Indian Institute of Technology Bhilai

IC202: Calculus II

2023-24-M Instructor: Dr. Raj Kumar Mistri

TUTORIAL SHEET-1

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1. Let $\mathbb{R}^n$ be the set of n-tuple of real numbers. Let $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We define the sum of x and y as

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n),$$

and the scalar multiplication of x with the scalar a as

$$ax = (ax_1, ax_2, \ldots, ax_n).$$

Using the properties of real numbers, prove that

   (a) $x + y = y + x$.
   (b) $a(x + y) = ax + ay$.
   (c) $a(bx) = (ab)x$.
   (d) $(a + b)x = ax + bx$.
   (e) $ax = 0$ for every $x \in \mathbb{R}^n$ if and only if $a = 0$.

2. Find the distance between the following two points in the given space.

   (a) $(\sqrt{5}, \sqrt{13})$ and $(\sqrt{13}, -\sqrt{5})$ in $\mathbb{R}^2$.
   (b) $(3, 5, 7)$ and $(5, 1, 2)$ in $\mathbb{R}^3$.
   (c) $(-3, -5, 7)$ and $(5, -1, 2)$ in $\mathbb{R}^3$.
   (d) $(1, 3, 2, 0)$ and $(3, 5, 1, 2)$ in $\mathbb{R}^4$.

3. Write the expressions for the following open balls in the given spaces.

   (a) Open ball with centre $(0, 0)$ and radius 3 in $\mathbb{R}^2$.
   (b) Open ball with centre $(0, 1, 3)$ and radius 5 in $\mathbb{R}^3$.
   (c) Open ball with centre $(0, 0)$ and radius δ in $\mathbb{R}^2$, where δ is a positive real number.
   (d) Open ball with centre $(0, 0, 0)$ and radius ϵ in $\mathbb{R}^3$, where ϵ is a positive real number.
   (e) Open ball with centre $(1, 1, 2)$ and radius $\epsilon/2$ in $\mathbb{R}^3$, where ϵ is a positive real number.

4. Write the expressions for the following:

   (a) δ-neighborhood of the point $(0, 0)$ in $\mathbb{R}^2$, where δ is a positive real number.
   (b) ϵ-neighborhood of the point $(0, 1, 3)$ in $\mathbb{R}^3$, where ϵ is a positive real number.
   (c) Deleted ϵ-neighborhood of the point $(0, 0)$ in $\mathbb{R}^2$, where δ is a positive real number.
   (d) Deleted ϵ-neighborhood of the point $(1, 1, 5)$ in $\mathbb{R}^3$, where ϵ is a positive real number.

5. Show that the open sphere with centre (a, b, c) in $\mathbb{R}^3$ and radius $r > 0$ is contained in the open cube

$$C_1 = \{(x, y, z) \in \mathbb{R}^3 : |x - a| < r, |y - b| < r, |z - c| < r\},$$

and contains the open cube

$$C_2 = \{(x, y, z) \in \mathbb{R}^3 : |x - a| < r/\sqrt{3}, |y - b| < r/\sqrt{3}, |z - c| < r/\sqrt{3}\}.$$

6. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are $2n$ real numbers. Prove the *Cauchy's Inequality*

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq \sqrt{(a_1^2 + a_2^2 + \cdots + a_n^2)}\,\sqrt{b_1^2 + b_2^2 + \cdots b_n^2}.$$

7. Let $x, y, z \in \mathbb{R}^n$. Prove the following:

   (a) $\|x - y\| = 0$ if and only if $x = y$.
   (b) *Triangle Inequality:* $\|x + y\| \leq \|x\| + \|y\|$.
   Hence prove that $\|x - y\| \leq \|x - z\| + \|y - z\|$.