Cardinality

- How can you count to infinity?
- We are familiar with infinite sets. Should we think of them as being of different sizes?
- We need to come up with a different strategy

Defining Cardinality

- Let $f : A \to B$ where there are more elements in set B than A, f can be injective but not surjective
- For injective $|A| \leq |B|$ and $|A| \geq |B|$ for surjective
- Therefore for a function to be bijective $|A| = |B|$
- We extend these ideas to infinite sets!

> **Definition**
> Two sets have the same **cardinality** if there exists a bijection between them
> *Notation*: If two sets have the same cardinality, then we write $|A| = |B|$

- If two sets have the equal cardinality can be said to be equinumerous
- denumerable: same cardinality as $\mathbb{N}$
- countable: set is finite
- uncountable: set is not countable
- Do $(0, 1)$ and $(0, 1]$ have the same cardinality? yes, this is hard as fuck to prove though

> **Definition**
> Let A and B be sets
> - We write $|A| \leq |B|$ if there is an injection from $A \to B$
> - We write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$
> - We write $|A| \geq |B|$ if there is a surjection from $A \to B$
> - We write $|A| > |B|$ if $|A| \geq |B|$ and $|A| \neq |B|$

Pigeon hole principle

> **Theorem**

If n objects are placed in k boxes
If $n > k$ then there exists at least one box which has two objects in it
If $n < k$ then there exists a box that is empty

Infinity Tower

Theorem

Let S be a set. Then the power set of S, denoted by $\mathcal{P}(S)$ satisfies
$|S| \leq |\mathcal{P}(S)|$

Cantor-Schroder-Berstein (CSB)

Theorem

Let A and B be sets. If $|A| \leq |B|$ and $|A| \geq |B|$ then $|A| = |B|$

Examples

Example

1. Lets show that $|(0,1)| = |0,2|$
2. Show that $|[3,\infty)| = |5,\infty)$

- 1. Define $f : (0,1) \to (0,2)$
  - let $f(x) = 2x$
  - Now lets prove that the function is bijective
  - Injective: $f(a) = f(b), a, b \in (0,1)$ $f(a) = 2a, f(b) = 2b \implies a = b$ so injective
  - Surjective: Given $c \in (0,2)$ then let $x = \frac{c}{2} \in (0,1)$ then $f\left(\frac{c}{2}\right) = \frac{2 \cdot c}{2} = c$
  - Since f is bijective then $|(0,1)| = |0,2|$
- 2. Define $g : [3,\infty) \to [5,\infty)$
  - Let $g(x) = x + 2$
  - find the inverse
  - $h : [5,\infty) \to [3,\infty)$
  - $h(y) = y - 2$

- Then $g \circ h(x) = h(g(x)) = x$
- And $h \circ g(y) = h(g(y)) = y$
- So h is a two sided inverse of g, so g is bijective. So $|[3, \infty)| = |[5, \infty)|$

Problems

> **Problem 1**
>
> Show that $|(0, 1)| = |(0, \infty)|$

- $\tan\left(\frac{\pi}{2}x\right)$ lmao
- Let $f : (0, 1) \to (0, \infty)$
- let $f(x) = \frac{1}{x}$
- $g : (1, \infty) \to (0, \infty), g(y) = y - 1$
- Then other things posted in the notes online

> **Problem 2**
> Prove or disprove: for every $n \in \mathbb{N}, \exists$ distinct $k, l \in \mathbb{N}$ s.t. $11 \mid (a^k - a^l)$

- send help lmao prof save me ;-;
- Scratch:
  - Boxes: number of of values in $\pmod{11}$, 11 of them
  - Birds, powers of a
  - So there must be a box with two powers in it
  - $a^k \equiv a^l \pmod{11}$

> **Proof**
> Consider $a^1, a^2, \ldots, a^{12}$.
> There are eleven equivalence classes in mod 11.
> By the PHP least two of $a^1, a^2, \ldots, a^{12}$ must be in the same equivalence class in mod 11.
> So $\exists k, l \in \mathbb{N}$ s.t. $k \neq l$ and $a^k \equiv a^l \pmod{11} \implies a^k - a^l \equiv 0 \pmod{11}$ so $11 \mid a^k - a^l$

> **Problem 3**
>
> Given an example of two sets A and B that are both uncountable but $|A| \neq |B|$

- Lets take $|A| = \mathbb{R}$ and $|B| = \mathcal{P}(S)$ then $|A| < |B|$ due to a theorem from class

Problem 4

Prove that $|(0, 1)| = |(0, 1]|$

- Lets use CSB
- for $(0, 1) \to (0, 1]$ we take $f(x) = x$ which is injective(gotta actually prove this) so $|(0, 1)| \leq |(0, 1]|$
- for $(0, 1] \to (0, 1)$ we take $g(x) = \frac{x}{2}$ which is injective so $|(0, 1]| \leq |(0, 1)|$
- Since $|(0, 1)| \leq |(0, 1]|$ and $|(0, 1]| \leq |(0, 1)|$ so by CSB theorem $|(0, 1)| = |(0, 1]|$