SHEET 3

EX 1

$S = \{1, 2, 3, 4\}$

$$P = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \qquad P_{ij} = P(X_{t+1} = j \mid X_t = i)$$

1) $X_0 = 1$

$$\pi_1 = P\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1/2\\0\\1/2\end{pmatrix}, \quad \pi_2 = P^2\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = P\begin{pmatrix}0\\1/2\\0\\1/2\end{pmatrix} = \begin{pmatrix}1/2\\0\\1/2\\0\end{pmatrix}$$

2)
$$\det(P - \lambda I) = \det\begin{pmatrix} -\lambda & 1/3 & 0 & 2/3 \\ 1/2 & -\lambda & 1/2 & 0 \\ 0 & 2/3 & -\lambda & 1/3 \\ 1/2 & 0 & 1/2 & -\lambda \end{pmatrix} =$$

LAPLACE

$$= -\lambda\left(\det\begin{pmatrix} -\lambda & 1/2 & 0 \\ 2/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{pmatrix}\right) - \frac{1}{3}\det\begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & -\lambda & 1/3 \\ 1/2 & 1/2 & -\lambda \end{pmatrix} - \frac{2}{3}\det\begin{pmatrix} 1/2 & -\lambda & 1/2 \\ 0 & 2/3 & -\lambda \\ 1/2 & 0 & 1/2 \end{pmatrix} =$$

$$= -\lambda\left(-\lambda\left(\lambda^2 - \frac{1}{6}\right) - \frac{1}{2}\left(-\frac{2}{3}\lambda\right)\right) - \frac{1}{3}\left(\frac{1}{2}\left(\lambda^2 - \frac{1}{6}\right) - \frac{1}{2}\left(-\frac{1}{6}\right)\right) -$$

$$- \frac{2}{3}\left(\frac{1}{2}\left(\lambda^2 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)\right) = \lambda^2(\lambda^2 - 1) = 0 \iff$$

$$\lambda = 0 \ \vee\ \lambda = 1 \ \vee\ \lambda = -1$$

Eigenvectors:

$\lambda = 1)$ $\quad (P - I)\underline{v}_1 = 0 \iff \underline{v}_1 = \begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$.

$\lambda = -1)$ $\quad (P + I)\underline{v}_2 = 0 \iff \underline{v}_2 = \begin{pmatrix}-1/4\\1/4\\-1/4\\1/4\end{pmatrix}$

Then the chain is periodic, indeed we see:

$\forall n \geq 1 \qquad \pi_{2n+1} = \pi_1, \quad \pi_{2n} = \pi_2$

$\Longrightarrow$ the limit $\nexists$ and the chain doesn't converge.

Ex 2

$$P = \begin{pmatrix} 1/2 & 2/3 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1/4 & 4/5 \\ 0 & 0 & 3/4 & 1/5 \end{pmatrix}$$

1) $X_0 = 1$

$S = S_1 \cup S_2 = \{1,2\} \cup \{3,4\}$, where S_1, S_2 are closed classes $\Longrightarrow$ if we start in 1 we remain in $\{1,2\}$, while if we start in $\{3,4\}$ we remain there.

$\Longrightarrow X_t$ may converge to $\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$ or in general to $\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix}$ $x, y \in \mathbb{R}^+ \mid x+y=1$

2) $X_0 = 3 \Longrightarrow X_t$ may converge to $\begin{pmatrix} 0 \\ 0 \\ z \\ \gamma \end{pmatrix}$ $z, \gamma \in \mathbb{R}^+ \mid z+\gamma=1$

3) ~~we expect to find ... $\in \mathbb{R}^4$~~

$Pp = p \qquad p = (x, y, z, \gamma) \in \mathbb{R}^4$

$$\begin{cases} \frac{x}{2} + \frac{2}{3} y = x \\ \frac{x}{2} + \frac{1}{3} y = y \\ \frac{1}{4} z + \frac{4}{5} \gamma = z \\ \frac{3}{4} z + \frac{1}{5} \gamma = \gamma \end{cases} \Longleftrightarrow \begin{cases} x = \frac{4}{3} y \\ \\ \gamma = \frac{15}{16} z \end{cases}$$

$$\Longrightarrow \quad \Pi = \left\{ \pi \in \mathbb{R}^4 \mid \pi = y \begin{pmatrix} 4/7 \\ 3/7 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ \frac{16}{29} \\ \frac{}{29} \end{pmatrix},\ z+y=1 \right\}$$

is the set ~~of~~ that characterises all the invariant measures

EX 3

$X \sim N(1,1) \quad W \sim N(0,1) \quad X, W$ independent

$Y = X^2 + W$

1) $X = x,\ Y = y \quad ?\, w \mid \quad P(X=x,\ Y=y) = P(X=x,\ W=w)$

$P(X=x,\ Y=y) \underset{\text{independence}}{=} P(X=x) \cdot P(W=w)$

$P(W = y - x^2) \Longleftrightarrow = P(W=w) \Longleftrightarrow w = y - x^2$

2) $\pi_{X \mid Y=y}(x) = \dfrac{\pi_{Y \mid X=x}(y)\ \pi_X(x)}{\underbrace{\pi_Y(y)}_{\text{normalization constant}}} \Longrightarrow$

$\pi_{X \mid Y=y}(x) \propto \pi_{Y \mid X=x}(y) \cdot \pi_X(x)$

$Y \mid X=x \sim N(x^2, 1)$ and $X \sim N(1,1)$

$\Longrightarrow \pi_{X \mid Y=y}(x) \propto \dfrac{1}{\sqrt{2\pi}} e^{-\frac{(y-x^2)^2}{2}} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} =$

$= \dfrac{1}{2\pi} e^{-\frac{1}{2}\left[(y-x^2)^2 + (x-1)^2\right]}$