Outline

- Introduction to Signal Flow Graphs
  - Definitions
  - Terminologies
  - Examples
- Mason's Gain Formula
  - Examples
- Signal Flow Graph from Block Diagrams
- Design Examples

Introduction

- Alternative method to block diagram representation, developed by Samuel Jefferson Mason.

- Advantage: the availability of a flow graph gain formula, also called Mason's gain formula.

- A signal-flow graph consists of a network in which nodes are connected by directed branches.

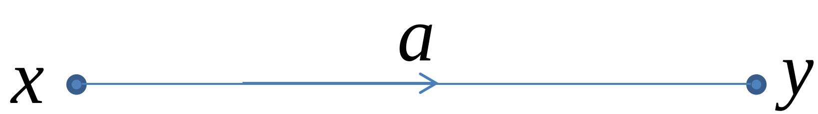
- It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

Fundamentals of Signal Flow Graphs

- Consider a simple equation below and draw its signal flow graph:

$$y = ax$$

- The signal flow graph of the equation is shown below;

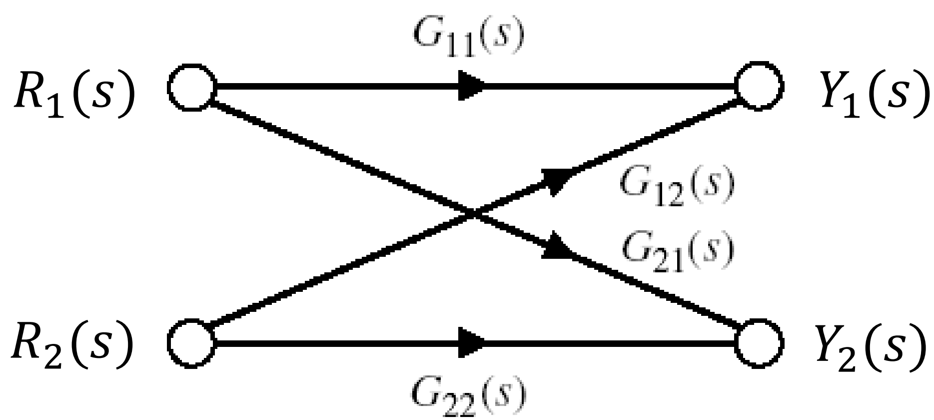


- Every variable in a signal flow graph is represented by a **Node**.
- Every transmission function in a signal flow graph is represented by a **Branch**.
- Branches are always **unidirectional**.
- The arrow in the branch denotes the **direction** of the signal flow.

Signal-Flow Graph Models

$$Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s)$$

$$Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s)$$

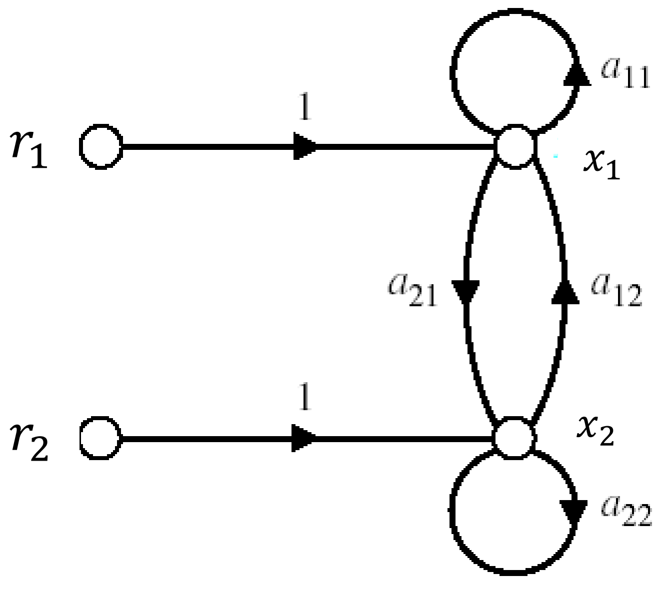


Signal-Flow Graph Models

r_1 and r_2 are inputs and x_1 and x_2 are outputs

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2$$



Signal-Flow Graph Models

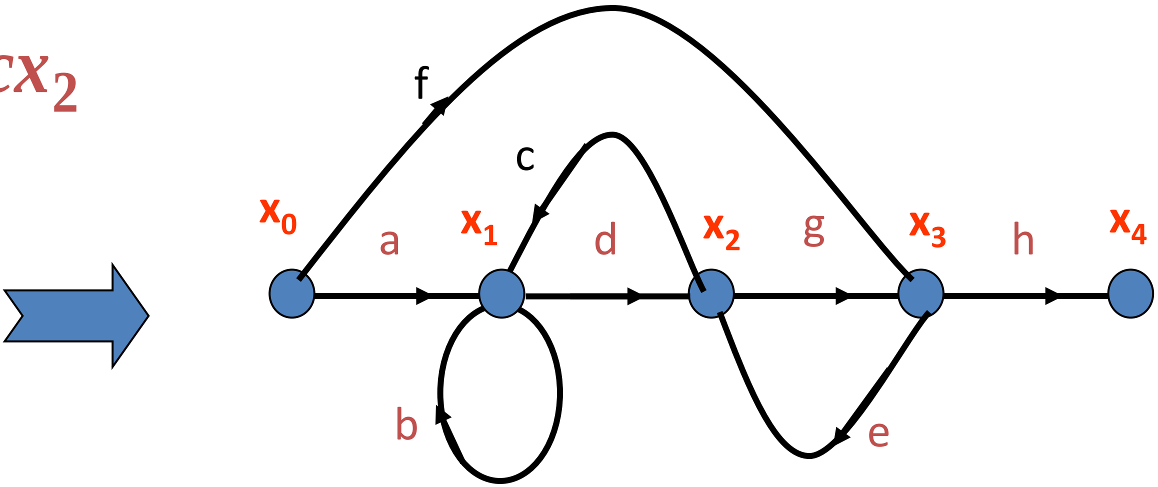
x_o is input and x_4 is output

$$x_1 = ax_0 + bx_1 + cx_2$$

$$x_2 = dx_1 + ex_3$$

$$x_3 = fx_0 + gx_2$$

$$x_4 = hx_3$$



Terminologies

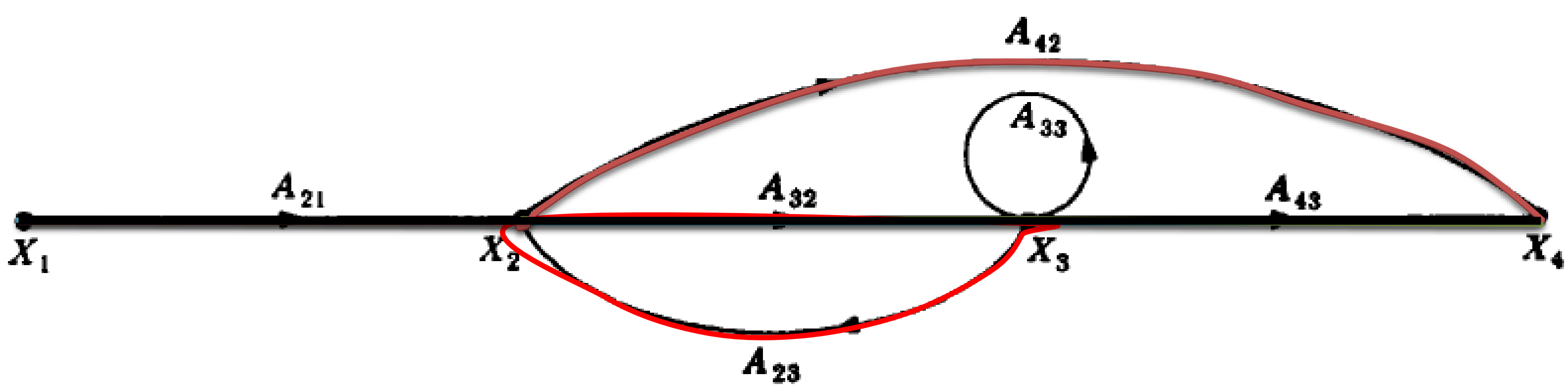
- An **input node** or source contain only the outgoing branches. i.e., X_1
- An **output node** or sink contain only the incoming branches. i.e., X_4
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

  X_1 *to* X_2 *to* X_3 *to* X_4 X_1 *to* X_2 *to* X_4 X_2 *to* X_3 *to* X_4

- A **forward path** is a path from the input node to the output node. i.e.,

  X_1 *to* X_2 *to* X_3 *to* X_4, and X_1 *to* X_2 *to* X_4, are forward paths.

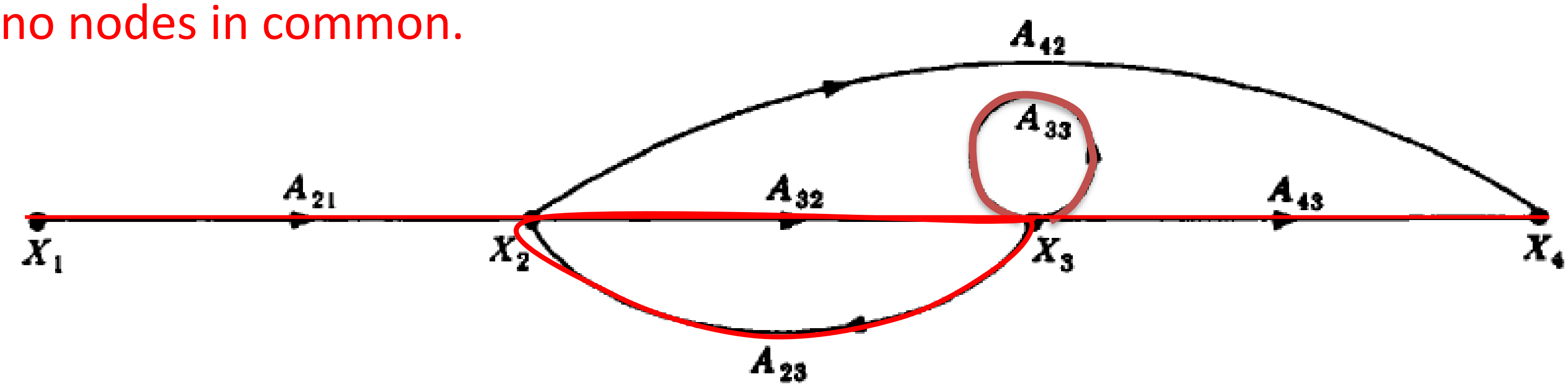
- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.; X_2 *to* X_3 **and back to** X_2 is a feedback path.



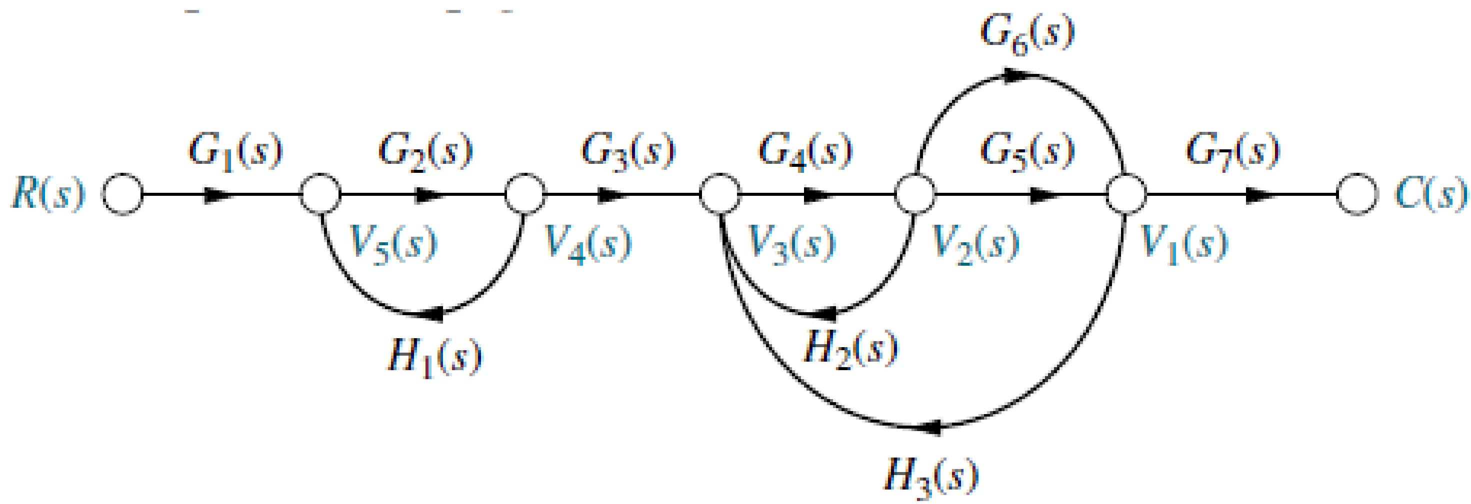
Terminologies

- A **self-loop** is a feedback loop consisting of a single branch. i.e.; A_{33} is a self loop.
- The **gain** of a branch is the transmission function of that branch.
- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path X_1 *to* X_2 *to* X_3 *to* X_4 is $A_{21}A_{32}A_{43}$
- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.

- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.

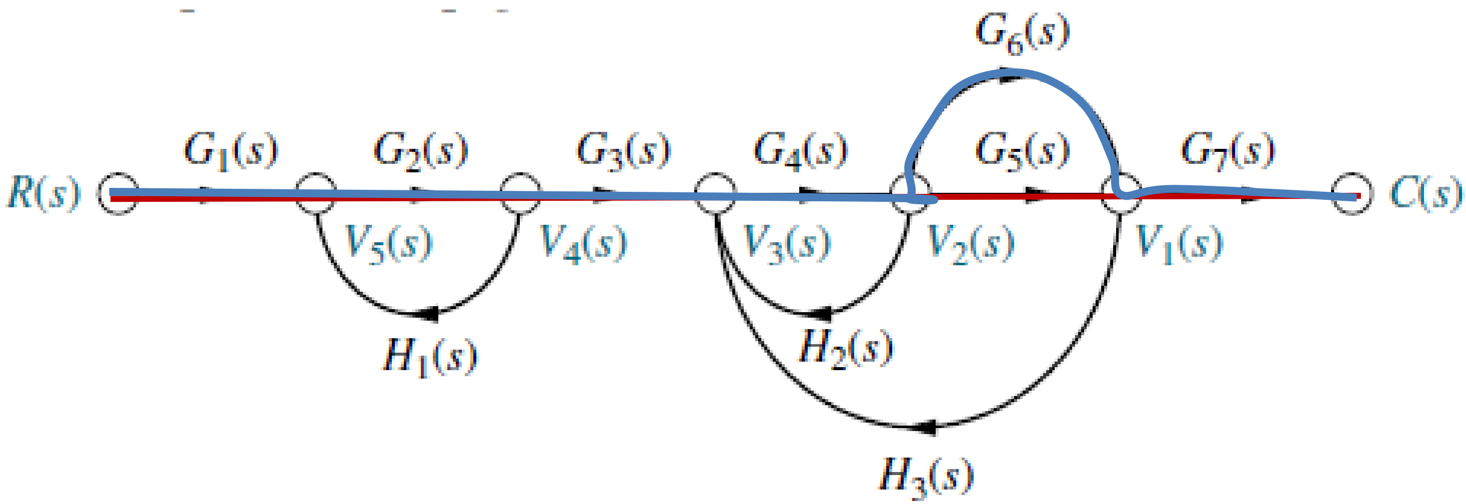


Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths (loops).
e) Determine the loop gains of the feedback loops.
f) Determine the path gains of the forward paths.
g) Non-touching loops

Consider the signal flow graph below and identify the following



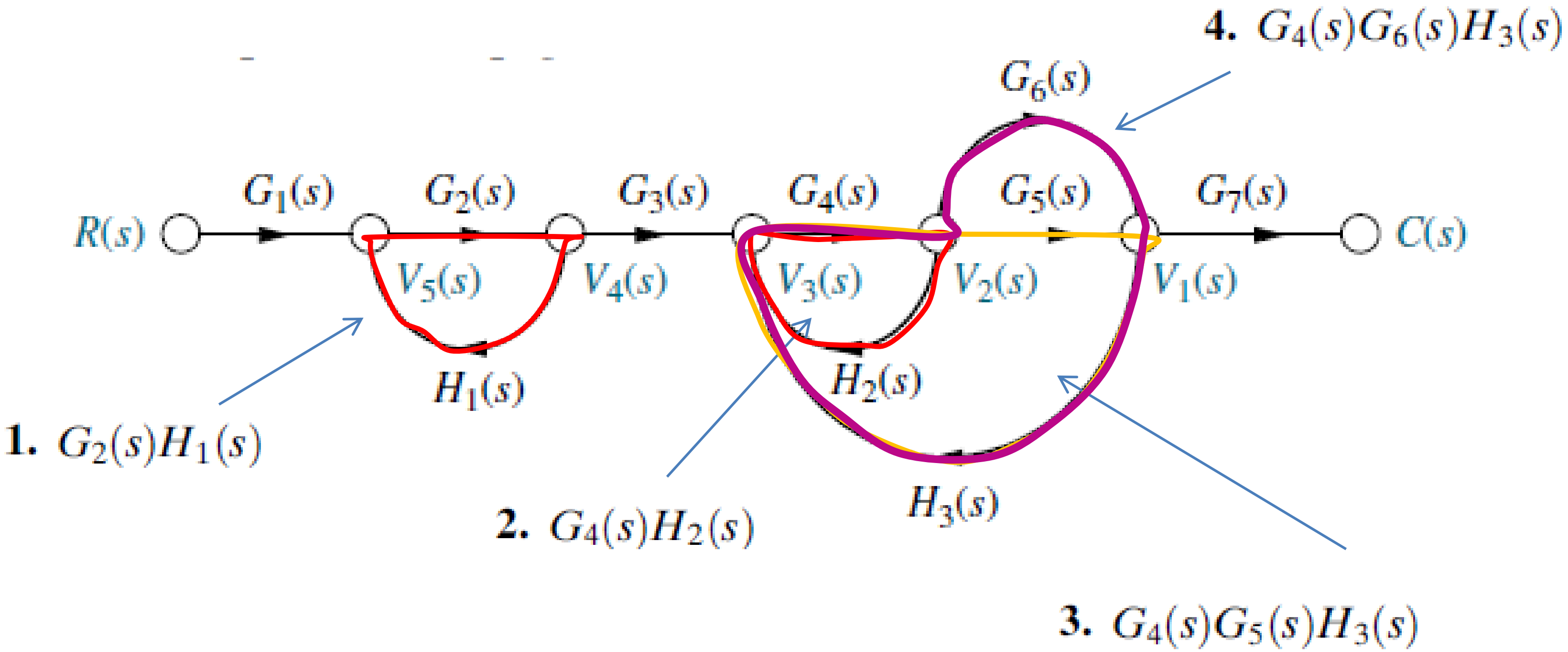
- There are two forward path gains;

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$

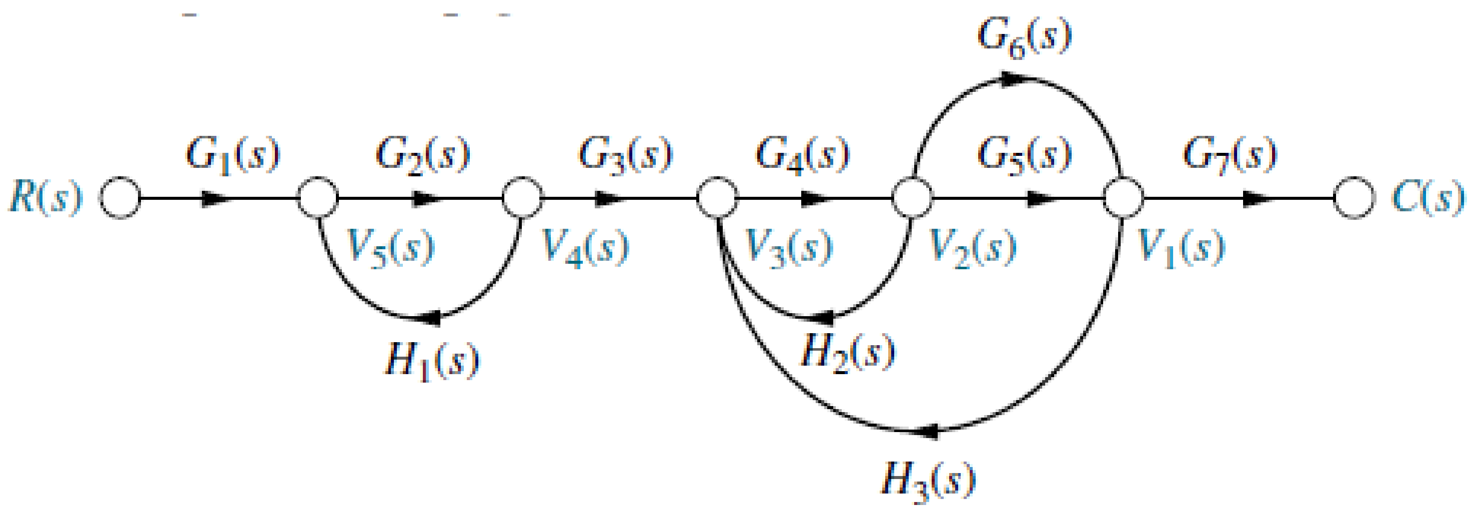
2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

Consider the signal flow graph below and identify the following

- There are four loops



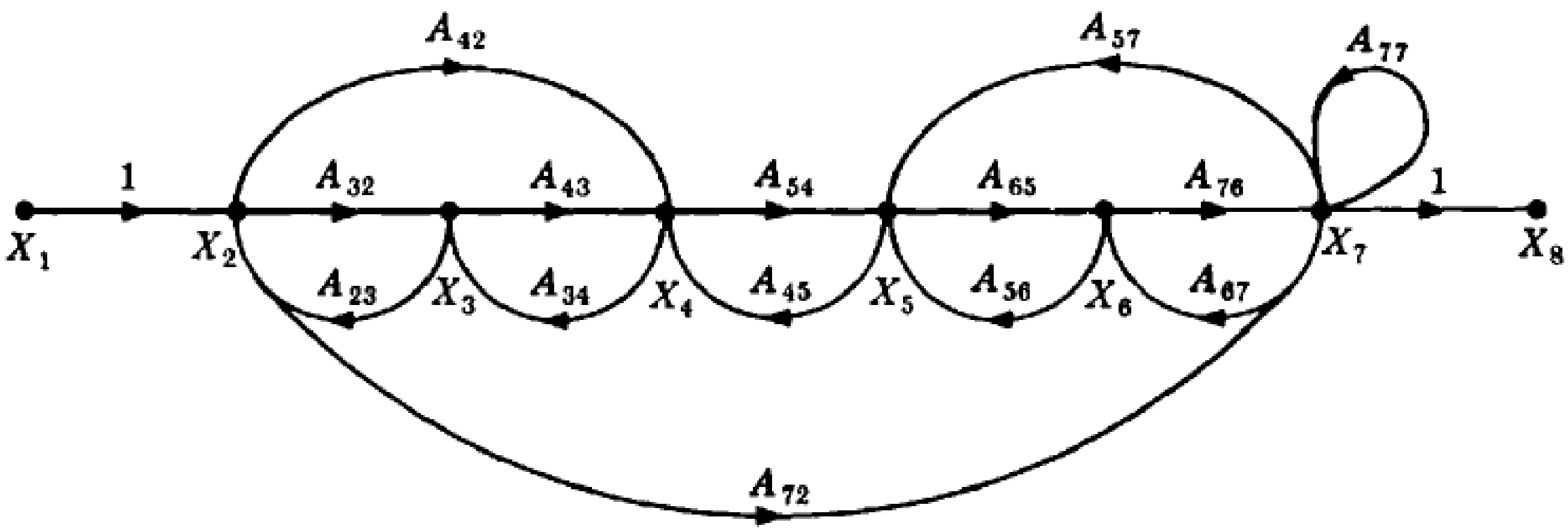
Consider the signal flow graph below and identify the following



- Nontouching loop gains;

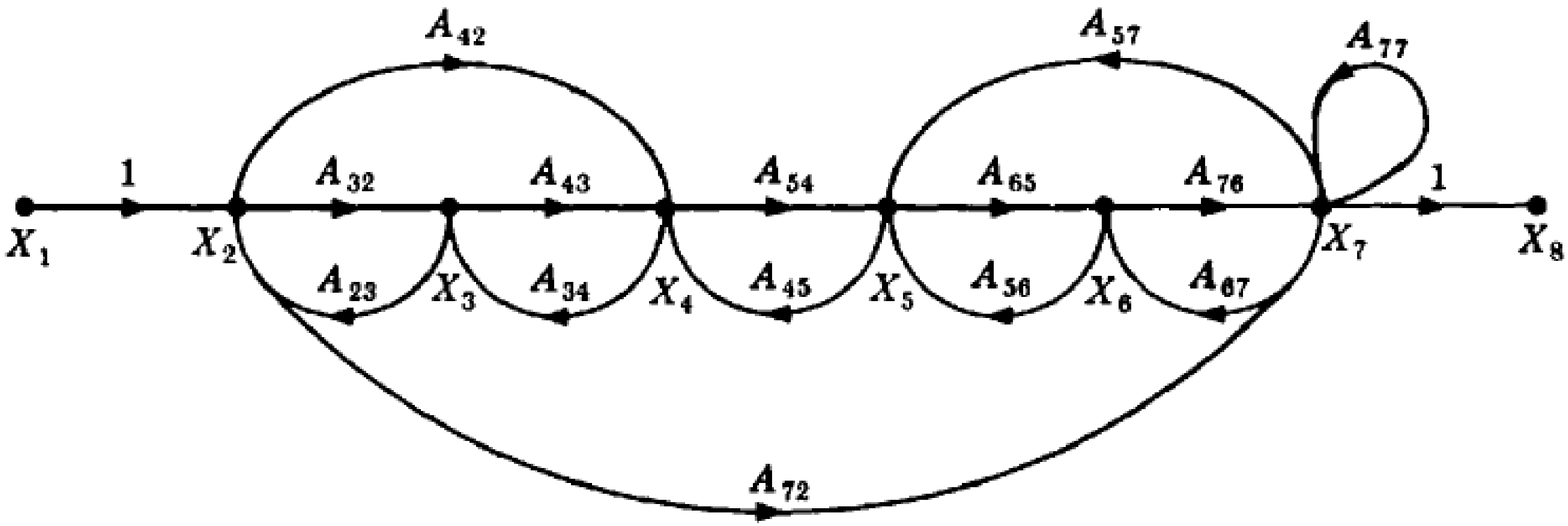
1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths.
e) Self loop.
f) Determine the loop gains of the feedback loops.
g) Determine the path gains of the forward paths.

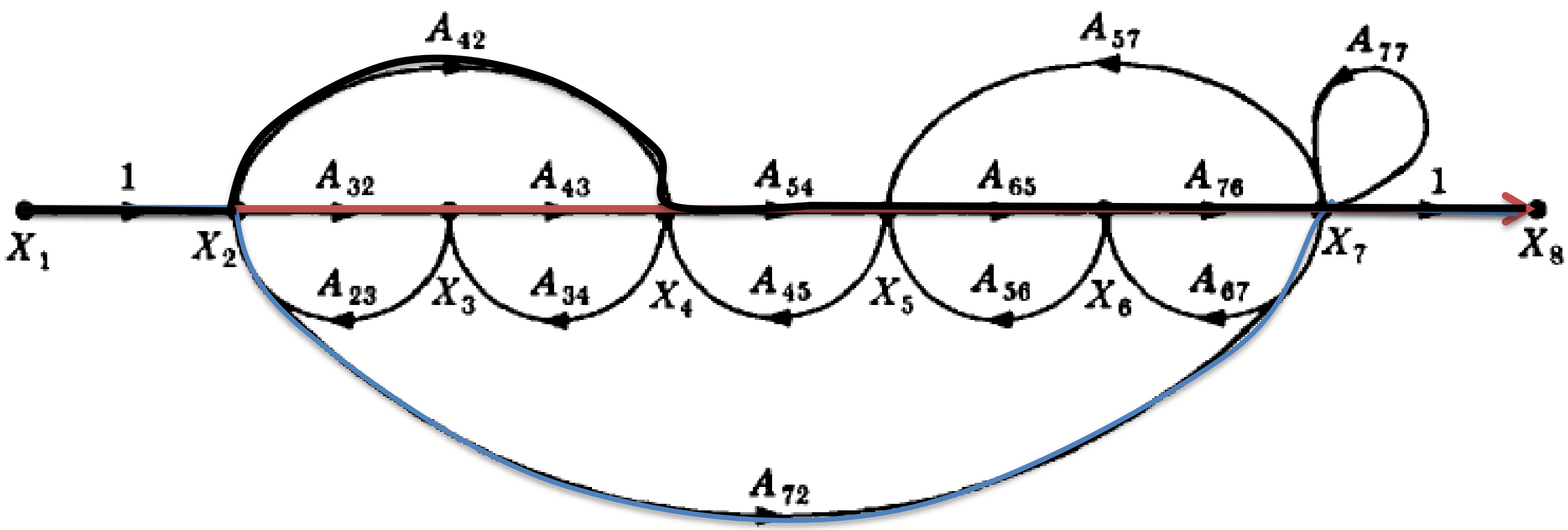
Input and output Nodes



a) Input node X_1

b) Output node X_8

(c) Forward Paths

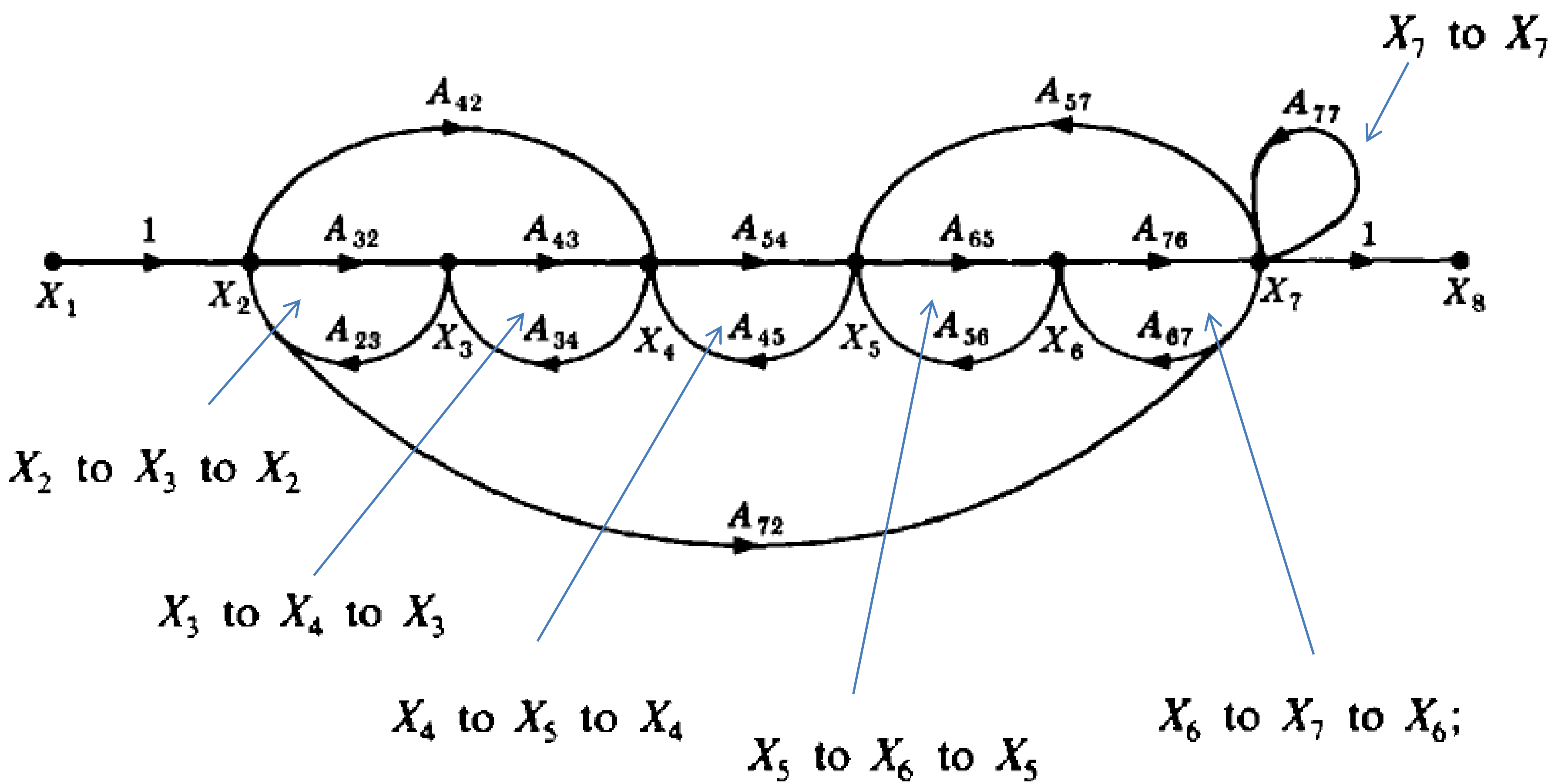


X_1 to X_2 to X_3 to X_4 to X_5 to X_6 to X_7 to X_8

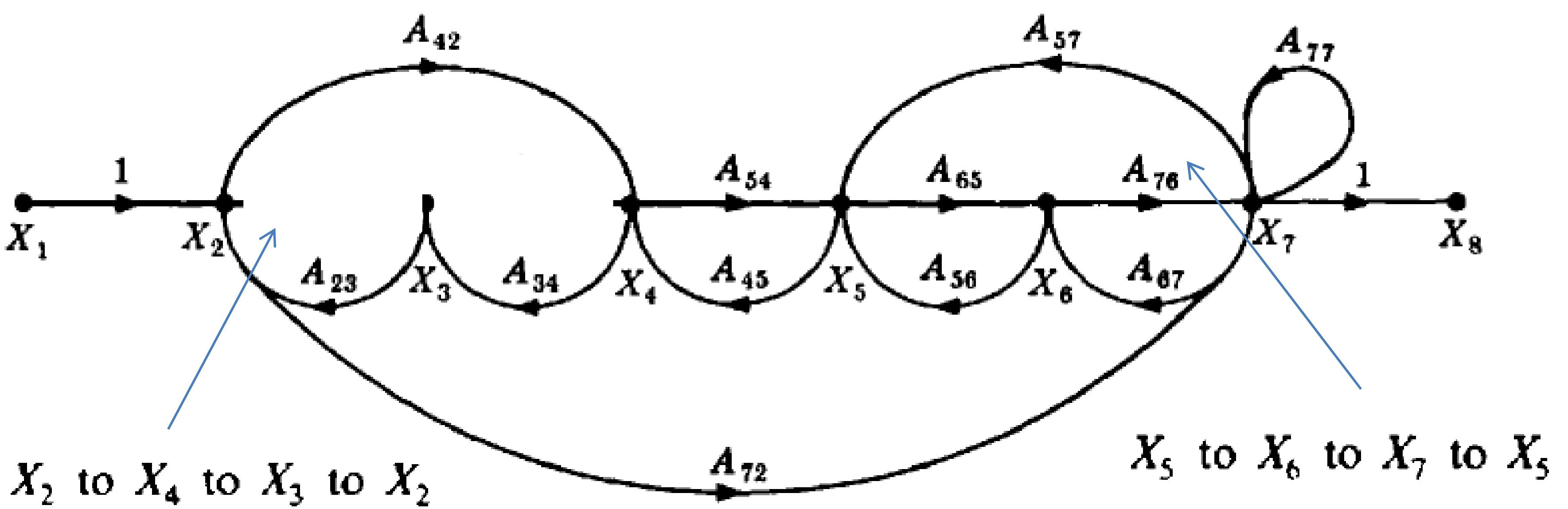
X_1 to X_2 to X_7 to X_8

X_1 to X_2 to X_4 to X_5 to X_6 to X_7 to X_8

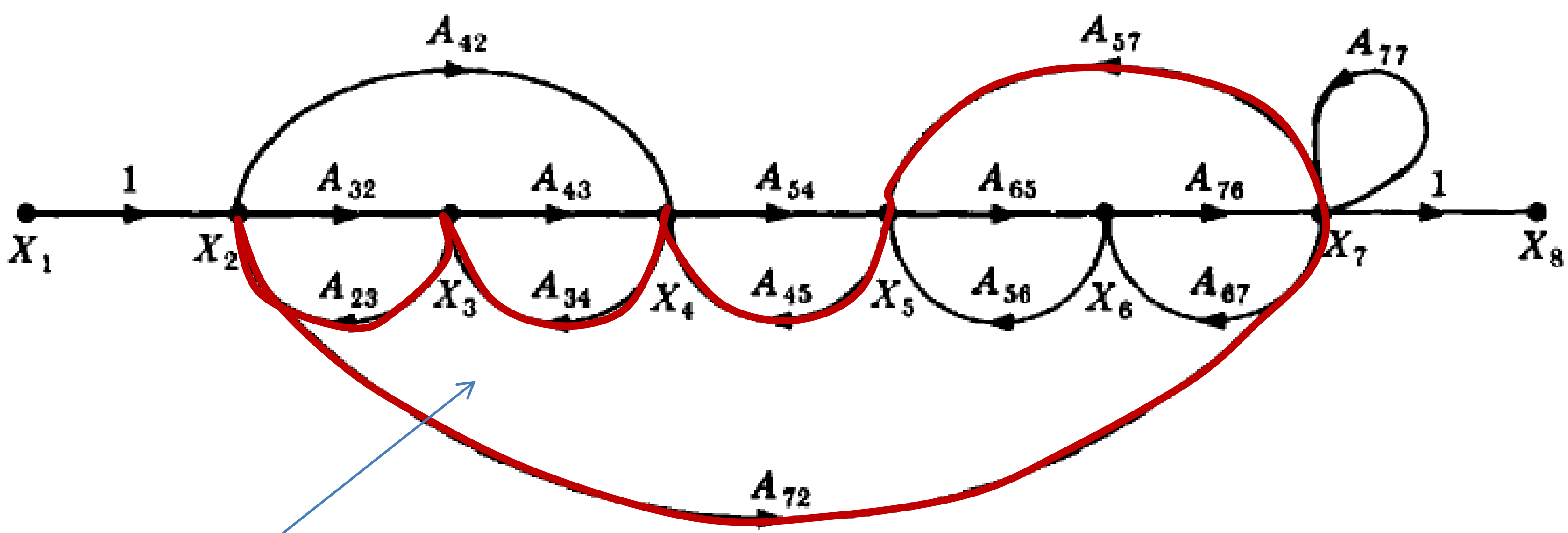
(d) Feedback Paths or Loops



(d) Feedback Paths or Loops

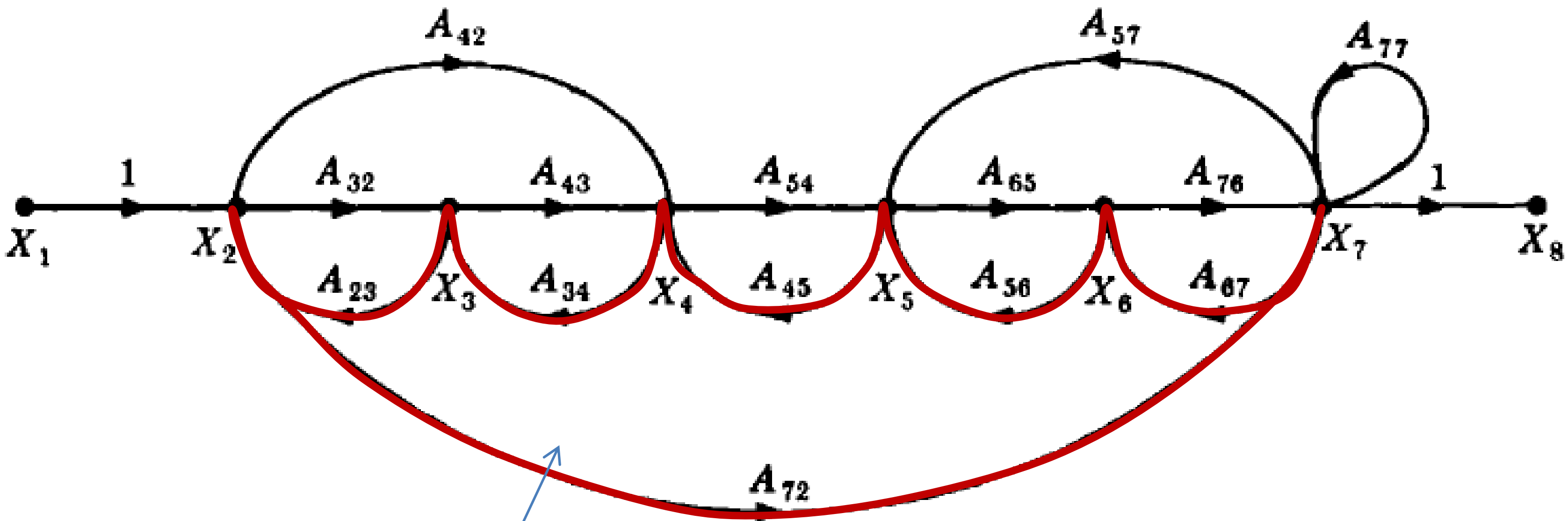


(d) Feedback Paths or Loops



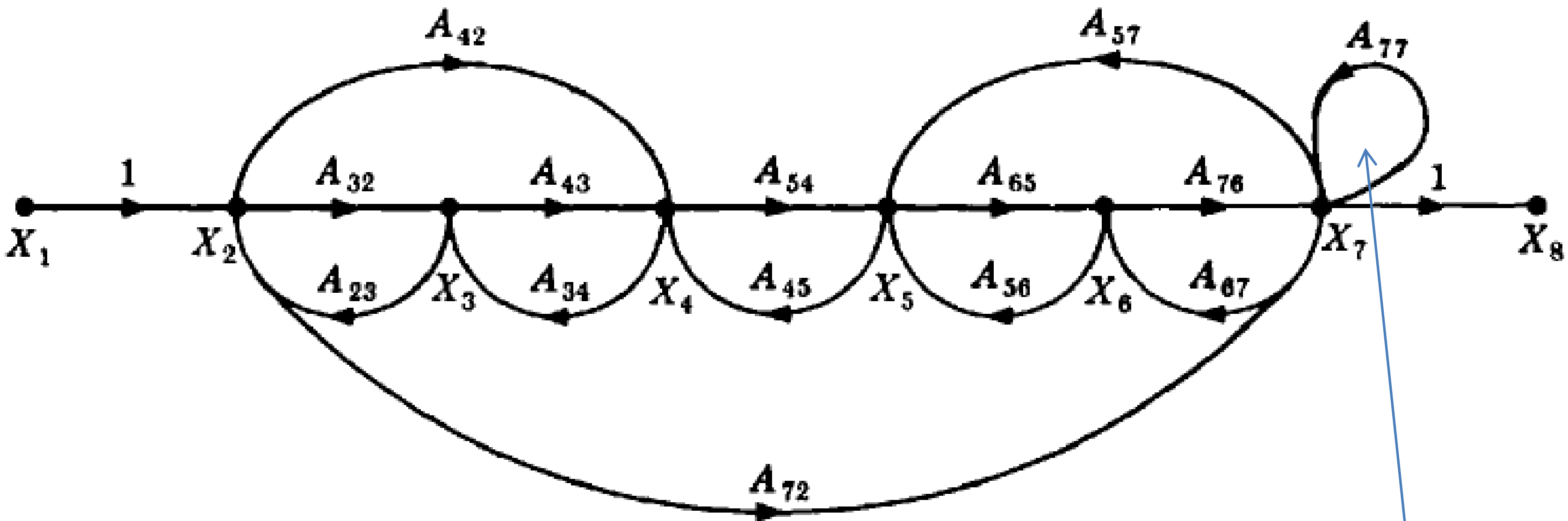
X_2 to X_7 to X_5 to X_4 to X_3 to X_2

(d) Feedback Paths or Loops



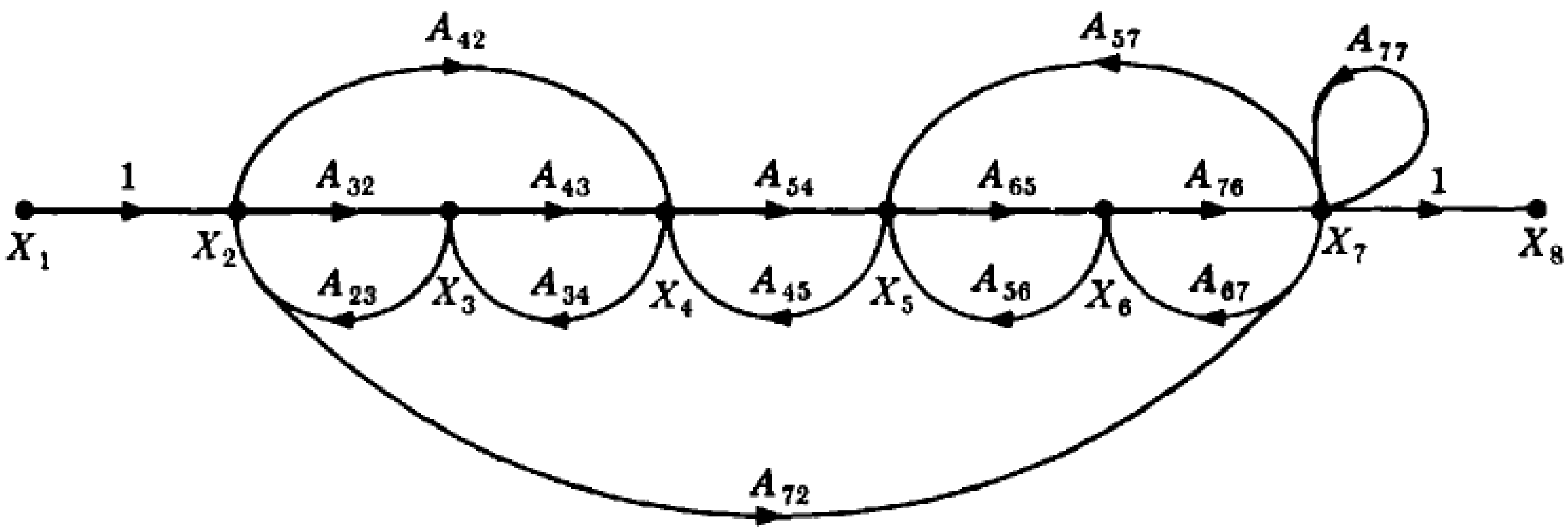
X_2 to X_7 to X_6 to X_5 to X_4 to X_3 to X_2

(e) Self Loop(s)



X_7 to X_7

(f) Loop Gains of the Feedback Loops



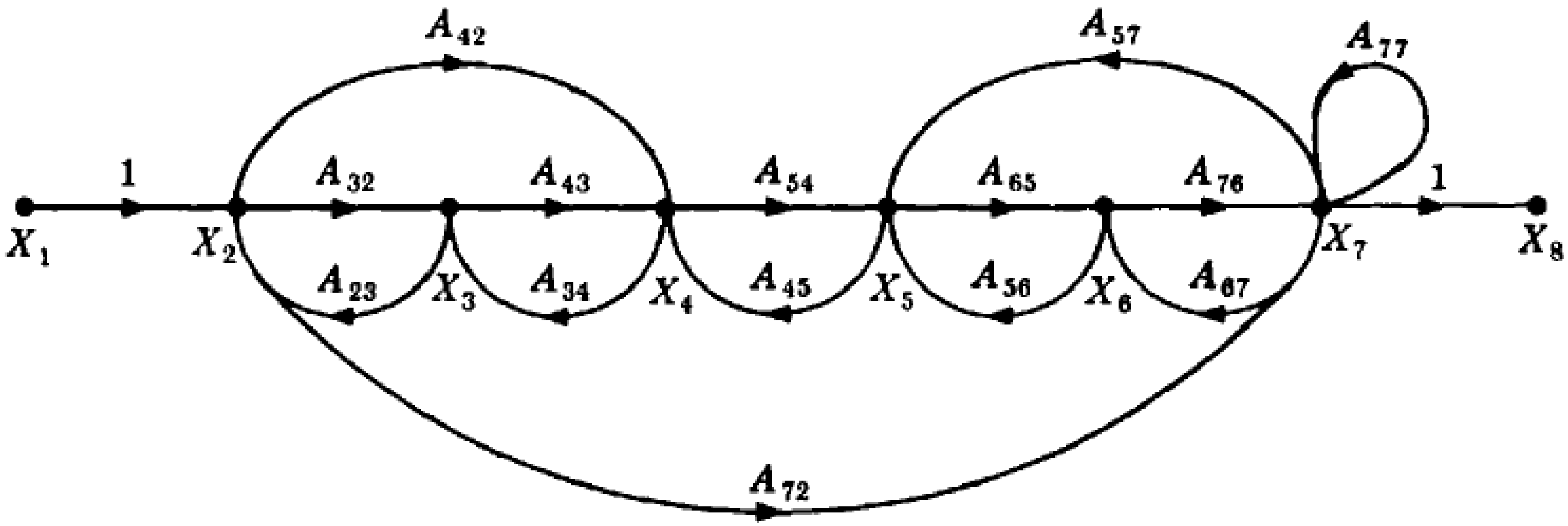
$A_{32}A_{23}$ $\quad$ $A_{76}A_{67}$; $\quad$ $A_{72}A_{57}A_{45}A_{34}A_{23}$;

$A_{43}A_{34}$; $\quad$ $A_{65}A_{76}A_{57}$; $\quad$ $A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$

$A_{54}A_{45}$; $\quad$ A_{77}

$A_{65}A_{56}$ $\quad$ $A_{42}A_{34}A_{23}$;

(g) Path Gains of the Forward Paths



$A_{32} A_{43} A_{54} A_{65} A_{76}$

A_{72}

$A_{42} A_{54} A_{65} A_{76}$

Mason’s Rule (Mason, 1953)

- The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.
- On the other hand, Mason’s rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.
- The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.

Mason's Rule:

- The transfer function, ***C(s)/R(s),*** of a system represented by a signal-flow graph is;

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

Where

n = number of forward paths.
P_i = the i^{th} forward-path gain.
Δ = Determinant of the system
Δ_i = Determinant of the i^{th} forward path

- Δ is called the signal flow graph determinant or characteristic function.

Mason's Rule:

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

Δ = 1- (sum of all individual loop gains) + (sum of the products of the gains of all possible two loops that do not touch each other) – (sum of the products of the gains of all possible three loops that do not touch each other) + … and so forth with sums of higher number of non-touching loop gains

Δ_i = value of Δ for the part of the block diagram that does not touch the i-th forward path (Δ_i = 1 if there are no non-touching loops to the i-th path.)

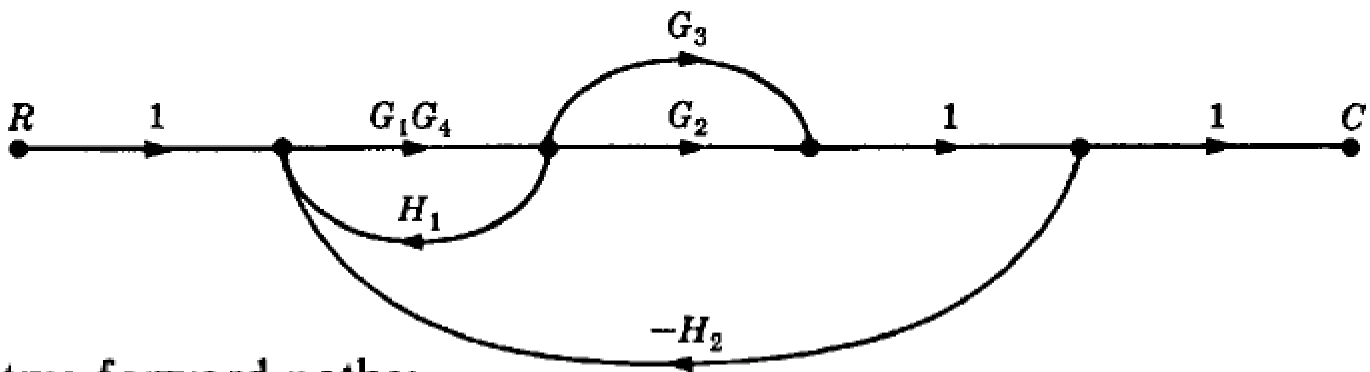
Or

Δ_i is obtained from Δ by removing the loops that are touching the i^{th} forward path.

Systematic approach

1. Calculate forward path gain P_i for each forward path *i.*
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate Δ from steps 2,3,4 and 5
7. Calculate Δ_i as portion of Δ not touching forward path P*i*

Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



There are two forward paths:

$P_1 = G_1G_2G_4 \qquad P_2 = G_1G_3G_4$

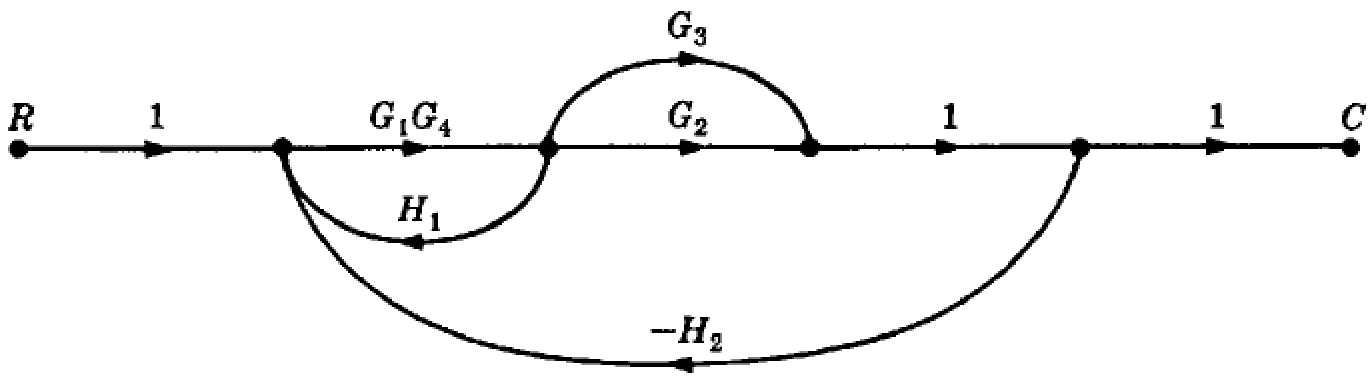
Therefore,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

There are three feedback loops

$$L_1 = G_1G_4H_1, \quad L_2 = -G_1G_2G_4H_2, \quad L_3 = -G_1G_3G_4H_2$$

Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



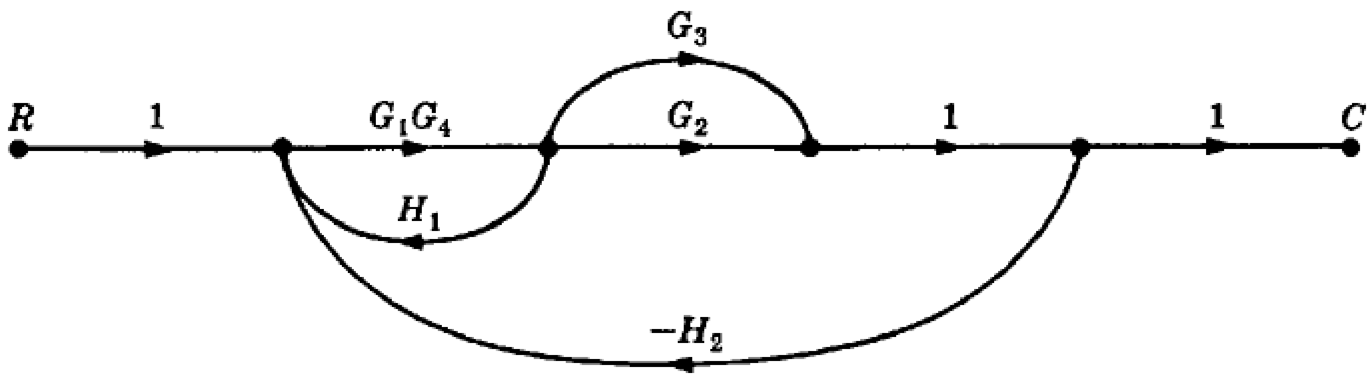
There are no non-touching loops, therefore

Δ = 1- (sum of all individual loop gains)

$$\Delta = 1 - \left(L_1 + L_2 + L_3\right)$$

$$\Delta = 1 - \left(G_1G_4H_1 - G_1G_2G_4H_2 - G_1G_3G_4H_2\right)$$

Example#1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



Eliminate forward path-1

Δ_1 = 1- (sum of all individual loop gains)+...

$\Delta_1 = 1$

Eliminate forward path-2

Δ_2 = 1- (sum of all individual loop gains)+...

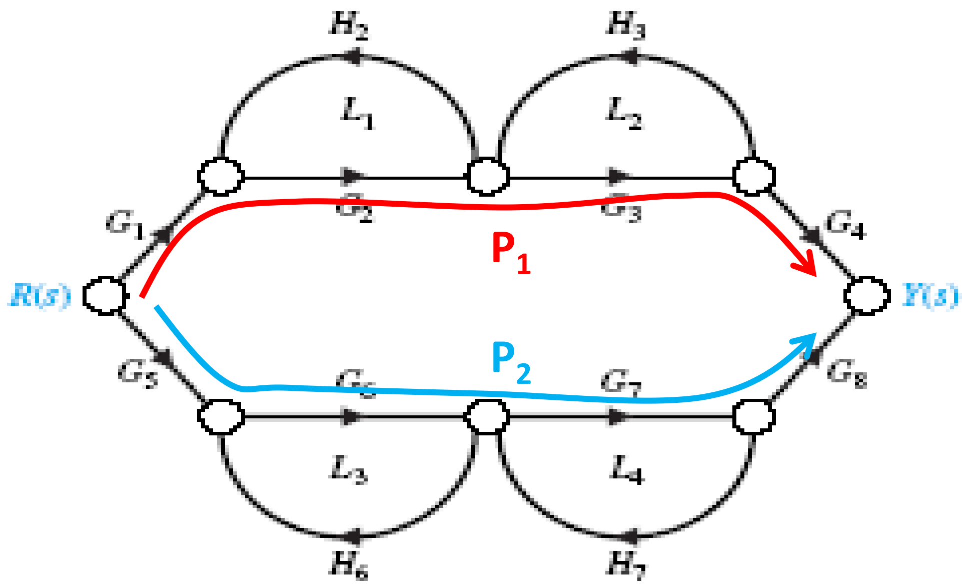
$\Delta_2 = 1$

Example#1: Continue

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

Example#2: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



1. Calculate forward path gains for each forward path.

$$P_1 = G_1G_2G_3G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path 2)}$$

2. Calculate all loop gains.

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad L_4 = G_7H_7$$

3. Consider two non-touching loops.

L_1L_3 L_1L_4

L_2L_4 L_2L_3

Example#2: continue

4. Consider three non-touching loops.
   None.

5. Calculate Δ from steps 2,3,4.

$$\Delta = 1 - \left(L_1 + L_2 + L_3 + L_4\right) + \left(L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4\right)$$

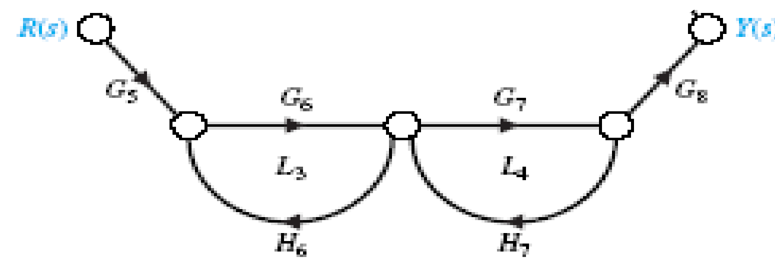
$$\Delta = 1 - \left(G_2 H_2 + H_3 G_3 + G_6 H_6 + G_7 H_7\right) + \\ \left(G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + H_3 G_3 G_6 H_6 + H_3 G_3 G_7 H_7\right)$$

Example#2: continue

Eliminate forward path-1

$$\Delta_1 = 1 - (L_3 + L_4)$$

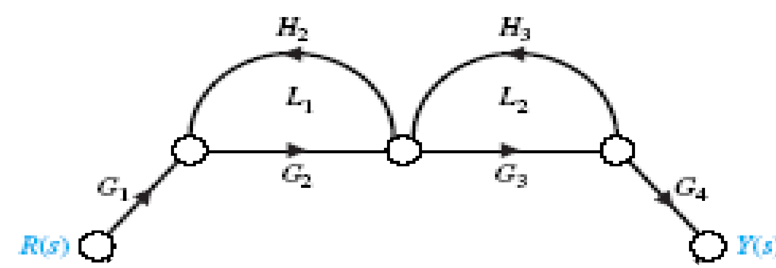
$$\Delta_1 = 1 - (G_6 H_6 + G_7 H_7)$$



Eliminate forward path-2

$$\Delta_2 = 1 - (L_1 + L_2)$$

$$\Delta_2 = 1 - (G_2 H_2 + G_3 H_3)$$



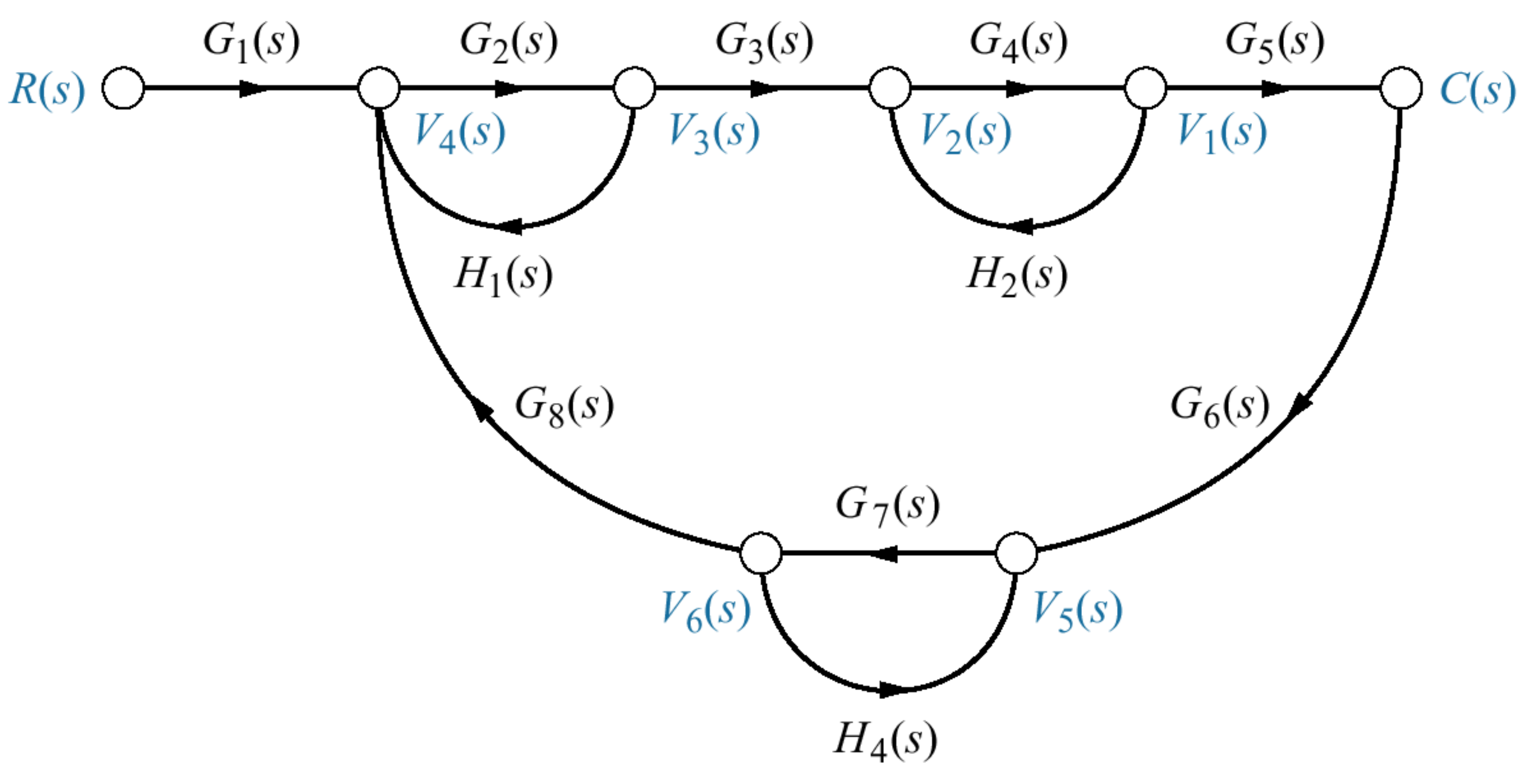
Example#2: continue

$$\frac{Y(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$\frac{Y(s)}{R(s)} = \frac{G_1G_2G_3G_4\left[1-\left(G_6H_6 + G_7H_7\right)\right] + G_5G_6G_7G_8\left[1-\left(G_2H_2 + G_3H_3\right)\right]}{1-\left(G_2H_2 + H_3G_3 + G_6H_6 + G_7H_7\right) + \left(G_2H_2G_6H_6 + G_2H_2G_7H_7 + H_3G_3G_6H_6 + H_3G_3G_7H_7\right)}$$

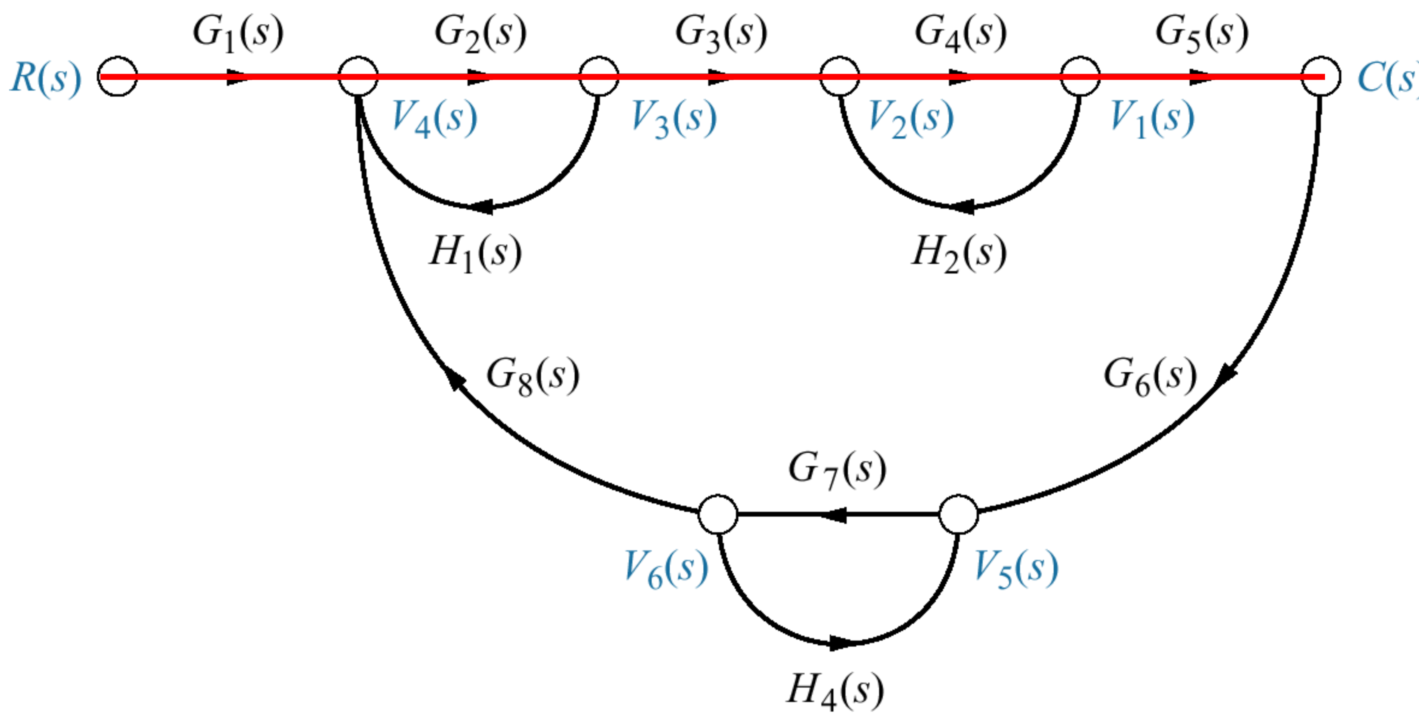
Example#3

- Find the transfer function, C(s)/R(s), for the signal-flow graph in figure below.



Example#3

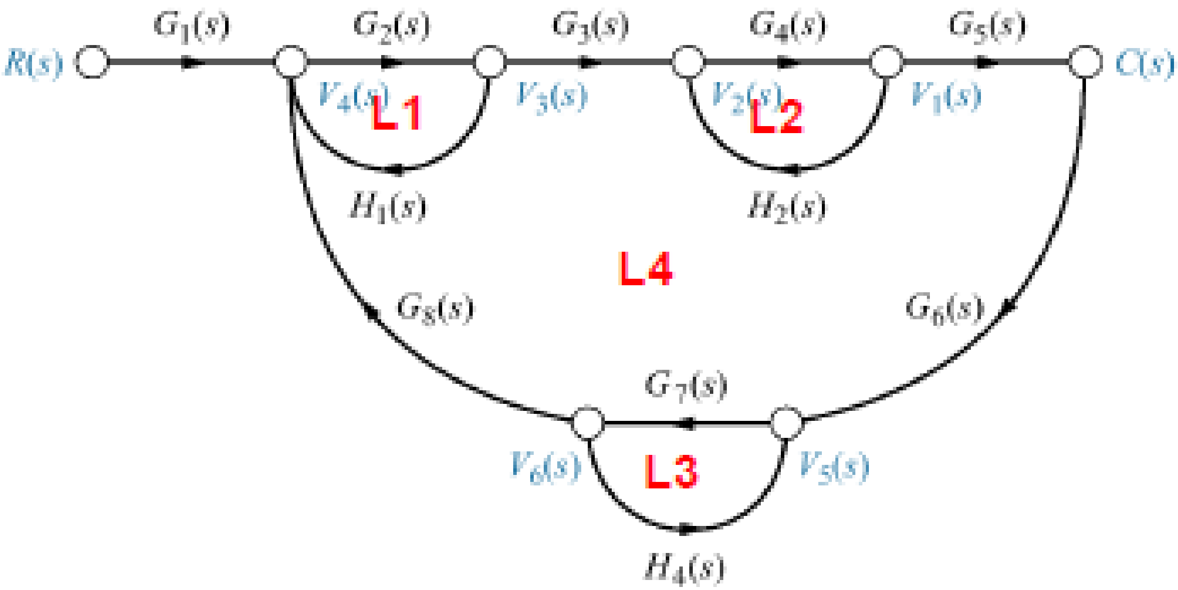
- There is only one forward Path.



$$P_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

Example#3

- There are four feedback loops.



L1. $G_2(s)H_1(s)$

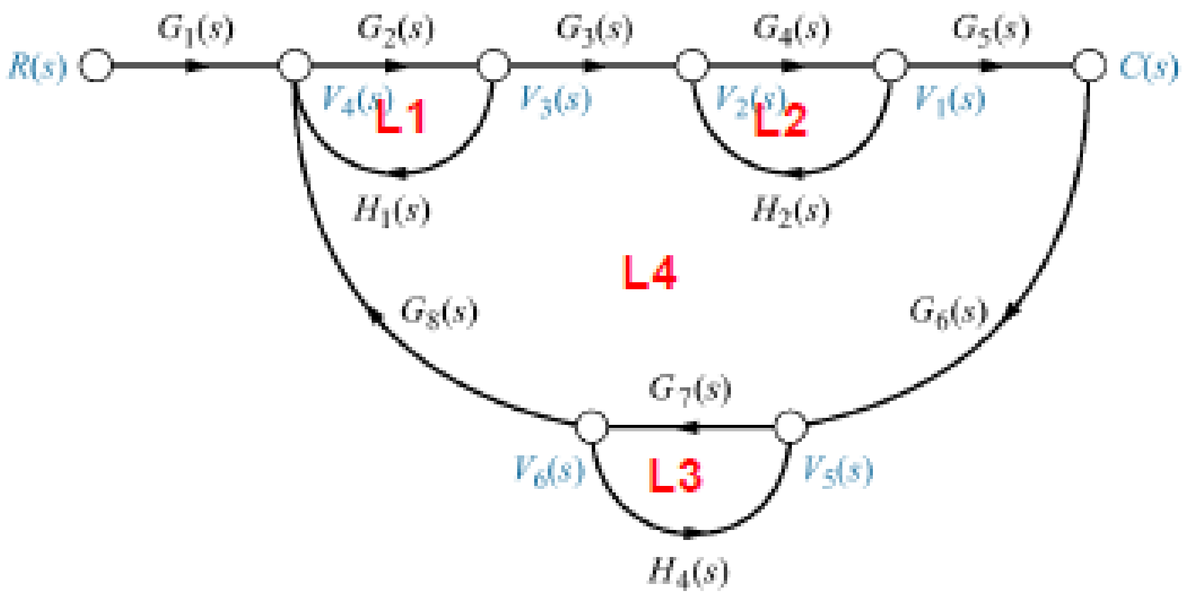
L2. $G_4(s)H_2(s)$

L3. $G_7(s)H_4(s)$

L4. $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

Example#3

- Non-touching loops taken two at a time.

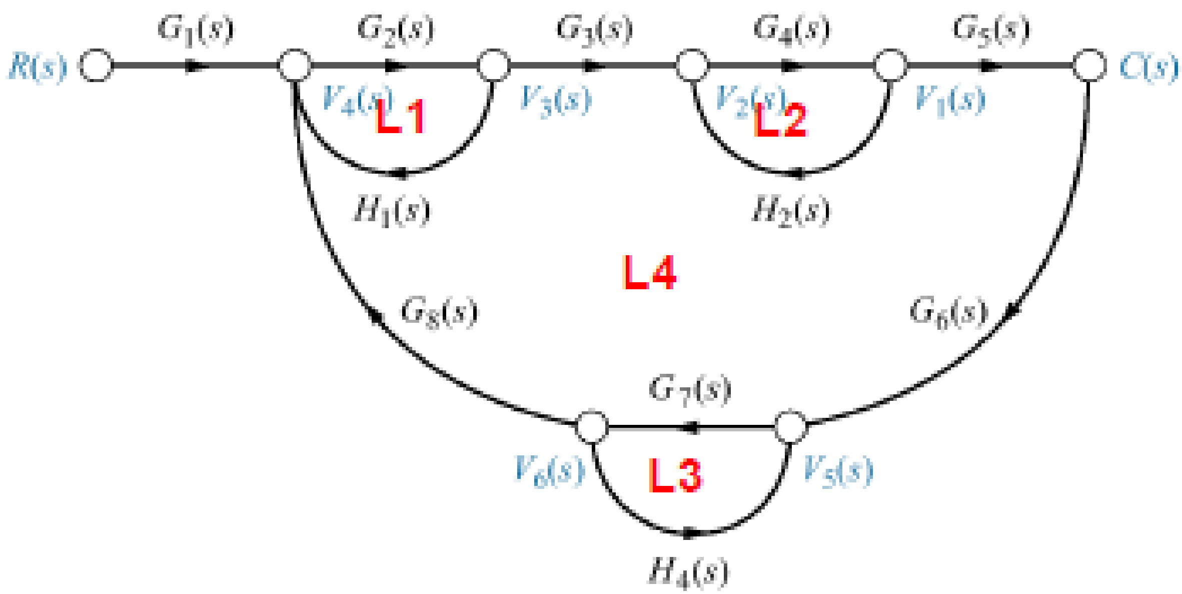


L1 and L2: $G_2(s)H_1(s)G_4(s)H_2(s)$ L2 and L3: $G_4(s)H_2(s)G_7(s)H_4(s)$

L1 and L3: $G_2(s)H_1(s)G_7(s)H_4(s)$

Example#3

- Non-touching loops taken three at a time.



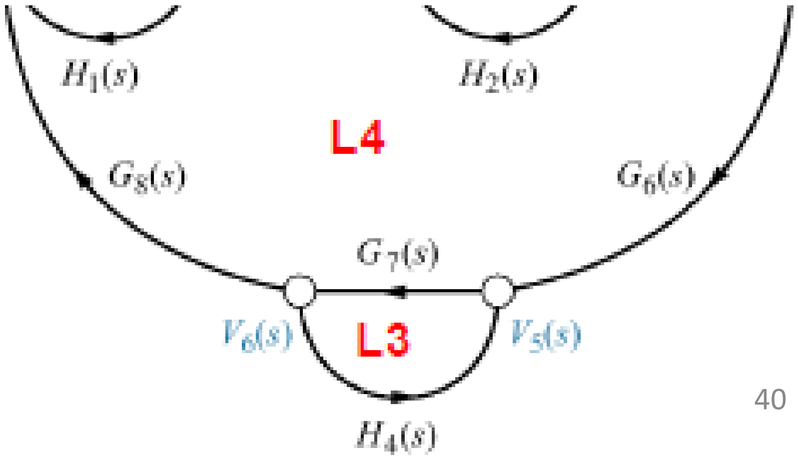
L1, L2, L3: $G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

Example#3

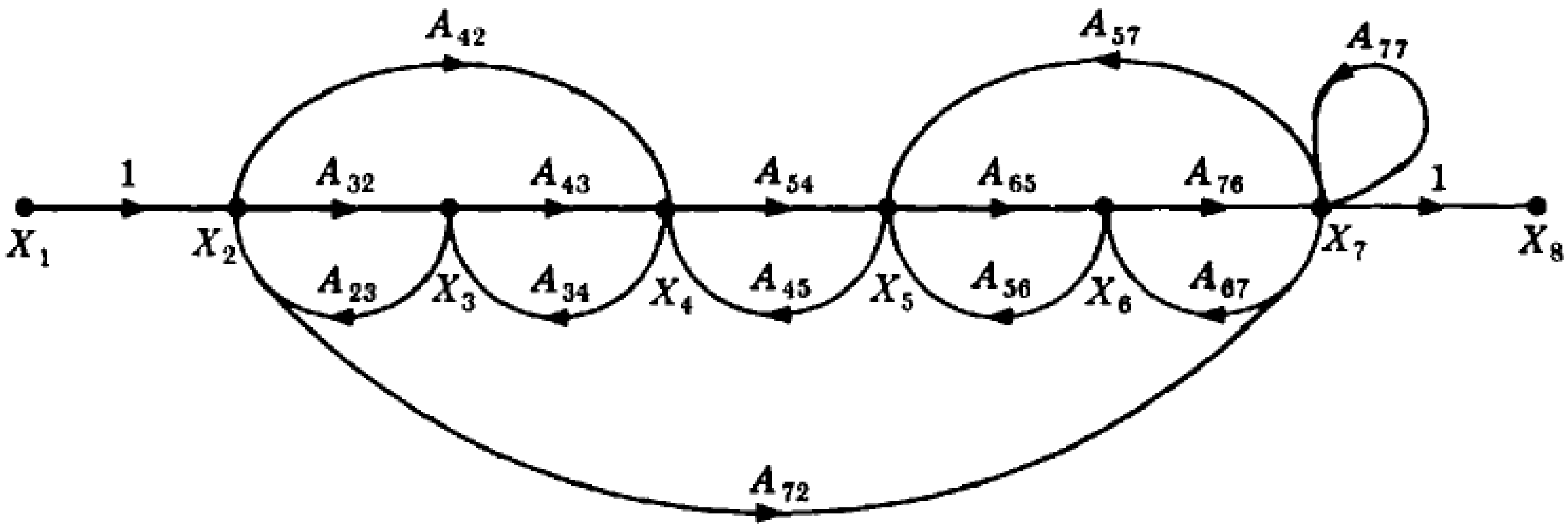
$$\Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s)$$
$$+ G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)]$$
$$+[G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s)$$
$$+ G_4(s)H_2(s)G_7(s)H_4(s)]$$
$$-[G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]$$

Eliminate forward path-1

$$\Delta_1 = 1 - G_7(s)H_4(s)$$



Example#4: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph

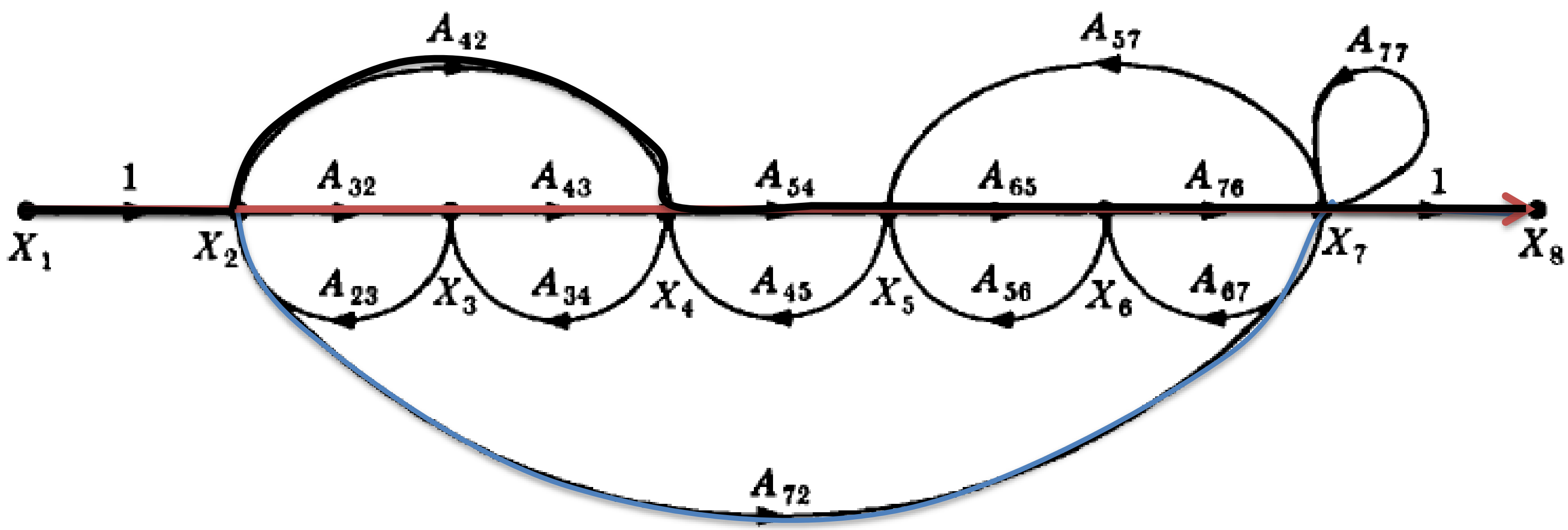


There are three forward paths, therefore n=3.

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{3} P_i\Delta_i}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

Example#4: Forward Paths

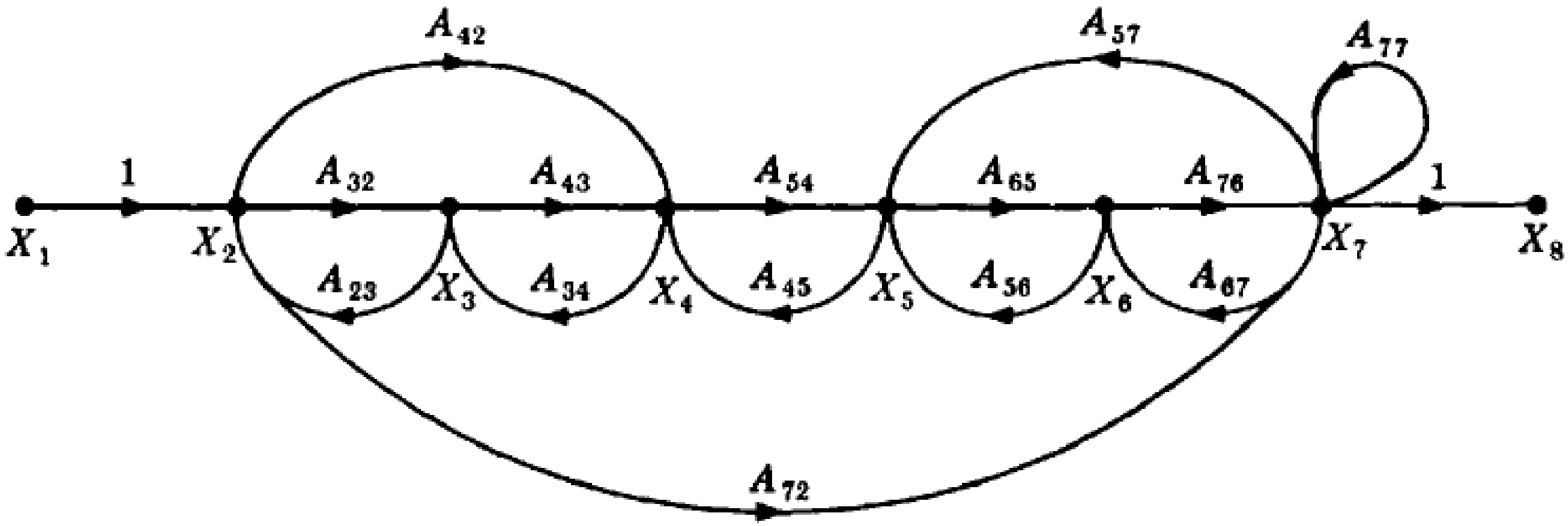
$P_3 = A_{42} A_{54} A_{65} A_{76}$



$P_1 = A_{32} A_{43} A_{54} A_{65} A_{76}$

$P_2 = A_{72}$

Example#4: Loop Gains of the Feedback Loops



$$L_1 = A_{32} A_{23}$$

$$L_2 = A_{43} A_{34}$$

$$L_3 = A_{54} A_{45}$$

$$L_4 = A_{65} A_{56}$$

$$L_5 = A_{76} A_{67}$$

$$L_6 = A_{77}$$

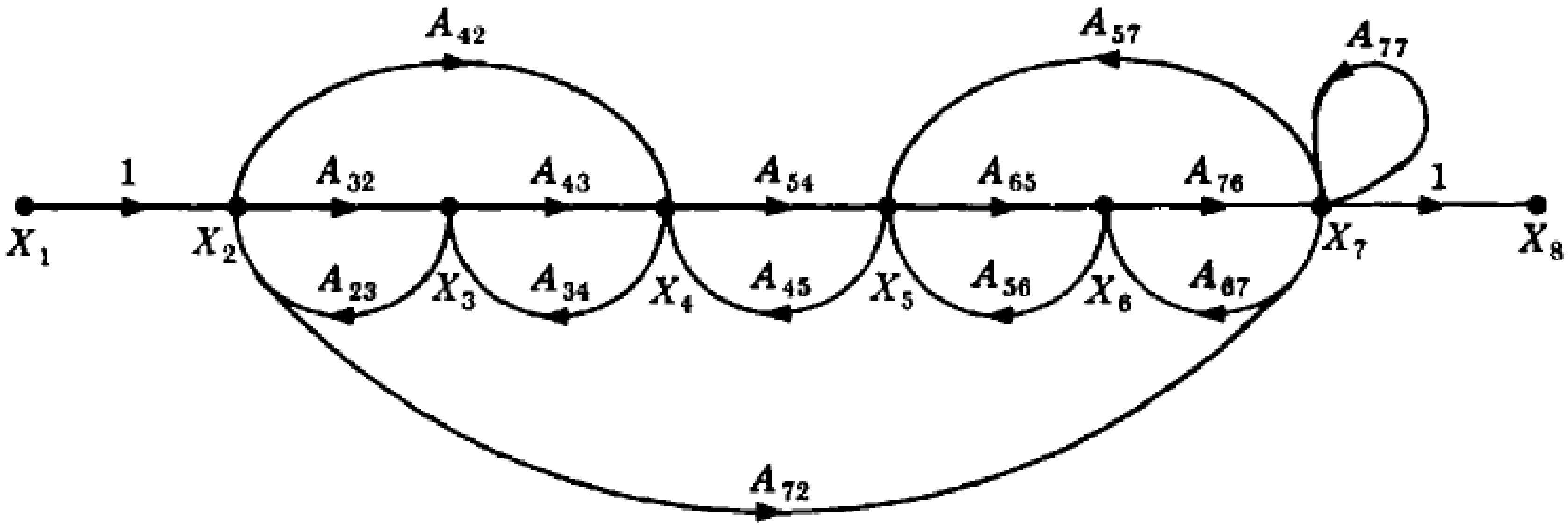
$$L_7 = A_{42} A_{34} A_{23}$$

$$L_8 = A_{65} A_{76} A_{67}$$

$$L_9 = A_{72} A_{57} A_{45} A_{34} A_{23}$$

$$L_{10} = A_{72} A_{67} A_{56} A_{45} A_{34} A_{23}$$

Example#4: two non-touching loops



L_1L_3 L_2L_4 L_3L_5 L_4L_6 L_5L_7 L_7L_8

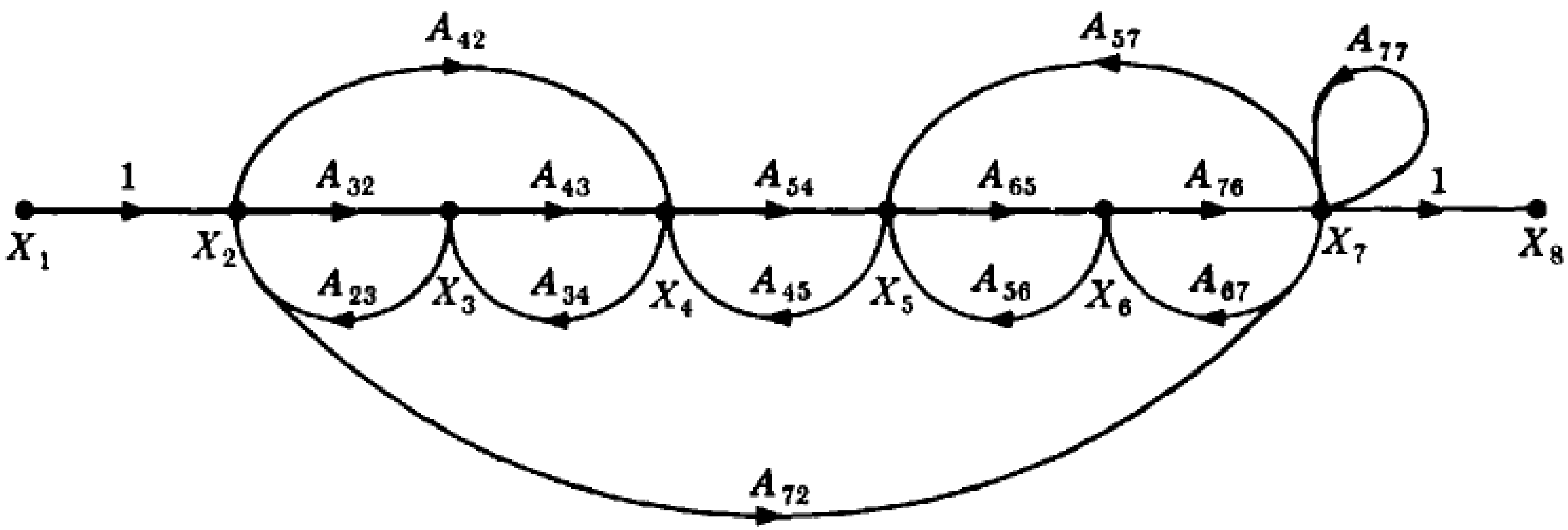
L_1L_4 L_2L_5 L_3L_6 L_4L_7

L_1L_5 L_2L_6

L_1L_6 L_2L_8

L_1L_8

Example#4: Three non-touching loops



L_1L_3 L_2L_4 L_3L_5 L_4L_6 L_5L_7 L_7L_8

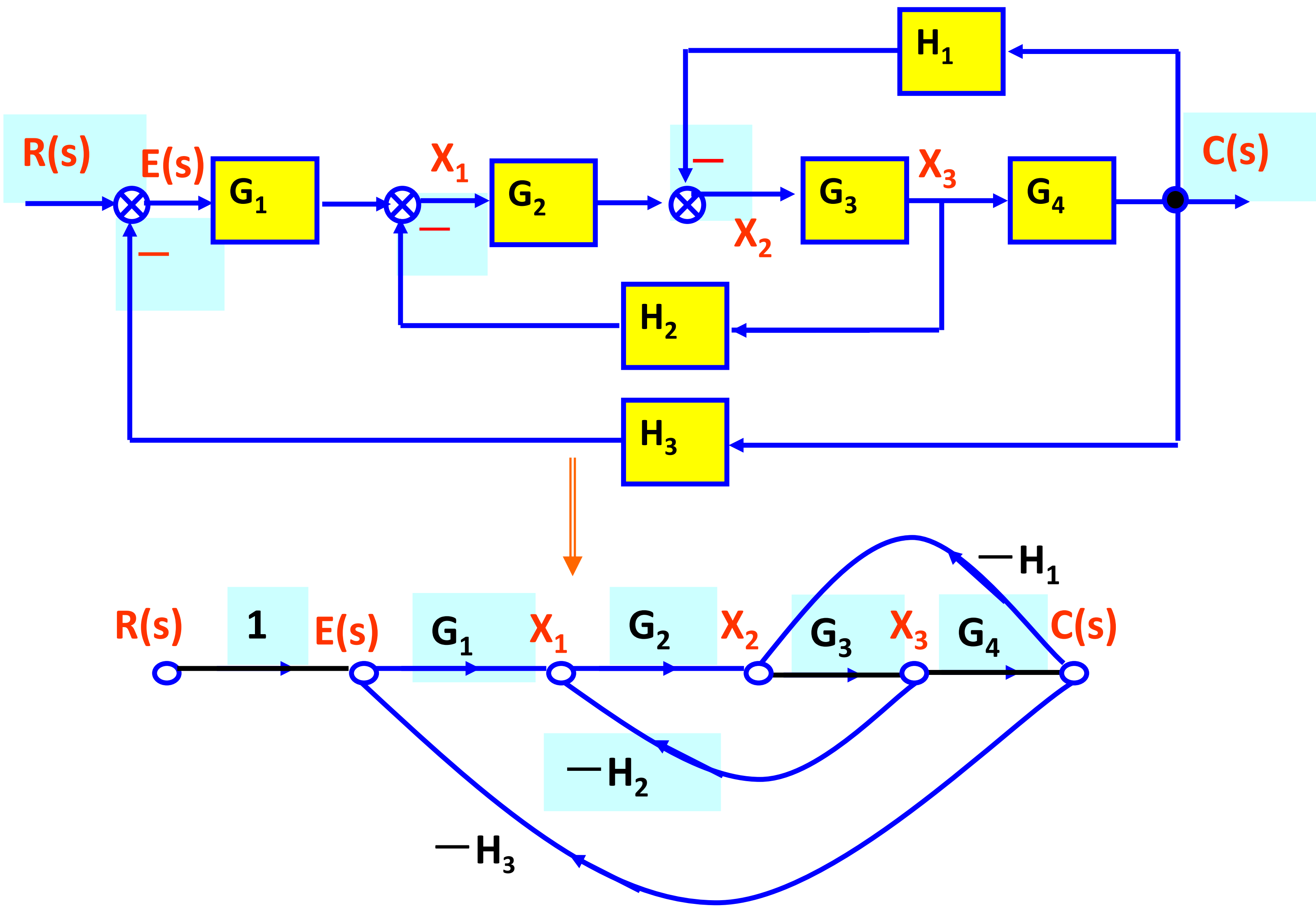
L_1L_4 L_2L_5 L_3L_6 L_4L_7

L_1L_5 L_2L_6

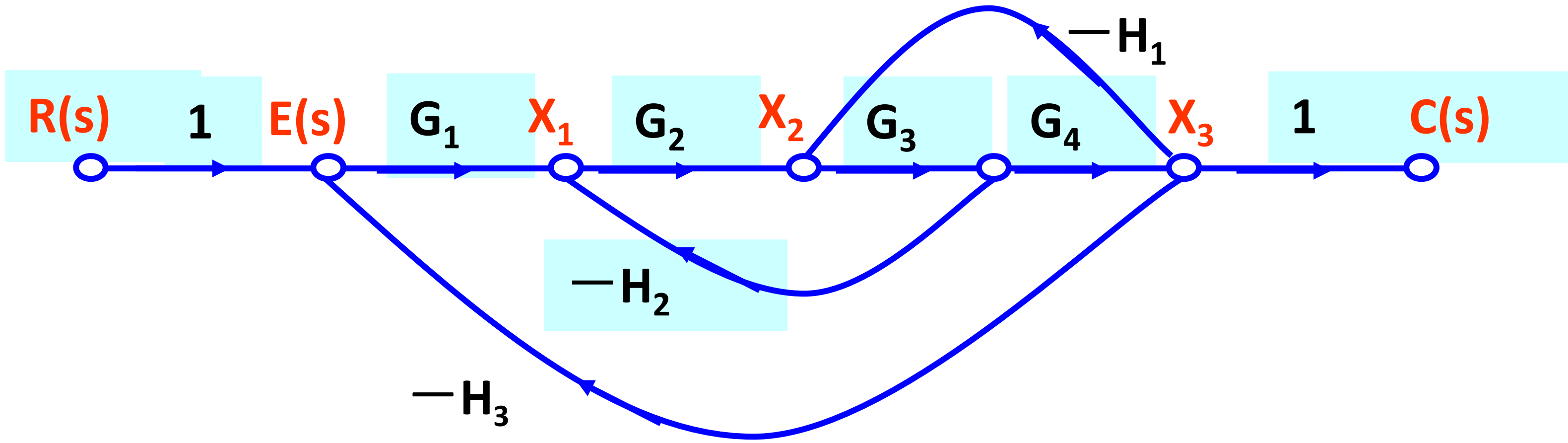
L_1L_6 L_2L_8

L_1L_8

From Block Diagram to Signal-Flow Graph Models
Example#5



From Block Diagram to Signal-Flow Graph Models
Example#5

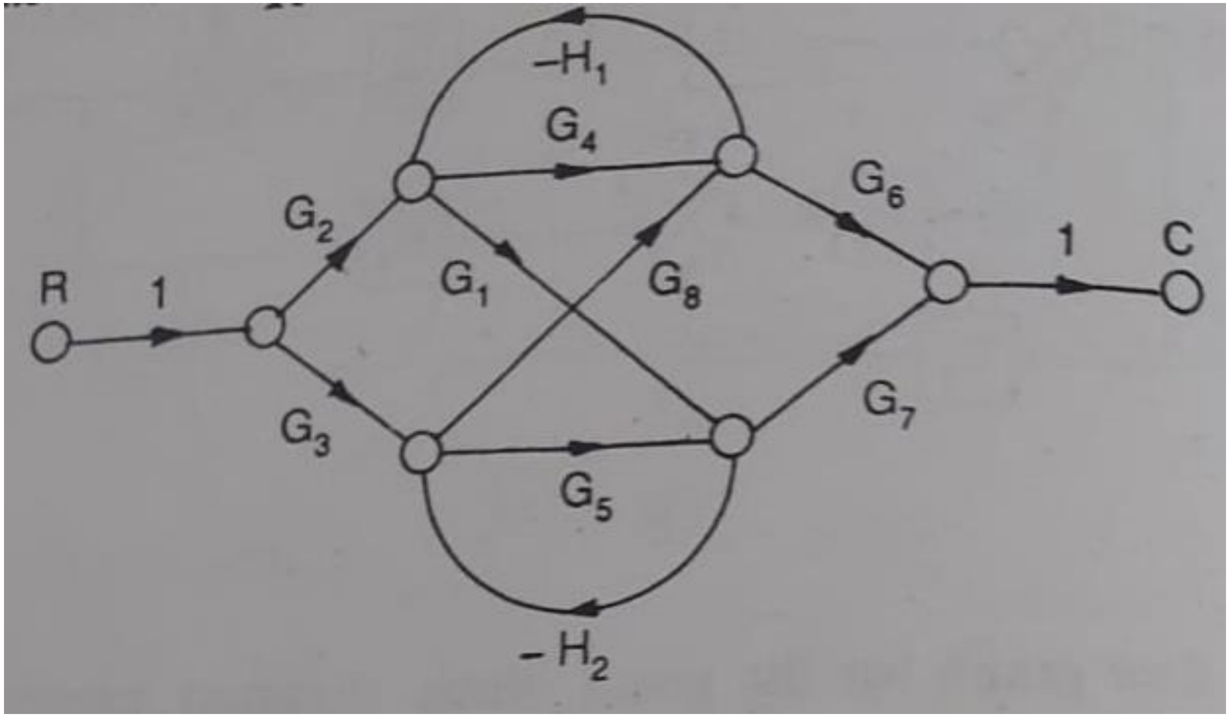


$$\Delta = 1 + (G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1)$$

$$P_1 = G_1G_2G_3G_4; \qquad \Delta_1 = 1$$

$$G = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1}$$

Find the gain of the given signal flow graph.



Solution

Forward Paths

$$P_1 = G_2G_4G_6$$
$$P_2 = G_3G_5G_3$$
$$P_3 = G_2G_1G_7$$
$$P_4 = G_3G_8G_6$$
$$P_5 = -\ G_2G_1H_2G_8G_6$$
$$P_6 = -\ G_3G_8H_1G_1G_7$$

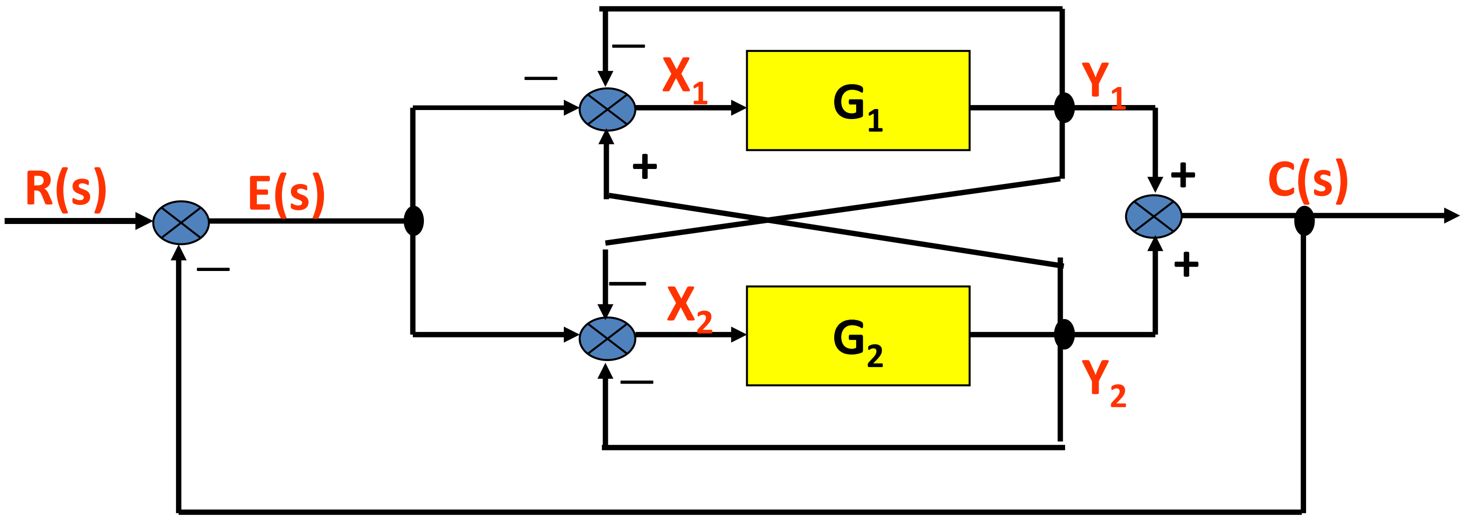
Loops

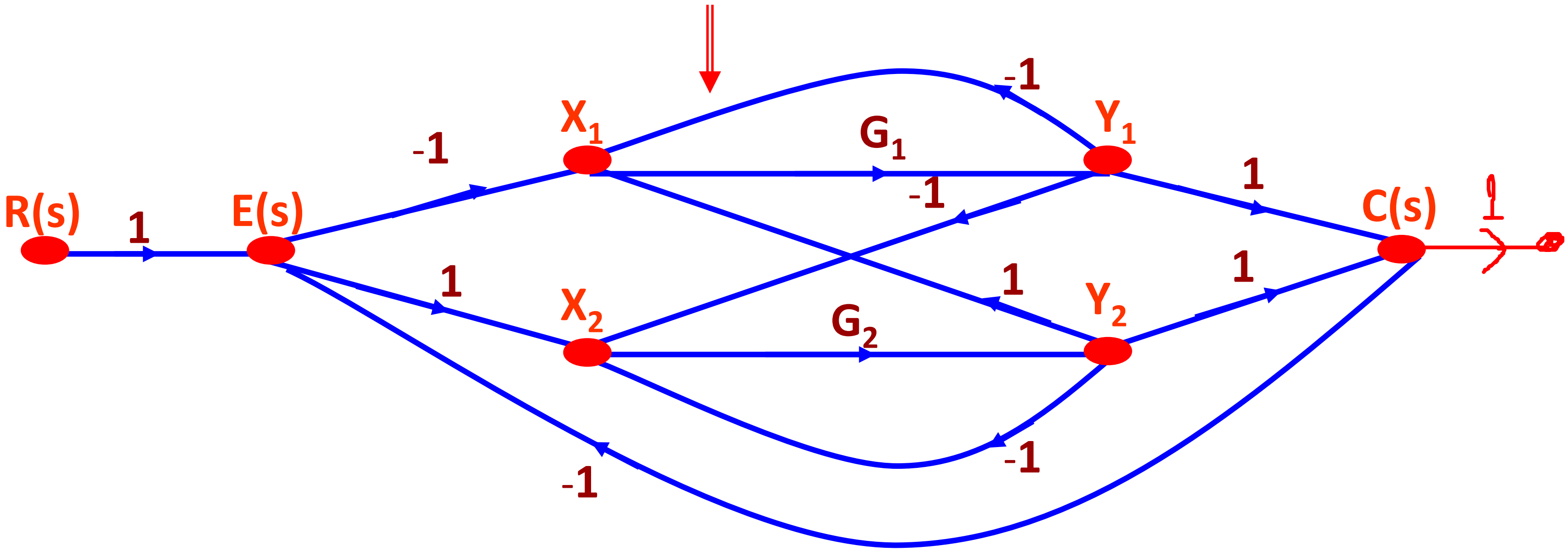
$$L_1 = -\ G_4H_1$$
$$L_2 = -\ G_5H_2$$
$$L_3 = G_1H_2G_8H_1$$

Nontouching Loops There is one pair having gain product

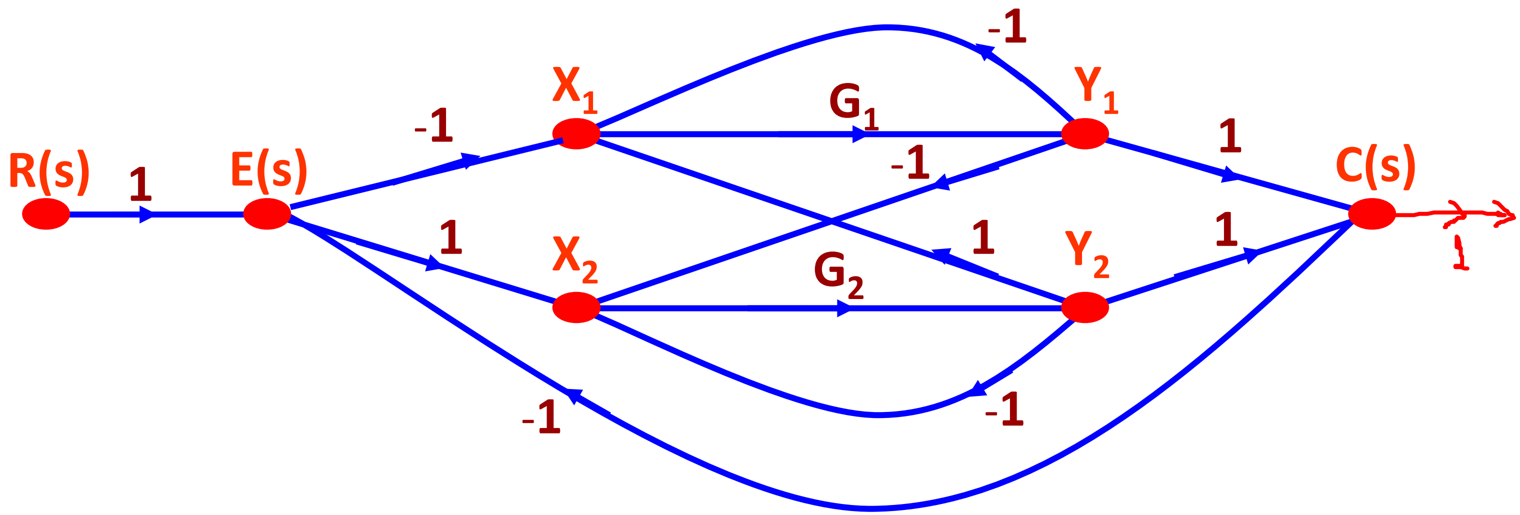
$$= G_4H_1G_5H_2$$
$$\Delta = 1 + G_4H_1 + G_5H_2 - G_1H_2G_8H_1 + G_4H_1G_5H_2$$
$$\Delta_1 = 1 + G_5H_2$$
$$\Delta_2 \quad 1 + G_4H_1$$
$$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$$
$$\therefore \quad T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6}{\Delta}$$
$$= \frac{\begin{array}{c} G_2G_4G_6\,(1 + G_5H_2) + G_3G_5G_7\,(1 + G_4H_1) \\ + G_2G_1G_7 + G_3G_8G_6 \\ - G_2G_6G_8G_1H_2 - G_3G_7G_8G_1H_1 \end{array}}{1 + G_4H_1 + G_5H_2 + G_4G_5H_1H_2 - G_1G_8H_1H_2}$$

Example#6





Example#6



7 loops:

$$[G_1 \cdot (-1)]; \quad [G_2 \cdot (-1)]; \quad [G_1 \cdot (-1) \cdot G_2 \cdot 1]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)];$$

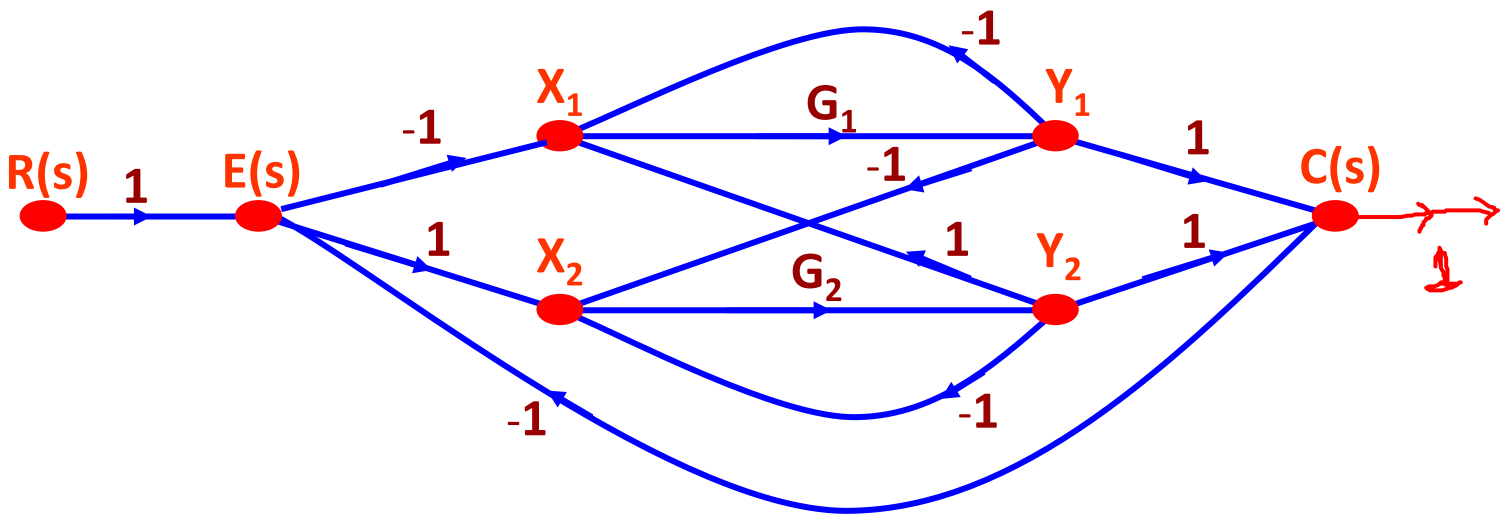
$$[(-1) \cdot G_1 \cdot (-1) \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \cdot (-1)].$$

3 '2 non-touching loops' :

$$[G_1 \cdot (-1)] \cdot [G_2 \cdot (-1)]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)] \cdot [G_2 \cdot (-1)];$$

$$[1 \cdot G_2 \cdot 1 \cdot (-1)] \cdot [G_1 \cdot (-1)].$$

Example#6



Then: $$\Delta = 1 + 2G_2 + 4G_1G_2$$

4 forward paths:

$$p_1 = (-1)\cdot G_1 \cdot 1 \qquad \Delta_1 = 1 + G_2$$

$$p_2 = (-1)\cdot G_1 \cdot (-1)\cdot G_2 \cdot 1 \qquad \Delta_2 = 1$$

$$p_3 = 1 \cdot G_2 \cdot 1 \qquad \Delta_3 = 1 + G_1$$

$$p_4 = 1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \qquad \Delta_4 = 1$$

Example#6

We have

$$\frac{C(s)}{R(s)} = \frac{\sum p_k \Delta_k}{\Delta}$$

$$= \frac{G_2 - G_1 + 2G_1G_2}{1 + 2G_2 + 4G_1G_2}$$