$$S_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu_N)^2$$

N = k + n

$$\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu_N)^2 = \frac{1}{N-1}[\sum_{i=1}^{k}(x_i - \mu_N)^2 + \sum_{i=1}^{n}(x_i - \mu_N)^2]$$

$$= \frac{1}{N-1}[\sum_{i=1}^{k}(x_i - \mu_k + \mu_k - \mu_N)^2 + \sum_{i=1}^{n}(x_i - \mu_n + \mu_n - \mu_N)^2]$$

$$= \frac{1}{N-1}[\sum_{i=1}^{k}((x_i - \mu_k)^2 + 2(x_i - \mu_k)(\mu_k - \mu_N) + (\mu_k - \mu_N)\text{^}2) + \sum_{i=1}^{n}((x_i - \mu_n)^2 + 2(x_i - \mu_n)(\mu_n - \mu_N) + (\mu_n - \mu_N)\text{^}2)]$$

$$= \frac{1}{N-1}[\sum_{i=1}^{k}((x_i - \mu_k)^2 + (\mu_k - \mu_N)\text{^}2) + \sum_{i=1}^{n}((x_i - \mu_n)^2 + (\mu_n - \mu_N)\text{^}2)]$$

$$= \frac{1}{N-1}[\sum_{i=1}^{k}(x_i - \mu_k)^2 + k(\mu_k - \mu_N)\text{^}2 + \sum_{i=1}^{n}(x_i - \mu_n)^2 + n(\mu_n - \mu_N)\text{^}2]$$

$$= \frac{1}{N-1}[(k-1)S_k^2 + k(\mu_k - \mu_N)^2 + (n-1)S_n^2 + n(\mu_n - \mu_N)\text{^}2]$$

$$= \frac{1}{N-1}[(k-1)S_k^2 + k\left(\mu_k - (\frac{k\mu_k + n\mu_n}{N})\right)^2 + (n-1)S_n^2 + n(\mu_n - (\frac{k\mu_k + n\mu_n}{N}))\text{^}2]$$

$$= \frac{1}{N-1}[(k-1)S_k^2 + k\left(\frac{n(\mu_k - \mu_n)}{N}\right)^2 + (n-1)S_n^2 + n(\frac{k(\mu_n - \mu_k)}{N})\text{^}2]$$

$$= \frac{(k-1)S_k^2}{N-1} + \frac{(n-1)S_n^2}{N-1} + \frac{1}{N-1}[k\left(\frac{n(\mu_k - \mu_n)}{N}\right)^2 + n\left(\frac{k(\mu_n - \mu_k)}{N}\right)^2]$$

$$= \frac{(k-1)S_k^2}{N-1} + \frac{(n-1)S_n^2}{N-1} + \frac{1}{N-1}[\frac{kn^2(\mu_k - \mu_n)^2}{N^2} + \frac{nk^2(\mu_n - \mu_k)^2}{N^2}]$$

$$= \frac{(k-1)S_k^2}{N-1} + \frac{(n-1)S_n^2}{N-1} + \frac{1}{N-1}[\frac{kn(\mu_k - \mu_n)^2}{N^2} * (n+k)]$$

$$= \frac{(k-1)S_k^2}{N-1} + \frac{(n-1)S_n^2}{N-1} + \frac{1}{N-1}[\frac{kn(\mu_k - \mu_n)^2}{N^2} * N]$$

$$= \frac{(k-1)S_k^2}{N-1} + \frac{(n-1)S_n^2}{N-1} + \frac{1}{N-1}[\frac{kn(\mu_k - \mu_n)^2}{N}]$$

$$= \frac{\boldsymbol{(k-1)}}{\boldsymbol{N-1}}\boldsymbol{S_k^2} + \frac{\boldsymbol{(n-1)}}{\boldsymbol{N-1}}\boldsymbol{S_n^2} + \frac{\boldsymbol{nk}}{\boldsymbol{N(N-1)}} * \boldsymbol{(\mu_k - \mu_n)^2}$$