תרגיל 2

2

$f'_x = x^n \qquad f''_x = n \cdot x^{n-1} \quad \leftarrow \quad f = \frac{x^{n+1}}{n+1}$

$$\frac{W_t^{n+1}}{n+1} - \frac{W_0^{n+1}}{n+1} = \int_0^t W_s^n \, dW_s + \frac{1}{2} n \int_0^t W_s^{n-1} \, ds$$

$$\frac{W_t^{n+1} - W_0^{n+1}}{n+1} = \int_0^t W_s^n \, dW_s + \frac{1}{2} n \cdot t \, W_t^{n-1}$$

$$\boxed{\int_0^t W_s^n \, dW_s = \frac{W_t^{n+1} - W_0^{n+1}}{n+1} - \frac{1}{2} n t \, W_t^{n-1}}$$

$$E\Big(\underbrace{\int_0^t W_s^n \, dW_s}_{0=}\Big) = \frac{E(W_t^{n+1}) - W_0^{n+1}}{n+1} - \frac{1}{2} n \cdot t \, E(W_t^{n-1})$$

$$\boxed{E(W_t^{n+1}) = W_0^{n+1} + \frac{1}{2} n \cdot (n+1) \cdot t \cdot E(W_t^{n-1})}$$

$n=1$ *

$$E(W_t^2) = W_0^2 + t$$

$n=3$ *

$$E(W_t^4) = W_0^4 + 6 \cdot t \, E(W_t^2)$$

$$\boxed{E(W_t^4) = W_0^4 + 6t W_0^2 + 6t^2}$$

$n=5$ *

$$E(W_t^6) = W_0^6 + 15 \cdot t \cdot E(W_t^4)$$

$$\boxed{E(W_t^6) = W_0^6 + 15t W_0^4 + 90 t^2 W_0^2 + 90 t^3}$$

3

$$S_t = S_0 \cdot e^X$$

(א)

$$E(S_t) = S_0 \cdot E(e^X) \underset{\lambda}{=} S_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t + \frac{\sigma^2 t}{2}} = \underline{\underline{S_0 e^{rt}}}$$

$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

[illegible] $t=1$ [illegible]
$\mu = (r - \frac{1}{2}\sigma^2)t$
$\sigma^2 = \sigma^2 t$

(ב)

$$V(S_t) = E(S_t^2) - E^2(S_t) = S_0^2 E(e^{2X}) - S_0^2 e^{2rt} =$$

$$= S_0^2 \cdot e^{2t(r - \frac{1}{2}\sigma^2) + 2\sigma^2 t} - S_0^2 e^{2rt} = \underline{S_0^2 \left(e^{2tr + \cancel{[illegible]} \sigma^2 t} - e^{2rt}\right)} =$$

$t=2$ [illegible]
$\mu = (r - \frac{1}{2}\sigma^2)t$
$\sigma^2 = \sigma^2 t$

$$= \boxed{S_0^2 \, e^{2rt} \left(e^{\sigma^2 t} - 1\right)}$$

$$P(S_t \geq \mu) = P(S_0 e^{X} \geq \mu) = P(\cancel{S_0} e^{X} \geq \cancel{S_0} e^{rt}) = P(e^{X} \geq e^{rt}) = \quad ⓒ$$

$S_0 > 0$

$$\overset{\ln}{=} P(X \geq rt) = 1 - P(X \leq rt) = 1 - P\left(Z \leq \frac{rt - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) =$$

$$1 - P\left(Z \leq \frac{\cancel{rt - rt}}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) = 1 - \Phi\left(\frac{1}{2}\sigma\sqrt{t}\right)$$

$\boxed{1}$

$$y = f \cdot g$$

$$y_x = g \cdot f_x + f \cdot g_x$$

$$y_t = g \cdot f_t + f \cdot g_t$$

$$y_{xx} = g \cdot f_{xx} + 2 g_x \cdot f_x + f \cdot g_{xx}$$

$$d(y_t) = \left(y_t + a y_x + \tfrac{1}{2} b^2 y_{xx}\right)dt + b \cdot y_x \, dW_t$$

$$d(f \cdot g) = \left(g \cdot f_t + f \cdot g_t + a(g \cdot f_x + f \cdot g_x) + \tfrac{1}{2} b^2 (g \cdot f_{xx} + 2 f_x g_x + f \cdot g_{xx})\right)dt +$$

$$+ \; b \cdot (g \cdot f_x + f \cdot g_x) \, dW_t$$

$$g \cdot df = g \cdot \left(f_t + a \cdot f_x + \tfrac{1}{2} b^2 f_{xx}\right)dt + b \cdot g \cdot f_x \, dW_t$$

$$f \cdot dg = f \cdot \left(g_t + a \cdot g_x + \tfrac{1}{2} b^2 g_{xx}\right)dt + b \cdot f \cdot g_x \, dW_t$$

$$g \cdot df + f \cdot dg = \left(g \cdot f_t + f \cdot g_t + a(g \cdot f_x + f \cdot g_x) + \tfrac{1}{2} b^2 (g \cdot f_{xx} + f \cdot g_{xx})\right)dt +$$

$$+ \; b \cdot (g \cdot f_x + f \cdot g_x) \, dW_t$$

$$d(f \cdot g) = \overbrace{\left(g \cdot f_t + f \cdot g_t + a(g \cdot f_x + f \cdot g_x) + \tfrac{1}{2} b^2 (g \cdot f_{xx} + f \cdot g_{xx})\right)dt + b(g \cdot f_x + f \cdot g_x)\,dW_t}^{= g\,df + f\,dg}$$

$$+ \; b^2 f_x g_x \, dt$$

$$\boxed{d(f \cdot g) = g\,df + f\,dg + b^2 f_x g_x \, dt}$$