(Pr 1) Consider the 2 sets

$X_1 := \{a_1 = 1, a_2 = 2, a_3 = 3\}$

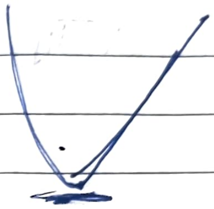
$X_2 = \{b_1 = 1.5, b_2 = 2, b_3 = -1\}$

with uniform probability mass:

$P(a_i) = P(b_i) = 1/3$ for $i = 1, 2, 3$.

A coupling is defined by a matrix $T \in \mathbb{R}^{3\times 3}$ with $t_{ij} \geq 0$

and $\sum_{i=1}^{3} t_{ij} = \sum_{j=1}^{3} t_{ij} = 1/3$



Find a coupling that minimizes

$$J(t) = \sum_{i,j=1}^{3} t_{ij} \, |b_i - a_j|^2$$

What do you notice about sparsity structure of the optimal coupling matrix T^*?

→ This problem is a special case of transport problem i.e. assignment problem

$$\begin{array}{c|ccc|c} & a_1 & a_2 & a_3 & \\ \hline b_1 & & & & 1/3 \\ b_2 & & & & 1/3 \\ b_3 & & & & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 & \end{array}$$

$D_{11} = |b_1 - a_1|^2 = 1/4 = 0.25$

$D_{12} = 1/4 = 0.25$

$D_{13} = (1.5)^2 = 9/4 = 2.25$

$D_{21} = 1$

$D_{22} = 0$

$D_{23} = 1$

$D_{31} = 4$

$D_{32} = 9$

$D_{33} = 16$

To minimise T_{ij} we can begin by setting $\frac{1}{3}$ probability to every element ~~to a row~~ at ~~a ti~~ a time in a row.
& then get respective probabilities of other elements. Doing this we will have 3 matrices & we can choose the minimal J yield value yielding one as optimum.

Lets take the 1st row

$$T_1 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & t_{22} & t_{23} \\ 0 & t_{32} & t_{33} \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & 1/3 & 0 \\ t_{21} & 0 & t_{23} \\ t_{31} & 0 & t_{33} \end{bmatrix} \quad T_3 = \begin{bmatrix} 0 & 0 & 1/3 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & 0 \end{bmatrix}$$

We can represent every other element wrt a single element in a matrix.

For T_1 $t_{23} = -t_{22} + 1/3$	$t_{23} = \frac{1}{3} - t_{21}$	$t_{21} = 1/3 - t_{22}$
$t_{32} = \frac{1}{3} - t_{22}$	$t_{31} = 1/3 - t_{21}$	$t_{31} = t_{22}$
$t_{33} = t_{22}$	$t_{33} = t_{21}$	$t_{32} = \frac{1}{3} - t_{22}$
$J(T_1) = \frac{1}{3}(b_1 - a_1)^2 + t_{22}(b_2 - a_2)^2 + t_{23}(b_2 - a_3)^2 + t_{32}(b_3 - a_2)^2 + t_{33}(b_3 - a_3)^2$	$\therefore J(T_2) = \frac{1}{3}(b_1 - a_2)^2 + t_{21}(b_2 - a_1)^2 + t_{23}(b_2 - a_3)^2 + t_{31}(b_3 - a_1)^2 + t_{33}(b_3 - a_3)^2$	$\therefore J(T_3) = \frac{1}{3}(b_1 - a_3)^2 + t_{21}(b_2 - a_1)^2 + t_{22}(b_2 - a_2)^2 + t_{31}(b_3 - a_1)^2 + t_{32}(b_3 - a_2)^2$
Now we simply	assign values of	b_i & a_i

$J(T_1) = \frac{1}{3} \times 0.25 + 0 + (\frac{1}{3} - t_{22})(1) + (\frac{1}{3} - t_{22})(9)$

$+ t_{22} \times (16)$

$= \frac{0.25}{3} + 0 + \frac{1}{3} - t_{22} + 3 - 9 t_{22} + 16 t_{22}$

$= \frac{1.25}{3} + 3 + 6 t_{22}$

$= \frac{10.25}{3} + 6 t_{22} \Rightarrow$ Min $(J(T_1))$ for $t_{22} = 0$ is 3.42

$J(T_2) = \frac{1}{3} \cdot D_{12} + t_{21} \cdot D_{21} + (\frac{1}{3} - t_{21}) \cdot D_{23}$

$+ (\frac{1}{3} - t_{21}) \cdot D_{31} + t_{21} \cdot D_{33}$

$= \frac{1}{3} \times 0.25 + \cancel{t_{21}} + \frac{1}{3} - \cancel{t_{21}} + \frac{4}{3} - 4 t_{21}$

$+ 16 t_{21}$

$= \frac{5.25}{3} + 12 t_{21}$ Min $(J(T_2))$ is for $t_{21} = 0$ is 1.75.

$J(T_3) = \frac{1}{3} D_{13} + (\frac{1}{3} - t_{22})(D_{21}) + \cancel{t_{22} \cdot D_{22}}$

$+ t_{22} \cdot D_{31} + (\frac{1}{3} - t_{22}) D_{32}$

$= \frac{1}{3} \times 2.25 + \frac{1}{3} - t_{22} + t_{22} \cdot 4 + 3 - 9 t_{22}$

$= \frac{12.25}{3} - 6 t_{22}$

$J(T_3)$ is min if t_{22} is max i.e. $\frac{1}{3}$

$\therefore J(T_3) = \frac{12.25}{3} - \frac{6}{3} = \frac{6.25}{3} \approx 2.08$

∴ We have max min for $J(T_2)$ for $t_{21} = 0$

∴ By putting respective elements wrt t_{21}

$$t_{23} = 1/3 \quad , t_{31} = 1/3 \quad t_{33} = 0$$

& T_2 optimal is $\begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \\ 1/3 & 0 & 0 \end{bmatrix}$

EXERCISE 2A

$X \sim Unif([0,1])$, $\pi_X(x) = \mathbb{1}_{[0,1]}(x)$

$\forall f \; \mathbb{R} \to \mathbb{R} \; (f(X) \in L^1)$

$$E[f(X)] = \int_{\mathbb{R}} f(x)\pi_X(x)dx = \int_0^1 f(x)\,dx$$

Given $M \in \mathbb{N}$ a quadrature rule is

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{M} b_i f(c_i)$$

$b_i, c_i \in \mathbb{R}$ $\forall i \in \{1, \ldots, M\}$

Gauss - Legendre quadrature rule

$$\int_{-1}^{1} f(\tilde{x})\,d\tilde{x} \approx \sum_{i=1}^{M} \hat{b}_i f(\tilde{c}_i),$$

where $\{\tilde{c}_i\}_{i=1}^{M}$ are the roots of $P_M(\tilde{x})$, M - degree

Legendre polynom, and

$$\tilde{b}_1 = \frac{2}{(1-\tilde{c}_1^2)(P'_M(\tilde{c}_1))^2}$$

$\boxed{1}$ Given $M=1$

$$E[f(X)] = \int_0^1 f(x)\,dx =$$

$$= \frac{1}{2}\int_{-1}^{1} f\left(\frac{\tilde{x}}{2}+\frac{1}{2}\right) d\tilde{x} \approx$$

$\hookrightarrow$ $x = \frac{\tilde{x}}{2} + \frac{1}{2}$ $\quad \tilde{x} \in [-1,1]$

$dx = \frac{d\tilde{x}}{2}$

$$\approx \frac{1}{2}\tilde{b}_1 f\left(\frac{\tilde{c}_1}{2}+\frac{1}{2}\right) = f\left(\frac{1}{2}\right)$$

$\hookrightarrow$ GAUSS-LEGENDRE QUADRATURE RULE $\rightarrow$ $\tilde{b}_1 = 2$, $\tilde{c}_1 = 0$ since $P_1(\tilde{x}) = \tilde{x}$

Then, for $M=1$, $b_1 = 1$ and

$c_1 = \frac{1}{2}$

This quadrature rule has order $p=2$ since $\forall f \in \Pi_1(\mathbb{R})$ $\int_0^1 f(x)\,dx = b_1 f(c_1)$

If $f(x) = a_0 \in \mathbb{R}$

$$\int_0^1 f(x)\,dx = a_0 = 1\, f(1/2) = b_1 f(c_1)$$

If $f(x) = a_0 + a_1 x$ $\quad a_i \in \mathbb{R}$ $\forall i \in \{0,1\}$

$$\int_0^1 f(x)\,dx = a_0 + \frac{a_1}{2} =$$

$$= 1\, f(1/2) = b_1 f(c_1)$$

$\boxed{2}$ If we consider $M=2$ quadrature points by proceeding as in $\boxed{1}$ we get

$$E[f(X)] = \int_0^1 f(x)\,dx \approx$$

$$\approx \frac{1}{2} \sum_{i=1}^{2} \tilde{b}_i \, f\left(\frac{1}{2}\tilde{c}_i + \frac{1}{2}\right) \quad (*)$$

$$P_2(\check{x}) = \frac{1}{2}(3\check{x}^2 - 1) = 0$$

$$\Leftrightarrow \check{x} = -\frac{1}{\sqrt{3}} = \tilde{c}_1 \quad \vee \quad \check{x} = \frac{1}{\sqrt{3}} = \tilde{c}_2$$

and $\tilde{b}_1 = \tilde{b}_2 = 1$

$$\Longrightarrow$$

$$(*) = \frac{1}{2}\left(f\left(\frac{1}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\right) + f\left(\frac{1}{2}\frac{1}{\sqrt{3}} + \frac{1}{2}\right)\right) =$$

$$= \frac{1}{2} f\left(\frac{1 - 1/\sqrt{3}}{2}\right) + \frac{1}{2} f\left(\frac{1/\sqrt{3} + 1}{2}\right)$$

Then, $b_1 = b_2 = 1/2$ and $c_1 = \frac{1 - 1/\sqrt{3}}{2}$,

$$c_2 = \frac{1 + 1/\sqrt{3}}{2}$$

The formula has order $p = 3$

since $\int_0^1 f(x)\,dx = \sum_{i=1}^{2} b_i f(c_i) \quad \forall f \in \Pi_2(\mathbb{R})$

If $f(x) = a_0 \in \mathbb{R}$

$$\int_0^1 f(x)\,dx = a_0 = \frac{1}{2}(2a_0) =$$

$$= b_1 f(c_1) + b_2 f(c_2)$$

If $f(x) = a_0 + a_1 x$ $\quad a_0, a_1 \in \mathbb{R}$

$$\int_0^1 f(x)\,dx = a_0 + \frac{a_1}{2}$$ and

$$b_1 f(c_1) + b_2 f(c_2) = \frac{1}{2}\Bigg(a_0 + a_1\left(\frac{1-1/\sqrt{3}}{2}\right) +$$

$$+ a_0 + a_1\left(\frac{1+1/\sqrt{3}}{2}\right)^2\Bigg) =$$

$$= \frac{1}{2}(2a_0 + a_1) = a_0 + \frac{a_1}{2}$$

If $f(x) = a_0 + a_1 x + a_2 x^2$ $\quad a_i \in \mathbb{R}$ $\quad i \in \{0,1,2\}$

$$\int_0^1 f(x)\,dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$ and

$$b_1 f(c_1) + b_2 f(c_2) = \frac{1}{2}\left(a_0 + a_1 \left(\frac{1 - 1/\sqrt{3}}{2}\right) + \right.$$

$$+ a_2 \left(\frac{1 - 1/\sqrt{3}}{2}\right)^2 + a_0 + a_1 \left(\frac{1 + 1/\sqrt{3}}{2}\right) +$$

$$\left. + a_2 \left(\frac{1 + 1/\sqrt{3}}{2}\right)^2 \right) =$$

$$= \frac{1}{2}\left(2a_0 + a_1 + \frac{a_2}{4}\left(\frac{8}{3}\right)\right) =$$

$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$

Problem 2b (30pts) [Problem 3.3 of the lecture notes] Determine the ANOVA decomposition for

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$

and compute the associated variances σ_1^2, σ_2 and σ_{12}^2. The underlying measure is uniform measure on $[0,1]^2$. (See also pp.71–72 for explanation about ANOVA decomposition.)

ANOVA decomposition:

1) $f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2)$

2) Let's compute each $f(x)$

$$f_0 = \mathbb{E}[f(x)] = \int_0^1\int_0^1 f(x_1,x_2)\,dx_1 dx_2 = \int_0^1 \left(\int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\,dx_1 \right) dx_2 = \int_0^1 \Big|_0^1 (6x_1^2 + 6x_1x_2 - 3x_1^2x_2)\,dx_2 =$$

$$= \Big|_0^1 6x_2 + \frac{3}{2}x_2^2 = \frac{15}{2}$$

3) $$f_1(x_1) = \int_0^1 f(x_1,x_2)\,dx_2 - f_0 = \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\,dx_2 - \frac{15}{2} =$$

$$= \Big|_0^1 (12x_1x_2 + 3x_2^2 - 3x_1x_2^2) - \frac{15}{2} = 9x_1 - \frac{9}{2}$$

4) $$f_2(x_2) = \int_0^1 f(x_1,x_2)\,dx_1 - f_0 = \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\,dx_1 - \frac{15}{2} =$$

$$= \Big|_0^1 (6x_1 + 6x_1x_2 - 3x_1^2x_2) - \frac{15}{2} = 6 + 6x_2 - 3x_2 - \frac{15}{2} = 3x_2 - \frac{3}{2}$$

5) $$f_{12}(x_1,x_2) = f(x_1,x_2) - f_1(x_1) - f_2(x_2) - f_0 =$$

$$= 12x_1 + 6x_2 - 6x_1x_2 - 9x_1 + \frac{9}{2} - 3x_2 + \frac{3}{2} - \frac{15}{2} =$$

$$= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}$$

□

e) According to the uniform distribution:

$$\boxed{E(X) = \frac{a+b}{2}}, \quad \boxed{var(X) = \frac{(b-a)^2}{12}}$$

$$\sigma_1^2 = var(f_1(x_1)) = var\left(9x_1 - \frac{9}{2}\right) = Var(9x_1) = \frac{(9\cdot 1 - 0)^2}{12} = \frac{81}{12} = \frac{27}{4}$$

$$\sigma_2^2 = Var(f_2(x_2)) = Var\left(3x_2 - \frac{3}{2}\right) = Var(3x_2) = \frac{9}{12} = \frac{3}{4}$$

$$\sigma_{12}^2 = Var(f_{12}(x_1, x_2)) = Var\left(3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}\right) =$$

$$= Var(3x_1+3x_2) + Var(6x_1x_2) - 2Cov(3x_1+3x_2, -6x_1x_2) =$$

$$= 9Var(x_1) + 9Var(x_2) - \underbrace{2Cov(3x_1, 3x_2)}_{0 \text{ as indep.}} + 36Var(x_1x_2) - 2Cov(3x_1+3x_2; 6x_1x_2) =$$

$$= 9Var(x_1) + 9Var(x_2) + 36Var(x_1x_2) - 36Cov(x_1 + x_2, x_1x_2) =$$

$$= [9var(x_1) + 9Var(x_2) + 36Var(x_1, x_2) - 36Cov(x_1, x_1x_2) - 36cov(x_2, x_1x_2)] \quad (*)$$

f) let's find σ_{12}^2:

$$Var(x_1, x_2) = E[x_1^2 x_2^2] - (E[x_1x_2])^2 =$$

$$= \int_0^1\int_0^1 x_1^2 x_2^2 \, dx_1 dx_2 - \underbrace{E[x_1]E[x_2]}_{\text{as independent } x_1 \text{ and } x_2}{}^2 = \frac{1}{9} - \left(\frac{1}{2}\cdot\frac{1}{2}\right)^2 =$$

$$= 4/9 - \frac{1}{16} = \frac{7}{144}$$

$$Cov(x_1, x_1x_2) = E[x_1 x_1x_2] - \underbrace{E[x_1]E[x_1x_2]}_{\text{independency}} =$$

$$= \int_0^1\int_0^1 x_1^2 x_2 \, dx_1 dx_2 - E[x_1]E[x_1]E[x_2] =$$

$$= \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{24}$$

Using computed (*) formula:

$$\sigma_{12}^2 = \frac{9}{12} + \frac{9}{12} + 36\cdot\frac{7}{144} - 36\cdot\frac{1}{24} - 36\cdot\frac{1}{24} = \frac{1}{4}$$

As a result: $\boxed{\sigma_1^2 = \frac{27}{4}, \sigma_2^2 = \frac{3}{4}, \sigma_{12}^2 = \frac{1}{4}}$