Selected Minicourses in *Beyond Uniform Hyperbolicity 2023*

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Contents

1 Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori (Yi Shi)

§1.1 Local Rigidity (Apr 25)

Definition 1.1.1. $f \in \mathrm{Diff}^1(M)$ is **Anosov** if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

Example 1.1.2 (Arnold's cat map)
$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

Question 1.1.3. Is every Anosov diffeomorphism transitive?

Question 1.1.4. Topological classification of Anosov diffeomorphism.

Theorem 1.1.5 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Theorem 1.1.6 (Franks-Newhouse)
Every codimension-1 Anosov diffeomorphism must be supported on $\mathbb{T}^d$.

Definition 1.1.7. $f \in \mathrm{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \to (0,1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0,1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \to M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.1.10 $E^s \oplus E^u$ integrable $\implies$ NOT accessible.

However, Dolgopyat-Wilkinsonm and Hertz-Hertz-Ures, etc. showed that "MOST" partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

$$\textbf{Geometric Rigidity} \iff \textbf{Dynamic Spectral Rigidity}$$

That is, $E^s \oplus E^u$ is integrable $\implies$ E^c has exponents rigidity.

Example 1.1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22}-close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \to \mathbb{T}^4$ which is C^1-along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \to \mathbb{T}^2$ be C^1-close to A. Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$$

is an Anosov diffeomorphism C^1-close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

NO exponents rigidity.

Main Theorem: Local Rigidity. Assume that $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K \to \infty} \frac{\#\{A \text{ is generic} : \|A\| \leqslant K\}}{\#\{A : \|A\| \leqslant K\}} = 1.$$

Denote
$$T\mathbb{T}^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$
the finest dominated splitting, then $\dim L_i^{s/u} \leqslant 2$.

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$

Assume that $l \geqslant 2$ and $1 \leqslant k < l$. Denote

$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$

Then
$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$$
makes f be an absolutely partially hyperbolic system.

Theorem 1.1.13 (Gogolev-Shi, arXiv: 2207.00704)

Assume $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \mathrm{Diff}^2(\mathbb{R}^d)$ be C^1-close to A, the following are equivalent:

1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws}:
$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \cdots, l.$$
3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws}.

Dimension 3 case.

Theorem 1.1.14 (Hammerlindl-Ures, 2014)

Let $f \in \mathrm{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \mathrm{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then

(1) either f is accessible, thus ergodic.
(2) or there exists an f-invariant minimal foliation $\mathscr{F}^{su}$ such that $T\mathscr{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_*.

Theorem 1.1.15 (Gan-Shi, 2020)

Let $f \in \mathrm{Diff}_m^{1+}(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:

1. $E^s \oplus E^u$ is integrable.
2. f has spectral rigidity in E^c: $\lambda^c(f) \equiv \lambda^c(f_*)$.

Both imply f is Anosov.

Corollary 1.1.16 Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.1.13 — spectral rigidity $\implies$ joint integrability. The case of all E_i^s are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \mathrm{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E_l^s \implies$ smooth conjugacy in $E_l^s \implies h(\mathscr{F}_{l-1}^s) = \mathscr{L}_{l-1}^s$ (+spectral rigidity in E_{l-1}^s) $\implies$ smooth conjugacy in $E_{l-1}^s \implies \cdots \implies h(\mathscr{F}_{k+1}^s) = \mathscr{L}_{k+1}^s$ (+spectral rigidity in E_{k+1}^s) $\implies h(\mathscr{F}^{ss}) = \mathscr{L}^{ss} \implies \mathscr{F}^{ss} \oplus \mathscr{F}^u = h^{-1}(\mathscr{L}^{ss} \oplus \mathscr{L}^u)$ joint integrability.

Proof of Theorem 1.1.13 — joint integrability $\implies$ spectral rigidity. Main ideas:

1. $E^{ss} \oplus E^u$ integrability $\implies h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathscr{F}^{ss} \implies$ spectral rigidity in E_{k+1}^s.
3. $E^{ss} \oplus E_{k+1}^s \oplus E^u$ is integrable, and play induction on E_{k+2}^s.

Lemma 1.1.17

For every $1 \leqslant i \leqslant l$, the conjugation h preserves the center foliation: $h(\mathscr{F}_{(i,l)}^s) = \mathscr{L}_{(i,l)}^s$. Here, $\mathscr{F}_{(i,l)}^s$ and $\mathscr{L}_{(i,l)}^s$ are the foliations tangent to $E_i^s \oplus \cdots \oplus E_l^s$ and $L_i^s \oplus \cdots \oplus L_l^s$, respectively.

Proof. Since f is C^1-close to A, we have

$$\|A_{L_{i-1}^s}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E_i^s(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \to \mathbb{R}^d$ be lifts of f and h, then $y \in \widetilde{\mathscr{F}}_{(i,l)}^s(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leqslant (\rho_i - \varepsilon)^{-n}\|x - y\| + C < (\|A_{L_{i-1}^s}\| + \varepsilon)^{-n}\|x - y\| + C,$$

iff $H(y) \in \widetilde{\mathscr{L}}_{(i,l)}^s(H(x))$. □

Lemma 1.1.18

If $\mathscr{F}$ is a C^0-foliation sub-foliated by a minimal linear foliation $\mathscr{L}$ on $\mathbb{T}^d$, then $\mathscr{F}$ is minimal and linear.

Proof. **Minimal.** every leaf $\mathscr{F}(x) \supset \mathscr{L}(x)$ is dense.
Linear. We will show that, on universal cover, $\widetilde{\mathscr{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \widetilde{\mathscr{F}}(0)$, there exists $v_n \to \widetilde{\mathscr{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \to x$. Since $\mathscr{F}$ is sub-foliated by $\mathscr{L}$ and $\mathscr{L}$ is linear, we have

$$y + k_n + v_n \in \widetilde{\mathscr{F}}(y + k_n) = \widetilde{\mathscr{F}}(k_n) = \widetilde{\mathscr{F}}(k_n + v_n).$$

Take $n \to \infty$, then $y + x \in \widetilde{\mathscr{F}}(x) = \widetilde{\mathscr{F}}(0)$. □

Lemma 1.1.19 If $E^{ss} \oplus E^u$ is integrable to $\mathscr{F}^{su}$, then $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.

Proof. Note that $h(\mathscr{F}^{su})$ is sub-foliated by $h(\mathscr{F}^u) = \mathscr{L}^u$, where $\mathscr{L}^u$ is linear and minimal on $\mathbb{T}^d$. Hence $h(\mathscr{F}^{su})$ is linear, A-invariant and transverse to $\mathscr{L}^{ws} = h(\mathscr{F}^{ws})$. This implies $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$. So

$$h(\mathscr{F}^{ss}) = h(\mathscr{F}^s \cap \mathscr{F}^{su}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{su}) = \mathscr{L}^s \cap \mathscr{L}^{su} = \mathscr{L}^{ss}.$$

□

Corollary 1.1.20

Recall that $T\mathcal{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$, then for $T\mathcal{F}_j^s = E_j^s$, we have

$$h(\mathcal{F}_j^s) = \mathcal{L}_j^s, \quad \forall j = k, k+1, \cdots, l.$$

Lemma 1.1.21 (Diophantine approximation of $\mathcal{F}^{ss}$)

There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathcal{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$-dense in $\mathbb{T}^d$.

Proof. Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of $\mathcal{L}^{ss}$. Here A is irreducible and $\mathcal{L}^{ss}$ is algebraic, hence Diophantine. □

Proof of Theorem 1.1.13. We will fist show that the Lyapunov exponent at every point is the same in the dim $E_{k+1}^s = 1$ case. Take $p, q \in \mathrm{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f.

Take

- $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$.
- Segments $J \subset \mathcal{F}_{k+1}^s(p)$ and $J_n \subset \mathcal{F}_{k+1}^s(x_n)$ such that $J_n = \mathrm{Hol}^{ss}(J)$ ($x_n = \mathrm{Hol}^{ss}(p)$). Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{L}_{k+1}^s) = \mathcal{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \quad \Longrightarrow \quad \exists C_0 > 0, |J_n| \geqslant C_0|J|.$$

Now we choose m_n, k_n such that

- x_n and q are very close in first k_n-steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathcal{F}_1^{ss}(p)$.

Then

$$|f^{m_n}(J_n)| \gtrsim \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n\lambda_{k+1}^s(q)]|J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geqslant \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \to \infty.$$

However, $J_n = \mathrm{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \mathrm{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathcal{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \mathrm{Hol}^{ss}(p)$, this contradicts to $\mathcal{F}^{ss}$ is C^1-smooth in $\mathcal{F}^{ss} \oplus \mathcal{F}_{k+1}^s(p)$.

For the case of dim $E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain

- For every periodic points p, q, we have $\min \lambda_{k+1}^s(p) = \min \lambda_{k+1}^s(q)$.
- Considering the growth of area of local disks, we have

$$\mathrm{Jac}(Df, E_{k+1}^s(p)) = \mathrm{Jac}(Df, E_{k+1}^s(q)), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda_{k+1}^s(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda_{k+1}^s(A)$. □

2 Methods for Studying Abelian Actions and Centralizers (Danijela Damjanović / Disheng Xu)

3 Dimension of Stationary Measures (Francios Ledrappier / Pablo Lessa)