Methods for Studying Abelian Actions and Centralizers

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Contents

§1 Definitions and examples (Danijela, May 1)

Plan for this minicourse

1. Many examples, invariant structures, main results.
2. Some methods in simple cases.
3. More methods and more about centralizer rigidity
4. More methods.

Setting

- M a closed C^∞-manifold.
- $f : M \to M$ a C^∞-diffeomorphism.
- $\mathscr{Z}(f) := \{g \in \mathrm{Diff}^\infty(M) : gf = gf\}$, the centralizer of f in $\mathrm{Diff}^\infty(M)$.

It is obvious that $\mathscr{Z}(f) \geqslant \langle f \rangle \cong \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. **Smale's question**:

Is it true that typically in C^r-topology, $\langle f \rangle = \mathscr{Z}(f)$?

This is confirmed to be true in C^1-topology by Bonatti-Crovisier-Wilkinson.

We are also interested in a typical situation that $\mathscr{Z}(f)$ is large. The main theme is a centralizer rigidity:

$$f \text{ has a complicated dynamics} + \mathscr{Z}(f) \text{ is large}$$
$$\Longrightarrow f \text{ is } C^\infty\text{-conjugate to an algebraic system}$$

Algebraic systems.
- $M = G/\Gamma$ where G is a Lie group and Γ is a cocompact lattice in G.
- $L_g : x \mapsto g.x$ the left translation for $g \in G$.
- $A : G \to G$ an automorphism preserving Γ, it induces $A : G/\Gamma \to G/\Gamma$.
- Affine maps $L_g \circ A$.
- Another examples of "algebraic systems" are bi-homogeneous examples. These are defined as translations on the symmetric space $L_g : K\backslash G/\Gamma \to K\backslash G/\Gamma$ where $K < G$ is a compact subgroup, by elements in G which commute with K.

Definition 1.1. An action is **smoothly algebraic** if it is C^∞-conjugate to an algebraic model.

Complicated dynamics. f is partially hyperbolic with $TM = E^s \oplus E^c \oplus E^u$. Assume in addition that E^c is integrable to a foliation $\mathscr{W}^c$ with C^1-leaves. Then we also say that f is **normally hyperbolic** to the foliation $\mathscr{W}^c$.

Definition 1.2. f is **accessible** if any $x, y \in M$ can be connected via a stable / unstable broken path.

Notation 1.3. For groups H_1, H_2, we denote $H_1 \doteq H_2$ if H_1 is virtually H_2, that means H_1 contains a finite index subgroup isomorphic to H_2.

Example 1.4 (Examples with small (rank-one) centralizers)
1. A hyperbolic automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, then $\mathscr{Z}(f) \doteq \mathbb{Z}$.
2. Geodesic flows $\varphi_t : \mathrm{SL}(2,\mathbb{R})/\Gamma \to \mathrm{SL}(2,\mathbb{R})/\Gamma$, it corresponds to the diagonal flows $A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$ acts by left translations. Then φ_t is partially hyperbolic and $\mathscr{Z}(\varphi_t) \doteq \mathbb{R}$.

Example 1.5 (Examples with larger centralizers)
1. For $A : \mathbb{T}^2 \to \mathbb{T}^2$ a hyperbolic automorphism, let $f = \begin{bmatrix} A & \\ & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$, then any $\begin{bmatrix} A^k & \\ & A^l \end{bmatrix}$ commutes with f for $k, l \in \mathbb{Z}$. Hence $\mathscr{Z}(f) > \mathbb{Z}^2$.
2. Product of geodesic flows on $\mathrm{SL}(2,\mathbb{R})/\Gamma$. Then $\mathscr{Z}(\varphi_t) > \mathbb{R}^2$.

Note that in the first example, the elements of the form $A^k \times \mathrm{id}$ or $\mathrm{id} \times A^l$ are not ergodic. Which means there is a factor in the system. The same holds for the second example. We want to avoid these cases.

Definition 1.6 (Rank one factor). Let $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ be an action with $k + l \geqslant 2$. We say it has a **C^s rank-one factor** if we have
- A C^∞-manifold $\overline{M}$ and a C^s-submersion $\pi : M \to \overline{M}$.
- A surjective homomorphism $\sigma : \mathbb{R}^k \times \mathbb{Z}^l \to H$ where $H \doteq \mathbb{Z}$ or $\mathbb{R}$.
- A locally free C^s-action $H : \overline{M} \to \overline{M}$ such that $\pi(g.x) = \sigma(g).\pi(x)$.

Definition 1.7. A smooth action $\mathbb{R}^k \times \mathbb{Z}^l : M \to M$ is called **(genuinely) higher-rank** if $k + l \geqslant 2$ and there is no C^∞-rank-one factors.

Example 1.8 (Higher-rank actions)
1. $A : \mathbb{T}^3 \to \mathbb{T}^3$ a hyperbolic automorphism with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \notin \mathbb{R} \setminus \{-1, 1\}$. Then $\mathscr{Z}(A) \doteq \mathbb{Z}^2 = \langle A, B \rangle$ where B is also a hyperbolic automorphism. Let V_i be the eigenspace of A corresponding to λ_i, then B preserves each V_i. Hence $A^k B^l|_{V_i} = \lambda_i^k \mu_i^l$.

Although there is not integers k, l such that $\lambda_i^k \mu_i^l = 1$, but there exists pairs of real numbers (s, t) such that $\lambda_i^s \mu_i^t$. These lines are very important.
Specifically, let

$$\chi_i(s, t) = s \log|\lambda_i| + t \log|\mu_i|.$$

Then $L_i := \ker \chi_i$ is a line in the plane for any $i = 1, 2, 3$. An algebraic fact shows that the lines are irrational (hence there is no integers k, l such that $(k, l) \in L_i$).

2. The diagonal flow on $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/\Gamma$ where Γ is an irreducible lattice. By Mautner's theorem, every line in the diagonal flow acts ergodically.
3. Weyl chamber flow. Let $M = \mathrm{SL}(3, \mathbb{R})/\Gamma$, we consider

$$\mathbb{R}^2 \cong \mathrm{Diag} := \left\{ \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} : t_1 + t_2 + t_3 = 0 \right\}$$

acting on M. Element in Diag acts on U_{12} by expansion / contraction by a factor $e^{t_1 - t_2}$. Any element in Diag for which $t_i \neq t_j$ for all $i \neq j$ acts normally hyperbolically with respect to the homogeneous foliation defined by the Diag action by left translations. But for an element in Diag for which $t_i = t_j$, it acts by isometries on U_{ij}. These elements act normally hyperbolically with respect to the homogeneous foliation given by the left translations by group $\langle \mathrm{Diag}, U_{ij} \rangle$. Moreover, any element in Diag on the line $t_i = t_j$ is accessible. This is a consequence of the group structure of $SL(3, \mathbb{R})$. Besides, every nontrivial element acts ergodically with respect to the Haar measure.

Exercise 1.9. For an $\mathbb{R}^2$ action on M, if every line in $\mathbb{R}^2$ is ergodic iff there is no rank-one factors (also refer to [**V22**]).

§2 Statements of the results in rigidity theory (Danijela, May 2)

Proposition 2.1

Let A be an irreducible matrix in $\mathrm{SL}(d, \mathbb{Z})$, then $\mathcal{Z}_{\mathrm{aff}}(A) \doteq \mathbb{Z}^{m+n-1}$, where m is the number of real eigenvalues and m is the number of pairs of complex eigenvalues (refer to [**KKS02**]). Moreover, every smooth diffeomorphism commuting with A is affine[**AP65**]. See also Example 4.4.

Now we back to to the first example in 1.8. The lines L_i divide the plane into 6 chambers. For an element not on the lines, it expands or contracts the space $V_i, i = 1, 2, 3$. Note that for elements in the same chamber, for each V_i, they expands or contracts this space simultaneously.

Definition 2.2. A $\mathbb{Z}^k$ action on M is **Anosov** if it contains an Anosov element. Furthermore, it is **totally Anosov** if all nontrivial elements are Anosov.

Definition 2.3. An $\mathbb{R}^k$ action on M is **Anosov** if it some $a \in \mathbb{R}^k$ acts normally hyperbolic to the $\mathbb{R}^k$-orbit foliation. It is **totally Anosov** if there is a dense set of elements normally hyperbolic to the orbit foliation.

Proposition 2.4

Let $\langle A, B\rangle$ be the pair given in the first example of 1.8. Then $\langle A, B\rangle$ is an exponentially mixing action: for every $\theta > 0$, there exists $\tau = \tau(\theta) > 0$ such that for every θ-Hölder functions ξ, η such that

$$\left|\left\langle \xi \circ A^k B^l, \eta\right\rangle\right| \leqslant C_{\xi,\eta} e^{-\tau(|k|+|l|)}.$$

Theorem 2.5 (Gorodnik-Spatzier[GS15])

For any $\mathbb{Z}^k$-action on a nilmanifold N/Γ by automorphisms, if there is no rank-one factor, then it is exponentially mixing (in the sense above).

Remark 2.6 The last two examples in 1.8 are also exponentially mixing. Mixing is shown by Moore's ergodicity. Exponential mixing due to R.Howe, or look up Zhenqi Wang's.

Example 2.7

Let $f : A \times R_\theta$ where $A : \mathbb{T}^3 \to \mathbb{T}^3$ is as before and R_θ is an irrational rotation. Then $\mathscr{Z}(f) \doteq \mathbb{Z}^2 \times \mathbb{T}$.

Definition 2.8. A **fibered partially hyperbolic system** is a partially hyperbolic $f : M \to M$ with compact leaves $\mathscr{W}_f^c$ and $M/\mathscr{W}_f^c$ is a topological manifold. It induces a map $\overline{f} : \overline{M} = M/\mathscr{W}_f^c \to \overline{M}$ satisfying $\pi \circ f = \overline{f} \circ \pi$.

Remark 2.9 Bohnet-Bonatti[BB16] , Gogolev[G11] , Avila-Viana-Wilkinson[AVW22] have studied the fibered partially hyperbolic systems.

Proposition 2.10

Let A, B be as in the first example of 1.8. Let $f : (x, y) \mapsto (Ax, y + \varphi(x))$ be a fibered partially hyperbolic map, assume that $\mathscr{Z}(f)$ contains $(Bx, y + \psi(x))$. Then f is smoothly conjugate to an affine map.

Proof. We have the cocycle equation

$$\varphi - \varphi \circ B = \psi - \psi \circ A. \tag{2.1}$$

Now we consider the map $(Ax, y + \varphi(x))$, it smoothly conjugate to $(Ax, y + c)$ if $\varphi = H - H \circ A$ (assume that $\int \varphi = 0$ and conjugate via $(x, y + H(x))$). Some possible solutions for H are

$$D_A^+(\varphi) := \sum_{k=0}^{\infty} \varphi \circ A^k \quad \text{or} \quad D_A^-(\varphi) := -\sum_{k=-\infty}^{-1} \varphi \circ A^k.$$

Note that D_A^+ has derivatives along $\mathscr{W}_A^s$ and D_A^- has derivatives along $\mathscr{W}_A^u$. Then if we can show $D_A^+ = D_A^-$, we are done. Let $D_A(\varphi) = \sum_{k\in\mathbb{Z}} \varphi \circ A^k$, then by (2.1), we have $D_A(\varphi) = D_A(\varphi \circ B^l)$ for every $l \in \mathbb{Z}$. Then for every Hölder function ξ, by exponentially mixing

$$\lim_{l\to\pm\infty} \left\langle D_A(\varphi \circ B^l), \xi\right\rangle = \lim_{l\to\pm} \sum_{k\in\mathbb{Z}} \left\langle \varphi \circ A^k B^l, \eta\right\rangle \to 0.$$

Hence $D_A(\varphi) = 0$ and H is C^∞. $\square$

Exercise 2.11 ("Higher rank trick" works without exponentially mixing)**.** Let

$$a(x,y) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \quad b(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \end{bmatrix},$$

where α, β are Diophantine irrational numbers. Note that a, b are commuting diffeomorphisms on $\mathbb{T}^2$. Then every φ, ψ satisfy (2.1) over $\langle a, b \rangle$ are C^∞-coboundaries.

Samples of local, global and semi-local rigidity results.

Theorem 2.12 (Local rigidity, Katok-Spatzier[**KS96**])

For actions in Example 1.8, they are locally rigid. That is, for every C^1-perturbation of the action, the action is smoothly algebraic.

Remark 2.13 Local rigidity was extended to

- Partially hyperbolic version of the Example 1.8.1 on $\mathbb{T}^d$, by Damjanović-Katok[**DK10**].
- Partially hyperbolic version of the Example 1.8.3, by Damjanović-Katok[**DK11**], and Vinhage-Wang[**VW19**].
- KAM method for partially hyperbolic affine actions, by Zhenqi Wang [**W10**].

Theorem 2.14 (Global rigidity, Fisher-Kalinin-Spatzier[**FKS13**], Hertz-Wang[**RW14**])

An Anosov $\mathbb{Z}^k$-action ($k \geqslant 2$) on a nilmanifold N/Γ is smoothly affine, providing it is homotopic to a higher rank action by an automorphism.

Remark 2.15 It implies that Example 1.8.1 is globally rigid.

Theorem 2.16 (Spatzier-Vinhage[**SV22**])

Example 1.8.3 is also globally rigid, (precise version will be stated later).

Theorem 2.17 (Semi-local, Damjanović-Wilkinson-Xu[**DWX23**])

Let $f_0 = A \times R_\theta$ as in Example 2.7. Let f be a volume preserving C^1 smooth perturbation of f_0 and assume that f is ergodic. Then

$$\mathscr{Z}(f) \doteq \begin{cases} \mathbb{Z}; \\ \mathbb{Z} \times \mathbb{T}; \\ \mathbb{Z}^2 \times \mathbb{T}, \text{ and } f \text{ is smoothly algebraic.} \end{cases}$$

§3 More methods in simple cases (Danijela, May 2)

Another simple case. Let A, B be given in Example 1.8.1. Let $\varphi, \psi : \mathbb{T}^3 \to \mathbb{T}^3$ be C^∞ maps. Let

$$F(x,y) = (Ax, Ay + \varphi(x)), \quad G(x,y) = (Bx, By + \psi(x))$$

be commuting maps. Then we have the cocycle equation

$$A \circ \psi - \psi \circ A = B \circ \varphi - \varphi \circ B. \tag{3.1}$$

Let $\mathbb{R}^3 = V_1 \oplus V_2 \oplus V_3$ where each V_i is an eigenspace. Split the equation into each V_i and let φ_i, ψ_i be the components of φ, ψ respectively. We have (for simplicity, just consider $i = 1$)

$$\lambda_1 \psi_1 - \psi_1 \circ A = \mu_1 \varphi_1 - \varphi_1 \circ B.$$

We want to find $H(x)$ such that $(x, y + H(x))$ conjugates (Ax, Ay) to $(Ax, Ay + \varphi(x))$. So we need to solve the equation

$$\varphi = A \circ H - H \circ A, \quad \text{i.e.} \quad \varphi_1 = \lambda_1 H_1 - H_1 \circ A.$$

Then we can take

$$D^+_{A,1} = \sum_{k=0}^{\infty} \lambda_1^{-(k+1)} \varphi_1 \circ A^k \text{ if } |\lambda_1| > 1, \quad \text{or} \quad D^-_{A,1} = \sum_{k=-\infty}^{-1} \lambda_1^{-(k+1)} \varphi_1 \circ A^k \text{ if } |\lambda_1| < 1.$$

Note that $D^\pm_{A,1}$ converge uniformly and hence are C^0. We can define $D^\pm_{B,1}$ similarly and we have $D^\pm_{A,1} = D^\pm_{B,1}$ when they are convergent. The problem is how to show that H is smooth.

Now we turn to considering general commuting toral diffeomorphisms. We will be back to this example later.

Commuting toral diffeomorphisms. Assume that $\langle f, g \rangle$ homotopic to $\langle A, B \rangle$ and f is Anosov. By Franks-Manning theorem, there exists a Hölder homeomorphism h such that $f = h \circ A \circ h^{-1}$. We apply Oseledets' decomposition for abelian actions on an ergodic measure preserving space. Then there exists an (a priori just measurable) invariant splitting

$$TM = \bigoplus_i E^i$$

and linear functions $\chi_i : \mathbb{Z}^2 \to \mathbb{R}$ such that $\chi_i(a)$ is the Lyapunov exponent of a in E^i. For each linear function χ, let

$$E^{[\chi]} := \bigoplus_{\chi_i = c\chi,\ c>0} E^i,$$

which is called the **coarse Lyapunov distribution**. Assume that $a \in \mathbb{Z}^2$ is Anosov, then the unstable / unstable distributions of a are Hölder continuity. That is, both

$$\bigoplus_{\chi(a)<0} E^{[\chi]}, \quad \bigoplus_{\chi(a)>0} E^{[\chi]}$$

are Hölder. If we have sufficiently many Anosov elements (one in each Weyl chamber), by taking intersection, we can obtain the Hölder continuity of the coarse Lyapunov distribution. [This is not the core of this minicourse. Another minicourse given by Disheng focuses on this topic. The notes can be found here.]

Proposition 3.1

One Anosov in each chamber $\implies$ $E^{[\chi]}$ are Hölder and integrate to Hölder foliations.

Remark 3.2 The Hölder continuity also implies that the distribution is independent with the choice of the measure.

Back to the example. The following figure illustrates a Weyl chamber picture, where $\pm$ denotes the sign of χ_i, characterizing whether the element contracts or expands V_i.

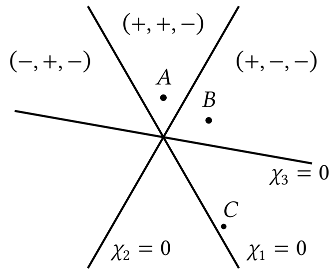

Figure 1: Weyl chamber picture

Then for the element A, we have the expression $H_1 = \sum_{k=0}^{\infty} \lambda_1^{-(k+1)} \varphi_1 \circ A^k$ since $\log \lambda_1 = \chi_1(A) > 0$. Note that A contracts V_3, we obtain H_1 is C^∞ along V_3. If we use the expression for B as $H_1 = \sum_{k=0}^{\infty} \mu_1^{-(k+1)} \psi_1 \circ B^k$, we can obtain that H_1 is C^∞ along $V_2 \oplus V_3$. But we can never get the regularity along V_1 by this method, since for every element C we can only get the regularity long V_i where $\chi_i(C)$ and $\chi_1(C)$ have different signs.

Here we need another trick by the exponentially mixing. We will take a C very close to $\chi_1 = 0$, and assume that $(x, y) \mapsto (Cx, Cy + \zeta(x)) \in \langle F, G \rangle$. Then by exponentially mixing

$$D_{C,1}(\zeta_1) := \sum_{k \in \mathbb{Z}} e^{-(k+1)\chi_1(C)} \zeta_1 \circ C^k$$

converges as a distribution. Furthermore, since $e^{\chi_1(C)}\zeta_1 - \zeta_1 \circ C = \mu_1 \psi_1 - \psi_1 \circ B$, we obtain

$$\mu_1^l D_{C,1}(\zeta_1) = D_{C,1}(\zeta_1 \circ B^l) = \sum_{k \in \mathbb{Z}} e^{-(k+1)\chi_1(C)} \zeta_1 \circ C^k B^l.$$

If $\chi_1(C)$ is smaller enough, the exponentially mixing will show that the distribution tends to 0 as $l \to \pm\infty$. Taking an appropriate direction such that $\mu_1^l \to \infty$, we obtain $D_{C,1}(\zeta_1) = 0$.

This trick tells us $D_{C,1}(\zeta_1)^- = D_{C,1}(\zeta_1)^+$, then H has two expressions. So we can choose a desired direction ($k \to +\infty$ or $k \to -\infty$) such that C^k contracts V_1. This gives the regularity of H along V_1.

How this derives the global rigidity Theorem 2.14. Given $\langle f, g \rangle$ commuting on $\mathbb{T}^d$ that h-conjugates to $\langle A, B \rangle$. Here h is a priori just Hölder continuous. The aim is to show that h is indeed C^∞.

1. One Anosov $\implies$ one Anosov element in each Weyl chamber. This is a highly nontrivial part, which is due to [Hertz-Wang[RW14]].
2. One Anosov in each chamber $\implies$ Anosov elements are somehow dense (projectively dense). Moreover, the Weyl chamber picture is the same for $\langle f, g \rangle$ and $\langle A, B \rangle$. [Fisher-Kalinin-Spatzier[FKS13]]
3. Use the exponentially mixing argument to upgrade the regularity.

Idea of showing Theorem 2.17. For the rigidity part, we consider the factor $\overline{f} : \mathbb{T}^3 \to \mathbb{T}^3$. Then $\mathcal{Z}(\overline{f}) \doteq \mathbb{Z}^2$ and we can apply a similar argument as before. For the other two cases, we apply a dichotomy of the disintegration of the volume by Avila-Viana-Wilkinson[AVW22]: if f is accessible, then the disintegration is

- either purely atomic,
- or Lebesgue.

Global rigidity in general manifolds.

Conjecture 3.3 (Katok-Spatzier)
If a higher rank action $\mathbb{Z}^k \times \mathbb{R}^l : M \to M$ contains an Anosov element, then it is smoothly algebraic.

But this conjecture in general is not true for $l \geqslant 2$, since

Theorem 3.4 (Vinhage[V22])
There exists a C^∞-time change of Example 1.5.2 (product of geodesic flows) that has no C^∞-rank-one factor, is Anosov and not C^∞-algebraic.

However, for $\mathbb{Z}^k$-actions of $k \geqslant 2$, Katok-Spatzier's conjecture still may be true. For the $\mathbb{R}^l$ cases, the "Anosov" condition of Katok-Spatzier's conjecture need to be replaced with a "totally Anosov" condition.

Theorem 3.5 (Global rigidity, Spatzier-Vinhage [SV22])
The global rigidity of $\mathbb{R}^l$-actions ($l \geqslant 2$) on ANY manifold M providing totally Anosov, coarse Lyapunov $E^{[\chi]}$ are 1d and no rank one factors.

Remark 3.6 It gives a new approach to construct algebraic structures on the manifold.

Theorem 3.7 (Damjanović-Spatzier-Vinhage-Xu[DSVX22])
A totally Anosov $\mathbb{R}^l$-action is smoothly algebraic providing

- Volume preserving.
- Weyl chamber walls are accessible (strongly accessible).
- Oseledets spaces admit measurable conformal structures.

Conjecture 3.8 (Extended Katok-Spatzier's conjecture, Damjanović-Wilkinson-Xu)
Let f be a fibered partially hyperbolic diffeomorphism, assume that $\mathcal{Z}(f)$ contains a k-dimensional Lie group of maps which are id on the base with $k = \dim \mathcal{W}^c_f$. If the projection of $\mathcal{Z}(f)$ onto to the base has no C^0-rank-one factor, then f is C^∞-fibration of a smoothly algebraic system.

Conjecture 3.9 (Semi-local conjectures, Damjanović-Wilkinson-Xu)

Assume $f_0 : \underline{G/\Gamma \to G/\Gamma}$ be an affine map where G/Γ is a connected homogeneous space. Assume that $\langle \mathrm{stable}(f_0), \mathrm{unstable}(f_0)\rangle \Gamma = G$ (it implies property-K by Dani) and $\mathcal{Z}(f_0)$ has no rank-one factors. Let f be a C^1-small perturbation of f_0, then

1. Is f smoothly affine?
2. If $\mathcal{Z}(f)$ also has no rank-one factors, is f smoothly affine?

§4 Centralizers of diffeomorphisms (Disheng, May 3)

Let M be a closed C^∞-manifold. Let

$$\mathcal{Z}^r(f) := \{ g \in \mathrm{Diff}^r(M) : gf = fg \}$$

for every $1 \leqslant r \leqslant \infty$. We abbreviate $\mathcal{Z}^\infty(f)$ to $\mathcal{Z}(f)$. We also denote

$$\mathcal{Z}^0(f) := \{ g \in \mathrm{Homeo}(M) : gf = fg \}.$$

Example 4.1

Let $f = \mathrm{id}$, then $\mathcal{Z}(f) = \mathrm{Diff}^\infty(M)$. Conversely, if $\mathcal{Z}(f) = \mathrm{Diff}^\infty(M)$, we can show that $f = \mathrm{id}$ by the following argument. If $f \neq \mathrm{id}$, take $x \in M$ such that $f(x) \neq x$. Then there exists $g \in \mathcal{Z}(f)$ such that $g(x) = x$ and $g(f(x)) \neq f(x)$. We get a contradiction. This argument works for any doubly transitive centralizer.

Example 4.2

Let $f = R_\alpha$ be a rotation on $\mathbb{T}^d$. Then

- If f is not minimal, $\mathcal{Z}(f)$ is a "∞-dimension Lie group".
- If f is minimal, then $\mathcal{Z}(f) \cong \mathbb{T}^d$. Furthermore, assuming f is minimal and $\mathcal{Z}(f) \curvearrowright M$ transitively, i.e., $\forall x, y \in M$, there exists $g \in \mathcal{Z}(f)$ such that $g(x) = y$, then f is C^∞-conjugate to some R_α on $\mathbb{T}^d$.

Remark 4.3 Without minimality assumption, Damjanović-Wilkinson-Xu[DWX23] classify f in the second case.

Example 4.4

Let $f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$. Then $\mathcal{Z}(f)$ is not trivial since $\sqrt{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} = \pm\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{Z}(f)$. In this case, $\mathcal{Z}(f)$ is **virtually trivial**: $\langle f \rangle$ is finite index in $\mathcal{Z}(f)$.

In general, let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an irreducible matrix. We have

Fact 4.5. $\mathcal{Z}_{\mathrm{GL}(d,\mathbb{Z})}(A)$ is an abelian group of rank $r + c - 1$, where r is the number of real eigenvalues and c is the number of pairs of complex eigenvalues.

Fact 4.6 (Adler-Palais[AP65]). If f is an ergodic automorphism on $\mathbb{T}^d$ (nilmanifolds by Walters [W70]), every element in $\mathcal{Z}^0(f)$ is affine.

Fact 4.7. If f is Anosov affine map on torus, then there exists only finitely many of translations commuting with f.

Theorem 4.8 (Bonatti-Crovisier-Wilkinson[**BCW09**])
For C^1-generic $f \in \mathrm{Diff}^1(M)$, $\mathcal{Z}^1(f) = \langle f \rangle$.

Remark 4.9 It shows that for every f_0, we can perturb it to make $\mathcal{Z}(f)$ small. Besides, for $f_0 = R_\alpha$ on $\mathbb{T}^d$, we can also perturb it to make $\mathcal{Z}(f)$ large.

Remark 4.10 On the circle, there is a C^1-dense set of $f \in \mathrm{Diff}^1(\mathbb{S}^1)$ such that $\mathcal{Z}(f)$ is large[**BF15**]. However, Kopell[**K67**] showed that if endowing $\mathrm{Diff}^\infty(\mathbb{S}^1)$ with the uniform C^r-topology ($r \geqslant 1$), then there is a C^r-open dense set of f whose centralizer is trivial.

Corollary 4.11 (Hertz-Wang)
Let $A \in \mathrm{SL}(d, \mathbb{Z}) : \mathbb{T}^d \to \mathbb{T}^d$ be irreducible and hyperbolic, if f is C^∞-close to A, then

$$\mathcal{Z}(f) = \begin{cases} \text{virtually } \langle f \rangle; \\ \text{virtually } \mathbb{Z}^{r+c-1>1}, \text{ and } f \text{ is } C^\infty\text{-conjugate to } A. \end{cases}$$

Remark 4.12 It might be quite complicated for the case that f is not irreducible. If without assumption of hyperbolicity, the n for genuinely partially hyperbolic f the problem becomes extremely difficult, we basically need a semi-local C^1-version of Damjanović-Katok[**DK10**].

Example 4.13
Let

$$f_0 = A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3,$$

then $\mathcal{Z}(f_0) \doteqdot \mathbb{Z} \times \mathrm{Diff}^\infty(\mathbb{T}^1)$. For f C^∞-close to f_0, the centralizer $\mathcal{Z}(f)$ can be also extremely large. But Burslem[**B04**] showed there exists an f C^∞-close to f_0 with trivial centralizer.

Theorem 4.14 (Damjanović-Wilkinson-Xu)
Let f be a volume preserving diffeomorphism C^1-close to f_0 as in Example 4.13 which is accessible. Then $\mathcal{Z}(f)$ is either virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^1 \times \mathbb{T}^1$. In the later case, f is C^∞-conjugate to $(x, y) \mapsto (\widetilde{f}(x), y + \alpha(x))$, which is commuting with $(x, y) \mapsto (x, y + \alpha)$ for every $\alpha \in \mathbb{T}^1$.

Proof. **Step 1.** By Hirch-Pugh-Shub[**HPS77**], there exists $\mathcal{W}^c_f$ which is an f-invariant foliation with compact leaves (E^c may not be uniquely integrable). Letting $\overline{M} = M/\mathcal{W}^c_f$ which is a topological manifold, then f induces $\overline{f} : \overline{M} \to \overline{M}$.

Step 2. For every $g \in \mathcal{Z}(f)$, we can show that $g\mathscr{W}_f^c = \mathscr{W}_f^c$. Then every $g \in \mathcal{Z}(f)$ induces $\overline{g} : \overline{M} \to \overline{M}$ which is well-defined.
Step 3. Let $P : \mathcal{Z}(f) \to \mathrm{Homeo}(\overline{M})$ be defined by $g \mapsto \overline{g}$. Then we obtain an exact sequence

$$0 \to \ker P \to \mathcal{Z}(f) \xrightarrow{P} \mathrm{Homeo}(\overline{M}).$$

Here

$$\ker P = \left\{ g \in \mathcal{Z}(f) \,:\, \forall x \in M, g(\mathscr{W}_f^c(x)) = \mathscr{W}_f^c(x) \right\} =: \mathrm{CZ}(f).$$

□

§5 Centralizer of diffeomorphisms II (Disheng, May 4)

Continued proof. **Step 2.** Let us discuss more about the second step. There is an open problem

Conjecture 5.1
Let f be a C^1 partially hyperbolic diffeomorphism and assume that there is $\mathscr{W}_f^c$ tangent to E_f^c which is f-invariant. Then for every $g \in \mathcal{Z}(f)$, do we have $g(\mathscr{W}_f^c) = \mathscr{W}_f^c$?

But we can show the desired conclusion in our case. Let $\hat{f}, \hat{g}$ be liftings of f, g on $\mathbb{R}^2 \times \mathbb{T}^1$ respectively. Let $\widehat{\mathscr{W}_f^c}$ be the corresponding foliation on $\mathbb{R}^2 \times \mathbb{T}^1$. Then for every $\hat{x}, \hat{y} \in \mathbb{R}^2 \times \mathbb{T}^1$, $\hat{y} \in \widehat{\mathscr{W}_f^c}(x)$ iff $d(\hat{f}^n(\hat{x}), \hat{f}^n(\hat{y}))$ is uniformly bounded for $n \in \mathbb{Z}$. This is equivalent to

$$d(\hat{g}\hat{f}^n\hat{x}, \hat{g}\hat{f}^n\hat{y}) = d(\hat{f}^n\hat{g}\hat{x}, \hat{f}^n\hat{g}\hat{y})$$

is uniformly bounded for $n \in \mathbb{Z}$. Hence $\hat{g}$ preserves $\widehat{\mathscr{W}_f^c}$.

Remark 5.2 In general, we do not know the induced action of f on the space of center leaves.

Step 3. Recall the induced map $P : g \mapsto \overline{g}$. Then $\overline{g}$ commutes with $\overline{f}$ in $\mathrm{Homeo}(\overline{M})$. We obtain a key short exact sequence

$$0 \to \mathrm{CZ}(f) \to \mathcal{Z}(f) \to \mathrm{Im}(P) \to 0,$$

where

$$\mathrm{CZ}(f) = \left\{ g \in \mathcal{Z}(f) \,:\, \forall x \in M, g(\mathscr{W}_f^c(x)) = \mathscr{W}_f^c(x) \right\}.$$

Step 4. We want to study $\mathrm{CZ}(f)$. Note that f is volume preserving and accessible, and hence ergodic by Burns-Wilkinson[**BW10**]. Then g is also volume preserving. By Avila-Viana-Wilkinson II[**AVW22**], the volume along $\mathscr{W}^c$ behaves like

$$\begin{cases} \text{Atomic disintegration;} \\ \text{Lebesgue disintegration, in this case } f \text{ is } C^\infty\text{-conjugate to } (\tilde{f}(x), y + \alpha(x)). \end{cases}$$

In the atomic case, there exists a subset $S \subset \mathbb{T}^3$ such that
- $\mathrm{Vol}(S) = 1$, and
- there is a positive integer k such that $\#\left(S \cap \mathscr{W}_f^c(x)\right) = k$ for every x.

Remark 5.3 This is known as the **pathological center foliation** [SW00; RW01; P04].

Note that $\mathrm{Vol}(g(S) \cap S) = 1$ and hence for almost every $x \in \mathbb{T}^3$, $g(S \cap \mathscr{W}^c_f(x)) = S \cap \mathscr{W}^c_f(x)$.

Observation 5.4. For every x_0, $\mathrm{CZ}(f) \cong \mathrm{CZ}(f)|_{\mathscr{W}^c_f(x_0)}$.

Lemma 5.5

Let h be an accessible partially hyperbolic diffeomorphism. Let g commute with h and $g\mathscr{W}^c_h(x) = \mathscr{W}^c_h(x)$ for every $x \in M$. If $g(x) = x$ for some $x \in M$, then $g = \mathrm{id}$.

Proof. Use the fact that g is commuting with *su*-holonomies. □

Back to our case, note that $g^{k!}$ is identity on $S \cap \mathscr{W}^c_f(x)$ since there are only k points on it! So $g^{k!}$ has a fixed point and $g^{k!} = \mathrm{id}$ on $\mathbb{T}^3$ by the lemma.

Step 5. Note that $\bar{f}$ is C^0-conjugate to $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2$, so $\mathrm{Im}(P) \subset \mathscr{Z}^0(\bar{f})$ which is virtually $\mathbb{Z}$. Hence in the atomic case, $\mathscr{Z}(f)$ is virtually $\mathbb{Z}$. □

Example 5.6 (Centralizer classification on the Heisenberg three-manifold)

Let $G = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$ be the Heisenberg group and the Lie algebra $\mathfrak{g}$ is generated by

$$X = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Consider the $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$-map φ on the Lie algebra given by $aX + bY + cZ \mapsto (2a+b)X + (a+b)Y + Z$. It induces an automorphism $\Phi : G \to G$ preserving the lattice

$$\Gamma = \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z}/2 \\ & 1 & \mathbb{Z} \\ & & 1 \end{bmatrix}.$$

Let Φ induce the map $f_0 : G/\Gamma \to G/\Gamma$. Let f be C^1-close to f_0, volume preserving. Then

$$\mathscr{Z}(f) = \begin{cases} \text{virtually } \mathbb{Z}; \\ f \text{ is isometric extension of a toral Anosov map.} \end{cases}$$

Theorem 5.7 (Damjanović-Wilkinson-Xu[DWX21])

Let $\phi_t : T^1 S \to T^1 S$ be the geodesic flow on a negatively curved surface. Let $f_0 = \phi_1$ and f be C^1-close to f_0, volume preserving. Then

$$\mathscr{Z}(f) = \begin{cases} \text{virtually } \mathbb{Z}; \\ \text{virtually } \mathbb{R}, \text{ and } f \text{ is the time-one map of a } C^\infty \text{ volume preserving flow.} \end{cases}$$

Proof. The argument is similar. The different point is that we apply Avila-Viana-Wilkinson I[AVW15] to obtain a dichotomy on $\mathrm{CZ}(f)$: either finite or we have a rigidity on f. □

§6 (Disheng, May 5)

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