Problem 2

$X \sim N(1,3)$, $f(x) = 1 + 2x + x^2$

- $\mathbb{E}[f(X)] = \mathbb{E}[1 + 2x + x^2] =$

$= \mathbb{E}[1] + \mathbb{E}[2x] + \mathbb{E}[x^2] = 2\mathbb{E}[x] + 1 +$
$+ \mathbb{E}[x^2]$.

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{\sqrt{3}}\right)^2} dx =$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{18\pi}} e^{\frac{-(x-1)^2}{18}} dx = 1$$

$\mathbb{E}[x^2] = Var[x] + (\mathbb{E}[x])^2 = 3 + 1 = 4$

$\Rightarrow \boxed{\mathbb{E}[f(x)] = 1 + 2 \cdot 1 + 4 = 7.}$

- $Var[f(X)] = Var[1 + 2x + x^2] =$
$= \{Var[a + x] = Var[x]\} = Var[2x + x^2] =$

$= Var[2x] + Var[x^2] + 2Cov(2x, x^2) =$

$= 4Var[x] + \mathbb{E}[x^4] - (\mathbb{E}[x^2])^2 + 2\mathbb{E}[2x \cdot x^2] -$
$- 2\mathbb{E}[2x]\mathbb{E}[x^2] =$

$$\cancel{= 2\cdot 3 + 4 - 1 + 2\mathbb{E}[x^3] - 2\cdot 1\cdot 4 =}$$

$$\cancel{= 9 - 1 + 2\mathbb{E}[x^3]}$$

$$= 4\cdot 3 + \mathbb{E}[x^4] - 4^2 + 2\cdot 2\cdot \mathbb{E}[x^3] -$$

$$- 4\cdot 1\cdot 4 = 12 + \mathbb{E}[x^4] + 4\,\mathbb{E}[x^3] - 32$$

$$\mathbb{E}[x^4] = Var(x^2) + \mathbb{E}[x^2]^2 =$$

$$= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 = 1 + 6\cdot 3 + 3\cdot 9 =$$

$$= 1 + 18 + 27 = \underline{46}$$

$$\mathbb{E}[x^3] = \mu^3 + 3\mu\sigma^2 = 1 + 3\cdot 1\cdot 3 = 10$$

$$\Rightarrow Var[f(X)] = 12 + 46 + 40 - 32 =$$

$$= 98 - 32 = 66.$$

4) $$\mathbb{E}[f_M] = \mathbb{E}\left[\frac{1}{M}\sum_{i=1}^{M} f(x_i)\right] =$$

$$= \frac{1}{M}\mathbb{E}\left[\sum_{i=1}^{M} f(x_i)\right] = \frac{1}{M}\cdot 7M = 7$$