LU Decomposition

Some Matrix Definitions

DEFINITION

Let A be a matrix and let U be the matrix obtained in row echelon form
the **rank** of A is given by the number of non-zero rows in U. We denote the rank of a matrix by rank(A)

DEFINITION

Let A be an $n \times m$ matrix with rank$(A)$$= r$
The system $Ax = b$ is **inconsistent** (i.e. no solution) if rank(A)<rank$(A|b)$ because that means there exists at least one row with $0 \quad 0 \quad 0 \quad \ldots \quad 0 \mid 1$ which implies that $0 = 1$
The system has a **unique solution** when rank(A) =rank$(A \mid b) = n$, in other words the system is consistent and the rank of A is equal to the number of variables in the system
The system has **infinitely many solutions** when the system is consistent but rank$(A) < n$

What is LU decomposition?

DEFINITION

A unit lower triangular matrix is a *square* matrix with ones on the diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

An upper triangle matrix is the same but in the top right section

THEOREM

Let E be the $m \times m$ matrix with ones along the diagonal, c in the entry row at i and column j with $i > j$ and all other entries are zeros

$$E = \begin{bmatrix} 1 & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & c & & \cdots & \\ & & & & 1 \end{bmatrix}$$

Then for any $m \times n$ matrix A, matrix multiplication EA applies to A the elementary row row operation: add c times row j to i

Furthermore, the inverse of E is given by

$$E^{-1} = \begin{bmatrix} 1 & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & -c & & \cdots & \\ & & & & 1 \end{bmatrix}$$

Where $-c$ is the entry at row i and column j.

EXAMPLE

Consider the following matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & -2 \\ -1 & 1 & 1 & 1 \\ -1 & 2 & 3 & 1 \\ 1 & -1 & 2 & 1 \end{bmatrix}$$

The elementary matrix which adds -1 times row 1 to row 4 is the following

Perform matrix multiplication to verify

$$EA = \begin{bmatrix} 1 & -1 & 1 & -2 \\ -1 & 1 & 1 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

THEOREM

If A can be reduced by Gaussian elimination to row echelon form *only* with operations "add c times row j to row i" (in other words, without scaling rows and without interchanging rows), then A has an **LU decomposition** of the form

$$A = LU$$

Where L is a unit lower triangular matrix. In particular, after performing Gaussian elimination on A, the matrix U is the corresponding row echelon form of A and L is given by

$$L = \begin{bmatrix} 1 & & & & \\ -c_{2,1} & 1 & & & \\ -c_{3,1} & -c_{3,2} & 1 & & \\ \dots & \dots & \dots & \dots & \\ -c_{m,1} & -c_{m,2} & \dots & -c_{m,m-1} & 1 \end{bmatrix}$$

where each entry corresponds to the elementary row operation add "$c_{i,j}$ times row j to row i" performed during Gaussian elemintation

EXAMPLE

Compute the **LU** decomposition of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 3 & 4 \end{bmatrix}$$

Add -1 times row 1 to row 2 and add -2 times row 1 to row 3

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Add row 2 to row 3 to get the following

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

In terms of elementary matrices we have just shown that

$$E_{3,2}E_{3,1}E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

We can rearrange the system to get $A = E_{2,1}^{-1}E_{3,1}^{-1}E_{3,2}^{-1}U$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

We can combine the matrices as in the proof of the **LU** decomposition to find $A = LU$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 2 \\ 4 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that we can construct the matrix L directly from the list of operations

1. Add -1 times row 1 to row 2
2. Add -2 times row 1 to row 3
3. Add 1 times row 2 to row 3

THEOREM

Suppose A has an **LU** decomposition $A = LU$

1. $\text{rank}(A) = \text{rank}(U)$
2. det(A) =det$(U) = u_{1,1} \dots u_{m,m}$ which are the diagonal entries of U
   Note not all matrices have na **LU** decomposition. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   does not have an **LU** decomposition, why? However if we allow partial pivoting (ie. interchanging rows during Gaussian elimination), then Gaussian elimination with partial pivoting computes for *any* matrix A a decomposition $A = PLU$ where P is a permutation matrix L is a unit lower triangle and U is an upper triangle.
   This is called the **LU decomposition with partial pivoting** and has similar computational advantages as the LU decomposition

Forward and Backward Substitution

DEFINITION

Let $A = LU$ be the LU decomposition of A, let $l_{i,j}$ denote the entries of L and let $u_{i,j}$ denote the entries of U.
Consider the system $Ax = b$ and let $y = Ux$

Forward Substitution is the process of solving the lower triangular system $Ly = b$ from top to bottom:

$$y_1 = b_1$$

$$y_2 = b_2 - l_{2,1} y_1$$

$$\vdots$$

$$y_n = b_n - l_{n,1}y_1 - \cdots - l_{n,n}y_{n-1}$$

Backward Substitution is the process of solving the upper triangular system $Ux = y$ from bottom to top

$$x_n = \frac{y_n}{u_{n,n}}$$

$$x_{n-1} = \frac{y_{n-1} - u_{n-1,n}x_n}{u_{n-1}, n-1}$$

$$\vdots$$

$$x_1 = \frac{y_1 - u_{1,2}x_2 - \cdots - u_{1,n}x_n}{u_{1,1}}$$

EXAMPLE

Solve the system $Ax = b$ where

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Solve $Ly = b$
$y_1 = -1$
$y_2 = 1 - 2(-1) = 3$
$y_3 = 2 - (-1) - 3 =$
Solve $Ux = y$
$x_3 = 0$
$x_2 = 3$
$x_1 = \dfrac{-1 - 4(3) - 0}{2} = -\dfrac{13}{2}$
Therefore $x = \begin{bmatrix} -\frac{13}{2} \\ 3 \\ 0 \end{bmatrix}$
Note. The LU decomposition is especially useful when solving many different systems with the *same* coefficient matrix A. For example to compute the inverse A^{-1} of a square matrix of size n we need to solve n different systems $Ax_k = e_k$ for $k = 1, \ldots, n$ where e_k is the kth column of the identity matrix I. The result is $A^{-1} = [x_1, \ldots, x_n]$ In other words the columns of A^{-1} are given by $x_1, \ldots, x_n$.