Linear systems

Complex eigenvalues

Theorem

If P is a real valued constant square matrix and it has complex eigenvalues $\lambda = a + ib$ with an associated eigenvector $\vec{v}$, then $\overline{\lambda} = a - ib$ is the complex conjugate eigenvalue with a complex conjugate eigenvector $\overline{\vec{v}}$

- Proof is in the online notes

Example

Transform the IVP $\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$ into a system of ODEs and solve

Let $x_1 = y, x_2 = y'$ $\begin{cases} x_1' = y' \\ x_2' = y'' \end{cases}$

$\begin{cases} y'' = -2y' - 2y = x_2' \\ x_2' = -2x_2 - 2x_1 \end{cases}$

so $x_1' = x_2$ and $x_2' = -2x_2 - 2x_1$

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The eigenvalues of P are $\lambda = -1 \pm i$

In the complex case we only need $\vec{v_1}$ which we will find using $\lambda_1 = -1 + i$

This gives the eigenvector $\vec{v_1} =< 1, -1 + i >$

Now we can apply the theorem

$\vec{v_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 + i \end{bmatrix} e^{(-1+i)t} = \begin{bmatrix} \cos t + i \sin t \\ (-1 + i)(\cos t + i \sin t) \end{bmatrix} e^{-t}$

This gives us

$\vec{x}(t) = C_1 e^{-t} \begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix}$

Now we can solve the initial condition which finally gives us

$\vec{x}(t) = e^{-t} \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix}$

$y = x_1 = e^{-t} \sin t$

Multiple Eigenvalues

- We still consider the $x' = Px$
- We covered the case where the eigenvalues are distinct, what happens if they are repeated

Geometric and Algebraic Multiplicity

Definition

The multiplicity of an eigenvalue λ_j as a root of the characteristic polynomial is called the algebraic multiplicity e.g. $(\lambda - 1)^1 = 0$, multiplicity of two
The geometric multiplicity of and eigenvalue is the number of linearly independent eigenvectors associated to the eigenvalue

- Geometric multiplicity will always be less than or equal to algebraic
- Let $P = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
- The eigenvalue 3 has an algebraic multiplicity of two and now we can choose two linearly independent vectors in this set of vectors
- We have $v_1 =< 0, 1 >, \vec{v_2} =< 1, 0 >$
- So the algebraic multiplicity is two and the geometric multiplicity is two

Theorem
If for each repeated eigenvalue of P the AM equals the GM then the general solution is similar to the distinct real eigenvalues case

- The solution to the problem above would be $\vec{x}(t) = C_1\vec{v}_1 e^{3t} + C_2\vec{v}_2 e^3 t$

Defective eigenvalues

Definition

If the GM is less than the AM then λ_j is called defective and the difference AM - GM is the defect

- We can't apply the eigenvalue method to get the general solution
- Let $P = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}$
- $\lambda = -1$ has an algebraic multiplicity of two
- However all the eigenvectors will have the form $\vec{v} =< \alpha, \alpha >$ where α is a constant
- It is impossible to get two linearly independent vectors

- Why can't we apply the eigenvalue theorem? real case
  Because we need to have n linearly independent vectors for an $n \times n$ matrix. Otherwise, for example, if we write the solution in a case where GM $<$ AM for a 2×2 system $\vec{x}(t) = C_1\vec{v_1}e^{\lambda t} + C_2\vec{v_2}e^{\lambda t} = (C_1 + C_2R)\vec{v_1}e^{\lambda t}$ we are left with one solution to work with while we are looking for two linearly independent solutions to build the general solution

We need to build an additional solution

- We already have one solution $\begin{cases} \vec{x_1} = \vec{v_1}e^{\lambda_1 t} \\ (P - \lambda_1 I)\vec{v_1} = 0 \end{cases}$
- We guess $\vec{x_2} = (t\vec{v_1} + \vec{v_2})e^{\lambda_2 t}$
- (1) $Px_2 = \vec{P}v_1te^{\lambda_2 t} + P\vec{v_2}e^{\lambda_2 t} = \lambda_1\vec{v_1}te^{\lambda_1 t} + P\vec{v_2}e^{\lambda_2 t}$
- (2) $x_2' = \vec{v_1}e^{\lambda_2 t} + t\lambda_2 v_1\vec{e}^{\lambda_2 t} + \vec{v_2}\lambda_2 e^{\lambda_2 t}$
- $(1) = (2) \implies \vec{v_1}e^{\lambda_2 t} = (P - \lambda_2 I)\vec{v_2}e^{\lambda_2 t}$
- So $\vec{x_2}$ is a solution of $(P - \lambda_2 I)\vec{v_2} = \vec{v_1}$. Also $\vec{v_2}$ is called a generalized eigenvector. Furthermore we have $(P - \lambda_2 I)(P - \lambda_2 I)\vec{v_2} = \vec{0} = (P - \lambda_2 I)\vec{v_1}$ using $(P - \lambda_1 I)\vec{v_1} = 0$ and the pervious relation. So we can get $\vec{v_2}$ solving $(P - \lambda_2 I)^2\vec{v_2} = 0$ with $(P - \lambda_2 I)\vec{v_2} \neq 0$ because $\vec{v_1}$ is an eigenvector
- And finally $\vec{v_1} = (P - \lambda_2 I)\vec{v_2}$

Now we can return to previous example

- We take $\vec{v_1} =< 1, 1 >, x_1 =< 1, 1 > e^{-t}$
- Now lets find x_2. It is a solution if $(P + I)\vec{v_2} = \vec{v_1}$
- $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $\vec{v_2} =< 0, 1 >$ works!
- The general solution is $\vec{x}(t) = C_1\vec{x_1}(t) + C_2\vec{x_2}(t)$ with $\vec{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$ and
  $\vec{x_2} = \left[t \begin{bmatrix} 1 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] e^{-t}$
- Another example is in the notes lecture 25-26 W9 2021