# UNIT-II

## Probability Distribution

### PART-B

1. Out of 20 tape recorders 5 are defective. Find the standard deviation of defective in the sample of 10 randomly chosen tape recorders. Find (i) $P(X=0)$ (ii) $P(X=1)$ (iii) $P(X=2)$ (iv) $P(1<X<4)$

Sol:- Given $n=10$, $P=\frac{5}{20}=\frac{1}{4}$, $q=1-P=1-\frac{1}{4}=\frac{3}{4}$

By Binomial distribution
$P(X=x)=n_{C_x}p^x q^{n-x}$

$$\text{Standard deviation } (\sigma) = \sqrt{npq}$$
$$= \sqrt{10\times\frac{1}{4}\times\frac{3}{4}}$$
$$= 1.369$$

$$P(X=0) = 10_{C_0}\left(\tfrac{1}{4}\right)^0\left(\tfrac{3}{4}\right)^{10} = 0.0563$$

$$P(X=1) = 10_{C_1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^{9} = 0.1877$$

2. A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand (ii) on which demand is refused.

Sol:- Given $\lambda = np = 1.5$

(i) $P(x=0) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.5}\times(1.5)^0}{0!} = 0.2231$

(ii) $P(x>2) = 1-\{P(x=0)+P(x=1)+P(x=2)\}$

$$= 1-\left\{\frac{e^{-1.5}(1.5)^0}{0!}+\frac{e^{-1.5}(1.5)^1}{1!}+\frac{e^{-1.5}(1.5)^2}{2!}\right\}$$

$$= 0.191$$

3. The average number of phone calls per minute coming into a switch board between 2 PM & 4 PM is 2.5. Determine the probability that during one particular time (i) 4 or fewer calls

ii, more than 6 calls.

Sol: Given $\lambda = 2.5 = np$

i, $P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{e^{-2.5}(2.5)^0}{0!} + \frac{e^{-2.5}(2.5)^1}{1!} + \frac{e^{-2.5}(2.5)^2}{2!} + \frac{e^{-2.5}(2.5)^3}{3!} + \frac{e^{-2.5}(2.5)^4}{4!}$$

$= 0.891$

ii, $P(X>6) = 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)\}$

$$= 1 - \left\{ \frac{e^{-2.5}(2.5)^0}{0!} + \frac{e^{-2.5}(2.5)^1}{1!} + \frac{e^{-2.5}(2.5)^2}{2!} + \frac{e^{-2.5}(2.5)^3}{3!} + \frac{e^{-2.5}(2.5)^4}{4!} + \frac{e^{-2.5}(2.5)^5}{5!} + \frac{e^{-2.5}(2.5)^6}{6!} \right\}$$

$= 0.014$

4. In 1000 sets of trials per an event of small probability the frequencies f of the number of x of successes are

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | total |
|---|---|---|---|---|---|---|---|---|---|
| f | 305 | 365 | 210 | 80 | 28 | 9 | 2 | 1 | 1000 |

Fit the expected frequencies.

Sol: Given $N = 1000 = \Sigma f$

| x | f | fx |
|---|---|---|
| 0 | 305 | 0 |
| 1 | 365 | 365 |
| 2 | 210 | 420 |
| 3 | 80 | 240 |
| 4 | 28 | 112 |
| 5 | 9 | 45 |
| 6 | 2 | 12 |
| 7 | 1 | 7 |
| | $\Sigma f = 1000$ | $\Sigma fx = 1201$ |

Mean $(\mu) = \frac{\Sigma fx}{\Sigma f} = 1.201 = \lambda$

$\therefore \lambda = 1.2$

By Poisson distribution

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Expected frequencies $= N \cdot P(X=x)$

i, Expected frequencies $= N \cdot P(X=0) = 1000 \cdot \frac{e^{-1.201}(1.201)^0}{0!} = 300.89$

ii, $E.F = NP(X=1) = 1000 \cdot \frac{e^{-1.201}(1.201)^1}{1!} = 361.37$

iii, $E.F = N.P(X=2) = 1000 \cdot \frac{e^{-1.201}(1.201)^2}{2!} = 217$

iv, $E.F = N \cdot P(X=3) = 1000 \cdot \frac{e^{-1.201}(1.201)^3}{3!} = 86.87$

v, $E.F = N.P(X=4) = 1000 \cdot \frac{e^{-1.201}(1.201)^4}{4!} = 26.08$

vi, $E.F = N \cdot P(X=5) = 1000 \cdot \frac{e^{-1.201}(1.201)^5}{5!} = 6.26$

vii, $E.F = N \cdot P(X=6) = 1000 \cdot \frac{e^{-1.201}(1.201)^6}{6!} = 1.25$

viii, $E.F = N \cdot P(X=7) = 1000 \cdot \frac{e^{-1.201}(1.201)^7}{7!} = 0.21$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $f$ | 305 | 365 | 210 | 80 | 28 | 9 | 2 | 1 |
| | | | | | | 6.26 | 1.25 | 0.21 |
| E.F | 300.89 | 361.37 | 217 | 86.87 | 26.08 | | | |

5. For a normally distributed variate with mean 1 and S.D 3.
Find i, $P(3.43 \le X \le 6.19)$ ii, $P(-1.43 \le X \le 6.19)$

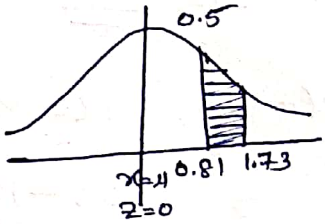
Sol:- Given Mean $(\mu) = 1$, $S.D(\sigma) = 3$.

i, consider $P(3.43 \le X \le 6.19) = P(x_1 \le x \le x_2)$

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{3.43-1}{3} = 0.81$$

$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{6.19-1}{3} = 1.73$$

Now $P(z_1 \le z \le z_2)$
$= P(0.81 \le z \le 1.73)$




$= P(0 \le z \le 1.73) - P(0 \le z \le 0.81)$

$= 0.4582 - 0.2910$

$= 0.1672$

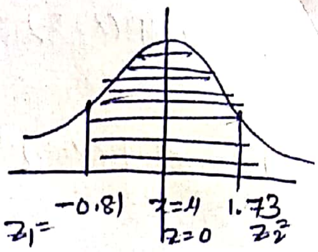
ii, consider $P(-1.43 \le X \le 6.19) = P(x_1 \le x \le x_2)$

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{-1.43-1}{3} = -0.81$$

$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{6.19-1}{3} = 1.73$$

Now $P(z_1 \le z \le z_2) = P(-0.81 \le z \le 1.73)$

$= P(-0.81 \le z \le 0) + P(0 \le z \le 1.73)$




$= P(0 \le z \le 0.81) + P(0 \le z \le 1.73)$

$= 0.2910 + 0.4582$

$= 0.7492$

6. If $x$ is a normal variate with mean 30 and S.D 5. Find

i, $P(26 \le x \le 40)$ ii, $P(x \ge 45)$.

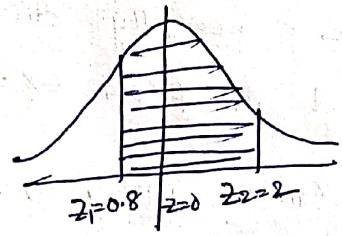
Sol:- Given $\mu = 30$, $\sigma = 5$

i, $P(26 \le x \le 40) = P(x_1 \le x \le x_2)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{26-30}{5} = -0.8$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40-30}{5} = 2$$

Now $P(z_1 \le z \le z_2)$




$= P(-0.8 \le z \le 2)$

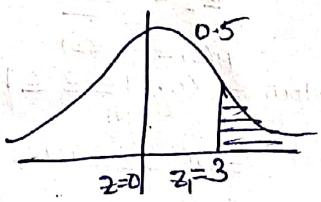
$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$

$= 0.2881 + 0.4772$

$= 0.7653$

ii, $P(x \ge 45) = P(x \ge x_1)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{45-30}{5} = 3$$




Now $P(z \ge z_1) = P(z \ge 3)$

$= 0.5 - P(0 \le z \le 3)$

$= 0.5 - 0.4987$

$= 0.0013$

7. 4 coins are tossed 160 times. Fit the Binomial distribution of getting number of heads.

Sol:- Given $n = 4$, $N = 160$, $p = \frac{1}{2}$, $q = \frac{1}{2}$

$x$ = Getting no. of heads = $\{0, 1, 2, 3, 4\}$

By Binomial distribution $P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

i, Expected frequencies of getting zero head $= N \cdot P(X=0)$

$= 160 \times {}^{4}C_{0} (1/2)^{0} (1/2)^{4-0}$

$= 10$

ii, E.F of getting '1' head $= N \cdot P(X=1)$

$= 160 \times {}^{4}C_{1} (1/2)^{1} (1/2)^{4-1}$

$= 40$

ii, E.F of getting "2" heads $= N \cdot P(X=2)$

$= 160 \times {}^{4}C_{2} (1/2)^{2} (1/2)^{4-2}$

$= 60$

iii, E.F of getting '3' heads $= N \cdot P(X=3)$

$= 160 \times {}^{4}C_{3} (1/2)^{3} (1/2)^{4-3}$

$= 40$

iv E.F of getting '4' heads $= N \cdot P(X=4)$

$= 160 \times {}^{4}C_{4} (1/2)^{4} (1/2)^{4-4}$

$= 10$

| $x$ | 0 | 1 | 2 | 3 | 4 | total |
|---|---|---|---|---|---|---|
| E.F | 10 | 40 | 60 | 40 | 10 | 160 |

8. The mean weight of 500 male students at a certain college is 75kg and the S.D is 7 kg assuming that the weights are normally distributed. Find how many students weight

i, Between 60 and 78 kgs ii, more than 92 kg.

Sol: Given $\mu = 75$, $\sigma = 7$

i, $P(60 \le x \le 78) = P(x_1 \le x \le x_2)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{60-75}{7} = -2.14$

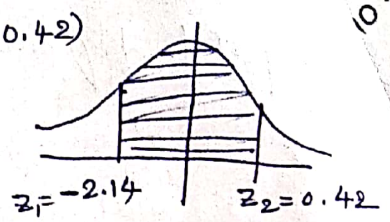
$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{78-75}{7} = 0.42$

$P(z_1 \le z \le z_2) = P(-2.14 \le z \le 0.42)$

$= P(-2.14 \le z \le 0) + P(0 \le z \le 0.42)$

$= 0.4838 + 0.1628$

$= 0.6466$

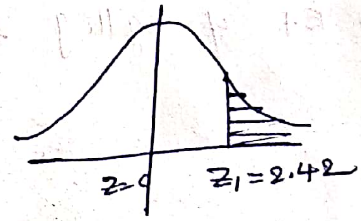



$\therefore$ For 500 students $= 500 \times 0.6466 = 323$

ii, $P(x > 92) = P(x \ge x_1)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{92-75}{7} = 2.42$

$P(z > z_1) = P(z > 2.42)$




$= 0.5 - P(0 \le z \le 2.42)$

$= 0.5 - 0.4922$

$= 0.0078$

$\therefore$ For 500 students $= 500 \times 0.0078 = 4$

9. The mean and S.D of the box obtained by 1000 students in an examination are respectively. 34.5 and 16.5. Assuming the normality of the distribution. Find the approximate number of students expected to obtain marks b/w 30 & 60.

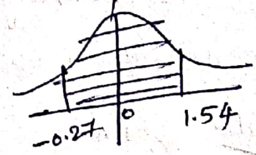
Sol: Given $\mu = 34.5$, $\sigma = 16.5$

$P(30 \le x \le 60) = P(x_1 \le x \le x_2)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{30 - 34.5}{16.5} = -0.27$

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{60 - 34.5}{16.5} = 1.54$

$P(z_1 \le z \le z_2) = P(-0.27 \le z \le 1.54)$




$= P(-0.27 \le z \le 0) + P(0 \le z \le 1.54)$

$= 0.5466$

$\therefore$ For 1000 students $= 1000 \times 0.5466$

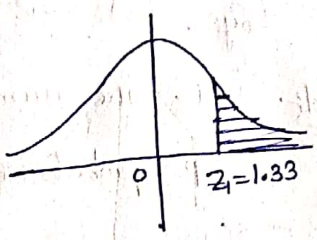
$= 546$

10. If the masses of 300 students are normally distributed with mean 68 kgs and S.D 3 kgs. How many students have masses i) greater than 72 kg ii) less than or equal to 64 kg iii) b/w 65 and 71 kg inclusive.

Sol: Let $\mu = 68$, $\sigma = 3$

i) $P(x > 72) = P(x > x_1)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{72 - 68}{3} = 1.33$$

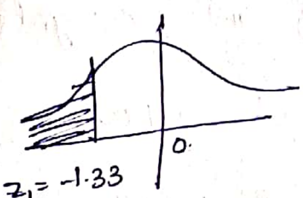



$$\therefore P(z > z_1) = P(z > 1.33)$$
$$= 0.5 - P(0 \le z \le 1.33)$$
$$= 0.5 - 0.4082$$
$$= 0.0918$$

For 300 students $= 300 \times 0.0918 = 28$

ii) $P(x \le 64) = P(x \le x_1)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{64 - 68}{3} = -1.33$$




$$\therefore P(z \le z_1) = P(z \le -1.33)$$
$$= 0.5 - P(-1.33 \le z \le 0)$$
$$= 0.5 - (0 \le z \le 1.33)$$
$$= 0.5 - 0.4082$$
$$= 0.0918$$

For 300 students $= 300 \times 0.0918 = 28$

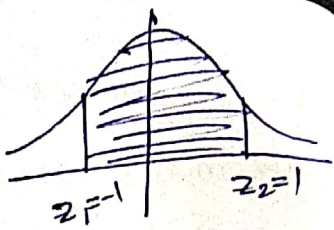
iii) $P(65 \le x \le 71) = P(x_1 \le x \le x_2)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{65 - 68}{3} = -1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{71 - 68}{3} = 1$$

$$\therefore P(z_1 \le z \le z_2) = P(-1 \le z \le 1)$$
$$= P(-1 \le z \le 0) + P(0 \le z \le 1)$$
$$= P(0 \le z \le 1) + P(0 \le z \le 1)$$

$= 2\,P(0 \le z \le 1)$




$= 2 \times 0.3413$

$= 0.6826$

For 300 students $= 300 \times 0.6826 = 205$

11. Out of 800 families with 5 children each, how many would you expect to have i, 3 boys ii, 5 girls iii, either 2 or 3 boys? Assume equal probabilities for boys and girls.

Sol: Let the number of boys in each family $= x$

$p =$ probability of each boy $= \frac{1}{2}$

$q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$, Number of children $n=5$

$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

i, $P(3 \text{ boys}) = P(X=3) = {}^{5}C_{3} \left(\frac{1}{2}\right)^{3} \left(\frac{1}{2}\right)^{5-3} = \frac{5}{16}$ per family

$\therefore$ For 800 families the probability of number of families having 3 boys $= \frac{5}{16}(800) = 250$ families

ii, $P(5 \text{ girls}) = P(\text{No boys}) = P(X=0) = {}^{5}C_{0} \left(\frac{1}{2}\right)^{0} \left(\frac{1}{2}\right)^{5-0} = \frac{1}{32}$

$\therefore$ For 800 families $= \frac{1}{32}(800) = 25$ families

iii $P(\text{either 2 or 3 boys}) = P(X=2) + P(X=3)$

$= {}^{5}C_{2} \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{5-2} + {}^{5}C_{3} \left(\frac{1}{2}\right)^{3} \left(\frac{1}{2}\right)^{5-3}$

$= \frac{5}{8}$ per family

$\therefore$ Expected no. of families with 2 or 3 boys $= \frac{5}{8}(800) = 500$ families

12. If a poisson distribution is such that $P(X=1) = \frac{3}{2} P(X=3)$. Then find i, $P(X \ge 1)$ ii, $P(X \le 3)$ iii, $P(2 \le X \le 5)$.

Sol: Given $P(X=1) = \frac{3}{2} P(X=3)$

$$\frac{e^{-\lambda} \lambda^{1}}{1!} = \frac{3}{2} \cdot \frac{e^{-\lambda} \lambda^{3}}{3!}$$

$\Rightarrow \lambda^3 - 9\lambda = 0$

$\lambda = 0, 3, -3$

$\therefore \lambda = 3 \quad (\because \lambda > 0)$

$\therefore$ By Poisson distribution $P(X=x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$

i) $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0) = 1 - \frac{e^{-3}\cdot 3^0}{0!} = 0.9502$

ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \frac{e^{-3}\cdot 3^0}{0!} + \frac{e^{-3}\cdot 3^1}{1!} + \frac{e^{-3}\cdot 3^2}{2!} + \frac{e^{-3}\cdot 3^3}{3!}$

$= 0.6472$

iii) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= \frac{e^{-3}\cdot 3^2}{2!} + \frac{e^{-3}\cdot 3^3}{3!} + \frac{e^{-3}\cdot 3^4}{4!} + \frac{e^{-3}\cdot 3^5}{5!}$

$= 0.7169$

13. Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents are i) atleast one ii) atmost one.

Sol: Given $\lambda = 1.8$, $P(X=x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$

i) $P(\text{atleast one}) = P(X \geq 1) = 1 - P(X=0) = 1 - \frac{e^{-1.8}\cdot(1.8)^0}{0!} = 0.8347$

ii) $P(\text{atmost one}) = P(X \leq 1) = P(X=0) + P(X=1)$

$= \frac{e^{-1.8}\cdot(1.8)^0}{0!} + \frac{e^{-1.8}\cdot(1.8)^1}{1!}$

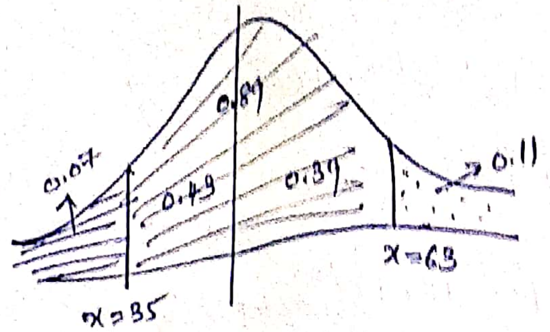
$= 0.4628$

14. In a Normal distribution, 7% of the items are under 35 and 89% are under 63. Find the mean and standard deviation of the distribution.

Sol: $P(X<35) = 0.07$, $P(X<63) = 0.89$

$x_1 = 35$, $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{35-\mu}{\sigma} = -z_1$ — (1)

$x = 63, \; z_2 = \frac{x_2 - \mu}{\sigma} = \frac{63 - \mu}{\sigma} = z_2$ —— (2)




From (1) & (2)

$-1.48 = \frac{35 - \mu}{\sigma} \Rightarrow \mu = 35 + 1.48\sigma$ —— (3)

$1.28 = \frac{63 - \mu}{\sigma} \Rightarrow \mu = -1.28\sigma + 63$ —— (4)

Solve (3) & (4), we get $\mu = 50.29$

$\sigma = 10.33$

15. A shipment of 20 tape recorders contains 5 defectives find the standard deviation of the probability distribution of the number of defectives in a sample of 10 randomly chosen for inspection.

Sol:- Given $n = 10, \; p = \frac{5}{20} = 1/4, \; q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Standard deviation $(\sigma) = \sqrt{npq}$

$= \sqrt{10 \times \frac{1}{4} \times \frac{3}{4}}$

$= 1.369$

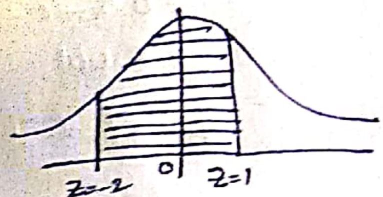
16. 1000 students have written an examination with the mean of test is 35 and standard deviation is 5. Assuming the distribution to be normal. find i, How many students marks like b/w 25 and 40? ii, How many students get more than 40? iii, How many students get below 20? How many students get more than 50.

Sol: ~~Here~~ $\mu = 35, \sigma = 5$

i, $P(25 \le x \le 40) = P(x_1 \le x \le x_2)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{25-35}{5} = -2$

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{40-35}{5} = 1$




Now $P(z_1 \le z \le z_2) = P(-2 \le z \le 1)$

$= P(-2 \le z \le 0) + P(0 \le z \le 1)$

$= P(0 \le z \le 2) + P(0 \le z \le 1)$

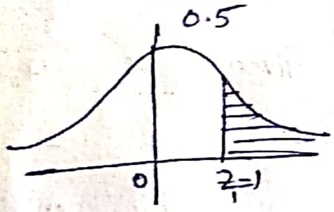
$= 0.4772 + 0.3415$

$= 0.8185$

$\therefore$ For 1000 students $= 1000 \times 0.8185 = \underline{819}$

ii, $P(x > 40) = P(x > x_1)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{40-35}{5} = 1$




$P(z > z_1) = P(z > 1)$

$= 0.5 - P(0 \le z \le 1)$

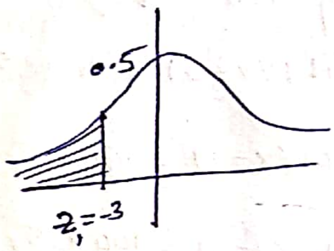
$= 0.5 - 0.3415$

$= 0.1585$

$\therefore$ For 1000 students $= 1000 \times 0.1585 = 159$

iii $P(x < 20) = P(x < x_1)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{20-15}{5} = -3$




$P(z < z_1) = P(z < -3)$

$= 0.5 - P(-3 \le z \le 0)$

$= 0.5 - P(0 \le z \le 3)$

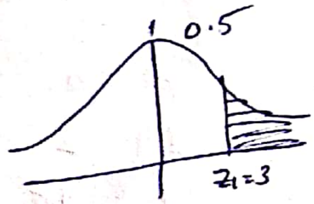
$= 0.5 - 0.499$

$= 0.001$

$\therefore$ For 1000 students $= 1000 \times 0.001 = 1$

iv, $P(x > 50) = P(x > x_1)$

$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{50-15}{5} = 3$



$\therefore P(x > 81) = P(z > 3)$

$= 0.5 - P(0 \le z \le 3)$

$= 0.5 - 0.999$

$= 0.001$

$\therefore$ No. of students whose marks are greater than 50

$= 1000 \times 0.001$

$= 1$

17. Fit a Binomial distribution to the following data

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f$ | 2 | 14 | 20 | 34 | 22 | 8 |

Sol:- Given

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | total |
|---|---|---|---|---|---|---|---|
| $f$ | 2 | 14 | 20 | 34 | 22 | 8 | $N = \Sigma f = 100$ |
| $fx$ | 0 | 14 | 40 | 102 | 88 | 40 | $\Sigma fx = 284$ |

$$\text{Mean}(\mu) = \frac{\Sigma fx}{\Sigma f} = \frac{284}{100} = 2.84 = np$$

$$n = 5, \quad p = \frac{2.84}{5} = 0.568, \quad q = 1 - p = 1 - 0.568 = 0.432$$

Binomial distribution $P(X = x) = {}^nC_x \, p^x q^{n-x}$

Expected frequency $= N \cdot P(X = x)$

i, $N \cdot P(X=0) = 100 \times {}^5C_0 (0.568)^0 (0.432)^{5-0} = 1.5$

ii, $N \cdot P(X=1) = 100 \times {}^5C_1 (0.568)^1 (0.432)^{5-1} = 9.89$

iii, $N \cdot P(X=2) = 100 \times {}^5C_2 (0.568)^2 (0.432)^{5-2} = 2.6$

iv, $N \cdot P(Y=3) = 100 \times {}^5C_3 (0.568)^3 (0.432)^{5-3} = 34.1$

v, $N \, P(X=4) = 100 \times {}^5C_4 (0.568)^4 (0.432)^{5-4} = 22.4$

v, $N \, P(X=5) = 100 \times {}^5C_5 (0.568)^5 (0.432)^{5-5} = 5.9$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f$ | 2 | 14 | 20 | 34 | 22 | 8 |
| E.F | 1.5 | 9.89 | 2.6 | 34.1 | 22.4 | 5.9 |

18. Show that the recurrence relation for the Poisson distribution is $P(x) = \frac{\lambda}{x} P(x-1)$

Sol

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad , \quad P(x-1) = \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}$$

$$= \frac{e^{-\lambda} \lambda^{x-1+1}}{x(x-1)!}$$

$$= \frac{\lambda}{x} \cdot \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}$$

$$P(x) = \frac{\lambda}{x} \cdot P(x-1)$$

19. The life of electronic tubes of a certain types may be assumed to be normal distributed with mean 155 hours and standard deviation 19 hours. Determine the probability that the life of a randomly chosen tube is

i) b/w 136 hrs and 174 hrs.

ii) less than 117 hrs

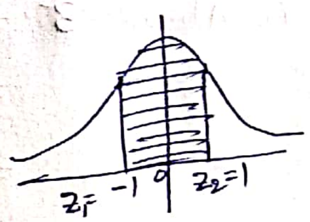
iii) will be more than 195 hrs.

Sol Given $\mu = 155$, $\sigma = 19$

i) $P(136 \le x \le 174) = P(x_1 \le x \le x_2)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{136-155}{19} = \frac{-19}{19} = -1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{174-155}{19} = \frac{19}{19} = 1$$




$$P(z_1 \le z \le z_2) = P(-1 \le z \le 1)$$

$$= P(-1 \le z \le 0) + P(0 \le z \le 1)$$

$$= P(0 \le z \le 1) + P(0 \le z \le 1)$$

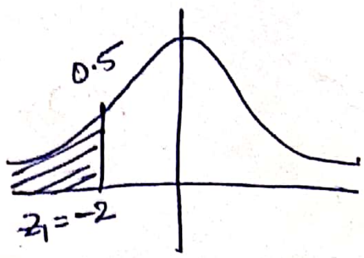
$$= 2P(0 \le z \le 1)$$

$$= 2 \times 0.343$$

$$= 0.686$$

ii, $P(x<117) = P(x<x_1)$

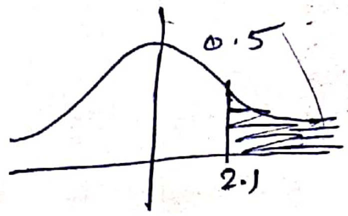
$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{117-155}{19} = \frac{-38}{19} = -2$$




$$P(z<z_1) = P(z<-2)$$
$$= 0.5 - P(-2 \le z \le 0)$$
$$= 0.5 - P(0 \le z \le 2)$$
$$= 0.5 - 0.4772$$
$$= 0.0228$$

iii, $P(x>195) = P(x>x_1)$

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{195-155}{19} = 2.1$$




$$P(z>z_1) = P(z>2.1)$$
$$= 0.5 - P(0 \le z \le 2.1)$$
$$= 0.5 - 0.4821$$
$$= 0.0179$$

20. The probability that a man hitting a target is $1/3$. If he fires 5 times, determine the probability that he fires.
i, Atmost 3 times ii, At least 2 times.

Sol: Given $p$ = The probability of hitting a target $= \frac{1}{3}$

$$q = 1-p = 1-\frac{1}{3} = \frac{2}{3} \quad , n=5$$

i, Atmost 3 $= P(X \le 3)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= 5C_0\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^{5-0} + 5C_1\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^{5-1} + 5C_2\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^{5-2}$$
$$+ 5C_3\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^{5-3}$$
$$= 0.954$$

ii, $P(X \ge 2) = 1 - P(X<2) = 1 - \{P(X=0) + P(X=1)\}$

$$= 1 - \left\{5C_0\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^{5-0} + 5C_1\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^{5-1}\right\}$$
$$= 0.5391$$

## Part - C

1. Prove that the poisson distribution is a limiting case of Binomial distribution.

Sol: Binomial distribution is $b(x,n,p) = P(X=x) = n_{c_x} p^x q^{n-x}$

$$= \frac{n!}{(n-x)!\, x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))\cancel{(n-x)!}}{\cancel{(n-x)!}\, x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots[n-(x-1)]}{x!} p^x q^{n-x}$$

Now $np = \lambda$, $\Rightarrow p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$

$$= \frac{n(n-1)(n-2)\cdots[n-(x-1)]}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\left[\frac{n(n-1)(n-2)\cdots[n-(x-1)]}{n^x}\right] \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!}\left[\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-(x-1)}{n}\right] \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!}\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

we have $n \to \infty$

$$= \underset{n\to\infty}{Lt}\left[\frac{\lambda^x}{x!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right\}\cdot\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}\Bigg]$$

$$= \frac{\lambda^x}{x!}\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^x}{x!}\left[\underset{n/\lambda\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^{\frac{n}{-\lambda}}\right]^{-\lambda} \quad \left[\because \underset{x\to\infty}{Lt}\left(1+\frac{1}{x}\right)^x = e\right]$$

$$= \frac{\lambda^x}{x!}\cdot e^{-\lambda}$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Derive variance of the poisson distribution

2. Derive variance of the poisson distribution

Sol: Variance of the poisson distribution

$f(x,\lambda) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad \forall x = 0,1,\dots\infty$

$$\text{Variance}(\sigma^2) = \sum x^2 f(x) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - \mu^2$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{x^2 \lambda^x}{x(x-1)!} - \mu^2$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{[x(x-1)+x]}{x(x-1)!} \lambda^x - \mu^2$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \frac{x(x-1)\lambda^x}{x(x-1)(x-2)!} + \sum_{x=1}^{\infty} \frac{x}{x(x-1)!}\lambda^x\right] - \mu^2$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}\right] - \mu^2$$

$$= e^{-\lambda}\left\{\left[\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \dots\right] + \left[\frac{\lambda}{0!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \dots\right]\right\} - \mu^2$$

$$= e^{-\lambda}\left[\lambda^2 e^{\lambda} + \lambda e^{\lambda}\right] - \lambda^2$$

$$\therefore \text{Variance}(\sigma^2) = \lambda$$

3. Prove that mode in Normal distribution

Sol: Mode of the normal distribution:- Mode is the value of $x$, for which $f(x)$ is maximum.

$$\therefore N(\mu,\sigma^2) = P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \forall -\infty < x < \infty$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \left(-\frac{1}{2}\right) 2 \left(\frac{x-\mu}{\sigma}\right) \cdot \left(\frac{1}{\sigma}\right)$$

$$f'(x) = f(x) \cdot \left(-\left(\frac{x-\mu}{\sigma^2}\right)\right)$$

$$\therefore f'(x) = 0 \quad \text{when } x = \mu$$

Now $f''(x) = -f'(x)\left(\frac{x-\mu}{\sigma^2}\right) - f(x)\cdot\frac{1}{\sigma^2}$

$f''(x) = -f(x)\left(\frac{x-\mu}{\sigma^2}\right) - f(x)\cdot\frac{1}{\sigma^2}$

$f''(x) < 0$, when $x = \mu$

$\therefore f(x)$ is maximum when $x = \mu$

$\therefore$ Mode $= \mu$

4. Derive median of the normal distribution.

Sol: Median of the normal distribution:-

Suppose M is a median of normal distribution then

$\int_{-\infty}^{M} f(x)\,dx$ (or) $\int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$

Consider $\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$

$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$ —— ①

Consider $\int_{-\infty}^{\mu} f(x)\,dx = \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

Let $z = \frac{(x-\mu)}{\sigma}$     $x : -\infty \longrightarrow \mu$

$x = \mu + \sigma z$     $z : -\infty \longrightarrow 0$

$dx = \sigma dz$

$= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-z^2/2}\cdot\sigma dz$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-z^2/2} dz$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$

Let $z^2/2 = t \Rightarrow z^2 = 2t$

$2z\,dz = 2\,dt$     $z : 0 \longrightarrow \infty$

    $t : 0 \longrightarrow \infty$

$\Rightarrow dz = \frac{dt}{\sqrt{2t}}$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \Gamma(1/2)$$

$$= \frac{1}{2\sqrt{\pi}} \times \sqrt{\pi}$$

$$\therefore \int_{-\infty}^{\mu} f(x)dx = \frac{1}{2}$$

From ①, $\frac{1}{2} + \int_{\mu}^{M} f(x)dx = \frac{1}{2}$

$$\int_{\mu}^{M} f(x)dx = 0$$

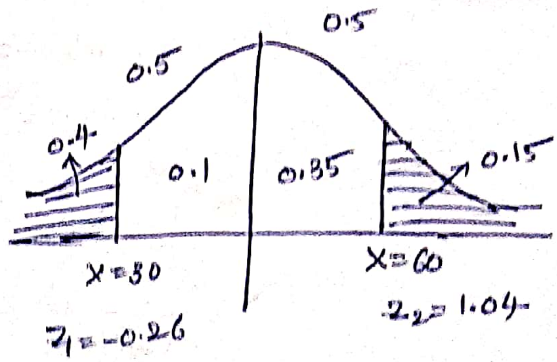
$$\Rightarrow M = \mu$$

$\therefore$ Median $= \mu$

5. The marks obtained in statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find mean and S.D.

Sol: $P(X \geq 60) = 15\% = 0.15$ ($x_1$)

$P(X \leq 30) = 40\% = 0.4$ ($x_2$)




$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{60-\mu}{\sigma}$ — ①

$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{30-\mu}{\sigma}$ — ②

From ① & ②

$\frac{60-\mu}{\sigma} = 1.04$ — ③ , $\frac{30-\mu}{\sigma} = -0.26$ — ④

Solve ③ & ④, we get $\mu = 35.81$, $\sigma = 23.26$

6. The variance and mean of a binomial variable X with parameters n and p are 4 and 3. Find
i, $P(X=1)$ ii, $P(X\geq 1)$ iii, $P(0<X<3)$

Sol: Given Mean $=4 \Rightarrow np=4$ — (1)

Variance $=3 \Rightarrow npq=3$ — (2)

From (1) & (2), $q=3/4$, $p=1-q=1-3/4=1/4$

Substitute p value in (1), we get $n=16$

$\therefore P(X=x)=n_{C_x}\, p^x q^{n-x}$

i, $P(X=1)=16_{C_1}\,(1/4)^1\,(3/4)^{16-1}$

ii, $P(X\geq 1)=1-P(X<1)$

$=1-P(X=0)$

$=1-16_{C_0}\,(1/4)^0\,(3/4)^{16-0}$

$=$

iii, $P(0<X<3)=P(X=1)+P(X=2)$

$=16_{C_1}\,(1/4)^1(3/4)^{16-1}+16_{C_2}\,(1/4)^2(3/4)^{16-2}$

$=$

7. Fit a Binomial distribution to the following data.

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | total |
|---|---|---|---|---|---|---|---|---|
| f | 13 | 25 | 52 | 58 | 32 | 16 | 4 | 200 |

Sol:

| x | f | fx |
|---|---|---|
| 0 | 13 | 0 |
| 1 | 25 | 25 |
| 2 | 52 | 104 |
| 3 | 58 | 174 |
| 4 | 32 | 128 |
| 5 | 16 | 80 |
| 6 | 4 | 24 |
| | $N=\Sigma f=200$ | $\Sigma fx=535$ |

Given $n=6$, $N=200$

$\text{mean}(\mu)=\dfrac{\Sigma fx}{\Sigma f}=\dfrac{535}{200}=2.675$

$np=2.675$

$\Rightarrow p=0.446$

$q=0.554$

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$P(X=x) = {}^nC_x\, p^x q^{n-x}$

$E.F = N.P(X=x) = N.\,{}^nC_x\, p^x q^{n-x}$

i, $E.F$ for $P(X=0) = N.P(X=0) = 200 \times {}^6C_0 (0.446)^0 (0.554)^{6-0}$
$= 5.84$

ii, $NP(X=1) = 200 \times {}^6C_1 (0.446)^1 (0.554)^{6-1} = 28.12$

iii, $NP(X=2) = 200 \times {}^6C_2 (0.446)^2 (0.554)^{6-2} = 56.36$

iv, $NP(X=3) = 200 \times {}^6C_3 (0.446)^3 (0.554)^{6-3} = 60.25$

v, $NP(X=4) = 200 \times {}^6C_4 (0.446)^4 (0.554)^{6-4} = 36.23$

vi, $NP(X=5) = 200 \times {}^6C_5 (0.446)^5 (0.554)^{6-5} = 11.62$

vii, $NP(X=6) = 200 \times {}^6C_6 (0.446)^6 (0.554)^{6-6} = 1.55$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | total |
|---|---|---|---|---|---|---|---|---|
| $f$ | 13 | 25 | 52 | 58 | 32 | 16 | 4 | 200 |
| E.F | 5.84 | 28.12 | 56.36 | 60.25 | 36.23 | 11.62 | 1.55 | 200 |

8. Prove that Mean in normal distribution.

Sol: <u>Mean of the normal distribution</u> :-

$$N(\mu, \sigma^2) = P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = z$ $\qquad x: -\infty \to \infty$

$\Rightarrow x = \mu + \sigma z$ $\qquad z: -\infty \to \infty$

$dx = \sigma\, dz$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{-\frac{1}{2}z^2} \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \mu\, e^{-z^2/2} dz + \int_{-\infty}^{\infty} \sigma z\, e^{-z^2/2} dz \right\}$$

$$= \frac{\mu}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} e^{-z^2/2} dz + 0 \right]$$

$$= \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2/2} dz$$

$$= \frac{\sqrt{2}\,\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2/2} dz$$

Let $z^2/2 = t \Rightarrow z^2 = 2t$

$2z\,dz = 2dt$

$\Rightarrow dz = \frac{dt}{z} = \frac{dt}{\sqrt{2t}}$

$$= \frac{\sqrt{2}\,\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^{\infty} e^{-t} \cdot t^{(1/2 - 1)} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \cdot \Gamma\tfrac{1}{2}$$

$$= \frac{\mu}{\sqrt{\pi}} \times \sqrt{\pi}$$

Mean $= \mu$

9. The marks obtained in mathematics by 1000 students are normally distributed with mean 78% and standard deviation 11%. Determine

i) How many students got marks above 90% marks.

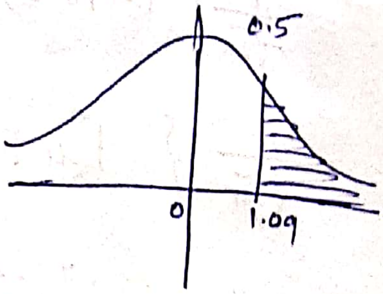
ii) What was the highest mark obtained by the lowest 10% of the students

iii) Within what limits did the middle of 90% of the students lie.

Sol: Given $\mu = 78\% = 0.78$, $\sigma = 11\% = 0.11$

i) $P(x > 90\%) = P(x > 0.9) = P(x > x_1)$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{0.9 - 0.78}{0.11} = 1.09$$




$P(z > z_1) = P(z > 1.09)$

$= 0.5 - P(0 \leq z \leq 1.09)$

$= 0.5 - 0.3621$

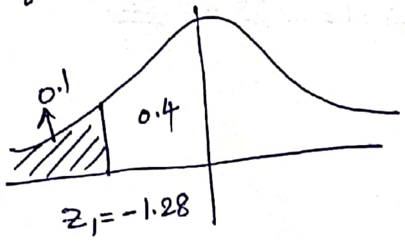
$= 0.1379$

$\therefore$ the number of students with marks more than 90%

$= 0.1379 \times 1000$

$= \underline{138}$

ii, 0.1 area to the left of z corresponds to the lowest 10% of the students




$$z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow -1.28 = \frac{x_1 - 0.78}{0.11}$$

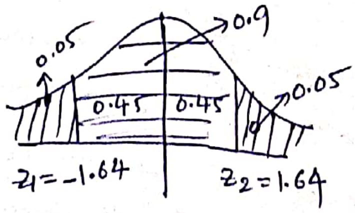
$$\Rightarrow x_1 = 0.6392$$

$\therefore$ The highest mark obtained by the lowest 10% of students

$= 0.6392 \times 1000$

$= 64\%$

iii, Middle 90% $= \frac{90}{100} = 0.9$




$z_1 = \frac{x_1 - \mu}{\sigma} \Rightarrow -1.64 = \frac{x_1 - 0.78}{0.11} \Rightarrow x_1 = 0.5996$ (or) 59.96%

$z_2 = \frac{x_2 - \mu}{\sigma} \Rightarrow 1.64 = \frac{x_2 - 0.78}{0.11} \Rightarrow x_2 = 0.9604$ (or) 96.04%

$\therefore$ The middle 90% have marks in b/w 60 to 96

10. Derive the mean of a Binomial distribution

Sol. Mean of the Binomial distribution:-

Mean $= \sum x \cdot f(x)$

$= \sum_{x=0}^{n} x \; n_{C_x} p^x q^{n-x}$

$= \sum_{x=0}^{n} \cancel{x} \dfrac{n(n-1)!}{(n-x)!\,\cancel{x}(x-1)!} p.\,p^{x-1} q^{n-x}$

$= \sum_{x=0}^{n} \dfrac{n(n-1)!}{(n-x)!\,(x-1)!} p.\,p^{x-1} q^{n-x}$

$= np \sum_{x=1}^{n} \dfrac{(n-1)!}{(n-x)!\,(x-1)!} p^{x-1} q^{n-x}$

Let $x-1=y, \Rightarrow x=y+1$

$= np \sum_{y=0}^{n-1} \dfrac{(n-1)!}{(n-1-y)!\,y!} p^y . q^{n-1-y}$

$= np \sum_{y=0}^{n-1} (n-1)_{C_y} p^y q^{(n-1)-y}$

$= np\,(q+p)^{n-1}$

$= np\,(1)^{n-1}$

$\therefore$ Mean $(\mu) = np$