Bayesian Inference and Data Assimilation- Exercise Sheet 1

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Exercise 1:
We will use the Forward Euler method.
We can rewrite the problem as:

$$\begin{cases} z(0) = (0.1, 0.1)^T \\ \dot{z}(t) = \begin{pmatrix} y(t) \\ -x(t) + (1 - x(t)^2)y(t) \end{pmatrix} = f(t, z(t)), \end{cases}$$

where then $f : [0,t] \times \mathbb{R}^2 \to \mathbb{R}^2$, with $t = 30$ (specific case of the exercise).
Idea behind Forward Euler method implementation:
Given $N \in \mathbb{N}$ and $\Delta t = \frac{t}{N}$, we divide $[0,t]$ in N sub-intervals of length Δt and bounds t_i, t_{i+1}, then we consider $\{z_k\}_{k=0}^N \subset \mathbb{R}^2$, where $z_k \approx z(t_k)$ $k \in \{0, \dots, N\}$ with $z_0 = z(0)$.
This way:

$$z'(t_k) = \lim_{\Delta t \to 0} \frac{z(t_{k+1}) - z(t_k)}{\Delta t} \approx \frac{z(t_{k+1}) - z(t_k)}{\Delta t} \text{ (assuming } \Delta t \text{ to be small).}$$

We can approximate both $z'(t_k)$ and $z'(t_{k+1})$ with $z(t_{k+1}) - z(t_k)$, consequently:

$$\frac{z(t_{k+1}) - z(t_k)}{\Delta t} \approx z'(t_k)\theta + (1-\theta)z'(t_{k+1}),\ \theta \in [0,1].$$

By replacing $z(t_k) \approx z_k$ and given that $z'(t) = f(t, z(t))$ we get:

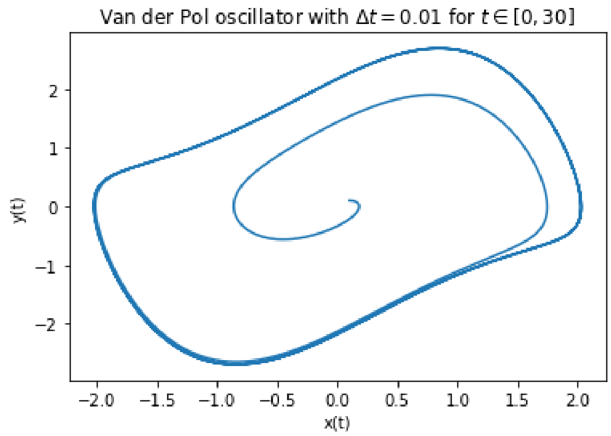
$$\frac{z_{k+1} - z_k}{\Delta t} \approx f(t_k, z_k)\theta + f(t_{k+1}, z_{k+1})(1-\theta),\ \theta \in [0,1].$$

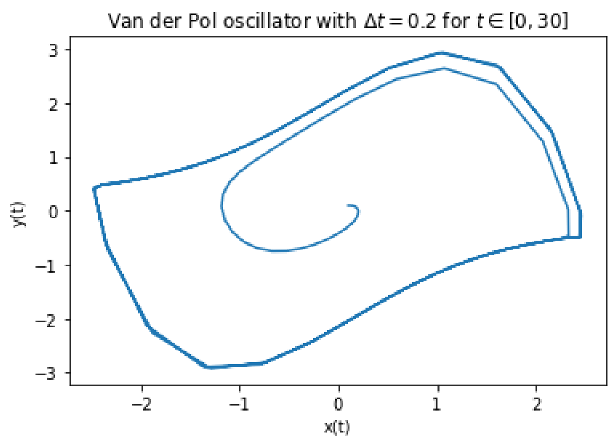
The Forward Euler method takes $\theta = 1$, therefore the method is:

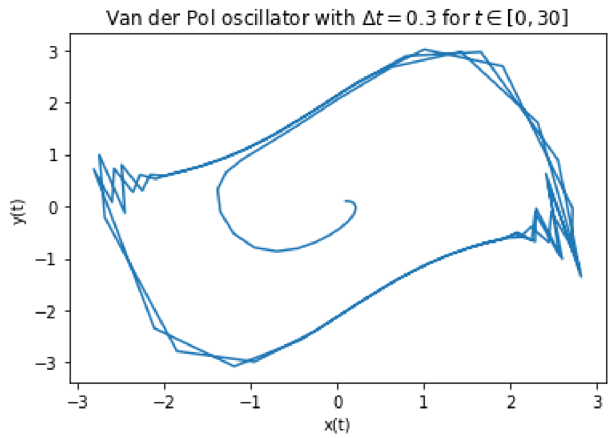
$$\begin{cases} z_0 = z(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\ z_{k+1} = z_k + \Delta t f(t_k, z_k) = \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \Delta t \begin{pmatrix} y_k \\ -x_k + (1 - x_k^2)y_k \end{pmatrix} \\ t_{k+1} = t_k + \Delta t,\ k \in \{1, \dots, N\}. \end{cases}$$

We can observe that this method is explicit, i.e to get information about t_{k+1} we use only the information available at time t_k.

Results after computing the solution with forward Euler method for $\Delta t = 0.01, 0.2, 0.3$ and $z_0 = (0.1, 0.1)^T$:







Observations: It's clearly visible that by increasing Δt the solution becomes more instable. This happens because if Δt increases it's not small enough so that this approximation (on which Euler method is built) is still good:

$$z'(t_k) = \lim_{\Delta t \to 0} \frac{z(t_{k+1}) - z(t_k)}{\Delta t} \approx \frac{z(t_{k+1}) - z(t_k)}{\Delta t}.$$

1 Nicolas Approach:

Given the following differential equations that describe the Van Der Pol oscillator, the task is to implement the first 30 timesteps using Euler Forward Scheme.

1a) $x'(t) = y(t)$
1b) $y'(t) = -x(t) + (1 - (x(t)^2)) * y(t)$

The Euler Scheme is an approximation of the actual function. Using a unifor discretization, the function is split into n equally sized parts. The bigger n is chosen, the better is the approximation of the underlying function. the parameter n is also called the step size for the Euler Scheme. In this exercise different stepsizes will be used to observe instability in increasing or decreasing step sizes.
Applying the Euler Scheme results in the following update function

$x_{k+1} = x_k + \Delta t * y_k$
$y_{k+1} = y_k + \Delta t * (-x_k) + (1 - (x_k^2)) * y_k)$
$t_{k+1} = t_k + \Delta t$

In a vectorized version this can be rewritten as:

$$z_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & \Delta t \\ (\Delta t * (-x_k + (1 - x_k^2) * y_k))/x_k & 1 \end{pmatrix}$$

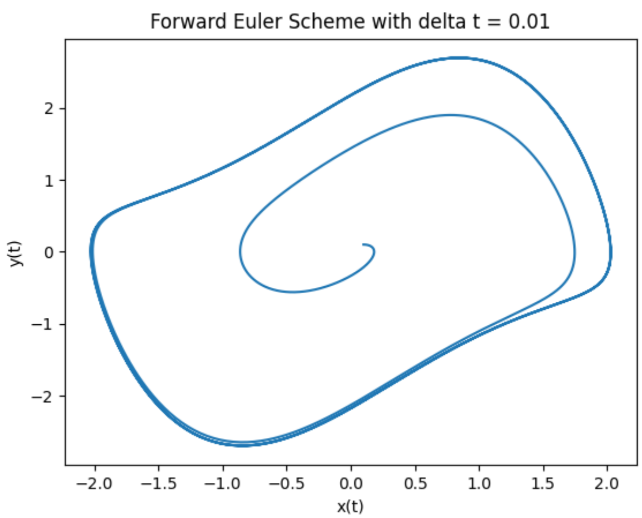
the updating function is then:

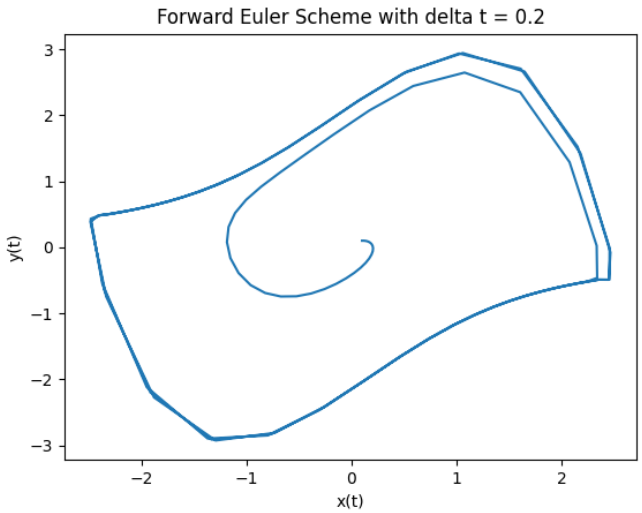
$$z_{k+1} = C * z_k$$

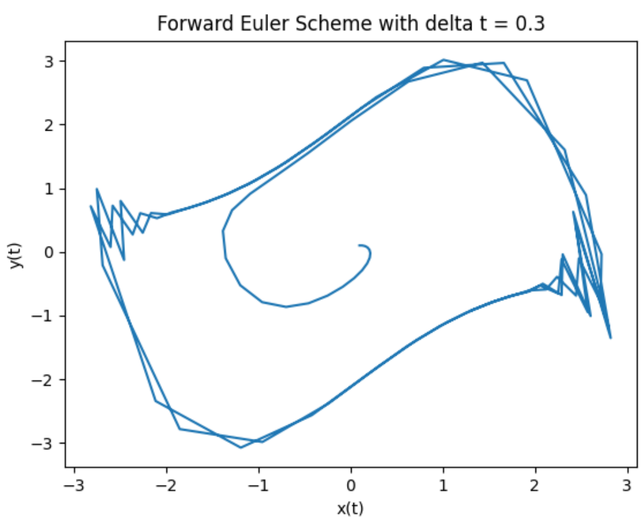
given $x(0) = y(0) = 0.1$ the intital state is:

$$z_0 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

Results after computing the solution with forward Euler method for $\Delta t = 0.01, 0.2, 0.3$ and $z_0 = (0.1, 0.1)^T$ with $t : 0 < t < 30$:







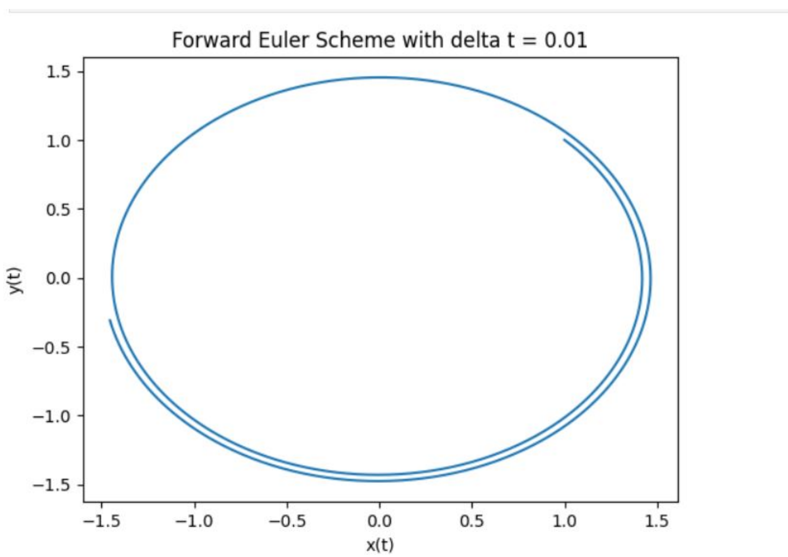
1.1 Observation

It appears that an increasing value of Δt results in higher instsability. Espechially in the corners of the figure, it seems that an increasing Δt results in high instability, while in the middle it is not as visible. The meaning of Δt is the step width for each step of the Euler-method. Thus it is explainable why the increasing step-width results in an higher instability in the corners of the figure. Since the Euler-method takes the slope of the given point z_k and moves Δt in the direction of the slope, making bigger steps in areas where the slope of the function changes rapidly, results in an instability by going too far in a particular direction. This can be seen in the top-left and bottom-right corner of the graph. Those corners are the ones with the most rapid change in solpe which can be translated to making sharp turns.

Exercise 2:
In this case $f(t, z(t)) = \begin{pmatrix} y(t) \\ -x(t) \end{pmatrix}$, then the Cauchy problem is:

$$\begin{cases} z(0) = (1,1)^T \\ \dot{z}(t) = f(t, z(t)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \end{cases}$$

Here f it's a linear application. **2A)**
After applying the Euler method (implemented as for exercise 1) we get the values of $y(5)$ and $y(10)$ by considering the second component in the output vector, $\tilde{z}$, and $k = \frac{\tilde{t}}{\Delta t}$ with $\tilde{t} \in \{5, 10\}$.
For $N = 10$ iterations we get:



2B)
With $\Delta t = 0.01$ the approximation of the solution is good, therefore we expect to obtain, as approximation of the initial condition, a point that is close to $(x_0, y_0) = (1, 1)$. To do so we solve

(basing on the exact solution) the following linear system:

$$\begin{pmatrix} y(5) \\ y(10) \end{pmatrix} = \begin{pmatrix} -sin(5) & cos(5) \\ -sin(10) & cos(10) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

2C) By increasing the step size the solution becomes more instable, thus the relative error on the estimate of the initial condition increases as well.