ODE Review

Separation of variables

- Any equation written as $y' = f(x, y, y', \ldots, y^n)$ is a differential equation, this of first order if $n = 1$
- The equation is linear if $f(x)$ is a linear function in y
- $\frac{dy}{dx} = f(x)g(y)$ is a separable equation, got to $\frac{dy}{g(y)} = f(x)dx$ and then integrate

> **Example**
> $\frac{dy}{dx} = \frac{x+2}{y-3}$ so then $f(x) = x + 2$ and $g(y) = y - 3$
> We get the following $\frac{dy}{y-3} = (x + 2)dx$ and then integrate
> $\frac{y^2}{2} = 3y = \frac{x^2}{2} + 2x + C$

Linear First order ODE

- Equations of the form $y' = P(x)y = Q(x)$ or $ay' + by = c$ where a, b, c are functions of x
- We use the integrating factor here, $\mu(x) = e^{\int P(x)dx}$
- With this integrating factor we write
  $\mu y' + \mu P(x)y = \mu Q(x) \implies (\mu y)' = \int \mu Q(x)dx + C$
- $y(x) = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x)dx + C$

> **Example**
> $xy' + y = 2x^3$
> $y' + \frac{1}{x}y = 2x^2 \implies P(x) = \frac{1}{x}$ so then $\mu = e^{\int \frac{1}{x}dx} = x$
> Now we have $\int (xy)' = \int 2x^3\, dx + C$
> Simplifying this gives $y = \frac{x^3}{2} + \frac{C}{x}$

Second Order ODE

Constant coefficients

- $ay'' + by' + cy = d = 0 \quad a, b, c, d \in \mathbb{R}$
- $y(x, r) = e^{rx}$

- $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \implies ar^2 + br + c = 0$
- $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ the stuff under the root is the discriminant
- $**b^2 - 4ac > 0 \implies$ two real distinct roots r_1, r_2
- $y(x) = C_1e^{r_1x} + C_2e^{r_2x}$
- $**b^2 - 4ac = 0 \implies$ repeated roots, $r = -\frac{b}{2a}$
- We know $y_1(x) = C_1e^{rx}$, and $y_2 = xe^{rx}$
- So $y(x) = C_1e^{rx} + C_2xe^{rx}$
- $**b^2 - 4ac < 0 \implies$ you get complex conjugate roots $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$
- $y(x) = C_1e^{(\alpha+\beta i)} + C_2e^{(\alpha-\beta i)} = e^{\alpha x}[C_1e^{\beta ix} + C_2e^{i\beta x}] \ldots$

> **Example**
> $y'' + 2y' + y = 0$
> $r^2 + 2r + 1 = 0 \implies r_{1,2} = -1$
> $y(x) = C_1e^{-x} + C_2xe^{-x}$

Cauchy-Euler ODE

- Of the form $x^2y'' + axy' + by = 0, y(x) = x^r$
- $(r^2(r-1) + ar + b)x^r = 0$ if $x^r \neq 0$
- So then the characteristic equation is $r^2(r-1) + ar + b = 0$
- $r = \frac{-(a-1) \pm \sqrt{(a-q)^2 - 4b}}{2}$
- **1.** positive discriminant gives two distinct roots, r_1, r_2
- $y(x) = C_1x^{r_1} + C_2x^{r_2}$
- **2.** negative discriminant, two complex roots, $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$
- $y(x) = x^\alpha[C_1x^{i\beta} + C_2x^{-i\beta}] = x^\alpha[C_1e^{i\beta \ln x} + C_2e^{-i\beta \ln x}]$ this gives us $x^\alpha[C_1\cos(\beta \ln x) + C_2\sin(\beta \ln x)]$
- **3.** discriminant is equal to zero, $r_1, r_2 = r$
- $y_1(x) = x^r, y_2(x) = \frac{\partial}{\partial r}y_1(x, r)$
  - $y(x, r) = x^r = e^{r \ln x} = \ln x \cdot x^r$

Power Series Method

- $P(x)y'' + Q(x)y' + R(x)y = 0$
- Typically use numerical methods
- Given a function $f(x)$ we approximate this function with the power series approximation $f(x) \approx \sum_{n=0}^{n} a_nx^n$
- The power series is $f(x) = \sum_{n=0}^{\infty} a_nx^n$
- If $f(x)$ is analytic $f(x) = \sum \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ this is the taylor series, if $x_0 = 0$ then $f(x) = \sum \frac{f^n(0)}{n!}x^n$
- This is only useful if the power series converges so lets look at the convergence

- $f(x) = \sum a_n x^n$ converges if $l = \lim_{N \to \infty} \sum_{n=0}^{N} a_n x^n$
- If $R = 0$ then the series only converges around x_0
- If l is finite then the power series converges in the interval $x_0 - R, x_0 + R$ where R is the radius of convergence, R is finite
- If $R \to \infty$ then the series converges everywhere
- We can prove the convergence with the **ratio test
- The power series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum b_n$ converges if $\lim_{n \to \infty} |\frac{b_{n+1}}{b_n}| = l < 1$

> **Example**
> $f(x) = \sum \frac{1}{n!} x^n$
> $b_n = \frac{x^n}{n!}; b_{n+1} = \frac{x^{n+1}}{(n+1)!}$
> $\lim_{n \to \infty} |\frac{n!}{(n+1)!}| \cdot |x| = 0$
> So then the series converges for all real numbers

> **Example**
>
> $f(x) = \sum \frac{(x-1)^{2n}}{4^n}$
> $\lim_{n \to \infty} |\frac{(x-1)^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{(x)^{2n}}$
> $|\frac{(x-1)^2}{4}| \lim_{n \to \infty} 1 \implies |\frac{(x-1)^2}{4}| < 1$
> Which means that $|x - 1| < 2$ so $R = 2$ centerd at $x_0 = 1$
> So the interval of convergence is $(-1, 3)$

- Now to solve the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ with the power series
- Let $y(x) = \sum a_n (x - x_0)^n, y' = \sum a_n n (x - x_0)^n, y'' = \sum a_n n(n-1)(x - x_0)^n$

> **Example**
> $y' - y = 0$, the solution to this is clearly $y = Ce^x$ but lets use power series
> $y(x) = \sum a_n x^n, y' = \sum a_n n x^{n-1}$
> So then we have $\sum_{n=1} a_n n x^{n-1} - \sum a_n x^n = 0$, let $m = n - 1$
> $\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m - \sum_{m=0}^{\infty} a_m x^m = 0$ Now we can combine these series into one series
> $\sum_{m=0}^{\infty} [a_{m+1}(m+1) - a_m] x^m = 0$, this is only possible if $a_{m+1}(m+1) - a_m = 0$
> $a_{m+1} = \frac{a_m}{m+1}$ for $m = 1, 2, 3, \ldots$, recursive relation
> a_0 is arbitrary
> $m = 0 \implies a_1 = a_0$
> $m = 1 \implies a_2 = \frac{a_1}{2} = \frac{a_0}{2}$
> $m = 2 \implies a_3 = \frac{a_2}{3} = \frac{a_1}{6} = \frac{a_0}{6}$
> $m = n \implies a_{m+1} = \frac{a_0}{(m+1)!}$

So $y(x) = a_0 + a_1 x + \frac{a_2 x}{2} + \dots = a_0 \left(1 + x + \frac{x^2}{2} + \dots\right)$
$y(x) = Ce^x$
Now you can very easily find the radius of convergance

☰ Example

$x^2 y' - y = 0$ we will solve this classically and with the power series
$\frac{dy}{y} = x^{-2} dx$
$\ln y = -\frac{1}{x} + C$
$y = Ae^{-1/x}$
Now lets try the power series method, $y(x) = \sum_{n=0}^{\infty} a_n x^n$
$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$
Now plug this into the equation:
$\sum_{n=1}^{\infty} a_n n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$, let $m = n + 1$
Now we have the following
$\sum_{m=2}^{\infty} a_{m-1}(m-1)x^m - \sum_{n=0}^{\infty} a_n x^n = 0$
$\sum_{m=2}^{\infty} a_{m-1}(m-1)x^m - \sum_{n=2}^{\infty} a_n x^n - a_0 x^0 - a_1 x^1 = 0$
$-a_0 - a_1 x^1 + \sum_{m=2}^{\infty} [a_{m-1}(m-1) - a_m] x^m$
this equation only holds if $a_0 = 0$ and $a_1 = 0$
$a_m = a_{m-1}(m-1), m = 2, 3, 4 \dots$
$m = 2 \implies a_2 = a_1 \cdot_1 = 0$
$a_m = 0 \forall m$
$y(x) = 0$
The difference between the results are caused by $x = 0$ being a singular point

- Power series method is only valid at ordinary points

Definition

Given $P(x)y'' + Q(x)y' + R(x)y = 0$
A point x_0 is an ordinary point of the equation if $p(x) = \frac{Q(x)}{P(x)}$ and $q_1(x) = \frac{R(x)}{P(x)}$\$ are analtic at $x0$
i.e. $p(x) = p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \dots,$
$q_1(x) = q_{1,0} + q_{1,1}(x - x_0) + q_{1,2}(x - x_0)^2$
If a point is an ordinary point then the power series will work!

☰ Example

$y'' - \sqrt{x} y' + xy = 0$
$p(x) = -\sqrt{x}, \quad q_1(x) = x$

$x = 0$ is a singular point as the derivative is not defined there

Example

$x^2y' - y = 0$
$x = 0$ is a singular point as the derrivatve is not defined there, $x = 1$ is an ordinary point, so we get the following
$x^2 \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum a_n (x-1)^n = 0$ now we complete the square for x^2
$x^2 = (x-1)^2 + 2x - 1 = (x-1)^2 + 2(x-1) + 1$
Now we have
$\sum_{n=1} a_n n(x-1)^{n+1} + \sum_{n=1} 2a_n n(x-1)^n + \sum_{n=1} a_n n(x-1)^{n-1} - \sum a_n(x-1)^n =$

- Given the equation $(x-x_0)^2y'' + \alpha(x-x_0)y' + y = 0$
- So we have $p(x) = \frac{1}{x-x_0}$ and $q(x) = \frac{1}{(x-x_0)^2}$
- $P(x)y'' + Q(x)y' + R(x)y = 0$
- $y'' + P(x)y' + q(x)y = 0$
- $(x-x_0)^2y'' + (x-x_0)[(x-x_0)p(x)]y' + (x-x_0)^2q(x)y = 0$

Definition

A point $x = x_0$ is a **regular singular point** if
$\lim_{x \to x_0}(x-x_0)p(x)$ and $\lim_{x \to x_0}(x-x_0)^2q(x)$ are finite

Example

$(1-x^2)y'' - 2xy' + \beta y$
$x_0 = \pm_1$ are singular points, now we must check if they are regular
At $x_0 = 1$, $\lim_{x \to 1} \left((x-1)[-\frac{2x}{1-x^2} \right] = 1$ and $\lim_{x \to 1}(x-1)^2 \left[\frac{\beta}{1-x^2}\right] = 0$

Definition

Forbenius series
If $x = x_0$ is RSP, $y(x) = (x-x_0)^r \sum a_n(x-x_0)^n$
$y(x) = \sum a_n(x-x_0)^{n+r}$

Example

$4xy'' + 2y' + y = 0$

$x = 0$ is a regular singular point

so

$y(x) = \sum a_n x^{n+r}, \quad y' = \sum a_n(n+r)x^{n+r-1}, \quad y'' = \sum a_n(n+r)(n+r-1)(x^{n+r-2})$

Now we can plug these derivatives to get the following

$\sum 4a_n(n+r)(n+r-1)x^{n+r-1} + \sum 2a_n(n+r)x^{n+r-1} + \sum a_n x^{n+r} = 0$

Now we will convert the final term, let $n + r = m + r - 1 \implies m = n + 1$

$\sum[4a_m(m+r)(m+r-1) + 2a_m(m+r)]x^{m+r-1} + \sum_{m=1} a_m x^{m+r-1} = 0$

The third term can become $\sum_{m=1}^{\infty} a_{m-1}x^{m+r-1}$

$[4a_0r(r-1) + 2a_0r]x^{r-1} \sum_{m=1}(2a_m(m+r)[2(m+4-1)+1] + a_{m-1})x^{n+r-1} = 0$

The coefficients of the lowest x power gives you the indicial equation

$a_o[4r(r-1) + 2r + 0] = 0$

$2r(r-1) + r = 0 \implies r_{1,2} = 0, \frac{1}{2}$

From the summation we found earlier we have $a_m = \frac{-a_{m-1}}{2(m+r)[2(m+r)+1]}$

This right here is the recurrence relation

If $r = 0$ then $a_m = -\frac{a_{m-1}}{2m(2m-1)}$

$a_1 = \frac{-a_0}{2 \cdot 1}$

$a_2 = -\frac{a_1}{4 \cdot 3} = \frac{a_0}{4!}$

$y_1(x) = x^r \sum a_n x^n = a_0 + a_1x + a_2x^2 + \ldots$

Now do the same with $r = \frac{1}{2}$

- Given r_1 and r_2
- Case 1: $r_1 - r_2$ is not an integer, the solution is $y(x) = C_1y_1(x) + C_2y_2(x)$
- Case 2: $r_1 - r_2 = 0$ $y_2(x) = \frac{\partial}{\partial r} y_1(x, r) \mid_{r=r_1}$
  - $y_1 = c^r \sum a_n x^n$
  - $x^r = e^{r \ln x}$
  - Differentiate to get $\ln x \cdot e^{r \ln x} = x^r \ln x$
  - $y_2(x) = y_1 \ln x + \sum b_n x^{n+r}$
- Case 3: $r_1 - r_2$ is an integer
  - $y_2(x) = \frac{\partial}{\partial r}((r - r_1)y_1(x))$
  - $y_2(x) = a \ln(x) y_1 + \sum bnx^{n+r}$
- $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ this is a bessel equation used mainly for vibrations
- $p(x) = \frac{1}{x}$
- $q(x) = \frac{x^2 - \alpha^2}{x^2}$
- $p_0 = \lim_{x \to 0} \frac{x}{x} = 1$
- $q_0 = \lim_{x \to 0} \frac{x^2(x^2 - \alpha^2)}{x^2} = -\alpha^2$
- $y(x) = \sum a_n x^{n+r}$
- $r(r-1) + rp_0 + q_0 = 0$
- $r(r-1) + r - \alpha^2 = 0 \implies r = \pm\alpha$

Summary

- Review first and second order odes
- Series solutions of odes
    - Taylor series
    - Forbenius series $\to$ regular singular points
- Bessel functions $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$, $x = 0$ is a regular singular point