5

Laplace Transformations

CHAPTER-5

Laplace Transformations

5.0 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

Laplace transform, in mathematics, a particular integral transform invented by the French mathematician pierre-simon marquis De laplace (1749-1827), and systematically developed by the British physicist oliver Heaviside (1850-1925), to simplify the solution of many differential equations that describe physical processes.

The methods of Laplace transform has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of finding the general solution and then evaluating from it the arbitrary constants.

5.1 DEFINITION

Let f(t) be a function defined for all positive values of t. Then the **laplace transform** of f(t) is denoted by $L\{f(t)\}$ and defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt, \qquad \text{.............. (1)}$$

provided the integral on the right hand side is exists. Here 's' is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of 's' is briefly written as $\bar{f}(s)$ or F(s).

i.e., $L\{f(t)\} = \bar{f}(s)$

Which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

In such a case f(t) is called the inverse Laplace transform of $\bar{f}(s)$. The symbol L, which transform f(t) into is called the Laplace transformation operator.

5.2 SUFFICIENT CONDITIONS FOR EXISTANCE OF LAPLACE TRANSFORM

The Laplace transform of f(t) i.e., $\int_0^\infty e^{-st} f(t)dt$ exists if

(i) The function f(t) should be piece-wise continuous in the given closed interval and

(ii) The function f(t) is of exponential order.

Now we shall define the terms "piece-wise continuity" and "exponential order".

1. **Piecewise Continuous Function :**

A function f(t) is said to be piece wise continuous on a finite interval [a, b] if f(t) is continuous

in every sub interval and has finite limits at end point of each of these sub intervals.

i.e., f(t) has a finite number of discontinuities in the given interval.

For Example : $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ is a picewise continuous function.

2. **Function of Exponential Order :**

A function f(t) is of exponential order 'a' if

$$\lim_{t\to\infty} e^{-at} f(t) = \text{finite quantity}$$

i.e., f(t) is of exponential order 'a' if there exists M, such that $\left|e^{-at} f(t)\right| < M \text{ or } \left|f(t)\right| < Me^{at}$.

For example, since $\lim_{t\to\infty} \frac{t^2}{e^{3t}} = \text{finite}$, $f(t) = t^2$ is of exponential order 3.

5.3 PROPERTIES OF LAPLACE TRANSFORM

1. $L\{f(t)+g(t)\} = L\{f(t)\} + L\{g(t)\}$

2. $L\{c\, f(t)\} = c\, L\{f(t)\}$, 'c' is constant

3. Linearity properties.

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}$$, where c_1 and c_2 are constants.

5.4 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

1. $L\{k\} = \frac{k}{s}$, $(s > 0)$, **k is constant.**

PROOF :

By definition,

$$L\{k\} = \int_0^\infty e^{-st} \cdot k \cdot dt$$

$$= k \cdot \int_0^\infty e^{-st} dt$$

$$= k \cdot \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-k}{s}\left[e^{-st}\right]_0^\infty$$

$$= \frac{-k}{s}\left[e^{-\infty} - e^0\right],$$

$$= -\frac{k}{s}(0-1) \qquad \left[\because e^{-\infty} = 0\right]$$

$$= \frac{k}{s} \text{ if } s > 0 \qquad \left[\because e^{-st} \to 0 \text{ as } t \to \infty, \text{ if } s > 0\right]$$

Thus $L\{k\} = \frac{k}{s}$, if $s > 0$

Note :

(i) If $k = 0$, $L\{0\} = 0$

(ii) If $k = 1$, $L\{1\} = \frac{1}{s}$

2. $L\{e^{at}\} = \frac{1}{s-a}$ **if** $s > a$

PROOF :

By definition,

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty$$

$$= -\frac{1}{s-a}\left[e^{-\infty} - e^{0}\right], \quad \text{if } (s-a) > 0$$

$$= \frac{-1}{s-a}(0-1) \qquad \left[\because \lim_{t\to\infty} e^{-(s-a)t} = 0, \text{ if } (s-a) > 0\right]$$

$$= \frac{1}{s-a}, \ s > a$$

Note : Laplace transform does not exist when $s \leq a$.

Similarly,

3. $L\{e^{at}\} = \dfrac{1}{s+a}$, if $s > -a$

4. $L\{\sin at\} = \dfrac{a}{s^2+a^2}$, if $s > 0$

PROOF :

By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + a^2}(-s \sin at - a\cos at)\right]_0^\infty$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c\right]$$

$$= \frac{1}{s^2+a^2}\left[e^{-\infty} \times \text{finite quantity} - e^{0}(0-a)\right], \ s > 0.$$

$$= \frac{1}{s^2+a^2}(0+a)$$

$$= \frac{a}{s^2+a^2}, s>0$$

$$\therefore \quad L\{\sin at\} = \frac{a}{s^2+a^2}, \text{if } s>0$$

5. $L\{\cos at\} = \frac{s}{s^2+a^2}$, **if** $s>0$

PROOF :

By definition,

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

$$L\{\cos at\} = \int_0^{\infty} e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^{\infty}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c\right]$$

$$= \frac{e^{-\infty}}{s^2+a^2}(\text{finite value}) - \frac{e^0}{s^2+a^2}(-s\cos 0 + a\sin 0)$$

$$= 0 - \frac{1}{s^2+a^2}(-s+0), s>0$$

$$= \frac{s}{s^2+a^2}, \text{if } s>0$$

$$\therefore \quad L\{\cos at\} = \frac{s}{s^2+a^2}, \text{if } s>0$$

6. $L\{\sin hat\} = \frac{a}{s^2-a^2}$, **if** $s>|a|$

PROOF :

We know that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh at = \frac{e^{at}-e^{-at}}{2}$$

$$\therefore \quad L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right], \text{ (by linearity property)}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right], \text{ if } s-a>0 \text{ and } s+a>0$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right], \text{ if } s>|a|$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right], \text{ if } s>|a|$$

$$= \frac{a}{s^2-a^2}, \text{ if } s>|a|$$

Similarly,

7. **$L\{\cos hat\} = \frac{s}{s^2-a^2}$, if $s > |a|$.**

8. **$L\{t^n\} = \frac{n!}{s^{n+1}}$ if n is a positive integer.**

PROOF :

By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{t^n\} = \int_0^\infty e^{-st}(t^n)\,dt$$

$$= \int_0^\infty t^n e^{-st}\,dt$$

$$= \left[t^n \int e^{-st}dt\right]_0^\infty - \int_0^\infty \left(\frac{d}{dt}(t^n)\int e^{-st}dt\right)dt \quad (\because \text{ by integration by parts})$$

$$= \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n \cdot t^{n-1} \cdot \frac{e^{-st}}{(-s)} dt$$

$$= \frac{-1}{s} \cdot \left[\lim_{t\to\infty} t^n \cdot e^{-st} - 0\right] + \frac{n}{s}\int_0^\infty t^{n-1} \cdot e^{-st} dt$$

$$= \frac{-1}{s}\left[\lim_{t\to\infty} \frac{t^n}{e^{st}}\right] + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \sqrt{b^2 - 4ac}$$

$$= \frac{-1}{s}(0) + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt \qquad \left[\because \lim_{t\to\infty} \frac{t^n}{e^{st}} = 0, \text{ by L.Hospital rule}\right]$$

$$= \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

$$= \frac{n}{s} L\{t^{n-1}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot L\left\{t^{n-2}\right\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots\cdots \frac{1}{s} \cdot L\{1\}, \text{ for n is an integer}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} \qquad \left[\because L\{1\} = \frac{1}{s}\right]$$

$$= \frac{n!}{s^{n+1}}$$

Note :

1. put $n = 0, L\{1\} = \frac{1}{s}$
2. put $n = 1, L\{t\} = \frac{1}{s^2}$
3. put $n = 2, L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$.

L Hospital Rule : If f(a) = 0 and g(a) = 0 and both f'(x) and g'(x) are exist, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}, \text{ if } f'(a) \neq 0 \,\&\, g'(a) \neq 0$$

Gamma Function :

Definition : if $n > 0$ then the Gamma function denoted by $\Gamma(n)$ and is defined as

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1} dx}$$

Properties of Gamma function :

(i) $\Gamma(n+1) = n \cdot \Gamma(n) \cdot (n > 0)$

(ii) $\Gamma(n+1) = n(n-1)\Gamma(n-1), n-1 > 0$

$= n(n-1)(n-2).......\Gamma(1)$ if n is a positive integer

$= n\,(n-1)\,(n-2)\,.....1 \qquad [\because \Gamma(1) = 1]$

$\therefore \quad \Gamma(n+1) = n!$, when n is a positive integer.

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(iv) $\Gamma(0), \Gamma(-1), \Gamma(-2),......$ are all not defined.

9. $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$**, where $s > 0$ and n is a real number > -1.**

Proof :

By definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}, \text{ on putting } st = x \Rightarrow dt = \frac{a}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^{(n+1)-1} dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1), \text{ if } S > 0 \text{ and } n+1 > 0.$$

$$= \frac{n!}{s^{n+1}}, \text{ if } n \text{ is positive integrer.}$$

Cor :

1. when $n = \frac{1}{2}$, we have

$$L\{\sqrt{t}\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} \qquad [\because \Gamma(n+1) = n \cdot \Gamma(n)]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \qquad \left[\because \Gamma\left(\frac{1}{3}\right) = \sqrt{\pi}\right]$$

2. when $n = -\frac{1}{2}$

$$L\left\{t^{-\frac{1}{2}}\right\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$$

$$L\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

Table

S.No.	f(t)	L{f(t)}
1.	k	$\frac{k}{s}$
2.	1	$\frac{1}{s}$
3.	t^n	$\begin{cases} \frac{n!}{s^{n+1}}, & \text{if n is positive integer} \\ \frac{\Gamma(n+1)}{s^{n+1}}, & \text{other wise} \end{cases}$
4.	t	$\frac{1}{s^2}$
5.	t^2	$\frac{2}{s^3}$
6.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
7.	$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
8.	e^{at}	$\frac{1}{s-a}$
9.	e^{-at}	$\frac{1}{s+a}$
10.	sin at	$\frac{a}{s^2+a^2}$
11.	cos at	$\frac{s}{s^2+a^2}$
12.	sin hat	$\frac{a}{s^2-a^2}$
13.	cos hat	$\frac{s}{s^2-a^2}$

SOLVED EXAMPLES

EXAMPLE-1

Find

(i) $L\{t^2 - 3t + 5\}$ [Apr. 2017]

(ii) $L\{(t^2 + 1)^2\}$ [Oct. 2020 ; Apr. 2016, 2017]

(iii) $L\{2e^{3t} - e^{-3}t\}$ [Apr. 2017]

(iv) $L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$ [Apr. 2019]

(v) $L\{\cos 2t + \sin ht + 1\}$ [Apr. 2019]

Solution :

(i) $L\{t^2-3t+5\} = L\{t^2\} - 3L\{t\} + 2\cdot L\{1\}$, using linearity properties

$$= \frac{2!}{s^{2+1}} - 3\cdot\frac{1}{s^{1+1}} + 2\cdot\frac{1}{s} \qquad \left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s}$$

(ii) $L\{(t^2+1)^2\} = L\{t^4 + 2t^2 + 1\}$ $[\because (a+b)^2 = a^2 + 2ab + b^2]$

$= L\{t^4\} + 2L\{t^2\} + L\{1\}$, using linearity properties

$$= \frac{4!}{s^{4+1}} + 2\cdot\frac{2!}{s^{2+1}} + \frac{1}{s} \qquad \left[\because L\left\{t^n\right\} = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s} \qquad [\because 4! = 4\times 3\times 2\times 1 = 24]$$

(iii) $L\{2e^{3t} - e^{-3t}\} = 2L\left\{e^{3t}\right\} - L\left\{e^{-3t}\right\}$, using Linearity property

$$= 2\cdot\frac{1}{s-3} - \frac{1}{s+3}$$

$$= \frac{2}{s-3} - \frac{1}{s+3}$$

(iv) $L\left\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\right\}$

$= L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3.L\{\cos 3t\}$, using linearity property

$$= \frac{1}{s-2} + 4\cdot\frac{3!}{s^{3+1}} - 2\cdot\frac{3}{s^2+3^2} + 3\cdot\frac{s}{s^2+3^2}$$

$$\left[\because L\{e^{at}\} = \frac{1}{s-a}, L\{t^n\} = \frac{n!}{s^{n+1}}, \\ L\{\sin at\} = \frac{a}{s^2+a^2} \text{ and } L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$$

(v) $L\{\cos 2t + \sin ht + 1\} = L\{\cos 2t\} + L\{\sinh t\} + L\{1\}$

$$= \frac{s}{s^2+2^2} + \frac{1}{s^2-1^2} + \frac{1}{s}$$

$$= \frac{s}{s^2+4} + \frac{1}{s^2-1} + \frac{1}{s}$$

EXAMPLE-2

Find :

(i) $L\{\sin(wt+\theta)\}$ *[Apr. 2017]*

(ii) $L\{\sin^2 at\}$ *[Apr. 2007]*

(iii) $L\{\cos^2 t\}$ *[Apr. 2016]*

(iv) $L\{\sin^3 3t\}$ *[Apr. 2017]*

(v) $L\{\cos^3 2t\}$

Solution :

(i) $L\{\sin(wt+\theta)\} = L\{\sin w + \cos\theta + \cos wt \sin\theta\}$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \cos\theta L\{\sin wt\} + \sin\theta L\{\cos wt\}$, $[\because$ by linearity properties$]$

$$= \cos\theta \cdot \frac{w}{s^2+w^2} + \sin\theta \cdot \frac{s}{s^2+w^2} \qquad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2} \text{ and } L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= \frac{w\cos\theta + s \cdot \sin\theta}{s^2+w^2}$$

(ii) $$L\{\sin^2 at\} = L\left\{\frac{1-\cos 2at}{2}\right\} \qquad \left[\because \sin^2 A = \frac{1-\cos 2A}{2}\right]$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos\ 2at\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2a)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

(iii) $$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} \qquad \left[\because \cos^2 A = \frac{1+\cos 2A}{2}\right]$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

(iv) **We know that**

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\Rightarrow \quad 4\sin^3 A = 3\sin A - \sin 3A$$

$$\Rightarrow \quad \sin^3 A = \frac{1}{4}[3\sin A - \sin 3A]$$

$$\therefore \quad L\{\sin^3 3t\} = L\left\{\frac{1}{4}(3\sin 3t - \sin 9t\right\}$$

$$= \frac{1}{4}\left[3L\{\sin 3t\} - L\{\sin 9t\}\right]$$

$$= \frac{1}{4}\left[3\cdot\frac{3}{s^2+3^2} - \frac{9}{s^2+9^2}\right]$$

$$= \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

$$= \frac{9}{4}\left[\frac{1}{s^2+9} - \frac{1}{s^2+81}\right]$$

(v) We know that

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\Rightarrow \quad 4\cos^3 A = 3\cos A + \cos A$$

$$\Rightarrow \quad \cos^3 A = \frac{1}{4}(3\cos A + \cos 3A)$$

$$\therefore \quad L\{\cos^3 2t\} = L\left\{\frac{1}{4}(3\cdot\cos 2t + \cos 6t)\right\}$$

$$= \frac{1}{4}\left[3L\{\cos 2t\} + L\{\cos 6t\}\right]$$

$$= \frac{1}{4}\left[3L\{\cos 2t\} + L\{\cos 6t\}\right]$$

$$= \frac{1}{4}\left[3\cdot\frac{s}{s^2+2^2} + \frac{s}{s^2+6^2}\right]$$

$$= \frac{1}{4}\left[\frac{3s}{s^2+4} + \frac{s}{s^2+36}\right]$$

EXAMPLE-3

Find :

(i) *L {sin2t cos2t}*

(ii) *L {sin 5t cos 3t}* *[Apr. 2016]*

(iii) *L {sin 2t cos 3 t}* *[Apr. 2017; Oct. 2008, 2016]*

(iv) *L{cos 5t cos 2t}* *[Oct. 2016; Apr. 2008]*

(v) *L{sin 8t sin 3t}* *[Oct. 2007]*

(vi) $L\{\sin t \sin 2t \sin 3t\}$ *[Apr. 2009]*

Solution :

(i) $$L[\sin 2t\cos 2t] = L\left\{\frac{2\sin 2t\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin 4t\} \qquad [\because \sin 2A = 2\sin A\cos A]$$

$$= \frac{1}{2}\cdot\left[\frac{4}{s^2+4^2}\right] \qquad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2}\right]$$

$$= \frac{2}{s^2+16}$$

(ii) $$L\{\sin 5t\cos 3t\} = L\left\{\frac{2\sin 5t\cos 3t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin(5t+3t)+\sin(5t-3t)\}$$

$[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)]$

$$= \frac{1}{2}L\{\sin 8t+\sin 2t\}$$

$$= \frac{1}{2}\left[L\{\sin 8t\}+L\{\sin 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{8}{s^2+8^2}+\frac{2}{s^2+2^2}\right] \qquad \left[\because L\{\sin at\}=\frac{a}{s^2+a^2}\right]$$

$$= \frac{4}{s^2+64}+\frac{1}{s^2+4}$$

(iii) $$L\{\sin 2t\cos 3t\} = L\left\{\frac{2\cos 3t\sin 2t}{2}\right\}$$

$$= \frac{1}{2}L\{\sin(3t+2t)-\sin(3t-2t)\}$$

$[\because 2\cos A\sin B = \sin(A+B)-\sin(A-B)]$

$$= \frac{1}{2}L\{\sin 5t-\sin t\}$$

$$= \frac{1}{2}\left[L\{\sin 5t\}-L\{\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2}-\frac{1}{s^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right]$$

(iv) $$L\{\cos 5t\cos 2t\} = L\left\{\frac{2\cos 5t\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{\cos(5t+2t)+\cos(5t-2t)\}$$

$[\because 2\cos A\cos B = \cos(A+B)+\cos(A-B)]$

$$= \frac{1}{2}L\{\cos 7t + \cos 3t\}$$

$$= \frac{1}{2}\left[L\{\cos 7t\} + L\{\cos 3t\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+7^2} + \frac{s}{s^2+3^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+49} + \frac{s}{s^2+9}\right]$$

(v) $L\{\sin 8t \sin 3t\} = L\left\{\frac{2\sin 8t \sin 3t}{2}\right\}$

$$= \frac{1}{2}L\{\cos(8t-3t) - \cos(8t+3t)\}$$

$$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\left[L\{\cos 5t - \cos 11t\}\right]$$

$$= \frac{1}{2}\left[L\{\cos 5t\} - L\{\cos 11t\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+11^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+25} - \frac{s}{s^2+121}\right]$$

(vi) Consider

$$\sin t \sin 2t \sin 3t = \frac{1}{2}(2\sin t \sin 2t)\sin 3t$$

$$= \frac{1}{2}[\cos(2t-t) - \cos(2t+t)]\sin 3t$$

$$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}[\cos t - \cos 3t]\sin 3t$$

$$= \frac{1}{4}[2\sin 3t \cos t - 2\sin 3t \cos 3t]$$

$$= \frac{1}{4}[\sin(3t+t)+\sin(3t-t)-\sin 2(3t)]$$

$$[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B) \text{ and } \sin 2A = 2\sin A\cos A]$$

$$= \frac{1}{4}[\sin 4t+\sin 2t-\sin 6t]$$

$$\therefore L\{\sin t\sin 2t\sin 3t\} = L\left\{\frac{1}{4}(\sin 4t+\sin 2t-\sin 6t)\right\}$$

$$= \frac{1}{4}[L\{\sin 4t\}+L\{\sin 2t\}-L\{\sin 6t\}]$$

$$= \frac{1}{4}\left[\frac{4}{s^2+4^2}+\frac{2}{s^2+2^2}-\frac{6}{s^2+6^2}\right] \qquad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{s^2+16}+\frac{1}{2(s^2+4)}-\frac{3}{2(s^2+36)}$$

EXAMPLE-4

Find L{sin h² 2t}

Solution :

We know that

$$\text{Sin h}^2\,x = \frac{\cosh 2x-1}{2}$$

$$\therefore \quad L\{\sinh^2 2t\} = L\left\{\frac{\cosh 4t-1}{2}\right\}$$

$$= \frac{1}{2}[L\{\cosh 4t\}-L\{1\}]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-4^2}-\frac{1}{s}\right] \qquad \left[\because L\{\cosh at\} = \frac{s}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2-16}-\frac{1}{s}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-s^2+16}{s(s^2-16)}\right]$$

$$= \frac{1}{2}\left[\frac{16}{s(s^2-16)}\right]$$

$$= \frac{8}{s(s^2-16)}$$

Alternative Method :

We know that,

$$\text{Sin h x} = \frac{e^x - e^{-x}}{2}$$

$$\therefore \quad \sin h^2\, 2t = \left[\frac{e^{2t} - e^{-2t}}{2}\right]^2$$

$$= \frac{1}{4}\left[e^{4t} - 2 + e^{-4t}\right], \qquad \left[\because (a+b)^2 = a^2 - 2ab + b^2\right]$$

$$\therefore \quad L\left\{\sinh^2 2t\right\} = L\left\{\frac{1}{4}\left[e^{4t} - 2 + e^{-4t}\right]\right\}$$

$$= \frac{1}{4}\left[L\left\{e^{4t}\right\} - 2L\{1\} + L\left\{e^{-4t}\right\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s-4} - \frac{2}{s} + \frac{1}{s+4}\right]$$

EXAMPLE-5

Find L {f(t)}, if

(i) $f(t) = \begin{cases} 0, 0 < t < 1 \\ 1, 1 < t < 2 \\ 2, t > 2 \end{cases}$ *[Apr. 2008]*

(ii) $f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$ *[Apr. 2005]*

(iii) $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$ *[Apr. 2009]*

Solution :

(i) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} o dt + \int_1^2 e^{-st} 1 \cdot dt + \int_2^\infty e^{-st} \cdot 2dt$$

$$= 0 + \int_1^2 e^{-st} dt + 2\int_2^\infty e^{-st} \cdot 2dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_1^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{-1}{s}\left[e^{-2s} - e^{-s}\right] - \frac{2}{3}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{e^{-s} - e^{-2s}}{s} - \frac{2}{s}\left(0 - e^{-2s}\right)$$

$$= \frac{1}{s}\left[e^{3} - e^{-2s} + 2e^{-2s}\right]$$

$$= \frac{1}{s}\left[e^{-s} + e^{-2s}\right]$$

(ii) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} \cdot e^{t} dt + \int_1^\infty e^{-st} \cdot 0 \cdot dt$$

$$= \int_0^1 e^{(1-s)t} dt$$

$$= \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1$$

$$= \frac{e^{(1-s)1}}{1-s} - \frac{e^{(1-3)0}}{1-s} = \frac{e^{1-s}}{1-s} - \frac{1}{1-s}$$

$$= \frac{e^{1-s}-1}{1-s}$$

(iii) By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)dt$$

$$= \int_0^{2\pi} e^{-st} \cdot f(t)dt + \int_{2\pi}^\infty e^{-st} f(t)dt$$

$$= \int_0^{2\pi} e^{-st} \cdot \cos t\, dt + \int_{2\pi}^\infty e^{-st} \cdot 0\, dt$$

$$= \int_0^{2\pi} e^{-st} \cos t\, dt$$

$$= \left[\frac{e^{-st}}{s^2+1}[-s\cos t + \sin t]\right]_0^{2\pi}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c\right]$$

$$= \frac{e^{-2\pi s}}{s^2+1}[-s\cos 2\pi + \sin 2\pi] - \frac{e^0}{s^2+1}(-s\cos 0 + \sin 0)$$

$$= \frac{e^{-2\pi s}}{s^2+1}(-s(1)+0) - \frac{1}{s^2+1}(-s(1)+0)$$

$$= \frac{-se^{-2\pi s}}{s^2+1} + \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1}\left[1 - e^{-2\pi s}\right]$$

EXERCISE-5.1

Find the Laplace transform of

1. (i) $t^3 - 3t^2 + 2$
   (ii) $t^2 + at + 6$
   (iii) $1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$
   (iv) $(t + 1)^2$
   (v) $(a + bt)^2$ [Apr. 2016]
2. (i) $3e^{2t} + 4e^{-3t}$ [Apr. 2018]
   (ii) $e^{3t} - e^{-3t}$ [Oct. 2018 ; Apr. 2017]
   (iii) $e^t - e^{-3t}$ [Apr. 2018]
   (iv) $e^{at} + 6$ [Oct. 2018]
   (v) $e^{at} + b$
3. (i) $t^3 + 5 \cos t$
   (ii) $5t + 2e^t + 8 \sin 3t$ [Apr. 2016]
   (iii) $e^{2t} - 4t^3 + 2 \sin 3t$ [Apr. 2017]
   (iv) $e^{2t} + 4t^3 - 3 \sin 2t$ [Apr. 2019]
   (v) $t^4 + e^{2t} + 2 \sin 2t$ [Apr. 2017]
   (vi) $t^4 + e^{2t} + 2 \sin 3t$ [Apr. 2019]
   (vii) $\sin at + t^2 + 5e^{-3t}$
   (viii) $e^{2t} + 4t^3 + \cos 3t$ [Apr. 2016]
   (ix) $3t^2 + 2 \cos 2t + e^{-t}$ [Apr. 2016]
   (x) $t^3 + \cos 3t - e3t$ [Apr. 2018]
   (xi) $4e^{2t} + 6t^3 - 2 \cos 5t$ [Apr. 2016]
   (xii) $3t^2 + 2 \cos 2t + e^{-t}$ [Apr. 2016]
   (xiii) $t^3 + \cos 3t - e^{3t}$
   (xiv) $3 \sin 4t - 2\cos 5t$ [Oct. 2018, 2016]
   (xv) $3 \sin 4t + 4 \cos 3t$ [Apr. 2018; 2016]

(xvi) $2 \sin 5t - 3 \sin 6t$ [Apr. 2019]

(xvii) $3 \cosh 5t - 4 \sinh 5t$

(xviii) $t^2 + \sinh 2t + \sin 2t$

(xix) $e^{-3t} + 4 \cosh t + \sin 2t$ [Apr. 2017]

(xx) $t^3 - 2 \sin 3t + 3 \cos 3t$ [Apr. 2018]

(xxi) $9e^{-2t} + 5 \cos 2t + 5 \sin 3t$ [Apr. 2016]

(xxii) $e^{2t} + 4t^3 - 3 \sin 2t + 2 \cos 2t$

(xxiii) $3t^2 + 4 \sin 2t + 3 \cos 2t - 1$ [Apr. 2017, 2014]

(xxiv) $4e^{5t} + 6t^3 - 3 \sin 4t + 2 \cos 2t$.

4. (i) $\cos(wt - \alpha)$

(ii) $\sin(at + b)$

5. (i) $\sin^2 t$ [Apr. 2017, 2016 ; Oct. 2016]

(ii) $\sin^2 3t$

(iii) $\cos^2 at$

(iv) $\cos^2 2t$ [Apr. 2016, 2009, 2008]

(v) $\cos^2 3t$ [Apr. 2016]

(vi) $\sin^3 2t$ [Apr. 2008]

(vii) $\sin^3 t + e^{2t}$ [Apr. 2009]

(viii) $\cos^3 2t$

(ix) $\cosh^2 2t$

(x) $\cosh^2 3t$

6. (i) $\sin 8t \cos 4t$ [Apr. 2016]

(ii) $\sin 61t \cos 23t$ [Oct. 2008]

(iii) $\sin 2t \cos t$ [Apr. 2016]

(iv) $\cos 4t \sin 2t$ [Apr. 2018, 2017, 2016, 2014, 2009, 2008]

(v) $\sin 2t \sin 3t$ [Apr. 2017]

(vi) $\cos ct \cos 3t$

7. (i) $\cos t \cos t\ 2t \cos 3t$

(ii) $\cos 6t \cos 4t \sin t$ [Oct. 2007]

8. (i) $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 2 & t > 2 \end{cases}$

(ii) $f(t) = \begin{cases} t, & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

(iii) $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

(iv) $f(t) = \begin{cases} e^{-t}, & 0 < t < 4 \\ 0, & t > 4 \end{cases}$

(v) $f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

ANWERS

1. (i) $\frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}$ (ii) $\frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s}$

(iii) $\frac{1}{s} + \frac{\sqrt{\pi}}{s^{\frac{3}{2}}} + \sqrt{\frac{\pi}{s}}$ (iv) $\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

(v) $\frac{a^2}{s} + \frac{2ab}{s^2} + \frac{2b^2}{s^3}$

2. (i) $\frac{3}{s-2} + \frac{4}{s+3}$ (ii) $\frac{1}{s-3} - \frac{1}{s+3}$

(iii) $\frac{1}{s-1} - \frac{1}{s+3}$ (iv) $\frac{1}{a}\left[\frac{1}{s+a} - \frac{1}{s}\right]$

(v) $\frac{e^b}{s-a}$

3. (i) $\frac{6}{s^4}+\frac{5s}{s^2+1}$ (ii) $\frac{5}{s^2}+\frac{2}{s-1}+\frac{24}{s^2+9}$

(iii) $\frac{1}{s-2}-\frac{24}{s^4}+\frac{6}{s^2+9}$ (iv) $\frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+4}$

(v) $\frac{24}{s^5}+\frac{1}{s-2}+\frac{4}{s^2+4}$ (vi) $\frac{24}{s^4}+\frac{1}{s-2}+\frac{6}{s^2+9}$

(vii) $\frac{a}{s^2+a^2}+\frac{2}{s^3}+\frac{5}{s+3}$ (viii) $\frac{1}{s-2}+\frac{24}{s^4}+\frac{s}{s^2+9}$

(ix) $\frac{6}{s^3}+\frac{2s}{s^2+4}+\frac{1}{s+1}$ (x) $\frac{6}{s^4}+\frac{s}{s^2+9}-\frac{1}{s-3}$

(xi) $\frac{4}{s-2}+\frac{36}{s^4}-\frac{2s}{s^2+25}$ (xii) $\frac{6}{s^3}+\frac{2s}{s^2+4}+\frac{1}{s+1}$

(xiii) $\frac{6}{s^4}+\frac{s}{s^2+9}-\frac{1}{s-3}$ (xiv) $\frac{12}{s^2+16}-\frac{2s}{s^2+25}$

(xv) $\frac{12}{s^2+16}+\frac{4s}{s^2+9}$ (xvi) $\frac{10}{s^2+25}-\frac{3s}{s^2+36}$

(xvii) $\frac{3s}{s^2-25}-\frac{20}{s^2-25}$ (xviii) $\frac{2}{s^3}+\frac{2}{s^2-4}+\frac{2}{s^2+4}$

(xix) $\frac{1}{s+3}+\frac{4s}{s^2-1}+\frac{2}{s^2+4}$ (xx) $\frac{6}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+9}$

(xxi) $\frac{9}{s+2}+\frac{5s}{s^2+4}+\frac{15}{s^2+9}$ (xxii) $\frac{1}{s-2}+\frac{24}{s^4}-\frac{6}{s^2+4}+\frac{2s}{s^2+4}$

(xxiii) $\frac{6}{s^3}+\frac{8}{s^2+4}+\frac{3s}{s^2+4}-\frac{1}{s}$ (xxiv) $\frac{4}{s-5}+\frac{36}{s^4}-\frac{12}{s^2+16}+\frac{25}{s^2+4}$

4. (i) $\frac{s\cos\alpha}{s^2+w^2}+\frac{w\sin\alpha}{s^2+w^2}$ (ii) $\frac{1}{s^2+a^2}(a\cos b+s\sin b)$

5. (i) $\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$ (ii) $\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+36}\right]$

(iii) $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4a^2}\right]$ (iv) $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+16}\right]$

(v) $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+36}\right]$ (vi) $\frac{48}{(s^2+4)(s^2+36)}$

(vii) $\frac{3}{4}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]+\frac{1}{s-2}$ (viii) $\frac{s(s^2+28)}{(s^2+4)(s^2+36)}$

(ix) $\frac{s^2-8}{s(s^2-16)}$ (x) $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-36}\right]$

6. (i) $\frac{6}{s^2+144}+\frac{1}{s^2+16}$ (ii) $\frac{42}{s^2+7056}+\frac{19}{s^2+1444}$

(iii) $\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]$ (iv) $\frac{3}{s^2+36}-\frac{1}{s^2+4}$

(v) $\frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+25}\right]$ (vi) $\frac{1}{2}\left[\frac{s}{s^2+81}+\frac{s}{s^2+9}\right]$

7. (i) $\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$ (ii) $\frac{1}{4}\left[\frac{11}{s^2+121}-\frac{9}{s^2+81}-\frac{3}{s^2+9}-\frac{1}{s^2+1}\right]$

8. (i) $\frac{1}{s}(1+e^{-2s})$ (ii) $\frac{1}{s^2}\left[1-e^{-s}(1+s)\right]$

(iii) $e^{-s}\left[\frac{1}{s}+\frac{1}{s^2}\right]-e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)$ (iv) $\frac{1}{s+1}\left[1-e^{-4(s+1)}\right]$

(v) $\frac{2(1-e^{-\pi s})}{s^2+4}$

5.4.1 FIRST SHIFTING THEOREM OR FIRST TRANSLATION THEOREM

THEOREM :

If $L\{f(t)\} = \overline{f}(s)$ then $L\{e^{at}f(t)\} = \overline{f}(s-a)$

PROOF :

By definition

$$L\{(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s) \qquad \text{..............} \quad (1)$$

$$\therefore \quad L\{e^{at}f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \bar{f}(s-a) \qquad [\because \text{from } (1)]$$

i.e., $\boxed{L\{e^{at}f(t)\} = \bar{f}(s-a)}$

Thus, if we know the transformation $\bar{f}(s)$ of f(t), we can find the transformation of e^{at} f(t) simply replacing 's' by s – a to get $\bar{f}(s-a)$.

i.e., $L\left\{e^{at}f(t)\right\} = L\{f(t)\}_{s\to s-a} = \left[\bar{f}(s)\right]_{s\to s-a} = f(s-a)$

Note : If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{e^{-at}f(t)\right\} = \bar{f}(s+a)$.

Application of this property leads us to the following useful results.

1. $L\left\{e^{at}\right\} = \dfrac{1}{s-a}$ $\qquad \left[\because L\{1\} = \dfrac{1}{s}\right]$
2. $L\left\{e^{at}t^n\right\} = \dfrac{n!}{(s-a)^{n+1}}$ $\qquad \left[\because L\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}\right]$
3. $L\left\{e^{at}\sin bt\right\} = \dfrac{b}{(s-a)^2+b^2}$ $\qquad \left[\because L\{\sin bt\} = \dfrac{b}{s^2+b^2}\right]$
4. $L\left\{e^{at}\cos bt\right\} = \dfrac{s-a}{(s-a)^2+b^2}$ $\qquad \left[\because L\{\cos bt\} = \dfrac{s}{s^2+b^2}\right]$
5. $L\left\{e^{at}\sinh bt\right\} = \dfrac{b}{(s-a)^2-b^2}$ $\qquad \left[\because L\{\sinh bt\} = \dfrac{b}{s^2-b^2}\right]$
6. $L\left\{e^{at}\cosh bt\right\} = \dfrac{s-a}{(s-a)^2-b^2}$ $\qquad \left[\because L\{\cosh bt\} = \dfrac{s}{s^2-b^2}\right]$

Similarly,

7. $L\left\{e^{-at}\right\} = \left[\dfrac{1}{s}\right]_{s\to s+a} = \dfrac{1}{s+a}$
8. $L\left\{e^{-at}t^n\right\} = \left[\dfrac{n!}{s^{n+1}}\right]_{s\to s+a} = \dfrac{n!}{(s+a)^{n+1}}$

9. $L\left\{e^{-at}\sin bt\right\}=\left[\frac{b}{s^2+b^2}\right]_{s\to s+a}=\frac{b}{(s+a)^2+b^2}$

10. $L\left\{e^{-at}\cos bt\right\}=\left[\frac{s}{s^2+b^2}\right]_{s\to s+a}=\frac{s+a}{(s+a)^2+b^2}$

11. $L\left\{e^{at}\sinh bt\right\}=\left[\frac{b}{s^2-b^2}\right]_{s\to s+a}=\frac{b}{(s+a)^2-b^2}$

12. $L\left\{e^{-at}\cosh bt\right\}=\left[\frac{s}{s^2-b^2}\right]_{s\to s+a}=\frac{s+a}{(s+a)^2-b^2}$

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L\left\{t^2e^{2t}\right\}$ *[Apr. 2018, 2017]*

(ii) $L\{t^3\, e^{-3t}\}$ *[Apr. 2018, 2017]*

(iii) $L\{e^{2t}\sin 3t\}$ *[Apr. 2018]*

(iv) $L\,\{e^{-2t}\cos 5t\}$ *[Apr. 2017]*

(v) $L\left\{e^{2t}(\cos 4t+3\sin 4t\right\}$ *[Apr. 2018, 2017, 2016, 2008]*

Solution :

(i) Let

$$f(t) = t^2$$

$$L\,\{f(t)\} = L\,\{t^2\}$$

$$= \frac{2}{s^3}=\bar{f}(s) \qquad \text{.............. (1)}$$

By first shifting theorem,

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

$$\therefore L\{e^{2t}\,t^2\} = \bar{f}(s-2)$$

$$= \frac{2}{(s-2)^3} \qquad [\because \text{ from (1)}]$$

(ii) Let

$$f(t) = t^3$$

$$L\{f(t)\} = L\{t^3\}$$

$$= \frac{3!}{s^{3+1}} \qquad \left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{6}{s^4} = \bar{f}(s)$$

By first shifting theorem,

$$L\{e^{-at}f(t)\} = \bar{f}(s+a)$$

$$\therefore \quad L\{t^3 e^{-3t}\} = \bar{f}(s+3)$$

$$= \frac{6}{(s+3)^4}$$

(iii) Let $\quad f(t) = \sin 3t$

$$L\{f(t)\} = L\{\sin 3t\}$$

$$= \frac{3}{s^2+3^2} \qquad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2}\right]$$

$$= \frac{3}{s^2+9} = \bar{f}(s)$$

By first shifting theorem,

$$L\{e^{at} f(t)\} = \bar{f}(s-a)$$

$$\therefore \quad L\left\{e^{2t}\sin 3t\right\} = \bar{f}(s-2)$$

$$= \frac{3}{(s-2)^2+9}$$

(iv) Let $\quad f(t) = \cos 5t$

$$L\{f(t)\} = L\{\cos 5t\}$$

$$= \frac{s}{s^2+5^2} \qquad \left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+25} = \bar{f}(s)$$

By first shifting theorem,

$$L\left\{e^{-at}f(t)\right\} = \bar{f}(s+a)$$

$$\therefore L\left\{e^{-2t}\cos 5t\right\} = \bar{f}(s+2)$$

$$= \frac{s+2}{(s+2)^2+25}$$

(v) Let $f(t) = \cos 4t + 3\sin 4t$

$$L\{f(t)\} = L\{\cos 4t + 3\sin 4t\}$$

$$= L\{\cos 4t\} + 3L\{\sin 4t\}$$

$$= \frac{s}{s^2+4^2} + 3\cdot\frac{4}{s^2+4^2}$$

$$= \frac{s}{s^2+16} + \frac{12}{s^2+16}$$

$$= \frac{s+12}{s^2+16} = \overline{f}(s)$$

By first shifting theorem

$$L\{e^{at}f(t)\} = \overline{f}(s-a)$$

$$\therefore \quad L\left\{e^{2t}(\cos 4t + 3\sin 4t)\right\} = \overline{f}(s-2)$$

$$= \frac{s-2+12}{(s-2)^2+16}$$

$$= \frac{s+10}{s^2-4s+20}$$

EXAMPLE-2

Find :

(i) $L\{e^{-t}\sin^2 t\}$ *[Apr. 2018; Oct. 2008]*

(ii) $L\{\cosh at \cos bt\}$

(iii) $L\{e^{3t}\sin 3t \cos 2t\}$ *[Apr. 2018]*

(iv) $L\{e^{-2t}\cos 5t \cos 2t\}$ *[Apr. 2019]*

Solution :

(i) Let $f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$$L\{f(t)\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \overline{f}(s)$$

By first shifting theorem

$$L\{e^{-at} f(t)\} = \overline{f}(s+a)$$

$$\therefore L\{e^{-t} \sin^2 t\} = \overline{f}(s+1)$$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]$$

(ii) $$L\{\cos h\ at \cos bt\} = L\left\{\frac{(e^{at}+e^{-at})}{2}\cos bt\right\} \qquad \left[\because \cosh ax = \frac{e^{ax}+e^{-ax}}{2}\right]$$

$$= \frac{1}{2}\left[L\left\{e^{at}\cos bt\right\} + L\left\{e^{-at}\cos bt\right\}\right]$$

$$= \frac{1}{2}\left[L\{\cos bt\}_{s\to s-a} + L\{\cos bt\}_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+b^2}\right)_{s\to s-a} + \left(\frac{s}{s^2+b^2}\right)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} + \frac{s+a}{(s+a)^2+b^2}\right]$$

(iii) Let $$f(t) = \sin 3t \cos 2t = \frac{1}{2}[\sin(3t+2t) + \sin(3t-2t)]$$

$$\left[\because \sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}\right]$$

$$= \frac{1}{2}[\sin 5t + \sin t]$$

$$L\{f(t)\} = L\left\{\frac{1}{2}(\sin 5t + \sin t)\right\}$$

$$= \frac{1}{2}\left[L\{\sin 5t\} + L\{\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] = \overline{f}(s)$$

By first shifting theorem,

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

$$\therefore\ L\left\{e^{3t}(\sin 3t\cos 2t)\right\} = \bar{f}(s-3)$$

$$= \frac{1}{2}\left[\frac{5}{(s-3)^2+25}+\frac{1}{(s-3)^2+1}\right]$$

(iv) Let $f(t) = \cos 5t \cos 2t$

$$= \frac{1}{2}[\cos(5t+2t)+\cos(5t-2t)]$$

$$\left[\because \cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]\right]$$

$$= \frac{1}{2}[\cos 7t+\cos 3t]$$

$$L\{f(t)\} = \frac{1}{2}\left[L\{\cos 7t\}+L\{\cos 3t\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+7^2}+\frac{s}{s^2+3^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+49}+\frac{s}{s^2+9}\right] = \bar{f}(s)$$

$\therefore$ B first shifting theroem,

$$L\{e^{-at}\, f(t) = \bar{f}(s+a)$$

$$\therefore\ L\left\{e^{-2t}\cos 5t\cos 2t\right\} = \bar{f}(s+2)$$

$$= \frac{1}{2}\left[\frac{s+2}{(s+2)^2+49}+\frac{s+2}{(s+2)^2+9}\right]$$

5.4.2 *UNIT STEP FUNCTION OR HEAVISIDES UNIT FUNCTION*

DEFINITION :

The unit step function is denoted by $u(t-a)$

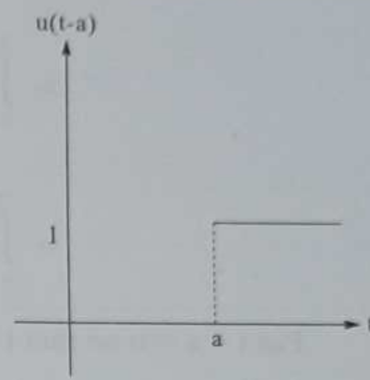
or $H(t-a)$ and is defined as

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

where 'a' is positive

Laplace transform of u(t – a) :

By Befinition,



$$L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt$$

$$= \int_0^\infty e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= \int_0^a e^{-st}\cdot 0dt + \int_a^\infty e^{-st}1\cdot dt$$

$$= \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{-1}{s}\left[e^{-\infty} - e^{-as}\right] = \frac{-1}{s}(0 - e^{-as})$$

$$= \frac{e^{-as}}{s}$$

Note : $f(t)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t) & \text{if } t > a \end{cases}$

5.4.3 SECOND SHIFTING THEOREM

THEOREM :

If $L\{f(t)\} = \overline{f}(s)$ and $g(t) = \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t > a \end{cases}$ then $L\{g(t)\} = e^{-as}\,\overline{f}(s)$.

PROOF :

By definition

$$L\{g(t)\} = \int_0^\infty e^{-st}g(t)dt$$

$$= \int_0^a e^{-st}g(t)dt + \int_a^\infty e^{-st}g(t)dt$$

$$= \int_0^a e^{-st} 0 \cdot dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put t – a = u so that t = u + a and dt = du

Limits : Lower limit when t = a ⇒ u = 0 and upper limit when t = ∞ ⇒ u = ∞.

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-s(u+a)} \cdot f(u)\,du$$

$$= \int_0^\infty e^{-su} \cdot e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)dt$$

$$= e^{-as} \cdot \overline{f}(s)$$

Another form of second shitting theorem :

If $L\{f(t)\} = \overline{f}(s)$ then $L\{f(t-a) \,.\, u(t-a)\} = e^{-as}\,\overline{f}(s) \cdot$ where u(t – a) is unit step funtion.

SOLVED EXAMPLES

EXAMPLE-1

Find L{g(t)}, where $g(t) = \begin{cases} 0, & \text{for } 0 < t < b \\ t-b, & \text{for } t > b \end{cases}$

Solution :

Here f(t–a) = t – b,

i.e., f(t) = t,

$\therefore\ L\{f(t)\} = L\{t\} = \dfrac{1}{s^2} = \overline{f}(s)$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\overline{f}(s)$$

$$= e^{-bs}\cdot\frac{1}{s^2}$$

$$= \frac{e^{-bs}}{s^2}$$

EXAMPLE-2

Find $L\{g(t)\}$ where $g(t) = \begin{cases} 0, & \text{if } t < \frac{\pi}{3} \\ \cos(t-\frac{\pi}{3}), & \text{if } t > \frac{\pi}{3} \end{cases}$

Solution :

Let $f(t-a) = \cos\left(t-\frac{\pi}{3}\right)$

i.e., $f(t) = \cos t$ and $a = \frac{\pi}{3}$

$$L\{f(t)\} = L\{(\cos t\} = \frac{s}{s^2+1} = \overline{f}(s)$$

By second shifting theorem,

$$L\{g(t)\} = e^{-as}\,\overline{f}(s)$$

$$= e^{-\frac{\pi}{3}s}\cdot\frac{s}{s^2+1}$$

$$= \frac{se^{-\frac{\pi s}{3}}}{s^2+1}$$

EXAMPLE-3

Find the Laplace transform of $g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ \sin t, & t > 0 \end{cases}$

Solution :

Given function g(t) can be written as

$$g(t) = \begin{cases} 0, & \text{if } t < \frac{\pi}{2} \\ \cos\left(\frac{\pi}{2}-t\right), & \text{if } t > \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} 0, & \text{if } t < \frac{\pi}{2} \\ \cos\left(t - \frac{\pi}{2}\right), & \text{if } t > \frac{\pi}{2} \end{cases}$$

Here $f(t-a) = \cos\left(t - \frac{\pi}{2}\right)$

$$\Rightarrow \qquad f(t) = \cos t \text{ and } a = \frac{\pi}{2}$$

$$L\{f(t)\} = L\{\cos t\}$$

$$= \frac{s}{s^2+1} = \overline{f}(s)$$

By second shifting theorem

$$L\{g(t)\} = e^{-as} \cdot \overline{f}(s)$$

$$= e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1} \cdot e^{\frac{-\pi s}{2}}$$

5.4.4 CHANGE OF SCALE PROPERTY

THEOREM :

If $L\{f(t)\} = \overline{f}(s)$ then $L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$.

PROOF :

By definition,

$$L\{f(at)\} = \int_0^\infty e^{-st} \cdot f(at)\,dt$$

Put $at = u \Rightarrow t = \frac{u}{a}$, then $dt = \frac{du}{a}$

Limits : Lower limit when $t = 0 \Rightarrow u = 0$

Upper limit when $t = \infty \Rightarrow u = \infty$

$$\therefore \qquad L\{f(at)\} = \int_0^\infty e^{-\frac{us}{a}} \cdot f(u) \cdot \frac{du}{a}$$

$$= \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)u} \cdot f(u)\,du$$

$$= \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)t} \cdot f(t)\,dt$$

$$\boxed{L\{f(at\} = \frac{1}{a}\cdot \bar{f}\left(\frac{s}{a}\right)}$$

Note : If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{f\left(\frac{t}{a}\right)\right\} = a \cdot \bar{f}(as)$

SOLVED EXAMPLES

EXAMPLE-1

If $L\{f(t)\} = \dfrac{s^2 - s + 1}{(2s+1)^2(s-1)}$, *find* $L\{f(2t)\}$

Solution :

Given that

$$L\{f(t)\} = \frac{s^2 - s + 1}{(2s+1)^2(s-1)} = \bar{f}(s)$$

By change of scale property,

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

$$L\{f(2t)\} = \frac{1}{2}\cdot \bar{f}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\left(\frac{s}{2}\right)^2 - \left(\frac{s}{2}\right) + 1}{\left(2\left(\frac{s}{2}\right)+1\right)^2\left(\frac{s}{2}-1\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2}{4} - \frac{s}{2} + 1}{(s+1)^2\left(\frac{s-2}{2}\right)}\right]$$

$$= \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$$

EXAMPLE-2

$$If\ L\left\{\frac{sint}{t}\right\} = tan^{-1}\left(\frac{1}{s}\right) find\ L\left\{e^t\frac{sin3t}{t}\right\}$$

Solution :

Let $$f(t) = \frac{\sin t}{t}$$

$$L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right) = \overline{f}(s)$$

By change scale property,

$$L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$$

$$\Rightarrow \quad L\{f(3t)\} = \frac{1}{3}\overline{f}\left(\frac{s}{3}\right)$$

$$\Rightarrow \quad L\left\{\frac{\sin 3t}{3t}\right\} = \frac{1}{3}\cdot\tan^{-1}\left(\frac{1}{\frac{s}{3}}\right)$$

$$\Rightarrow \frac{1}{3}\cdot L\left\{\frac{\sin 3t}{t}\right\} = \frac{1}{3}\cdot\tan^{-1}\left(\frac{3}{s}\right)$$

$$\Rightarrow \quad L\left\{\frac{\sin 3t}{t}\right\} = \tan^{-1}\left(\frac{3}{s}\right) = \overline{f}(s)$$

By first shiffing theorem,

$$L\left\{e^t\frac{\sin 3t}{t}\right\} = \overline{f}(s-1)$$

$$= \tan^{-1}\left(\frac{3}{s-1}\right)$$

EXERCISE-5.2

Find the Laplace transform of

I. First Shifting Theorem :

1. (i) te^t (ii) te^{-t}
(iii) te^{2t} (iv) te^{-2t}
(v) $t\,e^{3t}$ (vi) te^{5t}
(vii) $t^2\,e^{-2t}$ (viii) $t^3\,e^{2t}$ [Apr. 2019]
(ix) t^3e^{-2t} [Apr. 2018] (x) t^3e^{3t} [Apr.2018]
(xi) $t^7\,e^{5t}$ [Apr. 2019, 2016] (xii) $e^{-at}(1-at)$
(xiii) $e^t(t^2-6t+7)$ (xiv) $e^{-2t}(t^2-6t+7)$
(xv) $e^t(t+2)^2$ (xvi) $\sqrt{t}\,e^{3t}$

2. (i) $e^{-2t}\sin 4t$ [Apr. 2018, 2017, 2008, 2005]
(ii) $e^{-t}\sin 2t$ (iii) $e^{-t}\cos 2t$
(iii) $e^{-t}\cos 2tr$ (iv) $e^{-2t}\cos t$
(v) $e^{-2t}\cos 2t$ (vi) $e^{2t}\cos 3t$ [Apr. 2018, 2014, 2008]
(vii) $e^{2t}\cos 4t$ [Apr. 2018, 2016]
(viii) $e^{-3t}\cos 4t$ [Oct. 2018; Apr. 2009]
(ix) $e^{-t}\cosh t$

3. (i) $e^{-2t}(2\cos 3t-3\sin 3t)$ (ii) $e^{-2t}(3\sin 4t-4\cos 4t)$ [Apr. 2008]
(iii) $e^{2t}(3\sin 4t-4\cos 4t)$ (iv) $e^{-3t}(2\cos 5t-3\sin 5t)$ [Apr. 2019]
(v) $e^{-t}(3\sin h\,2t-5\cos h\,2t)$ [Apr. 2019]

4. (i) $e^{3t}\sin 2t$ (ii) $e^{-t}\cos 2t$ [APr. 2019]
(iii) $e^{2t}\cos 2t$

5. (i) $\sin h\,at\sin at$ (ii) $\cos h\,at\cos at$
(iii) $\sin h\,3t\cos 2t$

6. (i) $e^{4t}\sin 2t\cos t$ [APr. 2009, 2007; Oct. 2008]
(ii) $e^t\sin 3t\cos t$ (iii) $e^{-2t}\sin 5t\cos 3t$ [Oct. 2018]
(iv) $e^t\cos 4t\sin 2t$ (v) $e^{12t}\sin 5t\cos 7t$ [Oct. 2008]
(vi) $e^{-3t}\cos 5t\cos 2t$ [Arp. 2019] (viii) $e^{2t}\sin 3t\sin t$

II. Second Shifting Theorem :

7. (i) $g(t) = \begin{cases} 0, & \text{if } t < 4 \\ (t-4)^3, & \text{if } t > 4 \end{cases}$ (ii) $g(t) = \begin{cases} 0 & \text{if } t < a \\ e^{t-a} & \text{if } t > a \end{cases}$

(iii) $g(t) = \begin{cases} 0, & \text{if } t < \frac{2\pi}{3} \\ \sin\left(t - \frac{2\pi}{3}\right), & \text{if } t > \frac{2\pi}{3} \end{cases}$ (iv) $g(t) = \begin{cases} 0, & \text{if } t < \frac{2\pi}{3} \\ \cos\left(t - \frac{2\pi}{3}\right), & \text{if } t > \frac{2\pi}{3} \end{cases}$

III. Change of Scale Property :

8. (i) If $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$, find $L\{f(2t)\}$ **[Apr. 2019, 2017; Oct. 2008]**

(ii) If $L\{f(t)\} = \frac{1}{s} \cdot e^{-\frac{1}{s}}$, find (a) $L\{f(5t)\}$ and (b) $L\left\{e^{-t}f(3t)\right\}$

(iii) If $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$, find $L\{f(2t)\}$

(iv) Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$

(v) If $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$, find $L\left\{e^{-t}f(2t)\right\}$

ANSWERS

1. (i) $\frac{1}{(s-1)^2}$ (ii) $\frac{1}{(s+1)^2}$

(iii) $\frac{1}{(s-2)^2}$ (iv) $\frac{1}{(s+2)^2}$

(v) $\frac{1}{(s-3)^2}$ (vi) $\frac{1}{(s-5)^2}$

(vii) $\frac{2}{(s+2)^3}$

(viii) $\frac{6}{(s-2)^4}$

(ix) $\frac{6}{(s+2)^4}$

(x) $\frac{6}{(s-3)^4}$

(xi) $\frac{5040}{(s-15)^8}$

(xii) $\frac{1}{s+a}-\frac{a}{(s+a)^2}$

(xiii) $\frac{2}{(s-1)^3}-\frac{6}{(s-1)^2}+\frac{7}{s-1}$

(xiv) $\frac{2}{(s+2)^3}-\frac{6}{(s+2)^2}+\frac{7}{s+2}$

(xv) $\frac{2}{(s-1)^3}+\frac{4}{(s-1)^2}+\frac{4}{s-1}$

(xvi) $\frac{\sqrt{\pi}}{2(s-3)^{\frac{3}{2}}}$

2. (i) $\frac{4}{(s+2)^2+16}$

(ii) $\frac{2}{(s+1)^2+4}$

(iii) $\frac{s+1}{(s+1)^2+4}$

(iv) $\frac{s+2}{(s+2)^2+1}$

(v) $\frac{s+2}{(s+2)^2+4}$

(vi) $\frac{s-2}{(s+2)^2+9}$

(vii) $\frac{s-2}{(s-2)^2+16}$

(viii) $\frac{s+3}{(s+3)^2+16}$

(ix) $\frac{4}{(s-3)^2-16}$

(x) $\frac{s+1}{(s+1)^2-1}$

3. (i) $\frac{2s-5}{s^2+4s+13}$

(ii) $\frac{4(1-3)}{s^2+4s+20}$

(iii) $\frac{20-4s}{s^2-4s+120}$

(iv) $\frac{2s-9}{s^2+6s+34}$

(v) $\frac{1-5s}{s^2+2s-3}$

4. (i) $\frac{2}{(s-3)(s^2-6s+13)}$

(ii) $\frac{1}{2}\left[\frac{1}{s+1}+\frac{s+1}{s^2+2s+5}\right]$

(iii) $\frac{1}{2}\left[\frac{1}{s-2}+\frac{s-2}{(s-2)^2+4}\right]$

5. (i) $\frac{2a^2s}{s^4+4a^4}$ (ii) $\frac{s^3}{s^4+4a^4}$

(iii) $\frac{1}{2}\left[\frac{s^2-6s+11}{(s-3)(s^2-6s+13)}-\frac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$

6. (i) $\frac{1}{2}\left[\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right]$ (ii) $\frac{2}{(s-1)^2+16}$

(iii) $\frac{4}{(s+2)^2+64}+\frac{1}{(s+2)^2+4}$

(iv) $\frac{3}{(s-1)^2+36}-\frac{1}{(s-1)^2+1}$ (v) $\frac{6}{(s-12)^2+144}-\frac{1}{(s-12)^2+4}$

(vi) $\frac{1}{2}\left[\frac{s+3}{(s+3)^2+49}+\frac{s+3}{(s+3)^2+9}\right]$ (vii) $\frac{1}{2}\left[\frac{s-2}{(s-2)^2+36}-\frac{1}{(s-1)^2+4}\right]$

(viii) $\frac{1}{2}\left[\frac{s-2}{(s-2)^2+4}-\frac{s-2}{(s-2)^2+16}\right]$

7. (i) $\frac{6e^{-4s}}{s^4}$ (ii) $\frac{e^{-as}}{s-1}$

(iii) $\frac{e^{\frac{-2\pi s}{3}}}{s^2+1}$ (iv) $\frac{s}{s^2+1}e^{\frac{-2\pi s}{3}}$

8. (i) $\frac{60-4s}{s^2-12s+180}$ (ii) (a) $\frac{1}{s}e^{\frac{-5}{s}}$ (b) $\frac{e^{\frac{-3}{3+1}}}{s+1}$

(iii) $\frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$ (iv) $\tan^{-1}\left(\frac{a}{s}\right)$

(v) $\frac{4(9-s)}{s^2-6s+73}$

5.4.5 LAPLACE TANSFORM OF MULTIPLICATION BY t^n

1. Multiplication by 't'

THEOREM :

If $L\{f(t)\} = \bar{f}(s)$, then $L\{tf(t)\} = -\frac{d}{ds}\left[\bar{f}(s)\right]$

PROOF :

We know that

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

By Leibnitz's rule for differentiating under the integral sign.

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st} f(t))\,dt$$

$$= \int_0^\infty (-t)e^{-st} \cdot f(t)dt$$

$$= -\int_0^\infty e^{-st}(tf(t))\,dt$$

$$= -L\{tf(t)\}$$

$$\therefore \quad L\{tf(t)\} = \frac{-d}{ds}\left[\bar{f}(s)\right]$$

By Mathematical induction the result for nth derivative follows :

If $L\{f(t)\} = \bar{f}(s)$, then $\boxed{L\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]}$

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L\{te^{2t}\}$ [Apr. 2017]

(ii) $L\{t\cos at\}$

(iii) $L\{t\sin 3t\}$ [Apr. 2019, 2009, 2008 ; Oct. 2018, 2016]

(iv) $L\{t\sin^2 t\}$

(v) $L\{t\sin 3t\cos 2t\}$

Solution :

(i) Let $f(t) = e^{2t}$

$$L\{f(t)\} = L\{e^{2t}\} = \frac{1}{s-2} = \overline{f}(s)$$

$$\therefore \quad L\{tf(t)\} = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

$$L\{te^{2t}\} = -\frac{d}{ds}\left[\frac{1}{s-2}\right]$$

$$= -\left(\frac{-1}{(s-2)^2}\right)$$

$$= \frac{1}{(s-2)^2}$$

Alternative Method :

$$\text{Let } f(t) = t \Rightarrow L\{f(t)\} = \frac{1}{s^2} = \overline{f}(s)$$

By first shifting theorem,

$$L\{e^{at}f(t)\} = \overline{f}(s-a)$$

$$\therefore \quad L\{te^{2t}\} = \frac{1}{(s-2)^2}$$

(ii) $$\text{Let } f(t) = \cos at$$

$$L\{f(t)\} = L\{\cos at\}$$

$$= \frac{s}{s^2+a^2} = \overline{f}(s)$$

$$\therefore \quad L\{tf(t)\} = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

$$L\{t\cos at\} = \frac{-d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}\right] \qquad \left[\because \left(\frac{u}{v}\right)' = \frac{vu - uv'}{v^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(iii) $$\text{Let } f(t) = \sin 3t$$

$$L\{f(t)\} = L\{\sin 3t\}$$

$$= \frac{3}{s^2+9} = \overline{f}(s)$$

$$\therefore \quad L\{tf(t)\} = -\frac{d}{ds}\left[\overline{f}(s)\right]$$

$$L\{t\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -3\frac{d}{ds}\left[\left(s^2+9\right)^{-1}\right]$$

$$= -3\left(-(s^2+9)^{-2}\right)\cdot\frac{d}{ds}\left[(s^2+9)\right]$$

$$= \frac{3}{(s^2+9)^2}\cdot 2s$$

$$= \frac{6s}{(s^2+9)^2}$$

(iv) $$\text{Let } f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$$

$$L\{f(t)\} = \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \bar{f}(s)$$

Using Laplace transform of multiplication by 't', we have

$$L\{t\, f(t)\} = -\frac{d}{ds}\left[\bar{f}(s)\right]$$

$$L\{t \sin^2 t\} = -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{d}{ds}\left(\frac{1}{s}\right) - \frac{d}{ds}\left(\frac{s}{s^2+4}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{s^2} - \left(\frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right)\right] \qquad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$$

(v) $$\text{Let } f(t) = \sin 3t \cos 2t = \frac{1}{2}[\sin(3t+2t) + \sin(3t-2t)]$$

$$= \frac{1}{2}[\sin 5t + \sin t] \qquad \left[\because \sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))\right]$$

$$L\{f(t)\} = L\left\{\frac{1}{2}(\sin 5t + \sin t)\right\}$$

$$= \frac{1}{2}\left[L\{\sin 5t\} + L\{\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] = \bar{f}(s)$$

$$L\{tf(t)\} = -\frac{d}{ds}\left(\overline{f}(s)\right)$$

$$\therefore \quad = -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{d}{ds}\left(\frac{5}{s^2+25}\right)+\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[5\left(\frac{-1}{(s^2+25)^2}(2s)+\frac{-1}{(s^2+1)}(2s)\right)\right] \quad \left[\because \frac{d}{dx}\left[\frac{1}{f(x)}\right]=\frac{-1}{[f(x)]^2}\cdot f'(x)\right]$$

$$= \frac{5s}{(s^2+25)^2}+\frac{s}{(s^2+1)^2}$$

EXAMPLE-2

Find :

(i) *$L\{t^2 \cos t\}$* *[Apr. 2019; Oct. 2008]*

(ii) *$L\{t^2 \sin at\}$*

Solution :

(i) Let $f(t) = \cos t$

$$\therefore \quad L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1} = \overline{f}(s)$$

$$\therefore \quad L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\left[\overline{f}(s)\right]$$

$$L\{t^2 \cos t\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s+1}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)\cdot 1-s(2s)}{(s^2+1)^2}\right] \quad \left[\because \left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)^{2-1}\cdot(2s)}{(s^2+1)^4} \quad [\because \text{ by using quotient rule}]$$

$$= \frac{(s^2+1)\left[-2s(s^2+1)-4s(1-s^2)\right]}{(s^2+1)^3}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$$

$$= \frac{2s^3-6s}{(s^2+1)^3}$$

(ii) Let $f(t) = \sin at$

$$L\{f(t)\} = \{\sin at\}$$

$$= \frac{a}{s^2+a^2} = \overline{f}(s)$$

$$\therefore \quad L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\overline{f}(s)\right)$$

$$L\{t^2 \sin at\} = \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right]$$

$$= a\frac{d}{ds}\left(\frac{-1}{(s^2+a^2)}\cdot 2s\right) \quad \left[\because \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\cdot f'(x)\right]$$

$$= a\cdot\frac{d}{ds}\left(\frac{-2s}{(s^2+a)^2}\right)$$

$$= -2a\left[\frac{(s^2+a^2)^2\cdot 1 - s2(s^2+a^2)^{2-1}\cdot 2s}{(s^2+a^2)^4}\right] \quad \left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

EXAMPLE-3

Find :

(i) $L\{t\,e^{-3t}\cos 5t\}$ *[Apr. 2018]*

(ii) $L\{te^{-t}\sin 2t\}$ *[Oct. 2016]*

Solution :

(i) Let $f(t) = \cos 5t$

$$L\{f(t)\} = L\{\cos 5t\}$$

$$= \frac{s}{s^2+25}$$

$$L\{tf(t)\} = -\frac{d}{ds}\left(\frac{s}{s^2+25}\right)$$

$$L\{t\cos 5t\} = -\left[\frac{(s^2+25)\cdot 1 - s\cdot(2s)}{(s^2-25)^2}\right] \qquad \left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$$

$$= -\left[\frac{s^2+25-2s^2}{(s^2+25)^2}\right]$$

$$= \frac{s^2-25}{(s^2+25)^2} = \overline{f}(s)$$

By first shifting theorem

$$L\{e^{-3t}\,t\cos 5t\} = \overline{f}(s+3)$$

$$= \frac{(s+3)^2-25}{((s+3)^2+25)^2}$$

$$= \frac{s^2+6s-16}{(s^2+6s+34)^2}$$

(ii) Let $f(t) = \sin 2t$

$$L\{f(t)\} = L\{\sin 2t\}$$

$$= \frac{2}{s^2+4}$$

$$L\{tf(t)\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$L\{t \sin 2t\} = -2\left[\frac{-1}{(s^2-4)^2}\cdot 2s\right] \qquad \left[\because \frac{d}{dx}(f(x)^n = nf(x)^{n-1}f(x)\right]$$

$$= \frac{4s}{(s^2+4)^2} = \overline{f}(s)$$

By first shifting theorem,

$$L\{e^{-t} t \sin 2t\} = \overline{f}(s+1)$$

$$= \frac{4(s+1)}{\left[(s+1)^2+4\right]^2}$$

$$= \frac{4(s+1)}{(s^2+2s+5)^2}$$

5.4.6 LAPLACE TRANSFOR OF DIVISION BY t

Theorem : If $L\{f(t)\} = \overline{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)dx$

Provided the integral exists.

Proof : We have

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

Integrating both sides with respect to s from s = 0 to s = ∞.

$$\int_s^\infty \overline{f}(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds$$

By changing the order of integration we get

$$\int_s^\infty \overline{f}(s)ds = \int_0^\infty\int_s^\infty f(t)e^{-st}ds\,dt$$

$$= \int_0^\infty f(t)\left\{\int_s^\infty e^{-st}ds\right\}dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-\infty} - e^{-st}}{-t}\right]dt$$

$$= \int_0^\infty \left(0 + \frac{e^{-st}}{t}\right) f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{f(t)}{t}\right)dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

$$\boxed{\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)\,ds}$$

SOLVED EXAMPLES

EXAMPLE-1

Find $L\left\{\frac{\sin at}{t}\right\}$ *[Apr. 2007; 2005]*

Solution :

Let $f(t) = \sin at$

$$L\{f(t)\} = L\{\sin at\}$$

$$= \frac{a}{s^2 + a^2} = \overline{f}(s)$$

We know that,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)\,ds$$

$$\therefore \quad L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2 + a^2}\,ds$$

$$= a \cdot \frac{1}{a} \cdot \left[\mathrm{Tan}^{-1}\left(\frac{s}{a}\right)\right]_s^\infty \qquad \left[\because \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \mathrm{Tan}^{-1}(\infty) - \mathrm{Tan}^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right) \qquad \left[\because \tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}\right]$$

(or)

$$= \tan^{-1}\left(\frac{a}{s}\right) \qquad \left[\because \tan^{-1}(x) = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

EXAMPLE-2

Find :

(i) $L\left\{\frac{1-e^{2t}}{t}\right\}$ *[Apr. 2017]*

(ii) $L\left\{\frac{e^{at}-e^{bt}}{t}\right\}$

(iii) $L\left\{\frac{e^{-3t}-e^{-4t}}{t}\right\}$

Solution :

(i) Let $f(t) = 1 - e^{2t}$

$$L\{f(t)\} = L\{1-e^{2t}\}$$

$$= L\{1\} - L\{e^{2t}\}$$

$$= \frac{1}{s} - \frac{1}{s-2} = \bar{f}(s)$$

By L.T. of division by 't', we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

$$L\left\{\frac{1-e^{2t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-2}\right)dx$$

$$= [\log s - \log(s-2)]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= \lim_{\frac{1}{s}\to\infty} \log\left(\frac{s}{s-2}\right) - \log\left(\frac{s}{s-2}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{1}{1-\frac{2}{s}}\right) - \log\left(\frac{s}{s-2}\right)$$

$$= \log\left(\frac{1}{1-0}\right) - \log\left(\frac{s}{s-2}\right)$$

$$= \log 1 - \log\left(\frac{s}{s-2}\right)$$

$$= 0 - \log\left(\frac{s}{s-2}\right) \qquad [\because \log 1 = 0]$$

$$= \log\left(\frac{s-2}{s}\right)$$

(ii) Let $f(t) = e^{at} - e^{bt}$

$$L\{f(t)\} = L\{e^{at} - e^{bt}\}$$

$$= \frac{1}{s-a} - \frac{1}{s-b} = \bar{f}(s)$$

We know that,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

$$L\left\{\frac{e^{at} - e^{bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s-b}\right) ds$$

$$= [\log(s-a) - \log(s-b)]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{s-b}\right)\right]_s^{\infty}$$

$$= \lim_{s\to 0}\log\left(\frac{s-a}{s-b}\right) - \log\left(\frac{s-a}{s-b}\right)$$

$$= \lim_{\frac{1}{s}\to 0}\log\left(\frac{\left(1-\frac{a}{s}\right)}{\left(1-\frac{b}{s}\right)}\right) - \log\left(\frac{s-a}{s-b}\right)$$

$$= \log\left(\frac{1-0}{1-0}\right) - \log\left(\frac{s-a}{s-b}\right)$$

$$= \log 1 - \log\left(\frac{s-a}{s-b}\right)$$

$$= 0 - \log\left(\frac{s-a}{s-b}\right)$$

$$= \log\left(\frac{s-b}{s-a}\right)$$

(iii) Let $f(t) = e^{-3t} - e^{-4t}$

$$L\{f(t)\} = L\{e^{-3t} - e^{-4t}\}$$

$$= \frac{1}{s+3} - \frac{1}{s+4} = \overline{f}(s)$$

We know that

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty}\overline{f}(s)\,ds$$

$$L\left\{\frac{e^{-3t} - e^{4t}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s+3} - \frac{1}{s+4}\right)ds$$

$$= [\log(s+3) - \log(s+4)]_s^{\infty}$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^{\infty}$$

$$= \lim_{s\to\infty} \log\left(\frac{s+3}{s+4}\right) - \log\left(\frac{s+3}{s+4}\right)$$

$$= \lim_{\frac{1}{s}\to 0} \log\left(\frac{\left(1+\frac{3}{s}\right)}{\left(1+\frac{4}{s}\right)}\right) - \log\left(\frac{s+3}{s+4}\right)$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s+3}{s+4}\right)$$

$$= \log 1 - \log\left(\frac{s+3}{s+4}\right)$$

$$= 0 - \log\left(\frac{s+3}{s+4}\right) = \log\left(\frac{s+4}{s+3}\right)$$

EXAMPLE-3

Find : $L\left\{\frac{e^{at} - \cos bt}{t}\right\}$

Solution :

Let

$$f(t) = e^{at} - \cos bt$$

$$L\{f(t)\} = L\{e^{at} - \cos bt\}$$

$$= L\{e^{at}\} - L\{\cos bt\}$$

$$= \frac{1}{s-a} - \frac{s}{s^2+b^2} = \overline{f}(s)$$

We know that

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)\,ds$$

$$\therefore \quad L\left\{\frac{e^{at} - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{2}\cdot\frac{2s}{s^2+b^2}\right) ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$$= \left[\log(s-a) - \log(s^2+b^2)^{\frac{1}{2}}\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\frac{s-a}{\sqrt{s^2+b^2}} - \log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \lim_{\frac{1}{s}\to 0} \log\left(\frac{\left(1-\frac{a}{s}\right)}{s\sqrt{1+\frac{b^2}{s^2}}}\right) - \log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log\left(\frac{1-0}{\sqrt{1+0}}\right) - \log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log 1 - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right) \qquad \left(\because \log a - \log b = \log\frac{a}{b}\right)$$

EXAMPLE-4

Find :

(i) $L\left\{\frac{1-\cos at}{t}\right\}$

(ii) $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

(iii) $L\left\{\frac{\cos 4t \ \sin 2t}{t}\right\}$ *[Apr. 2008]*

Solution :

(i) Let $f(t) = 1 - \cos at$

$$\therefore \quad L\{f(t)\} = L\{1-\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2} = \bar{f}(s) \qquad \text{(say)}$$

$$\therefore \quad L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s)\,ds$$

$$= \int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds$$

$$= \int_s^{\infty}\left(\frac{1}{s} - \frac{1}{2}\cdot\frac{2s}{s^2+a^2}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^{\infty}$$

$$= \left[\log s - \log\left(s^2+a^2\right)^{\frac{1}{2}}\right]_s^{\infty}$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]_s^{\infty} \qquad \left[\because \log a = \log b = \log\frac{a}{b}\right]$$

$$= \lim_{s\to\infty}\log\frac{s}{\sqrt{s^2+a^2}} - \log\frac{s}{\sqrt{s^2+a^2}}$$

$$= \lim_{\frac{1}{s}\to 0}\log\left(\frac{s}{s\sqrt{1+\frac{a^2}{s^2}}}\right) - \log\frac{s}{\sqrt{s^2+a^2}}$$

$$= \log\frac{1}{\sqrt{1+0}} - \log\frac{s}{\sqrt{s^2+a^2}}$$

$$= \log 1 - \log\frac{s}{\sqrt{s^2+a^2}}$$

$$= \log\frac{\sqrt{s^2+a^2}}{s} \qquad \left[\because \log a - \log b = \log\frac{a}{b}\right]$$

(ii) Let $\quad f(t) = \cos at - \cos bt$

$$L\{f(t)\} = L\{\cos at - \cos bt\}$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = \bar{f}(s) \qquad \text{(say)}$$

$$\therefore \quad L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{\frac{1}{s}\to 0}\log\left(\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s\left(1+\frac{b^2}{s^2}\right)}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \qquad \left[\because \log a - \log b = \log\frac{a}{b}\right]$$

(iii) Let $f(t) = \cos 4t \sin 2t$

$$= \frac{1}{2}[2\cos 4t \sin 2t]$$

$$= \frac{1}{2}[\sin(4t+2t) - \sin(4t-2t)] \qquad \left[\begin{array}{l}\because 2\cos A \sin B = \\ \sin(A+B) - \sin(A-B)\end{array}\right]$$

$$= \frac{1}{2}(\sin 6t - \sin 2t)$$

$$L\{f(t)\} = L\left\{\frac{1}{2}(\sin 6t - \sin 2t\right\} = \frac{1}{2}\left(\frac{6}{s^2+6^2} - \frac{2}{s^2+2^2}\right) = \bar{f}(s)$$

We know that

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

$$L\left\{\frac{\cos 4t \sin 2t}{t}\right\} = \frac{1}{2}\int_s^\infty \left(\frac{6}{s^2+6^2} - \frac{2}{s^2+2^2}\right)ds$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{6}\right) - \tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left\{\left[(\tan^{-1}(\infty) - \tan^{-1}(\infty)\right] - \left(\tan^{-1}\left(\frac{s}{6}\right) - \tan^{-1}\left(\frac{s}{2}\right)\right)\right\}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right]$$

5.4.7 LAPLACE TRANSFORM OF DERIVATIVES

THEOREM-1 :

If $f(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$

PROOF :

By Definition,

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

$$= \left[e^{-st}\cdot f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st}\cdot f(t)dt \qquad [\because \text{Integration by parts}]$$

$$= \lim_{t\to\infty} e^{-st}\cdot f(t) - f(0) + s\int_0^\infty e^{-st} f(t)dt$$

Since f(t) is of exponential order, $\lim_{t \to \infty} e^{-st} f(t) = 0$

$$\therefore \quad L\{f'(t)\} = 0 - f(0) + sL\{f(t)\}$$

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

Result-1 : By applying the above theorem to f''(t), we have

$$L\{f''(t)\} = S \cdot L\{f'(t)\} - f'(0)$$

$$= s\left[SL\{f(t)\} - f(0)\right] - f'(0)$$

$$\therefore \quad \boxed{L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)}$$

Result-2 : Similarly, for L.T. of derivatives of order n :

$$\boxed{L\left\{f^n(t)\right\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots f^{n-1}(0)}$$

SOLVED EXAMPLES

EXAMPLE-1

Find the Laplace transform of the following using the theorem of Laplace transform of derivatives.

(i) t^2 *(ii)* $t \cos at$

Solution :

(i) Let $f(t) = t^2$

$f'(t) = 2t$

$f''(t) = 2$

and $f'(0) = 0$ and $f(0) = 0$

$\therefore$ By theorem of laplace transform on derivatives.

$$L\{f''(t)\} = s^2 L\{f(t)\} - s\, f(0) - f'(0)$$

$$L\{2\} = s^2 L\{t^2\} - s(0) - 0$$

$$2 \cdot \frac{1}{s} = s^2 L\left\{t^2\right\}$$

$$\therefore \quad L\left\{t^2\right\} = \frac{2}{s^3}$$

(ii) Let $f(t) = t\cos at$

$$f'(t) = 1\cdot\cos at + t\cdot(-a\sin at) \qquad [\because (uv)' = u'v + uv']$$

$$= \cos at - at\sin at$$

$$f''(t) = -a\sin at - a\;(1\cdot\sin at + ta\cos at)$$

$$= a\sin at - a\sin at - a^2 t\cos at$$

$$= -2a\sin at - a^2 t\cos at$$

and $f(0) = 0,\; f'(0) = 1$

we know that

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

$$\Rightarrow \quad L\{-2a\sin at - a^2 t\cos at\} = s^2 L\{t\cos at\} - s(0) - 1$$

$$\Rightarrow \quad -2a\, L\{\sin at\} - a^2 L\{t\cos at\} = s^2 L\{t\cos at\} - 1$$

$$\Rightarrow \quad S^2 L\{t\cos at\} + a^2 L\{t\cos at\} = 1 - 2a\,.\,L\{\sin at\}$$

$$\Rightarrow \quad (s^2 + a^2)\, L\{t\cos at\} = 1 - \frac{2a\cdot a}{s^2 + a^2}$$

$$\Rightarrow \quad L\{t\cos at\} = \frac{1}{s^2 + a^2}\left[\frac{s^2 + a^2 - 2a^2}{s^2 + a^2}\right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

EXAMPLE-2

Given $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{S^{\frac{3}{2}}}$, ***show that*** $L\left\{\frac{1}{\sqrt{t\pi}}\right\} = \frac{1}{\sqrt{s}}$

Solution : Let $f(t) = 2\sqrt{\frac{t}{\pi}}$

$$f'(t) = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{t\pi}}$$

and $f(0) = 0$

By Laplace transform for derivatives

$$L\{f'(t)\} = SL\{f(t)\} - f(0)$$

$$L\left\{\frac{1}{\sqrt{t\pi}}\right\} = S\cdot\frac{1}{S^{\frac{3}{2}}}-0$$

$$= \frac{1}{S^{\frac{1}{2}}} = \frac{1}{\sqrt{s}}$$

5.4.8 LAPLACE TRANSFORM OF INTEGRALS

Theorem : If $L\{f(t)\} = \overline{f}(s)$, then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\overline{f}(s)$

Proof : Let $\phi(t) = \int_0^t f(u)du$, then $\phi'(t) = f(t)$ and $\phi(0) = 0$.

By Laplace transform of derivatives

$$\therefore \quad L\{\phi'(t)\} = s\cdot L\{\phi(t)\} - \phi(0)$$

$$\therefore \quad L\{f(t)\} = s\cdot L\left\{\int_0^t f(u)du\right\} - 0$$

$$\overline{f}(s) = s\cdot L\left\{\int_0^t f(u)\,du\right\}$$

$$\boxed{\therefore \; L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\cdot\overline{f}(s)}$$

Result-1 : Similarly if $L\{f(t)\} = \overline{f}(s)$, then

$$L\left\{\int_0^t\int_0^t f(u)du\,du\right\} = \frac{1}{s^2}\overline{f}(s)$$

In general $$L\left\{\underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{(n\text{ times})} f(u)\underbrace{du\,du\,....du}_{(n\text{ times})}\right\} = \frac{1}{s^n}\overline{f}(s)$$

SOLVED EXAMPLES

EXAMPLE-1

Find $L\left\{\int_0^t \sin 2t\,dt\right\}$ *[Apr. 2017, 2009 ; Oct. 2016, 2008]*

Solution :

(i) Let $f(t) = \sin 2t$, then

$$L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} = \bar{f}(s)$$

By Laplace transform of integral, we have

$$L\left\{\int_0^t f(t)dt\right\} = \frac{\bar{f}(s)}{s}$$

$$\therefore L\left\{\int_0^t \sin 2t\,dt\right\} = \frac{1}{s}\cdot\frac{2}{s^2+4}$$

$$= \frac{2}{s(s^2+4)}$$

EXAMPLE-2

Find $L\left\{\int_0^t e^t \cos h t dt\right\}$

Solution :

(i) Let $f(t) = e^t \cos ht$, then

$$L\{f(t)\} = L\{e^t \cosh t\}$$

$$= L\{\cos h t\}_{s\to s-1}$$

$$= \frac{s-1}{(s-1)^2-1}$$

$$= \frac{s-1}{s^2-2s} = \bar{f}(s)$$

By L.T of integrals, we have

$$L\left\{\int_0^t f(t)dt\right\} = \frac{\bar{f}(s)}{s}$$

$$\therefore \quad L\left\{\int_0^t e^t \cosh t dt\right\} = \frac{1}{s}\cdot\frac{s-1}{s^2-2s}$$

$$= \frac{s-1}{s^2(s-2)}$$

EXAMPLE-3

Find $L\left\{\int_0^t te^{-t}\sin t\, dt\right\}$

Solution :

We knw that

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$\therefore \quad L\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\left(\frac{-1}{(s^2+1)}(2s)\right) \qquad \left[\because \frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{[f(x)]^2}\cdot f'(x)\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

By first shifting theorem,

$$L\{e^{-t}\, t \sin t\} = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{2(s+1)}{\left[(s+1)^2+1\right]^2}$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2} = \bar{f}(s)$$

By L.T of integrals

$$L\left\{\int_0^t te^{-t}\sin t\,dt\right\} = \frac{\overline{f}(s)}{s}$$

$$= \frac{1}{s}\frac{2(s+1)}{(s^2+2s+2)^2}$$

$$= \frac{2(s+1)}{s(s^2+2s+2)^2}$$

EXAMPLE-4

Find $L\left\{\int_0^t e^t \frac{\sin t}{t}\,dt\right\}$ *[Apr. 2019 ; Oct. 2008, 2016]*

Solution :

We have $L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore \quad L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

By first shifting theorem,

$$L\left\{e^t\frac{\sin t}{t}\right\} = \left[\cot^{-1}(s)\right]_{s\to s-1} = \cot^{-1}(s-1) = \overline{f}(s)$$

Using the laplace transform of integrals,

$$L\left\{\int_0^t e^t\frac{\sin t}{t}\,dt\right\} = \frac{\overline{f}(s)}{s}$$

$$= \frac{1}{s}\cdot\cot^{-1}(s-1)$$

EXAMPLE-5

Find $L\left\{\int_0^t \frac{1-e^{-t}}{t}dt\right\}$

Solution :

Let $\quad f(t) = 1 - e^{-t}$

$\therefore \quad L\{f(t)\} = L\{1-e^{-t}\}$

$$= \frac{1}{s} - \frac{1}{s+1} = \bar{f}(s)$$

By L.T. of division by 't', we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)dx$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds$$

$$= [\log s - \log(s+1)]_s^\infty$$

$$= \left[\log\left(\frac{s}{s+1}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s+1}\right) - \log\left(\frac{s}{s+1}\right)$$

$$= \lim_{\frac{1}{s}\to 0} \log\left(\frac{\cancel{s}}{\cancel{s}\left(1+\frac{1}{s}\right)}\right) - \log\left(\frac{s}{s+1}\right)$$

$$= \log\left(\frac{1}{1+0}\right) - \log\left(\frac{s}{s+1}\right)$$

$$= \log 1 - \log\left(\frac{s}{s+1}\right)$$

$$= 0 - \log\left(\frac{s+1}{s}\right)$$

$$= \log\left(\frac{s+1}{s}\right) = \bar{f}(s)$$

By L.T of integrals

$$L\left\{\int_0^t \left(\frac{1-e^t}{t}\right)dt\right\} = \frac{\bar{f}(s)}{s}$$

$$= \frac{1}{s}\cdot\log\left(\frac{s+1}{s}\right)$$

EXAMPLE-6

Find $L\left\{\int_0^t\int_0^t \cos at\,dt\,dt\right\}$

Solution :

Let $\quad f(t) = \cos at$

$$L\{f(t)\} = L\{\cos at\}$$

$$= \frac{s}{s^2+a^2} = \bar{f}(s)$$

By L.T. of integrals, we have

$$L\left\{\int_0^t\int_0^t f(t)dt\,dt\right\} = \frac{\bar{f}(s)}{s^2}$$

$$L\left\{\int_0^t\int_0^t \cos at\,dt\,dt\right\} = \frac{1}{s^2}\cdot\frac{s}{s^2+a^2} = \frac{1}{s(s^2+a^2)}$$

EXERCISE 5.3

Find Laplace transform of

I. Multiplication by power of 't' :

1. (i) $t\,e^t$

(ii) $t\,e^{-t}$

(iii) $t^2\,e^{2t}$ [Apr. 20016]

(iv) $t^2\,e^{-2t}$ [Apr. 2019, 2016]

(iv) $t\,e^{-t} + \sin 4t$ [Oct. 2018]

2. (i) $t \sin at$ [Apr. 2017]

   (ii) $t \sin 2t$ [Apr. 2016, 2008]

   (iii) $t \cos 3t$ [Apr. 2014]

   (iv) $t (\sin t + \cos t)$

   (v) $t(3 \sin 2t - 2 \cos 2t)$

   (vi) $t \sin 2t \cos t$

   (vii) $t \sin 2t \cos 3t$ [Apr. 2019]

   (viii) $t \cos 3t \cos t$ [Apr. 2016]

   (ix) $t \cos^2 t$ [Apr. 2018, 2016]

   (x) $t \sin^2 2t$

   (xi) $t \sin^2 3t$

3. (i) $t^2 \sin t$ [Apr. 2019]

   (ii) $t^2 \sin 3t$

   (iii) $t^2 \cos 2t$ [Apr. 2007]

   (iv) $t^2 \cos 3t$

   (v) $t^2 \cos h\, at$

4. (i) $te^{-t} \sin t$

   (ii) $te^{-t} \sin 3t$ [Apr. 2018, 2016, 2008]

   (iii) $te^{-t} \sin 4t$ [Oct. 2006]

   (iv) $t\, e^{-t} \sin 3t$

   (v) $t\, e^{-2}t \sin 4t$

   (vi) $t\, e^{-4t} \sin 3t$

   (vii) $t\, e^{2t} \sin 3t$

   (viii) $t\, e^{3t} \sin 2t$

   (ix) $te^{2t} \sin bt$ [Oct. 2005]

   (x) $t\, e^{2t} \cos 5t$ [Apr 2019]

   (xi) $te^{-t} \cos t$

   (xii) $t\, e^{2t} \sin ht$ [Apr. 2019]

II. Division by t

5. (i) $\dfrac{\sin t}{t}$

   (ii) $\dfrac{\sin 2t}{t}$

   (iii) $\dfrac{\sin 4t}{t}$ [Apr. 2018; 2008]

   (iv) $\dfrac{\sin ht}{t}$

6. (i) $\dfrac{1-e^{t}}{t}$ [Apr. 2017]

   (ii) $\dfrac{1-e^{-t}}{t}$

   (iii) $\dfrac{e^{-at}-e^{-bt}}{t}$

   (iv) $\dfrac{e^{-t}-e^{-2t}}{t}$ [Oct. 2018]

   (v) $\dfrac{e^{2t}-e^{3t}}{t}$

7. $\frac{e^t - \cos t}{t}$

8. (i) $\frac{1-\cos t}{t}$ [Apr. 2018, 2016, 2009]

(ii) $\frac{1-\cos 2t}{t}$

(ii) $\frac{\sin^2 t}{t}$ (iii) $\frac{\text{co2t} - \cos 3t}{t}$ [Oct. 2006]

(iv) $\frac{\sin 3t \cos t}{t}$ [Apr. 2009; Oct. 2008]

9. (i) $\frac{e^{-t}\sin t}{t}$ [Apr. 2017]

(ii) $\frac{e^t \sin t}{t}$

III Laplace transform of derivatives

10. It $L\left\{t^{\frac{1}{2}}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$, then prove that $L\left\{t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s}}$

11. Give $L\left\{\sin\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \cdot e^{\frac{1}{4s}}$ Prove that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}} \cdot e^{-\frac{1}{4s}}$

12. Using laplace transform of derivatives find the laplace transform of t sin at.

13. If $L\{t\sin at\} = \frac{2as}{(s^2+a^2)}$, then prove that $L\{\sin at + at\cos at\} = \frac{2as^2}{(s^2+a^2)^2}$

IV Laplace transform of integrals

Find the laplace transform of

14. (i) $\int_0^t \sin t\, dt$ [Apr. 2018]

(ii) $\int_0^t \cos t\, dt$ (iii) $\int_0^t \sin ht\, dt$

(iv) $\int_0^t \cosh t\, dt$

15. (i) $\int_0^t e^t \sin t\, dt$ (ii) $\int_0^t e^{-t} \sin t\, dt$ [Apr. 2018, 2017]

(iii) $\int_0^t e^{-t} \cos t\, dt$ [Oct. 2008]

(iv) $\int_0^t e^{at} \sinh bt\, dt$

16. (i) $\int_0^t t e^{-t} \sin 4t\, dt$ [Apr. 2008]

(ii) $\int_0^t t e^{-2t} \sin 3t\, dt$

17. (i) $\int_0^t \frac{1-e^{-t}}{t} dt$ (ii) $\int_0^t \left(\frac{1-e^{2t}}{t}\right) dt$

(iii) $\int_0^t \frac{\sin t}{t} dt$ (iv) $\int_0^t \frac{e^{-t}\sin t}{t} dt$

18. (i) $\int_0^t \int_0^t \sin at\, dt\, dt$ (ii) $\int_0^t \int_0^t \cosh at\, dt\, dt$ [Apr. 2009]

ANSWERS

1. (i) $\frac{1}{(s-1)^2}$ (ii) $\frac{1}{(s+1)^2}$

(iii) $\frac{2}{(s-2)^3}$ (iv) $\frac{2}{(s+2)^3}$

(v) $\frac{2}{(s+3)^3}$ (vi) $\frac{1}{(s+1)^2} + \frac{4}{s^2+16}$

2. (i) $\frac{2as}{(s^2+a^2)^2}$ (ii) $\frac{4s}{(s^2+4)^2}$

(iii) $\frac{s^2-9}{(s^2+9)^2}$ (iv) $\frac{s^2+2s-1}{(s^2+1)^2}$

(v) $\dfrac{8+12s-4s^2}{(s^2+4)^2}$ (vi) $\dfrac{3s}{(s^2+3^2)}+\dfrac{s}{(s^2+1)^2}$

(vii) $\dfrac{5s}{(s^2+25)^2}-\dfrac{s}{(s^2+1)^2}$ (viii) $\dfrac{1}{2}\left[\dfrac{s^2-16}{(s^2+16)^2}+\dfrac{s^2-4}{(s^2+4)^2}\right]$

(ix) $\dfrac{1}{2}\left[\dfrac{1}{s^2}+\dfrac{s^2-4}{(s^2+4)^2}\right]$ (x) $\dfrac{1}{2}\left[\dfrac{1}{s^2}-\dfrac{s^2-16}{(s^2+16)^2}\right]$

(xi) $\dfrac{1}{2}\left[\dfrac{1}{s^2}-\dfrac{s^2-36}{(s^2+36)^2}\right]$

3. (i) $\dfrac{2(3s^2-1)}{(s^2+1)^3}$ (ii) $\dfrac{18(s^2-3)}{(s^2+9)^3}$

(iii) $\dfrac{4s^3-18s}{(s^2+4)^3}$ (iv) $\dfrac{2s^3-54s}{(s^2+9)^3}$

(v) $\dfrac{2s(s^2+3a^2)}{(s^2-a^2)^3}$

4. (i) $\dfrac{2(s+1)}{(s^2+2s+2)^2}$ (ii) $\dfrac{6(s+1)}{(s^2+2s+10)^2}$

(iii) $\dfrac{8(s+1)}{(s^2+2s+17)^2}$ (iv) $\dfrac{6(s+2)}{(s^2+4s+13)^2}$

(v) $\dfrac{8(s+1)}{(s^2+4s+20)^2}$ (vi) $\dfrac{6(s+4)}{(s^2+8s+2s)^2}$

(vii) $\dfrac{6(s-2)}{(s^2-4s+13)^2}$ (viii) $\dfrac{4(s-3)}{(s^2-6s+13)^2}$

(ix) $\dfrac{2b(s-a)}{(s^2-2as+a^2+b^2)}$ (x) $\dfrac{s^2-5s-21}{(s^2-4s+29)^2}$

(xi) $\dfrac{s^2+2s}{(s^2+2s+2)^2}$ (xiii) $\dfrac{2(s-2)}{(s^2-4s+3)^2}$

5. (i) $\cot^{-1}(s)$ (ii) $\cot^{-1}\left(\dfrac{s}{2}\right)$

(iii) $\cos^{-1}\left(\dfrac{s}{4}\right)$ (iv) $\dfrac{1}{2}\log\left(\dfrac{s+1}{s-1}\right)$

6. (i) $\log\left(\frac{s-1}{s}\right)$ (ii) $\log\left(\frac{s+1}{s}\right)$

(iii) $\log\left(\frac{s+b}{s+a}\right)$ (iv) $\log\left(\frac{s+2}{s+1}\right)$

(v) $\log\left(\frac{s-3}{s-2}\right)$

7. $\frac{1}{2}\log\left(\frac{s^2+1}{(s-1)^2}\right)$

8. (i) $\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$ (ii) $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$

(iii) $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$ (iv) $\frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$

(v) $\frac{1}{2}\left[\pi-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$

9. (i) $\cot^{-1}(s+1)$ (ii) $\cot^{-1}(s-1)$

12. (i) $\frac{2as}{(s^2+a^2)^2}$

14. (i) $\frac{1}{s(s^2+1)}$ (ii) $\frac{1}{s^2+1}$

(iii) $\frac{1}{s(s^2-1)}$ (iv) $\frac{1}{s^2-1}$

15. (i) $\frac{1}{s(s^2-2s+2)}$ (ii) $\frac{1}{s(s^2+2s+2)}$

(iii) $\frac{s+1}{s(s^2+2s+2)}$ (iv) $\frac{b}{s\left((s-a)^2+b^2\right)}$

16. (i) $\frac{8(s+1)}{s(s^2+2s+17)^2}$ (ii) $\frac{6(s+2)}{s(s^2+4s+13)^2}$

17. (i) $\frac{1}{s}\log\left(\frac{s+1}{s}\right)$ (ii) $\frac{1}{s}\log\left(\frac{s-2}{s}\right)$

(iii) $\cot^{-1}$

(iv) $\frac{1}{s}\cot^{-1}(s+1)$

18. (i) $\frac{a}{s^2(s^2+a^2)}$

(ii) $\frac{1}{s(s^2-a^2)}$

5.5 EVALUATION OF DEFINITE INTEGRALS BY LAPLACE TRANSFORMS

Definite integrals having lower limit 0 and upper limit ∞ can be evaluate by using Laplace transform

SOLVED EXAMPLES

EXAMPLE-1

Evaluate :

(i) $\int_0^\infty te^{-2t}dt$ **[Apr. 2016]**

(ii) $\int_0^\infty e^{-2t}\sin 3t\,dt$ **[Apr. 2016]**

(iii) $\int_0^\infty e^{-4t}\cos 3t\,dt$ **[Apr. 2019, 2017; Oct. 2016, 2008]**

By using Laplace transform

Solution :

(i) Given integral is same as

$$\int_0^\infty e^{-st}\cdot t\,dt \quad \text{where } s = 2$$

But $\int_0^\infty e^{-st}t\cdot dt = L\{t\} = \frac{1}{s^2}$

$$\therefore \int_0^\infty te^{-2t}\,dt = \int_0^\infty te^{-st}dt = \frac{1}{s^2} \text{ where } s = 2$$

$$= \frac{1}{2^2}$$

$$= \frac{1}{4}$$

(ii) $$\int_0^\infty e^{-2t}\sin 3t\,dt = \int_0^\infty e^{-st}\sin 3t\,dt, \text{ where } s=2$$

$$= L\{\sin 3t\}, \text{ where } s=2$$

$$= \frac{3}{s^2+3^2}, \text{ where } s=2$$

$$= \frac{3}{2^2+3^2} = \frac{3}{13}$$

(iii) $$\int_0^\infty e^{-4t}\cos 3t\,dt = \int_0^\infty e^{-st}\cos 3t\,dt, \text{ where } s=4$$

$$= L\{\cos 3t\}, \text{ by definition}$$

$$= \left(\frac{s}{s^2+3^2}\right)_{s=4}$$

$$= \frac{4}{4^2+3^2}$$

$$= \frac{4}{16+9}$$

$$= \frac{4}{25}$$

EXAMPLE-2

Evaluate :

(i) $\int_0^\infty te^{-2t}\sin t\,dt$ *[Apr. 2019, 2009]*

(ii) $\int_0^\infty te^{-3t}\cos t\,dt$ *[Apr. 2019]*

by using Laplace transform

Solution :

(i) $$\int_0^\infty te^{-2t}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)dt, \text{ where } s=2$$

$$= L\{t\sin t\}, \text{ by definition } \& \ s=2$$

$$= \frac{-d}{ds}\left[\frac{1}{s^2+1}\right], \text{ where } s = 2$$

$$= -\left[\frac{-1}{(s^2+1)^2}\cdot\frac{d}{ds}(s^2+1)\right], \text{ where } s = 2$$

$$= \frac{2s}{(s^2+1)^2}, \text{ where } s = 2$$

$$= \frac{2(2)}{(2^2+1)^2}$$

$$= \frac{4}{25}$$

(ii) $$\int_0^\infty te^{-3t}\cos t\, dt = \int_0^\infty e^{-st}(t\cos t)dt, \text{ where } s = 3$$

$$= L\{t\cos t\}, \text{ By definition and } s = 3$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right), s = 3$$

$$= -\left[\frac{(s^2+1)\cdot 1 - s(2s)}{(s^2+1)^2}\right], s = 3$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right], s=3$$

$$= \frac{s^2-1}{(s^2+1)^2}, s=3$$

$$= \frac{3^2-1}{(3^2+1)^2}$$

$$= \frac{8}{100}$$

$$= \frac{2}{25}$$

EXAMPLE-3

Evaluate :

(i) $\int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right)dt$ (ii) $\int_0^\infty \left(\frac{\cos at-\cos bt}{t}\right)dt$

(iii) $\int_0^\infty \frac{\sin mt}{t}dt$ (iv) $\int_0^\infty e^{-t}\frac{\sin t}{t}dt$

by using Laplace transform

Solution :

(i) Let $f(t) = e^{at}-e^{-bt}$

$$L\{f(t)\} = L\{e^{-at}-e^{-bt}\}$$

$$= \frac{1}{s+a}-\frac{1}{s+b} = \overline{f}(s)$$

By L.T. of division by t

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)\,ds$$

$$\therefore \quad L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$= [\log(s+a)-\log(s+b)]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty}\log\left(\frac{s+a}{s+b}\right)-\log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{\frac{1}{s}\to\infty}\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)-\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{1+0}{1+0}\right)-\log\left(\frac{s+a}{s+b}\right)$$

$$= \log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

i.e., $$\int_0^{\infty} e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\left(\frac{s+b}{s+a}\right)$$

Taking $s = 0$, we get

$$\int_0^{\infty} e^{o}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\left(\frac{0+b}{0+a}\right)$$

$$\int_0^{\infty} \frac{e^{-at}-e^{-bt}}{t}dt = \log\frac{b}{a}$$

(ii) $$\text{Let } f(t) = \cos at - \cos bt$$

$$L\{f(t)\} = L\{\cos at - \cos bt\}$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = \overline{f}(s)$$

By L.T. of division by 't', we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \overline{f}(s)ds$$

$$\therefore \quad L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\int_s^{\infty}\left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right)ds$$

$$= \left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{\frac{1}{s}\to 0}\log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

i.e., $$\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Put $s = 0$, we get

$$\int_0^\infty\left(\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2}\log\frac{b^2}{a^2} = \frac{1}{2}\log\left(\frac{b}{a}\right)^2$$

$$= \frac{1}{2}\cdot 2\cdot\log\frac{b}{a}$$

$$= \log\frac{b}{a}$$

(iii) Let $f(t) = \sin mt$

$\therefore$ $$L\{f(t)\} = L\{\sin mt\} = \frac{m}{s^2+m^2} = \bar{f}(s)$$

$\therefore$ $$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

$$L\left\{\frac{\sin mt}{t}\right\} = \int_s^\infty \frac{m}{s^2+m^2}\,ds$$

$$= m\cdot\left[\frac{1}{m}\cdot\tan^{-1}\left(\frac{s}{m}\right)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{m}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{m}\right)$$

i.e., $$\int_0^{\infty} e^{-st}\left(\frac{\sin mt}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{m}\right)$$

Put, $s = 0$, we get

$$\int_0^{\infty} e^{0}\left(\frac{\sin mt}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$\therefore \quad \int_0^{\infty} \frac{\sin mt}{t} dt = \frac{\pi}{2}$$

(iv) we know that

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{s^2+1} ds$$

$$= \left[\tan^{-1}(s)\right]_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$\therefore \quad \int_0^{\infty} e^{-st}\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}(s)$$

Put $s = 1$, we get

$$\int_0^{\infty} e^{-t}\frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1}(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

EXERCISE 5.4

Evaluate the following using L.T.

1. (i) $\int_0^\infty te^{-3t}dt$ **[Apr. 2017, 2016]** (ii) $\int_0^\infty t^2e^{-3t}dt$

(iii) $\int_0^\infty t^3e^{-2t}dt$ **[Apr. 2009]**

2. (i) $\int_0^\infty e^{-2t}\cos t\,dt$ **[Apr. 2016]** (ii) $\int_0^\infty e^{-t}\cos 2t\,dt$

(iii) $\int_0^\infty e^{-2t}\cos 3t\,dt$

3. (i) $\int_0^\infty e^{-t}\sin 2t\,dt$ **[Apr. 2017]** (ii) $\int_0^\infty e^{-3t}\sin 2t\,dt$ **[Apr. 2018]**

(iii) $\int_0^\infty e^{-4t}\cos t\,dt$ **[Oct. 2018, 2016, 2008; Apr. 2009,2007]**

4. (i) $\int_0^\infty te^{-t}\cos t\,dt$ (ii) $\int_0^\infty te^{-2t}\cos t\,dt$ **[Apr. 2017]**

(iii) $\int_0^\infty te^{-2t}\cos 3t\,dt$

5. (i) $\int_0^\infty te^{-t}\sin t\,dt$

(ii) $\int_0^\infty te^{-2t}\sin 3t\,dt$ **[Apr. 2019; Oct. 2018, 2016, 2008]**

(iii) $\int_0^\infty te^{-4t}\sin 3t\,dt$ **[Apr 2018]** (iv) $\int_0^\infty te^{-3t}\sin t\,dt$

6. (i) $\int_0^\infty\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$ **[Apr. 2019, 2006]**

(ii) $\int_0^\infty \left(\frac{e^{-t}-e^{-3t}}{t}\right)dt$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-4t}}{t}\right)dt$

(iv) $\int_0^\infty \left(\frac{\cos bt-\cos 4t}{t}\right)dt$

(v) $\int_0^\infty \frac{\sin t}{t}dt$

(vi) $\int_0^\infty \frac{\sin 2t}{t}dt$

7. $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt$

ANSWERS

1. (i) $\frac{1}{9}$ (ii) $\frac{2}{27}$
   (iii) $\frac{3}{8}$
2. (i) $\frac{2}{3}$ (ii) $\frac{1}{5}$
   (iii) $\frac{2}{13}$
3. (i) $\frac{2}{5}$ (ii) $\frac{2}{13}$
   (iii) $\frac{3}{25}$
4. (i) 0 (ii) $\frac{3}{25}$
   (iii) $\frac{-5}{169}$
5. (i) $\frac{1}{2}$ (ii) $\frac{12}{169}$
   (iii) $\frac{24}{625}$ (iv) $\frac{3}{50}$
6. (i) log 2 (ii) log 3
   (iii) log 4 (iv) $\log\left(\frac{2}{3}\right)$
   (v) $\frac{\pi}{2}$ (vi) $\frac{\pi}{2}$
7. $\frac{1}{4}\log 5$

6

Inverse Laplace Transforms

CHAPTER-6

Inverse Laplace Transforms

6.1 INTRODUCTION

In the previous chapter, we have found the Laplace transform of a few functions, let us now determine the inverse transform of a given functions of s.

6.1.1 DEFINTION

If $L\{f(t)\} = \overline{f}(s)$ then f(t) is known as the Inverse Laplace transform of $\overline{f}(s)$ and is denoted by $L^{-1}\{\overline{f}(s)\}$.

i.e., $L\{f(t)\} = \overline{f}(s) \Leftrightarrow f(t) = L^{-1}\{\overline{f}(s)\}$

Here L^{-1} is known as the inverse Laplace transform operator.

For example, $L\{e^{at}\} = \frac{1}{s-a}$, then $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Note : Inverse Laplace transform of $\overline{f}(s)$ need not exists for all $\overline{f}(s)$

Inverse Laplace Transform of Some Standard Functions

S.No.	Laplace Transform $L\{f(t)\} = \overline{f}(s)$	Inverse Laplace Transform $L^{-1}\{\overline{f}(s)\} = f(t)$
1.	$L\{1\} = \frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\} = 1$
2.	$L\{t^{n-1}) = \frac{(n-1)!}{s^n}$	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$, n is positive integer
3.	$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
4.	$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
5.	$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}.\sin at$

6.	$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
7.	$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
8.	$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

6.2 PROPERTY OF INVERSE LAPLACE TRANSFORM

6.2.1 LINEARITY PROPERTY

If $L\{f(t)\} = \bar{f}(s)$ and $L\{g(t)\} = \bar{g}(s)$ then,

$$L^{-1}\{C_1\bar{f}(s)+C_2\bar{g}(s)\} = C_1L^{-1}\{\bar{f}(s)\}+C_2L^{-1}\{\bar{g}(s)\} = C_1f(t) + C_2g(t)$$

where C_1 and C_2 are any two constants.

PROOF :

From the linearity property for Laplace transform, we know that,

$$L\{C_1f(t) + C_2g(t) = C_1L\{f(t)\} + C_2L\{g(t)\}$$
$$= C_1\bar{f}(s)+C_2\bar{g}(s)$$

Taking inverse laplace transfom on both sides, we get

$$C_1f(t) + C_2g(t) = L^{-1}\{C_1\bar{f}(s)+C_2\bar{g}(s)\}$$

i.e.,
$$L^{-1}\{C_1\bar{f}(s)+C_2\bar{g}(s)\} = C_1f(t) + C_2g(t)$$
$$= C_1L^{-1}\{\bar{f}(s)\}+C_2L^{-1}\{\bar{g}(s)\}$$

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L^{-1}\left\{\frac{1}{s^2}\right\}$

(ii) $L^{-1}\left\{\frac{s}{s^2+9}\right\}$ *[Apr. 2018, 2019]*

(iii) $L^{-1}\left\{\frac{1}{s^2+4}\right\}$

(iv) $L^{-1}\left\{\frac{6}{s^2+4}+\frac{1}{s-6}+\frac{1}{s^2}\right\}$ [Apr. 2019, 2018]

Solution :

(i) $L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t^{2-1}}{(2-1)!}$ $\left[\because L^{-1}\left\{\frac{1}{s^n}\right\}=\frac{t^{n-1}}{(n-1)!}\right]$

$= t$

(ii) $L^{-1}\left\{\frac{s}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\}$

$= \cos 3t$ $\left[\because L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at\right]$

(iii) $L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$= \frac{1}{2}.\sin 2t$ $\left[\because L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at\right]$

(iv) $L^{-1}\left\{\frac{6}{s^2+4}+\frac{1}{s-6}+\frac{1}{s^2}\right\} = 6L^{-1}\left\{\frac{1}{s^2+2^2}\right\}+L^{-1}\left\{\frac{1}{s-6}\right\}+L^{-1}\left\{\frac{1}{s^2}\right\}$,

using linearity property

$= 6\frac{1}{2}.\sin 2t + e^{6t} + t$

$\left[\because L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at, L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at} \text{ and } L^{-1}\left\{\frac{1}{s^2}\right\}=t\right]$

$= 3\sin 2t + e^{6t} + t$

EXAMPLE-2

Find :

(i) $L^{-1}\left\{\frac{s^3+2s+2}{s^5}\right\}$ [Apr. 2019]

(ii) $L^{-1}\left\{\frac{2s-5}{s^2+36}\right\}$ [Apr. 2016]

(iii) $L^{-1}\left\{\frac{5s+10}{9s^2+16}\right\}$ [Oct. 2018]

(iv) $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$ [Apr.2017 ; Oct. 2008]

Solution :

(i) $$L^{-1}\left\{\frac{s^3+2s+2}{s^5}\right\} = L^{-1}\left\{\frac{s^3}{s^5}+\frac{2s}{s^5}+\frac{2}{s^5}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}+\frac{2}{s^4}+\frac{2}{s^5}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}\right\}+2L^{-1}\left\{\frac{1}{s^4}\right\}+2L^{-1}\left\{\frac{1}{s^5}\right\}, \text{ using linearity property}$$

$$= \frac{t^{2-1}}{(2-1)!}+2.\frac{t^{4-1}}{(4-1)!}+2.\frac{t^{5-1}}{(5-1)!} \qquad \left[\because L^{-1}\left\{\frac{1}{s^n}\right\}=\frac{t^{n-1}}{(n-1)!}\right]$$

$$= \frac{t^1}{1}+2.\frac{t^3}{3!}+2.\frac{t^4}{4!}$$

$$= t+2.\frac{t^3}{1.2.3}+2.\frac{t^4}{1.2.3.4} \qquad [\because n!=1\times.....\times n]$$

$$= t+\frac{t^3}{3}+\frac{t^4}{12}$$

(ii) $$L^{-1}\left\{\frac{2s-5}{s^2+36}\right\} = L^{-1}\left\{\frac{2s}{s^2+36}-\frac{5}{s^2+36}\right\}$$

$$= 2L^{-1}\left\{\frac{s}{s^2+6^2}\right\}-5L^{-1}\left\{\frac{1}{s^2+6^2}\right\}, \text{ using linearity propert}$$

$$= 2.\cos 6t-\frac{5}{6}\sin 6t$$

(iii) $$L^{-1}\left\{\frac{5s+10}{9s^2+16}\right\} = L^{-1}\left\{\frac{5s}{9\left(s^2+\frac{16}{9}\right)}+\frac{10}{9\left(s^2+\frac{16}{9}\right)}\right\}$$

$$= \frac{5}{9}L^{-1}\left\{\frac{s}{s^2+\left(\frac{4}{3}\right)^2}\right\}+\frac{10}{9}L^{-1}\left\{\frac{1}{s^2+\left(\frac{4}{3}\right)^2}\right\}$$, using linearity property

$$= \frac{5}{9}.\cos\frac{4}{3}t+\frac{10}{9}.\frac{1}{\frac{4}{3}}.\sin\frac{4}{3}t$$

$$= \frac{5}{9}\cos\frac{4t}{3}+\frac{10}{9}\times\frac{3}{4}\sin\frac{4t}{3}$$

$$= \frac{5}{9}\cos\frac{4t}{3}+\frac{5}{6}\sin\frac{4t}{3}$$

(iv) $$L^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\frac{2s}{s^2-4}-\frac{5}{s^2-4}\right\}$$

$$= 2L^{-1}\left\{\frac{S}{s^2-2^2}\right\}-5L^{-1}\left\{\frac{1}{s^2-2^2}\right\}$$, using linearity property

$$= 2.\cosh 2t-5.\frac{1}{2}.\sinh 2t$$

$$= 2\cosh 2t-\frac{5}{2}\sinh 2t$$

EXAMPLE-3

Solve :

Find $L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\}$

Solution :

$$L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\} = L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\} = L^{-1}\left\{\frac{s^2-9+9s}{s^3-9s}\right\}$$

$$= L^{-1}\left\{\frac{s^2-9}{s(s^2-9)}\right\}+L^{-1}\left\{\frac{9s}{s(s^2-9)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\right\}+L^{-1}\left\{\frac{9}{s^2-3^2}\right\}$$

$$= 1+9.\frac{1}{3}.\sinh 3t$$

$$= 1 + 3 \sinh 3t.$$

EXERCISE 6.1

1. Find the inverse Laplace tranform of

(i) $\frac{1}{s^3}$ (ii) $\frac{1}{s-3}$ (iii) $\frac{2}{s+3}$ (iv) $\frac{1}{s^2+25}$ (v) $\frac{s}{s^2+16}$

(vi) $\frac{1}{s^2-9}$ (vii) $\frac{s}{s^2-36}$ (viii) $\frac{1}{2s+5}$

2. (i) $\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2+36}$ **[Apr. 2016]**

(ii) $\frac{1}{s-4}+\frac{1}{s}+\frac{s}{s^2+2^2}$ **[Oct. 2016]**

(iii) $\frac{2}{s^2+4}+\frac{3s}{s^2+9}$

(iv) $\frac{2}{s-4}+\frac{3}{s^2-9}$ **[Apr. 2018, 2016]**

(v) $\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}$

(vi) $\frac{5}{s^2+4}+\frac{2s}{s^2+36}+\frac{s}{s^2-16}$

3. (i) $\frac{s^2+4}{s^3}$

(ii) $\frac{s^2+4s+20}{s^3}$ **[Apr. 2017]**

(iii) $\frac{s^2-s+2}{s^3}$ **[Apr. 2016]**

(iv) $\frac{s^2-3s+4}{s^3}$ **[Apr. 2018, 2017, 2008]**

(v) $\frac{s^2-3s+2}{s^4}$ **[Apr. 2017]**

(vi) $\frac{3(s^2-1)^2}{2s^5}$

4. (i) $\frac{s+1}{s^2+4}$ **[Oct. 2016]**

(ii) $\frac{2s-5}{s^2+4}$ **[Apr. 2016]**

(iii) $\frac{2s+5}{s^2+16}$ (iv) $\frac{2s+1}{s^2+25}$ [Apr. 2016]

(v) $\frac{2s-3}{s^2+36}$ (vi) $\frac{3s-12}{s^2+8}$

(vii) $\frac{3s-5}{s^2+64}$ **[Oct. 2016]** (viii) $\frac{3s-8}{4s^2+25}$ [Apr. 2014]

(ix) $\frac{2s-3}{s^2-4}$ **[Apr. 2017]** (x) $\frac{2s+1}{s^2-9}$ [Apr. 2017]

(xi) $\frac{3s+1}{s^2-9}$ (xii) $\frac{2s+3}{s^2-16}$ [Apr. 2018, 2016]

(xiii) $\frac{2s-5}{9s^2-25}$ [Oct. 2008]

ANSWERS

1. (i) $\frac{t^2}{2}$ (ii) e^{3t} (iii) $2e^{-3t}$ (iv) $\frac{1}{5}\sin 5t$ (v) $\cos 4t$

   (vi) $\frac{1}{3}\sinh 3t$ (vii) $\cosh 3t$ (viii) $\frac{1}{2}e^{-\frac{5t}{2}}$

2. (i) $e^{3t} + 1 + \cos 6t$ (ii) $e^{4t} + 1 + \cos 2t$

   (iii) $\sin 2t + 3\cos 3t$ (iv) $2e^{4t} + \sinh 3t$

   (v) $e^{3t} + 1 + \cosh 2t$ (vi) $\frac{5}{2}\sin 2t + 2\cos 6t + \cosh 4t$

3. (i) $1 + 2t^2$ (ii) $1 + 4t + 10t^2$

   (iii) $1 - t + t^2$ (iv) $1 - 3t + 2t^2$

   (v) $t - \frac{3t^2}{2} + \frac{2t^3}{3}$ (vi) $1 - 2t^2 + \frac{t^4}{6}$

   (vii) $\frac{3}{2}\left(1 - t^2 + \frac{t^4}{24}\right)$

4. (i) $\cos 2t + \frac{1}{2}\sin 2t$ (ii) $2\cos 2t - \frac{5}{2}\sin 2t$

(iii) $\cos 4t + \frac{5}{4}\sin 4t$ (iv) $2\cos 5t + \frac{1}{5}\sin 5t$

(v) $2\cos 6t - \frac{1}{2}\sin 6t$ (vi) $3\cos 2\sqrt{2}t - 3\sqrt{2}\sin 2\sqrt{2}t$

(vii) $3\cos 8t - \frac{5}{8}\sin 8t$ (viii) $\frac{3}{4}\cos\frac{5t}{2} - \frac{4}{5}\sin\frac{5t}{2}$

(ix) $2\cosh 2t - \frac{3}{2}\sinh 2t$ (x) $2\cosh 3t + \frac{1}{3}\sinh 3t$

(xi) $2\cosh 3t + \frac{7}{3}\sinh 3t$ (xii) $2\cosh 4t + \frac{3}{4}\sinh 4t$

(xiii) $\frac{2}{9}\cosh\frac{5t}{3} - \frac{1}{3}\sinh\frac{5t}{3}$

6.2.2 *FIRST SHIFTING THEOREM (OR) FIRST TRANSLATION THEOREM*

THEOREM :

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(s-a)\} = e^{at}L^{-1}\{\bar{f}(s)\} = e^{at}f(t)$

PROOF :

From the first shifting theorem on Laplace transform, if $L\{(t)\} = \bar{f}(s)$ then

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

$$\therefore L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t) = e^{at}L^{-1}\{\bar{f}(s)\}$$

NOTE-1 :

1. If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(s+a)\} = e^{-at}L^{-1}\{\bar{f}(s)\} = e^{-at}f(t)$

NOTE-2 :

(i) $L^{-1}\{\bar{f}(s)\} = e^{-at}L^{-1}\{\bar{f}(s-a)\}$

(ii) $L^{-1}\{\bar{f}(s)\} = e^{at}L^{-1}\{\bar{f}(s+a)\}$

NOTE-3 :

By Applying the above theorem, we have

(i) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}L^{-1}\left\{\frac{1}{s^n}\right\} = e^{at}\frac{t^{n-1}}{(n-1)!}$

(ii) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}L^{-1}\left\{\frac{1}{s^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

(iii) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = e^{at}\cos bt$

(iv) $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = e^{at}L^{-1}\left\{\frac{1}{s^2-b^2}\right\} = \frac{1}{b}e^{at}\sinh bt$

(v) $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}L^{-1}\left\{\frac{s}{s^2-b^2}\right\} = e^{at}\cosh bt$

(vi) $L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^n}\right\} = e^{-at}\frac{t^{n-1}}{(n-1)!}$

(vii) $L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^2+b^2}\right\} = \frac{1}{b}e^{-at}\sin bt$

(viii) $L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = e^{-at}\cos bt$

(ix) $L^{-1}\left\{\frac{1}{(s+a)^2-b^2}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^2-b^2}\right\} = \frac{1}{b}e^{-at}\sinh bt$

(x) $L^{-1}\left\{\frac{(s+a)}{(s+a)^2-b^2}\right\} = e^{-at}L^{-1}\left\{\frac{s}{s^2-b^2}\right\} = e^{-at}\cosh bt$

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L^{-1}\left\{\frac{1}{(s+a)^n}\right\}$ (ii) $L^{-1}\left\{\frac{2}{(s-1)^3}\right\}$

(iii) $L^{-1}\left\{\frac{s}{(s-1)^3}\right\}$ *[Oct. 2016]* (iv) $L^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\}$ *[Oct. 2006]*

Solution :

(i) $$L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = e^{-at}L^{-1}\left\{\frac{1}{s^n}\right\} \text{ by first shifting theorem}$$

$$= e^{-at}.\frac{t^{n-1}}{(n-1)!}$$

(ii) $$L^{-1}\left\{\frac{2}{(s-1)^3}\right\} = 2.L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2e^tL^{-1}\left\{\frac{1}{s^3}\right\}, \text{ by first shifting theorem}$$

$$= 2e^t.\frac{t^2}{2!} \qquad \left[\because L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}\right]$$

$$= 2e^t.\frac{t^2}{2}$$

$$= t^2e^t$$

(iii) $$L^{-1}\left\{\frac{s}{(s-2)^3}\right\} = L^{-1}\left\{\frac{(s-2)+2}{(s-2)^3}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{s+2}{s^3}\right\}, \text{ using first shifting theorem}$$

$$= e^{2t}L^{-1}\left\{\frac{s}{s^3}+\frac{2}{s^3}\right\}$$

$$= e^{2t}\left[L^{-1}\left\{\frac{1}{s^2}\right\}+2L^{-1}\left\{\frac{1}{s^3}\right\}\right], \text{ using linearity property}$$

$$= e^{2t}\left[t + 2.\frac{t^2}{2}\right]$$

$$= e^{2t}(t + t^2)$$

(iv) $$L^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\} = L^{-1}\left\{\frac{3s+3-2}{(s+1)^4}\right\}$$

$$= L^{-1}\left\{\frac{3(s+1)-2}{(s+1)^4}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{3s-2}{s^4}\right\} \qquad \left[\because L^{-1}\{\bar{f}(s+a)\} = e^{-at}L^{-1}\{\bar{f}(s)\}\right]$$

$$= e^{-t}L^{-1}\left\{\frac{3s}{s^4} - \frac{2}{s^4}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{3}{s^3} - \frac{2}{s^4}\right\}$$

$$= e^{-t}\left[3L^{-1}\left\{\frac{1}{s^3}\right\} - 2L^{-1}\left\{\frac{1}{s^4}\right\}\right]$$

$$= e^{-t}\left[3.\frac{t^2}{2!} - 2.\frac{t^3}{3!}\right]$$

$$= e^{-t}\left[\frac{3t^2}{1.2} - \frac{2.t^3}{1.2.3}\right]$$

$$= e^{-t}\left[\frac{3t^2}{2} - \frac{t^3}{3}\right]$$

EXAMPLE-2

Find :

(i) $L^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\}$ [Apr. 2019, 2017]

(ii) $L^{-1}\left\{\dfrac{s}{(s+2)^2-4}\right\}$ [Apr. 2016]

Solution :

(i) $$L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2+16}\right\} \qquad \left[\because L^{-1}\{\bar{f}(s+a)\} = e^{-at}L^{-1}\{\bar{f}(s)\}\right]$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= e^{-2t}.\frac{1}{4}.\sin 4t$$

$$= \frac{1}{4}e^{-2t}\sin 4t$$

(ii) $$L^{-1}\left\{\frac{s}{(s+2)^2-4}\right\} = L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2-4}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s-2}{s^2-2^2}\right\} \qquad [\because \text{ By first shifting theorem}]$$

$$= e^{-2t}\left[L^{-1}\left\{\frac{s}{s^2-2^2}\right\} - L^{-1}\left\{\frac{2}{s^2-2^2}\right\}\right]$$

$$= e^{-2t}[\cosh 2t - \sinh 2t]$$

EXAMPLE-3

Solve :

(i) $L^{-1}\left\{\dfrac{1}{s^2+6s+5}\right\}$ *[Apr. 2017]*

(ii) $L^{-1}\left\{\dfrac{s+2}{s^2+s+7}\right\}$

(iii) $L^{-1}\left\{\dfrac{3s-14}{s^2-4s+8}\right\}$ *[Apr. 2018, 2008 ; Oct. 2016]*

Solution :

(i) $$L^{-1}\left\{\frac{1}{s^2+6s+5}\right\} = L^{-1}\left\{\frac{1}{s^2+2.s.3+3^2+5-3^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+3)^2-4}\right\} \qquad \left[\because (a+b)^2 = a^2+2ab+b^2\right]$$

$$= e^{-3t} L^{-1}\left\{\frac{1}{s^2-2^2}\right\} \qquad \left[\because L^{-1}\{\bar{f}(s+a) = e^{-at} L^{-1}\{\bar{f}(s)\}\right]$$

$$= e^{-3t} . \frac{1}{2} . \sinh 2t$$

$$= \frac{1}{2} e^{-3t} \sinh 2t$$

(ii) $$L^{-1}\left\{\frac{s+2}{s^2+4s+7}\right\} = L^{-1}\left\{\frac{s+2}{s^2+2.s.2+2^2+7-2^2}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+3}\right\}$$

$$= e^{-2t} . L^{-1}\left\{\frac{s}{s^2+\left(\sqrt{3}\right)^2}\right\}, \text{ using first shifting theorem}$$

$$= e^{-2t} . \cos\sqrt{3}\, t$$

(iii) $$L^{-1}\left\{\frac{3s-14}{s^2-4s+8}\right\} = L^{-1}\left\{\frac{3s-6-8}{s^2-2.s.2+2^2+8-2^2}\right\}$$

$$= L^{-1}\left\{\frac{3(s-2)-8}{(s-2)^2+4}\right\}$$

$$= e^{2t} L^{-1}\left\{\frac{3s-8}{s^2+4}\right\}, \text{ using first shifting theorem}$$

$$= e^{2t} . \left[3L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 8.L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right]$$

[∵ By linearity proeperty]

$$= e^{2t} . \left[3\cos 2t - 8.\frac{1}{2}\sin 2t\right]$$

$$= e^{2t}[3\cos 2t - 4\sin 2t]$$

6.2.3 SECOND SHIFTING THEOREM

If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\{e^{-as}\overline{f}(s)\} = g(t)$, where $g(t) = \begin{cases} 0 & , \text{if } t<a \\ f(t-a), & \text{if } t>a \end{cases}$

PROOF :

By definition, we have

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

Put $t-a = x$, then $dt = dx$

Limits : Lower limit, when $t = a \Rightarrow x = 0$ and upper limit when $t = \infty \Rightarrow x = \infty$

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-s(x+a)} f(x)dx$$

$$= \int_0^\infty e^{-sx}.e^{-as} f(x)\,dx$$

$$= e^{-as}\int_0^\infty e^{-sx} f(x)\,dx$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)dt$$

$$= e^{as} L\{f(t)\}$$

$$= e^{-as}\overline{f}(s)$$

$$\therefore \quad g(t) = L^{-1}\left\{e^{-as}\overline{f}(s)\right\}$$

i.e., $$L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = g(t)$$

$$\therefore \quad L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} 0 & ,\text{ if } t>a \\ f(t-a), & \text{if } t>a \end{cases}$$

Note : Second shifting theorem also expressed as,

$$L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)u(t-a) \text{ where } u(t-a) = \begin{cases} 0, & \text{if } t<a \\ 1, & \text{if } t>a \end{cases}$$

SOLVED EXAMPLES

EXAMPLE-1

Solve : (i) $L^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$ (ii) $\dfrac{e^{-3s}}{(s+4)^2}$

Solution :

(i) Let $\bar{f}(s) = \dfrac{1}{s^2}$, then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= t = f(t) \text{ say}$$

$\therefore$ By second shifting theorem, we have

$$L^{-1}\left\{e^{-as}\cdot\bar{f}(s)\right\} = f(t-2)\ u(t-2)$$

$$\therefore\ L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = (t-2)\ u(t-2)$$

$$\text{(or) } L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \begin{cases} 0, & \text{if } t<2 \\ t-2, & \text{if } t>2 \end{cases}$$

(ii) Let $\bar{f}(s) = \dfrac{1}{(s+4)^2}$ then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$

$$= e^{-4t}L^{-1}\left\{\frac{1}{s^2}\right\}, \qquad [\because \text{By first shifting theorem}]$$

$$= e^{-4t}.t$$

$$= te^{-4t} = f(t)$$

$\therefore$ By second shifting theorem,

$$L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-3s}}{(s+4)^2}\right\} = \begin{cases} 0 & \text{, if } t<3 \\ (t-3)e^{-4(t-3)}, & \text{if} >3 \end{cases}$$

EXAMPLE-2

Solve :

(i) $L^{-1}\left\{\dfrac{e^{-as}}{s^2-\omega^2}\right\}$

(ii) $L^{-1}\left\{\dfrac{se^{\frac{-2\pi}{3}s}}{s^2+9}\right\}$

Solution :

(i) Let $\bar{f}(s) = \dfrac{1}{s^2-\omega^2}$, then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2-\omega^2}\right\}$$

$$= \frac{1}{\omega}\sinh \omega t = f(t)$$

$\therefore$ By second shifting theorem,

$$L^{-1}\{e^{-at}\bar{f}(s)\} = \begin{cases} 0 & \text{, if } t<a \\ f(t-a), & \text{if } t>a \end{cases}$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-as}}{s^2-\omega^2}\right\} = \begin{cases} 0 & \text{, if } t<a \\ \dfrac{1}{\omega}\sinh \omega(t-a), & \text{if } t>a \end{cases}$$

(ii) Let $\bar{f}(s) = \frac{s}{s^2+9}$, then

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

$$= \cos 3t = f(t)$$

$\therefore$ By second shifting theorem, we have

$$L^{-1}\{e^{-as}\bar{f}(s)\} = \begin{cases} 0 & , \text{if } t<a \\ f(t-a), & \text{if } t>a \end{cases}$$

$$\therefore \quad L^{-1}\left\{\frac{se^{\frac{-2\pi}{3}s}}{s^2+9}\right\} = \begin{cases} 0 & , \text{if } t<\frac{2\pi}{3} \\ \cos 3\left(t-\frac{2\pi}{3}\right), & \text{if } t>\frac{2\pi}{3} \end{cases}$$

$$= \begin{cases} 0 & , \text{if } t<\frac{2\pi}{3} \\ \cos 3t, & \text{if } t>\frac{2\pi}{3} \end{cases}$$

6.2.4 CHANGE OF SCALE PROPERTY

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

PROOF :

By definition,

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\bar{f}(as) = \int_0^\infty e^{-ast} f(t)\,dt$$

Put $at = u$, then $t = \frac{u}{a}$ and $dt = \frac{du}{a}$ and when $t = 0 \Rightarrow u = 0$ and $t = \infty \Rightarrow u = \infty$

$$\therefore \quad \bar{f}(as) = \int_0^\infty e^{-su} f\left(\frac{u}{a}\right)\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-st} f\left(\frac{t}{a}\right)dt$$

$$= \frac{1}{a}L\left\{f\left(\frac{t}{a}\right)\right\} = L\left\{\frac{1}{a}f\left(\frac{t}{a}\right)\right\}$$

$$\therefore \quad L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right)$$

Note : If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\bar{f}\left(\frac{s}{a}\right)\right\} = a.f(at)$

SOLVED EXAMPLES

EXAMPLE-1

Solve :

If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t}{2}\sin t$, *show that* $L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{t}{4}\sin\left(\frac{t}{4}\right)$.

Solution :

Given that,

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t}{2} = f(t) \text{ (say)}$$

By changing of scale property

$$L^{-1}\{\bar{f}(as)\} = \frac{1}{a}.f\left(\frac{t}{a}\right)$$

$$\therefore \quad L^{-1}\{\bar{f}(4s)\} = \frac{1}{4}f\left(\frac{t}{4}\right)$$

$$\text{i.e., } L^{-1}\left\{\frac{4s}{[(4s)^2+1]^2}\right\} = \frac{1}{4}.\frac{\frac{t}{4}}{2}\sin\frac{t}{4}$$

$$\Rightarrow L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = \frac{t}{32}\sin\frac{t}{4}$$

Multiplying with '8' on both sides, we get

$$8.L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = 8.\frac{t}{32}\sin\frac{t}{4}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}$$

EXAMPLE-2

Solve :

If $L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, *then find* $L^{-1}\left\{\frac{4s^2-1}{(4s^2+1)^2}\right\}$.

Solution :

Given that,

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = t\cos t = f(t) \text{ (say)}$$

By change of scale property,

$$L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right)$$

$$\therefore \quad L^{-1}\{\bar{f}(2s)\} = \frac{1}{2}f\left(\frac{t}{2}\right)$$

$$\text{i.e.,} \quad L^{-1}\left\{\frac{(2s)^2-1}{(2s)^2+1)^2}\right\} = \frac{1}{2}.\frac{t}{2}\cos\frac{t}{2}$$

$$\therefore \quad L^{-1}\left\{\frac{4s^2-1}{(4s^2+1)^2}\right\} = \frac{t}{4}\cos\frac{t}{2}$$

EXERCISE 6.2

1. Find the inverse Laplace transform of

I. First Shifting Theorem :

(i) $\frac{1}{(s-a)^n}$ (ii) $\frac{1}{(s+a)^3}$ (iii) $\frac{1}{(s+1)^3}$

(iv) $\frac{3}{(s+1)^4}$ (v) $\frac{1}{(s-2)^3}$ (vi) $\frac{7}{(2s+1)^3}$

2. (i) $\frac{s}{(s-2)^2}$ [Apr. 2018] (ii) $\frac{s}{(s-3)^5}$ [Apr. 2017]

(iii) $\frac{1}{(s+2)^2}$ [Apr. 2017] (iv) $\frac{s}{(s+3)^2}$ (v) $\frac{s}{(s-7)^4}$

(vi) $\frac{s}{(s+1)^5}$ (vii) $\frac{4s+5}{(s+1)^4}$ [Apr. 2016]

(viii) $\frac{2s+3}{(s-1)^3}$ [Apr. 2017] (ix) $\frac{s^2}{(s-2)^3}$ [Apr. 2016, 2009]

(x) $\frac{s^2}{(s-1)^3}$

3. (i) $\frac{2}{(s+1)^2+4}$ (ii) $\frac{1}{(s-3)^5}+\frac{2}{(s+1)^2+4}$ [Apr. 2008]

(iii) $\frac{1}{(s-2)^2+4}$ (iv) $\frac{s}{(s+2)^2+4}$ [Apr. 2019 ; Oct. 2016]

(v) $\frac{s}{(s+2)^2+25}$ [Oct. 2016] (vi) $\frac{s}{(s+3)^2+4}$ (vii) $\frac{s}{(s-3)^2+1}$

(viii) $\frac{s-4}{4(s-3)^2+16}$ [Apr. 2008]

4. (i) $\frac{1}{s^2+4s+20}$ (ii) $\frac{s}{s^2+4s+13}$

(iii) $\frac{s+2}{s^2+4s+8}$ [Oct. 2018]

(iv) $\frac{3s-14}{s^2+4s+8}$ [Apr. 2018]

(v) $\frac{3s+13}{s^2+4s+3}$ [Oct. 2016]

(vi) $\dfrac{s+2}{s^2-4s+20}$ **[Oct. 2016]** (vii) $\dfrac{3s-2}{s^2-4s+20}$

(viii) $\dfrac{20-4s}{s^2-4s+20}$ **[Oct. 2016]** (ix) $\dfrac{s+1}{s^2+6s+25}$

(x) $\dfrac{s+1}{s^2+6s-7}$ (xi) $\dfrac{s-1}{s^2+6s+10}$ **[Apr. 2016]**

(xii) $\dfrac{8s+20}{s^2-12s+32}$ (xiii) $\dfrac{3s-2}{s^2-2s+5}$ **[Apr. 2019, 2018, 2009]**

(xiv) $\dfrac{s+2}{s^2-2s-8}$ (xv) $\dfrac{3s+7}{s^2-2s-3}$

(xvi) $\dfrac{2s+3}{s^2+2s+2}$

II. Second Shifting Theorem :

5. (i) $\dfrac{e^{-3s}}{s^3}$ **[Apr. 2018]** (ii) $\dfrac{e^{-2s}}{s-5}$

(iii) $\dfrac{e^{-s}}{(s-1)^3}$

6. (i) $\dfrac{e^{-s}}{s^2+w^2}$ (ii) $\dfrac{e^{-\pi s}}{s^2+1}$

(iii) $\dfrac{se^{-\pi s}}{s^2+4}$ (iv) $\dfrac{se^{-as}}{s^2-9}$

7. (i) $e^{-2s}\left(\dfrac{3+5s}{s^2}\right)$

III. Change of Scale Property :

8. If $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t}{2}\sin t$, show that $L^{-1}\left\{\dfrac{8s}{(4s^2+1)^2}\right\} = \dfrac{t}{2}\sin\dfrac{t}{2}$.

9. If $L^{-1}\left\{\dfrac{s^2-1}{(s^2+1)^2}\right\} = t\cos t$, show that $L^{-1}\left\{\dfrac{9s^2-1}{(9s^2+1)^2}\right\} = \dfrac{t}{9}\cos\dfrac{t}{3}$

ANSWERS

1. First Shifting Theorem

1. (i) $e^{at}\frac{t^{n-1}}{(n-1)!}$ (ii) $\frac{t^2}{2}.e^{-at}$ (iii) $\frac{t^2}{2}e^{-t}$

(iv) $\frac{t^3}{2}e^{-t}$ (v) $\frac{t^2}{2}e^{2t}$ (vi) $\frac{7}{16}t^2e^{\frac{-t}{2}}$

2. (i) $e^{2t}(1+2t)$ (ii) $e^{3t}\left(\frac{t^3}{6}-\frac{t^4}{24}\right)$ (iii) $e^{-2t}(1-2t)$

(iv) $e^{-3t}(1-3t)$ (v) $e^{7t}\left(\frac{t^2}{2}+\frac{7}{6}t^3\right)$ (vi) $e^{-t}\left(\frac{t^3}{6}-\frac{t^4}{24}\right)$

(vii) $e^{-t}\left(2t^2+\frac{t^3}{6}\right)$ (viii) $e^{t}\left(2t+\frac{5t^2}{2}\right)$ (ix) $e^{2t}(1+4t+2t^2)$

(x) $e^{t}\left(1+2t+\frac{t^2}{2}\right)$

3. (i) $e^{-t}\sin 2t$ (ii) $\frac{t^4}{24}e^{3t}+e^{-t}\sin 2t$ (iii) $\frac{e^{2t}}{2}\sin 2t$

(iv) $e^{-2t}(\cos 2t-\sin 2t)$ (v) $e^{-2t}\left(\cos 5t-\frac{5}{2}\sin 5t\right)$ (vi) $e^{-3t}\left(\cos 2t-\frac{3}{2}\sin 2t\right)$

(vii) $e^{3t}(\cos t+3\sin t)$ (viii) $\frac{e^{3t}}{8}(2\cos 2t-\sin 2t)$

4. (i) $\frac{e^{-2t}}{4}\sin 4t$ (ii) $e^{-2t}\left(\cos 3t-\frac{2}{3}\sin 3t\right)$

(iii) $e^{-2t}\cos 2t$ (iv) $e^{-2t}(3\cos 2t-10\sin 2t)$

(v) $e^{-2t}(3\cosh t+7\sinh t)$ (vi) $e^{2t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)$

(vii) $e^{2t}\left(\cos 3t+\frac{4}{3}\sin 3t\right)$ (viii) $e^{2t}(3\sin 4t-4\cos 4t)$

(ix) $e^{-3t}\left(\cos 4t - \frac{1}{2}\sin 4t\right)$ (x) $e^{-3t}\left(\cosh 4t - \frac{1}{2}\sinh 4t\right)$

(xi) $e^{-3t}(\cos t - 4\sin t)$ (xii) $e^{6t}(8\cosh 2t + 34\sinh 2t)$

(xiii) $e^{t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right)$ (xiv) $e^{t}(\cosh 3t + \sinh 3t)$

(xv) $e^{t}(3\cosh 2t + 5\sinh 2t)$ (xvi) $e^{-t}(2\cos t + \sin t)$

5. (i) $\frac{(t-3)^2}{2}u(t-3)$ (ii) $e^{5(t-2)}u(t-2)$

(iii) $\frac{(t-1)^2}{2}e^{(t-1)}u(t-1)$

6. (i) $\frac{1}{w}\sin w(t-1)u(t-1)$ (ii) $\sin(t-\pi)$

(iii) $\cos 2u(t-\pi)$ (iv) $\cosh 3(t-a)u(t-a)$

7. $(3t-1)u(t-2)$

6.3 INVERSE LAPLACE TRANFORM OF MULTIPLICATION BY s^n

THEOREM :

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$, then

$$L^{-1}\{s\bar{f}(s)\} = \frac{d}{dt}[f(t)]$$

PROOF :

From the Laplace transform of derivatives, we have

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$= s\bar{f}(s) \qquad [\because f(0) = 0]$$

$$\therefore \quad L^{-1}\{s\bar{f}(s)\} = f'(t) = \frac{d}{dt}[f(t)]$$

Note : In general, $L^{-1}\{s^n\bar{f}(s)\} = f^n(t) = \frac{d^n}{dt^n}[f(t)]$, provided $f(0) = f(0) = f'(0) = f^{n-1}(0) = 0$

SOLVED EXAMPLES

EXAMPLE-1

Find

(i) $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$

(ii) $L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}$

Solution :

(i) $\text{Let } \bar{f}(s) = \frac{1}{(s+2)^2}$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} = te^{-2t} = f(t)$$

Here $f(0) = 0.e^{-2(0)} = 0$

$$\therefore \quad L^{-1}\{s.\bar{f}(s)\} = \frac{d}{dt}[f(t)]$$

$$L^{-1}\left\{\frac{s}{(s+2)}\right\} = \frac{d}{dt}(te^{-2t})$$

$$= 1.e^{-2t} + t.e^{-2t}(-2)$$

$$= e^{-2t} - 2te^{-2t}$$

$$= e^{-2t}(1-2t)$$

Alternative method :

$$L^{-1}\left\{\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{\frac{s+2-2}{(s+2^2}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{s-2}{s^2}\right\}\text{, using first shifting theorem}$$

$$= e^{-2t}L^{-1}\left\{\frac{s}{s^2} - \frac{2}{s^2}\right\}$$

$$= e^{-2t}\left[L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= e^{-2t}(1-2t)$$

(ii) Let $\bar{f}(s) = \dfrac{1}{(s+3)^2+4}$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{(s+3)^2+4}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{1}{s^2+4}\right\},$$ [∵ By shifting theorem]

$$= e^{-3t},\frac{1}{2}\sin 2t$$

$$= \frac{e^{-3t}}{2}\sin 2t = f(t)$$

and $f(0) = \dfrac{e^{-3(0)}}{2}\sin 2(0) = 0$

$\therefore$ $L^{-1}\{s\bar{f}(s)\} = f'(t)$

$$\because \quad L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\} = \frac{d}{dt}\left(\frac{e^{-3t}}{2}\sin 2t\right)$$

$$= \frac{1}{2}\left[e^{-3t}(-3)\sin 2t + e^{-3t}\cos 2t(2)\right],$$ (by product rule)

$$= \frac{1}{2}e^{-3t}[-3\sin 2t + 2\cos 2t]$$

Alternative Method :

$$L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\} = L^{-1}\left\{\frac{s+3-3}{(s+3)^2+4}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s-3}{s^2+4}\right\}$$ [∵ By first shifting theorem]

$$= e^{-3t} L^{-1}\left\{\frac{s}{s^2+2^2} - \frac{3}{s^2+2^2}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 3L^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right]$$

$$= e^{-3t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

$$= \frac{e^{-3t}}{2}[2\cos t - 3\sin 2t]$$

6.4 INVERSE LAPLACE TRANSFORM OF DIVISION BY 'S'

THEOREM :

If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$

PROOF :

By Laplace transform of integrals, we have

$$L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\overline{f}(s)$$

$$\therefore \quad L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$$

Note : In general $L^{-1}\left\{\frac{\overline{f}(s)}{s^n}\right\} = \int_0^t\int_0^t \ldots \int_0^t f(u)\,dt\,dt \ldots\ldots dt$

(n times)

SOLVED EXAMPLES

EXAMPLE-1

Solve :

(i) $L^{-1}\left\{\frac{1}{s^2-4s}\right\}$ *[Apr. 2018]*

(ii) $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

Solution :

(i) $$L^{-1}\left\{\frac{1}{s^2-4s}\right\} = L^{-1}\left\{\frac{1}{s(s-4)}\right\}$$

$$\text{Let } \bar{f}(s) = \frac{1}{s-4}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t} = f(t)$$

$$\therefore \quad L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$= \int_0^t e^{4t}dt$$

$$= \left(\frac{e^{4t}}{4}\right)_0^t$$

$$= \frac{1}{4}(e^{4t} - e^{\circ})$$

$$= \frac{1}{4}(e^{4t} - 1)$$

(ii) $$\text{Let } \bar{f}(s) = \frac{1}{s^2+9}, \text{ then}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin 3t = f(t)$$

By inverse L.T of Division by 's' we have

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$= \int_0^t \frac{1}{3}\sin 3t\,dt$$

$$= \frac{1}{3}\cdot\left[\frac{-\cos 3t}{3}\right]_0^t$$

$$= \frac{1}{9}[-\cos 3t + \cos 0]$$

$$= \frac{1}{9}[-\cos 3t + \cos 0]$$

$$= \frac{1}{9}[1 - \cos 3t]$$

Above problems can also determine be found by using partial fractions or convolution theorem.

EXAMPLE-2

Solve :

Find $L^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$

Solution :

Let $\bar{f}(s) = \dfrac{1}{s^2+1}$, then $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t = f(t)$

By I.L.T of division by 's' we have

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

$$= \int_0^t \sin t\,dt$$

$$= (-\cos t)_0^t$$

$$= -\cos t + \cos 0$$

$$= 1 - \cos t$$

Again by I.L.T of division by s, we have

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos t)\,dt$$

$$= [t - \sin t]_0^t$$

$$= (t - \sin t) - (0 - \sin 0)$$

$$= t - \sin t$$

Alternative Method : Using partial fractions

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{s^2+1-s^2}{s^2(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{s^2+1}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= t - \sin t$$

6.5 INVERSE LAPLACE TRANSFORM OF DERIVATIVES

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{\bar{f}^n(s)\} = (-1)^n t^n f(t)$

where $\bar{f}^n(s) = \frac{d^n}{ds^n}[\bar{f}(s)]$, $n = 1, 2, 3, \ldots\ldots$

Proof : From the Laplace transform of multiplication by t^n, we have

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$$

i.e., $$L\{t^n f(t)\} = (-1)^n . \bar{f}^n(s)$$

$\therefore$ $$L^{-1}\{(-1)^n f^n(s)\} = t^n f(t)$$

$$(-1)^n L^{-1}\{\bar{f}^n(s)\} = t^n f(t)$$

$\Rightarrow$ $$(-1)^{2n} L^{-1}\{\bar{f}^n(s)\} = (-1)^n t^n f(t)$$

$\therefore$ $$L^{-1}\{\bar{f}^n(s)\} = (-1)^n . t^n f(t)$$

Note : If $n = 1$, then $L^{-1}\{f'(s)\} = -t\, f(t)$

i.e., $$L^{-1}\left\{\frac{d}{ds}[f(s)]\right\} = -t . L^{-1}\{\bar{f}(s)\}$$

$\therefore$ $$L^{-1}\{\bar{f}(s)\} = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\}$$

Note : To find inverse Laplace transform of logarithmic functions and inverse trigonometric functions, use inverse Laplace transform of derivatives method.

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

(ii) $L^{-1}\left\{\log\left(\frac{s+3}{s-4}\right)\right\}$

(iii) $L^{-1}\left\{\log\left(\frac{s^2+w^2}{s^2}\right)\right\}$ (or) $L^{-1}\left\{\log\left(1+\frac{w^2}{s^2}\right)\right\}$

Solution :

(i) Let $\bar{f}(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a)-\log(s+b)$ $\left[\because \log\left(\frac{a}{b}\right)=\log a-\log b\right]$

Differentiating with respect to 's', we get

$$\bar{f}'(s) = \frac{d}{ds}[\log(s+a)]-\frac{d}{ds}[\log(s+b)]$$

$$= \frac{1}{s+a}-\frac{1}{s+b}$$

$$\therefore\ L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{1}{s+a}-\frac{1}{s+b}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+a}\right\}-L^{-1}\left\{\frac{1}{s+b}\right\}$$

$$= e^{-at}-e^{-bt} \qquad \text{.............} \quad (1)$$

We know that,

$$L^{-1}\{\bar{f}'(s)\} = (-1)tf(t)$$

$$= -t.L^{-1}\{\bar{f}(s)\}$$

$$L^{-1}\{\bar{f}(s)\} = \frac{-1}{t}.L^{-1}\{f'(s)\}$$

$$= \frac{-1}{t}[e^{-at}-e^{-bt}] \qquad [\therefore \text{ From } (1)]$$

$$\therefore\ L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = \frac{1}{t}[e^{-bt}-e^{-at}]$$

(ii) Let $\bar{f}(s) = \log\left(\frac{s+3}{s-4}\right) = \log(s+3) - \log(s-4)$ $\left[\because \log\frac{a}{b} = \log a - \log b\right]$

Differentiating with respect to 's' we get

$$f'(s) = \frac{d}{ds}[\log(s+3) - \log(s-4)]$$

$$= \frac{1}{s+3} - \frac{1}{s-4}$$

$$\therefore L^{-1}\{f'(s)\} = L^{-1}\left\{\frac{1}{s+3} - \frac{1}{s-4}\right\}$$

$$= e^{-3t} - e^{4t}$$

We know that

$$L^{-1}\{\bar{f}(s)\} = \frac{-1}{t}.L^{-1}\{f'(s)\}$$

$$= \frac{-1}{t}[e^{-3t} - e^{4t}]$$

$$= \frac{1}{t}(e^{4t} - e^{-3t})$$

(iii) Let

$$\bar{f}(s) = \log\left(\frac{s^2+\omega^2}{s^2}\right)$$

$$= \log(s^2+\omega^2) - \log s^2 \qquad \left[\because \log\frac{a}{b} = \log a - \log b\right]$$

$$= \log(s^2+\omega^2) - 2\log s \qquad \left[\because \log a^m = m\log a\right]$$

$$\therefore \bar{f}'(s) = \frac{d}{ds}[\log(s^2+\omega^2)] - 2\frac{d}{ds}[\log s]$$

$$= \frac{1}{s^2+\omega^2}\frac{d}{ds}(s^2+\omega^2) - 2\frac{1}{s} \qquad \left[\because \frac{d}{dx}\log[f(x)] = \frac{1}{f(x)}.f'(x)\right]$$

$$= \frac{1}{s^2+\omega^2}(2s) - \frac{2}{s}$$

$$= \frac{2s}{s^2+\omega^2} - \frac{2}{s}$$

$$L^{-1}\{\bar{f}'(s)\} = L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\}$$

$$= 2\cos\omega t - 2(1)$$

$$= 2(\cos\omega t - 1)$$

We know that

$$L^{-1}\{\bar{f}'(s)\} = t.L^{-1}\{\bar{f}(s)\}$$

$$\Rightarrow \quad L^{-1}\{\bar{f}'(s)\} = \frac{-1}{t}L^{-1}\{\bar{f}'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s^2+\omega^2}{s^2}\right)\right\} = \frac{-1}{t}.[2(\cos\omega t - 1)]$$

$$= \frac{2}{t}(1-\cos\omega t)$$

EXAMPLE-2

Find :

(i) $L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$ *(or)* $L^{-1}\{\cot^{-1}(s)\}$ ***[Apr. 2016]***

(ii) $L^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\}$

Solution :

(i) Let $\bar{f}(s) = \tan^{-1}\left(\frac{1}{s}\right) = \cot^{-1}(s)$, then

$$f'(s) = \frac{d}{ds}(\cot^{-1}(s))$$

$$= \frac{-1}{1+s^2}$$

We know that,

$$L^{-1}\{\bar{f}(s)\} = \frac{-1}{t}L^{-1}\{\bar{f}'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = \frac{-1}{t}L^{-1}\left\{\frac{-1}{1+s^2}\right\}$$

$$= \frac{1}{t}.L^{-1}\left\{\frac{1}{1+s^2}\right\}$$

$$= \frac{1}{t}.\sin t$$

$$\therefore \quad L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = L^{-1}\{\cot^{-1}(s)\} = \frac{\sin t}{t}$$

(ii) Let $\overline{f}(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$, then

$$\overline{f}'(s) = \frac{d}{ds}\left(\cot^{-1}\left(\frac{s+a}{b}\right)\right)$$

$$= \frac{-1}{1+\left(\frac{s+a}{b}\right)^2}.\frac{d}{ds}\left(\frac{s+a}{b}\right) \qquad \left[\because \frac{d}{dx}\cot^{-1}[f(x)] = \frac{-1}{1+f(x)^2}.f'(x)\right]$$

$$= \frac{-b}{b^2+(s+a)^2}.\frac{1}{b}$$

$$= \frac{-b}{(s^2+a^2)+b^2}$$

We know that,

$$L^{-1}\{\overline{f}(s)\} = \frac{-1}{t}.L^{-1}\{\overline{f}'(s)\}$$

$$L^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\} = \frac{-1}{t}.L^{-1}\left\{\frac{-b}{(s+a)^2+b^2}\right\}$$

$$= \frac{1}{t}.e^{-at}.L^{-1}\left\{\frac{b}{s^2+b}\right\} \qquad [\because \text{By first shifting theorem}]$$

$$= \frac{e^{-at}}{t}.\sin bt$$

EXAMPLE-3

Solve : Find

(i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$ (ii) $L^{-1}\left\{\dfrac{s}{(s^2-a^2)^2}\right\}$

Solution :

(i) Let $\bar{f}(s) = \dfrac{1}{s^2+a^2}$, then

$$\bar{f}'(s) = \frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{-1}{(s^2+a^2)^2}\cdot\frac{d}{ds}(s^2+a^2) \qquad \left[\because \frac{d}{dx}\left[\frac{1}{f(x)}\right]=\frac{-1}{[f(x)]^2}.f'(x)\right]$$

$$= \frac{-2s}{(s^2+a^2)^2}$$

By inverse Laplace transform of derivatives,

$$L^{-1}\{\bar{f}'(s)\} = (-1)t.L^{-1}\{\bar{f}(s)\}$$

$$\therefore L^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = -t.L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$-2.L^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = -t.\frac{1}{a}\sin at$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}.\sin at$$

(ii) Let $\bar{f}(s) = \dfrac{1}{s^2-a^2}$, then

$$f'(s) = \frac{d}{ds}\left(\frac{1}{s^2-a^2}\right)$$

$$= \frac{-1}{(s^2-a^2)^2}.2s$$

$$= \frac{-2s}{(s^2-a^2)^2}$$

We know that,

$$L^{-1}\{\bar{f}'(s)\} = -t L^{-1}\{\bar{f}(s)\}$$

$$L^{-1}\left\{\frac{-2s}{(s^2-a^2)^2}\right\} = t.L^{-1}\left\{\frac{1}{s^2-a^2}\right\}$$

$$-2L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = -t\frac{1}{a}.\sinh at$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\} = \frac{t}{2a}\sinh at$$

6.6 INVERSE LAPLACE TRANSFORM OF INTEGRALS

If $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\} = \frac{f(t)}{t}$

Proof : From the Laplace transform of division by 't', we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

Thus $L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\} = \frac{f(t)}{t} = \frac{1}{t}L^{-1}\{\bar{f}(s)\}$

The above result also be written as

$$L^{-1}\{\bar{f}(s)\} = t.L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\}$$

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L^{-1}\left\{\int_s^\infty \left(\frac{1}{s}-\frac{1}{s-1}\right)ds\right\}$

(ii) $L^{-1}\left\{\int_s^\infty \left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds\right\}$

Solution :

(i) Let $\overline{f}(s) = \frac{1}{s} - \frac{1}{s-1}$, then

$$L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s-1}\right\}$$

$$= 1 - e^t = f(t)$$

From inverse Laplace transform of integrals

$$L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$$

$$\therefore L^{-1}\left\{\int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds\right\} = \frac{1}{t}(1-e^t)$$

(ii) Let $\overline{f}(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$, then

$$L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}$$

$$= \cos at - \cos bt = f(t) \text{ (say)}$$

By inverse Laplace transform of integrals, we have

$$L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$$

$$\therefore L^{-1}\left\{\int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds\right\} = \frac{1}{t}.(\cos at - \cos bt)$$

EXAMPLE-2

Solve :

Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Solution :

Let $\overline{f}(s) = \frac{s}{(s^2+a^2)^2}$

By inverse Laplace Transform of integrals, we have

$$L^{-1}\{\bar{f}(s)\} = t.L^{-1}\left\{\int_s^\infty f(s)ds\right\}$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t.L^{-1}\left\{\int_s^\infty \frac{s}{(s^2+a^2)^2}ds\right\}$$

$$= t.L^{-1}\left\{\frac{1}{2}\int_s^\infty (s^2+a^2)^{-2}.(2s)ds\right\}$$

$$= \frac{t}{2}L^{-1}\left\{\left[\frac{(s^2+a^{2)-2+1}}{-2+1}\right]_s^\infty\right\} \qquad \left[\because \int (f(x))^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c\right]$$

$$= \frac{t}{2}L^{-1}\left\{\left(\frac{-1}{s^2+a^2}\right)_s^\infty\right\}$$

$$= \frac{t}{2}L^{-1}\left\{\frac{-1}{\infty}+\frac{1}{s^2+a^2}\right\}$$

$$= \frac{t}{2}L^{-1}\left\{0+\frac{1}{s^2+a^2}\right\}$$

$$= \frac{t}{2}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{t}{2}.\frac{1}{a}.\text{sinat}$$

$$= \frac{t}{2a}\sin at$$

EXERCISE 6.3

Find the inverse Laplace transfomr of

I. Multiplication by 's'

1. (i) $\frac{s}{(s+a)^3}$

   (ii) $\frac{s}{(s+2)^2}$

   (iii) $\frac{s}{s^2+8s+16}$ (or) $\frac{s}{(s+4)^2}$

2. (i) $\dfrac{s}{(s+2)^2+4}$

3. (i) $\dfrac{s^2}{(s-1)^2}$ (ii) $\dfrac{s^2}{(s-2)^3}$

II. Division by s

4. (i) $\dfrac{1}{s(s-a)}$ (ii) $\dfrac{1}{s(s+a)}$

(iii) $\dfrac{1}{s^2+s}$ *(iv)* $\dfrac{1}{s(s-3)}$

5. (i) $\dfrac{1}{s(s^2+a^2)}$ **[Apr. 2008, 2007]** (ii) $\dfrac{1}{s(s^2+4)}$ **[Apr. 2017]**

(iii) $\dfrac{1}{s(s^2-a^2)}$ (iv) $\dfrac{1}{s(s^2-16)}$

6. (i) $\dfrac{1}{s(s^2+2s+2)}$ (ii) $\dfrac{1}{s(s^2+4s+5)}$

III. Inverse Laplace transform of derivatives

7. (i) $\log\left(\dfrac{s+a}{s-a}\right)$ (ii) $\log\left(\dfrac{s+1}{s-1}\right)$

(iii) $\log\left(\dfrac{1+s}{s}\right)$ (iv) $\log\left(\dfrac{1+s^2}{s^2}\right)$

(v) $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2+b^2}\right)$

8. (i) $\cot^{-1}\left(\dfrac{s}{w}\right)$ (ii) $\tan^{-1}\left(\dfrac{2}{s}\right)$

IV. Inverse Laplace transform of integrals

9. (i) $\displaystyle\int_s^{\infty}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)ds$

(ii) $\displaystyle\int_s^{\infty}\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)ds$

ANSWERS

1. (i) $e^{-at}\left(t-\frac{a}{2}t^2\right)$ (ii) $e^{-2t}(1-2t)$

 (iii) $e^{-4t}(1-4t)$

2. (i) $e^{-2t}(\cos 2t-\sin 2t)$

3. (i) $\frac{e^t}{2}(t^2+4t+2)$ (ii) $e^{2t}(2t^2+4t+1)$

4. (i) $\frac{1}{a}(e^{at}-1)$ (ii) $\frac{1}{a}(1-e^{-at})$

 (iii) $1-e^{-t}$ (iv) $\frac{1}{3}(e^{3t}-1)$

5. (i) $\frac{1-\cos at}{a^2}$ (ii) $\frac{1-\cos 2t}{4}$

 (iii) $\frac{1}{a^2}[\cosh at-1]$ (iv) $\frac{1}{4}[\cosh 2t-1]$

6. (i) $\frac{1}{2}[1-e^{-t}(\sin t+\cos t)]$ (ii) $\frac{1}{5}[1-e^{-2t}(2\sin t+\cos t)]$

7. (i) $\frac{e^{at}-e^{-at}}{t}$ (or) $\frac{2}{t}\sinh at$ (ii) $\frac{e^t-e^{-t}}{t}$

 (iii) $\frac{1-e^{-t}}{t}$ (iv) $\frac{2}{t}(1-\cos t)$

 (v) $\frac{1}{t}(\cos bt-\cos at)$

8. (i) $\frac{\sin wt}{t}$ (ii) $\frac{\sin 2t}{t}$

9. (i) $\frac{1}{t}(e^{at}-e^{bt})$ (ii) $\frac{1}{t}(\cos 2t-\cos 3t)$

6.7 INVERSE LAPLACE TRANSFORM USING PARTIAL FRACTIONS

To find the inverse Laplace transform of $\overline{f}(s)$ which is of the form $\frac{p(s)}{q(s)}$, where $p(s)$ and $q(s)$ are polynomials of s.

When the degree of p(s) ≤ degree of q(s), then the rational fraction $\frac{p(s)}{q(s)}$ is called a proper fraction. It can be written as the sum of simpler rational fractions, depending on the nature of factors of the denominator q(s) as follows :

S. No.	Factor in th Denominator	Corresponing partial Fraction
1.	Non- repeated linear factor (as + b) (Occuring once)	$\frac{A}{as+b}$, where A is constant to be determined.
2.	Repeated linear factor $(as + b)^r$ (Occuring r times)	$\frac{A_1}{as+b}+\frac{A_2}{(as+b)^2}+......+\frac{A_r}{(as+b)^r}$, where A_1, A_2, ... A_r are constants to be determined
3.	Non-repeated irreducible quadratic factor $(as^2 + bs + c)$ (Occuring once)	$\frac{As+B}{as^2+bs+c}$, where A and B are constant to be found
4.	Repeated irreducible quadratic factor $(as^2 + bs + c)^r$ (Occuring r-times).	$\frac{As+B}{as^2+bs+c}+\frac{A_2s+B_2}{(as^2+bs+c)^2}+........+\frac{A_rS+B_r}{(as^2+bs+c)^r}$ Where A, B, A_2, B_2, ..., A_r, B_r are constant to be determined.

By finding the inverse L.T of each of these partial fractions, $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{p(s)}{q(s)}\right\}$ can be determined.

SOLVED EXAMPLES

EXAMPLE-1

Find :

(i) $L^{-1}\left\{\frac{1}{s^2-4s}\right\}$ *[Apr. 2018]* (ii) $L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

(iii) $L^{-1}\left\{\frac{1}{s^2+3s+2}\right\}$ *[Apr. 2019] (or)* $L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$ *[Apr. 2017]*

(iv) $L^{-1}\left\{\frac{5s+1}{(s+2)(s-1)}\right\}$

Solution :

(i) Let $\frac{1}{s(s-4)} = \frac{A}{s}+\frac{B}{s-4}$ .... (1)

$$\frac{1}{s(s-4)} = \frac{A(s-4)+Bs}{s(s-4)}$$

$$\Rightarrow \quad A = A(s-4)+Bs \qquad \text{.............. (2)}$$

To find A, Put s = 0 in (2), we get

$$1 = A(0-4)+B(0)$$

$$\Rightarrow \quad A = \frac{-1}{4}$$

To find B, Put s = 4 in (2), we get

$$1 = A(4-4)+B(4)$$

$$\Rightarrow \quad B = \frac{1}{4}$$

Substituting the vaues of A and B in (1), we get

$$\frac{1}{s(s-4)} = \frac{-1}{4}.\frac{1}{s}+\frac{1}{4}.\frac{1}{s-4}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s-4)}\right\} = L^{-1}\left\{\frac{-1}{4}.\frac{1}{s}+\frac{1}{4}.\frac{1}{s-4}\right\}$$

$$= \frac{-1}{4}L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{4}L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= \frac{-1}{4}.1+\frac{1}{4}.e^{4t}$$

$$= \frac{1}{4}[e^{4t}-1]$$

(ii) Let $\frac{1}{(s-a)(s-b)} = \frac{A}{s-a}+\frac{A}{s-b}$ .... (1)

$$\frac{1}{(s-a)(s-b)} = \frac{A(s-b)+B(s-a)}{(s-a)(s-b)}$$

$$\Rightarrow \quad 1 = A(s-b)+B(s-a) \qquad \text{.............. (2)}$$

To find A, put $s - a = 0$ i.e., $s = a$ in (2), we get

$$1 = A(a - b) + B(a - a)$$

$$\Rightarrow \quad A = \frac{1}{a-b}$$

To find B, put $s - b = 0$, i.e., $s = b$ in (2), we get

$$1 = A(b - b) + B(b - a)$$

$$\Rightarrow \quad B = \frac{1}{b-a}$$

Substituting the values of A and B in (1), we get

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\cdot\frac{1}{s-a} + \frac{1}{b-a}\cdot\frac{1}{s-b}$$

$$= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = L^{-1}\left\{\frac{1}{(a-b)}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]\right\}$$

$$= \frac{1}{a-b}\left[L^{-1}\left\{\frac{1}{s-a}\right\} - L^{-1}\left\{\frac{1}{s-b}\right\}\right]$$

$$= \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$$

(iii) Consider,

$$s^2 + 3s + 2 = s^2 + s + 2s + 2$$

$$= s(s + 1) + 2(s + 1)$$

$$= (s + 1)(s + 2)$$

Let $\dfrac{1}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$ .... (1)

$$1 = A(s + 2) + B(s + 1) \quad \text{............... (2)}$$

Put $s = -1$ in (2), we get

$$1 = A(-1 + 2) + B(-1 + 1)$$

$$\Rightarrow \quad A = 1$$

Put s = –2 in (2), we get

$$1 = A(-2+2) + B(-2+1)$$

$$\Rightarrow \quad B = -1$$

Substituting the values of A and B in (1), we get

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\because L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

(iv) Let $\frac{5s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$ .... (1)

$$\frac{5s+1}{(s+2)(s-1)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$5s + 1 = A(s-1) + B(s+2) \quad \text{.............. (2)}$$

Put s = 1 in (2), we get

$$5(1) + 1 = A(1-1) + B(1+2)$$

$$\Rightarrow \quad 6 = 3B$$

$$\Rightarrow \quad B = 2$$

Put s = –2 in (2), we get

$$5(-2) + 1 = A(-2-1) + B(-2+2)$$

$$-9 = -3A$$

$$\Rightarrow \quad A = 3$$

Substituting the values of A and B in (1), we get

$$\frac{5s+1}{(s+2)(s-1)} = \frac{3}{s+2} + \frac{2}{s-1}$$

$$\therefore L^{-1}\left\{\frac{5s+1}{(s+2)(s-1)}\right\} = L^{-1}\left\{\frac{3}{s+2} + \frac{2}{s-1}\right\}$$

$$= 3.e^{-2t} + 2e^{t}$$

EXAMPLE-2

Find : $L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$ *[Apr. 2019]*

Solution :

Let $$\frac{1}{s(s+1)(s+2)} = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2} \quad \text{.............} \quad (1)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1)(s+2)+Bs(s+2)+Cs(s+1)}{s(s+1)(s+2)}$$

$$\Rightarrow \quad 1 = A(s+1)(s+2)+Bs(s+2)+Cs(s+1) \quad \text{.............} \quad (2)$$

Put $s = 0$ in (2), we get

$$1 = A(0+1)(0+2)+B(0)(0+2)+C(0)(0+1)$$

$$1 = A(2)$$

$$\Rightarrow \quad A=\frac{1}{2}$$

Put $s = -1$ in (2), we get

$$1 = A(-1+1)(-1+2)+B(-1)(-1+2)+C(-1)(-1+1)$$

$$1 = 0+B(-1)+0$$

$$\Rightarrow \quad B = -1$$

Put $s = -2$ in (2), we get

$$1 = A(-2+1)(-2+2)+B(-2)(-2+2)+C(-2)(-2+1)$$

$$1 = C(+2)$$

$$\Rightarrow \quad C = \frac{1}{2}$$

Substituting the values of A, B and C in (1) we get

$$\frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s}-\frac{1}{s+1}+\frac{\frac{1}{2}}{s+2}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{2}(1)-e^{-t}+\frac{1}{2}e^{-2t}$$

$$= \frac{1}{2}[1-2e^{-t}+e^{-2t}]$$

EXAMPLE-3

Find : $L^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\}$

Solution :

Let $$\frac{1}{(s+2)^2(s-2)} = \frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{s-2} \qquad \text{.............. (1)}$$

$$\frac{1}{(s+2)^2(s-2)} = \frac{A(s+2)(s-2)+B(s-2)+C(s+2)^2}{(s+2)^2(s-2)}$$

$$1 = A(s+2)(s-2)+B(s-2)+C(s+2)^2 \qquad \text{.............. (2)}$$

Put $s = -2$ in (2), we get

$$1 = A(-2+2)(-2+-2)+B(-2-2)+C(-2+2)^2$$

$$1 = 0+B(-4)+0$$

$$B = \frac{-1}{4}$$

Put $s = 2$ in (2), we get

$$1 = A(2+2)(2-2)+B(2-2)+C(2+2)^2$$

$$1 = 0+0+C(16)$$

$$\Rightarrow \qquad C = \frac{1}{16}$$

Equating the coefficients s^2 on both sides of (2), we get

$$0 = A+C$$

$$\Rightarrow \qquad A = -C = \frac{-1}{16}$$

Substituting the values of A, B and C in (1), we get

$$\frac{1}{(s+2)^2(s-2)} = \frac{\frac{-1}{16}}{s+2}+\frac{\frac{-1}{4}}{(s+2)^2}+\frac{\frac{1}{16}}{s-2}$$

$$\therefore\ L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \frac{1}{16}.L^{-1}\left\{\frac{1}{s+2}\right\}-\frac{1}{4}L^{-1}\left\{\frac{1}{(s+2)^2}\right\}+\frac{1}{16}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{-1}{16}e^{-2t} - \frac{1}{4}.e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{16}.e^{2t}$$

$$= \frac{-1}{16}e^{-2t} - \frac{1}{4}e^{-2t}(t) + \frac{1}{16}e^{2t}$$

$$= \frac{-1}{16}e^{-2t} - \frac{t}{4}e^{-2t} + \frac{1}{16}e^{2t}$$

EXAMPLE-4

Find $L^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\}$

Solution :

Let $\frac{s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$ .... (1)

Multiplying both sides with $(s-1)(s^2+1)$, we get

$\Rightarrow \quad s = A(s^2+1) + (Bs+C)(s-1)$

$= As^2 + A + Bs^2 - Bs + Cs - C$

$os^2 + s + 0 = (A+B)s^2 + (-B+C)S + (A-C)$ .... (2)

Equating the coefficients of like powers of s, we get

$A + B = 0 \Rightarrow A = -B$ .... (3)

$-B + C = 1$ .... (4)

$A - C = 0 \Rightarrow A = C$ .... (5)

Substitute (3) in (4), we get

$A + C = 1$ .... (6)

From (5) and (6), we have

$C + C = 1$

$2C = 1$

$C = \frac{1}{2}$

From (5), $A = C = \dfrac{1}{2}$

From (3), $B = -A = \dfrac{-1}{2}$

Substituting the values of A, B and C in (1), we get

$$\frac{s}{(s-1)(s^2+1)} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{-1}{2}s + \frac{1}{2}}{s^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\left(\frac{s-1}{s^2+1}\right)$$

$$\therefore L^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(e^t) - \frac{1}{2}[\cos t - \sin t]$$

$$= \frac{1}{2}[e^t - \cos t + \sin t]$$

EXAMPLE-5

Find $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solution :

Let $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$ .... (1)

Multiplying both sides with $(s-1)(s^2+2s+5)$, we get

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$

$$= As^2 + 2As + 5A + Bs^2 - Bs + Cs - C$$

$$5s+3 = (A+B)s^2 + (2A-B+C)s + (5A-C) \quad \text{.............. (2)}$$

Equating the coefficients of like powers of s on both sides, we get,

$$A + B = 0 \Rightarrow A = -B \quad \text{.............. (3)}$$

$$5 = 2A - B + C \quad \text{.............. (4)}$$

$$3 = 5A - C \qquad \text{.............. (5)}$$

From (3) and (4) we get

$$5 = 2A + A + C$$

$$5 = 3A + C \qquad \text{.............. (6)}$$

Adding (5) and (6), we get

$$8 = 8A \Rightarrow A = 1$$

From (3), $B = -A = -1$

From (5), $C = 5A - 3 = 5(1) - 3 = 2$

Substituting the values of A, B and C, we get

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{(-1)s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{(s-2)}{s^2+2s+5}$$

$$\therefore L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{1}{s-1} - \frac{(s-2)}{s^2+2s+1+4}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s-2}{(s+1)^2+4}\right\}$$

$$= e^t - L^{-1}\left\{\frac{s+1-3}{(s+1)^2+2^2}\right\}$$

$$= e^t - e^{-t} L^{-1}\left\{\frac{s-3}{s^2+2^2}\right\}$$

[$\because$ By first shifting theorem]

$$= e^t - e^{-t}\left[L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - L^{-1}\left\{\frac{3}{s^2+2^2}\right\}\right]$$

$$= e^t - e^{-t}\left[\cos 2t - \frac{3}{2}\sin 2t\right]$$

EXAMPLE-6

Solve :

Find $L^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

Solution :

$$L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}L^{-1}\left\{\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{b^2-a^2}L^{-1}\left\{\frac{s^2+b^2-a^2-s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{b^2-a^2}L^{-1}\left\{\frac{(s^2+b^2)-(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{b^2-a^2}L^{-1}\left\{\frac{(s^2+b^2)}{(s^2+a^2)(s^2+b^2)}-\frac{(s^2+a^2)}{(s^2+a^2)(s^2+b^2)}\right\}$$

$$= \frac{1}{b^2-a^2}L^{-1}\left\{\frac{1}{s^2+a^2}-\frac{1}{s^2+b^2}\right\}$$

$$= \frac{1}{b^2-a^2}\left[L^{-1}\left\{\frac{1}{s^2+a^2}\right\}-L^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right]$$

$$= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at-\frac{1}{b}\sin bt\right]$$

EXERCISE 6.4

Solve :

1. (i) $\dfrac{1}{s(s+a)}$ (ii) $\dfrac{s-a}{s(s+a)}$

(iii) $\dfrac{1}{s(s-a)}$ (iv) $\dfrac{1}{s^2-3s}$

(v) $\dfrac{s+2}{s(s+3)}$ (vi) $\dfrac{s+12}{s^2+4s}$

2. (i) $\dfrac{1}{(s-1)(s-2)}$ (or) $\dfrac{1}{s^2-3s+2}$ [Apr. 2007]

(ii) $\dfrac{1}{(s-1)(s+2)}$ (iii) $\dfrac{s}{(s-1)(s+2)}$

(iv) $\dfrac{1}{(s-1)(s+3)}$ [Apr. 2007] (v) $\dfrac{1}{s^2-5s+6}$ (or) $\dfrac{1}{(s-2)(s-3)}$

(vi) $\dfrac{1}{(s+1)(s+3)}$ [Apr. 2018] (vii) $\dfrac{3s+13}{s^2+4s+3}$ (or) $\dfrac{3s+13}{(s+1)(s+3)}$ [Apr. 2009]

(viii) $\dfrac{s}{(s+1)(s+2)}$ [Apr. 2018, 2016, 2009, 2008]

(ix) $\dfrac{1}{s^2+6s+5}$ [Apr. 2017; Oct. 2008]

(x) $\dfrac{s-2}{s^2+5s+6}$ [Apr. 2008] (xi) $\dfrac{s+1}{s^2+6s-7}$ [Apr. 2016]

(xii) $\dfrac{3s}{s^2+2s-8}$ [Oct. 2008] (xiii) $\dfrac{3s+7}{s^2-2s-3}$ [Apr. 2008]

3. (i) $\dfrac{1}{(s+1)(s+2)(s+3)}$ (ii) $\dfrac{s^2}{(s+1)(s+2)(s+3)}$

4. (i) $\dfrac{4s+5}{(s-1)^2(s+2)}$

5. (i) $\dfrac{1}{(s+3)(s^2+2)}$ [Apr. 2008] (ii) $\dfrac{1}{(s+1)(s^2+1)}$

(iii) $\dfrac{s}{(s-1)(s^2+1)}$ (iv) $\dfrac{1}{(s+1)(s^2+2s+2)}$

7. (i) $\dfrac{1}{s^2(s^2+1)}$ (ii) $\dfrac{1}{(s^2+1)(s^2+4)}$

(iii) $\dfrac{1}{(s^2+1)(s^2+9)}$

8. $\dfrac{s}{(s^2+1)(s^2+4)}$

ANSWERS

1. (i) $\frac{1}{a}(1-e^{-at})$ (ii) $2e^{-at}-1$

(iii) $\frac{1}{a}(e^{at}-1)$ (iv) $\frac{1}{3}(e^{3t}-1)$

(v) $\frac{1}{3}(2+e^{-3t})$ (vi) $3-2e^{-4t}$

2. (i) $e^{2t}-e^{t}$ (ii) $\frac{1}{3}[e^{t}-e^{-2t}]$

(iii) $\frac{1}{3}[e^{t}+2e^{-2t}]$ (iv) $\frac{1}{4}(e^{t}-e^{-3t})$

(v) $e^{3t}-e^{2t}$ (vi) $\frac{1}{2}[e^{-t}-e^{-3t}]$

(vii) $5e^{-t}-2e^{-3t}$ (viii) $2e^{-2t}-e^{-t}$

(ix) $\frac{1}{4}(e^{-t}-e^{-5t})$ (x) $5e^{-3t}-4e^{-2t}$

(xi) $\frac{1}{4}[e^{t}+3e^{-7t}]$ (xii) $2e^{-4t}+e^{2t}$

(xiii) $4e^{3t}-e^{-t}$

3. (i) $\frac{1}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$ (ii) $\frac{1}{2}e^{-t}-4e^{-2t}+\frac{9}{2}e^{-3t}$

4. $\frac{1}{3}e^{t}+3te^{t}-\frac{1}{3}e^{-2t}$

5. (i) $\frac{1}{11}\left[\cos\sqrt{2}t-\frac{3}{\sqrt{2}}\sin\sqrt{2}t\right]-\frac{1}{11}e^{-3t}$ (ii) $\frac{1}{2}[e^{-t}-\cos t+\sin t]$

(iii) $\frac{1}{2}[e^{t}-\cos t+\sin t]$ (iv) $\frac{1}{18}[e^{-3t}-\cos 3t+\sin 3t]$

6. $e^{-t}(1-\cos t)$

7. (i) $t - \sin t$ (ii) $\frac{1}{3}\left[\sin t - \frac{1}{2}\sin 2t\right]$

(iii) $\frac{1}{8}\left[\sin t - \frac{1}{3}\sin 3t\right]$

8. $\frac{1}{3}[\cos t - \cos 2t]$

6.8 CONVOLUTION

Convolution is used to find the inverse laplace transform of product functions.

Convolution :

Suppose $f(t)$ and $g(t)$ be two functions defined for $t > 0$. The convolution of two functions $f(t)$ and $g(t)$, denoted by $f(t)*g(t)$ or $(f*g)(t)$ is defined as

$$(f*g)(t) = \int_0^t f(u)g(t-u)du$$

Provided the right hand side integral exists.

The relation $f(t)*g(t)$ is also called as resultant or falting of $f(t)$ and $g(t)$.

NOTE :

The convolution product is commutative

i.e., $f(t)*g(t) = g(t)*f(t)$

Convolution Theorem :

$$\text{If } L\{f(t)\} = \overline{f}(s) \text{ and } L\{g(t)\} = \overline{g}(s)\text{, then}$$

$$L\{f(t)*g(t)\} = \overline{f}(s).\overline{g}(s)$$

$$\text{(OR) } L^{-1}\{\overline{f}(s).\overline{g}(s)\} = f(t)*g(t)$$

PROOF :

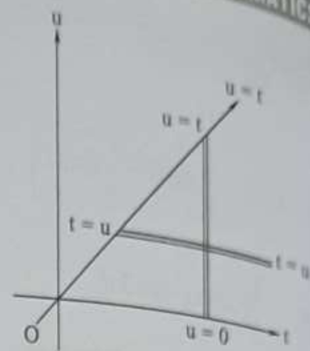
(Proof is for student reference only).

By definition of Laplace Transform,

$$L\{f(t)*g(t)\} = \int_0^\infty e^{-st}[f(t)*g(t)dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)du\right)dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(u)g(t-u)du\,dt$$



The domain of intergration for this double integral is the entire area lying betwen the lines u = 0 and u = t

On changing the order of intergration, we get

$$L\{f(t)*g(t)\} = \int_0^\infty\int_u^\infty e^{-st} f(u)g(t-u)dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_u^\infty e^{-s(t-u)} g(t-u)dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv} g(v)dv\right\}du \quad \text{(putting } t-u=v)$$

$$= \int_0^\infty e^{-su} f(u)\{\overline{g}(s)\}du$$

$$= \int_0^\infty e^{-su} f(u).du.\{\overline{g}(s)\}$$

$$= \overline{f}(s).\overline{g}(s)$$

(or) $\quad f(t)*g(t) = L^{-1}\{\overline{f}(s).\overline{g}(s)\}$, which completes the proof.

SOLVED EXAMPLES

EXAMPLE-1

Using convolution theorem, find

(i) $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$ *(ii)* $L^{-1}\left\{\dfrac{1}{(s+1)(s+2)}\right\}$ *[Apr. 2018, 2017, 2016, 2008]*

Solution :

(i) Let $\bar{f}(s) = \dfrac{1}{s+a}$ and $\bar{g}(s) = \dfrac{1}{s+b}$, then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\text{and } g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$$

$\therefore$ Convolution theorem, we get

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$\therefore L^{-1}\left\{\frac{1}{s+a}.\frac{1}{s+b}\right\} = \int_0^t f(u).g(t-u)du$$

$$= \int_0^t e^{-au}e^{-b(t-u)}du$$

$$= \int_0^t e^{-au}.e^{-bt}.e^{bu}du$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{-(a-b)t} - e^{-(a-b)0}\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{-(a-b)t} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

(ii) Let $\bar{f}(s) = \frac{1}{s+1}$ and $\bar{g}(s) = \frac{1}{s+2}$, then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

$$\text{and } g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

By convolution theorem, we have

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \int_0^t f(u)g(t-u)\,du$$

$$= \int_0^t e^{-u}.e^{-2(t-u)}\,du$$

$$= \int_0^t e^{-u}.e^{-2t}.e^{2u}du$$

$$= e^{-2t}\int_0^t e^{u}du$$

$$= e^{-2t}[e^{u}]_0^t$$

$$= e^{-2t}.[e^{t} - e^{0}]$$

$$= e^{-t} - e^{-2t}$$

Note : Above problems can be solved by using partial fractions.

EXAMPLE-2

Using convolution theorem, find

(i) $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ *[Apr. 2019, 2009]*

(ii) $L^{-1}\left\{\frac{1}{(s^2+9)(s+3)}\right\}$

Solution :

(i) Let $\overline{f}(s) = \frac{1}{s}$ and $\overline{g}(s) = \frac{1}{s^2+1}$, then

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

and $$g(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

By convolution theorem, we have

$$L^{-1}\{\overline{f}(s).\overline{g}(s)\} = f(t)*g(t) = g(t)*f(t)$$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t g(u).f(t-u)du$$

$$= \int_0^t \sin u \cdot 1 du$$

$$= [-\cos u]_0^t$$

$$= -\cos t + \cos 0$$

$$= 1 - \cos t$$

(ii) Let $\overline{f}(s) = \frac{1}{s^2+9}$ and $\overline{g}(s) = \frac{1}{s+3}$, then

$$f(t) = L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{S^2+3^2}\right\} = \frac{1}{3}\sin 3t$$

and $$g(t) = L^{-1}\{\overline{g}(s)\} = L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

By convolution theorem, we have

$$L^{-1}\{\overline{f}(s).\overline{g}(s)\} = f(t)*g(t)$$

$$L^{-1}\left\{\frac{1}{(s^2+a)}.\frac{1}{(s+3)}\right\} = \int_0^t f(u).g(t-u)du$$

$$= \int_0^t \frac{1}{3}\sin 3u.e^{-3(t-u)}du$$

$$= \frac{1}{3}\int_0^t e^{-3t}e^{3u}.\sin 3u\, du$$

$$= \frac{e^{-3t}}{3}\int_0^t e^{3u}\cdot \sin 3u\, du$$

$$= \frac{e^{-3r}}{3}\left[\frac{e^{3u}}{3^2+3^2}(3\sin 3u - 3\cos 3u)\right]_0^t$$

$$\left[\because \int e^{ax}\sin bx\, dx + \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos ax) + c\right]$$

$$= \frac{e^{-3t}}{3}\left[\frac{e^{3t}}{18}(3\sin 3t - 3\cos 3t) - \frac{e^{\circ}}{18}(3\sin 0 - 3\cos 0\right]$$

$$= \frac{e^{-3t}}{3}\left[\frac{e^{3t}}{18}(3(\sin 3t - \cos 3t) - \frac{1}{18}(0-3)\right]$$

$$= \frac{1}{18}(\sin 3t - \cos 3t) + \frac{e^{-3t}}{18}$$

$$= \frac{1}{18}[\sin 3t - \cos 3t + e^{-3t}]$$

EXAMPLE-3

Solve :

Using convolution theorem, find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Solution :

Let $\overline{f}(s) = \frac{1}{s^2}$ and $\overline{g}(s) = \frac{1}{(s+1)^2}$, then

$$f(t) = L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and $g(t) = L^{-1}\{\overline{g}(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}.L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-t}.t = te^{-t}$

From convolution theorem, we have

$$L^{-1}\{\overline{f}(s).\overline{g}(s)\} = f(t)*g(t) = g(t)*f(t)$$

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right\} = \int_0^t g(u).f(t-u)du$$

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}.(t-u)du$$

$$= \int_0^t e^{-u}(ut-u^2)du$$

$$= \left[(ut-u^2)\frac{e^{-u}}{-1}-(t-2u)\frac{e^{-u}}{(-1)^2}+(0-2)\frac{e^{-u}}{(-1)^3}\right]_0^t \text{, by Besnowlli's rule}$$

$$= \left[-(t.t-t^2)\frac{e^{-t}}{4}-(t-2t)e^{-t}+2e^{-t}\right]-\left[-(0-0)e^{o}-(t-2(0)]e^{-0}+2^{e-0}\right]$$

$$= (0+te^{-t}+2e^{-t})-(0-t+2)$$

$$= te^{-t}+2e^{-t}+t-2$$

$$= t-2+(t+2)e^{-t}$$

EXAMPLE-4

Using convolution theorem, find

(i) $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$ (ii) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

(iii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$

Solution :

(i) Let $\overline{f}(s) = \dfrac{1}{s^2+a^2}$ and $\overline{g}(s) = \dfrac{1}{s^2+a^2}$, then

$$f(t) = L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\text{sinat}$$

and $g(t) = L^{-1}\{\overline{g}(s)\} = L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}\text{sinat}$

By convolution theorem,

$$L^{-1}\{\overline{f}(s).\overline{g}(s)\} = f(t)*g(t)$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t f(u)g(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{1}{a}\sin au\,\frac{1}{a}\sin(at-au)\,du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au\sin(at-au)du$$

$$= \frac{1}{2a^2}\int_0^t [\cos[au-(at-au)]-\cos[au+at-au]]du$$

$$[\because 2\sin A\sin B = \cos(A-B)-\cos(A+B)]$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at)-\cos at]du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a}-(\cos at)u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin(2at-at)}{2a}-(\cos at)t\right)-\left(\frac{\sin(0-at)}{2a}-(\cos at).0\right)\right]$$

$$= \frac{1}{2a^2}\left[\left(\frac{\sin at}{2a}-t\cos at\right)-\left(\frac{-\sin at}{2a}-0\right)\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a}-t\cos at+\frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a}-t\cos at\right]$$

$$= \frac{1}{2a^3}[\sin at - at\cos at]$$

(ii) Let $\overline{f}(s) = \frac{s}{s^2+a^2}$ and $\overline{g}(s) = \frac{1}{s^2+a^2}$, then

$$f(t) = L^{-1}\{\overline{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\text{and } g(t) = L^{-1}\{\overline{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

By convolution theorem, we have

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$\therefore L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\} = \int_0^t f(u)g(t-u)du$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t 2\cos au \sin(at-au)du$$

$$= \frac{1}{2a}\int_0^t [\sin(au+at-au)-\sin(au-at+au)]du$$

$$[\because 2\cos A\sin B = \sin(A+B)-\sin(A-B)]$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\sin at\int_0^t du - \int_0^t \sin(2au-at)du\right]$$

$$= \frac{1}{2a}\left\{[\sin at(u)]_0^t + \left[\frac{\cos(2au-at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2a}\left\{[\sin at](t-0)+\frac{1}{2a}[\cos(2at-at)-\cos(0-at)]\right\}$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}(\text{cosat}-\text{cosat})\right]$$

$$= \frac{1}{2a}t\sin at$$

$$= \frac{t}{2a}\sin at$$

(iii) Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+a^2}$, then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\text{and } g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

By convolution theorem, we have

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$\therefore L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+a^2}\right\} = \int_0^t f(u)g(t-u)du$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\cos(a(t-u))du$$

$$= \frac{1}{2}\int_0^t 2\cos au.\cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+at-au)+\cos(2au-at)]du$$

$$= \frac{1}{2}\int_0^t [\cos at+\cos(2au-at)]du$$

$$= \frac{1}{2}\left[\cos at(u)+\frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[\left(\cos at(t)+\frac{\sin(2at-at)}{2a}\right)-\left(\text{coat}(0)+\frac{\sin(0-at)}{2a}\right)\right]$$

$$= \frac{1}{2}\left[\left(t\cos at+\frac{\sin at}{2a}\right)-\left(0-\frac{\sin at}{2a}\right)\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}[at\cos at + \sin at]$$

$$= \frac{1}{2a}[\sin at + at\cos at]$$

EXAMPLE-5

Solve : Using convolution theorem, find $L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

Solution :

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+b^2}$, then

$$f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

and $g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{b}\sin bt$

By convolution theorem, we have

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)}.\frac{1}{s^2+b^2}\right\} = \int_0^t f(u)g(t-u)du$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au\frac{1}{b}\sin b(t-u)du$$

$$= \frac{1}{2b}\int_0^t 2\cos au.\sin(bt-bu)du$$

$$= \frac{1}{2b}\int_0^t [\sin(au+bt-bu) - \sin(au-(bt-bu))]$$

$$[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2b}\int_0^t (\sin[(a-b)u+bt]-\sin[(a+b)u-bt])du$$

$$= \frac{1}{2b}\left\{\left(\frac{-\cos[(a-b)t+bt]}{a-b}+\frac{\cos[(a+bt)-bt]}{a+b}\right)\right.$$

$$\left.-\left(\frac{-\cos[(a-b).0+bt]}{(a-b)}+\frac{\cos[(a+b).0-bt)}{a+b}\right)\right\}$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos at}{a-b}+\frac{\cos at}{a+b}\right)-\frac{-\cos bt}{a-b}+\frac{\cos bt}{a+b}\right]$$

$$= \frac{1}{2b}\left[\cos at\left[\frac{-1}{a-b}+\frac{1}{a+b}\right]-\cos bt\left[\frac{-1}{a-b}+\frac{1}{a+b}\right]\right]$$

$$= \frac{1}{2b}\left[\cos at\left(\frac{-a-b+a-b}{a^2-b^2}\right)-\cos bt\left(\frac{-a-b+a-b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2b}\left[\cos at\left(\frac{-2b}{a^2-b^2}\right)-\cos bt\left(\frac{-2b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2b}\cdot\frac{-2b}{a^2-b^2}[\text{coat}-\cos bt]$$

$$= \frac{1}{b^2-a^2}[\cos at-\cos bt]$$

EXERCISE 6.5

Solve : Using Convolution Theorem, Find inverse Laplace transform of

1. (i) $\frac{1}{(s+1)(s+3)}$ **[Apr. 2009]** *(ii)* $\frac{1}{(s+9)(s+3)}$ *[Apr. 2019]*

   (iii) $\frac{1}{(s-1)(s+2)}$ (iv) $\frac{1}{s(s-1)}$ [Apr. 2019]

2. (i) $\frac{1}{s^2(s-a)}$ (ii) $\frac{1}{s^2(s+1)}$

   (iii) $\frac{1}{s^2(s^2-a^2)}$ (iv) $\frac{1}{s^2(s^2+a^2)}$

3. (i) $\frac{1}{(s+1)(s+9)^2}$ (ii) $\frac{1}{(s-2)(s+2)^2}$

4. (i) $\frac{s}{(s^2+1)^2}$ **[Oct. 2016 ; Apr. 2009, 2008]**

(ii) $\frac{s^2}{(s^2+4)^2}$

5. (i) $\frac{1}{(s^2+a^2)(s^2+b^2)}$ (ii) $\frac{1}{(s^2+1)(s^2+9)}$

6. (i) $\frac{s}{(s^2+1)(s^2+4)}$ (ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

ANSWERS

1. (i) $\frac{1}{2}[e^{-t}-e^{-3t}]$ (ii) $\frac{1}{6}[e^{-3t}-e^{-at}]$

(iii) $\frac{1}{3}(e^{t}-e^{-2t})$ (iv) $e^{t}-1$

2. (i) $\frac{1}{a^2}$ (iii) $\frac{1}{a^3}[\sinh at - at]$

(iv) $\frac{1}{a^3}[at-\sin at]$

6.9 APPLICATION OF LAPLACE TRANSFORM TO SOLVE ORDINARY DIFFERNTIAL EQUATIONS OF SECOND WITH INITIAL CONDITIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, specially usefull for solving linear differential equation with constant coefficient.

Working Rule : To solve a linear diffferential equation with constant coefficients by transform method:

Consider the initial value problem.

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by = r(t) \quad \text{.............. (1)}$$

$$\text{with } y(0)=k_0 \text{ and } y'(0)=k_1 \quad \text{.............. (2)}$$

Where a, b, k_0, k are constants and r(t) is a function of 't'.

Step-(i) :

Apply Laplace transform on both sides of the given differential equation (1)

Step-(ii) :

Use the formulae for Laplace transform of derivatives

(a) $L\{y'(t)\} = s\bar{y}(s) - y(0)$

(b) $L\{y''\} = s^2 y(s) - sy(0) - y'(0)$

(c) $L\{y'''\} = s^3\bar{y}(s) - s^2 y(0) - sy'(0) - y''(0)$

Step-(iii) :

Replace y(0), y'(0) using the given initial conditions.

Step-(iv) :

Solve for $\bar{y}(s)$ in terms of s.

Step-v :

Take inverse Laplace transform to $\bar{y}(s)$ obtained in the step (iv). This gives the required solution of 0.D.E (1) satisfying the initial conditions (2).

SOLVED EXAMPLES

EXAMPLE-1

Solve : Use Laplace transfrom method to solve

(i) $\frac{d^2y}{dt^2} + y = 0$ *with y(0) = 0 and y'(0) = 2* *[Apr. 2008]*

(ii) $y'' - 2y' - 8y = 0$, *if y(0) = 3 and y'(0) = 6* *[Apr. 2018, 2016]*

Solution :

(i) Given differential equation is

$$\frac{d^2y}{dt^2} + y = 0 \quad \text{.............. (1)}$$

with $y(0) = 0$

and $y'(0) = 2$ .... (2)

Taking Laplace transform on both sides of (1), we get

$$L\left\{\frac{d^2y}{dt^2}\right\}+L\{y\} = L\{0\}$$

$$[s^2L\{y\} - sy(0) - y'(0)] + L\{y\} = 0$$

$$s^2L\{y\} - s(0) - 2 + L\{y\} = 0 \qquad [\because \text{ From (2)}]$$

$$(s^2 + 1)L\{y\} = 2$$

$$L\{y\} = \frac{2}{s^2+1}$$

$$\because\ y = L^{-1}\left\{\frac{2}{s^2+1}\right\}$$

$$= 2\sin t$$

Which is the required solution.

(ii) Given differential equation is,

$$y'' - 2y' - 8y = 0 \qquad \text{.............. (1)}$$

$$y(0) = 3$$

$$\text{and} \quad y'(0) = 6 \qquad \text{.............. (2)}$$

Applying Laplace Transform on both sides of (1), we get

$$L\{y''\} - 2L\{y'\} - 8L\{y\} = L\{0\}$$

$$[s^2L\{y\} - sy(0) - y'(0)] - 2[sL\{y\} - y(0)] - 8L(y\} = 0$$

$$(s^2L\{y\} - s(3) - 6) - 2(sL\{y\} - 3) - 8L\{y\} = 0 \qquad [\because \text{ From (2)}]$$

$$s^2L\{y\} - 3s - 6 - 2sL\{y\} + 6 - 8L\{y\} = 0$$

$$(s^2 - 2s - 8)\, L\{y\} = 3s$$

$$L\{y\} = \frac{3s}{s^2-2s-8} \qquad \text{.............. (3)}$$

$$= \frac{3s}{s^2-4s+2s-8}$$

$$= \frac{3s}{(s+2)(s-4)}$$

$$= \frac{1}{s+2} + \frac{2}{s-4} \text{ (by partial fractions)}$$

$$\therefore \quad y = L^{-1}\left\{\frac{1}{s+2} + \frac{2}{s-4}\right\}$$

$$= L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= e^{-2t} + 2e^{4t}$$

EXAMPLE-2

Solve : Use Laplace transform method to solve

(i) $y'' + y = 2e^t$, $y(0) = 0$ *and* $y'(0) = 0$ [Apr. 2019, 2016]

(ii) $y'' + 4y' + 3y = e^{-t}$ *when* $y(0) = y'(0) = 1$ [Apr. 2019]

(iii) $y'' + 2y' - 3y = t$ *with* $y(0) = y'(0) = 0$ [Oct. 2018]

Solution :

(i) Given differential equation is,

$$y'' + y = 2e^t \quad \text{.............. (1)}$$

$$y(0) = 0$$

$$\text{and} \quad y'(0) = 0 \quad \text{.............. (2)}$$

Taking Laplace Transform on both sides of (1), we get

$$L\{y''\} + L\{y\} = L\{2e^t\}$$

$$[s^2L\{y\} - sy(0) - y'(0)] + L\{y\} = 2.\frac{1}{s-1}$$

$$s^2L\{y\} - 0 - 0 + L\{y\} = \frac{2}{s-1}$$

$$(s^2 + 1)L\{y\} = \frac{2}{s-1}$$

$$L\{y\} = \frac{2}{(s-1)(s^2+1)} \quad \text{.............. (3)}$$

$$\text{Let} \quad \frac{2}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \quad \text{.............. (4)}$$

$$\Rightarrow \quad 2 = A(s^2+1)+(Bs+C)(s-1)$$

$$= As^2 + A + Bs^2 - Bs + Cs - C$$

$$2 = (A+B)s^2 + (-B+C)s + (A-C) \qquad \text{..............} \quad (5)$$

Equating the coefficient of like powers of 's' on both sides, we get

$$A+B=0 \Rightarrow A=-B \qquad \text{..............} \quad (6)$$

$$-B+C=0 \Rightarrow B=C \qquad \text{..............} \quad (7)$$

$$A-C=2 \qquad \text{..............} \quad (8)$$

From (6) and (7) $A = -C$ substitute in (8), we get

$$-C-C=2$$

$$-2C=2$$

$$C=-1$$

$\therefore$ $A = -C = 1$ and $B = C = -1$

Substituting the values of A, B and C in (4) we get

$$\frac{2}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{-s-1}{s^2+1} \qquad \text{..............} \quad (9)$$

From (3) and (9), we get

$$L\{y\} = \frac{1}{s-1} - \frac{s+1}{s^2+1}$$

$$\therefore \quad y = L^{-1}\left\{\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^t - \cos t - \sin t$$

(ii) Given differential equation is,

$$y'' + 4y' + 3y = e^{-t} \qquad \text{..............} \quad (1)$$

with $y(0) = y'(0) = 1$ .... (2)

Taking Laplace Transform on both sides of (1), we get

$$\Rightarrow \quad L\{y''\} + 4L\{y'\} + 3L\{y\} = L\{e^{-t}\}$$

$$\Rightarrow \quad [s^2L\{y\} - sy(0) - y'(0)] + 4[sL\{y\} - y(0)] + 3L\{y\} = \frac{1}{s+1}$$

$$\Rightarrow \quad [s^2L\{y\} - s - 1] + 4[sL\{y\} - 1] + 3L\{y\} = \frac{1}{s+1}$$

$$\Rightarrow \quad s^2L\{y\} - s - 1 + 4sL\{y\} - 4 + 3L\{y\} = \frac{1}{s+1}$$

$$\Rightarrow \quad [s^2 + 4s + 3]L\{y\} = \frac{1}{s+1} + s + 5$$

$$(s+1)(s+3)L\{y\} = \frac{1+s^2+s+5s+s}{s+1}$$

$$= \frac{s^2+6s+6}{s+1}$$

$$\therefore \quad L\{y\} = \frac{s^2+6s+6}{(s+1)(s+1)(s+3)}$$

$$= \frac{s^2+6s+6}{(s+1)^2(s+3)} \qquad \text{.............. (3)}$$

Let $\dfrac{s^2+6s+6}{(s+1)^2(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+3}$ .... (4)

$$\Rightarrow \quad s^2 + 6s + 6 = A(s+1)(s+3) + B(s+3) + c(s+1)^2 \qquad \text{.............. (5)}$$

Put s = –1 in (5), we get

$$(-1)^2 + 6(-1) + 6 = A(-1+1)(-1+3) + B(-1+3) + C(-1+1)^2$$

$$1 = B(2) \Rightarrow B = \frac{1}{2}$$

Put s = –3 in (5), we get

$$(-3)^2 + 6(-3) + 6 = A(-3+1)(-3+3) + B(-3+3) + c(-3+1)^2$$

$$-3 = A(0) + B(0) + c(-2)^2$$

$$-3 = c(4)$$

$$c = \frac{-3}{4}$$

Compare the coefficients of s^2 on both sides of (5), we get

$$A + C = 1 \Rightarrow A = 1 - C = 1 + \frac{3}{4} = \frac{7}{4}$$

Substituting the values of A, B and C in (4), we get

$$\frac{s^2+6s+6}{(s+1)^2-(s+3)} = \frac{7}{4}.\frac{1}{s+1}+\frac{1}{2}.\frac{1}{(s+1)^2}-\frac{3}{4}.\frac{1}{s+3} \qquad \text{..............} \quad (6)$$

$\therefore$ From (3) and (6), we have

$$L\{y\} = \frac{7}{4}.\frac{1}{(s+1)}+\frac{1}{2}.\frac{1}{(s+1)^2}-\frac{3}{4}.\frac{1}{(s+3)}$$

$$\therefore \ y = L^{-1}\left\{\frac{7}{4}.\frac{1}{s+1}+\frac{1}{2}\frac{1}{(s+1)^2}-\frac{3}{4}.\frac{1}{s+3}\right\}$$

$$= \frac{7}{4}.L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{3}{4}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{7}{4}.e^{-t}+\frac{1}{2}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{3}{4}e^{-3t}$$

$$= \frac{7}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{3}{4}e^{-3t}$$ which is the required solution.

(iii) Given differential equation is

$$y'' + 2y' - 3y = t \qquad \text{..............} \quad (1)$$

with $y(0) = y'(0) \ = 0$ .... (2)

Applying Laplace transform on both sides of (1), we get

$$L\{y''\} + 2L'\{y'\} - 3L\{y\} = L\{t\}$$

$$[s^2L\{y\} - sy(0) - y'(0)] + 2[sL\{y\} - y(0)] - 3L\{y\} = \frac{1}{s^2}$$

$$[s^2L\{y\} - 0 - 0] + 2[sL\{y\} - 0] - 3L\{y\} = \frac{1}{s^2}$$

$$s^2L\{y\} + 2sL\{y\} - 3L\{y\} = \frac{1}{s^2}$$

$$[s^2 + 2s - 3]\, L\{y\} = \frac{1}{s^2}$$

$$L\{y\} = \frac{1}{s^2(s^2+2s-3)}$$

$$= \frac{1}{s^2(s+3)(s-1)} \qquad \text{.............. (3)}$$

$$\text{Let } \frac{1}{s^2(s+3(3-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{D}{s-1} \qquad \text{.............. (4)}$$

$$= \frac{As(s+3)(s-1) + B(s+3)(s-1) + Cs^2(s-1) + Ds^2(s+3)}{s^2(s+3)(s-1)}$$

$$\therefore \quad 1 = As(s+3)(s-1) + B(s+3)(s-1) + Cs^2(s-1) + Ds^2(s+3) \qquad \text{.............. (5)}$$

Put $s = 0$ in (5), we get

$$1 = A(0) + B(0+3)(0-1) + C(0) + D(0)$$

$$\Rightarrow \quad 3B = -1$$

$$\Rightarrow \quad B = \frac{-1}{3}$$

Put $s = 1$ in (5), we get

$$1 = A1(1+3)(1-1) + B(1+3)(1-1) + C1^2(1-1) + D(1)^2(1+3)$$

$$1 = 4D \Rightarrow D = \frac{1}{4}$$

Put $s = -3$ in (5), we get

$$1 = A(0) + B(0) + C(-3)^2(-3-1) + D(0)$$

$$1 = C(-36) \Rightarrow C = \frac{-1}{36}$$

Compare the coefficients of s^3 on both sides, we get

$$0 = A + C + D$$

$$\Rightarrow \quad A = -C - D$$

$$= \frac{1}{36} - \frac{1}{4}$$

$$= \frac{1-9}{36}$$

$$= \frac{-8}{36}$$

$$= \frac{-2}{9}$$

Substituting the values of A, B, C and D in (4), we get

$$\frac{1}{s^2(s+3)(s-1)} = \frac{-2}{9}.\frac{1}{s} - \frac{1}{3}.\frac{1}{s^2} - \frac{1}{36}\frac{1}{s+3} + \frac{1}{4}.\frac{1}{s-1} \quad \text{..............} \quad (6)$$

From (3) and (6), we have

$$L\{y\} = \frac{-2}{9}.\frac{1}{s} - \frac{1}{3}.\frac{1}{s^2} - .\frac{1}{s+3} + \frac{1}{4}.\frac{1}{s-1}$$

$$y = \frac{-2}{9}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{36}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{-2}{9}(1) - \frac{1}{3}t\frac{1}{36}e^{-3t} + \frac{1}{4}e^{t}$$

$$= \frac{-2}{9} - \frac{t}{3} - \frac{1}{36}e^{-3t} + \frac{1}{4}e^{t}$$

EXAMPLE-3

Solve $y'' + y = \sin 3t$ with $y(0) = y'(0) = 0$ by Laplace transform method.

Solution :

Given differential equation is

$$y'' + y = \sin 3t \quad \text{..............} \quad (1)$$

with $\quad y(0) = y'(0) = 0 \quad$ .... (2)

Apply L.T. on both sides of (1), we get

$$L\{y''\} + L\{y\} = L\{\sin 3t\}$$

$$\left[s^2L\{y\} - s\,y(0) - y'(0)\right] + L\{y\} = \frac{3}{s^2+3^2}$$

$$\left[s^2L\{y\} - 0 - 0\right] + L\{y\} = \frac{3}{s^2+3^2}$$

$$s^2L\{y\}+L\{y\} = \frac{3}{s^2+9}$$

$$(s^2+1)\,L\{y\} = \frac{3}{s^2+9}$$

$$L\{y\} = \frac{3}{(s^2+1)(s^2+9)}$$

$$= \frac{3}{8}\left[\frac{9-1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{3}{8}\left[\frac{(s^2+9)-(s^2+1)}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{3}{8}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]$$

$$y = L^{-1}\left\{\frac{3}{8}\left[\frac{1}{s^2+1}-\frac{1}{s^2+9}\right]\right\}$$

$$= \frac{3}{8}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\}-L^{-1}\left\{\frac{1}{s^2+9}\right\}\right]$$

$$= \frac{3}{8}\left[\sin t-\frac{1}{3}\sin 3t\right]$$

$$= \frac{3}{8}\sin t-\frac{1}{8}\sin 3t\,.$$

EXERCISE 6.6

Solve the following differential equations by Laplace transform method:

I. Homogeneous

1. $y'' + 4y' + 3y = 0$ given that $y(0) = 3$, $y'(0) = 1$. [May. 2022]
2. $y'' - 6y' + 9y = 0$, $y(0) = 2$ and $y'(0) = 9$
3. $y'' - 4y' + 5y = 0$ when $y(0) = 1$ and $y'(0) = 2$

II. Non-Homogenoeus

4. (i) $y'' + y = t$, $y(0) = 1$ and $y'(0) = 0$ [Apr. 2018, 2016]

   (ii) $y'' + y = t$, if $y(0) = 1$ and $y'(0) = 2$ [Apr. 2009]

   (iii) $y'' + y = 6 \cos 2t$ with $y(0) = 3$, $y'(0) = 1$ [Apr. 2018]

   (iv) $y'' + u = 4e^t$ given $y(0) = 0$ and $y'(0) = 0$ [Apr. 2008]

5. (i) $y'' + 3y' + 2y = e^{-t}$ with $y(0) = 0$ and $y'(0) = 1$ [Apr. 2019, 2018, 2016]

   (or)

   (ii) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = e^{-t}$ with $x(0) = 0$ and $x'(0) = 1$.

6. (i) $(D^2 + 2D + 1)y = 3te^{-t}$ given that $y(0) = 4$ and $y'(0) = 0$ [Apr. 2019]

   (ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = t$ with conditions $y(0) = 0$ and $y'(0) = 1$

   (iii) $y'' + 2y' + y = 3te^{-t}$ when $y(0) = 4$, $y'(0) = 2$ [Apr. 2019]

7. $(D^2 - 2D + 1)\, x = e^t$ with initial conditions $x = 2$, $\dfrac{dx}{dt} = -1$ at $t = 0$

8. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dt} - 3y = \sin t$, if $y(0) = y'(0) = 0$ [Apr. 2016, 2007]

ANSWERS

1. $y = -2e^{-3t} + 5e^{-t}$

2. $y = (3t + 2)e^{3t}$

3. $y = e^{2t}\cos t$

4. (i) $y = t - \sin t + \cos t$

   (ii) $y = t - 3\sin t + \cos t$

   (iii) $y = 5\cos t - 2\cos 2t + \sin t$

   (iv) $y = 2(e^{t} - \sin t - \cos t)$

5. $y = te^{-t}$

6. (i) $e^{-t}\left(\frac{t^2}{2} + 4t + 4\right)$

   (ii) $y = t + 2(e^{-t} + te^{-t} - 1)$

   (iii) $y = e^{-t}\left(\frac{t^3}{2} + 6t + 4\right)$

7. $x = \frac{e^t}{2}(t^2 - 6t + 4)$

8. $y = \frac{-1}{10}\cos t - \frac{1}{5}\sin t - \frac{1}{40}e^{-3t} + \frac{1}{8}e^{t}$

BOARD DIPLOMA EXAMINATION

MODEL PAPER FOR SEM END

ADVANCED ENGINEERING MATHEMATICS

For Second Year, IV Semester Examination

Time : 2 Hours | Max. Marks : 40

PART - A 8 × 1 = 8

Note : *Answer **all** questions. Each question carries **One** marks.*

1. Find the roots of auxiliary equation of the differential equation $(D^2 + 4D)\,y = 0$.
2. Define Fourier Series for the function f (x) in the interval $(c, c + 2\pi)$.
3. Find the Particular Integral of $(D^2 - 4D + 1)y = e^x$.
4. Find $L(e^{2t} + \cos 3t)$.
5. Find $L(t + 5\cosh t)$.
6. State the First Shifting theorem of Laplace Transforms.
7. Find $L^{-1}\left(\dfrac{1}{s-3} + \dfrac{s}{s^2+4}\right)$.
8. Find $L^{-1}\left(\dfrac{1}{2s+S}\right)$.

PART - B 4 × 3 = 12

Note : *Answer **ALL** questions. Each question carries **Three** marks.*

9. (a) Solve : $(D^2 + D + 1)\,y = 4e^{3x}$.

(or)

(b) Find $L(t\cos 3t)$.

10. (a) Find Half Range Sine Series of $f(x) = x$ in $(0, \pi)$.

(or)

(b) Find $L^{-1}\left(\dfrac{s+1}{s^2+6s-7}\right)$

11. (a) If $L\{f(t)\} = \dfrac{20-4s}{s^2-4s+20}$, find $L\{f(3t)\}$.

(or)

(b) Find $\int_0^\infty t.e^{-2t}\sin 3t\,dt$ using Laplace Transform Technique.

12. (a) Show that $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)=\frac{1-\cos at}{a^2}$

(or)

(b) Find $L^{-1}\left(\frac{s}{(s+2)^2+4}\right)$.

PART - C

$4 \times 5 = 20$

Note : *Answer any* ***Four*** *questions. Each question carries* ***Five*** *marks.*

13. (a) Solve : $(D^2+D-2)\,y=x+\sin x$.

(or)

(b) Find $L\{t\,e^t \sin 3t\}$.

14. (a) Expand $f(x)=x^2$ as a Fourier series in the interval $(-\pi,\pi)$.

(or)

(b) Find $L^{-1}\left(\frac{s}{(s+2)^2(s^2+1)}\right)$.

15. (a) Find $L\left(\frac{\sin 3t.\cos t}{t}\right)$

(or)

(b) Evaluate $L\left\{\int_0^t \frac{\sin t}{t}dt\right\}$

16. (a) Find $L^{-1}\left(\frac{1}{(s+1)(s+2)}\right)$ using convolution theorem.

(or)

(b) Solve the differential equation $y''-2y'-8y=\sin t$, when $y(0)=3$, $y'(0)=6$ by Laplace transform method.

K.N.K