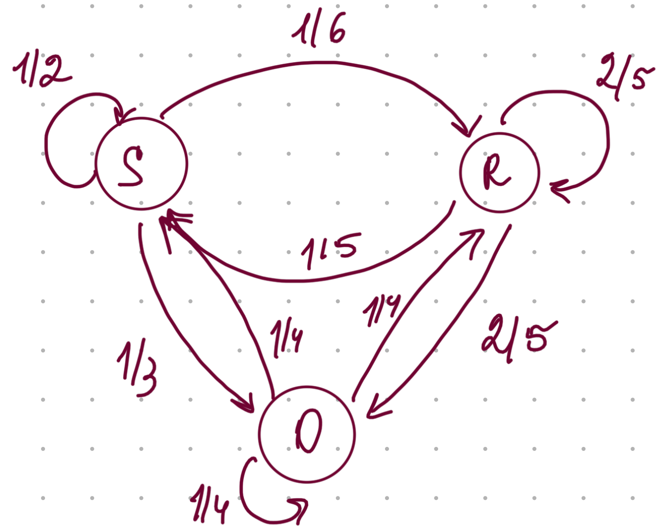
Problem 1 Finally we find a precise weather forecasting model: It models the weather as sunny, overcast or rainy, i.e., $S = \{\text{sunny}, \text{overcast}, \text{rainy}\}$. The probability of observing a certain weather condition in the next day only depends on today's weather, and the probability is given by the following table:

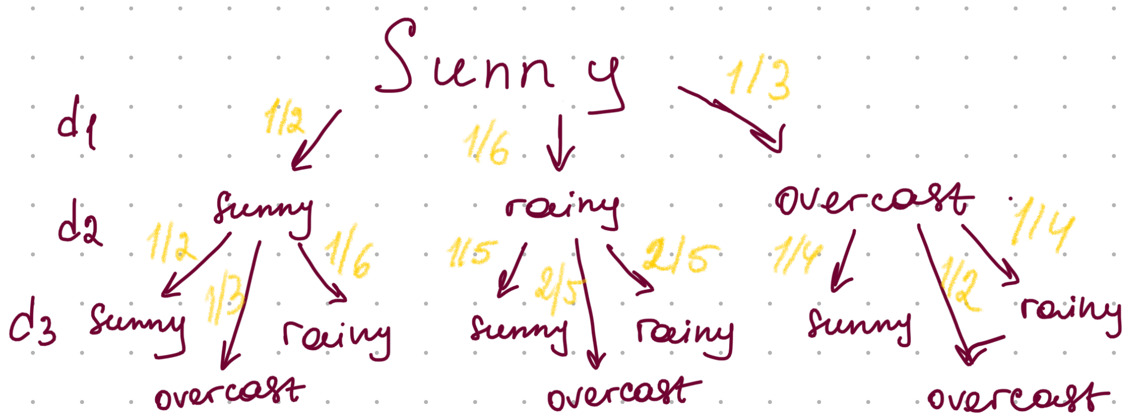
↓Tomorrow \Today→	sunny	overcast	rainy
sunny	1/2	1/4	1/5
overcast	1/3	1/2	2/5
rainy	1/6	1/4	2/5

Then the model is expressed with the following matrix

$$P := \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{6} & \frac{1}{4} & \frac{2}{5} \end{pmatrix}$$

1. Suppose the weather today is sunny. What is the probability that it will be sunny, overcast or rainy on the day after tomorrow?





① Probability that it will be sunny:

$$P(d_3 = \text{sunny} \mid d_1 = \text{sunny}) = 1/2 \cdot 1/2 + 1/6 \cdot 1/5 + 1/3 \cdot 1/4 = \frac{11}{30}$$

② Probability that it will be rainy the day after tomorrow:

$$P(d_3 = \text{rainy} \mid d_1 = \text{sunny}) = 1/2 \cdot 1/6 + 1/6 \cdot 2/5 + 1/3 \cdot 1/4 = \frac{7}{30}$$

③ Probability that it will be overcast!

$$P(d_3=\text{overcast}\mid d_1=\text{sunny}) = 1/2 \cdot 1/3 + 1/6 \cdot 2/5 + 1/3 \cdot 1/2 = \frac{12}{30} = \frac{2}{5}$$

$$P(1)+P(2)+P(3) = 30/30 = 1$$

2. Find the invariant measure of the Markov process. That is, solve for a probability vector p such that $p = Pp$.

Let $\pi(A) = \sum_{i\in A} \pi_i$, $A \subseteq I$ – a non-negative measure on I

A measure π is invariant for a transition matrix P if

$\pi P = \pi$, that is $\sum_{i\in I} \pi_i p_{ij} = \pi_j \quad \forall j \in I$

$$P := \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{6} & \frac{1}{4} & \frac{2}{5} \end{pmatrix}$$

$\pi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T$ – probability vector for the 1st sunny day.

$\pi_1 = \pi_0 P = \begin{pmatrix} 1/2 \\ 1/4 \\ 1/5 \end{pmatrix}^T$ – probability vector for the second day

$\pi_2 = \pi_1 P = \begin{pmatrix} 11/30 \\ 3/10 \\ 7/25 \end{pmatrix}^T$ – probability vector for the day after tomorrow

Let's find the invariant measure:

$$\det(P-\lambda I) = \det\begin{pmatrix} 1/2-\lambda & 1/4 & 1/5 \\ 1/3 & 1/2-\lambda & 2/5 \\ 1/6 & 1/4 & 2/5-\lambda \end{pmatrix} =$$

$$= -\lambda^3 + \frac{7\lambda^2}{5} - \frac{13\lambda}{30} + \frac{1}{30}$$

$$\left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = \frac{1}{30}(6+\sqrt{6}) \\ \lambda_3 = \frac{1}{30}(6-\sqrt{6}) \end{array}\right.$$

Eigenvector for $\lambda_1 = 1$:

$$\begin{pmatrix} -1/2 & 1/4 & 1/5 \\ 1/3 & -1/2 & 2/5 \\ 1/6 & 1/4 & -3/5 \end{pmatrix} \sim \frac{1}{2}\begin{pmatrix} 6 & -3 & 0 \\ -4 & 10 & -8 \\ 0 & -5 & 10 \end{pmatrix}$$

Getting: $v_1 = \left(\frac{6}{5}, \frac{8}{5}, 1\right)$

Normalized: $\pi^*_z = \left(\frac{6}{5\sqrt{5}}, \frac{8}{5\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ – the invariant measure.

if $\pi^* = P\pi^*$, then π^* is invariant measure = stationary distribution.

which means that $\lim\limits_{n\to\infty} P^n = \begin{pmatrix} \frac{6}{5\sqrt5} \\ 8/5\sqrt5 \\ 1/\sqrt5 \end{pmatrix}$

$$P\pi^* = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{6} & \frac{1}{4} & \frac{2}{5} \end{pmatrix} \cdot \begin{pmatrix} 6/5\sqrt5 \\ 8/5\sqrt5 \\ 1/\sqrt5 \end{pmatrix} = \begin{pmatrix} 6/5\sqrt5 \\ 8/5\sqrt5 \\ 1/\sqrt5 \end{pmatrix} = \pi^* \Rightarrow$$ converge to invariant measure

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I suppose that the upper solution is wrong, so

$$P = P^T = \begin{array}{c} \text{tomorrow} \rightarrow \\ \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/4 & 1/2 & 1/4 \\ 1/5 & 2/5 & 2/5 \end{pmatrix} \end{array} \begin{array}{l} \text{today} \\ \downarrow \end{array}$$

As the resulting state distribution should sums to 1.

$\pi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T$ - probability vector for the 1st sunny day.

$\pi_1 = \pi_0 P = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}^T$ - probability vector for the second day

$\pi_2 = \pi_1 P = \begin{pmatrix} 1/4 \\ 1/9 \\ 1/36 \end{pmatrix}^T$ - probability vector for the day after tomorrow

let's find the invar. measure:

$$\det(P - \lambda I) = \begin{pmatrix} 1/2-\lambda & 1/3 & 1/6 \\ 1/4 & 1/2-\lambda & 1/4 \\ 1/5 & 2/5 & 2/5-\lambda \end{pmatrix} = -\lambda^3 + \frac{7\lambda^2}{5} - \frac{13\lambda}{30} + \frac{1}{30}$$

$$\left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = \frac{1}{30}(6+\sqrt6) \approx 0{,}2816 \\ \lambda_3 = \frac{1}{30}(6-\sqrt6) \approx 0{,}11835 \end{array}\right.$$

Eigenvectors:

$$v_1 = \begin{pmatrix} -0{,}536 \\ -0{,}807 \\ 0{,}217 \end{pmatrix} \quad V_2 = \begin{pmatrix} -0{,}715 \\ 0{,}296 \\ -0{,}79 \end{pmatrix} \quad V_3 = \begin{pmatrix} -0{,}44 \\ 0{,}51 \\ 0{,}5727 \end{pmatrix}$$

Taking the first row of vector matrix

$$\text{Invar. measure} = \left(\frac{-0{,}536}{-0{,}536 + -0{,}715 - 0{,}44} ; \frac{-0{,}715}{-0{,}536 - 0{,}715 - 0{,}44} ; \frac{-0{,}44}{-1{,}691} \right) = \left(0{,}3169,\ 0{,}422,\ 0{,}26 \right)$$