Non-Homogeneous PDEs

Time independent non-homogeneous BC

Dirichlet BC

- $u_t = \alpha^2_{xx}$
- $u(0,t) = u_0, \quad u(L,t) - u_1$
- $u(x,0) = f(x)$
- $u(x,t) = v(x,t) + \bar{u}(x,t)$

Steady state problem

- Solution is independent of time
- $u_t = \alpha^2 u_{xx}$
- $u(0,t) = u_0, \quad u(L,t) = u_1$
- $u(x,0) = f(x)$
- **Step 1** find the steady state solution
- $u_t = u_{xx} = 0 \implies u_\infty(x) = Ax + B$, the solution in the long run
- $u_\infty(0) = B = u_0 \quad u_\infty(L) = AL + u_0 = u_1$
- $u_\infty(x) = \frac{u_1 - u_0}{L} x + u_0$
- $u(x,t) = v(x,t) + u_\infty(x)$
- $u_t = v_t$
- $u_{xx} = v_{xx} + 0 \implies v_t = v_{xx} \implies \text{PDE}$

Boundary and initial conditions

- $u(0,t) = v(0,t) + u_\infty(0)$
- $u_0 = v(0,t) + u_0 \implies v(0,t) = 0$
- $u(L,t) = v(L,t) + u_\infty(L)$
- $u_1 = v(L,t) = u_1 \implies v(L,t) = 0$
- $u(x,0) = v(x,0) + u_\infty(x)$
- $v(x,0) = f(x) = u_\infty(x)$

Solve the PDE

- As done in separation of variables
- $v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-a^2 (\frac{n\pi}{L})^2 t}$

- $b_n = \frac{1}{L}\int_{-L}^{L}[f(x) - u_\infty(x)]\sin\left(\frac{n\pi x}{L}\right)$
- $u(x,t) = u_\infty(x) + v(x,t)$

Types of Particular solutions

Dirichlet

- $u_\infty(x) = Ax + B$

Mixed type 1/2

- $u_\infty(x) + Ax + B$

Neumann

- $u_p(x,t) = Ax^2 + Bx + Ct$ particular solution
- $u_t = u_{xx}$
- $u_x(0,t) = q_0, \quad u_x(L,t) = q_1$
- Assume we have a steady state problem
- $0 = u_{xx} \implies u_\infty(x) = Ax + B$
- $u'_\infty(x) = A$
- $u'_\infty(0) = q_0 = A$
- $u'_\infty(L) = q_1 = A$
- This is only possible if $q_0 = q_1 = A$
- Since Neumann condition does not allow steady state unless $q_0 = q_1$ let $u_p(x,t) = Ax^2 + Bx + Ct$
- $\frac{\partial u_p}{\partial t} = c$
- $\frac{\partial^2 u_p}{\partial x^2} = 2A$
- So then we must have that $C = 2A$
- $u_{px}(x,t) = 2Ax + B$
- $u_{px}(0,t) = B = q_0$
- $u_{px}(L,t) = 2AL + B = q_1 \implies A = \frac{q_1 - q_0}{2L}$
- So now we can write $u_p(x,t) = \frac{q_1 - q_0}{2L}x^2 + q_0 x + \frac{q_1 - q_0}{L}t$

Eigen Function Expansion

- Given $u(x,t) = v(x,t) + u_p(x,t)$
- Let $v(x,t) = \sum_{n=0}^{\infty} V_n(t) b_n \sin\left(\frac{n\pi x}{L}\right)$
- $F(x,t) = \sum_{n=0}^{\infty} S_n(t)\sin\left(\frac{n\pi x}{L}\right)$ this is the source term
- Our aim is to get the coefficients

- $v_t = \sum_{n=0}^{\infty} V_n'(t) \sin\left(\frac{n\pi x}{L}\right)$
- $v_{xx} = \sum_{n=0}^{\infty} -\left(\frac{n\pi}{L}\right)^2 V_n(t) \sin\left(\frac{n\pi x}{L}\right)$
- Remember the solution is $v_t = v_{xx} + F(x,t) \implies v_t - v_{xx} - F(x,t) = 0$
- So then we have the following
- $\sum_{n=0}^{\infty} \left[V_n' + \left(\frac{n\pi}{L}\right)^2 V_n - S_n(t)\right] \sin\left(\frac{n\pi x}{L}\right) = 0$
- $S_n(t)$ is known
- $V_n'(t) + \left(\frac{n\pi}{L}\right)^2 V_n(t) = S_n(t)$ which is a first order ode

Examples

Example 1

- $u_t = u_{xx}$
- $u_x(0,t) = \ln t \quad u_x(1,t) = t^2$
- $u(x,0) = f(x)$
- **Find a particular solution**
- $u_p(x,t) = \frac{A(t)}{2}x^2 + B(t)x + C(t)t$
- $\frac{\partial u_p}{\partial x} = A(t)x + B(t)$
- $u_x(0,t) = \ln t = u_{px} = B(t)$
- $u_x(1,t) = t^2 = A(t) + \ln t$
- $A(t) = t^2 - \ln t$
- $u_p(x,t) = \frac{t^2 - \ln t}{2}x^2 + \ln(t)x$ we are ignoring $C(t)$ since this matches the boundary conditions
- $u(x,t) = v(x,t) = u_p(x,t)$
- $v_t + \frac{(2t - \frac{1}{t})}{2}c^2 + \frac{x}{t} = v_{xx} + t^2 - \ln t$
- $v_t = v_{xx} + t^2 - \ln t - \frac{x}{t} - \frac{2t - \frac{1}{t}}{2}x^2$
- $v_x(0,t) = v_x(1,t) = 0$
- $v(x,0) = f(x) - u_\infty(x,0)$
- This doesn't work for $\ln t$ lmao we need to change it to something like $\ln(t+1)$

Example 2

- $u_t = 16u_{xx} + \cos\left(\frac{7\pi x}{2}\right) \quad 0 < x < 2$
- $u_x(0,t) = 1 = u_x(2,t)$
- $u(x,0) = x^2 - 4$
- $u(x,t) = v(x,t) + u_p(x,t)$
- $u_p(x,t) = \frac{A(t)}{2}x^2 + B(t)x + C(t)$
- $\frac{\partial u_p}{\partial x} = A(t)x + B(t)$
- $u_{p_x}(0,t) = B = 1$

- $u_{p_x}(2,t) = 1 = 2A + 1 \implies A = 0$
- $u_p(x,t) = x$
- $u(x,t) = v(x,t) + x$, now plug these into the pde
- $v_t = 16v_{xx} + \cos\left(\frac{7\pi x}{2}\right)$
- $v_x(0,t) = v_x(2,t) = 0$
- $u(x,0) = v(x,0) + u_p(x,0) \implies v(x,0) = x^2 - 4 - x$
- Let $v(x,t) = \frac{V_0}{2} + \sum_{n=1}^{\infty} V_n \cos\left(\frac{n\pi x}{2}\right)$
- For a neumann $\lambda_n \in \left\{0, \left(\frac{n\pi}{L}\right)^2\right.$
- $\cos\left(\frac{7n\pi}{2}\right) = \frac{S_0}{2} + \sum S_n(t) \cos\left(\frac{n\pi x}{2}\right)$
- $S_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{2}\right)$
- $S_n = \delta_{n7}$
- $v_t = \frac{V_0}{2} + \sum_{n=1}^{\infty} V_n' \cos\left(\frac{7\pi x}{2}\right)$
- $v_{xx} = \sum_{n=1}^{\infty} -\left(\frac{n\pi}{2}\right)^2 V_n \cos\left(\frac{n\pi x}{2}\right)$
- $v_t - 16v_{xx} - \cos\left(\frac{7\pi x}{2}\right) = 0$
- $\left[\frac{V_{t_0}}{2} - 16[0] - 0\right] + \sum_{n=1}^{\infty}\left[V_n' + 16\left(\frac{n\pi}{2}\right)^2 V_n = \delta_{n_7}\right]\cos\left(\frac{n\pi x}{2}\right)$
- $\frac{\frac{\partial V_0}{\partial t}}{2} = 0\,(1)$
- $V_n + \left(\frac{4n\pi}{2}\right)^2 V_n - \delta_{n_7} = 0\,(2)$
- From (1) we have that $V_0 = C_0$
- From (2) we have $V_{x_n} + (2\pi n)^2 V_n = \delta_{n_7}$
- Here we must use integrating factor, $r = e^{\int (2n\pi)^2 dt} = e^{(2n\pi)^2 t}$
- $V_n = e^{-(2n\pi)^2 t}\left[\int_0^t e^{(2n\pi)^2 t}\delta_{n7} dz + C\right]$
- $V_n = e^{-(2n\pi)^2 t}\left[\frac{1}{(2n\pi)^2} e^{(2n\pi)^2 t}\delta_{n7} + C\right]$
- $V_n(t) = \frac{1}{(2n\pi)^2} S_{n7} + Ce^{-(2n\pi)^2 t}$
- $v(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{(2n\pi)^2}\delta_{n7} + C_n e^{-(2n\pi)^2 t}\right]\cos\left(\frac{n\pi x}{2}\right)$
- $v(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left[\frac{\delta_{n7}}{(2n\pi)^2} + C_n\right]\cos\left(\frac{n\pi x}{2}\right)$
- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad L = 2$
- $c_n = a_n - \frac{\delta_{n7}}{(2n\pi)^2}$

Example 3

- $u_t = u_{xx} + e^{-t}\sin(x), \quad 0 < x < \frac{\pi}{2}$
- $u_0(0,t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = e^{-t}$
- $u(x,0) = x$
- We have mixed boundary conditions
- $\lambda_n - \mu_n^2 \implies \mu_n = 2n + 1$
- $X_n(x) = \sin((2n+1)x)$
- $u_p(x,t) = A(t)x + B(t)$
- $u_p(0,t) = B(t) = 0$

- $u_{p_x}(x,t) = A(t) \implies A(t) = e^{-t}$
- $v_t = e^{-t}x = v_{xx} + 0 + e^{-t}\sin(x)$
- $v(t) = v_{xx} + e^{-t}[x + \sin x]$
- $u(x,0) = v(x,0) + u_p(x,0)$
- $x = v(x,0) + x \implies v(x,0) = 0$
- $v(x,t) = \sum_{n=0}^{\infty} V_n(t) + \sin((2n+1)x)$
- $e^{-t}[x + \sin x] = \sum_{n=0}^{\infty} S_n(t)\sin((2n+1)x)$
- So we know that $S_n(t) = \frac{2}{\frac{\pi}{2}}\int_0^{\pi/2}[x + \sin x]\sin((2n+1)x)dx$
- $\int_0^{\pi/2} x\sin[(2n+1)x]\,dx$
- $dv = \sin[(2n+1)x]$
- $v = -\frac{1}{2n+1}\cos[(2n+1)x]$
- $u = x, \quad du = 1$
  $\int_0^{\pi/2} x\sin[(2n+1)x]\,dx = -\frac{x}{2n+1}\cos((2n+1)x)\,|_0^{\frac{\pi}{2}} + \frac{1}{2n+1} + \frac{1}{2n+1}\int_0^{\pi/2}\cos((2n+1)x)\,dx$
- This gives us $0 + \frac{(-1)^n}{(2n+1)^2} + \frac{\pi}{4}$
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