Revised simplex method

Recall: The standard for of an LPP is; Max $z = c.x$ such that $Ax = b$ with $x \geq 0$. The following notations are important

1. the coefficient vector c in the objective function is known as the cost vector
2. $A = (a_1,\ a_2, \cdots,\ a_n)$ where a_i s are the coefficient of the variables x_i in the matrix A
3. x_B denotes the vector consisting the basic variables,
4. c_B denotes the co-efficients of the basic vectors in the objective function,
5. B denotes the basis matrix (square matrix),
6. $c_j - z_j = c_j - c_B B^{-1} a_j$,
7. the updated column vector $\overline{a_i}$ corresponding to the variable x_i in each iteration (or simplex table) is given by $\overline{a_i} = B^{-1} a_i$.

Example: Consider the problem; Maximize, $z = 5x_1 + 2x_2 + 2x_3$

subject to

$$
\begin{gathered}
x_1 + 2x_2 - 2x_3 \leq 30 \\
x_1 + 3x_2 + x_3 \leq 36 \\
x_1, x_2, x_3 \geq 0.
\end{gathered}
$$

Adding slack variables x_4 and x_5 the lpp reduces to the standard form
Maximize, $z = 5x_1 + 2x_2 + 2x_3 + 0x_4 + 0x_5$

subject to

$$
\begin{gathered}
x_1 + 2x_2 - 2x_3 + x_4 = 30 \\
x_1 + 3x_2 + x_3 + x_5 = 36 \\
x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{gathered}
$$

The coefficients of the variables x_i in the coefficient matrix are denoted by a_i.

Iteration 1: Initially x_4 and x_5 will act as basic variables. Hence we have the following

$$x_{B_1} = (x_4,\ x_5) = (30,\ 36),\ c_{B_1} = (0,\ 0),\ B_1 = I$$

Now we calculate $c_j - z_j$ for non-basic variables only.

$$c_1 - z_1 = c_1 - c_{B_1} {B_1}^{-1} a_1 = 5 - (0,\ 0).a_1 = 5$$

In a similar way

$$c_2 - z_2 = 2,\ c_3 - z_3 = 2.$$

since $c_1 - z_1$ is the maximum positive element, x_1 is the entering variable. To decide the leaving variable, we need to get the min ratio and for that we need to calculate the column vectors for x_1 in the table. In this case

$$\overline{a_1} = B_1^{-1} a_1 = a_1 = (1,\ 1).$$

The min ration θ is given by the minimum quantity such that

$$30 - \theta \geq 0,\ 36 - \theta \geq 0.$$

Hence $\theta = 30$ and this occurs in the first constraint, so x_4 is the leaving variable.

Iteration 2: Now x_1 and x_5 will act as the basic variables. We have the following

$$x_{B_2} = (x_1,\ x_5) = (30,\ 6),\ c_{B_2} = (5,\ 0),\ B_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Now we need to know $c_{B_2} B_2^{-1}$ to proceed further. Note that B_2 can be written as

$$B_2 = B_1 E_1$$

where E_1 is the eta matrix of same order whose columns are the part of identity matrix except one column. In this case

$$B_1 = I,\ E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $y = c_{B_2} B_2^{-1}$. We can re-write this in the form of system of equations

$$(y_1,\ y_2)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = (5,\ 0)$$

solving this we get $y = c_{B_2} B_2^{-1} = (5, 0)$. Now we are ready to calculate $c_j - z_j$ for non-basic variables.

$$c_2 - z_2 = c_2 - c_{B_2} B_2^{-1} a_2 = 2 - (5,\ 0).(2,\ 3) = -8$$

Similarly $c_3 - z_3 = 12,\ c_4 - z_4 = -5$. The maximum positive element is $c_3 - z_3 = 12$. Hence x_3 is the entering variable.
To find the leaving variable we need to find the min ratio and for that we need to calculate the column vector ($\overline{a_3}$) in the updated table for the variable x_3 which is given by

$$\overline{a_3} = B_2^{-1} a_3.$$

This can be obtained by solving the system of equations

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}p_1\\ p_2\end{pmatrix} = \begin{pmatrix}-2\\ 1\end{pmatrix}$$

solving we get $\overline{a_3} = \begin{pmatrix}-2\\ 3\end{pmatrix}$. The minimum ratio θ is given by $6 - 3\theta \geq 0$ and hence $\theta = 2$ occurs corresponding to the last constraint i.e., the variable x_5 will leave the basis.
Iteration 3: In this iteration the basis variables are x_1 and x_3. We have the following

$$x_{B_3} = (x_1,\ x_3) = (34,\ 2),\ c_{B_3} = (5,\ 2),\ B_3 = \begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$$

Now we need to know $c_{B_3} B_3^{-1}$ to proceed further. Note that B_3 can be written as

$$B_3 = B_2 E_2 = B_1 E_1 E_2 = E_1 E_2$$

where $E_2 = \begin{pmatrix}1 & -2\\ 0 & 3\end{pmatrix}$. Let $y = c_{B_3} B_3^{-1},\ yB_3 = c_{B_3},\ (yE_1)E_2 = c_{B_3}$. Let $u = yE_1$. Then we have two system of linear equations

$$uE_2 = c_{B_3},\ yE_1 = u.$$

Solving these we get $y = (1,\ 4)$. Now we are ready to calculate $c_j - z_j$.

$$c_2 - z_2 = 2 - y.(2,\ 3) = -12,\ c_4 - z_4 = 0 - y.(1,\ 0) = -1,\ c_5 - z_5 = -4$$

All $c_j - z_j$ are negative. Hence optimality condition satisfied. The solution is $z_{max} = 174$ occurs at $x_1 = 34,\ x_2 = 0,\ x_3 = 2$.