Problem 1 In this exercise, we will do a very simple filtering and smoothing step by hand. The forward map is given as

$$X_{n+1} = \frac{1}{2}X_n + 1 + \Xi_n$$

where $\Xi_n \sim N(0,1)$. The observation operator is given by

$$Y_n = X_n + \sqrt{2}\Sigma_n$$

with $\Sigma_n \sim N(0,1)$. Assume $X_0 \sim N(-1,2)$. All noise processes are independent.

1. Prediction step: What is the distribution of X_1?
2. Filtering step: What is the distribution of X_1 conditioned on $Y_1 = 2$?
3. Smoothing step: What is the distribution of X_0 conditioned on $Y_1 = 2$?
4. Now we want to implement this in pseudo code. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim N(m,1)$$

with the same observation model and noise parameter. You also observe that $Y_1 = y$. Write a pseudo code, that given the inputs α, β, m and y will output the distribution of (i) X_1, (ii) X_1 conditioned on $Y_1 = y$, and (iii) X_0 conditioned on $Y_1 = y$.

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The properties:

if $X \sim N(a, b)$ then
$\alpha + X \sim N(\alpha + a, b)$
$\alpha X \sim N(\alpha a, \alpha^2 b)$
$X + Y \sim N(a+c, b+d)$ where $Y \sim N(c, d)$

$$X_1 = \frac{1}{2}X_0 + 1 + \Xi_0$$

1) $X_0 \sim N(-1, 2) \Rightarrow \frac{1}{2}X_0 \sim N(-\frac{1}{2}; \frac{1}{2})$

2) $(\frac{1}{2}X_0 + 1) \sim N(\frac{1}{2}; \frac{1}{2})$

3) $\Xi \sim N(0, 1) \Rightarrow (\frac{1}{2}X_0 + 1 + \Xi) \sim N(\frac{1}{2}; \frac{3}{2})$

$\Rightarrow X_1 \sim N(\frac{1}{2}; \frac{3}{2})$

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1) From the previous step the next mean and variance are:
$\mu_1 = \frac{1}{2}, \sigma_1 = \frac{3}{2}$

2) Kalman Gain $= \dfrac{Var[X_1]}{Var[X_1] + Var[\sqrt{2}\,\Sigma_n]}$, where $\sqrt{2}\,\Sigma_n$ - observation noise

3) $\Sigma_n \sim N(0,1) \Rightarrow \sqrt{2}\,\Sigma_n \sim N(0,2)$
$\Downarrow$
Kalman gain $= \dfrac{3/2}{3/2+2} = \frac{3}{7}$

4) $\mu_{new} = \mu_1 - \text{Kalman gain}(\mu_1 - y_1) = \frac{1}{2} - \frac{3}{7}\left(\frac{1}{2} - 2\right) = \frac{8}{7}$

$\sigma_{new} = \sigma_1(1 - \text{kalman gain}) = \frac{3}{2}\left(1 - \frac{3}{7}\right) = \frac{6}{7}$

$\boxed{(X_1 | y_1 = 2) \sim N\left(\frac{8}{7}, \frac{6}{7}\right)}$

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1) $X_0 \sim N(-1, 2)$

2) $y_1 = X_1 + \sqrt{2}\,\Sigma_1$, $\sqrt{2}\,\Sigma_1 \sim N(0,2)$, $X_1 \sim N\left(\frac{1}{2}; \frac{3}{2}\right)$

$y_1 \sim N\left(\frac{1}{2} + 0; \frac{3}{2} + 2\right) \Rightarrow y_1 \sim N\left(\frac{1}{2}; \frac{7}{2}\right)$

3) Also $y_1 = X_1 + \sqrt{2}\,\Sigma_1 = \frac{1}{2}X_0 + 1 + \Xi_0 + \sqrt{2}\,\Sigma_1$

if $y_1 = 2$, then $2 = \frac{1}{2}X_0\big|_{y=2} + 1 + \Xi_0 + \sqrt{2}\,\Sigma_1$

$\frac{1}{2}X_0\big|_{y=2} = 2 - 1 - \Xi_0 - \sqrt{2}\,\Sigma_1$

$\frac{1}{2}X_0|y=2 = 1 - \Xi_0 - \sqrt{2}\,\Sigma_1$

$\boxed{\alpha + X \sim N(\alpha + a, b)}$

4) $(-1)\Xi_0 \sim N(0,1)$
$(-1)\cdot\sqrt{2}\,\Sigma_1 \sim N(0,2)$ $\Rightarrow \frac{1}{2}X_0|y=2 \sim N(1+0+0, 1+2) = N(1,3)$

$\boxed{\text{Thus we get: } X_0|y=2 = N\left(\frac{1}{2}\cdot 1; \frac{1}{2^2}\cdot 3\right) = N\left(\frac{1}{2}; \frac{3}{4}\right)}$

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4. Now we want to implement this in pseudo code. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim N(m, 1)$$

with the same observation model and noise parameter. You also observe that $Y_1 = y$. Write a pseudo code, that given the inputs α, β, m and y will output the distribution of (i) X_1, (ii) X_1 conditioned on $Y_1 = y$, and (iii) X_0 conditioned on $Y_1 = y$.

1

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def_X1(α, β, m, y, noise_mean=0, noise_var=1):

    meanX0 = α • m
    var_X0 = α**2 • 1

    mean_X1 = α*m + β·noise_mean
    var_X1 = var_X0 + β**2

    return (mean_X1, var_X1)
```

2

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def_X1_y1 (mean_X1, var_X1, y, noise_mean=0, noise_var=2):
    Kalman_gain=(var_X1/(var_X1+noise_var))

    mean_X1_y1 = mean_X1 - Kalman_gain(mean_X1-y)
    var_X1_y1= var_X1(1-Kalman_gain)

    return (mean_X1_y1, var_X1_y1)
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3 $\Xi_1 \sim N(0,1)$, $\Sigma_n \sim N(0,1)$, $X_0 \sim N(m,1)$

```
def_X0_y1 (m, var_X0 = 1, noise_mean_0=0, noise_var_0=1, noise_mean_1=0,
                                                           noise_var_1=1)
mean_X0_y1=(1+ noise_mean_0·(-1) + noise_mean_1·√2)·1/2
var_X0_y1=(noise_var_0+noise_var_1·2)·1/4
return (mean_X0_y1, var_X0_y1)
```