2010 Exam solution

Question 1

Given a closed system of 2 tanks containing a salt solution with constant flow between them of 10 L/min, find the amount of salt in each tank at time t if the first tank constains 30L, the second 50L, and the initial amount of salt in tank 1 is 10g and in tank 2 15g.

Write system of DEs in matrix form:

$$\vec{x}' = A\,\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{5} \end{bmatrix}$$

Find eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{3} - \lambda & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{5} - \lambda \end{vmatrix} = \lambda^2 + \frac{8}{15}\lambda = \lambda\left(\lambda + \frac{8}{15}\right)$$

Roots are $\lambda_{1,2} = 0, -\frac{8}{15}$. Find eigenvectors. For $\lambda = 0$:

$$A - \lambda_1 I = \begin{bmatrix} -\frac{1}{3} & \frac{1}{5} \\ \frac{1}{3} & -\frac{1}{5} \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{3} \end{bmatrix}$$

such that $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

For $\lambda = -\frac{8}{15}$:

$$A - \lambda_2 I = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Pick

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

such that $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$.

General solution is

$$\vec{x} = c_1 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{3} \end{bmatrix} + c_2\, \mathrm{e}^{-\frac{8}{15}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Find c_1, c_2 that satisfy ICs.

$$\vec{x}(0) = \begin{bmatrix} 10 \\ 15 \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{3} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 150 \\ 225 \end{bmatrix} = \begin{bmatrix} 3c_1 + 15c_2 \\ 5c_1 - 15c_2 \end{bmatrix}$$

$$c_1 = \frac{375}{8}, c_2 = \frac{5}{8}$$

Unique solution is

$$\vec{x} = \frac{375}{8} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{3} \end{bmatrix} + \frac{5}{8} \mathrm{e}^{-\frac{8}{15}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question 2

Solve the DE

$$(1+t^2)\, y'' + 2t\, y' + \frac{3}{t^2} = 0$$

with ICs $y(1) = 2, y'(1) = -1$. Note: Use

$$\frac{A + B\, t}{t\,(t^2+1)} = \frac{1}{t} + \frac{B - A\, t}{t^2+1}$$

This is a separable equation in y':

$$((1+t^2)\, y')' = -\frac{3}{t^2}$$

Integrate both sides:

$$(1+t^2)\, y' = \frac{3}{t} + c_1$$

Therefore:

$$y = \int \left(\frac{3}{t(1+t^2)} + \frac{c_1}{1+t^2} \right) \mathrm{d}t$$

Simplify:

$$\frac{3}{t(1+t^2)} = \frac{A}{t} + \frac{B - A\, t}{t^2+1}$$

$$A(t^2+1) + B\, t - A\, t^2 = 3$$

$$A = 3, B = 0$$

$$y = \int \left(\frac{3}{t} - \frac{3t}{t^2+1} + \frac{c_1}{t^2+1} \right) \mathrm{d}t = 3 \ln |t| - \frac{3}{2} \ln |t^2+1| + c_1 \arctan t + c_2$$

Find c_1, c_2 via ICs.

$$y'(1) = -1 \rightarrow (1+1) \cdot (-1) = \frac{3}{1} + c_1 \rightarrow c_1 = -5$$

$$y(1) = 2 \rightarrow 2 = -\frac{3}{2} \ln 2 + -5 \arctan 1 + c_2 \rightarrow c_2 = 2 + \frac{3}{2} \ln 2 + 5 \arctan 1$$

Unique solution is:

$$y = 3\ln t - \frac{3}{2}\ln(t^2+1) - 5\arctan t + 2 + \frac{3}{2}\ln 2 + 5\arctan 1$$

Question 3

Find the general solution to the DE:

$$2y^{(3)} - 2y'' + 25y' = \sin 2x + x$$

Solve associated homogeneous equation. Characteristic equation is

$$2\lambda^3 - 2\lambda^2 + 25\lambda = 0$$

$$\lambda(2\lambda^2 - 2\lambda + 25) = 0$$

Roots are

$$\lambda_{1,2,3} = 0, \frac{2 \pm 14\mathrm{i}}{4} = \frac{1}{2} \pm \frac{7}{2}\mathrm{i}$$

Therefore,

$$y_h = c_1 + c_2\,\mathrm{e}^{\frac{1}{2}x}\cos\frac{7x}{2} + c_3\,\mathrm{e}^{\frac{1}{2}x}\sin\frac{7x}{2}$$

Find particular solutions for each part of the RHS. For sine part, guess solution of the form: $y_{p1} = A\cos 2x + B\sin 2x$.

$$\begin{aligned} y'_{p1} &= -2A\sin 2x + 2B\cos 2x \\ y''_{p1} &= -4A\cos 2x - 4B\sin 2x \\ y^{(3)}_{p1} &= 8A\sin 2x - 8B\cos 2x \end{aligned}$$

Input in ODE:

$$16A\sin 2x - 16B\cos 2x + 8A\cos 2x + 8B\sin 2x - 50A\sin 2x + 50B\cos 2x = \sin 2x$$

$$\sin 2x\,(16A + 8B - 50A) + \cos 2x\,(-16B + 8A + 50B) = \sin 2x$$

Equate coefficients on both sides:

$$\begin{cases} \cos 2x\text{:} \quad 34B = 8A \to A = \frac{34}{8}B \\ \sin 2x\text{:} \quad -34A + 8B = 1 \to B = -\frac{2}{273}, A = -\frac{17}{546} \end{cases}$$

So:

$$y_{p1} = -\frac{17}{546}\cos 2x - \frac{2}{273}\sin 2x$$

For polynomial part, guess solution of the form: $y_{p2} = A\,x^3 + B\,x^2 + C\,x$.

$$\begin{aligned} y'_{p2} &= 3A\,x^2 + 2B\,x + C \\ y''_{p2} &= 6A\,x + 2B \\ y^{(3)}_{p2} &= 6A \end{aligned}$$

Input in ODE:

$$12A - 12A\,x - 4B + 75A\,x^2 + 50B\,x + 25C = x$$

$$75A\,x^2 + (-12A + 50B)x + (12A - 4B + 25C) = x$$

Equate coefficients on both sides:

$$A = 0, B = \frac{1}{50}, C = \frac{2}{625}$$

So:

$$y_{p2} = \frac{1}{50}x^2 + \frac{2}{625}x$$

General solution to the DE is:

$$c_1 + c_2\,\mathrm{e}^{\frac{1}{2}x}\cos\frac{7x}{2} + c_3\,\mathrm{e}^{\frac{1}{2}x}\sin\frac{7x}{2} - \frac{17}{546}\cos 2x - \frac{2}{273}\sin 2x + \frac{1}{50}x^2 + \frac{2}{625}x$$

where $c_{1,2,3} \in \mathbb{R}$, $t > 0$.

Question 4

Find the solution to the following system of DEs with ICs: $\vec{x}(0) = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

$$\vec{x}' = A\,\vec{x}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Notice that the sum along each row is the same and equals 6. Therefore, A has an eigenvalue $\lambda = 6$ with corresponding eigenvector $\vec{v} = [1, 1, 1]^T$.

Find the rest of the eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)\,[15 - 8\lambda + \lambda^2] - [5 - 2\lambda] + [-10 + 3\lambda] = 0$$

$$15 - 8\lambda + \lambda^2 - 15\lambda + 8\lambda^2 - \lambda^3 - 5 + 2\lambda - 10 + 3\lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 18\lambda = -\lambda(\lambda - 6)(\lambda - 3) = 0$$

$$\lambda_{1,2,3} = 0, 6, 3$$

Find eigenvectors. For $\lambda_1 = 0$:

$$A - \lambda_1 I = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Pick

$$\vec{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

s.t. $(A-\lambda_1 I)\vec{v}_1=\vec{0}$. For $\lambda_3=3$:

$$A-\lambda_3 I=\begin{bmatrix}-2 & 2 & 3\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\xrightarrow{R_1\to R_1+2R_2}\begin{bmatrix}0 & 4 & 5\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Pick:

$$\vec{v}_3=\begin{bmatrix}1\\ -5\\ 4\end{bmatrix}$$

s.t. $(A-\lambda_3 I)\vec{v}_3=0$. General solution to system of DEs is:

$$\vec{x}=c_1\begin{bmatrix}-5\\ 1\\ 1\end{bmatrix}+c_2\,\mathrm{e}^{6t}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+c_3\,\mathrm{e}^{3t}\begin{bmatrix}1\\ -5\\ 4\end{bmatrix}$$

Find c_1, c_2, c_3 that satisfy ICs. Define

$$\vec{c}=\begin{bmatrix}c_1\\ c_2\\ c_3\end{bmatrix}, B=\begin{bmatrix}-5 & 1 & 1\\ 1 & 1 & -5\\ 1 & 1 & 4\end{bmatrix}, \vec{b}=\begin{bmatrix}0\\ 3\\ 2\end{bmatrix}$$

such that

$$B\vec{c}=\vec{b}$$

Find B^{-1}.

$$\det B=-5\cdot(4+5)-1\cdot(4-1)+1\cdot(-5-1)=-5\cdot 9-3-6=-54$$

$$B^{-1}=-\frac{1}{54}\begin{bmatrix}(4+5) & -(4+5) & 0\\ -(4-1) & (-20-1) & -(-5-1)\\ (-5-1) & -(25-1) & (-5-1)\end{bmatrix}=-\frac{1}{54}\begin{bmatrix}9 & -9 & 0\\ -3 & -21 & 6\\ -6 & -24 & -6\end{bmatrix}=\begin{bmatrix}-\frac{1}{6} & \frac{1}{6} & 0\\ \frac{1}{18} & \frac{7}{18} & -\frac{1}{9}\\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9}\end{bmatrix}$$

Therefore,

$$\vec{c}=B^{-1}\vec{b}=\begin{bmatrix}-\frac{1}{6} & \frac{1}{6} & 0\\ \frac{1}{18} & \frac{7}{18} & -\frac{1}{9}\\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9}\end{bmatrix}\begin{bmatrix}0\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{17}{18}\\ \frac{14}{9}\end{bmatrix}$$

Unique solution is:

$$\vec{x}=\frac{1}{2}\begin{bmatrix}-5\\ 1\\ 1\end{bmatrix}+\frac{17}{18}\mathrm{e}^{6t}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\frac{14}{9}\mathrm{e}^{3t}\begin{bmatrix}1\\ -5\\ 4\end{bmatrix}$$

Question 5

Show that the eigenvalue problem

$$y''+\lambda y'+y=0$$

with boundary conditions $y(0)=y(1)=0$ has no real eigenvalues by considering the following cases:

1. Show that if $\lambda\in\mathbb{R}$ and $|\lambda|<2$ then λ is not an eigenvalue.
2. Show tha if $\lambda=\pm 2$ then λ is not an eigenvalue.
3. Show that if $\lambda\in\mathbb{R}$ and $|\lambda|>2$ then λ is not an eigenvalue.

Characteristic equation is:

$$u^2+\lambda\,u+1=0$$

$$u=\frac{-\lambda\pm\sqrt{\lambda^2-4}}{2}$$

Three cases to examine:

1. If $\lambda^2-4>0$ (i.e. $|\lambda|>2$), there are two real roots:

   $$u_{1,2}=\frac{-\lambda\pm\sqrt{\lambda^2-4}}{2}$$

   and the general solution is

   $$y=c_1\,\mathrm{e}^{\frac{-\lambda+\sqrt{\lambda^2-4}}{2}t}+c_2\,\mathrm{e}^{\frac{-\lambda-\sqrt{\lambda^2-4}}{2}t}$$

   Check if there are c_1,c_2 that satisfy ICs.

   $$\begin{aligned} y(0)=0 &= c_1+c_2 \\ y(1)=0 &= c_1\,\mathrm{e}^{\frac{-\lambda+\sqrt{\lambda^2-4}}{2}}+c_2\,\mathrm{e}^{\frac{-\lambda-\sqrt{\lambda^2-4}}{2}} \end{aligned}$$

   These two conditions hold iff $c_1=c_2=0$, i.e. there is no non-trivial solution and no eigenvalue in this case.

2. If $\lambda^2=4$ (i.e. $\lambda=\pm 2$), there is one double root ($\lambda=-2$ or $\lambda=2$), and the solution is:

   $$y_{\pm}=c_1\,\mathrm{e}^{\pm\lambda t}+c_2\,t\,\mathrm{e}^{\pm\lambda t}$$

   Check if there are c_1,c_2 that satisfy ICs.

   $$\begin{aligned} y(0)=0 &= c_1 \\ y(1)=0 &= c_1\,\mathrm{e}^{\pm 2}+c_2\,\mathrm{e}^{\pm 2}\rightarrow c_2=0 \end{aligned}$$

   Again, there is no non-trivial solution and no eigenvalue in this case.

3. If $\lambda^2-4<0$ (i.e. $|\lambda|<2$), there are two non-real roots:

   $$u_{1,2}=\frac{-\lambda\pm\sqrt{4-\lambda^2}\mathrm{i}}{2}$$

   and the general solution is:

   $$y=c_1\,\mathrm{e}^{-\frac{\lambda}{2}t}\cos\left(\frac{\sqrt{4-\lambda^2}}{2}t\right)+c_2\,\mathrm{e}^{-\frac{\lambda}{2}t}\sin\left(\frac{\sqrt{4-\lambda^2}}{2}t\right)$$

Check if there are c_1, c_2 that satisfy ICs.

$$\begin{aligned} y(0)=0 &= c_1 \\ y(1)=0 &= c_2\,\mathrm{e}^{-\frac{\lambda}{2}}\sin\left(\frac{\sqrt{4-\lambda^2}}{2}\right) \end{aligned}$$

Either $c_2=0$ (which means no eigevalue in this case), or

$$\sin\left(\frac{\sqrt{4-\lambda^2}}{2}\right)=0$$

$$\frac{\sqrt{4-\lambda^2}}{2}=\pi n, \quad n=1,2,3,\ldots$$

$$\lambda=\pm\sqrt{4-4\pi^2 n^2}$$

This doesn't yield a **real** eigenvalue.

In conclusion, the eigenvalue problem doesn't have any real eigenvalue.

Question 6

Solve the following BVP:

$$9y_{xx}=y_{tt}, \quad x\in(0,1), t>0$$

with homogeneous BCs and ICs: $y(x,0)=\sin\pi x$, $y_t(x,0)=3\sin(2\pi x)-4\sin(3\pi x)$.

Split the problem into a zero initial velocity and a zero initial position case.

Zero initial velocity, i.e. ICs are: $y(x,0)\equiv f(x)=\sin\pi x$, $y_t(x,0)\equiv 0$.

In this case, solution is given by:

$$y_1(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

where

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

Calculate c_n.

$$c_n=2\int_0^1 \sin(\pi x)\cdot\sin(n\pi x)\,\mathrm{d}x$$

As sines of different frequencies are orthogonal, integral is non-zero only for $n=1$.

$$c_1=2\int_0^1 \sin^2(\pi x)\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{\pi x}{2}-\frac{\sin(2\pi x)}{4}\right]_0^1=\frac{2}{\pi}\cdot\frac{\pi}{2}=1$$

Solution is therefore,

$$y_1(x,t)=\sin(\pi x)\cos(3\pi t)$$

Zero initial position, i.e. ICs are $y(x,0) \equiv 0, y_t(x,0) \equiv g(x) = 3\sin(2\pi x) - 4\sin(3\pi x)$.

In this case, solution is given by:

$$y_2(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \sin\frac{n\pi a t}{L}$$

where

$$k_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\frac{n\pi x}{L}\, \mathrm{d}x$$

Calculate k_n.

$$k_n = \frac{2}{3n\pi}\left[3\int_0^1 \sin(2\pi x)\sin(n\pi x)\,\mathrm{d}x - 4\int_0^1 \sin(3\pi x)\sin(n\pi x)\,\mathrm{d}x\right]$$

By means of orthogonality,

$$k_n = k_2 + k_3 = \frac{6}{6\pi}\int_0^1 \sin^2(2\pi x)\,\mathrm{d}x - \frac{8}{9\pi}\int_0^1 \sin^2(3\pi x)\,\mathrm{d}x$$

$$k_2 = \frac{1}{\pi}\cdot\frac{1}{2\pi}\left[\frac{2\pi x}{2} - \frac{\sin(4\pi x)}{4}\right]_0^1, \qquad k_3 = -\frac{8}{9\pi}\cdot\frac{1}{3\pi}\left[\frac{3\pi x}{2} - \frac{\sin(6\pi x)}{4}\right]_0^1$$

$$k_2 = \frac{1}{2\pi}, \quad k_3 = -\frac{4}{9\pi}$$

Therefore,

$$y_2(x,t) = \sin(\pi x)\cos(3\pi t) + \frac{1}{2\pi}\sin(2\pi x)\sin(6\pi t) - \frac{4}{9\pi}\sin(3\pi x)\sin(9\pi t)$$