Sturm-Liouville

$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 = y(L) \end{cases} \implies \lambda = \left(\frac{n\pi}{L}\right)^2, \quad y(x) = \sin\left(\frac{n\pi}{L}x\right)$

$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = 0 = y'(1) \end{cases} \implies \begin{cases} \lambda = \left(\frac{n\pi}{L}\right)^2 \\ y(x) = \cos\left(\frac{n\pi}{L}x\right) \end{cases}$

Every nice function can be written as a $f(x) = \sum_{n=0}^{\infty} c_n y_n(x)$

$\int_0^L y_n(x) y_m(x)\, dx = 0$ if $m \neq n$

Definition

$\mathcal{L}y = -\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x)$ The SL form

$= -\left(\frac{dP}{dx} \cdot \frac{dy}{dx} + p(x)\frac{d^2y}{dx^2}\right) + q(x)y(x)$

Assume:P is continuous diff, $q(x)$ is continuous and $r(x)$ is continuous

SL problem: $\mathcal{L}y = \lambda r(x)y$

$\alpha_1 y(0) + \alpha_2 y'(0) = 0$

$\beta_1 y(L) + \beta_2 y'(L) = 0$

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

α_1 or $\alpha_2 \neq 0$

β_1 or $\beta_2 \neq 0$

Examples 1

1

$P \equiv 1, q \equiv 0, r \equiv 1$

$\alpha_2 = \beta_2 = 0$

$\begin{cases} \frac{d^2y}{dx^2} + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$

2

$P \equiv 1, q \equiv 0, r \equiv -1$

$\alpha_1 = \beta_1 = 0$

$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = y'(L) = 0 \end{cases}$

Not SL

Periodic boundary conditions are not SL problems
e.g. $y'' + \lambda y = 0$
$y(0) = y(L)$
$y'(0) = y'(L)$

Examples 2

$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0$
$P_0(x) \neq 0$ on $[0, L]$
P_i is continuous
$y'' + \frac{P_1}{P_0}y' + \frac{P_2}{P_0}y + \lambda \frac{R}{P_0}y = 0$
$y'' + a(x)y' = \frac{d}{dx}(p(x)y')$
$\hat{P}(x) = e^{\int \frac{P_1}{P_2}dx}$
$\implies -\frac{d}{dx}(-\hat{P}(x)y') + \hat{P}\frac{P_2}{P_0}y = -\lambda \hat{P}\frac{R}{P_0}y$
$P(x) = -\hat{P}(x)$
$q(x) = \hat{P}\frac{P_2}{P_0}$
$r(x) = -\hat{P}\frac{R}{P_0}$

Example 1

$\phi'' + x\phi' + \lambda\phi = 0$
$e^{\int x dx} = e^{\frac{x^2}{2}}$
$-\frac{d}{dx}(-e^{\frac{x^2}{2}}\phi') = -\lambda e^{\frac{x^2}{2}}\phi$
$P(x) = -e^{\frac{x^2}{2}}$
$q(x) = 0$
$r(x) = e^{\frac{-x^2}{2}}$

Example 2

$-y'' + x^4y' = \lambda y$
$\implies y'' - x^4y = -\lambda y$
$e^{-\int x^4 dx} = e^{\frac{x^5}{5}}$
$-\frac{d}{dx}(-e^{\frac{-x^5}{5}}y') = -\lambda e^{\frac{-x^5}{5}}y$

Regular SL Problem

If $L < \infty, P > 0, r > 0$ on $[0, L]$
then $\mathcal{L}y = \lambda r y$ is a regular SL problem
Otherwise it is a singular SL problem

Properties of regular SL

Eigenvalues

I All eigenvalues are real
II There are infinitely many eigenvalues, can be written as
$\lambda_0 < \lambda_1 < \lambda_2, \dots \to \infty$
III If $\frac{\alpha_1}{\alpha_2} < 0, \frac{\beta_1}{\beta_2} > 0$ and $q_1 > 0$ on $[0, L]$ then $\lambda_0 > 0$

Eigenfunctions

I For each λ, there is a unique (up to rescaling) ϕ_j solution to SL problem
II ϕ can be taken to be real valued, can be normalized to satisfy $\int_0^L \phi_j^2(x) r(x)\, dx = 1$
III If $i \neq j$ then $\int_0^L \phi_j \phi_i r\, dx = 0$, orthogonality
IV ϕ_j has exactly j zeros on $(0, L)$ *In notes says $j - 1$ zeros because of indexing*
V Collection of all ϕ_j is a spanning set i.e. if the function $f : [0, L] \to \mathbb{R}$ is nice enough
then $\exists c_j \in \mathbb{R}$ s.t. $f(x) = \sum_{j=0}^{\infty} c_j \phi(x)$
Using orthogonality we can find out what c_j has to be
$\int_0^L f(x) \phi_k(x) r(x)\, dx = \sum_{j=0}^{\infty} c_j \int_0^L \phi_j \phi_k r\, dx = c_k \int_0^L \phi_k^2 r\, dx$
So $c_k = \frac{\int_0^L f(x)\phi_k(x) r(x)\, dx}{\int_0^L \phi_k^2(x) r(x)\, dx}$

Langrage's Identity

$\int_0^L v\mathcal{L}u - u\mathcal{L}v\, dx = -P(u'v - uv') \mid_0^L$
$\int_0^L v\mathcal{L}u\, dx = \int_0^L v\left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu\right] dx$
$= -\int_0^L v\frac{d}{dx}\left(p\frac{du}{dx}\right) d + \int_0^L quv\, dx$
$= -[vpu']_0^L \int_0^L \frac{dv}{dx} p \frac{du}{dx}\, dx + \dots$
$= -vpu' \mid_0^L + p\frac{dv}{dx} \cdot u \mid_0^L$
AAAAAAAAAAAAAA Im not writing this all down, look it up online
So if u, v satisfies the boundary conditions
$\int_0^L u\mathcal{L}v\, dx = \int_0^L v\mathcal{L}u\, dx$
A linear operator that satisfies this identity is called symmetric

Real examples

Example 1

Consider $(xy')' + \frac{2}{x}y = -\lambda\frac{1}{x}y$

$y'(1) = y'(2) = 0$

First we expand the first term $xy'' + xy' + \frac{2}{x}y = -\lambda\frac{1}{x}y$

$\implies x^2y'' + xy' + (\lambda + 2)y = 0$

$y = x^r$

$(r(r-1) + r + (\lambda + 2))y = 0$

$r = \pm\sqrt{-(\lambda + 2)}$

Case I $\lambda + 2 < 0, y(x) = A^{\sqrt{-(\lambda+2)}} + B^{-\sqrt{-(\lambda+2)}}$

Case II $\lambda + 2 = 0, y(x) = A + B\ln(x)$

$y'(1) = 0 = \frac{B}{x} \implies B = 0$

$y = A$

Case III $\lambda + 2 > 0\ y(x) = A\cos(\sqrt{\lambda+2}\ln x) + B\sin(\sqrt{\lambda+2}\ln x)$

$y' = -\frac{A}{x}\sqrt{\lambda+2}\sin(\sqrt{\lambda+2}\ln x) + \frac{B}{x}\sqrt{\lambda+2}\cos(\sqrt{\lambda+2}\ln x)$

$y'(1) = B\sqrt{\lambda+2} = 0 \implies B = 0$

$y'(2) = -\frac{A\sqrt{\lambda+2}}{2}\sin(\sqrt{\lambda+2}\ln 2)$

$\sqrt{\lambda+2} = \frac{n\pi}{\ln 2}$

$\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2, \quad n = 0, 1, 2, \ldots$

$y_n = \cos\left(\frac{n\pi}{\ln 2}\ln x\right)$

Recall that $r(x) = \frac{1}{x}$

$\int_1^2 \cos\left(\frac{n\pi}{\ln 2}\ln x\right)\cos\left(\frac{m\pi}{\ln 2}\ln x\right) \cdot \frac{1}{x}\,dx$

Let $z = \frac{\pi}{\ln 2}\ln x$

$dz = \frac{\pi}{\ln 2} \cdot \frac{dx}{x}$

So then we have $\frac{\ln 2}{\pi} / \int_0^\pi \cos(nz)\cos(mz)\,dz$

$= \frac{\ln 2}{\pi} \cdot \frac{\pi}{2}\delta_{mn} = \frac{\ln 2}{2}\delta_{mn}$

$\begin{cases} \delta_{mn} = 1 \text{ if } n = m \\ \delta_{mn} = 0 \text{ if} n \neq m \end{cases}$

$f(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi}{\ln 2}\ln x\right)$

$\int_1^2 f(x)\cos\left(\frac{n\pi}{\ln 2}\ln x\right) \cdot \frac{1}{x}\,dx$

$= \sum_{n=0}^{\infty} c_n \int_1^2 \cos\left(\frac{n\pi}{\ln 2}\ln x\right)\cos\left(\frac{m\pi}{\ln 2}\ln x\right)\frac{dx}{x}$

$= \frac{c_k(\ln 2)}{2}$

$c_k = \frac{2}{\ln 2}\int_1^2 f(x)\cos\left(\frac{k\pi}{\ln 2}\ln x\right)\frac{dx}{x}$