Indian Institute of Technology Bhilai

IC202: Calculus II

2023-24-M Instructor: Dr. Raj Kumar Mistri

TUTORIAL SHEET-3

1. Let a be a real number satisfying $0 \le a < \epsilon$ for every real number $\epsilon > 0$. Show that $a = 0$.

2. Let n be a fixed positive integer. Show that if a function $f : \mathbb{R}^n \to \mathbb{R}$ has a limit at a point $\mathbf{a} \in \mathbb{R}^n$, then it is unique.

3. Show that the limits of the following functions do not exist as $(x, y) \to (0, 0)$.

   (a) $\frac{xy}{x^2+y^2}$.

   (b) $\frac{x^2}{x^2+y}$.

   (c) $\frac{x^2-y^2+2xy}{x^2+y^2}$.

   (d) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$

4. In the following problems, how close to the origin should we take the point (x, y) or (x, y, z) to make

$$|f(x,y) - f(0,0)| < \epsilon,$$

   or

$$|f(x,y,z) - f(0,0,0)| < \epsilon$$

   for the given ϵ?

   (a) $f(x,y) = xy, \epsilon = 0.0004$.

   (b) $f(x,y,z) = x^2 + y^2 + z^2, \epsilon = 0.01$.

5. (a) For the function $f(x,y) = \frac{xy}{x^2+y^2}$, prove that the simultaneous limit does not exist at $(0, 0)$, while the two repeated limits exist and are equal.

   (b) For the function $f(x,y) = \frac{(y-x)(1+x^2)}{(y+x)(1+y^2)}$, show that

$$\lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = -1 \text{ and } \lim_{y\to 0}(\lim_{x\to 0} f(x,y)) = 1.$$

   Decide the existence of simultaneous limit of $f(x, y)$ as $(x, y) \to (0, 0)$.

   (c) Let $f(x,y) = \begin{cases} 1 + xy, & \text{if } xy \neq 0; \\ 0, & \text{if } xy = 0. \end{cases}$

   Then prove that

$$\lim_{x\to 0}(\lim_{y\to 0} f(x,y)) = 1 = \lim_{y\to 0}(\lim_{x\to 0} f(x,y))$$

   but $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

6. Show that the following functions are continuous at $(0, 0, 0)$.

   (a) $f(x,y,z) = x \sin y + y \sin z + z \sin x$ using $\epsilon - \delta$ definition of continuity.

   (b) $f(x,y,z) = e^x \cos y + e^y \cos z + e^z \cos x$ by directly evaluating the limit.

   (c) $f(x,y,z) = |x| + |y| + |z|$ using $\epsilon - \delta$ definition of continuity.

   (d) $f(x,y,z) = \ln(1 + x^2 + y^2 + z^2)$ by writing f as a composition of two functions.

7. Show that the following functions are not continuous at $(0, 0)$.

(a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^2}, & \text{if } (x,y) \neq (0,0); \\ 2, & \text{if } (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} y\sin\frac{1}{x} + x\sin\frac{1}{y}, & \text{if } x \neq 0, y \neq 0; \\ 1, & \text{otherwise.} \end{cases}$

8. Let $\pi_j : \mathbb{R}^n \to \mathbb{R} : \pi_j(x_1, x_2, \ldots, x_n) = x_j$ be the j-th projection, $1 \leq j \leq n$. Prove that π_j is a continuous function using $\epsilon - \delta$ definition of continuity.