Lecture-2

(1) All the online lecture-pdf files will be available at my webpage under the teaching.

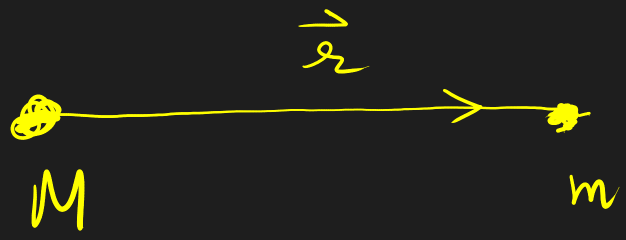
(2) At the end of every lecture pdf there will be problems - home work problems - you have to solve them. For each week upload the solutions as a single file at the link on my webpage.

Each week the link will be active until Sunday 9PM
Deadline

Motion of bodies in central forces.

(1) Motion of a body in the gravitational force

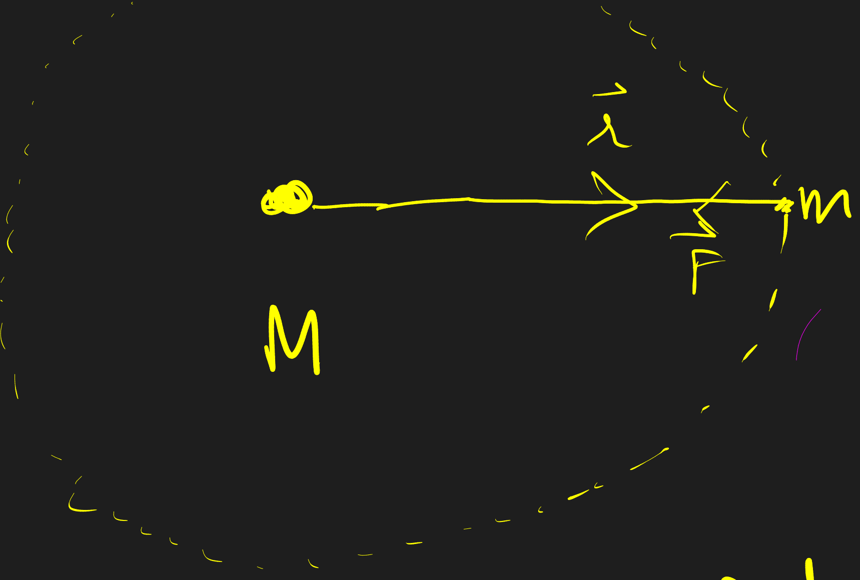
$M >> m$



$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$\vec{F} = -\frac{GMm}{r^3}\vec{r}$$

$$|\vec{F}| = F = \frac{GMm}{r^2}$$



Kinetic energy of the body of mass m

$$K.E. = \frac{1}{2} m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2} m\left(\frac{dy}{dt}\right)^2$$

only two coordinates ; because the motion is in always 2-D. [We can formally prove it]

Angular momentum. $\vec{L} = m(\vec{r} \times \vec{v})$

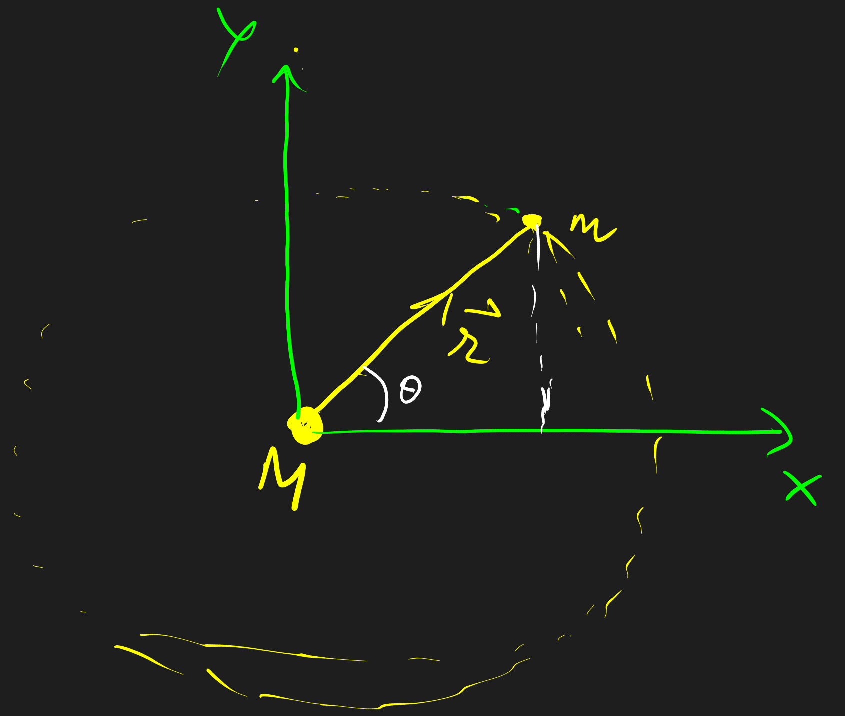
$$\frac{d}{dt}(\vec{L}) = m\left(\vec{r} \times \frac{d\vec{v}}{dt}\right) + m\left(\frac{d\vec{r}}{dt} \times \vec{v}\right)$$

$$= \vec{r} \times \cancel{(m\vec{a})} + m\,\cancel{(\vec{v} \times \vec{v})}$$

$m\vec{a} = \vec{F}$

$$\frac{d\vec{L}}{dt} = 0 \qquad \vec{L} \text{ is conserved. (constant)}$$

This proves that the motion is 2D - confined to a plane



$$K.E. = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\dot{x} = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \frac{d}{dt}(r\sin\theta) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\boxed{\begin{aligned} K.E. &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\ P.E. &= -\frac{GMm}{r} \\ \mathcal{L} &= mrv = mr^2\omega = mr^2\frac{d\theta}{dt} = mr^2\dot{\theta} \end{aligned}}$$

$$E = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}$$

$$\mathcal{L} = m r^2 \dot{\theta}$$

$$\frac{2}{m}(E) = \frac{2}{m}\left(\frac{1}{2} m \dot{r}^2\right) + \frac{2}{m}\left(\frac{mr^2}{2}\left(\frac{\mathcal{L}}{mr^2}\right)^2\right) - \frac{GMm}{r}\,\frac{2}{m}$$

$$\dot{r}^2 = \frac{2E}{m} - \frac{\cancel{2}\,\mathcal{L}^2}{\cancel{2}\, m r^2\, m} + \frac{GM\cancel{m}\,2}{r\,\cancel{m}}$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2GM}{r}}$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2}$$

$$\frac{d\theta}{dr} = \frac{\frac{L}{mr^2}}{\sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} + \frac{2GM}{r}}} = \frac{d\theta}{dr}$$

$$\int \frac{\frac{\ell}{mr^2}\,dr}{\sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} + \frac{2GM}{r}}} = \int d\theta$$

$$y = \frac{1}{r} \Rightarrow dy = -\frac{1}{r^2}\,dr$$

$$\int \frac{(-dy)}{\sqrt{\frac{2E\cancel{m^2}}{\cancel{m}\ell^2} - \frac{\cancel{\ell^2}\cancel{m^2}}{\cancel{m^2}\cancel{\ell^2}}y^2 + \frac{2GM\,m^2}{\ell^2}y}} = \int d\theta$$

$$-\int \frac{dy}{\sqrt{\frac{2Em}{L^2} - y^2 + \frac{2GMm^2}{L^2}\,y}} = \int d\theta$$

$$-\int \frac{dy}{\sqrt{\underbrace{\frac{2Em}{L^2} + \frac{G^2M^2m^4}{L^4}}_{K^2} - \underbrace{\left(y - \frac{GMm^2}{L^2}\right)^2}_{K^2\cos^2\alpha}}} = \int d\theta$$

$$y - \frac{GMm^2}{L^2} = K\cos\alpha \quad \text{and} \quad K = \sqrt{\frac{2Em}{L^2} + \frac{G^2M^2m^4}{L^4}}$$

$$-\int \frac{-K\sin\alpha \, d\alpha}{\sqrt{K^2 - K^2\cos^2\alpha}} = \int d\theta$$

$$\int \frac{\sin\alpha \, d\alpha}{\sin\alpha} = \int_0^\theta d\theta$$

$$\alpha = \theta$$

$$\cos^{-1}\left[\frac{1}{K}\left(y-\frac{GMm}{L^2}\right)\right]=\theta$$

$$\frac{1}{K}\left(y-\frac{GMm}{L^2}\right)=\cos\theta$$

$$y=\frac{GMm}{L^2}+K\cos\theta$$

$$y = \frac{GMm^2}{\mathcal{L}^2} + \sqrt{\frac{2Em}{\mathcal{L}^2} + \frac{G^2M^2m^4}{\mathcal{L}^4}} \cos\theta$$

$$\frac{1}{r} = \frac{GMm^2}{\mathcal{L}^2}\left[1 + \sqrt{\frac{2Em}{\mathcal{L}^2}\frac{\mathcal{L}^4}{G^2M^2m^4} + 1} \cos\theta\right]$$

$$= \frac{GMm^2}{\mathcal{L}^2}\left[1 + \sqrt{1 + \frac{2E\mathcal{L}^2}{G^2M^2m^3}} \cos\theta\right]$$

$$\frac{1}{r} = \frac{GMm^2}{\mathcal{L}^2}\left[1 + \epsilon \cos\theta\right]$$

$$\boxed{r = \frac{\mathcal{L}^2}{GMm^2} \frac{1}{1+\epsilon\cos\theta}}$$

$$\epsilon = \sqrt{1 + \frac{2E\mathcal{L}^2}{G^2M^2m^3}}$$

$$\frac{1}{r} = \frac{GMm^2}{\mathcal{L}^2}\left[1+\epsilon\cos\theta\right]$$

$G, M, m, \mathcal{L}, \epsilon, E$ are constants. what type of orbit is this.

$$\frac{1}{r} = \frac{GMm^2}{L^2}\left[1+\epsilon\cos\theta\right]$$

when is r minimum ; at $\theta = 0$ because then $[1+\epsilon\cos\theta]$ is max.

$$\frac{1}{r_{min}} = \frac{GMm^2}{L^2}\left[1+\epsilon\right]$$

similarly $$\frac{1}{r_{max}} = \frac{GMm^2}{L^2}\left[1-\epsilon\right]$$

① If $\epsilon = 0$ then $r_{max} = r_{min}$

then the equation represents a circle

$$\frac{1}{r} = \frac{GMm^2}{L^2} \Rightarrow \boxed{r = \frac{L^2}{GMm^2}}$$

circle

② If $0 < \epsilon < 1$

The solution represents an Ellipse

③ If $\epsilon = 1$ then $\frac{1}{r_{min}} = \frac{GMm^2}{L^2}(1+1)$ but $\frac{1}{r_{max}} = 0$

If $\epsilon = 1$

$r_{max} \rightarrow \infty$ } Parabola

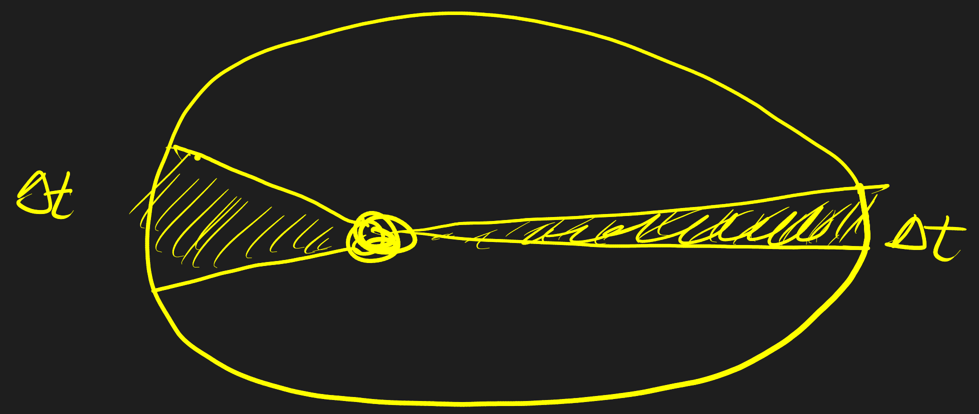
(4) $\epsilon > 1 \rightarrow$ Hyperbola

So the orbits ~~is~~ in the central force are conic sections (elliptical orbits)

KEPLER's 1st Law

KEPLER's 2nd LAW

in a constant time period the area swept is constant



$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{GMm}{r}$$

$$E = \frac{1}{2} m \dot{r}^2 + \left(\frac{m r^2 \ell^2}{2\, m^2 r^4} - \frac{GMm}{r} \right)$$

$$E = \frac{1}{2} m \dot{r}^2 + V_{eff}$$

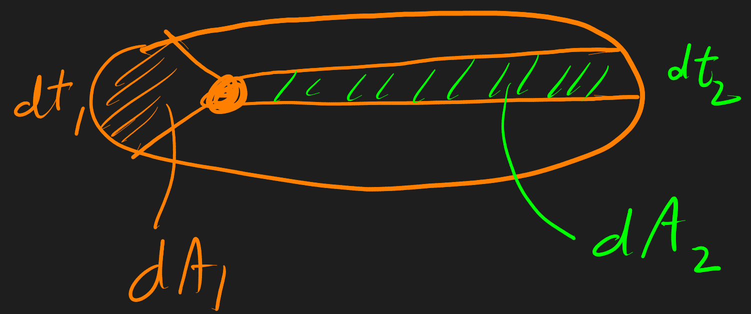
$$\boxed{V_{eff} = \frac{\ell^2}{2 m r^2} - \frac{GMm}{r}}$$

Problems:

(1) Prove the Kepler's 2nd law that the orbiting planet sweeps out a constant area in constant time interval no matter which part of the orbit it is covering

If $dt_1 = dt_2$

then prove $dA_1 = dA_2$



(2) Beginning from the orbital equation $\frac{1}{r} = \frac{GMm^2}{L^2}(1+\epsilon\cos\theta)$

{Kepler's 3rd law} Prove that cube of semi major axis in elliptical orbit is proportional to the square of time period: $a^3 \propto T^2$

(3) Derive the equations of the orbit for following two cases

(I) $V = \alpha r^2$

(II) $V = \alpha / r^2$

where α is a constant.

(4) Find out the eccentricity ϵ for the earth's orbit around sun. Comment whether the orbit is closer to being elliptical or circular.