Sheet 3 Sol's

Q2)

i) $E[z^T A z] = E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} z_i A_{ij} z_j\right]$

Expanding the sum in 2 parts i.e. for $i=j$ & $i \neq j$.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E[z_i A_{ij} z_j] = \sum_{i=1}^{n} A_{ii} E[z_i z_i] + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} E[z_i z_j]$$

$$= \sum_{i=1}^{n} A_i E[z_i^2] + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} E[z_i z_j] \quad -(1)$$

$\because Var(z_i) = E[z_i^2] - u_i^2$ and $E[z_i z_j] = \Sigma_{ij}$

Simplifying (1) further

Assuming z_i, z_j are ind.

$$= \sum_{i=1}^{n} A_{ii}\left(Var(z_i) + u_i^2\right) + \cancel{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} \Sigma_{ij}} \; 0$$

$$= \underbrace{\sum_{i=1}^{n} A_{ii} \Sigma_{ii}} + \sum_{i=1}^{n} A_{ii} u_i^2$$

$$= tr(A\Sigma) + u^T A . u$$

ii) $Cov(Az) = A \Sigma A^T$

Using $Cov(x) = E[(x-u_x)(x-u_x)^T]$ where $E(x) = u_x$

& $E[Az] = Au$ where $u = E[z]$ ε

$$Cov(Az) = E[(Az - Au)(Az - Au)^T]$$

Distributing A:

$\Rightarrow E[A(z-u)(z-u)^T A^T]$

$\Rightarrow A\, E[(z-u)(z-u)^T] A^T$ .. using linearity

$\Rightarrow A \Sigma A^T$

) Given: $\hat{\beta} = (X^T X)^{-1} X^T y$; $y = X\beta + \varepsilon$ & $cov(y) = \sigma^2 I_{n\times n}$

To show $Cov(\hat{\beta}) = \sigma^2 I (X^T X)^{-1}$

$\hat{\beta} = (X^T X)^{-1} X^T y$ & $\because Cov(y) = \sigma^2 I_{n\times n}$ ... Var in ε is const. or dispersion of residuals is same for all values of prediction var.

$\because cov(AB) = A \cdot cov(B) \cdot A^T$

$Cov(\hat{\beta}) = (X^T X)^{-1} X^T \cdot cov(y) \cdot X (X^T X)^{-1}$

$= (X^T X)^{-1} \cdot X^T \cdot \sigma^2 I_{n\times n} \cdot X (X^T X)^{-1}$

$= \sigma^2 (X^T X)^{-1} \cdot \underbrace{X^T X \cdot (X^T X)^{-1}}_{I}$

$= \sigma^2 (X^T X)^{-1}$

Q3)

As per the given Q: Likelihood is

$L(p) = [(1-p)]^{2x} \cdot [2p(1-p)]^{y} [p^2]^{z}$

$\log(L(p)) = 2x \log(1-p) + y \log 2p + y \log(1-p) + 2z \log p$

$\dfrac{\partial \log(L(p))}{\partial p} = 0$ to find MLE, we get

$-\dfrac{2x}{1-p} + \dfrac{y}{p} - \dfrac{y}{1-p} + \dfrac{2z}{p} = 0$

Multiplying with $p(1-p)$ on both sides

$y(1-p) + 2z(1-p) = 2px + yp$

$y - py + 2z - 2zp = 2px + yp$

$y + 2z = 2px + 2py + 2zp$

$\therefore p = \dfrac{y+2z}{2x+2y+2z}$

$$\therefore \frac{\partial^2 \log(L(p))}{\partial p^2} = -\frac{2x}{(1-p)^2} - \frac{y}{p^2} - \frac{y}{(1-p)^2} - \frac{2z}{p^2} < 0$$

as x, y & z are counts of people & deno's are square $\therefore$ the critical pt value of p denotes the maximum.

$\therefore$ MLE is $\dfrac{y+2z}{2x+2y+2z}$.