# Reading Seminar on Homogeneous Dynamics (2023 Fall)

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# §1 Introduction (Weikun He, Sep 22)

Let $M$ be a hyperbolic surface, that is $M = \Gamma\backslash\mathbb{H}$ where $\Gamma$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ (**Fuchsian group**) and $\mathbb{H}$ is the hyperbolic space with the constant curvature $-1$. Let $\Delta$ be the **Laplace-Beltrami operator** on $\mathbb{H}$ given by (using the upper half plane model of $\mathbb{H}$)

$$\Delta f(x+iy) = -y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x+iy), \quad \forall f \in C^\infty(\mathbb{H}).$$

Then $\Delta$ induces a Laplace-Beltrami operator $\Delta_M$ on $M$ with $\Delta_M : L^2(M,\mathrm{Vol}) \to L^2(M,\mathrm{Vol})$. Then $\Delta_M$ satisfies

$$\langle \Delta_M f, f\rangle = \int \|\nabla f\|^2 \,\mathrm{d}\,\mathrm{Vol}, \quad \forall f \in L^2(M).$$

Consider eigenvalues $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots$ and eigenfunctions $f_i \in L^2(M)$ of $\Delta_M$ with

$$\Delta_M f_i = \lambda_i f_i, \quad \|f_i\|_{L^2(M,\mathrm{Vol})} = 1.$$

**Theorem 1.1** (Quantum ergodicity, Šnirel'man, Zelditch, Colin de Verdière)
Along a subsequence of density 1

$$|f_i|^2 \mathrm{d}\,\mathrm{Vol} \xrightarrow{\text{weak}^*} \mathrm{Vol},$$

provided that the geodesic flow $(g^t)$ on $(T^1M, \mu)$ is ergodic, where $\mu$ is the Liouville measure on $T^1M$.

In our case, the geodesic flow on $T^1M = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is given by

$$g^t(\Gamma x) = \Gamma x a^t, \quad a^t = \begin{bmatrix} e^t & \\ & e^{-t}\end{bmatrix},$$

and $\mu$ is induced by Haar. The ergodicity follows from Howe-Moore. Actually for QE, working on $T^1M$, $|f_i|^2 \mathrm{d}\,\mathrm{Vol}$ can be lifted to a measure $\mu_i$ on $T^1M$ (this operation is called a **microlocal lift**). A weak * limit of a subsequence $(\mu_i)$ is called a **quantum limit**.

**Theorem 1.2** (Šnirel'man) A quantum limit is $(g^t)$-invariant.

**Conjecture 1.3** (Quantum unique ergodicity, Rudnick-Sarnak)
For compact Riemannian manifold $M$ with negative curvature, the Liouville measure is the unique quantum limit.

**Remark 1.4** QUE fails for the billiard model of some regions $\Omega \subset \mathbb{R}^2$, which satisfies QE.

**Arithmetic QUE.** Let $M = \Gamma\backslash\mathbb{H}$ and $\Gamma$ is arithmetic. Then $\Delta_M$ commutes with Hecke operators $T_n, n \in \mathbb{Z}$. For $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, Hecke operators are given by

$$T_n\psi(z) = \sum_{ad=n,\ b\in\mathbb{Z}/d\mathbb{Z}} \psi\left(\frac{az+b}{d}\right), \quad \forall \psi : \Gamma\backslash\mathbb{H} \to \mathbb{R}.$$

A Hecke eigenform is an eigenfunction joint for $\Delta_M$ and $T_n, n \in \mathbb{Z}$. From now on, a **quantum limit** means a limit Hecke eigenforms.

**Theorem 1.5** (Lindenstrauss[Lin06]) QUE holds for compact arithmetic $\Gamma\backslash\mathbb{H}$.

He classified $(a_t, 1)$-invariant measures on $\mathrm{SL}(2,\mathbb{Z}[1/p])\backslash(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{Q}_p))/\mathrm{PSL}(2,\mathbb{Z}_p)$ for prime numbers $p$. A QUE for compact case follows from this and

**Fact 1.6** (Rudnick-Sarnak)**.** A closed geodesic cannot be a quantum limit.

For non compact cases, the proof needs additionally "non escape of mass". This is shown by Soundararajan.

**Some important results.**

1. QUE for continuous spectrum (Eisenstein series). Jakobson: subconvexity bounds for some $L$-functions
$$|\zeta(\frac{1}{2}+it)| \ll (1+|t|)^{\frac{1}{4}-\delta}.$$
2. Entropy bound. Anantharaman: If $\beta$ is a quantum limit of a compact negatively curved manifold, then
$$h(T^1M, \beta, g^t) > 0.$$
3. Control of eigenfunctions. Dyatlov-Jin, Dyatlov-Jin-Nonnenmacher: Let $M$ be an Anosov surface ($(T^1M, g^t)$ is an Anosov flow). Let $\Omega \subset M$ be an open subset, then there exists $C_\Omega > 0$ such that for every eigenfunction $f$,
$$\int_\Omega |f|^2 \mathrm{d\,Vol} \geqslant C_\Omega |f|^2_{L(M)}.$$
This uses a **fractal uncertainty principle**.

**Essential spectral gap.** The limit set of $\Gamma$ is $\Lambda_\Gamma := \overline{\Gamma x} \cap \partial\mathbb{H}$, where $\partial\mathbb{H} = \mathbb{S}^1$ (using the disk model). Let $\mathrm{Conv}(\Lambda_\Gamma)$ be the convex hull of $\Lambda_\Gamma \subset \mathbb{H}$, which is the union of geodesics connecting two points in $\Lambda_\Gamma$. We say $\Gamma\backslash\mathbb{H}$ is **convex cocompact** if $\Gamma\backslash\mathrm{Conv}(\Lambda_\Gamma)$ is compact.

**Example 1.7** Schottky surfaces are convex cocompact.

**Selberg zeta function**: for $s \in \mathbb{C}$, let
$$Z_M(s) := \prod_\ell \prod_{k=1}^{\infty} (1 - e^{-(\ell+k)s}),$$
where $\ell$ takes over lengths of primitive closed geodesics. Then $Z_M$ extends to $\mathbb{C}$ meromorfically. In fact, $\#\{Z_M = 0\} \cap \{\mathrm{Re} > 1/2\} < \infty$, which corresponds to small eigenvalues of $\Delta_M$.

We say $M$ has an **essential spectral gap** if there exists $\beta > 0$ such that
$$\#\{Z_M = 0\} \cap \left\{\mathrm{Re} > \frac{1}{2} - \beta\right\} < \infty.$$
Patterson and Sullivan showed that $M$ has an essential spectral gap with $\beta = \frac{1}{2} - \delta$ provided $0 < \delta(\Gamma) < \frac{1}{2}$, where $\delta(\Gamma)$ is the critical exponent of $\Gamma$.

Naud showed that $\beta$ can be $\frac{1}{2} - \beta + \varepsilon$ for some $\varepsilon > 0$. This implies a similar version of the prime number theorem. Let $\mathcal{N}(L) := \#\{\text{closed primitive geodesics of length } \leqslant L\}$, then
$$\mathcal{N}(L) = \mathrm{Li}(e^{\delta T}) + O(e^{(\delta-\varepsilon)T}),$$

where $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$. Jakobson-Naud conjectured that the optimal $\beta$ is $(1-\delta)/2$.

Another result is due to Dyatlov-Zahl. They showed that $\beta > 0$ for $\frac{1}{2}$. They established

$$\text{Fractal uncertainty principle} \implies \text{Essential spectral gap.}$$

Then Bourgain-Dyatlov showed the fractal uncertainty principle for every $\delta \in (0,1)$ and hence gave an essential spectral gap.

**Fractal uncertainty principle.** Let $h > 0$ be a small number. Consider the semiclassical Fourier transform

$$\mathscr{F}_h f(\xi) = (2\pi h)^{-\frac{1}{2}} \int e^{-i\xi x/h} f(x) \mathrm{d}x, \quad \forall f : \mathbb{R} \to \mathbb{C}.$$

For every $X \subset \mathbb{R}$, let $X^{(h)}$ denotes the $h$-neighborhood of $X$. We say $(X, Y)$ satisfies **uncertainty principle** if

$$\|\mathbb{1}_{X^{(h)}} \mathscr{F}_h \mathbb{1}_{Y^{(h)}}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leqslant O(h^\beta)$$

for some $\beta > 0$.

Assume that $X, Y$ are $\delta$-Ahlfors-David regular, that is there exists $\nu$ supported on $X$ and $C_R > 1$ such that

- $\forall x \geqslant \mathbb{R}$ and $r > 0$, $\nu(B(x,r)) \leqslant C_R r^\delta$.
- $\forall x \in X$ and $r > 0$, $\nu(B(x,r)) \geqslant C_R^{-1} r^\delta$.

**Example 1.8**

$X = \Lambda_\Gamma$ is $\delta$-Ahlfors-David regular by taking $\nu$ to be Patterson-Sullivan measure.

Then we have a trivial bound

$$\|\mathbb{1}_{X^{(h)}} \mathscr{F}_h \mathbb{1}_{Y^{(h)}}\|_{L^2 \to L^2} \leqslant h^{\frac{1}{2} - \delta}.$$

- $\|\mathbb{1}_{Y^{(h)}}\|_{L^2 \to L^1} \leqslant |Y^{(h)}|^{\frac{1}{2}} \ll h^{(1-\delta)/2}$, by $\delta$-Ahlfors-David regularity.
- $\|\mathscr{F}_h\|_{L^1 \to L^\infty} \ll h^{-\frac{1}{2}}$.
- $\|\mathbb{1}_{Y^{(h)}}\|_{L^\infty \to L^2} \leqslant |Y^{(h)}|^{\frac{1}{2}} \ll h^{(1-\delta)/2}$.

Dyatlov-Jin uses Dolgopyat method showed an FUP. Jin-Zhang gives an effective bound for $\beta = \beta(\delta)$. Backus-Leng-Tao showed for higher dimension.

## §2 Preliminaries on spectral theories (Yao Ma, Oct 13)

**Theorem 2.1**

Let $M$ be a compact Riemannian manifold and $\Delta$ is the Laplacian. Then the spectrum of $\Delta$ has the form

$$\sigma(\Delta) = \{0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots, \quad \lambda_n \to \infty\}.$$

**Theorem 2.2**

Suppose $A$ is a self-adjoint compact operator on a Hilbert space $\mathscr{H}$. Then there exists an orthonormal basis $\{\phi_i\}$ such that $A\phi_j = \lambda_i \phi_j, \lambda_j \in \mathbb{R}$, and the only limit point of $\{\lambda_j\}$ is 0.

**Definition 2.3.** A **finite meromorphic family** is a function $E : U \subset \mathbb{C} \to \mathscr{B}(\mathscr{H}, \mathscr{H})$ such that for every $a \in U$, $E$ can be expressed as $E(s) = \sum_{k=-m}^{\infty}(s-a)^k A_k$ in a neighborhood of $a$, where $A_k$ has finite rank for $k < 0$ and is compact for $k \geqslant 0$.

**Theorem 2.4** (Analytic Fredholm)

Suppose $E(s)$ is a finite meromorphic family of compact operators. If $I - E(s)$ is invertible for at least one $s \in U$ then $(I - E(s))^{-1}$ exists as a finite meromorphic family.

*Proof.* It suffices to prove the result in a neighborhood of $s_0 \in U$. Decompose $E(s) = A(s) + F(s)$ where $F(s)$ is a meromorphic family of finite-rank operators and $A(s)$ is a holomorphic family of compact operators. Assume $U$ is small enough, we can find a finite-rank operator $R$ such that $\|A(s) - R\| \leqslant 1$. Let $G(s) = (F(s)+R)(I - A(s)+R)^{-1}$, then $I - E(s) = (I - G(s))(I - A(s)+R)$. Since we can write $G(s)$ as

$$G(s) = \sum_{j,k=1}^{N} \gamma_{jk}(s)\psi_j \langle \phi_j, \cdot \rangle,$$

where $\gamma_{jk}$'s are meromorphic, we obtain the conclusion. □

**Theorem 2.5** (Weyl's law)

Suppose $\{\lambda_0 \leqslant \cdots\}$ is the spectrum of a self-adjoint positive elliptic operator $A$ on $M^n$ with the expression $A = a(x, D) = \sum_{k=0}^{m} a_m(x, D) = \sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha \in C^\infty(M)[D]$ and $a_m \neq 0$. Then

$$N(t) := \#\{\lambda_j \leqslant t\} \sim \frac{1}{(2\pi)^n}\int_M \int_{a_m(x,\xi)<t} \mathrm{d}\xi\mathrm{d}x = \frac{1}{n}\left(\int_M \mathrm{d}x \int_{|\xi'|=1} a_m^{-n/m}\mathrm{d}\xi'\right)t^{n/m}.$$

*Proof.* Define $A^z = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma^z}{\gamma I - A}\mathrm{d}\gamma$. Let $A_z(x, y)$, we consider $\xi(z) = \int_M A_z(x, x)\mathrm{d}x$.

**Theorem 2.6** (Trace formula) $\xi(z) = \sum_{k=0}^{\infty} \lambda_k^z$.

Write $\xi(z) = \int_0^\infty t^z N(t)$, assume that $N(t) = ct^\alpha + o(t^\alpha)$. Then $\xi(z) = \frac{c\alpha}{z+\alpha} + f(z)$.

**Theorem 2.7**

$A_z(x, x)$ could be extended to a meromorphic function on $\mathbb{C}$ with poles only at $z_j = (j - n)/m$, $j = 0, 1, \cdots$ with residues $\gamma_j$. Here

$$\gamma_0 = -\frac{1}{m}\int_{|\xi|=1} a_m^{-n/m}(x, \xi)\mathrm{d}\xi'.$$

Then we obtain $c = n/m$ and $\alpha = \gamma_0/c$. □

## §3 Micro-local lifts and quantum ergodicity (Yuxiang Jiao, Oct 20)

This lecture is based on two lecture notes on this topic: [Gorodnik] and [Einsiedler-Ward].

Setting

- $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2,\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of $G$.
- $\mathbb{H}$ the real hyperbolic plane, $\mathbb{H} \cong G/K$, $T^1\mathbb{H} \cong \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{id}\}$.
- $\Gamma$ a uniform lattice or a congruence lattice over $\mathbb{Q}$ (e.g. $\mathrm{SL}(2,\mathbb{Z})$) in $G$.
- $M = \Gamma\backslash\mathbb{H} = \Gamma\backslash G/K$ a hyperbolic surface with the volume form $\mathrm{Vol}_M$.
- $X = \Gamma\backslash G$ with a right $G$-action, $m_X$ the Haar measure on $X$.
- $u_x = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \in U, a_y = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix} \in A$, where $x + iy \in \mathbb{H}$. Let $k_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \in K$.

**Theorem 3.1** (Micro-local lifts)
Let $(\phi_i)$ be an $L^2$-normalized eigenfunctions of $\Delta_M$ in $C^\infty(M) \cap L^2(M)$ with corresponding eigenvalues $\lambda_i \to \infty$. Assume that $|\phi_i|^2 \mathrm{d}\,\mathrm{Vol}_M \to \mu$ in the weak* sense. Then there exists lifted functions $\tilde{\phi}_i$ on $X$ with a weak* limit $\tilde{\mu}$ of $(|\tilde{\phi}_i|^2 \mathrm{d}m_X)$ satisfying:
(1) $\tilde{\mu}$ projects to $\mu$ on $M$.
(2) $\tilde{\mu}$ is $A$-invariant.
The measure $\tilde{\mu}$ is called a **micro-local lift** of $\mu$, or a **quantum limit** of $(\phi_i)$.

**Remark 3.2** There is a natural way to lift $\phi$ to $\tilde{\phi}$ as a $K$-invariant functions. But it is hard to show the $A$-invariance of $\tilde{\mu}$ using these lifts.

**Remark 3.3** A subtle point of this theorem is that the construction of $\tilde{\phi}_i$ only depends on $\phi_i$ but not on the entire sequence $(\phi_i)$. So we can focus on studying these $\tilde{\phi}$'s on $\Gamma\backslash G$.

**Differential operators.**

**Definition 3.4.** For every $v \in \mathfrak{g}$, we define the differential operator $D_v : C^\infty(X) \to C^\infty(X)$ as

$$D_v f(x) = \left[\frac{\partial}{\partial t} f(x\exp(tv))\right]_{t=0}.$$

**Lemma 3.5** $D_v D_w - D_w D_v = D_{[v,w]}$ for every $v, w \in \mathfrak{g}$.

*Proof.* We have

$$\begin{aligned}
D_v D_w f(x) &= \left[\frac{\partial^2}{\partial t_1 \partial t_2} f(x\exp(t_2 v)\exp(t_1 w))\right]_{t_1=t_2=0} \\
&= \left[\frac{\partial^2}{\partial t_1 \partial t_2} f(x\exp(t_1 w)\exp(t_2 \mathrm{Ad}_{\exp(-t_1 w)}(v))\right]_{t_1=t_2=0} \\
&= \left[\frac{\partial}{\partial t_1} D_{v_{t_1}} f(x\exp(t_1 w))\right]_{t_1=0} \qquad \text{where } v_{t_1} = \mathrm{Ad}_{\exp(-t_1 w)}(v) \\
&= D_w D_v f(x) + D_? f(x).
\end{aligned}$$

Here $? = (\partial/\partial t)(\mathrm{Ad}_{\exp(-tw)} v)|_{t=0} = [v, w]$. ☐

By this identity, the differential operator can be extended to the universal enveloping algebra of $\mathfrak{g}$. There is a very special element in the universal enveloping algebra, the Casimir element, which is fixed by the adjoint representation. In fact, it induces the Laplacian on $M$. We consider

- $H = \begin{bmatrix} 1/2 & \\ & -1/2 \end{bmatrix}$, the direction of geodesic flow.
- $U^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, U^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, the directions of horocycle flow.
- $W = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = U^+ - U^-$, the direction of $K = \mathrm{SO}(2, \mathbb{R})$.

**Definition 3.6.** $\Omega = D_H D_H + \frac{1}{2} D_{U^+} D_{U^-} + \frac{1}{2} D_{U^-} D_{U^+}$ is called the **Casimir operator**.

**Fact 3.7.** $\Omega$ commutes with $D_v$ for every $v \in \mathfrak{g}$.

**Proposition 3.8**

For every $f \in C^\infty(M)$, we regard $f$ as a $K$-invariant smooth function on $X$. Then we have

$$\Omega f = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = -\Delta f.$$

**Fourier analysis.** First we recall the Fourier expansion on the torus $K = \mathrm{SO}(2, \mathbb{R})$. For $f \in C^\infty(K)$ and $x = k_\theta \in K$, we have

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{in\theta} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(k_\eta) e^{in(\theta - \eta)} \frac{\mathrm{d}\eta}{2\pi} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x k_\eta) e^{-in\eta} \frac{\mathrm{d}\eta}{2\pi}.$$

For $f \in C^\infty(X)$, we may define the Fourier expansion of $f$ along $K$-direction. Let

$$f_n(x) = \int_0^{2\pi} f(x k_\eta) e^{-in\eta} \frac{\mathrm{d}\eta}{2\pi}.$$

Note that $f_n$ satisfying $f_n(x k_\theta) = e^{in\theta} f_n(x)$.

We consider the subspace of ***K*-eigenfunctions of weight *n*** as

$$\mathscr{A}_n := \{ f \in C^\infty(X) : D_W f = inf \} = \left\{ f \in C^\infty(X) : f(x k_\theta) = e^{in\theta} f(x) \right\}.$$

We have $\mathscr{A}_n \mathscr{A}_m \subset \mathscr{A}_{n+m}$ and $\mathscr{A}_n \perp \mathscr{A}_m$ in the $L^2(X)$ sense for every $n \neq m$.

A function is said to be ***K*-finite** if it lies in a span of finitely many $\mathscr{A}_n$'s. For every $f \in C^\infty(X)$ and $L \geqslant 0$, let

$$f_{[-L,L]} = \sum_{n=-L}^{L} f_n \in \bigoplus_{n=-L}^{L} \mathscr{A}_n,$$

which is a $K$-finite function.

**Lemma 3.9** ($K$-finite approximation)

Let $f \in C_c^\infty(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$. Then $f_{[-L,L]} \to f$ uniformly as $L \to \infty$. Moreover, $D_H f_{[-L,L]}$ converges uniformly to $D_H f$.

The adjoint representation of $W$ on $\mathfrak{g}$ can be diagonalized in $\mathfrak{g}(\mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})$. Letting

$$E^+ := \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad \text{and} \quad E^- := \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix},$$

we have $[W, E^\pm] = \pm 2i E^\pm$. Therefore (extending $D_\cdot$ to $\mathfrak{g}(\mathbb{C})$ complex linearly),

$$D_W D_{E^\pm} f = D_{E^\pm} D_W f + D_{[W, E^\pm]} f = i(n \pm 2) D_{E^\pm} f.$$

This implies $D_{E^\pm} \mathscr{A}_n = \mathscr{A}_{n \pm 2}$. Note that $\{ E^+, E^-, W \}$ forms a basis of $\mathfrak{g}(\mathbb{C})$, and

$$\Omega = D_{E^+} D_{E^-} - \frac{1}{4} D_W D_W + \frac{i}{2} D_W = D_{E^-} D_{E^+} - \frac{1}{4} D_W D_W - \frac{i}{2} D_W$$

**Micro-local lifts.** Now we are at the stage to construct the micro-local lift. Let $\phi \in C^\infty(M)$ be an eigenfunction of $\Delta$ with eigenvalue $\lambda = \frac{1}{4} + r^2$, $\|\phi\|_2 = 1$. We aim to construct a probability measure on $X$ which projects to $|\phi|^2 \mathrm{d}\,\mathrm{Vol}_M$ and is asymptotically $A$-invariant as $\lambda \to \infty$.

We define inductively by $\phi_0 = \phi(xK) \in \mathscr{A}_0$, and

$$\phi_{2n+2} = \frac{1}{ir + \frac{1}{2} + n} D_{E^+}\phi_{2n} \in \mathscr{A}_{2n+2}, \quad n \geqslant 0, \tag{3.1}$$

$$\phi_{2n-2} = \frac{1}{ir + \frac{1}{2} - n} D_{E^-}\phi_{2n} \in \mathscr{A}_{2n-2}, \quad n \leqslant 0. \tag{3.2}$$

Since $\Omega$ commutes with $D_{E^\pm}$, $\Omega\phi_{2n} = \lambda\phi_{2n}$. Besides,

$$\begin{aligned}\|D_{E^+}\phi_{2n}\|^2 &= \left\langle D_{E^+}^* D_{E^+}\phi_{2n}, \phi_{2n}\right\rangle = -\langle D_{E^-}D_{E^+}\phi_{2n}, \phi_{2n}\rangle \\ &= -\left\langle (\Omega + \frac{1}{4}D_W^2 + \frac{i}{2}D_W)\phi_{2n}, \phi_{2n}\right\rangle = \left| ir + \frac{1}{2} + \frac{1}{2}n\right|^2 \|\phi_{2n}\|^2.\end{aligned}$$

Hence $\phi_n$ is $L^2$-normalized. We also mention that (3.1) and (3.2) hold for every $n \in \mathbb{Z}$.

Now we consider the $L^2$-normalized function

$$\psi = \psi_\lambda = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \phi_{2n},$$

where $N = N(\lambda)$ to be chosen later.

**Lemma 3.10** (Almost lifts)

If $f \in C_c^\infty(M)$ (regarding $f$ as a function in $\mathscr{A}_0$), then

$$\int f|\psi|^2 \mathrm{d}m_X = \int f|\phi|^2 \mathrm{d}\,\mathrm{Vol}_M + O_f(Nr^{-1}).$$

More generally, if $f$ is a $K$-finite function in $C_c^\infty(X)$, then

$$\int f|\psi|^2 \mathrm{d}m_X = \left\langle f \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle_{L^2(X)} + O_f(\max\{N^{-1}, Nr^{-1}\}).$$

*Proof.* Assume that $f \in \bigoplus_{\ell=-2L}^{2L} \mathscr{A}_n$ is a $K$-finite function. Then

$$\int f|\psi|^2 \mathrm{d}m_X = \langle f\psi, \psi\rangle_{L^2(X)} = \frac{1}{2N+1} \sum_{m,n=-N}^{N} \langle f\phi_{2m}, \phi_{2n}\rangle .$$

For $|m-n| > 2L$, $\langle f\phi_{2m}, \phi_{2n}\rangle = 0$. For $|m-n| \leqslant 2L$, we have

$$\begin{aligned}\langle f\phi_{2m}, \phi_{2n}\rangle &= \frac{1}{ir + m - \frac{1}{2}} \langle fD_{E^+}\phi_{2m-2}, \phi_{2n}\rangle \\ &= \frac{1}{ir + m - \frac{1}{2}} \left[ \langle D_{E^+}(f\phi_{2m-2}), \phi_{2n}\rangle - \underbrace{\langle \phi_{2m-2}D_{E^+}f, \phi_{2n}\rangle}_{O_f(1)} \right] \\ &= \frac{1}{ir + m - \frac{1}{2}} \langle f\phi_{2m-2}, E_+^*\phi_{2n}\rangle + O_f(r^{-1}) = -\frac{-ir - n + \frac{1}{2}}{ir + m - \frac{1}{2}} \langle f\phi_{2m-2}, \phi_{2n-2}\rangle + O_f(r^{-1}) \\ &= \langle f\phi_{2m-2}, \phi_{2n-2}\rangle + O_f(r^{-1}) = \cdots = \left\langle f\phi_{2(n-m)}, \phi_0\right\rangle + O_f(Nr^{-1}).\end{aligned}$$

Hence,

$$\langle f\psi, \psi\rangle = \sum_{\ell=-L}^{L} \frac{2N+1-|\ell|}{2N+1} \langle f\phi_{2\ell}, \phi_0\rangle + O_f(Nr^{-1}) = \left\langle f \sum_{\ell=-N}^{N} \phi_{2\ell}, \phi_0 \right\rangle + O_f(N^{-1}) + O_f(Nr^{-1}).$$

□

**Theorem 3.11** (Almost $A$-invariance, Zelditch)
If $f \in C_c^\infty(X)$ is a $K$-finite function and $N$ is sufficiently large (depending on $f$), then

$$\left\langle (rD_H + \mathscr{L})(f) \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = 0,$$

where $\mathscr{L}$ is a fixed degree-two differential operator. In particular,

$$\left\langle D_H(f) \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = O_f(N^{1/2}r^{-1}).$$

*Proof.* Assume that $f \in \bigoplus_{\ell=-2L}^{2L} \mathscr{A}_n$. Let $\tilde{\phi} = \sum_{n=-N}^{N} \phi_{2n}$. Recall that $D_{E^-}D_{E^+}\phi_0 = \Omega\phi_0 = \lambda\phi_0$. We have

$$\begin{aligned}\lambda\left\langle f\tilde{\phi}, \phi_0\right\rangle &= \left\langle f\tilde{\phi}, D_{E^-}D_{E^+}\phi_0\right\rangle = \left\langle D_{E^-}D_{E^+}(f\tilde{\phi}), \phi_0\right\rangle \\ &= \left\langle D_{E^-}D_{E^+}(f)\tilde{\phi}, \phi_0\right\rangle + \left\langle D_{E^+}(f)D_{E^-}(\tilde{\phi}), \phi_0\right\rangle + \left\langle D_{E^-}(f)D_{E^+}(\tilde{\phi}), \phi_0\right\rangle + \left\langle fD_{E^-}D_{E^+}(\tilde{\phi}), \phi_0\right\rangle.\end{aligned}$$

For the last term of the right hand side, we have (recalling $\Omega\phi_{2n} = \lambda\phi_{2n}$)

$$\left\langle fD_{E^-}D_{E^+}(\tilde{\phi}), \phi_0\right\rangle = \left\langle f\cdot(\Omega + \frac{1}{4}D_W^2 + \frac{i}{2}D_W)(\tilde{\phi}), \phi_0\right\rangle = \lambda(f\tilde{\phi}, \phi_0) + \frac{1}{4}\left\langle f\cdot(D_W^2 + 2iD_W)(\tilde{\phi}), \phi_0\right\rangle.$$

Then two terms $\lambda(f\tilde{\phi}, \phi_0)$ cancel out. For dealing with other terms, we use the fact that $D_{E^\pm}\tilde{\phi} \approx (ir \mp \frac{1}{2}D_W - \frac{1}{2})\tilde{\phi}$. The difference is a sum of $K$-eigenfunctions of weight about $\pm 2N$. Assuming $N$ large enough comparing to $L$, we have

$$\left\langle D_{E^\pm}(f)D_{E^\mp}(\tilde{\phi}), \phi_0\right\rangle = \left\langle D_{E^\pm}(f)\cdot(ir \pm \frac{1}{2}D_W - \frac{1}{2})\tilde{\phi}, \phi_0\right\rangle.$$

Recalling $E^+ + E^- = 2H$, we obtain

$$2ir\left\langle D_H(f)\tilde{\phi}, \phi_0\right\rangle + \langle\square, \phi_0\rangle = 0,$$

where $\square$ is independent with $r$. To show that $\langle\square, \phi_0\rangle$ is indeed of the form $\left\langle\mathscr{L}(f)\tilde{\phi}, \phi_0\right\rangle$, we note that it is of a special form: the only differential operator acting on $\tilde{\phi}$ is $D_W$. Recall $D_W\phi_0 = 0$. Therefore, for every $f_1, f_2$, we have

$$0 = \langle f_1f_2, D_W(\phi_0)\rangle = -\langle D_W(f_1f_2), \phi_0\rangle = -\langle D_W(f_1)f_2, \phi_0\rangle - \langle f_1D_W(f_2), \phi_0\rangle.$$

We obtain $\langle\square, \phi_0\rangle = \left\langle\mathscr{L}(f)\tilde{\phi}, \phi_0\right\rangle$, where $\mathscr{L}$ is an explicit second order differential operator. □

Finally, we take $N = \lceil r^{1/2}\rceil \approx \lambda^{1/4}$, which guarantees $N^{-1}, Nr^{-1}, N^{1/2}r^{-1} \to 0$. Then the weak* limit $\tilde{\mu}$ (passing to a subsequence if necessary) projects to $\mu$ by Lemma 3.10 and $A$-invariant by Theorem 3.11. It is worth noting that we rely on Lemma 3.9 to verify this for not only $K$-finite function but also for every $f \in C_c^\infty(X)$. With this, we conclude the proof of Theorem 3.1.

**Quantum ergodicity.** Now we will show some idea of the proof of the quantum ergodicity, Theorem 1.1.

**Theorem 3.12**

For every $K$-finite function $f \in C^\infty(X)$, we have

$$\frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} \left| \int f |\phi_\lambda|^2 \mathrm{d}m_X - \int f \mathrm{d}m_X \right| \to 0, \tag{3.3}$$

where $N(L) = \#\{\lambda \in \mathrm{Spec}(\Delta) : \lambda \leqslant L\}$.

**Lemma 3.13** (General Weyl law / Trace formula)

For every $K$-finite $f$, we have

$$\frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} \int f |\psi_\lambda|^2 \mathrm{d}m_X \to \int f \mathrm{d}m_X.$$

*Proof of Theorem 3.12.* For every $K$-finite $f$, let

$$A_T(f)(x) = \frac{1}{T} \int_0^T f(x \exp(tH)) \mathrm{d}t.$$

By Lemma 3.10 and Theorem 3.11,

$$|\langle f, |\psi_\lambda|^2 \rangle| = |\langle A_T(f), |\psi_\lambda|^2 \rangle| + O_f(T\lambda^{-1/4}).$$

Assuming $\int f \mathrm{d}m_X = 0$. For every $T > 0$, we have

$$\limsup \frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} |\langle f, |\phi_\lambda|^2 \rangle| \leqslant \limsup \frac{1}{N(L)} \sum \langle |A_T(f)|, |\phi_\lambda|^2 \rangle \leqslant \int |A_T(f)| \mathrm{d}m_X.$$

By the ergodicity of geodesic flow, we have $\int |A_T(f)| \mathrm{d}m_X \to 0$ as $T \to \infty$. □

To show Theorem 1.1, it remains two steps.

- First, for each $f$, extract a density 1 subsequence converging to $\int f \mathrm{d}m_X$ using (3.3). This needs an estimate on the spectral density, see Section 5 in [Zel87].
- Secondly, there is a density-1 subsequence independent with the choice of $f$, see Section 6 in [Zel87].

# §4 Statement of main theorems and basic definitions (Disheng Xu, Oct 27)

Setting

- $L$ is an $S$-algebraic group, $S \subset \left\{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p\right\}$.
- $G = \mathrm{SL}(2,\mathbb{R}) \times L$, $H$ the $\mathrm{SL}(2,\mathbb{R})$ factor of $G$.
- $K$ a compact subgroup of $L$, $\Gamma$ a discrete subgroup of $G$.
- $X = \Gamma \backslash G / K$.
- $A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$ the diagonal subgroup.

**Example 4.1**

1. $L = \mathrm{SL}(2, \mathbb{Q}_p), K = \mathrm{SL}(2, \mathbb{Z}_p)$ and $\Gamma$ is the diagonal embedding of $\mathrm{SL}(2, \mathbb{Z}[1/p])$ in $G$. Then $\Gamma\backslash G/K \cong \mathrm{SL}(2, \mathbb{Z})\backslash\mathrm{SL}(2, \mathbb{R})$.
2. $L = \mathrm{SL}(2, \mathbb{R})$ and $K = \{e\}$ and $\Gamma$ is an irreducible lattice.

Invariant measures are impossible to classify in these two cases. The cases are similar to "rank one" hyperbolic dynamics, which do not have measure rigidity.

**Theorem 4.2**

Assume that $\Gamma \cap L$ is finite. Let $\mu$ be an $A$-invariant probability measure on $\Gamma$. Assume that

(1) All ergodic components of $\mu$ has positive entropy.
(2) $\mu$ is $L/K$ recurrent.

Then $\mu$ is a combination of $H$-invariant algebraic measure.

We consider $\Gamma$ as the following two cases:

(1) $\Gamma$ is a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$;
(2) $\Gamma$ derived from Eichler orders in an $\mathbb{R}$-split quaternion algebra over $\mathbb{Q}$. In this case, $\Gamma$ is cocompact.

We call these lattices congruence lattices over $\mathbb{Q}$.

**Theorem 4.3**

Let $M = \Gamma\backslash\mathrm{SL}(2, \mathbb{R})$ where $\Gamma$ is a congruence lattice over $\mathbb{Q}$. Then every "arithmetic quantum limit" is $c\,\mathrm{Vol}_{\Gamma\backslash\mathrm{SL}(2,\mathbb{R})}$, where $c = 1$ for the cocompact case and $0 \leqslant c \leqslant 1$ for general cases.

Here "arithmetic quantum limit" requires each $\phi_i$ to be eigenfunctions for both $\Delta$ and Hecke operators. It is conjectured $\Delta$ has simple spectrums and hence the "arithmetic quantum limit" coincides the "quantum limit".

Theorem 4.3 follows from Theorem 4.2 by the following results:

(1) Any arithmetic quantum limit $\mu$ has positive entropy: every $A$-ergodic component has entropy $\geqslant 2/9$ [BL03].
(2) $\mathrm{SL}(2, \mathbb{Q}_p)/\mathrm{SL}(2, \mathbb{Z}_p)$-recurrence.

Two other consequences of Theorem 4.2 are the following.

**Theorem 4.4**

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. Let $A(\mathbb{A})$ be the diagonal group of $\mathrm{SL}(2, \mathbb{A})$ and let $\mu$ be an $A(\mathbb{A})$-invariant probability measure on $\mathrm{SL}(2, \mathbb{Q})\backslash\mathrm{SL}(2, \mathbb{A})$ then $\mu$ is $\mathrm{SL}(2, \mathbb{A})$-invariant.

**Theorem 4.5**

Let $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ and $H$ as before. Let $\Gamma$ be a discrete subgroup of $G$ satisfying its projection to each $\mathrm{SL}(2, \mathbb{R})$ factor is finite. Let $\mu$ be an ergodic invariant measure under the action of $B = \left\{\begin{bmatrix} * & \\ & * \end{bmatrix} \times \begin{bmatrix} * & \\ & * \end{bmatrix}\right\}$. Then

- either $\mu$ is an algebraic measure,
- or $\mu$ has zero entropy with respect to every one-parameter subgroup of $B$.

$(G, T)$**-spaces.**

- $X$ locally compact separable metric space.
- $T$ locally compact separable metric space with a distinguished point $e \in T$.
- $G$ a locally compact second countable group.
- A continuous transitive action $G \curvearrowright T$.

**Definition 4.6.** $X$ is called a **$(G, T)$-(foliated) space** if there is an open cover $\mathfrak{T}$ of $X$ by relatively compact sets, and for every $U \in \mathfrak{T}$ a continuous map $t_U : U \times T \to X$ satisfying:

(1) For every $x \in U$, $t_U(x, e) = x$.
(2) For every $x \in U$ and $y \in t_U(x, T)$, and $V \in \mathfrak{T}$, there exists $\theta \in G$ such that $t_V(y, \cdot) \circ \theta = t_U(x, \cdot)$. In particular, $t_U(x, T) = t_V(y, T)$.
(3) There is some $r_U > 0$ so that for every $x \in U$, $t_U(x, \cdot)$ is injective on $\overline{B^T_{r_U}(e)}$.

**Definition 4.7.** We say a Radon measure $\mu$ on a $(G, T)$ space is **recurrent**, if for every measurable set $B$ with $\mu(B) > 0$, for every $x \in B, x \in U \in \mathfrak{T}$ and for every compact $K \subset T$, there is $t \in T \setminus K$ such that $t_U(x, t) \in B$.

## §5 Hecke-Maass forms (Pengyu Yang, Nov 3)

Recall

$$\mathbb{Q}_p = \left\{ \sum_{i=\ell}^{\infty} a_i p^i : 0 \leqslant a_i \leqslant p-1 \right\}, \quad \mathbb{Z}_p = \left\{ \sum_{i=\ell}^{\infty} a_i p^i : 0 \leqslant a_i \leqslant p-1 \right\}.$$

For $a = a_\ell p^\ell + \cdots$ where $a_\ell \neq 0$, the $p$-adic norm is $|a| = p^{-\ell}$.

Consider $\mathbb{Z}[1/p] = \{ n/p^m : n \in \mathbb{Z}, m \geqslant 0 \}$. Then $\mathbb{Q}_p = \mathbb{Z}_p + \mathbb{Z}[1/p]$ and $\mathbb{Z}_p \cap \mathbb{Z}[1/p] = \mathbb{Z}$. Now we show the isomorphism

$$\mathrm{SL}(2,\mathbb{Z}) \backslash \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{Z}[1/p]) \backslash \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{Q}_p)/\mathrm{SL}(2,\mathbb{Z}_p).$$

We consider the map $[g_\infty] \mapsto [(g_\infty, 1)]$. This is well-defined a map.

**Injectivity.** If $[(g_\infty, 1)] = [(g'_\infty, 1)]$, then there exists $\gamma_p \in \mathrm{SL}(2,\mathbb{Z}[1/p])$ and $k_p \in \mathrm{SL}(2,\mathbb{Z}_p)$ such that $(g_\infty, 1) = (\gamma_p g'_\infty, \gamma_p k_p)$. Therefore, $\gamma_p = k_p^{-1} \in \mathrm{SL}(2,\mathbb{Z}[1/p]) \cap \mathrm{SL}(2,\mathbb{Z}_p) = \mathrm{SL}(2,\mathbb{Z})$. Hence $[g_\infty] = [g'_\infty]$.

**Surjectivity.** It suffices to show $\mathrm{SL}(2,\mathbb{Q}_p) = \mathrm{SL}(2,\mathbb{Z}[1/p])$. Note that $\mathrm{SL}(2,\mathbb{Q}_p)$ can be decomposed as a finite product of unipotent subgroups. Therefore $\mathrm{SL}(2,\mathbb{Z}[1/p])$ is dense in $\mathrm{SL}(2,\mathbb{Q}_p)$. Combining with $\mathrm{SL}(2,\mathbb{Z}_p)$ is open in $\mathrm{SL}(2,\mathbb{Q}_p)$, we obtain the desired conclusion. □

**Maass forms.** Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

**Definition 5.1.** $f \in C^\infty(\mathbb{H})$ is a **Maass form** for $\Gamma$ if

(i) $f(\gamma z) = f(z)$, for every $\gamma \in \Gamma$.
(ii) $\Delta f = \lambda f$.
(iii) $f(x + iy) = O(y^N)$ for some $N > 0$.

We call $f$ a **Maass cusp form** if $\int_0^1 f(z + x) \mathrm{d}x = 0$ for every $z$.

**Fourier expansion.** Since $f(x + 1) = f(x)$, we have

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(y) e^{2\pi i r x}.$$

Write $a_r(y) = \sqrt{y}k(2\pi|r|y)$. Assume that $\lambda = \frac{1}{4} - \nu^2$. Then we have

$$\left(y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - (y^2+\nu^2)\right)k = 0.$$

This ODE has two fundamental solutions $I_\nu, K_\nu$ where $I_\nu$ is exponentially growth and $K_\nu$ is rapid decay. By the definition of Maass forms, $k$ is a multiple of $K_\nu$, where

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y\frac{t+t^{-1}}{2}}t^\nu\frac{\mathrm{d}t}{t}.$$

Therefore,

$$f(z) = \sum_r a(r)\sqrt{y}K_\nu(2\pi|r|y)e^{2\pi irx}.$$

It is a cusp form iff $a(0) = 0$.

**Hecke correspondence.** Let

$$X_2 = \mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R}) = \mathrm{PGL}(2,\mathbb{Z}[1/p])\backslash\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{Q}_p)/\mathrm{PGL}(2,\mathbb{Z}_p).$$

There are four equivalent definitions of Hecke correspondence:

1. For every $\mathbb{H}\in z$, let $T_pz = \{pz, z/p, (z+1)/p, \cdots, (z+p-1)/p\}$.
2. Let $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$ and $\gamma_p = \mathrm{diag}(p,1)$. We have
$$\Gamma\gamma_p\Gamma = \Gamma\begin{bmatrix} p & \\ & 1\end{bmatrix} \sqcup \bigsqcup_{i=0}^{p-1}\Gamma\begin{bmatrix} 1 & i \\ & p\end{bmatrix}.$$
Let $T_p : \Gamma g \mapsto \Gamma\gamma_p\Gamma g$.
3. Using the fact $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R})$ is $\{\text{lattices in } \mathbb{R}^2\}/(\Lambda\sim\lambda\Lambda : \lambda\in\mathbb{R}^\times)$, then $T_p(\Lambda) = \{\Lambda' : [\Lambda:\Lambda'] = p\}$.
4. We have
$$\mathrm{PGL}(2,\mathbb{Z}_p)\begin{bmatrix} p & \\ & 1\end{bmatrix}\mathrm{PGL}(2,\mathbb{Z}_p) = \begin{bmatrix} p & \\ & 1\end{bmatrix} \sqcup \bigsqcup_{i=0}^{p-1}\begin{bmatrix} 1 & i \\ & p\end{bmatrix}\mathrm{PGL}(2,\mathbb{Z}_p).$$
Let $T_p([(g_\infty, g_p)]) = [(g_\infty, g_p\mathrm{diag}(p,1))]$.

**Bruhat-Tits building.** (gluing infinitely many euclidean spaces). Recall that for a real Lie group $\mathbb{G}(\mathbb{R})$, it acts on the symmetric space $\mathbb{G}/\mathbb{K}$ by isomorphisms. We want to define this notion similarly for $p$-adic groups $\mathbb{G}(\mathbb{Q}_p)$.

We here only give the example for $\mathrm{SL}(2,\mathbb{Q}_p)$ and list some properties. $\mathrm{SL}(2,\mathbb{Q}_p)$ acts on the $(p+1)$-regular tree $T$. The stabilizer of each vertex is a maximal compact subgroup. Let

$$\pi : \mathrm{SL}(2,\mathbb{Z}_p)\to\mathrm{SL}(2,\mathbb{F}_p), \quad \begin{bmatrix} a & b \\ c & d\end{bmatrix}\mapsto\begin{bmatrix} a \bmod p & b \bmod p \\ c \bmod p & d \bmod p\end{bmatrix}.$$

Then $\ker\pi = \begin{bmatrix} 1+\mathscr{P} & \mathscr{P} \\ \mathscr{P} & 1+\mathscr{P}\end{bmatrix}$ is the stabilizer of $\{v\in T : d(v,v_0)\leqslant 1\}$ for some $v_0\in T$. An **apartment** is a maximal flat geodesic subspace in the BT-tree. Then we have a 1-1 correspondence

$$\{\text{apartments}\}\longleftrightarrow\{\text{maximal split tori}\},$$

denoted by $\mathscr{A}\mapsto T(\mathscr{A})$. There are two properties of apartments:

(i) For every apartments $\mathscr{A},\mathscr{A}'$, there exits $g\in G$ such that $g\mathscr{A} = \mathscr{A}'$ and $g|_{\mathscr{A}\cap\mathscr{A}'} = \mathrm{id}$.

(ii) For every distinct vertices $x, x'$, there exits an apartment $\mathscr{A}$ such that $x, x'\in\mathscr{A}$. Moreover, if $x' = gx$ then there exists $a\in T(\mathscr{A})$ such that $x' = ax$.

Use these properties, we can prove Cartan decomposition $G = KAK$, where $K = G_o$ the stabilizer of $o$ and $A = T(\mathscr{A})$ where $o \in \mathscr{A}$.

*Proof.* For every $g \in G$, there exists $\mathscr{A}'$ such that $o, go \in \mathscr{A}'$. Then there exists $g_1 \in G$ such that $\mathscr{A} = g_1\mathscr{A}'$ and $g_1 o = o$, hence $g_1 \in G_o$. Note that $g_1^{-1}go, o \in \mathscr{A}$, there exists $a \in T(\mathscr{A})$ such that $g_1^{-1}go = ao$. Therefore, $a^{-1}g_1^{-1}g \in G_o$. □

**Hecke operators.** For $N \in \mathbb{Z}_+$, let

$$T_N f(\Lambda) = \frac{1}{\sqrt{N}} \sum_{[\Lambda:\Lambda']=N} f(\Lambda').$$

We have $T_M T_N = T_N T_M$ and $T_N \Delta = \Delta T_N$. We say $f$ is a **Hecke-Maass form** if $f$ is a Maass cusp form and is an eigenform for all $T_n$. Write

$$f = \sum_n a(n)\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi i n x}.$$

Assume that $a(1) = 1$ then $T_n(f) = a(n)f$.

## §6 Positivity of the entropy of quantum limits (Jiesong Zhang, Nov 10)

This lecture is devoted to prove the positivity of entropies for arithmetic quantum limits, which is based on [BL03].

**Theorem 6.1**

Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a congruence lattice and $\mu$ be a quantum limit on $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. There exists $\tau_0 > 0(\tau_0 = 1/50)$ and $\kappa' > 0(\kappa' = 2/9)$ such that the following holds. For every compact subset $K$ of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ and every $x \in K$, we have

$$\mu(xB(\varepsilon,\tau_0)) \ll_K \varepsilon^{\kappa'}.$$

Here, $B(\varepsilon,\tau) = a((-\tau,\tau))u^-((-\varepsilon,\varepsilon))u^+((-\varepsilon,\varepsilon))$, where $u^+ = \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}, u^- = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, a = \begin{bmatrix} a^t & \\ & a^{-t} \end{bmatrix}$.

**Corollary 6.2**

For every almost every ergodic component $\mu_0$ of $\mu$, we have $h(\mu_0) \geqslant \frac{\kappa'}{2}h(\mathrm{Haar})$.

Theorem 6.1 is a direct consequence of the following theorem.

**Theorem 6.3**

Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a cocompact lattice or a congruence lattice, $\Phi \in L^2(\Gamma\backslash\mathrm{SL}(2,\mathbb{R}))$ be an $L^2$-normalized eigenfunction of all Hecke operators. Then for every compact subset $\Omega$ and $x \in \Omega, r > 0$,

$$\int_{xB(r,\tau_0)} |\Phi(y)|^2 \mathrm{d}\,\mathrm{Vol}(y) \ll r^{\kappa'}.$$

To show this theorem, we need two following results.

**Corollary 6.4** ([BL03, Corollary 3.7])

Let $\Phi$ be as above. Let $n = p_1 p_2 \cdots p_k$ be a square free positive integer. Take $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $\alpha_i = 1$ if $T_{p_i}\Phi = \lambda_{p_i}\Phi$ with $\lambda_{p_i} > \sqrt{p_i}/10$ and $\alpha_i = 2$ otherwise. Then for all $x \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, we have

$$|\Phi(x)|^2 \ll_k \sum_{y \in T_m(x)} |\Phi(y)|^2.$$

**Theorem 6.5** ([BL03, Theorem 3.5])

For any set of prime numbers $\mathscr{P}$, $x \in \Lambda\backslash \mathrm{SL}(2,\mathbb{R})$ and $\varepsilon > 0$, there is a set $W$ of cube free integers with the following properties:

(1) $n \in W$ has bounded number of prime factors (uniformly in $x, \varepsilon$).

(2) For every $n \in W$, $p^2 | n$ iff $p | n$ and $p > \mathscr{P}$.

(3) $\{ yB(\varepsilon, \tau_0) : y \in T_n(x), n \in W \}$ are pairwise disjoint.

(4) $\#W \gg \varepsilon^{-\kappa'/4}$.

*Proof of Theorem 6.3.* Let $\mathscr{P}$ be the set of all primes for which $\lambda_p < \sqrt{p}/10$. Let $x \in \Omega$. By the theorem above, there exists $W$ satisfying the conditions. Then we have

$$(\#W) \int_{xB(\varepsilon,\tau_0)} |\Phi(y)|^2 \mathrm{d}\,\mathrm{Vol}(y) \ll \sum_{n \in W} \sum_{z \in T_n(x)} \int_{zB(\varepsilon,\tau_0)} |\Phi(y)|^2 \mathrm{d}\,\mathrm{Vol}(y) \leqslant -\!\!\!\!\!\!\int |\Phi(y)|^2 = 1.$$

Using $\#W \gg \varepsilon^{-\kappa'/4}$ we obtain the desired conclusion. $\square$

Now we show the idea to prove Theorem 6.5. Let $A$ be a finite set of integers and $\mathscr{P}$ be a set of primes. For every $y \geqslant 0$, we let $P(y) := \prod_{p \leqslant y, p \in \mathscr{P}} p$. Let

$$A_d := \#\{ a \in A : a \equiv 0 \bmod d \}.$$

$$S(A, \mathscr{P}, y) := \{ a \in A : \gcd(a, P(y)) = 1 \}.$$

**Proposition 6.6**

Let $\omega : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function. Assume that

(1) $A_d = X\omega(d)/d + R(d)$.

(2) $\sum_{p \leqslant y} \omega(p)/p \ll \log\log y$.

Then there exists $\alpha > 0$ and for every $M > 0$, we have

$$S(A, \mathscr{P}, y) = X \prod_{p \leqslant y} (1 - \frac{\omega(p)}{p}) \left( 1 + O\left( \frac{1}{\log^M y} \right) \right) + O(y^{\alpha \log\log y}).$$

**Example 6.7**

We estimate the number of prime numbers in $[Y, X+Y]$, denoted by $\pi(X,Y)$. Let $A = [Y, X+Y] \cap \mathbb{Z}$ and $\mathscr{P}$ be the set of all prime numbers. Take $\omega(p) = 1$. Then $A_d = X/d + O(1)$

and $\sum_{p \leqslant y} 1/p \ll \log\log y$. Therefore

$$\pi(X,Y) \leqslant S(A,\mathscr{P},y) + y = X\prod_{p\leqslant y}(1-\frac{1}{p})\left(1+O\left(\frac{1}{\log^M y}\right)\right) + O(y^{\alpha \log\log y}).$$

Taking $\log y = \log X/(\log\log X)$, we obtain $\pi(X,Y) \ll \frac{X(\log\log X)}{\log X}$.

**Example 6.8** (Upper bound of twin prime numbers)

Let $\pi_2(X)$ be the number of twin prime numbers at most $X$. Let $A = \{x \leqslant X : x(x+2)\}$ and $\mathscr{P}$ be the set of all prime numbers. Let $\omega(p) := \begin{cases} 1, & p = 1 \\ 2, & p \geqslant 2 \end{cases}$. We can show that $A_d = X\omega(d)/d + O(1)$. We have $\pi_2(X) \leqslant S(A,\mathscr{P},y) + y$. Taking $\log y = \log X/(\log\log X)$, we obtain

$$S(A,\mathscr{P},y) \ll \frac{X(\log\log X)^2}{(\log X)^2}.$$

# §7 Leafwise measures (Weikun He, Nov 17)

Recall the notion of $(G,T)$-space:

- $G$ a locally compact topological space,
- $T$ a locally compact separable metric space with a distinguished point $e$,
- $G \curvearrowright T$ transitive.

We further assume that $G$ acts on $T$ by isometries. Recall that a $(G,T)$-structure on $X$ is a collection $(U,t_U)_{u\in\mathfrak{T}}$ where $\{U\}_{U\in\mathfrak{T}}$ is an open cover of $X$ and $t_U : U\times T \to X$ continuous, satisfying:

(1) For every $x \in U$, $t_U(x,e) = x$.
(2) For every $x \in U$ and $y = t_U(x,t_0)$, and $V \in \mathfrak{T}(y)$, there exists $\theta \in G$ such that $t_V(y,\theta\cdot) = t_U(x,t)$. In particular, $t_U(x,T) = t_V(y,T)$. We also assume that $\theta t_0 = e$.
(3) There is some $r_U > 0$ so that for every $x \in U$, $t_U(x,\cdot)$ is injective on $\overline{B^T_{r_U}(e)}$.

For every $x \in X$, we let $B^T_r(x) := t_U(x,B^T_r) \subset X$, which is independent with the choice of $U \in \mathfrak{T}(x)$. The "T-leaf" of $x$ is $T_x(x,T) = B^T_\infty(x)$.

**Example 7.1**

Let $X$ with a right $G$-action. Let $T = G, e = 1_G$. Let $t_U(x,g) = xg$. For every $y = xg_0$, we have $xg = y(g_0^{-1}g)$.

In our case, we take

- $H = \mathrm{SL}(2,\mathbb{R})$, $L = \mathrm{SL}(2,\mathbb{Q}_p)$, $K = \mathrm{SL}(2,\mathbb{Z}_p) < L$.
- $T = L/K$, $e = K \in L/K$.
- $X = \Gamma\backslash H\times L/(1\times K)$, where $\Gamma$ is a discrete subgroup of $H\times L$.

We have a $(L,T)$-structure on $X$. For every $x \in U \subset X$ where $x = \Gamma(h,\ell)K$, we want to take $t_U(x,t) = \Gamma(h,\ell g)K$ where $t = gK$. But this is not well-defined. We need to fix $\ell_U : U \to L$ and $h_U : U \to H$ such that for every $x \in U$ with $x = \Gamma(h_U(x),\ell_U(x))K$ and $t = gK$, we take $t_U(x,t) = \Gamma(h_U(x),\ell_U(x)gK)$.

**Conditional measures.** Recall $(X, \mathscr{B}, \mu)$ is a standard probability space. Let $\mathscr{A} \subset \mathscr{B}$ be a sub-$\sigma$-algebra. There exists $(\mu_x^{\mathscr{A}})_{x \in X}$, where $\mu_x^{\mathscr{A}} \in \mathscr{P}(X, \mathscr{B})$ such that for every $f \in L^1(X, \mu)$,

$$\int f \mathrm{d}\mu_x^{\mathscr{A}} = \mathbb{E}[f|\mathscr{A}](x),$$

where $\mathbb{E}[f|\mathscr{A}]$ is the conditional expectation.

**Remark 7.2** For every $f \in L^1(X, \mu)$, $x \mapsto \int f \mathrm{d}\mu_x^{\mathscr{A}}$ is $\mathscr{A}$-measurable.

**Definition 7.3.** We say $\mathscr{A}$ is **countably generated** if it is generated as a $\sigma$-algebra by a countable set $\{ A_i \}_{i \in \mathbb{N}} \subset \mathscr{A}$.

**Definition 7.4.** If $\mathscr{A}$ is countably generated. For every $x \in X$, the **$\mathscr{A}$-atom of $x$** is

$$[x]_{\mathscr{A}} = \bigcap_{A \in \mathscr{A} : x \in A} A = \left( \bigcap_{i : x \in A_i} A_i \right) \cap \left( \bigcap_{i : x \notin A_i} (x \setminus A_i) \right),$$

which is measurable.

**Remark 7.5** If $\mathscr{A}$ is countably generated then $\mu_x^{\mathscr{A}}([x]_{\mathscr{A}}) = 1$ for almost every $x$.

**Example 7.6**

$X$ is a separable metric space and $\mathscr{B}$ is the Borel $\sigma$-algebra, then $\mathscr{B}$ is compactly generated.

**Example 7.7**

$\varphi : (Y, \mathscr{C}) \to (X, \mathscr{B})$ is measurable. If $\mathscr{B}$ is countably generated then so is $\mathscr{C}$.

**Example 7.8**

$X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $(X, \mathscr{B})$ is Borel. Let $\varphi_t : \mathbb{T}^2 \to \mathbb{T}^2$ be the irrational flow. Let $\mathscr{E} = \{ A \in \mathscr{B} : \forall t, \varphi_t A = A \}$. Then $\mathscr{E}$ is **NOT** compactly generated.

We can use conditional measures to show this fact. Let $\mu$ be the Lebesgue measure on $\mathbb{T}^2$. Then $\mu_x^{\mathscr{E}}([x]_{\mathscr{E}}) = 1$ by discussions above. But we can show that $\mu_x^{\mathscr{E}}$ is $\varphi_t$-invariant. Therefore $\mu_x^{\mathscr{E}}$ is Lebesgue by the unique ergodicity, which contradicts $\mu_x^{\mathscr{E}}([x]_{\mathscr{E}}) = 1$.

**Remark 7.9** If $\mu$ is $f$-invariant. Let $\mathscr{I} = \left\{ A \in \mathscr{B} : f^{-1}A = A \right\}$. Then $\int \mu_x^{\mathscr{I}} \mathrm{d}\mu(x) = \mu$ is the ergodic decomposition.

**Remark 7.10** If $\mathscr{A} \doteq \mathscr{A}'$ mod $\mu$ countably generated, then $\mu_x^{\mathscr{A}}([x]_{\mathscr{A}}) = \mu_x^{\mathscr{A}'}([x]_{\mathscr{A}'})$.

In Example 7.8, $\mathscr{E} = \{ \emptyset, X \}$ mod Leb. But we do not have $\mu^{\mathscr{E}} = \mu^{\{ \emptyset, X \}}$. This shows some limitation of the conditional measure. To deal with these cases, we consider the leafwise measure.

**Leafwise measures.** We illustrate the construction of leafwise measures using Example 7.8. We regard $\varphi_t$ as an $T = \mathbb{R}$ action on $X = \mathbb{T}^2$. Consider the probability measure $\mu = \mathrm{Leb}|_Q/\mathrm{Leb}(Q)$ where $Q$ is a region in $X$. The leafwise measure is $(\mu_x^T)_{x\in X}$, a collection of Radon measures on $T = \mathbb{R}$. It is given by

$$\mu_x^T = \mathbb{1}_{\{t\in T:\varphi_t(x)\in Q\}} \cdot \mathrm{Leb}.$$

**Notation 7.11.** $\mu \propto \nu$ if there exists $c > 0$ such that $\mu = c\nu$.

We need to mention that $\mu_x^T(T) = \infty$ and there is no canonical way to normalize $\mu_x^T$. Therefore, $(\mu_x^T)_{x\in X}$ is defined up to a proportion and up to a null set.

Now we construct leafwise measures for general spaces. Let $X$ be a $(G,T)$-space.

**Definition 7.12.** $A \subset X$ is an **open $T$-plaque** if for every $x \in A$,

(1) $\exists r > 0$, $A \subset B_r^T(x)$,

(2) $t_U(x,\cdot)^{-1}A$ is open on $T$ for every $U \in \mathfrak{T}(x)$.

**Definition 7.13.** $(\mathscr{A}, U)$ is an **$(r,T)$-flower with center $B \subset U$** if $U \subset X$ and $\mathscr{A}$ is a countably generated $\sigma$-algebra on $U$ satisfying

(♣-1) $B \subset U$ and $\overline{U}$ is compact.

(♣-2) For every $y \in U$, $[y]_{\mathscr{A}} = U \cap B_{4r}^T(y)$.

(♣-3) For every $y \in B$, $B_r^t(y) \subset [y]_{\mathscr{A}}$.

**Theorem 7.14** (The existence of leafwise measures)

Let $\mu \in \mathscr{P}(X)$ where $X$ is a $(G,T)$-space. Assume for $\mu$-almost every $x \in X$, $B_\infty^T(x)$ is embedded (here "embedded" means $t \mapsto t_U(x,t)$ is injective). Then for every $V \in \mathfrak{T}$, there exists $(\mu_{x,T}^V)_{x\in V}$ where $\mu_{x,T}^V$ are Radon measures on $T$ such that

(1) For almost every $x \in V$, $\mu_{x,T}^V(B_1^T) = 1$.

(2) If $(\mathscr{A}, U)$ is an $(r,T)$-flower then for $\mu$-almost every $x \in U$ and every $V \in \mathfrak{T}(x)$, we have

$$t_V(x,\cdot)_*^{-1}\mu_x^{\mathscr{A}} \propto \mu_{x,T}^V|_{t_V(x,\cdot)^{-1}[x]_{\mathscr{A}}}.$$

These two conditions determine $(\mu_{x,T}^V)_{x\in V}$ up to a null set.

Moreover, for every $x, U \in \mathfrak{T}(x)$ and $y \in B_\infty^T(x)$, $V \in \mathfrak{T}(y)$, we have

$$\theta_*\mu_{x,T}^U \propto \mu_{y,T}^V,$$

where $\theta \in G$ satisfying $t_U(y,\theta t) = t_U(x,t)$.

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