IM/2020/058 - Yudha T

PMAT - Tutorial - 05

01

i) $\lim_{w \to -4} \frac{\sin \pi w}{w^2 - 16}$

$\lim_{w \to -4} \sin \pi w = \sin -4\pi = 0$

$\lim_{w \to -4} w^2 - 16 = 16 - 16 = 0$

limit has an ~~inter~~ indeterminate form of type $0/0$

∴ by using L's hospital rule

$$\lim_{w \to -4} \frac{\sin \pi w}{w^2 - 16} = \lim_{w \to -4} \frac{\frac{d \sin \pi w}{dw}}{\frac{d\, w^2 - 16}{dw}} = \frac{\pi \cos w \pi}{2w}$$

$$= \lim_{w \to -4} \frac{\pi \cos \pi}{2w} = \frac{\pi \cos -4\pi}{-8} = \frac{-\pi}{8}$$

ii) $\lim_{t \to \infty} \frac{\ln(3t)}{t^2}$

$\lim_{t \to \infty} \ln(3t) = \infty$

$\lim_{t \to \infty} t^2 = \infty$

limit has an interdeterminate form of type $\frac{\infty}{\infty}$

∴ by using L' hospital's rule

$$\lim_{t \to \infty} \frac{\ln(3t)}{t^2} = \lim_{t \to \infty} \frac{3 \cdot \frac{1}{3t}}{2t} = \lim_{t \to \infty} \frac{1}{2t^2} = 0$$

iii) $\lim_{x \to -\infty} \frac{x^2}{e^{1-x}}$

$\lim_{x \to -\infty} x^2 = \infty$

$\lim_{x \to -\infty} e^{1-x} = \infty$

limit has an interdeterminate form of type $\frac{\infty}{\infty}$
$\therefore$ using L' hospital's rule

$$\lim_{x \to -\infty} \frac{x^2}{e^{1-x}} = \lim_{x \to -\infty} \frac{2x}{e^{1-x} \times (-1)} = \lim_{x \to \infty} \frac{2}{e^{1-x}} = 0$$

iv) $\lim_{z \to \infty} \frac{z^2 + e^{4z}}{2z - e^z}$

$\lim_{z \to \infty} z^2 + e^{4z} = \infty$

$\lim_{z \to \infty} 2z - e^z = \infty$

limit has an interdeterminate form of type $\frac{\infty}{\infty}$
$\therefore$ using L' Hospital's rule

$$\lim_{z \to \infty} \frac{z^2 + e^{4z}}{2z - e^z} = \lim_{z \to \infty} \frac{2z + 4e^{4z}}{2 - e^z} = \lim_{z \to \infty} \frac{2 + 16e^{4z}}{1 - e^z}$$

Since $\lim_{z \to \infty} z + 16^{4z} = \infty$ and $\lim_{z \to \infty} 1 - e^z = \infty$ it is in the form of $\frac{\infty}{\infty}$ again

$\therefore$ apply L' Hospital's Rule

$$\lim_{z \to \infty} \frac{2 + 16e^{4z}}{1 - e^z} = \lim_{z \to \infty} \frac{6 + e^{4z}}{-e^z}$$

here it is unable to gain a limit that is not in the form of $\frac{\infty}{\infty}$ $\therefore$ This limit do not exist.

v) $\lim_{t\to\infty} t\ln\left(t+\frac{3}{t}\right) = \lim_{t\to\infty} \frac{\ln(t+3/t)}{1/t}$

$\lim_{t\to\infty} \frac{1}{\infty t} = 0$ and $\lim_{t\to\infty} \ln(t+3/t) = 0$

~~Since lim~~

limit has an interdeterminate form type of $0/0$

$\therefore$ Using L' Hospital's rule

$$\lim_{t\to\infty} \frac{\ln(t+3/t)}{1/t} = \frac{\lim_{t\to\infty} \frac{1}{t+3/t}\left(1+(-1)3/t^2\right)}{1/t^2}$$

$$= \lim_{t\to\infty} \frac{\frac{t^2-3}{t^2} \times \frac{t}{t^2+3}}{1/t^2} = \lim_{t\to\infty} \frac{t(t^2-3)}{t^2+3}$$

apply L' Hospital's rule again.

$$\lim_{t\to\infty} \frac{t(2t)+(t^2-3)}{2t} = \lim_{t\to\infty} \frac{3t^2-3}{2t} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{t\to\infty} \frac{6t}{2}$$

$$= \lim_{t\to\infty} 3t = \infty$$

vi) $\lim_{w\to 0^+} w^2\ln(4w^2) = \lim_{w\to 0^+} \frac{\ln(4w^2)}{1/w^2}$

Using L' Hospital's rule

$$\lim_{w\to 0^+} \frac{\ln(4w^2)}{1/w^2} = \lim_{w\to 0^+} \frac{\frac{1}{4w^2}\times 8w}{1/w^2} = \lim_{w\to 0^+} 2w^2 = 0$$

vii) $\lim_{x \to \infty} [e^x + x]^{1/x}$

$\lim_{x \to \infty} e^x + x = \infty$ and $\lim_{x \to \infty} 1/x = 0$ and it is in the form

of ∞^0, so let

$y = [e^x + x]^{1/x}$

Let $\ln y = \ln([e^x + x]^{1/x}) = \frac{1}{x} \ln[e^x + x] = \frac{\ln[e^x + x]}{x}$

apply L'Hospital's rule,

$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln[e^x + x]}{x} = \lim_{x \to \infty} \frac{1}{e^x + x}(e^x + 1)$

$= \lim_{x \to \infty} \frac{e^x + 1}{e^x + x} \quad \left(\frac{\infty}{\infty}\right)$

again apply L'Hospital's rule

$\lim_{x \to \infty} \frac{e^x + 1}{e^x + x} = \lim_{x \to \infty} \frac{e^x}{e^x + 1} \quad \left(\frac{\infty}{\infty}\right)$

Unable to get a limit that is not in the form of ∞/∞

$\therefore$ This limit does not exist

(b2) Get the critical points

i) $f(x) = 8x^3 + 81x^2 - 42x - 8$

$f'(x) = 24x^2 + 162x - 42$

$f'(x) = 6(4x^2 + 27x - 7)$

$f'(x) = 6(x+7)(4x-1)$

$f'(x) = 0$ when $x + 7 = 0$ or $4x - 1 = 0$

critical numbers are $x = -7$, $x = 1/4$

ii) $Q(x) = (2-8x)^4 (x^2-9)^3$

$Q'(x) = -32(2-8x)^3(x^2-9)^3 + 6x(x^2-9)^2(2-8x)^4$

$Q'(x) = 2(2-8x)^3(x^2-9)^2 \{-16(x^2-9) + x(20-8x)\}$

$Q'(x) = 2(2-8x)^3(x^2-9)^2(-24x^2+2x+144)$

$Q'(x) = -4(2-8x)^3(x^2-9)^2(12x^3-x-72)$

$Q'(x)=0$ when $2-8x=0$, $x^2-9=0$ or $12x^2-x-72=0$

critical numbers are

$4, 3, -3, \dfrac{\sqrt{3457}+1}{24}, \dfrac{-\sqrt{3457}+1}{24}$

iii) $F(x) = 5xe^{9-2x}$

$F'(x) = 5xe^{9-2x} \times (-2) + xe^{9-2x} \; 5$

$F'(x) = 5e^{9-2x} - 10xe^{9-2x}$

$f'(x) = 5e^{9-2x}(1-2x)$

$F'(x) = 0$ when $x = 1/2$

$\therefore$ critical value $\frac{1}{2}$

iv) $R(x) = \ln(x^2+4x+14)$

$R'(x) = \dfrac{2x+4}{x^2+4x+14}$

$R'(x) = 0$ when $x = -2$

$R'(x)$ do not exist when $x^2+4x+14 = 0$

$x = 3\sqrt{2}+2$, $x = -3\sqrt{2}+2$

critical values $-2, (3\sqrt{2}+2), (-3\sqrt{2}+2)$

v) $A(t) = 3t - 7\ln(8t+2)$

$A'(t) = 3\left(\dfrac{-7\times 8}{8t+2}\right) = \dfrac{3(8t+2)-56}{8t+2} = \dfrac{24t-50}{8t+2}$

$A'(t) = 0$ when $t = \dfrac{25}{12}$

$A'(t)$ do not exist $t = -\frac{1}{4}$

$\therefore$ Critical values $\frac{25}{12}, \left(-\frac{1}{4}\right)$

(03) i) $f(x) = \frac{x}{x^2+1}$ $[0,2]$

$f'(x) = \frac{(x^2+1)-2x}{(x^2+1)^2} = \frac{(x-1)^2}{(x^2+1)^2}$

critical value is 1 and this lie in the interval

$f(1) = \frac{1}{2} = 0.5$

for endpoints → $f(0) = 0/1 = 0$

$f(2) = 2/5 = 0.4$

absolute max $f(1) = 0.5$

absolute min $f(0) = 0$

ii) $f(x) = \frac{\ln x}{x}$ $[1,3]$

$f'(x) = \frac{1 - \ln x}{x^2}$

$f'(x) = 0$ when $\ln x = 1$, $x = 2.7181$ and $f'(x)$ do not exist when $x = 0$

critical values 2.7181, 0

$f(0) =$ undefined

$f(2.7181) = 0.3678$

end points → $f(1) = 0$, $f(3) = 0.3662$

absolute max → $f(e) = 0.3678$, absolute min → $f(1) = 0$

iii) $f(x) = x^3 - 2x^2 + 5$ $[-2,2]$

$f'(x) = x(3x-4)$

Critical values 0, $4/3$, in the interval

$f(0) = 5$, $f(4/3) = 10.92$

endpoints → $f(-2) = -11$, $f(2) = 5$

absolute max → 10.92

absolute min $= -11$

ix) $f(x) = \frac{1}{x+4}$ $[0,1]$

$f'(x) = \frac{-1}{(x+4)^2}$

critical value $= -4$, not in the given interval

endpoints $f(0) = 1/4$, $f(1) = 1/5$

absolute max $= 1/4$ absolute min $= 1/5$

(64)

i) $f(x) = x^2 - 2x - 8$, $[-1, 3]$

To satisfy the Rolle's theorem

$f(-1) = f(3)$

$f(-1) = -5$

$f(3) = -5$

Since $f(x)$ is polynomial, it is continuous & differentiable on $[-1, 3]$

also there is c in $(-1, 3)$ such that $f'(c) = 0$

$f(c) = c^2 - 2c - 8$

$f'(c) = 2c - 2 = 0$

$c = 1$

ii) $g(t) = 2t - t^2 - t^3$ $[-2, 1]$

to satisfy Rolle's theorem

$f(-2) = f(1)$

$f(-2) = 0$, $f(1) = 0$

$g(t)$ is a polynomial so it is continuous and differentiable

also there should be a c in $[-2, 1]$ such that $f'(c) = 0$

$f(c) = 2c - c^2 - c^3$

$f'(c) = 2 - 2c - 3c^2 = 0$

$3c^2 + 2c - 2 = 0$

$c = \frac{-2 \pm \sqrt{4 - 4(3)(-2)}}{2(3)}$

$= \frac{-1 \pm \sqrt{7}}{3}$

$c = -1.213$ and $c = 0.546$

(05)

i) $h(x) = 4x^3 - 8x^2 + 7x - 2$ $[2,5]$

To satisfy the mean value thorem $h(x)$ should differentiable and continuous at $[2,5]$

Since $h(x)$ is polynomial it is satify also there should be a c in $(2,5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5-2} \quad \text{—} \ ①$$

$$f(c) = 4c^3 - 8c^2 + 7c - 2$$

$$f'(c) = 12c^2 - 16c + 7 \quad \text{—} \ ②$$

$$① = ② \Rightarrow 12c^2 - 16c + 7 = \frac{f(5) - f(2)}{3}$$

$$12c^2 - 16c + 7 = 107$$

$$12c^2 - 16c - 100 = 0$$

$$c = 3.629, \qquad c = -2.296$$

ii) $A(t) = 8t + e^{-3t}$ $[-2,3]$

Since the function $A(t)$ is a sum of polynomial ~~it is~~ and exponential function or it is continuous & differentiable on $[-2,3]$

$$A(-2) = -16 + e^{6}$$

$$A(3) = 24 + e^{-9}$$

There should be a c in $(-2,3)$ such that

$$F'(c) = \frac{F(3) - F(-2)}{3-(-2)}$$

$$= \frac{40 + e^{-9} - e^{6}}{5} \quad \text{—} \ ①$$

$$F(c) = 8c + e^{-3c}$$

$$F'(c) = 8 - 3e^{-3c} \quad \text{—} \ ②$$

$$① = ② \Rightarrow \frac{40 + e^{-9} - e^{6}}{5} = 8 - 3e^{-3c}$$

$$c \simeq 1.09$$

(06) i) $\int [\sin(x) + 10\csc^2(x)]\,dx$

$= \int \sin x\,dx + \int 10\csc^2(x)\,dx$

$= -\cos x + 10(-\cot x) + C$

$= -\cos x - 10\cot x + C$

ii) $\int \left[4e^z + 15 - \frac{1}{6z}\right] dz$

$= \int 4e^z\,dz + \int 15\,dz - \int \frac{1}{6z}\,dz$

$= 4e^z + 15z - \frac{1}{6}\ln|z| + C$