Subspaces

Big Idea: Subspace of $\mathbb{R}^n$ include lines, planes and hyperplanes through the origin. A basis of a subspace is a linearly independent set of spanning vectors. The Rank-Nullity Theorem describes the dimensions of the nullspace and range of a matrix

Subspaces

DEFINTION

A subset $U \subseteq \mathbb{R}^n$ is a subspace *if*

1. U contains the zero vector $\vec{0}$
2. $\vec{u_1} + \vec{u_2} \in U \, \forall \vec{u_1}, \vec{u_2} \in U$
3. $c\vec{u} \in U \, \forall c \in \mathbb{R}, \vec{u} \in U$

Condition 2 is called closed under addition and condition 3 is called closed under scalar multiplication. Condition 3 with $c = 0$ implies condition 1

Example

- The zero subspace $\{0\}$ and the entire space $\mathbb{R}^n$ are both subspaces of $\mathbb{R}^n$
- Subspaces of $\mathbb{R}^2$ include any line through the origin
- Subspaces of $\mathbb{R}^3$ include any line or plane through the origin
- In general, subspaces of R^n are any hyperplanes of any dimension through the origin

EXAMPLE

Consider the set

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \geq 0 \right\}$$

Then U contains the zero vector
U is also closed under vector addition since
$\vec{\forall} u_1, \vec{u_2} \in U \quad \vec{u_1} + \vec{u_2} = \vec{u} \in U$ since $\vec{u} = [x, y]$ where $y \geq 0$
But U is not closed under scalar multiplication as $[1, 1] \in U$ but
$(-1)[1, 1] = [-1, -1] \notin U$
Therefore U is not a subset of $\mathbb{R}^2$

Linear Independence and Span

DEFINITION

A linear combination of vectors $\vec{u}_1, \ldots, \vec{u}_2 \in R^n$ is a vector

$$c_1\vec{u}_1 + \cdots + c_m\vec{u}_m$$

where $c_1, \ldots, c_m \in \mathbb{R}$. The **span** of vectors $\vec{u}_1, \ldots, \vec{u}_m \in R^n$ is the set of all linear combinations

$$\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\} = \{c_1\vec{u}_1 + \cdots + c_m\vec{u}_m \in \mathbb{R}^n : c_1, \ldots c_m \in \mathbb{R}\}$$

THEOREM

Let $\vec{u}_1, \ldots, \vec{u}_m \in R^n$. Then $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$ is a subspace of $\mathbb{R}^n$

EXAMPLE

The span of a single non zero vector $\vec{u}$ is a line with direction $\hat{u}$. The span of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a plane as long as the two vectors are **not** collinear

DEFINITION

A set of vectors $\{\vec{u}_1, \ldots, \vec{u}_m\} \subset \mathbb{R}^n$ forms a linearly independent set if the vectors satisfy the property

$$c_1\vec{u}_1 + \cdots + c_m\vec{u}_m = \vec{0} \iff c_1 = \cdots = c_m = 0$$

In other words $\{\vec{u}_1, .., \vec{u}_m\}$ is a linearly independent set if no vector in the set can be expressed as a linear combination of the others

How do we know if a set of vectors is linearly independent? Create a matrix where the columns are the given vectors

$$A = [\vec{u}_1 \quad \ldots \quad \vec{u}_m]$$

Then the vectors are linearly independent iff the linear system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$

Basis and Dimension

DEFINITION

Let $U \subseteq \mathbb{R}^n$ be a subspace. A set of vectors forms a **Basis** of U if:

1. $\{\vec{u_1}, \ldots, \vec{u_m}\}$ is a linearly independent set
2. $\text{span}\{\vec{u_1}, \ldots, \vec{u_m}\} = U$

The dimension of U is the number m of vectors in a basis

Nullspace and Range

DEFINITION

The nullspace of a $m \times n$ matrix A is

$$N(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

THEOREM

Let A be an $m \times n$ matrix. The nullspace $N(A)$ is a subspace of $\mathbb{R}^n$

THEOREM

Let A be an $m \times n$ matrix and let $A = LU$ be the LU Decomposition (if it exists). Then $N(A) = N(U)$.

DEFINITION

The range of an $m \times n$ matrix is:

$$R(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

The range is also called the image or column space of A

Matrix multiplication can be written as

$$A\vec{x} = \begin{bmatrix} \vec{a_1} & \ldots & \vec{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{a_1} + \cdots + x_n\vec{a_n}$$

Therefore the range of A is equal to the span of the columns

$$R(A) = \operatorname{span}\{\vec{a}_1, .., \vec{a}_n\}$$

and that's why $R(A)$ is sometimes called the column space

THEOREM

Let A be an $m \times n$ matrix. The range $R(A)$ is a subspace of $\mathbb{R}^m$

THEOREM

Let A be an $m \times n$ matrix. Then

$$\dim(R(A)) = \operatorname{rank}(A)$$

PROOF

The rank of A is the number of nonzero rows in the row echelon form of A. The dimension of $R(A)$ is the number of linearly independent columns in A which also equal the number of nonzero rows in A.

THEOREM

Let $A = LU$ be the LU Decomposition of A if it exists and let $r = \operatorname{rank}(\mathrm{A})$. Then

$$R(A) = \operatorname{span}\{\vec{l}_1, \ldots, \vec{l}_r\}$$

where $\vec{l}_1, \ldots, \vec{l}_r$ are the first r columns of L In particular, $\vec{l}_1, \ldots, \vec{l}_r$ is a basis of $R(A)$

PROOF

If $\operatorname{rank}(A) = r$ then only the first r entries of the vector $U\vec{x}$ are nonzero

$$U\vec{x} = \begin{bmatrix} * & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & * & * \\ 0 & \cdots & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \end{bmatrix}$$

Therefore

$$LU\vec{x} = \begin{bmatrix}\vec{l_1}, \dots, \vec{l_n}\end{bmatrix}\begin{bmatrix}* \\ \vdots\end{bmatrix} = (*)\vec{l_1} + \cdots + (*)\vec{l_r}$$

Rank Nullity Theorem

THEOREM

Let A be an $m \times n$ matrix. Then

$$\dim(R(A)) + \dim(N(A)) = n$$

PROOF

The dimension of $N(A)$ is equal to the number of columns in row echelon form of A without a leading nonzero entry
$\operatorname{rank}(A) = \dim(R(A))$ is equal to the number of columns of the row echelon form with a leading non zero entry and there are n total columns