HANDOUT FOR OPEN BOOK EXAMINATION
FOR C-21 IV SEMESTER
SC-401-ADVANCED ENGINEERING MATHEMATICS
Unit – I & II

Differential Equations:

Homogeneous Linear Differential Equations with Constant Coefficients.

a) Homogeneous Differential Equations of Second order with Constant Coefficients.

Given a differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, where a, b, c are real numbers.

Or $(aD^2 + bD + c)\,y = 0$ ie., $f(D)y = 0$

We write the general solution as follows:

Roots	General Solution
1. Roots are real and distinct. Say m_1 and m_2	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
2. Roots are real and equal. Say $m_1 = m_2 = m$	$y=(c_1+c_2x)e^{mx}$
3. Roots are complex. Say $m_1 = \alpha+i\beta, m_2=\alpha-i\beta$	$y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

b) Homogeneous Linear Differential Equations of Higher order with Constant Coefficients.

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+.....+a_ny=0$$

Where a' s are constants.

We write the general solution as follows:

Roots	General Solution
1. Roots are real and different.	$y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+c_4e^{m_4x}+.....+c_ne^{m_nx}$
2. Roots are real and some of them are equal.	i. $m_1 = m_2$ $y=(c_1+c_2x)e^{mx}+c_3e^{m_3x}+c_4e^{m_4x}+.....+c_ne^{m_nx}$ ii. $m_1 = m_2 = m_3$

	$y=(c_1+c_2x+c_3x^2)e^{mx}+c_4e^{m_{14}x}+c_5e^{m_5x}+.....+c_ne^{m_nx}$
3. Some Roots are imaginary.	say $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)+c_3e^{m_3x}+c_4e^{m_4x}+.....+c_ne^{m_nx}$
4. If two pairs of imaginary roots are equal	say $m_1=m_2=\alpha+i\beta$ and $m_3=m_4=\alpha-i\beta$ $y=e^{\alpha x}[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x]c_5e^{m_5x}+....+c_ne^{m_nx}$
5. Some roots are irrational	say $m_1=\alpha+\sqrt{\beta}, m_2=\alpha-\sqrt{\beta}$ $y=e^{\alpha x}[(c_1\cosh\sqrt{\beta}x+c_2\sinh\sqrt{\beta}x)+c_3e^{m_3x}+....+c_ne^{m_nx}$

Non-Homogeneous Linear Differential Equations of nth order with Constant Coefficients.

$f(D)y=X$, where X = f(x)

Or

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+.....+a_ny=X$$

The solution of the equation $f(D)y=X$, is

y = complementary function + particular integral

y= C.F + P.I

Method of calculating C.F:

Consider f(D)y=0 and calculate Complimentary Function as given in solution of Homogeneous equation.

Calculation of particular integrals

Sl. no.	X=f(x)	P.I.	Observation
1.	X=K (constant)	$P.I.=\frac{k}{f(D)}=\frac{k}{f(0)}$, if f(0) $\neq 0$	Replace D by '0' in f(D)
2.	X=e^{ax}	$P.I.=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$, if f(a) $\neq 0$	Replace D by 'a' in f(D)
		Case of failure: i. If f(a)=0, then write $f(D)=(D-a)^m\phi(D)$, Where $\phi(a)\neq 0$ $\therefore P.I.=\frac{1}{f(D)}e^{ax}=\frac{1}{(D-a)^m\phi(D)}e^{ax}=\frac{x^m}{m!}.\frac{e^{ax}}{\phi(a)}, \phi(a)\neq 0$	Repeated factor 'm' times

		i. If f(a)=0, and f(D) = $(D-a)^n$ then
		$\therefore P.I. = \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$
		Another method:
		If f(a)=0 then
		$\therefore P.I. = \frac{1}{f(D)} e^{ax} = x.\frac{1}{f^i(a)} e^{ax}$ provided $f^i(a) \neq 0$
		If, $f^i(a) = 0$ then
		$\therefore P.I. = \frac{1}{f(D)} e^{ax} = x^2.\frac{1}{f^{ii}(a)} e^{ax}$,provided $f^{ii}(a) \neq 0$
			All 'n' roots are real and equal.

3.	X=Sin ax (or) X=cos ax	i. $P.I. = \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$, provided $f(-a^2) \neq 0$ ii. $P.I. = \frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$, provided $f(-a^2) \neq 0$	Replace D^2 by $-a^2$ in f (D)
		Case of failure: **i.** When $f(-a^2) = 0$ then $(a) \frac{\sin ax}{D^2 + a^2} = \frac{-x}{2a} \cos ax$ $(b) \frac{\cos ax}{D^2 + a^2} = \frac{x}{2a} \sin ax$	
4.	$X = x^m$	$P.I. = \frac{1}{f(D)} x^m$ Express f(D) in the form of $1 \pm \phi(D)$. Take it to the numerator so that it takes the form $[1 \pm \phi(D)]^{-1}$. Now expand it in the ascending powers of D using Binominal Theorem.	

Note : useful Binomial Expansions:

1. $(1-D)^{-1} = 1 + D + D^2 + D^3 + ...$
2. $(1+D)^{-1} = 1 - D + D^2 - D^3 + ...$
3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + ...$
4. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + ...$
5. $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + ...+ 15D^4$
6. $(1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + ...15D^4$

Unit – III & IV
FOURIER SERIES

Formulas:

1. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$
2. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$
3. $\sin n\pi = \sin 2n\pi = 0$
4. $cos\ n\pi = (-1)^n, cos\ 2n\pi = 1$
5. $sin\ (2n+1)\frac{\pi}{2} = (-1)^n$
6. $cos\ (2n+1)\frac{\pi}{2} = 0$
7. $sin\ \frac{n\pi}{2} = \begin{cases} (-1)^{\frac{n-1}{2}}, if\ n\ is\ odd \\ 0,\ if\ n\ is\ even \end{cases}$
8. $cos\ \frac{n\pi}{2} = \begin{cases} 0, if\ n\ is\ odd \\ (-1)^{\frac{n-1}{2}},\ if\ n\ is\ even \end{cases}$

   Where n € Z and Z is the set of all integers.

1. Fourier Series:

Suppose that a given function f(x) defined in an interval [c,c+2 π] of length 2 π , and which satisfies certain conditions can be expressed in the trigonometric series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Euler's Formulae:

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx$$

2. Suppose that a given function f(x) defined in an interval [-π, π] and of length 2 π
   Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Values of a_0, a_n and b_n in the interval (-π,π)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

3. Suppose that a given function f(x) defined in an interval [0, 2π] and of length 2 π
   Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then the Values of a_0, a_n and b_n in the interval (0, 2π)

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\, dx$$

<u>FOURIER SERIES OF EVEN AND ODD FUNCTIONS:</u>

1. Fourier series for Even Functions in (-π,π)
   i. If f(x) is an even function in (-π,π), then b_n=0

   $\therefore$ The Fourier series of an even function f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

2. Fourier series for Odd Functions in $(-\pi, \pi)$
   ii. If f(x) is an odd function in $(-\pi, \pi)$, then $a_n=0$

   $\therefore$ The Fourier series of an odd function f(x) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

Fourier Half-Range Series:

1. The half-range sine series expansion of f(x) in the interval $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

2. The half-range Cosine series expansion of f(x) in the internal $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

Laplace Transforms

Definition:

Let f(t) be a function of t defined for all real numbers $t \geq 0$. Then, the Laplace Transform of f(t) is denoted by L{f(t)} and defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Provided the integral on the right hand side exists. Here 's' is a parameter which may be real or complex number.

s. no.	$L(f(t)) = \overline{f}(s)$
1	$L(k) = \frac{k}{s}$ where k is constant
2	$L(e^{at}) = \frac{1}{s-a}$
3	$L(e^{-at}) = \frac{1}{s+a}$
4	$L(t^n) = \frac{n!}{s^{n+1}}$ or $\frac{\Gamma(n+1)}{s^{n+1}}$
5	$L(\sin at) = \frac{a}{s^2+a^2}$
6	$L(\cos at) = \frac{s}{s^2+a^2}$
7	$L(\sinh at) = \frac{a}{s^2-a^2}$
8	$L(\cosh at) = \frac{s}{s^2-a^2}$

Properties of Laplace Transforms:

s. no.	Property	$L(f(t)) = \overline{f}(s)$
1	Linearity	$L[(c_1 f_1(t) + c_2 f_2(t)]$, c_1 and c_2 are constants $= (c_1\overline{f}_1(s) + c_1\overline{f}_2(s))$, c_1 and c_2 are constants
2	First shifting	$L[e^{at} f(t)] = \overline{f}(s-a)$ or $[\overline{f}(s)]_{s\to s-a}$ $L[e^{-at} f(t)] = \overline{f}(s+a)$ or $[\overline{f}(s)]_{s\to s+a}$

3	Second shifting	$If\ g(t) = \begin{cases} o, if\ t < a \\ f(t-a), if\ t > a \end{cases}$ Then, $L(g(t)) = e^{-as}\,\bar{f}(s)$ (Or) If, $g(t) = f(t-a)\,u(t-a)$, then $L(g(t)) = e^{-as}\,\bar{f}(s)$
4	Change of scale	$L[f(at)) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$
5	Derivative	$L(y) = \overline{y(s)}$
		$L(y') = s\bar{y}(s) - y(0)$
		$L(y'') = s^2\bar{y}(s) - sy(0) - y'(0)$
6	Integral	$L(\int_0^t f(t)dt)$ or $L(\int_0^t f(u)du) = \frac{1}{s}\bar{f}(s)$
7	Multiplication by t	$L\ (t(f(t))) = (-1)\frac{d}{ds}[\bar{f}(s))]$
8	Multiplication by t^n	$L\ (t^n(f(t))) = (-1)^n\frac{d^n}{ds^n}[\bar{f}(s))]$
9	Division by t	$L(\frac{f(t)}{t}) = \int_s^\infty \bar{f}(s)ds$

Formulae of Inverse Laplace Transform:

s. NO.	
1	$L^{-1}\left[\frac{K}{S}\right] = K$ where K is Constant
2	$L^{-1}\left[\frac{1}{S^n}\right] = \frac{t^{n-1}}{(n-1)!}$ where n is integer
3	$L^{-1}\left[\frac{1}{S^{n+1}}\right] = \frac{t^n}{n!}$ where n is integer
4	$L^{-1}\left[\frac{1}{S-a}\right] = e^{at}$
5	$L^{-1}\left[\frac{1}{S+a}\right] = e^{-at}$
6	$L^{-1}\left[\frac{1}{S^2+a^2}\right] = \frac{1}{a}\sin at$
7	$L^{-1}\left[\frac{S}{S^2+a^2}\right] = \cos at$

8	$L^{-1}\left[\frac{1}{S^2-a^2}\right] = \frac{1}{a}\sinh at$
9	$L^{-1}\left[\frac{S}{S^2-a^2}\right] =$ coshat

Properties of Inverse Laplace Transform:

1	Linearity	$L^{-1}[(c_1\bar{f}_1(s)+c_1\bar{f}_2(s)] = c_1 f_1(t) + c_2 f_2(t)$, c_1 and c_2 are constants
2	First shifting	$L^{-1}[\bar{f}(s-a)] = e^{at} f(t)$
		$L^{-1}[\bar{f}(s+a)] = e^{-at} f(t)$
3	Second shifting	$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\, u(t-a)$
4	Change of scale	$L^{-1}[\bar{f}(as)] = \frac{1}{a} f\left(\frac{t}{a}\right)$
5	Derivative	$L^{-1}\frac{d}{ds}[\bar{f}(s)] = (-1)\, t\, f(t)$
		$L^{-1}\frac{d^n}{ds^n}[\bar{f}(s)] = (-1)^n t^n f(t)$
6	Integral	$L^{-1}\left(\int_s^{\infty}\bar{f}(s)ds\right) = \frac{f(t)}{t}$
7	Multiplication by s	$L^{-1}[s\,\bar{f}(s)] = f'(t)$ if $f(0) = 0$
8	Multiplication by s^2	$L^{-1}[s^2\,\bar{f}(s)] = f''(t)$ if $f(0) = f'(0) = 0$
9	Division by s	$L^{-1}[\frac{\bar{f}(s)}{s})] = \int_0^t f(u)du$
9	Division by s^2	$L^{-1}[\frac{\bar{f}(s)}{s^2})] = \int_0^t\int_0^t f(u)du\, du$
10	Convolution Theorem	$L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t) * g(t) = \int_0^t f(u)g(t-u)du$

Some important formulae

1. Integration by using partial fractions.

To resolve given proper rational fraction into partial fractions using the following table

S. No.	Form of the proper fraction	Form of the partial fraction
1	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}$
2	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
3	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
4	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$

Note: If the given fraction is improper, then divide the numerator with denominator and use the following:

$$\frac{Numerator}{Denominator} = quotient + \frac{Remaindor}{Divisor}$$

INTEGRATION BY PARTS

1. $\int UV\,dx = U\int V dx - \int\left(\frac{dU}{dx}\int V dx\right)dx$

To choose U we follow the Order **'ILATE'**

ILATE means

I-Inverse trigonometric function (Ex: $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc)

L-Logarithmic functions (Ex: $\log x$, $\log x^2$ etc)

A-Algebraic functions (Ex: x^0, x^2, $(x-x^2)$ etc)

T-Trigonometric function (Ex: sin x, cos x, $\tan\theta$, $\sec^2 \alpha$ etc)

E-Exponential function (Ex: e^x, a^x, e^{2x} etc)

2. Bernoulli"s Rule

$$\int UV dx = UV_1 - U'V_2 + U''V_3 - U'''V_4 + \cdots$$

1. Where U′,U'',U'''……are successive derivatives of U ,
2. V_1, V_2, V_3 ….are successive integrals of V
3. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$
4. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

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