# Homework 8

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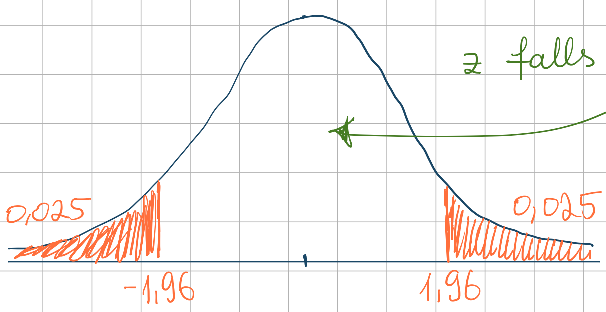
1. **Exercise 1 (12 Points)** Let $X_i \sim \mathcal{N}(\theta, \sigma)$ be iid random variable where $\sigma^2 = 4$. Consider the problem $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$. Employ a (two-sided) Z-test for this, that is $H_0$ is rejected if $\frac{|\overline{x} - \theta_0|}{\frac{\sigma}{\sqrt{N}}} > z_{\alpha/2}$.

   (a) Test if $H_0$ is accepted or rejected for $\alpha = 0.05$ for observed $\overline{x} = 1.5$ with $n = 15$.

$$z = \frac{|\overline{x} - \theta_0|}{\frac{\sigma}{\sqrt{N}}} = \frac{|1{,}5 - 1|}{\frac{2}{\sqrt{15}}} = \frac{0{,}5 \cdot \sqrt{15}}{2} \cong 0{,}968$$

Since $\alpha = 0{,}05 \Rightarrow \frac{\alpha}{2} = 0{,}025 \Rightarrow z_{.025} = 1{,}96$

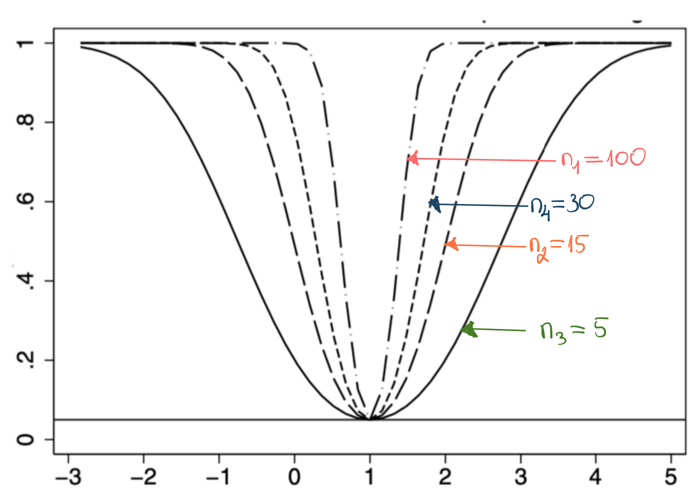
Now we compare $z$ and $z_{\alpha/2}$: $0{,}968 < 1{,}96$




z falls here, not under the tails.

$\boxed{H_0 \text{ is NOT rejected.}}$

   (b) In the figure 1, the associated power functions for different number of samples are given. Determine which power function is associated with which sample sizes: $n_1 = 100, n_2 = 15, n_3 = 5$ and $n_4 = 30$. Provide reasoning for each of your each assignments.




With a small sample size, the distribution tends to have a larger standard deviation. In other words, we get a wider bell curve. For this reason, $n_3 = 5$ corresponds to the curve with the uninterrupted line.

As we increase the sample size, we become more "confident" in the value of the mean. That is, the distribution is more concentrated around the mean, with a smaller variance (std). Therefore, we assign $n_2 = 15$ to the long-dashed line and $n_4 = 30$ to the short-dashed one. Finally, we assign $n_1 = 100$ to the curve with the highest concentration around the mean (smallest std, most precise).

(c) Determine a confidence interval for for the error probability $\alpha = 0.05$ if the empirical mean $\overline{x} = 1.5$ was calculated for a sample of size $n = 15$.. That is, determine an interval $(\overline{x} - a, \overline{x} + a)$ which contains the true parameter $\theta$ with probability $\alpha$.

$$\bar{x} - \underbrace{z_\alpha \frac{\sigma}{\sqrt{n}}}_{a} \leq \theta \leq \bar{x} + \underbrace{z_\alpha \frac{\sigma}{\sqrt{n}}}_{a} \longrightarrow CI = (\bar{x} - a, \bar{x} + a)$$

To find $z_\alpha$: $A_L = \frac{CL + 1}{2} = \frac{.95 + 1}{2} = \frac{1.95}{2} = .975$

$z_\alpha = 1.96 \longrightarrow$ from the z-score table

$$\bar{x} \pm 1.96 \cdot \frac{2}{\sqrt{15}} = \bar{x} \pm 1.01 = 1.5 \pm 1.01$$

$$CI = (1.5 - 1.01,\ 1.5 + 1.01) = (0.48,\ 2.51)$$

NOTE: although the t-test is preferred for small samples, we used the z-test because we were given the population standard deviation $\sigma$.

(d) What effect does an increase in the sample size have on the confidence interval calculated in (c), assuming that the empirical mean and the variance $\sigma^2$ remain unchanged?

As explained in part (b), increasing the sample size accounts for reduced standard error ($\frac{\sigma}{\sqrt{n}}$), which leads to a narrower confidence interval. In other words, the precision is increased and the range within which the true $\theta$ lies shrinks.

2. **Exercise 1 (12 Points)** As part of a large-scale study on Women and Pregnancy, the age of women at the birth of their first child is of interest. It is assumed that the average age of first-time mothers is over 25 years. To test this hypothesis, 49 mothers are randomly selected and asked about their age at the birth of their first child. The mean age was $\overline{x} = 26$ years.

(a) Assume that the age of primiparous women is normally distributed with variance $\sigma^2 = 9$ known from experience. Check the statistical null hypothesis $H_0 : \theta \leq 25$ against the alternative $H_1 : 0 > 25$ at level $\alpha = 0.05$ using a (one-sided) Z-test. That is $H_0$ is rejected if $\frac{\overline{x}-\theta}{\frac{\sigma}{\sqrt{N}}} > z_\alpha$. Interpret your result.

$n = 49 \quad \bar{x} = 26 \quad \sigma^2 = 9 \quad (\sigma = 3) \quad H_0: \theta \leq 25 \quad H_1: \theta > 25$

$\alpha = 0.05$

$$z = \frac{\bar{x} - \theta}{\underbrace{\frac{\sigma}{\sqrt{N}}}_{\text{std error}}} = \frac{26-25}{\frac{3}{\sqrt{49}}} = \frac{1}{\frac{3}{7}} = \frac{7}{3} = 2.33$$

From the z-table, the critical value for $\alpha = 0.05$ is 1.64.
We now compare z with $z_\alpha$: $2.33 > 1.64$.

Since our observed value of z is greater than the critical value, we **reject the null hypothesis $H_0$**.

We can find the critical $\bar{x}$ value from the z-score formula:

$$\bar{x}_c = \bar{x} + \frac{\sigma}{\sqrt{n}} \cdot z_c = 25 + \frac{3}{7} \cdot 1.64 = 25.70.$$

← The boundary beyond which we reject the null hypothesis in favor of the alternative.

(b) How is the first type of error defined and what does it mean here?

A Type I error in hypothesis testing is defined as an incorrect rejection of the null hypothesis in favor of the alternative hypothesis.

In task 2(a), Type I error means rejecting that the average age of first-time mothers is $\leq 25$ years, when it is, in fact, true.

(c) Determine the probability of making a type 2 error under the assumption that $\theta = 27$ is the true age of primiparae.

Earlier we calculated $z = 2{,}33$ and $\bar{x}_{critical} = \bar{x}_c = 25{,}70$.
That is, we fail to reject $H_0$ for values smaller than $25{,}70$.

To find $\beta$ (probability of type II error), we calculate the following:

$$\mathbb{P}(\bar{x}_c \leq 25{,}70 \mid \theta = 27) = \mathbb{P}\left( z \leq \frac{25{,}7 - 27}{3/7} \right) = \mathbb{P}\left(z \leq -3{,}0216\right)$$

$$= 0{,}0013$$

and then $\beta = 1 - 0{,}013 = 0{,}9987$ (Type II error probability)

→ Such a high $\beta$ indicates that there is a very high probability of failing to detect that the true mean is greater than 25, when it is, in fact, 27.

(d) Determine a 95% confidence interval for the age of the primiparae.

$$CI = \left(\bar{x} - z_c \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_c \cdot \frac{\sigma}{\sqrt{n}}\right) = \left(26 - 1.96 \cdot \frac{3}{7}, 26 + 1.96 \frac{3}{7}\right)$$

$$= (25.16, 26.84)$$