## Orthogonal Complement

**Big Idea**. The orthogonal complement $U^\perp$ of a subspace $U$ is the collection of all vectors which are orthogonal to every vector in $U$

## Orthogonal Vectors

DEFINTION

The **inner product** of vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k = x_1 y_1 + \cdots + x_n y_n$$

Lets summarize various properties of the inner product:

1. The inner product is symmetric

$$\langle x|y\rangle = \langle y|x\rangle \forall \vec{x}, \vec{y} \in \mathbb{R}^n$$

2. The inner product of column vectors is the same as matrix multiplication

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \boldsymbol{x}^T \boldsymbol{y}$$

3. The inner product satisfies the usual distributive rule of multiplication

$$\langle \boldsymbol{x}|c\boldsymbol{y} + d\boldsymbol{z}\rangle = c\langle \boldsymbol{x}|\boldsymbol{y}\rangle + d\langle \boldsymbol{x}|\boldsymbol{z}\rangle$$

4. The square root of the inner product of a vector with itself is equal to the 2-norm

$$\sqrt{\langle \boldsymbol{x}|\boldsymbol{x}\rangle} = \|\boldsymbol{x}\|$$

5. We can also write the inner product in terms of the angle between vectors

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \|\boldsymbol{x}\|\boldsymbol{y}\|\| \cos\theta$$

6. Let $A$ be an $m \times n$ matrix, ket $\boldsymbol{u} \in \mathbb{R}^n$ and let $\boldsymbol{v} \in \mathbb{R}^n$. Then

$$\langle A\boldsymbol{u}|\boldsymbol{v}\rangle = \langle \boldsymbol{u}|A^T\boldsymbol{v}\rangle$$

7. Let us have $A = \begin{bmatrix} \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_m^T \end{bmatrix} \in \mathbb{R}^{m\times n}$, where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in \mathbb{R}^n$, that is $\boldsymbol{u}_i^T$ is the $i\text{-th}$ row of $A$. Then for any $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$A\boldsymbol{x} = \begin{bmatrix} \langle \boldsymbol{u}_1 | \boldsymbol{x} \rangle \\ \vdots \\ \langle \boldsymbol{u}_m | \boldsymbol{x} \rangle \end{bmatrix}$$

which follows directly from the definition of the matrix vector product

DEFINTION

Vectors are orthogonal if the inner product between them is zero $\langle \boldsymbol{x} | \boldsymbol{y} \rangle = 0$
More generally vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ are orthogonal if
$\langle \boldsymbol{x}_i | \boldsymbol{x}_j \rangle = 0$ for all $i \neq j$
In other words each vector is orthogonal to every other vector in the set
Furthermore the are orthonormal if $\langle \boldsymbol{x}_i | \boldsymbol{x}_i \rangle = 1 = \|\boldsymbol{x}\|$ if each vector is orthogonal to the others and is a unit vector

Vectors $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal iff the acute angle between them is $\pi/2$ radians or $90^\circ$

THEOREM

Let the $\sum_n \boldsymbol{x}_n$ form an orthogonal basis, then

$$\|\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_m\|^2 = \|\boldsymbol{x}_1\|^2 + \cdots + \|\boldsymbol{x}_m\|^2$$

This is called the Pythagorean theorem

## Orthogonal subspaces

DEFINTION

let $U_1, U_2 \subseteq \mathbb{R}^n$ be subspaces. Then $U_1$ and $U_2$ are orthogonal if $\langle \boldsymbol{x}_1 | \boldsymbol{x}_2 \rangle = 0$ for all $\boldsymbol{x}_1 \in U_1$ and $\boldsymbol{x}_2 \in U_2$. If $U_1$ and $U_2$ are orthogonal subspaces then we write $U_1 \perp U_2$

THEOREM

Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_2\}$ be a basis of a subspace $U_1 \subseteq \mathbb{R}^n$ and let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l\}$ be a basis of a subspace $U_2 \subseteq \mathbb{R}^n$. Then $U_1 \perp U_2 \iff \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle = 0 \quad \forall i, j$
In other words, every $\boldsymbol{u}_i$ in the $U_1$ basis is orthogonal to each $\boldsymbol{v}_j$ in the basis $U_2$

EXAMPLE

Let $U_1 \subset \mathbb{R}^3$ and $U_2 \subset \mathbb{R}^3$ be 2-dimensional subspaces, planes. Is it possible that $U_1 \perp U_2$?
No!

## Orthogonal Complement

DEFINTION

Let $U \subseteq \mathbb{R}^n$ be a subspace. The orthogonal complement of $U$ is given by

$$U^\perp = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x} | \boldsymbol{y} \rangle = 0 \,\forall\, \boldsymbol{y} \in U\}$$

Note

- If $U \subseteq \mathbb{R}^n$ is any subspace then $U = (U^\perp)^\perp$ and also $U \cap U^\perp = \{\vec{0}\}$
- $\{\mathbf{0}\}^\perp = \mathbb{R}^n$

THEOREM

Let $U \subseteq R^n$ is a subspace, then $U^\perp \subseteq \mathbb{R}^n$ is a subspace

PROOF

Let us verify that $U^\perp$ satisfies the properties of a subspace
clearly $\langle \vec{0} | \boldsymbol{x} \rangle = 0 \,\forall \boldsymbol{x} \in U$ therefore $\vec{0} \in U^\perp$
Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in U^\perp$. Then

$$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2 | \boldsymbol{y} \rangle = \langle \boldsymbol{x}_1 | \boldsymbol{y} \rangle + \langle \boldsymbol{x}_2 | \boldsymbol{y} \rangle = 0 + 0 = 0 \forall \boldsymbol{y} \in U$$

Therefore $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in U^\perp$
Let $c \in \mathbb{R}, \quad \boldsymbol{u} \in U^\perp$. Then

$$\langle c\boldsymbol{x} | \boldsymbol{y} \rangle = c \langle \boldsymbol{x} | \boldsymbol{y} \rangle = c(0) = 0 \,\forall \boldsymbol{y} \in U$$

Therefore $c\boldsymbol{x} \in U^\perp$
Therefore $U^\perp$ is a subspace

## Fundamental subspaces

DEFINITION

Let $A$ be a $m \times n$ matrix, The fundamental subspaces of $A$ are
$N(A)$, $R(A)$ , $N(A^T)$ and $R(A^T)$

THEOREM

Let $A$ be an $m \times n$ matrix. Then $N(A) = R(A^T)^\perp$ and $R(A) = N(A^T)^\perp$

PROOF

The *second* equality follows from the first by replacing $A$ with $A^T$ therefore it is sufficient to prove $N(A) = R(A^T)^\perp$
A general strategy to prove equality of sets is to show that each set contains the other therefore lets prove $N(A) \subseteq R(A^T)^\perp$ and then the reverse

Let $\boldsymbol{x} \in N(A)$. Then $A\boldsymbol{x} = 0$ and so $\langle A\boldsymbol{x}|\boldsymbol{y}\rangle = 0$ for all $\boldsymbol{y} \in \mathbb{R}^m$
Using properties of the inner product we see that $\langle \boldsymbol{x}|A^T\boldsymbol{y}\rangle = 0\, \forall \boldsymbol{y} \in \mathbb{R}^m$
$\therefore \boldsymbol{x} \in R(A^T)^\perp$

Let $\boldsymbol{x} \in R(A^T)^\perp$. Then $\langle \boldsymbol{x}|A^T\boldsymbol{y}\rangle = 0$ and so $\langle A\boldsymbol{x}|\boldsymbol{y}\rangle = 0\, \forall \boldsymbol{y} \in \mathbb{R}^m$. Choose $\boldsymbol{y} = A\boldsymbol{x} \in \mathbb{R}^m$
and then $\langle A\boldsymbol{x}|A\boldsymbol{x}\rangle = 0 \implies \|A\boldsymbol{x}\| = 0 \implies A\boldsymbol{x} = \vec{0}$ and finally $\boldsymbol{x} \in N(A)$

THEOREM

Let $U \subseteq \mathbb{R}^n$ be a subspace. Then

$$\dim(U) + dim(U^\perp) = n$$

PROOF

Let $\dim(U) = m$ and let $\sum_m \boldsymbol{u}_m$ be a basis of $U$ and define

$$A = \begin{bmatrix} \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_m^T \end{bmatrix}$$

Then $U = R(A^T)$ and $U^\perp = R(A^T)^\perp = N(A)$ and we know
$\operatorname{rank}(A) = m = \dim(U)$
therefore

$$\dim(U) + \dim(U^{\perp}) = \mathrm{rank}(A) + \dim(N(A)) = n$$

by the Rank Nullity Theorem

EXAMPLE

Let $A$ be a matrix such that its LU Decomposition is of the form

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

where $*$ denotes a non zero number. Find the dimension of each subspace $N(A), R(A), N(A^T)$ and $R(A^T)$

Clearly $\dim(N(A)) = 1$ and $\dim(R(A)) = 3$ therefore

$$\dim(N(A^T)) = \dim(R(A)^{\perp}) = 3 - 3 = 0$$

and

$$\dim(R(A^T)) = \dim(N(A)^{\perp}) = 4 - 1 = 3$$