A GLANCE AT THE TITS ALTERNATIVE

YUXIANG JIAO

ABSTRACT. Tits alternative asserts that every finitely generated linear group over an arbitrary field contains either a solvable subgroup of finite index or a non-abelian free subgroup. In this essay, we show the idea of the proof for a special case which takes the field to be $\mathbb{C}$.

CONTENTS

1. INTRODUCTION

In 1972, Tits published his paper *Free subgroups in linear groups* [Tit72] in the Journal of Algebra. In which he showed a new phenomenon, now known as Tits alternative for linear groups. He asserted that every finitely generated linear group over an arbitrary field contains either a solvable subgroup of finite index or a non-abelian free subgroup.

In this essay, we will show the idea of the proof for a special case: the Tits alternative for a linear group over $\mathbb{C}$.

Definition 1.1. Let G be a group. G is said to be *virtually solvable* if there is a solvable subgroup $H < G$ of finite index.

Theorem 1. *Let G be a finitely generated subgroup of* $\mathrm{GL}(n, \mathbb{C})$, *then*

(1) either G is virtually solvable,
(2) or G contains a non-abelian free subgroup.

Remark 1.2. Note that a non-abelian free subgroup does not contain any nontrivial solvable subgroup, this theorem indeed gives a dichotomy.

As a corollary of Theorem 1, it can be shown that the theorem also holds for a non-finitely generated group.

Theorem 2. *Let G be a subgroup of* $\mathrm{GL}(n,\mathbb{C})$, *then*

(1) either G is virtually solvable,

(2) or G contains a non-abelian free subgroup.

Theorem 1 can be generalized to any field $\Bbbk$, we refer to [Tit72, Corollary 1].

Theorem 3. *Let G be a finitely generated subgroup of* $\mathrm{GL}(n,\Bbbk)$ *for some field* $\Bbbk$, *then*

(1) either G is virtually solvable,

(2) or G contains a non-abelian free subgroup.

Remark 1.3. It is also mentioned in Tits' paper that the theorem fails for a non-finitely generated group $G < \mathrm{GL}(n,\Bbbk)$ for a field of characteristic different from 0.

For the group other than linear groups, we also concern about whether Tits alternative holds. For example, $\mathrm{Homeo}(\mathbb{S}^1)$, the group of homeomorphisms on $\mathbb{S}^1$. Unfortunately, Tits alternative fails for such a large group. But Margulis proved a conjecture of Ghys [Mar00], which is an alternative version of Tits alternative in $\mathrm{Homeo}(\mathbb{S}^1)$.

Theorem 4. *Let G be a subgroup of* $\mathrm{Homeo}(\mathbb{S}^1)$, *then*

(1) either G preserves a common probability measure on $\mathbb{S}^1$,

(2) or G contains a non-abelian free subgroup.

Remark 1.4. Note that every virtually solvable group is amenable, then this theorem makes sense since (1) indeed contains the case that G is virtually solvable. But two cases of this theorem can happen simultaneously, so this theorem does not give a dichotomy.

2. The way to construct a free group: the Pingpong lemma

For proving Tits alternative, the aim is to find a general way to construct a free group. The pingpong lemma helps a lot.

Proposition 2.1 (Pingpong Lemma)**.** *Let G be a group acting on a set Ω, let H_1, H_2 be subgroups of G. Assume that there exists two disjoint nonempty sets $\Delta_1, \Delta_2 \subset \Omega$ such that*

$$h_i(\Delta_{3-i}) \subset \Delta_i, \quad \forall h_i \neq e \in H_i,\ \forall i = 1,2.$$

Then the group $\langle H_1, H_2\rangle$ generated by H_1 and H_2 is a free product $H_1 * H_2$.

Now we consider a baby case that $G < \mathrm{GL}(n,\mathbb{C})$ contains an element g such that there is an eigenvalue λ of g which has a unique maximal absolute value. Let

$$A_g = \ker(g - \lambda \mathrm{Id})$$

and A'_g be the unique g-invariant space such that $\mathbb{C}^n = A_g \oplus A'_g$. Let a_g be the corresponding point of A_g in $\mathbb{P}^{n-1}$, and a'_g be the hyperplane in $\mathbb{P}^{n-1}$ corresponding to A_g. Then for every open neighborhood $U \ni a_g$ and compact set $K \subset \mathbb{P}^{n-1} \setminus a'_g$, there exists $N > 0$ such that

$$g^n(K) \subset U, \quad \forall n \geqslant N.$$

We also assume that g^{-1} has the same property, write $r_g = a_{g^{-1}}$ and $r'_g = a'_{g^{-1}}$. Then for every open neighborhood $U \ni r_g$ and compact set $K \subset \mathbb{P}^{n-1} \setminus r'_g$, there exists $N > 0$ such that

$$g^{-n}(K) \subset U, \quad \forall n \geqslant N.$$

If there exists another element $h \in G$ satisfying the same condition such that

$$a_g, r_g \notin a'_h \cup r'_h, \quad a_h, r_h \notin a'_g \cup r'_g,$$

then we can choose Δ_1 be a neighborhood of $\{a_g, r_g\}$ and Δ_2 be a neighborhood of $\{a_h, r_h\}$. For some m large enough, we have

$$g^m \Delta_2 \subset \Delta_1, \quad g^{-m}\Delta_2 \subset \Delta_1, \quad h^m \Delta_1 \subset \Delta_2, \quad h^{-m}\Delta_1 \subset \Delta_2.$$

Hence g^m and h^m generates a non-abelian free group.

The general idea of the proof is to generalize the argument above. But there are some problems need to solve. For example, G is a compact (under the Euclidean topology in $\mathrm{GL}(n, \mathbb{C})$) semisimple group which is obvious not virtually solvable. But we may not find an element possess a unique maximal eigenvalue. Hence we need to consider G acting on some different spaces and more general argument.

Definition 2.2. Let $\Bbbk$ be a field. An *absolute value* on $\Bbbk$ is a function $|\cdot| : \Bbbk \to \mathbb{R}_+$ such that

(i) $|x| = 0$ iff $x = 0$.
(ii) $|xy| = |x||y|$ for every $x, y \in \Bbbk$.
(iii) $|x + y| \leqslant |x| + |y|$.

A field equipped with an absolute value $(\Bbbk, |\cdot|)$ is naturally a metric space by setting $d(x, y) = |x - y|$ for every $x, y \in \Bbbk$.

Definition 2.3. $(\Bbbk, |\cdot|)$ is said to be *locally compact* if it is locally compact as a metric space.

Consider G acts linearly on a vector space $V = \Bbbk^n$ where $\Bbbk$ is a locally compact field. For every $g \in G$, denote by $\overline{g}$ to be the representative in $\mathrm{GL}(V)$.

Definition 2.4. For $g \in G$, if $\overline{g}$ admits a maximal eigenvalue λ, that is, $|\lambda|$ takes the unique maximum value. Let $A_g = \ker(\overline{g} - \lambda)$ and $a_g = \mathbb{P}(A_g) \in \mathbb{P}(V)$. Then a_g is called an *attractor* of g. If g^{-1} has an attractor $a_{g^{-1}}$, we call $r_g = a_{g^{-1}}$ a *repellor* of g.

Lemma 2.5. *Let $G < \mathrm{GL}(n, \Bbbk)$ be a linear group over a locally compact field $\Bbbk$. Assume that the Zariski closure of G is Zariski connected in $\mathrm{GL}(n, \Bbbk)$ and the action of G on $\Bbbk^n$ is irreducible. Assume that G possesses a diagonalizable element g with an attractor and a repellor. Then G has a non-abelian free group.*

Sketch of Proof. The idea is to choose g' has the form hgh^{-1} for some $h \in G$, such that g^m and $(g')^m$ generates a non-abelian free group. Note that

$$a_{g'} = ha_g, \quad a'_{g'} = ha'_g, \quad r_{g'} = hr_g, \quad r'_{g'} = hr_g,$$

it suffices to find h to move away those subspaces. This can achieve because G do not preserves any non trivial subspace, hence the map

$$\varphi_{x,y^*} : h \mapsto y^*(hx)$$

is not identically zero for every $x \neq 0 \in V, y^* \neq 0 \in V^*$. Then $\{h : \varphi_{x,y^*} \neq 0\}$ is Zariski open and non empty. Since the closure of G is Zariski connected, we can always find $h \in G$ such that $\varphi_{x,y^*}(h) \neq 0$ for finitely many φ_{x,y^*} simultaneously. $\square$

3. Reduction to the semisimple case

3.1. Reduction to the semisimple case.

Let G be a subgroup of $\mathrm{GL}(V)$ for some vector space V. Denote $\mathbb{G}$ to be the Zariski closure of G which is an algebraic group. Let $\mathbb{G}^0$ be the identity component of $\mathbb{G}$. In this section, we will reduce Theorem 1 to the case that the algebraic closure of G is semisimple.

Lemma 3.1. *Let G be an algebraic group, then G possesses a unique largest normal solvable subgroup, which is closed.*

Definition 3.2. The *radical* of an algebraic group $\mathbb{G}$ is the identity component of the largest normal solvable subgroup.

Definition 3.3. A connected algebraic group with a trivial radical is said to be *semisimple*.

Theorem 5. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated, non trivial subgroup such that the Zariski closure $\mathbb{G}$ is semisimple. Then G contains a non-abelian free subgroup.*

Lemma 3.4. *Theorem 1 follows by Theorem 5.*

Proof. Let $\mathbb{G}$ be the Zariski closure of G in $\mathrm{GL}(V)$, let $G^0 = G \cap \mathbb{G}^0$. Note that $[\mathbb{G} : \mathbb{G}^0] < \infty$, hence $[G : G^0] < \infty$. If G is not virtually solvable, then G^0 is not solvable and hence $\mathbb{G}^0$ is also not solvable. Let $\mathbb{J}$ be the radical of $\mathbb{G}^0$, which is a proper subgroup. Since $\mathbb{G}^0$ is connected, $\mathbb{G}^0/\mathbb{J}$ is a nontrivial semisimple algebraic group.

Note that G^0 is a finite index subgroup of a finitely generated group, hence G^0 is finitely generated. Hence $G^0/(G^0 \cap \mathbb{J})$ is finitely generated and is dense in $\mathbb{G}^0/\mathbb{J}$. By Theorem 5, $G^0/(G^0 \cap \mathbb{J})$ contains a non-abelian free subgroup. And hence G also contains a non-abelian free subgroup. □

3.2. Properties of semisimple groups.

Definition 3.5. An algebraic group is called *simple* if it is connected, non-abelian and it has no nontrivial closed connected proper normal subgroup.

Proposition 3.6. *Let $\mathbb{G}$ be a semisimple algebraic group and let $\mathbb{G}_i (i \in I)$ be the simple subgroups of $\mathbb{G}$. Then*

1. *I is finite and $\mathbb{G} = \prod_{i \in I} \mathbb{G}_i$.*
2. *For each i, $\mathbb{G}_i \cap (\prod_{j \neq i} \mathbb{G}_j)$ is finite.*
3. *The commutator $[\mathbb{G}_i, \mathbb{G}_j] = e$ for every $i \neq j$.*
4. *Any closed connected normal subgroup of $\mathbb{G}$ is a the product of some $\mathbb{G}_i$.*

Definition 3.7. A group G is said to be *perfect* if $[G, G] = G$.

Corollary 3.8. *A semisimple algebraic group is perfect.*

Corollary 3.9. *If $\mathbb{G} < \mathrm{GL}(n, \mathbb{k})$ is a semisimple algebraic group, then $\mathbb{G} < \mathrm{SL}(n, \mathbb{k})$.*

Let $\mathbb{k}$ be a field and let $\mathbb{K}$ be the algebraic closure of $\mathbb{k}$. Recall that a linear endomorphism over $\mathbb{k}$ is called semisimple if it is diagonalizable over $\mathbb{K}$. The following proposition allows us to find large diagonalizable elements in a semisimple algebraic group.

Proposition 3.10. *Let $\mathbb{G}$ be a semisimple algebraic group. Then the set of semisimple elements in $\mathbb{G}$ contains a (Zariski) dense open subset of $\mathbb{G}$.*

4. Constructing a proximal element

4.1. **Elements of infinite order.**

For constructing a free subgroup in G, it is necessary to find some elements of infinite order. For example, if we want to find a free group in a compact semisimple group. The first step will be finding an element g with infinite order. And then, we construct a representation such that g admits an attractor and a repellor.

Lemma 4.1. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated group acting irreducible on $\mathbb{C}^n$. Then there exists a basis $\{e_1, \cdots, e_{n^2}\}$ of $M_n(\mathbb{C})$ such that*

$$G \subset \left\{ \sum_{i=1}^{n^2} t_i e_i : t_i \in \mathrm{tr}(G) \right\}.$$

Proof. By the classification of semisimple algebra over algebraic closed field, we know that G linearly spans $M_n(\mathbb{C})$. We can choose $g_1, \cdots, g_{n^2}$ in G which forms a basis of $M_n(\mathbb{C})$. Consider a non-degenerated bilinear form $M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{C}$ given by $(x, y) \mapsto \mathrm{tr}(xy)$. Let $\{e_i\}$ be a dual basis to $\{g_i\}$ with respect to this bilinear form, then $\{e_i\}$ is satisfied. $\square$

Lemma 4.2. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated group and let F be the set of elements of finite order in G. Then $\mathrm{tr}(F)$ is finite.*

Proof. Let $\mathbb{k}$ be the finite extension of $\mathbb{Q}$ contains all entries in G. Then every eigenvalue of an element in G is a root of a polynomial over $\mathbb{k}$ of degree n. Let ξ be a root of unity satisfying a polynomial over $\mathbb{k}$ of degree less than n. Let T be a transcendence basis of $\mathbb{k}/\mathbb{Q}$ and $\mathbb{k}_a$ be the algebraic closure in $\mathbb{k}(\xi)$. Then

$$[\mathbb{Q}(\xi) : \mathbb{Q}] \leqslant [\mathbb{k}_a : \mathbb{Q}] = [\mathbb{k}_a(T) : \mathbb{Q}(T)] \leqslant [\mathbb{k}(\xi) : \mathbb{Q}(T)] \leqslant n[\mathbb{k} : \mathbb{Q}(T)] < \infty.$$

Hence there are only finite possible values of ξ. The trace of a finite order element in G is a sum of n such roots of unity, hence $\mathrm{tr}(F)$ is finite. $\square$

Proposition 4.3. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated group acting irreducible on $\mathbb{C}^n$. Let F be the set of elements of finite order in G. If F is Zariski dense in G, then G is finite.*

Proof. Since $\mathrm{tr}(F)$ is finite hence Zariski closed in $\mathbb{C}$, then $\mathrm{tr}^{-1}(\mathrm{tr}(F))$ is Zariski closed. By assumption, F is Zariski dense in G, thus $G = \mathrm{tr}^{-1}(\mathrm{tr}(F))$. Which follows that $\mathrm{tr}(G)$ is finite. By Lemma 4.1, G is finite. $\square$

4.2. **Choosing an appropriate absolute value.**

We have found an element of infinite order g, but the eigenvalues of g might fall on the unit circle. In such case, we need to choose an appropriate absolute value such that this the absolute value of some eigenvalue of g is not 1.

For every prime number p, let $|\cdot|_p$ be the p-adic absolute value and $\mathbb{Q}_p$ be the completion of $(\mathbb{Q}, |\cdot|_p)$. Besides, we use $|\cdot|_\infty$ to denote the archimedean absolute value over $\mathbb{Q}$. The following proposition shows that for each algebraic element λ over $\mathbb{Q}$ which is not a root of unity, there always exists a p-adic absolute value such that $|\lambda|_p \neq 1$.

Proposition 4.4. *Let $\mathbb{k}$ be a finite algebraic extension of $\mathbb{Q}$, let $x \in \mathbb{k}^\times$. Then $|x|_p = 1$ for every p if and only if x is a root of unity.*

Lemma 4.5. *Let $\Bbbk \subset \mathbb{C}$ be a finite field extension of $\mathbb{Q}$ and let $\lambda \in \Bbbk^\times$ be an element of infinite order. Then there exists an extension of $\Bbbk$ to a locally-compact field $\Bbbk'$ endowed with an absolute value $|\cdot|$ such that $|\lambda| \neq 1$.*

Proof. Let $\Bbbk_a$ be the algebraic closure of $\mathbb{Q}$ in $\Bbbk$. If λ is transcendental over $\mathbb{Q}$, choose a transcendence basis T of $\Bbbk/\mathbb{Q}$ such that $\lambda \in T$. Let $\tau : \Bbbk \hookrightarrow \mathbb{C}$ be an embedding fix $\Bbbk_a$ and $|\tau(\lambda)|_\infty \neq 1$. Setting $\Bbbk' = \mathbb{C}$ is satisfied.

If λ is algebraic over $\mathbb{Q}$, there exits p such that $|\lambda|_p$. Let $\Bbbk_p$ be the completion of $\Bbbk_a$ with respect to $|\cdot|_p$. Since the transcendental degree of $\Bbbk_p/\Bbbk_a$ is infinite, there exists an embedding $\Bbbk_a(T) \hookrightarrow \Bbbk_p$ fix $\Bbbk_a$ where T is a finite transcendence basis. Since $\Bbbk$ is a finite algebraic extension of $\Bbbk_a(T)$, then there exist a finite extension $\Bbbk'/\Bbbk_p$ such that $\Bbbk$ can be regard as a subfield of $\Bbbk$ and $|\lambda| \neq 1$. □

4.3. **Proximal elements.**

Definition 4.6. Let $\mathbb{G}$ be an algebraic group. A *rational representation* of G is a morphism between algebraic groups $\rho : \mathbb{G} \to \mathrm{GL}(n, \Bbbk)$ for some field $\Bbbk$.

We call a rational representation is *irreducible* if there is no proper nontrivial invariant subspace of $\Bbbk^n$ invariant under the action. Let $\mathbb{K}$ be the algebraic closure of $\Bbbk$, the representation is called *absolutely irreducible* if $\rho(\mathbb{G})$ is irreducible on $\mathbb{K}^n$.

Lemma 4.7. *Let $\mathbb{G}$ be a perfect algebraic group with a nontrivial rational representation $\rho : \mathbb{G} \to \mathrm{GL}(n, \Bbbk)$. Then $\mathbb{G}$ possesses a nontrivial irreducible rational representation.*

Proof. WLOG, assume ρ is the nontrivial rational representation with a lowest degree. Assume that ρ is reducible, then there is a nontrivial $\rho(\mathbb{G})$ invariant subspace $W \neq \Bbbk^n$. Note that W is dimensional 1 and $\rho|_W$ is trivial. Then $\rho(\mathbb{G})$ has the form

$$\begin{bmatrix} 1 & * \\ 0 & \rho_{\Bbbk^n/W} \end{bmatrix}.$$

Besides, the representation $\rho_{\Bbbk^n/W}$ is also trivial. Then $\rho(\mathbb{G})$ has the form of upper triangular matrices. It follows that $\rho(\mathbb{G})$ is solvable, contradicts with $\mathbb{G}$ is perfect. □

Lemma 4.8. *Let $\rho : \mathbb{G} \to \mathrm{GL}(n, \Bbbk)$ be a rational representation of an algebraic group over $(\Bbbk, |\cdot|)$. Let $g \in G$ be an element such that*

(i) $\rho(g)$ is diagonalizable,
(ii) the number of eigenvalues of $\rho(g)$ with the maximal absolute is less than n.

Then there exists an absolutely irreducible representation ρ' of $\mathbb{G}$ such that $\rho'(g)$ is diagonalizable and g has an attractor.

Proof. Let $d < n$ be the number of eigenvalues of $\rho(g)$ with the maximal absolute. Consider the action $\bigwedge^d \rho$, then g has an attractor. A similar argument as the previous lemma shows that there exists an absolutely irreducible representation ρ' such that $\rho'(g)$ is diagonalizable and g has an attractor. □

Proposition 4.9. *Let $\Bbbk$ be a locally compact field and let $\mathbb{G}$ be a Zariski connected subgroup of $\mathrm{GL}(n, \Bbbk)$ acting irreducibly on $\Bbbk^n$. Assume that there exists a diagonalizable element in $\mathbb{G}$ with an attractor, then the set*

$$X = \{g \in \mathbb{G} : g \text{ has an attractor and a repellor}\}$$

is Zariski dense in $\mathbb{G}$.

Sketch of Proof. For $g \in \mathrm{GL}(n, \Bbbk)$ acting on $\mathbb{P}(\Bbbk^n)$, if there exists a compact set $K \subset \mathbb{P}(\Bbbk^n)$ such that $gK \subset K^\circ$, one can show that g has an attractor in K. Note that this is an open property. Hence there is a nonempty Zariski open (and hence dense) set in G satisfying an appropriate condition (just move away the invariant subspaces). Let g_0 be the element with an attractor, then we compose those elements with g_0^m for some m large enough. We can see a compact set K such that $gK \subset K^\circ$. But here are something subtle is that m may depend on the element in this (Zariski) open set. This can be solved by an algebraic trick, see [Tit72, Proposition 3.1]. □

Remark 4.10. In a view of dynamical system, Margulis and Goldsheid showed that if a finitely generated subgroup $G < \mathrm{SL}(n, \mathbb{C})$ is Zariski dense in $\mathrm{SL}(n, \mathbb{C})$, then G contains lots of proximal elements [GM89]. Indeed, they showed that the top Lyapunov exponent of random walk on G tends to infinity if G is Zariski dense in $\mathrm{SL}(n, \mathbb{C})$. For a more general result, in Xu's master thesis [XS12], they generalized this result to the case that Zariski closure of G is a semisimple algebraic group over $\mathbb{C}$. But this is not enough to deduce Tits alternative in the case that the Zariski closure of G is a compact semisimple group.

5. Proof of the main theorem

By lemma 3.4, it suffices to show Theorem 5. We restate the theorem here for the convenience of reading.

Theorem. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a finitely generated, non trivial subgroup such that the Zariski closure $\mathbb{G}$ is semisimple. Then G contains a non-abelian free subgroup.*

Proof. By lemma 4.7, WLOG, we can assume that $\mathbb{G}$ acts irreducibly on $\mathbb{C}^n$. By proposition 4.3, the elements of finite order is not dense in $\mathbb{G}$. Since G is dense in $\mathbb{G}$ and semisimple elements is dense in $\mathbb{G}$ by proposition 3.10, there exists a semisimple element of infinite order in G.

Let g be a such element and λ be an eigenvalue of g which is not a root of unity. Since G is finitely generated, then $G < \mathrm{GL}(n, \Bbbk)$ for some field $\Bbbk$ which is a finite extension of $\mathbb{Q}$. Note that this does not change the Zariski topology on G. By lemma 4.5, there exists an extension of $\Bbbk$ to a locally-compact field $\Bbbk'$ such that $|\lambda| \neq 1$.

By corollary 3.9, $g \in \mathbb{G} \subset \mathrm{SL}(n, \mathbb{C})$. Then $\det g = 1$ implies that the number of eigenvalues with a maximal absolute value of g is less than n. By lemma 4.8, we can assume that G acts absolutely irreducibly on ${\Bbbk'}^n$ and g has an attractor. By proposition 4.9, the elements with an attractor and a repellor is dense in $\mathbb{G}$. It allows us to take such an element $h \in G$. Then there exists a finite extension of $\Bbbk'$ such that h is diagonalizable and G still acts irreducibly. By lemma 2.5, G contains a non-abelian free subgroup. □

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