Linearization

5.1 Linearization, critical points and equilibrium

- We focus on first order non linear autonomous systems of of the form $\vec{x} = f(\vec{x})$ where f does not explicitly depend on t
- To find critical points we do $f = \vec{0}$
- More on critical points in Two Dimensional systems and their vector fields

Linearization

- Lets consider a two dimensional system
- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$
- Assume that (x_0, y_0) is a critical point, we define the the deviation $u(t), v(t)$ as:
- $x = x_0 + u, y = y_0 + v$ so $u = x - x_0$ and $v = y - y_0$ where x x and y are the solution of the system
- We want to find the ODEs governing the variation of u, v
- Now there is a derivation using the taylor series however this does not concern us

> **Definition**
> The linearization of a system is given by:
> $\begin{bmatrix} u' \\ v' \end{bmatrix} = J_f(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$
> Where J_f is the Jacobian matrix evaluated at (x_0, y_0) and defined by:
> $J_f(x_0, y_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix}$
> The linearization is an approximate solution of $\vec{x} = f(\vec{x})$

Stability of critical points

- A critical point is isolated if it is the only critical point in some sufficiently small open rectangle in 2D
- A system at a critical point is almost linear if the critical point is isolated and the Jacobian matrix at the critical point is invertible i.e. $\det(J_f) \neq 0$

- A critical point is
  - Stable if every real solution that starts with $\vec{x}(t=0)$ sufficiently close to the point remains arbitrarily close to it for all $t>0$
  - Unstable if at least one solution doesn't satisfy the above
  - Asymptotically stable if it is stable and every solution starts sufficiently close has $\lim_{t\to\infty}\vec{x}=(x_0,y_0)$

Examples

- $\begin{cases} x' = \sin(x+y) \\ y' = e^x - 1 \end{cases}$
- The critical points are $x=0$ $y=\pm n\pi$
- Linearization $\begin{bmatrix} \cos\pm\pi & \cos\pm n\pi \\ 1 & 0 \end{bmatrix}$
- if even $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ the eigenvalues are $\lambda_{+,-}=\frac{1}{2}(1\pm\sqrt{5})$ saddle(source) the eigenvectors are $v_-=\begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{bmatrix}$ and $v_+=\begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix}$
- if odd $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ the eigenvalues are $\lambda=\frac{1}{2}(-1\pm i\sqrt{3})$ spiral sink (source)
- Direction $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ counterclockwise
- x nullcline $\sin(x+y)=0 \implies x+y=\pm\pi$
- y nullcline $e^x-1=0 \implies x=0$
- Another example
- Conservative equations, there the energy is a conserved quantity
- $E(x,y)=\frac{1}{2}y^2+F(x)$
- y "momentum abd $y=x'$ solutions are such that $E(x,y)=C$
- Differential equation? $\frac{d}{dt}E=0$ apply the chain rule $F'+y'=0$
- So we get $x''=-f(x)$; $f(x)=F'(x)$ "Newtons equation"
- Equivalently $\begin{cases} x' = y \\ y' = -f(x) \end{cases}$ "Hamiltons Equation"
- Critical points are $y=0$ and $f(x)=0$
- Take the Jacobian $J=\begin{bmatrix} 0 & 1 \\ -f'(x) & 0 \end{bmatrix}$
- Eigenvalues are $\lambda^2+f'(x_0)=0$
- Two possibilities: $f'(x_0)>0$ then $\lambda=\pm i\sqrt{f'(x_0)}$ center; $f'(x_0)\leq_0$ then $\lambda=\pm\sqrt{-f(x_0)}$ saddle
- Another Example!
- $x''+x-x^2=0 \iff \begin{cases} x' = y \\ \\ y' = x^2-x \end{cases}$ $f(x)=x-x^2$ $E(x,y)=\frac{1}{2}y^2+\frac{1}{2}x^2-\frac{1}{3}x^3$
- Critical points are $(0,0)$;$(1,0)$

- The Jacobian $J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x-1 & 0 \end{bmatrix}$
- At $(0,0)$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ eigenvalues are $\lambda = \pm i$ center
- At $(1,0)$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ eigenvalues are $\lambda = \pm 1$ saddle $\vec{v_+} =< 1, 1 >, \vec{v_-} =< 1, -1 >$