

Problem 1 In this exercise, we will do a very simple filtering and smoothing step by hand. The forward map is given as

$$X_{n+1} = \frac{1}{2}X_n + 1 + \Xi_n$$

where $\Xi_n \sim N(0, 1)$. The observation operator is given by

$$Y_n = X_n + \sqrt{2}\Sigma_n$$

with $\Sigma_n \sim N(0, 1)$. Assume $X_0 \sim N(-1, 2)$. All noise processes are independent.

1. Prediction step: What is the distribution of X_1 ?
2. Filtering step: What is the distribution of X_1 conditioned on $Y_1 = 2$?
3. Smoothing step: What is the distribution of X_0 conditioned on $Y_1 = 2$?
4. Now we want to implement this in pseudo code. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim N(m, 1)$$

with the same observation model and noise parameter. You also observe that $Y_1 = y$.

Write a pseudo code, that given the inputs α, β, m and y will output the distribution of

(i) X_1 , (ii) X_1 conditioned on $Y_1 = y$, and (iii) X_0 conditioned on $Y_1 = y$.

①

The properties:

$$\begin{aligned} \text{if } X \sim N(\mu, \sigma^2) \text{ then } & \mu + X \sim N(\mu + \alpha, \sigma^2) \\ & \alpha X \sim N(\alpha\mu, \alpha^2\sigma^2) \\ & X + Y \sim N(\mu + c, \sigma^2 + d) \text{ where } Y \sim N(c, d) \end{aligned}$$

$$X_1 = \frac{1}{2}X_0 + 1 + \Xi_0$$

$$1) X_0 \sim N(-1, 2) \Rightarrow \frac{1}{2}X_0 \sim N\left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$2) \left(\frac{1}{2}X_0 + 1\right) \sim N\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$3) \Xi_0 \sim N(0, 1) \Rightarrow \left(\frac{1}{2}X_0 + 1 + \Xi_0\right) \sim N\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$\Rightarrow X_1 \sim N\left(\frac{1}{2}, \frac{3}{2}\right)$$

(2)

1) From the previous step the next mean and variance are:

$$\mu_1 = \frac{1}{2}, \sigma_1^2 = \frac{3}{2}$$

$$2) \text{ Kalman Gain} = \frac{\text{Var}[X_1]}{\text{Var}[X_1] + \text{Var}[\sqrt{2}\sum_n]}, \text{ where } \sqrt{2}\sum_n \text{- observation noise}$$

$$3) \sum_n \sim N(0, 1) \Rightarrow \sqrt{2}\sum_n \sim N(0, 2)$$

$$\text{Kalman gain} = \frac{\frac{3}{2}}{\frac{3}{2} + 2} = \frac{3}{7}$$

$$4) \mu_{\text{new}} = \mu_1 - \text{Kalman gain}(\mu_1 - y_1) = \frac{1}{2} - \frac{3}{7}(\frac{1}{2} - 2) = \frac{8}{7}$$

$$\sigma_{\text{new}}^2 = \sigma_1^2(1 - \text{Kalman gain}) = \frac{3}{2}(1 - \frac{3}{7}) = \frac{6}{7}$$

$$(X_1 | Y_1=2) \sim N(\frac{8}{7}, \frac{6}{7})$$

(3)

$$1) X_0 \sim N(-1, 2)$$

$$2) Y_1 = X_1 + \sqrt{2}\sum_1, \sqrt{2}\sum_1 \sim N(0, 2), X_1 \sim N\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$Y_1 \sim N\left(\frac{1}{2} + 0, \frac{3}{2} + 2\right) \Rightarrow Y_1 \sim N\left(\frac{1}{2}, \frac{7}{2}\right)$$

$$3) \text{Also } Y_1 = X_1 + \sqrt{2}\sum_1 = \frac{1}{2}X_0 + 1 + \sum_{j=2}^1 + \sqrt{2}\sum_1$$

$$\text{if } y_1 = 2, \text{ then } 2 = \frac{1}{2}x_0 + 1 + \sum_{j=2}^1 + \sqrt{2}\sum_1$$

$$\frac{1}{2}x_0 = 2 - 1 - \sum_{j=2}^1 - \sqrt{2}\sum_1$$

$$\frac{1}{2}x_0|y_1=2 = 1 - \sum_{j=2}^1 - \sqrt{2}\sum_1$$

$$X + \alpha \sim N(\alpha + \mu, \sigma^2)$$

$$4) (-1)\sum_{j=2}^1 \sim N(0, 1)$$

$$(-1) \cdot \sqrt{2}\sum_1 \sim N(0, 2) \Rightarrow \frac{1}{2}x_0|y_1=2 \sim N(1+0+0, 1+2) = N(1, 3)$$

$$\text{Thus we get: } X_0|y_1=2 = N\left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1, \frac{3}{2}\right) = N\left(\frac{1}{2}, \frac{3}{4}\right)$$

(4)

4. Now we want to implement this in pseudo code. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim N(m, 1)$$

with the same observation model and noise parameter. You also observe that $Y_1 = y$. Write a pseudo code, that given the inputs α, β, m and y will output the distribution of (i) X_1 , (ii) X_1 conditioned on $Y_1 = y$, and (iii) X_0 conditioned on $Y_1 = y$.

① `def X1(d, β, m, y, noise_mean=0, noise_var=1):`

$$\begin{aligned} \text{mean}_{X_0} &= d \cdot m \\ \text{var}_{X_0} &= d^{**2} \cdot 1 \end{aligned}$$

$$\begin{aligned} \text{mean}_{X_1} &= d \cdot m + \beta \cdot \text{noise_mean} \\ \text{var}_{X_1} &= \text{var}_{X_0} + \beta^{**2} \end{aligned}$$

`return (mean_{X_1}, var_{X_1})`

② `def X1_Y1 (mean_{X_1}, var_{X_1}, y, noise_mean=0, noise_var=2):`

$$\text{kalman_gain} = (\text{var}_{X_1} / (\text{var}_{X_1} + \text{noise_var}))$$

$$\text{mean}_{X_1_Y_1} = \text{mean}_{X_1} - \text{kalman_gain} (\text{mean}_{X_1} - y)$$

$$\text{var}_{X_1_Y_1} = \text{var}_{X_1} (1 - \text{kalman_gain})$$

`return (mean_{X_1_Y_1}, var_{X_1_Y_1})`