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# Homework Sheet 4

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## Task 1

1. **8 points**
A square matrix $M$ is called idempotent if $M^2 = M$.

(a) Show that $C = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ is idempotent, where $I_n$ is the identity matrix and $\mathbf{1}$ is a vector of ones. Also show that $C\mathbf{1} = \mathbf{0}$.

$C = I_n - \frac{1}{n}\vec{1}\vec{1}^T$ where $\vec{1} = [1\ 1\ \cdots\ 1]^T_n$, $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$

$C^2 = \left(I_n - \frac{1}{n}\vec{1}\vec{1}^T\right)\cdot\left(I_n - \frac{1}{n}\cdot\vec{1}\vec{1}^T\right) =$

$= I_n^2 - I_n \cdot \frac{1}{n}\cdot\vec{1}\vec{1}^T - \frac{1}{n}\vec{1}\vec{1}^T \cdot I_n + \frac{1}{n}\vec{1}\vec{1}^T \cdot \frac{1}{n}\vec{1}\vec{1}^T$

$= I_n - \frac{1}{n}\cdot\vec{1}\vec{1}^T - \frac{1}{n}\vec{1}\vec{1}^T + 1$ — what is 1 in this context !!!

$= I_n - \frac{2}{n}\vec{1}\vec{1}^T + 1$

$= I_n - 2 + 1$ — you can not substract a number and a matrix

$= I_n - 1\cdot\frac{1}{n}\vec{1}\vec{1}^T$ → n- square matrix !

$= I_n - \frac{1}{n}\vec{1}\vec{1}^T$

We proved that $C^2 = C$. Therefore, C is idempotent.

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- Next part is to show that $C\cdot\vec{1} = \vec{0}$

$C = I_n - \frac{1}{n}\vec{1}\vec{1}^T \ /\cdot\vec{1}$ where $\vec{1} = [1\ 1\ \cdots\ 1]^T_n$

$C\cdot\vec{1} = \left(I_n - \frac{1}{n}\vec{1}\vec{1}^T\right)\cdot\vec{1}$

$C\cdot\vec{1} = I_n\cdot\vec{1} - \frac{1}{n}\cdot n$

$C\cdot\vec{1} = \vec{1} - 1$ (scalar) — No!

You can not substract a scalar and a vector !! !
if 1 is a scalar most certainly the answer will not be zero !!

$C\cdot\vec{1} = \vec{0}$ ; where $\vec{0} = [0\ 0\ \cdots\ 0]^T_n$

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(b) Let $a \in \mathbb{R}^{n\times 1}$, show that

$$Ca = \begin{bmatrix} a_1 - \bar{a} \\ \vdots \\ a_n - \bar{a} \end{bmatrix}.$$

$C = I_n - \frac{1}{n}\vec{1}\vec{1}^T \ |\cdot\vec{a}$

$C\cdot\vec{a} = \left(I_n - \frac{1}{n}\cdot\vec{1}\vec{1}^T\right)\cdot\vec{a}$

$= I_n\cdot\vec{a} - \frac{1}{n}\cdot\vec{1}\vec{1}^T\vec{a}$

$= \vec{a} - \frac{1}{n}\sum_{i=1}^{n} a_i$ ($\bar{a}$) — same remark as above careful with your matrix dimension )

$= \vec{a} - \bar{a}$ (vector − scalar) $\Rightarrow C\vec{a} = \begin{pmatrix} a_1 - \bar{a} \\ a_2 - \bar{a} \\ \vdots \\ a_n - \bar{a} \end{pmatrix}$

This is not a

Let $A \in \mathbb{R}^{n\times k}$, show that

$$CA = \begin{bmatrix} a_{11}-\bar{a}_1 & \cdots & a_{1k}-\bar{a}_k \\ \vdots & \vdots & \vdots \\ a_{n1}-\bar{a}_1 & \cdots & a_{nk}-\bar{a}_k \end{bmatrix}$$

row vector, where j-th member is the mean of the j-th column of the A matrix

$$C\cdot A = \left(I_n - \frac{1}{n}11^T\right)A = I_n A_{n\times k} - \frac{1}{n}11^T A = A_{n\times k} - \left[\frac{1}{n}\sum_{j=1}^{k} a_j\right]_{1\times k}$$

$$\Rightarrow C\cdot A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} - \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_k \end{bmatrix}$$

→ you can not substract these two matrices . the dimensions does not work !!

$$\Rightarrow C\cdot A = \begin{bmatrix} a_{11}-\bar{a}_1 & a_{12}-\bar{a}_2 & \cdots & a_{1k}-\bar{a}_k \\ \vdots & & & \\ a_{n1}-\bar{a}_1 & a_{n2}-\bar{a}_2 & \cdots & a_{nk}-\bar{a}_k \end{bmatrix}$$

(c) Show that

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = x^T C x, \quad \text{where} \quad x=[x_1,\ldots,x_n]^T.$$

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$$x^T C x = \vec{x}^T\left(I_n - \frac{1}{n}\vec{1}\vec{1}^T\right)\vec{x} = \vec{x}^T I_n \vec{x} - \vec{x}^T\cdot\frac{1}{n}\vec{1}\vec{1}^T\vec{x} =$$

$$= \vec{x}^T\vec{x} - \frac{1}{n}\vec{x}^T\vec{1}\vec{1}^T x = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}n^2\bar{x}^2$$

$$\left(\sum_{i=1}^{n} x_i\cdot x_i\right)_1 = \left(\sum_{i=1}^{n} x_i^2\right)_1$$

$$= \sum_{i=1}^{n} x_i^2 - n\cdot\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n}\bar{x}^2 = \sum_{i=1}^{n}(x_i^2-\bar{x}^2)$$

$$= \sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2 + 2x_i\bar{x} - 2\bar{x}^2\right)$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2\cdot\bar{x}\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{0}$$

$$\boxed{= \sum_{i=1}^{n}(x_i-\bar{x})^2}$$

(d) Let $z_1, \ldots, z_n \in \mathbb{R}^p$ be realisations of a multivariate random variable $Z$. Express the sample mean $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ and the sample covariance $S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n}(z_i - \bar{z})\sum_{j=1}^{n}(z_j - \bar{z})^T\right)$ as matrix operation, using the above results and the design matrix $T = (x_1 \ldots x_n) \in \mathbb{R}^{p \times n}$.

- Sample mean

$$\bar{z} = \frac{1}{n} \cdot T \cdot \vec{1}$$

observations

$$T \rightarrow \begin{matrix} \text{features} \\ \\ \\ \end{matrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & x_{p3} & \cdots & x_{pn} \end{pmatrix} \times \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{p \times 1} \leftarrow \vec{1}$$

$$x_{11} \cdot 1 + x_{12} \cdot 1 + \ldots + x_{1n} \cdot 1 = \sum_{i=1}^{n} x_{1i}$$

- Sample covariance

$$S^2 = \frac{1}{n-1} \cdot \bar{T} \cdot \bar{T}^{-1},$$ where $\bar{T}$ is the centered matrix $T - \bar{z} \cdot \vec{1}$

Q.2

$$\sum \hat{\varepsilon}_i^2 = (y - X\hat{\beta})^T (y - X\hat{\beta}) = y^T y - y^T X\hat{\beta} - \hat{\beta}^T X^T y + \hat{\beta}^T X^T X\hat{\beta}$$

$$= y^T y - 2\hat{\beta}^T X^T y + \hat{\beta}^T X^T X\hat{\beta}$$

why?

$$\bar{y} = \frac{\sum y_i}{n}$$

$$\sum (y_i - \bar{y})^2 = \sum (y_i^2 - 2y_i\bar{y} + \bar{y}^2)$$

$$= y^T y - 2n\bar{y}\bar{y} + n\bar{y}^2$$

$$= y^T y - n\bar{y}^2$$

✓

$$1 - \frac{\sum \hat{\varepsilon}_i^2}{\sum (y_i - \bar{y})} = 1 - \left[\frac{y^T y - 2\hat{\beta}^T X^T y + \hat{\beta}^T X^T X\beta}{y^T y - n\bar{y}^2}\right]$$

$$= \frac{2\hat{\beta}^T X^T y - \hat{\beta}^T X^T X\beta - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$= \frac{2\hat{\beta}^T X^T y - \hat{\beta}^T X^T (y - \varepsilon) - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$= \frac{\hat{\beta}^T X^T y + \hat{\beta}^T X^T \varepsilon - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

but $X^T \varepsilon = 0$

$$= \frac{\hat{\beta}^T X^T y - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

✓