③

(1) Let $f \mid \|f\|_\infty \le 1$, then

$$|\mathbb{E}_{\pi_1}[f] - \mathbb{E}_{\tilde{\pi}_1}[f]| = \left|\int_{\mathbb{R}} f(x)\,(\pi_1(x) - \tilde{\pi}_1(x))\,dx\right| =$$

$$= \left|\int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} P(x,x')\,(\pi_0(x') - \tilde{\pi}_0(x'))\,dx'\right)dx\right| =$$

$$= \left|\int_{\mathbb{R}} (\pi_0(x') - \tilde{\pi}_0(x'))\underbrace{\left(\int_{\mathbb{R}} f(x)\,P(x,x')\,dx\right)}_{= g(x')} dx'\right|$$

$$g(x') := \int_{\mathbb{R}} f(x)\,P(x,x')\,dx = \mathbb{E}_{P(x,x')}[f] \implies \|g\|_\infty \le 1$$

$\|f\|_\infty \le 1$

we have shown that $\forall f \mid \|f\|_\infty \le 1$

$\exists\, g \mid \|g\|_\infty \le 1$ such that:

$$|\mathbb{E}_{\pi_1}[f] - \mathbb{E}_{\tilde{\pi}_1}[f]| = |\mathbb{E}_{\pi_0}[g] - \mathbb{E}_{\tilde{\pi}_0}[g]|$$

$\implies \forall f \mid \|f\|_\infty \le 1$

$$\sup_{\|h\|_\infty \le 1} |\mathbb{E}_{\pi_0}[h] - \mathbb{E}_{\tilde{\pi}_0}[h]| \ge |\mathbb{E}_{\pi_0}[g] - \mathbb{E}_{\tilde{\pi}_0}[g]| =$$

$$= |\mathbb{E}_{\pi_1}[f] - \mathbb{E}_{\tilde{\pi}_1}[f]|$$

since this holds $\forall f \mid \|f\|_\infty \le 1$ we have:

$$\sup_{\|h\|_\infty \le 1} |\mathbb{E}_{\pi_0}[h] - \mathbb{E}_{\tilde{\pi}_0}[h]| \ge \sup_{\|f\|_\infty \le 1} |\mathbb{E}_{\pi_1}[f] - \mathbb{E}_{\tilde{\pi}_1}[f]|$$

$$\iff d_{TV}(\pi_1, \tilde{\pi}_1) \le d_{TV}(\pi_0, \tilde{\pi}_0)$$

Q.E.D

(2) We can prove the statement by induction:

Base case: $n=0$ $\quad d_{TV}(\tilde{\pi}_0, \pi_0) \leq d_{TV}(\tilde{\pi}_0, \pi_0)$

Inductive hp: $\quad d_{TV}(\tilde{\pi}_n, \pi_n) \leq d_{TV}(\tilde{\pi}_0, \pi_0) \quad \forall n \geq 0$

$$d_{TV}(\tilde{\pi}_{n+1}, \pi_{n+1}) \underset{\substack{\downarrow \\ \text{same proof of} \\ \text{(1) with } n+1 \\ \text{instead of } 1 \text{ and} \\ n \text{ instead of } 0.}}{\leq} d_{TV}(\tilde{\pi}_n, \pi_n) \underset{\substack{\downarrow \\ \text{Inductive} \\ \text{hp.}}}{\leq} d_{TV}(\tilde{\pi}_0, \pi_0)$$

$\Rightarrow$

$$\forall n \geq 0 \qquad d_{TV}(\tilde{\pi}_n, \pi_n) \leq d_{TV}(\tilde{\pi}_0, \pi_0)$$