Transfer Operators
(Mini-courses at Suzhou University)

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Contents

1 Introduction to Transfer Operators

Notes of a mini course at Suzhou University, taught by **Huyi Hu**.

§1.1 Definitions

Setting

- X compact metric space, $\mathscr{B}$ the Borel σ-algebra, ν a probability measure on $(X, \mathscr{B})$.
- $T : (X, \mathscr{B}) \to (X, \mathscr{B})$ measurable.
- $\mathcal{B}$ a Banach space consisting of functions defined on X. Let $\mathcal{B}^*$ be the dual space of $\mathcal{B}$, where for $g \in \mathcal{B}^*$, we can write

$$\langle g, f \rangle = \int f \cdot g \mathrm{d}\nu, \quad \forall f \in \mathcal{B}.$$

  ν is called a **reference measure**.

Example 1.1.1

1. $\mathcal{B} = L^p(X, \nu)$ and $\mathcal{B}^* = L^q(X, \nu)$ for $1 \leqslant p < \infty$.
2. $\mathcal{B} = C(X)$ and $\mathcal{B}^* = \{\text{signed measures on } X\}$.
3. $\mathcal{B} = H^s(X)$ and $\mathcal{B}^* = H^{-s}(X)$, the Sobolev spaces.

Definition 1.1.2. The **Koopman operator** $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is defined by

$$\mathcal{F} f = f \circ T, \quad \forall f \in \mathcal{B}.$$

Definition 1.1.3. The **(Perron-Frobenius-Ruelle) transfer operator** $\mathcal{L} : \mathcal{B}^* \to \mathcal{B}^*$ is the dual operator of $\mathcal{F}$, that is, for any $g \in \mathcal{B}^*$,

$$\langle \mathcal{L} g, f \rangle = \langle g, \mathcal{F} f \rangle, \quad \forall f \in \mathcal{B}.$$

Annotation 1.1.4 We can always think transfer operators acting on the measure space. Even though a function we should view as a density with respect to the reference measure.

Proposition 1.1.5

1. $\mathcal{L}$ is a linear operator.
2. $\mathcal{L}$ preserves integral: $\int (\mathcal{L} f) \mathrm{d}\nu = \int f \mathrm{d}\nu$.
3. $\mathcal{L}$ is a contraction, that is, $\|\mathcal{L}\|_{L^1} \leqslant 1$.
4. $\mathcal{L}$ is a positive operator.
5. $\mathcal{L}^k$ is a transfer operator associated to T^k.

Note that $L^\infty(X,\nu)^* \supsetneqq L^1(X,\nu)$, then if we want to define the transfer operator on L^1, we need to check whether it is well-defined.

Lemma 1.1.6

Suppose T is nonsingular with respect to ν, i.e. $\forall E \in \mathscr{B}$ with $\nu(E) = 0$, $\nu(T^{-1}E) = 0$. Then $\forall f \in L^1(X,\nu), \mathcal{L}f \in L^1(X,\nu)$.

Proof. For every $E \in \mathscr{B}$, let

$$\mu(E) = \int_E \mathcal{L}f \mathrm{d}\nu = \int_{T^{-1}E} f \mathrm{d}\nu.$$

Because T is nonsingular, we have μ is absolutely continuous with respect to ν. Then μ admits a density, that is, $\mathcal{L}f \in L^1(X,\nu)$. □

Proposition 1.1.7

For any nonnegative $h \in L^1(X,\nu)$, $\mathcal{L}h = h$ iff the measure $\mathrm{d}\mu = h\mathrm{d}\nu$ is T-invariant. In particular, $\mathcal{L}c = c$ for any nonzero constant c iff ν is T-invariant.

Proposition 1.1.8

Assume for ν-a.e. $x \in X$, there exists $E \in \mathscr{B}$ with $x \in E, \nu(E) < 0$ and $T^{-1}E = \bigcup_{i=1}^N E_i$, $1 \leqslant N \leqslant \infty$, such that $\{E_i\}$ are pairwise disjoint and $T|_{E_i}$ is one-one onto E, and $\frac{\mathrm{d}(\nu\circ T)}{\mathrm{d}\nu}$ exists and not equal to 0. Then for every $f \in L^1(X,\nu)$,

$$\mathcal{L}f(x) = \sum_{y\in T^{-1}x} f(y)\frac{1}{\frac{\mathrm{d}(\nu\circ T)}{\mathrm{d}\nu}(y)}$$

for ν-a.e. $x \in X$.

Remark 1.1.9 — If $X \subset \mathbb{R}^n$ and ν is the Lebesgue measure, and T is piecewise differentiable. By Sard's theorem, the Jacobian

$$|\det D_yT| \neq 0, \quad \nu-\text{a.e. } x \in X, y \in T^{-1}y.$$

So,

$$\mathcal{L}f(x) = \sum_{y\in T^{-1}x} f(y)\frac{1}{|\det D_yT|}$$

is well-defined. In particular, if T is injective, then

$$\mathcal{L}f(x) = f(T^{-1}x)\frac{1}{|\det D_{T^{-1}x}T|}.$$

Definition 1.1.10. Given a **potential function** $\phi : X \to \mathbb{R}$, the **transfer operator** $\mathcal{L}_\phi$ is defined by

$$\mathcal{L}_\phi f(x) = \sum_{y\in T^{-1}x} e^{\phi(y)}f(y).$$

Remark 1.1.11 — If we take $\phi(x) = -\log \frac{\mathrm{d}(\nu\circ T)}{\mathrm{d}\nu}(x)$, then $\mathcal{L}_\phi$ coincides with the earlier definition of transfer operators.

Lemma 1.1.12

Suppose X is a compact metric space and $T : X \to X$ is continuous. For every $\phi \in C(X)$, let $\mathcal{L}_\phi : C(X) \to C(X)$ be the transfer operator and $\mathcal{L}_\phi^* : \mathcal{M}(X) \to \mathcal{M}(X)$ be the dual. Then there exists $\lambda \in \mathbb{R}_+$ and $\nu \in \mathcal{M}(X)$ such that $\mathcal{L}_\phi^*\nu = \lambda\nu$.

Suppose $\mathcal{L}_\phi^*\nu = \lambda\nu$. Then $\frac{1}{\lambda}\mathcal{L}_\phi$ is the transfer operator $\mathcal{L}$ defined earlier for (X,ν). In fact, for every $f, g \in C(X)$,

$$\int \left(\frac{1}{\lambda}\mathcal{L}_\phi g\right) \cdot f \mathrm{d}\nu = \int \frac{1}{\lambda}\mathcal{L}_\phi\left(g \cdot f\circ T\right)\mathrm{d}\nu = \int \mathcal{F}f \cdot g \mathrm{d}\nu.$$

Definition 1.1.13. The measure ν satisfying $\mathcal{L}_\phi^*\nu = \nu$ is called a **conformal measure** for $\mathcal{L}_\phi$.

If we assume that $\exists \nu \in \mathcal{M}(X)$ and $\lambda > 0$ such that $\mathcal{L}_\phi^*\nu = \lambda\nu$, we further assume that $\exists h \in L^1(\nu)$ with $h \geqslant 0$ such that $\mathcal{L}_\phi h = \lambda h$. Then we get $\mathrm{d}\mu = h\mathrm{d}\nu$ is a T-invariant measure. This is because for every $f \in C(X)$,

$$\langle h\nu, f\circ T\rangle = \langle \mathcal{L}^*\nu, h(f\circ T)\rangle = \langle \nu, \mathcal{L}(h(f\circ T))\rangle = \langle \nu, (\mathcal{L}h)f\rangle = \langle h\nu, f\rangle.$$

At this time, we define

$$\psi(x) = \phi(x) + \log h(x) - \log h(Tx) - \log\lambda,$$

then

$$\mathcal{L}_\psi g(x) = \frac{1}{\lambda h(x)}\mathcal{L}_\phi(hg)(x).$$

In particular, if we take $g = 1$, then we get $\mathcal{L}_\psi 1 = 1$. Let $\mathrm{d}\mu = h\mathrm{d}\nu$, we also have $\mathcal{L}_\psi^*\mu = \mu$.

Definition 1.1.14. $\mathcal{L}_\psi$ is called the **normalized transfer operator** of $\mathcal{L}_\phi$.

§1.2 The Ruelle's Perron-Frobenius theorem

Let M be a compact Riemannian manifold. A map $T : M \to M$ is called **expanding** if there exists $\rho \in (0,1)$ such that for every $x, y \in M$ close enough,

$$d(x,y) \geqslant \rho^{-1} d(x,y).$$

Notation 1.2.1. $C^\alpha, C^{1+\alpha}$: α-Hölder (Df is α-Hölder, resp) continuous functions/maps.

Theorem 1.2.2 (Ruelle's Perron-Frobenius Theorem)

Let $T : X \to X$ be a $C^{1+\alpha}$ expanding map and $\phi \in C^\alpha$. Then $\exists \lambda \in \mathbb{R}_+, \nu \in \mathcal{M}(X)$ and $h \in C^\alpha$ with $h > 0$ such that

$$\mathcal{L}_\phi^*\nu = \lambda\nu, \quad \mathcal{L}_\phi h = \lambda h, \quad \int h\mathrm{d}\nu = 1.$$

Consequently, the measure μ given by $\mathrm{d}\mu = h\mathrm{d}\nu$ is invariant.

Sketch of the proof. Let ν be the fixed point of the map

$$\mu \mapsto \overline{\mathcal{L}}_\phi^* \mu = \frac{\mathcal{L}_\phi^* \mu}{\|\mathcal{L}_\phi^* \mu\|},$$

this shows the existence of ν and λ. Denote $\overline{\mathcal{L}}_\phi = \frac{1}{\lambda}\mathcal{L}_\phi$, take the space

$$\mathcal{H} = \mathcal{H}_J = \left\{ f \in C^\alpha : f > 0, \int f \mathrm{d}\nu = 1, \frac{f(x)}{f(y)} \leqslant e^{Jd(x,y)^\alpha}, \forall x, y \in M \right\},$$

where J is fixed constant. Then $\mathcal{H}$ is compact by Arzela-Ascoli. Besides, $\overline{\mathcal{L}}_\phi \mathcal{H} \subset \mathcal{H}$ for J large enough. By Schauder fixed point theorem, $\mathcal{L}_\phi$ has a positive eigenfunction. □

Remark 1.2.3 — The invariant measure μ is a Gibbs measure and an equilibrium of ϕ. That is,

$$P(T, \phi) = h_\mu(T) + \int \phi \mathrm{d}\mu.$$

Also, the system has exponential decay of correlations with respect to μ. (See §1.5, §1.6)

Remark 1.2.4 — If $\phi = -\log|\det D_x T|$, then ν is the Lebesgue measure and $\mathrm{d}\mu = h\mathrm{d}\nu$ is the absolutely continuous invariant measure.

Remark 1.2.5 — The same arguments still apply on invariant subsets $S \subset M$ if T has a Markov partition on S.

Remark 1.2.6 — Since $\|\mathcal{L}\|_{L^1(\nu)} \leqslant 1$ and $\mathcal{L}h = h$ means that 1 is the largest eigenvalue in $L^1(\nu)$. It can also be proved that:

1. If T is topologically transitive, then μ is ergodic and 1 is simple.
2. If T is topologically mixing, then μ is mixing and 1 is the unique eigenvalue on $\mathbb{S}^1$.

§1.3 The Lasota-Yorke inequality

Let $B_w = L^1(X, \nu)$ and denote $\|\cdot\|_w = \|\cdot\|_{\mathcal{B}_w} = \|\cdot\|_{L^1(\nu)}$ be a weaker norm. Suppose there is a stronger norm $\|\cdot\| = \|\cdot\|_{\mathcal{B}}$ on $L^1(X, \nu)$, and assume that

$$\mathcal{B} := \{f \in \mathcal{B}_w : \|f\|_{\mathcal{B}} < \infty\}$$

forms a Banach space. We assume that $\mathcal{B}$ satisfying the following assumptions.

Assumption (Compactness). The inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_w$ is compact, that is, the unit ball in $\mathcal{B}$ is compact in $\mathcal{B}_w$.

Assumption (Lower semicontinuity). For any sequence $\{f_n\} \subset \mathcal{B}$ with $f_n \to f$ ν-a.e.,

$$\|f\| \leqslant \liminf_n \|f_n\|.$$

Assumption (Openness). For any nonnegative function $f \in \mathcal{B}$, the set $\{f > 0\}$ is almost open with respect to the reference measure.

Definition 1.3.1. A set S is **almost open** w.r.t. ν if $\exists$ open set $U \supset S$ such that $\nu(U \setminus S) = 0$.

Example 1.3.2

1. $\mathcal{B}$ is the space of α-Hölder continuous functions with the norm

$$\|f\| = \|f\|_{L^1} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

2. $\mathcal{B}$ is the space of C^1 functions with the norm

$$\|f\| = \|f\|_{L^1} + \|Df\|.$$

3. $X = I = [0,1]$, $\mathcal{B}$ is the space of functions with bounded variation. Take the norm

$$\|f\| = \|f\|_{L^1} + \bigvee_0^1 f.$$

Let $\mathcal{B}_w = L^1(X, \nu)$ and $\mathcal{L}$ be a transfer operator w.r.t. a reference measure ν.

Theorem 1.3.3

Suppose there exists a Banach space $\mathcal{B}$ satisfying the Compactness assumption. And there exists $\theta \in (0,1), B > 0$ such that

$$\|\mathcal{L}f\|_{\mathcal{B}} \leqslant \theta \|f\|_{\mathcal{B}} + B \|f\|_{\mathcal{B}_w}, \quad \forall f \in \mathcal{B}. \tag{LY}$$

Then there exists $h \in \mathcal{B}_w$ such that $\mathcal{L}h = h$. Further, if $\mathcal{B}$ satisfies Lower semicontinuity assumption, then $h \in \mathcal{B}$.

Definition 1.3.4. The inequality (LY) is called the **Lasota-Yorke inequality**.

Proof. By (LY), inductively,

$$\|\mathcal{L}^n f\|_{\mathcal{B}} \leqslant \theta^n \|f\|_{\mathcal{B}} + B \left(\sum_{i=0}^{n-1} \theta^i \right) \|f\|_{\mathcal{B}_w}.$$

Hence, $\{\mathcal{L}^n f\}$ is bounded in $\mathcal{B}$. Let

$$f_n = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}^i f,$$

then there exists $h \in \mathcal{B}_w$ which is a limit point of $\{f_n\}$ in $\mathcal{B}_w$. Then $\mathcal{L}h = h$. Furthermore, there exists a sequence in $\{f_n\}$ converges to h ν-a.e.. Then if $\mathcal{B}$ satisfies Lower semicontinuity assumption, $h \in \mathcal{B}$. □

Another form of Lasota-Yorke inequality used frequently is

$$\|\mathcal{L}^n f\|_{\mathcal{B}} \leqslant A\theta^n \|f\|_{\mathcal{B}} + B \|f\|_{\mathcal{B}_w}, \quad \forall n \geqslant 0,$$

where $\theta \in (0,1), A, B > 0$ are constants.

Application

Recall the BV space we have mentioned in example 1.3.2, we can check that $\mathcal{B}$ satisfies the assumptions of Compactness and Lower semicontinuity.

Let $T : I \to I$ be a piecewise C^2 map, such that $\exists S = \{x_1, x_2, \cdots, x_n\} \subset I$ with:

(i) $\Delta := \inf_{x\in I\setminus S} |T'(x)| > 2$,

(ii) $\sup_{x\in I\setminus S} \frac{|T''(x)|}{|T'(x)|^2} < K < \infty$.

Let ν be the Lebesgue measure.

Theorem 1.3.5

Then T has an absolutely continuous invariant measure with $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \in \mathcal{B}$.

Proof. Take $\theta = 2/\Delta$, $B = 2K + 2c$ where $c = \min\{\nu(T([x_{i-1}, x_i))\}$, we can show that

$$\bigvee_0^1 \mathcal{L}f \leqslant \theta \bigvee_0^1 f + B\int_0^1 |f|\mathrm{d}\nu.$$

Hence $\mathcal{L}$ satisfies (LY), the statement follows. □

§1.4 Quasi-compactness

Let $\mathcal{B}$ and $\mathcal{B}_w$ be Banach spaces with norm $\|\cdot\|$ and $\|\cdot\|_w$, respectively. Let $\mathcal{P} : \mathcal{B}_w \to \mathcal{B}_w$ be an operator with $\mathcal{P}\mathcal{B}_w \subset \mathcal{B}_w$. Suppose that

(i) For $\{x_n\} \subset \mathcal{B}$ with $\{x_n\} \leqslant 1$, $\lim_{n\to\infty} \|x_n - x\|_w = 0$, implies $x \in \mathcal{B}$ and $\|x\| \leqslant 1$.

(ii) $\exists H > 0$ such that $\|\mathcal{P}^n\|_w \leqslant H, \forall n \geqslant 0$.

(iii) $\exists \theta \in (0, 1), B > 0$ such that $\|\mathcal{P}f\| \leqslant \theta\|f\| + B\|f\|_w, \forall f \in \mathcal{B}$.

(iv) The map $\mathcal{P} : \mathcal{B} \to \mathcal{B}_w$ is compact, that is, $\mathcal{P}$ maps bounded sets to compact sets.

Theorem 1.4.1 (Ionescu-Tulcea and Marinescu's Theorem)

Then $\mathcal{P}$ is quasi-compact in $\mathcal{B}$, and $\sigma_{\text{ess}}(\mathcal{P}|_{\mathcal{B}}) \leqslant \theta$. In particular,

1. the operator $\mathcal{P} : \mathcal{B} \to \mathcal{B}$ has finitely many number of eigenvalues $\lambda_1, \cdots, \lambda_p$ of modulus 1 of finitely multiplicity;
2. there is a decomposition of $\mathcal{P}$ into

$$\mathcal{P} = \sum_{i=0}^{p} \lambda_i \pi_i + \mathcal{Q},$$

where π_i are projection to the eigenspace of λ_i and $\mathcal{Q}$ is a bounded operator such that $\mathcal{Q}\pi_i = \pi_i\mathcal{Q} = 0, \pi_i^2 = \pi_i, \pi_i\pi_j = 0$ for $i \neq j$ and $\rho(\mathcal{Q}) < 1$.

Remark 1.4.2 — The inequality $\|\mathcal{P}f\| \leqslant \theta\|f\|+B\|f\|_w$ was used by W.Doeblin and R.Fortet (1937) and is called the **Doeblin-Fortet inequality**.

Remark 1.4.3 — Lasota and Yorke (1973) obtained the inequality for the transfer operators of piecewise expanding maps on the unit interval with the Banach space $\mathcal{B} = BV$.

Sketch of the proof. We have $\|\mathcal{P}^n f\| \leqslant \theta^n \|f\| + B(1-\theta)^{-1} \|f\|_w$. So $\|\mathcal{P}^n\|_{\mathcal{B}}, \|\mathcal{P}^n\|_w$ are uniformly bounded and hence $\sigma(\mathcal{P}) \leqslant 1$. Let λ be an eigenvalue with module 1, take

$$\mathcal{B}_\lambda = \{x \in \mathcal{B} : \|x\|_w \leqslant 1, \mathcal{P}x = \lambda x\}.$$

Then $\mathcal{B}_\lambda$ is bounded in $\mathcal{B}$ and hence compact in $\mathcal{B}_w$. Therefore, $\mathcal{B}_\lambda$ is finitely dimensional.

It takes some work to prove that there are only finite number of eigenvalues on $\mathbb{S}^1$, we just accept this conclusion. Then for every $\lambda \in \mathbb{S}^1$, take

$$\pi_\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\lambda^k} \mathcal{P}^k.$$

The statement follows. □

Definition 1.4.4. An operator $\mathcal{P}$ on a Banach space $\mathcal{B}$ is called **quasi-compact** if there exists $\theta \in [0, \rho(\mathcal{P}))$ and closed subspaces F and H that satisfy the following conditions:

(i) $\mathcal{B} = F \oplus H$;

(ii) $\mathcal{P}F \subset F$ and $0 < \dim F < \infty$;

(iii) $\mathcal{P}H \subset H$ and $\rho(\mathcal{P}|_H) \leqslant \theta$.

Assumption (Assumption P). Suppose $\mathcal{P} : \mathcal{B}_w \to \mathcal{B}_w$ is an operator satisfying the following:

(i) There exists $H > 0$ such that for any $f \in \mathcal{B}_w, n \in \mathbb{N}$, $\|\mathcal{P}^n f\|_w \leqslant H \|f\|$.

(ii) There exists $\theta \in (0,1), A, B > 0$ such that for any $f \in \mathcal{B}, n \in \mathbb{N}$,

$$\|\mathcal{P}^n f\| \leqslant A\theta^n \|f\| + B \|f\|_w.$$

Theorem 1.4.5

Suppose $\mathcal{P} : \mathcal{B}_w \to \mathcal{B}_w$ is an operator of a Banach space $\mathcal{B}_w$. Assume there is a Banach space $\mathcal{B} \subset \mathcal{B}_w$ with satisfies the assumptions of Compactness, Lower semicontinuity and Openness. Besides, $\mathcal{P}\mathcal{B} \subset \mathcal{B}$ and $\mathcal{P}$ satisfies the assumption above. Then $\mathcal{P}|_{\mathcal{B}}$ is quasi-compact and $\rho(\mathcal{P}|_H) \leqslant \theta$.

Recall a spectral decomposition of an Axiom A system.

Theorem 1.4.6

Let $f : M \to M$ be an Axiom A diffeomorphism, and $\Omega(f)$ be the non wandering set. One can write $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_s$, where the Ω_i are pairwise disjoint closed sets, such that

(i) $f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive.

(ii) $\Omega_i = X_{i,1} \cup \cdots \cup X_{i,n_i}$ with $X_{i,j}$ are pairwise disjoint closed sets, $f(X_{i,j}) = X_{i,j}$ and $f^{n_i}|_{X_{i,j}}$ is topologically mixing.

Theorem 1.4.7

Suppose $\mathcal{B}$ and $\mathcal{B}_w$ satisfy the assumptions of Compactness, Lower semicontinuity and Openness, and the transfer operator $\mathcal{L}$ satisfies Assumption P. Then

1. T has a finite number of ergodic absolutely continuous invariant measures $\mu_1, \cdots, \mu_n$ with density functions $h_1, \cdots, h_n \in \mathcal{B}$, respectively.
2. For each $1 \leqslant i \leqslant n$, there exists a finite collection of disjoint sets $\{\Lambda_{i,j}\}_{j=1}^{n_i}$ such that
   (i) $\mathrm{d}\mu_i = h_i \mathrm{d}\nu$, and $\{h_i > 0\} = \Lambda_{i,1} \cup \cdots \cup \Lambda_{i,n_i}$;
   (ii) for $j = 1, \cdots, n_i$, $\mathcal{L}h_{i,j} = h_{i,j+1}$, where $h_{i,j} = h_i \mathbb{1}_{\Lambda_{i,j}}$;
   (iii) for $j = 1, \cdots, n_i$, $T\Lambda_{i,j} = \Lambda_{i,j+1}$, and $(\Lambda_{i,j}, T^{n_i}|_{\Lambda_{i,j}}, h_{i,j}\mathrm{d}\nu)$ is exact, and therefore is mixing.

Sketch of the proof. A fact shows that all the eigenvalues of $\mathcal{L}$ on $\mathbb{S}^1$ are roots of unity. Take $k > 0$ such that $\lambda_i = 1$ for every eigenvalue $\lambda_i \in \mathbb{S}^1$. Then 1 is the only eigenvalue of $\mathcal{L}^k$ on $\mathbb{S}^1$. Write $\mathcal{L}^k = \pi + \mathcal{Q}^k$ where π is the projection to the eigenspace of eigenvalue 1. Denote $\mathcal{E} = \pi(\mathcal{B}), \mathcal{E}_+ = \{h \in \pi(\mathcal{B}) : h \geqslant 0\}$ and $\mathcal{E}_+(1) = \left\{h \in \mathcal{E}_+ : \int h\mathrm{d}\nu = 1\right\}$. $\mathcal{E}_+$ is convex.

For every distinct extreme points h_1, h_2 of $\mathcal{E}_+(1)$, we have $\min\{h_1, h_2\} = 0$, ν-a.e.. Besides, $\mathcal{E}_+(1)$ only has finitely many extremal points $\{h_i\}$. Let $\Lambda_i = \{h_i > 0\}$, the statement follows. ☐

§1.5 The Spectral gap

Definition 1.5.1. A linear operator $\mathcal{P}$ has a **spectral gap** if 1 is a simple eigenvalue and

$$\sigma(\mathcal{P}) = \{1\} \cup \mathcal{S}, \text{ where } \mathcal{S} \subset \{z \in \mathbb{C} : |z| \leqslant z < 1\}.$$

If $\mathcal{P}$ have a spectral gap, then we can write $\mathcal{P} = \pi + \mathcal{Q}$, where π is the projection onto the eigenspace of 1, and $\sigma(\mathcal{Q}) < 1$. Let h be an eigenvector of 1. For every $f \in \mathcal{B}$, write $f = ah + g$, where $ah = \pi f$ and $g = \mathcal{Q}g$. Then $\mathcal{P}^n g = \mathcal{Q}^n g \to 0$ exponentially fast.

From spectral gap to decay of correlation

In probability, the **covariance** of two random variable ξ and η are given by

$$\mathrm{Cov}(\xi, \eta) = \mathbf{E}[(\xi - \mathbf{E}\,\xi)(\eta - \mathbf{E}\,\eta)] = \mathbf{E}(\xi\eta) - \mathbf{E}\,\xi\,\mathbf{E}\,\eta.$$

The correlation of ξ and η are given by

$$\mathrm{Cor}(\xi, \eta) = \frac{\mathrm{Cov}(\xi, \eta)}{\sigma_\xi \sigma_\eta} = \frac{\mathbf{E}(\xi\eta) - \mathbf{E}\,\xi\,\mathbf{E}\,\eta}{\sigma_\xi \sigma_\eta},$$

where σ_ξ^2 and σ_η^2 are variances of ξ, η, respectively. In particular, if ξ and η are independent, then the covariance of ξ and η is zero.

In a dynamical system (M, T), for functions $f, g : M \to \mathbb{R}$, we usually study the $\mathrm{Cov}(f \circ T^n, g)$. It naturally to guess that

$$\mathrm{Cov}(f \circ T^n, g) = \int f \circ T^n \cdot g\mathrm{d}\mu - \int f\mathrm{d}\mu \int g\mu \to 0.$$

Which corresponds to the mixing property. **Decay of correlations** concerns the rates of convergence. Let $f \in \mathcal{B}$ and $g \in \mathcal{B}^*$, then

$$\mathrm{Cov}(f \circ T^n, g) = \left|\int f \cdot \mathcal{L}^n g\mathrm{d}\mu - \int f\mathrm{d}\mu \int g\mathrm{d}\mu\right| \leqslant \|f - \langle\mu, f\rangle\|_{\mathcal{B}} \|\mathcal{L}^n g - \langle\mu, g\rangle\|_{\mathcal{B}^*}.$$

Then the rate of $\mathcal{L}^n g - \langle\mu, g\rangle \to 0$ leads to a rate of decay of correlations.

Definition 1.5.2. A system (T, μ) has **exponential decay of correlations** for functions in $\mathcal{B}$ and $\mathcal{B}^*$, if there exists $\tau \in (0,1)$ such that $\forall f \in \mathcal{B}, g \in \mathcal{B}^*$,

$$\|\mathrm{Cov}(f \circ T^n, g)\| \leqslant C \cdot \tau^n,$$

where $C = C(f, g)$ is a constant depending on f, g.

Let $\mathcal{L}$ be a normalized transfer operator, that is $\mathcal{L}c = c$. We assume that $\mathcal{L}$ has a spectral gap. Then we can write

$$\mathcal{L} = \pi + \mathcal{Q},$$

such that $\sigma(\mathcal{Q}) < 1$. Besides, $\pi(g) = \langle \mu, g \rangle$, hence $\mathcal{L}$ has an exponential decay of correlations.

> Annotation 1.5.3 A spectral gap leads to an exponential speed of mixing.

§1.6 The Hilbert metric

Let V be a vector space.

Definition 1.6.1. A **cone** $\mathcal{C} \subset V$ is a subset such that

(i) If $f \in \mathcal{C}$, then $\alpha f \in \mathcal{C}, \forall \alpha \in \mathbb{R}_+$.

(ii) If $f, f \in \mathcal{C}$, then $f + g \in \mathcal{C}$.

We require further $0 \notin \mathcal{C}$ and $\mathcal{C} \cap -\mathcal{C} = \varnothing$.

Example 1.6.2

1. $V = \mathbb{R}^2$ and $\mathcal{C} = \{x = (x_1, x_2) : 0 < x_2 \leqslant 2x_1\}$.
2. $V = \mathbb{R}^n$ and $\mathcal{C} = \{x = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}$.
3. $V = C^0(X)$ and $\mathcal{C} = \left\{f \in C^0(X) : f(x) > 0, \forall x \in X\right\}$.

Definition 1.6.3. The **projective metric (Hilbert metric)** $\Delta(\,\cdot\,,\,\cdot\,)$ on a given cone $\mathcal{C}$ is given by

$$\Delta(f, g) = \log \frac{a(f, g)}{b(f, g)}, \quad \forall f, g \in \mathcal{C},$$

where $a(f, g) = \inf\{a : ag - f \in \mathcal{C}\}$, $b(f, g) = \sup\{b : f - bg \in \mathcal{C}\}$.

> **Remark 1.6.4** — $\Delta(\,\cdot\,,\,\cdot\,)$ is usually viewed as a metric defined on the **projective space** of $\mathcal{C}$.

Lemma 1.6.5

1. $\Delta(f, g) = 0$ iff $f = \alpha g$ for some $\alpha \in \mathbb{R}_+$.
2. $\Delta(f, g) = \Delta(g, f)$.
3. $\Delta(f, h) \leqslant \Delta(f, g) + \Delta(g, h)$.

Theorem 1.6.6

Let V_1, V_2 be two vector spaces and $\mathcal{A} : V_1 \to V_2$ be a linear map such that $\mathcal{A}(\mathcal{C}_1) \subset \mathcal{C}_2$ for cones $\mathcal{C}_i \subset V_i, \mathcal{C}_i \cap -\mathcal{C}_i = \varnothing$. Let Δ_i be the Hilbert metric corresponding to $\mathcal{C}_i$. Set $\Gamma = \sup_{f,g\in\mathcal{A}(\mathcal{C}_i)} \Delta_2(f,g)$, we have

$$\Delta_2(\mathcal{A}f, \mathcal{A}g) \leqslant \tanh\left(\frac{\Gamma}{4}\right)\Delta_1(f,g), \quad \forall f, g \in \mathcal{C}_1.$$

If we take $V = V_1 = V_2$, then we have $\forall f, g \in \mathcal{C}$,

$$\Delta(\mathcal{A}^n f, \mathcal{A}^n g) \leqslant \left(\tanh\frac{\Gamma}{4}\right)^n \Delta(f,g) \to 0$$

exponentially fast.

Application

Let $T : M \to M$ be an expanding map with the expanding constant $\rho^{-1} > 1$. Let ϕ be an α-Hölder potential function. Then there exists $\nu \in \mathcal{M}(M)$ and $h \in C^\alpha$ with $h > 0$ such that

$$\mathcal{L}_\phi^* \nu = \lambda\nu, \quad \mathcal{L}_\phi h = \lambda h, \quad \int h \mathrm{d}\nu = 1.$$

Let $\mathcal{L} = \mathcal{L}_\psi$ be the normalized transfer operator. We have $\mathcal{L}1 = 1$ and $\mathcal{L}^*\mu = \mu$ and μ is an absolutely continuous invariant measure. Let $\mathcal{B} = C^\alpha(M)$.

> **Aim 1.6.7.** To show that $\forall f \in \mathcal{B}$, $\|\mathcal{L}f - \langle\mu, f\rangle\| \leqslant Cr^n, r < 1$.

Let $J_0 > 0$ such that $|\phi(x) - \phi(y)| \leqslant J_0 d(x,y)^\alpha$. Take $J > 0$ sufficiently large such that $\rho^{-\alpha}(1 + J_0/J) < 1$. Denote $J' = \rho^{-\alpha}(J_0 + J)$. Define

$$\mathcal{C} = C_J(M) = \{f \in C^\alpha : f > 0, |\log f(x) - \log f(y)| \leqslant Jd(x,y)^\alpha, \forall x, y \in M\},$$

and we fix the Hilbert metric Δ on $\mathcal{C}$.

Lemma 1.6.8 $\mathcal{L}C_J(M) \subset C_{J'}(M)$.

Lemma 1.6.9 $\exists\Gamma > 0$ such that $\Delta(f, 1) \leqslant \Gamma$ for every $f \in C_{J'}(M)$.

Then for every $f \in C_J$, we have

$$\Delta(\mathcal{L}^n f, 1) \leqslant \left(\tanh\frac{\Gamma}{4}\right)^n \Delta(f,1) = r^n\Delta(f,1).$$

Assume that $\Delta(\mathcal{L}^n f, 1) = \log(a_n/b_n)$, by definition

$$a_n - \mathcal{L}^n f, \mathcal{L}^n f - b_n \in C_J.$$

In particular, $a_n > \mathcal{L}^n f > b_n$. Because $|\log a_n - \log b_n| \leqslant Cr^n$, we get

$$|\mathcal{L}^n f(x) - \langle\mu, f\rangle| \leqslant C'r^n, \quad \forall x \in M.$$

This implies that $\mathrm{Cov}(f \circ T^n, g) \to 0$ exponentially fast for every $f \in C_J$. For every $f \in C^\alpha$, we can choose $a, b \in \mathbb{R}, a \neq 0$ such that $af + b \in C_J(M)$, then the conclusion also holds.

§1.7 Hyperbolic cases

Till now, we have only considered the expanding maps. Now we consider the hyperbolic cases.

Let $T : M \to M$ be a C^r Anosov diffeomorphism, $r \geqslant 1$. That is, there exists constant $\lambda > 1, C > 0$, and a splitting $TM = E^u \oplus E^s$ such that for every $x \in M, n \geqslant 0$,

$$\begin{cases} \|DT_x^n v\| \geqslant C^{-1}\lambda^n \|v\|, & \forall v \in E_x^u; \\ \|DT_x^n v\| \leqslant C\lambda^n \|v\|, & \forall v \in E_x^s. \end{cases}$$

For each $x \in M$, take $\gamma^s(x)$ such that $\gamma(x) \subset W^s_{\mathrm{loc}}(x)$ and $T(\gamma^s(x)) \subset \gamma^s(T(x))$. Denote

$$\Gamma^s := \{\gamma^s(x) : x \in M\}.$$

For any $f \in C^1(M)$, define the (weaker) norm by

$$\|f\|_w = \sup_{\gamma \in \Gamma^s} \sup_{\substack{\varphi \in C^1(\gamma), \\ \|\varphi\|_{C^1(\gamma)} \leqslant 1}} \left| \int_\gamma f\varphi \mathrm{d}m_\gamma \right|,$$

where m_γ is the Lebesgue measure on γ. Denote by $\partial^u f$ be the directional derivative along W^u, and set

$$\|f\| = \|f\|_w + \|\partial^u f\|_w .$$

Let $\mathcal{B}$ and $\mathcal{B}_w$ be the completion of $C^1(M)$ with respect to $\|\cdot\|$ and $\|\cdot\|_w$, respectively.

Lemma 1.7.1

1. The unit ball of $\mathcal{B}$ is compact in $\mathcal{B}_w$.
2. $\forall \{f_n\} \subset \mathcal{B}$ with $f_n \to f$ ν-a.e., $\|f\|_{\mathcal{B}} \leqslant \liminf_{n\to\infty} \|f_n\|_{\mathcal{B}}$.

Recall that for a diffeomorphism $T : M \to M$, we consider the transfer operator

$$\mathcal{L}f(x) = f(T^{-1}x)\frac{1}{|\det D_{T^{-1}x}T|} = \frac{f(T^{-1}x)}{JT(T^{-1}x)}.$$

Theorem 1.7.2

1. There exists $H > 0$ such that for any $f \in C^1(X)$,

$$\|\mathcal{L}^n f\|_w \leqslant H \|f\|_w , \quad \forall n \geqslant 0.$$

2. $\exists \theta \in (0,1), A > 0$ and $B_n > 0$ for each n, such that for any $f \in C^1(X)$,

$$\|\mathcal{L}^n f\| \leqslant A\theta^n \|f\| + B_n \|f\|_w , \quad \forall n \geqslant 0.$$

2 Positive Transfer Operators with Dini Continuous Potentials

Notes of a mini course at Suzhou University, taught by **Yunping Jiang**. [Lecture Notes]

§2.1 Introduction

- Dynamics: $f : X \to X$, induces a group/semigroup $\{f^n\}_{n\in\mathbb{Z}/\mathbb{N}}$.

Aim 2.1.1. For $x_0 \in X$, the forward limit $f^n(x_0) \to ?$ (study the future of x_0).

We consider the ω-limit set $\omega(x_0)$. In a invertible case, for example

Example 2.1.2
If x_0 is a fixed point, then $\omega(x_0) = \{x_0\}$. If x_0 is periodic, then $\omega(x_0) = \mathrm{Orb}(x_0)$.

In a non-invertible case, we construct the space

$$X_{\infty,f} = \{(\cdots, x_n, x_{n-1}, \cdot, x_1, x_0) : x_n \in X, fx_n = x_{n-1}\}$$

to record the history of x_0. Which is called an **inverse limit**.

Example 2.1.3
Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, $f : \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto z^2$. The inverse limit can be viewed as

$$\Sigma^- = \prod_{-\infty}^{-1} \{0, 1\}.$$

Given a dynamical system $f : X \to X$, let

$$P_n := \{\text{the set of all periodic points of period } n\}, \quad F_n = \{x \in X : f^n(x) = x\}.$$

Then $F_n = \bigcup_{d|n} P_d$, hence $\sharp F_n = \sum_{d|n} \sharp P_d$. By the Möbius inversion, we have

$$\sharp P_n = \sum_{d|n} \mu(d) \sharp F_{n/d}.$$

Definition 2.1.4. The **topological entropy** is

$$h(f) := \lim_{n\to\infty} \frac{1}{n} \log \sharp F_n.$$

Definition 2.1.5. The **zeta function** of the system is defined as

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\sharp F_n}{n} z^n\right).$$

Fact 2.1.6. $\zeta(z)$ is analytic in the disk of radius e^{-h} centered at 0.

Definition 2.1.7. A partition $X = X_0 \cup \cdots \cup X_{d-1}$ is called a **Markov partition** for f if

(i) each $X_i \neq \varnothing$, compact,

(ii) $X_i \cap X_j = \varnothing$,

(iii) $f|_{X_i}$ is one-one,

(iv) $f(X_i) = \bigcup_{k=1}^{m_i} X_{i_k}$.

If f has a Markov partition η_0, define $A = (a_{ij})_{d\times d}$, where

$$a_{ij} = \begin{cases} 1, & f(X_i) \supset X_j; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\Sigma_A := \left\{\omega = (i_0, i_1, \cdots) : i_n \in \{0, 1, \cdots, d-1\}, a_{i_{n-1}i_n} = 1\right\},$$

let $\sigma_A : \Sigma_A \to \Sigma_A$ be the shift map. We have (if it is a conjugate)

$$\sharp F_{n,f} = \sharp F_{n,\sigma_A} = \operatorname{tr}(A^n),$$

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{tr}(A^n)}{n} z^n\right) = \exp\left(-\operatorname{tr}(\log(\mathrm{Id} - zA))\right) = \frac{1}{\det(\mathrm{Id} - zA)}.$$

Theorem 2.1.8

For a mixing map $f : X \to X$ with a Markov partition. Then the unweighted zeta function $\zeta(z)$ is a rational function with a unique smallest pole $1/\rho(A)$ where $h(f) = \log \rho(A)$.

Theorem 2.1.9 (Perron-Frobenius)

If A is an $n \times n$ positive matrix, then A has a unique simple, positive, maximal eigenvalue ρ with a positive eigenvalue.

Remark 2.1.10 — $\rho = \rho(A)$ is the spectral radius, $\rho = \lim_{n\to\infty} \sqrt[n]{\|A^n\|}$.

Example 2.1.11

Let $I = [0, 1], I_0 = [0, a], I_1 = [b, 1]$, where $a < b$. Let $f : I_0 \to I, I_1 \to I$ linearly respectively. Then f admits a Markov partition $\{I_0, I_1\}$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$\sharp F_{n,f} = 2^n, \quad \zeta(z) = \frac{1}{1 - 2z}.$$

In general, we would like to consider a weighted zeta function for a weight ψ as follows

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\left(\sum_{x\in F_n}\prod_{i=0}^{n-1}\psi(f^i x)\right)\right).$$

Note that if $\psi \equiv 1$, then $\zeta(z)$ coincides with the unweighted zeta function. Define the pressure

$$P(\log\psi) = \limsup_{n\to\infty}\frac{1}{n}\log\left(\sum_{x\in F_n}\exp\left(\sum_{i=0}^{n-1}\log\psi(f^i(x))\right)\right).$$

Fact 2.1.12. $\zeta(z)$ is analytic in a disk of radius e^{-P} centered at 0.

§2.2 The Ruelle's Perron-Frobenius theorem

Setting

- X a compact metric space.
- $f : X \to X$ a **locally expanding map**, that is, $\exists C > 0, 0 < a < 1, \lambda > 1$ such that
$$d(f^n x, f^n y) \geqslant C\lambda^n d(x, y),$$
$\forall x, y \in X, d_n(x, y) = \max_{0\leqslant i\leqslant n-1} d(f^i x, f^i y) \leqslant a$ (the **Bowen metric**).
- f is **mixing**: $\forall U \subset X$ open, there exists n such that $f^n(U) = X$.
- $C(X) = \{$the space of all continuous functions φ with the norm $\|\varphi\| = \sup_{x\in X}|\varphi(x)|\}$.

For every $\varphi \in C(X)$, we define the **modulus of continuity**

$$\omega_\varphi(t) = \max_{d(x,y)\leqslant t}|\varphi(x) - \varphi(t)|.$$

Then we know that $\omega_\varphi(t) \to (0)(t \to 0)$.

- If $\omega_\varphi(t) \leqslant Ct^\alpha$ for some $C > 0, 0 < \alpha \leqslant 1$, then φ is called **α-Hölder**.
- We call φ is **Dini** if
$$\int_0^a \frac{\omega_\varphi(t)}{t}\mathrm{d}t < \infty.$$

Then we have

$$C^{\mathrm{H}}(X) = \bigcup_{0<\alpha\leqslant 1} C^\alpha(X) \subset C^{\mathrm{Dini}}(X).$$

Let ω be a modulus of continuity, that is, ω is nonnegative, increasing and $\omega(t) \to 0(t \to 0^+)$. We also call it **Dini** if

$$\int_0^a \frac{\omega(t)}{t}\mathrm{d}t < \infty.$$

For a modulus of continuity, we define the space

$$C^\omega := \{\varphi \in C(X) : \exists C > 0, \omega_\varphi(t) \leqslant C\omega(t)\}.$$

If ω is Dini, let

$$\widetilde{\omega}(t) = \sum_{n=1}^{\infty}\omega(\lambda^{-n}t),$$

which is also a modulus of continuity. $C^{\widetilde{\omega}}$ plays an important role in later studies. (See §2.3)

Definition 2.2.1. A positive $\psi \in C(X)$ is called a **potential**. Given a potential, the **transfer operator** is defined by

$$\mathcal{L}\varphi(x) = \sum_{y \in f^{-1}x} \psi(y)\phi(y).$$

Fact 2.2.2. If $\psi \in C^\alpha(X)$, then $\mathcal{L} : C^\alpha \to C^\alpha$.

Fact 2.2.3. If $\psi \in C^\omega$ and ω is Dini, then $\mathcal{L} : C^\omega \to C^\omega$.

Theorem 2.2.4 (Ruelle's Frobenius-Perron Theorem)

Suppose $f : X \to X$ is locally expanding and mixing. For any $\psi \in C^\alpha(X)$, $\mathcal{L}$ has a unique, maximal, positive and simple eigenvalue $\rho = \rho(\mathcal{L})$ (the spectral radius) with a positive eigenfunction $\varphi_0 \in C^\alpha(X)$.

For $K > 0, s > 0$, let

$$C^\alpha_{K,s} = \{\varphi \in C^\alpha(X) : \varphi(x) \geqslant s, |\log\varphi(x) - \log\varphi(y)| \leqslant Kd(x,y)^\alpha, \forall x, y, d(x,y) \leqslant a\}.$$

Lemma 2.2.5

Any bounded sequence in $C^\alpha_{K,s}$ has a convergent subsequence in $C(X)$ whose limit is in $C^\alpha_{K,s}$.

WLOG, assume $\min_{x\in X}\psi(x) = 1$ and let

$$K_0 = \sup_{d(x,y)\leqslant a} \frac{|\log\psi(x) - \log\psi(y)|}{d(x,y)^\alpha}.$$

Lemma 2.2.6

Let $0 < s < 1$, $K > \frac{K_0}{\lambda^\alpha - 1}$. Then $\forall \varphi \in C^\alpha(X)$ nonnegative, $\|\varphi\| = 1$, there exists $N > 0$, such that $\mathcal{L}^n\varphi \in C^\alpha_{K,s}$.

Proof. Since $\|\varphi\| = 1$, take $y_0 \in X, \varphi(y_0) = 1$, choose an open neighborhood of $U \ni y_0$ such that $\varphi|_U \geqslant s$. By mixing, there exists $n_0 \geqslant 0$ such that $f^{n_0}(U) = X$. Consider

$$\widetilde{\varphi}(x) = \mathcal{L}^{n_0}\varphi(x) = \sum_{y\in f^{-n_0}x} \left(\prod_{i=0}^{n_0-1} \psi(f^i y)\right) \varphi(y) \geqslant s.$$

Besides, for every $x, y \in X$ closed enough,

$$\mathcal{L}\widetilde{\varphi}(x) = \sum_{x'\in f^{-1}x} \psi(x')\widetilde{\varphi}(x') \leqslant \sum_{y'\in f^{-1}y} \psi(y')\widetilde{\varphi}(y')e^{(K_0+K')d(x',y')^\alpha}.$$

Note that $d(x,y) \geqslant \lambda d(x',y')$, hence $(K_0 + K')d(x',y')^\alpha \leqslant \lambda^{-\alpha}(K_0 + K')d(x,y)^\alpha$. By induction, we have

$$\mathcal{L}^n\widetilde{\varphi}(x) \leqslant \mathcal{L}^n\widetilde{\varphi}(y)e^{K_n d(x,y)^\alpha},$$

where $K_n = K_0\sum_{i=1}^n \lambda^{-i\alpha} + \lambda^{-n\alpha}K' \to \frac{K_0}{\lambda - 1}(n \to \infty)$. $\square$

Remark 2.2.7 — This lemma shows that a transfer operator can improve the Hölder norm in some sense.

By this lemma, we only need to find positive eigenvalue in $C^\alpha_{K,s}$. Define

$$S = \left\{\mu > 0 : \exists \varphi \in C^\alpha_{K,s}(X), \mathcal{L}\varphi \geqslant \mu\varphi\right\}.$$

Lemma 2.2.8 $S \neq \varnothing$ and bounded.

Proof of RFP Theorem. Take $\rho = \sup S$, there exists $\lambda_n \to \rho, \lambda_n \in S$. Then there exists φ_n such that $\mathcal{L}\varphi_n \geqslant \lambda_n\varphi_n$. Assume that $\min\varphi_n = s$, then $\{\varphi_n\} \subset C^\alpha_{K,s}$ is bounded. Take a subsequence converges to φ_0. Then $\varphi_0 \geqslant \rho\varphi_0$. By mixing, we have

$$\mathcal{L}\varphi_0 = \rho\varphi_0, \text{ and } \dim E_\rho = 1 \text{ where } E_\rho = \{\varphi \in C^\alpha : \mathcal{L}\varphi = \rho\varphi\}.$$

The maximal of ρ follows by the previous lemma. □

Application

Suppose M is a compact Riemannian manifold, $f : M \to M$ is $C^{1+\alpha}$. A probability measure ν is called f-invariant if $\nu(f^{-1}A) = \nu(A)$ for every $A \in \mathscr{B}$. We say ν is **smooth** if

$$\nu(A) = \int_A p(y)\mathrm{d}y$$

for some $p(y) \in C(M)$.

Lemma 2.2.9

Suppose $f : M \to M$ is C^1 and $J(f) \neq 0$, and suppose $\mathrm{d}\nu = \alpha(y)\mathrm{d}y$. Then ν is f-invariant iff

$$\sum_{y\in f^{-1}x} \frac{p(y)}{J(f)(y)} = p(x).$$

Theorem 2.2.10 (Krzyzewski-Szlenk)

Suppose $f : M \to M$ is $C^{1+\alpha}$, locally expanding and mixing. Then f has a unique smooth f-invariant probability measure with an α-Hölder continuous density.

Proof. Take $\psi = \frac{\|J(f)\|}{|J(f)|}$, then $\min\psi = 1$ and $\psi \in C^\alpha_{K_0,1}$. Consider the transfer operator

$$\mathcal{L}\varphi(y) = \sum_{x\in f^{-1}y} \psi(x)\varphi(x).$$

Then there exists $p(x) \in C^\alpha$ such that

$$\sum_{y\in f^{-1}x} \frac{p(y)}{J(f)(y)} = \mu_0 p(y),$$

where $\mu_0 = \frac{\rho}{\|J(f)\|} = 1$. Normalize p, the statement follows. □

For $\psi(x) > 0$, let

$$\mathcal{L}\varphi(x) = \sum_{y \in f^{-1}x} \psi(y)\varphi(y).$$

Then

$$\mathcal{L}^n\varphi(x) = \sum_{y \in f^{-n(x)}} \left(\prod_{i=0}^{n-1} \psi(f^i y) \right) \varphi(y),$$

write $G_n(y) = \prod_{i=0}^{n-1} \psi(f^i y)$, then

$$\mathcal{L}^n\varphi(x) = \sum_{y \in f^{-1}x} G_n(y)\varphi(y).$$

§2.3 A generalized RPFT of the Dini case

Suppose $\omega(t)$ is a modulus of continuity.

Example 2.3.1

1. $\omega(t) = t^\alpha, 0 < \alpha \leqslant 1$, α-Hölder.
2. $\omega(t) = \frac{1}{|\log t|^\alpha}, \alpha > 1.$
3. $\omega(t) = e^{-\alpha|\log\log t|^\beta}, \beta > 1.$

Recall $\widetilde{\omega}(t) = \sum_{n=1}^\infty \omega(\lambda^{-n}t)$, where λ is the expanding constant of f.

Lemma 2.3.2

Suppose ω is Dini, then

$$\widetilde{\omega}(t) \leqslant \frac{1}{\log\lambda} \int_0^t \frac{\omega(s)}{s} \mathrm{d}s \leqslant \sum_{n=0}^\infty \omega(\lambda^{-n}t).$$

It follows that $\widetilde{\omega}$ is also a modulus of continuity. But in general, $\widetilde{\omega}(t)$ is not Dini anymore.

Theorem 2.3.3 (A Generalized RPFT)

Assume $f : X \to X$ is locally expanding and mixing. Assume ψ is a Dini potential and $\omega(t) = \omega_\psi(t)$. Then the spectral radius $\rho = \rho(\mathcal{L})$ is not an eigenvalue of $\mathcal{L} : C^\omega(X) \to C^\omega(X)$. Instead, ρ is the unique maximum, positive and simple eigenvalue of $\mathcal{L} : C^{\widetilde{\omega}}(X) \to C^{\widetilde{\omega}}(X)$.

Proof. Firstly, we check that $\mathcal{L} : C^{\widetilde{\omega}}(X) \to C^{\widetilde{\omega}}(X)$. For every $\varphi \in C^{\widetilde{\omega}}(X)$, we have

$$\begin{aligned} |\mathcal{L}\varphi(y) - \mathcal{L}\varphi(x)| &\leqslant \sum_{x' \in f^{-1}x, y' \in f^{-1}y} |\psi(y')\varphi(y') - \psi(x')\varphi(x')| \\ &\leqslant \sum_{x',y'} |\psi(y') - \psi(x')||\varphi(y')| + |\varphi(y') - \varphi(x')||\psi(x')| \\ &\leqslant K(\omega(\lambda^{-1}d(x,y)) + \widetilde{\omega}(\lambda^{-1}d(x,y))) = K\widetilde{\omega}(d(x,y)). \end{aligned}$$

Hence $\mathcal{L}\varphi \in C^{\widetilde{\omega}}(X)$.

Recall $G_n(x) = \prod_{i=0}^{n-1} \psi(f^i x)$, then

$$\mathcal{L}^n \varphi(x) = \sum_{x' \in f^{-n}x} G_n(x')\varphi(x').$$

Proposition 2.3.4 (Dini distortion property)

Assume $\psi \in C^\omega$, then $\exists K_0 > 0$ such that for every $x, y \in X$, $d_n(x,y) \leqslant a$,

$$\left|\log \frac{G_n(x)}{G_n(y)}\right| \leqslant K_0 \widetilde{\omega}(d(f^n x, f^n y)).$$

Then for every $\varphi \in C^\omega, x, y \in X, d_n(x,y) \leqslant a$, we have

$$\begin{aligned}
\mathcal{L}^n \varphi(x) &= \sum_{x' \in f^{-n}x} G_n(x')\varphi(x') \\
&\leqslant \sum_{y' \in f^{-n}y} G_n(y') e^{K_0 \sum\limits_{i=1}^{n} \omega(\lambda^{-i} d(x,y))} \varphi(y') e^{K' \omega(\lambda^{-n} d(x,y))} \\
&\leqslant \mathcal{L}^n \varphi(y) \exp\left(K_0 \sum_{i=1}^{n} \omega(\lambda^{-i} d(x,y)) + K' \omega(\lambda^{-n} d(x,y))\right).
\end{aligned}$$

This implies that if ψ is only Dini continuous, even $\mathcal{L}^n \varphi$ is in $C^\omega(X)$, the limit point will not be in $C^\omega(X)$ in general. In particular, every non-negative eigenvector of $\mathcal{L}$ can only be in $C^{\widetilde{\omega}}(X)$.

For $K, s > 0$, define

$$C^{\widetilde{\omega}}_{K,s} = \left\{\varphi \in C(X) : \varphi(x) \geqslant s, |\log \varphi(x) - \log \varphi(y)| \leqslant K \widetilde{\omega}(d(x,y))\right\}.$$

Lemma 2.3.5 Any bounded sequence in $C^{\widetilde{\omega}}_{K,s}$ has a convergent subsequence.

Assume $\min_{x \in X} \psi(x) = 1$.

Lemma 2.3.6 Let $0 < s < 1, K \geqslant K_0$, then $\mathcal{L}(C^{\widetilde{\omega}}_{K,s}) \subset C^{\widetilde{\omega}}_{K,s}$.

Let $S = \left\{\mu : \exists \varphi \in C^{\widetilde{\omega}}_{K,s}, \mathcal{L}\varphi \geqslant \mu\varphi\right\}$. Then $S \neq \varnothing$ and S is bounded. Let $\rho = \sup S$, then there exists $\varphi_0 \in C^{\widetilde{\omega}}_{K,s}$ such that $\mathcal{L}\varphi_0 = \rho\varphi_0$. $\square$

Theorem 2.3.7 (Generalized KS Theorem)

Assume $f : M \to M$ is a $C^{1+\omega}$ locally expanding and mixing map where ω is Dini. Then f has a unique smooth f-invariant probability measure whose density is in $C^{\widetilde{\omega}}(M)$.

Geometric Interpretation

For $\psi \in C^\alpha(X)$, let

$$\mathcal{D}_\alpha = \left\{\varphi \in C^\alpha(X) : \varphi > 0, \|\varphi\| = 1\right\},$$

and define $F(\varphi) = \frac{\mathcal{L}\varphi}{\|\mathcal{L}\varphi\|} : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$. Then RPFT implies F has a unique fixed point in $\mathcal{D}_\alpha$ and moreover,

$$\|F^n\varphi - \varphi_0\| \leqslant C\mu^n\|\varphi - \varphi_0\|, \quad \forall n.$$

Now we consider

$$\mathcal{D}_\omega = \{\varphi \in C^\omega(X) : \varphi > 0, \|\varphi\| = 1\},$$

then $F : \mathcal{D}_\omega \to \mathcal{D}_\omega$. By generalized RPFT, F has no fixed point in $\mathcal{D}_\omega$. Instead, F has a fixed point in $\partial\mathcal{D}_\omega \subset \mathcal{D}_{\widetilde{\omega}}$. Furthermore, $\forall\varphi \in \mathcal{D}_\omega$,

$$\|F^n\varphi - \varphi_0\| \to 0, \quad n \to \infty.$$

Two convergences above are very similar with the following complex analysis theorem.

Theorem 2.3.8 (Denjoy-Wolff)
Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$, let $F : \mathbb{D} \to \mathbb{D}$ be an analytic map and is not elliptic Möbius transformation (not an injection). Then

(1) f has a unique fixed point $p \in \mathbb{D}$ such that for every $z \in \mathbb{D}$, $f^n(z) \to p$ exponentially fast.

(2) $\exists p \in \partial\mathbb{D}$ such that $f^n(z) \to p$ for every $z \in \mathbb{D}$.

Annotation 2.3.9 This conclusion reminds me of the Poincaré disc model.

Indeed, the proof of this theorem applying the hyperbolic metric on $\mathbb{D}$. Can we do this for $\mathcal{L}$? Yes. The hyperbolic metric on $\mathbb{D}$ can also be viewed as a cross ratio. We can also define a "cross ratio" on $\mathcal{L}$ similarly. That is the **Hilbert metric**. For some detailed discussion, see section The Hilbert metric.

§2.4 A probability point of view

Let $\mathscr{B}(X)$ be the Borel σ-algebra on X. Let $\mathcal{M}(X)$ be the space of all finite Borel measures on X. By Riesz representation theorem, $\mathcal{M}(X)$ corresponds to the space of all positive functionals on $C(X)$, given by

$$\langle \mu, f\rangle = \int f \mathrm{d}\mu, \quad \forall f \in C(X), \mu \in \mathcal{M}(X).$$

Endowing $\mathcal{M}(X)$ with the weak* topology, then every bounded closed set is compact in $\mathcal{M}(X)$.

Let $\mathrm{Prob}(X)^f$ the space of all f-invariant probability measures on X. Then $\mathrm{Prob}(X)^f$ is non-empty, convex. Every extremal point of $\mathrm{Prob}(X)^f$ is an ergodic f-invariant measure.

For a transfer operator $\mathcal{L}$ with a Dini continuous potential φ. Take $\varphi_0 \in C^{\widetilde{\omega}}(X)$ such that $\mathcal{L}\varphi_0 = \rho\varphi_0$, where $\rho = \rho(\mathcal{L})$. Consider

$$\widetilde{\psi} = \frac{\psi\varphi_0}{\rho\varphi_0 \circ f}, \quad \widetilde{\mathcal{L}}\varphi(x) = \sum_{x' \in f^{-1}x} \widetilde{\psi}(x')\varphi(x'),$$

then $\widetilde{\mathcal{L}}1 = 1$, $\widetilde{\mathcal{L}}$ is called the **normalized transfer operator** of $\mathcal{L}$. Let $\mathcal{L}^*, \widetilde{\mathcal{L}}^*$ be the dual operators of $\mathcal{L}, \widetilde{\mathcal{L}}$, respectively. Then μ is f-invariant iff $\widetilde{\mathcal{L}}^*\mu = \mu$. In fact,

$$\mathcal{L}^n\varphi = \rho^n\varphi_0\widetilde{\mathcal{L}}^n\left(\frac{\varphi}{\varphi_0}\right), \quad \mathcal{L}^{*n}\nu = \frac{\rho^n}{\varphi_0}\widetilde{\mathcal{L}}^{*n}(\varphi_0\nu).$$

Probability view

We consider
$$P_n\varphi(x) = (\widetilde{\mathcal{L}}^n\varphi)(f^n x) = \sum_{x'\in f^{-n}(f^n x)} \widetilde{G}_n(x')\varphi(x').$$
Then $P_n 1 = 1$ for every $n \geqslant 1$.

Annotation 2.4.1 It looks like taking a conditional expectation with respect to a partition given by local stable manifolds.

Lemma 2.4.2
$\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is a compatible chain of Markovian projections, i.e.

(i) $P_m P_n = P_n P_m = P_m$, for every $m \geqslant n \geqslant 1$.

(ii) Let $\Gamma_n = \operatorname{Im} P_n$, then for every $\varphi \in C(X), \chi \in \Gamma_n$,
$$P_n(\varphi\chi) = \chi P_n\varphi.$$

Let $P_n^* : M(X) \to M(X)$ be the dual of P_n, then $P_n(\operatorname{Prob}(X)) \subset \operatorname{Prob}(X)$. Let
$$\mathcal{G}_n = \{\mu \in \operatorname{Prob}(X) : P_n^*\mu = \mu\},$$
then $\mathcal{G}_m \subset \mathcal{G}_n$ for every $m \geqslant n$. Let $\mathcal{G}_\infty = \bigcap_{n=0}^{\infty} \mathcal{G}_n$, then $\mathcal{G}_\infty \neq \varnothing$.

Annotation 2.4.3 For every f-invariant measure μ given by the transfer operator, we have $\langle \mu, P_n\varphi\rangle = \langle \mu, \widetilde{\mathcal{L}}^n\varphi\rangle = \langle \mu, \varphi\rangle$, hence $\mu \in \mathcal{G}_n, \forall n \geqslant 0$.

Lemma 2.4.4
Any $\mu \in \mathcal{G}_\infty$ is a **Gibbs measure** associated with ψ, i.e. $\exists C = C(r) > 0$ such that
$$C^{-1} \leqslant \frac{\mu(B_n(x,r))}{\rho^{-n}G_n(x)} = \frac{\mu(B_n(x,r))}{\exp\left(-n\log\rho + \sum_{i=0}^{n-1}\log\psi(f^i x)\right)} \leqslant C,$$
where $B_n(x,r) = \{y \in X : d_n(x,y) \leqslant r\}$.

Let $\mathscr{B}_n$ be the σ-algebra generated by Γ_n, then $\mathscr{B}_{n+1} \subset \mathscr{B}_n$, write $\mathscr{B}_\infty = \bigcap_{n=1}^{\infty}\mathscr{B}_n$.

Definition 2.4.5. A measure $\mu \in \mathcal{G}_\infty$ is called **ergodic** if $\mu(B) = 0$ or 1 for every $B \in \mathscr{B}_\infty$.

Take $\mu \in \mathcal{G}_\infty$, for every $\chi \in \Gamma_n$, we have
$$\langle \mu, \chi P_n\varphi\rangle = \langle \mu, P_n(\varphi\chi)\rangle = \langle \mu, \varphi\chi\rangle,$$
hence $P_n\varphi = \mathbf{E}(\varphi|\mathscr{B}_n),\ \mu-\text{a.e.}$ the conditional expectation. Then $\{P_n\varphi\}_{n\geqslant 0}$ forms a backward martingale, hence
$$P_n\varphi \to \mathbf{E}(\varphi|\mathscr{B}_\infty), \quad \mu-\text{a.e. and } L^1.$$

Lemma 2.4.6

1. If $\mu_1, \mu_2 \in \mathcal{G}_\infty$ are ergodic, then either $\mu_1 = \mu_2$ or $\mu_1 \perp \mu_2$.
2. $\mu \in \mathcal{G}_\infty$ is ergodic iff μ is an extremal point of $\mathcal{G}_\infty$.

Lemma 2.4.7 $\sharp\mathcal{G}_\infty = 1$.

Proof. By the inequality given by lemma 2.4.4, we can show that for every $\mu_1, \mu_2 \in \mathcal{G}_\infty$, μ_1 and μ_2 are mutually absolutely continuous. □

Lemma 2.4.8
Suppose $\mathcal{P} = \{P_n\}_{n=1}^\infty$ is a compatible chain of Markovian projections, then the followings are equivalent:

(1) $\sharp\mathcal{G}_\infty = 1$.

(2) $\forall \varphi \in C(X)$, $P_n\varphi$ pointwise converge to a constant.

(3) $\forall \varphi \in C(X)$, $P_n\varphi$ uniformly converge to a constant.

Theorem 2.4.9
Suppose X is a compact metric space and $f : X \to X$ is a locally expanding and mixing map. For any $\psi \in C^{\mathrm{Dini}}(X)$, $\omega(t) = \omega_\psi(t)$, we have

1. $\rho = \rho(\mathcal{L})$ is the unique maximal positive simple eigenvalue of $\mathcal{L} : C^{\widetilde{\omega}}(X) \to C^{\widetilde{\omega}}(X)$ with a positive eigenfunction φ_0, $\mathcal{L}\varphi_0 = \rho\varphi_0$.
2. there is a unique probability measure ν_0 on X such that $\mathcal{L}^*\nu_0 = \rho\nu_0$.
3. Take $\langle \nu_0, \varphi_0 \rangle = 1$, then for every $\varphi \in C(X)$,
$$\rho^{-n}\mathcal{L}^n\varphi \rightrightarrows \langle \nu_0, \varphi \rangle \varphi_0.$$
4. The probability measure $\mu_0 = \varphi_0\nu_0$ is the unique Gibbs measure associated with ψ and is ergodic.