[1] Branch and Bound :

Branch and Bound is a problem solving strategy similar Back tracking. and generate ~~space~~ state space tree for solving this problems.

Branch and Bound is a general method for solving optimisation problems. This method is used for finding minimisation problems but not maximisation problems.

comparison between Back tracking and Branch and Bound :

Back tracking	Branch and Bound
(1). This method is used to provide solutions to decision problems	This technique is used to solve the optimisation problems.
(2). In this technique depth first search method is used.	In this type breadth first search method is used.
(3). This method is used for solving n-queen's problem, graph coloring problems and sum of subsets problems.	This method is used for solving Travelling sales person problem and job scheduling problems
(4). Here possibility to obtain bad solutions	NO bad solutions are obtained
(5). A state space tree is not searched completely and searching terminates as soon as solution is obtained.	A state space tree is searched completely and possibility of obtaining optimum solution.

general Method :

The Brand and Bound method searches a state space tree using search mechanisms like BFS and DFS. In BFS, the state space tree search is called FIFO search, Here Queue is used. In DFS, the state space tree search is called LIFO search Here stack is used.

The 3 types of search strategies in brand and bound are

(1) FIFO search

(2). LIFO search

(3). LC (least cost) search.

Branch & Bound.

State space Tree

Bound Function

Ex: (1). FIFO search

(2). LIFO search

Upper & lower bounds

Ex LC-search.

(1). FIFO search:

Breadth first search with Queue based Branch and Bound is called FIFO Branch and bound.

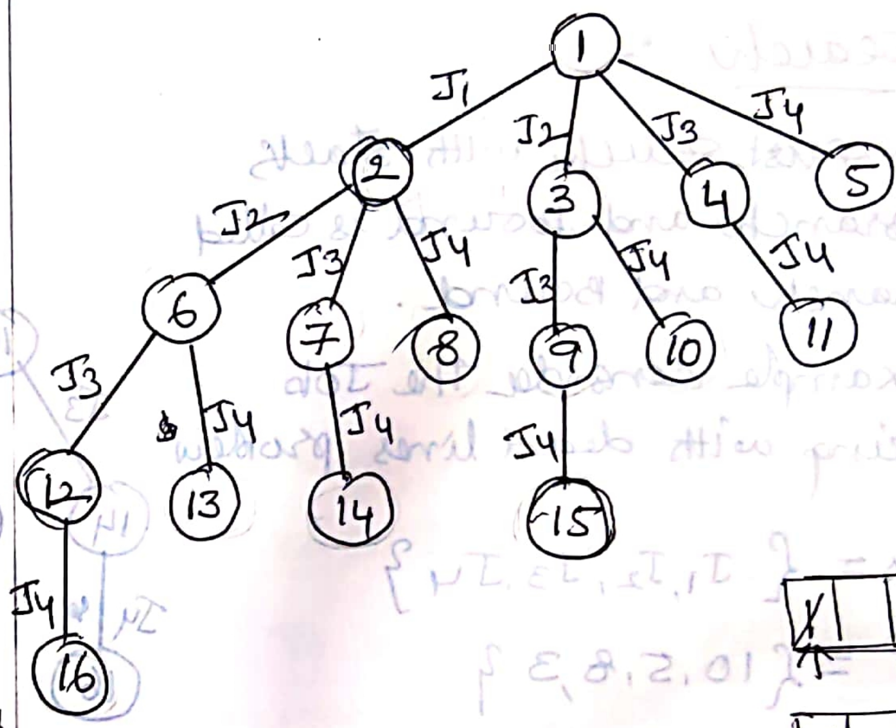
For example consider Job sequencing with dead lines problem

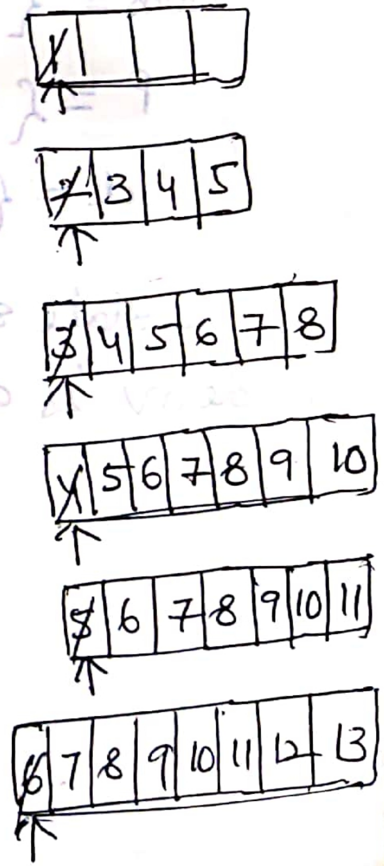
Jobs = $\{J_1, J_2, J_3, J_4\}$

$P = \{10, 5, 8, 3\}$

$d = \{1, 2, 1, 2\}$

The state space tree for the given problem is as follows.





LIFO Search :-

Depth first search with stack based Branch and Bound is called LIFO Branch and Bound.

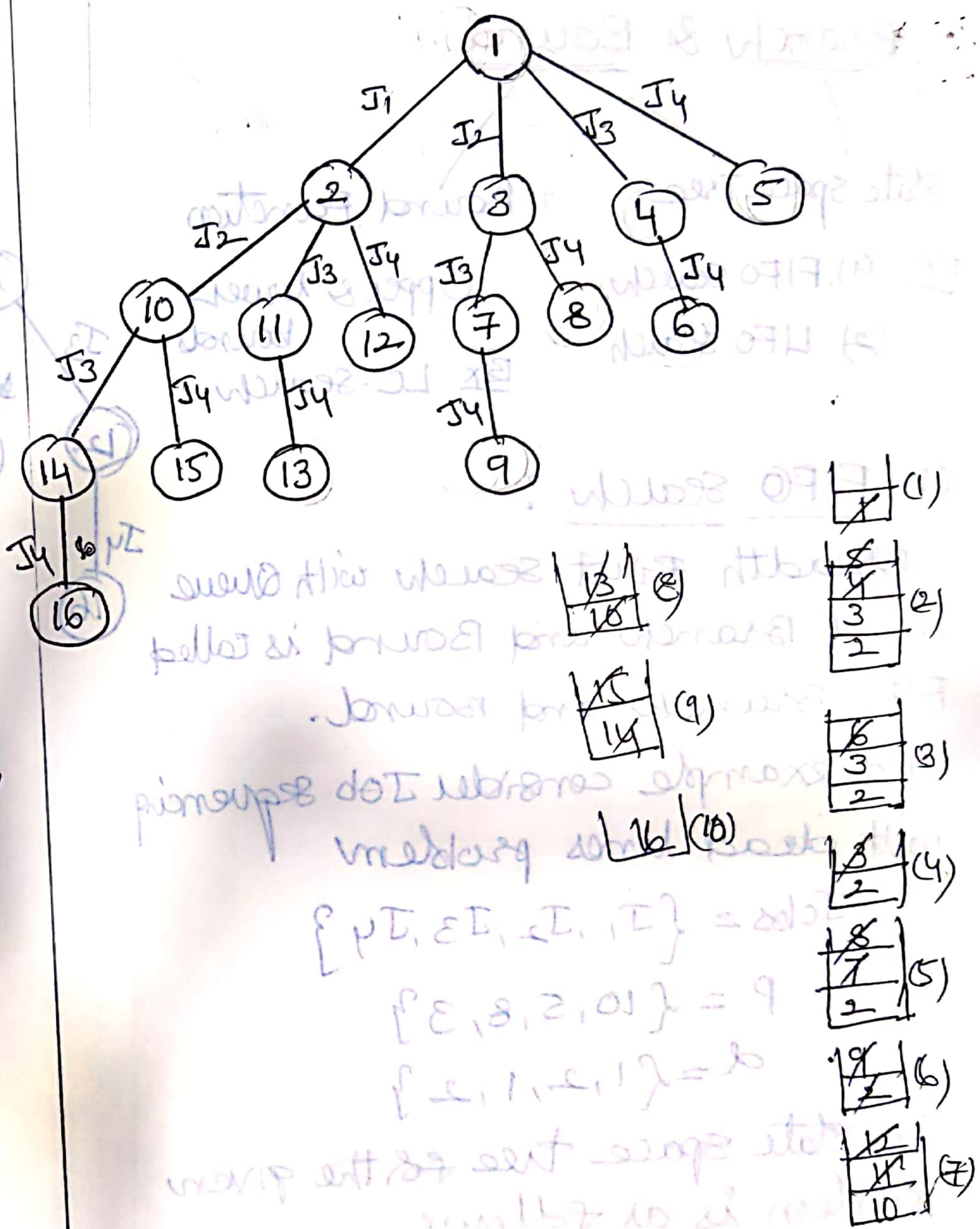
For example consider the Job sequencing with dead lines problem

Jobs = { J_1, J_2, J_3, J_4 }

$P = \{10, 5, 8, 3\}$

$d = \{1, 2, 1, 2\}$

The state space tree for the given problem is as follows.



(3). Least cost (LC) search.

LC search is a kind of search in which least cost is involved for reaching the node. The bounding functions are used to avoid the generations of sub trees. In bounding lower bounds and upper bounds are generated at each node.

0/1 Knapsack problem using LCBB.:

Here Knapsack is a bag. we have 'n' no. of objects, each object having a profit 'pi', weight 'wi' and capacity 'm'. The objective is to place the objects in a Knapsack to get the maximum profit.

we get $\sum p_i x_i$ is max iff $-\sum p_i x_i$ is mini

$\therefore$ Min $-\sum_{i=1}^{n} p_i x_i$ subject to $\sum_{i=1}^{n} w_i x_i$

such That $\sum w_i x_i \le m$ and $x = 0$ & 1

where $1 \le i \le n$.

①

The Knapsack problem is a maximisation problem. But the Branch and Bound problem is a minimisation problem. ~~So to solve the knapsack problem, the process is converted into negatives.~~

To convert the problem into minimisation, take the negative signs for upper bounds and lower bounds.

Example:

$n = 4$ (no of objects)

profits - $(P1, P2, P3, P4) = (10, 10, 12, 18)$

weights - $(w1, w2, w3, w4) = (2, 4, 6, 9)$

capacity $= m = 15$

$n = 4$, $m = 15$

$(P_1, P_2, P_3, P_4) = (10, 10, 12, 18)$

$(w_1, w_2, w_3, w_4) = (2, 4, 6, 9)$

To solve this problem calculate the lower and upper bounds of each node.

In lower bound – fractions are allowed

– upper bound – not allowed.

consider

node = 1

$m = 15$

6	$\frac{3}{9} \times 18$
12	6
10	4
10	2

38 LB

12	6
10	4
10	2

32 . UB

$LB = \hat{c} = 38$ $\quad UB = \hat{u} = 32$

node = 2

$x_1 = 1$

6	$\frac{3}{9} \times 18$
12	6
10	4
10	2

38 LB

12	6
10	4
10	2

32 UB

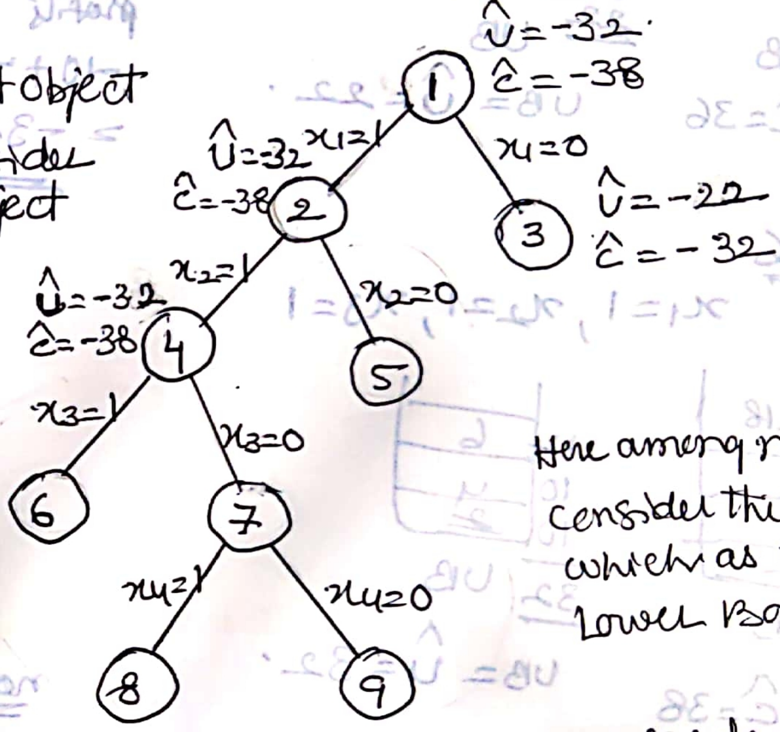
$LB = \hat{c} = 38$ $\quad UB = \hat{u} = 32$

Example: Draw the portion of the state space tree generated by LCBB for the following 0/1 knapsack problem $n = 4$, $(P_1, P_2, P_3, P_4) = (10, 10, 12, 18)$, $(w_1, w_2, w_3, w_4) = (2, 4, 6, 9)$, and $m = 15$.

Here

$x_1 = 1$ – consider 1st object

$x_1 = 0$ – not consider the 1st object



Here among nodes 2 & 3 consider the node which as the mini lower Bound.

node = 3

$x_1 = 0$ Here 1st object is not considered.

10	$\frac{5}{9} \times 18$
12	6
10	4

32 LB

12	6
10	4

22 UB

$LB = \hat{c} = 32$ $\quad UB = \hat{u} = 22$

node = 4

$x_1 = 1$, $x_2 = 1$

6	$\frac{3}{9} \times 18$
12	6
10	4
10	2

38 LB

12	6
10	4
10	2

32 UB

$LB = \hat{c} = 38$ $\quad UB = \hat{u} = 32$

node = 5

$x_1 = 1$, $x_2 = 0$

	LB
14	$\frac{7}{9} \times 18$
12	6
10	2

m=15

36 LB

	UB
12	16
10	2

m=11 - 15

22 UB

$LB = \hat{c} = 36$

$UB = \hat{U} = 22$.

node = 6

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$

	LB
6	$\frac{3}{9} \times 18$
12	6
10	4
10	2

38 LB

	UB
12	6
10	4
10	2

32 UB

$LB = \hat{c} = 38$

$UB = \hat{U} = 32$.

node = 7

$x_1 = 1$, $x_2 = 1$, $x_3 = 0$

	LB
18	9
10	4
10	2

38 LB

	UB
18	9
10	4
10	2

38 UB

$LB = \hat{c} = 38$

$UB = \hat{U} = 38$

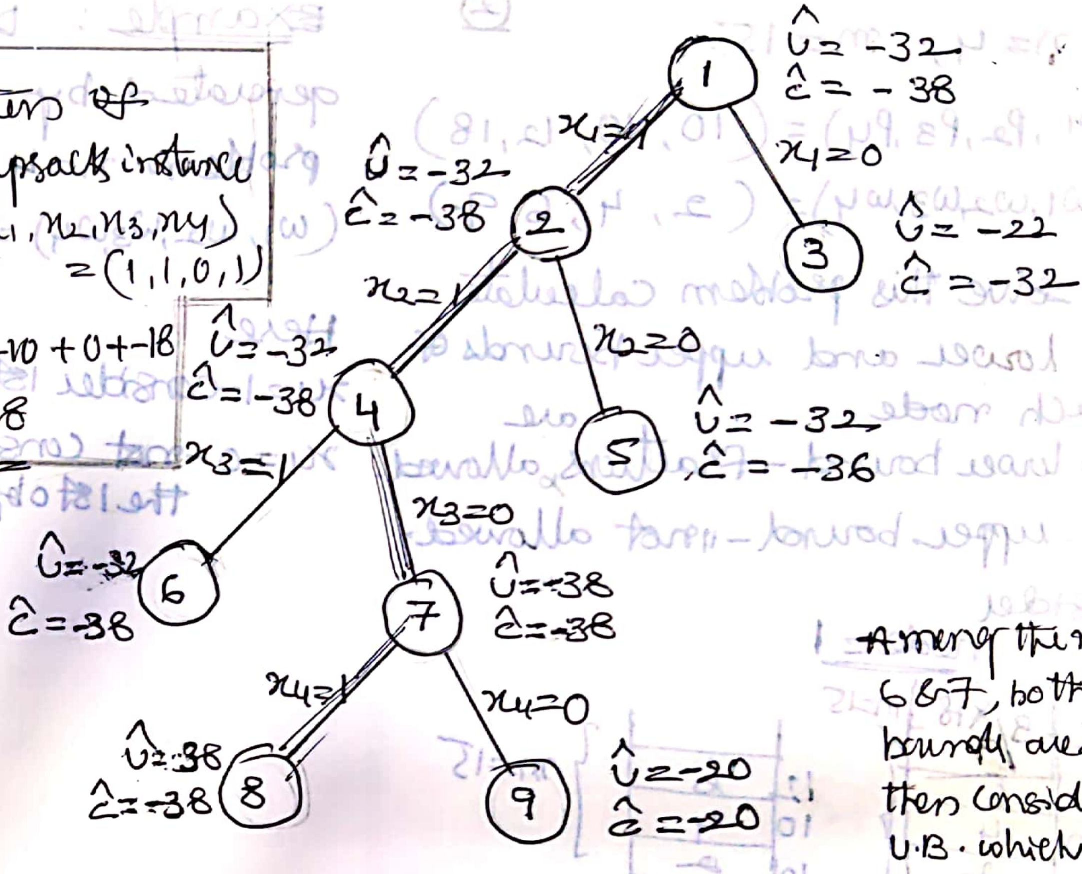
③

solutions of ∴ Knapsack instance is $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

profits

$-10 + -10 + 0 + -18$

$= -38$



Among the nodes 6 & 7, both lower bounds are same then consider the U.B. which node has mini U.B

node = 8

$x_1 = 1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

	LB
18	9
10	4
10	2

38 LB

	UB
18	9
10	4
10	2

38 UB

$LB = \hat{c} = 38$

$UB = \hat{U} = 38$

node = 9

$x_1 = 1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$

	LB
10	4
10	2

20 LB

	UB
10	4
10	2

20 UB

$LB = \hat{c} = 20$

$UB = \hat{U} = 20$

Example 2 :

Draw the portion of the state space tree generated by LCBB for the following ~~0/1 Kn~~ 0/1 Knapsack problem $n = 5$

$(P_1, P_2, P_3, P_4, P_5) = (10, 15, 6, 8, 4)$

$(w_1, w_2, w_3, w_4, w_5) = (4, 6, 3, 4, 2)$

and $m = 12$.

node = 1

4	$\frac{2}{3} \times 6$
15	6
10	4
29	LB

15	6
10	4
25	UB

$\therefore LB = \hat{c} = 29$ $\quad \therefore UB = \hat{u} = 25$

node = 2 ⇒ $x_1 = 1$

4	$\frac{2}{3} \times 6$
15	6
10	4
29	LB

15	6
10	4
25	UB

$\therefore LB = \hat{c} = 29$ $\quad \therefore UB = \hat{u} = 25$

node = 3

$x_1 = 0$

6	$\frac{3}{4} \times 8$
6	3
15	6
27	LB

6	3
15	6
21	UB

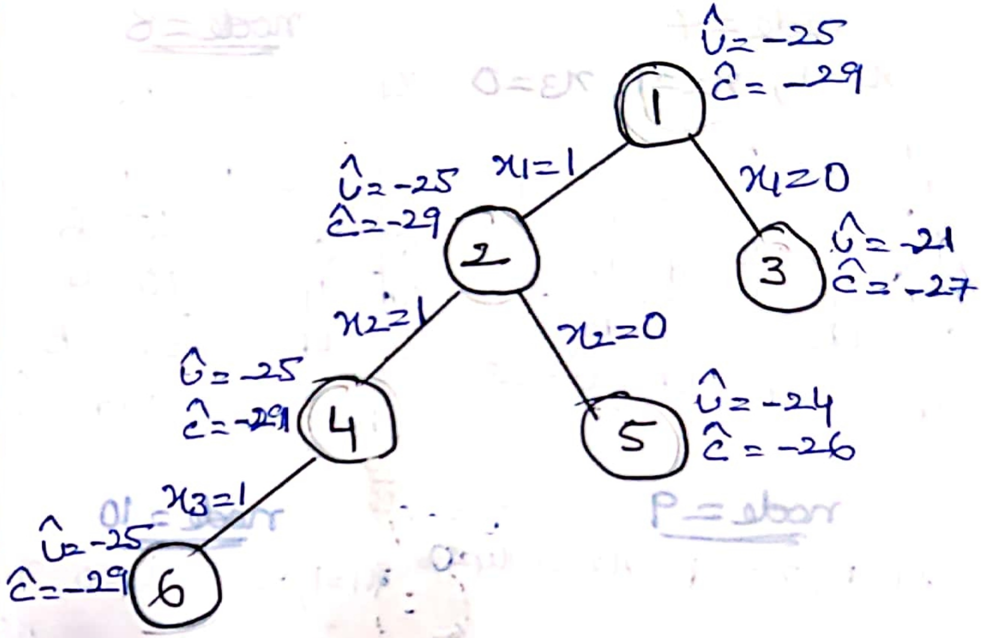
$\therefore LB = \hat{c} = 27$ $\quad \therefore UB = \hat{u} = 21$

node = 4 $\quad x_1 = 1, x_2 = 1$

4	$\frac{2}{3} \times 6$
15	6
10	4
29	LB

15	6
10	4
25	UB

$\therefore LB = \hat{c} = 29$ $\quad UB = \hat{u} = 25$



node = 5

$x_1 = 1, x_2 = 0$

2	$\frac{1}{2} \times 4$
8	4
6	3
10	4
26	LB

8	4
6	3
10	4
24	UB

$\therefore LB = \hat{c} = 26$ $\quad \therefore UB = \hat{u} = 24$

node = 6

$x_1 = 1, x_2 = 1, x_3 = 1$

4	$\frac{2}{3} \times 6$
15	6
10	4
29	LB

15	6
10	4
25	UB

$\therefore LB = \hat{c} = 29$ $\quad \therefore UB = \hat{u} = 25$

node = 7

$x_1 = 1,\ x_2 = 1,\ x_3 = 0$

4 | $\frac{2}{4} \times 8$
15 | 6
10 | 4

29 LB

15 | 6
10 | 4

25 UB

$\therefore LB = \hat{c} = 29$ $\therefore UB = \hat{u} = 25$

node = 8

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$

4 | $\frac{2}{4} \times 8$
15 | 6
10 | 4

29 LB

15 | 6
10 | 4

25 UB

$\therefore LB = \hat{c} = 29$ $\therefore UB = \hat{u} = 25$

node = 9

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0$

4 | 2
15 | 6
10 | 4

29 LB

4 | 2
15 | 6
10 | 4

29 UB

$\therefore LB = \hat{c} = 29$ $\therefore UB = \hat{u} = 29$

node = 10

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0$
$x_5 = 1$

4 | 2
15 | 6
10 | 4

29 LB

4 | 2
15 | 6
10 | 4

29 UB

$\therefore LB = \hat{c} = 29$ $UB = \hat{c} = 29$

node = 11

$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0$

15 | 6
10 | 4

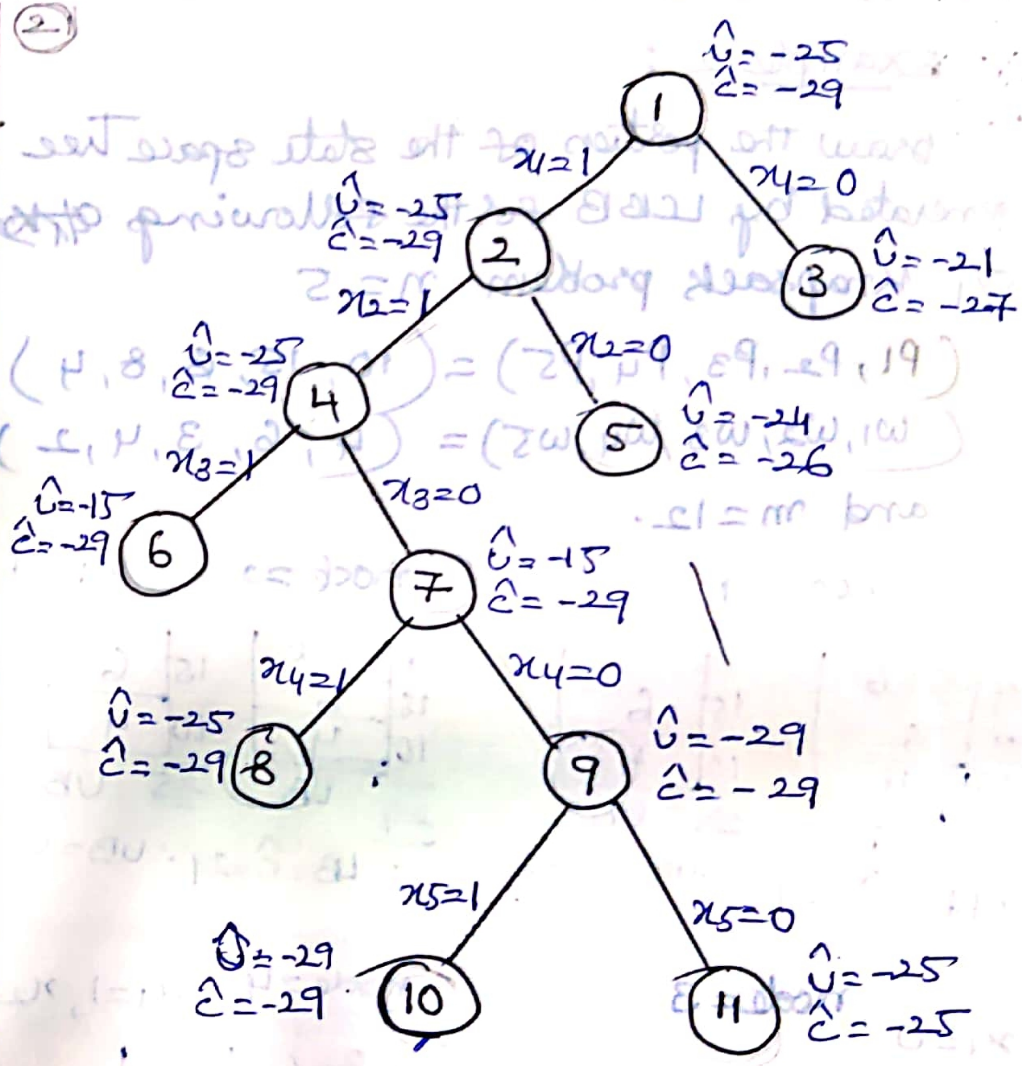
25 LB

15 | 6
10 | 4

25 UB

$\therefore LB = \hat{c} = 25$ $UB = \hat{u} = 25$

(2)



Solution of knapsack instance is

$(x_1, x_2, x_3, x_4, x_5) = (1, 1, 0, 0, 1)$.

$\therefore$ profit $= -10 + -15 + 0 + 0 + -4$

$= -29$

Travelling sales person problem using LCBB

In travelling sales person problem starting at first vertex and visit all the vertices of the graph exactly once. Finally reach the first vertex with minimum cost.

Example: solve the following instance of travelling sales person problem using LCBB.

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

To solve this problem first convert the matrix into Reduced matrix. A R.M is one in which the matrix contains atleast one zero in each row and column.

Row reduction is performed to make atleast one "zero" in each row.

$$\begin{array}{c|c} \begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix} & \begin{matrix} 10 \\ 2 \\ 2 \\ 3 \\ 4 \end{matrix} \\ \hline & \underline{\underline{21}} \end{array}$$

∴ Row reduction is

$$\begin{bmatrix} \infty & 10 & 20 & 0 & 1 \\ 13 & \infty & 14 & 2 & 0 \\ 1 & 3 & \infty & 0 & 2 \\ 16 & 3 & 15 & \infty & 0 \\ 12 & 0 & 3 & 12 & \infty \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 0 & 3 & 0 & 0 \end{matrix} \rightarrow \underline{\underline{4}}$$

∴ column reduction is

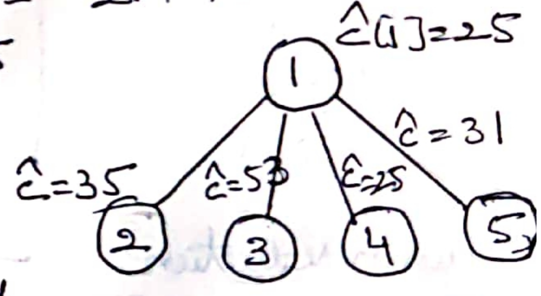
Reduction matrix I.A =

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

∴ Total reduction cost } = row reduction cost + column red cost

= 21 + 4 = 25

∴ $\hat{c} = 25$



Now calculate path (i,j) by replacing row i and column j to ∞ and position (j,1) to ∞.

consider path(1,2)

1st row → ∞

2nd column → ∞

(2,1) → ∞

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \quad \underline{0}$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 \end{array} = 0$$

$\therefore$ reduction cost $\} = r = 0$

$\hat{c}[2] = \hat{c}[1] + A(1,2) + r$

$= 25 + 10 + 0 = 35$

$\hat{c}[2] = 35$

path (1,3)

1st row → ∞

3rd col → ∞

(3,1) → ∞

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 15 & 3 & \infty & \infty & 0 \\ 11 & 0 & \infty & 12 & \infty \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \quad \underline{0}$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 11 & 0 & 0 & 0 & 0 \end{array} = 11$$

column reduction:

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{array}\right]$$

$\therefore r = 0 + 11 = 11$

$\therefore \hat{c}[3] = \hat{c}[1] + A(1,3) + r$

$= 25 + 17 + 11 = 53$

$\hat{c}[3] = 53$

path (1,4)

1st row - ∞

4th col → ∞

(4,1) → ∞

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{array}\right] = 0$$

$$\underbrace{\qquad\qquad\qquad}_{0}$$

$r = 0 + 0 = 0$ $\quad \therefore r = 0$

$\therefore \hat{c}[4] = \hat{c}[1] + A(1,4) + r$

$= 25 + 0 + 0 = 25$

$\hat{c}[4] = 25$

path(1,5)

1st row → ∞

5th col → ∞

(5,1) → ∞

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & 2 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 15 & 3 & 12 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{array}\right] \begin{array}{c} \\ 2 \\ \\ 3 \\ \\ \end{array} \quad \underline{5}$$

Row reduction : 4

$$\left[\begin{array}{ccccc} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{array}\right]$$

$\therefore r = 5 + 0 = 5$

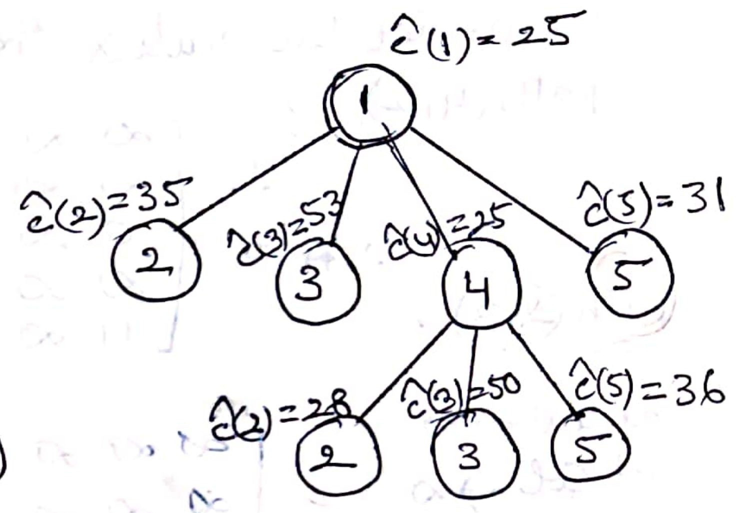
$\hat{c}[5] = \hat{c}[1] + A(1,5) + r$

$= 25 + 1 + 5$

$\hat{c}[5] = 31$

Now consider the matrix that occurs at path (1,4).

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$



path (4,2) :

4th row $\to \infty$
2nd col $\to \infty$
$(2,1) \to \infty$

$$A = \begin{array}{c} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 \end{matrix} = 0 \end{array}$$

$\underline{0}$

$$\hat{c}(2) = \hat{c}(4) + A(4,2) + r$$
$$= 25 + 3 + 0$$
$$= 28$$

path (4,3) :

4th row $\to \infty$
3rd col $\to \infty$
$(3,1) \to \infty$

$$A = \begin{array}{c} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & \infty & 0 \\ \infty & 3 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & 0 & \infty & \infty & \infty \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 11 & 0 & 0 & 0 & 0 \end{matrix} \end{array}$$

$\underline{2}$

$\underline{11} \leftarrow 11$

$r = 2 + 11 = 13$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$

$$\hat{c}(3) = \hat{c}(4) + A(4,3) + r$$
$$= 25 + 12 + 13$$
$$= 50$$

path (4,5)

4th row $\to \infty$
5th col $\to \infty$
$(5,1) \to \infty$

$$A = \begin{array}{c} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix} \begin{matrix} 0 \\ \to 11 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \end{array} = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

$\underline{11}$

$$\therefore r = 11 + 0 = 11$$

$$\hat{c}(5) = \hat{c}(4) + A(4,5) + r$$
$$= 25 + 0 + 11$$
$$= 36$$

Now consider the matrix that occurs at path (4,2)

$$\therefore A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

Path (2,3):

2nd row → ∞
3rd col → ∞
(3,1) → ∞

$$A = \begin{array}{c} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & \infty & \infty & \infty \end{bmatrix} \begin{array}{c} \\ \\ 2 \\ \\ 11 \end{array} \\ \begin{array}{ccccc} 11 & 0 & 0 & 0 & 2 \end{array} \end{array}$$

$r = 2 + 11 = 13$

$\hat{c}(3) = \hat{c}(2) + A(2,3) + r$
$= 28 + 11 + 13 = 52$

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix}$$

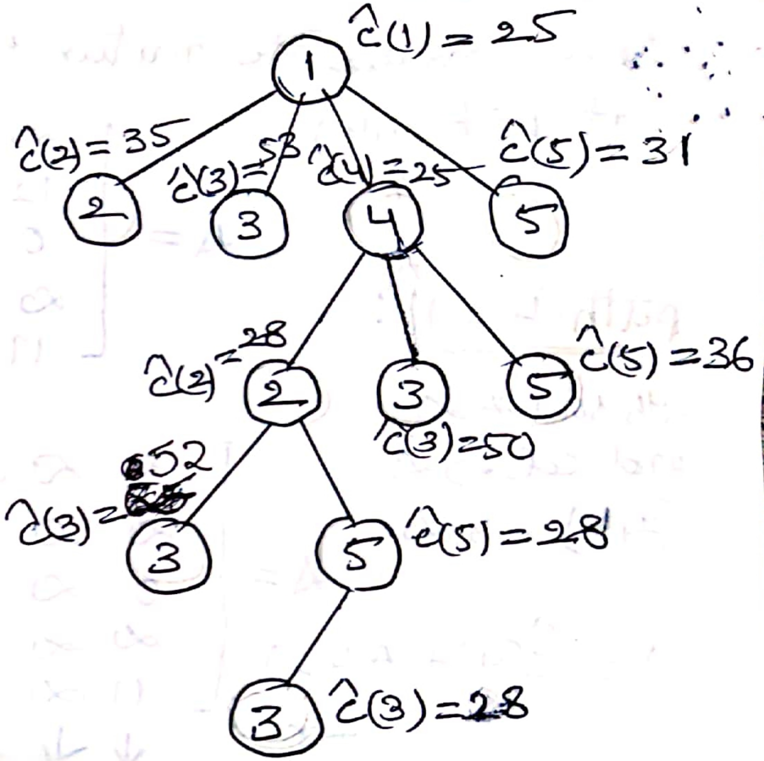
Path (2,5)

2nd row → ∞
5th col → ∞
(5,1) → ∞

$$A = \begin{array}{c} \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \end{array}$$

$\therefore r = 0$

$\hat{c}(5) = \hat{c}(2) + A(2,5) + r$
$= 28 + 0 + 0$
$= 28$



Now consider the path (5,3)

5th row → ∞
3rd col → ∞
(3,1) → ∞

$$\therefore A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

$r = 0$

$\hat{c}(3) = \hat{c}(5) + A(5,3) + r$
$= 28 + 0 + 0$
$= 28$

Final solution path ~~of~~ the matrix is

1 – 4 – 2 – 5 – 3 – 1

Travelling sales person problem using LCBB:-

Example 2:

solve the following instance of travelling sales person problem using LCBB.

$$\text{row reductions}\left\{\begin{bmatrix} \infty & 7 & 3 & 12 & 8 \\ 3 & \infty & 6 & 14 & 9 \\ 5 & 8 & \infty & 6 & 18 \\ 9 & 3 & 5 & \infty & 11 \\ 18 & 14 & 9 & 8 & \infty \end{bmatrix}\right.$$

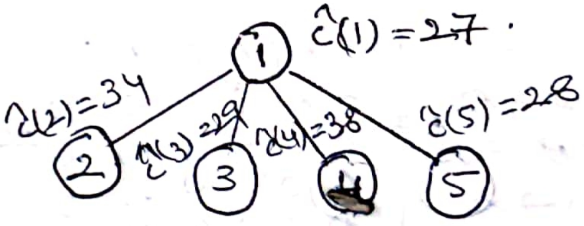
$$\text{row reduction} = \begin{array}{c} \begin{bmatrix} \infty & 7 & 3 & 12 & 8 \\ 3 & \infty & 6 & 14 & 9 \\ 5 & 8 & \infty & 6 & 18 \\ 9 & 3 & 5 & \infty & 11 \\ 18 & 14 & 9 & 8 & \infty \end{bmatrix} \end{array} \begin{array}{l} \to 3 \\ \to 3 \\ \to 5 \\ \to 3 \\ \to 8 \\ \hline 22 \end{array}$$

$$\text{column reduction} = \begin{bmatrix} \infty & 4 & 0 & 9 & 5 \\ 0 & \infty & 3 & 11 & 6 \\ 0 & 3 & \infty & 1 & 13 \\ 6 & 0 & 2 & \infty & 8 \\ 10 & 6 & 1 & 0 & \infty \end{bmatrix}$$

$$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$$
$$0 \quad 0 \quad 0 \quad 0 \quad 5 = 5$$

$$\text{Reduction matrix } A = \begin{bmatrix} \infty & 4 & 0 & 9 & 0 \\ 0 & \infty & 3 & 11 & 1 \\ 0 & 3 & \infty & 1 & 8 \\ 6 & 0 & 2 & \infty & 3 \\ 10 & 6 & 1 & 0 & \infty \end{bmatrix}$$

$\therefore r = 22+5$
$= 27$



path(1,2)

1st row → ∞
2nd col → ∞
(2,1) → ∞

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & 11 & 1 \\ 0 & \infty & \infty & 1 & 8 \\ 6 & \infty & 2 & \infty & 3 \\ 10 & \infty & 1 & 0 & \infty \end{bmatrix} \begin{array}{l} \\ \to 1 \\ \\ \to 2 \\ \\ \hline 3 \end{array} = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 2 & 10 & 0 \\ 0 & \infty & \infty & 1 & 8 \\ 4 & \infty & 0 & \infty & 1 \\ 10 & \infty & 1 & 0 & \infty \end{bmatrix} = 0$$

$\therefore r = 3+0 = 3$

$\hat{c}(2) = \hat{c}(1) + A(1,2) + r$
$= 27+4+3 = 34$

path(1,3)

1st row → ∞
3rd col → ∞
(3,1) → ∞

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 1 \\ \infty & 3 & \infty & 1 & 8 \\ 6 & 0 & \infty & \infty & 3 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix} \begin{array}{l} \\ \\ \to 1 \\ \\ \\ \hline 1 \end{array} = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 1 \\ \infty & 2 & \infty & 0 & 7 \\ 6 & 0 & \infty & \infty & 2 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 0 \\ \infty & 2 & \infty & 0 & 6 \\ 6 & 0 & \infty & \infty & 2 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix}$$

$r = 1+1 = 2$

$\hat{c}(3) = \hat{c}(1) + A(1,3) + r$
$= 27+0+2 = 29$

path(1,4)

1st row → ∞
4th col → ∞
(4,1) → ∞

$$A = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 3 & \infty & 1 \\ 0 & 3 & \infty & \infty & 8 \\ \infty & 0 & 2 & \infty & 3 \\ 10 & 6 & 1 & \infty & \infty \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \to 1 \\ \hline 1 \end{array} = \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 3 & \infty & 1 \\ 0 & 3 & \infty & \infty & 8 \\ \infty & 0 & 2 & \infty & 3 \\ 9 & 5 & 0 & \infty & \infty \end{bmatrix}$$
$$\downarrow 1$$

$r = 1+1 = 2$

$\hat{c}(4) = \hat{c}(1) + A(1,4) + r$
$= 27+9+2 = 38$

path (1,5)

1st row → ∞
5th col → ∞
(5,1) → ∞

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&3&11&\infty\\0&3&\infty&1&\infty\\6&0&2&\infty&\infty\\\infty&6&1&0&\infty\end{bmatrix}\begin{matrix}0\\0\\0\\0\\0\end{matrix}=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&2&11&\infty\\0&3&\infty&1&\infty\\6&0&1&\infty&\infty\\\infty&6&0&0&\infty\end{bmatrix}$$

$$\begin{matrix}0&0&1&0&0\end{matrix}=1 \qquad \underline{0}$$

$r=0+1=1$

$$\hat{c}(5)=\hat{c}(1)+A(1,5)+r$$
$$=27+0+1=28$$

consider path (1,5) matrix as

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&2&11&\infty\\0&3&\infty&1&\infty\\6&0&1&\infty&\infty\\\infty&6&0&0&\infty\end{bmatrix}$$

path (5,2):

5th row → ∞
2nd col → ∞
(2,1) → ∞

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&2&11&\infty\\0&\infty&\infty&1&\infty\\6&\infty&1&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}\begin{matrix}\\\to 2\\\\\to 1\\\\\end{matrix}=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&0&9&\infty\\0&\infty&\infty&1&\infty\\5&\infty&0&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

$\underline{3}$ $\qquad$ 1

$r=3+1=4$

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&0&8&\infty\\0&\infty&\infty&0&\infty\\5&8&0&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

$$\hat{c}(2)=\hat{c}(5)+A(5,2)+r$$
$$=28+6+4=\underline{38}$$

path (5,3):

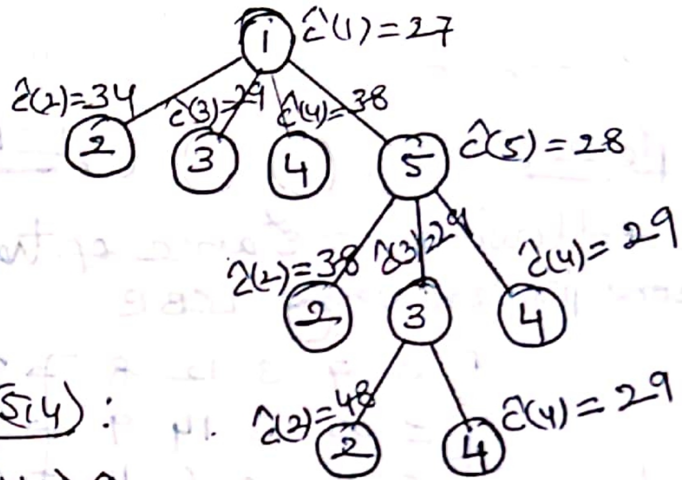
5th row → ∞
3rd col → ∞
(3,1) → ∞

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&\infty&11&\infty\\\infty&3&\infty&1&\infty\\6&0&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}\begin{matrix}\\\\\to 1\\\\\\\end{matrix}=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&\infty&11&\infty\\\infty&2&\infty&0&\infty\\6&0&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

$\underline{1}$

$r=1+0=1$

$$\hat{c}(3)=\hat{c}(5)+A(5,3)+r$$
$$=28+0+1=\cancel{30}=29$$



path (5,4):

5th row → ∞
4th col → ∞
(4,1) → ∞

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&2&\infty&\infty\\0&3&\infty&\infty&\infty\\\infty&0&1&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&1&\infty&\infty\\0&3&\infty&\infty&\infty\\\infty&0&0&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

1

$$\hat{c}(4)=\hat{c}(5)+A(5,4)+r \qquad r=0+1=1$$
$$=28+0+1=29$$

consider the path matrix (5,3) as

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\0&\infty&\infty&10&\infty\\\infty&2&\infty&0&\infty\\6&0&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

path (3,2)

3rd row → ∞
2nd col → ∞
(2,1) → ∞

$$\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&11&\infty\\\infty&\infty&\infty&\infty&\infty\\6&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}\begin{matrix}\\\to 11\\\\\to 6\\\\\end{matrix}=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&0&\infty\\\infty&\infty&\infty&\infty&\infty\\0&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

$\underline{17}$

$$\hat{c}(2)=\hat{c}(3)+A(3,2)+r$$
$$=29+2+17=48$$

path (3,4)

3rd row → ∞
4th col → ∞
(4,1) → ∞

$$A=\begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\\\infty&0&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\end{bmatrix}$$

$r=0+0=0$

$$\hat{c}(4)=\hat{c}(3)+A(3,4)+r$$
$$=29+0+0$$
$$=29$$