Exercise 3

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Task 2:

Show that $E[z^T A z] = \mu^T A \mu + Tr(A\Sigma)$

$z^T A z$ is a quadratic expression since $A \in \mathbb{R}^{n \times n}$

- for the quadratic form it holds true that it can be represented in terms of the outer product $z^T z$ and the matrix A. It holds true that

$z^T A z = Tr(z^T A z)$

- Therfore:

$E[z^T A z] = E[Tr(z^T A z)]$

using cyclic property of the trace we obtain

$E[Tr(A z^T z)]$

Since the trace is a scalar it is its own expectation

$E[Tr(A z^T z)] = Tr(E[A z^T z])$

linearity of Expectation:

$Tr(A \cdot E[z^T z])$

$E[z^T z] = cov(z) + E[z]^T \cdot E[z]$

Puting it all together set $E[z] = \mu$ and $cov(z) = \Sigma$ we obtain

$Tr(A \cdot E[z^T z]) = Tr(A \cdot (cov(z) + E[z^T] \cdot E[z]))$

$\Leftrightarrow Tr(A \cdot cov(z) + E[z^T] A E[z])$

$\Leftrightarrow Tr(A \cdot cov(z)) + Tr(A E[z^T] E[z])$

$\Leftrightarrow Tr(A \cdot cov(z)) + Tr(E[z^T] A E[z])$

$\Rightarrow Tr(A\Sigma) + \mu^T A \mu$

2. Show that $cov(Az) = A \Sigma A^T$

Coming from univariate case we know that $var(ax) = a^2 var(x)$
$A A^T = A^2$

$cov(Az) = E[(Az - E(Az))(Az - E(Az))^T]$

$\Leftrightarrow E[(Az - AE(z))(Az - AE[z])^T]$

$\Leftrightarrow E[(A(z - E[z]))(A(z - E[z]))^T]$

$\Leftrightarrow A E[(z - E[z])(z - E[z])^T] A^T$

$\Leftrightarrow A \, cov(z) A^T$
$\Leftrightarrow A \Sigma A^T$

3. Show that $cov(\hat{\beta}) = (\sigma^2 X^T X)^{-1}$

we start with $\hat{\beta}$:

$\hat{\beta} = (X^T X)^{-1} X^T Y$ we replace Y with $X\beta + \varepsilon$

$\hat{\beta} = (X^T X)^{-1} X^T (X\beta + \varepsilon)$

$= (X^T X^{-1}) X^T X \beta + (X^T X)^{-1} X^T \varepsilon$ | we know that $(X^T X)^{-1} X^T X = I$

$\Rightarrow \hat{\beta} = \beta + (X^T X)^{-1} X^T \varepsilon$

$cov(\hat{\beta}) = cov(\beta + (X^T X)^{-1} X^T \varepsilon)$ | linearity

$= cov(\beta) + cov((X^T X)^{-1} X^T \varepsilon)$ | $cov(\beta) = 0$

$= cov((X^T X)^{-1} X^T \varepsilon)$ | $cov(cz)$ wher c is constant $= c^2 cov(z)$

$= (X^T X)^{-1} cov(X^T \varepsilon)(X^T X)^{-1}$ | $cov(X^T \varepsilon) = X^T cov(\varepsilon) X = X^T \sigma^2 I X = \sigma^2 X^T X$

$= (X^T X)^{-1} (X^T X \sigma^2)(X^T X)^{-1}$ | $(X^T X)^{-1}(X^T X \sigma^2) = I\sigma^2$

$= \sigma^2 (X^T X)^{-1}$

Task 3:

we can modle this as a binomial distribution:

$p(AA) = (1-p)^2$
$p(Aa) = 2p(1-p)$
$p(aa) = p^2$

$n = x + y + z$ wher x, y, z is the number of observations of AA, Aa and aa

likelihood $L(AA, Aa, aa; p) = p(AA)^{x} \cdot p(Aa)^{y} \cdot p(aa)^{z}$

take log

log likelihood $l(L(\cdot)) =$

$l(p) = 2x \log(1-p) + y \log(2p(1-p)) + 2z \log(p)$

Max-log-likelihood:

$\frac{dl}{dp} - \frac{2x}{1-p} + \frac{y}{p} + \frac{2z}{p} = 0$