Exercise 4

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Task 2:

Show that:

$$1 - \frac{\sum \hat{\varepsilon}_i^2}{\sum (y_i - \bar{y})^2} = \frac{\hat{\beta}^T X^T y - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

Properties
$\hat{\beta} \Rightarrow (X^T X)^{-1} X^T y)^T$
$y = X^T \hat{\beta} + \hat{\varepsilon}$
$\bar{y} = E[X^T \hat{\beta}]$
$\hat{\varepsilon} = y - \hat{y}$
$X\varepsilon = 0$

$$1 - \frac{\sum \hat{\varepsilon}_i^2}{\sum (y_i - \bar{y})^2}$$

we have following equalities

$\sum_{i=1}^{N} (y_i - \bar{y})^2$ = Sum of total squares and can be expressed as:

$y^T y - n\bar{y}^2$

$$1 - \frac{\sum \hat{\varepsilon}_i^2}{y^T y - n\bar{y}^2}$$ | consider $\sum \varepsilon_i^2$ as $\hat{\varepsilon}\hat{\varepsilon}^T$ $(y - \hat{y})(y - \hat{y})^T$

when we rewrite it as matrix we have

$$y^T y - y\hat{y}^T - \hat{y}^T y + \hat{y}\hat{y}^T$$

now we can express $\hat{y}$ as $X\hat{\beta}$

$$\Rightarrow y^T y - yX\hat{\beta} - X\hat{\beta}y + \underbrace{(X\hat{\beta})(X\hat{\beta})^T}_{X\hat{\beta}X^T\hat{\beta}^T \Rightarrow \hat{\beta}X X^T\hat{\beta}^T}$$

$$= 1 - \left[\frac{y^T y - 2yX\hat{\beta} + \hat{\beta}XX^T\hat{\beta}^T}{y^T y - n\bar{y}^2} \right] \quad \Big| \cdot \frac{y^T y - n\bar{y}^2}{y^T y - n\bar{y}^2} = 1$$

$$= \frac{\cancel{y^T y} - n\bar{y}^2 \cancel{- y^T y} - 2yX\hat{\beta} + \hat{\beta}XX^T\hat{\beta}^T}{y^T y - n\bar{y}^2}$$

$$= \frac{2yX\hat{\beta} - \hat{\beta}XX^T\hat{\beta}^T - n\bar{y}^2}{y^T y - n\bar{y}^2}$$ | $X^T\hat{\beta}^T = y + \varepsilon$ where ε = residuals

$$= \frac{2yX\hat{\beta} - \hat{\beta}X(y + \varepsilon) - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$= \frac{yX\hat{\beta} + \hat{\beta}X\varepsilon - n\bar{y}^2}{y^T y - n\bar{y}^2}$$ | Property X is orthogonal to residuals, thus $\hat{\beta}X\varepsilon = 0$

$$= \frac{yX\hat{\beta} - n\bar{y}^2}{y^T y - n\bar{y}^2}$$