[\sqrt{12} \int_0^1 x^2 \left(x - \frac{1}{2} \right) \mathrm{d}x \right] \sqrt{12} \left(x - \frac{1}{2} \right) \\
&= x^2 - \left[\frac{1}{3} x^3 \right]_0^1 - 12 \left[\frac{1}{4} x^4 - \frac{1}{6} x^3 \right]_0^1 \left(x - \frac{1}{2} \right) \\
&= x^2 - \frac{1}{3} - 12 \left(\frac{1}{4} - \frac{1}{6} \right) \left(x - \frac{1}{2} \right) = x^2 - x + \frac{1}{6}. \\
\|\psi_3\| &= \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6} \right) \left(x^2 - x + \frac{1}{6} \right) \mathrm{d}x} \\
&= \sqrt{\int_0^1 \left(x^4 - x^3 + \frac{1}{6} x^2 - x^3 + x^2 - \frac{1}{6} x + \frac{1}{6} x^2 - \frac{1}{6} x + \frac{1}{36} \right) \mathrm{d}x} \\
&= \sqrt{\int_0^1 \left(x^4 - 2 x^3 + \frac{4}{3} x^2 - \frac{1}{3} x + \frac{1}{36} \right)} \\
&= \sqrt{\left[\frac{1}{5} x^5 - \frac{1}{2} x^4 + \frac{4}{9} x^3 - \frac{1}{6} x^2 + \frac{1}{36} x \right]_0^1} = \sqrt{\frac{1}{180}}. \\
u_3 &= \frac{\psi_3}{\|\psi_3\|} = \sqrt{180} \left(x^2 - x + \frac{1}{6} \right).
\end{aligned}$$

An orthonormal basis for V is

$$\left\{ 1, \sqrt{12} \left(x - \frac{1}{2} \right), \sqrt{180} \left(x^2 - x + \frac{1}{6} \right) \right\}.$$

Question 4.

a) Find an orthogonal complement in $\mathbb{R}^3$ for the subspace

$$U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

Well,

$$U^{\perp} = \{ v \in \mathbb{R}^3 | \langle v, u \rangle = 0, \forall u \in U \}.$$

Given a general vector $v = [x, y, z]^T$ in $\mathbb{R}^3$, we demand

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = 0,$$

that is,

$$x + 2\,y + 3\,z = 0 \quad \Longrightarrow \quad x = -2\,y - 3\,z.$$

A general vector in $U^{\perp}$ has the form $[-2\,y - 3\,z, y, z]$ for $y, z \in \mathbb{R}$. A basis for $U^{\perp}$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

b) Write a decomposition of the vector $[-1, 3, 0]^T$ as a sum of a vector in U and a vector in $U^{\perp}$.

We need to find the coefficients $a, b, c \in \mathbb{R}$ such that

$$\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

This translates to solving the system

$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}.$$

Solve the augmented system.

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 5 & 6 & 5 \\ 0 & 6 & 10 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2/5} \left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & \frac{6}{5} & 1 \\ 0 & 6 & 10 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 6R_2]{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{5} & 1 \\ 0 & 1 & \frac{6}{5} & 1 \\ 0 & 0 & \frac{14}{5} & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 \cdot \frac{5}{14}} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{5} & 1 \\ 0 & 1 & \frac{6}{5} & 1 \\ 0 & 0 & 1 & -\frac{15}{14} \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + \frac{3}{5}R_3]{R_2 \to R_2 - \frac{6}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{14} \\ 0 & 1 & 0 & \frac{16}{7} \\ 0 & 0 & 1 & -\frac{15}{14} \end{array}\right] \Longrightarrow a, b, c = \frac{5}{14}, \frac{16}{7}, -\frac{15}{14}.$$

The vector in U is $a\,[1,2,3]^T$ and the vector in $U^{\perp}$ is $b\,[-2,1,0]^T + c\,[-3,0,1]^T$.

$$\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \left(\frac{5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) + \left(\frac{16}{7}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{15}{14}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{5}{14} \\ \frac{5}{7} \\ \frac{15}{14} \end{bmatrix}}_{\in U} + \underbrace{\begin{bmatrix} -\frac{19}{14} \\ \frac{16}{7} \\ -\frac{15}{14} \end{bmatrix}}_{\in U^{\perp}}$$