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Homogeneous Linear Differential Equations with Constant Coefficients

CHAPTER-1

Homogeneous Linear Differential Equations with constant Coefficients

1.0 INTRODUCTION

In the third semester, we discussed solutions of linear differential equations of first order. In this semester, we consider linear differential equations of second and higher order with constant coefficients.

A differential equation in which the dependent variable and all its derivatives occur in the first degree only and are not multiplied together is called **Linear Differential Equation.**

Differential Operator : Let D stands for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$ and so on. The symbols D, D^2, etc are called differential operators. For example, $D^2 x^3 = 6x$.

1.1 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A differential equation of the form :

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+........+a_n y = X \quad \text{.............. (1)}$$

Where 'X' is a function of 'x' only and $a_0 \neq 0, a_1, a_2, ..., a_n$ are constants, is called a linear differential equation of n^{th} order.

Using the symbols $D, D^2, ..., D^n$, (1) be comes

$$a_0D^n y+a_1D^{n-1}y+a_2D^{n-2}y+........+a_n y = X$$

(or) $$(a_0 D^n + a_1D^{n-1} + a_2 D^{n-2}+ ....+ a_n)\, y = X$$

(or) $$f(D)y = X, \quad \text{.............. (2)}$$

where $f(D) = a_0D^n + a_1 D^{n-1} + ... + a_n$.

If $X = 0$, then (1) becomes

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}}{dx^{n-1}}+a_2\frac{d^{n-2}}{dx^{n-2}}+......\ a_n y = 0 \quad \text{.............. (3)}$$

This is a homogeneous linear differential equation of order 'n' with constant coefficients.

1.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS

The general form of the homogeneous linear differential equation of second order is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0 \qquad \text{.............. (1)}$$

where a, b, c are constants.

Note : If the right-hand side is non zero, and is a function of 'x', then the differential equation is said to be non homogeneous linear differential equation of second order.

THEOREM :

If $y = y_1(x)$ and $y = y_2(x)$ are two independent solutions of a differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, then $y = c_1 y_1 + c_2 y_2$ is also a solution of the given differential equation, where c_1 and c_2 are arbitrary constants.

PROOF :

Given differential equation is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0 \qquad \text{.............. (1)}$$

Since $y = y_1(x)$ and $y = y_2(x)$ are solutions of (1), we have

$$a\frac{d^2y_1}{dx^2}+b\frac{dy_1}{dx}+cy_1 = 0 \qquad \text{.............. (2)}$$

and $a\frac{d^2y_2}{dx^2}+b\frac{dy_2}{dx}+cy_2 = 0$ .... (3)

Now substituting $y = c_1y_1 + c_2y_2$ in equation(1), we have

$$\begin{aligned}
a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy &= a\frac{d^2}{dx^2}(c_1y_1+c_2y_2)+b\frac{d}{dx}(c_1y_1+c_2y_2)+c(c_1y_1+c_2y_2)\\
&= c_1\left[a\frac{d^2y_1}{dx^2}+b\frac{dy_1}{dx}+cy_1\right]+c_2\left[a\frac{d^2y_2}{dx^2}+b\frac{dy_2}{dx}+cy_2\right]\\
&= c_1[0]+c_2[0] \qquad [\because \text{ using (2) and (3)}]\\
&= 0
\end{aligned}$$

$\therefore$ $y = c_1 y_1 + c_2 y_2$ is a solution of (1).

Thus $y = c_1y_1 + c_2 y_2$ will be a complete solution or general solution of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ because it contains two arbitrary constants and equation is of second order.

Symbolic form of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

The symbolic representation of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ is

$aD^2 y + bDy + cy = 0$ where, $D \equiv \frac{d}{dx}$

$\Rightarrow (aD^2 + bD + c)\, y = 0$

$\Rightarrow f(D)y = 0$ where $f(D) = aD^2 + bD + c$

1.2.1 GENERAL SOLUTION OF $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

Consider the equation,

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0 \qquad \text{..............} \quad (1)$$

i.e., $(aD^2 + bD + c)\, y = 0$ where, $\frac{d}{dx} = D$

i.e., $f(D)y = 0$ where $f(D) = aD^2 + bD + C$

Let us take a trial solution $y = e^{mx}$ of (1) for some value of m. Then

$$a\frac{d^2}{dx^2}(e^{mx})+b\frac{d}{dx}(e^{mx})+c(e^{mx}) = 0$$

$\Rightarrow am^2e^{mx} + bme^{mx}+ ce^{mx} = 0$

$\Rightarrow [am^2 + bm + c]\, e^{mx} = 0$

$\Rightarrow am^2 + bm + c = 0 \qquad \left[\because e^{mx} \neq 0\right]$

Which is quadratic in 'm' gives two values for 'm'

The above equation is called Auxiliary Equation or characteristic Equation of (1).

Note : The auxiliary equation is obtain by putting $D = m$ in $f(D) = 0$. That is, the auxiliary equation of $f(D)\, y = 0$ is $f(m) = 0$.

The general solution of homogeneous differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = 0$ is depending on the nature of the roots of auxiliary equation $am^2 + bm + c = 0$ as follows :

Case-1 :

Roots are Real and Distinct : Let m_1 and m_2 be two distinct roots of auxiliary equation then $y = e^{m_1 x}$ and $y = e^{m_2 x}$ are distinct solutions of (1)

Therefore $y=c_1 e^{m_1x} + c_2 e^{m_2x}$ is the general solution of the given differential equation.

Case-2 :

Roots of Auxiliary equation are real and equal

Let m_1 be the real root of auxiliary equation which is repeated twice.

$\therefore\ am^2 + bm + c = a(m - m_1)^2$

and also $(aD^2 + bD + c)y = 0$

$\Rightarrow\ a(D - m_1)^2 y = 0$

$\Rightarrow\ (D - m_1)(D - m_1) y = 0$

Put $(D - m_1)\ y = v,$ .... (2)

Then $(D - m_1)\ v = 0$

$$\Rightarrow \frac{dv}{dx} - m_1 v = 0$$

$$\Rightarrow \frac{dv}{dx} = m_1 v$$

$$\Rightarrow \frac{dv}{v} = m_1\, dx$$

Integrating,

$$\int \frac{dv}{v} = \int m_1 dx$$

$$\log v = m_1 x + \log c_1$$

$$\Rightarrow \log V - \log c_1 = m_1 x$$

$$\Rightarrow \log \frac{v}{c_1} = m_1 x$$

$$\Rightarrow \frac{v}{c_1} = e^{m_1 x}$$

$$\Rightarrow v = c_1\, e^{m_1 x}$$

From (2), $(D - m_1)\, y = c_1 e^{m_1 x}$

$$\Rightarrow \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

It is linear in 'y', Here $P = -m_1$ and $Q = e^{m_1 x}$

$$I.F = e^{\int p\,dx} = e^{\int -m_1 dx} = e^{-m_1 x}$$

The general solution is given by

$$y(I.F) = \int Q(I.F)\,dx$$

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} e^{-m_1 x} dx + c_2$$

$$\Rightarrow \quad ye^{-m_1 x} = c_1 \int dx + c_2$$

$$\Rightarrow \quad ye^{-m_1 x} = c_1 x + c_2$$

$$\Rightarrow \quad y = (c_1 + c_2 x) e^{m_1 x}$$

$\therefore$ If the auxiliary equation has repeated roots m, m (say), the solution can be written as $y = (c_1 + c_2 x)\, e^{mx}$, where c_1 and c_2 are arbitrary constants.

Case-3 :

When the roots of auxiliary equation are complex pair

Let $\alpha \pm i\beta$ be the two complex roots of auxilliary equation. Therefore the solution of (1) is given by

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x}\left[c_1 e^{i\beta} + c_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[c_1[\cos\beta x + i\sin\beta x] + c_2[\cos\beta x - i\sin\beta x]\right] \quad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + (c_1 - c_2) i \sin\beta x\right]$$

$$= e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$ where $c_1 = c_1 + c_2$ are $c_2 = (c_1 - c_2)$.

$\therefore$ If the roots of auxilliary equation are complex pair $m = \alpha \pm i\beta$ (say), then the solution is given by $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$, where c_1 and c_2 are arbitrary constants.

Working rule to solve $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

Step-1 : Write the given differential equation in symbolic form or operator form as $(aD^2 + bD + c)\,y$ i.e., $f(D)y = 0$. where $f(D) = aD^2 + bD + c$.

Step-2 : Write down its auxiliary equation as $f(m) = 0$ i.e., $am^2 + bm + c = 0$. (Replace D by m in $f(D) = 0$). Solve it for 'm'.

Step-3 : From the roots of auxiliary equation, write the general solution as follows :

S.No.	Nature of Roots	General Solution
1.	Two real and distinct roots say m_1, m_2	$y = c_1e^{m_1x}+c_2e^{m_2x}$
2.	Two real and equal roots say m, m	$y = (c_1+c_2x)e^{mx}$
3.	Two roots are complex pair say $m=\alpha\pm i\beta$	$y = e^{\alpha x}\left[c_1\cos\beta x+c_2\sin\beta x\right]$
4.	Two roots are pair of surd say $m=\alpha\pm\sqrt{\beta}$	$y = c_1e^{(\alpha+\sqrt{\beta})x}+c_2e^{(\alpha-\sqrt{\beta})x}$ (or) $y = e^{\alpha x}\left[c_1\cos h\sqrt{\beta}x+c_2\sin h\sqrt{\beta}x\right]$

SOLVED EXAMPLES

EXAMPLE-1

Solve :

(a) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$ *or* $(D^2-5D+4)y=0$ ***[Apr. 2019, 2017, 2011, 2009, 2002]***

(b) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0$ *or* $y''+y'-12y=0$ ***[Apr. 2016, 2014, 2004]***

(c) $(D^2-4D+1)y=0$ *where* $D=\frac{d}{dx}$ ***[Apr. 2016; Oct. 2006]***

Solution :

(a) The given differential equation in symbolic form is

$(D^2-5D+4)\,y=0$ .... (1)

$\Rightarrow$ $f(D)\,y = 0$, where $f(D) = D^2-5D+4$

The auxilliary equation of (1) is $f(m) = 0$

i.e., $m^2-5m+4 = 0$

$$\Rightarrow \quad (m-1)(m-4) = 0$$

$$\Rightarrow \quad m = 1 \text{ or } m = 4$$

Since the roots of auxilliary equation are real and distinct, therefore, the general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

i.e.,

$$y = c_1e^{x} + c_2e^{4x}$$

(b) Given differential equation is

$$y'' + y' - 12y = 0 \qquad \text{.............. (1)}$$

Its operator form is

$$(D^2 + D - 12)y = 0$$

$$f(D)y = 0 \quad \text{where } f(D) = D^2 + D - 12$$

The auxiliary equation is $f(m) = 0$

i.e.,

$$m^2 + m - 12 = 0$$

$$\Rightarrow \quad m^2 + 4m - 3m - 12 = 0$$

$$\Rightarrow \quad m(m+4) - 3(m+4) = 0$$

$$\Rightarrow \quad (m-3)(m+4) = 0$$

$$\Rightarrow \quad m = 3 \text{ or } m = -4$$

Since the roots of auxiliary equation are real and distinct. Therefore, the general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

i.e.,

$$y = c_1e^{3x} + c_2e^{-4x}$$

(c) The given differential equation is

$$(D^2 - 4D + 1)\,y = 0 \qquad \text{.............. (1)}$$

$$f(D)y = 0 \quad \text{where } f(D) = D^2 - 4D + 1$$

The auxilliary equation of (1) is $f(m) = 0$

i.e., $m^2 - 4m + 1 = 0$.

Compare it with $ax^2 + bx + c = 0$, here $a = 1$, $b = -4$, $c = 1$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.1}}{2.1}$$

$$= \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

Since the roots of auxilliary equation are real and distinet therefore, the general solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ i.e., $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

(or) $\quad y = e^{2x}\left[c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x\right]$

EXAMPLE-2

Solve :

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$ *[Apr. 2018, 2017; Oct. 2011]*

(b) $4y'' - 4y' + y = 0$

Solution :

(a) Given differential equation is

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0 \quad \text{.............} \quad (1)$$

Its operator form is $(D^2 + 4D + 4)\, y = 0$

$\Rightarrow \quad f(1)y = 0$, where $f(D) = D^2 + 4D + 4$

The auxilliary equation is $f(m) = 0$

i.e., $m^2 + 4m + 4 = 0$

$\Rightarrow \quad (m + 2)\,(m + 2) = 0$

$\Rightarrow \quad m = -2, -2$

$\therefore$ The roots of auxilliary equation are real and equal. Hence the general solution of (1) is

$$y = (c_1 + c_2 x)e^{mx}$$

$$\Rightarrow \quad y = (c_1 + c_2 x)e^{-2x}$$

(b) Given differential equation in the symbolic form is $(4D^2 - 4D + 1)\, y = 0$ .... (1)

$\Rightarrow$ f(D) y = 0 where $f(D) = 4D^2 - 4D + 1$

The auxilliary equation of (1) is f(m) = 0

i.e., $4m^2 - 4m + 1 = 0$

$\Rightarrow (2m - 1)^2 = 0$

$\Rightarrow 2m - 1 = 0$ or $2m - 1 = 0$

$\Rightarrow m = \frac{1}{2}, \frac{1}{2}$

Since the roots of auxilliary equation are real and equal. Therefore, the general solution of (1) is $y = (c_1 + c_2 x)e^{mx}$

i.e., $y = (c_1 + c_2 x)e^{\frac{1}{2}x}$

$y = (c_1 + c_2 x)\, e^{\frac{x}{2}}$

EXAMPLE-3

Solve :

(a) $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$ *[Apr. 2018, 2017, 2016; Oct. 2008]*

(b) $(D^2 + D + 1)\, y = 0$

(c) $y'' + 4y = 0$ *[May 2022 ; Apr. 2019, 2014, 2013]*

Solution :

(a) Given differential equation is

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0 \quad \text{.............. (1)}$$

Its symbolic form is

$$(D^2 + 4D + 13)\, y = 0$$

i.e., f(D) y = 0, where $f(D) = D^2 + 4D + 13$.

Auxilliary equation of (1) is f(m) = 0.

i.e., $m^2 + 4m + 13 = 0$.

Compare it with $ax^2 + bx + c = 0$, here a = c, b = 4, c = 13.

$\therefore$ The roots are
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{4^2 - 4.1.13}}{2}$$
$$= \frac{-4 \pm \sqrt{-36}}{2}$$
$$= \frac{-4 \pm 6i}{2},$$
$$= -2 \pm 3i = \alpha \pm i\beta$$

Since the roots of auxilliary equation are complex pair.

$\therefore$ The general solution of (1) is
$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

i.e.,
$$y = e^{-2x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

(b) Given differential equation is
$$(D^2 + D + 1)\, y = 0 \qquad \text{.............. (1)}$$

$\Rightarrow$ $f(D)\, y = 0$, where $f(D) = D^2 + D + 1$

The auxilliary equation is f(m) = 0

i.e., $m^2 + m + 1 = 0$

The roots are
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$
$$= \frac{-1 \pm \sqrt{-3}}{2}$$
$$= \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$
$$= \alpha \pm i\beta, \quad \alpha = \frac{-1}{2} \;\&\; \beta = \frac{\sqrt{3}}{2}$$

The roots of auxilliary equation are complex pair

$\therefore$ The general solution of (1) is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$= e^{-\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

(c) The given equation in symbolic form is

$$(D^2 + 4)y = 0$$

$$\Rightarrow \quad f(D)y = 0, \quad \text{where } f(D) = D^2 + 4$$

The auxilliary equation is $f(m) = 0$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4} = \pm 2i$$

$$= \alpha \pm i\beta, \text{ here } \alpha = 0, \beta = 2$$

Since the roots of auxilliary equation are complex pair, therefore, the general solution of (1) is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$= e^{0.x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$= c_1 \cos 2x + c_2 \sin 2x.$$

EXAMPLE-4

Solve : $(D^2 + 4)\, y = 0$ given that $y(0) = 0$ and $y'(0) = 1$. ***[Mar. 2003]***

Solution :

Given differential equation is

$$(D^2 + 4)\, y = 0 \qquad \text{.............. (1)}$$

and $y(0) = 0$ and $y'(0) = 1$

Auxilliary equation of (1) is $f(m) = 0$.

i.e., $m^2 + 4 = 0$.

$$\Rightarrow \quad m^2 = -4$$

$$\Rightarrow \quad m = \sqrt{-4}$$

$$= \pm 2i$$

$$= \alpha \pm i\beta, \quad \alpha = 0 \text{ and } \beta = 2$$

The roots of auxilliary equation are complex pair, therefore the general solution is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$= e^{ox}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$= c_1 \cos 2x + c_2 \sin 2x \quad \text{.............} \quad (2)$$

Differentiating w.r.t 'x', we get

$$y' = -2c_1 \sin 2x + 2c_2 \cos 2x \quad \text{.............} \quad (3)$$

Given that $y = 0$ when $x = 0$ and $y' = 1$ where $x = 0$.

$\therefore$ From (2) and (3),

$$0 = -c_1 \cos 2(0) + c_2 \sin 2(0)$$

$$\Rightarrow \quad 0 = c_1 (1) + 0 \Rightarrow c_1 = 0$$

and $$1 = -2 c_1 \sin 2(0) + 2c_2 \cos 2(0)$$

$$1 = 0 + 2c_2$$

$$\Rightarrow \quad c_2 = \frac{1}{2}$$

Substituting the values of c_1 and c_2 in (2), we get

$$y = 0.\cos 2x + \frac{1}{2} \sin 2x$$

$$y = \frac{1}{2}\sin 2x$$

(or) $$y = \frac{1}{2} \cdot 2\sin x \cos x = \sin x \cos x,$$ which is th required solution.

EXERCISE 1.1

1. **Write the nature of the roots of auxiliary equation of differential equation $(D^2 - 2D)\ y = 0$** **[May 202**

2. (i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ (or) $y'' + 3y' + 2y = 0$ (or) $(D^2 + 3D + 2)\ y = 0$ **[Apr. 2019, 2013, 2009; Oct. 201**

   (ii) $y'' - 3y' + 2y = 0$ **[Apr. 2019, 2017; Oct. 201**

   (iii) $(D^2 - 4D + 3)\ y = 0$ **[Apr. 2018, 201**

   (iv) $(D^2 + 4D + 3)\ y = 0$ **[Oct. 2018; Apr. 201**

   (v) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ **[Oct. 2008; Apr. 2016, 200**

   (vi) $(D^2 + 5D + 6)\ y = 0$ **[Apr. 2016; Oct. 2010, 200**

   (vii) $(D^2 - 6D + 5)\ y = 0$

   (viii) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$ **[Apr. 201**

   (ix) $\frac{d^2y}{dx} + 5\frac{dy}{dx} + 4y = 0$ **[Oct. 201**

   (x) $(D^2 - 18D + 77)\ y = 0$ **[Apr. 2016; Oct. 2014, 200**

   (xi) $(D^2 - 6D + 8)\ y = 0$ **[Oct. 201**

   (xii) $(D^2 - 8D + 12)\ y = 0$ **[Apr. 2017, 2013; Oct. 2014, 2012, 201**

   (xiii) $y'' + 40\ y' + 111y = 0$ **[Oct. 201**

   (xiv) $y'' - 40\ y' + 111y = 0$ **[Oct. 2018, 201**

3. (i) $(D^2 - 2D - 3)\ y = 0$ **[Apr. 201**

   (ii) $(D^2 + 2D - 3)\ y = 0$ **[Apr. 201**

   (iii) $(D^2 + D - 12)\ y = 0$ **[Apr. 2018, 2017, 201**

   (iv) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 54y = 0$ **[Apr. 2018, 2017, 201**

   (v) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$ **[Apr. 2017**

(vi) $(D^2 - 1)\, y = 0$

(vii) $(D^2 - 9)\, y = 0$

(viii) $(D^2 - 16)\, y = 0$

(ix) $(D^2 - 25)\, y = 0$ **[Apr. 2018, 2017, 2012]**

4. (i) $(D^2 + 6D + 4)\, y = 0$ **[Apr. 2017]**

(ii) $(D^2 - 6D + 4)\, y = 0$ **[Apr. 2016]**

5. (i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ **[Apr. 2017, 2011; Oct. 2010, 2005]**

(ii) $(D^2 + 2D + 1)\, y = 0$ **[Apr. 2019]**

(iii) $(D^2 - 4D + 4)\, y = 0$ **[Apr. 2018, 2016; Oct. 2016]**

(iv) $(D^2 + 6D + 9)\, y = 0$ **[Apr. 2019, 2016, 2010, 2008; Oct. 2018, 2016, 2010]**

(v) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ **[Apr. 2019]**

(vi) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ **[Apr. 2016]**

(vii) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$ **[Apr. 2016]**

(viii) $(D^2 + 10\, D + 25)\, y = 0$

(ix) $(D^2 - 10D + 25)\, y = 0$ **[Oct. 2016]**

(x) $\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 64y = 0$ **[Oct. 2013]**

(xi) $9D^2 y - 30\, Dy + 25y = 0$

6. (i) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$ **[Apr. 2014, 2009]**

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ **[Oct. 2016, 2014, 2011, 2010; Apr. 2012, 2011, 2010, 2008]**

(iii) $(D^2 + 2D + 5)\, y = 0$ **[Oct. 2016]**

(iv) $(D^2 - 2D + 10)\, y = 0$ **[Apr. 2016]**

(v) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$ [Apr. 2006, 2009]

(vi) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+7y=0$ [Apr. 2017, 2014, 2010]

(vii) $(D^2+4D+13)\,y=0$ [Apr. 2018, 2017, 2016, 2009]

(viii) $(D^2+D+1)\,y=0$

(ix) $(D^2-D+1)\,y=0$ [Apr. 2018]

(x) $(D^2-3D+5)\,y=0$

(xi) $2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$ [Mar. 2005; Oct. 2011]

(xii) $y''+8y'+25y=0$ (Oct. 2007)

(xiii) $(D^2+1)\,y=0$ [Apr. 2017, 2010]

(xiv) $(D^2+9)\,y=0$ [Apr. 2019, 2018, 2017, 2010]

(xv) $(D^2+16)\,y=0$

(xvi) $(D^2+25)y=0$

7. (i) $\frac{d^2x}{dt^2}-8\frac{dx}{dt}+16x=0$ given that $x=1$ and $\frac{dx}{dt}=0$ when $t=0$

(ii) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$ given that $y=1$ and $\frac{dy}{dx}=0$ when $x=0$.

(iii) $y''-2y'+10y=0$ and $y(0)=4$ and $y'(0)=1$ [Apr. 2011, 2008]

ANSWERS

1. Real and not equal

2. (i) $y=c_1e^{-x}+c_2e^{-2x}$ (ii) $y=c_1e^{x}+c_2e^{2x}$

(iii) $y=c_1e^{x}+c_2e^{3x}$ (iv) $y=c_1e^{-x}+c_2e^{-3x}$

(v) $y=c_1e^{2x}+c_2e^{3x}$ (vi) $y=c_1e^{-2x}+c_2e^{-3x}$

(vii) $y=c_1e^{x}+c_2e^{5x}$ (viii) $y=c_1e^{-x}+c_2e^{-5x}$

(ix) $y=4e^{-x}+c_2e^{-4x}$ (x) $y=c_1e^{7x}+c_2e^{11x}$

(xi) $y=c_1e^{2x}+c_2e^{4x}$ (xii) $y=c_1e^{2x}+c_2e^{6x}$

(xiii) $y=c_1e^{-3x}+c_2e^{-37x}$ (xiv) $y=4e^{3x}+c_2e^{37x}$

3. (i) $y = c_1 e^{-x} + c_2 e^{3x}$ (ii) $y = c_1 e^{x} + c_2 e^{-3x}$

(iii) $y = c_1 e^{3x} + c_2 e^{-4x}$ (iv) $y = c_1 e^{6x} + c_2 e^{-9x}$

(v) $y = c_1 e^{-x} + c_2 e^{4x}$ (vi) $y = 4e^{x} + c_2 e^{-x}$

(vii) $y = c_1 e^{3x} + c_2 e^{-3x}$ (viii) $y = 4e^{4x} + c_2 e^{-4x}$

(ix) $y = c_1 e^{5x} + c_2 e^{-5x}$

4. (i) $y = c_1 e^{(-3+\sqrt{5})x} + c_2 e^{(-3-\sqrt{5})x}$ (or) $y = e^{-3x}[c_1 \cosh\sqrt{5x} + c_2 \sinh\sqrt{5}x$

(ii) $y = c_1 e^{(3+\sqrt{5})x} + c_2 e^{(3-\sqrt{5})x}$ (or) $y = e^{3x}\,[c_1 \cosh\sqrt{5}x + c_2 \sinh\sqrt{5}x]$

5. (i) $y = (c_1 + c_2 x)e^{-4x}$ (ii) $y = (c_1 + c_2 x)\, e^{-x}$

(iii) $y = (c_1 + c_2 x)\, e^{2x}$ (iv) $y = (c_1 + c_2 x)e^{-4x}$

(v) $y = (c_1 + c_2 x)\, e^{3x}$ (vi) $y = (c_1 + c_2 x)\, e^{4x}$

(vii) $y = (c_1 + c_2 x)e^{-4x}$ (viii) $y = (c_1 + c_2 x)\, e^{-5x}$

(ix) $y = (c_1 + c_2 x)\, e^{-5x}$ (x) $y = (c_1 + c_2 x)\, e^{8x}$

(xi) $y = (c_1 + c_2 x)\; e^{\frac{5}{3}x}$

6. (i) $y = e^{x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right]$ (ii) $y = e^{x}[c_1 \cos 2x + c_2 \sin 2x]$

(iii) $y = e^{-x}\,[c_1 \cos 2x + c_2 \sin\sqrt{2}\,x]$ (iv) $y = e^{x}\,[c_1 \cos 3x + c_2 \sin 3x]$

(v) $y = e^{-2x}\left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$ (vi) $y = e^{-2x}\,[c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x]$

(vii) $y = e–2x\;[c_1 \cos 3x + c_2 \sin 3x]$ (viii) $y = e^{-\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

(ix) $y = e^{\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$ (x) $y = e^{\frac{3}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

(xi) $y = e^{\frac{5}{4}x}\left[c_1 \cos\frac{\sqrt{7}}{4}x + c_2 \sin\frac{\sqrt{7}}{4}x\right]$ (xii) $y = e^{-4x}\,[c_1 \cos 3x + c_2 \sin 3x]$

(xiii) $y = c_1 \cos x + c_2 \sin x$ (xiv) $y = c_1 \cos 3x + c_2 \sin 3x$

(xv) $y = c_1 \cos 5x + c_2 \sin 5x$

7. (i) $x = (1 - 4t)e^{4t}$ (ii) $y = e^{dx}[\cos x - 2 \sin x]$

(iii) $y = e^{x}\,[4 \cos 3x - \sin 3x]$

1.3 LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

1.3.1 DEFINITIONS

1. Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and not multiplied to gether.

   The general linear differential equation of the n^{th} order is of the form

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + p_n y = X$$

   where $p_0, p_1, p_2 ...... p_n$ are either functions of x or constants and X is a functions of 'x' alone or constant.

2. **Linear differential equations with constant coefficients :** The general form of the linear differential equation of n^{th} order with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = X \quad .... (1)$$

   where $a_0, a_1, a_2, ..., a_n$ are constants and X is a function of x alone.

   If $X = 0$, then the equation (1) is said to be homogeneous linear differential equation otherwise it is said to be non homogeneous linear differential equation.

1.3.2 LINEAR DEPENDENCE AND INDEPENDENCE OF TWO FUNCTIONS

Two functions $y_1(x)$ and $y_2(x)$ are said to be linearly dependent if one can be expressed as scalar multiple of other (i.e., $y_1 = ky_2$). Otherwise they are linearly independent.

Condition for linear dependence or Independence

Define the determinant or wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = y_1 y_2^1 - y_1^1 y_2$$

(i) If $w \neq o$ then y_1 and y_2 are linearly independent

(ii) If $w = 0$ then y_1 and y_2 are linearly dependent.

THEOREM :

If $y = y_1, y = y_2, ..., y_2 = y_n$ are n linearly independent solutions of the differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = 0, \qquad \ldots\ldots\ldots\ldots \quad (1)$$

then $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \ldots\ldots\ldots + c_n y_n$ is complete or general solution of (1). Where $c_1, c_2, \ldots\ldots\ c_n$ are arbitrary constants.

1.3.3 AUXILIARY EQUATION

Consider n^{th} order homogeneous linear differential equation with constant coefficients.

$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n) y = 0$ .... (1)

$\Rightarrow$ i.e., $f(D)y = 0$, where $f(D) = a_0 D^n + a_1 D^{n-1} + \ldots\ldots\ a_n$

Let $y = e^{mx}$ be a solution of equation (1). Then

$$\Rightarrow a_1 D^n e^{mx} + a_1 D^{n-1} e^{mx} + \ldots\ldots + a_n e^{mx} = 0$$

$$\Rightarrow a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \ldots\ldots\ldots + an\, e^{mx} = 0$$

$$\Rightarrow (a_0 m^n + a_1 m^{n-1} + \ldots\ldots + a_n) e^{mx} = 0$$

i.e., $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots\ldots + a_n = 0$ $\qquad [\because e^{mx} \neq o)$

(or) $f(m) = 0$ is called the auxiliary equation.

Note : Replace D by m in f(D) and equate it to zero, to get the auxiliary equation, $f(m) = 0$.

Working rule to find the general solution of f(D) y = 0.

Consider n^{th} order homogeneous linear differential equation.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{ny}} + \ldots\ldots\ldots + a_n y = 0 \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Step-1 : Write the equation in the symbolic form as

$$(a_o D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots a_n)\ y = 0$$

(or) $f(D)\ y = 0$, where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots + a_n$

Step-2 : Write down the auxiliary equation (A.E) as $f(m) = 0$

(or) $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0$

i.e., A.E of (1) is obtained by replacing D by m in f(D) and equate to zero.

Step-3 : Solve the auxiliary equation for m.

Let $m_1, m_2, \ldots\ldots\ m_n$ be its roots.

Step-4 : According to the nature of the roots of A.E. write the general solution of (1), given in the following table.

S.No.	Nature of roots of A.E i.e., $f(m) = 0$	General solution
1.	Real and distanct roots $m_1, m_2 \ldots\ldots m_m$	$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots\ldots + c_ne^{m_nx}$
2.	$m_1, m_1, m_3, m_4, \ldots\ldots, m_n$ (i.e., two roots are equal and remaining (n–2) are real and different).	$y = (c_1 + c_2 x)\, e^{m_1x} + c_3e^{m_3x}$.
3.	$m_1, m_1, m_1, m_4, m_5, \ldots\ldots m_n$ (i.e., three roots are real and equal and $n - 3$ are real and different).	$y = (c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x}$ $+ \ldots\ldots + c_ne^{m_nx}$
4.	Two roots are conjugate complex pair (i.e., $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$) and the remaining n–2 are real and distinct.	$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3e^{m_3x} + c_4e^{m_4x}$ $+ \ldots\ldots + c_ne^{m_nx}$.
5.	Two equal complex conjugate pairs (i.e., $\alpha \pm i\beta$ are repeated twice) and remaining n-4 roots are real and different.	$y = e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x]$ $+ c_5e^{m_5x} + \ldots\ldots\ldots + c_ne^{m_nx}$.

SOLVED EXAMPLES

EXAMPLE-1

Solve : $y''' + 6y'' + 11y' + 6y = 0$

Solution :

Given differential equation in symbolic form is

$$(D^3 + 6D^2 + 11D + 6)y = 0 \quad \ldots\ldots\ldots\ldots (1)$$

i.e., $f(D)\, y = 0$, where $f(D) = D^3 + 6D^2 + 11D + 6$

Auxiliary equation of (1) is $f(m) = 0$

$$\text{i.e., } m^3 + 6m^2 + 11m + 6 = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Put $m = -1$ in (2), we get

$$(-1)^3 + 6(-1)^2 + 11(-1) + 6 = 0$$

$$-1 + 6 - 11 + 6 = 0$$

$$0 = 0$$

$\therefore$ $m = -1$ is a root of A. E

By synthetic division

$$m=-1\left|\begin{array}{cccc} 1 & 6 & 11 & 6 \\ 0 & -1 & -5 & -6 \\ \hline 1 & 5 & 6 & \end{array}\right.$$

$\therefore$ (m + 1) (m + 5m + 6) = 0

$\Rightarrow$ (m + 1) (m + 2) (m + 3) = 0

$\Rightarrow$ m = – 1, –2, – 3

Since the roots of A.E. are real and distinct.

Hence the general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$$

$$= c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$$

EXAMPLE-2

Solve : $(D^3 - 5D^2 + 8D - 4)\, y = 0$ ***[Apr. 2018, 2017]***

Solution :

Given differential equation is

$$(D^3 - 5D^2 + 8D - 4)\, y = 0 \quad \text{..............} \quad (1)$$

$f(D)\, y = 0$, where $f(D) = D^3 - 5D^2 + 8D - 4$

Auxiliary equation of (1) is f(m) = 0

$$m^3 - 5m^2 + 8m - 4 = 0 \quad \text{..............} \quad (2)$$

Put m = 1, in (2), we get

$1^3 - 5(1)^2 + 8(1) - 4 = 0$

$0 = 0$

$\therefore$ m = 1 is a root of A.E.

By synthetic division

$$m=1\left|\begin{array}{cccc} 1 & -5 & 8 & -4 \\ 0 & 1 & -4 & 4 \\ \hline 1 & -4 & 4 & 0 \end{array}\right.$$

$\therefore (m - 1)(m^2 - 4m + 4) = 0$

$(m-1)(m-2)^2 = 0$

$\therefore$ m = 1, 2, 2

$\therefore$ The roots of A.E. are out of which two are equal and one is distinct.

$\therefore$ the general solution of (1) is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x}$$
$$= (c_1 + c_2 x)\, e^{2x} + c_3 e^{x}.$$

EXAMPLE-3

Solve : $(D^3 - 3D^2 + 3D - 1)\, y = 0$ *[Apr. 2019, 2018, 201*

Solution :

f(D) y = 0, where $f(D) = 3D^2 + 3D - 1$ .... (1)

Auxiliary equation of (1) is f(m) = 0.

i.e., $m^3 - 3m^2 + 3m - 1 = 0$ .... (2)

Put m = 1 in (2), we get

$1^3 - 3(1)^2 + 3(1) - 1 = 0$

$\therefore$ m = 1 is a root of (2), by synthetic division

$$\begin{array}{r|rrrr} m=1 & 1 & -3 & +3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$\therefore (m-1)(m^2 - 2m + 1) = 0$

$\therefore (m - 1)(m-1)^2 = 0$

$\Rightarrow (m-1)^3 = 0$

$\Rightarrow$ m = 1, 1, 1.

$\therefore$ The roots of A.E. are real and equal

$\therefore$ The general solution of (1) is

$$y = (c_1 + c_2 x + c_3 x^2)e^{mx}$$
$$= (c_1 + c_2\, x + c_3\, x^2)\, e^{x}.$$

EXAMPLE-4

Solve : $(D^3 + D^2 + 4D + 4)\, y = 0$ *[Oct. 2018, 200*

Solution :

Given differential equation is

$(D^3 + D^2 + 4D + 4)\, y = 0$ .... (1)

$f(D)\,y = 0$, where $f(D) = D^3 + D^2 + 4D + 4$

Auxiliary equation of (1) is $f(m) = 0$

i.e., $m^3 + m^2 + 4m + 4 = 0$ .... (2)

Put $m = -1$ in (2), we get

$(-1)^3 + (-1)^2 + 4(-1) + 4 = 0$

$\therefore$ $m = -1$ is a root of (2), by if it is synthetic division

$$m = 1\left|\begin{array}{cccc} 1 & 1 & 4 & 4 \\ 0 & -1 & 0 & -4 \\ \hline 1 & 0 & 4 & 0 \end{array}\right.$$

$\therefore$ $(m + 1)(m^2 + 4) = 0$

$$m = -1,\ m^2 = -4 \Rightarrow m = \pm 2i = \alpha \pm i\beta$$

$\therefore$ The roots of A.E are, one is real and two are compex pair. Therefore the general solution of (1) is

$$y = c_1 e^{m_1 x} + e^{\alpha x}\left[c_2 \cos\beta x + c_3 \sin\beta x\right]$$

i.e., $$y = c_1 e^{-x} + e^{ox}\left[c_2 \cos 2x + c_3 \sin 2x\right]$$

$$= c_1 e^{-x} + c_2 \cos_2 2x + c_3 \sin 2x\,.$$

EXAMPLE-5

Solve : $\dfrac{d^3 y}{dx^3} + y = 0$ *[Oct. 2018, 2009]*

Solution :

Given differential equation in symbolic form

$$(D^3 + 1)\,y = 0$$ .... (1)

i.e., $f(D)\,y = 0$, where $f(D) = D^3 + 1$

Auxiliary equation of (1) is $f(m) = 0$

i.e., $m^3 + 1 = 0$

$(m + 1)(m^2 - m + 1) = 0$ $[\because a^3 + b^3 = (a+b)(a^2-ab+b^2)]$

$m + 1 = 0$, $m^2 - m + 1 = 0$

$m = -1$, $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4.1.1}}{2.1}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$ The general solution of (1) is

$$y = c_1 e^{m_1 x} + e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

$$\text{i.e.,} \quad y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right]$$

EXAMPLE-6

Solve : $(D^4 - 5D^2 + 4)\, y = 0$ *[Apr. 2009, 2008]*

Solution :

Given differential equation is

$(D^4 - 5D^2 + 4)\, y = 0$ .... (1)

$f(D)\, y = 0$, where $f(D) = D^4 - 5D^2 + 4$

Auxiliary equation of (1) is $f(m) = 0$

$$m^4 - 5m^2 + 4 = 0$$

$$(m^2)^2 - 4m^2 - m^2 + 4 = 0$$

$$m^2(m^2 - 4) - 1(m^2 - 4) = 0$$

$$(m^2 - 1)(m^2 - 4) = 0$$

$$\Rightarrow \quad m = \pm 1, \pm 2$$

Since the roots of A.E. are real and distinct.

$\therefore$ The general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$$

$$\text{i.e.,} \quad y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x}$$

EXAMPLE-7

Solve : $(D^4 - 18D^2 + 81)\, y = 0$ *[Oct. 2016]*

Solution :

Given differential equation is

$(D^4 - 18D^2 + 81)\, y = 0$ .... (1)

Auxiliary equation of (1) is $m^4 - 18m^2 + 81 = 0$

i.e., $(m^2 - 9)^2 = 0$

$m^2 - 9 = 0,\ m^2 - 9 = 0$

$m = \pm 3,\ m = \pm 3$

$m = 3, 3, -3, -3$

The general solution of (1) is

$$y = (c_1 + c_2 x)e^{3x} + (c_3 + c_4 x)e^{-3x}$$

EXAMPLE-8

Solve : $(D^4 - 16)\, y = 0$ *[Oct. 2016]*

Solution :

Given differential equation is

$(D^4 - 16)\, y = 0$ .... (1)

Auxiliary equation of (1) is $m^4 - 16 = 0$

i.e., $(m^2 - 4)(m^2 + 4) = 0$

$m^2 - 4 = 0,\quad m^2 + 4 = 0$

$\Rightarrow m = \pm 2,\ m = \pm 2i$

Hence the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + e^{\alpha x}\left[c_3 \cos\beta x + c_4 \sin\beta x\right]$$

$$\text{i.e., } y = c_1 e^{2x} + c_2 e^{-2x} + e^{ox}\left[c_3 \cos 2x + c_4 \sin 2x\right]$$

$$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2 + c_4 \sin 2x$$

EXAMPLE-9

Solve : $(D^4 + 18D^2 + 81)\, y = 0$

Solution :

Given differential equation is

$(D^4 + 18D^2 + 81)\, y = 0$ .... (1)

A.E of (1) is $m^4 + 18m^2 + 81 = 0$

$\Rightarrow \quad (m^2 + 9)^2$

$\Rightarrow \quad m = \pm 3i, \pm 3i$

Hence the general solution of (1) is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$$

$$\text{i.e., } y = e^{0x}[(c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x]$$

$$= (c_1 + c_2 x)\cos 3x + (c_3 + c_4 x)\sin 3x.$$

EXERCISE 1.2

Solve :

1. (i) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ **[Apr. 2017; Oct. 2012]**

(ii) $(D^3 - 4D^2 + D + 6)y = 0$ **[Oct. 2016]**

(iii) $y''' - 9y'' + 23y' - 15\,y = 0$ **[Oct. 2008]**

(iv) $(D^3 - 2D^2 - 3D)\,y = 0$ **[Apr. 2019, 2016]**

(or)

$y''' - 2y'' - 3y' = 0$ **[Oct. 2016]**

(v) $\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - 8\frac{dx}{dt} = 0$

(vi) $(D^3 - 2D^2 - D + 2)\,y = 0$ **[Apr. 2019]**

(vii) $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$ **[Apr. 2017]**

2. (i) $(D^3 - D^2 - 8D + 12)\,y = 0$ **[Apr. 2017]**

(ii) $(D^3 - 2D^2 - 4D + 8)\,y = 0$

(iii) $(D^3 + 3D^2 - 4)\,y = 0$ **[Apr. 2017, 2008]**

(iv) $(D^3 - D^2 - D + 1)\,y = 0$ **[Apr. 2018, 2017, 2016; Oct. 2009]**

(v) $(D^3 - 5D^2 + 7D - 3)\,y = 0$

(vi) $(D^3 - 7D^2 + 160 - 12)\,y = 0$

(vii) $(D^3 + 3D^2 + 3D + 1)\, y = 0$

(viii) $(D^4 - D^3 - 9D^2 - 11\, D - 4)\, y = 0$ **[Apr. 2011]**

(ix) $(D^4 + 4D^3 - 5D^2 - 36\, D - 36)\, y = 0$

(x) $(D^2 - 1)^2\, y = 0$

3. (i) $(D^3 + 3D^2 + 3D + 2)\, y = 0$

(ii) $\frac{d^3y}{dx^3} - y = 0$

(iii) $(D^4 - 81)\, y = 0$ **[Apr. 2019]**

4. $(D^4 + 8D^2 + 16)\, y = 0$

ANSWERS

1. (i) $y = c_1\, e^x + c_2\, e^{2x} + c_3\, e^{3x}$ (ii) $y = c_1 e^{-x} + c_2\, e^{2x} + c_3\, e^{3x}$

(iii) $y = c_1\, e^x + c_2\, e^{3x} + c_3\, e^{5x}$ (iv) $x = c_1 + c_2\, e^{-t} + c_3\, e^{3t}$

(v) $y = c_1 + c_2\, e^{2x} + c_3 e^{-4x}$ (vi) $y = c_1 e^{-x} + c_2\, e^x + c_3\, e^{2x}$

(vii) $y = c_1 e^x + c_2 e^{2x} + c_3\, e^{-3x}$ (viii) $y = c_1\, e^{-x} + c_2\, e^x + c_3\, e^{2x}$

2. (i) $y = (c_1 + c_2 x)\, e^{2x} + c_3 e^{-3x}$ (ii) $y = (c_1 - c_2\, x) e^{2x} + c_3\, e^{-2x}$

(iii) $y = c_1\, e^x + (c_2 + c_3\, x) e^{-2x}$ (iv) $y = (c_1 + c_2\, x) e^x + c_3\, e^{-x}$

(v) $y = (c_1 + c_2\, x\,) e^x + c_3\, e^{3x}$ (vi) $y = (c_1 + c_2\, x)\, e^{2x} + c_3\, e^{3x}$

(vii) $y = (c_1 + c_2 + c_3\, x^2)\, e^{-x}$ (viii) $y = (c_1 + c_2 x + c_3\, x^2)\, e^{-x} + c_4 e^{4xx}$

(ix) $y = c_1 e^{3x} + c_2\, e^{-3x} + (c_3 + c_4\, x) e^{-2x}$ (x) $y = (c_1 + c_2 x) e^{-x} + (c_3 + c_4\, x)\, e^x$

3. (i) $y = c_1 e^{-2x} + e^{-\frac{x}{2}} \left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right]$

(ii) $y = c_1 e^x + e^{\frac{-x}{2}} \left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right]$

(iii) $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 3x + c_4 \sin 3x$

4. $y = (c_1 + c_2\, x) \cos 2x + (c_3 + c_4\, x) \sin 2x$

2

Non-Homogeneous Linear Differential Equations with Constant Coefficients

CHAPTER-2

Non - Homogeneous Linear Differential Equations with constant Coefficients

2.1 INTRODUCTION

Consider n^{th} order non-homogeneous linear differential equation with cons coefficients.

$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)y = Q(x)$ ....

i.e., $f(D)\, y = Q(x)$, where $f(D) = a_0 D^n + a_1 D^{n-1} + ... + a_n$

Then $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n)\, y = 0$ ....

(or) $f(D)\, y = 0$ is the corresponding homogeneous differential equation of (1) obtaine putting $Q(x) = 0$ in (1).

2.2 GENERAL SOLUTION OF NON-HOMOGENEOUS LINEAR DIFFERENT EQUATION WITH CONSTANT COEFFICIENTS

THEOREM :

Suppose $y = u(x)$ be the general solution of $f(D)y = 0$ and $y = v(x)$ be any partic solution (containing no arbitrary constant) of $f(D)\, y = Q(x)$ then $y = u(x) + v(x)$ will b general solution or complete solution of $f(D)\, y = Q(x)$.

PROOF :

Since $y = u(x)$ is the general solution of $f(D)\, y = 0$ then $(a_0 D^n + a_1 D^{n-1} + ... a_n)\, u = ($

$$\text{i.e., } a_0 \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + ... + a_n u = 0 \quad ....$$

and $y = v(x)$ be any particular solution of $f(D)\, y = Q(x)$

then $(a_0 D^n + a_1 D^{n-1} + ... + a_n)\, v = Q(x)$

$$\text{i.e., } a_0 \frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + ... + a_n v = Q(x) \quad ....$$

Adding (3) and (4), we have

$$a_0 \frac{d^n}{dx^n}(u+v) + a_1 \frac{d^{n-1}}{dx^{n-1}}(u+v) + ... + a_n (u+v) = Q(x).$$

This shows that $y = u + v$ is the general solution or complete solution of $f(D)\, y = X$.

Hence the general solution of f(D) y = Q(x), consists two parts. The first part 'u' which contains the number of arbitrary constants is equal to the order of the differential equation is called the complementary function (C.F) of f(D) y = Q(x) and the second part 'v' which contains no arbitrary constants is called the particular integral (P.I) of f(D) y = Q(x).

$\therefore$ The complete (or) general solution of f(D) y = Q(x) is

$$\boxed{y = CF + PI}$$

While finding the general solution of f(D) y = Q(x).

First find the C.F. i.e., the general solution of f(D) y = 0 and then the P.I. i.e., particular solution of f(D) y = Q(x).

Methods of finding C.F. already discussed in chapter1. Now will discuss to find P.I.

2.3 INVERSE OPERATOR

Since 'D' stand for differentiation, then $\frac{1}{D}$ or D^{-1} stands for integration. Here $\frac{1}{D}$ or D^{-1} is known as inverse operator of D.

Note : $\frac{1}{f(D)}$ stands for inverse operator of f(D).

For example, $\frac{1}{D}x = \int x\,dx = \frac{x^2}{2}$,

While finding the particular integral no need to take the arbitrary constant 'c'.

2.3.1 *TO FIND P.I. OF f(D)y = Q(x)*

Given differential equation is

$$f(D)\,y = Q(x) \qquad \text{.............. (1)}$$

Substituting $y = \frac{1}{f(D)}Q(x)$ in (1), we get

$$f(D)\left\{\frac{1}{f(D)}Q(x)\right\} = Q(x)$$

or $\qquad Q(x) = Q(x)$

Thus $\frac{1}{f(D)}Q(x)$ satisfies the equation $f(D)\,y = Q(x)$. Therefore $\frac{1}{f(D)}Q(x)$ is the Particular Integral (P.I) of $f(D)\,y = Q(x)$.

$$\therefore \text{ P.I} = \frac{Q(x)}{f(D)}$$

i.e., to find P.I. of $f(D)\,y = Q(x)$, we find the value of $\frac{1}{f(D)}Q(x)$

THEOREM :

If $Q(x)$ is any function of 'x' and 'α' is constant then $\frac{1}{D-\alpha}Q(x) = e^{\alpha x}\int Q(x)e^{-\alpha x}dx$.

PROOF :

Let, $$\frac{1}{D-\alpha}Q(x) = y \qquad \text{.............. (1)}$$

Operating both sides by $(D-\alpha)$, we get

$$(D-\alpha)\left[\frac{1}{D-\alpha}Q(x)\right] = (D-\alpha)\,y$$

$$\Rightarrow \quad Q(x) = (D-\alpha)\,y$$

$$\Rightarrow \quad (D-\alpha)\,y = Q(x)$$

$$\Rightarrow \quad \frac{dy}{dx} - \alpha y = Q(x) \qquad \text{.............. (2)}$$

Which is linear in y, Here $P = -\alpha$ and $Q = Q(x)$.

$$\text{I.F} = e^{\int p\,dx} = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

$\therefore$ The general solution of (2) is

$$y\,(\text{I.F}) = \int Q(x).(\text{I.F})dx$$

$$ye^{-\alpha x} = \int Q(x).e^{-\alpha x}dx$$

$$\Rightarrow \quad y = e^{\alpha x}\int Q(x)\cdot e^{-\alpha x}dx \qquad \text{.............. (3)}$$

From (1) and (3), we have

$$\boxed{\frac{1}{D-\alpha}Q(x) = e^{\alpha x}\int Q(x)e^{-\alpha x}dx}$$

EXAMPLE-1

Solve : $(D^2 - 3D + 2)\, y = e^{3x}$

Solution :

Given differential equation is

$$(D^2 - 3D + 2)\, y = e^{3x} \qquad \text{.............. (1)}$$

i.e., $f(D)\, y = Q(x)$

where, $f(D) = D^2 - 3D + 2$ and

$Q(x) = e^{3x}$

To find C.F : Auxiliary equation of (1) if $f(m) = 0$

i.e., $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$\therefore$ The complementary function of (1) is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{x} + c_2 e^{2x}$$

To find P.I : P. I of (1) is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{(D-1)(D-2)} e^{3x}$$

$$= \frac{1}{D-1}\left\{\frac{1}{D-2} e^{3x}\right\}$$

$$= \frac{1}{D-1}\left\{e^{2x}\int e^{3x}\cdot e^{-2x}dx\right\} \qquad \left[\because \frac{1}{D-\alpha}Q(x) = e^{\alpha x}\int Q(x)e^{-\alpha x}dx\right]$$

$$= \frac{1}{D-1}\left\{e^{2x}\cdot\int e^{x}dx\right\}$$

$$= \frac{1}{D-1}\left\{e^{2x} \cdot e^{x}\right\}$$

$$= \frac{1}{D-1} \cdot e^{3x}$$

$$= e^{x} \cdot \int e^{3x} \cdot e^{-x} dx$$

$$= e^{x} \cdot \int e^{2x} dx$$

$$= e^{x} \cdot \frac{e^{3x}}{2}$$

$$= \frac{e^{3x}}{2}$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}$$

Note : To find P.I, there is an alternative easy method.

2.4 METHODS OF FINDING PARTICULAR INTEGRAL OF f(D) y = Q(x) SPECIAL CASES

The particular integral of f(D) y = Q(x) can be obtained by methods that are much shor than the general method provided x is one of the following special forms.

1. When $Q(x) = e^{ax}$ or constant, where a is constant.
2. When Q(x) = Sin ax or cos ax, where a is constant.
3. When $Q(x) = x^{m}$ where m is a positive integrar
4. When $Q(x) = e^{ax}v$, where v is a function of x (not included in syllabus)

2.4.1 TO FIND P.I OF f(D) y = Q(x) WHEN $Q(x) = e^{ax}$, WHERE a IS CONSTANT

The particular integral of f(D) y = Q(x) is $\frac{Q(x)}{f(D)}$

i.e., $P.I = \frac{e^{ax}}{f(D)}$

We know that $D(e^{ax}) = ae^{ax}$

$$D^2(e^{ax}) = a^2 e^{ax}$$

$$\vdots$$

$$D^n(e^{ax}) = a^n e^{ax}$$

Now, $f(D)\, e^{ax} = (a_0 D^n + a_1 D^{n-1} + \ldots\ldots + a_n)\, e^{ax}$

$$= (a_0 a^n + a_1 a^{n-1} + \ldots\ldots + a_n)\, e^{ax}$$

$$= f(a)\, e^{ax}$$

Operating both sides with $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}\left[f(D)e^{ax}\right] = \frac{1}{f(D)}\left[f(a)e^{ax}\right]$$

$$e^{ax} = f(a) \cdot \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \quad \frac{e^{ax}}{f(a)} = \frac{e^{ax}}{f(D)}$$

$$\therefore \quad \text{P.I.} = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}, \text{ provided } f(a) \neq 0.$$

[i.e., replace D by a in f(D) provided $f(a) \neq 0$]

Case of Failure : To evaluate $\frac{1}{f(D)} e^{ax}$, when $f(a) = 0$.

If $f(a) = 0$ then the above method fails and 'a' is a root of f(D), (D – a) is a factor of f(D).

Let $f(D) = (D-a)\, \phi(D)$, where $\phi(a) \neq 0$.

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} \quad \text{(by using above results)}$$

$$= \frac{1}{\phi(a)} e^{ax} \int e^{ax} \cdot e^{-ax} dx \qquad \left[\because \frac{1}{D-\alpha} Q(x) = e^{\alpha x} \cdot \int Q(x)^{-\alpha x} dx\right]$$

$$= \frac{1}{\phi(a)} \cdot e^{ax} \cdot \int dx$$

$$= \frac{1}{\phi(a)} e^{ax} \cdot x$$

$$= x \cdot \frac{e^{ax}}{\phi(a)}, \phi(a) \neq 0$$

Similarly, $\phi(a) = 0$ then $f(D) = (D-a)^2 \phi(D)$ with $\phi(a) \neq 0$, then

$\frac{e^{ax}}{f(D)} = \frac{1}{\phi(D)(D-a)} e^{ax} = \frac{1}{\phi(a)} \cdot \frac{x^2}{2!} e^{ax}$ and also if $f(D) = (D-a)^2 \phi(D)$ with $\phi(a) \neq 0$, the

$$\boxed{P.I = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)^2 \phi(D)} = \frac{1}{\phi(a)} \cdot \frac{x^2}{2!} e^{ax}}$$

Alternative method to find P.I. when case of failure i.e., case (1) when $f(a) = 0$ then

$$P.I = \frac{e^{ax}}{f(D)} = \frac{x \cdot e^{ax}}{f'(a)}, \quad f'(a) \neq 0$$

i.e., multiply numerator with 'x' and differentiate denominator with respect to D and repla D by a.

Similarly if $f'(a) = 0$, then

$$P.I = \frac{e^{ax}}{f(D)} = \frac{x^2 e^{ax}}{f''(a)}, \; f''(a) \neq 0$$

$f''(a)$ is second derivative of f(D) then replace D by a and also if $f^{r-1}(a) = 0$ then

$$P.I = \frac{e^{ax}}{f(D)} = \frac{x^r}{f^r(a)} e^{ax}, \text{ provided } f^r(a) \neq 0$$

and $\quad f^r(a) = \frac{d^r}{dD^r}[f(D)] \text{at } D = a$

Working rule to solve $f(D)y = Q(x)$ when $Q(x) = e^{ax}$

Step-1 : Write the auxiliary equation as $f(m) = 0$

Step-2 : Solve the auxiliary equation $f(m) = 0$ for m

Step(iii) : Write the complementary function (C.F)

Step(iv) : Find particular Integral (P.I) by

$$P.I = \frac{1}{f(D)} \cdot e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}, \; f(a) \neq 0$$

i.e., replace D by a in f(D), provided $f(a) \neq 0$ if $f(a) = 0$ then use

$$\frac{e^{ax}}{f(D)} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{\phi(a)} \cdot \frac{x^r}{r!} e^{ax}, \phi(a) \neq 0$$

(or) $$\frac{e^{ax}}{f(D)} = \frac{xe^{ax}}{f'(a)}, \; f'(a) \neq 0$$

Note :

1. When $x = Q(x)$ (constant) then write $x = e^{ox}$
2. When $Q(x) = \sin hx$ or $\cosh x$

   use $\sin ax = \frac{e^{ax} - e^{-ax}}{2}$ and $\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

SOLVED EXAMPLES

EXAMPLE-1

Find the complementry function of $(D^2 + 3D + 2)\, y = 4e^{4x}$

Solution :

Given differential equation is

$(D^2 + 3D + 2)\, y = 4e^{4x}$ .... (1)

i.e., $f(D)\, y = Q(x)$

where, $f(D) = D^2 + 3D + 2$ and $Q(x) = 4e^{4x}$.

Auxiliary equation of (1) is $f(m) = 0$

i.e., $m^2 + 3m + 2 = 0$

$\Rightarrow (m + 1)(m + 2) = 0$

$\Rightarrow m = -1, m = -2$

Since the roots of A.E. are real and distinct

$\therefore$ The complementary function of (1) is

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

EXAMPLE-2

Find the particular integral of $(D^2 + 4D + 5)\, y = 13\, e^x$ [Apr. 2019]

Solution :

Given differential equation is

$(D^2 + 4D + 5)\, y = 13\, e^x$ .... (1)

i.e., $f(D)\, y = Q(x)$

where $f(D) = D^2 + 4D + 5$ and $Q(x) = 13\, e^x$.

$\therefore$ P.I. of (1) is

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2+4D+5} 13e^x$$

$$= 13 \frac{1}{1^2+4(1)+5} e^x \qquad (\because \text{ By replacing D by a } (a=1)]$$

$$= \frac{13e^x}{10}$$

EXAMPLE-3

Find particular integral of $(D^3-1)\, y = 18$ [Apr. 2019]

Solution :

Given differential equation is

$(D^3 - 1)y = 18$ .... (1)

i.e., $f(D)\, y = Q(x)$

where $f(D) = D^3 - 1$ and $Q(x) = 18$

$$\text{Particular integral} = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^3-1} \cdot 18$$

$$= \frac{1}{D^3-1} 18 \cdot e^{ox}$$

$$= 18 \cdot \frac{1}{D^3 - 1} e^{0.x}$$

$$= 18 \cdot \frac{1}{O^3 - 1} e^{0.x}$$

$$= 18 \cdot \frac{1}{-1}$$

$$= -18$$

EXAMPLE-4

Find particular integral of $(D^2-3D+2)\, y = e^x$

Solution :

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{e^x}{D^2 - 3D + 2}$$

$$= \frac{e^x}{(1)^2 - 3(1) + 2} \qquad \text{(Put D =1)}$$

$$= \frac{e^x}{O} \qquad \text{(case of failure)}$$

$$\therefore \qquad P.I = \frac{e^x}{D^2 - 3D + 2}$$

$$= \frac{e^x}{(D-1)(D-2)}$$

$$= \frac{1}{(1-2)} \left[\frac{1}{D-1} e^x\right]$$

$$= \frac{1}{-1} \cdot \frac{x}{1} e^x \qquad \left[\because \frac{1}{(D-a)^r} e^{ax} = \frac{r^r}{r!} e^{ax}\right]$$

$$= -xe^x$$

Alternative Method :

$$P.I = \frac{e^x}{D^2 - 3D + 2}$$

$$= \frac{xe^x}{2D-3} \qquad \left[\because P\,I - \frac{xe^x}{f'(D)} \text{ and } f'(D) = 2D-3\right]$$

$$= \frac{xe^x}{2(-1)-3}$$

$$= \frac{xe^x}{-1}$$

$$= -xe^x.$$

EXAMPLE-5

Solve $(D^2 - 5D + 6)\,y = 3e^{5x}$ *[Apr. 2014, 201*

Solution :

Given differential equation is

$(D^2 - 5D + 6)\,y = 3e^{5x}$ .... (1

i.e., $f(D)y = Q(x)$

Where $f(D) = D^2 - 5D + 6$ and $Q(x) = 3e^{5x}$

Auxiliary equation of (1) is $f(m) = 0$

i.e.,$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

Since the roots of A.E are real and distinct. Therefore the complementary function of (1) is

$$C.F = c_1e^{m_1x} + c_2e^{m_2x}$$

$$= c_1e^{2x} + c_2e^{3x}$$

Now particular integral of (1) is

$$P.I = \frac{1}{f(D)}Q(x)$$

$$= \frac{1}{D^2-5D+6}3e^{5x}$$

$$= 3\cdot\frac{1}{5^2-5(5)+6}e^{5x}$$

$$= 3 \cdot \frac{1}{6} e^{5x}$$

$$= \frac{1}{2} e^{5x}$$

Hence the general solution of (1) is

$$y = C.F + P.I$$

$$= c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{5x}$$

EXAMPLE-6

$(D^2 + 4D + 4)\, y = 5 + e^{-2x}$ *[Apr. 2019, 2017]*

Solution :

Given differential equation is

$(D^2 + 4D + 4)\, y = 5 + e^{-2x}$ .... (1)

i.e., $f(D)y = Q(x)$

Where $f(D) = D^2 + 4D + 4$ and $Q(x) = 5 + e^{-2x}$.

To find complementary function (C.F) :

Auxiliary equation of (1) is $f(m) = 0$

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow \quad (m + 2)^2 = 0$$

$$\Rightarrow \quad m = -2, -2$$

Since the roots of A.E. are real and equal. Therefore the complementary function of (1) is

$$y_c = (c_1 + c_2 x)\, e^{mx}$$

$$= (c_1 + c_2 x)\, e^{-2x}$$

To find particular integral (P.I) :

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 + 4D + 4}(5 + e^{-2x})$$

$$= \frac{1}{(D+2)} 5 \cdot e^{ox} + \frac{1}{(D+2)^2} e^{-2x}$$

$$= 5\cdot\frac{1}{(0+2)^2}e^{0x} + \frac{x^2}{2!}e^{-2x} \qquad \left[\because \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}\right]$$

$$= \frac{5}{4} + \frac{x^2}{2}e^{-2x}$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p$$

$$= (c_2 + c_2 x)e^{-2x} + \frac{5}{4} + \frac{x^2}{2}e^{-2x}$$

EXAMPLE-7

Solve :

(i) $(D-1)\, y = \sin h\, x$ *[Apr. 2018; 2016*

(ii) $(D^2 - D - 6)\, y = e^x \cosh 2x$ *[Apr. 2016*

Solution :

(i) Given differential equation is

$$(D^2 - 1)y = \sin hx$$

ie., $\quad f(D)\, y = Q(x)$ .... (1

where $f(D) = D^2 - 1$ and $Q(x) = \sin hx$.

Auxiliary equation of (1) is $f(m) = 0$

i.e., $\quad m^2 - 1 = 0$

$(m+1)(m-1) = 0$

$m = -1, 1$

$\therefore$ The complementary function of (1) is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$= c_1 e^{-x} + c_2 e^{x}$$

Now, $\quad P.I = \frac{1}{f(D)}\cdot Q(x)$

$$y_p = \frac{1}{D^2 - 1}\cdot \sinh x$$

$$= \frac{1}{D^2-1}\cdot\left[\frac{e^x - e^{-x}}{2}\right] \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-1}e^{x} - \frac{1}{D^2-1}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1)^2-1}e^{x} - \frac{1}{(-1)^2-1}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{O}e^{x} - \frac{1}{O}e^{-x}\right] \qquad \text{(case of failure)}$$

$$= \frac{1}{2}\left[\frac{x}{2D}e^{x} - \frac{x}{2D}e^{-x}\right] \qquad \left[\because \frac{xe^{ax}}{f'(D)} \text{ when } f(a)=0\right]$$

$$= \frac{1}{2}\left[\frac{x}{2(1)}e^{x} - \frac{x}{2(-1)}e^{-x}\right] = \frac{1}{2}\left[\frac{xe^{x}}{2} + \frac{x}{2}e^{-x}\right]$$

$$= \frac{x}{2}\left[\frac{e^{x}+e^{-x}}{2}\right] = \frac{x}{2}\cosh x$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p$$

$$= c_1e^{-x} + c_2e^{x} + \frac{x}{2}\cosh x$$

(ii) Given differential equation is

$$(D^2 - D - 6)y = e^x \cosh 2x \qquad \text{..............} \quad (1)$$

i.e., $f(D)\, y = Q(x)$

where, $f(D) = D^2 - D - 6$ and $Q(x) = e^x \cosh 2x$.

Auxiliary equation is $f(m) = 0$

i.e., $m^2 - m - 6 = 0$

$$m^2 - 3m + 2m - 6 = 0$$

$$m(m-3) + 2(m-3) = 0$$

$$m = 3, -2$$

Hence the complementary function of (1) is

$$C.F = c_1e^{m_1x} + c_2e^{m_2x}$$

$$= c_1e^{3x} + c_2e^{-2x}$$

Now, $$\text{P.I} = \frac{1}{f(D)}Q(x)$$

$$= \frac{1}{D^2 - D - 6}e^x \cdot \cosh 2x$$

$$= \frac{1}{D^2 - D - 6}e^x \cdot \left[\frac{e^{2x} + e^{-2x}}{2}\right]$$

$$= \frac{1}{D^2 - D - 6}\left[\frac{e^{3x} + e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - D - 6}(e^{3x} + e^{-x})\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - D - 6}e^{3x} + \frac{1}{D^2 - D - 6}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D+2)(D-3)}e^{3x} + \frac{1}{(-1)^2 - (-1) - 6}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(3+2)} \cdot \frac{xe^{3x}}{1} + \frac{1}{-4}e^{-x}\right] \qquad \left[\because \frac{e^{ax}}{(D-a)^r} = \frac{x^r}{r!}e^{ax}\right]$$

$$= \frac{xe^{3x}}{10} - \frac{1}{8}e^{-x}$$

$\therefore$ The general solution of (1) is

$$y = \text{C.F} + \text{P.I}$$

$$= c_1e^{3x} + c_2e^{-2x} + \frac{xe^{3x}}{10} - \frac{1}{8}e^{-x}$$

$$= c_1e^{3x} + c_2e^{-2x} + \frac{xe^{3x}}{10} - \frac{1}{8}e^{-x}$$

EXAMPLE-8

Solve : $(D^2 - 2D + 1)\, y = e^x$ *if* $y = 2,\ \dfrac{dy}{dx} = -1$ *at* $x = 0$ *[Apr. 2017]*

Solution :

Given that

$$(D^2 - 2D + 1)y = e^x \qquad \text{.............. (1)}$$

and $y = 2, \frac{dy}{dx} = -1$ at $x = 0$

Auxiliary equation is $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$\therefore$ The complementary function is

$$CF = (c_1 + c_2x)e^{m_1x}$$

$$= (c_1 + c_2x)e^x$$

Now

$$P.I = \frac{1}{D^2 - 2D + 1} \cdot e^x$$

$$= \frac{1}{(D-1)^2}e^x$$

$$= \frac{x^2}{2}e^x \qquad \left[\because \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}\right]$$

$\therefore$ The general solution is $y = C.F + P.I$

$$y = (c_1 + c_2x)e^x + \frac{x^2}{2}e^x \qquad \text{..............} \quad (2)$$

Differentiating w.r.t 'x', we get

$$\frac{dy}{dx} = (O + c_2)e^x + (c_1 + c_2x)e^x + \frac{1}{2}(2xe^x + x^2e^x) \qquad \text{..............} \quad (3)$$

$$\left[\because \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\right]$$

$$= c_2e^x + (c_1 + c_2x)e^x + \frac{1}{2}(2xe^x + x^2e^x)$$

Given that, $y = 2, \frac{dy}{dx} = -1$ at $x = 0$

$$\therefore \quad 2 = (c_1 + c_2 \cdot O)e^o + \frac{O^2}{2}e^o \Rightarrow c_1 = 2$$

$$-1 = c_2e^o + (c_1 + c_2(O))e^o + \frac{1}{2}(oe^o + o^2e^o) \Rightarrow c_1 + c_2 = -1$$

$$\Rightarrow c_2 = -1 - c_1$$

$$= -1 - 2$$
$$= -3$$

Substitute the values of c_1 and c_2 in (2), we get

$$y = (2-3x)e^x + \frac{x^2}{2}e^x,$$

which is the required solution.

EXERCISE-2.1

1. Find the complementary function of

(i) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = e^{5x}$ [Apr. 201

(ii) $(D^2 + 4D + 5)\, y = 13\, e^x$

(iii) $(D^3 - 1)\, y = 18$

2. Find the particular integral of

(i) $(D^2 + 1)\, y = 5e^{2x}$ [Apr. 20

(ii) $(D^2 + 4D + 4)\, y = e^{3x}$ [Apr. 2019, 2015; Oct.20

(iii) $(D^2 - 4D + 4)\, y = e^{2x}$

(iv) $(D^2 - 2D + 1)\, y = \cosh x$ [Apr. 20

3. **Solve :**

(i) $(D^2 - 3D + 2)\, y = e^{3x}$ [Apr. 20

(ii) $(D^2 - 3D + 2)\, y = 5e^{3x}$ [Apr. 20

(iii) (a) $(D^2 - 4D + 3)\, y = e^{2x}$ [Apr. 20

(b) $(D^2 - 4D + 3)\, y = e^{5x} + e^{2x}$ [Apr. 20

(iv) $(D^2 - 4D + 3)y = e^{-4x}$ [Apr. 20

(v) $(D^2 - 4D + 3)\, y = e^{4x}$ [Oct. 20

(vi) $(D^2 + 5D + 6)\, y = e^x$ [Apr. 2014; Oct. 20

(vii) $(D^2 + 5D + 6)\, y = e^{2x}$ [Apr. 20

(viii) $(D^2 - 5D + 6)\, y = e^{4x}$ [Apr. 2016, 2014, 20

(ix) $(D^2 - 5D - 6)\, y = 5e^{5x}$ [Apr. 2016]

(x) $(D^2 - 5D - 6)\, y = e^{2x} + e^{3x}$ [Oct. 2016]

(xi) $(D^2 + D - 6)\, y = e^{3x}$ [Apr. 2017]

(xii) $(D^2 + D - 6)y = e^{-2x} + 5$ [Apr. 2016]

(xiii) $(D^2 - D - 6)y = e^{x}$ [Apr. 2014, 2013]

(xiv) $(D^2 - D - 12)\, y = e^{2x} + e^{3x}$ [Oct. 2017, 2013]

(xv) $(D^2 - 5D + 4)\, y = 2e^{-x}$ [Apr. 2010]

(xvi) $(D^2 + 6D + 5)y = e^{2x}$ [Apr. 2017]

(xvii) $(D^2 - 6D + 5)\, y = 16e^{3x}$ [Apr. 2017]

(xviii) $(D^2 - 7D + 6)\, y = e^{2x}$ [Apr. 2017, 2011, 2010]

(xix) $(D^2 - 7D + 6)\, y = e^{3x}$ [Apr. 2016]

(xx) $(D^2 - 15D + 36)\, y = e^{4x}$ [Oct. 2011]

(xxi) $(D^2 - 9D + 18)\, y = 6 + e^{4x}$ [Apr. 2016]

(xxii) $(D^2 + 2D - 8)\, y = e^{-3x} + e^{-2x}$ [Apr. 2016]

(xxiii) $(D^2 + 6D + 4)\, y = 4 + e^{2x}$ [Apr. 2018]

(xxiv) $(D^2 - 1)\, y = e^{-2x}$ [Apr. 2011]

(xxv) $(D^2 - 4)\, y = e^{5x}$ [Apr. 2014]

(xxvi) $(4D^2 + 4D - 3)\, y = e^{2x}$ [Oct. 2014, 2013, 2012 ; Apr. 2014, 2011]

4. (i) $(D^2 - 2D + 1)\, y = e^{3x}$ [Apr. 2018]

(ii) $(D^2 + 2D + 1)\, y = 4e^{3x}$ [Apr. 2014; Oct. 2013]

(iii) $(D^2 + 4D + 4)\, y = e^{2x}$ [Oct. 2017]

(iv) $(D^2 - 6D + 9)\, y = e^{-3x}$ [Apr. 2016]

(v) $(D^2 + 6D + 9) = e^{3x}$ [Apr. 2018]

5. (i) $(D^2 + 1)\, y = 17$ [Apr. 2012]

(ii) $(D^2 + 1)y = e^{-x}$ [Apr. 2017]

(iii) $(D^2 + 1)\, y = 3 + 5e^{x}$ [Oct. 2018]

(iv) $(D^3 + 1)\, y = 3 + 5e^x$

(v) $(D^2 + 4)\, y = e^{-3x}$ [Oct. 20

(vi) $(D^2 + D + 1)\, y = e^{4x}$

(vii) $(D^2 + D + 1)\, y = 2e^{4x}$ [Apr. 20 [Oct. 20

(viii) $(D^2 + D + 3)\, y = e^{2x}$ [Apr. 2017, 20

(ix) $(D^2 + D + 4)\, y = e^{2x}$ [Oct. 20

(x) $(D^2 - 4D + 5)\, y = e^{3x}$ [Apr. 20

(xi) $(D^2 + 4D + 6)\, y = e^{2x}$ [Apr. 2019, 20

(xii) $(D^2 + 4D + 8)\, y = e^{2x}$ [Apr. 2014, 20

6. (i) $(D^2 - D + 1)\, y = \sin hx$

(ii) $(D^2 + 4D + 5)\, y = 2 \cosh x$ [Apr. 20

(iii) $(D^2 - 1)\, y = \cosh 2x$ [Oct. 2014, 2011; Apr. 20

(iv) $(D^2 - 1)\, y = 1 + \cosh 2x$ [Apr. 20

(v) $(D^2 - 8D + 16)\, y = \cosh 2x$ [Apr. 20

(vi) $(D^2 + 36)\, y = \sinh 3x$ [Apr. 20

7. (i) $(D^2 - 1)\, y = e^x$ [Apr. 20

(ii) $(D^2 - 1)\, y = \dfrac{e^x}{2}$ [Oct. 20

(iii) $(D^2 - 9)\, y = e^{3x}$ [Apr. 201

(iv) $(D^2 - 36)\, y = e^{6x}$ [Apr. 201

(v) $(D^2 - 5D + 6)\, y = e^{2x} + e^{3x}$ [Apr. 201

(vi) $(D^2 - 5D + 6)\, y = (e^x + 1)^2$ [Apr. 201

(vii) $(D^2 + 5D + 6)\, y = e^{-2x}$ [Oct. 201

(ix) $(D^2 + 3D + 2)\, y = e^{-x} + e^{-2x}$ [Apr. 201

(x) $(D^2 - 3D + 2)\, y = (e^x + 1)^2$ [Apr. 201

(xi) $(D^2 - 4D + 3)\, y = e^{3x}$ [Apr. 201

(xii) $(D^2 - 4D + 3)\, y = e^x + e^{3x}$

(xiii) $(D^2 + 4D + 3)y = e^{-3x}$

(xiv) $(D^2 - 7D + 10)y = 3e^{5x}$ [Oct. 2016]

(xv) $(D^2 + 2D - 8)\ y = e^{-3x} + e^{-4x}$ [Apr. 2018, 2017, 2008]

(xvi) $(D^2 + D - 6)\ y = 5e^{2x}$ [Apr. 2018, 2017]

(xvii) $(D^2 + D - 6)\ y = 5e^{2x} + 6$

(xviii) $(D^2 + D - 6)\ y = 5e^{2x} + e^{-3x}$ [Apr. 2018]

(xix) $(D^2 + D - 6)\ y = 1 + e^{-3x}$ [Apr. 2013]

(xx) $(D^2 + D - 6)\ y = e^{3x} + e^{-3x}$ [Oct. 2012, 2009, 2008; Apr. 2017, 2012, 2009]

(xxi) $(D^2 - D - 6)\ y = e^{-2x}$ [Oct. 2016]

(xxii) $(2D^2 + D - 6)\ y = e^{-2x}$ [Oct. 2016]

(xxiii) $(D^2 - 4)\ y = (1 + e^x)^2$

(xxiv) $(D^3 + 1)\ y = e^{-x}$ [Apr. 2010]

8. (i) $(D^2 - 2D + 1)y = e^x + 1$ [Apr. 2017]

(ii) $(D^2 + 4D + 4)\ y = e^{-2x}$ [Apr. 2016]

(iii) $(D^2 + 4D + 4)\ y = 2e^{-3x}$

(iv) $(D^2 - 14D + 49)\ y = e^{7x}$ [Oct. 2009]

9. (i) $(D^2 - 1)\ y = \cosh x$

(ii) $(D^2 - 9D + 18)\ y = \cosh 3x$ [Apr. 2007; Oct. 2008]

(iii) $(D^2 - 16)\ y = \cosh 4x$ [Oct. 2016]

(iv) $(D^2 - 3D + 2)\ y = \cosh 2x$ [Apr. 2016]

(v) $(D^2 + 40 + 4)\ y = 2 \sin h\ 2x$

10. (i) $(D^2 - 6D + 9)\ y = e^x \sin hx$ [Apr. 2016]

(ii) $(D^2 - 2D + 1)\ y = e^x \cosh 2x$ [Apr. 2013]

11. $(D^2 - 3D + 2)\ y = e^x$ if $y = 3$ and $\dfrac{dy}{dx} = 3$ when $x = 0$ [Oct. 2006; 2009]

ANSWERS

1. (i) $\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

(ii) $e^{-2x}\left[c_1 \cos x + c_2 \sin x\right]$

(iii) $c_1 e^{x} + e^{-\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

2. (i) e^{2x} (ii) $\frac{e^{3x}}{25}$

(iii) $\frac{x^2 e^{2x}}{2}$

(iv) $\frac{x^2}{4}e^{x} + \frac{e^{-x}}{8}$

3. (i) $c_1e^{x} + c_2e^{2x} + \frac{e^{3x}}{2}$ (ii) $c_1e^{x} + c_2e^{2x} + \frac{5e^{3x}}{2}$

(iii) $c_1e^{x} + c_2e^{3x} - e^{2x}$ (iv) $c_1e^{x} + c_2e^{3x} + \frac{e^{-4x}}{2}$

(v) $c_1e^{x} + c_2e^{3x} + \frac{e^{4x}}{3}$ (vi) $c_1e^{-2x} + c_2e^{-3x} + \frac{e^{x}}{12}$

(vii) $c_1e^{-2x} + c_2e^{-3x} + \frac{e^{2x}}{20}$ (viii) $c_1e^{2x} + c_2e^{3x} + \frac{e^{4x}}{2}$

(ix) $c_1e^{2x} + c_2e^{3x} + \frac{5e^{5x}}{6}$ (x) $c_1e^{-x} + c_2e^{6x} - \frac{e^{2x}}{12} - \frac{e^{3x}}{12}$

(xi) $c_1e^{2x} + c_2e^{-3x} + \frac{e^{3x}}{6}$ (xii) $c_1e^{2x} + c_2e^{-3x} - \frac{e^{-2x}}{4} - \frac{5}{6}$

(xiii) $c_1e^{-2x} + c_2e^{3x} - \frac{e^{x}}{6}$ (xiv) $c_1e^{-3x} + c_2e^{4x} - \frac{e^{2x}}{10} - \frac{e^{3x}}{6}$

(xv) $c_1e^{x} + c_2e^{4x} + \frac{e^{-x}}{5}$ (xvi) $c_1e^{-x} + c_2e^{-5x} + \frac{e^{2x}}{21}$

(xvii) $c_1e^{-x}+c_2e^{-5x}+\frac{e^{3x}}{2}$ (xviii) $c_1e^{x}+c_2e^{6x}-\frac{1}{4}e^{2x}$

(xix) $c_1e^{x}+c_2e^{6x}-\frac{e^{3x}}{6}$ (xx) $c_1e^{3x}+c_2e^{12x}-\frac{e^{4x}}{8}$

(xxi) $c_1e^{3x}+c_2e^{6x}+\frac{1}{3}-\frac{e^{4x}}{2}$ (xxii) $c_1e^{2x}+c_2e^{-4x}-\frac{1}{5}e^{-3x}-\frac{1}{8}e^{-2x}$

(xxiii) $c_1e^{(-3+\sqrt{5})x}+c_2e^{(-3-\sqrt{5})x}+1+\frac{e^{2x}}{20}$

(xxiv) $c_1e^{x}+c_2e^{x}+\frac{e^{-2x}}{3}$ (xxv) $c_1e^{2x}+c_2e^{-2x}+\frac{e^{5x}}{21}$

(xxvi) $c_1e^{\frac{x}{2}}+c_2e^{-\frac{3}{2}x}+\frac{e^{2x}}{21}$

4. (i) $(c_1+c_2x)e^{x}+\frac{e^{3x}}{4}$ (ii) $(c_1+c_2x)e^{-x}+\frac{e^{3x}}{4}$

(iii) $(c_1+c_2x)e^{-2x}+\frac{e^{2x}}{16}$ (iv) $(c_1+c_2x)e^{3x}+\frac{e^{-3x}}{4}$

(v) $(c_1+c_2x)e^{-3x}+\frac{e^{3x}}{36}$

5. (i) $c_1\cos x+c_2\sin x+17$ (ii) $c_1\cos x+c_2\sin x+\frac{e^{-x}}{2}$

(iii) $c_1\cos x+c_2\sin x+3+\frac{5e^{x}}{2}$ (iv) $c_1e^{-x}+e^{\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+3+\frac{5e^{x}}{2}$

(v) $c_1(o)2x+c_2\sin 2x+\frac{e^{-3x}}{13}$ (vi) $(c_1+c_2x)e^{x}+\frac{e^{3x}}{8}+\frac{e^{-x}}{8}$

(vii) $e^{-\frac{x}{2}}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]+\frac{2e^{4x}}{21}$

(viii) $e^{-\frac{x}{2}}\left[c_1\cos\frac{\sqrt{11}}{2}x+c_2\sin\frac{\sqrt{11}}{2}x\right]+\frac{e^{2x}}{9}$

(ix) $e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{15}}{2}x + c_2 \sin\frac{\sqrt{15}}{2}x\right) + \frac{e^{2x}}{10}$

(x) $e^{2x}[c_1 \cos x + c_2 \sin x] + \frac{e^{3x}}{2}$ (xi) $e^{-2x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{e^{2x}}{10}$

(xii) $e^{-2x}[c_1 \cos 2x + c_2 \sin 2x] + \frac{e^{2x}}{20}$

6. (i) $e^{\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + \frac{e^{x}}{2} - \frac{e^{-x}}{6}$

(ii) $e^{-2x}[c_1 \cos x + c_2 \sin x] + \frac{e^{x}}{10} + \frac{x}{2}$ (iii) $c_1e^{x} + c_2e^{-x} + \frac{1}{6}(e^{2x} + e^{-2x})$

(iv) $c_1e^{x} + c_2e^{-x} - 1 + \frac{1}{6}(e^{2x} + e^{-2x})$ (v) $(c_1 + c_2x)e^{4x} + \frac{e^{2x}}{8} + \frac{e^{-2x}}{72}$

(vi) $c_1 \cos 6x + c_2 \sin 6x + \frac{1}{90}(e^{3x} - e^{-3x})$

7. (i) $c_1e^{x} + c_2e^{-x} + \frac{xe^{x}}{2}$ (ii) $c_1e^{x} + c_2e^{-x} + \frac{xe^{x}}{4}$

(iii) $c_1e^{3x} + c_2e^{-3x} + \frac{xe^{3x}}{6}$ (iv) $c_1e^{6x} + c_2e^{-6x} + \frac{xe^{6x}}{12}$

(v) $c_1e^{2x} + c_2e^{3x} - xe^{2x} + xe^{3x}$ (vi) $c_1e^{2x} + c_2e^{3x} - xe^{2x} + e^{x} + \frac{1}{6}$

(vii) $c_1e^{-2x} + c_2e^{-3x} + xe^{-2x}$ (viii) $c_1e^{-x} + c_2e^{-2x} + xe^{-x} - xe^{-2x}$

(ix) $c_1e^{x} + c_2e^{2x} + xe^{2x} - 2e^{x} + \frac{1}{2}$ (x) $c_1e^{x} + c_2e^{3x} + \frac{xe^{3x}}{2}$

(xi) $c_1e^{x} + c_2e^{3x} - \frac{xe^{x}}{2} + \frac{xe^{3x}}{2}$ (xii) $c_1e^{-x} + c_2e^{-3x} - \frac{xe^{-3x}}{2}$

(xiii) $c_2e^{2x} + c_2e^{5x} + xe^{5x}$ (xiv) $c_1e^{2x} + c_2e^{-4x} - \frac{e^{-3x}}{5} - \frac{xe^{-4x}}{6}$

(xv) $c_1e^{2x}+c_2e^{-3x}+xe^{2x}$ (xvi) $c_1e^{2x}+c_2e^{-3x}+xe^{2x}-1$

(xvii) $c_1e^{2x}+c_2e^{-3x}+xe^{2x}-\frac{xe^{-3x}}{5}$ (xviii) $c_1e^{2x}+c_2e^{-3x}-\frac{1}{6}-\frac{x}{5}e^{-3x}$

(xix) $c_1e^{2x}+c_2e^{-3x}+\frac{e^{3x}}{6}-\frac{xe^{-3x}}{5}$ (xx) $c_1e^{-2x}+c_2e^{3x}-\frac{xe^{-2x}}{5}$

(xxi) $c_1e^{\frac{3}{2}x}+c_2e^{-2x}-\frac{xe^{-2x}}{7}$ (xxii) $c_1e^{2x}+c_2e^{-2x}-\frac{1}{4}-\frac{2}{3}e^{x}+\frac{xe^{2x}}{4}$

(xxiii) $c_1e^{-x}+e^{\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{xe^{-x}}{3}$

8 (i) $(c_1+c_2x)e^{x}+\frac{x^2}{2}e^{x}+1$

(ii) $(c_1+c_2x)e^{-2x}+\frac{x^2}{2}e^{-2x}$

(iii) $(c_1+c_2x)^{-3x}+x^2e^{-3x}$

(iv) $(c_1+c_2x)e^{7x}+\frac{x^2}{2}e^{7x}$

9. (i) $c_1e^{x}+c_2e^{-x}+\frac{1}{4}(xe^{x}-xe^{-x})$ (ii) $c_1e^{3x}+c_2e^{6x}-\frac{x}{6}e^{3x}+\frac{e^{-3x}}{108}$

(iii) $c_1e^{4x}+c_2e^{-4x}+\frac{x}{8}\sinh 4x$ (iv) $c_1e^{x}+c_2e^{2x}+\frac{x}{2}e^{2x}+\frac{e^{-2x}}{24}$

(v) $(c_1+c_2x)e^{-2x}+\frac{e^{2x}}{16}-\frac{x^2}{2}e^{-2x}$

10. (i) $(c_1+c_2x)e^{3x}+\frac{e^{2x}}{2}-\frac{1}{18}$

(ii) $(c_1+c_2x)e^{x}+\frac{e^{3x}}{8}+\frac{e^{-x}}{8}$

(iii) $2x^{x}+e^{2x}-xe^{x}$

2.5 P.I OF $f(D)\,y = Q(x)$ WHEN $Q(x) = \sin ax$ or $\cos ax$

where a is any constant

The particular integral of $f(D)\,y = Q(x)$ is

$$P.\,I = \frac{Q(x)}{f(D)} = \frac{\sin ax}{f(D)}$$

We know that

$$D(\sin ax) = a\cos ax\,; \qquad D^2(\sin ax) = -a^2 \sin ax$$

$$D^3(\sin ax) = -a^3 \cos ax\ : \ D^4(\sin ax) = a^4 \sin ax = (-a^2)^2 \sin ax$$

$$D^5(\sin ax) = a^5 \cos ax\ : \quad D^6(\sin ax) = -a^6 \sin ax = (-a^2)^3 \sin ax.$$

•

•

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Hence $\quad D^2(\sin ax) = (-a)^2 \sin ax$

$$(D^2)^2(\sin ax) = (-a)^2 \sin ax$$

$$(D^2)^3 \sin ax = (-a^2)^3 \sin ax$$

•

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•

$$(D^2)^n \sin ax = (-a^2)^n \sin ax$$

$$\Rightarrow \qquad f(D^2)\sin ax = f(-a^2)\sin ax$$

Operating $\frac{1}{f(D^2)}$ on both sides, we get

$$\frac{1}{f(D^2)}\left\{f(D^2)\sin ax\right\} = \frac{1}{f(D^2)}\left\{f(-a^2)\sin ax\right\}$$

$$\Rightarrow \qquad \sin ax = f(-a^2)\cdot\frac{1}{f(D^2)}\sin ax$$

$$\Rightarrow \qquad \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax, \quad f(-a^2) \neq 0$$

Therefore $P.I = \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax, \quad f(-a^2) \neq 0$

Similarly $P.I = \frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax, \quad f(-a^2) \neq 0$

Note : Replace D^2 by $-a^2$, D^3 by $D^2 D = -a^2 D$, $D^4 = (D^2) = (-a^2)^2$ etc., Now with these substitutions only first powers of D will be left in f(D), and f(D) shall be reduce to a linear factor of the form $lD \pm m$ then multiply numerator and denominator by a conjugate factor $lD \mp m$, then denominator shall be reduce to $l^2m^2 - m^2$ in which replace D^2 by $-a^2$ and it will become a constant and the numerator shall be $(lD \mp m,)$ sin ax. Here D stands for differentiation and hence it is easily evaluated.

2.5.1 CASE OF FAILURE

1. To evaluate $\frac{1}{f(D^2)}\sin ax$ or $\frac{1}{f(D^2)}\cos ax$ when $f(-a^2) = 0$.

If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $f(D^2)$

$\therefore f(D^2) = (D^2 + a^2)\ \phi(D^2)$ with $\phi(-a^2) \neq 0$.

Thus,
$$P.I = \frac{1}{f(D)^2}\sin ax (\text{or}) \cos ax$$
$$= \frac{1}{(D^2+a^2)\phi(D^2)}\sin ax\ (\text{or})\ \cos ax$$
$$= \frac{1}{\phi(-a^2)}\cdot\frac{1}{D^2+a^2}\sin ax (\text{or})\ \cos ax$$

To evaluate $\frac{\sin ax}{D^2+a^2}$ or $\frac{\cos ax}{D^2+a^2}$:

We know that
$$e^{iax} = \cos ax + i \sin ax$$
i.e., cos ax real part of e^{iax}

sin ax = imaginary part of e^{iax}

Now,
$$\frac{e^{iax}}{D^2+a^2} = \frac{e^{iax}}{(D+ai)(D-ai)}$$
$$= \frac{1}{(ai+ai)}\left\{\frac{e^{iax}}{D-ai}\right\}$$
$$= \frac{1}{2ai}\cdot\frac{xe^{iax}}{1} \qquad \left[\because \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}\right]$$
$$= \frac{xe^{iax}}{2ai}$$

$$= \frac{ixe^{iax}}{2ai^2}$$

$$= -\frac{ix}{2a}e^{iax} \qquad [\because i^2 = -1]$$

i.e., $$\frac{\cos ax + i\sin ax}{D^2+a^2} = -\frac{ix}{2a}[\cos ax + i\sin ax]$$

$$= -\frac{ix\cos ax}{2a} - \frac{i^2 x\sin ax}{2a}$$

$$\frac{\cos ax}{D^2+a^2} + i\frac{\sin ax}{D^2+a^2} = \frac{x\sin ax}{2a} - i\frac{x\cos ax}{2a}$$

Comparing real and imaginary parts on both sides, we get

$$\boxed{\frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax} \quad \text{and} \quad \boxed{\frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax}$$

(or)

(i) $$\frac{1}{D^2+a^2}\cos ax = \frac{x}{\frac{d}{dD}(D^2+a^2)}\cos ax = \frac{x}{2D}\cos ax = \frac{x}{2}\left(\frac{1}{D}\cos ax\right) = \frac{x}{2}\int \cos ax = \frac{x}{2a}\sin ax$$

(ii) $$\frac{1}{D^2+a^2}\sin ax = \frac{x}{\frac{d}{dD}(D^2+a^2)}\sin ax = \frac{x}{2D}\sin ax = \frac{x}{2}\left[\frac{1}{D}\sin ax\right] = \frac{x}{2}\int \sin ax\,dx = \frac{-x}{2a}\cos ax$$

Working Rule :

Case-1 : When $f(-a^2) \neq 0$. then,

(i) $$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$$

(ii) $$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$$

i.e., replace D^2 by $-a^2$ in $f(D^2)$

Case-2 : When $f(-a^2) = 0$. Then use

$$\frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax \quad \text{and} \quad \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$$

Note : 1. To find P.I. for $\sin^2 x$, $\cos^2 x$, $\sin^3 x$ and $\cos^3 x$. then use (i) $\sin^2 x = \frac{1-\cos 2x}{2}$,

(ii) $\cos^2 x = \frac{1+\cos 2x}{2}$, (iii) $\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$ and (iv) $\cos^3 x = \frac{1}{4}\left[3\cos x + \cos 3x\right]$

2. To find P.I for the functions of the form sin A cosB, cosA sinB, cosA cos B and sin A sin B, then use

(i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \cos B = \cos (A + B) - \sin (A - B)$

(iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \sin B = \cos (A - B) - \cos(A + B)$

SOLVED EXAMPLES

EXAMPLE-1

Find particular integral of

(i) $(D^2 + 16)\, y = \sin 3x$ *[Apr. 2019, 2018, 2017]*

(ii) $(D^2 - 9)\, y = \cos 3x$ *[Apr. 2019]*

(iii) $(D^2 + 1)\, y = 8 \sin x$ *[Apr. 2016]*

(iv) $(D^2 + 4)\, y = \cos 2x$ *[Apr. 2018]*

(v) $(D^2 + 2D + 1)\, y = \sin x$ *[Apr. 2016]*

Solution :

(i) Particular integral $= \frac{Q(x)}{f(D)}$

$$= \frac{\sin 3x}{D^2+16}$$

$$= \frac{\sin 3x}{-3^2+16} \qquad [\text{put } D^2 = -3^2]$$

$$= \frac{\sin 3x}{-9+16}$$

$$= \frac{\sin 3x}{7}$$

(ii) Given differential equation is

$$(D^2 - 9)\,y = \cos 3x$$

Here $f(D) = D^2 - 9$ and $X = \cos 3x$

$$\text{Particular Integral} = \frac{Q(x)}{f(D)}$$

$$= \frac{\cos 3x}{D^2 - 9}$$

$$= \frac{\cos 3x}{-3^2 - 9} \qquad [\because \text{ put } D^2 = -a^2 = -3^2]$$

$$= \frac{\cos 3x}{-9-9}$$

$$= \frac{\cos 3x}{-18}$$

$$= \frac{-1}{18}\cos 3x$$

(iii)

$$\text{P.I} = \frac{Q(x)}{f(D)}$$

$$= \frac{8\sin x}{D^2 + 1}$$

$$= 8\frac{\sin x}{-1^2 + 1}$$

$$= \frac{8\sin x}{O} \quad \text{case of failure}$$

$$\therefore \quad \text{P.I.} = \frac{8\sin x}{D^2 + 1}$$

$$= 8\left(\frac{-x}{2(1)}\cos x\right) \qquad \left[\because \frac{1}{D^2 + a^2}\sin ax = \frac{-x}{2a}\cos ax\right]$$

$$= \frac{-8x}{2}\cos x$$

$$= -4x\cos x.$$

(iv)

$$\text{P.I} = \frac{Q(x)}{f(D)}$$

$$= \frac{\cos 2x}{D^2+4}$$

$$= \frac{\cos 2x}{D^2+2^2}$$

$$= \frac{\cos 2x}{-2^2+2^2}$$

$$= \frac{\cos 2x}{0}, \text{ case of failure}$$

$$\therefore \text{P.I} = \frac{\cos 2x}{D^2+2^2}$$

$$= \frac{x}{2(2)}\sin 2x \qquad \left[\because \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax\right]$$

$$= \frac{x}{4}\sin 2x$$

(v) Given differential equation is

$$(D^2+2D+1)\,y = \sin x \qquad \text{..............} \quad (1)$$

i.e., $f(D)y = Q(x)$

where $f(D) = D^2+2D+1;\quad Q(x) = \sin x$

$$\text{P.I.} = \frac{Q(x)}{f(D)}$$

$$= \frac{\sin x}{D^2+2D+1}$$

$$= \frac{\sin x}{-1^2+2D+1} \qquad [\text{Put } D^2 = -a^2 = -1^2]$$

$$= \frac{\sin x}{2D}$$

$$= \frac{1}{2}\cdot\left(\frac{1}{D}\cdot\sin x\right)$$

$$= \frac{1}{2}\left(\int \sin x\,dx\right)$$

$$= \frac{1}{2}\cdot(-\cos x)$$

$$= -\frac{1}{2}\cos x$$

EXAMPLE-2

Solve :

(i) $(D^2 - 4)\, y = \cos 3x$ *[Apr. 2015]*

(ii) $(D^2 + 1)\, y = \cos 3x$ *[Oct. 2014; Apr. 2013]*

(iii) $(D^2 + 4)\, y = 5 + \sin 2x$ *[Oct. 2016]*

Solution :

(i) Given differential equation is

$$(D^2 - 4)\, y = \cos 3x \qquad \ldots\ldots\ldots\ldots (1)$$

i.e., $f(D)y = Q(x)$

where, $f(D) = D^2 - 4$ and $Q(x) = \cos 3x$.

Auxiliary equation of (1) is $f(m) = 0$.

$$m^2 - 4 = 0$$

$$(m + 2)(m - 2) = 0$$

$$m = \pm 2$$

Thus complementary function of (1) is

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

Particular Integral of (1) is

$$y_p = \frac{Q(x)}{f(D)}$$

$$= \frac{\cos 3x}{D^2 - 4}$$

$$= \frac{\cos 3x}{-3^2 - 4} \qquad [\because \text{ put } D^2 = -a^2 = -3^2]$$

$$= \frac{\cos 3x}{-9 - 4}$$

$$= \frac{-1}{13}\cos 3x$$

∴ The general solution of (1) is

$$y = y_c + y_p$$

$$\text{i.e.,} = c_1e^{2x} + c_2e^{-2x} - \frac{1}{13}\cos 3x$$

(ii) Given differential equation is

$$(D^2 + 1)\, y = \cos 3x \quad \text{.............. (1)}$$

Here $f(D) = D^2 + 1$ and $Q(x) = \cos 3x$

Auxiliary equation of (1) is $f(m) = 0$

$$\Rightarrow \quad m^2 + 1 = 0$$

$$\Rightarrow \quad m^2 = -1$$

$$\Rightarrow \quad m = \pm i$$

∴ The complementary function of (1) is

$$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

$$= e^{ox}[c_1 \cos x + c_2 \sin\beta x]$$

$$= c_1 \cos x + c_2 \sin x$$

Particular Integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{\cos 3x}{D^2 + 1}$$

$$= \frac{\cos 3x}{-3^2 + 1} \qquad [\because \text{Put } D^2 = -a^2 = -3^2]$$

$$= \frac{\cos 3x}{-9 + 1}$$

$$y = \frac{-1}{8}\cos 3x$$

∴ The general solution of (1) is

$$y = C.F + P.I$$

i.e., $$y = c_1 \cos x + c_2 \sin x - \frac{1}{8}\cos 3x$$

(iii) Given differential equation is

$$(D^2 + 4)\, y = 5 + \sin 2x$$

Here $f(D) = D^2 + 4$ and $Q(x) = 5 + \sin 2x$

Auxiliary equation is $f(m) = 0$

$$m^2 + 4 = 0$$

$$m = \pm 2i \quad (\alpha = 0 \ \&\ \beta = 2)$$

$\therefore$ The complementary function of (1) is

$$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$$

$$= e^{ox}[c_1 \cos 2x + c_2 \sin 2x]$$

$$= c_1 \cos 2x + c_2 \sin 2x$$

Particular integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^2+4}\,[5 + \sin 2x]$$

$$= \frac{1}{D^2+4} 5e^{ox} + \frac{1}{D^2+2^2}\sin 2x$$

$$= \frac{5}{D^2+4} e^{ox} + \frac{-x}{2(2)}\cos 2x \quad \left[\because \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax\right]$$

$$= \frac{5}{4} - \frac{x}{4}\cos 2x$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

i.e., $$y = c_1 \cos 2x + c_2 \sin 2x + \frac{5}{4} - \frac{x}{4}\cos 2x.$$

(iv) Given differential equation is

$$(D^3 + 4D)\, y = 5 + \sin 2x \qquad \text{.........}$$

Compare (1) with f(D) y = Q(x), we have

$$f(D) = D^3 + 4D \text{ and } Q(x) = 5 + \sin 2x$$

Auxiliary equation of (1) is f(m) = 0

$$m^3 + 4m = 0$$

$$m(m^2 + 4) = 0$$

$$m = 0, \quad m = \pm 2i$$

∴ The complementary function of (1) is

$$y_c = c_1 e^{ox} + e^{ox}[c_2 \cos 2x + c_3 \sin 2x]$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x$$

Particular integral of (1) is

$$y_p = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^3 + 4D}(5 + \sin 2x)$$

$$= \frac{1}{D^3 + 4D}5 + \frac{1}{D^3 + 4D}\sin 2x$$

$$= \frac{1}{D(D^2 + 4)}5.e^{ox} + \frac{1}{D(D^2 + 4)}\sin 2x$$

$$= 5 \cdot \frac{1}{D^2 + 4} \cdot \frac{1}{D}e^{ox} \cdot + \frac{1}{D^2 + 4} \cdot \int \sin 2x\, dx$$

$$= \frac{5}{4} \cdot \frac{x}{1}e^{ox} + \frac{1}{D^2 + 4}\left(-\frac{\cos 2x}{2}\right) \quad \left[\because \int \sin ax\, dx\right]$$

$$= \frac{5x}{4} - \frac{1}{2} \cdot \frac{1}{D^2 + 4}(\cos 2x)$$

$$= \frac{5x}{4} - \frac{1}{2} \cdot \frac{x}{2(2)}\sin 2x \qquad \left[\because \frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax\right]$$

$$= \frac{5x}{4} - \frac{x}{8}\sin 2x$$

∴ The general solution of (1) is

$$y = y_c + y_p$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x + \frac{5x}{4} - \frac{x}{8}\sin 2x$$

EXAMPLE-3

Solve :

(i) $(D^2 + 3D + 2)\, y = \sin 3x$ [Apr. 20

(ii) $(D^2 + 2D - 8)\, y = e^{-4x} + \cos x$ [Apr. 20

(iii) $(D^2 - 4)\, y = \cos^2 x$ [Oct. 2014, 2012; Apr. 20

(iv) $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$ [Apr. 2017, 20

Solution :

(i) Given differential equation is

$$(D^2 + 3D + 2)\, y = \sin 3x$$

Here $f(D) = D^2 + 3D + 2$ and $Q(x) = \sin 3x$

Auxiliary equation of (1) is $f(m) = 0$

$$m^2 + 3m + 2 = 0$$

$$(m + 1)(m + 2) = 0$$

$$m = -1, -2$$

$\therefore$ Complementary function of (1) is

$$C.F = c_1 e^{-x} + c_2 e^{-2x}$$

Particular Integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{\sin 3x}{D^2 + 3D + 2}$$

$$= \frac{\sin 3x}{-3^2 + 3D + 2} \qquad [\because \text{ Replace } D^2 \text{ by } -3^2]$$

$$= \frac{\sin 3x}{-9 + 3D + 2}$$

$$= \frac{\sin 3x}{3D - 7}$$

$$= \frac{1}{3D - 7} \times \frac{3D + 7}{3D + 7} \sin 3x$$

$$= \frac{(3D+7)}{9D^2-49}(\sin 3x)$$

$$= \frac{1}{9(-3^2)-49}\cdot(3D+7)\sin 3x$$

$$= \frac{1}{-81-49}\ [3D(\sin 3x)+7\sin 3x]$$

$$= \frac{1}{-130}[3(\cos 3x\cdot 3)+7\sin 3x]$$

$$= \frac{1}{-130}\ [9\cos 3x+7\sin 3x]$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

$$= c_1e^{-x}+c_2e^{-2x}-\frac{1}{130}\ (9\cos 3x + 7\sin 3x)$$

(ii) Given differential equation is

$(D^2 + 2D - 8)\,y = e^{-4x} + \cos x$ .... (1)

Here $f(D) = D^2 + 2D - 8$ and $Q(x) = e^{-4x} + \cos x$

Auxiliary equation of (1) is $f(m) = 0$

$$m^2 + 2m - 8 = 0$$

$$m^2 + 4m - 2m - 8 = 0$$

$$m(m + 4) - 2(m + 4) = 0$$

$$(m - 2)\,(m + 4) = 0$$

$$m = 2, -4$$

$\therefore$ The complementary function of (1) is

$$C.F = c_1e^{2x} + c_2e^{-4x}$$

The particular integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^2+2D-8}(e^{-4x}+\cos x)$$

$$= \frac{1}{D^2+2D-8}e^{-4x} + \frac{1}{b^2+2D-8}\cos x$$

$$= \frac{1}{(D-2)(D+4)}e^{-4x} + \frac{1}{-1^2+2D-8}\cos x$$

$$= \frac{1}{(-4-2)}\cdot\frac{x}{1!}e^{-4x} + \frac{1}{2D-9}\cos x \qquad \left[\because \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}\right.$$

$$= \frac{-x}{6}e^{-4x} + \frac{1}{2D-9}\times\frac{2D+9}{2D+9}\cos x$$

$$= \frac{-x}{6}e^{-4x} + \frac{2D+9}{4D^2-81}\cos x$$

$$= -\frac{x}{6}e^{-4x} + \frac{[2D(\cos x)+9\cos x]}{4(-1^2)-81}$$

$$= -\frac{x}{6}e^{-4x} + \frac{2(-\sin x)+9\cos x}{-85}$$

$$= -\frac{x}{6}e^{-4x} + \frac{2\sin x - 9\cos x}{85}$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

$$= c_1e^{2x} + c_2e^{-4x} - \frac{x}{6}e^{-4x} + \frac{1}{85}(2\sin x - 9\cos x)$$

(iii) Given differential equation is

$$(D^2-4)\,y = \cos^2 x$$

Here $f(D) = D^2 - 4$ and $Q(x) = \cos^2 x$

Auxiliary equation of (1) is $f(m) = 0$

$$m^2 - 4 = 0$$

$$m = \pm 2$$

$\therefore$ Complementary function is

$$C.F = c_1e^{-2x} + c_2e^{2x}$$

Particular integral is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^2-4}\cdot\cos^2 x$$

$$= \frac{1}{D^2-4}\left(\frac{1+\cos 2x}{2}\right) \quad \left[\because \cos^2 A = \frac{1+\cos 2A}{2}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-4}(1) + \frac{1}{D^2-4}(\cos 2x)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{O^2-4}e^{ox} + \frac{1}{-2^2-4}\cos 2x\right\}$$

$$= \frac{-1}{8}e^{ox} - \frac{1}{16}\cos 2x$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

$$= c_1e^{-2x} + c_2e^{2x} - \frac{1}{8} - \frac{1}{16}\cos 2x$$

(iv) Given differential equation is

$(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$ .... (1)

Compare (1) with $f(D)\, y = Q(x)$, we have

$f(D) = D^2 - 4D + 3, \quad Q(x) = \sin 3x \cos 2x.$

A.E. is $f(m) = 0$

$m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$

$\therefore$ Complementary function of (1) is

$$C.F = c_1e^x + c_2\, e^{3x}$$

Particular integral of (1) is

$$P.I = \frac{Q(x)}{f(D)} = \frac{\sin 3x \cos 2x}{D^2-4D+3}$$

$$= \frac{1}{D^2-4D+3}\left\{\frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)]\right\}$$

$$\left[\because \sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{D^2-4D+3}(\sin 5x) + \frac{1}{D^2-4D+3}\sin x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{-5^2-4D+3}\sin 5x + \frac{1}{-1^2-4D+3}\sin x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{-22-4D}\sin 5x + \frac{1}{2-4D}\sin x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{-2(11+2D)}\sin 5x + \frac{1}{2(1-2D)}\sin x\right\}$$

$$= -\frac{1}{4}\left\{\frac{1}{11+2D}\times\frac{11-2D}{11-2D}\sin 5x - \frac{1}{1-2D}\times\frac{1+2D}{1+2D}\sin x\right\}$$

$$= -\frac{1}{4}\left\{\frac{(11-2D)}{121-4D^2}\sin 5x - \frac{(1+2D)}{1-4D^2}\sin x\right\}$$

$$= -\frac{1}{4}\left\{\frac{[(11\sin 5x - 2D(\sin 5x)]}{11-4(-5^2)} - \frac{[\sin x + 2D(\sin x)]}{1-4(-1)^2}\right\}$$

$$= -\frac{1}{4}\left\{\frac{11\sin 5x - 2\cdot\cos 5x(5)}{121+100} - \frac{(\sin x + 2\cos x)}{1+4}\right\}$$

$$= -\frac{1}{884}(11\sin 5x - 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

Hence the general solution of (1) is

$$y = \text{C.F} + \text{P.I}$$

$$= c_1e^{x} + c_2e^{3x} - \frac{1}{884}(11\sin 5x - 10\cos 5x) + \frac{1}{20}(\sin x + 2\cos x)$$

EXAMPLE-4

Solve : $(D^4 - 81)\,y = \cos 3x + \sin h\, 3x$ [Oct. 201

Solution :

Given differential equation is

$$(D^4 - 81)y = \cos 3x + \sin h\, 3x \quad \text{.............} (1$$

Here $f(D) = D^4 - 81$ and $Q(x) = \cos 3x + \sinh 3x$

A.E. of (1) is $f(m) = 0$

i.e., $m^4 - 81 = 0$

$(m^2+9)(m^2-9) = 0$

$m = \pm 3, \pm 3i.$

$\therefore$ The complementary function of (1) is

$$C.F = c_1 e^{-3x} + c_2 e^{3x} + c_3 \cos 3x + c_4 \sin 3x$$

Particular Integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^4-81}(\cos 3x + \sinh 3x)$$

$$= \frac{1}{D^4-81}\cos 3x + \frac{1}{D^4-81}\sinh 3x$$

$$= \frac{1}{(D^2-9)(D^2+9)}\cos 3x + \frac{1}{D^4-81}\left(\frac{e^{3x}-e^{-3x}}{2}\right)$$

$$= \frac{1}{(-3^2-9)}\frac{1}{(D^2+3^2)}\cos 3x + \frac{1}{2}\left\{\frac{1}{(4D^4-81)}\cdot e^{3x} - \frac{1}{D^4-81}e^{-3x}\right\}$$

$$= \frac{1}{-9-9}\cdot\frac{x}{2(3)}\sin 3x + \frac{1}{2}\left\{\frac{1}{(3)^4-81}e^{3x} - \frac{1}{(-3)^4-81}e^{-3x}\right\}$$

$$\left[\because \frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\sin ax\right]$$

$$= \frac{1}{-108}\times\sin 3x + \frac{1}{2}\left\{\frac{1}{O}e^{3x} - \frac{1}{O}e^{-3x}\right\}$$

$$= \frac{-1}{108}x\sin 3x + \frac{1}{2}\left\{\frac{x}{\frac{d}{dD}(D^4-81)}e^{3x} - \frac{x}{\frac{d}{dD}(D^4-81)}e^{-3x}\right\}$$

$$= -\frac{x}{108}\sin 3x + \frac{1}{2}\left\{\frac{x}{4D^3}e^{3x} - \frac{x}{4D^3}e^{-3x}\right\}$$

$$= -\frac{x}{108}\sin 3x + \frac{1}{2}\left\{\frac{x}{4(3)^3}e^{3x} - \frac{x}{4(-3)^3}e^{-3x}\right\}$$

$$= \frac{x}{108}\sin 3x + \frac{1}{2}\left\{\frac{x}{108}e^{3x} + \frac{x}{108}e^{-3x}\right\}$$

$$= -\frac{x}{108}\sin 3x + \frac{x}{108}\left[\frac{e^{3x}+e^{-3x}}{2}\right]$$

$$= -\frac{x}{108}\sin 3x + \frac{x}{108}\cosh 3x = \frac{x}{108}(\cosh 3x - \sin 3x)$$

$\therefore$ The required general solution of (1) is

$$y = C.F + P.I$$

$$= c_1e^{-3x} + c_2e^{3x} + c_3\cos 3x + c_4\sin 3x - \frac{x}{108}(\cosh 3x - \sin 3x)$$

EXERCISE 2.2

1. Find the complementary function of
   (i) $(D^2 - 4)\,y = \cos 2x$
   (ii) $(D^4 - 1)\,y = \cos x$
2. Find the particular integral of
   (i) $(D^2 - 4)\,y = \cos 2x$ [Oct. 201
   (ii) $(D^2 - 25)\,y = \sin 5x$ [Apr. 201
   (iii) $(D^2 - 36)\,y = \sin 6x$ [Apr. 201
   (iv) $(D^2 + 1)\,y = \cos x$ [Apr. 201
   (v) $(D^2 - D - 2)\,y = \sin 2x$
   (vi) $(D^2 + 1)y = e^{2x} + \sin 3x$ [Oct. 200
3. Solve
   (i) $(D^2 - 1)y = \cos 3x$ [Apr. 201
   (ii) $(D^2 - 4)\,y = \sin x$ [Apr. 200
   (iii) $(D^2 - 4)\,y = \sin 2x$ [Oct. 2016, 2013; Apr. 201

(iv) $(D^2 - 9)\ y = \sin 2x$ [Apr. 2016]

(v) $(D^2 + 4)\ y = \sin 3x$ [Apr. 2014]

(or) $y'' + 4y = \sin 3x$ [Apr. 2014; Oct. 2010, 2009]

(vi) $(D^2 + 1)\ y = \cos 4x$ [Apr. 2016]

(vii) $(D^2 + 4)\ y = \cos x$ [Oct. 2014]

(viii) $(D^2 + 1)\ y = e^{2x} + \sin 3x$ [Oct. 2009]

4. (i) $(D^2 + 1)\ y = \cos x$ [Apr. 2016; Oct. 2013]

(ii) $(D^2 + 1)\ y = \sin x$ [Oct. 2010]

(iii) $(D^2 + 4)\ y = \sin 2x$ [Oct. 2018, 2013 ; Apr. 2018, 2017, 2010]

(iv) $(D^2 + 4)\ y = \sin 2x + \cos 2x$ [Apr. 2016]

(v) $(D^2 + 4)\ y = \sin 2x + \cos 2x + 2$ [Oct. 2018]

(vi) $(D^2 + 4)\ y = e^{2x} + \sin 2x + \cos 2x$ [Apr. 2009]

(vii) $(D^2 + 4)\ y = e^{x} + \sin 2x$ [Apr. 2019]

(viii) (a) $(D^2 + 9)\ y = \cos 3x$ [Apr. 2016, 2012]

(b) $(D^2 + 9)\ y = \cos 3x + e^{-3x}$ [Oct. 2016]

(ix) $(D^2 + 9)\ y = \sin 3x$ [Oct. 2016]

(x) $(D^2 + 16)\ y = \sin 4x$ [Apr. 2019, 2014, 2012 ; Oct. 2012]

(xi) (a) $(D^2 + 16)y = \cos 4x$ [Apr. 2011]

(b) $(D^2 + 16)\ y = 8 \cos 4x$ [Apr. 2017]

(xii) $(D^2 + 16)\ y = 5 + \cos 4x$ [Oct. 2016]

(xiii) $(D^2 + 25)\ y = \sin 5x$ [Apr. 2018]

(xiv) $(D^2 + 36)y = \cos 6x$ [Oct. 2016, 2010; Apr. 2012]

(xv) $(D^2 + 49)\ y = \sin 7x$ [Apr. 2019]

5. (i) $(D^2 + D)\ y = \cos x$ [Oct. 2012]

(ii) $(D^2 + D)y = \cos 4x$ [Apr. 2017]

(iii) $(D^3 + D)y = \sin 2x$ [Oct. 2016]

(iv) $(D^3 + 4D)\ y = \sin 2x$

6. (i) $(D^2 - 3D + 2)\,y = \cos 3x$ **[Apr. 2018, 2017, 2016, 20**

(or) $y'' - 3y' + 2y = \cos 3x$ **[Apr. 20**

(ii) $(D^2 - 3D + 2)\,y = \sin 3x$ **[Oct. 20**

(iii) $(D^2 + D - 6)\,y = \cos x$ **[Oct. 20**

(iv) $(D^2 - D - 2)\,y = \sin 2x$ **[Apr. 2018, 20**

(v) $(D^2 - 4D - 5)\,y = \cos 2x$ **[Oct. 2014, 20**

(vi) $(D^2 - 4D - 3)\,y = \sin 3x$

(vii) $(D^2 + 5D + 6)\,y = \sin 5x$ **[Apr. 2019, 20**

(viii) $(D^2 + 5D - 6)\,y = \cos 4x$ **[Apr. 20**

(ix) $(D^2 + 2D + 1)\,y = \sin 3x$ **[Apr. 20**

(x) $(D^2 - 4D + 4)\,y = \cos 2x$ **[Apr. 2017, 2016, 2011, 20**

(xi) $(D^2 + D + 4)\,y = \cos 2x$ **[Apr. 2019, 2017, 2016, 2014, 2014 ; Oct. 2016, 20**

(xii) $(D^2 + D + 1)\,y = \cos 3x$ **[Apr. 20**

(xiii) $(D^2 - 4D + 1)\,y = 3\cos 2x$ **[Apr. 20**

(xiv) $(D^2 - 4D + 13)\,y = 3\sin 3x$ **[Oct. 2013; Apr. 20**

(xv) $(D^2 - 4D + 13)\,y = 8\sin 3x$ **[Apr. 20**

7. (i) $(D^2 + D - 2)\,y = \sin x + e^{2x} + 4$ **[Apr. 20**

(ii) $(D^2 + D + 1)\,y = e^{x} + \sin 2x$ **[Oct. 200**

(iii) $(D^2 - 4D + 3)\,y = e^{2x} - \sin 3x$ **[Oct. 2018, 2011; Apr. 2010, 200**

(iv) $(D^2 - 6D + 9)\,y = e^{3x} + \cos 2x$ **[May 202**

(v) $(D^2 - 4D + 4)\,y = e^{x} + \cos 2x$ **[Apr. 2018]**

(vi) $(D^2 - 4D + 4)\,y = 5 + \cos 2x$ **[Apr. 2018]**

(vii) $(D^2 - 2D + 2)\,y = e^{3x} + \cos 2x$ **[Apr. 2011]**

(viii) $(D^2 - 4D - 5)\,y = e^{3x} + 4\cos 3x$ **[Nov. 2002]**

(ix) $(D^2 + 5D + 6)\,y = e^{-2x} + \sin x$ **[Oct. 2016]**

(x) $(D^2 + 5D + 6)\,y = e^{2x} + \sin x$ **[Apr. 2017]**

(xi) $(D^2 - D - 2)\, y = e^x + \cos x$ [Apr. 2016]

(xii) $(D^2 + D + 9)\, y = 25 + \sin 3x$ [Apr. 2018]

(xiii) $(D^2 + D + 9)\, y = \sin 3x + \cos 2x$ [Oct. 2016]

8. (i) $(D^2 + 4)\, y = \sin^2 x$

(ii) $(D^2 + 9)\, y = \cos^2 x$ [Oct. 2016]

(iii) $(D^2 + 36)\, y = \sin^2 x$ [Apr. 2016]

9. (i) $(D^2 + 4)\, y = \cos 3x \cos x$

(ii) $(D^2 - 3D + 2)\, y = \cos 3x \cos 2x$

(iii) $(D^2 + 1)\, y = \sin 2x \sin x$

(iv) $(D^2 + 5D - 6)\, y = \sin 4x \sin x$

10. (i) $(D^4 - 1)\, y = 4 \sin x$

ANSWERS

1. (i) $c_1 e^{-2x} + c_2 e^{2x}$ (ii) $c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

2. (i) $-\frac{1}{8}\cos 2x$ (ii) $-\frac{1}{50}\sin 5x$ (iii) $\frac{-1}{72}\sin 6x$

(iv) $\frac{x}{2}\sin x$ (v) $\frac{1}{20}(\cos 2x - 3\sin 2x)$ (vi) $\frac{e^{2x}}{5} - \frac{1}{8}\sin 3x$

3. (i) $y = c_1 e^x + c_2 e^{-x} - \frac{1}{10}\cos 3x$ (ii) $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5}\sin x$

(iii) $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}\sin 2x$ (iv) $y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{13}\sin 2x$

(v) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x$

(vi) $y = c_1 \cos x + c_2 \sin x - \frac{1}{15}\cos 4x$

(vii) $y = c_1 \cos 2 + c_2 \sin 2x + \frac{1}{3}\cos x$

(viii) $y = c_1 \cos x + c_2 \sin x + \frac{e^{2x}}{5} - \frac{1}{8}\sin 3x$

4. (i) $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x$

(ii) $y = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x$

(iii) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{4}\sin 2x$

(iv) $y = c_1 \cos 2x + c_2 \sin 2x \frac{-x}{4}\cos 2x + \frac{x}{4}\sin 2x$

(v) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{4}\cos 2x + \frac{x}{4}\sin 2x + \frac{1}{2}$

(vi) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{2x}}{8} - \frac{x}{4}\cos 2x + \frac{x}{4}\sin 2x$

(vii) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{x}}{5} - \frac{x}{4}\cos 2x$

(viii) (a) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6}\sin 3x$

(b) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6}\sin 3x + \frac{1}{18}e^{-3x}$

(ix) $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{6}\sin 3x$

(x) $y = c_1 \cos 4x + c_2 \sin 4x - \frac{x}{8}\cos 4x$

(xi) (a) $y = c_1 \cos 4x + c_2 \sin 4x + \frac{x}{8}\sin 4x$

(b) $y = c_1 \cos 4x + c_2 \sin 4x + x \sin 4x$

(xii) $y = c_1 \cos 4x + c_2 \sin 4x + \frac{5}{16} + \frac{x}{8}\sin 4x$

(xiii) $y = c_1 \cos 5x + c_2 \sin 5x - \frac{x}{10} \cos 5x$

(xiv) $y = c_1 \cos 6x + c_2 \sin 6x + \frac{x}{12} \sin 6x$

(xv) $y = c_1 \cos 7x + c_2 \sin 7x - \frac{x}{14} \cos 7x$

5. (i) $y = c_1 + c_2 e^{-x} + \frac{1}{2}(\sin x - \cos x)$

(ii) $y = c_1 + c_2 e^{-x} - \frac{1}{68}(4\cos 4x - \sin 4x)$

(iii) $y = c_1 + c_2 \cos x + c_3 \sin x + \frac{1}{6} \cos 2x$

(iv) $y = c_1 + c_2 \cos 2x + c_3 \sin 2x - \frac{x}{8} \sin 2x$

6. (i) $y = c_1 e^{x} + c_2 e^{2x} - \frac{1}{130}(7\cos 3x + 9\sin 3x)$

(ii) $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{130}(9\cos 3x - 7\sin 3x)$

(iii) $y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{50}(\sin x - 7\cos x)$

(iv) $y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{20}(\cos 2x - 3\sin 2x)$

(v) $y = c_1 e^{-x} + c_2 e^{5x} - \frac{1}{145}(8\sin 2x + 9\cos 2x)$

(vi) $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{30}(2\cos 3x - \sin 3x)$

(vii) $y = c_1 e^{-2x} + c_2 e^{-3x} - \frac{1}{186}(25\cos 5x - 19\sin 5x)$

(viii) $y = c_1 e^{x} + c_2 e^{-6x} + \frac{1}{442}(10\sin 4x - 11\cos 4x)$

(ix) $y=(c_1+c_2x)e^{-x}-\frac{1}{25}(3\cos 3x+4\sin 3x)$

(x) $y=(c_1+c_2x)e^{2x}-\frac{1}{8}\sin 2x$

(xi) $y=e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)-\frac{2}{73}(3\cos 3x+8\sin 3x)$

(xii) $y=e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)+\frac{1}{73}(3\sin 3x-8\cos 3x)$

(xiii) $y=c_1e^{(2+\sqrt{3})x}+c_2e^{(2-\sqrt{3})x}-\frac{3}{73}(8\sin x+3\cos 2x)$

7. (i) $y=c_1e^{x}+c_2e^{-2x}-\frac{1}{10}(\cos x+3\sin x)+\frac{1}{4}e^{2x}-2$

(ii) $y=e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)+\frac{e^x}{3}-\frac{1}{3}(2\cos 2x+3\sin 2x)$

(iii) $y=c_1e^{x}+c_2e^{3x}-e^{2x}-\frac{1}{30}(2\cos 3x-8\sin 3x)$

(iv) $y=(c_1+c_2x)e^{3x}+\frac{x^2}{2}e^{3x}+\frac{1}{169}(5\cos 2x-12\sin 2x)$

(v) $y=(c_1+c_2x)e^{2x}+e^{x}-\frac{1}{8}\sin 2x$

(vi) $y=(c_1+c_2x)e^{2x}+\frac{5}{4}-\frac{1}{8}\sin 2x$

(vii) $y=e^{x}(c_1\cos x+c_2\sin x)+\frac{1}{5}e^{3x}-\frac{1}{10}(\cos 2x+2\sin 2x)$

(viii) $y=c_1e^{-x}+c_2e^{5x}-\frac{1}{8}e^{3x}-\frac{2}{85}(6\sin 3x+7\cos 3x)$

(ix) $y=c_1e^{-2x}+c_2e^{-3x}+xe^{-2x}+\frac{1}{10}(\sin x-\cos x)$

(x) $y=c_1e^{-2x}+c_2e^{-3x}+\frac{1}{20}e^{2x}-\frac{1}{10}(\cos x-\sin x)$

(xi) $y=c_1e^{-x}+c_2e^{2x}-\frac{e^x}{2}-\frac{1}{10}(\sin x+3\cos x)$

(xii) $y=e^{-\frac{x}{2}}\left(c_2\cos\frac{\sqrt{35}}{2}x+c_2\sin\frac{\sqrt{35}}{2}x\right)-\frac{1}{3}\cos 3x+\frac{25}{9}$

(xiii) $y=e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{35}}{2}x+c_2\sin\frac{\sqrt{35}}{2}x\right)-\frac{1}{3}\cos 3x+\frac{1}{29}(2\sin 2x+5\cos x)$

8. (i) $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}(1-x\sin 2x)$

(ii) $y=c_1\cos 3x+c_2\sin 3x+\frac{1}{18}+\frac{1}{10}\cos 2x$

(iii) $y=c_1\cos 6x+c_2\sin 6x+\frac{1}{72}-\frac{1}{64}\cos 2x$

9. (i) $y=c_1\cos 2x+c_2\sin 2x-\frac{1}{24}\cos 4x+\frac{x}{8}\sin 2x$

(ii) $y=c_1e^x+c_2e^{2x}-\frac{1}{1508}(23\cos 5x+15\sin 5x)+\frac{1}{20}(\cos x-3\sin x)$

(iii) $y=c_1\cos x+c_2\sin x+\frac{x}{4}\sin x+\frac{1}{16}\cos 3x$

(iv) $y=c_1e^x+c_2e^{-6x}+\frac{1}{60}(\sin 3x-\cos 3x)+\frac{1}{3172}(31\cos 5x-25\sin 5x)$

10. (i) $y=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x+x\cos x$

2.6 P.I OF F(D) y = Q(x) WHEN Q=(x), M IS POSITIVE INTEGER

Consider the differential equation be

$$f(D)\,y = x^m \qquad \text{.............. (1)}$$

$$\therefore \quad P.I = \frac{1}{f(D)}x^m$$

Working Rule to Evaluate P.I

Step-1: Write f(D) as ascending powers of D and make the first term unity by taking lowest degree term common so that the denominator is of the form $[1 \pm \phi(D)]^n$.

Step-2 : Take $[1 \pm \phi(D)]^n$ in the numerator so that it takes the form $[1 \pm \phi(D)]^{-n}$

Step-3 : Expand $[1 \pm \phi(D)]^{-n}$ by Binomial Theorem in ascending powers of D upto term containing D^m, since $D^{m+1} x^m = 0$, $D^{m+2} x^m = 0$, and so on.

Step-4 : Differentiate term by term

Remember the Following Binomial Expansions :

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$
2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$
3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$
4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots$

SOLVED EXAMPLES

EXAMPLE-1

Find particular integral of $(D^2 + 31)y = x$ [Apr. 201

Solution :

Given differential equation is

$(D^2 + 31)\, y = x$ ....

Here $f(D) = D^2 + 31$ and $y = Q(x) = x$

$$P.I = \frac{Q(x)}{f(D)} = \frac{x}{D^2+31}$$

$$= \frac{x}{31+D^2}$$

$$= \frac{x}{31\left(1+\frac{D^2}{31}\right)}$$

$$= \frac{1}{31}\left[1+\frac{D^2}{31}\right]^{-1} x$$

$$= \frac{1}{31}\left[1-\frac{D^2}{31}\right]x \qquad \left[\because (1+x)^{-1} = 1-x+x^2\ldots\ldots\right]$$

$$=\frac{1}{31}\left[x-\frac{D^2}{31}(x)\right]$$

$$=\frac{1}{31}(x-0)$$

$$=\frac{x}{31}$$

EXAMPLE-2

Solve $(D^2 + D - 6)\,y = x$ **[Oct. 2016, 2013, 2010 ; Apr. 2013, 2011, 2010]**

Solution :

Given differential equation is

$$(D^2 + D - 6)\,y = x \qquad \text{..............} \quad (1)$$

Compare (1) with $f(D)y = x$, we have

$f(D) = D^2 + D - 6$ and $Q(x) = x$

Auxiliary equation of (1) is $f(m) = 0$

$$m^2 + m - 6 = 0$$

$(m + 3)\,(m - 2) = 0 \;\Rightarrow m = 2, -3$

$\therefore$ The complementary function of (1) is

$$C.F = c_1e^{2x} + c_2e^{-3x}$$

Particular integral is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{x}{D^2+D^2+D-6}$$

$$= \frac{x}{-6+D+D^2}$$

$$= \frac{x}{-6\left[1-\left(\dfrac{D+D^2}{6}\right)\right]}$$

$$= \frac{1}{6}\left[1-\left(\frac{D+D^2}{6}\right)\right]^{-1} x$$

$$= -\frac{1}{6}\left[1+\left(\frac{D+D^2}{6}\right)\right] x \qquad (\because (1-x)^{-1} = 1 + x + x^2 \ .....)$$

[Neglecting higher powers of D]

$$= -\frac{1}{6}\left[x+\left(\frac{D+D^2}{6}\right) x\right]$$

$$= \frac{1}{6}\left[x+\frac{1}{6}(Dx+D^2x\right]$$

$$= -\frac{1}{6}\left[x+\frac{1}{6}(1+0)\right]$$

$$= -\frac{1}{6}\left(x+\frac{1}{6}\right)$$

$\therefore$ The general solution of (1) is $y = C.F + P.I$

$$= c_1e^{2x} + c_2e^{-3x} - \frac{1}{6}\left(x+\frac{1}{6}\right)$$

EXAMPLE-3

Solve : $(D^2 + 2D)\ y = x^2$

[Apr. 2018, 2017, 2014, 2013, 2012]

Solution :

Auxiliary equation is $m^2 + 2m = 0 \Rightarrow m(m+2) = 0$

$\therefore \quad m = 0, -2$

$\therefore$ Complementary function of (1) is

$$C.F = c_1e^{m_1x} + c_2e^{m_2x}$$

$$= c_1e^{ox} + c_2e^{-2x}$$

$$= c_1 + c_2e^{-2x}$$

Particular integral of (1) is

$$P.I = \frac{Q(x)}{f(D)}$$

$$= \frac{x^2}{D^2+2D}$$

$$= \frac{x^2}{2D+D^2}$$

$$= \frac{x^2}{2D(1+\frac{D^2}{2D})}$$

$$= \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x^2$$

$$= \frac{1}{2D}\left[1-\frac{D}{2}+\left(\frac{D}{2}\right)^2-\left(\frac{D}{2}\right)^3\right]x^2 \qquad \left[\because (1+x)^{-1}=1-x+x^2-.....\right]$$

$$= \frac{1}{2}\left[\frac{1}{D}-\frac{1}{D}\cdot\frac{D}{2}+\frac{1}{D}\cdot\frac{D^2}{4}-\frac{1}{D}\cdot\frac{D^3}{8}\right]x^2$$

$$= \frac{1}{2}\left[\frac{1}{D}-\frac{1}{2}+\frac{D}{4}-\frac{D^2}{8}\right]x^2$$

$$= \frac{1}{2}\left[\frac{1}{D}x^2-\frac{1}{2}(x^2)+\frac{1}{4}Dx^2-\frac{1}{8}\cdot D^2(x^2)\right]$$

$$= \frac{1}{2}\left[\int x^2dx-\frac{1}{2}x^2+\frac{1}{4}(2x)-\frac{1}{8}(2)\right]$$

$$= \frac{1}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{4}\right)$$

$\therefore$ The general solution of (1) is

$$y = C.F + P.I$$

$$= c_1+e_2e^{-2x}+\frac{1}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{4}\right]$$

EXAMPLE-4

Solve : $(D^2 + 1)\, y = x^2 + 2x + 1$ *[Apr. 2019,*

Solution :

Auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow \quad m = \pm i = 0 \pm i = \alpha \pm i\beta$

Thus complementary function is

$$y_c = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$= e^{\alpha x}\left[c_1 \cos x + c_2 \sin x\right]$$

$$= c_1 \cos x + c_2 \sin x$$

Now particular integral is

$$= \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^2+1}(x^2+2x+1)$$

$$= \frac{1}{1+D^2}(x^2+2x+1)$$

$$= (1+D^2)^{-1}\,(x^2 + 2x + 1)$$

$$= \left[1-(D^2)+(D^2)^2.......\right](x^2+2x+1)$$

$$= \left[1-D^2\right](x^2+2x+1) \qquad \text{(Neglecting higher powers of D)}$$

$$= (x^2 + 2x + 1) - D^2\,(x^2 + 2x + 1)$$

$$= x^2 + 2x + 1 - \frac{d^2}{dx^2}(x^2+2x+1)$$

$$= x^2 + 2x + 1 - 2$$

$$= x^2 + 2x - 1$$

$\therefore$ The general solution is $y = y_c + y_p$

$$= c_1 \cos x + c_2 \sin x + x^2 + 2x - 1$$

EXAMPLE-5

Solve : $(D^2 - 2D + 1)\, y = x^4 + \cos x$ *[Oct. 2011]*

Solution :

Auxiliary equation is $m^2 - 2m + 1 = 0$

i.e., $(m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore$ Complementary function is

$$y_c = (c_1 + c_2 x)\, e^x$$

Now particular integral is

$$y_p = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^2 - 2D + 1}(x^4 + \cos x)$$

$$= \frac{1}{1 - 2D + D^2}(x^4) + \frac{1}{D^2 - 2D + 1}\cos x$$

$$= \frac{1}{(1-D)^2}(x^4) + \frac{1}{-1^2 - 2D + 1}\cos x \qquad \text{(Put } D^2 = -a^2 = -1^2\text{)}$$

$$= (1-D)^{-2}(x^4) + \frac{1}{-1 - 2D + 1}\cos x$$

$$= (1 + 2D + 3D^2 + 4D^3 + 5D^4)x^4 + \frac{1}{-2D}\cos x \quad \text{(Neglecting higher power of D)}$$

$$= x^4 + 2D(x^4) + 3D^2(x^4) + 4D^3(x^4) + 5D^4(x^4) - \frac{1}{2}\int \cos x\, dx$$

$$= x^4 + 2(4x^3) + 3(12x^2) + 4(24x) + 5(24) - \frac{1}{2}\sin x$$

$$= x^4 + 8x^3 + 36x^2 + 96x + 120 - \frac{1}{2}\sin x$$

$\therefore$ The general solution of the given differential equation is

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x + x^4 + 8x^3 + 36x^2 + 96x + 120 - \frac{1}{2}\sin x$$

EXAMPLE-6

Solve : $(D^3 + 4D)\, y = \cos 2x + x$

Solution :

Given differential equation is

$$(D^3 + 4D)\, y = \cos 2x + x$$

Compare (1) with f(D) y = Q(x), we have

$$f(D) = D^3 + 4D \text{ and } Q(x) = \cos 2x + x$$

Auxiliary equation of (1) is f(m) = 0

i.e., $m^3 + 4m = 0$

$$m(m^2 + 4) = 0$$

$$m = 0,\ \pm 2i$$

∴ The complementary function of (1) is

$$y_c = c_1 e^{ox} + e^{ox}(c_2 \cos 2x + c_3 \sin 2x)$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x$$

Now, particular integral of (1) is

$$y_p = \frac{Q(x)}{f(D)}$$

$$= \frac{1}{D^3 + 4D}(\cos 2x + x)$$

$$= \frac{1}{D^3 + 4D}\cos 2x + \frac{1}{4D + D^3}x$$

$$= \frac{1}{D(D^2 + 4)}\cos 2x + \frac{1}{4D\left(1 + \frac{D^2}{4}\right)}x$$

$$= \frac{1}{D^2 + 4}\int \cos 2x\,dx + \frac{1}{4D}\left(1 + \frac{D^2}{4}\right)^{-1}x$$

$$= \frac{1}{D^2 + 4}\left(\frac{\sin 2x}{2}\right) + \frac{1}{4D}\left(1 - \frac{D^2}{4}\right)x \text{ (neglecting higher powers of D)}$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}\sin 2x\right] + \frac{1}{4D}\left(x - \frac{D^2 x}{4}\right)$$

$$= \frac{1}{2}\left[\frac{-x}{2(2)}\cos 2x\right] + \frac{1}{4D}[x-0] \quad \left[\because \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos 2x\right]$$

$$= \frac{-x}{8}\cos 2x + \frac{1}{4}\left(\frac{x^2}{2}\right)$$

$$= \frac{x}{8}\cos 2x + \frac{x^2}{8}$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x - \frac{x}{8}\cos 2x + \frac{x^2}{8}$$

EXERCISE - 2.3

1. Find the complementary function of
   (i) $(D^2 - 1)\, y = x$
   (ii) $(D^2 - D)y = 2x - 1 - 3e^x$
2. Find the particular solution of
   (i) $(D^2 - 1)\, y = x$ [Oct. 2018]
   (ii) $(D^2 - 1)\, y = x^2$ [Apr. 2016]
   (iii) $(D + 1)^2\, y = x$ [Apr. 2017]
   (iv) $(D^2 + D + 3)\; y = x + \sin 2x$ [Apr. 2016]
3. Solve :
   (i) $(D^2 - 1)\, y = x$ [Oct. 2016]
   (ii) $(D^2 - 1)y = x^2$ [Apr. 2014; Oct. 2013, 2012]
   (iii) $(D^2 + 1)\, y = x$ [Apr. 2014; Oct.2013, 2012]
   (iv) $(D^2 + 1)\, y = x^2$
   (v) $(D^2 + 1)\, y = x^2 + 2$ [Apr. 2018]

(vi) $(D^2 + 1) y = x^2 + 3x$ [Apr. 2016]

(vii) $(D^2 + 4) y = x^2$ [Apr. 2018, 2017, 2015; Oct. 2014, 2013]

(viii) $(D^2 + 4) y = x^3 + 3$

(x) $(D^2 + 4) y = x + \sin x$ [Apr. 2019, 2006]

(xi) $(D^2 + 4) y = x^3$ [Apr. 2017, 2016]

(xii) $(D^2 + 4) y = x^4$ [Oct. 2014, 2012; Apr. 2013]

(xiii) $(D^2 - 4) y = x^3$ [Oct. 2014; Apr. 2013]

(xiv) $(D^2 - 4) y = 2x^3$ [Oct. 2018]

(xv) $(D^2 + 9) y = x^4$ [Apr. 2016]

(xvi) $(D^2 - 9) y = x^3$ [Arp 2017]

(xvii) $(D^2 + 16) y = x^3$ [Apr. 2018]

(xviii) $(D^2 + 31) y = x$ [Apr. 2017, 2012]

4. (i) $(D^2 - 3D + 2) y = x$ [Apr. 2011]

(ii) $(D^2 + 3D + 2) y = x$ [Apr. 2014, 2013; Oct. 2013, 2010]

(iii) $(D^2 - 3D + 2) y = 2x^2$ [Apr. 2018, 2016, 2009]

(iv) $(D^2 - 3D + 2) y = 5x^2$ [Oct 2014, 2012]

(v) $(D^2 - 3D + 2) y = e^{4x} + x + x^2$

(vi) $(D^2 + 3D + 2) y = x^2$ [Oct. 2018, 2013, 2012, 2011; Apr. 2016]

(vii) $(D^2 + 3D + 2) y = x^2 + \sin x$ [Apr. 2016]

(viii) $(D^2 + D + 2) y = x$ [Apr. 2017; Oct. 2016, 2011]

(ix) $(D^2 + D + 2) y = x^2$ [Oct. 2016]

(x) $(D^2 + D - 2) y = e^x + x$ [Apr. 2017]

(xi) $(D^2 + D - 2) y = e^{2x} + x$ [Apr. 2019]

(xii) $(D^2 + D - 2) x = \sin x$ [Apr. 2019, 2006]

(xii) $(D^2 + D - 6) y = x^2$ [Oct. 2013]

(xiii) $(D^2 - 4D + 3) y = x^2$ [Apr. 2019, 2011 ; Oct. 2014]

(xiv) $y'' + 5y' + 4y = 3 - 2x$ (or) $(D^2 + 5D + 4)y = 3 - 2x$

(xv) $(D^2 + 5D + 4y = 3 - 2x$ [Apr. 2014]

(xv) $(D^2 + 5D + 4)y = x^2 + 9$ [Apr. 2014, 2011, 2010]

(xvi) $(D^2 + 5D + 4)y = x^2 + 7x + 9$ [Apr. 2009, 2008; Oct 2008]

5. (i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$

(or)

$(D^2 + 2D + 1)y = x$ (or) $(D + 1)^2 y = x$

(ii) $(D^2 + 2D + 1)y = 2x^2$ [Apr. 2013]

(iii) $(D^2 + 2D + 1)y = x^2 + 2x$ [Apr. 2013, 2012 ; Oct. 2012]

(iv) $(D + 1)^2 y = e^{-x} + x^2$ [Oct. 2008]

(v) $(D^2 + 2D+1)y = x^3$

(vi) $(D^2 + 2D + 1)y = x^4$ [Apr. 2016]

(vii) $(D^2 + 4D + 4)y = e^{-2x} + x^2$

(viii) $(D^2 - 4D + 4)y = \cos 2x + x^2$

(ix) $(D^2 - 4D + 4)y = x^3$ [Oct. 2016, 2009]

(x) $(D^2 - 6D + 9)y = x^2$ [Apr. 2019, 2018, 2011]

(xi) $(D^2 - 6D + 9)y = e^{2x} + e^{3x} + x^2$ [Apr. 2016]

6. (i) $(D^3 - 3D + 2)y = x$

(ii) $(D^4 - 2D^3 + D^2)y = x$

ANSWERS

1. (i) $y_c = c_1 e^x + c_2 e^{-x}$ (ii) $y_c = c_1 + c_2 e^x$

2. (i) $y_p = -x$ (ii) $y_p = -(x^2 + 2)$

(iii) $y_p = x$ (iv) $y_p = \frac{1}{3}\left(x - \frac{1}{3}\right) - \frac{1}{5}(2\cos 2x + \sin 2x)$

3. (i) $y = c_1e^x + c_2e^{-x} - x$ (ii) $y = c_1e^x + c_2e^{-x} - (x^2 + 2)$

(iii) $y = c_1 \cos x + c_2 \sin x + x$ (iv) $y = c_1 \cos x + c_2 \sin x + x^2 - 2$

(v) $y = c_1 \cos x + c_2 \sin x + x^2$ (vi) $y = c_1 \cos x + c_2 \sin x + x^2 + 3x - 2$

(vii) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)$

(viii) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^2 + 5)$

(ix) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} + \frac{1}{3}\sin x$ (x) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^3 - 3x)$

(xi) $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^4 - 6x^2 + 3)$

(xii) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{8}(2x^2 + 3)$ (xiii) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}\left(x^3 + \frac{3x}{2}\right)$

(xiv) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{729}(81x^4 - 108x^2 + 24)$

(xv) $y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{27}(3x^3 + 2x)$

(xvi) $y = c_1 \cos 4x + c_2 \sin 4x + \frac{x}{128}(8x^2 - 3)$

(xvii) $y = c_1 \cos \sqrt{31}x + c_2 \sin \sqrt{31}x + \frac{x}{31}$

4. (i) $y = c_1 e^{x} + c_2 e^{2x} + \frac{x}{2} + \frac{3}{4}$ (ii) $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x}{2} - \frac{3}{4}$

(iii) $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}\left(2x^2 + 6x + 7\right)$ (iv) $y = c_1 e^{x} + c_2 e^{2x} + \frac{5}{4}(2x^2 + 6x + 7)$

(v) $y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{4x}}{6} + \frac{1}{2}(x^2 + 4x + 5)$

(vi) $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4}(2x^2 - 6x + 7)$

(vii) $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{4}(2x^2 - 6x + 7) - \frac{1}{10}(3\cos x - \sin x)$

(viii) $y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{7}}{2}x + c_2\sin\frac{\sqrt{7}}{2}x\right) + \frac{1}{2}\left(x - \frac{1}{2}\right)$

(ix) $y = y = c_1 + c_2x + (c_3 + c_4x)e^{-x} + x^2 + \frac{x^3}{6}$

(x) $y = c_1e^{x} + c_2e^{-2x} + \frac{x}{3}e^{x} - \frac{1}{2}\left(x + \frac{1}{2}\right)$

(xi) $y = c_1e^{x} + c_2e^{-2x} + \frac{e^{2x}}{4} - \frac{1}{2}\left(x + \frac{1}{2}\right)$

(xii) $y = c_1e^{x} + c_2e^{-2x} - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(3\sin x + \cos x)$

(xiii) $y = c_1e^{2x} + c_2e^{-3x} + \frac{1}{18}(18x^2 + 6x + 7)$

(xiv) $y = c_1e^{x} + c_2e^{3x} + \frac{x^2}{3} + \frac{8x}{9} + \frac{26}{27}$

(xv) $y = c_1e^{-x} + c_2e^{-4x} + \frac{1}{8}(11 - 4x)$

(xvi) $y = c_1e^{-x} + c_2e^{-4x} + \frac{1}{32}(8x^2 - 20x + 21)$

(xvii) $y = c_1e^{-x} + c_2e^{-4x} + \frac{1}{32}(8x^2 - 20x + 93)$

(xviii) $y = c_1e^{-x} + c_2e^{-4x} + \frac{1}{32}(8x^2 + 36x + 23)$

5. (i) $y = (c_1 + c_2x)e^{-x} + x - 2$ (ii) $y = (c_1 + c_2x)e^{-x} + 2(x^2 - 4x + 6)$

(iii) $y = (c_1 + c_2)e^{-x} + x^2 - 2x + 2$ (iv) $y = (c_1 + c_2x)e^{-x} + \frac{x^2}{2}e^{-x} + x^2 - 4x + 6$

(v) $y=(c_1+c_2x)e^{-x}+x^3-6x^2+18x-24$

(vi) $y=(c_1+c_2x)e^{-x}+x^4-8x^3+36x^2+96x+120$

(vii) $y=(c_1+c_2x)e^{-2x}+\frac{x^2}{2}e^{-2x}+\frac{1}{8}(2x^2-4x+3)$

(viii) $y=(c_1+c_2x)e^{-2x}-\frac{1}{8}\sin 2x+\frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)$

(ix) $y=(c_1+c_2)e^{-2x}+\frac{1}{4}\left(x^3+3x^2+\frac{9}{2}x+3\right)$

(x) $y=(c_1+c_2x)e^{3x}+\frac{1}{27}(3x^2+4x+2)$

(xi) $y=(c_1+c_2x)e^{3x}+e^{2x}+\frac{x^2}{2}e^{3x}+\frac{1}{27}(3x^2+4x+2)$

6. (i) $y=(c_1+c_2x)e^{x}+c_3e^{-2x}+\frac{1}{4}(2x+3)$

(ii) $y=c_1+c_2x+(c_3+c_4x)e^{-x}+x^2+\frac{x^3}{6}$