Problem 2 We consider a real-valued random variable X that has a probability density.

1. Show that $c = \mathbb{E}[X]$ minimizes the mean-squared error $\mathbb{E}\left[(X-c)^2\right]$.
2. Show that the median minimizes $\mathbb{E}\left[|X-c|\right]$. The median is defined as a number c such that $\mathbb{P}[X < c] = \mathbb{P}[X > c] = 0.5$.

(1)

Let's transform the formula, taking into account that $E[X] = \mu$:

$$\mathbb{E}\left[(X-c)^2\right] = \mathbb{E}\left[(X-\mu+\mu-c)^2\right] = \mathbb{E}\left[\left((X-\mu)+(\mu-c)\right)^2\right] =$$

$$= \mathbb{E}\left[(X-\mu)^2 + 2(\mu-c)(X-\mu) + (\mu-c)^2\right] =$$

$$= \mathbb{E}\left[(X-\mu)^2\right] + 2(\mu-c)\,\mathbb{E}[X-\mu] + (\mu-c)^2 =$$

$$= \mathbb{E}\left[(X-\mu)^2\right] + 2(\mu-c)\underbrace{\left(\mathbb{E}[X]-\mu\right)}_{=0} + (\mu-c)^2 =$$

$$= \{\mathbb{E}[X] = \mu\} = \mathbb{E}\left[(X-\mu)^2\right] + (\mu-c)^2$$

- The minimum of mean squared error will be if $(\mu-c)^2 = 0$, i.e. $c = \mu$.
- From the definition $E[X] = \mu \Rightarrow c = \mu = E[X]$ minimizes MSE

$\blacksquare$

(2)

Let's $F(c) = \mathbb{E}\left[|X-c|\right]$ and $f(x)$ is pdf.

1) $F(c) = \mathbb{E}\left[|X-c|\right] = \int_{\mathbb{R}} |x-c| f(x)\,dx = \int_{-\infty}^{c} (c-x) f(x)\,dx + \int_{c}^{\infty} (x-c) f(x)\,dx$

let's compute the derivative:

$$\frac{dF}{dc} = (c-x)f(x)\Big|_{x=c} + \int_{-\infty}^{c} f(x)\,dx + (x-c)f(x)\Big|_{x=c^-}$$

$$-\int_{c}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx - \int_{c}^{\infty} f(x)\,dx = 0$$

equals to 0 as we are searching for extreme

$\Downarrow$

The minimum if:

$$\int_{-\infty}^{c} f(x)\,dx = \int_{c}^{\infty} f(x)\,dx \iff P(X<c) = P(X>c) = 0{,}5$$

$\Rightarrow$ the median minimises $E[|X-c|]$

EX 3

$$X = (X_1, X_2) \sim N(0, I) \quad , \quad W \sim N(0, 6)$$
$$Y = X_1 + X_2 + W$$

By conservation of independence X_1 and X_2 are independent from W.

1) $Y = y$ is given.

First we observe that

$$\pi_X (x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1, x_2)^T (x_1, x_2)\right) =$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x_2^2}{2}}$$

Then $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$.

Now we observe that:

$$X_1 = y - \underbrace{X_2 - W}_{X_2 + W \sim N(0, 6+1)} \Longrightarrow X_1 \sim N(y, 6+1)$$

linear transformation of a normal is still normal due to independence

By symmetry we have

$$X_2 \sim N(y, 6+1)$$

2) The expectation of X_1 and X_2 is y, which doesn't depend on 6

$$\Longrightarrow \lim_{6 \to 0} y = \lim_{6 \to +\infty} y = y$$

While

$$\lim_{6 \to 0} 6 + 1 = 1$$

and

$$\lim_{6 \to \infty} 6 + 1 = +\infty$$