Selected Minicourses in *Beyond Uniform Hyperbolicity 2023*

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Contents

1 Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori (Yi Shi)

§1.1 Local Rigidity (Apr 25)

Definition 1.1.1. $f \in \mathrm{Diff}^1(M)$ is **Anosov** if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

> **Example 1.1.2** (Arnold's cat map)
> $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

> **Question 1.1.3.** Is every Anosov diffeomorphism transitive?

> **Question 1.1.4.** Topological classification of Anosov diffeomorphism.

> **Theorem 1.1.5** (Franks-Manning)
> Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

> **Theorem 1.1.6** (Franks-Newhouse)
> Every codimension-1 Anosov diffeomorphism must be supported on $\mathbb{T}^d$.

Definition 1.1.7. $f \in \mathrm{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \to (0, 1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0, 1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \to M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.1.10 $E^s \oplus E^u$ integrable $\implies$ NOT accessible.

However, Dolgopyat-Wilkinsonm and Hertz-Hertz-Ures, etc. showed that "MOST" partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

<p align="center"><strong style="color:red">Geometric Rigidity $\iff$ Dynamic Spectral Rigidity</p>

That is, $E^s \oplus E^u$ is integrable $\implies E^c$ has exponents rigidity.

Example 1.1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22}-close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \to \mathbb{T}^4$ which is C^1-along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \to \mathbb{T}^2$ be C^1-close to A. Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$$

is an Anosov diffeomorphism C^1-close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

<p align="center"><strong style="color:red">NO exponents rigidity.</p>

Main theorem: local rigidity. Assume that $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K \to \infty} \frac{\#\{ A \text{ is generic} : \|A\| \leqslant K \}}{\#\{ A : \|A\| \leqslant K \}} = 1.$$

Denote

$$TT^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$

the finest dominated splitting, then $\dim L_i^{s/u} \leqslant 2$.

Let $f \in \text{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$TT^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$

Assume that $l \geqslant 2$ and $1 \leqslant k < l$. Denote

$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$

Then

$$TT^d = E^{ss} \oplus E^{ws} \oplus E^u$$

makes f be an absolutely partially hyperbolic system.

Theorem 1.1.13 (Local rigidity, Gogolev-Shi, arXiv: 2207.00704)

Assume $A \in \text{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \text{Diff}^2(\mathbb{R}^d)$ be C^1-close to A, the following are equivalent:
1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws}:

$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \cdots, l.$$

3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws}.

Dimension 3 case.

Theorem 1.1.14 (Hammerlindl-Ures, 2014)

Let $f \in \text{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \text{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then
- either f is accessible, thus ergodic.
- or there exists an f-invariant minimal foliation $\mathcal{F}^{su}$ such that $T\mathcal{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_*.

Theorem 1.1.15 (Gan-Shi, 2020)

Let $f \in \text{Diff}_m^{1+}(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:
- $E^s \oplus E^u$ is integrable;
- f has spectral rigidity in E^c: $\lambda^c(f) \equiv \lambda^c(f_*)$.
Both imply f is Anosov.

Corollary 1.1.16   Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.1.13 — spectral rigidity $\implies$ joint integrability.   The case of all E_i^s are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \mathrm{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E_l^s \implies$ smooth conjugacy in $E_l^s \implies h(\mathscr{F}_{l-1}^s) = \mathscr{L}_{l-1}^s$ (+spectral rigidity in $E_{l-1}^s) \implies$ smooth conjugacy in $E_{l-1}^s \implies \cdots \implies h(\mathscr{F}_{k+1}^s) = \mathscr{L}_{k+1}^s$ (+spectral rigidity in $E_{k+1}^s) \implies h(\mathscr{F}^{ss}) = \mathscr{L}^{ss} \implies \mathscr{F}^{ss} \oplus \mathscr{F}^u = h^{-1}(\mathscr{L}^{ss} \oplus \mathscr{L}^u)$ joint integrability.

Proof of Theorem 1.1.13 — joint integrability $\implies$ spectral rigidity.   Main ideas:

1. $E^{ss} \oplus E^u$ integrability $\implies h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathscr{F}^{ss} \implies$ spectral rigidity in E_{k+1}^s.
3. $E^{ss} \oplus E_{k+1}^s \oplus E^u$ is integrable, and play induction on E_{k+2}^s.

> **Lemma 1.1.17**
>
> For every $1 \leqslant i \leqslant l$, the conjugation h preserves the center foliation: $h(\mathscr{F}_{(i,l)}^s) = \mathscr{L}_{(i,l)}^s$. Here, $\mathscr{F}_{(i,l)}^s$ and $\mathscr{L}_{(i,l)}^s$ are the foliations tangent to $E_i^s \oplus \cdots \oplus E_l^s$ and $L_i^s \oplus \cdots \oplus L_l^s$, respectively.

Proof. Since f is C^1-close to A, we have

$$\|A_{L_{i-1}^s}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E_i^s(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \to \mathbb{R}^d$ be lifts of f and h, then $y \in \widetilde{\mathscr{F}}_{(i,l)}^s(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leqslant (\rho_i - \varepsilon)^{-n}\|x - y\| + C < (\|A_{L_{i-1}^s}\| + \varepsilon)^{-n}\|x - y\| + C,$$

iff $H(y) \in \widetilde{\mathscr{L}}_{(i,l)}^s(H(x))$. $\square$

> **Lemma 1.1.18**
>
> If $\mathscr{F}$ is a C^0-foliation sub-foliated by a minimal linear foliation $\mathscr{L}$ on $\mathbb{T}^d$, then $\mathscr{F}$ is minimal and linear.

Proof. **Minimal.** every leaf $\mathscr{F}(x) \supset \mathscr{L}(x)$ is dense.
Linear. We will show that, on universal cover, $\widetilde{\mathscr{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \widetilde{\mathscr{F}}(0)$, there exists $v_n \to \widetilde{\mathscr{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \to x$. Since $\mathscr{F}$ is sub-foliated by $\mathscr{L}$ and $\mathscr{L}$ is linear, we have

$$y + k_n + v_n \in \widetilde{\mathscr{F}}(y + k_n) = \widetilde{\mathscr{F}}(k_n) = \widetilde{\mathscr{F}}(k_n + v_n).$$

Take $n \to \infty$, then $y + x \in \widetilde{\mathscr{F}}(x) = \widetilde{\mathscr{F}}(0)$. $\square$

> **Lemma 1.1.19**   If $E^{ss} \oplus E^u$ is integrable to $\mathscr{F}^{su}$, then $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.

Proof. Note that $h(\mathscr{F}^{su})$ is sub-foliated by $h(\mathscr{F}^u) = \mathscr{L}^u$, where $\mathscr{L}^u$ is linear and minimal on $\mathbb{T}^d$. Hence $h(\mathscr{F}^{su})$ is linear, A-invariant and transverse to $\mathscr{L}^{ws} = h(\mathscr{F}^{ws})$. This implies $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$. So

$$h(\mathscr{F}^{ss}) = h(\mathscr{F}^s \cap \mathscr{F}^{su}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{su}) = \mathscr{L}^s \cap \mathscr{L}^{su} = \mathscr{L}^{ss}.$$

$\square$

> **Corollary 1.1.20**
>
> Recall that $T\mathscr{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$, then for $T\mathscr{F}_j^s = E_j^s$, we have
>
> $$h(\mathscr{F}_j^s) = \mathscr{L}_j^s, \quad \forall j = k, k+1, \cdots, l.$$

> **Lemma 1.1.21** (Diophantine approximation of $\mathscr{F}^{ss}$)
>
> There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathscr{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$-dense in $\mathbb{T}^d$.

Proof. Since $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of $\mathscr{L}^{ss}$. Here A is irreducible and $\mathscr{L}^{ss}$ is algebraic, hence Diophantine. $\qquad\square$

Proof of Theorem 1.1.13. We will fist show that the Lyapunov exponent at every point is the same in the dim $E_{k+1}^s = 1$ case. Take $p, q \in \mathrm{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f.

Take
- $x_n \in \mathscr{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$.
- Segments $J \subset \mathscr{F}_{k+1}^s(p)$ and $J_n \subset \mathscr{F}_{k+1}^s(x_n)$ such that $J_n = \mathrm{Hol}^{ss}(J)$ ($x_n = \mathrm{Hol}^{ss}(p)$). Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and $h(\mathscr{L}_{k+1}^s) = \mathscr{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \quad \Longrightarrow \quad \exists C_0 > 0, |J_n| \geqslant C_0 |J|.$$

Now we choose m_n, k_n such that
- x_n and q are very close in first k_n-steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathscr{F}_1^{ss}(p)$.

Then
$$|f^{m_n}(J_n)| \gtrsim \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n\lambda_{k+1}^s(q)]|J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geqslant \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \to \infty.$$

However, $J_n = \mathrm{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \mathrm{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathscr{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \mathrm{Hol}^{ss}(p)$, this contradicts to $\mathscr{F}^{ss}$ is C^1-smooth in $\mathscr{F}^{ss} \oplus \mathscr{F}_{k+1}^s(p)$.

For the case of dim $E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain
- For every periodic points p, q, we have $\min \lambda_{k+1}^s(p) = \min \lambda_{k+1}^s(q)$.
- Considering the growth of area of local disks, we have

$$\mathrm{Jac}(Df, E_{k+1}^s(p)) = \mathrm{Jac}(Df, E_{k+1}^s(q)), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda_{k+1}^s(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda_{k+1}^s(A)$. $\qquad\square$

§1.2 Global Rigidity (Apr 26)

In the last lecture, we have shown a local rigidity result. That is, we only consider diffeomorphisms f that is C^1-close to A. Today we will consider the global rigidity, i.e., the relation between f and $f_* \in \mathrm{GL}(d, \mathbb{Z})$.

> **Question 1.2.1.** What happens if f is not close to $A = f_*$?

> **Theorem 1.2.2** (Gogolev-Farell)
>
> For $d \geqslant 10$, let $A \in \mathrm{GL}(d, \mathbb{Z})$ be a hyperbolic matrix. Then
>
> $$\mathscr{A}_A^{1+}(\mathbb{T}^d) := \left\{ f \in \mathrm{Diff}^{1+}(\mathbb{T}^d) \ : \ f \text{ is Anosov}, \ f_* = A \right\}$$
>
> has infinitely many connected components.

> **Theorem 1.2.3** (Full leaf conjugacy, Gogolev-Shi, arXiv: 2207.00704)
>
> Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with absolutely partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$:
>
> $$\|Df_{E^{ss}}\| < \mu < m(Df|_{E^{ws}}) < \|Df|_{E^{ws}}\| < 1 < m(Df|_{E^u}).$$
>
> If $E^{ss} \oplus E^u$ is integrable, then
> 1. $A = f_* \in \mathrm{GL}(d, \mathbb{Z})$ is partially hyperbolic:
>
> $$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u, \quad \dim L^\sigma = \dim E^\sigma, \ \ \sigma = ss, ws, u.$$
>
> 2. f is dynamically coherent and fully conjugate to A:
>
> $$h(\mathscr{F}^\sigma) = \mathscr{L}^\sigma, \quad \sigma = ss, ws, u.$$
>
> Here $h \circ f = A \circ h$.

> **Question 1.2.4.** Let $f = \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$.
> - Is $f_* \in \mathrm{GL}(d, \mathbb{Z})$ partially hyperbolic?
> - Is f dynamically coherent or not? If yes, does f leaf conjugate to A.

> **Lemma 1.2.5**
>
> Let $\mathscr{F}$ be a C^0-foliation on $\mathbb{T}^d$ with C^1-leaves. If there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to $\mathrm{Id}_{\mathbb{T}^d}$ such that $h(\mathscr{F}) = \mathscr{L}$ is a linear foliation, then $\mathscr{F}$ is quasi-isometric:
>
> $$d_{\widetilde{\mathscr{F}}}(x, y) \leqslant a \cdot d(x, y) + b, \quad \forall x \in \mathbb{R}^d, y \in \widetilde{\mathscr{F}}(x).$$
>
> Here $a, b > 0$ and $\widetilde{\mathscr{F}}$ is the lift of $\mathscr{F}$ in $\mathbb{R}^d$.

Proof of Theorem 1.2.3. Since $h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is sub-foliated by minimal linear foliation $h(\mathscr{F}^u) = \mathscr{L}^u$ is linear. We have $\mathscr{L}^{ss} := h(\mathscr{F}^{ss}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear.

Brin's argument shows that $E^{ws} \oplus E^u$ integrate to $\mathcal{F}^{cu}$ and $h(\mathcal{F}^{cu})$ is linear and minimal. Then $\mathcal{F}^{ws}$ integrate to $\mathcal{F}^{ws}$ and $\mathcal{L}^{ws} := h(\mathcal{F}^{ws})$ is A-invariant and linear.

Note that $\mathcal{L}^{ws}$ and $\mathcal{L}^{ss}$ are transverse in $\mathcal{L}^s$, then A admits an invariant splitting $T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u$. We need to show this is a dominated splitting. This follows from the above lemma and the fact that h is homotopic to $\mathrm{Id}_{\mathbb{T}^d}$. $\qquad \square$

> **Theorem 1.2.6** (Global rigidity, Gogolev-Shi, arXiv: 2207.00704)
>
> Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be Anosov and irreducible. Assume that f is absolutely partially hyperbolic $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$ and center bunching. If $E^{ss} \oplus E^u$ is integrable, then
>
> 1. f has a finest dominated splitting on E^{ws} with the same dimensions for $A|_{L^{ws}}$:
>
> $$E^{ws} = E_1^{ws} \oplus \cdots \oplus E_k^{ws}, \quad \dim E_i^{ws} = \dim L_i^{ws}.$$
>
> 2. f is spectrally rigid along every E_i^{ws}:
>
> $$\lambda(E_i^{ws}, f) \equiv \lambda(L_i^{ws}, A), \quad \forall i = 1, \cdots, k.$$

> **Remark 1.2.7** • Here f need NOT to be C^1-close to $A = f_*$.
> • To get dominated splitting, we usually need some C^1-robust property like: robustly transitive, far from homoclinic bifurcations.
> • If $A = f_*$ satisfies the generic assumption in the last lecture, then the conjugacy h is C^{1+}-smooth along $\mathcal{F}^{ws}$.
> • The center bunching condition
>
> $$\|Df|_{E^{ws}(x)}\| < m(Df|_{E^{ws}(x)}) \cdot m(Df|_{E^u(x)})$$
>
> is a technical condition, which guarantees C^{1+}-smoothness of $\mathcal{F}^{su}$.

> **Corollary 1.2.8**
>
> Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be codimension one with real simple spectrum. For every Anosov $f \in \mathrm{Diff}_m^2(\mathbb{T}^d)$ with $f_* = A$ and
>
> $$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u, \quad \dim E^{ss} = 1,$$
>
> if
> • $E^{ss} \oplus E^u$ is integrable;
> • the metric entropy $h_m(f) = h_m(A)$;
> then f is C^{1+}-conjugate to A.

Main idea for showing Theorem 1.2.6. Play the game similar to the last lecture. We will use the Diophantine approximation of $\mathcal{F}^{ss}$ to show the rigidity of smallest exponent in E^{ws}:

$$\lambda_{\min}^{ws}(p) = \lambda_{\min}^{ws}(q), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we will show the dimension of $\lambda_{\min}^{ws}$ for each periodic point is constant. Next, we define the Pesin stable foliation $\mathcal{F}_{\min}^{ws}$ and show it is $\mathcal{F}^{su}$-holonomy invariant, that is Hol^{su} : $\mathcal{F}^{ws}(p) \to \mathcal{F}^{ws}(q)$ preserves $\mathcal{F}_{\min}^{ws}$, for every $p, q \in \mathrm{Per}(f)$. Finally, we show a uniform spectral exponents gap and extract out $\mathcal{F}_{\min}^{ws}$.

> **Lemma 1.2.9**
>
> Let $\text{Hol}_{x,y}^{su} : \mathcal{F}(x) \to \mathcal{F}(y)$ be the holonomy map of $\mathcal{F}^{su}$ with $\text{Hol}_{x,y}^{su}(x) = y$ for every $x \in \mathbb{T}^d$ and $y \in \mathcal{F}^{su}(x)$. Then
>
> $$\text{Hol}_{x,y}^{su}(K) = h^{-1} \circ T_{h(x),h(y)} \circ h(K).$$
>
> Here $T_{h(x),h(y)} : \mathbb{T}^d \to \mathbb{T}^d$ is the linear translation send $h(x)$ to $h(y)$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $K \subset \mathcal{F}^{ws}(x)$ with $\text{diam}(K) > \varepsilon$, then
>
> $$\text{diam}(\text{Hol}_{x,y}^{su}(K)) > \delta, \quad \forall y \in \mathcal{F}^{su}(x).$$

Remark 1.2.10 The same holds for $\text{Hol}_{x,y}^{ss} : \mathcal{F}^{ws}(x) \to \mathcal{F}^{ws}(y)$ where $y \in \mathcal{F}^{ss}(x)$.

Proof. It follows immediately from f is fully conjugate to A. □

Proof of Theorem 1.2.6. We fist show that

Claim 1.2.11. $\lambda_{\min}^{ws}(p) = \lambda_{\min}^{ws}(q), \quad \forall p, q \in \text{Per}(f).$

Proof. Assume that $\lambda_{\min}^{ws}(p) < \lambda_{\min}^{ws}(q)$. Take $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(x_n, p) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$. Take disk $D \subset \mathcal{F}_{\min}^{ws}(p)$, the Pesin stable manifold associated to $\lambda_{\min}^{ws}(p)$. Let $D_n = \text{Hol}^{ss}(D) \subset \mathcal{F}^{ws}(x_n)$, then $\text{diam}(D_n) \gg \text{diam}(D)$. Applying a similar (k_n, m_n)-argument, we get a contradiction since $\mathcal{F}^{ss}$ is C^1-smooth in $\mathcal{F}^{ws}(p)$. □

Now we have $\lambda_{\min}^{ws} := \lambda_{\min}^{ws}(p)$ for every $p \in \text{Per}(f)$. We define the Pesin stable foliation associated to $\lambda_{\min}^{ws}$ for each periodic point.

Claim 1.2.12. $\mathcal{F}_{\min}^{ws}$ is Hol^{su}-invariant.

Proof. Let $\mathcal{L}_{\min}^{ws}|_{\mathcal{L}^{ws}(p)} := h(\mathcal{F}_{\min}^{ws}|_{\mathcal{L}^{ws}(p)})$, it suffices to show

$$T_{h(p),h(x)}(\mathcal{L}_{\min}^{ws}(p)) \subset \mathcal{L}_{\min}^{ws}(x)$$

for every $p, q \in \text{Per}(f)$ and $x \in \mathcal{F}^{ws}(q)$. Otherwise, take a disk $D \subset \mathcal{F}_{\min}^{ws}(p)$, then $T_{h(p),h(x)}(h(D))$ is transverse to $\mathcal{L}_{\min}^{ws}|_{\mathcal{L}_{\text{loc}}^{ws}(q)}$ at $h(x)$. Take $x_n \in \mathcal{F}^{ss}$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, x) \ll K_n^{-\alpha}$, then

$$D_n := \text{Hol}_{p,x_n}^{ss}(D) \to h^{-1} \circ T_{h(p),h(x)} \circ h(D).$$

It follows that $\text{Hol}_{\text{loc}}^u(D)$ is "uniformly transverse" (the angle will not tend to zero) to $\mathcal{L}_{\min}^{ws}$ in $\mathcal{F}_{\text{loc}}^{ws}(q)$, where $\text{Hol}_{\text{loc}}^u(D) : \mathcal{F}^{ws}(x_n) \to \mathcal{F}^{ws}(q)$ is C^{1+}-smooth. Since the transverse direction has a weaker contracting rate, we play the (k_n, m_n)-game and get a contradiction. □

Let $\mathcal{L}_{\min}^{ws} := h(\mathcal{L}_{\min}^{ws})$, then the density of $\text{Per}(f)$ and minimality of $\mathcal{F}^{ws}$ imply $T_{x,y}(\mathcal{L}_{\min}^{ws}(x)) \subset \mathcal{L}_{\min}^{ws}(y)$. By the translation invariance and the A-invariance, we have
- $\mathcal{L}_{\min}^{ws}$ is a linear foliation on $\mathbb{T}^d$, and
- $L_{\min}^{ws} := T\mathcal{L}_{\min}^{ws}$ associate to an eigenspace of A.

Also by an estimate of the growth, we get $\lambda(A, L_{\min}^{ws}) \equiv \lambda_{\min}^{ws}$.

Finally, we establish the induction step. Following the idea of [Bonatti-Díaz-Pujals, 2003], consider the quotient cocycle $D\widetilde{f} : E^{ws}/E_{\min}^{ws} \to E^{ws}/E_{\min}^{ws}$ which is Hölder continuous over f. Again by a (k_n, m_n)-game, we can show that λ_2^{ws} is uniformly larger than $\lambda_{\min}^{ws}$. Then the splitting $T\mathbb{T}^d = (E^{ss} \oplus E_{\min}^{ws}) \oplus F \oplus E^u$ is an absolutely partially hyperbolic splitting. The joint integrability follows from $h(\mathcal{F}^{ss} \oplus \mathcal{F}_{\min}^{ws})$ is linear. □

§1.3 Rigidity on $\mathbb{T}^4$ (Apr 27)

Let us recall some results shown in last two lectures. We remark that the key point is that

$$E^{ss} \oplus E^u \text{ is integrable} \implies h(\mathscr{F}^{ss} = \mathscr{L}^{ss}) \text{ is linear.}$$

Question 1.3.1. Let f be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_j^u \oplus \cdots \oplus E_m^u.$$

What happens if $E_k^s \oplus E_j^u$ is jointly integrable? Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$?

Theorem 1.3.2 (Gogolev-Kalinin-Sadovskya)

Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$ implies $h(\mathscr{F}_k^s) = \mathscr{L}_k^s$ and $h(\mathscr{F}_j^u) = \mathscr{L}_j^u$ hence $E_k^s \oplus E_j^u$ is jointly integrable.

The work of Avila-Viana.

Theorem 1.3.3 (Avila-Viana, 2010)

For every symplectic f which is C^∞-close to A with splitting

$$T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u,$$

then
- either f is accessible and non-uniformly hyperbolic;
- or $E^s \oplus E^u$ is integrable and $\exists h \in \mathrm{Diff}_m^\infty(\mathbb{T}^4)$ such that

$$h \circ f = A \circ h.$$

In particular, f is Bernoulli.

Main theorem.

Theorem 1.3.4 (Gogolev-Shi, arXiv: 2207.00704)

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an irreducible Anosov automorphism with dominated splitting

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}, \quad \text{with} \quad \dim L^{ws} = \dim L^{wu} = 1.$$

For $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:
- $E^{ss} \oplus E^{uu}$ is integrable;
- f is spectral rigid along E^{ws} and E^{wu}.

> **Corollary 1.3.5**
>
> Let $A \in \mathrm{Sp}(4, \mathbb{Z})$ be hyperbolic and irreducible with dominated splitting
>
> $$T\mathbb{T}^4 = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}.$$
>
> For symplectic $f \in \mathrm{Diff}^2_\omega(\mathbb{T}^4)$ be C^1-close to A with
>
> $$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$
>
> the following are equivalent:
> - $E^{ss} \oplus E^{uu}$ is integrable;
> - f is C^{1+}-smoothly conjugate to A.

Proof of corollary. If $E^{ss} \oplus E^{uu}$ is integrable, then we have spectral rigidity in $E^{ws} \oplus E^{wu}$, h is smooth along $E^{ws} \oplus E^{wu}$ and $h(\mathscr{L}^{ss}) = \mathscr{L}^{ss}, h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$. Since h is smooth along $\mathscr{F}^{ws}$ and $\mathscr{F}^{wu}$, the holonomy map $\mathrm{Hol}^{su}_{\mathscr{F}}$ is C^{1+}. Then we use the symplectic structure that $E^c = E^{ws} \oplus E^{wu}$ is perpendicular to E^{su} (with respect to ω). Hence $\mathscr{F}^{ws} \oplus \mathscr{F}^{wu}$ is C^{1+}. Then we can show that h is absolutely continuous in $\mathscr{F}^{su}$ and hence h is C^{1+}. $\qquad\square$

Proof of main theorem. Main problem is whether $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$ is the linear one? Or equivalently, whether we have $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ or $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$? This is nontrivial.

> **Lemma 1.3.6**
>
> If one of $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ is integrable, then f is spectral rigid in $E^{ws} \oplus E^{wu}$.

Proof. If $E^{ss} \oplus E^u$ is integrable, then $h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear and hence $h(\mathscr{F}^{ss}) = h(\mathscr{F}^{ss} \oplus \mathscr{F}^u) \cap \mathscr{L}^s = \mathscr{L}^{ss}$ is linear. Then both $h(\mathscr{F}^{su})$ and $h(\mathscr{F}^{uu})$ are linear. Then we obtain a spectral rigidity by Theorem 1.1.13. $\qquad\square$

The solvable action. Let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$ and $L^c(0) = L^{ws}(0) \oplus L^{wu}(0) \subset \mathbb{R}^d$. Define the linear action

$$\alpha_0 : \Gamma \times L^c(0) \to L^c(0), \quad \alpha_0(k, n)(x) = L^{su}(A^k(x) + n) \cap L^c(0).$$

If we write $n = n^s + n^c + n^u \in L^s \oplus L^c \oplus L^u$, then $\alpha_0(k, n)(x) = A^k x + n^c$.

For $F : \mathbb{R}^4 \to \mathbb{R}^4$ be the lift of f and $F(0) = 0$, then
 - $F^k(x + n) = F^k(x) + A^k n, \forall x \in \mathbb{R}^d$ and $\forall n \in \mathbb{Z}^d$.
 - $F(\widetilde{\mathscr{F}}^c(0)) = \widetilde{\mathscr{F}}^c(0)$.

Then $\Gamma \curvearrowright \widetilde{\mathscr{F}}^c(0)$ given by

$$\alpha(k, n)(x) = \widetilde{\mathscr{F}}^{su}(F^k(x) + n) \cap \widetilde{\mathscr{F}}^c(0), \quad \forall (k, n) \in \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d, x \in \widetilde{\mathscr{F}}^c(0).$$

> **Lemma 1.3.7** This is a group action by the solvable group Γ.

Main idea. If both $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ are not integrable, then we can find a free subgroup by a pingpong argument, which contradicts Γ is solvable.

> **Lemma 1.3.8**
>
> If $\alpha(0, n)(\widetilde{\mathscr{F}}^{ws}(0)) \subset \widetilde{\mathscr{F}}^{ws}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$, then both $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$ are linear. The same holds if $\alpha(0, n)(\widetilde{\mathscr{F}}^{wu}(0)) \subset \widetilde{\mathscr{F}}^{wu}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$.

Proof. Note that $\bigcup_{n \in \mathbb{Z}^d} \widetilde{\mathscr{F}}^{ws}(n)$ is dense in $\mathbb{R}^d$ and hence $E^{ss} \oplus E^{ws} \oplus E^{uu}$ jointly integrates to $\mathscr{F}^{su} \oplus \mathscr{F}^{ws}$. Then we deduce the linearity. $\qquad\square$

Proof of Theorem 1.3.4. Assume for a contradiction that there exists $n_1, n_2 \in \mathbb{Z}^d$ such that

- $\alpha(0, n_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(\alpha(0, n_1)(0))$;
- $\alpha(0, n_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(\alpha(0, n_1)(0))$.

Lemma 1.3.9

There exists $m_1, m_2 \in \mathbb{Z}^d$ such that

- $\alpha(0, m_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0, m_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Lemma 1.3.10

For l large enough, $n = A^l m_1 - A^{-l} m_2 \in \mathbb{Z}^d$ satisfies

- $\alpha(0, n)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0, n)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Now we consider $F : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0)$ and

$$G : \alpha(0, n) \circ \alpha(1, 0) \circ \alpha(0, -n) : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0).$$

Then F is saddle-like dynamics at $\widetilde{\mathscr{F}}^{ws}(0) \cup \widetilde{\mathscr{F}}^{ws}(0)$ near 0. The map G is also saddle-like near $\alpha(0, n)0$. By a pingpong-argument, we can show that $\left\{ F^k, G^k \right\}$ generates a free group for a sufficiently large k. This contradicts that Γ is solvable. $\qquad\square$

§1.4 Anosov Maps (Apr 28)

Cone-field. Let f be an Anosov diffeomorphism with splitting $TM = E^s \oplus E^u$. Then there are cone-fields C^s, C^u containing E^s, E^u such that

$$Df(\overline{C^u(x)}) \subset C^u(fx), \quad Df^{-1}(\overline{C^s(x)}) \subset C^s(f^{-1}x).$$

Then $E^s(x)$ is determined by $\mathrm{Orb}^+(x)$ as

$$E^s(x) = \bigcap_{n \geqslant 0} Df^{-n}(C^s(f^n x)),$$

and $E^u(x)$ is determined by $\mathrm{Orb}^-(x)$ as

$$E^u(x) = \bigcap_{n \geqslant 0} Df^n(C^u(f^{-n}x)).$$

Theorem 1.4.1 (Anosov, 1967)

The Arnold's cat map $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is structurally stable. That is, for every $f : \mathbb{T}^2 \to \mathbb{T}^2$ C^1-close to A, there exists a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ close to $\mathrm{Id}_{\mathbb{T}^2}$ such that $h \circ f = A \circ h$.

> **Remark 1.4.2** Every Anosov diffeomorphism is structurally stable.

> **Remark 1.4.3** If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is continuous with $f_* = A$, then there exists a surjective $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h \circ f = A \circ h$.

By a cone-argument, we can show that a small perturbation of an Anosov diffeomorphism is also Anosov. In general, we have Franks-Manning's global classification of Anosov diffeomorphisms.

> **Theorem 1.4.4** (Franks-Manning)
> Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Anosov maps.

Definition 1.4.5. A local diffeomorphsim $f : M \to M$ is **Anosov**, if there exists a continuous, Df invariant subbundle $E^s \subset TM$ such that
- $\|Df(v^s)\| < 1$ for every $v^s \in E^s$ with $\|v^s\| = 1$;
- Df induces an expanding map $D\widetilde{f} : TM/E^s \to TM/E^s$, that is

$$\|D\widetilde{f}(\tilde{v}^u)\| > 1, \quad \forall \tilde{v}^u \in TM/E^s, \|\tilde{v}^u\| = 1.$$

In this lecture, the Anosov map always refers to the non-invertible Anosov map.

> **Remark 1.4.6** Since $\mathrm{Orb}^-(x)$ is not unique, there may be no $E^u(x)$.

> **Theorem 1.4.7** (Mañe-Pugh, 1974)
> $f : M \to M$ is an Anosov map iff $\widetilde{f} : \widetilde{M} \to \widetilde{M}$ is an Anosov diffeomorphism.

Definition 1.4.8 (Przytycki, 1976). A local diffeomorphsim $f : M \to M$ is an **Anosov map**, if in the orbit space

$$\tilde{x} = (x_i)_{i \in \mathbb{Z}} \in M_f := \{ (x_i) : f(x_i) = x_{i+1}, \ \forall x \in \mathbb{Z} \},$$

there exists a splitting

$$T_{x_i}M = E^s(x_i) \oplus E^u(x_i), \quad \forall i \in \mathbb{Z}$$

which is Df-invariant

$$D_{x_i} f(E^s(x_i)) = E^s(x_{i+1}), \quad D_{x_i} f(E^u(x_i)) = E^u(x_{i+1}), \quad \forall i \in \mathbb{Z},$$

and for every $v^{s/u} \in E^{s/u}(x_i)$ with $\|v^{s/u}\| = 1$:

$$\|D_{x_i} f(v^s)\| < 1, \quad \|D_{x_i} f(v^u)\| > 1.$$

Example 1.4.9

For every $n \geqslant 3$, the map

$$A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

is an Anosov map.

Remark 1.4.10 Every Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^d$ has a hyperbolic linearization $f_* \in M(\mathbb{Z}, d)$.

Unlike the Anosov diffeomorphisms, the Anosov map is not structurally stable.

Theorem 1.4.11 (Mañé-Pugh, 1974; Przytycki, 1976)

Let $A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ with $n \geqslant 3$. Then A is **NOT** structurally stable. That is, for every $\varepsilon > 0$, there exists an Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ with $d_{C^\infty}(f, A) < \varepsilon$ such that there is no $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to $\mathrm{Id}_{\mathbb{T}^2}$ with $h \circ f = A \circ h$.

Remark 1.4.12 Every non-invertible Anosov map is not structurally stable unless it is expanding.

Proof. Take $p \neq 0$ such that $A(p) = 0$. Let U, U' be disjoint neighborhoods of 0 and p. Let (x_i) be an A-orbit satisfying

$$x_0 = p, \quad x_i = 0, \forall i > 0, \quad \text{and} \quad x_i \notin U', \forall i < 0.$$

Take a C^∞ ε-perturbation of f on U': push p along the stable leaf.

Then there exists an f-orbit $\{y_i\}$ satisfying

$$y_0 = p, \quad \text{and} \quad y_i = x_i, \forall i < 0.$$

Then $y_i \in \mathscr{F}_\varepsilon^s(0)$ for every $i > 0$, where $\mathscr{F}^s$ is the stable leaf of A. Then the A-orbit x_i shadows the f-orbit y_i and hence the conjugacy $h(y_i) = 0$. But there is no homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h(y_i) = 0$ for every $i > 0$. $\qquad\square$

Theorem 1.4.13 (Przytycki, 1976)

An Anosov map $f : M \to M$ is structurally in the orbit space (M_f, σ_f), where $\sigma_f : (x_i) \mapsto (x_{i+1})$. That is, for every $g : M \to M$ C^1-close to f, there exists a homeomorphism $\overline{h} : M_g \to M_f$ such that $\overline{h} \circ \sigma_g = \sigma_f \circ \overline{h}$.

Question 1.4.14.

- Assume that $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov map with $f_* = A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}, n \geqslant 3$. When f topologically conjugate to A?
- Assume that $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ are Anosov maps with $f_* = g_*$. When f topologically conjugates to g?

∎

Example 1.4.15 (Przytycki, 1976)

Let

$$A = \begin{bmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & n \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3, \quad n \geqslant 2$$

be a **special Anosov map** (E^u does not depend on the choice of the inverse orbit). When n is big enough, for every $x \in \mathbb{T}^3$, there exists an f C^1-close to A such that

$$\left\{ D\pi(E^u(x_0)) : \tilde{x} = (x_i) \in M_f \text{ with } x_0 = x \right\} \subset \mathscr{G}^2(T_x\mathbb{T}^3)$$

contains a curve in the Grassmannian $\mathscr{G}^2(T_x\mathbb{T}^3)$.

Theorem 1.4.16 (Micena-Tahzibi, 2016)

Let $f : M \to M$ be a transitive Anosov map, then
- either f has an integrable E^u (f is special),
- or there exists a residue set $\mathscr{R} \subset M$ such that x has infinitely many unstable directions for every $x \in \mathscr{R}$.

Main theorems.

Theorem 1.4.17 (An-Gan-Gu-Shi, arXiv: 2205.13144)

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^{1+}-Anosov map, then the following are equivalent:
- f topologically conjugate to $f_* = A$;
- f is spectral rigid in stable bundle:

$$\lambda^s(p, f) \equiv \log \|A|_{L^s}\|, \quad \forall p \in \mathrm{Per}(f).$$

Remark 1.4.18 The same holds if $f : \mathbb{T}^d \to \mathbb{T}^d$ is an irreducible Anosov map with $\dim E^s = 1$.

Theorem 1.4.19 (An-Gan-Gu-Shi, arXiv: 2205.13144)

Let $A \in M(d, \mathbb{Z})$ be Anosov, irreducible and $|\det(A)| > 1$. If A has real simple spectrum in the stable bundle:

$$T\mathbb{T}^d = L_1^s \oplus L_2^s \oplus \cdots \oplus L_k^s \oplus L^u, \quad \dim L_i^s = 1,$$

then for every f C^1-close to A, the following are equivalent:
- f topologically conjugates to A,
- f is spectral rigidity in stable bundle, i.e. f admits dominated splitting

$$T\mathbb{T}^d = E_1^s \oplus E_2^s \oplus \cdots \oplus E_k^s \oplus E^u$$

and

$$\lambda(E_i^s, f) \equiv \log \|A|_{L_i^s}\|, \quad \forall i = 1, \cdots, k.$$

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Main philosophy.   For every $y, z \in \mathbb{T}^d$, they are in the same "strongest stable manifold" if

$$f^n(y) = f^n(z), \quad \text{for some } n > 0.$$

Then f topologically conjugates to $A \iff E^u$ does not depend on $\mathrm{Orb}^-(x)$. Hence we have $E^u(x) = E^u(y)$ if $f^n(y) = f^n(z)$. This is equivalent to E^u is "jointly integrable" with

$$\mathscr{F}^{ss}(x) := \{ z \ : \ f^n(x) = f^n(z), \text{ for some } n > 0 \}.$$

This leads to a spectral rigidity in E^s, which is the weak stable direction in this view.

Topological classification.

> **Theorem 1.4.20** (Gu-Shi, arXiv: 2212.11457)
>
> Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be homotopic C^{1+}-Anosov maps, then the following are equivalent:
> - f topologically conjugates to g;
> - for every $p \in \mathrm{Per}(f)$ and corresponding $p' \in \mathrm{Per}(g)$,
>
> $$\lambda^s(p, f) \equiv \lambda^s(p', g).$$

> **Remark 1.4.21** Since there is no a priori conjugacy, we should explain the meaning of "corresponding point". This can be given by a (stable) leaf conjugacy, which is defined a priori. Note that each periodic stable leaf admits a unique periodic point since f is uniformly contracting on the stable leaf. The corresponding point can be defined in this way.

> **Corollary 1.4.22** (Gu-Shi, arXiv: 2212.11457)
>
> Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. Then h is C^r-smooth along the stable foliation.

> **Theorem 1.4.23** (Gu-Shi, arXiv: 2212.11457)
>
> Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. If
>
> $$\mathrm{Jac}(f^{\pi(p)}(p)) = \mathrm{Jac}(g^{\pi(p)}(h(p))), \quad \forall p \in \mathrm{Per}(f),$$
>
> then h is C^{r_*}-smooth. Here $r_* = \begin{cases} r - 1 + \mathrm{Lip}, & r \in \mathbb{N} \\ r, & r \notin \mathbb{N} \end{cases}$.

2 Methods for Studying Abelian Actions and Centralizers (Danijela Damjanović / Disheng Xu)

§2.1 Lecture 1 (May 1)

Plan

1. Many examples, invariant structures, main results.
2. Some methods in simple cases.
3. More methods and more about centralizer rigidity
4. More methods.

Setting

- M a closed C^∞-manifold.
- $f : M \to M$ a C^∞-diffeomorphism.
- $\mathscr{Z}(f) := \left\{ g \in \mathrm{Diff}^\infty(M) : gf = gf \right\}$, the centralizer of f in $\mathrm{Diff}^\infty(M)$.

It is obvious that $\mathscr{Z}(f) > \langle f \rangle \cong \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. **Smale's question**:

Is it true that typically in C^r-topology, $\langle f \rangle = \mathscr{Z}(f)$?

This is confirmed to be true in C^1-topology by Bonatti-Crovisier-Wilkinson.

We also interest in a typical situation that $\mathscr{Z}(f)$ is large. The main theme is a centralizer rigidity:

<p style="text-align:center;color:red">f has a complicated dynamics $+$ $\mathscr{Z}(f)$ is large</p>

<p style="text-align:center;color:red">$\Longrightarrow f$ is C^∞-conjugate to an algebraic system</p>

Algebraic systems.

- $M = G/\Gamma$ where G is a Lie group and Γ is a cocompact lattice in G.
- $L_g : x \mapsto g.x$ the left translation for $g \in G$.
- $A : G \to G$ an automorphism preserving Γ, it induces $A : G/\Gamma \to G/\Gamma$.
- Affine maps $L_g \circ A$.
- Another example of "algebraic systems" are the translations on the symmetric space $L_g : K\backslash G/\Gamma \to K\backslash G/\Gamma$ where $K < G$ is a compact subgroup.

Definition 2.1.1. An action is **smoothly algebraic** if it is C^∞-conjugate to an algebraic model.

Complicated dynamics. f is partially hyperbolic with $TM = E^s \oplus E^c \oplus E^u$. Assume that E^c is also integrable to a foliation $\mathscr{W}^c$ with C^1-leaves.

Definition 2.1.2. f is **accessible** if any $x, y \in M$ can be connected via a stable / unstable broken path.

Notation 2.1.3. For groups H_1, H_2, we denote $H_1 \doteq H_2$ if H_1 is virtually-H_2, that means H_1 contains a finite index subgroup isomorphic to H_2.

Example 2.1.4 (Examples with rigid centralizers)
1. A hyperbolic automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, then $\mathcal{Z}(f) \doteq \mathbb{Z}$.
2. Geodesic flows $\varphi_t : \mathrm{SL}(2, \mathbb{R})/\Gamma \to \mathrm{SL}(2, \mathbb{R})/\Gamma$, it corresponds to the diagonal flows $A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$ acts by left translations. Then φ_t is partially hyperbolic and $\mathcal{Z}(\varphi_t) \doteq \mathbb{R}$.

Example 2.1.5 (Examples with larger centralizers)
1. For $A : \mathbb{T}^2 \to \mathbb{T}^2$ a hyperbolic automorphism, let $f = \begin{bmatrix} A & \\ & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$, then any $\begin{bmatrix} A^k & \\ & A^l \end{bmatrix}$ commutes with f for $k, l \in \mathbb{Z}$. Hence $\mathcal{Z}(f) > \mathbb{Z}^2$.
2. Product of geodesic flows on $\mathrm{SL}(2, \mathbb{R})/\Gamma$. Then $\mathcal{Z}(\varphi_t) > \mathbb{R}^2$.

Note that in the first example, the elements of the form $A^k \times \mathrm{Id}$ or $\mathrm{Id} \times A^l$ are not ergodic. Which means there is a factor in the system. The same holds for the second example. We want to avoid these cases.

Definition 2.1.6 (Rank one factor). Let $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ be an action with $k + l \geqslant 2$. We say it has a C^s **rank-one factor** if we have
- A C^∞-manifold $\overline{M}$ and a C^s-submersion $\pi : M \to \overline{M}$.
- A surjective homomorphism $\sigma : \mathbb{R}^k \times \mathbb{Z}^l \to H$ where $H \doteq \mathbb{Z}$ or $\mathbb{R}$.
- A locally free C^s-action $H : \overline{M} \to \overline{M}$ such that $\pi(g.x) = \sigma(g).\pi(x)$.

Definition 2.1.7. A smooth action $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ is called **higher-rank** if $k + l \geqslant 2$ and there is no C^∞-rank-one factors.

Example 2.1.8 (Higher-rank actions)
1. $A : \mathbb{T}^3 \to \mathbb{T}^3$ a hyperbolic automorphism with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{S}^1$. Then $\mathcal{Z}(A) \doteq \mathbb{Z}^2 = \langle A, B \rangle$ where B is also a hyperbolic automorphism. Let V_i be the eigenspace of A corresponding to λ_i, then B preserves each V_i. Hence $A^k B^l|_{V_i} = \lambda_i^k \mu_i^l$. Although there is not integers k, l such that $\lambda_i^k \mu_i^l = 1$, but there exists pairs of real numbers (s, t) such that $\lambda_i^s \mu_i^t$. These lines are very important.
   Specifically, let
   $$\chi_i(s, t) = s \log |\lambda_i| + t \log |\mu_i|.$$
   Then $\ker \chi_i = L_i$ is a line in the plane for any $i = 1, 2, 3$. An algebraic fact shows that the lines are irrational (hence there is no integers k, l such that $(k, l) \in L_i$).
2. The diagonal flow on $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/\Gamma$ where Γ is an irreducible lattice. By Mautner's theorem, every line in the diagonal flow acts ergodically.
3. Weyl chamber flow. Let $M = \mathrm{SL}(3, \mathbb{R})/\Gamma$, we consider
   $$\mathbb{R}^2 \cong \left\{ \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} : t_1 + t_2 + t_3 = 0 \right\}$$
   acting on M. For an element, it acts on U_{12} by factor $e^{t_1 - t_2}$. We care about the lines on the plane that $t_i = t_j$ for $i \neq j$. All these lines acting normally hyperbolic and hence ergodic.

Proposition 2.1.9

For an $\mathbb{R}^2$ action on M, if every line in $\mathbb{R}^2$ is ergodic iff there is no rank-one factors.

3 Dimension of Stationary Measures (Francios Ledrappier / Pablo Lessa)