Practice final 1

1 Carefully define or restate each statement

a) A rational number $q \in \mathbb{Q}$

I don't know the actual answer lmao

Solution: $q \in \mathbb{Q}$ if there exists co-prime $a, b \in \mathbb{Z}$ where $b \neq =$ such that $q = \frac{a}{b}$

b) Bezouts lemma

For some $a, b \in \mathbb{Z}, \quad \exists x, y \in \mathbb{Z}$ s.t. $ax + by = \gcd(a, b)$

c) The Fundamental Theorem of Arithmetic

IDK

Solution: Let $n \in \mathbb{N}$. Then n can be uniquely factorized into a product of prime powers $p_1^{e_1} p_2^{e_2} \dots p_n e^{e_n}$ up to order where p_i are distinct primes and $e_i \in \mathbb{Z}$

d) A convergent sequence $(x_n)_{n \in \mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

IDK!!!!! confusing ass questions

Solution: $(x_n)_{n \in \mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0 \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \varepsilon$.

e)The principle of mathematical induction

Given a base case, assume that the statement holds for some value k, show that that implies that it also holds for $k + 1$

Solution: Let $l \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq l\}$. If $P(l)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$

2 Write the negation of each the following and prove or disprove the original

statement

a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$

Take the negation $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ s.t. $(x + y < z) \wedge (x - y \leq z)$
Let $x = 0$, then let $z = |y| + 1$, then $x + y = y < |y| + 1$ and $x - y = -y \leq |y| < |y| + 1$ showing that the negation is true, thus the original statement is false.

b) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$

Take the negation, $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t. $\exists z \in \mathbb{R}$ s.t. $xy \leq z$
Let $y = 0$ and let $z = 1$, then $xy = 0 < 1 = z$ so $xy \leq z$ holds and the original statement is false

3 Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

a) $\forall U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

Solution: Assume the original statement, we show each inclusion in turn

- Assume $x \notin f^{-1}(g^{-1}(U))$, so $f(x) \notin g^{-1}(U)$ and $g(f(x)) \notin U$. It follows that $x \notin (g \circ f)^{-1}(U)$, so by contrapositive, $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$
- Assume $x \in f^{-1}(g^{-1}(U))$. Then $f(x) \in g^{-1}(U)$ and $g(f(x)) \in U$ so $g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$