6 Duality Theory

In general 'dual', means two or double. In linear programming, duality implies that each linear programming problem can be analysed in two different ways but would have equivalent solutions. Any LPP (either maximization and minimization) can be stated in another equivalent form based on the same data. The new LPP is called dual linear programming problem or in short dual.

Remark 6.1 *In general, it is immaterial which of the two problems is called primal or dual, since the dual of the dual is primal ! (check it).*

An example: Let us consider the following diet problem.
The daily requirement of a patient is 20 and 30 units of vitamins v_1 and v_2 respectively. The food F_1 contains 3 units of v_1 and 4 units of v_2. Another food F_2 contains 2 units of v_1 and 3 units of v_2. If F_1 and F_2 cost Rs. 7 and Rs. 5 per unit then find the minimum cost of buying the vitamins.
We formulate the corresponding LPP. Let x_1 and x_2 units of the foods F_1 and F_2 are required to fulfil the need of vitamin of the patient. Then the LPP is

$$\text{Minimize, } z = 7x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 20 \\ 4x_1 + 3x_2 &\geq 30 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Now let us consider another scenario related to the above problem.

A dealer sells the above mentioned vitamins v_1 and v_2 separately. His problem is to fix the cost of v_1 and v_2 in such a way that the price of F_1 and F_2 do not exceed the amount mentioned above. He wants to get maximum amount of selling of these two vitamins.

Let w_1 and w_2 are the price per unit of v_1 and v_2 respectively. Therefore the LPP is

$$\text{Maximize, } z^* = 20w_1 + 30w_2$$

subject to

$$\begin{aligned} 3w_1 + 4w_2 &\leq 7 \\ 2w_1 + 3w_2 &\leq 5 \\ w_1, w_2 &\geq 0. \end{aligned}$$

The initial problem (primal problem) can be written as

$$\text{Minimize, } z = c.x$$

subject to

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \end{aligned}$$

and the second problem (dual problem) is to written as

$$\text{Maximize, } z^* = b^t.w$$

subject to

$$\begin{aligned} A^t w &\leq c^t \\ w &\geq 0. \end{aligned}$$

The above two problems is an example of a primal-dual problem.

6.1 Standard primal-dual form

Before converting a primal problem into its dual form we have to first reduce the problem in standard form which is

i) all constraints involve the sign $\leq$ in a problem of maximization,

ii) all constraints involve the sign $\geq$ in a problem of minimization.

Let us consider the following maximization LPP

$$\text{Maximize } \; z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leq & b_2 \\
& \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \\
x_j \geq 0, \;\; j = 1, 2, \cdots, n. & &
\end{array}$$

This form is known as standard symmetric primal. The corresponding dual problem is

$$\text{minimize} \;\; z^* = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$$

Subject to

$$\begin{array}{rcl}
a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m & \geq & c_1 \\
a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m & \geq & c_2 \\
& \vdots & \\
a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m & \geq & c_n \\
w_j \geq 0, \;\; j = 1, 2, \cdots, m. & &
\end{array}$$

The rule of transformation are

1. Coefficient matrix of the dual is the transpose of the coefficient matrix of the primal
2. interchange cost vector and requirement vector
3. change the direction of the sign of the inequalities
4. change the nature of optimisation (minimisation and maximisation) of the objective function.

Next we mention some results which are important in this chapter (without proof)!

Theorem 6.2 *Dual of the dual is the primal itself.*

Theorem 6.3 *If either the primal or the dual problem has finite optimal solution then the other will also have finite optimal solution. Furthermore the optimal values of the objective functions in both the problem will be same.*

Theorem 6.4 *If either the primal or the dual problem has unbounded solution then the other will also have no feasible solution.*

Theorem 6.5 *If either the primal problem has feasible solution and the dual has no feasible solution then the primal problem is said to have unbounded solution and vice versa.*

Here we mention the rules for obtaining the dual optimal solution form the final simplex table of the primal problem and vice versa.

1. The optimal values of the objective functions are same,
2. The value of $z_j - c_j$ corresponding to the columns of slack (surplus) vector in the final simplex table of the primal (dual) problem are the values of the corresponding dual (primal) optimal variable provided the problem is solved as a maximization problem.

6.2 Importance of duality theory

When the number of constraints are greater than the number of variables then duality theory is very helpful to solve the problem by simplex method.

Example 6.6 *Solve the following LPP by solving its dual problem by simplex method.*

$$\textit{Minimize, } z = 3x_1 + x_2$$

subject to

$$\begin{gathered} 2x_1 + x_2 \geq 14 \\ x_1 - x_2 \geq 4 \\ x_1, x_2 \geq 0. \end{gathered}$$

Solution: The dual of the above LPP is

$$\text{Maximize, } z^* = 14w_1 + 4w_2$$

subject to

$$\begin{aligned} 2w_1 + w_2 &\le 3 \\ w_1 - w_2 &\le 1 \\ w_1, w_2 &\ge 0. \end{aligned}$$

To solve the above problem by simplex method we introduce two slack variables w_3 and w_4 for both the constraints. Then the reduced standard form is

$$\text{Maximize, } z^* = 14w_1 + 4w_2 + 0.w_3 + 0.w_4$$

subject to

$$\begin{aligned} 2w_1 + w_2 + w_3 &= 3 \\ w_1 - w_2 + w_4 &= 1 \\ w_1, w_2, w_3, w_4 &\ge 0. \end{aligned}$$

		c	14	4	0	0	
Basis	c_B	b	a_1	a_2	$a_3(e_1)$	$a_4(e_2)$	Min. ratio
a_3	0	3	2	1	1	0	$\frac{3}{2}$
a_4^*	0	1	1*	−1	0	1	$\frac{1}{1} = 1^*$
$c_j - z_j$		0	14*	4	0	0	
a_3^*	0	1	0	3*	1	−2	$\frac{1}{3}$
a_1	14	1	1	−1	0	1	...
$c_j - z_j$		14	0	18*	0	−14	
a_2	4	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	
a_1	14	$\frac{4}{3}$	1	0	$\frac{1}{3}$	$\frac{1}{3}$	
$c_j - z_j$		20	0	0	−6	−2	

Since all $z_j - c_j \ge 0$, the optimal solution of the problem is max $z^* = 20$ at $w_1 = \frac{4}{3}$ and $w_2 = \frac{1}{3}$. Now the value of $z_j - c_j$ corresponding to the slack variables w_3 and w_4 are $z_3 - c_3 = 6$ and $z_4 - c_4 = 2$. Therefore the optimal solution of the primal problem is $x_1 = 6$ and $x_2 = 2$ and min $z = 20$.