

Homework Sheet 4

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Task 1

1. 8 points

A square matrix M is called idempotent if $M^2 = M$.

(a) Show that $C = I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T$ is idempotent, where I_n is the identity matrix and $\mathbf{1}$ is a vector of ones. Also show that $C \mathbf{1} = \mathbf{0}$.

$$C = I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \quad \text{where} \quad \mathbf{1} = [1 \ 1 \ \dots \ 1]^T, \quad I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}$$

$$\begin{aligned} C^2 &= \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \cdot \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) = \\ &= \underbrace{I_n^2}_{I_n} - I_n \cdot \frac{1}{n} \mathbf{1} \mathbf{1}^T - \frac{1}{n} \mathbf{1} \mathbf{1}^T \cdot I_n + \frac{1}{n} \mathbf{1} \mathbf{1}^T \cdot \frac{1}{n} \mathbf{1} \mathbf{1}^T \\ &= I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T - \frac{1}{n} \mathbf{1} \mathbf{1}^T + 1 \\ &= I_n - \frac{2}{n} \mathbf{1} \mathbf{1}^T + 1 \\ &= I_n - 2 + 1 \\ &= I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \\ &= I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \end{aligned}$$

We proved that $C^2 = C$. Therefore, C is idempotent.

• Next Part is to show that $C \mathbf{1} = \mathbf{0}$

$$\begin{aligned} C &= I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \quad / \cdot \mathbf{1} \quad \text{where} \quad \mathbf{1} = [1 \ 1 \ \dots \ 1]^T \\ C \cdot \mathbf{1} &= \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \cdot \mathbf{1} \\ C \cdot \mathbf{1} &= I_n \cdot \mathbf{1} - \frac{1}{n} \cdot n \\ C \cdot \mathbf{1} &= \mathbf{1} - \text{1 scalar} \\ C \cdot \mathbf{1} &= \mathbf{0} ; \quad \text{where} \quad \mathbf{0} = [0 \ 0 \ \dots \ 0]^T \end{aligned}$$

(b) Let $a \in \mathbb{R}^{n \times 1}$, show that

$$Ca = \begin{bmatrix} a_1 - \bar{a} \\ \vdots \\ a_n - \bar{a} \end{bmatrix}.$$

$$C = I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \quad | \cdot \vec{a}$$

$$\begin{aligned} C \vec{a} &= \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \vec{a} \\ &= I_n \cdot \vec{a} - \frac{1}{n} \mathbf{1} \mathbf{1}^T \vec{a} \end{aligned}$$

$$= \vec{a} - \frac{1}{n} \sum_{i=1}^n a_i \vec{a}$$

$$\begin{aligned} &\text{vector} \quad \vec{a} - \bar{a} \text{ scalar} \Rightarrow C \vec{a} = \begin{pmatrix} a_1 - \bar{a} \\ a_2 - \bar{a} \\ \vdots \\ a_n - \bar{a} \end{pmatrix} \end{aligned}$$

Let $A \in \mathbb{R}^{n \times k}$, show that

$$CA = \begin{bmatrix} a_{11} - \bar{a}_1 & \cdots & a_{1k} - \bar{a}_k \\ \vdots & & \vdots \\ a_{n1} - \bar{a}_1 & \cdots & a_{nk} - \bar{a}_k \end{bmatrix}$$

row vector, where j -th member is the mean of the j -th column of the A matrix

$$C \cdot A = \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) A = I_n \cdot A_{n \times k} - \frac{1}{n} \mathbf{1} \mathbf{1}^T A = A_{n \times k} - \left[\frac{1}{n} \sum_{j=1}^k a_j \right]_{1 \times k}$$

$$\Rightarrow C \cdot A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} - \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_k \end{bmatrix}$$

$$\Rightarrow C \cdot A = \begin{bmatrix} a_{11} - \bar{a}_1 & a_{12} - \bar{a}_2 & \cdots & a_{1k} - \bar{a}_k \\ \vdots & \vdots & & \vdots \\ a_{n1} - \bar{a}_1 & a_{n2} - \bar{a}_2 & \cdots & a_{nk} - \bar{a}_k \end{bmatrix}$$

(c) Show that

$$\sum_{i=1}^n (x_i - \bar{x})^2 = x^T C x, \quad \text{where } x = [x_1, \dots, x_n]^T.$$

$$x^T C x = \bar{x}^T \left(I_n - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \vec{x} = \bar{x}^T I_n \vec{x} - \bar{x}^T \cdot \frac{1}{n} \mathbf{1} \mathbf{1}^T \vec{x} =$$

$$= \underbrace{\bar{x}^T \vec{x}}_{\sum_{i=1}^n x_i \cdot x_i} - \frac{1}{n} \bar{x}^T \mathbf{1} \mathbf{1}^T \vec{x} = \sum_{i=1}^n x_i^2 - \frac{1}{n} \left(\sum_{i=1}^n x_i \right)^2 = \sum_{i=1}^n x_i^2 - \frac{1}{n} n^2 \bar{x}^2$$

$$\sum_{i=1}^n x_i \cdot x_i = \sum_{i=1}^n x_i^2$$

$$= \sum_{i=1}^n x_i^2 - n \bar{x}^2 = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n \bar{x}^2 = \sum_{i=1}^n (x_i^2 - \bar{x}^2)$$

$$= \sum_{i=1}^n (x_i^2 - 2x_i \bar{x} + \bar{x}^2 + 2x_i \bar{x} - 2\bar{x}^2)$$

$$= \sum_{i=1}^n (x_i - \bar{x})^2 + 2 \cdot \bar{x} \underbrace{\sum_{i=1}^n (x_i - \bar{x})}_0$$

$$= \sum_{i=1}^n (x_i - \bar{x})^2$$

(d) Let $z_1, \dots, z_n \in \mathbb{R}^p$ be realisations of a multivariate random variable Z . Express the sample mean $\bar{z} = \frac{1}{n} \sum_{i=1}^n z_i$ and the sample covariance $S^2 = \frac{1}{n-1} \left(\sum_{i=1}^n (z_i - \bar{z}) \sum_{j=1}^n (z_j - \bar{z})^T \right)$ as matrix operation, using the above results and the design matrix $T = (x_1 \dots x_n) \in \mathbb{R}^{p \times n}$.

• Sample mean

$$\bar{z} = \frac{1}{n} \cdot T \cdot \vec{1}$$

observations

features

$$\begin{matrix} & \begin{matrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & x_{p3} & \dots & x_{pn} \end{matrix} & \times & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ & & & \text{p} \times 1 \end{matrix}$$

$\vec{1}$

• Sample covariance

$$x_{11} \cdot 1 + x_{12} \cdot 1 + \dots + x_{1n} \cdot 1 = \sum_{i=1}^n x_{1i}$$

$$S^2 = \frac{1}{n-1} \cdot \bar{T} \cdot \bar{T}^{-1}, \quad \text{where } \bar{T} \text{ is the centered matrix } T - \bar{z} \cdot \vec{1}$$

$$Q.2 \quad \hat{\Sigma}^2 = (y - X\hat{\beta})^T (y - X\hat{\beta}) = y^T y - y^T X \hat{\beta} - \hat{\beta}^T X^T y + \hat{\beta}^T X^T X \hat{\beta}$$

$$= y^T y - 2\hat{\beta}^T X^T y + \hat{\beta}^T X^T X \hat{\beta}$$

$$\sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n (y_i^2 - 2y_i \bar{y} + \bar{y}^2)$$

$$= y^T y - 2n\bar{y}\bar{y} + n\bar{y}^2$$

$$= y^T y - n\bar{y}^2$$

$$\bar{y} = \frac{\sum y_i}{n}$$

$$1 - \frac{\hat{\Sigma}^2}{\sum (y_i - \bar{y})^2} = 1 - \frac{y^T y - 2\hat{\beta}^T X^T y + \hat{\beta}^T X^T X \hat{\beta}}{y^T y - n\bar{y}^2}$$

$$= \frac{2\hat{\beta}^T X^T y - \hat{\beta}^T X^T X \hat{\beta} - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$= \frac{2\hat{\beta}^T X^T y - \hat{\beta}^T X^T (y - \varepsilon) - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$= \frac{\hat{\beta}^T X^T y + \hat{\beta}^T X^T \varepsilon - n\bar{y}^2}{y^T y - n\bar{y}^2}$$

$$\text{but } X^T \varepsilon = 0$$

$$= \frac{\hat{\beta}^T X^T y - n\bar{y}^2}{y^T y - n\bar{y}^2}$$