The grand reveal

- Questions on the "things to integrate" line: PDEs
  1. We know $\text{curl}(\text{grad} f) = \vec{0}$ but if $\text{curl}\vec{F} = 0$ does there exist f such that $\vec{\nabla} f = \vec{F}$
  2. We know $\text{div}(\text{curl}\vec{G}) = \vec{0}$ but if $\text{div}\vec{F} = \vec{0}$ does there exist $\vec{G}$ such that $\vec{\nabla} \times \vec{G} = \vec{F}$
- Answers to 1 and 2 are equivalent to the answers to the analogous questions 1 and 2
- Questions on the "places to integrate" line: Topology
  1. We know that $\partial(\partial S) = \varnothing$ but if $\partial C = \varnothing$ does there exist S with $\partial S = C$?
  2. We know that $\partial(\partial E) = \varnothing$ but if $\partial S = \varnothing$ does there exist E with $\partial E = S$?
- All of this begs the question: is there some unifying principal? YES!! TO see it requires us to translate this all into a different and better language. Differential Forms

Differential Forms

- Let $x_i, \dots x_n$ be coordinate on $\mathbb{R}^n$. Introduce symbols $dx_1, \dots, dx_n$ and the "wedge product" a satisfying the one rule $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Let $U \subset \mathbb{R}^n$ be an open set
- A differential k-form on U is a formal linear combination of k-fold wedge products of the dx_i's where the coefficients are functions on U
- For example let $U \subset \mathbb{R}^3$ and let x, y, z be the usual coordinates on $\mathbb{R}^3$. Then for example 1-forms are given by expressions of the form $Pdx + Qdy + Rdz$ where P, Q, R are functions on U.
- Since our rules tell us $dx \wedge dy = -dy \wedge dx$, etc we find all 2-forms can be written in the form $Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ note the rule forces $dx \wedge dx = 0, dy \wedge dy = 0, \dots$
- Similarly, since $dx \wedge dy \wedge dz = -dx \wedge dz \wedge dy = dz \wedge dx \wedge \wedge dy$ etc... all 3-forms can be written as $f(x, y, z)dx \wedge dy \wedge dz$. f is a function on U. There can't be any k-forms for $k \geq 4$ and 0-forms are just functions. We denote the set of k-forms on U by $\Omega^k(u)$
  - $f(x, y, z) \quad \Omega^0 \longleftrightarrow$ functions f
  - $Pdx + Qdy + Rdz \quad \Omega^1(u) \longleftrightarrow$ vector fields $< P, Q, R >$
  - $Pdy \wedge dz + Qdz \wedge dz + Rdx \wedge dy \quad \Omega^2(u) \longleftrightarrow$ Vector fields $< P, Q, R >$
  - $fdx \wedge dy \wedge dz \quad \Omega^3(u) \longleftrightarrow$ functions f
- Wedge product extends to forms $\wedge : \Omega^k(u) \times \Omega^l(u) \longrightarrow \Omega^{k+l}$ you just multiply, expand, then apply the rule

> **Example**
> $(F_1 dx + F_2 dy + F_3 dz) \wedge (G_1 dx + G_2 dy + G_3 dz)$
> $= F_1 G_1 dx \wedge dx + F_1 G_2 dx \wedge dy + F_1 G_3 dx \wedge dz$

$+F_2G_1dy \wedge dx + F_2G_2dy \wedge dy + F_2G_3dy \wedge dz$
$+F_3G_1dz \wedge dx + F_3G_2dz \wedge dy + F_3G_3dz \wedge dz$
$= (F_2G_3 - F_3G_2)dy \wedge dz + (F_3G_1 - F_1G_3)dz \wedge dx + (F_1G_2 - F_2G_1)dx \wedge dy \longleftarrow$
$< F_1, F_2, F_3 > \times < G_1, G_2, G_3 >$

- So $\wedge : \Omega^1(u) \times \Omega^1(u) \to \Omega^2(u)$ corresponds to the cross-product of vector fields under our dictionary

> **Example**
> $(F_1dx + F_2dy + F_3dz) \wedge (G_1dy \wedge dz + G_2dz \wedge dx + G_3dx \wedge dz)$
> $= \ldots$
> $= (F_1G_1 + F_2G_2 + F_3G_3)dx \wedge dy \wedge dz$

- So $\wedge : \Omega^1(u) \times \Omega^2(u) \longrightarrow \Omega^3(u)$ corresponds to dot product of vector fields under our dictionary
- In general, if $w \in \Omega^k(u)$ and $n \in \Omega^l(u)$ $w \wedge n = (-1)^k n \wedge w$

The One Derivative to Rule Them All

- We now define the One Derivative to Rule Them All:$d : \Omega^k(u) \to \Omega^{k+1}(u)$ for $k = 0$ we define $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ for all other k, apply d to each coefficient and then use wedge product to expand

> **Example**
> $w = Pdx + Qdy + Rdz$ then $dw = dP \wedge dx + dQ \wedge dy + dR \wedge dz$
> $= (P_x dx + P_y dy + P_z dz) \wedge dx$
> $+(Q_x dx + Q_y dy + Q_z dz) \wedge dy$
> $+(R_x dx + R_y dy + R_z dz) \wedge dz$
> $= (R_y - Q_z)dy \wedge dz + (P_z - R_x)dz \wedge dx + (Q_x - P_y)dx \wedge dy$

- So $d : \Omega^1(u) \to \Omega^2(u)$ corresponds to curl under our dictionary
- Similarly $d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) = (P_x + Q_y + R_z)dx \wedge dy \wedge dz$ so $d : \Omega^2(u) \longrightarrow \Omega^3(u)$ corresponds to div under our dictionary
- $\Omega^0 \quad \longrightarrow \quad \Omega^1(u) \quad \longrightarrow \quad \Omega^2(u) \quad \longrightarrow \Omega^3(u)$
- functions $\longrightarrow$(grad) vector fields $\longrightarrow$(curl) vector fields $\longrightarrow$(div) functions
- In general we have $d^2 = 0$
- We also have $\alpha \in \Omega^k(u)$ then $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$

> **Example**

α is a 0-form, f is a function

β is a 1-form, $\vec{F}$ is a vector field

$d(\alpha \wedge \beta)$,(curl$(f\vec{F})$) = $d\alpha \wedge \beta(\vec{\nabla} f \times \vec{F})$ + $\alpha \wedge d\beta(f\text{curl}\vec{F})$

Example

α is a 1-form, $\vec{F}$ is a function

β is a 1-form, $\vec{G}$ is a vector field

$d(\alpha \wedge \beta)$,(div$(\vec{F} \times \vec{G})$) = $d\alpha \wedge \beta(\text{curl}\vec{F} \cdot \vec{G})$ - $\alpha \wedge d\beta(\vec{F} \cdot \text{curl}\vec{G})$

Excursion into $\mathbb{R}^4$. Maxwell's equations

- $\vec{B} = < B_1, B_2, B_3 >$ magnetic field
- $\vec{E} = < E_1, E_2 E_3 >$ electric field
- $\vec{J} = < j_1, j_2, j_3 >$ current density
- ρ
- $\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$
- ε_0, μ_0, c are electric constant, magnetic constant, speed of light, we can use units where these are all 1
- This formulation makes it look like $\vec{E}$ and $\vec{B}$ are 2 different things, but under a Lorentz transformation, $\vec{E}$ and $\vec{B}$ get scrambled. They are really two components of a single two form
- $\mathbb{R}^4$ has coordinates x, y, z, t the electromagnetic 2-form F is given by $F = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy + (E_1 dx + E_2 dy + E_3 dz) \wedge dt$
- $J = \rho dx \wedge dy \wedge dz - (j_1 dy \wedge dz + j_2 dz \wedge dx + j_3 dx \wedge dy) \wedge dt$ current 3-form
- $F \in \Omega^2(\mathbb{R}^4)$ $J \in \Omega^3(\mathbb{R}^4)$
- Hodge star operator $\star : \Omega^2(\mathbb{R}^4) \longrightarrow \Omega^2(\mathbb{R}^4)$
  - For any coordinate 2-form α, define $\star\alpha$ to be the coordinate 2-form such that $\alpha \wedge \star\alpha = dt \wedge dx \wedge dy \wedge dz$
  - $\star(dx \wedge dy) = dt \wedge dz$
  - $\star(dy \wedge dz) = dt \wedge dx$
  - $\star(dz \wedge dx) = dt \wedge dy$
  - $\star(dx \wedge dt) = dy \wedge dz$
  - $\star(dy \wedge dt) = dz \wedge dx$
  - $\star(dz \wedge dt) = dx \wedge dy$
  - $\circ \wedge \circ$
- Maxwell's equations: $dF = 0 \qquad d(\star F) = J$

- Note: if the domain is $\mathbb{R}^4$, then $dF = 0 \Rightarrow \exists A \in \Omega^1(\mathbb{R}^4)$ such that $F = dA$. A is called the electromagnetic gauge field. It is the fundamental object. It is a 1-form, unique up to a gauge transformation $A \longmapsto A + df$ where f is a function
- Note $d^2 = 0 \Rightarrow dJ = 0$ the so called continuity equation
- The above description allows for a coordinate free description. Very important in understanding the symmetries of the equations.

Integration, change of variables, (orientation)

- A k-form can be integrated on a k-dimensional (oriented) manifold
- We've already seen the integral of a 1-form over a 1-manifold C
- (i.e. a curve): $\int_C Pdx + Qdy + Rdz$ this was one of our ways of writing a work integral
- Also the integral of a 3-form $fdx \wedge dy \wedge dz$ over a 3-manifold E (a.k.a a solid) is just the usual triple integral $\int\int\int_E fdxdydz$
- A 0-manifold is a collection of points $p_1, \ldots, p_k$ (orientation associates a sign to each point)
- The integral of a 0-form f over $\{p_1, \ldots, p_k\}$ is a (signal) sum: $\sum_{i=1}^k \pm f(p_i)$
- Consider a surface S and a 2-form $w = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. To define the integral $\int\int_S w$, we use the chain rule to write the 2-form in terms of the variables in a parameterization of S
- Let $\vec{r}(u, v) =< x(u, v), y(u, v), z(u, v) > (u, v) \in D \subset \mathbb{R}^2$ then $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$, $dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$, $dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$ so:
- $dy \wedge dz = (y_u du + y_v dv) \wedge (z_u du + z_v dv) = (y_u z_v - y_v z_u) du \wedge dv$
- $dz \wedge dx = (z_u x_v - z_v x_u) du \wedge dv$
- $dx \wedge dy = (x_u y_v - x_v y_u) du \wedge dv$
- and so $Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \{P(y_u z_v - y_v z_u) + Q(z_u x_v - z_v x_u) + R(x_u y_v - x_v y_u)\}$ $< P, Q, R > \cdot (\vec{r_u} \times \vec{r_v})$
- Thus $\int\int_S Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \int\int_D < P, Q, R > \cdot (\vec{r_u} \times \vec{r_v}) dudv$ which was exactly our notion of flux
- The way we just defined integration of 2-forms over surfaces basically works in general: we can define integration of n-forms over n-manifolds in the same way.
- **Generalized Stoke's Theorem**: Suppose $w \in \Omega^{k-1}(u)$ and M is a k-manifold then $\int_M dw = \int_{\partial M} w$
  - $\Omega^0 \quad \longrightarrow \quad \Omega^1(u) \quad \longrightarrow \quad \Omega^2(u) \quad \longrightarrow \Omega^3(u)$
  - functions $\longrightarrow$(grad) vector fields $\longrightarrow$(curl) vector fields $\longrightarrow$(div) functions
  - points $\longleftarrow$ curves $\longleftarrow$ surfaces $\longleftarrow$ solids

  - |Sums (FTLI)| line integrals (Stoke's theorem)| flux integrals (divergence theorem) | triple integrals
- A differential form $w \in \Omega^k(u)$ is called closed if $dw = 0$. It is called exact if there is some $\alpha \in \Omega^{k-1}(u)$ such that $w = d\alpha$
- What are all the closed k-forms? In other words , what are all the solutions the equation $dw = 0$ (really a system of PDEs). There are of course the "trivial" solutions: pick any $k-1$ from α and let $w = d\alpha$. The non-trivial solutions are given by closed k-form which are not exact. (For $k = 1$ this is equivalent to asking for a vector field $\vec{F}$ such that curl$\vec{F} = \vec{0}$ but $\vec{F}$ is not conservative)
- The k-th DeRham cohomology can be thought of the space of solutions to PDE $dw = 0$ modulo the space of trivial solutions: $H^k_{DR}(u) = \frac{\{\text{closed } k\text{-froms}\}}{\{\text{exact } k\text{-forms}\}}$
- A k-manifold M in U is closed if $\partial M = \varnothing$. A closed k-manifold M is called a boundary if there is a $(k+1)$-manifold E such that $\partial E = M$. the k-th homology of U is $H_k(u) = \frac{\{\text{closed } k\text{-manifolds in U}\}}{\{\text{boundaries}\}}$ To fully make sense of this quotient, we need to make these sets into vector spaces: we do this by taking formal sums of manifolds
- For example $H_1(u) = 0$ if every loop bounds a disk. If $H_1(u) \neq 0$ then U is not simply connected
- The Derham theorem: $H^k_{DR}(u) \cong H_k(u)$ Integration is the connection between forms (H^k_{DR}) and manifolds (H_k) Stoke's theorem makes this work