On the dimension of limit sets on $\mathbb{RP}^2$

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Contents

§1 Introduction (Nov 14)

This minicourse is based on [LPX, arXiv:2311.10265] and [JLPX, arXiv:2311.10262].

Rigidity of quasi-circles (Bowen 70s). Let $\Gamma = \pi_1(S_g)$, where S_g is the closed surface of genus g. Let $\rho_0 : \Gamma \to \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}_+(\mathbb{H}^2)$ be the natural embedding. Let $\iota_0 : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{C})$ be the embedding. Therefore,

$$\iota_0 \circ \rho_0 \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C})),$$

where $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is a finite dimensional space. We consider **deformations** of of $\iota_0 \circ \rho_0$, that is, $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ close to $\iota_0 \circ \rho_0$. Such ρ are quasi-Fuchsian representations.

Note that $\iota_0 \circ \rho_0(\Gamma)$ preserves a minimal invariant closed set in $\mathbb{CP}^1$, the $\mathbb{RP}^1$, which is called the **limit set** of $\iota_0 \circ \rho_0$ and denoted by $L(\iota_0 \circ \rho_0)$. If ρ is closed to $\iota_0 \circ \rho_0$, then there also exists a ρ-limit set $L(\rho)$ on $\mathbb{CP}^1$, which is a topological circle.

Theorem 1.1 (Bowen) $\dim_{\mathrm{H}} L(\rho) > 1$ if $\rho(\Gamma)$ is not contained in a conjugate of $\mathrm{PSL}_2(\mathbb{R})$.

Some related results:

- (Ruelle) $\dim_{\mathrm{H}} L(\rho)$ depends analytically on ρ.
- (McMullen, Bridgemen) They studied the Hessian of $\dim_{\mathrm{H}} L(\rho)$. This also relates to the Weil-Peterson metric on Teichmüller spaces.
- (Sullivan, Lax-Philips) $\#\{\gamma : \|\rho(\gamma)\| \leqslant T\} = C \cdot T^{\dim_{\mathrm{H}} L(\rho)}(1 + O(T^{-\varepsilon}))$.
- Critical exponents (Patterson, Sullivan): consider the Poincaré series

$$P_\rho(s) = \sum_{\gamma\in\Gamma} e^{-s\cdot d_{\mathbb{H}^3}(o,\rho(\gamma)o)}.$$

  Let $s(\rho) := \min\{s > 0 : P_s(\rho) < \infty\}$. Then $s(\rho) = \dim_{\mathrm{H}} L(\rho)$.

Limit sets on $\mathbb{RP}^2$. Let Γ be as above. Let $\rho_1 : \Gamma \to \mathrm{SL}_2(\mathbb{R})$ and $\iota_1 : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})$ be the embedding. Similarly, $\iota_1 \circ \rho_1(\Gamma)$ acts on $\mathbb{RP}^2$ preserving the limit set $L(\iota_1 \circ \rho_1) = \mathbb{RP}^1$. We consider ρ close to $\iota_1 \circ \rho_1$. Sullivan showed that the limit set $L(\rho) \subset \mathbb{RP}^2$ is a topological circle.

Theorem 1.2 (Li-Pan-Xu, 2023)

For every $\varepsilon > 0$, there exists an open neighborhood $\mathcal{O} \subset \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$ of $\iota_1 \circ \rho_1$ such that for every $\rho \in \mathcal{O}$:

- either $\rho(\Gamma)$ reducibly on $\mathbb{RP}^2$,
- or $|\dim_{\mathrm{H}} L(\rho) - 3/2| < \varepsilon$.

Conjecture 1.3 If $\dim_{\mathrm{H}} L(\rho) = 1$ then $L(\rho) = \mathbb{RP}^1$.

To compute the Hausdorff dimension of limit sets, we recall the **affinity exponent**. For every $\rho(\gamma) \in \mathrm{SL}_3(\mathbb{R})$, let $\rho(\gamma) = k_1 a k_2$ be its Cartan decomposition, where $a = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. For $s > 0$, let

$$P_\rho(s) = \begin{cases} \displaystyle\sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)^s (\rho(\gamma)), & 0 < s \leqslant 1; \\ \displaystyle\sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)(\rho(\gamma)) \left(\frac{\sigma_3}{\sigma_1}\right)^{s-1} (\rho(\gamma)), & 1 < s \leqslant 2. \end{cases}$$

The affinity exponent is defined to be $s_{\mathrm{A}}(\rho) := \min\left\{ s > 0 : P_\rho(s) < \infty \right\}$.

Theorem 1.4 (Li-Pan-Xu) $s(\rho) = \dim_{\mathrm{H}} L(\rho)$.

Why affinity exponent? Recall the definition of Hausdorff dimensions. For a set X, we have

$$\mathscr{H}^s(X) = \lim_{\varepsilon \to 0} \inf\left\{ \sum (\mathrm{diam}\, U_i)^s : \bigcup U_i \supset X, \mathrm{diam}\, X \leqslant \varepsilon \right\}.$$

Then $\dim_{\mathrm{H}}(X) := \min\{ s > 0 : \mathscr{H}^s(X) < 0 \}$.

To cover $L(\rho)$, we consider the image of a unit ball on $\mathbb{RP}^2$ by $\rho(\gamma)$. This is an ellipse with two axes of lengths σ_2/σ_1 and σ_3/σ_1. We can cover this ellipse by two ways: use a ball of radius σ_2/σ_1 or use σ_2/σ_3 balls of radius σ_3/σ_1. If such ellipses is not too much, the first way is more optimal. This corresponds to the case $s \leqslant 1$, where $P_\rho(s) = \sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)^s (\rho(\gamma))$. For the case when there are much ellipses, we use the second way to cover each ellipse. This gives the expression of series for $s > 1$ as $\sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)(\rho(\gamma)) \left(\frac{\sigma_3}{\sigma_1}\right)^{s-1} (\rho(\gamma))$.

Anosov representations.

Definition 1.5. Let $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$ be a homomorphism. We say ρ is **Anosov** if there exists $c > 0$ such that for every $\gamma \in \Gamma$,

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) < Ce^{-|\gamma|/C},$$

where $|\gamma|$ is the word length of γ with respect to a fixed symmetric generating set.

This concept was introduced by Labourie (2000s). He studied the Hitchin component, which is related to higher Teichmüller theory.

Property. Anosov representations form an open subset in $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$.

Note that $\iota_1 \circ \rho_1(\Gamma)$ is Anosov since $\rho_1(\Gamma)$ is cocompact in $\mathrm{SL}_2(\mathbb{R})$. Consequently, $\rho(\Gamma)$ is also Anosov. Theorem 1.4 holds for Zariski dense Anosov representations.

The crucial part of Theorem 1.4 is the lower bound of $\dim_{\mathrm{H}} L(\rho)$. We make use of stationary measures. A basic fact is $\dim_{\mathrm{H}} X \geqslant \dim_{\mathrm{H}} \mu$ if $\operatorname{supp} \mu \subset X$. The lower bound of $\dim_{\mathrm{H}} L(\rho)$ is given by the following two ingredients.

- **Variational principle of affinity exponents**: $s(\rho) = \sup_{\operatorname{supp} \mu \subset L(\rho)} \{ \dim_{\mathrm{LY}} \mu \}$. This part is joint with Jiao.
- **Dimension formula of stationary measures**: $\dim_{\mathrm{H}} \mu = \dim_{\mathrm{LY}} \mu$.

Definition 1.6 (Stationary measures)**.** ν a finitely supported probability measure on $\mathrm{SL}_3(\mathbb{R})$. A probability measure μ on $\mathbb{RP}^2$ is **ν-stationary** if

$$\mu = \nu * \mu = \int_{\mathrm{SL}_3(\mathbb{R})} g_* \mu \mathrm{d}\nu(g).$$

Definition 1.7 (Lyapunov exponents)**.** The **Lyapunov exponents** of ν are given by

$$\lambda_1(\nu) = \lim_{n \to \infty} \int \log \|g_1 \cdots g_n\| \mathrm{d}\nu(g_1) \cdots \nu(g_n),$$

$$\lambda_2(\nu) = \lim_{n \to \infty} \int \log \frac{\| \wedge^2 (g_1 \cdots g_n)\|}{\|g_1 \cdots g_n\|} \mathrm{d}\nu(g_1) \cdots \nu(g_n),$$

and $\lambda_3 = -\lambda_1 - \lambda_2$.

If $\langle \operatorname{supp} \nu \rangle$ is Zariski dense in $\mathrm{SL}_3(\mathbb{R})$ then $\lambda_1(\nu) > \lambda_2(\nu) > \lambda_3(\nu)$.

Definition 1.8. The **Furstenberg entropy** is given by

$$h_{\mathrm{F}}(\mu, \nu) = \int \log \frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(\xi) \left(\frac{\mathrm{d}g\mu}{\mathrm{d}\mu}(\xi) \right) \mathrm{d}\nu(g) \mathrm{d}\mu(\xi). \tag{1.1}$$

Definition 1.9. The **Lyapunov dimension** of μ is

$$\dim_{\mathrm{LY}} \mu = \begin{cases} \dfrac{h_{\mathrm{F}}(\mu, \nu)}{\lambda_1(\nu) - \lambda_2(\nu)}, & \text{if } h_{\mathrm{F}}(\mu, \nu) \leqslant \lambda_1(\nu) - \lambda_2(\nu); \\ 1 + \dfrac{h_{\mathrm{F}}(\mu, \nu) - (\lambda_1(\nu) - \lambda_2(\nu))}{\lambda_1(\nu) - \lambda_3(\nu)}, & \text{otherwise.} \end{cases}$$

Theorem 1.10 (Li-Pan-Xu)

If ν is finitely supported, $\langle \operatorname{supp} \nu \rangle$ is Zariski dense and exponential separation, then

$$\dim_{\mathrm{H}} \mu = \dim_{\mathrm{LY}} \mu.$$

The equality $\dim_{\mathrm{H}} \mu = \dim_{\mathrm{LY}} \mu$ was conjectured by Kaplan-Yorke and Douady-Oesterlé.

Ledrappier-Young formula and projections. The following is shown by Ledrappier-Lessa and Rapaport that there exists $\gamma_1, \gamma_2 > 0$ such that for almost every direction V and points y, the projection $\dim \pi_{V^\perp}\mu = \gamma_1$ and the fiber $\dim \mu_y^V = \gamma_2$. Moreover, we have the **Ledrappier-Young formula**

$$\dim \mu = \gamma_1 + \gamma_2, \quad h_{\mathrm{F}} = (\lambda_1 - \lambda_2)\gamma_1 + (\lambda_1 - \lambda_3)\gamma_2.$$

Theorem 1.10 is then a direct consequence of the following result on the dimension of projection measures.

Theorem 1.11 (Li-Pan-Xu) $\gamma_1 = \min\{1, h_{\mathrm{F}}/(\lambda_1 - \lambda_2)\}$.

§2 Entropy growth argument of Hochman (Nov 16)

Last time we have mentioned that we want to compute the dimension of projection measures. The entropy growth argument of Hochman is a powerful tool to compute these quantities.

Recall the example of standard 1/3-Cantor set. Let $f_1, f_2 : [0,1] \to [0,1]$ where $f_1(x) = x/3$ and $f_2(x) = (x+2)/3$. The standard Cantor set

$$C_3 = \bigcap_{n\geqslant 1} \bigcup_{i_1,\cdots,i_n} f_{i_1} \cdots f_{i_n}[0,1].$$

Consider the random walk $\nu = \frac{1}{2}\delta_{f_1} + \frac{1}{2}\delta_{f_2}$. We have the weak convergence $\nu^{*n} * \delta_0 \to \mu_3$. For almost every $x \in C_3$, we have

$$\dim \mu_3 = \lim_{n\to\infty} \frac{\log \mu_3(B(x, 1/3^n))}{\log(1/3^n)} = \frac{\log 2}{\log 3} = \frac{\text{Entropy}}{\text{Lyapunov exponent}}.$$

Bernoulli convolution. For $\lambda \in (0,1)$, we consider two matrices

$$A_1 = \begin{bmatrix} \lambda & 1 \\ & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & -1 \\ & 1 \end{bmatrix}$$

with the fraction action on $\mathbb{R}$. That is, $A_1 x = \lambda x + 1$ and $A_2 x = \lambda x - 1$ for $x \in \mathbb{R}$. Let $\nu = \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2}$. Then

$$\mu^{*n} * \delta_0 \to \mu_\lambda,$$

where μ_λ is called **Bernoulli convolution**. Here μ_λ is supported on $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$. The difficulty of studying μ_λ comes from that $A_1 I_\lambda$ and $A_2 I_\lambda$ have some overlapping. Erdös first studied the absolutely continuity of μ_λ's.

Theorem 2.1 (Erdös)

$|\hat{\mu}_\lambda(\xi)| \not\to 0$ ($|\xi| \to \infty$) if $1/\lambda$ is a Pisot number (x is Pisot if x is an algebraic integer and all its Galois conjugates have absolute values less than 1).

Conjecture 2.2

μ_λ is absolutely continuous if $\lambda > 1/2$ and $1/\lambda$ is not Pisot.

Theorem 2.3 (Solomyak) For almost every $\lambda > 1/2$, μ_λ is absolutely continuous.

Theorem 2.4 (Shmerkin) $\dim_{\mathrm{H}}\{\lambda > 1/2 : \mu_\lambda \text{ is not absolutely continuous}\} = 0$.

An application of Hochman's argument is computing the dimension of μ_λ.

Theorem 2.5 (Hochman)
If λ is an algebraic number then $\dim \mu_\lambda = \min\{1, -h_\lambda/(log\lambda\}$, where h_λ is the Garcia entropy given by $h_\lambda = \lim_{n\to\infty} H(\nu^{*n}) = h_{\mathrm{RW}}(\nu)$.

In fact, Hochman's result requires such **exponential separation condition**: there exists $C > 0$ such that for every n large enough and $g_1 \neq g_2 \in \operatorname{supp} \nu^{*n}$, $d(g_1, g_n) > C^{-n}$.

Theorem 2.6 (Varjú) If λ is transcendental then $\dim \mu_\lambda = 1$.

The entropy argument. We first recall some notions of the dimension of measures.

Definition 2.7. μ is called **exact dimensional** if there exists $\alpha \geqslant 0$ such that for μ-almost every x,

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha,$$

and α is called the **exact dimension** of μ.

Feng-Hu showed that μ_λ is exact dimensional.
Let μ be a probability measure on $\mathbb{R}$ and ϑ be a measurable partition of $\mathbb{R}$. The entropy

$$H(\mu, \vartheta) = \sum_{I\in\vartheta} -\mu(I) \log \mu(I).$$

Let ϑ_n be the dyadic partition on $\mathbb{R}$. The following is a basic fact of exact dimensional measures (log is taking in base 2).

Proposition 2.8
If μ is an exact dimensional probability measure on $\mathbb{R}$, then

$$\dim \mu = \lim_{n\to\infty} \frac{1}{n} H(\mu, \vartheta_n).$$

By stationarity, we have

$$H(\mu_\lambda, \vartheta_n) = H(\nu^{*m} * \mu, \vartheta_n).$$

We identify ν^{*m} as probability measure on $\mathbb{R}$ by the map

$$\mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}, \quad \begin{bmatrix} \lambda^n & \pm 1 \pm \lambda \pm \cdots \pm \lambda^{n-1} \\ 0 & 1 \end{bmatrix} \mapsto \pm 1 \pm \lambda \pm \cdots \pm \lambda^{n-1}.$$

Then we have

$$\nu^{*m} = \sum_{I\in\vartheta_{n'}} \mu^{*m}(I) \cdot \mu_I^{*m},$$

where $\mu_I^{*m} = \mu^{*m}|_I / \mu^{*m}(I)$.

Fix q sufficiently large. We choose $n' = [n\log(1/\lambda)]$ and $m = n$. We have

$$\begin{aligned}
&\frac{1}{qn}H(\mu, \vartheta_{qn+n'}|\vartheta_{n'}) = \frac{1}{qn}H(\nu^{*n} * \mu, \vartheta_{qn+n'}|\vartheta_{n'}) \\
&\geqslant \sum_{I\in\vartheta_{n'}} \nu^{*n}(I)\frac{1}{qn}H(\nu_I^{*n} * \mu, \vartheta_{qn+n'}|\vartheta_{n'}) \quad \text{(concavity of entropy)} \\
&\geqslant \sum_{I\in\vartheta_{n'}} \nu^{*n}(I)\frac{1}{qn}H((\nu_I^{*n} * \delta_0) \boxplus (S_{\lambda^n})_*\mu, \vartheta_{qn+n'}).
\end{aligned}$$

Here $S_\lambda : x \mapsto rx$. The last inequality comes from $\operatorname{diam}\operatorname{supp}(\nu_I^{*n} * \mu) \approx \operatorname{diam}\operatorname{supp}(\nu_I^{*n} * \delta_0) \approx \lambda^n$. By our choice of n, $\nu_I^{*n} * \mu$ only intersects finite items of $\vartheta_{n'}$. If we consider the trivial bound of the above computation, we have

$$\frac{1}{qn}H((\nu_I^{*n} * \delta_0) \boxplus (S_{\lambda^n})_*\mu, \vartheta_{qn+n'}) \geqslant \frac{1}{qn}H((S_{\lambda^n})_*\mu, \vartheta_{qn+n'}) - o(1) \approx \frac{1}{qn}H(\mu, \vartheta_{qn}) \to \dim\mu.$$

This tells nothing to us. But if we assume for a contradiction that $\dim\mu$ is strictly less than the expected value then there will be an entropy growth.

Theorem 2.9 (Hochman, 14)

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every η_1, η_2 on $\mathbb{R}$ satisfying

(1) $\operatorname{diam}\operatorname{supp}\eta_1, \operatorname{diam}\operatorname{supp}\eta_2 \approx 2^{-k}$,

(2) $\frac{1}{n}H(\eta_1, \vartheta_{n+k})$,

(3) η_2 is ε-entropy porous and $\dim\eta_2 < 1 - \varepsilon$.

Then $\frac{1}{n}H(\eta_1 \boxplus \eta_2, \vartheta_{n+k}) \geqslant \frac{1}{n}H(\mu, \vartheta_{n+k}) + \delta$.

Why positive entropy of $\nu_I^{*n} * \delta_0$? Here we will take $\eta_1 = \nu_I^{*n} * \delta_0$ to obtain an entropy growth. The positivity of $H(\nu_I^{*n} * \delta_0|\vartheta_{n'})$ comes from the exponential separation and the contradiction hypothesis. Assume that $\dim\mu < \min\{1, -h_\lambda/\log\lambda\}$ then

$$\begin{aligned}
&\frac{1}{qn}H(\nu^{*n}, \vartheta_{qn+n'}|\vartheta_{n'}) = \frac{1}{qn}(H(\nu^{*n}, \vartheta_{qn+n'}) - H(\nu^{*n}, \vartheta_{n'})) \\
&\approx \frac{1}{qn}(H(\nu^{*n}) - H(\nu^{*n} * \delta_0, \vartheta_{n'})) \approx \frac{1}{qn}(nh_\lambda - n'\dim\mu) > 0.
\end{aligned}$$

§3 Variational principle of affinity exponents (Nov 21)

Recall. $\Gamma = \pi_1(S_g)$ where S_g is the closed surface with genus $g \geqslant 2$. $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$ is an Anosov representation: there is $C > 0$ such that for every $\gamma \in \Gamma$,

$$\frac{\sigma_2}{\sigma_1}(\rho(\gamma)) < Ce^{-|\gamma|/C},$$

where $|\gamma|$ is the word length and $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ are singular values. The affinity exponent is the critical exponent $s_A(\rho) = \min\{s > 0 : P_\rho(s) < \infty\}$ where

$$P_\rho(s) = \begin{cases} \displaystyle\sum_{\gamma\in\Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)^s(\rho(\gamma)), & 0 < s \leqslant 1; \\ \displaystyle\sum_{\gamma\in\Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)(\rho(\gamma))\left(\frac{\sigma_3}{\sigma_1}\right)^{s-1}(\rho(\gamma)), & 1 < s \leqslant 2. \end{cases}$$

We further assume that $\rho(\Gamma)$ is Zariski dense in $\mathrm{SL}_3(\mathbb{R})$. We have

$$s_{\mathrm{A}}(\rho) \geqslant \dim_{\mathrm{H}} L(\rho) \geqslant \sup\{\, \mu \;:\; \mu \text{ is } \nu\text{-stationary}\,\},$$

where ν is taken over all finitely supported probability measure on $\rho(\Gamma)$ whose support generates a Zariski dense subgroup. We have shown that for these stationary measures μ we have the dimension formula $\dim \mu = \dim_{\mathrm{LY}} \mu$ [which is the most difficult part of the proof].

Recall. The Lyapunov dimension of μ is

$$\dim_{\mathrm{LY}} \mu = \begin{cases} \dfrac{h_{\mathrm{F}}(\mu,\nu)}{\lambda_1(\nu)-\lambda_2(\nu)}, & \text{if } h_{\mathrm{F}}(\mu,\nu) \leqslant \lambda_1(\nu)-\lambda_2(\nu); \\[2ex] 1 + \dfrac{h_{\mathrm{F}}(\mu,\nu)-(\lambda_1(\nu)-\lambda_2(\nu))}{\lambda_1(\nu)-\lambda_3(\nu)}, & \text{otherwise.} \end{cases}$$

Here $h_{\mathrm{F}}(\mu,\nu)$ is the Furstenberg entropy and λ_i are Lyapunov exponents.

In practice, we want to replace the Furstenberg entropy with a more computable entropy, the **random walk entropy**

$$h_{\mathrm{RW}}(\nu) = \lim_{n\to\infty} \frac{1}{n} H(\nu^{*n}).$$

A natural bound is $h_{\mathrm{F}}(\mu,\nu) \leqslant h_{\mathrm{RW}}(\nu)$. Conversely, let $\mu_{\mathscr{F}}$ be the unique ν-stationary measure on the flag variety $\mathscr{F}(\mathbb{R}^3) = \left\{0 \subset V_1 \subset V_2 \subset \mathbb{R}^3 \;:\; \dim V_1 = 1, \dim V_2 = 2\right\}$, then

$$h_{\mathrm{RW}}(\nu) = h_{\mathrm{F}}(\mu_{\mathscr{F}},\nu) \geqslant h_{\mathrm{F}}(\mu,\nu),$$

providing that $\langle \operatorname{supp}\nu\rangle$ is a discrete subgroup of $\mathrm{SL}_3(\mathbb{R})$ [Kaimanovich-Ledrappier].

Let us use the random walk on $\mathrm{SL}_2(\mathbb{R})$ to explain why the Furstenberg entropy occurs. Let μ be a ν-stationary measure on $\mathbb{RP}^1$. For $x \in \mathbb{RP}^1$, write $x = g_{i_1}\cdots g_{i_n} y$. Then

$$\mu(B(g_{i_1}\cdots g_{i_n}y,r)) = \frac{\mu(B(g_{i_1}\cdots g_{i_n}y,r))}{\mu(g_{i_1}^{-1}B(g_{i_1}\cdots g_{i_n}y,r))} \times \frac{\mu(g_{i_1}^{-1}B(g_{i_1}\cdots g_{i_n}y,r))}{\mu(g_{2_1}^{-1}g_{i_1}^{-1}B(g_{i_1}\cdots g_{i_n}y,r))} \times \cdots.$$

The right hand side can be replace with

$$\frac{\mathrm{d}g_{i_1}^{-1}\mu}{\mathrm{d}\mu}(g_{i_1}^{-1}x) \times \frac{\mathrm{d}g_{i_2}^{-1}\mu}{\mathrm{d}\mu}(g_{i_2}^{-1}g_{i_1}^{-1}x) \times \cdots.$$

Using the ergodic theorem, the Furstenberg entropy occurs.

The identity between the Furstenberg entropy and the random walk entropy follows from properties of Anosov representations. For a representation ρ, there are Γ-equivariant maps $\iota : \partial\Gamma \to \mathbb{RP}^2$ and $\iota_{\mathscr{F}} : \partial\Gamma \to \mathscr{F}(\mathbb{R}^3)$ satisfying the following naturality commuting diagram

$$\begin{tikzcd} \partial\Gamma \arrow[r, "\iota"] \arrow[dr, "\iota_{\mathscr{F}}"'] & \mathbb{RP}^2 \\ & \mathscr{F}(\mathbb{R}^3) \arrow[u, "\pi"'] \end{tikzcd} .$$

We note that the image of ι and $\iota_{\mathscr{F}}$ are exactly the limit sets of $\rho(\Gamma)$ on $\mathbb{RP}^2$ and $\mathscr{F}(\mathbb{R}^3)$ respectively. Moreover, $\pi|_{\iota_{\mathscr{F}}(\partial\Gamma)}$ is injective. Therefore, $\pi_*\mu_{\mathscr{F}} = \mu$ has the trivial fiber property. This gives the desired identity

$$h_{\mathrm{F}}(\mu,\nu) = h_{\mathrm{F}}(\mu_{\mathscr{F}},\nu).$$

Idea of finding good random walks. We want to find ν supported on $\rho(\Gamma_N)$ where $\Gamma_N = \{\gamma : |\gamma| = N\}$ satisfying

- $\operatorname{supp}\nu$ freely generates a free semigroup. Then for ν equally weighted on $\operatorname{supp}\nu$, we have $h_{\mathrm{RW}}(\nu) = \log(\#\operatorname{supp}\nu)$.
- There exists $x \in \mathfrak{a}^{++}$, a positive Weyl chamber of $\mathfrak{sl}_3(\mathbb{R})$, such that for every $\gamma_{i_1}\cdots\gamma_{i_\ell} \in \operatorname{supp}\nu$,
$$|\kappa(\rho(\gamma_1,\cdots,\gamma_{i_\ell})) - \ell x| < \varepsilon\ell,$$
where $\kappa(\cdot)$ is the Cartan projection given by $g \mapsto \log a$, where $k_1'ak_2 \in KA^+K$ is the Cartan decomposition of g. Then the Lyapunov vector is near x.

In this case, we can estimate the Lyapunov dimension of the ν-stationary measure.

The approximation of affinity exponent by Lyapunov exponent follows by the expression of affinity exponents. For every $\varepsilon > 0$, let $s = s_{\mathrm{A}}(\rho) - \varepsilon$. Then the series $P_\rho(s)$ diverges. Consequently, there is a sequence $N_k \to \infty$ such that

$$S_{N_k} = \{\gamma \in \Gamma : \psi_s(\rho(\gamma)) \in [N_k - 1, N_k]\}$$

has the cardinality at least $e^{(1-\varepsilon)N_k}$. Here, ψ_s is a linear form on $\mathfrak{a} = \{\lambda = \operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)\} \subset \mathfrak{sl}_3(\mathbb{R})$ given by

$$\psi_s(\lambda) = \begin{cases} s(\lambda_1 - \lambda_2), & s \leqslant 1; \\ (\lambda_1 - \lambda_2) + (s-1)(\lambda_1 - \lambda_3), & s > 1. \end{cases}$$

Applying a geometric group theoretic argument. We can find $S'_{N_k} \subset S_{N_k}$ (not really contained in this S_{N_k}) satisfying two conditions above and $\#S'_{N_k} \geqslant e^{-\varepsilon N_k}\#S'_{N_k}$.

Another application: the dimension gap of Anosov representations.

Theorem 3.1 (Ledrappier-Lessa, 23)
For every Zariski dense Anosov representation $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$, we have

$$\dim_{\mathrm{H}} L_{\mathscr{F}}(\rho) \leqslant 5/2 < 3 = \dim \mathscr{F}(\mathbb{R}^3),$$

where $L_{\mathscr{F}}$ is the limit set on the flag variety.

One of key ingredients of this result is the Ledrappier-Young formula. Every probability measure on $\mathrm{SL}_3(\mathbb{R})$ gives three random walks: on $\mathbb{P}(\mathbb{R}^3)$, $\mathbb{P}(\wedge^2\mathbb{R}^3)$ and $\mathscr{F}(\mathbb{R}^3)$. By applying Ledrappier-Young formula, we obtain

$$\begin{aligned} h_{\mathrm{RW}} &= r_1(\lambda_1 - \lambda_2) + r_2(\lambda_1 - \lambda_3) = r_1'(\lambda_2 - \lambda_2) + r_2'(\lambda_1 - \lambda_3) \\ &= r_1''(\lambda_1 - \lambda_2) + r_2''(\lambda_2 - \lambda_3) + r_3''(\lambda_1 - \lambda_3). \end{aligned}$$

Noting that $r_1, r_2, r_1', r_2' \leqslant 1$, we obtain an upper bound of h_{RW}. By a direct computation, we have

$$\dim_{\mathrm{LY}} \mu_{\mathscr{F}} = \max\{r_1'' + r_2'' + r_3'' : 0 \leqslant r_1'', r_2'', r_3'' \leqslant 1\} \leqslant \frac{5}{2}.$$

Applying the variational principle on the flag variety, we have

$$\dim_{\mathrm{H}} L_{\mathscr{F}}(\rho) \leqslant s_{\mathrm{A}}(\rho) \leqslant \sup\{\dim_{\mathrm{LY}} \mu_{\mathscr{F}}\} \leqslant \frac{5}{2}.$$