PDEs

Types of PDEs

- The order of a PDE is the value of the highest order partial derivative occurring in the equation
- The degree is the power of the highest derivative
- $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ is a second order liner PDE, the PDE is said to be homogeneous if $G = 0$
- Analogous to characterizing quadratic equations $AX^2 + BXY + CY^2 + DX + EY = k$ as hyperbolic, parabolic or elliptic, determined by the sign of $\Delta = B^2 - 4AC$
  - $\Delta > 0$: Hyperbolic, example $u_{tt} = c^2u_{xx}$ **wave equation
  - $\Delta = 0$: Parabolic, example $u_t = u_{xx}$, **heat/diffusion equation
  - $\Delta < 0$: Elliptic, example $u_{xx} + u_{yy} = f$, Laplace's equation if $f = 0$, otherwise Poisson equation
- Given the flux density relationship $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$, we can guess the solution $u(x,t) = e^{at+bx}$
- Plug this into the equation to get $e^{at+bx}(a + cb) = 0$

Dirichlet

- $\lambda_n = -\mu_n^2, \quad \mu_n = \frac{n\pi}{L}$
- $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Neumann

- $\lambda_0 = 0, \quad \mu_n = \frac{n\pi}{L}$
- $X_0 = 1, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

Periodic

- $\lambda_0 = 0, \quad \lambda_n = -\mu_n^2, \quad \mu = \frac{n\pi}{L}$
- $X_n \in \{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\}$

Mixed type A

- $u_t = \alpha^2 u_{xx}$
- $u(0,t) = 0 = u_x(0,t)$

- $\mu_n = \frac{2n+1}{2L}\pi$
- $X_n = \sin(\mu_n x)$

Mixed type B

- $u_t = \alpha^2 u_{xx}$
- $\mu_n = \frac{(2n+1)\pi}{2L}$
- $X_n = \cos(\mu_n x)$
- $u_x(0,t) = 0 = u(L,t)$

Numerical (finite difference method)

- Approximating derivatives with difference operators
- Recall that $f'(x) = \lim_{x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
- Then we can approximate the derivative to $f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$ with a sufficiently small Δx
- Using the Taylor series we can say that $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \ldots$
- $f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \ldots$
- If we subtract the two equations we get $f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + \frac{2\Delta x^3}{3!} + \ldots$
- So we can get the approximate error of the derivative, the following is called a central difference scheme
- $f'(x) = \frac{f(x+\Delta x)-f(x-\Delta x)}{\Delta x} + 0(\Delta x)^2$, this is second order accurate
- $f'(x) - \frac{f(x+\Delta x)-f(x)}{\Delta x} - 0\Delta x$, this is a forward scheme and is first order accurate
- Now we can sum up the two equations and make f'' the subject
- $f(x+\Delta x) + f(x-\Delta x) = 2f(x) + \frac{2\Delta x^2}{2} f''(x) + \ldots$
- Then we get $f''(x) = \frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{\Delta x^2} + 0(\Delta x)^2$, central difference scheme for f''

1D Heat Equation

- The heat equation is given by $u_t = \alpha u_{xx}$ types of pdes
- We need the boundary conditions for the x domain and we need the initial conditions
- For example the Dirichlet $u(0,t) = A, u(h,t) = B$
- The Neumann boundary conditions $u_x(0,t) = A, u_x(L,t) = B$
- The Robin type conditions $u(0,t) + u_x(0,t) = A$
- Now we will solve an example
- $u_t = a^2 u_{xx}$

- Boundary Condition: $u(0,t)=0, \quad u(L,t)=0, \quad u(x,0)=f(x)$
- Step 1 Now it is time to discretize, given the domain of x ranging from $0 \to L$
- We split it into n components where $x_0 = 0$ and $L = x_n$
- Now we do the same with t splitting into k components
- So $x_{n+1} - x_n = \Delta x$ and $t_{k+1} - t_k = \Delta k$
- Step 2 Discretize the PDE
- Approximate u_t with first forward difference and u_{xx} with central difference scheme
- $u_n^k = u(x_n, t_k)$
- So now we have $\frac{u_n^{k+1} - u_n^k}{\Delta t} = \alpha^2 \left[\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}\right]$
- Rearrange the equation to get $u_n^{k+1} = (1-2r)u_n^k + ru_{n+1}^k + ru_{n-1}^k$ where $r = \frac{\alpha^2 \Delta t}{\Delta x^2}$

$$\begin{bmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_N \end{bmatrix}^{k+1} = \begin{bmatrix} r & 1-2r & r & 0 & 0 & \dots \\ 0 & r & 1-2r & r & 0 & \dots \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \end{bmatrix} \begin{bmatrix} u_0 \\ . \\ . \\ . \\ . \\ u_N \end{bmatrix}^k$$

- $u(0,t) = 0 \implies u_0^k = 0$
- Therefore by recursive protocol $u_0^{k+1} = u_0^k = 0$
- Now if we have the Neumann boundary condition, $\frac{\partial u(0,t)}{\partial x} = q_0, \quad \frac{\partial u(L,t)}{\partial x} = p_0$
- We know that $u_0^{k+1} = ru_1^k + (1-2r)u_0^k + ru_{-1}^k$
- This creates a ghost node at -1, we will use the central difference scheme to amend this
- $u_x(0,t) = \frac{u_1^k - u_{-1}^k}{2\Delta x}$
- $u_{-1}^k = -2\Delta x q_0 + u_1^k$
- So then we have $u_0^{k+1} = ru_1^k + (1-2r)u_0^k + r[u_1^k - 2\Delta x q_0]$
- Now for the upper boundary, $N+1$ is the ghost node
- $u_N^{k+1} = ru_{N+1}^k + (1-2r)u_N^k + ru_{N-1}^k$
- We replace the term using the equation $u_x(L,t) = \frac{u_{N+1}^k - u_{N-1}^k}{2\Delta x}$, as we did for the lower ghost node

Wave equation

- $u_{tt} = \alpha^2 u_{xx}$ types of pdes
- B.C. $u(0,t) = u(L,t) = 0$
- I.C. $u(x,0) = f(x); \quad u_t(x,0) = g(x)$
- Use central difference scheme for time and space
- $u_n^k = u(x_n, t_k)$
- $\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} = \frac{\alpha^2(u_{n+1}^k - 2u_n^k + u_{n-1}^k)}{\Delta x^2}$
- $c = \frac{\alpha \Delta t}{\Delta x}$

- Now we can rearrange the equation to get: $u_n^{k+1} = c^2 u_{n+1}^k + 2(1-c^2)u_n^k + c^2 u_{n-1}^k - u_n^{k-1}$
- What does this look like in matrix form?
  $$\vec{u}^k = \begin{bmatrix} c^2 & 2(1-c^2) & c^2 & 0 & 0 & \dots \\ 0 & c^2 & 2(1-c^2) & 0 & 0 & \dots \\ . & . & . & . & . & \dots \end{bmatrix} \vec{u}^k + \begin{bmatrix} -1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ . & . & . & \dots \end{bmatrix} \vec{u}^{k-1}$$
- $u_n^1 = c^2 u_{n+1}^0 + 2(1-c^2)u_n^0 + c^2 u_{n-1}^0 - u_n^{-1}$
- u_n^{-1} is a ghost node!, we must discretize
- $\frac{u_n^1 - u_n^{-1}}{2\Delta t} = g(x) \implies \quad u_n^{-1} = u_n^1 - 2\Delta t g(x)$

Laplace Equation

- $u_{xx} + u_{yy} = 0$ types of pdes
- $u(0,y) = f_1(y); \quad u(L,y) = f_2(y)$
- $u(x,0) = g_1(x); \quad u(x,L) = g_2(x)$
- $\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{\Delta x^2} + \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{\Delta y^2} = 0$
- $2[\Delta x^2, u_{m,n} + \Delta y^2 u_{m,n}] = \Delta y^2(u_{m+1,n} + u_{m-1,n}) + \Delta x^2(u_{m,n+1} + u_{m,n-1})$
- $u_{m,n} = \frac{1}{2(\Delta x^2 + \Delta y^2)}$
- Now what happens if we let $\Delta x = \Delta y$?
- $u_{m,n} = \frac{1}{4}(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1})$
- It is the average of the neighbouring nodes!
- Jacobian Iteration Scheme
- $u_{m,n}^{()} \to u_{m,n}^{(1)} \to \dots \to u_{m,n}^{(k)}$
- $u_{m,n}^{(k+1)} = \frac{1}{4}[u_{m+1,n}^{(k)} + u_{m-1,n}^{(k)} + u_{m,n+1}^{(k)} + u_{m,n-1}^{(k)}]$

Fourier

- $f(x) = \sum_{n=0}^{\infty} b_n[\sin(nx) + a_n \cos(nx)]$
- Laplace Language

Separation of Variables

- $u(x,t) = T(t) \cdot X(x)$
- Systems of ODEs, Linear PDE
- Step 1 Assume $u(x,t) = T(t) \cdot X(x)$
- Step 2 Plug into the PDE to get ODEs
- Step 3 ODE in space gives us the Boundary Value Problem, eigenvalue problem
- BVP$\mapsto \begin{cases} X''(x) + 2X = 0 \\ X(0) = X(L) = 0 \end{cases}$
- Always take your domain as $2L$

Heat equation with Dirichlet B.C. (Sine series)

- $u_t = \alpha^2 u_{xx} \quad 0 < x < L$
- B.C. $u(0,t) = u(L,t) = 0$
- I.C. $u(x,0) = f(x)$
- 1 Let $u(x,t) = X(x) \cdot T(t)$
- $u_t = X \cdot T'$
- $u_{xx} = X'' \cdot T$
- 2 plug these back into the equation
- $XT' = \alpha^2 X'' T$
- $\frac{T'}{T} = \frac{\alpha^2 X''}{X}$
- This is only possible if $\frac{T'}{\alpha^2 T} = \frac{X''}{X} = \lambda$
- So now we have the two equations $T' - \lambda\alpha^2 T = 0$
- $X'' - \lambda X = 0$
- $T(t) = -e^{\alpha^2 \lambda t}$
- Now we have an eigenvalue problem
- $\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(L) \end{cases}$
- let $X = e^{rx}$
- $r^2 = \lambda = 0 \implies$ we can get different solutions
- Case 1: $\lambda > 0$, let $\lambda = \mu^2$
- $r^2 - \mu^2 = 0 \implies r = \pm\mu$
- So then $y(x) = Ae^{\mu x} + Be^{-\mu x} \to y(x) = A\cosh(\mu x) + B\sinh(\mu x)$
- Now lets check the boundary conditions $y(x) = a\sin(x) + b\cos(x)$
- $y(0) = 0 = b$
- Going down this road only gives a trivial solution which is not what we want
- More found in the notes on canvas
- Case 2: $\lambda = 0$
- $X'' = 0 \implies X(x) = Ax + B$
- $X(0) = B = 0, \quad X(L) = AL = 0 \implies A = 0$ so we have another trivial solution
- Case 3 $\lambda < 0$
- $r^2 + \mu^2 = 0 \implies R = \pm i\mu$
- $X(x) = A\cos(\mu x) + B\sin(\mu x)$
- $X(0) = A = 0$
- $X(L) = B\sin(\mu L) = 0 \implies \sin(\mu L) = 0$, this is only possible if $\mu L = n\pi$
- $\mu_n = \frac{n\pi}{L}$,the eigenvalues are $-\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2$
- $X_n(x) = B_n \sin(\mu_n x), n = 1, 2, 3, \ldots$
- $U_n(x,t) = X_n(x)T(t) = e^{-(\frac{n\pi}{L}\alpha)^2 t} \sin(\mu_n x)$
- $u(x,t) = \sum_{n=1}^{\infty} b_n U_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$
- $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$
- So the series is the Fourier sine series of $f(x)$

- To find b_n, we employ the orthogonal property of sin and cos
- $\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$
- $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$
- $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0$
- Now we are going to multiply $f(x)$ by $\sin\left(\frac{m\pi x}{L}\right)$
- This gives us the following
- $\int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$
- $\int_{-L}^{L} b_1 \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) + \int_{-L}^{L} b_2 \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) + \ldots$
- There will be some index where we get $\int_{-L}^{L} b_m \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$ which is the only term that is not equal to zero
- So now we have $\int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = b_m L$
- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$
- Now lets look at the boundary and initial conditions again
- B.C: $u(0,t) = u(1,t) = 0$
- I.C.: $u(x,0) = x$
- $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$
- $b_n = \int_{-1}^{1} x \sin(n\pi x) dx$
- Time to integrate by parts, $u = x \implies du = 1, dv = \sin(n\pi x) \implies v = -\frac{1}{n\pi}\cos(n\pi x)$
- $b_n = \left[-\frac{x}{n\pi}\cos(n\pi x)\right] \Big|_{-1}^{1} + \frac{1}{n\pi} \int \cos(n\pi x) dx$
- $b_n = -\frac{1}{n\pi}[\cos(n\pi) + \cos(n\pi)] + 0 = -\frac{2}{n\pi}\cos(n\pi), n = 1, 2, 3, ..$
- $\cos(n\pi) = (-1)^n, \implies b_n = \frac{2}{n\pi}(-1)^{n+1}, n = 1, 2, 3, \ldots$
- So now we have $f(x) = x = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin(n\pi x)$
- $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
- Lets look at $f\left(\frac{1}{2}\right)$, we want an x value that doesn't make the values disappear
- $f\left(\frac{1}{2}\right) = \frac{1}{2} = \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} + \ldots\right]$
- $\frac{1}{2} = \frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{1}{2} \checkmark$

Neumann B.C. (Fourier Cosine Series)

- $u_t = \alpha u_{xx}$
- $u_x(0,t) = 0, u_x(L,t) = 0, u(x,0) = 0$
- $u_t = XT', \quad u_{xx} = X''T$
- $XT' = \alpha^2 X''T \implies \frac{T'}{\alpha^2 T} = \frac{X''}{X} = \lambda$
- $T' - \lambda\alpha^2 T = 0, \qquad X'' - \lambda X = 0$
- $X'(0) = X'(L) = 0$ these two lines give us the eigenvalue problem
- $T = Ce^{\lambda\alpha^2 t}$
- Let $X(x) = e^{rx}$
- Then plugging that in gives us the equation $r^2 - \lambda = 0$

- Case 1: $\lambda > 0$, let $\lambda = \mu^2$
- $r^2 - \mu^2 = 0 \implies r = \pm\mu$
- $X(x) = A\cosh(\mu x) + B\sinh(\mu x)$
- $X'(x) = A\mu\sinh(\mu x) + B\mu\cosh(\mu x)$
- $X'(0) = 0 = B\mu \implies B = 0$
- $X'(L) = 0 = A\mu\sinh(\mu L) \implies A = 0$
- Which gives us a trivial solution
- Case 2: $\lambda = 0$
- $X'' = 0 \implies X = Ax + B$
- $X'(0) = 0 = A \implies A = 0$
- $X'(L) = 0 = A \implies A = 0$
- So we have the arbitrary solution B
- $\lambda_0 = B, X_0(x) = B \equiv 1$ since the coefficient will later get absorbed
- Case 3: $\lambda < 0$
- $r^2 - \lambda = 0, \quad \lambda = -\mu^2$
- $r^2 + \mu^2 = 0$
- $r = \pm i\mu$
- $X = A\sin(\mu x) + B\cos(\mu x)$
- $X' = A\mu\cos(\mu x) - B\mu\sin(\mu x)$
- $X'(0) = X'(L) = 0$
- $A = 0$
- $X_n(x) = B\cos(\mu_n x)$
- $\lambda n = -\mu_n^2, \quad X_n(x) = \cos(\mu_n x)$
- $u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)$
- $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$
- $u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$ this is the Fourier Cosine Series
- When $n = 0$, $\int_{-L}^{L} f(x)\cos(0x)dx = \int_{-L}^{L} A_0\cos(0x)dx = 2A_0L$
- So $A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$
- $A_n = \frac{1}{L}\int_{L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, n = 1, 2, 3, \ldots$
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$
- $a_n - \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots$
- $u_t = u_{xx}, \quad 0 < x < 1$
- $u_x(0,t) = u_x(1,t) = 0$
- $u(x,0) = x$

  $A_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{2x}{n\pi}\sin(n\pi x)\mid_{-1}^{1} - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx = 0 + \frac{2}{(n\pi)^2}[\cos(n\pi) - 1]$
- $a_n = \frac{2}{(n\pi)^2} \cdot \begin{cases} 0, \text{if is even} \\ -2, \text{if n is odd} \end{cases}$
- $a_{2k+1} = -\frac{4}{((2k+1)\pi)^2}$
- $f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x)$
- $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x)$

- Now we can check
- $f(0) = 0 = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$
- $\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$

Heat equation in a Ring

- Full range Fourier series
- $u_t = \alpha^2 u_{xx}$
- $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots$
- Δ is the Laplace operator
- $u_t = \alpha^2 \Delta u = a^2 [u_{xx} + u_{yy}]$
- $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$
- in 1D $\Delta u = u_{xx}$
- $x = r \cos \theta, \quad y = r \sin \theta$
- Heat flow in angular direction
- $\Delta u = \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial (r\theta)^2}$
- Let $x = r\theta \implies \frac{\partial^2 u}{\partial (r\theta)^2} = \frac{\partial^2 u}{\partial x^2}$
- $u_t = \alpha^2 u_{xx}$
- Let the circumference of the ring be L
- $u\left(-\frac{L}{2}, t\right) = u\left(\frac{L}{2}, t\right)$
- $u_x\left(-\frac{L}{2}, t\right) = u_x\left(\frac{L}{2}, t\right)$
- For the next little while $L = \frac{L}{2}$ because I'm lazy
- $u(x, 0) = f(x)$
- $u(x, t) = X(x) \cdot T(t)$
- (1) $T' = \lambda \alpha^2 T \implies T(t) = e^{\lambda \alpha^2 t}$
- $\begin{cases} X'' - \lambda X = 0 \\ X(-L) = X(L) \\ X'(-L) = X(L) \end{cases}$
- Case 1: $\lambda > 0 \implies$ trivial solution
- Case 2: $\lambda = 0 \implies X_0(x) = 1$
- Case 2: $\lambda < 0 \implies r_{1,2} = \pm i\mu$
- $X(x) = A \cos(\mu x) + B \sin(\mu x)$
- $X' = -A\mu \sin(\mu x) + B\mu \cos(\mu x)$
- $X(-L) = A \cos(\mu L) - B \sin(\mu L)$
- $X(L) = A \cos(\mu L) + B \sin(\mu L)$
- So then $B = 0$ or $\mu L = n\pi \implies \mu_n = \frac{n\pi}{L}$
- $X'(-L) = A\mu \sin(\mu L) + +B\mu \cos(\mu L)$
- $X'(L) = -A\mu \sin(\mu L) + B\mu \cos(\mu L)$
- So then either $A = 0$ or $\mu_n = \frac{n\pi}{L}$
- $\lambda < 0, \quad \mu_n = \frac{n\pi}{L}$
- $X_n \in \{cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\}$

- $u_n(x,t) = X_n(x) \cdot T_n(t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\sin\left(\frac{n\pi x}{L}\right)\right)\right] T_n(t)$
- $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t} [a_n \cos(\mu_n x) + b_n \sin(\mu_n x)]$
- $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3$
- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3$
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$
- $L = \frac{1}{2}$
- $f(x) = x$,assume the function to be periodic
- $a_0 = \frac{1}{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x dx = 2 \int_0^1 x dx = 1$
- $a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} x \cos\left(\frac{n\pi x}{L}\right) dx = 2 \int_0^1 x \cos\left(\frac{n\pi x}{L}\right) dx$

Prove Orthogonality

- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- Add them together to get $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
- let $A = \frac{m\pi x}{L}, \quad B = \frac{n\pi x}{L}$
- let $m = 0 = n$
- $\frac{1}{2} \int_{-L}^{L} (1+1) = 2L$
- $m = n \neq 0$
- $A = B = \frac{m\pi x}{L}$
- $\frac{1}{2} \int_{-L}^{L} \left[\cos\left(\frac{2m\pi x}{L}\right) + 1\right] dx$
- $\int_{-L}^{L} \cos\left(\frac{2m\pi x}{L}\right) dx = \sin\left(\frac{2m\pi x}{L}\right) |_{-L}^{L} = 0$