Riemann Sufaces
(Spring 2022, Bohan Fang)

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§1.i. Riemann surfaces

Definition 1.1. A **Riemann surface** X is a connected one dimensional complex manifold.

Example 1.2 (Projective Line)
Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider

$$\mathbb{P} = (\mathbb{C}^2 \setminus \{(0,0)\})/\mathbb{C}^* : (z_1, z_2) \sim (\lambda z_1, \lambda z_2), \lambda \in \mathbb{C}^*.$$

Equip with a homogeneous coordinate $[z_0 : z_1] = [\lambda z_0 : \lambda z_1], \lambda \in \mathbb{C}^*$. Let $U_0 = \{[1 : z_1]\}$ and $U_1 = \{[z_0 : 1]\}$, then $\mathbb{P} = U_0 \cup U_1$.

Example 1.3 (Complex Tori)
Let $\omega_1, \omega_2 \in \mathbb{C}$ be two complex numbers which are linearly independent on $\mathbb{R}$. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, which is a subgroup of $\mathbb{C}$. Consider $X = \mathbb{C}/L$ as a quotient space. Then

$$\pi : \mathbb{C} \to X = \mathbb{C}/L$$

is a open map. It makes $\mathbb{C}/L$ a complex manifold.

Definition 1.4. Let $f = f(z, w)$ be a polynomial in two variables. Define $X = \{(z, w) : f(z, w) = 0\} \subset \mathbb{C}^2$ as an **affine plane curve**.

For a point $p = (z, w) \in \mathbb{C}^2$, we say X is non-singular at p if

$$\frac{\partial f}{\partial z} \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial w} \neq 0,$$

We say X is **smooth** if X is non-singular at every point.

Fact 1.5. f irreducible $\iff$ X connected.

Fact 1.6. If f is irreducible and non-singular, then X is a Riemann surface.

Definition 1.7. The **projective plane** is

$$\mathbb{P}^2 := (\mathbb{C}^3 \subset \{(0,0,0)\})/\mathbb{C}^*$$

equipped the homogeneous coordinate $[z_0 : z_1 : z_2] = [\lambda z_0 : \lambda z_1 : \lambda z_2], \forall \lambda \in \mathbb{C}^*$, that makes $\mathbb{P}^2$ as a 2 dimensional compact complex manifold. The local charts $U_i = \{[z_0, z_1, z_2] : z_i = 1\}$, for $i = 0, 1, 2$.

For every homogeneous polynomial F in degree d, that is

$$F(\lambda z_1, \lambda z_2, \lambda z_3) = \lambda^d F(z_1, z_2, z_3), \quad \forall \lambda \in \mathbb{C}.$$

We say $X = \{F = 0\} \subset \mathbb{P}^2$ a **projective plane curve**.

Definition 1.8. We say F **non-singular** if

$$\frac{\partial F}{\partial z_0} = \frac{\partial F}{\partial z_1} = \frac{\partial F}{\partial z_2} = F = 0$$

has no solutions.

Proposition 1.9 If F is non-singular, then X is a compact Riemann surface.

Proof. This proposition follows by the following lemma and fact. □

Lemma 1.10

Let F be a homogeneous polynomial. F is non-singular iff each $X_i = X \cap U_i$ as an affine plane curve is smooth.

Corollary 1.11

F is non-singular $\iff$ X is a smooth one dimensional complex manifold.

Fact 1.12. If F is homogeneous non-singular, then F is irreducible.

Complete intersection in $\mathbb{P}^n$. Let F be a homogeneous polynomial in $n+1$ variables. Then $\{F=0\}$ is a hypersurface. Now we consider $F_1, \cdots, F_{n-1}$ are $(n+1)$ variables homogeneous polynomials. Let $X = \bigcap \{F_i = 0\}$, which is called a **complete intersection**.

Definition 1.13. We call X a **smooth complete intersection** in $\mathbb{P}^n$ if $\left[\frac{\partial F_i}{\partial z_j}\right]$ is rank $(n-1)$ at every point in X.

Theorem 1.14

If X is a smooth complete intersection of $(n-1)$ polynomials, then X is a compact Riemann surface.

Local complete intersection. $X = \cap_\alpha \{F_\alpha = 0\} \subset \mathbb{P}^n$ where F_α's are homogeneous polynomials. Near each point $p \in X$, X is given by $(n-1)$ polynomials $\{F_{\alpha_i} = 0\}, i = 1, 2, \cdots, n-1$.

Fact 1.15. Any compact Riemann surface is a local complete intersection.

§1.ii. Functions

Definition 1.16. Define $\mathcal{O}_X(X)$ to be the holomorphic functions on a Riemann surface X. Let $W \subset X$ be an open subset, define $\mathcal{O}_X(W)$ to be the holomorphic functions on W.

Let f be a holomorphic function on a neighborhood of p. Then we can discuss that p is a (removable/pole/essential) singularity. We say f is meromorphic at p if p is a (removable/pole) singularity. We say f is **meromorphic** if f is meromorphic everywhere. Denote

$$\mathcal{M}_X(W) = \{f : W \to \mathbb{C} : f \text{ is meromorphic}\}.$$

For every $f \in \mathcal{M}_X(X)$ and $p \in X$, we can define the order $\operatorname{ord}_p(f)$ as the order of $f \circ \phi^{-1}$ at $\phi(p)$ for a local chart $\phi : U \ni p \to \mathbb{C}$.

Some properties of meromorphic functions:

- f has discrete zeros and poles.

- If $f = g$ on $S \subset W$ and S has limit points in W, then $f = g$ in W.
- If there exists $p \in W$ such that $|f(x)| \leqslant |f(p)|$ for every $x \in W$, then f is constant. In particular, $\mathcal{O}_X = \mathbb{C}$ for every compact Riemann surface X.

Example 1.17 (Meromorphic functions on projective line)

Let $X = \mathbb{P}^1$. For homogeneous polynomials $p(x,w), q(z,w)$, we consider $r = p/q$. Then r is a meromorphic function on $\mathbb{P}^1$ iff $\deg f = \deg g$. Let $[a_i : b_i]$ be all of poles and zeros, let $e_i = \operatorname{ord}_{[a_i:b_i]} f$. Consider

$$r = \Pi_i (b_i z - a_i w)^{e_i},$$

then $\sum e_i = 0$. Moreover, every $f \in \mathcal{M}_{\mathbb{P}^1}$ is of this form up to a constant.

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§2.i. Examples of meromorphic functions

Example 2.1 (Meromorphic functions on complex tori)

Let $X = \mathbb{C}/L$ be the complex torus, where $L = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, the upper half plane. There is no nontrivial holomorphic function on X.

Now we define a function on $\mathbb{C}$ as

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i (n^2\tau + 2nz)},$$

which converge locally uniformly. Hence $\theta(z)$ is a holomorphic function on $\mathbb{C}$. Then

$$\theta(z+1) = \theta(z), \quad \theta(z+\tau) = e^{-\pi i(\tau + 2z)}\theta(z).$$

Zeros of θ are $\frac{1}{2} + \frac{\tau}{2} + L$. Let x be a complex number, we consider

$$\theta^{(x)}(z) = \theta(z - \frac{1}{2} - \frac{\tau}{2} - x).$$

Then

$$\theta^{(x)}(z+1) = \theta(z), \quad \theta^{(x)}(z+\tau) = -e^{-2\pi i(z-x)}\theta^{(x)}(z).$$

Consider the ratio

$$R(z) = \frac{\prod_{i=1}^m \theta^{(x_i)}(z)}{\prod_{i=1}^n \theta^{(y_i)}(z)}.$$

We want $R(z)$ to be a meromorphic function on $\mathbb{C}/L$. Thus we need

$$m = n, \quad \text{and} \quad \sum_{i=1}^m x_i = \sum_{i=1}^n y_i + \mathbb{Z}.$$

Then zeros of R are $\{x_i\}$ and poles are $\{y_i\}$. In particular, the number of zeros equals to the number of poles. Moreover, every $f \in \mathcal{M}_{\mathbb{C}/L}$ is of this form up to a constant (see Example 3.3).

Example 2.2 (Meromorphic functions on smooth plane curves)

Let $X = \{f(x, y) = 0\}$ be a smooth plane curve in $\mathbb{C}^2$. Take two coprime polynomials g, h in $\mathbb{C}^2$. We want g/h to be a meromorphic function on X. Thus we need $h \not\equiv 0$ on X. By Hilbert Nullstellensatz, it is equivalent to $f \nmid h$.

Example 2.3 (Meromorphic functions on projective plane curves)

Let $X = \{F(x, y) = 0\}$ be a smooth plane curve in $\mathbb{P}^2$. Take two coprime homogeneous polynomials G, H in $\mathbb{C}^3$ with the same degree. We want G/H to be a meromorphic function on X. Thus we need $H \not\equiv 0$ on X. It implies that $F \nmid H$. We will show later that all of meromorphic functions on X is of the form G/H by the compactness of X.

Definition 2.4. Let X be a Riemann surface, we say X is a **smooth projective curve** if X can be holomorphically embedded in a projective space $\mathbb{P}^n$.

The following fact will be shown later.

Fact 2.5. All compact Riemann surfaces are smooth projective curves.

Example 2.6 A local complete intersection curve is a smooth projective curve.

§2.ii. Holomorphic maps

Let X, Y be two Riemann surfaces.

Definition 2.7. We say a map $F : X \to Y$ is **holomorphic at $p \in X$** if there are charts $\phi_1 : U_1 \ni p \to V_1$ on X and $\phi_2 : U_2 \ni F(p) \to V_2$ on Y such that $\phi_2 \circ F \circ \phi_1$ is holomorphic. We say F is a **holomorphic map** if F is holomorphic everywhere.

Let $F : X \to Y$ be a holomorphic map and W be an open set in Y. Then F induces

$$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}W).$$

For meromorphic functions, we should be a little bit careful. That is, if F is not a constant then

$$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}W).$$

Definition 2.8. We say X and Y are isomorphism if there exists a bijective holomorphic map $F : X \to Y$ such that $F^{-1} : Y \to X$ is isomorphism.

There are several basic properties of holomorphic maps.

- $F : X \to Y$ is a non-constant holomorphic map, then F is an open map.
- If $F : X \to Y$ is injective, then F is an isomorphism onto $F(X)$.
- If $\{x : F(x) = G(x)\}$ contains a limit point, then $F = G$.

Corollary 2.9

Let $F : X \to Y$ be a non-constant holomorphic map, then for every $y \in Y$, $F^{-1}(y)$ is discrete. In particular, if X is compact then $F^{-1}(y)$ is finite.

Proposition 2.10

Let X be a compact Riemann surface. $F : X \to Y$ is a non-constant holomorphic map. Then Y is compact and Y is onto.

Meromorphic functions. For every $f \in \mathcal{M}_X(X)$, we construct

$$F : X \to \mathbb{P}^1, \quad p \mapsto \begin{cases} [1 : f(p)], & p \text{ is not a pole;} \\ [0 : 1], & p \text{ is a pole.} \end{cases}$$

By the Lorentz series at a pole, we know that $F : X \to \mathbb{P}^1$ is indeed a holomorphic map:

$$\mathcal{M}_X(X) \xleftrightarrow{1-1} \left\{F : X \to \mathbb{P}^1 : \text{ holomorphic}\right\}.$$

Proposition 2.11 (Local normal form)

Let $F : X \to Y$ be a non-constant holomorphic map, let $p \in X$ with $F(p) = q$. Then there exists a unique positive integer m such that for every local chart $\phi_2 : U_2 \ni q \to V_2$, there exists a chart $\phi_1 : U_1 \ni p \to V_1$ such that

$$\phi_2 \circ F \circ \phi_1^{-1} : V_1 \to V_2, \quad z \mapsto z^m.$$

Definition 2.12. The unique integer m given above is called the **multiplicity** at p, denote it by $\operatorname{mult}_p F = m$.

For $p \in X$, take a local chart such that $z(p) = 0$. Locally, F is given by

$$z \mapsto c + \sum_{n \geqslant m} c_n z^n$$

where $c_n \neq 0$. Then $\operatorname{mult}_p F = m$. Or, if F is locally given by $z \mapsto h(z)$, then

$$\operatorname{mult}_p F = 1 + \operatorname{ord}_{z_0} \left(\frac{\mathrm{d}h}{\mathrm{d}z}\right)$$

where $z_0 = z(p)$.

Definition 2.13. Let $F : X \to Y$ be a non-constant holomorphic map. We say $p \in X$ is a **ramification point** for F if $\operatorname{mult}_p F \geqslant 2$. A point $y \in Y$ is a **branch point** if $F^{-1}(y)$ contains a ramification point.

Example 2.14

Let $X = \{f(x, y) = 0\}$ be a smooth affine plane curve. We consider the holomorphic map

$$\pi : X \to \mathbb{C}, \quad (x, y) \mapsto x.$$

Then π is ramified at $p \in X$ iff $(\partial f/\partial y)(p) = 0$.

Example 2.15

Let $X = \{F = 0\}$ be a smooth projective plane curve. We consider the holomorphic map

$$G : X \to \mathbb{P}^1, \quad [x : y : z] \mapsto [x : z].$$

Then G is ramified at $p \in X$ iff $(\partial F/\partial y)(p) = 0$.

Proposition 2.16 The set of ramification points is discrete.

Degree of a map. Let $F : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces. For every $y \in Y$, we define

$$d_y F = \sum_{x \in F^{-1}(y)} \operatorname{mult}_x F.$$

Proposition 2.17 $d_y F$ is a constant, independent of y.

Definition 2.18. This constant is called the **degree** of F, denoted $\deg F$.

Remark 2.19 — We supplement the definition for a constant function as 0.

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Proof of Proposition 2.17. It suffices to show that $\deg_y F$ is a locally constant function. Locally, f is given by $z \mapsto z^m$. Then for every $w \neq 0$, $\# f^{-1}(w) = m$. Then $\sum_{p \in f^{-1} w} \operatorname{mult}_p f$ is locally constant. Since $f^{-1} y$ is discrete for every $y \in Y$, by this normal form, we know that $\deg_y F$ is locally constant. □

Example 3.1

Assume that X is compact. Let $F : X \to Y$ be a holomorphic map with $\deg F = 1$. Then F is a bijection. Since every bijective holomorphic map has a holomorphic inverse, we know that F is isomorphism.

Example 3.2

Let f be a meromorphic function on a compact space X. Assume that f has only one pole. Then $f : X \to \mathbb{P}^1$ has degree 1 and hence $X \simeq \mathbb{P}^1$.

Let X be a compact Riemann surface, let $f \in \mathcal{M}_X(X)$. Regard f as a holomorphic map $f : X \to \mathbb{P}^1$. Let x_i's be zeros of f and y_i's be poles of f. Then

$$\begin{aligned} \deg f &= \sum_i \operatorname{mult}_{x_i} f = \sum_i \operatorname{ord}_{x_i} f \\ &= \sum_j \operatorname{mult}_{y_j} f = \sum_j -\operatorname{ord}_{y_j} f. \end{aligned}$$

Which implies that $\sum_i \operatorname{ord}_{x_i} f + \sum_j \operatorname{ord}_{y_j} f = 0$.

Example 3.3 (Meromorphic functions on complex tori)

Let $X = \mathbb{C}/L$ be a complex torus and f is a meromorphic function on X. Let $p_1, \cdots, p_n$ be zeros of f and $q_1, \cdots, q_n$ be poles of f. Note that X is also an abelian group, we want to show that $\sum_{i=0}^n p_i = \sum_{i=0}^n q_i$.

If not, we can take $p_0, q_0 \in \mathbb{C}$ such that $\sum p_i = \sum q_i$. By the construction in Example 2.1, we can choose $R \in \mathcal{M}_X(X)$ with zeros x_i and poles y_i. Then R/f is a meromorphic function on $\mathbb{C}/L$ with only one pole. It follows that $\mathbb{C}/L \simeq \mathbb{P}^1$, a contradiction.

Topology of a compact Riemann surface. The "topological invariant" for compact Riemann surfaces is genus g. Euler number $2-2g$.

Theorem 3.4 (Hurwitz Formula)

Let $F : X \to Y$ be a holomorphic map between compact Riemann surfaces. Then we have

$$2g(X) - 2 = (\deg F) \cdot (2g(Y) - 2) + \sum_{p \in X} (\operatorname{mult}_p F - 1).$$

§3.i. Examples of Riemann surfaces

Line. Any line in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

Conics. We consider

$$F(x, y, z) = [x, y, z] \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = v^t A_F v.$$

Then F is non-singular iff $\det A_F \neq 0$. Note that if $B = T^t A T$, then the projective curves $\{v^t A v = 0\}$ and $\{v^t B v = 0\}$ are isomorphic. But for every complex symmetric matrix A, there exists T such that $A = T^t T$ where $\operatorname{rank} T = \operatorname{rank} A$. In particular, every conic is isomorphic to

$$\{x^2 + y^2 + z^2 = 0\}.$$

Now we consider a particular conic given by $C = \{xz = y^2\}$, which is a smooth curve. Then every point on C can be written as $[r^2 : rs : s^2]$. We consider the map

$$C \to \mathbb{P}^1, \quad [r^2 : rs : s^2] \mapsto [r : s],$$

which gives an isomorphism between C and $\mathbb{P}^1$. Hence every smooth conic is isomorphic to $\mathbb{P}^1$.

In general, we can consider non-smooth conics. If $\operatorname{rank} A = 1$, then the conic is a double line. If $\operatorname{rank} A = 2$, then the conic is two intersecting lines.

Hyperelliptic curves. First, we need some preparation. For two Riemann surfaces X, Y, let $U \subset X$ and $V \subset Y$ be two open sets. Let $\phi : U \to V$ be an isomorphism. Then we can define the space $X \coprod_\phi Y$ by gluing up U, V via ϕ.

Proposition 3.5

Let X, Y be two Riemann surfaces, if $X \coprod_\phi Y$ is Hausdorff then it is a Riemann surface.

Now we consider a polynomial $h(x)$ with $\deg h = 2g+1+\epsilon$ where $\epsilon = 0, 1$. Let $X = \left\{y^2 = h(x)\right\}$, which is a smooth plane curve. We consider $U = \{x \neq 0\} \cap X \subset X$. We also take $k(z) = z^{2g+2}h(1/z)$ and $Y = \left\{w^2 = k(z)\right\}$. Let $V = \{z \neq 0\} \cap Y \subset Y$ and

$$\phi : U \to V, \quad (x, y) \mapsto (z, w) = \left(\frac{1}{x}, \frac{y}{x^{g+1}}\right),$$

which is an isomorphism. Then $Z = X \coprod_\phi Y$ is Hausdorff and compact since $Z = \{|x| \leqslant 1\} \cup \{|z| \leqslant 1\}$. Which implies that Z is a compact Riemann surface. The function x on X extends to a holomorphic map $\pi : Z \to \mathbb{P}^1$. Then $\deg \pi = 2$.

The branch point of π is at 0 or ∞. If $\epsilon = 0$, it gives $2g+1$ ramification points at $\{h = 0\}$ and one ramification point at ∞. If $\epsilon = 1$, then there are $2g+2$ ramification points at $\{h = 0\}$. By Hurwitz formula,

$$2g(Z) - 2 = 2(g(\mathbb{P}^1) - 2) + (2g+2).$$

Hence $g(Z) = g$.

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§4.i. Examples of Riemann surfaces

Let us recall the smooth plane curve $X = \left\{y^2 = h(x)\right\} \subset \mathbb{C}^2$ with $\deg h = 2g+1$ or $2g+2$. The space Z we constructed above is a compact Riemann surface. A Riemann surface constructed in this way is called a **hyperelliptic Riemann surface**.

Now we consider an involution map $\sigma : Z \to Z$ given by $\sigma(x, y) = (x, -y)$. It is called the **hyperelliptic involution**. For every $f \in \mathcal{M}_Z(Z)$, we consider the pullback $\sigma^* f = f \circ \sigma$.

Definition 4.1. A function $f \in \mathcal{M}_Z(Z)$ is called an involution invariant function if $\sigma^* f = f$.

Proposition 4.2

Every involution invariant function on Z is of the form $f = \pi^* r$ where $\pi : Z \to \mathbb{P}^1$ is the projection defined above and $r \in \mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1)$.

For general meromorphic functions $f \in \mathcal{M}_Z(Z)$, we can separate f into $f = f^+ + f^-$ where

$$f^+ = \frac{1}{2}(f + \sigma^* f), \quad f^- = \frac{1}{2}(f - \sigma^* f).$$

Then f can be written as $f = r + ys$ since y is anti σ-invariant.

Maps between complex tori. We consider a holomorphic map $F : \mathbb{C}/L \to \mathbb{C}/M$ where L, M are rank 2 lattices in $\mathbb{C}$. After a translation if necessary, we can assume that $F(0) = 0$. By Hurwitz formula, F is unramified. If F is not a constant, then F is a covering map. Let $G : \mathbb{C} \to \mathbb{C}$ be the corresponding map on the universal cover, we also assume that $G(0) = 0$. Then

$$G(z + l) \equiv G(z) \bmod M, \quad \forall z \in \mathbb{C}, l \in L.$$

We consider $\omega(z, l) = G(z + l) - G(z)$, which is constant with respect to z. We abbreviate it into $\omega(l)$. If $\omega(l) = 0$ then G is periodic and hence constant. Moreover, we can show that every G is of the form γz for some $\gamma \in \mathbb{C}$ such that $\gamma L \subset M$.

Proposition 4.3

Any holomorphic map $F : \mathbb{C}/L \to \mathbb{C}/M$ can be lift to $G = \gamma z + a$. The degree of F equals to $[M : \gamma L]$.

Now we want to determine the automorphisms on $X = \mathbb{C}/L$. Write $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\operatorname{Im}\tau > 0$. If $F : X \to X$ is an isomorphism, then $\deg F = 1$. Hence $\gamma L = L$. Note that $\|\gamma\|$ is forced to be 1, and we can only consider the case that $\gamma \notin \mathbb{R}$. Take $l \neq 0 \in L$ with the minimal length, then $\{l, \gamma l\}$ is a basis of L. We consider $G^2(l) = \gamma^2 l$, which can be written as $\gamma^2 l = m\gamma l + nl$ for some $m, n \in \mathbb{Z}$. Hence γ is a root of $z^2 - mz - n = 0$. Combining with $\|\gamma\| = 1$, γ can only 4-th or 6-th roots of unity. Then there are only three cases:

(1) L is square, then $\operatorname{Aut}(\mathbb{C}/L) = \mathbb{Z}/4\mathbb{Z}$.
(2) L is hexagonal, then $\operatorname{Aut}(\mathbb{C}/L) = \mathbb{Z}/6\mathbb{Z}$.
(3) $\operatorname{Aut}(\mathbb{C}/L) = \{\pm \operatorname{id}\}$.

In general, let $\mathbb{C}/L$ and $\mathbb{C}/L'$ be two complex tori with $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ and $L' = \mathbb{Z} \oplus \tau'\mathbb{Z}$. They are isomorphism iff there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ such that $(a\tau + b)/(c\tau + d) = \tau'$.

Plugging holes in Riemann surfaces.

Definition 4.4. X a Riemann surface. A **hole chart** on X is a chart $\phi : U \subset X \to V$ such that

(i) V contains an open punctured disc $D_0 = \{z : 0 < \|z - z_0\| < \varepsilon_0\}$.
(ii) $\overline{\phi^{-1}(D_0)} \subset U$ and $\phi(\overline{\phi^{-1}(D_0)}) = \{z : 0 < \|z - z_0\| \leqslant \varepsilon_0\}$.

Example 4.5

1. $\mathbb{C} \setminus \{0\}$ has a hole chart near 0.
2. $\mathbb{C}$ has a a hole chart near ∞.
3. $D = \{z : \|z\| < 1\}$ has no hole chart near $\|z\| = 1$.

If X has a hole chart, then we can construct $\widehat{X} = X \sqcup \{\mathrm{pt}\}$ such that $\widehat{U} = U \sqcup \{\mathrm{pt}\}$ is open and has a corresponding chart $\phi : \widehat{U} \to V \cup \{z_0\}$. This is the operation of plugging a hole.

Example 4.6 The projective line $\mathbb{P}^1$ can be obtained by plugging the hole ∞ on $\mathbb{C}$.

Nodes of a plane curve. Let X be an affine plane curve given by $f(z, w) = 0$. A point p is called a **node** if $\partial f/\partial z(p) = \partial f/\partial w(p) = 0$ but the Hessian is nonsingular at p.

Example 4.7 $f = (z - z_0)(w - w_0)$.

If X has a node $p = (z_0, w_0)$. Then we can write f as

$$f(z, w) = l_1(z - z_0, w - w_0)l_2(z - z_0, w - w_0) + \text{ higher order terms}$$

where l_i are distinct linear homogeneous polynomials. Then we can write $f = gh$ locally. In particular, $f = 0$ iff $g = 0$ or $h = 0$. Then we can separate X into $X_g = \{g = 0\}$ and $X_h = \{h = 0\}$ near p. We can delate p in both X_g and X_h. Then we plugging two points in X_g, X_h respectively. This is a process that we resolve a node. Such process can also be preformed for a projective plane curve.

Proposition 4.8

Let $F(x, y, z)$ be an irreducible homogeneous polynomial. Let $X = \{F = 0\} \subset \mathbb{P}^2$. Assume that F has only finitely many singularities and all of them are nodes. Then the Riemann surface obtained by resolving nodes of X is a compact Riemann surface.

Genus of projective plane curves. Let X be a non-singular projective plane curve with degree d. We will show that the genus of X equals to $(d-1)(d-2)/2$, which is known as Plücker's formula.

Example 4.9 The Fermat curve $X = \left\{x^d + y^d + z^d = 0\right\}$ has genus $(d-1)(d-2)/2$.

More general, for a nodal projective projective curve, we have a formula for the genus of resolved curve.

Theorem 4.10 (Plücker's formula)

Let X be a projective plane curve of degree d with n nodes and no other singularities. Then

$$g(X) = \frac{(d-1)(d-2)}{2} - n.$$

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§5.i. Forms

Let X be a Riemann surface. A **holomorphic form** ω on X is a collection of $\{\omega_i = f_i \mathrm{d}z_i\}$ on an atlas $\{U_i\}$ where f_i are holomorphic such that they agree on $V_{i,j} = U_i \cap U_j$. Similarly, a **meromorphic form** is a collection of $\{\omega_i = f_i \mathrm{d}z_i\}$ where f_i are meromorphic.

Let ω be a meromorphic form on X. Let $p \in X$ and $p \in U \subset X$. Assume that $\omega = f(z)\mathrm{d}z$ on U. We define the **order** of ω at p as $\operatorname{ord}_p \omega := \operatorname{ord}_p f$. This definition is independent with the choice of coordinate charts.

Differential forms. Regarding $\mathbb{C}$ as a real manifold, we write $z = x + \sqrt{-1}y$. Then $\mathrm{d}z = \mathrm{d}x + \sqrt{-1}\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - \sqrt{-1}\mathrm{d}y$, both of them lie in $T^*\mathbb{C}$. Now we consider the tangent bundle $T\mathbb{C}$ with basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. Then the dual basis of $\mathrm{d}z, \mathrm{d}\bar{z}$ in $T\mathbb{C}$ is given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right).$$

Let $f \in C^\infty(\mathbb{C})$ be a complex valued function, then f is holomorphic iff $\frac{\partial}{\partial \bar{z}} f = 0$ [Cauchy-Riemann function]. Similarly, we consider a C^∞ 1-form ω on $\mathbb{C}$, that is, ω is a C^∞ section of $T^*\mathbb{C} \otimes \mathbb{C}$. Locally, we write

$$\omega = f(z, \bar{z})\mathrm{d}z + g(z, \bar{z})\mathrm{d}\bar{z}.$$

Let $z = T(w)$ be a change of coordinate, we have

$$\omega = f(T(w), \overline{T(w)})T'(w)\mathrm{d}w + g(T(w), \overline{T(w)})\overline{T'(w)}\mathrm{d}w.$$

We say an element in $\langle \mathrm{d}z \rangle$ a **$(1,0)$-form** and an element in $\langle \mathrm{d}\bar{z} \rangle$ a **$(0,1)$-form**. Then

$$T^*X \otimes \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X.$$

Now we consider the 2-form on $\mathbb{C}$, i.e., a section in $\bigwedge^2(T^*\mathbb{C})\otimes\mathbb{C}$. Since $\bigwedge^2(T^*\mathbb{C})\otimes\mathbb{C}$ is spanned by $\mathrm{d}z\wedge\mathrm{d}\bar{z}$, we also call them $(1,1)$-form. The transition formula is given by

$$\mathrm{d}z\wedge\mathrm{d}\bar{z} = \left\|T'(w)\right\|^2 \mathrm{d}w\wedge\mathrm{d}\bar{w}$$

where $z = T(w)$.

Operations. Let f be a C^∞ function. We define differentiations

$$\partial f = \frac{\partial}{\partial z} f\mathrm{d}z, \quad \bar{\partial} f = \frac{\partial}{\partial \bar{z}} f\mathrm{d}\bar{z}.$$

Then they are $(1,0)$-form and $(0,1)$-form respectively. We define the 1-form $\mathrm{d}f := \partial f + \bar{\partial} f$.

For a 1-form $\omega = f\mathrm{d}z + g\mathrm{d}\bar{z}$, we define

$$\mathrm{d}\omega = \mathrm{d}f\mathrm{d}z + \mathrm{d}g\mathrm{d}\bar{z} = \left(\frac{\partial}{\partial z}g - \frac{\partial}{\partial \bar{z}}f\right)\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

Note that $\partial\partial = 0$, $\bar{\partial}\bar{\partial} = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$, we have $d^2 = 0$.

Definition 5.1. A C^∞ function f is called **harmonic** if $\partial\bar{\partial} f = 0$. A C^∞ 1-form ω is called d-closed (resp. ∂-closed, $\bar{\partial}$-closed) if $\mathrm{d}\omega = 0$ (resp. $\partial\omega = 0$, $\bar{\partial}\omega = 0$).

Then a $(1,0)$-form ω is holomorphic iff $\mathrm{d}\omega = \bar{\partial}\omega = 0$.

Pull back. Let $F : X \to Y$ be a holomorphic map. Locally it is given by $z \mapsto w(z)$. For a 1-form $\omega = f\mathrm{d}w + g\mathrm{d}\bar{w}$ on Y. We define the pull back

$$F^*\omega = f(w(z), \overline{w(z)})w'(z)\mathrm{d}z + g(w(z), \overline{w(z)})\overline{w'(z)}\mathrm{d}\bar{z}$$

on X, which is 1-form. Note that F^* commutes with all differentiations. F^* also preserves holomorphicity and the type of forms.

Let $F : X \to Y$ be a holomorphic map and ω be a meromorphic form on Y. For $p \in X$, we have

$$\operatorname{ord}_p(F^*\omega) = (1 + \operatorname{ord}_p(\omega)) \operatorname{mult}_p F - 1.$$

Notation. We use the following notation in later discussion.

$$\mathcal{E}^{\square}(U) = \{C^\infty\ \square\text{-forms on } U\},$$

where $\square = (0), (1), (1,0), (0,1), (2)$, where C^∞ 0-form is the C^∞ function.

$$\mathcal{O}(U) = \{\text{holomorphic functions } f : U \to \mathbb{C}\}, \quad \Omega^1(U) = \{\text{holomorphic forms on } U\}.$$

$$\mathcal{M}(U) = \{\text{meromorphic functions } f : U \to \mathbb{C}\}, \quad \mathcal{M}^{(1)}(U) = \{\text{meromorphic forms on } U\}.$$

Proposition 5.2 (Poincaré's Lemma)

Let ω be a 1-form with $\mathrm{d}\omega = 0$ on an open set U. Let $p \in U$. Then there exists an open neighborhood $V \ni p$ and a C^∞ function f on V such that $\omega = \mathrm{d}f$ on V.

Proposition 5.3 (Dolbeault's Lemma)

Let ω be a C^∞ $(0,1)$-form on an open set U. Let $p \in U$. Then there exists an open neighborhood $V \ni p$ and a C^∞ function f on V such that $\omega = \bar{\partial} f$ on V.

§5.ii. Integral

Let $\omega \in \mathcal{E}^1(X)$ and $\gamma : [a,b] \to X$ be a path. The integral is defined as

$$\int_\gamma \omega = \sum_i \int_{a_i}^{b_i} (f_i(z(t), \overline{z(t)})z'(t) + g_i(z(t), \overline{z(t)})\overline{z'(t)})$$

where $\omega = f_i \mathrm{d}z + g_i \mathrm{d}\bar{z}$ is the local representation.

Now we consider a meromorphic form ω, let p be a pole and γ be a small path enclosing p and no other poles. We define the **Residue** as

$$\operatorname{Res}_p \omega = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \omega.$$

Lemma 5.4 Let $f \in \mathcal{M}(X)$, then

$$\operatorname{Res}_p \frac{\mathrm{d}f}{f} = \operatorname{ord}_p f.$$

We can also define the integral for 2-forms similarly.

Theorem 5.5 (Stoke's Theorem)
Let D be a triangulable closed set on X and $\omega \in \mathcal{E}^1(X)$, then

$$\int_{\partial D} \omega = \int_D \mathrm{d}\omega.$$

Theorem 5.6 (The Residue Theorem)
Let ω be a meromorphic 1-form on a compact Riemann surface. Then

$$\sum_{p \in X} \operatorname{Res}_p \omega = 0.$$

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§6.i. Divisors

Let X be a Riemann surface. Let $\mathbb{Z}^X$ be the set of all functions $X \to \mathbb{Z}$. For every $D \in \mathbb{Z}^X$, define the **support** $\operatorname{supp} D = \{x \in X : D(x) \neq 0\}$.

Definition 6.1. We say $D \in X^{\mathbb{Z}}$ a **divisor** if $\operatorname{supp} D$ is discrete.

For a divisor D, we write

$$D = \sum_{p \in \operatorname{supp} D} D(p)p,$$

where $D(p) \in \mathbb{Z}$. In the case X is compact, we define

$$\deg D = \sum_{p \in X} D(p),$$

which is finite. Then

$$\mathrm{Div}(X) = \{\text{divisors on } X\} \subset \mathrm{Div}_0(X) = \{\text{divisors on } X \text{ with } \deg = 0\}.$$

Divisors of meromorphic functions. For every $f \in \mathcal{M}(X)$, we define

$$\mathrm{div}(f) = \sum_p \mathrm{ord}_p(f) p \in \mathrm{Div}(X).$$

If X is compact, then $\mathrm{div}(f) \in \mathrm{Div}_0(X)$.

Definition 6.2. We define the family of **principle divisors**

$$\mathrm{PDiv}(X) = \{\mathrm{div}(f) : f \in \mathcal{M}(X)\}.$$

Example 6.3 If $X = \mathbb{P}^1$ then $\mathrm{PDiv}(X) = \mathrm{Div}_0(X)$.

For $f \in \mathcal{M}(X)$, we denote

$$\mathrm{div}_0(f) = \sum_{\mathrm{ord}_p(f)>0} \mathrm{ord}_p(f) p, \quad \mathrm{div}_\infty(f) = -\sum_{\mathrm{ord}_p(f)<0} \mathrm{ord}_p(f) p.$$

Then $\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$.

Divisors of meromorphic forms. For every $\omega \in \mathcal{M}^1(X)$, we define

$$\mathrm{div}(\omega) = \sum_p \mathrm{ord}_p(\omega) p \in \mathrm{Div}(X).$$

But the degree is not necessary zero. For example, we consider the $\omega \in \mathcal{M}^1(\mathbb{P}^1)$ as $\omega = \mathrm{d}z$. Then $\omega = -w^{-2}\mathrm{d}w$ near ∞. Hence $\mathrm{div}(\omega) = -2 \cdot \infty$ and $\deg \omega = -2$.

Similarly, we define

$$\mathrm{KDiv}(X) = \left\{\mathrm{div}(\omega) : \omega \in \mathcal{M}^1(X)\right\}.$$

Note that for every given $\omega_0, \omega_1 \in \mathcal{M}^1(X)$, the quotient $\omega_1/\omega_0 \in \mathcal{M}(X)$. Hence

$$\mathrm{KDiv}(X) = \mathrm{div}(\omega) + \mathrm{PDiv}(X)$$

for some $\omega \in \mathcal{M}^1(X)$.

Definition 6.4. A divisor in $\mathrm{KDiv}(X)$ is called a **canonical divisor**.

Let X be a compact Riemann surface. Let $f : X \to \mathbb{P}^1$ be a non-constant holomorphic map. Let $\omega = \mathrm{d}z \in \mathcal{M}^1(\mathbb{P}^1)$, then $\deg \omega = 2$. Assume that $\deg f = d$, then by Hurwitz formula

$$2g - 2 = d(-2) + \sum_p (\mathrm{mult}_p f - 1).$$

Recall that $\mathrm{ord}_p f^*\omega = (1 + \mathrm{ord}_{f(p)} \omega) \mathrm{mult}_p f - 1$. Let $\eta = f^*\omega$, we have

$$\begin{aligned}
\deg \eta &= \sum_p ((1 + \mathrm{ord}_{f(p)} \omega) \mathrm{mult}_p f - 1) \\
&= \sum_p (\mathrm{mult}_p f - 1) - 2 \sum_{p \in f^{-1}(\infty)} \mathrm{mult}_p f = 2g - 2.
\end{aligned}$$

Proposition 6.5

Let X be a compact Riemann surface, then for every $\omega \in \mathrm{KDiv}(X)$, $\deg \omega = 2g - 2$.

Divisors of holomorphic maps. Let $F : X \to Y$ be a non-constant holomorphic map.

Definition 6.6. For every $q \in Y$, we define the **inverse image divisor** of q as

$$F^*q := \sum_{p \in F^{-1}q} (\operatorname{mult}_p F)p.$$

More general, for a divisor $D \in \operatorname{Div}(Y)$, we can define the pull back F^*D to be a divisor in $\operatorname{Div}(X)$. Then

$$F^* : \operatorname{Div}(Y) \to \operatorname{Div}(X), \quad \operatorname{PDiv}(X) \to \operatorname{PDiv}(Y).$$

And $\deg F^*D = \deg F \cdot \deg D$.

Definition 6.7. The **ramification divisor** of F is a divisor on X as

$$R_F := \sum_{p \in X} (\operatorname{mult}_p F - 1)p.$$

The **branch divisor** of F is a divisor on Y as

$$B_F := \sum_{q \in Y} \left(\sum_{p \in F^{-1}(q)} (\operatorname{mult}_p F - 1) \right) q.$$

Then we have

$$\operatorname{div}(F^*\omega) = F^*(\operatorname{div}(\omega)) + R_F.$$

Combining with $\deg \omega = 2g - 2$, this gives a more precise version of Hurwitz formula.

Intersection divisors. Let X be a smooth projective curve, that is, $X \hookrightarrow \mathbb{P}^n$ for some n. Let G be a homogeneous polynomial with $G \not\equiv 0$ on X. We want to define a corresponding divisor of G. For every p with $G(p) \neq 0$, we need $\operatorname{div}(G)(p) = 0$. For every p with $G(p) = 0$, we choose a homogeneous polynomial H with $\deg G = \deg H$ and $H(p) \neq 0$. Then G/H is a meromorphic function on X. Then we define

$$\operatorname{div}(G)(p) := \operatorname{ord}_p(G/H).$$

This is well-defined.

Definition 6.8. The divisor $\operatorname{div}(G)$ is called the **intersection divisor** of G.

Note that for every G_1, G_2 with the same degree, we have

$$\operatorname{div}(G_1) - \operatorname{div}(G_2) = \operatorname{div}(G_1/G_2) \in \operatorname{PDiv}(X).$$

Partial ordering on divisors. For two divisors, we define $D_1 \geqslant D_2$ if $D_1(p) \geqslant D_2(p)$ for every $p \in X$. For a meromorphic function f, f is holomorphic iff $\operatorname{div}(f) \geqslant 0$.

§6.ii. Linear equivalence of divisors

For $D_1, D_2 \in \operatorname{Div}(X)$, we define $D_1 \sim D_2$ if $D_1 - D_2 \in \operatorname{PDiv}(X)$.

Example 6.9

1. On $\mathbb{P}^1$, $D_1 \sim D_2$ iff $\deg D_1 = \deg D_2$.
2. On complex torus $X = \mathbb{C}/L$, then $D = \sum n_i \cdot p_i \in \operatorname{PDiv}(X)$ iff $\sum n_i p_i = 0$ (regarding X as a abelian group).

Theorem 6.10 (Abel's Theorem)

Let X be complex torus, then $D \in \mathrm{PDiv}(X)$ if and only if $\deg D = 0$ and $A(D) = 0$, where $A : \mathrm{Div}(X) \to X$ is the **Abel-Jacobi** map given by $D = \sum n_i \cdot p_i \mapsto \sum n_i p_i$.

Degree of smooth projective curve. Let X be a smooth projective curve. We define the **degree of X** as

$$\deg X := \deg \operatorname{div} H$$

for any linear homogeneous polynomial H with $H|_X \not\equiv 0$. Such $\operatorname{div} H$ is called **hyperplane divisor**. For every homogeneous polynomials G_1, G_2 with $\deg G_1 = \deg G_2$, we have $\operatorname{div}(G_1) \sim \operatorname{div}(G_2)$.

Proposition 6.11

Let X be a smooth projective plane curve given by $F(x, y, z) = 0$. Then $\deg X = \deg F$.

Proof. Assume that $G = x$ and $[0 : 0 : 1] \notin X$. Then $h = x/y$ is a meromorphic function and $\operatorname{div}(G) = \operatorname{div}_0(h)$. Indeed, $\operatorname{div}_0(h) = \deg F$ since it has d solutions. □

Theorem 6.12 (Bezout's Theorem)

Let X be a smooth projective curve. Let G be a homogeneous polynomial with $G|_X \not\equiv 0$. Then

$$\deg \operatorname{div}(G) = \deg(G) \deg(X).$$

Plücker's formula. Let X be a smooth projective plane curve given by $F(x, y, z) = 0$. Assume that $[0, 1, 0] \notin X$ and let $\pi : X \to \mathbb{P}^1$, $[x, y, z] \mapsto [x : z]$.

Proposition 6.13 $\operatorname{div}(\partial F/\partial y) = R_\pi$.

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Let $F = F(x, y, z)$ be a homogeneous polynomial with degree d, then $\deg \operatorname{div}(\partial F/\partial y) = d(d-1)$ by Bezout's theorem. Note that the degree of π equals d, combining with Hurwitz formula, we get

Theorem 7.1 (Plücker's formula)

Let X be a smooth projective plane curve of degree d. Then

$$g(X) = \frac{(d-1)(d-2)}{2}.$$

§7.i. Space associated to a divisor

First we define $\operatorname{div} 0 = +\infty$. For every divisor D, we define the space

$$L(D) := \{f \in \mathcal{M}(X) : \operatorname{div} f + D \geqslant 0\},$$

which is a complex vector space. Note that $L(0) = \mathcal{O}(X)$ and $L(D_1) \subset L(D_2)$ provided $D_1 \leqslant D_2$. If X is compact, then $L(D) = \{0\}$ for every D with negative degree.

Linear system. Let D be a divisor and we define the **complete linear system**

$$|D| := \{E \in \mathrm{Div}(X) : E \sim D, E \geqslant 0\}.$$

Note that
- If X is compact and $\deg D < 0$, then $|D|$ is empty.
- If $f \in L(D)$, then $E = \mathrm{div}(f) + D \geqslant 0$ hence $E \in |D|$.

Lemma 7.2

If X is compact, then $S : \mathbb{P}(L(D)) \to |D|$ given by $f \mapsto \mathrm{div}(f) + D$ is a 1-1 correspondence.

A general **linear system** is a subset of a complete linear system corresponding to a linear subspace of $L(D)$.

Linear equivalent divisors. If $D_1 \sim D_2$, then there exists $h \in \mathcal{M}(X)$ such that $D_1 = D_2 + \mathrm{div}(h)$. Then $L(D_1) \simeq L(D_2)$ given by $f \mapsto hf$.

The space $L^{(1)}(D)$. Now we define the space

$$L^{(1)}(D) := \left\{\omega \in \mathcal{M}^{(1)}(X) : \mathrm{div}\,\omega + D \geqslant 0\right\}.$$

If K is a canonical divisor, that is $K = \mathrm{div}\,\omega$ for some $\omega \in \mathcal{M}^1(X)$. Then

$$\mu_\omega : L(D+K) \to L^{(1)}(D), \quad f \mapsto f\omega$$

gives an isomorphism.

Example 7.3 (On projective line)

We consider divisors on the projective line $\mathbb{P}^1$. Let $D = \sum_{i=1}^n e_i\lambda_i + e_\infty \cdot \infty$. We consider

$$f_D = \prod_{i=1}^n (z - \lambda_i)^{-e_i}.$$

Claim 7.4. $L(D) = \{g(z)f_D(z) : g \text{ is a polynomial with degree at most } \deg D\}$.

Example 7.5 (On complex tori)

Let D be a divisor on a complex torus $X = \mathbb{C}/L$. We have

(1) If $\deg D < 0$, then $L(D) = \{0\}$.
(2) If $\deg D = 0$ and $D \sim 0$, then $L(D) \cong \mathbb{C}$.
(3) If $\deg D = 0$ and $D \not\sim 0$, then $L(D) = \{0\}$.
(4) If $\deg D > 0$, then $\dim L(D) = \deg D$. This can be shown by induction on $\deg D$.

In general, we have

Lemma 7.6

Let X be a compact Riemann surface and D be a divisor, $p \in X$ be a point. Then either $L(D-p) = L(D)$ or $L(D-p)$ has codimension one in $L(D)$.

Corollary 7.7

Let X be a compact Riemann surface and D be a divisor, then both $L(D)$ and $L^{(1)}(D)$ are of finite dimensional.

§8. Mar 27

§8.i. Maps to the projective space

Definition 8.1. Let X be a Riemann surface. We call a map $\phi : X \to \mathbb{P}^n$ is **holomorphic at $p \in X$** if there are holomorphic functions $g_0, \cdots g_n$ defined near p, not all zero at p, such that $\phi(x) = [g_0(x) : \cdots : g_n(x)]$ near p.

Let X be a Riemann surface and $f = (f_0, \cdots f_n)$ where $f_i \in \mathcal{M}(X)$. We define

$$\phi_f(p) := [f_0(p) : f_1(p) : \cdots : f_n(p)] \in \mathbb{P}^n.$$

A priori, ϕ_f is defined at p if

- p is not a zero of every f_i, and
- p is not a pole of any f_i.

Moreover, ϕ_f is holomorphic at such p's. In fact, ϕ_f can be extended to all points, in such a way that ϕ_f is holomorphic. For $p \in X$, let $m = \min \mathrm{ord}_p f_i$. Then functions $z^m f_i$ satisfy above two conditions at p. We define

$$\phi_f := [z^{-m} f_0 : \cdots : z^{-m} f_n]$$

near p. This definition corresponds to the original definition at other points by the homogeneous property of $\mathbb{P}^n$.

Remark 8.2 — If $\phi : X \to \mathbb{P}^n$ is a holomorphic map, then $\phi = \phi_f$ for some f.

4Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map given by $\phi = [f_0 : \cdots : f_n]$ where $f_i \in \mathcal{M}(X)$. Let

$$D = -\min_i \mathrm{div}(f_i).$$

Then $-D \leqslant \mathrm{div}(f_i)$ for each i and hence $f_i \in L(D)$. Let

$$V_f := \left\{ \sum_i a_i f_i : a_i \in \mathbb{C} \right\}$$

which is a subspace of $L(D)$. We define the **linear system of ϕ** as

$$|\phi| := \{\mathrm{div}(g) + D : g \in V_f\} \subset |D|.$$

This definition is independent with the choice of $f_0, \cdots, f_n$. In fact, if $\phi = [g_0, \cdots, g_n]$, then there exists $\lambda \in \mathcal{M}(X)$ such that $g_i = \lambda f_i$.

Fact 8.3. For every $p \in X$, there exists $E \in |\phi|$ such that $p \notin \operatorname{supp} E$.

A linear system with dimension n whose divisors all have degree d is called a g_d^n.

> **Question 8.4.** Which g_d^n's can be the linear systems of a holomorphic map?

Definition 8.5. Let Q be a linear system, a point p is called a **base point** of Q if $E \geqslant p$ for every $E \in Q$. We say Q is **base-point-free (or free)** if it has no base points.

In particular, $|\phi|$ is free.

Let $Q \subset |D|$ be a linear system and $V \subset L(D)$ be the vector space corresponds to Q. If p is a base point of Q, then for every $f \in V$, we have $f \in L(D-p)$.

Lemma 8.6

A point $p \in X$ is a base point of $Q \subset |D|$ iff $V \subset L(D-p)$ where V is the vector space corresponding to Q. In particular, p is a base point of $|D|$ iff $L(D) = L(D-p)$.

Proposition 8.7

Let X be a compact Riemann surface. Then $p \in X$ is a base point of D iff $\dim L(D) = \dim L(D-p)$.

Example 8.8

If X is a complex torus, then $\dim L(D) = \deg D$ if $\deg D \geqslant 1$. Then $L(D)$ is base-point-free if $\deg D \geqslant 2$.

The hyperplane divisor of a holomorphic map to $\mathbb{P}^n$. Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map where X is a compact Riemann surface. Let H be a hyperplane in $\mathbb{P}^n$ given by $\{L = 0\}$ with $\deg L = 1$.

For every $p \in X$, let M be a linear homogeneous function with $M(p) \neq 0$. Let $h = (L/M) \circ \phi$ which is a meromorphic function on X. We define the divisor $\phi^*(H)$ as $\phi^*(H)(p) = \operatorname{ord}_p h$. This definition is independent with the choice of M. Such divisor is called the **hyperplane divisor** for the map ϕ.

Proposition 8.9

If $\phi = [f_0 : f_1 : \cdots : f_n]$ and $H = \{\sum_i a_i x_i = 0\}$, then

$$\phi^*(H) = \operatorname{div}\left(\sum_i a_i f_i\right) - \min_i \{\operatorname{div} f_i\}.$$

Corollary 8.10 $\{\phi^*(H) : H \text{ is a hyperplane}\} = |\phi|$.

Defining a holomorphic map via a linear system.

Proposition 8.11

Let $Q \subset |D|$ be a base-point-free linear system of (projective) dimension n on a compact Riemann surface. Then there exists a holomorphic map $\phi : X \to \mathbb{P}^n$ such that $Q = |\phi|$. Moreover, such ϕ is unique up to the choice of coordinates in $\mathbb{P}^n$.

Let $|D|$ be a complete linear system, which may has base points. Let $F = \min\{E : E \in |D|\}$. Then we have $|D| = F + |D - F|$ and $L(D - F) = L(D)$. Hence $|D - F|$ is a base-point-free linear system which corresponds to the same linear space with $|D|$.

By the previous discussions, we can construct a holomorphic map $\phi_D : X \to \mathbb{P}^n$ corresponding to a complete linear system $|D|$ without base points. We want to study when ϕ_D is an embedding.

Proposition 8.12

Let X be a compact Riemann surface and $|D|$ be a complete linear system without base points. Then there exists $p \neq q$ such that $\phi_D(p) = \phi_D(q)$ iff $L(D - p - q) = L(D - p) = L(D - q)$.

Corollary 8.13

ϕ_D is 1-1 if $\dim L(D - p - q) = \dim L(D) - 2$ for every pair of distinct points p and q.

§9. Mar 29

§9.i. Maps to the projective space

Even if ϕ_D is 1-1, the image of ϕ_D may not be a holomorphically embedded Riemann surface.

Example 9.1

Consider the map $\mathbb{C} \to \mathbb{P}^3$ given by $z \mapsto [1 : z^2 : z^3]$. Then it corresponds to $\{x^3 = y^2\} \subset \mathbb{C}^2 \hookrightarrow \mathbb{P}^3$.

Lemma 9.2

Assume that ϕ_D is 1-1. For every $p \in X$, the image of ϕ_D is holomorphically embedded near $\phi_D(p)$ iff $L(D - 2p) \neq L(D - p)$.

Proposition 9.3

Let X be a compact Riemann surface and $|D|$ be a complete linear system without base points. The ϕ_D is a holomorphic embedding iff $\dim L(D - p - q) = \dim L(D) - 2$ for every $p, q \in X$.

Definition 9.4. A divisor $D \in \mathrm{Div}(X)$ is called **very ample** if D is base-point-free and ϕ_D is a holomorphic embedding. D is called **ample** if there exists $m > 0$ such that mD is very ample.

The degree of the image and the map. Suppose that $\phi : X \to \mathbb{P}^n$ is a holomorphic map such that $\phi(X) = Y$ is a smooth projective curve. Let $H \subset \mathbb{P}^n$ be a hyperplane.

Proposition 9.5 $\deg \phi^*(H) = \deg \phi \cdot \deg Y$.

Corollary 9.6 If D is a very ample divisor, then $\deg \phi(X) = \deg D$.

§9.ii. Algebraic curves

Definition 9.7. A compact Riemann surface X is called an **algebraic curve** if it satisfies the following two conditions:

- **Separating points.** For every $p \neq q \in X$, there exists $f \in \mathcal{M}(X)$ such that $f(p) \neq f(q)$.
- **Separating tangents.** For every $p \in X$, there exists $f \in \mathcal{M}(X)$ such that $\mathrm{mult}_p f = 1$.

An algebraic curve refers to the compact Riemann surfaces with enough meromorphic functions. The following result is highly nontrivial.

Theorem 9.8 Every compact Riemann surface is an algebraic curve.

Constructing functions on algebraic curves. Let X be an algebraic curve. The for every $p \in X$ and $N \in \mathbb{Z}$, there exists $f \in \mathcal{M}(X)$ with $\mathrm{ord}_p(f) = N$. Now we construct functions on X by Laurent tails.

Definition 9.9. A Laurent polynomial $r(z) = \sum_{i=n}^{m} c_i z^i$ is called a **Laurent tail** of a Laurent series $h(z)$ if $h(z) - r(z)$ has all of its terms higher than the top degree term of r.

Lemma 9.10

Fix a point $p \in X$ and a local coordinate centered at p. Fix any Laurent polynomial $r(z)$, then there exists $f \in \mathcal{M}(X)$ whose Laurent series at p has $r(z)$ as a Laurent tail.

Lemma 9.11

For every $p \neq q \in X$, there exists $f \in \mathcal{M}(X)$ such that p is a zero and q is a pole of f.

Proposition 9.12 (Laurent series approximation)

Fix a finite number of points $p_1, \cdots, p_n \in X$, choose local coordinates z_i at each p_i and Laurent polynomials $r_i(z_i)$. Then there exists $f \in \mathcal{M}(X)$ such that f has $r_i(z_i)$ as a Laurent tail at p_i for every i.

The function field $\mathcal{M}(X)$.

Proposition 9.13

Let X be an algebraic curve, then $\mathcal{M}(X)$ is a finitely generated extension field of $\mathbb{C}$ of transcendence degree 1.

Proof (transcendence degree). The transcendence degree is at least one since $\mathcal{M}(X)$ contains a non constant function. Assume that there exists $f, g \in \mathcal{M}(X)$ which are algebraically independent. Take a divisor D such that $f, g \in L(D)$. Then for every $i, j \geqslant 0$ and $i + j \leqslant n$, we have $f^i g^j \in L(nD)$. It follows that

$$\dim L(nD) \geqslant \frac{n^2 + 3n + 2}{2}.$$

But $\dim L(nD) \leqslant 1 + \deg(nD) \leqslant 1 + n \deg D$, which leads to a contradiction. $\square$

Proof (finite generation). Take a non constant function $f \in \mathcal{M}(X)$. It suffices to show $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$.

Lemma 9.14

Let $A \in \mathrm{Div}(X)$ and $D = \mathrm{div}_\infty(f)$, then there exists a positive integer $m > 0$ and a meromorphic function g such that $A - \mathrm{div}(g) \leqslant mD$. Moreover, g can be taken to be a polynomial of f.

Corollary 9.15

For every $h, f \in \mathcal{M}(X)$, there exists a polynomial $r(t) \in \mathbb{C}[t]$ and $m > 0$ such that $r(f)h \in L(mD)$ where $D = \mathrm{div}_\infty(f)$.

In fact, we will show that $[\mathcal{M}(X) : \mathbb{C}(f)] \leqslant \deg D$, where $D = \mathrm{div}_\infty(f)$. Assume that $k = [\mathcal{M}(X) : \mathbb{C}(f)]$ and let $g_1, \cdots, g_k \in \mathcal{M}(X)$ be linearly independent over $\mathbb{C}(f)$. Then there exists $m_0 > 0$ and $r_i \in \mathbb{C}[t]$ such that $h_i = r_i(f) g_i \in L(m_0 D)$ for every i. Hence for every $m \geqslant m_0$, $f^j h_i \in L(mD)$ for every $j \leqslant m - m_0$ and $i \leqslant k$. Hence $\dim L(mD) \geqslant (m - m_0 + 1)k$. On the other hand, $\dim L(mD) \leqslant 1 + \deg(mD) \leqslant 1 + m \deg D$. We get a contradiction if $k > \deg D$. $\square$

Fact 9.16. $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg \mathrm{div}_\infty(f)$.

§10. Apr 3

§10.i. Laurent tail divisors

Let X be a compact Riemann surface. For every $p \in X$, we fix at once a local coordinate z_p centered at p.

Definition 10.1. A **Laurent tail divisor** is a finite formal sum $\sum_p r_p(z_p) \cdot p$ where $r_p(z_p)$ is a Laurent polynomial in the coordinate z_p. The set of Laurent tail divisors is denoted by $\mathcal{T}(X)$.

Given a divisor $D \in \mathrm{Div}(X)$, we define

$$\mathcal{T}[D](X) := \left\{ \sum r_p \cdot p \in \mathcal{T}(X) : \text{ the top term of } r_p \text{ is at most } -D(p) \right\}.$$

For every $D_1 \leqslant D_2$, there exists a natural truncation map

$$t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X).$$

For every $f \in \mathcal{M}(X)$ and $D \in \mathrm{Div}(X)$, there is a multiplication operator

$$\mu_f = \mu_f^D : \mathcal{T}[D](X) \to \mathcal{T}[\mathcal{D} - \mathrm{div}(f)](X).$$

Note that μ_f^D is an isomorphism, the inverse is $\mu_{1/f}^{D-\mathrm{div}(f)}$.

For every $D \in \mathrm{Div}(X)$, there is also a map

$$\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$$

defined by $f \mapsto \sum r_p \cdot p$ where r_p is the truncation of the Laurent series $f(z_p)$ removing all the terms higher than $-D(p)$. In fact, $\ker \alpha_D = L(D)$. Then for $D_1 \leqslant D_2$,

$$\alpha_{D_2} : \mathcal{M}(X) \xrightarrow{\alpha_{D_1}} \mathcal{T}[D_1](X) \xrightarrow{t_{D_2}^{D_1}} \mathcal{T}[D_2](X).$$

For every $D \in \mathrm{Div}(X)$ and $f \in \mathcal{M}(X)$,

$$\mu_f(\alpha_D(g)) = \alpha_{D-\mathrm{div}(f)}(fg).$$

Mittag-Leffler Problem.

> **Question 10.2.** Given a Laurent tail divisor $Z \in \mathcal{T}[D](X)$, does $Z \in \mathrm{Im}\, \alpha_D$?

We first define the first cohomology group

$$H^1(D) := \mathrm{coker}\, \alpha_D = \mathcal{T}[D](X) / \mathrm{Im}(\alpha_D).$$

Then we immediately find an exact sequence

$$0 \to L(D) \to \mathcal{M}(X) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \to H^1(D) \to 0,$$

which can be written as a short exact sequence

$$0 \to \mathcal{M}(X)/L(D) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \to H^1(D) \to 0.$$

For $D_1 \leqslant D_2$, note that $L(D_1) \hookrightarrow L(D_2)$ and there is a map $t : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$. Then the short exact sequence induces a map $H^1(D_1) \to H^1(D_2)$. By the snake lemma, we obtain

$$0 \to \ker(\mathcal{M}(X)/L(D_1) \to \mathcal{M}(X)/L(D_2)) \to \ker(t_{D_2}^{D_1}) \to \ker(H^1(D_1) \to H^1(D_2)) \to 0.$$

We define

$$H^1(D_1/D_2) := \ker(H^1(D_1) \to H^1(D_2)).$$

By calculating the dimension of the short exact sequence, we obtain

$$\dim H^1(D_1/D_2) = [\deg D_2 - \dim L(D_2)] - [\deg D_1 - \dim L(D_1)].$$

Lemma 10.3

Let X be an algebraic curve. Let $f \in \mathcal{M}(X)$ and $D = \operatorname{div}_\infty(f)$. Then $\dim H^1(0/mD)$ is bounded for $m \in \mathbb{Z}_+$.

Proof. Recall that $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg D$, hence $\dim L(mD) \geqslant (m - m_0 + 1)\deg D$. It follows that $\dim H^1(0/mD) \leqslant 1 + (m_0 - 1)\deg D$. $\square$

Lemma 10.4

Let X be an algebraic curve. Then there exists $M > 0$ such that for every $A \in \operatorname{Div}(X)$,

$$\deg A - \dim L(A) \leqslant M.$$

Proof. Choose an $f \in \mathcal{M}(X)$ and let $D = \operatorname{div}_\infty(f)$. For every $A \in \operatorname{Div}(X)$, there exists $m > 0$ and $g \in \mathcal{M}(X)$ such that $B = A - \operatorname{div}(g) \leqslant mD$. Note that $\deg B = \deg A$ and $L(B) \cong L(A)$, the conclusion follows by

$$\deg A - \dim L(A) = \deg B - \dim L(B) \leqslant \deg(mD) - \dim L(mD) \leqslant M.$$

$\square$

Then there exists $A_0 \in \operatorname{Div}(X)$ maximizing $\deg A_0 - \dim L(A_0)$,

Claim 10.5. $H^1(A_0) = 0$.

Proof. Assume for a contradiction that $Z \in \mathcal{T}[A](X)$ but $Z \notin \operatorname{Im}\alpha_{A_0}$. Take $B \geqslant A$ such that $t(Z) = 0$. Then $[Z] \in \ker(H^1(A) \to H^1(B))$. Which leads to $\deg B - \dim L(B) > \deg A_0 - \dim L(A)$. $\square$

Proposition 10.6

Let X be an algebraic curve, then $H^1(D)$ is finite dimensional for every $D \in \operatorname{Div}(X)$.

Proof. Take A_0 as above, write $D - A_0 = P - N$ where $P, N \geqslant 0$. Then $H^1(A_0 + P) = 0$ and hence

$$H^1(D) = H^1(A_0 + P - N) \cong H^1(A_0 + P - N/A_0 + P)$$

which is finite dimensional. $\square$

§11. Apr 10

§11.i. The Riemann-Roch theorem

Combine the identities $\dim H^1(0/D) = \dim H^1(0) - \dim H^1(D)$ and $\dim H^1(0/D) = \deg D - \dim L(D) + 1$, we obtain

Theorem 11.1 (The Riemann-Roch theorem: first form)

Let D be a divisor on an algebraic curve, then

$$\dim L(D) - \dim H^1(D) = \deg D + 1 - \dim H^1(0).$$