Error Analysis

Big Idea

The condition number if a nonsingular matrix A is $\mathrm{cond}(A) = ||A||||A^{-1}||$.
Given a linear system $Ax = b$, the condition number of A quantifies how sensitive the solution x is relative to changes in b

Vector norms

DEFINITION

The **Euclidean** norm of a vector $x \in \mathbb{R}^n$ is

$$||x|| = \sqrt{|x_1|^2 + \cdots + |x_n|^2} = \sqrt{\sum_{k=1}^{n} |x_k|^2}$$

Note the Euclidean norm assigns a magnitude or length to a vector but it turns out that there are many different ways to define the "magnitude" of a vector!

DEFINITION

A norm on $\mathbb{R}^n$ is a function $|| \cdot ||$ such that:

1. $||x|| \geq 0 \forall x \in \mathbb{R}^n$
2. $||x|| = 0 \iff x = 0$
3. $||cx|| = |c|||x||$ for any $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$
4. $||x + y|| \leq ||x|| + ||y|| \forall x, y \in \mathbb{R}^n$

Condition 4 is called the triangle inequality

DEFINITION

Let $1 \leq p < \infty$ the p-norm of a vector $x \in \mathbb{R}^n$ is given by

$$||x||_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$

In particular the 1-norm is given by

$$||x||_1 = |x_1| + \cdots + |x_n|$$

and the 2-norm is the familiar Euclidean norm given by

$$||x||_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

The ∞-norm of a vector $x \in \mathbb{R}^n$ is given by

$$||x||_\infty = \max\{|x_1|, \ldots, |x_2|\}$$

Matrix norms

DEFINITION

A matrix norm is a function on matrices that satisfies the properties:

1. $||A|| > 0 \forall A \neq 0$
2. $||A|| = 0 \iff A = 0$
3. $||cA|| = |c|||A||$ for any $c \in \mathbb{R}$
4. $||A + B|| \leq ||A|| + ||B||$
5. $||AB|| \leq \|A\|\|B\|$

Frobenius norm

The **Frobenius norm** of a matrix A is given by

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} ||a_{i,j}||^2}$$

where $a_{i,j}$ are entries of A

Operator norm

The operator norm of a matrix A is given by

$$||A|| = \max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{||Ax||}{||x||}$$

where $|| \cdot ||$ is the 2-norm

Note the operator norm satisfies the property $||Ax|| \leq \|A\|\|x\| \forall x \in \mathbb{R}^n$

THEOREM

Let A be a nonsingular matrix. Then

$$||A|| = \max_{||x||=1} ||Ax|| \quad \text{and} \quad ||A^{-1}|| = \frac{1}{\min_{||x||=1} ||A||}$$

In other words, $||A||$ is the *maximum* stretch of a unit vector by the linear transformation A and $||A^{-1}||$ is the reciprocal of the minimum stretch of a unit vector by the linear transformation A

PROPOSITION

Let D be a diagonal matrix and let d be the vector of diagonal entries of D: Then $||D|| = ||d||_\infty = \max\{|d_1|, \ldots, |dn|\}$ and $||D||_F = ||d||_2$

Note. How do we compute the matrix norm $||A||$ for a general matrix? This is a nontrivial problem we will later see how to use the singular values of A to determine the matrix norm

Condition number

The condition number of a nonsingular square matrix A is

$$\text{cond} = ||A||A^{-1}|||$$

By convention we define $\text{cond}(A) = \infty$ if $\det(A) = 0$

Note if A is nonsingular we have

$$\text{cond} = ||A||A^{-1}||| = \frac{\text{maximum stretch of a unit vector}}{\text{minimum stretch of a unit vector}}$$

Relative Errors

Let A be a non singular matrix and consider the linear system $Ax = b$. If a small change Δb corresponds to a change Δx in the sense that $A(x + \Delta x) = b + \Delta b$ then

$$\frac{||\Delta x||}{||x||} \leq \text{cond}(A)\frac{||\Delta b||}{||b||}$$

Given a vector b and small change Δb the relative change (or relative error) is

$$\frac{||\Delta b||}{||b||}$$

Note the error bound

$$\frac{||\Delta x||}{||x||} \leq \mathrm{cond}(A)\frac{||\Delta b||}{||b||}$$

implies that if A has a large condition number then a small changes in b may result in *very* large changes in the solution x. In other words, the solution x is sensitive to errors in Δb