Selected Minicourses in *Beyond Uniform Hyperbolicity* Będlewo, 2023

Yuxiang Jiao

Contents

1 Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori (Yi Shi)

§1.1 Local Rigidity (Apr 25)

Definition 1.1.1. $f \in \mathrm{Diff}^1(M)$ is **Anosov** if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

Example 1.1.2 (Arnold's cat map)
$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

Question 1.1.3. Is every Anosov diffeomorphism transitive?

Question 1.1.4. Topological classification of Anosov diffeomorphism.

Theorem 1.1.5 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Theorem 1.1.6 (Franks-Newhouse)
Every codimension-1 Anosov diffeomorphism must be supported on $\mathbb{T}^d$.

Definition 1.1.7. $f \in \mathrm{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \to (0,1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0,1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \to M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.1.10 $E^s \oplus E^u$ integrable $\implies$ NOT accessible.

However, Dolgopyat-Wilkinsonm and Hertz-Hertz-Ures, etc. showed that "MOST" partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

Geometric Rigidity $\iff$ Dynamic Spectral Rigidity

That is, $E^s \oplus E^u$ is integrable $\implies$ E^c has exponents rigidity.

Example 1.1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22}-close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \to \mathbb{T}^4$ which is C^1-along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \to \mathbb{T}^2$ be C^1-close to A. Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$$

is an Anosov diffeomorphism C^1-close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

NO exponents rigidity.

Main theorem: local rigidity. Assume that $A \in GL(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K \to \infty} \frac{\#\{ A \text{ is generic} : \|A\| \leqslant K \}}{\#\{ A : \|A\| \leqslant K \}} = 1.$$

Denote

$$T\mathbb{T}^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$

the finest dominated splitting, then $\dim L_i^{s/u} \leqslant 2$.

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$

Assume that $l \geqslant 2$ and $1 \leqslant k < l$. Denote

$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$

Then

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$$

makes f be an absolutely partially hyperbolic system.

Theorem 1.1.13 (Local rigidity, Gogolev-Shi, arXiv: 2207.00704)

Assume $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \mathrm{Diff}^2(\mathbb{R}^d)$ be C^1-close to A, the following are equivalent:

1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws}:

$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \cdots, l.$$

3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws}.

Dimension 3 case.

Theorem 1.1.14 (Hammerlindl-Ures, 2014)

Let $f \in \mathrm{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \mathrm{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then

- either f is accessible, thus ergodic.
- or there exists an f-invariant minimal foliation $\mathscr{F}^{su}$ such that $T\mathscr{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_*.

Theorem 1.1.15 (Gan-Shi, 2020)

Let $f \in \mathrm{Diff}_m^{1+}(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:

- $E^s \oplus E^u$ is integrable;
- f has spectral rigidity in E^c: $\lambda^c(f) \equiv \lambda^c(f_*)$.

Both imply f is Anosov.

Corollary 1.1.16 Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.1.13 — spectral rigidity $\Longrightarrow$ joint integrability. The case of all E_i^s are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \mathrm{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E_l^s \Longrightarrow$ smooth conjugacy in $E_l^s \Longrightarrow h(\mathscr{F}_{l-1}^s) = \mathscr{L}_{l-1}^s$ (+spectral rigidity in E_{l-1}^s) $\Longrightarrow$ smooth conjugacy in $E_{l-1}^s \Longrightarrow \cdots \Longrightarrow h(\mathscr{F}_{k+1}^s) = \mathscr{L}_{k+1}^s$ (+spectral rigidity in E_{k+1}^s) $\Longrightarrow h(\mathscr{F}^{ss}) = \mathscr{L}^{ss} \Longrightarrow \mathscr{F}^{ss} \oplus \mathscr{F}^u = h^{-1}(\mathscr{L}^{ss} \oplus \mathscr{L}^u)$ joint integrability.

Proof of Theorem 1.1.13 — joint integrability $\Longrightarrow$ spectral rigidity. Main ideas:
1. $E^{ss} \oplus E^u$ integrability $\Longrightarrow h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathscr{F}^{ss} \Longrightarrow$ spectral rigidity in E_{k+1}^s.
3. $E^{ss} \oplus E_{k+1}^s \oplus E^u$ is integrable, and play induction on E_{k+2}^s.

Lemma 1.1.17
For every $1 \leqslant i \leqslant l$, the conjugation h preserves the center foliation: $h(\mathscr{F}_{(i,l)}^s) = \mathscr{L}_{(i,l)}^s$. Here, $\mathscr{F}_{(i,l)}^s$ and $\mathscr{L}_{(i,l)}^s$ are the foliations tangent to $E_i^s \oplus \cdots \oplus E_l^s$ and $L_i^s \oplus \cdots \oplus L_l^s$, respectively.

Proof. Since f is C^1-close to A, we have

$$\|A_{L_{i-1}^s}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E_i^s(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \to \mathbb{R}^d$ be lifts of f and h, then $y \in \widetilde{\mathscr{F}}_{(i,l)}^s(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leqslant (\rho_i - \varepsilon)^{-n}\|x - y\| + C < (\|A_{L_{i-1}^s}\| + \varepsilon)^{-n}\|x - y\| + C,$$

iff $H(y) \in \widetilde{\mathscr{L}}_{(i,l)}^s(H(x))$. □

Lemma 1.1.18
If $\mathscr{F}$ is a C^0-foliation sub-foliated by a minimal linear foliation $\mathscr{L}$ on $\mathbb{T}^d$, then $\mathscr{F}$ is minimal and linear.

Proof. **Minimal.** every leaf $\mathscr{F}(x) \supset \mathscr{L}(x)$ is dense.
Linear. We will show that, on universal cover, $\widetilde{\mathscr{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \widetilde{\mathscr{F}}(0)$, there exists $v_n \to \widetilde{\mathscr{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \to x$. Since $\mathscr{F}$ is sub-foliated by $\mathscr{L}$ and $\mathscr{L}$ is linear, we have

$$y + k_n + v_n \in \widetilde{\mathscr{F}}(y + k_n) = \widetilde{\mathscr{F}}(k_n) = \widetilde{\mathscr{F}}(k_n + v_n).$$

Take $n \to \infty$, then $y + x \in \widetilde{\mathscr{F}}(x) = \widetilde{\mathscr{F}}(0)$. □

Lemma 1.1.19 If $E^{ss} \oplus E^u$ is integrable to $\mathscr{F}^{su}$, then $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.

Proof. Note that $h(\mathscr{F}^{su})$ is sub-foliated by $h(\mathscr{F}^u) = \mathscr{L}^u$, where $\mathscr{L}^u$ is linear and minimal on $\mathbb{T}^d$. Hence $h(\mathscr{F}^{su})$ is linear, A-invariant and transverse to $\mathscr{L}^{ws} = h(\mathscr{F}^{ws})$. This implies $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$. So

$$h(\mathscr{F}^{ss}) = h(\mathscr{F}^s \cap \mathscr{F}^{su}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{su}) = \mathscr{L}^s \cap \mathscr{L}^{su} = \mathscr{L}^{ss}.$$

□

Corollary 1.1.20

Recall that $T\mathcal{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$, then for $T\mathcal{F}_j^s = E_j^s$, we have

$$h(\mathcal{F}_j^s) = \mathcal{L}_j^s, \quad \forall j = k, k+1, \cdots, l.$$

Lemma 1.1.21 (Diophantine approximation of $\mathcal{F}^{ss}$)

There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathcal{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$-dense in $\mathbb{T}^d$.

Proof. Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of $\mathcal{L}^{ss}$. Here A is irreducible and $\mathcal{L}^{ss}$ is algebraic, hence Diophantine. □

Proof of Theorem 1.1.13. We will fist show that the Lyapunov exponent at every point is the same in the dim $E_{k+1}^s = 1$ case. Take $p, q \in \mathrm{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f.

Take

- $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$.
- Segments $J \subset \mathcal{F}_{k+1}^s(p)$ and $J_n \subset \mathcal{F}_{k+1}^s(x_n)$ such that $J_n = \mathrm{Hol}^{ss}(J)$ ($x_n = \mathrm{Hol}^{ss}(p)$). Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{L}_{k+1}^s) = \mathcal{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \quad \Longrightarrow \quad \exists C_0 > 0, |J_n| \geqslant C_0 |J|.$$

Now we choose m_n, k_n such that

- x_n and q are very close in first k_n-steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathcal{F}_1^{ss}(p)$.

Then

$$|f^{m_n}(J_n)| \gtrsim \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n \lambda_{k+1}^s(q)]|J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geqslant \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \to \infty.$$

However, $J_n = \mathrm{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \mathrm{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathcal{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \mathrm{Hol}^{ss}(p)$, this contradicts to $\mathcal{F}^{ss}$ is C^1-smooth in $\mathcal{F}^{ss} \oplus \mathcal{F}_{k+1}^s(p)$.

For the case of dim $E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain

- For every periodic points p, q, we have $\min \lambda_{k+1}^s(p) = \min \lambda_{k+1}^s(q)$.
- Considering the growth of area of local disks, we have

$$\mathrm{Jac}(Df, E_{k+1}^s(p)) = \mathrm{Jac}(Df, E_{k+1}^s(q)), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda_{k+1}^s(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda_{k+1}^s(A)$. □

§1.2 Global Rigidity (Apr 26)

In the last lecture, we have shown a local rigidity result. That is, we only consider diffeomorphisms f that is C^1-close to A. Today we will consider the global rigidity, i.e., the relation between f and $f_* \in \mathrm{GL}(d, \mathbb{Z})$.

Question 1.2.1. What happens if f is not close to $A = f_*$?

Theorem 1.2.2 (Gogolev-Farell)

For $d \geqslant 10$, let $A \in \mathrm{GL}(d, \mathbb{Z})$ be a hyperbolic matrix. Then

$$\mathscr{A}_A^{1+}(\mathbb{T}^d) := \left\{ f \in \mathrm{Diff}^{1+}(\mathbb{T}^d) \;:\; f \text{ is Anosov, } f_* = A \right\}$$

has infinitely many connected components.

Theorem 1.2.3 (Full leaf conjugacy, Gogolev-Shi, arXiv: 2207.00704)

Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with absolutely partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$:

$$\|Df_{E^{ss}}\| < \mu < m(Df|_{E^{ws}}) < \|Df|_{E^{ws}}\| < 1 < m(Df|_{E^u}).$$

If $E^{ss} \oplus E^u$ is integrable, then

1. $A = f_* \in \mathrm{GL}(d, \mathbb{Z})$ is partially hyperbolic:

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u, \quad \dim L^\sigma = \dim E^\sigma, \;\; \sigma = ss, ws, u.$$

2. f is dynamically coherent and fully conjugate to A:

$$h(\mathscr{F}^\sigma) = \mathscr{L}^\sigma, \quad \sigma = ss, ws, u.$$

Here $h \circ f = A \circ h$.

Question 1.2.4. Let $f = \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$.

- Is $f_* \in \mathrm{GL}(d, \mathbb{Z})$ partially hyperbolic?
- Is f dynamically coherent or not? If yes, does f leaf conjugate to A.

Lemma 1.2.5

Let $\mathscr{F}$ be a C^0-foliation on $\mathbb{T}^d$ with C^1-leaves. If there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to $\mathrm{id}_{\mathbb{T}^d}$ such that $h(\mathscr{F}) = \mathscr{L}$ is a linear foliation, then $\mathscr{F}$ is quasi-isometric:

$$d_{\widetilde{\mathscr{F}}}(x, y) \leqslant a \cdot d(x, y) + b, \quad \forall x \in \mathbb{R}^d, y \in \widetilde{\mathscr{F}}(x).$$

Here $a, b > 0$ and $\widetilde{\mathscr{F}}$ is the lift of $\mathscr{F}$ in $\mathbb{R}^d$.

Proof of Theorem 1.2.3. Since $h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is sub-foliated by minimal linear foliation $h(\mathscr{F}^u) = \mathscr{L}^u$ is linear. We have $\mathscr{L}^{ss} := h(\mathscr{F}^{ss}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear.

Brin's argument shows that $E^{ws} \oplus E^u$ integrate to $\mathscr{F}^{cu}$ and $h(\mathscr{F}^{cu})$ is linear and minimal. Then $\mathscr{F}^{ws}$ integrate to $\mathscr{F}^{ws}$ and $\mathscr{L}^{ws} := h(\mathscr{F}^{ws})$ is A-invariant and linear.

Note that $\mathscr{L}^{ws}$ and $\mathscr{L}^{ss}$ are transverse in $\mathscr{L}^s$, then A admits an invariant splitting $T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u$. We need to show this is a dominated splitting. This follows from the above lemma and the fact that h is homotopic to $\mathrm{id}_{\mathbb{T}^d}$. □

Theorem 1.2.6 (Global rigidity, Gogolev-Shi, arXiv: 2207.00704)

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be Anosov and irreducible. Assume that f is absolutely partially hyperbolic $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$ and center bunching. If $E^{ss} \oplus E^u$ is integrable, then

1. f has a finest dominated splitting on E^{ws} with the same dimensions for $A|_{L^{ws}}$:

$$E^{ws} = E_1^{ws} \oplus \cdots \oplus E_k^{ws}, \quad \dim E_i^{ws} = \dim L_i^{ws}.$$

2. f is spectrally rigid along every E_i^{ws}:

$$\lambda(E_i^{ws}, f) \equiv \lambda(L_i^{ws}, A), \quad \forall i = 1, \cdots, k.$$

Remark 1.2.7 • Here f need NOT to be C^1-close to $A = f_*$.

- To get dominated splitting, we usually need some C^1-robust property like: robustly transitive, far from homoclinic bifurcations.
- If $A = f_*$ satisfies the generic assumption in the last lecture, then the conjugacy h is C^{1+}-smooth along $\mathscr{F}^{ws}$.
- The center bunching condition

$$\|Df|_{E^{ws}(x)}\| < m(Df|_{E^{ws}(x)}) \cdot m(Df|_{E^u(x)})$$

is a technical condition, which guarantees C^{1+}-smoothness of $\mathscr{F}^{su}$.

Corollary 1.2.8

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be codimension one with real simple spectrum. For every Anosov $f \in \mathrm{Diff}^2_m(\mathbb{T}^d)$ with $f_* = A$ and

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u, \quad \dim E^{ss} = 1,$$

if

- $E^{ss} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then f is C^{1+}-conjugate to A.

Main idea for showing Theorem 1.2.6. Play the game similar to the last lecture. We will use the Diophantine approximation of $\mathscr{F}^{ss}$ to show the rigidity of smallest exponent in E^{ws}:

$$\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we will show the dimension of $\lambda^{ws}_{\min}$ for each periodic point is constant. Next, we define the Pesin stable foliation $\mathscr{F}^{ws}_{\min}$ and show it is $\mathscr{F}^{su}$-holonomy invariant, that is $\mathrm{Hol}^{su} : \mathscr{F}^{ws}(p) \to \mathscr{F}^{ws}(q)$ preserves $\mathscr{F}^{ws}_{\min}$, for every $p, q \in \mathrm{Per}(f)$. Finally, we show a uniform spectral exponents gap and extract out $\mathscr{F}^{ws}_{\min}$.

Lemma 1.2.9

Let $\mathrm{Hol}^{su}_{x,y} : \mathscr{F}(x) \to \mathscr{F}(y)$ be the holonomy map of $\mathscr{F}^{su}$ with $\mathrm{Hol}^{su}_{x,y}(x) = y$ for every $x \in \mathbb{T}^d$ and $y \in \mathscr{F}^{su}(x)$. Then

$$\mathrm{Hol}^{su}_{x,y}(K) = h^{-1} \circ T_{h(x),h(y)} \circ h(K).$$

Here $T_{h(x),h(y)} : \mathbb{T}^d \to \mathbb{T}^d$ is the linear translation send $h(x)$ to $h(y)$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $K \subset \mathscr{F}^{ws}(x)$ with $\mathrm{diam}(K) > \varepsilon$, then

$$\mathrm{diam}(\mathrm{Hol}^{su}_{x,y}(K)) > \delta, \quad \forall y \in \mathscr{F}^{su}(x).$$

Remark 1.2.10 The same holds for $\mathrm{Hol}^{ss}_{x,y} : \mathscr{F}^{ws}(x) \to \mathscr{F}^{ws}(y)$ where $y \in \mathscr{F}^{ss}(x)$.

Proof. It follows immediately from f is fully conjugate to A. □

Proof of Theorem 1.2.6. We fist show that

Claim 1.2.11. $\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f)$.

Proof. Assume that $\lambda^{ws}_{\min}(p) < \lambda^{ws}_{\min}(q)$. Take $x_n \in \mathscr{F}^{ss}(p)$ such that $d^{ss}(x_n, p) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$. Take disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, the Pesin stable manifold associated to $\lambda^{ws}_{\min}(p)$. Let $D_n = \mathrm{Hol}^{ss}(D) \subset \mathscr{F}^{ws}(x_n)$, then $\mathrm{diam}(D_n) \gg \mathrm{diam}(D)$. Applying a similar (k_n, m_n)-argument, we get a contradiction since $\mathscr{F}^{ss}$ is C^1-smooth in $\mathscr{F}^{ws}(p)$. □

Now we have $\lambda^{ws}_{\min} := \lambda^{ws}_{\min}(p)$ for every $p \in \mathrm{Per}(f)$. We define the Pesin stable foliation associated to $\lambda^{ws}_{\min}$ for each periodic point.

Claim 1.2.12. $\mathscr{F}^{ws}_{\min}$ is Hol^{su}-invariant.

Proof. Let $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)} := h(\mathscr{F}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)})$, it suffices to show

$$T_{h(p),h(x)}(\mathscr{L}^{ws}_{\min}(p)) \subset \mathscr{L}^{ws}_{\min}(x)$$

for every $p, q \in \mathrm{Per}(f)$ and $x \in \mathscr{F}^{ws}(q)$. Otherwise, take a disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, then $T_{h(p),h(x)}(h(D))$ is transverse to $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}_{\mathrm{loc}}(q)}$ at $h(x)$. Take $x_n \in \mathscr{F}^{ss}$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, x) \ll K_n^{-\alpha}$, then

$$D_n := \mathrm{Hol}^{ss}_{p,x_n}(D) \to h^{-1} \circ T_{h(p),h(x)} \circ h(D).$$

It follows that $\mathrm{Hol}^{u}_{\mathrm{loc}}(D)$ is "uniformly transverse" (the angle will not tend to zero) to $\mathscr{L}^{ws}_{\min}$ in $\mathscr{F}^{ws}_{\mathrm{loc}}(q)$, where $\mathrm{Hol}^{u}_{\mathrm{loc}}(D) : \mathscr{F}^{ws}(x_n) \to \mathscr{F}^{ws}(q)$ is C^{1+}-smooth. Since the transverse direction has a weaker contracting rate, we play the (k_n, m_n)-game and get a contradiction. □

Let $\mathscr{L}^{ws}_{\min} := h(\mathscr{L}^{ws}_{\min})$, then the density of $\mathrm{Per}(f)$ and minimality of $\mathscr{F}^{ws}$ imply $T_{x,y}(\mathscr{L}^{ws}_{\min}(x)) \subset \mathscr{L}^{ws}_{\min}(y)$. By the translation invariance and the A-invariance, we have

- $\mathscr{L}^{ws}_{\min}$ is a linear foliation on $\mathbb{T}^d$, and
- $L^{ws}_{\min} := T\mathscr{L}^{ws}_{\min}$ associate to an eigenspace of A.

Also by an estimate of the growth, we get $\lambda(A, L^{ws}_{\min}) \equiv \lambda^{ws}_{\min}$.

Finally, we establish the induction step. Following the idea of [Bonatti-Díaz-Pujals, 2003], consider the quotient cocycle $D\tilde{f} : E^{ws}/E^{ws}_{\min} \to E^{ws}/E^{ws}_{\min}$ which is Hölder continuous over f. Again by a (k_n, m_n)-game, we can show that λ^{ws}_2 is uniformly larger than $\lambda^{ws}_{\min}$. Then the splitting $T\mathbb{T}^d = (E^{ss} \oplus E^{ws}_{\min}) \oplus F \oplus E^u$ is an absolutely partially hyperbolic splitting. The joint integrability follows from $h(\mathscr{F}^{ss} \oplus \mathscr{F}^{ws}_{\min})$ is linear. □

§1.3 Rigidity on $\mathbb{T}^4$ (Apr 27)

Let us recall some results shown in last two lectures. We remark that the key point is that

$$E^{ss} \oplus E^u \text{ is integrable } \implies h(\mathscr{F}^{ss} = \mathscr{L}^{ss}) \text{ is linear.}$$

Question 1.3.1. Let f be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_j^u \oplus \cdots \oplus E_m^u.$$

What happens if $E_k^s \oplus E_j^u$ is jointly integrable? Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$?

Theorem 1.3.2 (Gogolev-Kalinin-Sadovskya)
Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$ implies $h(\mathscr{F}_k^s) = \mathscr{L}_k^s$ and $h(\mathscr{F}_j^u) = \mathscr{L}_j^u$ hence $E_k^s \oplus E_j^u$ is jointly integrable.

The work of Avila-Viana.

Theorem 1.3.3 (Avila-Viana, 2010)
For every symplectic f which is C^∞-close to A with splitting

$$T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u,$$

then

- either f is accessible and non-uniformly hyperbolic;
- or $E^s \oplus E^u$ is integrable and $\exists h \in \mathrm{Diff}_m^\infty(\mathbb{T}^4)$ such that

$$h \circ f = A \circ h.$$

In particular, f is Bernoulli.

Main theorem.

Theorem 1.3.4 (Gogolev-Shi, arXiv: 2207.00704)
Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an irreducible Anosov automorphism with dominated splitting

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}, \quad \text{with} \quad \dim L^{ws} = \dim L^{wu} = 1.$$

For $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is spectral rigid along E^{ws} and E^{wu}.

Corollary 1.3.5
Let $A \in \mathrm{Sp}(4,\mathbb{Z})$ be hyperbolic and irreducible with dominated splitting

$$T\mathbb{T}^4 = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}.$$

For symplectic $f \in \mathrm{Diff}^2_\omega(\mathbb{T}^4)$ be C^1-close to A with

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:
- $E^{ss} \oplus E^{uu}$ is integrable;
- f is C^{1+}-smoothly conjugate to A.

Proof of corollary. If $E^{ss} \oplus E^{uu}$ is integrable, then we have spectral rigidity in $E^{ws} \oplus E^{wu}$, h is smooth along $E^{ws} \oplus E^{wu}$ and $h(\mathscr{L}^{ss}) = \mathscr{L}^{ss}, h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$. Since h is smooth along $\mathscr{F}^{ws}$ and $\mathscr{F}^{wu}$, the holonomy map $\mathrm{Hol}^{su}_{\mathscr{F}}$ is C^{1+}. Then we use the symplectic structure that $E^c = E^{ws} \oplus E^{wu}$ is perpendicular to E^{su} (with respect to ω). Hence $\mathscr{F}^{ws} \oplus \mathscr{F}^{wu}$ is C^{1+}. Then we can show that h is absolutely continuous in $\mathscr{F}^{su}$ and hence h is C^{1+}. □

Proof of main theorem. Main problem is whether $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$ is the linear one? Or equivalently, whether we have $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ or $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$? This is nontrivial.

Lemma 1.3.6
If one of $E^{ss} \oplus E^{u}$ and $E^{s} \oplus E^{uu}$ is integrable, then f is spectral rigid in $E^{ws} \oplus E^{wu}$.

Proof. If $E^{ss} \oplus E^{u}$ is integrable, then $h(\mathscr{F}^{ss} \oplus \mathscr{F}^{u})$ is linear and hence $h(\mathscr{F}^{ss}) = h(\mathscr{F}^{ss} \oplus \mathscr{F}^{u}) \cap \mathscr{L}^{s} = \mathscr{L}^{ss}$ is linear. Then both $h(\mathscr{F}^{su})$ and $h(\mathscr{F}^{uu})$ are linear. Then we obtain a spectral rigidity by Theorem 1.1.13. □

The solvable action. Let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$ and $L^c(0) = L^{ws}(0) \oplus L^{wu}(0) \subset \mathbb{R}^d$. Define the linear action

$$\alpha_0 : \Gamma \times L^c(0) \to L^c(0), \quad \alpha_0(k,n)(x) = L^{su}(A^k(x) + n) \cap L^c(0).$$

If we write $n = n^s + n^c + n^u \in L^s \oplus L^c \oplus L^u$, then $\alpha_0(k,n)(x) = A^k x + n^c$.

For $F : \mathbb{R}^4 \to \mathbb{R}^4$ be the lift of f and $F(0) = 0$, then
- $F^k(x+n) = F^k(x) + A^k n$, $\forall x \in \mathbb{R}^d$ and $\forall n \in \mathbb{Z}^d$.
- $F(\widetilde{\mathscr{F}}^c(0)) = \widetilde{\mathscr{F}}^c(0)$.

Then $\Gamma \curvearrowright \widetilde{\mathscr{F}}^c(0)$ given by

$$\alpha(k,n)(x) = \widetilde{\mathscr{F}}^{su}(F^k(x) + n) \cap \widetilde{\mathscr{F}}^c(0), \quad \forall (k,n) \in \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d, x \in \widetilde{\mathscr{F}}^c(0).$$

Lemma 1.3.7 This is a group action by the solvable group Γ.

Main idea. If both $E^{ss} \oplus E^{u}$ and $E^{s} \oplus E^{uu}$ are not integrable, then we can find a free subgroup by a pingpong argument, which contradicts Γ is solvable.

Lemma 1.3.8
If $\alpha(0,n)(\widetilde{\mathscr{F}}^{ws}(0)) \subset \widetilde{\mathscr{F}}^{ws}(\alpha(0,n)0)$ for all $n \in \mathbb{Z}^d$, then both $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$ are linear. The same holds if $\alpha(0,n)(\widetilde{\mathscr{F}}^{wu}(0)) \subset \widetilde{\mathscr{F}}^{wu}(\alpha(0,n)0)$ for all $n \in \mathbb{Z}^d$.

Proof. Note that $\bigcup_{n\in\mathbb{Z}^d} \widetilde{\mathscr{F}}^{ws}(n)$ is dense in $\mathbb{R}^d$ and hence $E^{ss} \oplus E^{ws} \oplus E^{uu}$ jointly integrates to $\mathscr{F}^{su} \oplus \mathscr{F}^{ws}$. Then we deduce the linearity. □

Proof of Theorem 1.3.4. Assume for a contradiction that there exists $n_1, n_2 \in \mathbb{Z}^d$ such that
- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(\alpha(0,n_1)(0))$;
- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(\alpha(0,n_1)(0))$.

Lemma 1.3.9

There exists $m_1, m_2 \in \mathbb{Z}^d$ such that
- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Lemma 1.3.10

For l large enough, $n = A^l m_1 - A^{-l} m_2 \in \mathbb{Z}^d$ satisfies
- $\alpha(0,n)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,n)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Now we consider $F : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0)$ and

$$G : \alpha(0,n) \circ \alpha(1,0) \circ \alpha(0,-n) : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0).$$

Then F is saddle-like dynamics at $\widetilde{\mathscr{F}}^{ws}(0) \cup \widetilde{\mathscr{F}}^{ws}(0)$ near 0. The map G is also saddle-like near $\alpha(0,n)0$. By a pingpong-argument, we can show that $\left\{F^k, G^k\right\}$ generates a free group for a sufficiently large k. This contradicts that Γ is solvable. □

§1.4 Anosov Maps (Apr 28)

Cone-field. Let f be an Anosov diffeomorphism with splitting $TM = E^s \oplus E^u$. Then there are cone-fields C^s, C^u containing E^s, E^u such that

$$Df(\overline{C^u(x)}) \subset C^u(fx), \quad Df^{-1}(\overline{C^s(x)}) \subset C^s(f^{-1}x).$$

Then $E^s(x)$ is determined by $\mathrm{Orb}^+(x)$ as

$$E^s(x) = \bigcap_{n\geqslant 0} Df^{-n}(C^s(f^n x)),$$

and $E^u(x)$ is determined by $\mathrm{Orb}^-(x)$ as

$$E^u(x) = \bigcap_{n\geqslant 0} Df^{n}(C^u(f^{-n} x)).$$

Theorem 1.4.1 (Anosov, 1967)

The Arnold's cat map $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is structurally stable. That is, for every $f : \mathbb{T}^2 \to \mathbb{T}^2$ C^1-close to A, there exists a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ close to $\mathrm{id}_{\mathbb{T}^2}$ such that $h \circ f = A \circ h$.

Remark 1.4.2 Every Anosov diffeomorphism is structurally stable.

Remark 1.4.3 If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is continuous with $f_* = A$, then there exists a surjective $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h \circ f = A \circ h$.

By a cone-argument, we can show that a small perturbation of an Anosov diffeomorphism is also Anosov. In general, we have Franks-Manning's global classification of Anosov diffeomorphisms.

Theorem 1.4.4 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Anosov maps.

Definition 1.4.5. A local diffeomorphsim $f : M \to M$ is **Anosov**, if there exists a continuous, Df invariant subbundle $E^s \subset TM$ such that

- $\|Df(v^s)\| < 1$ for every $v^s \in E^s$ with $\|v^s\| = 1$;
- Df induces an expanding map $\widetilde{Df} : TM/E^s \to TM/E^s$, that is

$$\|\widetilde{Df}(\tilde{v}^u)\| > 1, \quad \forall \tilde{v}^u \in TM/E^s, \|\tilde{v}^u\| = 1.$$

In this lecture, the Anosov map always refers to the non-invertible Anosov map.

Remark 1.4.6 Since $\mathrm{Orb}^-(x)$ is not unique, there may be no $E^u(x)$.

Theorem 1.4.7 (Mañe-Pugh, 1974)
$f : M \to M$ is an Anosov map iff $\tilde{f} : \widetilde{M} \to \widetilde{M}$ is an Anosov diffeomorphism.

Definition 1.4.8 (Przytycki, 1976)**.** A local diffeomorphsim $f : M \to M$ is an **Anosov map**, if in the orbit space

$$\tilde{x} = (x_i)_{i \in \mathbb{Z}} \in M_f := \{ (x_i) : f(x_i) = x_{i+1}, \ \forall x \in \mathbb{Z} \},$$

there exists a splitting

$$T_{x_i}M = E^s(x_i) \oplus E^u(x_i), \quad \forall i \in \mathbb{Z}$$

which is Df-invariant

$$D_{x_i}f(E^s(x_i)) = E^s(x_{i+1}), \quad D_{x_i}f(E^u(x_i)) = E^u(x_{i+1}), \quad \forall i \in \mathbb{Z},$$

and for every $v^{s/u} \in E^{s/u}(x_i)$ with $\|v^{s/u}\| = 1$:

$$\|D_{x_i}f(v^s)\| < 1, \quad \|D_{x_i}f(v^u)\| > 1.$$

Example 1.4.9

For every $n \geqslant 3$, the map

$$A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

is an Anosov map.

Remark 1.4.10 Every Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^d$ has a hyperbolic linearization $f_* \in M(\mathbb{Z}, d)$.

Unlike the Anosov diffeomorphisms, the Anosov map is not structurally stable.

Theorem 1.4.11 (Mañe-Pugh, 1974; Przytycki, 1976)

Let $A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ with $n \geqslant 3$. Then A is **NOT** structurally stable. That is, for every $\varepsilon > 0$, there exists an Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ with $d_{C^\infty}(f, A) < \varepsilon$ such that there is no $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to $\mathrm{id}_{\mathbb{T}^2}$ with $h \circ f = A \circ h$.

Remark 1.4.12 Every non-invertible Anosov map is not structurally stable unless it is expanding.

Proof. Take $p \neq 0$ such that $A(p) = 0$. Let U, U' be disjoint neighborhoods of 0 and p. Let (x_i) be an A-orbit satisfying

$$x_0 = p, \quad x_i = 0, \forall i > 0, \quad \text{and} \quad x_i \notin U', \forall i < 0.$$

Take a C^∞ ε-perturbation of f on U': push p along the stable leaf.

Then there exists an f-orbit $\{ y_i \}$ satisfying

$$y_0 = p, \quad \text{and} \quad y_i = x_i, \forall i < 0.$$

Then $y_i \in \mathcal{F}^s_\varepsilon(0)$ for every $i > 0$, where $\mathcal{F}^s$ is the stable leaf of A. Then the A-orbit x_i shadows the f-orbit y_i and hence the conjugacy $h(y_i) = 0$. But there is no homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h(y_i) = 0$ for every $i > 0$. $\square$

Theorem 1.4.13 (Przytycki, 1976)

An Anosov map $f : M \to M$ is structurally in the orbit space (M_f, σ_f), where $\sigma_f : (x_i) \mapsto (x_{i+1})$. That is, for every $g : M \to M$ C^1-close to f, there exists a homeomorphism $\overline{h} : M_g \to M_f$ such that $\overline{h} \circ \sigma_g = \sigma_f \circ \overline{h}$.

Question 1.4.14.

- Assume that $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov map with $f_* = A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}, n \geqslant 3$. When f topologically conjugate to A?
- Assume that $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ are Anosov maps with $f_* = g_*$. When f topologically conjugates to g?

Example 1.4.15 (Przytycki, 1976)
Let

$$A = \begin{bmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & n \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3, \quad n \geqslant 2$$

be a **special Anosov map** (E^u does not depend on the choice of the inverse orbit). When n is big enough, for every $x \in \mathbb{T}^3$, there exists an f C^1-close to A such that

$$\left\{ D\pi(E^u(x_0)) \,:\, \tilde{x} = (x_i) \in M_f \text{ with } x_0 = x \right\} \subset \mathscr{G}^2(T_x\mathbb{T}^3)$$

contains a curve in the Grassmannian $\mathscr{G}^2(T_x\mathbb{T}^3)$.

Theorem 1.4.16 (Micena-Tahzibi, 2016)
Let $f : M \to M$ be a transitive Anosov map, then
- either f has an integrable E^u (f is special),
- or there exists a residue set $\mathscr{R} \subset M$ such that x has infinitely many unstable directions for every $x \in \mathscr{R}$.

Main theorems.

Theorem 1.4.17 (An-Gan-Gu-Shi, arXiv: 2205.13144)
Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^{1+}-Anosov map, then the following are equivalent:
- f topologically conjugate to $f_* = A$;
- f is spectral rigid in stable bundle:

$$\lambda^s(p, f) \equiv \log \|A|_{L^s}\|, \quad \forall p \in \mathrm{Per}(f).$$

Remark 1.4.18 The same holds if $f : \mathbb{T}^d \to \mathbb{T}^d$ is an irreducible Anosov map with $\dim E^s = 1$.

Theorem 1.4.19 (An-Gan-Gu-Shi, arXiv: 2205.13144)
Let $A \in M(d, \mathbb{Z})$ be Anosov, irreducible and $|\det(A)| > 1$. If A has real simple spectrum in the stable bundle:

$$T\mathbb{T}^d = L_1^s \oplus L_2^s \oplus \cdots \oplus L_k^s \oplus L^u, \quad \dim L_i^s = 1,$$

then for every f C^1-close to A, the following are equivalent:
- f topologically conjugates to A,
- f is spectral rigidity in stable bundle, i.e. f admits dominated splitting

$$T\mathbb{T}^d = E_1^s \oplus E_2^s \oplus \cdots \oplus E_k^s \oplus E^u$$

  and

$$\lambda(E_i^s, f) \equiv \log \|A|_{L_i^s}\|, \quad \forall i = 1, \cdots, k.$$

Main philosophy. For every $y, z \in \mathbb{T}^d$, they are in the same "strongest stable manifold" if

$$f^n(y) = f^n(z), \quad \text{for some } n > 0.$$

Then f topologically conjugates to $A \iff E^u$ does not depend on $\mathrm{Orb}^-(x)$. Hence we have $E^u(x) = E^u(y)$ if $f^n(y) = f^n(z)$. This is equivalent to E^u is "jointly integrable" with

$$\mathscr{F}^{ss}(x) := \{ z : f^n(x) = f^n(z), \text{ for some } n > 0 \}.$$

This leads to a spectral rigidity in E^s, which is the weak stable direction in this view.

Topological classification.

Theorem 1.4.20 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be homotopic C^{1+}-Anosov maps, then the following are equivalent:

- f topologically conjugates to g;
- for every $p \in \mathrm{Per}(f)$ and corresponding $p' \in \mathrm{Per}(g)$,

$$\lambda^s(p, f) \equiv \lambda^s(p', g).$$

Remark 1.4.21 Since there is no a priori conjugacy, we should explain the meaning of "corresponding point". This can be given by a (stable) leaf conjugacy, which is defined a priori. Note that each periodic stable leaf admits a unique periodic point since f is uniformly contracting on the stable leaf. The corresponding point can be defined in this way.

Corollary 1.4.22 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. Then h is C^r-smooth along the stable foliation.

Theorem 1.4.23 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. If

$$\mathrm{Jac}(f^{\pi(p)}(p)) = \mathrm{Jac}(g^{\pi(p)}(h(p))), \quad \forall p \in \mathrm{Per}(f),$$

then h is C^{r_*}-smooth. Here $r_* = \begin{cases} r - 1 + \mathrm{Lip}, & r \in \mathbb{N} \\ r, & r \notin \mathbb{N} \end{cases}$.

2 Methods for Studying Abelian Actions and Centralizers (Danijela Damjanović / Disheng Xu)

§2.1 Definitions and examples (Danijela, May 1)

Plan for this minicourse

1. Many examples, invariant structures, main results.
2. Some methods in simple cases.
3. More methods and more about centralizer rigidity
4. More methods.

Setting

- M a closed C^∞-manifold.
- $f : M \to M$ a C^∞-diffeomorphism.
- $\mathscr{Z}(f) := \{g \in \mathrm{Diff}^\infty(M) : gf = gf\}$, the centralizer of f in $\mathrm{Diff}^\infty(M)$.

It is obvious that $\mathscr{Z}(f) \geqslant \langle f \rangle \cong \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. **Smale's question**:

Is it true that typically in C^r-topology, $\langle f \rangle = \mathscr{Z}(f)$?

This is confirmed to be true in C^1-topology by Bonatti-Crovisier-Wilkinson.

We also interest in a typical situation that $\mathscr{Z}(f)$ is large. The main theme is a centralizer rigidity:

$$f \text{ has a complicated dynamics} + \mathscr{Z}(f) \text{ is large}$$
$$\implies f \text{ is } C^\infty\text{-conjugate to an algebraic system}$$

Algebraic systems.

- $M = G/\Gamma$ where G is a Lie group and Γ is a cocompact lattice in G.
- $L_g : x \mapsto g.x$ the left translation for $g \in G$.
- $A : G \to G$ an automorphism preserving Γ, it induces $A : G/\Gamma \to G/\Gamma$.
- Affine maps $L_g \circ A$.
- Another examples of "algebraic systems" are the translations on the symmetric space $L_g : K\backslash G/\Gamma \to K\backslash G/\Gamma$ where $K < G$ is a compact subgroup.

Definition 2.1.1. An action is **smoothly algebraic** if it is C^∞-conjugate to an algebraic model.

Complicated dynamics. f is partially hyperbolic with $TM = E^s \oplus E^c \oplus E^u$. Assume that f is **normally hyperbolic**: E^c is integrable to an invariant foliation $\mathscr{W}^c$ with C^1-leaves.

Definition 2.1.2. f is **accessible** if any $x, y \in M$ can be connected via a stable / unstable broken path.

Notation 2.1.3. For groups H_1, H_2, we denote $H_1 \doteq H_2$ if H_1 is virtually-H_2, that means H_1 contains a finite index subgroup isomorphic to H_2.

Example 2.1.4 (Examples with rigid centralizers)

1. A hyperbolic automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, then $\mathscr{Z}(f) \doteq \mathbb{Z}$.
2. Geodesic flows $\varphi_t : \mathrm{SL}(2,\mathbb{R})/\Gamma \to \mathrm{SL}(2,\mathbb{R})/\Gamma$, it corresponds to the diagonal flows $A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$ acts by left translations. Then φ_t is partially hyperbolic and $\mathscr{Z}(\varphi_t) \doteq \mathbb{R}$.

Example 2.1.5 (Examples with larger centralizers)

1. For $A : \mathbb{T}^2 \to \mathbb{T}^2$ a hyperbolic automorphism, let $f = \begin{bmatrix} A & \\ & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$, then any $\begin{bmatrix} A^k & \\ & A^l \end{bmatrix}$ commutes with f for $k, l \in \mathbb{Z}$. Hence $\mathscr{Z}(f) > \mathbb{Z}^2$.
2. Product of geodesic flows on $\mathrm{SL}(2,\mathbb{R})/\Gamma$. Then $\mathscr{Z}(\varphi_t) > \mathbb{R}^2$.

Note that in the first example, the elements of the form $A^k \times \mathrm{id}$ or $\mathrm{id} \times A^l$ are not ergodic. Which means there is a factor in the system. The same holds for the second example. We want to avoid these cases.

Definition 2.1.6 (Rank one factor). Let $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ be an action with $k + l \geqslant 2$. We say it has a C^s **rank-one factor** if we have

- A C^∞-manifold $\overline{M}$ and a C^s-submersion $\pi : M \to \overline{M}$.
- A surjective homomorphism $\sigma : \mathbb{R}^k \times \mathbb{Z}^l \to H$ where $H \doteq \mathbb{Z}$ or $\mathbb{R}$.
- A locally free C^s-action $H : \overline{M} \to \overline{M}$ such that $\pi(g.x) = \sigma(g).\pi(x)$.

Definition 2.1.7. A smooth action $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ is called **higher-rank** if $k + l \geqslant 2$ and there is no C^∞-rank-one factors.

Example 2.1.8 (Higher-rank actions)

1. $A : \mathbb{T}^3 \to \mathbb{T}^3$ a hyperbolic automorphism with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{R} \setminus \{-1, 1\}$. Then $\mathscr{Z}(A) \doteq \mathbb{Z}^2 = \langle A, B \rangle$ where B is also a hyperbolic automorphism. Let V_i be the eigenspace of A corresponding to λ_i, then B preserves each V_i. Hence $A^k B^l|_{V_i} = \lambda_i^k \mu_i^l$. Although there is not integers k, l such that $\lambda_i^k \mu_i^l = 1$, but there exists pairs of real numbers (s,t) such that $\lambda_i^s \mu_i^t$. These lines are very important.
   Specifically, let
   $$\chi_i(s,t) = s \log|\lambda_i| + t \log|\mu_i|.$$
   Then $L_i := \ker \chi_i$ is a line in the plane for any $i = 1, 2, 3$. An algebraic fact shows that the lines are irrational (hence there is no integers k, l such that $(k,l) \in L_i$).
2. The diagonal flow on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Gamma$ where Γ is an irreducible lattice. By Mautner's theorem, every line in the diagonal flow acts ergodically.
3. Weyl chamber flow. Let $M = \mathrm{SL}(3,\mathbb{R})/\Gamma$, we consider
   $$\mathbb{R}^2 \cong \left\{ \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} : t_1 + t_2 + t_3 = 0 \right\}$$
   acting on M. For an element, it acts on U_{12} by factor $e^{t_1 - t_2}$. We care about the lines on the plane that $t_i = t_j$ for $i \neq j$. All these lines acting normally hyperbolic and hence ergodic.

Exercise 2.1.9. For an $\mathbb{R}^2$ action on M, if every line in $\mathbb{R}^2$ is ergodic iff there is no rank-one factors.

§2.2 Statements of the results in rigidity theory (Danijela, May 2)

Proposition 2.2.1

Let A be an irreducible matrix in $\mathrm{SL}(d,\mathbb{Z})$, then $\mathscr{Z}(A) \doteq \mathbb{Z}^{m+n-1}$, where m is the number of real eigenvalues and m is the number of pairs of complex eigenvalues.

Now we back to to the first example in 2.1.8. The lines L_i divide the plane into 6 chambers. For an element not on the lines, it expands or contracts the space $V_i, i = 1,2,3$. Note that for elements in the same chamber, for each V_i, they expands or contracts this space simultaneously.

Definition 2.2.2. A $\mathbb{Z}^k$ action on M is **Anosov** if it contains an Anosov element. Furthermore, it is **totally Anosov** if all nontrivial elements are Anosov.

Definition 2.2.3. An $\mathbb{R}^k$ action on M is **Anosov** if it some $a \in \mathbb{R}^k$ acts normally hyperbolic to the $\mathbb{R}^k$-orbit foliation. It is **totally Anosov** if there is a dense set of normally hyperbolic elements.

Proposition 2.2.4

Let $\langle A, B\rangle$ be the pair given in the first example of 2.1.8. Then $\langle A, B\rangle$ is an exponentially mixing action: for every $\theta > 0$, there exists $\tau > 0$ such that for every θ-Hölder functions ξ, η such that

$$\left|\left\langle \xi \circ A^k B^l, \eta\right\rangle\right| \leqslant C_{\xi,\eta} e^{-\tau(|k|+|l|)}.$$

Theorem 2.2.5 (Gorodmk-Spatzier)

For any $\mathbb{Z}^k$-action on a nilmanifold N/Γ by automorphisms, if there is no rank-one factor, then it is exponentially mixing.

Remark 2.2.6 The last two examples in 2.1.8 are not exponentially mixing.

Example 2.2.7

Let $f : A \times R_\theta$ where $A : \mathbb{T}^3 \to \mathbb{T}^3$ is as before and R_θ is an irrational rotation. Then $\mathscr{Z}(f) \doteq \mathbb{Z}^2 \times \mathbb{T}$.

Definition 2.2.8. A **fibered partially hyperbolic system** is a partially hyperbolic $f : M \to M$ with compact leaves $\mathscr{W}^c_f$ and $M/\mathscr{W}^c_f$ is a topological manifold. It induces a map $\overline{f} : \overline{M} = M/\mathscr{W}^c_f \to \overline{M}$ satisfying $\pi \circ f = \overline{f} \circ \pi$.

Remark 2.2.9 Bohnet-Bonatti, Gogolev, Avila-Viana-Wilkinson have studied the fibered partially hyperbolic systems.

Proposition 2.2.10
Let A, B be as in the first example of 2.1.8. Let $f : (x, y) \mapsto (Ax, y + \varphi(x))$ be a fibered partially hyperbolic map, assume that $\mathscr{Z}(f)$ contains $(Bx, y + \psi(x))$. Then f is smoothly conjugate to an affine map.

Proof. We have the cocycle equation

$$\varphi - \varphi \circ B = \psi - \psi \circ A. \tag{2.2.1}$$

Now we consider the map $(Ax, y+\varphi(x))$, it smoothly conjugate to $(Ax, y+c)$ if $\varphi = H - H \circ A$ (assume that $\int \varphi = 0$ and conjugate via $(x, y + H(x))$). Some possible solutions for H are

$$D_A^+(\varphi) := \sum_{k=0}^{\infty} \varphi \circ A^k \quad \text{or} \quad D_A^-(\varphi) := -\sum_{k=-\infty}^{-1} \varphi \circ A^k.$$

Note that D_A^+ has derivatives along $\mathscr{W}_A^s$ and D_A^- has derivatives along $\mathscr{W}_A^u$. Then if we can show $D_A^+ = D_A^-$, we are done. Let $D_A(\varphi) = \sum_{k \in \mathbb{Z}} \varphi \circ A^k$, then by (2.2.1), we have $D_A(\varphi) = D_A(\varphi \circ B^l)$ for every $l \in \mathbb{Z}$. Then for every Hölder function ξ, by exponentially mixing

$$\lim_{l \to \pm\infty} \left\langle D_A(\varphi \circ B^l), \xi \right\rangle = \lim_{l \to \pm} \sum_{k \in \mathbb{Z}} \left\langle \varphi \circ A^k B^l, \eta \right\rangle \to 0.$$

Hence $D_A(\varphi) = 0$ and H is C^∞. $\square$

Exercise 2.2.11 ("Higher rank trick" works without exponentially mixing)**.** Let

$$a(x, y) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \quad b(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \end{bmatrix},$$

where α, β are Diophantine irrational numbers. Note that a, b are commuting diffeomorphisms on $\mathbb{T}^2$. Then every φ, ψ satisfy (2.2.1) over $\langle a, b \rangle$ are C^∞-coboundaries.

Samples of local, global and semi-local rigidity results.

Theorem 2.2.12 (Local rigidity, Katok-Spatzier)
For actions in Example 2.1.8, they are locally rigid. That is, for every C^1-perturbation of the action, the action is smoothly algebraic.

Remark 2.2.13 Local rigidity was extended to
- Partially hyperbolic version of the Example 2.1.8.1 on $\mathbb{T}^d$, by Damjanović-Katok.
- Partially hyperbolic version of the Example 2.1.8.3, by Damjanović-Katok, Vinhage-Wang.
- KAM method for partially hyperbolic affine actions, by Zhenqi Wang.

Theorem 2.2.14 (Global rigidity, Fisher-Kalinin-Spatzier, Hertz-Wang)
An Anosov $\mathbb{Z}^k$-action ($k \geqslant 2$) on a nilmanifold N/Γ is smoothly affine, providing it is homotopic to a higher rank action by an automorphism.

Remark 2.2.15 It implies that Example 2.1.8.1 is globally rigid.

Theorem 2.2.16 (Spatzier-Vinhage)
Example 2.1.8.3 is also globally rigid, (precise version will be stated later).

Theorem 2.2.17 (Semi-local, Damjanović-Wilkinson-Xu)
Let $f_0 = A \times R_\theta$ as in Example 2.2.7. Let f be a volume preserving C^1 smooth perturbation of f_0 and assume that f is ergodic. Then

$$\mathscr{Z}(f) = \begin{cases} \mathbb{Z}; \\ \mathbb{Z} \times \mathbb{T}; \\ \mathbb{Z}^2 \times \mathbb{T}, \text{ and } f \text{ is smoothly algebraic.} \end{cases}$$

§2.3 More methods in simple cases (May 2)

Another simple case. Let A, B be given in Example 2.1.8.1. Let $\varphi, \psi : \mathbb{T}^3 \to \mathbb{T}^3$ be C^∞ maps. Let

$$F(x,y) = (Ax, Ay + \varphi(x)), \quad G(x,y) = (Bx, By + \psi(x))$$

be commuting maps. Then we have the cocycle equation

$$A \circ \psi - \psi \circ A = B \circ \varphi - \varphi \circ B. \tag{2.3.1}$$

Let $\mathbb{R}^3 = V_1 \oplus V_2 \oplus V_3$ where each V_i is an eigenspace. Split the equation into each V_i and let φ_i, ψ_i be the components of φ, ψ respectively. We have (for simplicity, just consider $i = 1$)

$$\lambda_1 \psi_1 - \psi_1 \circ A = \mu_1 \varphi_1 - \varphi_1 \circ B.$$

We want to find $H(x)$ such that $(x, y + H(x))$ conjugates (Ax, Ay) to $(Ax, Ay + \varphi(x))$. So we need to solve the equation

$$\varphi = A \circ H - H \circ A, \quad \text{i.e.} \quad \varphi_1 = \lambda_1 H_1 - H_1 \circ A.$$

Then we can take

$$D^+_{A,1} = \sum_{k=0}^{\infty} \lambda_1^{-(k+1)} \varphi_1 \circ A^k \text{ if } |\lambda_1| > 1, \quad \text{or} \quad D^-_{A,1} = \sum_{k=-\infty}^{-1} \lambda_1^{-(k+1)} \varphi_1 \circ A^k \text{ if } |\lambda_1| < 1.$$

Note that $D^\pm_{A,1}$ converge uniformly and hence are C^0. We can define $D^\pm_{B,1}$ similarly and we have $D^\pm_{A,1} = D^\pm_{B,1}$ when they are convergent. The problem is how to show that H is smooth.

Now we turn to considering general commuting toral diffeomorphisms. We will be back to this example later.

Commuting toral diffeomorphisms. Assume that $\langle f, g\rangle$ homotopic to $\langle A, B\rangle$ and f is Anosov. By Franks-Manning theorem, there exists a Hölder homeomorphism h such that $f = h \circ A \circ h^{-1}$. We apply Oseledets' decomposition for abelian actions on an ergodic measure preserving space. Then there exists an (a priori just measurable) invariant splitting

$$TM = \bigoplus_i E^i$$

and linear functions $\chi_i : \mathbb{Z}^2 \to \mathbb{R}$ such that $\chi_i(a)$ is the Lyapunov exponent of a in E^i. For each linear function χ, let

$$E^{[\chi]} := \bigoplus_{\chi_i = c\chi,\ c>0} E^i,$$

which is called the **coarse Lyapunov distribution**. Assume that $a \in \mathbb{Z}^2$ is Anosov, then the unstable / unstable distributions of a are Hölder continuity. That is, both

$$\bigoplus_{\chi(a)<0} E^{[\chi]}, \qquad \bigoplus_{\chi(a)>0} E^{[\chi]}$$

are Hölder. If we have sufficiently many Anosov elements (one in each Weyl chamber), by taking intersection, we can obtain the Hölder continuity of the coarse Lyapunov distribution. [This is not the core of this minicourse. Another minicourse given by Disheng focuses on this topic. The notes can be found here.]

Proposition 2.3.1

One Anosov in each chamber $\implies$ $E^{[\chi]}$ are Hölder and integrate to Hölder foliations.

Remark 2.3.2 The Hölder continuity also implies that the distribution is independent with the choice of the measure.

Back to the example. The following figure illustrates a Weyl chamber picture, where $\pm$ denotes the sign of χ_i, characterizing whether the element contracts or expands V_i.

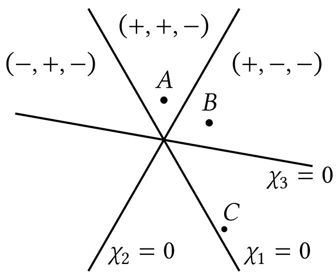

Figure 2.1: Weyl chamber picture

Then for the element A, we have the expression $H_1 = \sum_{k=0}^{\infty} \lambda_1^{-(k+1)} \varphi_1 \circ A^k$ since $\log \lambda_1 = \chi_1(A) > 0$. Note that A contracts V_3, we obtain H_1 is C^∞ along V_3. If we use the expression for B as $H_1 = \sum_{k=0}^{\infty} \mu_1^{-(k+1)} \psi_1 \circ B^k$, we can obtain that H_1 is C^∞ along $V_2 \oplus V_3$. But we can never get the regularity along V_1 by this method, since for every element C we can only get the regularity long V_i where $\chi_i(C)$ and $\chi_1(C)$ have different signs.

Here we need another trick by the exponentially mixing. We will take a C very close to $\chi_1 = 0$, and assume that $(x, y) \mapsto (Cx, Cy + \zeta(x)) \in \langle F, G \rangle$. Then by exponentially mixing

$$D_{C,1}(\zeta_1) := \sum_{k \in \mathbb{Z}} e^{-(k+1)\chi_1(C)} \zeta_1 \circ C^k$$

converges as a distribution. Furthermore, since $e^{\chi_1(C)}\zeta_1 - \zeta_1 \circ C = \mu_1 \psi_1 - \psi_1 \circ B$, we obtain

$$\mu_1^l D_{C,1}(\zeta_1) = D_{C,1}(\zeta_1 \circ B^l) = \sum_{k \in \mathbb{Z}} e^{-(k+1)\chi_1(C)} \zeta_1 \circ C^k B^l.$$

If $\chi_1(C)$ is smaller enough, the exponentially mixing will show that the distribution tends to 0 as $l \to \pm\infty$. Taking an appropriate direction such that $\mu_1^l \to \infty$, we obtain $D_{C,1}(\zeta_1) = 0$.

This trick tells us $D_{C,1}(\zeta_1)^- = D_{C,1}(\zeta_1)^+$, then H has two expressions. So we can choose a desired direction ($k \to +\infty$ or $k \to -\infty$) such that C^k contracts V_1. This gives the regularity of H along V_1.

How this derives the global rigidity Theorem 2.2.14. Given $\langle f, g \rangle$ commuting on $\mathbb{T}^d$ that h-conjugates to $\langle A, B \rangle$. Here h is a priori just Hölder continuous. The aim is to show that h is indeed C^∞.

1. One Anosov $\implies$ one Anosov element in each Weyl chamber. This is a highly non-trivial part, which is due to [Hertz-Wang].
2. One Anosov in each chamber $\implies$ Anosov elements are somehow dense (projectively dense). Moreover, the Weyl chamber picture is the same for $\langle f, g \rangle$ and $\langle A, B \rangle$. [Fisher-Kalinin-Spatzier]
3. Use the exponentially mixing argument to upgrade the regularity.

Idea of showing Theorem 2.2.17. For the rigidity part, we consider the factor $\overline{f} : \mathbb{T}^3 \to \mathbb{T}^3$. Then $\mathscr{Z}(\overline{f}) \doteq \mathbb{Z}^2$ and we can apply a similar argument as before. For the other two cases, we apply a dichotomy of the disintegration of the volume by Avila-Viana-Wilkinson: if f is accessible, then the disintegration is

- either purely atomic,
- or Lebesgue.

Global rigidity in general manifolds.

Conjecture 2.3.3 (Katok-Spatzier)
If a higher rank action $\mathbb{Z}^k \times \mathbb{R}^l : M \to M$ contains an Anosov element, then it is smoothly algebraic.

But this conjecture in general is not true for $l \geqslant 2$, since

Theorem 2.3.4 (Vinhage)
There exists a C^∞-time change of Example 2.1.5.2 (product of geodesic flows) that has no C^∞-rank-one factor, is Anosov and not C^∞-algebraic.

However, for $\mathbb{Z}^k$-actions of $k \geqslant 2$, Katok-Spatzier's conjecture still may be true. For the $\mathbb{R}^l$ cases, the "Anosov" condition of Katok-Spatzier's conjecture need to be replaced with a "totally Anosov" condition.

Theorem 2.3.5 (Global rigidity, Spatzier-Vinhage)
The global rigidity of $\mathbb{R}^l$-actions ($l \geqslant 2$) on ANY manifold M providing totally Anosov, coarse Lyapunov $E^{[\chi]}$ are 1d and no rank one factors.

Remark 2.3.6 It gives a new approach to construct algebraic structures on the manifold.

Theorem 2.3.7 (Damjanović-Spatzier-Vinhage-Xu)
A totally Anosov $\mathbb{R}^l$-action is smoothly algebraic providing

- Hyperplanes in Weyl chamber pictures have dense orbits.
- Weyl chamber walls are accessible (strongly accessible).
- Oseledets spaces admit measurable conformal structures.

Conjecture 2.3.8 (Extended Katok-Spatzier's conjecture, Damjanović-Wilkinson-Xu)
Let f be a fibered partially hyperbolic diffeomorphism, assume that $\mathscr{Z}(f)$ contains a k-dimensional Lie group of maps which are id on the base with $k = \dim \mathscr{W}^c_f$. If the projection of $\mathscr{Z}(f)$ onto to the base has no C^0-rank-one factor, then f is C^∞-fibration of a smoothly algebraic system.

Conjecture 2.3.9 (Semi-local conjectures, Damjanović-Wilkinson-Xu)
Assume $f_0 : G/\Gamma \to G/\Gamma$ be an affine map where G/Γ is a connected homogeneous space. Assume that $\overline{\langle \text{stable}(f_0), \text{unstable}(f_0)\rangle}\,\Gamma = G$ (it implies property-K by Dani) and $\mathscr{Z}(f_0)$ has no rank-one factors. Let f be a C^1-small perturbation of f_0, then

1. Is f smoothly affine?
2. If $\mathscr{Z}(f)$ also has no rank-one factors, is f smoothly affine?

3 Dimension of Stationary Measures (Françios Ledrappier / Pablo Lessa)

§3.1 Generalities about dimension and statement of results (Françios, May 1)

We will follow the paper [LL23].

Let (X, d) be a separable metric space and μ be a Radon measure on X. The local dimension for $x \in X$ is defined as

$$\overline{\dim}_x(\mu) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \underline{\dim}_x(\mu) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

Definition 3.1.1. We say μ is **exact dimensional** if there is a constant δ such that for μ almost every x,

$$\overline{\dim}_x(\mu) = \underline{\dim}_x(\mu) = \delta.$$

This is also related to the Hausdorff dimension. For a subset $A \subset X$ and $\alpha > 0$, the Hausdorff outer measure

$$H_\alpha(A) := \lim_{\varepsilon\to 0} \inf\Big\{ \sum \varepsilon_i^\alpha \;:\; A \subset \bigcup B(x_i, \varepsilon_i), \varepsilon_i < \varepsilon \text{ for every } i \Big\}.$$

The Hausdorff dimension of A is defined as

$$\dim_{\mathrm{H}} A := \inf\{ \alpha \geqslant 0 \;:\; H_\alpha(A) = 0 \}.$$

Fact 3.1.2. If μ is exact dimensional with dimension δ, then

$$\delta = \inf\{ \dim_{\mathrm{H}}(A) \;:\; \mu(A) > 0 \} = \inf\{ \dim_{\mathrm{H}} \;:\; \mu(X \setminus A) = 0 \}.$$

Example 3.1.3

Graph of the Weierstrass function

$$\phi(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$$

where $b \in \mathbb{N}$ and $\lambda \in (\frac{1}{b}, 1)$.

- Besicovitch-Ursell (1937): $\dim_{\mathrm{H}}\{ (x, \phi(x)) \} \leqslant 2 + \log\lambda / \log b$.
- W. Shen (2018): $\dim_{\mathrm{H}}\{ (x, \phi(x)) \} = 2 + \log\lambda / \log b$.

Let $(X_1, d_1, \mu_1), (X_2, d_2, \mu_2)$ be two spaces with $\dim \mu_i = d_i$. Then $\mu_1 \otimes \mu_2$ is exact dimensional on $(X_1 \times X_2, \max\{d_1, d_2\})$ and $\dim(\mu_1 \otimes \mu_2) = \delta_1 + \delta_2$.

Let (X, d_X, μ) be a space and $\pi(X, d_X) \to (Y, d_Y)$ be a Lipschitz map. Then

$$\overline{\dim}_{\pi(x)}(\mu_* \mu) \leqslant \overline{\dim}_x(\mu), \quad \underline{\dim}_{\pi(x)}(\mu_* \mu) \leqslant \underline{\dim}_x(\mu).$$

Moreover, there exists a family of $y \mapsto \mu_y$ of disintegration, that is

$$\int f(x)\mathrm{d}\mu(x) = \int_Y \int_{\pi^{-1}(y)} f(x)\mathrm{d}\mu_y(x)\mathrm{d}\mu(y).$$

Assume that for μ almost every y, μ_y is exact dimensional with dimension δ. If (X,δ) is Lipschitz equivalent to an Euclidean space, then

$$\underline{\dim}_x(\mu) \geqslant \underline{\dim}_{\pi(x)}(\mu_* \mu) + \delta.$$

Example 3.1.4

1. The Cantor measure is exact dimensional and with dimension $\log 2/\log 3$.
2. Let μ_p be the Bernoulli measure with law $(p, 1-p)$ on $\{0,1\}^{\mathbb{N}} \approx [0,1]$, then $\dim \mu_p = -p\log p - (1-p)\log(1-p)$.
3. Consider μ_p on $\{0,1\}^{\mathbb{N}}$ isomorphic to the Cantor set embedded into $[0,1]$, then $\dim \mu_p = [-p\log p - (1-p)\log(1-p)]/\log 3$.
4. In general, push μ_p on $\{0,1\}^{\mathbb{N}}$ to the (λ,ρ)-Cantor set (the limit set given by $(x \mapsto \lambda x)$ and $(x \mapsto \rho x + (1-\rho))$ on $[0,1]$), also denoted by μ_p. Then the dimension is

$$\dim \mu_p = \frac{-p\log p - (1-p)\log(1-p)}{-p\log\lambda - (1-p)\log\rho}.$$

Random walk on matrices. Let μ be a probability measure on $\mathrm{SL}(d,\mathbb{R})$. Let $(\Omega, m) := (\mathrm{SL}(d,\mathbb{Z}),\mu)^{\mathbb{Z}}$ and σ be the left shift map on it. Let $g_n : \Omega \to \mathrm{SL}(d,\mathbb{R})$ be the projection onto its n-th coordinate. Let

$$X_n(\omega) = \begin{cases} g_{n-1}(\omega)\cdots g_0(\omega), & n \geqslant 0; \\ g_n^{-1}(\omega)\cdots g_{-1}^{-1}(\omega), & n < 0. \end{cases}$$

Then $X_{m+n}(\omega) = X_m(\sigma^n\omega)X_n(\omega)$.

Assume that $\int \log\|g\|\,\mathrm{d}\mu(g) < \infty$. By the Oseledets' theorem, there exists a splitting

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_N(\omega)$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n}\log\|X_n(\omega)v\| = \chi_i, \quad \forall v \neq 0 \in E_i(\omega),$$

where $\chi_1 > \chi_2 > \cdots > \chi_N$ are all the different Lyapunov exponents. Let $d_i = \dim E_i$, then

$$\sum_{i=1}^N d_i = d, \quad \sum_{i=1}^N d_i\chi_i = 0.$$

Let

$$\mathcal{X}(\omega) = (E_1(\omega), \cdots E_n(\omega)) \in \prod_{i=1}^N \mathcal{G}_{d_i}(\mathbb{R}^d) =: \mathcal{X},$$

where $\mathcal{G}_{d_i}(\mathbb{R}^d)$ is the Grassmannian.

Theorem 3.1.5 (Main Theorem) The distribution of $\mathcal{X}(\omega)$ is exact dimensional.

§3.2 Stationary measures and entropies (Françios, May 2)

More precisely, let M be the distribution of $\mathscr{X}(\omega)$, that is

$$M(A) = m(\{\omega : \mathscr{X}(\omega) \in A\}), \quad \forall A \in \mathscr{B}(\mathscr{X}).$$

Then

$$\dim M = \Delta = \sum_{i \neq j} \gamma_{i,j}$$

where $0 \leqslant \gamma_{i,j} \leqslant d_i d_j$ will be explained later.

We also consider the flag variety on $\mathbb{R}^d$ as

$$\mathscr{F} = \left\{ \{0\} \subset U_1 \subset U_2 \subset \cdots \subset U_N = \mathbb{R}^d : U_i \text{ are subspaces of } \mathbb{R}^d, \ \dim U_j = \sum_{i \leqslant j} d_i \right\}.$$

For every $\omega \in \Omega$, let

$$f(\omega) = \left\{ \{U_j(\omega)\} : U_j(\omega) = \bigoplus_{i \leqslant j} E_i(\omega) \right\} \in \mathscr{F}.$$

Then

$$v \in U_j(\omega) \iff \limsup_{n \to -\infty} \frac{1}{|n|} \log \|X_n(\omega) v\| \leqslant -\chi_j.$$

Note that $f(\omega)$ only depends on the negative coordinates of ω, or equivalently, $f(\omega)$ is $\sigma(g_n(\omega) : n < 0)$-measurable.

We also consider another flag variety

$$\mathscr{F}' = \left\{ \{0\} \subset U_1' \subset U_2' \subset \cdots \subset U_N' = \mathbb{R}^d : U_i' \text{ are subspaces of } \mathbb{R}^d, \ \dim U_k' = \sum_{i > N-k} d_i \right\}.$$

Let

$$f'(\omega) = \left\{ \{U_k'(\omega)\} : U_k'(\omega) = \bigoplus_{i > N-k} E_i(\omega) \right\} \in \mathscr{F}'.$$

Then

$$v \in U_k(\omega)' \iff \limsup_{n \to +\infty} \frac{1}{n} \log \|X_n(\omega) v\| \leqslant \chi_{N-k+1}.$$

Similarly, $f'(\omega)$ is $\sigma(g_n(\omega) : n \geqslant 0)$-measurable.

Let ν be the distribution of $f(\omega)$ and ν' be the distribution of $f'(\omega)$ on the flag varieties respectively.

Theorem 3.2.1 (Ledrappier-Lessa) $(\mathscr{F}, \nu)$ is exact dimensional with $\dim \nu = \sum_{i<j} \gamma_{i,j}$.

We can show the cocycle invariance of $f(\omega)$ as $f(\sigma\omega) = g_0(\omega) f(\omega)$. It follows that ν is a μ-stationary measure on $\mathscr{F}$.

Remark 3.2.2 We have no further assumptions on μ (such as the usual Zariski dense condition). So the μ-stationary measure on $\mathscr{F}$ might not be unique. But we only consider this specific stationary measure.

> **Example 3.2.3**
> For the case of $d = 3$ and $d_i = 1$, we have two projection $(f(\omega) \mapsto U_1(\omega))$ and $(f(\omega) \mapsto U_2(\omega))$. These projections give two stationary measures $(\mathscr{L}, \nu_{\mathscr{L}})$ and $(\mathscr{P}, \nu_{\mathscr{P}})$. Rapaport (2021) has show that these projection measures are exact dimensional.

Definition 3.2.4. Let (Y, ν) be a (G, μ)-space with $\mu * \nu = \nu$, the **Furstenberg entropy** is

$$K(\mu, \nu) := \int_{G\times Y} \log \frac{\mathrm{d}g_*\nu}{\mathrm{d}\nu}(gy)\mathrm{d}\mu(g)\mathrm{d}\nu(y).$$

Observation 3.2.5. $K(\mu, \nu) = I(g_{-1}, f)$.

Here

$$I(\mathscr{A}, \mathscr{B}) = \sup_{A\in\mathscr{A}, B\in\mathscr{B}} H(A) + H(B) - H(A \vee B)$$

for two sub-algebras $\mathscr{A}, \mathscr{B}$.

> **Example 3.2.6**
> Back to Example 3.2.3, we have $K(\mu, \nu_{\mathscr{F}}) - K(\mu, \nu_{\mathscr{L}}) = I(g_{-1}, f|E_1)$. Note that the projections are indeed fiber bundles with disintegrations $\nu_{\mathscr{F}}^{\mathscr{L}}$ over $\nu^{\mathscr{L}}$ and disintegrations $\nu_{\mathscr{F}}^{\mathscr{P}}$ over $\nu_{\mathscr{P}}$.
> **Theorem (Lessa).** *Both $\nu_{\mathscr{F}}^{\mathscr{L}}$ and $\nu_{\mathscr{F}}^{\mathscr{P}}$ are exact dimensional and*
>
> $$\dim \nu_{\mathscr{F}}^{\mathscr{L}} = \frac{K(\mu, \nu_{\mathscr{F}}) - K(\mu, \nu_{\mathscr{L}})}{\chi_2 - \chi_3}.$$