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Exercice 2 Hand In

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For simplicity:

Task 1: $\boxed{4/4}$

Please include the plots and answer to question 3 on the same file !

Derive the log likelihood function and the maximum likelihood estimate (in general and for the specific sample).

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likelihood: $L(\theta) = P(T_1 = t_1; \theta) \cdot P(T_2 = t_2; \theta) \cdot \ldots \cdot P(T_N = t_N; \theta) = \prod_{i=1}^{N} \frac{1}{2\theta\sqrt{t_i}} \exp\left(-\frac{\sqrt{t_i}}{\theta}\right)$

log likelihood $\ell(\theta) := \log(L(\theta)) = \sum_{i=1}^{N} \log\left(\frac{1}{2\theta\sqrt{t_i}}\right) - \frac{\sqrt{t_i}}{\theta}$

$\Rightarrow \sum_{i=1}^{N} \log(1) - \log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta} \quad \Big| \log(1) = 0$

$\Rightarrow \sum_{i=1}^{N} -\log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta}$

$$\ell(\theta) = \sum_{i=1}^{N} -\log(2\theta\sqrt{t_i}) - \frac{\sqrt{t_i}}{\theta}$$

✓

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Maximum likelihood := $\arg\max_{\theta} \ell(\theta)$ $\boxed{\Leftrightarrow}$ $\left(\frac{\partial}{\partial\theta}\ell(\theta) = 0 \quad \wedge \quad \ell''(\theta) < 0\right)$

$\ell'(\theta) = \frac{\partial}{\partial\theta}\ell(\theta) = \sum_{i=1}^{N} -\frac{1}{2\theta\sqrt{t_i}} \cdot 2\sqrt{t_i} - \frac{\sqrt{t_i}}{\theta}$

$\Leftrightarrow \sum_{i=1}^{N} -\frac{2\sqrt{t_i}}{2\theta\sqrt{t_i}} + \frac{\sqrt{t_i}}{\theta^2}$

$\Leftrightarrow \sum_{i=1}^{N} -\frac{1}{\theta} + \frac{\sqrt{t_i}}{\theta^2} = 0 \quad \Big| \theta^2$

$= \sum_{i=1}^{N} -\theta + \sqrt{t_i}$

$= \sum_{i=1}^{N} \sqrt{t_i} = \sum_{i=1}^{N} \theta$

$= \sum \sqrt{t_i} = N\theta \quad \Big| \frac{1}{N}$

$= \frac{1}{N}\sum \sqrt{t_i} = \theta$

$\Rightarrow$ we find that our estimator $\hat{\theta}_{ML} = \frac{1}{N}\sum\sqrt{t_i}$ ✓

✓

Now: To prove, that our estimate $\hat{\theta}_{ML}$ is a maximum, we have to show that $\ell''(\hat{\theta}_{ML}) < 0$:

$\ell''(\theta) = \frac{\partial}{\partial\theta}\ell'(\theta) = \sum_{i=1}^{N} \frac{1}{\theta^2} - \frac{2\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \sum_{i=1}^{N} \frac{1}{\theta^2} - \frac{2\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \frac{N}{\theta^2} - \frac{2\sum\sqrt{t_i}}{\theta^3}$

$\Leftrightarrow \frac{\theta^3 N - 2\theta^2\sum\sqrt{t_i}}{\theta^2 \cdot \theta^3}$

$$\Leftrightarrow \frac{\theta^2(\theta N - 2\sum\sqrt{t_i})}{\theta^2 \cdot \theta^3}$$

$$\Leftrightarrow \frac{\theta N - 2\sum\sqrt{t_i}}{\theta^3} \quad \Big| \text{ Replace } \theta \text{ by } \hat{\theta}_{ML}$$

$$= \frac{N\hat{\theta}_{ML} - 2\sum\sqrt{t_i}}{(\hat{\theta}_{ML})^3}$$

We know, that $\forall t_i > 0$, $\hat{\theta}_{ML} > 0$ and thus it follows, that $(\hat{\theta}_{ML})^3$ is also postive $\forall t_i > 0$.

This means we can focus on the numerator discard the denominator, which leaves us with:

$$N\hat{\theta}_{ML} - 2\sum\sqrt{t_i} \quad \Big| \text{ we replace } \hat{\theta}_{ML} \text{ with } \frac{1}{N}\sum_{i=1}^{N}\sqrt{t_i}$$

$$= \frac{N}{N}\sum_{i=1}^{N}\sqrt{t_i} - 2\sum_{i=1}^{N}\sqrt{t_i}$$

$$\Leftrightarrow -\sum_{i=1}^{N}\sqrt{t_i} < 0$$

We find, that the Nominator of $l''(\hat{\theta}_{ML})$ is Negative $\forall t_i > 0$ and thus concludi that our Maximum likelihood estimator $\hat{\theta}_{ML}$ is in fact a point of maximum.

Using the discrete samples: $\{t_1 = 11300, t_2 = 5000, t_3 = 4300, t_4 = 8500, t_5 = 7900\}$

we obtain:

$$\frac{1}{5}\left(11300^{\frac{1}{2}} + 5000^{\frac{1}{2}} + 4300^{\frac{1}{2}} + 8500^{\frac{1}{2}} + 7900^{\frac{1}{2}}\right) = \underline{\underline{84.17327}}$$

$$\theta_{MLE} = \frac{1}{5}\cdot\left(\frac{t_1}{11300} + \frac{t_2}{5000} + \frac{t_3}{4300} + \frac{t_4}{8500} + \frac{t_5}{7900}\right)$$

(Q2) Consider a Poisson distributed r.v. with p.d.f.

$f_\theta(x) = \frac{\theta^x \cdot e^{-\theta}}{x!}$ & independent samples $x_1, \dots x_n$

Derive the MLE $\hat{\theta}$ of θ. Is the estimate unbiased & consistent?

The joint pdf for $x_1 \dots x_n$ samples are:

$$f_{\theta x_1 \dots x_n}(x_1 \dots x_n) = \frac{\theta^{x_1} \cdot e^{-\theta}}{x_1!} \cdot \frac{\theta^{x_2} \cdot e^{-\theta}}{x_2!} \cdots \frac{\theta^{x_n} \cdot e^{-\theta}}{x_n!}$$

$$= \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \cdot e^{-n\theta}$$

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Taking log of the joint pdf to obtain the log likelihood:

$\log(f_{X_1 \dots X_n}(x_1 \dots x_n))$

$$= \log \frac{1}{\prod_{i=1}^{n} x_i!} + \sum_{i=1}^{n} x_i \log\theta - n\theta.$$

Deriving wrt θ & equating to 0 to find MLE

$$\frac{\partial\, l(f_{X_1 \dots X_n}(x_1 \dots x_n))}{\partial\theta} = 0 + \frac{\sum_{i=1}^{n} x_i}{\theta} - n$$

$\therefore \hat{\theta} = \sum_{i=1}^{n} \frac{x_i}{n} = \hat{\theta} = \text{MLE}$ $\because 2^{nd}$ derivative of this log likelihood is < 0 i.e. $-\frac{1}{\theta^2}\sum x \dots$

To check:

a) Unbiasedness: We need to ensure $E[\hat{\theta}] = \theta$.

$$E[\hat{\theta}] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n} E\left[\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i]$$

$\because x_i$ follows poisson distr^n $E[x_i] = \theta$ $\therefore \sum_{i=1}^{n} E[x_i] = n\theta$

$$\therefore E[\hat{\theta}] = \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \theta$$

$\therefore$ MLE is unbiased.

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b) To check for consistency:

We need to show that as the sample size $n \to \infty$, the MLE $\hat{\theta}$ converges in probability to θ.

That means for any $\varepsilon > 0$.

$$\lim_{n \to \infty} P(|\hat{\theta} - \theta| > \varepsilon) = 0.$$

For Poisson distr^n as per law of large no's, it is ensured that as n grows, the sample mean $\Sigma x_i / n$ converges to the true parameter θ. Therefore, the MLE is consistent.