Bayesian inference and Data assimilation

Exercise 3

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Problem 1 Observe that

$$\begin{aligned}\hat{X} - X &= KY + b - X \\ &= K(HX + W) + b - X \\ &= b + (KH - 1)X + KW\end{aligned}$$

Therefore,

$$\begin{aligned}(\hat{X} - X)^2 = b^2 + (KH - 1)^2 X^2 + K^2 W^2 + 2b(KH - 1)X \\ + 2bKW + 2(KH - 1)KXW\end{aligned}$$

Since $\mathbb{E}[W] = \mathbb{E}[XW] = 0$ and $\mathbb{E}[X^2] = Q + m^2$, we obtain

$$\mathbb{E}\big[(\hat{X} - X)^2\big] = b^2 + 2b(KH - 1)m + (KH - 1)^2(Q + m^2) + K^2 R$$

Therefore, the error is minimized at

$$b^* = -(KH - 1)m$$

With this choice, we have

$$\hat{X} - X = (KH - 1)(X - m) + KW$$

Its mean is always zero and the mean squared error becomes

$$\mathbb{E}\big[(\hat{X} - X)^2\big] = K^2(H^2 Q + R) - 2KHQ + Q$$

Thus, it is minimized at

$$K^* = \frac{HQ}{H^2 Q + R}$$

and the corresponding optimal value is

$$\min_{K,b} \mathbb{E}\big[(\hat{X} - X)^2\big] = Q - \frac{H^2 Q^2}{H^2 Q + R}$$

Notes: Completion of square trick:

$$ax^2 + 2bx + c = a\Big(x + \frac{b}{a}\Big)^2 + c - \frac{b^2}{a}$$

If $a > 0$, it is minimized at $x = -b/a$ and the minimum value is $c - b^2/a$.

Problem 2 Since $|\Sigma| = \sigma_{11}^2 \sigma 22^2 - \sigma_{12}^2 \sigma_{21}^2$, it is straightforward to guess

$$\sigma_c = \frac{\sqrt{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2}}{\sigma_{22}} = \sqrt{\sigma_{11}^2 - \frac{\sigma_{12}^2 \sigma_{21}^2}{\sigma_{22}}}$$

It remains to show the choice admit $\bar{x}_c$ such that

$$\exp\Big(-\frac{1}{2}(z-\bar{z})^{\mathrm{T}} \Sigma^{-1} (z-\bar{z})\Big) = \exp\Big(-\frac{1}{2\sigma_c^2}(x_1 - \bar{x}_c)^2 - \frac{1}{2\sigma_{22}^2}(x_2 - \bar{x}_2)^2\Big)$$

After a tedious but straightforward calculation, one obtains

$$\bar{x}_c = \bar{x}_1 + \frac{\sigma_{12}^2}{\sigma_{22}^2}(x_2 - \bar{x}_2)$$

Problem 3

$$\mathbb{E}[Z] = (m, Hm), \quad \mathrm{var}(Z) = \begin{pmatrix} Q & QH \\ HQ & H^2Q + R \end{pmatrix}$$

Now the conditional distribution of X given Y is also a Gaussian with mean

$$\mathbb{E}[X \mid Y] = m + \frac{HQ}{H^2Q + R}(Y - Hm)$$

and variance

$$\mathbb{E}[(X - \mathbb{E}[X \mid Y])^2 \mid Y] = Q - \frac{H^2Q^2}{H^2Q + R}$$

which are exactly the best estimator $\hat{X}$ and its mean squared error value.

Problem 4 The marginal probability density $\pi_{X_2}(x_2)$ is given by

$$\begin{aligned}
\pi_{X_2}(x_2) &= \int \pi_{X_1,X_2}(x_1, x_2)\,\mathrm{d}x_1 \\
&= \int_{-\infty}^{\infty} \frac{1}{Z} \exp\big(-x_1^2 - x_2^2 - x_1^2 x_2^2\big)\,\mathrm{d}x_1 \\
&= \frac{\exp(-x_2^2)}{Z} \int_{-\infty}^{\infty} \exp\big(-x_1^2(x_2^2 + 1)\big)\,\mathrm{d}x_1
\end{aligned}$$

Since $x_2^2 + 1 > 0$, we can compute the integral by converting the integrand to the probability density function of a Gaussian. Let

$$\sigma := \frac{1}{\sqrt{2(x_2^2 + 1)}}$$

Because

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\big(-\frac{x_1^2}{2\sigma^2}\big)\,\mathrm{d}x_1 = 1$$

$$\implies \int_{-\infty}^{\infty} \exp\big(-x_1^2(x_2^2+1)\big)\,\mathrm{d}x_1 = \sqrt{\frac{\pi}{x_2^2+1}}$$

Therefore,

$$\pi_{X_2}(x_2) = \frac{1}{Z}\sqrt{\frac{\pi}{x_2^2+1}}\exp(-x_2^2)$$

and

$$\pi_{X_1|X_2}(x_1 \mid a) = \frac{\pi_{X_1,X_2}(x_1,x_2)}{\pi_{X_2}(x_2)} = \sqrt{\frac{x_2^2+1}{\pi}}\exp(-x_1^2 - x_1^2x_2^2)$$

Now

$$\begin{aligned}\mathbb{E}[X_1^2X_2 \mid X_2 = a] &= \int_{-\infty}^{\infty} x_1^2 a\sqrt{\frac{a^2+1}{\pi}}\exp(-x_1^2 - x_1^2a^2)\,\mathrm{d}x_1 \\ &= a\sqrt{\frac{a^2+1}{\pi}}\int_{-\infty}^{\infty} x_1^2\exp(-x_1^2(a^2+1))\,\mathrm{d}x_1\end{aligned}$$

We use the same trick as before:

$$\int_{-\infty}^{\infty} \frac{x_1^2}{\sqrt{2\pi}\sigma}\exp\big(-\frac{x_1^2}{2\sigma^2}\big)\,\mathrm{d}x_1 = \sigma^2$$

$$\implies \int_{-\infty}^{\infty} x_1^2\exp(-x_1^2(a^2+1))\,\mathrm{d}x_1 = \sqrt{2\pi}\sigma^3$$

Therefore

$$\mathbb{E}[X_1^2X_2 \mid X_2 = a] = \frac{a}{2(a^2+1)}$$