Problem 2 We consider a real-valued random variable X that has a probability density.

1. Show that $c = \mathbb{E}[X]$ minimizes the mean-squared error $\mathbb{E}\big[(X-c)^2\big]$.
2. Show that the median minimizes $\mathbb{E}\big[|X-c|\big]$. The median is defined as a number c such that $\mathbb{P}[X < c] = \mathbb{P}[X > c] = 0.5$.

①

Let's transform the formula, taking into account that $E[X] = \mu$:

$$\mathbb{E}\left[(X-c)^2\right] = \mathbb{E}\left[(X-\mu+\mu-c)^2\right] = \mathbb{E}\left[\left((X-\mu)+(\mu-c)\right)^2\right] =$$

$$= \mathbb{E}\left[(X-\mu)^2 + 2(\mu-c)(X-\mu) + (\mu-c)^2\right] =$$

$$= \mathbb{E}\left[(X-\mu)^2\right] + 2(\mu-c)\underbrace{\mathbb{E}[X-\mu]}_{=0} + (\mu-c)^2 =$$

$$= \mathbb{E}\left[(X-\mu)^2\right] + 2(\mu-c)\left(\mathbb{E}[X]-\mu\right) + (\mu-c)^2 =$$

$$= \{\mathbb{E}[X] = \mu\} = \mathbb{E}\left[(X-\mu)^2\right] + (\mu-c)^2$$

- The minimum of mean squared error will be if $(\mu-c)^2 = 0$, i.e. $c = \mu$.
- From the definition $E[X] = \mu \Rightarrow c = \mu = E[X]$ minimizes MSE

$\blacksquare$

②

Let's $F(c) = \mathbb{E}\left[|X-c|\right]$ and $f(x)$ is pdf.

1) $$F(c) = \mathbb{E}\left[|X-c|\right] = \int_{\mathbb{R}} |x-c| f(x)\,dx = \int_{-\infty}^{c} (c-x) f(x)\,dx + \int_{c}^{\infty} (x-c) f(x)\,dx$$

let's compute the derivative:

$$\frac{dF}{dc} = (c-x)f(x)\Big|_{x=c} + \int_{-\infty}^{c} f(x)\,dx + (x-c)f(x)\Big|_{x=c^-}$$

$$-\int_{c}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx - \int_{c}^{\infty} f(x)\,dx = 0$$

equals to 0 as we are searching for extreme

$\Downarrow$

The minimum if:

$$\int_{-\infty}^{c} f(x)\,dx = \int_{c}^{\infty} f(x)\,dx \iff P(X<c) = P(X>c) = 0{,}5$$

$\Rightarrow$ the median minimises $E[|X-c|]$