Exercice 2 Hand In

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Task 1:

Derive the log likelihood function and the maximum likelihood estimate (in general and for the specific sample).

likelihood: $L(\theta) = P(T_1 = t_1; \theta) \cdot P(T_2 = t_2; \theta) \cdot \ldots \cdot P(T_N = t_N; \theta) = \prod_{i=1}^{N} \frac{1}{2\theta\sqrt{t_i}} \exp\left(-\frac{\sqrt{t_i}}{\theta}\right)$

log likelihood $\ell(\theta) := \log(L(\theta)) = \sum_{i=1}^{N} \log\left(\frac{1}{2\theta\sqrt{t_i}}\right) - \frac{\sqrt{t_i}}{\theta}$

$\Rightarrow \sum_{i=1}^{N} \log(1) - \log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta} \quad \Big| \log(1) = 0$

$\Rightarrow \sum_{i=1}^{N} -\log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta}$

$$\underline{\underline{\ell(\theta) = \sum_{i=1}^{N} -\log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta}}}$$

Maximum likelihood $:= \arg\max_{\theta} \ell(\theta) \Leftrightarrow \frac{\partial}{\partial\theta} \ell'(\theta) = 0$

$\ell'(\theta) = \frac{\partial}{\partial\theta} \ell(\theta) = \sum_{i=1}^{N} -\frac{1}{2\theta\sqrt{t_i}} \cdot 2\sqrt{t_i} - \frac{\sqrt{t_i}}{\theta}$

$\Leftrightarrow \sum_{i=1}^{N} -\frac{2\sqrt{t_i}}{2\theta\sqrt{t_i}} + \frac{\sqrt{t_i}}{\theta^2}$

$\Leftrightarrow \sum_{i=1}^{N} -\frac{1}{\theta} + \frac{\sqrt{t_i}}{\theta^2} = 0 \quad \Big| \theta^2$

$= \sum_{i=1}^{N} -\theta + \sqrt{t_i}$

$= \sum_{i=1}^{N} \sqrt{t_i} = \sum_{i=1}^{N} \theta$

$= \sum \sqrt{t_i} = N\theta \quad \Big| \frac{1}{N}$

$= \frac{1}{N} \sum \sqrt{t_i} = \theta$

$\Rightarrow$ we find that our estimator $\underline{\underline{\hat{\theta}_{ML} = \frac{1}{N} \sum \sqrt{t_i}}}$

Now: To prove, that our estimate $\hat{\theta}_{ML}$ is a maximum, we have to show that $\ell''(\hat{\theta}_{ML}) < 0$:

$\ell''(\theta) = \frac{\partial}{\partial\theta} \ell'(\theta) = \sum_{i=1}^{N} \frac{1}{\theta^2} - \frac{2\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \sum_{i=1}^{N} \frac{1}{\theta^2} - \frac{2\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \frac{N}{\theta^2} - \frac{2\sum\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \frac{\theta^3 N - 2\theta^2 \sum\sqrt{t_i}}{\theta^2 \cdot \theta^3}$

$$\Leftrightarrow \frac{\theta^2\left(\theta N - 2\sum \sqrt{t_i}\right)}{\theta^2 \cdot \theta^3}$$

$$\Leftrightarrow \frac{\theta N - 2\sum \sqrt{t_i}}{\theta^3} \quad \Big| \text{ Replace } \theta \text{ by } \hat{\theta}_{ML}$$

$$= \frac{N\hat{\theta}_{ML} - 2\sum \sqrt{t_i}}{(\hat{\theta}_{ML})^3}$$

We know, that $\forall t_i > 0$, $\hat{\theta}_{ML} > 0$ and thus it follows, that $(\hat{\theta}_{ML})^3$ is also postive $\forall t_i > 0$.

This means we can focus on the numerator discard the denominator, which leaves us with:

$$N\hat{\theta}_{ML} - 2\sum \sqrt{t_i} \quad \Big| \text{ we replace } \hat{\theta}_{ML} \text{ with } \frac{1}{N}\sum_{i=1}^{N} \sqrt{t_i}$$

$$= \frac{N}{N}\sum_{i=1}^{N} \sqrt{t_i} - 2\sum_{i=1}^{N} \sqrt{t_i}$$

$$\Leftrightarrow -\sum_{i=1}^{N} \sqrt{t_i} < 0$$

We find, that the Nominator of $l''(\hat{\theta}_{ML})$ is Negative $\forall t_i > 0$ and thus concludi that our Maximum likelihood estimator $\hat{\theta}_{ML}$ is in fact a point of maximum.

Using the discrete samples: $\{t_1 = 11300, t_2 = 5000, t_3 = 4300, t_4 = 8500, t_5 = 7900\}$

we obtain:

$$\frac{1}{5}\left(11300^{1/2} + 5000^{1/2} + 4300^{1/2} + 8500^{1/2} + 7900^{1/2}\right) = 84.7327$$