KELVIN

Stars (Like Sun) are stable; where the pressure supports them against gravity.

$$E = U + V$$

Virial Equilibrium }

$$E = \frac{V}{2} = -U$$

$$-20 = \frac{-40}{2} = -20$$

$$\left.\begin{aligned} E &= -20 \\ V &= -40 \\ U &= +20 \end{aligned}\right\}$$

$$V = -\frac{GM}{R}$$

$$U \propto kT$$

because of loss of 2 units of energy through radiation (luminosity)

$$-22 = -\frac{44}{2} = -22 \quad \left\} \begin{aligned} E &= -22 \\ V &= -44 \\ U &= +22 \end{aligned} \right.$$

$$\begin{array}{c|ccc} E & (-20) & \longrightarrow & -22 \\ V & -40 & \longrightarrow & -44 \\ U & +20 & \longrightarrow & +22 \end{array}$$

$$V = -\frac{GM}{R}$$

$$U \propto kT$$

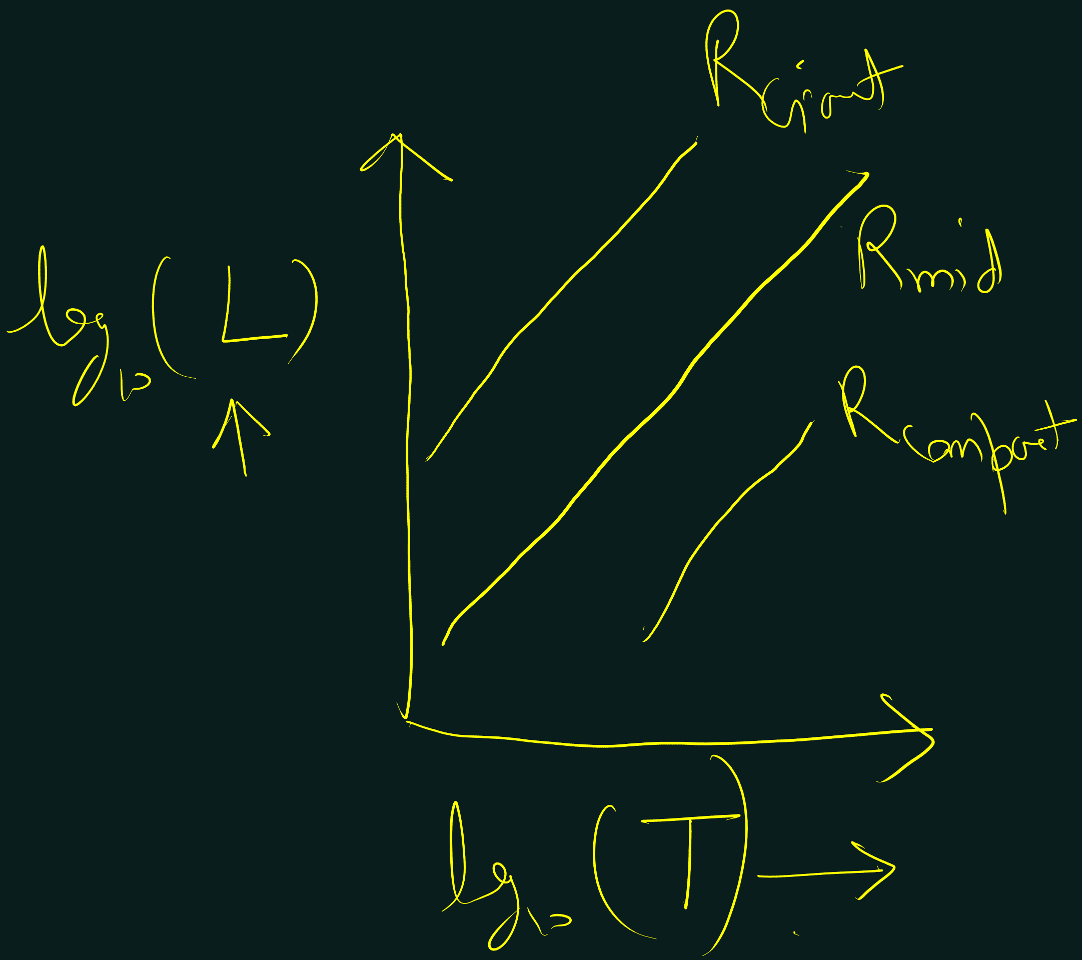
+2 units of energy generated by fusion (negligible / no loss through radiation)

$$E = -18$$

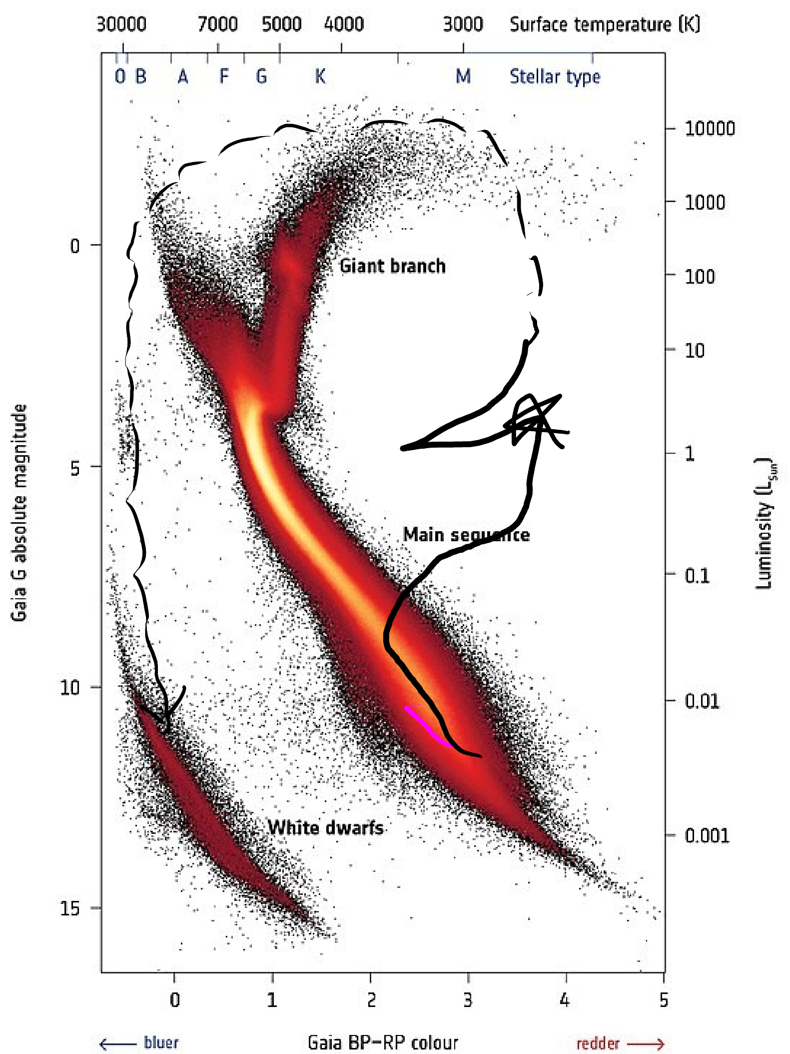
$$V = -36$$

$$U = +18$$

$$L = 4\pi R^2 \sigma T^4$$

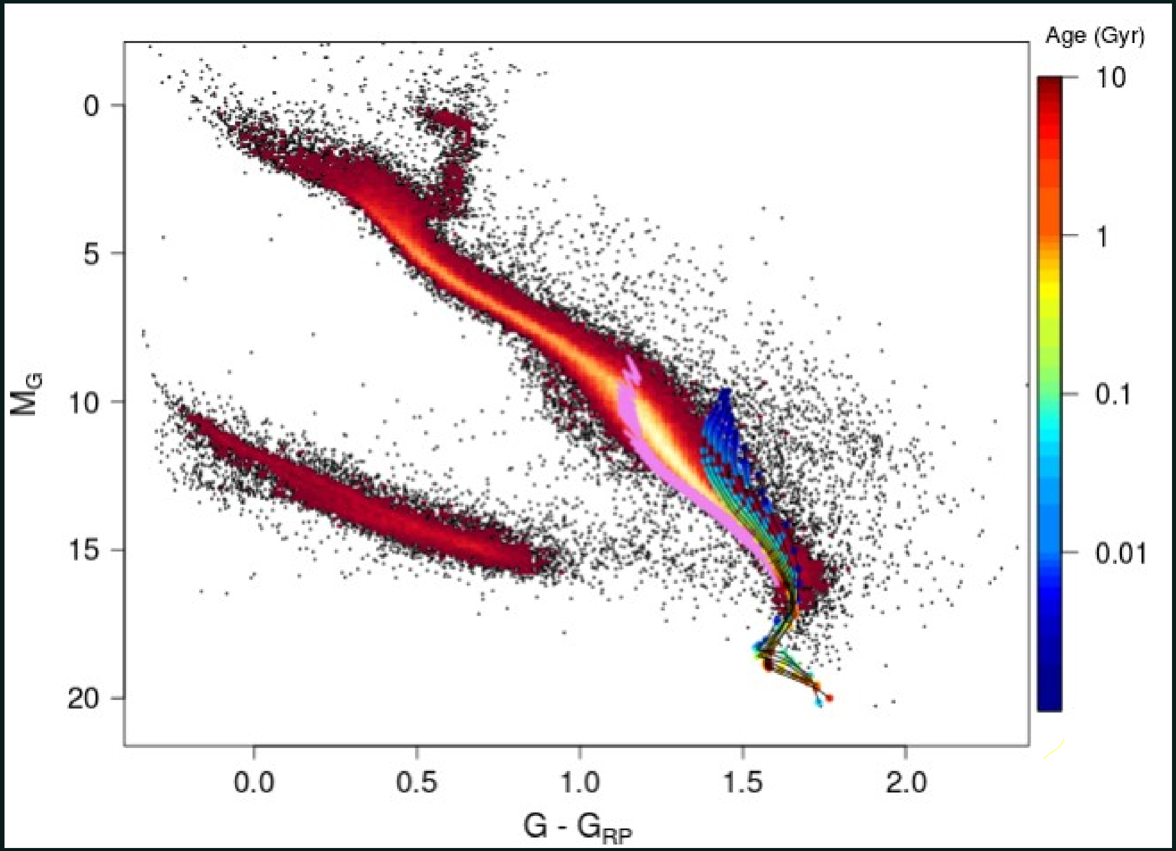


→ GAIA'S HERTZSPRUNG-RUSSELL DIAGRAM



Courtesy. GAIA

H-R Diagram

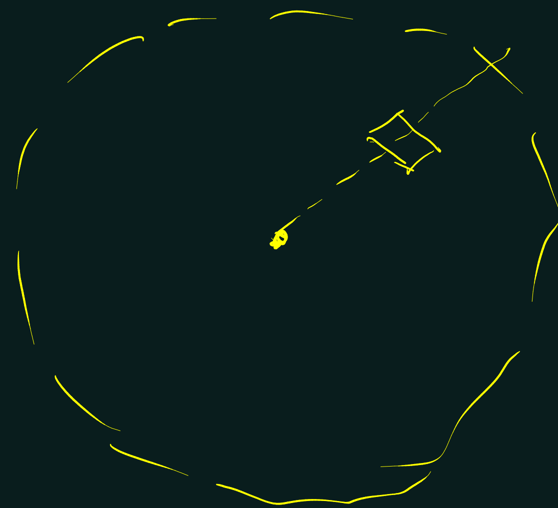


GAIA

Main Sequence stars

They are stable because

Pressure balances Gravity



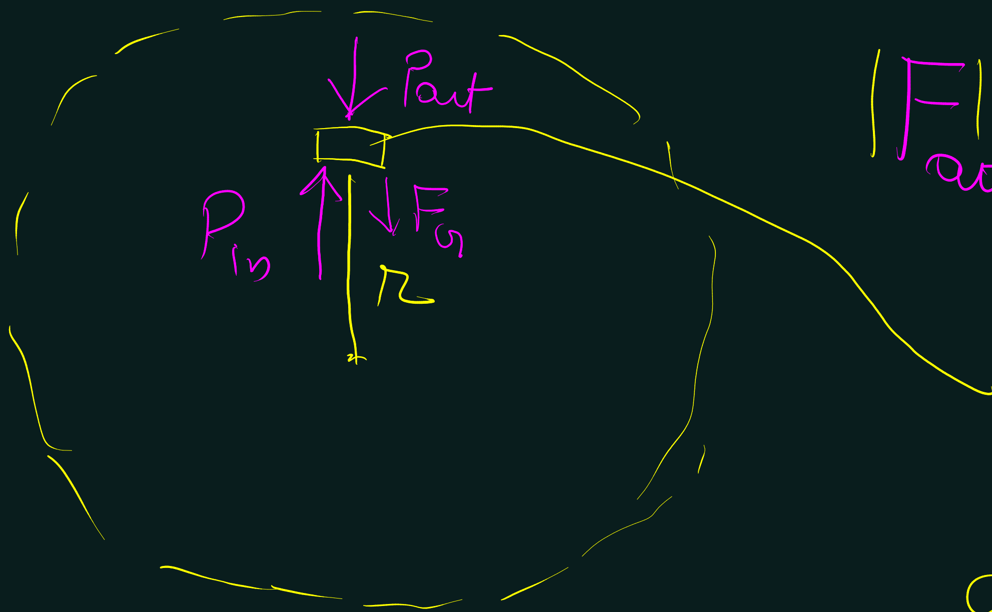
$$M = \frac{4}{3}\pi r^3 \rho$$

If the density is not constant

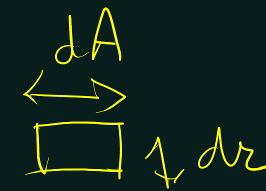
$$M = \int_0^r 4\pi r^2 \rho(r)\, dr$$

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho}$$

$$\rho \equiv \rho(r)$$



$$|F_{out}| + |F_G| = |F_{in}|$$



$$dm = \rho \, dA \, dr$$

$$F_{out} = P_{out} \, dA$$

$$F_{in} = P_{in} \, dA$$

$$F_G = -\frac{G M \, dm}{r^2}$$

$$P_{out}\, dA + \frac{GM\, dm}{r^2} = P_{in}\, dA$$

$$(P_{out} - P_{in})\, \cancel{dA} = -\frac{GM}{r^2}\, \rho\, \cancel{dA}\, dr$$

$$dP = -\frac{GM}{r^2}\, \rho\, dr$$

$$\boxed{\frac{dP}{dr} = -\frac{GM}{r^2}\, \rho}$$

$P = P(r)$

$M = M(r)$

$\rho = \rho(r)$

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho}$$

Three variables P, M, ρ

Only Two equations

we need a third equation to solve

(1) Ideal gas law : $P = \frac{\rho k_B T}{\mu m_p}$

$$P \propto \rho T$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \qquad (1)$$

$$\frac{dP}{dr} = - \frac{GM}{r^2}\rho \qquad (2)$$

M, P, ρ variables (three)

but solve for a hypothetical star which has $\rho = \text{constant}$.

Find out M as a function of r; and P as a function of r

Then Plot.

Equations of stellar structure

$$\frac{dM}{dr} = 4\pi r^2 \rho$$
$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$M \equiv M(r)$
$\rho \equiv \rho(r)$
$P \equiv P(r)$

$$P \propto \rho T \qquad P = \frac{\rho k_B T}{\mu m_p}$$

$T \equiv T(R)$

We can choose to solve for a general polytropic eq^n of state

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$M \equiv M(r)$

$P \equiv P(r)$

$\rho \equiv \rho(r)$

$$P = K\rho^{\gamma}$$

n	γ
1	2
3/2	5/3
2	3/2
3	4/3

$$\gamma = 1 + \frac{1}{n}$$

$$\rho = \rho_c \theta^n$$

$$P = K\rho^{1+\frac{1}{n}} = K \rho_c^{1+\frac{1}{n}} (\theta^n)^{\frac{n+1}{n}}$$

$$P = K \rho_c^{\frac{n+1}{n}} \theta^{n+1}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho} \frac{dP}{dr}\right) = -G\frac{dM}{dr} = -4\pi G r^2 \rho$$

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{K\rho_c^{\frac{n+1}{n}} (n+1)\theta^n \frac{d\theta}{dr}}{\rho_c \theta^n} = K\rho_c^{1/n}(n+1)\frac{d\theta}{dr}$$

$$\frac{d}{dr}\left[r^2 K \rho_c^{1/n} (n+1) \frac{d\theta}{dr} \right] = -4\pi G r^2 \rho_c \theta^n$$

$$\left[\frac{K(n+1)\rho_c^{\frac{1}{n}}}{4\pi G\rho_c}\right]\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -r^2\theta^n$$

$$\left[\frac{K(n+1)\rho_c^{1/n}}{4\pi G\rho_c}\right] = r_0^2$$

$$r_0^2\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -r^2\theta^n$$

$$r = r_0\,\xi$$

$$\frac{\cancel{r_0^2}}{\cancel{r_0}} \frac{d}{d\xi}\left(\cancel{r_0^2}\, \xi^2 \frac{1}{\cancel{r_0}} \frac{d\theta}{d\xi} \right) = -\cancel{r_0^2}\, \xi^2 \theta^n$$

$$\boxed{\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi} \right) = -\xi^2 \theta^n}$$

Lane-Emden Equation

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2 \theta^n$$

$\rho = \rho_c \theta^n$ and $r_0 = \sqrt{\frac{K(n+1)\rho_c^{1/n - 1}}{4\pi G}}$

$$\frac{dM}{dr} = 4\pi r^2 \rho \Rightarrow M = \int_0^{r_{max}} 4\pi r^2 \rho \, dr$$

$$M = 4\pi r_0^3 \rho_c \int_0^{\xi_{max}} \xi^2 \theta^n \, d\xi$$

$$M = r_0^3 \rho_c \left(4\pi I_{\text{constant}, n}\right)$$

$$r_0^3 \rho_c = \left(\frac{K(n+1)}{4\pi G}\right)^{3/2} \qquad \rho_c^{\frac{3}{2}\left(\frac{1}{n}-1\right)} \qquad \rho_c$$

$$= \left[\frac{K(n+1)}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-3n}{2n}+1} \qquad = \frac{3-3n+2n}{2n}$$

$$= \left[\frac{K(n+1)}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}}$$

$$\rho = \rho_c \theta^n$$

$n = \frac{3}{2}$; $\gamma = \frac{5}{3}$

$$P = K \rho^{5/3}$$

$$p = \frac{\rho k_B T}{\mu m_p}$$

$$P \propto \rho T$$

$$\rho \propto \frac{p}{T} \Rightarrow P = K \left(\frac{p}{T}\right)^{5/3}$$

$$P^{2/3} \propto T^{5/3}$$

$$P \propto T^{5/2}$$

$$p \propto U \propto \int B_\nu \, d\nu$$

$$\propto \sigma T^4$$

$$P \not\propto T^4$$

$$p \propto \rho^{4/3}$$

Matter (mono-atomic gas (hydrogen)) : $P_{gas} \propto \rho^{5/3}$

Radiation / Photons : $P_{rad} \propto \rho^{4/3}$

Total pressure $= P = P_{gas} + P_{rad}$

$= K_1 \rho^{5/3} \quad K_2 \rho^{4/3}$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$$P = K\rho^{\gamma}$$

$$P = K_1 \rho^{5/3} + K_2 \rho^{4/3}$$

Eddington's Solar Model

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

And some Eq^n of state

General: $P = K\rho^\gamma$

$\rho = \rho_c \Theta^n$

$r = r_0 \xi$

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\Theta}{d\xi}\right) = -\xi^2 \Theta^n$$

$$r_0 = \sqrt{\frac{K(n+1)\rho^{\frac{1}{n}-1}}{4\pi G}}$$

What for a system that has both radiation & the gas pressure

$$P_{tot} = P_{gas} + P_{rad}$$

$$= \frac{\rho k_B T}{\mu m_p} + \frac{4\sigma}{3c} T^4$$

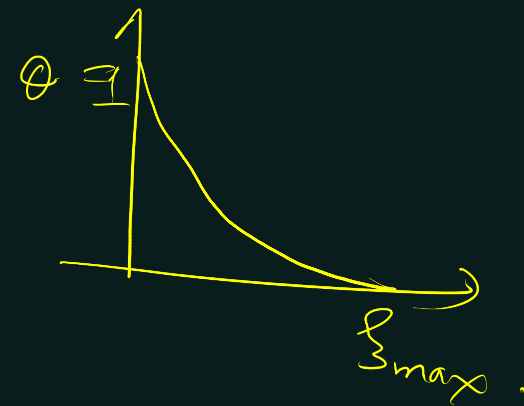
$$M = \int_0^{r_{max}} 4\pi r^2 \rho \, dr$$

$$r = r_0 \xi$$

$$\rho = \rho_c \theta^n$$

$$M = 4\pi r_0^3 \rho_c \int_0^{\xi_{max}} \xi^2 \Theta^n \, d\xi$$

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2 \theta^n$$



$$M = 4\pi r_0^3 \rho_c \quad I(n)$$

$$r_0 = \sqrt{\frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G}}$$

$$r_0^3 \rho_c = \left[\frac{K(n+1)}{4\pi G}\right]^{3/2} \rho_c^{\frac{3}{2}\left(\frac{1}{n}-1\right)} \quad \rho_c$$

For $n=3$; $\quad \gamma = 1 + \frac{1}{n} = \frac{4}{3}$

$$r_0^3 \rho_c = \left(\frac{4K}{4\pi G}\right)^{3/2} \rho_c^{-1} \rho_c^{1} = \left(\frac{K}{\pi G}\right)^{3/2}$$

$$M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} I(3)$$

$I(3) \approx 2$

$$\boxed{M = 8\pi \left(\frac{K}{\pi G}\right)^{3/2}}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$$P = K\rho^{\gamma}$$

$n = 3$

$M =$ constant.

Instead we have $P_{tot} = P_{gas} + P_{rad}$

$$P_{tot} = P_{gas} + P_{rad} \quad (1)$$

$$P_{tot} = \frac{\rho k_B T}{\mu m_p} + \frac{4\sigma}{3c} T^4$$

$$\frac{P_{rad}}{P_{gas}} = \frac{\beta P_{tot}}{(1-\beta) P_{tot}} \Rightarrow \frac{P_{rad}}{P_{gas}} = \frac{\beta}{1-\beta}$$

$$P_{rad} = \beta P_{tot}$$

$$P_{gas} = (1-\beta) P_{tot}$$

$$\frac{P_{rad}}{P_{gas}} = \frac{\frac{4\sigma}{3c} T^4}{\rho k_B T} \frac{\mu m_p}{} = \frac{4\sigma \mu m_p}{3 c k_B} \frac{T^3}{\rho}$$

$$\frac{4\sigma \mu m_p}{3c\, k_B} \frac{T^3}{\rho} = \frac{\beta}{1-\beta}$$

$$\Rightarrow \quad T^3 = \left(\frac{\beta}{1-\beta}\right)\left(\frac{3c\,k_B}{4\sigma \mu m_p}\right)\rho$$

$$P_{tot} = \frac{\rho\, k_B T}{\mu m_p} + \frac{4\sigma}{3c} T^4$$

$$P_{tot} = \frac{\rho\, k_B}{\mu m_p}\left(\frac{\beta}{1-\beta}\right)^{1/3}\left(\frac{3c k_B}{4\sigma \mu m_p}\right)^{1/3}\rho^{1/3} + \frac{4\sigma}{3c} T^4$$

$$P_{rad} = \beta \frac{P_{tot}}{\cancel{\beta}} \Rightarrow P_{tot} = (1/\beta) P_{rad} = \cancel{\beta}\,\frac{1}{\beta}\frac{4\sigma}{3c} T^4$$

$$P_{tot} = (1/\beta)\frac{4\sigma}{3c}\left(\frac{\beta}{1-\beta}\right)^{4/3}\left(\frac{3ck_B}{4\sigma\mu m_p}\right)^{4/3}\rho^{4/3}$$

$$P_{tot} = K\rho^{4/3}$$

$$K = \beta^{-1}\left(\frac{\beta}{1-\beta}\right)^{4/3}\frac{4\sigma}{3c}\left(\frac{3ck_B}{4\sigma\mu m_p}\right)^{4/3}$$

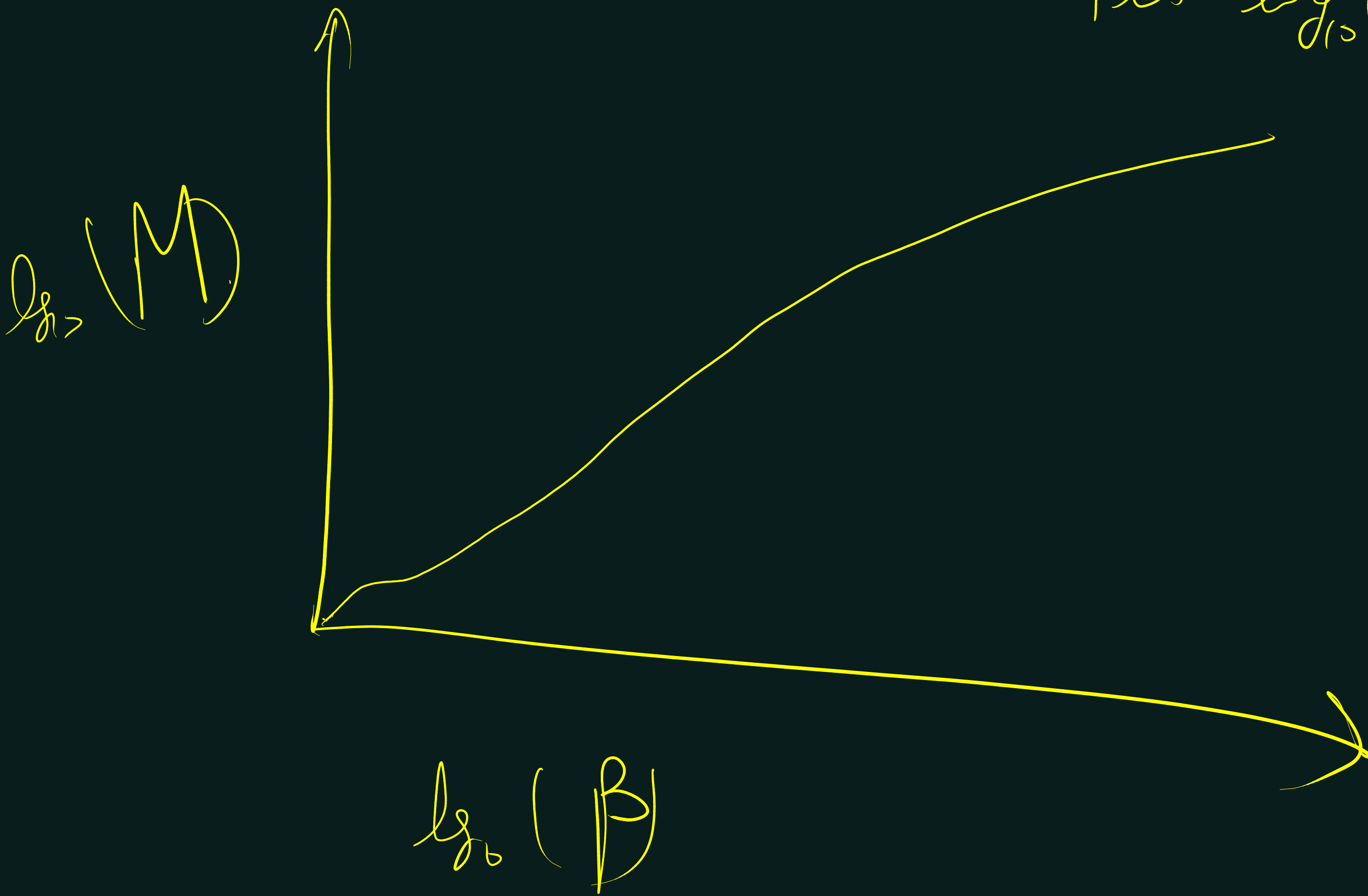
$$n = 3; \quad \gamma = \frac{4}{3}.$$

$$M = 8\pi \left(\frac{K}{\pi G}\right)^{3/2} = \frac{8\pi}{(\pi G)^{3/2}} K^{3/2}$$

$$= \frac{8\pi}{(\pi G)^{3/2}} \left[\beta^{-1} \left(\frac{\beta}{1-\beta}\right)^{4/3} \frac{4\sigma}{3c} \left(\frac{3ck_B}{4\sigma\mu m_p}\right)^{4/3} \right.$$

$$= \frac{8\pi}{(\pi G)^{3/2}} \left[\frac{1}{\beta} \left(\frac{\beta}{1-\beta}\right)^{4/3} \frac{4\sigma}{3c} \left(\frac{3ck_B}{4\sigma\mu m_p}\right)^{4/3} \right]^{3/2}$$

Plot $\log_{10}(M)$ with $\log_{10}(\beta)$



$$M = \frac{8\pi}{(\pi G)^{3/2}} \frac{(4\sigma)^{3/2}}{(3c)^{3/2}} \left(\frac{3c \quad k_B}{4\sigma \, \mu m_p}\right)^2 \frac{\sqrt{\beta}}{(1-\beta)^2}$$

$$M = 8\pi\sqrt{\frac{1}{(\pi G)^3}\left(\frac{4\sigma}{3c}\right)^3\left(\frac{3c}{4\sigma}\right)^4\frac{k_B^4}{m_p^4}}\;\frac{1}{\mu^2}\;\frac{\sqrt{\beta}}{(1-\beta)^2}$$

$$= \left[8\pi\sqrt{\frac{3c\,k_B^4}{4\sigma\, m_p^4\,\pi^3 G^3}}\right]\frac{1}{\mu^2}\;\frac{\sqrt{\beta}}{(1-\beta)^2}$$

Check the above maths & then you should get

$$\boxed{M \approx \underbrace{\left[20\, M_\odot\right]}_{\text{About } 17.8\, M_\odot \text{ to be precise}}\;\frac{1}{\mu^2}\;\frac{\sqrt{\beta}}{(1-\beta)^2}}$$

$$\log_{10}\left[\frac{M}{20\,M_\odot}\right] = -2\log_{10}\mu + \frac{1}{2}\log_{10}\beta - 2\log_{10}(1-\beta)$$

Eq. (1)

For $\beta \ll 1$; $\log_{10}(1-\beta) \approx 0$

$$\Rightarrow \log_{10}\left(\frac{M}{20\,M_\odot}\right) = -2\log_{10}\mu + \frac{1}{2}\log_{10}\beta$$

But; do not make the approximation

Instead define an array of $\log_{10}\beta$ from -5 to 1 and plot the actual.

Go to the following webpage

https://sites.google.com/site/mahavir44/teaching

then click on the link 'Attendance'