Unit-4

PushDown Automata

Pushdown Automata (PDA):-

A pushdown automata is a way to implement a content free grammar in a similar way we design finite automata for regular language or regular grammar.

→ It is more powerful than FSM

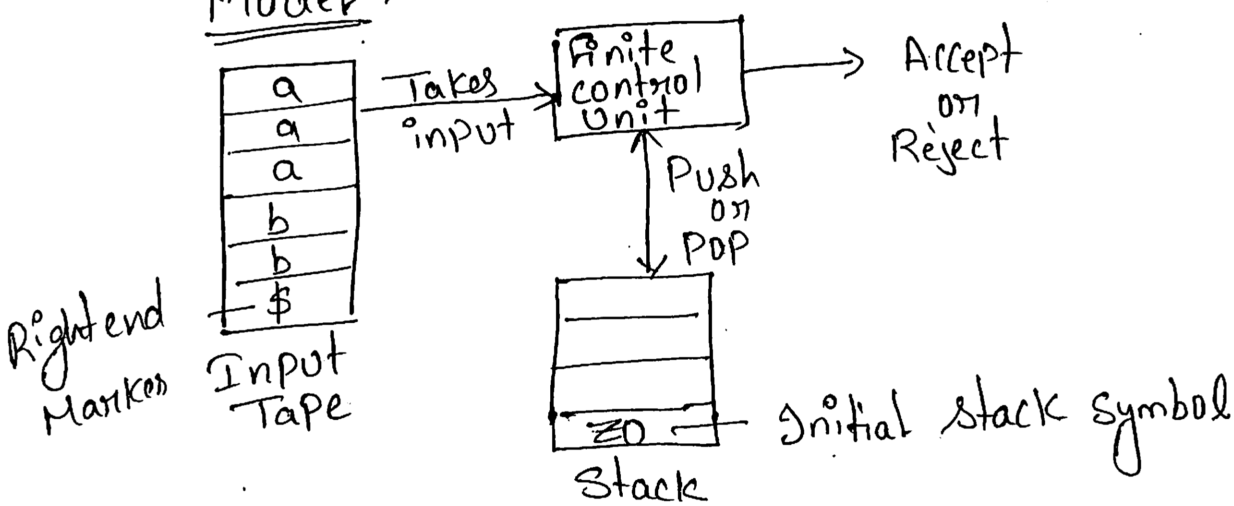
→ FSM has a very limited memory but PDA has more memory.

→ PDA = Finite State Machine + a Stack

A Pushdown Automata has 3 components:

1) An input Tape
2) A finite Control unit
3) A stack with infinite size

Model :-



A PDA is defined by 7 Tuples as below

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ Γ [Uppercase gama or capital gaama]

Q = A finite set of states

Σ = A finite set of input symbols

Γ = A finite stack alphabet

δ = The trasition function

q_0 = The start state

Z_0 = The start symbol of stack

F = The set of final states

where δ takes a triple argument $\delta(q, a, x)$

i) q is a state in Q

ii) a is either an input symbol in Σ or $a = \epsilon$

iii) x is a stack symbol, that is a member of Γ

$$\boxed{\therefore\ q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \rightarrow q \times \Gamma^*}$$

The output of δ is finite set of Pairs (P, γ)

where: i.e $\delta(q, a, x) = (P, \gamma)$

P is a new state

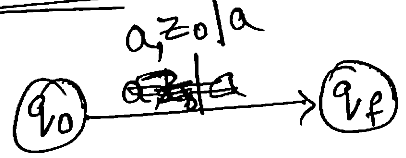
γ is a string of stack symbol that replaces x at the top of the stack.

eg:- If $\gamma = \epsilon$ then the stack is Popped

If $\gamma = x$ then the stack is unchanged

If $\gamma = YZ$ then x is replaced by Z and Y is pushed onto the stack.

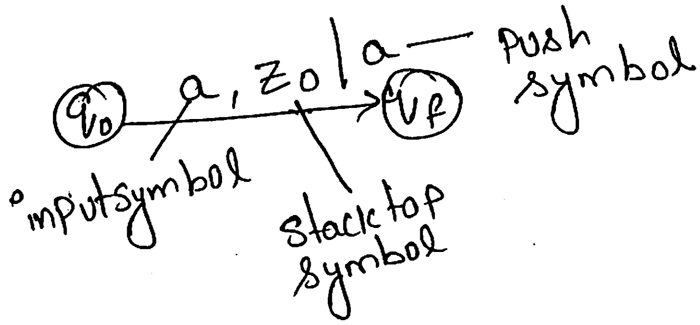
Graphical notation of PDA :-



a → Input symbol (may be ε)

b → symbol on top of the stack. ~~This symbol is popped.~~ if symbol is 'ε' that means the stack is neither read nor popped.

c → This symbol is pushed onto the stack, 'ε' means nothing is pushed.



Instantaneous Description:

It is the term related to PDA where (ID) Instantaneous descriptor of a PDA is represented by a triplet (q, w, s) where

q is a state
w is unconsumed input
s is the stack contents

The process of transition (ID) is denoted by the 'Turnstile' symbol " $\vdash$ "

For a PDA a turnstile notation can be represented as

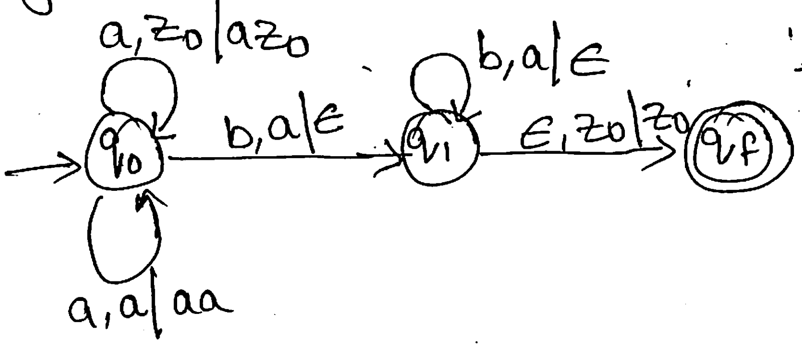
$\delta(q, aw, z) \vdash (P, w, \alpha)$

This implies that while taking a transition from state 'q' to state 'P', the input symbol 'a' is consumed & the top of the stack is 'z' is replaced by a new string 'α'.

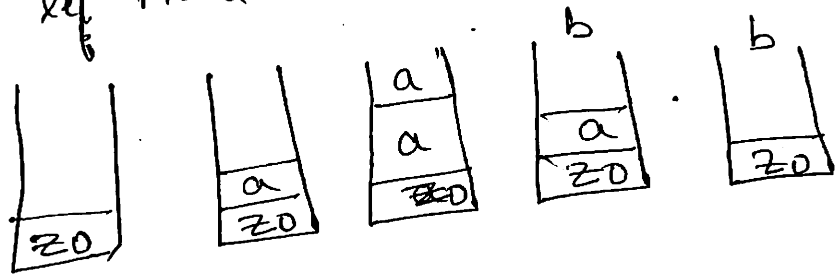
<u>Note</u>: If we want zero or more moves of PDA we have to use the symbol ($\vdash^*$).

Design a PDA which accepts $L = \{a^n b^n \mid n \geq 1\}$

$= \{ab, aabb, aaabbb, \ldots\}$



let n = 2



<u>Transition function</u>:

$\delta(q_0, a, z_0) = (q_0, az_0)$

$\delta(q_0, a, a) = (q_0, aa)$. $(q_0, b, a) = (q_1, \epsilon)$

$\delta(q_1, b, a) = (q_1, \epsilon)$

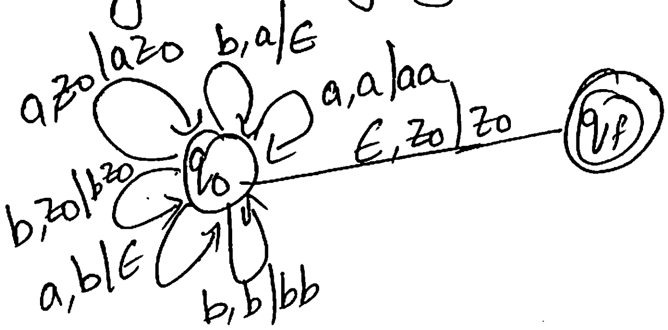
$\delta(q_1, \epsilon, z_0) = (q_f, z_0)$

Consider i/p - aaabbb for above PDA

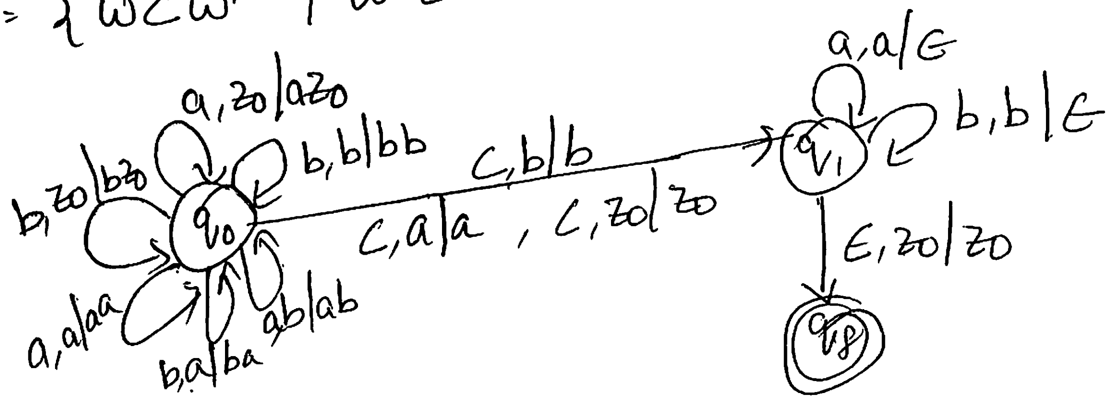
$(q_0, aaabbb, z_0) \vdash (q_0, aabbb, az_0)$
$\vdash (q_0, abbb, aaz_0)$
$\vdash (q_0, bbb, aaaz_0)$
$\vdash (q_0, bb, aaz_0)$
$\vdash (q_0, b, az_0)$
$\vdash (q_0, \epsilon, z_0)$
(q_f, z_0)

⇒ $L = \{ w \mid n_a(w) = n_b(w) \mid a^n b^n \; n \geq 1 \}$

given language $L = \{ab, aabb, abab, abba, baba, \ldots\}$



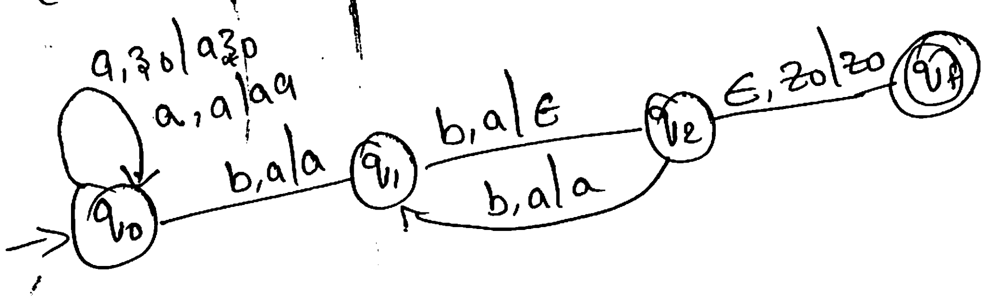
$L = \{ wcw^R \mid w \in (a+b)^* \}$ $L: \{aabaa, baab, \ldots$



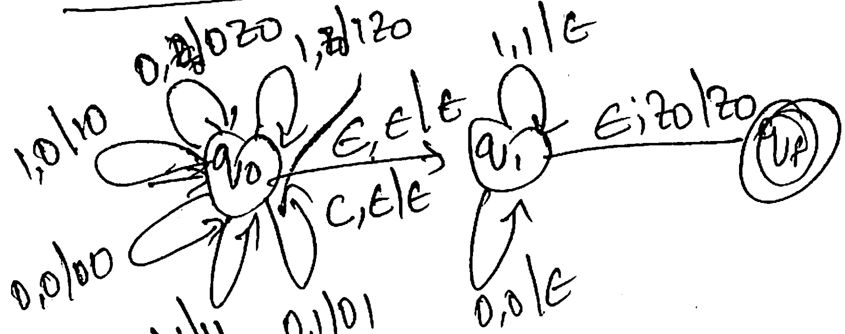
Acceptance of a CFL:-

Consider $L = a^n b^n \; n \geq 1$ & i/p aaabbb check the acceptance of given lang

$L = \{a^n b^{2n} \mid n \geq 1\}$



NPDA for Even length Palindrome

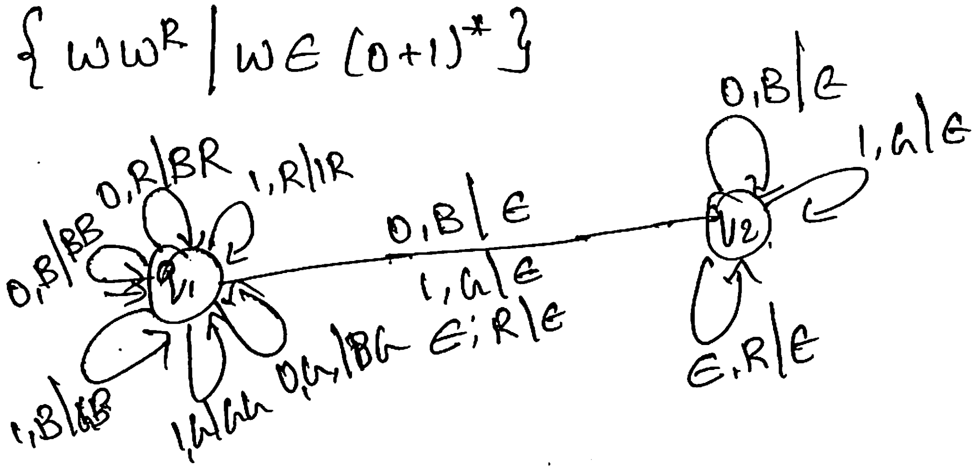


$M = (\{q_1, q_2\}, \{0,1\}, \{R, B, G\}, \delta, q_1, R, \emptyset\}$

B is used to represent 0 in the stack

G is used to represent 1 in the stack

$\{ww^R \mid w \in (0+1)^*\}$



This is NDPDA

$\delta(q_1, 0, R) = (q_1, BR)$

$\delta(q_1, 0, B) = (q_1, BB)\ (q_2, \epsilon)$

$\delta(q_1, 0, G) = (q_1, BG)$

$\delta(q_1, 1, R) = (q_1, GR)$

$\delta(q_1, 1, B) = (q_1, GB)$

$\delta(q_1, 1, G) = (q_1, GG)\ (q_2, \epsilon)$

$\delta(q_2, 0, B) = (q_2, \epsilon)$

$\delta(q_2, 1, G) = (q_2, \epsilon)$

$\delta(q_2, \epsilon, R) = (q_2, \epsilon),\ \delta(q_1, \epsilon, R) = (q_2, \epsilon)$

Equivalence between CFG & PDA :-

$A \to \alpha$

$\delta(q, \epsilon, A) = (q, \alpha) \mid A \to \alpha$ } Production function for
$\delta(q, a, a) = (q, \epsilon)$ } PDA.

If the given CFG is in GNF then these above Production functions can be applied to convert to PDA.

Eg: $S \to 0BB \quad B \to 0S \mid 1S \mid 0$

$\delta(q, \epsilon, S) = (q, 0BB)$

$\delta(q, \epsilon, B) = (q, 0S)\ (q, 1S)\ (q, 0)$

$\delta(q, 0, 0) = (q, \epsilon)$

$\delta(q, 1, 1) = (q, \epsilon)$ – is equivalent FA

$(q, 010000, S)$

$\vdash (q, 010000, 0BB)$

$\vdash (q, 10000, BB)$

$\vdash (q, 10000, 1SB)$

$\vdash (q, 0000, SB)$

$\vdash (q, 0000, 0BBB)$

$\vdash (q, 000, 0\cancel{S}BB)$

$\vdash (q, 00, \cancel{S}BB)$

$\vdash (q, 00, 0B) \cdot \vdash (q, 0, B) \vdash (q, 0, 0)$

$\vdash (q, \epsilon, \epsilon)$ //

U-3
5

$S \to 0A$, $A \to 0AB \mid 1$, $B \to 1$

$(q, \epsilon, S) = (q, 0A)$

$(q, \epsilon, A) = (q, 0AB) \mid (q, 1)$

$(q, \epsilon, B) = (q, 1)$

$(q, 0, 0) = (q, \epsilon)$

$(q, 1, 1) = (q, \epsilon)$

$(q, 0011, S) \vdash (q, 0011, 0A)$
$\vdash (q, 011, A)$
$\vdash (q, 011, 0AB)$
$\vdash (q, 11, AB)$
$\vdash (q, 11, 1B)$
$\vdash (q, 1, B)$
$\vdash (q, 1, 1)$
$\vdash (q, \epsilon, \epsilon)$ //

$\Rightarrow$ $S \to aABB \mid aAA$
$A \to aBB \mid a$
$B \to bBB \mid \epsilon$

$\to$ $S \to aABB \mid aAB \mid aA \mid aAA$
$A \to aBB \mid aB \mid a$
$B \to bBB \mid bB \mid b$

$\delta(q, \epsilon, S) = (q, aABB), (q, aAB) (q, aA) (q, aAA)$

$\delta(q, \epsilon, A) = (q, aBB), (q, aB) (q, a)$

$\delta(q, \epsilon, B) = (q, bBB), (q, bB), (q, b)$

$\delta(q, a, a) = (q, \epsilon)$

$\delta(q, b, b) = (q, \epsilon)$

$(q, aaaabbbb, S) \vdash (q, \cancel{a}aaabbbb, \cancel{a}ABB)$

$\vdash (q, \cancel{a}aaabbbb, \cancel{a}AB)$

$\vdash (q, aaabbbb, AA)$

$\vdash (q, \cancel{a}aabbbb, \cancel{a}A)$

$\vdash (q, aabbbb, A)$

Conversion from PDA to CFG:-

For the given PDA $M=(Q,\Sigma,q_0,\delta,Z_0,\Gamma,\emptyset)$, we will construct a grammar G such that $L(G)=L(M)$ for converting the PDA to CFG we use the following rules:

1. The Production for the start symbol 'S' are given by $S \rightarrow [q_0 Z_0 q]$ for
   where q indicates states from Q $q \in Q$
   & q_0 is initial state.

2. If there is a transition or move PDA is $\delta(q,a,Z)=(q',\epsilon)$ then add Production as $[q,Z,q'] \rightarrow a$

   If there exists a move of PDA
   $\delta(q,a,Z)=(q_1,Z_1,Z_2,\ldots Z_n)$ then add
   Production as $[q,Z,q'] \rightarrow a[q_1 Z_1 q_2][q_2 Z_2 q_3]\ldots[q_m Z_m q']$

where $q', q_1, q_2 \ldots q_m$ are states from Q.

→ Construct a CFG from the following PDA

$\delta(q_0, \epsilon, Z_0) = (q_0, \epsilon)$

$\delta(q_0, b, Z) = (q_0, ZZ)$

$\delta(q_0, a, Z) = (q_1, Z)$

$\delta(q_1, b, Z) = (q_1, \epsilon)$

$\delta(q_1, a, Z_0) = (q_0, Z_0)$

$G = (V\ T\ P\ S)$

$V = \{ [q_0 Z_0 q_0]\ [q_0 Z_0 q_1]\ [q_0 Z q_0]\ [q_0 Z q_1]\ [q_1 Z_0 q_0]$
$[q_1 Z_0 q_1]\ [q_1 Z q_0]\ [q_1 Z q_1] \}$

$T = \{a, b\}$

$P = \delta(q_0, \epsilon, Z_0) = (q_0, \epsilon)$

$[q_0, Z_0, q_0] \longrightarrow \epsilon$

$\rightarrow \delta(q_0, b, Z) = (q_0, ZZ)$

$[q_0, Z, q_0] \longrightarrow b[q_0, Z\, q_0]\ [q_0 Z q_0]$

$[q_0, Z, q_0] \rightarrow b[q_0, Z, q_1][q_1 Z q_0]$

$[q_0, Z, q_1] \rightarrow b[q_0, Z, q_0][q_0 Z q_1]$

$[q_0, Z, q_1] \rightarrow b[q_0, Z, q_1][q_1, Z, q_1]$

$\rightarrow \delta(q_0, a, Z) = (q_1, Z)$

$[q_0, Z, q_1] \rightarrow a[q_1, Z, q_1]$

$[q_0, Z, q_0] \rightarrow a[q_1, Z, q_0]$

$\rightarrow \delta(q_0, b, z) = (q_1, \epsilon)$

$[q_0, z, q_1] \rightarrow b$ —— ✓

$\rightarrow \delta(q_1, a, z_0) = (q_0, z_0)$

$[q_1, z_0, q_0] \rightarrow a[q_0, z_0, q_0]$

$[q_1, z_0, q_1] \rightarrow a[q_0, z_0, q_1]$

Ex: $M = (\{q_0, q_1\}, \{0,1\}, \{x, z_0\}, \delta, q_0, z_0, \emptyset\}$

$\delta(q_0, 0, z_0) = \{(q_0, xz_0)\}$

$\delta(q_0, 0, x) = (q_0, xx)$

$\delta(q_0, 1, x) = (q_1, \epsilon)$

$\delta(q_1, 1, x) = (q_1, \epsilon)$

$\delta(q_1, \epsilon, x) = (q_1, \epsilon)$

$\delta(q_1, \epsilon, z_0) = (q_1, \epsilon)$

$S \rightarrow [q_0, z_0, q_0]$, $S \rightarrow [q_0, z_0, q_1]$ — acceptable

$\rightarrow \delta(q_0, 0, z_0) = (q_0, xz_0)$

$[q_0, z_0, q_0] = 0\,[q_0, x, q_0]\,[q_0, z_0, q_0]$ $\times$

$[q_0, z_0, q_0] = 0\,[q_0, x, q_1]\,[q_1, z_0, q_0]$ $\times$

$[q_0, z_0, q_1] = 0\,[q_0, x, q_0]\,[q_0, z_0, q_1]$ $\times$

$[q_0, z_0, q_1] = 0\,[q_0, x, q_1]\,[q_1, z_0, q_1]$ — acceptable

$\rightarrow \delta(q_0, 0, x) = (q_0, xx)$

$[q_0, x, q_0] = 0\,[q_0, x, q_0]\,[q_0, x, q_0]$ $\times$

$[q_0, x, q_0] = 0\,[q_0, x, q_1]\,[q_1, x, q_0]$ $\times$

$[q_0, x, q_1] = 0\,[q_0, x, q_0]\,[q_0, x, q_1]$ $\times$

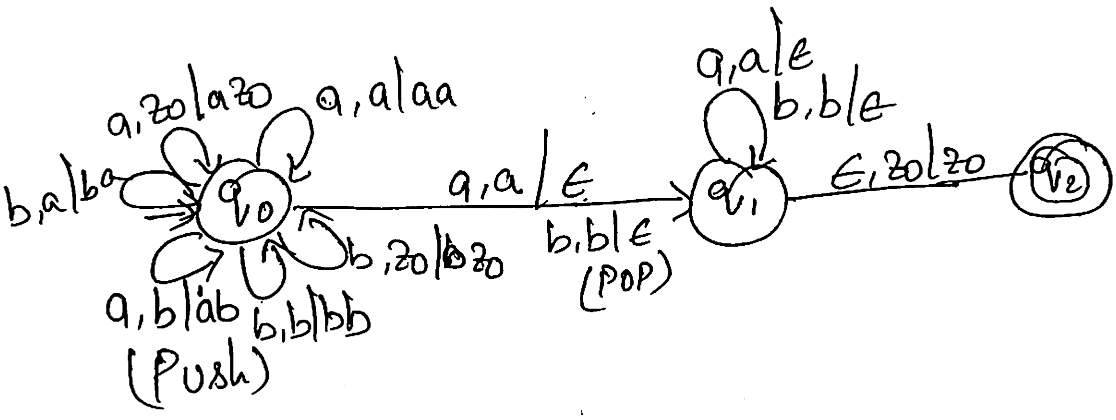
$[q_0, x, q_1] = 0\,\underbrace{[q_0, x, q_1][q_1, x, q_1]}$ — acceptable

both derives terminal symbols

$\rightarrow \delta(q_0, 1, x) = (q_1, \epsilon)$

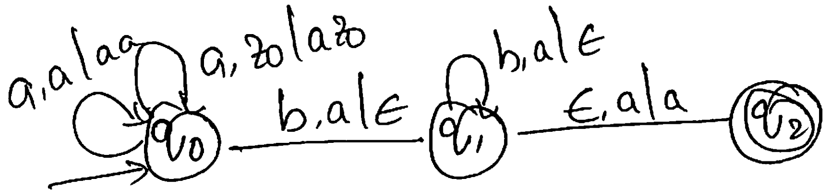
$[q_0, x, q_1] \Rightarrow 1$ — Terminal

Case 1: If top of stack is equal to input symbol it is a chance that it is a center point but not in all cases



4) $L = \{a^m b^n \mid m \geq n,\ m, n > 0\}$

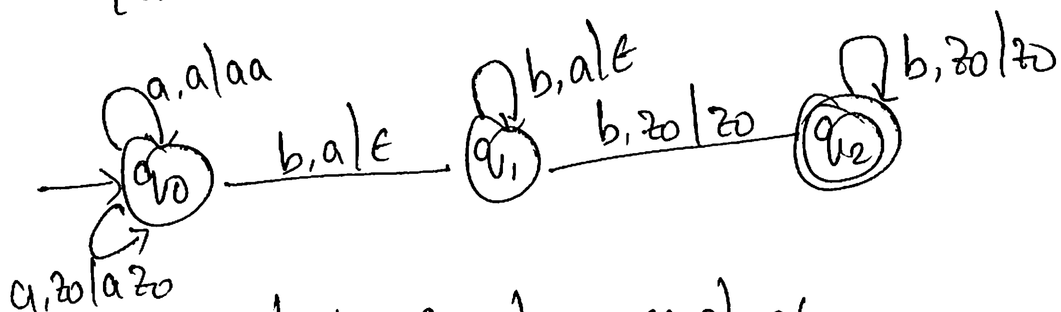
$= \{aab, aaab, aaaabbb, \ldots\}$



It is final state acceptance

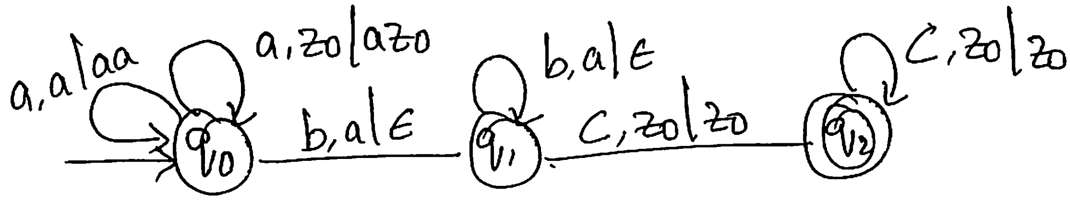
$L = \{a^m b^n \mid m \leq n,\ m, n \geq 0\}$

$\{abb, aabbb, aaabbbb, \ldots\}$



Stack Empty acceptance

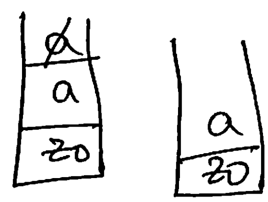
1) $\{a^n b^n c^m \mid n, m \geq 1\}$ DPDA



For dead configuration PDA will halt the transition

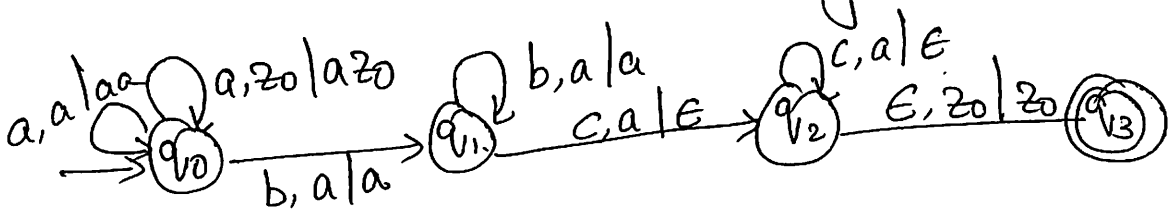
aabcc

PDA halts in q_1



2) $\{a^n b^m c^n \mid n, m \geq 1\}$.

a has to be matched against c & leave b's.



3) $L = \{WW^R \mid W \in (a,b)^*\}$ Palindrome

In this we don't know the center point i.e when to push & when to pop. so to perform PDA guess the center point. These are even length palindrome

$w = aba \quad w^R = aba \Rightarrow abaaba$

$w = bba \quad w^R = abb = bbaabb$