Lecture - 3

① Orbits — ② Spectroscopy

When solving for the orbit of a body of mass m around another massive body of mass M; we derived the following general equation for the orbit

$$\frac{1}{r} = \frac{GMm^2}{\mathcal{L}^2}\left[1 + \epsilon \cos\theta\right]$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{\mathcal{L}^2}{m r^2} - \frac{GMm}{r}$$

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{V_{eff}}$$

$$V_{eff} = \frac{\mathcal{L}^2}{2mr^2} - \frac{GMm}{r}$$

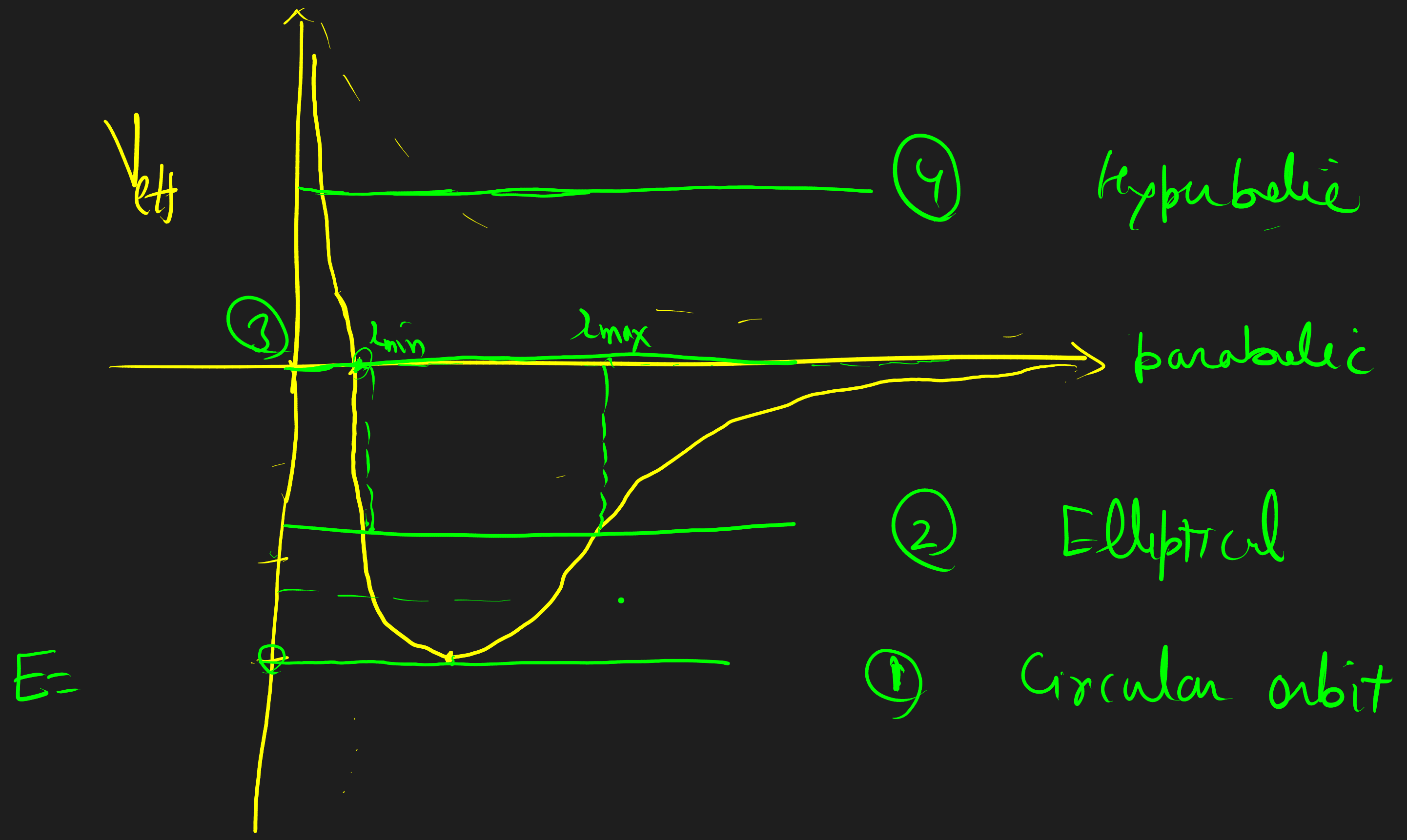
$$V_{eff} = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

at what r there will be a minima of this V_{eff}

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow -\frac{2L^2}{2mr^3} + \frac{GMm}{r^2} = 0$$

$$\frac{L^2}{2m\,GMm} = r \Rightarrow r = \frac{2L^2}{2GMm^2} = \frac{L^2}{GMm^2}$$

$$V_{eff} = 0 \Rightarrow \frac{L^2}{2mr^2} - \frac{GMm}{r} = 0 \Rightarrow r = \frac{L^2}{2GMm^2}$$



$$\epsilon = \sqrt{1 + \frac{2E\mathcal{L}^2}{G^2M^2m^3}}$$

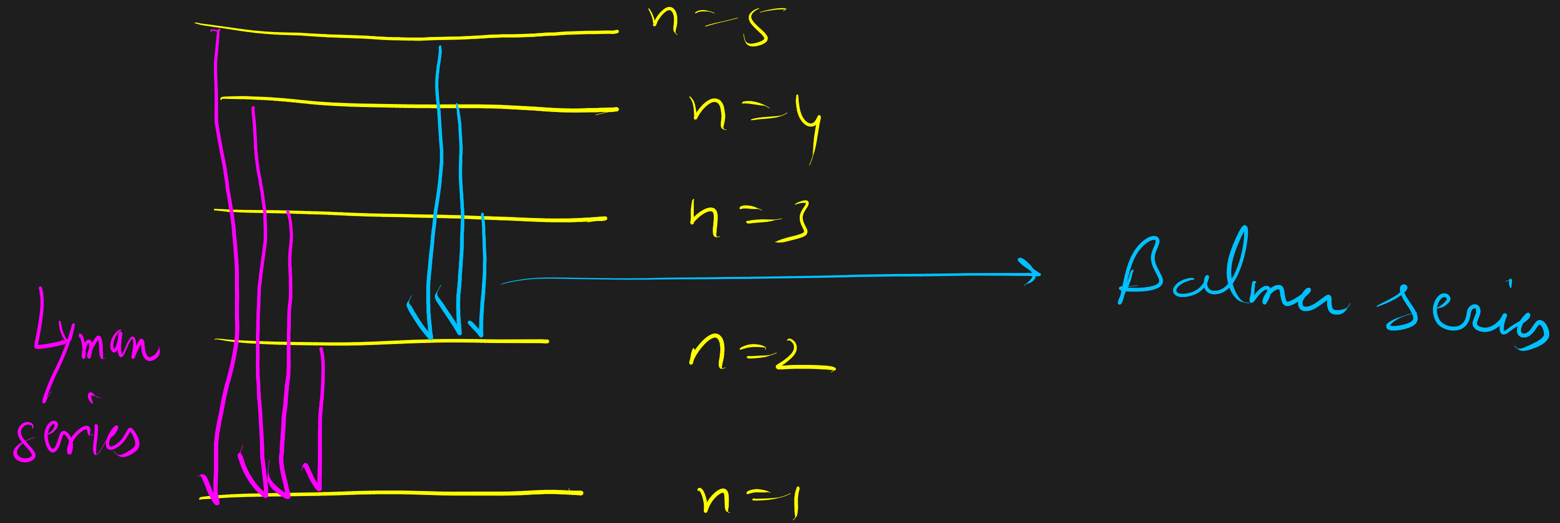
Minima of V_{eff} occurs at $r = \frac{\mathcal{L}^2}{GMm^2}$

The minimum value $= \frac{\mathcal{L}^2}{2m}\left(\frac{GMm^2}{\mathcal{L}^2}\right)^3 - \frac{GMm}{\mathcal{L}^2} \quad GMm^2$

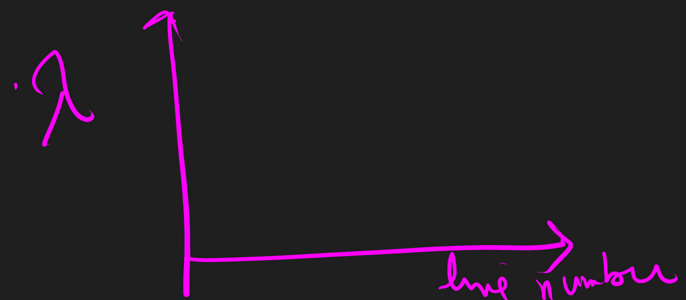
Spectroscopy

Another important branch on which Astronomy heavily relies

In Hydrogen atom



Experimentalists when they measure Lyman series of lines

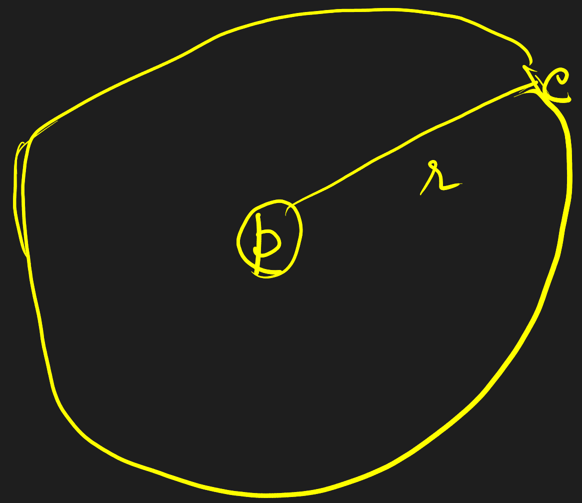


$$\frac{1}{\lambda} \propto \left(1-\frac{1}{n^2}\right)$$

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{n^2}\right)$$

R was experimentally found to be 109 732 700 cm^{-1}

Bohr claimed that electrons move around protons in planetary like orbits ; but with an important addition



F is coulomb force

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r^2}$$

But de Broglie told that there is a wave description of every elementary particle

orbit total length $= 2\pi r = \lambda = \frac{h}{mv}$

$$\Rightarrow 2\pi r = \frac{h}{mv}$$

$$mvr = \frac{h}{2\pi}$$

In nth orbit $\quad mvr = \frac{nh}{2\pi} = n\hbar$

Angular momentum, $L = n\hbar$

In the earth Sun system orbits are given by $\frac{1}{r} = \frac{GMm^2}{L^2}(1+\epsilon\cos\theta)$

the force law was $F = \frac{GMm}{r^2}$

In electron proton sytem the force law is $F = \frac{e^2}{4\pi\epsilon_0}\frac{1}{r^2}$

so the orbits

$$\frac{1}{r} = \frac{e^2 m}{4\pi\epsilon_0 n^2\hbar^2}(1+\epsilon\cos\theta)$$

For simplest $\epsilon = 0$; circular orbits

the radius $\boxed{r = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2}} = n^2\left(\frac{\hbar^2 4\pi\epsilon_0}{me^2}\right)$

the innermost orbit ; $(n=1)$ $\quad r_{n=1} = \dfrac{4\pi\epsilon_0 \hbar^2}{me^2} = a$

$a \equiv$ this is Bohr's radius. $\quad \simeq 0.58\ \mathring{A}$ } verify

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

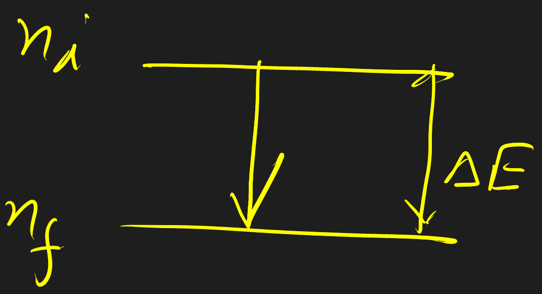
But for a circular orbit r is constant so $\dot{r} = 0$

$$E = \frac{L^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r} = \frac{n^2\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

$$E = \frac{n^2\hbar^2(me^2)^2}{2m\left(4\pi\epsilon_0 n^2\hbar^2\right)^2} - \frac{e^2}{4\pi\epsilon_0}\,\frac{me^2}{4\pi\epsilon_0 n^2\hbar^2}$$

$$= \frac{me^4}{2(4\pi\epsilon_0)^2 n^2\hbar^2} - \frac{me^4}{(4\pi\epsilon_0)^2 n^2\hbar^2}$$

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2 n^2\hbar^2}$$



$$\Delta E = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) =$$

$$\Delta E = \frac{hc}{\lambda}$$

$$\frac{1}{\lambda} = \frac{\Delta E}{hc} = \frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2 hc}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

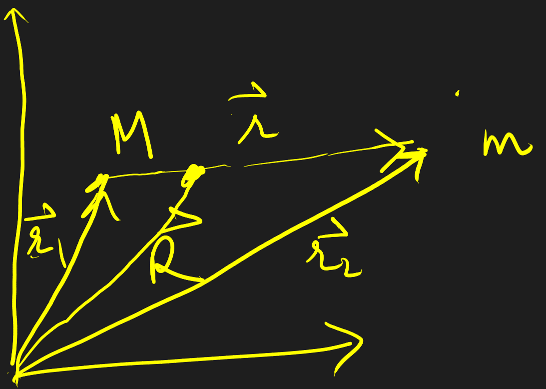
$$R = \frac{me^4}{8\epsilon_0^2 h^3 c} \approx 109.xxx\text{-}xxx.6 \ \text{cm}^{-1}$$

R comes out to be the same Rydberg constant as was determined experimentally.

Both the problems that we did are 2-body problem but if you noticed that we actually ignored the motion of the second body. ($M >>> m$)

So, actually , we solved only 1-body problem

2-Body Problem



$$K.E. = \frac{1}{2} M \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2}$$

$$\vec{R} = \frac{M\vec{r}_1 + m\vec{r}_2}{M+m}$$

$$K.E = \frac{1}{2} M \dot{r}_1^2 + \frac{1}{2} m \dot{r}_2^2$$

$$\vec{R} = \frac{M\vec{r}_1 + m\vec{r}_2}{M+m}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$(M+m)\vec{R} = M\vec{r}_1 + m\vec{r}_2$$

$$m\vec{r} = -m\vec{r}_1 + m\vec{r}_2$$

$$(M+m)\vec{R} - m\vec{r} = (M+m)\vec{r}_1 \Rightarrow \vec{r}_1 = \vec{R} - \frac{m}{M+m}\vec{r}$$

$$\vec{r}_2 = \vec{r} + \vec{r}_1 = \vec{r} + \vec{R} - \frac{m}{M+m}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{M}{M+m}\vec{r}$$

$$\vec{r}_1 = \vec{R} - \frac{m}{M+m}\vec{r}$$

$$K.E = \frac{1}{2} m \dot{\vec{r}}_2^{\,2} + \frac{1}{2} M \dot{\vec{r}}_1^{\,2} = \frac{1}{2} m \left(\dot{\vec{R}}^2 + \frac{M^2}{(M+m)^2} \dot{\vec{r}}^2 + 2 \frac{M}{m+M} \dot{\vec{R}} \dot{\vec{r}} \right)$$

$$+ \frac{1}{2} M \left(\dot{\vec{R}} + \frac{m^2}{(M+m)^2} \dot{\vec{r}}^2 - 2 \frac{m}{m+M} \dot{\vec{R}} \dot{\vec{r}} \right)$$

$$= \frac{1}{2} m \dot{\vec{R}}^2 + \ldots$$

$$K.E. = \frac{m\dot{\vec{R}}^2}{2} + \frac{mM^2}{2(m+M)^2}\dot{\vec{r}}^2 + \cancel{\frac{Mm}{m+M}\dot{\vec{R}}\,\dot{\vec{r}}}$$

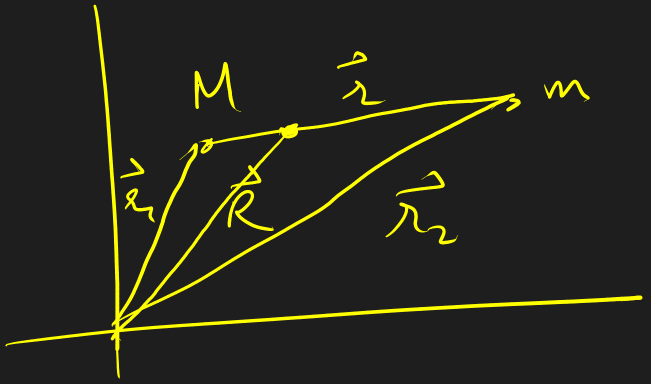
$$+ \frac{M\dot{\vec{R}}^2}{2} + \frac{Mm^2}{2(m+M)^2}\dot{\vec{r}}^2 - \cancel{\frac{Mm}{m+M}\dot{\vec{R}}\,\dot{\vec{r}}}$$

$$= \frac{(m+M)\dot{\vec{R}}}{2} + \frac{mM\cancel{(m+M)}}{(m+M)^{\cancel{2}}}\dot{\vec{r}}^2$$

$$= \frac{(m+M)}{2}\dot{\vec{R}} + \frac{1}{2}\left(\frac{mM}{m+M}\right)\dot{\vec{r}}^2 = \frac{(m+M)}{2}\dot{\vec{R}} + \frac{\mu\dot{\vec{r}}}{2}$$

$$K.E = \frac{1}{2} M \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2} = \cancel{\frac{1}{2}(m+M)\dot{\vec{R}}^{2}} + \frac{1}{2}\mu \dot{\vec{r}}^{\,2}$$

$$\mu = \frac{mM}{m+M} \approx m \text{ for } M >> m$$



$$K.E = \frac{1}{2}\mu \dot{\vec{r}}^{\,2}$$

✓

Only 1-body whose radial vector is r and mass is μ

So $K.E. = \frac{1}{2} M \dot{\vec{r}}_1^2 + \frac{1}{2} m \dot{\vec{r}}_2 = \frac{1}{2} \mu \dot{\vec{r}}^2$

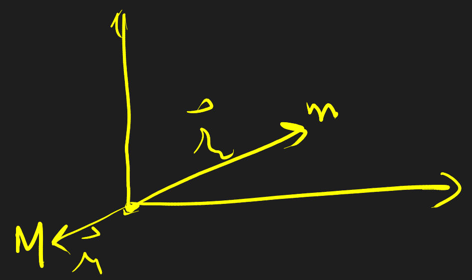
$\underbrace{\frac{1}{2} M \dot{\vec{r}}_1^2 + \frac{1}{2} m \dot{\vec{r}}_2}_{\text{2-Body Problem}} \qquad \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^2}_{\text{1-Body problem}}$

3-Body problem can be reduced to 2-Body problem

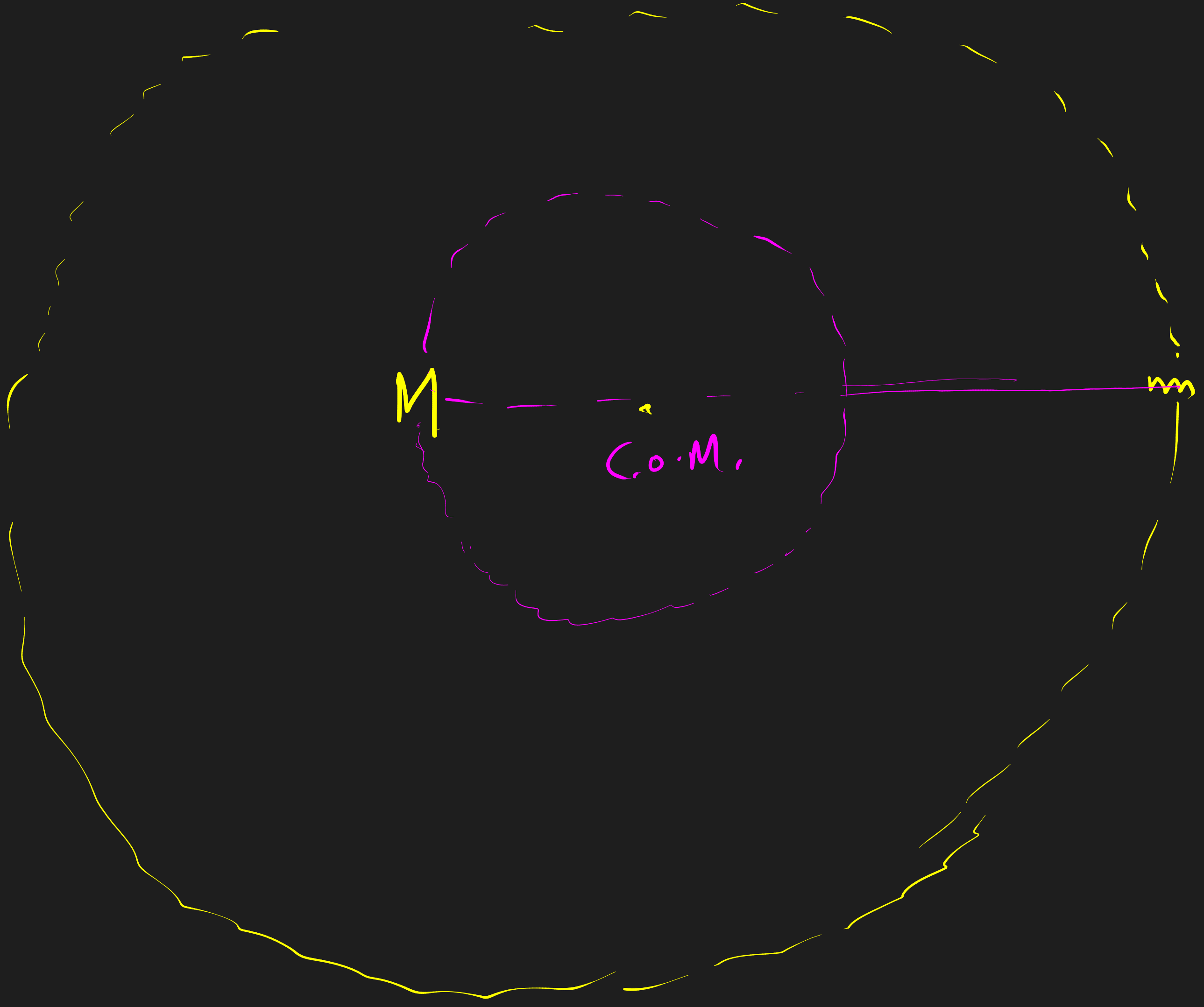
~~So the re~~

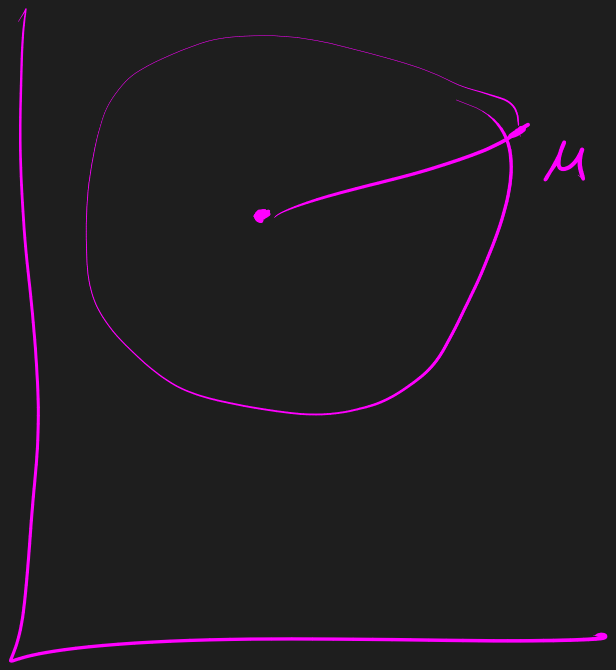
$$E_n = -\left[\frac{\mu e^4}{8 \epsilon_0^2 h^3 c} hc\right] \frac{1}{n^2}$$

Rydberg constant depends on -



$$\frac{1}{2} M \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2} \approx \frac{1}{2} \mu \dot{\vec{r}}^{\,2}$$





$$KE = \frac{1}{2} m \dot{r}_1 +$$

Problems :-

(1) Calculate the position of the centre of mass in the (I) Sun-Earth system (II) in the Earth Moon system.

(2) In the Bohr model of hydrogen atom, calculate the velocity of electron in the innermost orbit and compare it with the speed of light.

(3) Calculate the eccentricity of Bohr orbits considering them to be elliptical. (for $n = 1, 2, 3$ and 4)

(4) Calculate the wavelength of photon emitted when electron jumps from $n=2$ to $n=1$ in a hydrogen atom; then repeat the same calculation for a Deuterium atom.

What is the difference in wavelength that you calculated in hydrogen and deuterium.