Bayesian inference and Data assimilation

Exercise 9

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Problem 1 Let p_n be the probability distribution of X_n (in column vector). Then we have

$$p_{n+1} = Pp_n$$

The Markov chain has a unique invariant measure $[1/4, 1/4, 1/4, 1/4]^\top$.

1. If $X_0 = 1$, that is, $p_0 = [1, 0, 0, 0]^\top$,

$$p_1 = Pp_0 = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \end{bmatrix}^\top$$

   and

$$p_2 = Pp_1 = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \end{bmatrix}^\top$$

2. It does NOT converge to any probability distribution. **Counterexample:** By induction, it is straightforward that for all $n \geq 1$,

$$p_n = \begin{cases} \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \end{bmatrix}^\top & \text{if } n \text{ is even} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \end{bmatrix}^\top & \text{if } n \text{ is odd} \end{cases}$$

   Discussion: One can show that $p_{n+1}(1) + p_{n+1}(3) = p_n(2) + p_n(4)$ and vice-versa. Therefore, if $p_0(1) + p_0(3) \neq p_0(2) + p_0(4)$, then p_n will never converge.

Problem 2 Observe that $p_{n+1}(1) + p_{n+1}(2) = p_n(1) + p_n(2)$. Intuitively, this means that no probability mass can be conveyed from $\{1, 2\}$ to $\{3, 4\}$.

1. If $X_0 = 1$, that is, $p_0 = [1, 0, 0, 0]^\top$, then $p_n(3) = p_n(4) = 0$ for all $n \geq 0$ and therefore it behaves like a Markov chain on $\{1, 2\}$, and its state transition matrix is given by

$$P_1 = \begin{pmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{pmatrix}$$

   Let the invariant measure be $\frac{1}{a+b}[a, b]^\top$ and then it solves

$$\frac{1}{2}a + \frac{2}{3}b = a$$

   An easy way to solve this is as follows: (1) set some number for a and then (2) normalize later. Let $a = 2 \cdot 2 = 4$ and then we have $b = 3$. Thus the invariant measure for P_1 is $\left[\frac{4}{7}, \frac{3}{7}\right]$.

   Therefore, the original Markov chain converges to

$$p_n \quad \longrightarrow \quad \begin{bmatrix} \frac{4}{7} & \frac{3}{7} & 0 & 0 \end{bmatrix}^\top$$

   provided that $p_0 = [1, 0, 0, 0]^\top$.

2. If $X_0 = 3$, then $p_n(1) = p_n(2) = 0$ for all $n \geq 0$ and therefore it behaves like a Markov chain on $\{3, 4\}$, and its state transition matrix is given by

$$P_2 = \begin{pmatrix} 1/4 & 4/5 \\ 3/4 & 1/5 \end{pmatrix}$$

Again set the invariant measure be $\frac{1}{a+b}[a, b]^\top$ and then it solves

$$\frac{1}{4}a + \frac{4}{5}b = a$$

Let $a = 16$ and then we have $b = 15$. Thus the invariant measure for P_2 is $\left[\frac{16}{31}, \frac{15}{31}\right]$. Therefore, the original Markov chain converges to

$$p_n \quad \longrightarrow \quad \begin{bmatrix} 0 & 0 & \frac{16}{31} & \frac{15}{31} \end{bmatrix}^\top$$

provided that $p_0 = [0, 0, 1, 0]^\top$.

3. In general case, $c := p_0(1) + p_0(2)$ can take any values in $0 \leq c \leq 1$. Then it converges to

$$p_n \quad \longrightarrow \quad \begin{bmatrix} \frac{4}{7}c & \frac{3}{7}c & \frac{16}{31}(1-c) & \frac{15}{31}(1-c) \end{bmatrix}^\top$$

Problem 3 Given information is $X \sim N(1, 1)$ and $Y = X^2 + W$ with $W \sim N(0, 1)$. It is assumed that X and W are independent.

1. Given $X = x$, Y is determined by W, shifted by x^2. Therefore the following events are equivalent:

$$[X = x, Y = y] = [X = x, W = y - x^2]$$

2. Recall the Bayes rule:

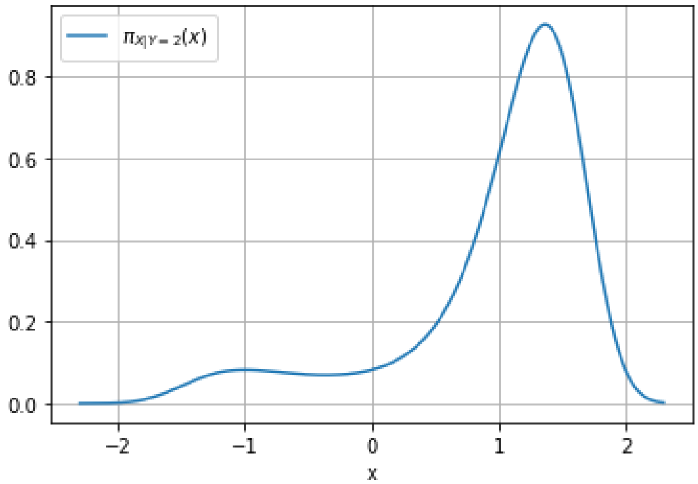
$$\pi_{X|Y}(x \mid y) = \frac{\pi_{Y|X}(y \mid x)\pi_X(x)}{\pi_Y(y)}$$

We assert that $\pi_{Y|X}(y \mid x) = \pi_W(y - x^2)$. Therefore,

$$\pi_{X|Y}(x \mid y) = \frac{1}{Z} \exp\Big(-\frac{1}{2}(y - x^2)^2\Big) \exp\Big(-\frac{1}{2}(x-1)^2\Big)$$

where Z is the normalization factor.

3. The plot for $y = 2$ is as follows:



It attains its maximum value at $x = 1.366$.

Code is here:

```
import numpy as np
import matplotlib.pyplot as plt

x = np.arange(-2.3,2.3,0.001)
y = np.exp(-0.5*(2-x**2)**2-0.5*(x-1)**2)

plt.plot(x,y,label=r'$\pi_{X|Y=2}(x)$')
plt.grid()
plt.xlabel('x')
plt.legend()

print('It attains its maximum value at {:5.3f}'.format(x[y.argmax()]))
```