Non-homogeneous systems

Theorem

Let $\vec{x}' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is a particular solution. Then every solution can be written as $\vec{x} = \vec{x}_c + \vec{x}_p$ where $\vec{x}_c$ is the complementary solution

Variation of parameters

- Consider $x' = Px + f\,(\star)$
- Assume we have x_c and obtained a fundamental matrix solution $X(t) = \left(x_1 + \cdots + x_n\right)$
- The general solution of the homogeneous system is $X(t)\vec{c}, \vec{c} = (c_{1,\ldots}c_n)$
- We assume that $x_p = X(t)\vec{u}$
- Plugging into $(\star)$ gives us $x_p' = Px_p + f \implies X\vec{u}' + X'\vec{u} = PX\vec{u} + f$
- Note, we know that $X' = PX$ since every column of X verifies $x' = Px$
- The previous relation becomes $X\vec{u}' = f \implies \vec{u}' = X^{-1}f$
- After integrating we get $\vec{u} = \int X^{-1}f\,dt$
- And finally $x_p = Xu = X\int X^{-1}f\,dt$

Example

Solve $x' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

$x_c = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-1}$

$X = (x_1 + x_2)$ this is a fundamental matrix

$X = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}$

Note, the formula for the inverse of a 2×2 matrix is $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(X) = 2e^{-4t}$

$X^{-1} = \frac{1}{2e^{-4t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix}$

$x^{-1}f = \frac{1}{2} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{bmatrix} = \begin{bmatrix} e^{2t} & -\frac{3}{2}te^{3t} \\ 1 & \frac{3}{2}te^{t} \end{bmatrix}$

Now lets solve $\vec{u}$

$u_1 = \int e^{2t - \frac{3}{2}te^{3t}}\,dt \Rightarrow u_1 = \frac{e^{2t}}{2} - \frac{te^{3t}}{2} + \frac{1}{6}e^{3t} + K_1$

$u_2 = \int_1 + \frac{3}{2} te^{t\, dt} \Rightarrow u_2 = t + \frac{3}{2} te^t - \frac{3}{2} e^t + K_2$

Choose $K_1 = K_2 = 0$

Now finally $x_p = Xu$

$$x_p = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{e^{2t}}{2} - \frac{te^{3t}}{2} + \frac{1}{6} e^{3t} \\ t + \frac{3}{2} te^t - \frac{3}{2} e^t \end{bmatrix}$$

After some matrix multiplication and simplification we get,

$$x_p = e^{-t} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + te^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \end{bmatrix}$$