Reading Seminar on Homogeneous Dynamics (2023 Spring)

Yuxiang Jiao

Contents

§1 Introduction (Pengyu Yang, Mar 17)

Arithmetic & Super-rigidity

Let $\mathbb{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group.

Theorem 1.1 (Borel-Harish-Chandra) $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$.

Definition 1.2. We say $\Gamma, \Gamma' \subset \mathbb{G}$ are **commensurable** if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty$$

Definition 1.3 (Restriction of scalar). Let $[k : \mathbb{Q}] = d$ and $\mathbb{G}$ be a k-group. The restriction $R_{k/\mathbb{Q}}\mathbb{G}$ is a $\mathbb{Q}$-group such that for every $k \subset K$,

$$R_{k/\mathbb{Q}}\mathbb{G}(K) \cong \mathbb{G}^{\sigma_1}(K) \times \mathbb{G}^{\sigma_i}(K) \times \cdots \times \mathbb{G}^{\sigma_d}(K)$$

where $\sigma_i : k \hookrightarrow \mathbb{C}$ are embeddings.

Remark 1.4 — $R_{k/\mathbb{Q}}\mathbb{G}(\mathbb{Q}) \cong \mathbb{G}(k)$, $R_{k/\mathbb{Q}}(\mathbb{Z}) \cong \mathbb{G}(\mathcal{O}_k)$.

Definition 1.5. Let G be a connected semisimple real Lie group with trivial center and no compact factor. Let $\Gamma \subset G$ be a lattice. We say Γ is **arithmetic** if there exists a semisimple algebraic $\mathbb{Q}$-group $\mathbb{H}$ such that there is a surjective $\varphi : \mathbb{H}(\mathbb{R})^0 \to G$ with compact kernel such that $\varphi(\mathbb{H}(\mathbb{Z}) \cap \mathbb{H}(\mathbb{R})^0)$ is commensurable with Γ.

Example 1.6

1. $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ or congruence subgroups.
2. $G = \mathrm{Sp}(2n, \mathbb{R})$ and $\Gamma = \mathrm{Sp}(2n, \mathbb{Z})$.
3. $B = \mathbb{Q}(2,3) := \langle i, j | i^2 = 2, j^2 = 3, ij = -ji \rangle$. Then $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{Mat}(2, \mathbb{R})$. Let

   $$\mathbb{G} = B^{(1)} := \left\{ a + bi + cj + dij : a^2 - 2b^2 - 3c^2 + 6d^2 = 1 \right\}.$$

   Then $\mathbb{G}(\mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$ given by $i \mapsto \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix}$ and $j \mapsto \begin{bmatrix} & 1 \\ 3 & \end{bmatrix}$. Then $\mathbb{G}(\mathbb{Z})$ is a cocompact arithmetic lattice in $\mathrm{SL}(2, \mathbb{R})$, which is not commensurable with $\mathrm{SL}(2, \mathbb{Z})$.
4. $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$, we consider the embedding $\Gamma \hookrightarrow G$ given by $A \mapsto (A, {}^{\sigma}A)$. The restriction of scalar $\mathbb{G} = R_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SL}(2, \mathbb{Q}(\sqrt{2}))$.
5. $G = \mathrm{SL}(2, \mathbb{C})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{-1}])$, $\mathbb{G} = R_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\mathrm{SL}(2, \mathbb{Q}(\sqrt{-1}))$.
6. Let $J = x_1^2 + x_2^2 + (1 - \sqrt{2})x_3^2$. Let $G = \mathrm{SO}(J)(\mathbb{R})^0 \cong \mathrm{SO}(2,1)^0$. Let $\mathbb{H} = R_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SO}(J)$. Then $\mathbb{H}(\mathbb{R}) \approx G \times \mathrm{SO}(x_1^2 + x_2^2 + (1 + \sqrt{2})x_3^2)(\mathbb{R}) \cong G \times \mathrm{SO}(3)$.

Theorem 1.7 (Margulis Arithmeticity)

Let G be a semisimple real Lie group with $\mathrm{rank}_{\mathbb{R}}\, G \geqslant 2$ without compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is a arithmetic.

Theorem 1.8 (Margulis Super-rigidity)

Let G be a semisimple real Lie group with $\operatorname{rank}_{\mathbb{R}} G \geqslant 2$. Assume that G is with trivial center and no compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Let $H = \mathbb{H}(k)$ be a connected simple k-group where k is a local field ($\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \cdots$). Let $\varphi : \Gamma \to H$ be a homomorphism such that $\varphi(\Gamma)$ is Zariski dense and unbounded. Then φ extends to G, that is, $\exists \psi : G \to H$ continuous such that $\psi|_\Gamma = \varphi$.

Remark 1.9 — Margulis Super-rigidity $\implies$ Margulis Arithmeticity.

Real rank one case

$X = G/K$	$\mathbb{H}^n$	$\mathbb{CH}^n$	$\mathbb{HH}^n$	$\mathbb{OH}^n$
G	$\mathrm{SO}(n,1)^0$	$\mathrm{SU}(n,1)$	$\mathrm{Sp}(n,1)$	F_4^{-20}
K	$\mathrm{SO}(n)$	$U(n)$	$\mathrm{Sp}(n)$	$\mathrm{Spin}(9)$

$\mathrm{SO}(2,1)$ **case.** $G = \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(2,1)^0 \cong \mathrm{Isom}(\mathbb{H}^2)^+$. Let $\Gamma = \pi_1(\Sigma_g)$. We consider

$$\mathcal{M}_g := \mathrm{Hom}(\Gamma, G)/\sim = \{\text{hyperbolic structure on } S\} = \{\text{complex structure on } S\}.$$

$\mathcal{M}_g$ is a complex orbifold of complex dimension $3g-3$. There is no rigidity.

$\mathrm{SO}(n,1)$ **case for** $n \geqslant 3$. There is some rigidity.

Theorem 1.10 (Mostow strong rigidity)

Let M, N be compact hyperbolic n-manifolds. Let $\varphi : M \to N$ be a homotopy equivalence. Then there exists an isometry $\psi : M \to N$ which is homotopic to φ.

Theorem 1.11 (Gromov-Piatetski-Shapiro)

For every $n \geqslant 3$, $\mathrm{SO}(n,1)$ contains infinitely many commensurable classes of non-arithmetic lattices.

$\mathrm{Sp}(n,1)$ **case and** F_4^{-20} **case.**

Theorem 1.12 (Corlette, Gromov-Shoen)

Let $G = \mathrm{Sp}(n,1)$ or F_4^{-20}. Every lattice $\Gamma < G$ is arithmetic.

$\mathrm{SU}(\mathrm{n},1)$ **case.** The only known non-arithmetic lattices are for $n = 2, 3$. For the $\mathrm{SU}(2,1)$ case, Mostow constructed reflection groups which are non-arithmetic. For the $\mathrm{SU}(3,1)$ case, Deligne-Mostow constructed non-arithmetic lattices.

This semester

Theorem 1.13 (Bader-Fisher-Miller-Stover, [BFMS21, Theorem 1.1])
Let $\Gamma \subset \mathrm{SO}(n,1)^0$ be a lattice. Suppose $K\backslash G/\Gamma$ contains infinitely many maximal totally geodesic subspace of $\dim \geqslant 2$. Then Γ is arithmetic.

Theorem 1.14 ([BFMS21, Theorem 1.5])
Let $W = \mathrm{SO}(m,1)^0 < G = \mathrm{SO}(n,1)^0$ where $1 < m < n$. If there exists $\{\mu_i\}$ a sequence of W-invariant ergodic probability measure on G/Γ such that $\mu_i \xrightarrow{w*} \mu_{G/\Gamma}$. Then Γ is arithmetic.

Theorem 1.15 (Super-rigidity, [BFMS21, Theorem 1.6])
Let $W = \mathrm{SO}(m,1)^0 < G = \mathrm{SO}(n,1)^0$ where $1 < m < n$. Let k be a local field. Let $\mathbb{H}$ be a connected k-algebraic group. Assume that $(k, \mathbb{H})$ is compatible with G. Let $\rho : \Gamma \to \mathbb{H}(k)$ be a homomorphism with unbounded and Zariski dense image. If there exists $\mathbb{H} \to \mathrm{SL}(V)$ a k-representation on a k-vector space V and a W-invariant probability measure ν on

$$(G \times \mathbb{P}(V))/\Gamma : \left\{(g,v) \sim (g\gamma, \rho(\gamma)^{-1}v)\right\}$$

such that ν projects to $\mu_{G/\Gamma}$. Then ρ extends to $G \to \mathbb{H}(k)$.

■ There are two good surveys about rigidity theory [Spa04] and [Fis22].
■ We will follow a textbook by Zimmer [Zim13] at the beginning in this semester.

§2 Ergodic theory (Yuxiang Jiao, Mar 31)

Setting
- G locally compact second countable group.
- S a Borel space (isomorphic to a complete separable metric space with Borel σ-algebra).
- S is a G-space: G acts on S (measurably).
- A quasi-invariant measure μ on S, that is, for every $A \subset S, g \in G$, $\mu(Ag) = 0$ iff $\mu(A) = 0$.

Definition 2.1. The action is called **ergodic** if every G-invariant measurable subset of S is either null or conull.

Example 2.2
1. $S = M$ a smooth manifold, $G \subset \mathrm{Diff}(M)$, $\mu \approx \mathrm{Leb}$ which is quasi-invariant.
2. $H < G$ a closed subgroup, $X = G/H$. Then $G \curvearrowright (X, \mu_X)$ is ergodic (by transitivity).
3. $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n$ is ergodic (by Fourier analysis).
4. $X = \prod_{\mathbb{Z}} \{\pm 1\}$ a compact abelian group. $H = \{x \in X : x_i = 1 \text{ for all but finitely many } i\}$. Then $H \curvearrowright X$ is ergodic (by Fourier analysis).
5. $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{RP}^1$. Ergodic? Regard $\mathbb{RP}^1 \cong \mathrm{SL}(2,\mathbb{R})/P$ where $P = \{g : g.\infty = \infty\}$. We remark that there is no Γ-invariant measure on $\mathbb{RP}^1$. Proposition 2.3 helps to deal with this action.

Moore's ergodicity theorem

Proposition 2.3 ([Zim13, Corollary 2.2.3])
Let H_1, H_2 be closed subgroups of G. Then $H_1 \curvearrowright G/H_2$ ergodic $\iff$ $H_2 \curvearrowright G/H_1$ ergodic.

Proof. Let S be a G-space and $H \subset G$ a closed subgroup. Then $H \curvearrowright S$ is ergodic iff $G \curvearrowright (S \times G/H)$ is ergodic. □

Definition 2.4. Let G be a connected semisimple Lie group with finite center and $\Gamma < G$ is a lattice. We say Γ **irreducible** if for every normal subgroup $H \subset G$, $\Gamma N/N$ is dense in G/N.

Theorem 2.5 (Moore's ergodicity theorem, [Zim13, Theorem 2.2.6])
Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let $\Gamma < G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and H is not compact. Then H is ergodic on G/Γ.

Example 2.6 $\mathrm{SL}(n, \mathbb{Z})$ acts ergodically on $\mathbb{RP}^{n-1}$.

Example 2.7 $\mathrm{SL}(n, \mathbb{Z})$ acts ergodically on $(\mathbb{R}^n, \mathrm{Leb})$. Since $\mathbb{R}^n \setminus \{0\} \cong \mathrm{SL}(n, \mathbb{R})/H$.

Definition 2.8. Let G be a finite product of connected non-compact simple Lie groups with finite center. Let S be an ergodic G-space with finite invariant measure. We say the action is **irreducible** if for every non-central normal subgroup $N \subset G$, N is ergodic on S.

Proposition 2.9 $\Gamma < G$ is an irreducible lattice $\iff$ $G \curvearrowright G/\Gamma$ is irreducible.

Example 2.10
G as above. Assume that $G \hookrightarrow H$ where H is a simple Lie group. Let $\Gamma < H$ be a lattice (hence irreducible). By Moore's ergodicity, H/Γ is an ergodic G-space. Furthermore, it is an irreducible G-space.

Theorem 2.11 (Moore's ergodicity theorem, general version, [Zim13, Theorem 2.2.15])
Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let S be an irreducible ergodic G-space with finite invariant measure. If $H \subset G$ is a closed subgroup and H is not compact. Then H is ergodic on S.

Relation with unitary representations

Let us show the idea of proof of Moore's ergodicity theorem. Note that $G \curvearrowright S$ induces an action $G \curvearrowright L^2(S)$. Since we assume that μ is G-invariant, then G acts by unitary operators. Denote as $\pi : G \to \mathcal{U}(L^2(S))$. We equip $\mathcal{U}(L^2(S))$ with strong operator topology, then π is continuous.

Denote $L^2_0(S)$ to be the orthogonal complement of $\mathbb{C}$ in $L^2(S)$ which is G-invariant.

Proposition 2.12 ([Zim13, Corollary 2.2.17])

G acts ergodically on $S \iff$ there is no non-trivial G-invariant vectors in $L_0^2(S)$.

Combining this proposition, it suffices to show

Theorem 2.13 ([Zim13, Theorem 2.2.19])

Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let π be a unitary representation of G such that $\pi|_{G_i}$ has no invariant vectors. If $H \subset G$ is a closed subgroup and $\pi|_H$ has non-trivial invariant vectors, then H is compact.

This theorem follows from the following result.

Theorem 2.14 (Vanishing of matrix coefficients, [Zim13, Theorem 2.2.20])

Let G, G_i, π be as above. For every unit vectors $v, w \in \mathcal{H}$, the Hilbert space where G acts on. The matrix coefficient $f_{v,w}(g) = (\pi(g)v, w)$ tends to zero as g tending to infinity.

If there exists an H-invariant vector v, then H is compact since $(\pi(h)v, v) \equiv 1$ for $h \in H$.

Remark 2.15 — Vanishing of matrix coefficients can be viewed as "mixing", which is stronger than ergodicity.

"Mixing" in $\mathrm{SL}(2,\mathbb{R})$

Theorem 2.16

Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation without invariant vectors. Then for every $\varphi, \psi \in \mathcal{H}$ and (g_n) divergent in $\mathrm{SL}(2,\mathbb{R})$, we have $(g_n.\varphi, \psi) \to 0$.

Proof. By KAK decomposition, it suffices to consider $g_n \in A$. Let $g_n = a_{t_n} = \mathrm{diag}(e^{t_n}, e^{-t_n})$ with $t_n \to \infty$. Assume for a contradiction that $(g_n.\varphi, \psi) \not\to 0$, we can assume that $(g_n.\varphi, \psi) \to c \neq 0$. Take a countable dense set $\mathcal{A} \subset \mathcal{H}$ containing φ, ψ above. Passing to a subsequence if necessary, we can assume that $(g_n.\varphi, \psi)$ convergent for every $\varphi, \psi \in \mathcal{A}$. Define

$$f(\varphi, \psi) = \lim_{n\to\infty} (g_n.\varphi, \psi),$$

which forms a nonzero sesquilinear form on $\mathcal{H}$. By Riesz representation theorem, there exists $E \in \mathscr{L}(\mathcal{H})$ such that $f(\varphi, \psi) = (E\varphi, \psi)$.

We want to show that every vector in $\mathrm{Im}\, E$ is fixed by $\mathrm{SL}(2,\mathbb{R})$. For every $u = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$, we have $g_n^{-1} u g_n \to \mathrm{id}$. Then

$$(u.E\varphi, \psi) = \lim_{n\to\infty} (ug_n.\varphi, \psi) = \lim_{n\to\infty} (g_n.\varphi, \psi) = (E\varphi, \psi).$$

Hence $u \circ E = E$. It follows that $\mathrm{Im}\, E$ is fixed by U. Similarly, $E \circ v = E$ for every $v = \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$. This does not lead to $\mathrm{Im}\, E$ are fixed by V directly.

We use a trick of considering the adjoint operator. Note that $E^* = \lim g_n^{-1}$ in the weak sense. By the commutativity, we have

$$(E\varphi, E\varphi) = \lim_k \lim_l (g_k.\varphi, g_l.\varphi) = \lim_k \lim_l (g_l^{-1}.\varphi, g_k^{-1}.\varphi) = (E^*\varphi, E^*\varphi).$$

Then $\ker E^* = \ker E$. Hence $\mathrm{Im}(\mathrm{id} - v) \subset \ker E = \ker E^*$. It follows that $E^* \circ v = E^*$ and hence $v^* \circ E = E$. Since $v^* = v^{-1}$ run over V, we get the V-invariance.

Because U, V generates $\mathrm{SL}(2,\mathbb{R})$, we have $\mathrm{Im}\, E$ is fixed by $\mathrm{SL}(2,\mathbb{R})$ and hence is trivial. We get a contradiction. □

In the case of $\mathrm{SL}(n,\mathbb{R})$, we can similarly define U^+, U^- as

$$U^+ = \left\{u : g_n^{-1} u g_n \to \mathrm{id}\right\}, \quad U^- = \left\{u : g_n u g_n^{-1} \to \mathrm{id}\right\}.$$

By some calculation on the Lie algebra, we can show that U^+ and U^- together generate $\mathrm{SL}(n,\mathbb{R})$.

§3 Preparation on algebraic groups I (Yuxiang Jiao, Mar 31)

Setting

- G a locally compact second countable group and S a measurable G-space.
- $k \subset K$ where k is a local field (where $\mathrm{char}\, k = 0$) and K is algebraic closed.
- $\mathbb{G}$ a linear algebraic group defined over k, $\mathbb{G}_k$ is its k-points.
- Regard $\mathbb{G} \subset \mathrm{GL}(n,\mathbb{K})$, it then $\mathbb{G}_k$ has a locally compact topology (the usual topology given by $\mathrm{GL}(n,k)$). We call it the Hausdorff topology.

Theorem 3.1 (Chevalley, [Zim13, Proposition 3.1.4])

If $\mathbb{H} \subset \mathbb{G}$ is a k-subgroup of $\mathbb{G}$, then there is a k-rational representation $\mathbb{G} \to \mathrm{GL}(n,K)$ and a point $x \in \mathbb{P}^{n-1}(k)$ such that $\mathbb{H}_k = \mathrm{Stab}_{\mathbb{G}_k}(x)$.

There are several definitions.

- A set is called **locally closed** if it is open in its closure.
- A Borel space is called **countably separated** if there exists a countable family of Borel sets $\{A_i\}$ which separate points.
- A Borel space is called **countably generated** if we additionally requires that $\{A_i\}$ generates the Borel σ-algebra.
- Let S be a Borel G-space which is countably separated, we call the action is smooth if S/G is countably separated.

Proposition 3.2

If G acts smoothly on S. Then every quasi-invariant measure on S is supported on an orbit (measurable support).

Theorem 3.3 ([Zim13, Theorem 2.1.4])

Suppose G acts continuously on a complete separable metrizable space S. Then the following are equivalent

(1) All orbits are locally closed.
(2) The action is smooth.
(3) For every $s \in S$, $G/\mathrm{Stab}_G(s) \to \mathrm{Orb}(s)$ is a homeomorphism.

Fact 3.4. Let V, W be varieties and $f : V \to W$ is a regular map. Then $f(V)$ contains an open set in its closure (in Zariski topology).

Now we consider an algebraic group $\mathbb{G}$ acts algebraically on a variety V. Then for every $x \in V$, the orbit $\mathbb{G}.x$ contains an open subset $U \subset \overline{\mathbb{G}.x}^{\mathrm{Zar}}$. Hence $\mathbb{G}.x = \mathbb{G}.U$ which is open in $\overline{\mathbb{G}.x}^{\mathrm{Zar}}$. Since a Zariski topology is coarser than Hausdorff topology, we deduce (general version needs to show a certain Galois cohomology group is finite)

Theorem 3.5 (Borel-Serre, [Zim13, Theorem 3.1.3])

If k is a local field of characteristic 0, and a k-group $\mathbb{G}$ acts k-algebraically on a k-variety V. Then every $\mathbb{G}_k$-orbit in V_k is locally closed in the Hausdorff topology.

Group actions on the measure space

Let $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(k)$ be the projective space. Let $G = \mathrm{PGL}(n, k)$ with a natural action on $\mathbb{P}^{n-1}(k)$. It induces an action on $\mathrm{Prob}(\mathbb{P}^{n-1})$, the family of probability measures on $\mathbb{P}^{n-1}$. We equips $\mathrm{Prob}(\mathbb{P}^{n-1})$ with the weak* topology, which makes it a compact metrizable space.

Theorem 3.6 ([Zim13, Theorem 3.2.4])

For any $\mu \in \mathrm{Prob}(\mathbb{P}^{n-1})$, the stabilizer $\mathrm{Stab}_G(\mu)$ has a normal subgroup of finite index which is k-almost algebraic (a compact extension of the k-points of a k-group). In particular, if $k = \mathbb{R}$, $\mathrm{Stab}_G(\mu)$ is the real points of an $\mathbb{R}$-group.

Theorem 3.7 ([Zim13, Theorem 3.2.6])

Every G-orbit in $\mathrm{Prob}(\mathbb{P}^{n-1})$ is locally closed, hence $G \curvearrowright \mathrm{Prob}(\mathbb{P}^{n-1})$ is smooth.

§4 Preparation on algebraic groups II (Yuxiang Jiao, Apr 7)

Let us sketch the proof of Theorem 3.6 here.

Lemma 4.1 (Furstenberg)

Let $(g_n) \subset G$ such that $g_n.\mu \to \nu$ where $\mu, \nu \in \mathrm{Prob}(\mathbb{P}^{n-1})$, then

(1) either (g_n) is bounded in G,

(2) or there exists proper subspaces $V, W \subset k^n$ such that $\operatorname{supp} \nu \subset [V] \cup [W]$.

Corollary 4.2 ([Zim13, Corollary 3.2.2])

Let $\mu \in \mathrm{Prob}(\mathbb{P}^{n-1})$, then

(1) either $\mathrm{Stab}_G(\mu)$ is compact,

(2) or there exists a proper subspace $V_0 \subset k^n$ such that $\mu([V_0]) > 0$ and $\mathrm{Stab}_G(\mu).[V_0] = [V_0] \cup [V_1] \cup \cdots \cup [V_r]$, a finite union of proper subspaces.

Proof of Theorem 3.6. Decompose μ into a sum of countably many $\mu_i \in \mathrm{Prob}(\mathbb{P}^{n-1})$, such that for each μ_i:

(i) μ_i is invariant under $\mathrm{Stab}_G(\mu)$.
(ii) $\mathrm{supp}\,\mu_i \subset [V_{i0}] \cup [V_{i1}] \cup \cdots \cup [V_{ir_i}]$, a finite union of subspaces with same dimension.
(iii) for each $V \subset k^n$ with $\dim V < \dim V_{i0}$, $\mu_i(V) = 0$.

Then $\mathrm{Stab}_G(\mu) = \bigcap_i \mathrm{Stab}_G(\mu_i)$. For each i, we consider

$$H_i = \{g \in G : g.[V_{i0}] \subset [V_{i1}] \cup \cdots \cup [V_{ir_i}]\}, \quad N_i = \{g \in G : g|_{V_{i0}} \text{ is a scalar}\}.$$

Then $\bigcap_i N_i \subset \mathrm{Stab}_G(\mu) \subset \bigcap_i H_i$. Since H_i, N_i are algebraic, the intersection can be replaced by a finite intersection. By previous lemma, we have

$$\bigcap_{i \in F} N_i \subset_{\text{Cocompact}} \mathrm{Stab}_G(\mu) \cap \bigcap_{i \in F} H_i' \subset_{\text{Finite index}} \mathrm{Stab}_G(\mu) \subset \bigcap_{i \in F} H_i,$$

where $H_i' := \{g \in G : g.[V_{ij}] = [V_{ij}], \forall j\}$. □

Theorem 4.3 (Borel density theorem)

Let $\mathbb{G}$ be a connected semisimple $\mathbb{R}$-group, $G = \mathbb{G}_{\mathbb{R}}^0$ and assume that G has no compact factor. Let Γ be a closed subgroup such that G/Γ has a finite G-invariant measure. Then

1. Γ is Zariski dense in $\mathbb{G}$.
2. Γ^0 is normal in G. In particular, if G is simple and Γ is a proper subgroup, then Γ is discrete.

Proof. Let $\mathbb{H}$ be the Zariski closure of Γ and $H = \mathbb{H} \cap G$. Since G is Zariski dense in $\mathbb{G}$ [Zim13, Theorem 3.1.9], it suffices to show $H = G$. By Chevalley's theorem (Theorem 3.1), there is a $\mathbb{R}$-regular homomorphism $\mathbb{G} \to \mathrm{GL}(n, \mathbb{C})$ such that $H = \mathrm{Stab}_G(x)$ for some $x \in \mathbb{P}^{n-1}(\mathbb{R})$. WLOG, we assume that $G.x$ linearly spans $\mathbb{P}^{n-1}(\mathbb{R})$. The conclusion follows if $n = 1$.

Assume that $n \geqslant 2$. Since G/H has a finite G-invariant measure, there is also a G-invariant measure μ on $G.x \subset \mathbb{P}^{n-1}(\mathbb{R})$. Note that G has no compact factor and hence there is a proper subspace V with $\mu([V]) > 0$ and $\mu([V']) = 0$ for every proper subspace $V' \subset V$. Then $G.V$ is a finite union of proper subspaces, by connectedness, $G.V = V$. But $G.x \cap [V] \neq 0$ since $\mu(G.x) = 1$ and $\mu([V]) > 0$, hence $G.x \subset [V]$. We get a contradiction. □

Theorems 3.6 and 3.7 together gives a clear description of the action of $\mathrm{PGL}(n, k)$ on $\mathrm{Prob}(\mathbb{P}^{n-1}(k))$. There are several corollaries as below.

Corollary 4.4 ([Zim13, Corollary 3.2.12])

If $\mathbb{G} \subset \mathrm{PGL}(G, K)$ is a k-group, then the action of $\mathbb{G}_k$ on $\mathrm{Prob}(\mathbb{P}^{n-1}(k))$ is smooth.

Proof. It suffices to consider $\mathbb{G}_k$-orbits on $G.\mu$ and note that $G.\mu \cong G/\mathrm{Stab}_G(\mu)$. □

Corollary 4.5 ([Zim13, Corollary 3.2.17])

If $\mathbb{H} < \mathbb{G}$ are k-groups such that $\mathbb{G}_k/\mathbb{H}_k$ is compact, then $\mathbb{G}_k$ acts smoothly on $\mathrm{Prob}(\mathbb{G}_k/\mathbb{H}_k)$.

Corollary 4.6 ([Zim13, Corollary 3.2.18])

If $\mathbb{H} < \mathbb{G}$ are $\mathbb{R}$-groups such that $\mathbb{G}_{\mathbb{R}}/\mathbb{H}_{\mathbb{R}}$ is compact, then for every $\mu \in \mathrm{Prob}(\mathbb{G}_{\mathbb{R}}/\mathbb{H}_{\mathbb{R}})$, $\mathrm{Stab}_{\mathbb{G}_{\mathbb{R}}}(\mu)$ is the real points of an $\mathbb{R}$-group.

Group actions on the function space

Let X be a σ-finite measure space and V be a locally compact space. Denote $F(X,V)$ be the space of measurable maps $f : X \to V$. We endow $F(X,V)$ with the topology in the sense of converging in measure. Then $F(X,V)$ is a complete separable metrizable space.

Proposition 4.7

Let $\mathbb{G}$ be a k-group and V be a k-variety, $\mathbb{G}$ acts k-regularly on V. Then the action of $\mathbb{G}_k$ on $F(X, V_k)$ is smooth and the stabilizers are k-points of a k-group.

Let V be an $\mathbb{R}$-variety. Define

$$\mathrm{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{C})) := \{f \text{ is the restriction to } V_{\mathbb{R}} \text{ of an } \mathbb{R}\text{-rational function } f : V \to \mathbb{P}^m(\mathbb{C})\}\,.$$

Proposition 4.8

Let $\mathbb{G}, \mathbb{H}$ be $\mathbb{R}$-groups acting on $\mathbb{P}^n(\mathbb{C}), \mathbb{P}^m(\mathbb{C})$ respectively. Let $V \subset \mathbb{P}^n(\mathbb{C})$ be a closed $\mathbb{G}$-invariant $\mathbb{R}$-subvariety, such that $V_{\mathbb{R}}$ is Zariski dense in V. Then $\mathbb{G}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}$ induces an action on $\mathrm{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{R}))$. We have

1. The $\mathbb{G}_{\mathbb{R}}, \mathbb{H}_{\mathbb{R}}$ and $\mathbb{G}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}$ actions on $\mathrm{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{R}))$ are smooth.
2. The stabilizers are real points of algebraic $\mathbb{R}$-groups.

§5 Margulis' super-rigidity theorem I (Jiesong Zhang, Apr 7)

Let $\mathbb{G}$ be a connected semisimple $\mathbb{R}$-group, $G = \mathbb{G}^0_{\mathbb{R}}$ and assume that G has trivial center and no compact factors. Let $\Gamma \subset G$ be an irreducible lattice. Let $H = \mathbb{H}_k$ be the k-points of a k-group (take $k = \mathbb{R}$ today), which is center-free. Let $\varphi : \Gamma \to H$ be a homomorphism such that

1. $\varphi(\Gamma)$ is Zariski dense and,
2. unbounded.

Today's main result is the following lemma.

Lemma 5.1

There are proper algebraic $\mathbb{R}$-subgroups $\mathbb{P} \subset \mathbb{G}, \mathbb{L} \subset \mathbb{H}$ and a Γ-map $\psi : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \to \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}$.

Definition 5.2. We say $\mathbb{H} \subset \mathbb{G}$ is **parabolic** if $\mathbb{G}/\mathbb{H}$ is a projective variety.

Proposition 5.3

If $\mathbb{G}$ is a k-group and $\mathbb{H}$ is a parabolic subgroup of $\mathbb{G}$. Then $\mathbb{G}_k/\mathbb{H}_k$ is compact.

Definition 5.4. Let G be a topological group. We say G is **amenable**, if every continuous G-action on a compact metrizable space admits a G-invariant probability measure.

Proposition 5.5

Let $\mathbb{P}$ be a minimal parabolic subgroup of $\mathbb{G}$ and $\Gamma \subset G$ is a lattice. Then $\mathbb{P}$ is an amenable group and Γ acts amenably on $\mathbb{G}/\mathbb{P}$.

The definition of amenable action, see

Proposition 5.6

Let S be an amenable Γ-space and X be a compact G-space. Then there is a measurable Γ-map $S \to \mathrm{Prob}(X)$.

We will skip the definition of an amenable action. We proof the following result directly.

Proposition 5.7

If $\Gamma \curvearrowright X$ where X is a compact metrizable space. Then there exists a Γ-map $\omega : \mathbb{G}/\mathbb{P} \to \mathrm{Prob}(X)$.

Proof. Let μ be the Haar measure on $\mathbb{G}$. Consider the action

$$(\Gamma \times \mathbb{G}) \curvearrowright (\mathbb{G} \times X), \quad (\gamma, g)(h, x) = (\gamma h g^{-1}, \gamma x).$$

Let $p : \mathbb{G} \times X \to \mathbb{G}$ be the projection. Let Q be the family of Borel measures τ on $G \times X$ satisfying $p_* \tau = \mu$ and $(\gamma, 1)_* \tau = \tau$. We claim that Q is nonempty. In fact, let D be a fundamental domain of Γ and $x_0 \in X$, let $\phi : \mathbb{G} \to \mathbb{G} \times X$ given by $g \mapsto (g, \gamma_g x_0)$ where $\gamma_g \in \Gamma$ is the unique element such that $g \in \gamma_g D$. Then ϕ is Γ-equivalent and hence $\phi_* \mu \in Q$.

Note that Q is a compact and convex set and Q is $(\Gamma \times \mathbb{G})$-invariant. Recall that $\mathbb{P}$ is amenable, then there exists a $(1, \mathbb{P})$-invariant element $\tau \in Q$. Write

$$\tau = \int_{\mathbb{G}} \delta_g \otimes \nu_g \mathrm{d}\mu(g), \quad \nu_g \in \mathrm{Prob}(X).$$

We can see that $\nu_g = \gamma_* \nu_{\gamma^{-1} gp} = \nu_{gp}$ for almost every g. It induces a Γ-map $\omega : gp \mapsto \nu_g$. □

Proof of Lemma 5.1. Let $\mathbb{P} \subset \mathbb{G}$ be a minimal parabolic group and $\mathbb{P}' \subset \mathbb{H}$ be a parabolic subgroup. Then $\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}}$ is a compact space. Note that Γ acts amenably on $\mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}}$. Hence there is a Γ-map

$$\varphi : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \to \mathrm{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}}).$$

It induces a map $\widetilde{\varphi} : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \to \mathrm{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$, which is Γ-invariant. Recall that the action $\mathbb{H}_{\mathbb{R}} \curvearrowright \mathrm{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}})$ is smooth. Hence $\widetilde{\varphi}$ is essential constant. Hence $\varphi(\mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}})$ falls in an orbit $\mathbb{G}.\mu$. Take $\mathbb{L}_{\mathbb{R}} = \mathrm{Stab}_{\mathbb{H}_{\mathbb{R}}}(\mu)$, the conclusion follows. □

§6 Margulis' super-rigidity theorem II (Bohan Yang, Apr 14)

Let us recall the Margulis' superrigidity theorem.

Theorem 6.1 (Margulis' superrigidity)

Let $\mathbb{G}$ be a connected semiseimple algebraic $\mathbb{R}$-group with $\mathbb{R}$-rank at least 2. Assume that $\mathbb{G}^0_{\mathbb{R}}$ has no compact factors. Let $\Gamma \subset \mathbb{G}^0_{\mathbb{R}}$ be an irreducible lattice. Let $\mathbb{H}$ be a connected simple algebraic $\mathbb{R}$-group and $\mathbb{H}_{\mathbb{R}}$ is not compact. Assume that $\pi : \Gamma \to \mathbb{H}_{\mathbb{R}}$ is a homomorphism with $\pi(\Gamma)$ Zariski dense. Then π extends to a rational homomorphism $\mathbb{G} \to \mathbb{H}$ defined over $\mathbb{R}$.

Throughout this section, we will use the notation in Zimmer's book [Zim13], which is terrible. There, $G/\Gamma = \{\Gamma \cdot g : g \in G\}$ and the action $G \curvearrowright X$ is always an right action $(g, x) \mapsto xg$. This means that G has a natural (right) action on G/Γ.

Lemma 6.2 ([Zim13, Lemma 5.1.3])

Suppose $\mathbb{P} \subset \mathbb{G}$ and $\mathbb{L} \subset \mathbb{H}$ are proper algebraic $\mathbb{R}$-subgroups, and $\varphi : \mathbb{G}/\mathbb{P} \to \mathbb{H}/\mathbb{L}$ is a rational Γ-map, then π extends to a rational homomorphism $\mathbb{G} \to \mathbb{H}$.

Hence it suffices to find such rational Γ-map φ. We will use the map constructed last time (Lemma 5.1). The aim is to show the constructed map is (essentially) rational (Step 2 in [Zim13]).

Definition 6.3. Let V be a complex variety and W be an $\mathbb{R}$-variety. Let $A \subset V_{\mathbb{R}}$ be a set of positive measure. We say $f : A \to V$ is **essentially rational** if there exists a rational map $R : W \to V$ such that $R = f$ on A.

We want to show that φ is rational. On criterion for rationality is a unipotent representation of a unipotent group. So want to replace $\mathbb{G}^0_{\mathbb{R}}/P_0$ by a such group, where $P_0 = \mathbb{P} \cap \mathbb{G}^0_{\mathbb{R}}$.

Lemma 6.4 ([Zim13, Lemma 5.1.4])

There exists a connected unipotent $\mathbb{R}$-subgroup $U \subset \mathbb{G}$ such that the product map $m : U \times \mathbb{P} \to \mathbb{G}$ is injective and the image is a Zariski dense $\mathbb{R}$-open set. Furthermore, it induces a map $U_{\mathbb{R}} \to \mathbb{G}^0_{\mathbb{R}}/P_0$ which is a measure space isomorphism.

In our case, $\mathbb{G} = \mathrm{SL}(n, \mathbb{C})$ and $\mathbb{P}$ is the triangular matrices. Then we can take U to be the lower triangular matrices with diagonal entries equal to 1.

Lemma 6.5 ([Zim13, Lemma 5.1.5])

It suffices to show for some $g \in \mathbb{G}^0_{\mathbb{R}}$, the map $u \mapsto \varphi(ug)$ is essentially rational on $U_{\mathbb{R}}$.

Definition 6.6. For every $t \in A \subset \mathbb{G}$, let C_t be the centralizer of t in $\mathbb{G}$. Let $C^u_t = C_t \cap U$.

Lemma 6.7 ([Zim13, Lemma 5.1.6])

There exists $t_1, \cdots, t_n \in A^0_{\mathbb{R}}$, $t_i \neq e$ and connected subgroups $U_i \subset C^u_{t_i}$ such that

(1) $\prod_{i=1}^r U_i \to U$ is an $\mathbb{R}$-isomorphism.

(2) For each r, $\prod_{i=1}^r U_i \subset U$ is a subgroup and $\prod_{i=r+1}^n U_i$ is normal in $\prod_{i=r}^n U_i$.

Lemma 6.8 ([Zim13, Lemma 5.1.7])

To prove Step 2, it suffices to prove if $e \neq t \in A^0_{\mathbb{R}}$, $V \subset C^0_t$ is a connected algebraic $\mathbb{R}$-group, then for almost every $g \in \mathbb{G}^0_{\mathbb{R}}$, $u \mapsto \varphi(ug)$ is essentially rational on $V_{\mathbb{R}}$.

Proof. Induction on $n - r$, we prove that $\varphi : u \mapsto \varphi(ug)$ is essentially rational on $\prod_{i=r}^n (U_i)_{\mathbb{R}}$. If $r = n$, then this is the "suffices to show" part. Suppose we have $u \mapsto \varphi(ug)$ is essentially rational on $\prod_{i=r}^n (U_i)_{\mathbb{R}}$. We define $\varphi_g(c, u) = \varphi(cug)$, $c \in U_{r-1}$. It suffices to show φ_g is essentially rational for almost every g.

By the "suffices to show" part, for every u and almost every g, $c \mapsto \varphi(cug)$ is essentially rational. By Fubini, for almost every g, $c \mapsto \varphi_g(c, u)$ is essentially rational. On the other hand, $\varphi(cug) = \varphi((cuc^{-1})cg)$, then for every c, for almost every g, $u \mapsto \varphi(cug)$ is essentially ration. By another Fubini and Theorem [Zim13, Theorem 3.4.4], we have for almost every g, φ_g is essentially rational. $\square$

Lemma 6.9 ([Zim13, Lemma 5.1.8])

To prove Step 2, it suffices to prove if $e \neq t \in A^0_{\mathbb{R}}$, then for almost every g, there exists

(1) an $\mathbb{R}$-subvariety $W_g \in \mathbb{H}/\mathbb{L}$ such that $\varphi_g : (C_t)^0_{\mathbb{R}} \to \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}, c \mapsto cg$ satisfies $\varphi_g(c) \in W_g$ for almost all c;
(2) an $\mathbb{R}$-algebraic group Q_g which acts $\mathbb{R}$-regularly on W_g;
(3) a measurable homomorphism $h_g : (C_t)^0_{\mathbb{R}} \to (Q_g)_{\mathbb{R}}$;
(4) a point $x_g \in W_g \cap \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}$;

such that $\varphi_g(c) = x_g h_g(c)$ for almost all $c \in (C_t)^0_{\mathbb{R}}$.

Proof. Let $V \subset C^u_t$ be a connected algebraic $\mathbb{R}$-group. If $\varphi_g = x_g h_g$ holds for all c, then $h_g|_{V_{\mathbb{R}}}$ is unipotent by [Zim13, Proposition 3.4.2] and hence $\varphi_g|_{V_{\mathbb{R}}}$ is rational. But $V_{\mathbb{R}}$ is of measure zero in $(C_t)^0_{\mathbb{R}}$, we need a further argument. For each $u \in V_{\mathbb{R}}$ and almost all $g \in \mathbb{G}^0_{\mathbb{R}}, c \in (C_t)^0_{\mathbb{R}}$, we have

$$\varphi(ucg) = x_g h_g(uc) = x_g h_g(u) h_g(c).$$

By Fubini, there exists a fixed c such that the equation holds for almost every g and almost every $u \in V_{\mathbb{R}}$. Therefore, $u \mapsto x_g h_g(u) h_g(c)$ is rational. Hence $u \mapsto \varphi(ug)$ is essentially rational. □

Proposition 6.10 ([Zim13, Proposition 3.5.2])

Let C be a locally compact group and $\varphi \in F(C, \mathbb{H}_k/\mathbb{L}_k)$. For every $g \in C$, let $\varphi_g \in F(C, \mathbb{H}_k/\mathbb{L}_k), \varphi_g(c) = \varphi(cg)$. Assume that almost every φ_g lie in a single $\mathbb{H}_k$-orbit of $F(C, \mathbb{H}_k/\mathbb{L}_k)$, then there exists (1)(2)(3)(4) as above.

Proof of Step 2. By the above proposition, we should check that for almost every $g \in \mathbb{G}^0_{\mathbb{R}}$, for almost every $c \in C = (C_t)^0_{\mathbb{R}}$, $(\varphi_g)_c$ lies in a common $\mathbb{H}_{\mathbb{R}}$-orbit. By a Fubini argument, it suffices to show that almost every φ_g lies in a same $\mathbb{H}_{\mathbb{R}}$-orbit.

Define $\Phi : \mathbb{G}^0_{\mathbb{R}} \to F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}), g \mapsto \varphi_g$. Let $T = \{t^n\} \subset A$, which is unbounded. Then

$$\varphi_{tg}(c) = \varphi(ctg) = \varphi(tcg) \overset{T \subseteq P_0}{=} \varphi(cg) = \varphi_g(c).$$

Hence Φ is a T-invariant measurable map, which induces $T : \mathbb{G}^0_{\mathbb{R}}/T \to F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})$. Recall φ is a Γ-map, then $\varphi_{g\gamma} = \varphi_g \pi(\gamma)$. Note that $\pi(\gamma) \in \mathbb{H}_{\mathbb{R}}$, consider the induced map

$$\overline{\Phi} : \mathbb{G}^0_{\mathbb{R}}/T \to F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$$

which is essentially Γ-invariant. Since T is unbounded, $\Gamma \curvearrowright \mathbb{G}^0_{\mathbb{R}}/T$ is ergodic. Combining with $F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$ is countably separated, $\overline{\Phi}$ is essentially constant. This complete the proof. □

§7 Margulis' arithmeticity theorem (Apr 21)

Definition 7.1 (Restriction of scalar). Let $[k : \mathbb{Q}] = d$ and $\mathbb{G}$ be a k-algebraic group. We define the $\mathbb{Q}$-algebraic group $R_{k/\mathbb{Q}}\mathbb{G}$ such that

$$R_{k/\mathbb{Q}}\mathbb{G} \cong \prod_{i=1}^{d} \mathbb{G}^{\sigma_i},$$

where $\sigma_1, \cdots, \sigma_d$ are the $\mathbb{Q}$-embeddings of $k \hookrightarrow \mathbb{C}$.

Proposition 7.2 $(R_{k/\mathbb{Q}}(\mathbb{G}))_{\mathbb{Q}} \cong \mathbb{G}_k$ and $(R_{k/\mathbb{Q}}(\mathbb{G}))_{\mathbb{Z}} \cong \mathbb{G}_{\mathcal{O}_k}$.

Theorem 7.3 (Margulis Arithmeticity)
Let G be a semisimple real Lie group with $\operatorname{rank}_{\mathbb{R}} G \geqslant 2$ without compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is a arithmetic.

The aim is to put Γ into some $\mathbb{G}_k \cong (R_{k/\mathbb{Q}}\mathbb{G})_{\mathbb{Q}}$. Then we consider the Zariski closure $\overline{\alpha(\Gamma)} = \mathbb{H}$. Taking the restriction of scalar and considering the integral points, $(R_{k/\mathbb{Q}}(\mathbb{H}))_{\mathbb{Z}}$ will be a desired construction. First we want to find an algebraic extension $k/\mathbb{Q}$ such that $\Gamma \subset \mathbb{G}_k$.

Note that G can be equipped with an algebraic structure. We assume that G is a connected semisimple algebraic $\mathbb{Q}$-group with trivial center and $\Gamma \subset G^0_{\mathbb{R}}$ is an irreducible lattice. Let $L(G)$ be the Lie algebra of G, which also admits an $\mathbb{Q}$-structure (if $G \subset \mathrm{GL}(n, \mathbb{C})$ then $G \subset M(n, \mathbb{C})$ admits a basis in $M(n, \mathbb{Q})$).

Lemma 7.4 ([Zim13, Lemma 6.1.8]) There exists an embedding $\pi : \Gamma \to \mathrm{GL}(m, k)$.

Proof. For every $g \in G$, we define $T(g) = \operatorname{tr}(\operatorname{Ad}(g))$, then T is a polynomial. Let V be the linear space of $\{gT\}$ which is finite dimensional with an G-action on it. It induces a G representation which is faithful. Since Γ is Zariski dense in G, there is $\{\gamma_1, \cdots, \gamma_m\} \subset \Gamma$ such that $\{\pi(\gamma_i)T\}$ is a basis of V. We need the following fact.

Fact 7.5 ([Zim13, Lemma 6.1.6]). For every $\gamma \in \Gamma$, $\operatorname{tr}(\operatorname{Ad}(\gamma))$ is algebraic.

Proof. It suffices to show for every $\gamma \in \Gamma$, $\operatorname{Aut}(\mathbb{C})(\operatorname{tr}(\operatorname{Ad}(\gamma)))$ is bounded. Note that for every σ, we have $\sigma(\operatorname{tr}(\operatorname{Ad}(\gamma))) = \operatorname{tr}(\operatorname{Ad}(\sigma(\gamma)))$. It suffices to show the following fact.

Fact 7.6. $\{\operatorname{tr}(\operatorname{Ad}(\sigma(\gamma))) : \sigma \in \operatorname{Aut}(\mathbb{C})\}$ is bounded.

Proof. Let $G = \prod H_i$ and $L(G) = \sum L(H_i)$ be the Lie algebras. Let $p_i : G \to H_i$ be the projection, then

$$\operatorname{tr}(\operatorname{Ad}(\sigma(\gamma))) = \sum_i \operatorname{tr}(\operatorname{Ad}_{H_i}(p_i(\sigma(\gamma)))).$$

By Borel density theorem, both Γ and $\sigma(\Gamma)$ are Zariski dense in G. Hence $(p_i \circ \sigma)(\Gamma)$ is Zariski dense in H_i. By Margulis' super-rigidity, either $(p_i \circ \sigma)(\Gamma)$ is contained in a compact subgroup or $p_i \circ \sigma|_{\Gamma}$ extends to a rational homomorphism $\pi : G \to H_i$. In the first case, every eigenvalue of $\operatorname{Ad}_{H_i}(p_i \circ \sigma(\gamma))$ is on the unit circle and hence $|\operatorname{tr}(\operatorname{Ad}_{H_i}(p_i \circ \sigma(\gamma)))| \leqslant \dim H_i$. If $(p_i \circ \sigma)$ extends to π, then $d\pi : L(G) \to L(H_i)$ is surjective and $\operatorname{Ad}_{H_i}(\pi(g)) \circ \mathrm{d}\pi = \mathrm{d}\pi \circ \operatorname{Ad}(g)$. Hence any eigenvalue of $\operatorname{Ad}_{H_i}(\pi(g))$ is an eigenvalue of $\operatorname{Ad}(g)$. Taking $g = \gamma$, we obtain an estimate $|\operatorname{tr}(\operatorname{Ad}_{H_i}(p_i \circ \sigma(\gamma)))| \leqslant e(\gamma) \dim H_i$, where $e(\gamma)$ only depends on $\operatorname{Ad}(\gamma)$. □

□

Then for every $\gamma, \pi(\gamma) \in \mathrm{GL}(n, k)$. This can be shown by the following way. Let c_{ij} be coefficient of matrices. Then for every $\gamma \in \Gamma$, we have

$$\pi(\gamma)(\pi(\gamma_i)T) = \sum_{j=1}^{m} c_{ij}(\pi(\gamma_j)T).$$

Expanding T into $\operatorname{tr}(\operatorname{Ad})$, which is algebraic for every element in Γ, the conclusion follows. □

Indeed, Γ is finitely generated. Hence k is a finite algebraic extension. In later discussions, we can assume that G is defined over k and $\Gamma \subset G_k$.

Proof of Theorem 7.3. Let $[k:\mathbb{Q}] = d$. We take the restriction of scalar, let $\alpha : G_k \to (R_{k/\mathbb{Q}}G)_\mathbb{Q}$ be the map given by $g \mapsto (\sigma_1(g), \cdots, \sigma_d(g))$ where $\sigma_1 = \mathrm{id}$. Let $H = \overline{\alpha(\Gamma)}^{\mathrm{Zar}}$, which is an algebraic $\mathbb{Q}$-group. Let $p : R_{k/\mathbb{Q}}(G) \to G$ such that $(p \circ \alpha)|_{G_k} = \mathrm{id}$. Note that Γ is Zariski dense in G, we have $p(H) = G$. Since G is semisimple and center-free, we have $p(\mathrm{Rad}(H)) = \mathrm{id}$ and $p(C(G)) = \mathrm{id}$. Combining with G is connected, we can also assume that H is semisimple, center-free and connected.

Claim 7.7. $(\ker p)_\mathbb{R}$ is compact.

Proof. Let F be a simple factor of $\ker p$, it suffices to check $F_\mathbb{R}$ is compact. Assume that $F_\mathbb{R}$ is non-compact, by Margulis' super-rigidity theorem, the map $G \xrightarrow{\alpha} H \xrightarrow{\text{projection}} F$ extends to a rational homomorphism $h : G \to F$. Writing $H \cong G \times F \times F'$, then $\{(g, h(g), f') : g \in G, f' \in F'\}$ contains Γ. It Contradicts that $\alpha(\Gamma)$ is Zariski dense in H. $\square$

For a prime p, let $\mathbb{Q}_p$ be the p-adic field. We have an embedding $H_\mathbb{Q} \to H_{\mathbb{Q}_p}$, which induces $\alpha : \Gamma \to H_{\mathbb{Q}_p}$. Since $\mathbb{Q}_p$ is totally disconnected, by Margulis' super-rigidity, $\alpha(\Gamma)$ is bounded. Hence the powers of each prime appearing in the denominators of the matrix entries of $\alpha(\gamma) \in H_\mathbb{Q}$ are uniformly bounded over $\gamma \in \Gamma$. Moreover, we can show that $\Gamma \cap H_\mathbb{Z}$ is of finite index in Γ. Applying p, we get $\Gamma \cap p(H_\mathbb{Z})$ is of finite index in Γ. Since $(\ker p)_\mathbb{R}$ is compact, $p(H_\mathbb{Z})$ is a lattice in $G_\mathbb{R}$. Then $\Gamma \cap p(H_\mathbb{Z}) < p(H_\mathbb{Z})$ is an inclusion of two lattices, hence of finite index. We obtain that Γ and $p(H_\mathbb{Z})$ are commensurable. We are done. $\square$

§8 Geodesic submanifolds and properly supported measures (Chengyang Wu, Apr 27)

Notation

- $G = \mathrm{SO}(n,1)^0$ the identity connected component of
$$\mathrm{SO}(n,1) = \left\{ A \in \mathrm{SL}(n+1, \mathbb{R}) : A^t \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} A = \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} \right\}.$$
- $K = \mathrm{SO}(n) < G$, the maximal connected compact subgroup.
- The maximal connected $\mathbb{R}$-diagonalizable subgroup
$$A = \left\{ a_t = \begin{bmatrix} \cosh t & & \sinh t \\ & I_n & \\ \sinh t & & \cosh t \end{bmatrix} : t \in \mathbb{R} \right\}.$$
- $M = C_G(A) \cap K \cong \mathrm{SO}(n-1)$, the maximal connected compact subgroup of $C_G(A)$.
- For every $m \leqslant n$, let $W_m = \begin{bmatrix} I_{n-m} & \\ & \mathrm{SO}(m,1)^0 \end{bmatrix} \subset G$ be the natural embedding.
- The hyperbolic n-space
$$\mathbb{H}^n := \left\{ v \in \mathbb{R}^{n+1} : Q_{n,1}(v) = -1, v_{n+1} > 0 \right\},$$
where $Q_{n,1} = v_1^2 + \cdots + v_n^2 - v_{n+1}^2$. Then $\mathbb{H}^n \cong K \backslash G$ since $G = \mathrm{Isom}(\mathbb{H}^n, Q_{n,1})^0$ and $K = \mathrm{Stab}_G(e_{n+1})$.
- Let $\Gamma < G$ be a lattice, let $X_\Gamma = K\backslash G/\Gamma$. Denote $\pi : G/\Gamma \to X_\Gamma$ the quotient.

Definition 8.1. A finite measure μ on G/Γ is called **homogeneous** if there exists a closed subgroup $S < G$ such that μ is Haar measure on an S-orbit on G/Γ. Such a homogeneous measure is called **W-ergodic** if there is a closed subgroup W of S such that μ is W-ergodic.

Proposition 8.2

For $X_\Gamma = K\backslash G/\Gamma$, the following are equivalent:

(1) X_Γ contains infinitely many maximal totally geodesic subspaces of dimension $\geqslant 2$.

(2) For some $1 < m < n$, there exists an infinite sequence (μ_i) of W_m-invariant ergodic measures of proper support for which the Haar measure on G/Γ is the weak* limit of (μ_i).

(3) For some $1 < m < n$, there exists an infinite sequence (μ_i) of homogeneous, W_m-ergodic measures of proper support for which the Haar measure on G/Γ is the weak* limit of (μ_i).

It is obvious that (3) is stronger than (2). Since W_m is generated by unipotent elements, by Ratner's theorem, (2) implies (3). Our aim today is to show the equivalence between (1) and (3).

Lemma 8.3

Fix $1 < m \leqslant n$, then we have

(1) Let $S \leqslant G$ be a closed subgroup containing W_m and $h \in G$ be such that $Sh\Gamma/\Gamma \subset G/\Gamma$ be a closed S-orbit. Then the subspace $Z = \pi(Sh\Gamma/\Gamma) \subset X_\Gamma$ is a closed, totally geodesic, m'-dimensional subspace for some $m' \geqslant m$, and up to normalization, the m'-volume of Z is the push-forward of the corresponding homogeneous measure on G/Γ.

(2) Furthermore, under the assumptions above, $m' = n$ if and only if $S = G$, and $m' = m$ if and only if all unipotent elements of S are contained in W_m. In the latter case, S is a subgroup of the normalizer $N_m = N_G(W_m)$ and $N_m h\Gamma/\Gamma \subset G/\Gamma$ is also closed with projection $\pi(N_m h\Gamma/\Gamma) = Z$.

(3) Conversely, every m-dimensional, closed, totally geodesic subspace $Z \subset X_\Gamma$ has finite m-volume, and moreover, $Z = \pi(Sh\Gamma/\Gamma)$ for some closed intermediate subgroup $W_m \leqslant S \leqslant N_m$ and some homogeneous W_m-ergodic subspace $Sh\Gamma/\Gamma \subset G/\Gamma$.

Theorem 8.4

Suppose that $W < G$ is a closed connected semisimple subgroup that is generated by unipotent elements. Let $\{\mu_i\}$ be a sequence of homogeneous, W-ergodic probability measures on G/Γ that weak* converges in the space of all finite Radon measures to a measure μ. Then μ is a homogeneous, W-ergodic probability measure on G/Γ and there exists a sequence $\{g_i\}$ in G and a natural number i_0 such that for every $i \geqslant i_0$, the measure $g_i\mu$ is a homogeneous, W-ergodic probability measure on G/Γ whose support contains $\operatorname{supp}\mu_i$.

Proof. **Step 1.** *μ is a probability measure.*

This is trivial for G/Γ is compact. In general, we consider the one-point compactification space $G/\Gamma \cup \{\infty\}$. Then $\mu_i \xrightarrow{w*} \mu \in \mathrm{Prob}(G/\Gamma \cup \{\infty\})$. It suffices to show $\mu(\infty) = 0$. We apply "Dani-Margulis": *Given a compact $F \subset X_\Gamma$ and $\varepsilon > 0$, there exists a compact set $F' \subset G/\Gamma$ such that for every $x \in F$, for every $\{u_t\}_{t\in\mathbb{R}} \subset G$ unipotent, we have*

$$\forall T > 0, \quad \frac{1}{T}\mathrm{Leb}\left\{t \in [0,T] : u_t x \in F'\right\} > 1 - \varepsilon.$$

By Birkhoff's ergodic theorem, if μ is ergodic with respect to a unipotent flow then $\mu(F) > 0 \implies \mu(F') > 1 - \varepsilon$.

Fix a one-parameter unipotent subgroup $\{u_t\} \subset W$. Since μ_i is W-ergodic, by Moore's ergodic theorem, μ_i is also ergodic with respect $\{u_t\}$. It suffices to find a compact $F \subset G/\Gamma$ such that $\mu_i(F) > 0$ for every $i > 0$.

Assume that $W = W_m$ for some $m > 1$. Using a "**compact core lemma**", we fix a compact set $C_1 \subset X_\Gamma$ that meets every closed, totally geodesic subspace of dimension $\geqslant 2$.

Proof of "compact core lemma". A compact core is a compact subset C_1 such that $X_\Gamma \setminus C_1$ is contained in the cusps. Note that each closed, totally geodesic subspace Z of dimension $\geqslant 2$ has fundamental group $\pi_1(Z)$ as a lattice in some W_m and injects into $\pi_1(X/\Gamma)$. However each cusp of X_Γ has a solvable fundamental group, so $\pi_1(Z)$ cannot inject into it. Hence $Z \cap C_1 \neq \varnothing$. □

Let C_2 be a compact set that $C_1 \subset C_2^\circ$. Let $F = \pi^{-1}(C_2) \leqslant G/\Gamma$. Note that for every i, $\pi_*\mu_i$ is the unit renormalization m'-volume on a closed, totally geodesic subspace Z. Hence $\mu_i(F) = \pi_*\mu_i(C_2) > 0$ since $C_2 \cap Z$ contains a nonempty open set. It follows that $\mu(F') > 1-\varepsilon$ and hence μ is a probability measure on G/Γ.

Step 2. *μ is homogeneous, W-ergodic.*

We first conclude that μ is a homogeneous, S-ergodic where S is the subgroup of the stabilizer of μ generated by unipotent elements. This is due to

Theorem 8.5 (Moses-Shah)

Let $\{U_i = \{u_i(t)\}\}$ be a sequence of one-parameter unipotent subgroups of G and $\{\mu_i\}$ be a sequence of probability measures on G/Γ such that μ_i is U_i-ergodic. Suppose that $\mu_i \xrightarrow{w^*} \mu$ a probability measure on G/Γ and $x \in \operatorname{supp}\mu$. Then the following holds:

(1) $\operatorname{supp}\mu = \Lambda(\mu).x$ where $\Lambda(\mu) := \{g \in G : g_*\mu = \mu\}$.

(2) Let $g_i \to \mathrm{id}_G$ be a sequence in G such that $g_i x \in \operatorname{supp}\mu_i$ and $\{u_i(t)g_i x : t > 0\}$ is uniformly distributed with respect to μ_i, then there exists $i_0 \geqslant 0$ such that $\operatorname{supp}\mu_i \subset g_i \operatorname{supp}\mu$ for every $i \geqslant i_0$.

(3) Let L be the subgroup generated by $g_i^{-1}U_i g_i (i \geqslant i_0)$, then μ is L-ergodic.

Let $\mathcal{Q}(G/\Gamma)$ be the set of probability measures μ on G/Γ that is ergodic with respect to the subgroup of $\Lambda(\mu)$ generated by one-parameter unipotent groups. Then Ratner's theorem shows that for every $\mu \in \mathcal{Q}(G/\Gamma)$, $\operatorname{supp}\mu$ is a closed $\Lambda(\mu)$-orbit. Moreover, by Moses-Shah's theorem, $\mathcal{Q}(G/\Gamma)$ is a weak* closed subset of $\operatorname{Prob}(G/\Gamma)$.

In conclusion, μ is homogeneous. Since S contains W, so S is not unipotent, therefore it must be semisimple. Then by Moore's ergodic theorem, μ is W-ergodic.

Step 3. *Complete the proof.*

Let $Y_i = \operatorname{supp}\mu_i$ and $Y = \operatorname{supp}\mu$. Fix $U_1, \cdots, U_k$ be one-parameter unipotent subgroups of W that generate W. Then μ_i is U_i-ergodic by Moore's ergodic theorem. By Birkhoff's ergodic theorem,

$$Y_i' := \{y \in Y_i : y \text{ is } \mu_i\text{-generic with respect to } U_i\}$$

is of full μ_i-measure in Y_i. Hence Y_i' is dense in Y_i.

Fix $y \in Y_i$, then we can find $y_i' \in Y_i'$ such that $y_i' \to y$. Choose $\{g_i\} \subset G$ such that $g_i \to \mathrm{id}_G$ and $g_i y_i' = y$. By Mozes-Shah's theorem, for every $1 \leqslant j \leqslant k$, there exists i_j such that $\operatorname{supp}\mu_i \subset g_i \operatorname{supp}\mu$ and μ is ergodic with respect to $g_i^{-1}U_j g_i$ for $i \geqslant i_j$. Taking $i_0 = \max_{1\leqslant j\leqslant k} i_j$, we obtain the conclusion. □

Proof of Proposition 8.2. **(3) $\Longrightarrow$ (1).** Fix $1 < m < n$ and let (μ_i) be a sequence of homogeneous, W_m-ergodic measures of proper support that converges to the Haar measure μ on G/Γ. Then

$\overline{\mu}_i := \pi_*\mu_i$ is supported on a closed totally geodesic subspace of X_Γ. Let Z_i be a maximal totally geodesic subspace of X_Γ containing $\operatorname{supp}\overline{\mu}_i$. Then $\bigcup_{i=1}^\infty Z_i$ is dense in X_Γ since $\mu_i \xrightarrow{w*} \mu$. It follows that $\{Z_i\}$ consists of infinitely many maximal totally geodesic subspaces.

(1) $\Longrightarrow$ (3). Let $\{Z_i\}$ be a sequence of distinct closed maximal totally geodesic subspaces of X_Γ. Without loss of generality, we assume that $\dim Z_i = m$ for some $1 < m < n$ for every i. By Lemma 8.3, each Z_i comes from the projection of a Y_i which is a homogeneous W_m-invariant ergodic subspace of G/Γ. Let μ_i be the homogeneous W_m-ergodic probability measure on Y_i with the stabilizer $S_i \supset W_m$. Without loss of generality, $\mu_i \to \mu$. Then μ is a homogeneous, W_m-ergodic probability measure on G/Γ. We now show that $\mu = \mu_{G/\Gamma}$ the Haar measure.

Assume that $\mu \neq \mu_{G/\Gamma}$, let $S = \Lambda(\mu) \supset W_m$ and $Y_\infty = \operatorname{supp}\mu = Sh\Gamma/\Gamma$ for some $h \in G$. By Theorem 8.4, there exists $(g_i) \subset G$ and $i_0 \geqslant 1$ such that g_iY_∞ is a homogeneous, W_m-invariant subspace of G/Γ that contains Y_i.

For every $i \geqslant i_0$, then $\pi(g_iY_\infty)$ is a closed geodesic subspace. Since $Y_\infty \neq G$, we have $\dim \pi(g_iY_\infty) < n$. Combining with $Z_i \subset \pi(g_iY_\infty)$ and Z_i is maximal, we conclude that $Z_i = \pi(g_iY_\infty)$. In particular $\dim \pi(g_iY_\infty) = m$.

Note that $\Lambda(g_i\mu) = g_iSg_i^{-1} \geqslant W_m$. By Lemma 8.3, the subgroup of $g_iSg_i^{-1}$ generated by unipotent elements is W_m and $W_m \leqslant g_iSg_i^{-1} \leqslant N_m$. Note that $g_iW_mg_i^{-1} \subset g_iSg_i^{-1}$, then both $g_iW_mg_i^{-1}$ and W_m are the subgroup of $g_iSg_i^{-1}$ generated by unipotent elements. Therefore $g_i \in N_m$.

Applying Lemma 8.3 to the closed $g_iSg_i^{-1}$-orbit g_iY_∞, then N_m-orbit $N_mg_ih\Gamma/\Gamma$ is also closed in G/Γ and $Z_i = \pi(N_mg_ih\Gamma/\Gamma)$. However, $g_i \in N_m$ and hence Z_i is independent with the choice of i. We get a contradiction. $\square$

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