EX 3

$$X = (X_1, X_2) \sim N(0, I) \quad , \quad W \sim N(0, 6)$$
$$Y = X_1 + X_2 + W$$

By conservation of independence X_1 and X_2 are independent from W.

1) $Y = y$ is given.

First we observe that

$$\pi_X(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1, x_2)^T (x_1, x_2)\right) =$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}$$

Then $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$.

Now we observe that:

$$X_1 = y - \underbrace{X_2 - W}_{X_2 + W \sim N(0, 6+1)} \implies X_1 \sim N(y, 6+1)$$

linear transformation of a normal is still normal due to independence

By symmetry we have

$$X_2 \sim N(y, 6+1)$$

2) The expectation of X_1 and X_2 is y, which doesn't depend on 6

$$\implies \lim_{6 \to 0} y = \lim_{6 \to +\infty} y = y$$

While

$$\lim_{6 \to 0} 6+1 = 1$$

and

$$\lim_{6 \to +\infty} 6+1 = +\infty$$