Problem 1 Consider the Markov chain that takes values on $S = \{1, 2, 3, 4\}$. The transition probability from state j to state i is given by

$$[P_{ij}] = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

1. Let $X_0 = 1$. Compute the probability mass function of X_2.
2. Does the probability distribution of X_n converges as n goes to infinity? If so, find the limit. If not, explain why.

(1)

We know that $X_0 = 1 \Rightarrow$ row vector $\pi^{(0)} = (1, 0, 0, 0)$

Let $\pi(A) = \sum_{i \in A} \pi_i$, $A \subseteq I$ - a non-negative measure on I

A measure π is invariant for a transition matrix P if $\pi P = \pi$, that is $\sum_{i \in I} \pi_i p_{ij} = \pi_j \; \forall j \in I$

Definition:

$\pi^{(n)} = (P(X_n = 1), P(X_n = 2), \dots P(X_n = r))$

$\pi^{(2)} = (P(X_2 = 1), P(X_2 = 2), P(X_3 = 3), P(X_4 = 4))$

$\pi^{(1)} = \pi^{(0)} P$

$\pi^{(2)} = \pi^{(1)} P = \pi^{(0)} P^2$

Therefore for we need $X_2 \sim \pi^T P^2$

$\Downarrow$

$$\pi^T P^2 = (1\ 0\ 0\ 0) \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

$= \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$

Thus, $P(X_2 = 1) = 0{,}5$, $P(X_2 = 2) = 0$, $P(X_2 = 3) = 0{,}5$, $P(X_2 = 4) = 0$.

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Let's find the invariant measure:

$$\begin{cases} \pi_1 = 1/3\,\pi_2 + 2/3\,\pi_4 \\ \pi_2 = 1/2\,\pi_1 + 1/2\,\pi_3 \\ \pi_3 = 2/3\,\pi_2 + 1/3\,\pi_4 \\ \pi_4 = 1/2\,\pi_1 + 1/2\,\pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \begin{cases} \pi_2 = \pi_4 \\ \pi_1 = 1/3\,\pi_2 + 2/3\,\pi_2 \\ \pi_3 = 2/3\,\pi_2 + 1/3\,\pi_2 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \pi_2 = \pi_4 \\ \pi_1 = \pi_2 \\ \pi_3 = \pi_2 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases} \Rightarrow \quad \pi_1 = \pi_2 = \pi_3 = \pi_4 = 1/4.$$

$\pi = (1/4,\ 1/4,\ 1/4,\ 1/4)$ – the invariant measure

$$P\pi^* = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ The probability distribution of X_n converges to invariant measure.