UNIT- I

Probability and Random Variables

Part - B

1. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that the red ball drawn is from bag B.

Sol:- Let E_1 & E_2 denote the events of selecting bag A and bag B respectively.

$\therefore P(E_1) = \frac{1}{2}, \; P(E_2) = \frac{1}{2}$

Let A denote the event of drawing a red ball.

$\therefore P(A/E_1) = \frac{3}{5}, \; P(A/E_2) = \frac{5}{9}$

$\therefore$ The probability that the red ball drawn is from bag B.

$$\therefore P(E_2/A) = \frac{P(A/E_2) \cdot P(E_2)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{\frac{5}{9} \cdot \frac{1}{2}}{\frac{3}{5} \cdot \frac{1}{2} + \frac{5}{9} \cdot \frac{1}{2}}$$

$$= \frac{25}{52}$$

2. Suppose 5 men out of 100 and 25 women out of 10000 are colour blind. A colour blind person is chosen at random. What is the probability of the person being a male (Assume male and female to be in equal numbers)?

Sol:- Given The probability that the choosen person is male

$P(E_1) = \frac{1}{2}$

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The probability that the choosen person is female

$P(E_2) = \frac{1}{2}$

& also given 5 men out of 100 and 25 women out of 10000 are color blind.

A color blind person is choosen at random

Let A be the event of a blind person

$$P(A/E_1) = \frac{5}{100}, \quad P(A/E_2) = \frac{25}{10000}$$

∴ the probability that the choosen person is male is given by

$$P(E_1/A) = \frac{P(A/E_1) \cdot P(E_1)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{\frac{5}{100} \cdot \frac{1}{2}}{\frac{5}{100} \cdot \frac{1}{2} + \frac{25}{10000} \cdot \frac{1}{2}}$$

$$= 0.95$$

3. In a bolt factory machines A, B, C manufacture 20%, 30% and 50% of the total of their output and 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. Find the probabilities that it is manufactured from (i) Machine A (ii) Machine B (iii) Machine C.

Sol:- Let E_1, E_2, E_3 are events that the bolts are manufactured by the machines A, B, C respectively.

$$\therefore P(E_1) = \frac{20}{100}, \quad P(E_2) = \frac{30}{100}, \quad P(E_3) = \frac{50}{100}$$

Let A denotes that the bolt is defective.

$$P(A/E_1) = \frac{6}{100}, \quad P(A/E_2) = \frac{3}{100}, \quad P(A/E_3) = \frac{2}{100}$$

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i, If bolt is defective, then the probability that it is from machine A

$$\therefore P(E_1/A) = \frac{P(A/E_1)\cdot P(E_1)}{P(A/E_1)\cdot P(E_1) + P(A/E_2)\cdot P(E_2) + P(A/E_3)\cdot P(E_3)}$$

$$= \frac{\frac{6}{100}\cdot\frac{20}{100}}{\frac{6}{100}\cdot\frac{20}{100} + \frac{3}{100}\cdot\frac{30}{100} + \frac{2}{100}\cdot\frac{50}{100}}$$

$$= \frac{12}{31}$$

ii, If bolt is defective, then the probability that it is from machine B

$$\therefore P(E_2/A) = \frac{P(A/E_2)\cdot P(E_2)}{P(A/E_1)\cdot P(E_1) + P(A/E_2)\cdot P(E_2) + P(A/E_3)\cdot P(E_3)}$$

$$= \frac{\frac{3}{100}\cdot\frac{30}{100}}{\frac{6}{100}\cdot\frac{20}{100} + \frac{3}{100}\cdot\frac{30}{100} + \frac{2}{100}\cdot\frac{50}{100}}$$

$$= \frac{9}{31}$$

iii, From Machine C

$$\therefore P(E_3/A) = \frac{P(A/E_3)\cdot P(E_3)}{P(A/E_1)\cdot P(E_1) + P(A/E_2)\cdot P(E_2) + P(A/E_3)\cdot P(E_3)}$$

$$= \frac{\frac{2}{100}\cdot\frac{50}{100}}{\frac{6}{100}\cdot\frac{20}{100} + \frac{3}{100}\cdot\frac{30}{100} + \frac{2}{100}\cdot\frac{50}{100}}$$

$$= \frac{10}{31}$$

4. Bag I contains 2 white, 3 red balls and bag II contains 4 white, 5 red balls, one ball is drawn at random from one of the bag it found to be red. Find the probabability that red ball is drawn from bag I.

Sol:- Let E_1, E_2 are events of selecting bags.

$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$

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A be the event of selecting red ball

$P(A/E_1) = \frac{3}{5}$, $P(A/E_2) = \frac{5}{9}$

$\therefore$ the probability that the red ball is drawn from bag I

$$P(E_1/A) = \frac{P(A/E_1) \cdot P(E_1)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{3}{5}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{5}{9}}$$

$$= \frac{27}{52}$$

5. In a certain college 25% are boys 10% are girls are studying statistics, the girls constitute 60% of class room.
a) What is the probability that statistics is being studied?
b) If a student is selected at random and is found to be studying statistics, find the probability that the student is a girl?

Sol:-
a) Let E_1, E_2 are events of girls & boys.

$P(E_1) = 0.6$, $P(E_2) = 0.4$

Let A be the event of studying ~~mathematics~~ statistics

$P(A/E_1) = 0.1$, $P(A/E_2) = 0.25$

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)$$

$$= 0.6 \times 0.1 + 0.4 \times 0.25$$

$$= \cancel{0.07}\ 0.16$$

b) $$P(E_1/A) = \frac{P(A/E_1) \cdot P(E_1)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{0.1 \times 0.6}{0.1 \times 0.6 + 0.25 \times 0.4}$$

$$= \cancel{6/16}\ 3/8$$

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6. The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the function

$$f(x) = \begin{cases} A e^{-x/5}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

i, Find the value of A that makes f(x) a probability density function.

ii, What is the probability that the number of minutes that she will take over the phone is more than 10 minutes?

Sol:- i, W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} A e^{-x/5}\,dx = 1$$

$$A\left[\frac{e^{-x/5}}{-1/5}\right]_0^{\infty} = 1$$

$$-5 \times A\left[e^{-\infty} - e^{-0}\right] = 1$$

$$-5A[0-1] = 1$$

$$\Rightarrow A = \frac{1}{5}$$

ii, $P(x > 10) = \int_{10}^{\infty} f(x)\,dx$

$$= \int_{10}^{\infty} A e^{-x/5}\,dx$$

$$= \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{10}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-2}\right]$$

$$= \frac{1}{e^2} \qquad [\because e^{-\infty} = 0]$$

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7. If x denote the sum of the two numbers that appear when a pair of fair dice is tossed. Determine i) Distribution function ii) Mean and iii) Variance.

Sol:- Given $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$= 6^2$

$= 36$

X denote the sum of the two numbers

$X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$P(X=2) = \{(1,1)\} = \frac{1}{36}$

$P(X=3) = \{(1,2), (2,1)\} = \frac{2}{36}$

$P(X=4) = \{(1,3), (3,1), (2,2)\} = \frac{3}{36}$

$P(X=5) = \{(1,4), (4,1), (2,3), (3,2)\} = \frac{4}{36}$

$P(X=6) = \{(1,5), (5,1), (2,4), (4,2), (3,3)\} = \frac{5}{36}$

$P(X=7) = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\} = \frac{6}{36}$

$P(X=8) = \{(2,6), (6,2), (3,5), (5,3), (4,4)\} = \frac{5}{36}$

$P(X=9) = \{(3,6), (6,3), (4,5), (5,4)\} = \frac{4}{36}$

$P(X=10) = \{(4,6), (6,4), (5,5)\} = \frac{3}{36}$

$P(X=11) = \{(5,6), (6,5)\} = \frac{2}{36}$

$P(X=12) = \{(6,6)\} = \frac{1}{36}$

i) Distribution function

$X=x$	2	3	4	5	6	7	8	9	10	11	12	total
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Mean $(\mu) = \Sigma x P(X=x)$

$= 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36}$

$+ 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$

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Mean $(\mu) = 7$

iii, Variance $(\sigma^2) = \Sigma x^2 \cdot P(X=x) - \mu^2$

$$= 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + 6^2 \times \frac{5}{36}$$
$$+ 7^2 \times \frac{6}{36} + 8^2 \times \frac{5}{36} + 9^2 \times \frac{4}{36} + 10^2 \times \frac{3}{36} + 11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36}$$
$$- (7)^2$$

$= 5.83$

8. Is the function density function $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, If so determine the probability that the variate having this density will fall in the interval $(1,2)$? Find the cumulative probability $F(2)$?

Sol:- Given $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

i, Clearly $f(x) \geq 0, \forall x$ in $(1,2)$ &

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx$$
$$= -\left[e^{-x}\right]_0^{\infty}$$
$$= -\left[e^{-\infty} - e^{-0}\right]$$
$$= -(0-1)$$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\therefore$ The function $f(x)$ is a density function

ii, $P(1 \leq x \leq 2) = \int_1^2 f(x)\,dx$

$$= \int_1^2 e^{-x}\,dx$$
$$= -\left(e^{-x}\right)_1^2$$
$$= e^{-1} - e^{-2}$$

iii, Cumulative probability function

$$F(2) = \int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^2 e^{-x}\,dx$$
$$= -\left(e^{-x}\right)_0^2$$
$$= 1 - e^{-2}$$

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9. If probability density function $f(x) = \begin{cases} Kx^3, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$. Find the value of K and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Sol:-

We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 Kx^3\,dx = 1$$

$$K\left[\frac{x^4}{4}\right]_0^3 = 1$$

$$\Rightarrow K = \frac{4}{81}$$

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx$$

$$= \int_{1/2}^{3/2} K x^3\,dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{1/2}^{3/2}$$

$$= \frac{4}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{5}{81}$$

10. A random variable X has the following probability function.

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Find i) k ii) $P(X<6)$ iii) $P(X \geq 6)$.

Sol:- i) $\sum_{x=0}^{7} P(x) = 1$

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$\Rightarrow k = \frac{1}{10} = 0.1$$

ii) $P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 0+k+2k+2k+3k+k^2$$

$$= 8k + k^2$$

$$= 0.81$$

iii) $P(X \geq 6) = 1 - P(X < 6)$
$= 1 - 0.81$
$= 0.19$

11. Let X denotes the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine i) Discrete probability distribution ii) Expectation iii) Variance.

Sol:- Let $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

X denotes the minimum of two numbers

$X = \{1, 2, 3, 4, 5, 6\}$

$P(X=1) = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (6,1), (5,1), (4,1), (3,1), (2,1)\} = \frac{11}{36}$

$P(X=2) = \{(2,2), (2,3), (3,2), (2,4), (4,2), (2,5), (5,2), (2,6), (6,2)\} = \frac{9}{36}$

$P(X=3) = \{(3,3), (3,4), (4,3), (3,5), (5,3), (3,6), (6,3)\} = \frac{7}{36}$

$P(X=4) = \{(4,4), (4,5), (5,4), (4,6), (6,4)\} = \frac{5}{36}$

$P(X=5) = \{(5,5), (5,6), (6,5)\} = \frac{3}{36}$

$P(X=6) = \{(6,6)\} = \frac{1}{36}$

i) ∴ Discrete Probability distribution

X	1	2	3	4	5	6	total
P(X)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$	1

ii) Expectation = $\underset{(\mu)}{\text{Mean}} = \Sigma x \cdot P(x)$

$= 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36}$

$= 2.52$

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iii) Variance $= \sum x^2 p(x) - \mu^2$

$= 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + 3^2 \cdot \frac{7}{36} + 4^2 \cdot \frac{5}{36} + 5^2 \cdot \frac{3}{36} + 6^2 \cdot \frac{1}{36} - (2.52)$

$= 1.97$

12. A random variable X has the following probability function

X	-3	-2	-1	0	1	2	3
P(X)	k	0.1	k	0.2	2k	0.4	2k

then find i) k ii) Mean iii) Variance.

Sol:- i) W.K.T $\sum P(X) = 1$

$k + 0.1 + k + 0.2 + 2k + 0.4 + 2k = 1$

$\Rightarrow k = \frac{0.1}{2}$

ii) Mean $(\mu) = \sum X \cdot P(X)$

$= (-3)(k) + (-2)(0.1) + (-1)(k) + (0)(0.2) + (1)(2k)$
$+ (2)(0.4) + (3)(2k)$

$= 0.8$

iii) Variance $(\sigma^2) = \sum x^2 P(X) - \mu^2$

$= (-3)^2(k) + (-2)^2(0.1) + (-1)^2(k) + 0 + 1^2(2k) + 2^2(0.4)$
$+ 3^2(2k) - (0.8)^2$

$= 2.86$

13. A continuous random variable has the probability density

$$f(x) = \begin{cases} kx e^{-\lambda x}, & \text{for } x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find i) k ii) Mean iii) Variance

Sol:- W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} kx\, e^{-\lambda x}\,dx = 1$

$k\int_{0}^{\infty} x\, e^{-\lambda x}\,dx = 1$

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$$K\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right)-1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^\infty=1$$

$$K\left[(0-0)-\left(0-\frac{1}{\lambda^2}\right)\right]=1$$

$$\Rightarrow K=\lambda^2$$

(ii) Mean $(\mu) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$=\int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx$$

$$=0+\int_0^\infty x \cdot kx\, e^{-\lambda x} dx$$

$$=K\int_0^\infty x^2 e^{-\lambda x} dx$$

$$=\lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right)-2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right)+2\left(\frac{e^{-\lambda x}}{\lambda^3}\right)\right]_0^\infty$$

$$=\lambda^2\left[(0-0+0)-\left(0-0-\frac{2}{\lambda^3}\right)\right]$$

$$=\frac{2}{\lambda}$$

(iii) Variance $(\sigma^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$$=\int_{-\infty}^{0} x^2 f(x)\, dx + \int_0^\infty x^2 f(x)\, dx - \left(\frac{2}{\lambda}\right)^2$$

$$=0+\int_0^\infty x^2 \cdot kx\, e^{-\lambda x} dx - \frac{4}{\lambda^2}$$

$$=K\int_0^\infty x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2}$$

$$=\lambda^2\left[x^3\left(\frac{e^{-\lambda x}}{-\lambda}\right)-3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right)+6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right.$$
$$\left.-6\left(\frac{e^{-\lambda x}}{-\lambda^4}\right)\right]_0^\infty-\frac{4}{\lambda^2}$$

$$=\lambda^2\left[(0-0+0-0)-\left(0-0+0-\frac{6}{\lambda^4}\right)\right]-\frac{4}{\lambda^2}$$

$$=\frac{2}{\lambda^2}$$

14. If the probability density function of random variable is $f(x) = k(1-x^2)$, $0 < x < 1$, then calculate

(i) k (ii) $P(0.1 < x < 0.2)$ (iii) $P(x > 0.5)$

Sol:- (i) W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 k(1-x^2)\,dx = 1$$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow k = \frac{3}{2}$$

(ii) $P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x)\,dx$

$$= \int_{0.1}^{0.2} k(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{(0.2)^3}{3}\right) - \left(0.1 - \frac{(0.1)^3}{3}\right)\right]$$

$$= 0.2965$$

(iii) $P(x > 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$

$$= \frac{3}{2}\int_{0.5}^{1} (1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left\{\left[1 - \frac{1^3}{3}\right] - \left[0.5 - \frac{(0.5)^3}{3}\right]\right\}$$

$$= 0.3125$$

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15. A random variable x has the following probability function

X	4	5	6	8
P(X)	0.1	0.3	0.4	0.2

Determine i) Expectation ii) Variance iii) standard deviation.

Sol:- i) Expectation $[E(x)] = \sum x f(x)$

$= 4\times0.1+5\times0.3+6\times0.4+8\times0.2$

$= 5.9$

ii) Variance $(\sigma^2) = \sum x^2 f(x) - \mu^2$

$= 4^2\times0.1+5^2\times0.3+6^2\times0.4+8^2\times0.2-(5.9)^2$

$= 1.49$

iii) Standard deviation $(\sigma) = \sqrt{1.49} = 1.22$

16. If x is a continuous random variable whose density function is $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1\le x<2 \\ 0, & \text{else where} \end{cases}$ find $E(25x^2+30x-5)$.

Sol:-

$\therefore E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^1 x.x\,dx + \int_1^2 x.(2-x)\,dx$

$= \int_0^1 x^2 dx + \int_1^2 (2x-x^2)\,dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$

$= \frac{1}{3} + \left[(2^2-1)-\left(\frac{8}{3}-\frac{1}{3}\right)\right]$

$= 3$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_0^1 x^2.x\,dx + \int_1^2 x^2(x-2)\,dx$

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$$= \int_0^1 x^3 dx + \int_1^2 (x^3 - 2x^2) dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - 2\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{2^4}{4} - \frac{1}{4}\right) - 2\left(\frac{2^3}{3} - \frac{1}{3}\right)\right]$$

$$= -\frac{2}{3}$$

$$\therefore E(25x^2 + 30x - 5) = 25 \cdot E(x^2) + 30\,E(x) - E(5)$$

$$= 25\left(-\frac{2}{3}\right) + 30(3) - 5$$

$$= 68.33$$

17. The cumulative distribution function for a continuous random Variable X is

$$F(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find i, Density function $f(x)$ ii, Mean iii, Variance

Sol: i, Density function $f(x) = \frac{d}{dx}\{F(x)\}$

$$\therefore f(x) = \begin{cases} \frac{1}{2} e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

ii) $\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{0} x\, f(x) dx + \int_0^{\infty} x\, f(x)\, dx$$

$$= 0 + \int_0^{\infty} x \cdot \frac{1}{2} e^{-2x} dx$$

$$= \frac{1}{2}\left\{x\left(\frac{e^{-2x}}{-2}\right) - 1\left(\frac{e^{-2x}}{4}\right)\right\}_0^{\infty}$$

$$= \frac{1}{2}\left\{(0-0) - \left(0 - \frac{1}{4}\right)\right\}$$

$$= \frac{1}{8}$$

iii, $\text{Variance} = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2 = \int_{-\infty}^{0} x^2 f(x) dx + \int_0^{\infty} x^2 f(x)\, dx - \mu^2$

$$= 0 + \int_0^{\infty} x^2 \cdot \frac{1}{2} e^{-2x} dx - \left(\frac{1}{8}\right)^2$$

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$$= \frac{1}{2}\left[x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)\right]_0^\infty - \frac{1}{64}$$

$$= \frac{1}{2}\left[(0-0+0) - \left(0-0+2\left(\frac{1}{-8}\right)\right)\right] - \frac{1}{64}$$

$$= \frac{7}{64}$$

18. Two coins are tossed simultaneously. Let x denotes the number of heads then find i, $E(x)$ ii, $E(x^2)$ iii, $E(x^3)$ iv, $V(x)$

Sol:- Given X = No. of Heads

$X = \{0,1,2\}$

$P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{2}{4}$, $P(X=2) = \frac{1}{4}$

$X=x$	0	1	2	total
$f(x)=P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	1

i, $E(x) = \Sigma x f(x)$

$= 0\times\frac{1}{4} + 1\times\frac{2}{4} + 2\times\frac{1}{4}$

$= 1$

ii, $E(x^2) = \Sigma x^2 f(x)$

$= 0^2\times\frac{1}{4} + 1^2\times\frac{2}{4} + 2^2\times\frac{1}{4}$

$= \frac{1}{2} + 1$

$= 3/2$

iii, $E(x^3) = \Sigma x^3 f(x)$

$= 0^3\times\frac{1}{4} + 1^3\times\frac{2}{4} + 2^3\times\frac{1}{4}$

$= \frac{1}{2} + 2$

$= 5/2$

iv, $V(X) = E(x^2) - [E(x)]^2$

$= 3/2 - (1)^2$

$= 1/2$

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19. Is the function defined by $f(x) = \begin{cases} 0, & x<2 \\ \frac{1}{18}(2x+3), & 2\le x\le 4 \\ 0, & x>4 \end{cases}$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2\le x\le 3$.

Sol:- i, For all points x in $-\infty \le x \le \infty$, $f(x) \ge 0$ &

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} 0\cdot dx + \int_{2}^{4} \frac{1}{18}(2x+3) + \int_{4}^{\infty} 0\cdot dx$$

$$= \frac{1}{18}\int_{2}^{4}(2x+3)\,dx$$

$$= \frac{1}{18}\left[2\frac{x^2}{2} + 3x\right]_2^4$$

$$= \frac{1}{18}\left[(4^2-2^2) + 3(4-2)\right]$$

$$= 1$$

$\therefore$ $f(x)$ is a probability density function

ii, $P(2\le x\le 3) = \int_{2}^{3} f(x)\,dx$

$$= \frac{1}{18}\int_{2}^{3}(2x+3)\,dx$$

$$= \frac{1}{18}\left[2\frac{x^2}{2} + 3x\right]_2^3$$

$$= \frac{1}{18}\left[(3^2-2^2) + 3(3-2)\right]$$

$$= \frac{4}{9}$$

20. The probability density function of a random variable X is $f(x) = \frac{k}{x^2+1}$, $-\infty < x < \infty$, Find k & distribution function $F(x)$

Sol:- we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} \frac{k}{x^2+1}\,dx = 1$$

$$\Rightarrow k\left(\tan^{-1}x\right)_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1 \Rightarrow k = \frac{1}{\pi}.$$

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Distribution function $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{x} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}x - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right]$$

Part - C

1. A box contains 2 red, 3 blue and 4 black, three balls are drawn from the box at random. Find probability that
   i, Three balls are different colours
   ii, Three balls are same colour.
   iii, Two are same and third is different.

Sol: i, Given 2 red, 3 blue, 4 black in that 3 balls drawn randomly

$$n = {}^9C_3$$

i, E = 3 balls are different

$m = {}^2C_1 \times {}^3C_1 \times {}^4C_1$

$$P(E) = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3}$$

ii, E = 3 balls are same

$m = {}^3C_3 + {}^4C_3$

$$P(E) = \frac{{}^3C_3 + {}^4C_3}{{}^9C_3}$$

iii, E = 2 balls are same and 3rd is different

$m = {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1$

$$\therefore P(E) = \frac{{}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1}{{}^9C_3}$$

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2. A businessman goes to hotels X, Y, Z, 20%, 50% & 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbing. What is the probability that business man's room having faulty plumbing is assigned to hotel Z?

Sol:- Let $P(X) = \frac{20}{100}$, $P(Y) = \frac{50}{100}$, $P(Z) = \frac{30}{100}$

Let E be the event that the hotels room has faulty plumbing

$$P(E/X) = \frac{5}{100},\ P(E/Y) = \frac{4}{100},\ P(E/Z) = \frac{9}{100}$$

The probability that the business man's room having faulty plumbing is assigned to hotel Z.

$$P(Z/E) = \frac{P(Z) \cdot P(E/Z)}{P(Z) \cdot P(E/Z) + P(Y) \cdot P(E/Y) + P(X) \cdot P(E/X)}$$

$$= \frac{\frac{30}{100} \times \frac{9}{100}}{\frac{30}{100} \cdot \frac{9}{100} + \frac{50}{100} \cdot \frac{4}{100} + \frac{20}{100} \cdot \frac{5}{100}}$$

$$= \frac{4}{9}$$

3. In a factory, machine A produces 40% of the output & machine B produces 60%. On the average, 9 items in 1000 produced by A are defective and 1 item in 250 produced by B is defective. An item drawn at random from a day's output is defective. What is the probability that it was produced by A or B?

Sol:- Given E_1, E_2 are events of machine A & B.

$$P(E_1) = 0{\cdot}\frac{40}{100},\ P(E_2) = \frac{60}{100}$$

A is an event of defective item

$$P(A/E_1) = \frac{9}{1000},\ P(A/E_2) = \frac{1}{250}$$

∴ Probability of product produced by E_1 & E_2

$$P(E_1/A) = \frac{P(A/E_1) \cdot P(E_1)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{\frac{9}{1000} \times \frac{40}{100}}{\frac{9}{1000} \times \frac{40}{100} + \frac{1}{250} \times \frac{60}{100}}$$

$$= 0.6$$

$$P(E_2/A) = \frac{P(A/E_2) \cdot P(E_2)}{P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2)}$$

$$= \frac{\frac{1}{250} \times \frac{60}{100}}{\frac{9}{1000} \times \frac{40}{100} + \frac{1}{250} \times \frac{60}{100}}$$

$$= 0.4$$

4. A fair die is tossed. Let the random variable X denote the twice the number appearing on the die. i) Write the probability distribution of X, ii) Mean iii) Variance.

Sol:- Let X denote twice the number appearing on the face when a die is thrown.

i) Probability distribution

$X=x$	2	4	6	8	10	12
$f(x) = P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

ii) Mean $= \Sigma x f(x)$

$= 2 \times \frac{1}{6} + 4 \times \frac{1}{6} + 6 \times \frac{1}{6} + 8 \times \frac{1}{6} + 10 \times \frac{1}{6} + 12 \times \frac{1}{6}$

$= 7$

iii) Variance $= \Sigma x^2 f(x) - \mu^2$

$= 2^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} + 8^2 \times \frac{1}{6} + 10^2 \times \frac{1}{6} + 12^2 \times \frac{1}{6} - (7)^2$

$= (60.67) - (7)^2$

$= 11.67$

9) If $f(x) = k e^{-|x|}$ is probability density function in the interval, $-\infty < x < \infty$, then find (i) k (ii) Mean (iii) Variance (iv) $P(0 < x < 4)$.

Sol: (i) we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} k\, e^{-|x|}\,dx = 1$$

$$\boxed{\because |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}}$$

$$k\left[\int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{\infty} e^{-|x|}\,dx\right] = 1$$

$$k\left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right] = 1$$

$$k\left[\left(e^{x}\right)_{-\infty}^{0} + \left(\frac{e^{-x}}{-1}\right)_{0}^{\infty}\right] = 1$$

$$k\left[(1-0) - (0-1)\right] = 1$$

$$\Rightarrow k = \frac{1}{2}$$

(ii) Mean $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= k\int_{-\infty}^{\infty} x\, e^{-|x|}\,dx$$

$$= k\left[\int_{-\infty}^{0} x\, e^{x}\,dx + \int_{0}^{\infty} x\, e^{-x}\,dx\right]$$

$$= k\left[\left(x e^{x} - e^{x}\right)_{-\infty}^{0} + \left(x\left(\frac{e^{-x}}{-1}\right) - \frac{e^{-x}}{1}\right)_{0}^{\infty}\right]$$

$$= k\left[-1+1\right]$$

$$= 0$$

(iii) Variance $= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$= k\int_{-\infty}^{\infty} x^2\, e^{-|x|}\,dx - \mu^2$$

$$= k\left[\int_{-\infty}^{0} x^2 e^{x}\,dx + \int_{0}^{\infty} x^2 e^{-x}\,dx\right] - \mu^2$$

$$= k\left[\left(x^2 e^{x} - 2x\, e^{x} + 2e^{x}\right)_{-\infty}^{0} + \left(x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{-1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right)_{0}^{\infty}\right] - \mu^2$$

$$= k(2+2) - 0$$

$$= \frac{1}{2}(4)$$

$$= 2$$

iv) $P(0 < X < 4) = \int_0^4 f(x)\,dx$

$= \int_0^4 k\, e^{-|x|}\,dx \quad , \quad |x| = x, \text{ if } x > 0$

$= k\left[\frac{e^{-x}}{-1}\right]_0^4$

$= \frac{1 - e^{-4}}{2}$

6. The function $f(x) = Ax^2,\ 0 < x < 1$ is valid probability density function then find the value of A.

Sol:- W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^1 f(x)\,dx = 1$

$\int_0^1 A x^2\,dx = 1$

$A\left[\frac{x^3}{3}\right]_0^1 = 1$

$A\left[\frac{1}{3} - 0\right] = 1$

$\therefore A = 3$

7. The density function of a random variable X is

$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$, Find $E(X)$, $E(X^2)$, $V(X)$.

Sol:-

i) $E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$= \int_{-\infty}^{0} x\, f(x)\,dx + \int_0^{\infty} x\, f(x)\,dx$

$= 0 + \int_0^{\infty} x\, e^{-x}\,dx$

$= \left[x\left(\frac{e^{-x}}{-1}\right) - 1.\,(e^{-x})\right]_0^{\infty}$

$= \left[(0-0) - (0-1)\right]$

$= 1$

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(ii) $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$

$= 0 + \int_0^{\infty} x^2 e^{-x}\,dx$

$= \left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\cdot(e^{-x}) + 2\,\frac{e^{-x}}{-1} \right]_0^{\infty}$

$= [(0-0+0)-(0-0-2)]$

$= 2$

(iii) $V(x) = E(x^2) - [E(x)]^2$

$= 2-(1)^2$

$= 1$

8. If $E(X)=10$, $V(X)=1$ then find $E\{2x(x+20)\}$.

Sol: Given $E(X)=10$, $V(X)=1$

$E\{2x(x+20)\} = E\{2x^2+40x\}$

$= 2E(x^2) + 40E(x)$

$= 2\{V(x) + (E(x))^2\} + 40E(x)$

$= 2(1+100) + 40\times 10$

$= 602$

9. A discrete random variable X has the following probability distribution.

X	1	2	3	4	5	6	7	8
$P(X=x)$	2k	4k	6k	8k	10k	12k	14k	4k

Find (i) k (ii) $P(X<3)$ (iii) $P(X\geq 5)$

Sol: (i) we have $\Sigma f(x) = 1$

$2k+4k+6k+8k+10k+12k+14k+4k=1$

$\Rightarrow k = \frac{1}{60}$

(ii) $P(X<3) = P(X=1) + P(X=2)$ ~~+ P(X<3)~~

$= 2k + 4k$

$= \frac{1}{60} \times 6$

$= \frac{1}{10}$

iii) $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$= 10k + 12k + 14k + 4k$

$= 40k$

$= 40 \times \frac{1}{160}$

$= 2/3$

10. For the continuous random variable X whose probability density function is given by $f(x) = \begin{cases} cx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

Find i) c ii) Mean iii) Variance of X.

Sol: i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^2 cx(2-x)\,dx = 1$

$c\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$

$c\left[(2^2 - 0) - \left(\frac{2^3}{3} - 0\right)\right] = 1$

$\Rightarrow c = 3/4$

ii) Mean $= E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$= \int_0^2 x\, c\, x(2-x)\,dx$

$= 3/4\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2$

$= \frac{3}{4}\left[2\left(\frac{2^3}{3} - 0\right) - \left(\frac{2^4}{4} - 0\right)\right]$

$= 1$

iii) Variance $(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \int_0^2 x^2\, cx(2-x)\,dx - (1)^2$

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$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx - 1$$

$$= \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1$$

$$= \frac{3}{4}\left[\left(\frac{2^4}{2} - 0\right) - \left(\frac{2^5}{5} - 0\right)\right] - 1$$

$$= \frac{1}{5}$$

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