Functions

Definition

For non empty sets A and B, a **function** f from A to B written $f : A \to B$ is a subset of $A \times B$ with two further properties

- for every a in A there is some b in B such that $(a, b) \in f$
- If $(a, b) \in f$ and $(a, c) \in f$ then $b = c$
  In other words, any input has exactly one output

- A is the domain and B is the co-domain
- The range of f is the set of elements in B that are mapped to by f: range$(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$
- *Image* and *pre-image*, Let $f : A \to B$ be a function, and let $C \subseteq A$ and let $D \subseteq B$
  - The set $f(C) = \{f(x) \mid x \in C\}$ is the **image** of C in B
  - The set $f^{-1}(D) = \{x \in A \mid f(x) \in D\}$ is the **preimage** of D in A
  - f^{-1} does not necessarily represent the inverse function

Example
$A = \{a, b, c, d, e\}, B = \{1, 2, 3, 4, 5\}$
$a \to 1, b \to 1, c \to 4, d, e \to 5$
Image of $\{a, b, c\} : f(\{a, b, c\}) = \{1, 4\}$
Image of $(a, b) = \{1\}$
Pre-Image of $\{4, 5\} : f^{-1}(\{4, 5\}) = \{c, d, e\}$
Pre-Image of $\{2, 3\} : f^{-1}(\{2, 3\}) = \varnothing$
Pre-Image of $\{1, 2\} : f^{-1}(\{1, 2\}) = \{a, b\}$

Problem 1

Suppose that $f : A \to B$ is a function and let C be a subset of A. Prove that $f(A) - f(C) \subseteq f(A - C)$
Let $y \in f(A) - f(C)$. We want to show $y \in f(A - C)$ Lowkey don't know how to do this
So $y \in f(A)$ and $y \notin f(C)$
Since $y \in f(A)$ then $\exists x \in A$ s.t. $f(x) = y$
Since $y \notin f(C)$, there does not exist $z \in C$ s.t. $f(z) = y$
We know $x \in A$ and $x \notin C$ given the line above, So $x \in A - C$

Therefore $f(x) \in f(A - C)$

Since $f(x) \in y$, then $y \in f(A - C)$

Problem 2

Suppose that $f : A \to B$ is a function and let C be a subset of A.

Prove that $f(A) - f(C) \supseteq f(A - C)$ does not always hold

Let $A = \{1, 2, 3\}, B = \{4, 5, 6\}, C = \{3\}$

Define f to be such that $f(1) = 5, f(2) = f(3) = 6$

Then $f(A) = \{5, 6\}, f(C) = \{6\}$ and $f(A) - f(C) = \{5\} \not\supseteq f(A - C) = 5, 6$

- Injective $\Rightarrow$ no overlap, everything maps to a distinct value, nothing in the range has more than one corresponding domain value, $f(a_1) = f(a_{2)} \Rightarrow a_1 = a_2$ and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
- Surjective $\Rightarrow$ every point in B is mapped to by f, $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$

Example

(a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective

(b) Surjective but not injective

(c) What would happen if we replaced $\mathbb{Z}$ with a finite set in the questions above

- (a) $f(x) = 5x$, ✓
- (b) $g(x) = x^3 - 4x$, ✓
- (c) No clue

Answer

(a) Define $f(n) = 2n, \forall n \in \mathbb{Z}$

<u>Prove that it is injective</u>: Assume $f(a_1) = f(a_2)$ for some $a_1, a_2 \in \mathbb{Z}$. Then $2a_1 = 2a_2 \Rightarrow a_1 = a_2$ as required so f is injective

<u>Prove that it is not surjective</u>: Consider $1 \in \mathbb{Z}$. If $1 \in \text{ range}(f) \ldots$

(b) $g(m) \begin{cases} m, m \leq 0 \\ m - 1, m > 0 \end{cases}$

<u>Prove that it is surjective</u>: Given $y \in \mathbb{Z}$, we will show that $\exists x \in \mathbb{Z}$ s.t. $g(x) = y$

Case 1: $y \leq 0$: Let $x = y$, then $g(x) = g(y) = y$✓

Case 2: $y > 0$: Let $x = y + 1$, then $g(x) = g(y + 1)$✓

<u>Prove that it is not injective</u>: Note that $g(0) = 0$ and $g(1) = 0$

(c) Not possible if domain $=$ co-domain

- Bijections: If f is both surjective and injective then f is bijective

Example

(a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is bijective

(b) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$ is not bijective

- (a) x^3
  - **Injective**: Given $a, b \in \mathbb{R}$ s.t. $f(a) = f(b)$, we have $a^3 = b^3 \Rightarrow a = b$ so f is injective
    - **Surjective**: Given $y \in \mathbb{R}$, let $x = \sqrt[3]{y} = y$, so f is surjective
- (b) x^4
  - **Injective**: $f(-1) = f(1) = 1$ so $\exists a, b \in \mathbb{R}$ s.t. $f(a) = f(b) \land a \neq b$ so f is not injective

Problem 3

Prove that the function $f : \mathbb{R} - 1 \to \mathbb{R} - \{2\}$ given by $f(x) = \frac{2x}{x-1}$ is bijective

Injective: Assume $f(a) = f(b)$ for some $a, b \in \mathbb{R} - \{1\}$

$\Rightarrow \frac{2a}{a-1} = \frac{2b}{b-1}$

$\Rightarrow 2a(b-1) = 2b(a-1)$

$\Rightarrow ab - a = ab - b$

$\Rightarrow a = b$ so f is injective

Surjective: Given $y \in \mathbb{R} - \{2\}$ let $x = \frac{y}{y-2}$

$\Rightarrow x - 1 = \frac{2}{y-2}$

$\Rightarrow \frac{1}{x-1} = \frac{y-2}{2}$ We can divide by $y - 2$ as $y \neq 2$

$\Rightarrow \frac{y}{2} = 1 + \frac{1}{x-1}$

$\Rightarrow \frac{y}{2} = \frac{x}{x-1}$

$\Rightarrow y = \frac{2x}{x-1} = f(x)$, so f is surjective

Since f is injective and surjective, f is bijective

Scratch for problem 3

We want x s.t. $y = \frac{2x}{x-1} \Rightarrow \frac{y}{2} = \frac{x}{x-1}$ Now we do something funny $\frac{y}{2} = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$ $\frac{y-2}{2} = \frac{1}{x-1} \Rightarrow x - 1 = \frac{2}{y-2}$ so we have $x = \frac{y}{y-2}$

Problem 4

(a) Let $f : A \to B$ be a surjection and let $D_1, D_2 \subseteq B$. Show that if $f^{-1}(D_1) \subseteq f^{-1}(D2)$ then $D_1 \subseteq D_2$.

Let $y \in D_1$. Consider $f^{-1}(\{y\})$. Since $\{y\} \subseteq D_1$, we have $f^{-1}(\{y\}) \subseteq f^{-1}(D_1)$, and since $f^{-1}(D_1) = f^{-1}(D_2)$, we have $f^{-1}(\{y\}) \subseteq f^{-1}(D_2)$, let $x \in f^{-1}(\{y\})$. Then $f(x) = y$, since f is a surjection, $f^{-1}(\{y\}) \neq \{\}$ $x \in f^{-1}(D_2)$, so let $f(x) \in D_2$. So $y \in D_2$. So $D_1 \subseteq D_2$

(b), posted online

Definition

Compositions

Let $f : A \to B$ and $g : B \to C$. The **composition** of f and g is
$g \circ f : A \to C$
where $(g \circ f)(a) = g(f(a)), \forall a, \in A$

- **Order Matters!!**, $f(x) = x^3$ & $g(x) = 2x$, $g \circ f = g(f(x)) = 2x^3$ and $f \circ g = 8x^3$

Inverses

Definition

Left Inverse

Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$, then $f : A \to B$ gives $f(a) = 1$ and $f(b) = 2$
We can go in the reverse direction with $g : B \to A$, $g(1) = a$ and $g(2) = b$
What we are looking for is $g \circ f(x) = x$ But 3 doesn't map to anything so g is not a function right now, lets let $g(3) = b$
Does g act as an inverse?
$g \circ f(a) = g(1) = a$
$g \circ f(b) = g(2) = b$
So we have that $g \circ f$ has the domain A and co-domain A
So $g \circ f : A \to A$, $g \circ f$ is the identity function on A
So g is a left inverse of f as we can put it on the left to be an inverse

- We say an inverse because there can be more than one inverse
- Another left inverse of f, $h : B \to A$, $h(1) = a$, $h(2) = b$ and $h(3) = a$ so $h \neq g$ but they
- are both left inverses of f

Definition
Right Inverse
$S = \{c, d, e\}, T = \{4, 5\}$
$f : S \to T$, $f(c) = 4$, $f(d) = 5$, $f(e) = 5$
This f does **not** have a left inverse! Because f is not injective
Let $g : T \to S, g(4) = c, g(5) = d$
So we have $f \circ g(4) = 4$ and $f \circ g(5) = 5$
So g is a right inverse of f

- Once again there is another right inverse, $h : T \to S$ with $h(4) = c$ and $h(5) = e$
- $g \neq h$ but they are still both right inverses of f
- Formally we have i_A be the identity function on A and i_B on B
- let $f : A \to B$ and $g : B \to A$
- if $g \circ f = i_A$ then g is a left inverse of f
- if $f \circ g = i_B$ then g is a right inverse of f
- if g is both the right and left inverse of f then g is the inverse of f
- A function only has an if and only if it is bijective
- A function has a left inverse iff it is injective
- A function has a right inverse iff it is surjective

> **Problem 5**
> Let $f : A \to B$ be a function. Prove:
> (a) If there is a function $g : B \to A$ that is a left inverse of $f, \forall x \in A$
> Then f is injective
> (b) Prove that if f is injective then there is a function $g : B \to A$ such that
> $g \circ f(x) = x, \forall x \in A$

- (a)
  - Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$. We want to show $x_1 = x_2$
  - Apply g : $g(f(x_1)) = g(f(x_2))$
  - Since $g \circ f(x) = x : x_1 = x_2$
  - So f is injective
- (b)
  - We can construct a function $g : B \to A$
  - Given $b \in B$
  - **Case 1**: $b \in \text{range}(f)$
    - Prove is in notes 20 on pages
  - **Case 2**: $b \in$ range(f)
    - Fix $z \in A$ independant of b
    - Define $g(b) = z$
  - This defines our function g
  - Consider $g \circ f(x) = g(f(x))$ for any $x \in A$
  - Since $f(x) \in \text{range}(f)$, by definition of g, we have $g(f(x)) = x, \forall x \in A$.
  - So g is a left inverse for f

Practice

- Find a function has a right inverse but no left inverse
- Let $f(x) = x^2$, $f : \mathbb{R} \to [0, \infty)$ so f is surjective on its domain
- Therefore It has a right inverse

- Let $g(x) = \sqrt{x}, g : [0, \infty) \to \mathbb{R}$
- Then $f \circ g(x) = f(g(x)) = f(\sqrt{x}) = x$. So g is a right inverse of f
- f does not have a left inverse : Let's claim that $h : [0, \infty) \to \mathbb{R}$ is a left inverse of f
- Then we must have $h \circ f(x) = x, \forall x \in \mathbb{R}$
- But we must have $h \circ f(-1) = h(1) = -1$ and $h \circ f(1) = h(1) = 1$ which doesn't make sense and is not possible
- So h doesn't exist