5 Artificial Variable Technique : Big M Method

Before using Simplex algorithm all the constraints are required to be converted into equations using slack and surplus variables. Next task is to find an initial BFS which will make our job easy if it is a unit matrix. If the constraints are of the form

$$Ax \leq b, \; x \geq 0$$

then the initial basis matrix (identity matrix) can be found using the slack variable. But this is not always possible if the constraints are of the forms $Ax \geq b$ and $Ax = b$. In this case we introduce artificial variable technique. The method for this is known as Big M method which we discuss below.

5.1 Big M method:

1. After reducing the LPP in standard form if the co-efficient matrix A does not contain an initial basis matrix (identity matrix) then we have to introduce minimum number of artificial variables as per requirement to the left hand side of the constraints to get an identity matrix as a BFS.

2. It is interesting to note that an equation is being kept as equation even after introducing an artificial variable on one side (the left hand side) of the equation is not correct form the mathematical point of view. It only makes sense if all the artificial variable are equal to zero at the optimal stage.

3. For this reason we assign a very high negative price or cost $-M$ where $M > 0$, in the objective function corresponding to the artificial variables. This guaranties that the value of the objective function can not be improved until all the artificial variables are equal to zero. Hence at the optimal stage if any artificial variable is present in the basis matrix then we conclude that the LPP has no feasible solution.

4. If the optimality condition satisfied and all the artificial variable are not present in the basis matrix at the optimal stage then the solution is the optimal for the LPP.

5. It may happen that all the artificial variable are at zero level but at least one artificial variable present in the basis matrix at the optimal stage. In this case the solution is called the optimal solution for the LPP but some of the constraints may be redundant.

Remark 5.1 *The cost $-M$ is so high that the sign α determines the sign of $\alpha M + \beta$.*

Remark 5.2 *In any stage, once an artificial vector corresponding to the artificial variable leaves the basis, we forget all about it for ever and never consider it as a vector to enter into the basis matrix at any stage afterwards. Once an artificial vector corresponding to the artificial variable leaves the basis we make the column corresponding to this variable empty for subsequent steps.*

Example 5.3 *Use Big M method to solve the LPP*

$$\textit{Minimize, } z = 4x_1 + 3x_2$$

subject to

$$\begin{gathered} x_1 + 2x_2 \geq 8 \\ 3x_1 + 2x_2 \geq 12 \\ x_1, x_2 \geq 0. \end{gathered}$$

Solution: We convert the problem to a maximization problem. Let $z_1 = -z$. Then min $z = -$ max $(-z) = -$ max z_1. Hence the problem is reduced to

$$\text{Maximize, } z_1 = -4x_1 - 3x_2$$

subject to

$$\begin{gathered} x_1 + 2x_2 \geq 8 \\ 3x_1 + 2x_2 \geq 12 \\ x_1, x_2 \geq 0. \end{gathered}$$

Remark 5.4 *Since the components of the vector b are positive so we do not have to multiply the constraints with -1.*

Introducing two surplus variables x_3 and x_4 to each of the constraints we obtain the following

$$\begin{gathered} x_1 + 2x_2 - x_3 = 8 \\ 3x_1 + 2x_2 - x_4 = 12 \\ x_1, x_2, x_3, x_4 \geq 0. \end{gathered}$$

As we can see the coefficient matrix does not contain a unit basis matrix, we have to introduce two artificial variables x_5 and x_6. Then the above equations are reduced to

$$\begin{gathered} x_1 + 2x_2 - x_3 + x_5 = 8 \\ 3x_1 + 2x_2 - x_4 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{gathered}$$

Now we can take x_5 and x_6 can be taken as basis variables and the initial BFS is $[x_5, x_6] = [8, 12]$. The corresponding objective function is given by

$$z_1 = -4x_1 - 3x_2 + 0.x_3 + 0.x_4 - Mx_5 - Mx_6$$

Now we can proceed for the simplex tables

Basis Vector (B)	c_B	$c \rightarrow$ b	-4 x_1	-3 x_2	0 x_3	0 x_4	-M x_5	-M x_6	Min. Ratio
x_5 *	-M	8	1	2*	-1	0	1	0	8/2
x_6	-M	12	3	2	0	-1	0	1	12/2
$c_j - z_j$			4M−4	4M−3*	−M	−M	0	0	
x_2	-3	4	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	-	0	8/1
x_6 *	-M	4	2*	0	1	-1	-	1	4/2
$c_j - z_j$			2M−5/2*	0	M − 3/2	−M	-	0	
x_2	-3	3	0	1	-3/4	1/4	-	-	-
x_1	-4	2	1	0	1/2	-1/2	-	-	-
$c_j - z_j$		-17	0	0	−1/4	−5/4	-	-	-

All $c_j - z_j \leq 0$ for all j and no artificial variables are present tin the final basis. Therefore all the artificial variable are at zero level at final stage. The optimal solution obtained is

$$x_1 = 2,\ x_2 = 3,\ x_3 = x_4 = x_5 = x_6 = 0$$

and the value of the objective function at the optimal solution is $z_1 = -17$. Therefore min $z = -z_1 = 17$.

Remark 5.5 *Before introducing artificial variables just check that whether the coefficient matrix contains the unit basis matrix or whether there exists vectors which will serve to form a unit basis matrix. As an example let the constraints are of the form*

$$\begin{gathered} x_1 + 2x_2 + x_3 \geq 8 \\ 2x_2 + x_3 \geq 12 \\ x_1, x_2, x_3 \geq 0. \end{gathered}$$

For simplex method we have to make all the constraints into equations. Introducing two surplus variables x_4 and x_5 the above system is reduced to

$$\begin{gathered} x_1 + 2x_2 + x_3 - x_4 = 8 \\ 2x_2 + x_3 - x_5 = 12 \\ x_1, x_2, x_3, x_4, x_5 \geq 0. \end{gathered}$$

We can see that the coefficient matrix A of the above system does not contain a unit basis matrix. Therefore we have to introduce artificial variables for the same. But the coefficient of x_1 is

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This contributes to form a unit basis matrix. Therefore we only need to introduce only one artificial variable x_6 to get the vector

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to form the initial unit basis matrix. Hence the system is further reduced to

$$\begin{gathered} x_1 + 2x_2 + x_3 - x_4 = 8 \\ 2x_2 + x_3 - x_5 + x_6 = 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{gathered}$$

and a_1, a_6 form the initial basis matrix.