Totally geodesic submanifolds and arithmeticity (Manfred Einsiedler, Winter 2024)

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Contents

§1 Lecture 1

1. Arithmeticity.

This minicourse focus on two following theorems about the arithmeticity of lattices.

Theorem 1.1 (Margulis) A lattice $\Gamma < \mathrm{SL}(3, \mathbb{R})$ is arithmetic.

Theorem 1.2 (Bader-Fisher-Miller-Stover)
Let $\Gamma < \mathrm{SO}(d, 1)(\mathbb{R})$ be a lattice. Suppose that $M = \Gamma \backslash \mathbb{H}^d$ contains infinitely many maximal proper totally geodesic closed submanifolds of dimension at least two. Then Γ is arithmetic.

Reminders on arithmetic lattices.

Example 1.3

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-subgroup of $\mathrm{SL}(d,\mathbb{C})$. Then $\Gamma = \mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{R}) \cap \mathrm{SL}(d,\mathbb{Z})$ is a lattice in $G = \mathbf{G}(\mathbb{R})$. For instance, $\mathrm{SL}(d,\mathbb{Z}) < \mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(d,1)(\mathbb{Z}) < \mathrm{SO}(d,1)(\mathbb{R})$.

Example 1.4 (Restriction of scalar)

Let $F/\mathbb{Q}$ be a number field and fix a basis of F over $\mathbb{Q}$. For any $\lambda \in F$, we let A_λ be the representation of the $\mathbb{Q}$-linear map $\lambda\cdot : x \in F \mapsto \lambda x \in F$. Let $\mathscr{A}_F$ be the image of F under the map $\lambda \mapsto A_\lambda \in \mathscr{A}_F \subset \mathbb{Q}^{d\times d}$. Then $\mathscr{A}_F$ is a subalgebra defined over $\mathbb{Q}$. For example, $F = \mathbb{Q}(\sqrt{a})$ for some $a \in \mathbb{Q}$ not a square. Then $\{1, \sqrt{a}\}$ form a $\mathbb{Q}$-basis of F. We have

$$\mathscr{A}_F = \left\{ \begin{bmatrix} x & ya \\ y & x \end{bmatrix} : x, y \in \mathbb{Q} \right\}.$$

Now let $\mathbf{G}$ be an algebraic subgroup of $\mathrm{SL}(n,\mathbb{C})$ defined over F. The restriction of scalar $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ is the following algebraic subgroup of $\mathrm{SL}(nd,\mathbb{C})$ defined over $\mathbb{Q}$:

$$\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} = \left\{ \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} : A_{ij} \in \mathscr{A}_F \text{ satisfy as blocks all equations that } \mathbf{G} \text{ satisfies} \right\}.$$

For example,

$$\mathrm{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : A_{ij} = \begin{bmatrix} x_{ij} & y_{ij}a \\ y_{ij} & x_{ij} \end{bmatrix}, A_{11}A_{22} - A_{12}A_{21} = \mathrm{id} \right\}.$$

Claim 1.5. $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{C}) \cong \mathbf{G}(\mathbb{C})^d$.

This claim follows from the following observation. Considering $\mathscr{A}_F$ as a linear variety in $\mathbb{C}^{d\times d}$. Then

(1) the $\mathbb{Q}$-points of $\mathscr{A}_F$ are isomorphic to F;
(2) the $\mathbb{R}$-points of $\mathscr{A}_F$ are isomorphic to $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$;
(3) the $\mathbb{C}$-points of $\mathscr{A}_F$ are isomorphic to $F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^d$.

Recall that $F = \mathbb{Q}(\lambda)$ for some $\lambda \in F$. Note that the characteristic polynomial of A_λ is the minimal polynomial of λ. Hence the eigenvalue of A_λ are the Galois conjugates of λ in $\mathbb{R}$ or in $\mathbb{C}$. We can diagonalize A_λ by some $g \in \mathrm{GL}(d,\mathbb{R})$ as

$$g^{-1}A_\lambda g = \mathrm{diag}(\varphi_1(\lambda), \cdots, \varphi_r(\lambda), \psi_1(\lambda), \cdots, \psi_s(\lambda)),$$

where $\varphi_i : F \to \mathbb{R}$ and $\psi_i : F \to \mathbb{C}$, $\psi_i(\lambda)$ can be viewed as 2×2-real matrix.

Now we conjugate $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$ by $\mathrm{diag}(g, \cdots, g)$, we obtain the following.

Claim 1.6. $\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{R}) \cong \prod_{\varphi:F\to\mathbb{R}} \mathbf{G}^\varphi(\mathbb{R}) \times \prod_{\text{pairs of } \varphi:F\to\mathbb{C}} \mathbf{G}^\varphi(\mathbb{C})$, where $\mathbf{G}^\varphi$ is the algebraic group defined by the polynomials f^φ for all relations f that $\mathbf{G}$ satisfies.

Example 1.7

Let F be a totally real number field and $\lambda \in F$ such that $\varphi(\lambda) > 0$ for precisely one Galois

embedding. Let
$$Q(x_1,\cdots,x_n,y)=x_1^2+\cdots+x_n^2-\lambda y^2.$$
Then $\mathbf{G}=\mathrm{SO}(Q)$ is a semisimple algebraic group defined over F if $n\geqslant 2$. Hence
$$\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbb{R})\cong\mathrm{SO}(n,1)(\mathbb{R})\times\mathrm{SO}(n+1,\mathbb{R})^{d-1},$$
which is also semisimple. Using the first example we know that $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{Z})$ is a lattice and hence the projection to $\mathrm{SO}(n,1)(\mathbb{R})$ is also a lattice.

Definition 1.8. Let G be a Lie group and Γ be a lattice. We say that Γ is **arithmetic** if there exists an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R})=G\times K$ for a compact group K, $\mathbb{G}(\mathbb{Z})<\mathbb{G}(\mathbb{R})$ is a lattice, and Γ is commensurable to a conjugate of the projection of $\mathbf{G}(\mathbb{Z})$ module K to G.

It is also worth noting that $\mathrm{SO}(n,1)(\mathbb{R})$ contains some non-arithmeticity lattices. An approach to construct non-arithmetic lattices is the following. We begin with two non compact arithmetic hyperbolic spaces $M_i=\Gamma_i\backslash\mathbb{H}^n$ and assume that they contain a same hyperbolic submanifold N. We then divide these them along N respectively and glue them back with exchanged pieces such that the resulting hyperbolic manifold M is still non compact. The non arithmeticity of M can be deduced from the following: the trace field for non-cocompact arithmetic lattices is $\mathbb{Q}$ and hence the length of closed geodesics are in $\exp(\mathbb{Q})$, but this is not always true for some weird ways of gluing manifolds.

§2 Lecture 2

2. Finite generation.

Theorem 2.1 (Garland-Raghunathan)

If G is a semisimple Lie group and $\Gamma<G$ is a lattice, then Γ is finitely generated.

We do not prove this theorem in this lecture. We will show the following proposition instead, which is easier to establish.

Proposition 2.2

If G is compactly generated and $\Gamma<G$ is a cocompact lattice, then Γ is finitely generated.

Proof. Let $Q\subset G$ be a compact subset such that $G=\bigcup_{n=1}^{\infty}Q^n$. Let $B\subset G$ be compact such that $\Gamma B=G$. Define $S:=\Gamma\cap(B\cup BQB^{-1})$, which is a finite set.

Claim 2.3. $BQ\subset SB$.

Indeed, let $b\in B, g\in Q$ then $bg=\gamma b_1$ with $\gamma\in\Gamma, b_1\in B$. Then $\gamma=bgb_1^{-1}\in S$.

Therefore, $BQ^n\subset S^nB$ and hence $G\subset\langle S\rangle B$. For any $\gamma\in\Gamma$, there exists some $\eta\in\langle S\rangle$ and $b\in B$ with $\gamma=\eta b$. Note that $b=\eta^{-1}\gamma\in\Gamma\cap B\subset S$, hence $\gamma\in\langle S\rangle$. □

3. Trace fields.

Proposition 2.4

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})$ has no compact factors. Let $\Gamma < G$ be a lattice. Then

$$F := \mathbb{Q}(\{ \operatorname{tr}(\operatorname{Ad}_\gamma) : \gamma \in \Gamma \})$$

is a finitely generated field. Moreover, there exists an algebraic group $\mathbf{G}^{\mathrm{ad}}$ defined over F and an algebraic isogeny $\varphi : \mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ defined over $\mathbb{R}$ such that $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(F)$.

Proof. We define the map $T : h \in \mathbf{G} \mapsto \operatorname{tr}(\operatorname{Ad}_h)$, which is a polynomial function on $\mathbf{G}$. For every $g \in \mathbf{G}$, we have that $g.T : h \in \mathbf{G} \mapsto \operatorname{tr}(\operatorname{Ad}_{hg})$ is another polynomial of the same degree. Hence $V = \langle g.T : g \in \mathbf{G} \rangle$ is finite dimensional.

Claim 2.5. $V = \langle \gamma.T : \gamma \in \Gamma \rangle$.

Proof. Because the right hand side $W = \langle \gamma.T : \gamma \in \Gamma \rangle$ satisfies $\gamma.W = W$ for all $\gamma \in \Gamma$. By Borel density (Γ is Zariski dense in $\mathbf{G}$), this implies that W is invariant for every $g \in \mathbf{G}$ and hence $W = W$ by the definition of V. □

Let $\gamma_1, \cdots, \gamma_n \in \Gamma$ be such that $\{ \gamma_i.T \}$ forms a basis of V. We define $\varphi(g)$ to be the matrix representation of g. on V with respect to the basis $\{ \gamma_i.T \}$. Then $\varphi(g) \in \mathrm{GL}(n, \mathbb{C})$ and we take $\mathbf{G}^{\mathrm{ad}}$ to be the image of φ.

Exercise 2.6. (1) Use Borel density to show that $\{ \gamma_i.T|_\Gamma \}$ is linearly independent.
(2) Moreover, there exists $s_1, \cdots, s_n \in \Gamma$ such that $\{ \gamma_i.T \}$ is linearly independent restricted to $\{ s_1, \cdots, s_n \}$.

Consequently, $A = \left[\operatorname{tr}(\operatorname{Ad}_{s_i\gamma_j})\right]_{1 \leqslant i,j \leqslant n} \in \mathrm{GL}(n, \mathbb{C})$. Fix j and conclude that

$$\gamma\gamma_j.T = \sum_i \varphi(\gamma)_{ij} \gamma_i.T.$$

Now we evaluate this polynomial on s_k, we obtain

$$\gamma\gamma_i.T(s_k) = \sum_i \varphi(\gamma)_{ij} \gamma_i.T(s_k) = \sum_i \varphi(\gamma)_{ij} A_{kj}.$$

On the other hand, $\gamma\gamma_i.T(s_k) = \operatorname{tr}(\operatorname{Ad}_{s_k\gamma\gamma_i}) \in F$. Hence $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(F)$. By Borel density,

$$\overline{\mathbf{G}^{\mathrm{ad}}(F)}^{\mathrm{Zar}} = \varphi(\overline{\Gamma}^{\mathrm{Zar}}) = \varphi(\mathbf{G}) = \mathbf{G}^{\mathrm{ad}}.$$

Hence $\mathbf{G}^{\mathrm{ad}}$ is defined over F.

Finally, recall that Γ is finitely generated by some $S \subset \Gamma$. Let $L \subset F$ be the field generated by the matrix entries of $\varphi(\gamma)$ for $\gamma \in S$. Then L is finitely generated and $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(L)$. This implies that both $\mathbf{G}^{\mathrm{ad}}$ and its Lie algebra are defined over L. We conclude that $\operatorname{tr}(\operatorname{Ad}_\gamma)$ calculated after applying the derivative of φ inside the Lie algebra of $\mathbf{G}^{\mathrm{ad}}$ gives values in L. We obtain $F \subset L \subset F$ and hence $L = F$. □

4. Margulis's strategy for arithmeticity.

Suppose $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$ and $\Gamma \subset \mathbf{G}(F)$, F is finitely generated satisfying $F \subset \mathbb{R}$. Let $\Bbbk$ be a local field and $\varphi : F \to \Bbbk$ be a Galois embedding. Let $\mathbf{H} = \mathbf{G}^{\varphi}$ be the algebraic $\Bbbk$-group obtained by applying φ to the coefficient of the elements of $\mathbf{G}$. Then $\varphi(\Gamma) \subset \mathbf{H}(\Bbbk)$ is Zariski dense by Borel density theorem.

Claim 2.7. Suppose for any such $\Bbbk$ and any group homomorphism $\varphi_\Gamma : \Gamma \to H = \mathbf{H}(\Bbbk)$ one of the followings holds:

- φ_Γ has a continuous extension to G, or
- φ_Γ has bounded image, i.e. $\overline{\varphi_\Gamma(\Gamma)} \subset \mathbf{H}(\Gamma)$ is compact in H.

Then Γ is arithmetic.

Notation 2.8. $\overline{V}^{\mathrm{Zar}}$ is the closure in Zariski topology and $\overline{V}$ is the closure in Hausdorff topology induced by the local field.

§3 Lecture 3

This time, we aim to show the claim mentioned at the end of last course.

Step 1: F is a number field. Suppose for a contradiction that $\mathrm{tr}(\mathrm{Ad}_{\gamma_0}) = x_0 \in F$ is transcendental for some $\gamma_0 \in \Gamma$. Pick p to be a prime and some transcendental $x_0' \in \mathbb{Q}_p$ with $\|x_0'\|_p > 1$. We can find a finite field extension $\Bbbk/\mathbb{Q}_p$ and $\varphi : F \to \Bbbk$ with $x_0 \mapsto x_0'$.

We apply the assumption for this φ and $\Bbbk$:

- φ_Γ cannot have a continuous extension $\varphi_G : G \to H$. Notice that G° is connected and hence $\varphi_G(G^\circ) = \{\,\mathrm{id}\,\}$. This implies that $\varphi_G(G)$ is finite, which contradicts the Zariski density.
- $\overline{\varphi_\Gamma(\Gamma)}$ cannot be compact because $\|\,\mathrm{tr}\,\mathrm{Ad}_{\varphi_\Gamma(\gamma_0)}\|_p = \|x_0'\|_p > 1$ and hence $\mathrm{Ad}_{\varphi_\Gamma(\gamma_0)}$ has an eigenvalue larger than 1.

Step 2: Γ is "almost integral". For simplicity, we assume that $F = \mathbb{Q}$. As Γ is finitely generated, there exists primes $p_1, \cdots, p_\ell \in \mathbb{N}$ such that $\Gamma \subset \mathbf{G}(\mathbb{Z}[1/(p_1 \cdots p_\ell)])$. Applying the assumption for $\varphi : \mathbb{Q} \to \mathbb{Q}_p$ with $p = p_j$, we have that $\overline{\varphi_\Gamma(\Gamma)}$ is compact. This means that all $\gamma \in \Gamma$ have entries where the powers of p in the denominator is bounded. In other words, since $\mathbb{H}(\mathbb{Z}_p)$ is compact open and $\overline{\varphi_\Gamma(\Gamma)}$ is compact, we have $\overline{\varphi_\Gamma(\Gamma)} \cap \mathbb{H}(\mathbb{Z}_p)$ has finite index in $\overline{\varphi_\Gamma(\Gamma)}$.

Applying this for all primes $p = p_j$, we obtain that $[\Gamma : \Gamma \cap \mathbf{G}(\mathbb{Z})] < \infty$. For general fields F, this argument shows that $[\Gamma : \Gamma \cap \mathbf{G}(\mathcal{O}_F)] < \infty$.

Step 3: Informations from the real and complex φ's.
Case 1. φ_Γ has a continuous extension.

Claim 3.1. In this case $\varphi = \mathrm{id} : F \hookrightarrow \mathbb{R}$.

Proof. If $\Bbbk = \mathbb{R}$ then φ is clearly an isogeny. Hence calculating the trace in the Lie algebras of H and G gives the same. This gives $\varphi(\mathrm{tr}\,\mathrm{Ad}_\gamma) = \mathrm{tr}\,\mathrm{Ad}_\gamma$ for $\gamma \in \Gamma$ and hence $\varphi = \mathrm{id}$ as F is generated by traces.

Suppose $\Bbbk = \mathbb{C}$. Let $\mathfrak{m}$ be the image of the real Lie algebra of G under the derivative of φ_G. Let $\mathfrak{h}$ be the complex Lie algebra of H. Then $\mathfrak{m}$ is an $\mathbb{R}$ Lie subalgebra of $\mathfrak{h}$. Note that $\mathfrak{m}_\mathbb{C}$ is preserved by $\varphi_\Gamma(\Gamma)$ and hence preserved by $\mathbf{H}$. Therefore $\mathfrak{m}_\mathbb{C}$ is an ideal in $\mathfrak{h}$. Since $\varphi_\Gamma(\Gamma)$ is Zariski dense in $\mathbf{H}$, $\mathfrak{m}_\mathbb{C}$ must be $\mathfrak{h}$ itself. Now we take an $\mathbb{R}$-basis of the Lie algebra of G. The pushforward of this basis under the derivative φ_G is an $\mathbb{R}$-basis of $\mathfrak{m}$ and hence a $\mathbb{C}$-basis of $\mathfrak{m}_{\mathbb{C}} = \mathfrak{h}$. Applying the same argument with the case $\Bbbk = \mathbb{R}$, we obtain that $\varphi|_F = \mathrm{id}$. $\square$

Case 2. $\overline{\varphi_\Gamma(\Gamma)}$ is compact in H.

Claim 3.2. $\varphi(F) \subset \mathbb{R}$ and H is compact.

Proof. Let $M = \overline{\varphi_\Gamma(\Gamma)} \subset H$, which is compact by the assumption. Let $\mathfrak{m}$ be the real Lie algebra of M. Then $\mathfrak{m}_{\mathbb{C}} = \mathfrak{h}$ by the same argument. Moreover, if $\Bbbk = \mathbb{C}$ then $\mathfrak{m} \cap (i\mathfrak{m}) = \varnothing$. This is because for every $v \in \mathfrak{m} \cap (i\mathfrak{m})$, the exponential map $\exp : t \mapsto \exp(tv) \in M$ is a bounded entire function over $\mathbb{C}$ and hence $v = 0$.

We obtain that if $\Bbbk = \mathbb{C}$ then $\mathfrak{h} = \mathfrak{m} \oplus i\mathfrak{m}$. Using a $\mathbb{R}$-basis of $\mathfrak{m}$ as a $\mathbb{C}$-basis of $\mathfrak{h}$, we see that $\varphi(F) \subset \mathbb{R} \subset \mathbb{C}$. So we can assume without loss of generality that $\Bbbk = \mathbb{R}$.

As in the real world, every compact subgroups are algebraic. Because M is compact and Zariski dense in $\mathbf{H}$, we obtain that H is compact. $\square$

We conclude what we have obtained from different embeddings.

(1) F is a number field.
(2) $\Gamma \cap \mathbf{G}(\mathcal{O}_F)$ is finite index in Γ.
(3) For $\varphi : F \to \mathbb{R}/\mathbb{C}$ with continuous extensions: $\varphi|_F = \mathrm{id}$.
(4) For $\varphi : F \to \mathbb{R}/\mathbb{C}$ with compact closures, $\varphi(F) \subset \mathbb{R}$ and $\mathbf{G}^\varphi(\mathbb{R})$ is compact.

Now we apply $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ and obtain a new semisimple algebraic group $\mathbb{Q}$. Its group of $\mathbb{R}$-points is isomorphic to $\mathbf{G}^{\mathrm{id}}(\mathbb{R}) \times K$, where $K = \prod_{\varphi \neq \mathrm{id}, \varphi: F \to \mathbb{R}} \mathbf{G}^\varphi(\mathbb{R})$ is compact. Moreover (by choosing a $\mathbb{Z}$-basis of $\mathcal{O}_F$ in the construction of $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$) we can ensure that $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})(\mathbb{Z}) \cong \mathbf{G}(\mathcal{O}_K)$. Finally, projecting module K we obtain the arithmetic lattice $\mathbf{G}^{\mathrm{id}}(\mathcal{O}_F) \subset G$. As $\Gamma \cap \mathbf{G}^{\mathrm{id}}(\mathcal{O}_F)$ has finite index in Γ, we obtain that Γ is arithmetic. $\square$

§4 Lecture 4

5. Superrigidity.

Theorem 4.1 (Margulis's Superrigidity)
Let $G = \mathrm{SL}(3,\mathbb{R})$ and Γ be a lattice. Let $\Bbbk$ be a local field and $\mathbf{H}$ be a simple adjoint algebraic group over $\Bbbk$. Let $\varphi : \Gamma \to H = \mathbf{H}(\Bbbk)$ a homomorphism with a Zariski dense image. Then one of the following must hold:

(1) φ has a continuous extension $\varphi_G : G \to H$, or
(2) $\overline{\varphi(\Gamma)}$ is compact in H.

This theorem implies the arithmeticity by Claim 2.7.

6. Getting started for $\mathrm{SL}(3,\mathbb{R})$.

Let $U < \mathrm{SL}(3,\mathbb{R})$ be a root subgroup, for example $\left\{ \begin{bmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{bmatrix} \right\}$. Then U acts ergodically on $X = \Gamma\backslash G$ by Moore's ergodic theorem. Let $x_0 \in X$ be a U-generic point for U, that is

$$\frac{1}{T}\int_0^T \delta_{x_0 u_t}\, \mathrm{d}t \xrightarrow{w*} m_X \quad \text{as } T \to \infty.$$

Let V be an irreducible representation of H over $\Bbbk$. Then H acts on $\mathbb{P}(V)$ without fixed points. Restricting on $\varphi(\Gamma)$ this remains true.

Although this superrigidity also states for $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, we can keep in mind that H is a p-adic Lie group but G is a real Lie group. In this case, φ is the only thing links these two group. So we may consider the space

$$\widetilde{X} = \Gamma \backslash (G \times \mathbb{P}(V)),$$

where γ acts on $G \times \mathbb{P}(V)$ as $(g, [v]) \mapsto (\gamma g, \varphi(\gamma)[v])$ diagonally. Note that the projection $\widetilde{X} \to X, \Gamma(g, [v]) \mapsto \Gamma g$ is a nice factor map (projecting to $\mathbb{P}(V)$ is not nice since Γ acting on $\mathbb{P}(V)$ is not properly discontinuously).

Let $\widetilde{x}_0$ be any point in $\widetilde{X}$ mapping to x_0. Let

$$\mu_T = \frac{1}{T} \int_0^T \delta_{\widetilde{x}_0 u_t} \, \mathrm{d}t.$$

Suppose $\mu_T \to \mu$ along a subset of T's, then μ satisifies

- μ is U-invariant, and
- μ is a probability measure projecting to m_X.

7. Getting started for $\mathrm{SO}(d,1)(\mathbb{R})$.

We skip this part for the moment. This will be discussed in Lecture 8.

8. A measure-valued map.

We are given a subgroup $S < G$ (for example, $S = U$), an extension $\widetilde{X} = \Gamma \backslash (G \times \mathbb{P}(V))$ and an S-invariant measure μ on $\widetilde{X}$ projecting to m_X.

We unfold $\widetilde{X}$ to create an infinite Γ invariant measure $\widetilde{\mu}$ on $G \times \mathbb{P}(V)$. We want to use conditional measures for the σ-algebra $\mathscr{C} = \mathscr{B}_G \times \mathscr{W}_{\mathbb{P}(V)}$, where $\mathscr{W}_{\mathbb{P}(V)}$ is the trivial σ-algebra on $\mathbb{P}(V)$. This way we get a measurable map

$$g \times G \to \delta_g \times \nu_g,$$

where ν_g is a probability measure on $\mathbb{P}(V)$. Moreover, $\widetilde{\mu}$ is invariant under S. Hence the conditional measure satisfy a resulting compatibility. In this case, we obtain $\nu_{gs} = \nu_g$ for $s \in S$ and almost every g.

Also $\widetilde{\mu}$ is Γ invariant. Then the conditional measure also satisfies

$$\delta_{\gamma g} \times \nu_{\gamma g} = \gamma_* (\delta_g \times \nu_g) = \delta_{\gamma g} \times (\varphi(\gamma)_* \nu_g),$$

and hence $\nu_{\gamma g} = \varphi(\gamma)_* \nu_g$ for γ and almost every $g \in G$. We can interpret this as a measurable Γ-equivariant map

$$\phi : G/S \to \mathcal{M}^1(\mathbb{P}(V)), \quad gS \mapsto \nu_g.$$

9. Locally closed orbits.

Let $V = \mathbf{V}(\Bbbk)$ be a variety over a local field $\Bbbk$. Let $H = \mathbf{H}(\Bbbk)$ act algebraically on V. We want to understand H-orbits and H-ergodic measures on V.

Claim 4.2. *H-orbits are locally close, i.e. for any $v \in V$ there exists a neighborhood B of v so that $B \cap \overline{Hv} = B \cap Hv$.*

Proof. $h \in H \mapsto h.v \in V$ is an algebraic map (possibly with a non-trivial stabilizer). Using that a polynomial regular map will only miss points from a lower dimension subvariety of the Zariski closure of the image, one can choose B. □

Corollary 4.3

Let μ be a measure on V that is H-ergodic. Then there is some $v \in V$ such that μ gives full measure to $H.v$.

§5 Lecture 5

Proof. Let $B_1, B_2, \cdots$ be a basis of the topology of V. For any n we apply the assumed ergodicity to $H.B_n$. Hence we have $\mu(H.B_n) = \varnothing$ or $\mu(V \setminus H.B_n) = \varnothing$. We take the union of these null sets and suppose v_0, v_1 do not belong to these null sets.

Claim 5.1. $H.v_0 = H.v_1$.

Proof. By the local closeness, we can take $B_{n_0} \ni v_0$ such that $B_{n_0} \cap \overline{H.v_0} = B_{n_0} \cap H.v_0$. Since v_0 does not belong to these null set, we have $\mu(H.B_{n_0}) > 0$. Consequently, $\mu(V \setminus H.B_{n_0}) = 0$ and hence $v_1 \in H.B_{n_0}$. Then we can take some $h_1 \in H$ such that $h_1 v_1 \in B_{n_0}$.

Assume that $h_1 v_1 \notin H.v_0 \cap B_{n_0}$. By the local discreteness of the orbits, we can take some $B_{n_1} \ni h_1 v_1$ such that $B_{n_1} \subset B_{n_0}$ and $B_{n_1} \cap H.v_0 = \varnothing$. A same deduction as above, we have $\mu(H.B_{n_1}) > 0$ and $v_0 \in H.B_{n_1}$. This contradicts $B_{n_1} \cap H.v_0 = \varnothing$. □

□

Proposition 5.2

Let $H = \mathbf{H}(\Bbbk)$ act algebraically on $V = \mathbf{V}(\Bbbk)$. Let μ be a Borel probability measure on V. Then $\mathrm{Stab}_H(\mu) = \{ h \in H : h_* \mu = \mu \}$ is a compact extension of the $\Bbbk$-points of the algebraic group $\mathrm{Fix}_{\mathbf{H}}(\mu) = \{ h : h.v = v, \forall v \in \operatorname{supp} \mu \}$.

Sketch of the proof. Without loss of generality, we can assume that $\operatorname{supp} \mu$ is Zariski dense in $\mathbf{V}$. If $h \in \mathrm{Stab}_H(\mu)$ then h normalizes $\mathrm{Fix}_{\mathbf{H}}(\mu)$. By taking the quotient we can assume that $\mathrm{Fix}_{\mathbf{H}}(\mu) = \{ \mathrm{id} \}$. Then we can find a finite set $\{ v_1, \cdots, v_n \} \subset \operatorname{supp} \mu$ such that $\dim \mathrm{Fix}_{\mathbf{H}}(v_1, \cdots, v_n) = 0$.

> **Exercise 5.3.** For all $(v_1', \cdots, v_n')$ in a sufficiently small neighborhood of $(v_1, \cdots, v_n)$, we have $\dim \mathrm{Fix}_{\mathbf{H}}(v_1', \cdots, v_n') = 0$.

Assuming by contradiction that $\mathrm{Stab}_{\mathbf{H}}(\mu)$ is non compact, we can apply Poincaré recurrence. Choose $(v_1', \cdots, v_n')$ near $(v_1, \cdots, v_n)$ which is infinitely recurrent under the $\mathrm{Stab}_{\mathbf{H}}(\mu)$ action. But the orbits are locally closed. Therefore there are infinitely many $h \in \mathrm{Stab}_{\mathbf{H}}(\mu)$ fixing $(v_1', \cdots, v_n')$. This contradicts $\dim \mathrm{Fix}_{\mathbf{H}}(v_1', \cdots, v_n') = 0$. □

Proposition 5.4 (Zimmer)

The H-actions on $\mathcal{M}^1(\mathbb{P}(V))$ has locally closed orbits.

10. Creating a map with values in H/L.

Recall that we have a Γ-equivariant map

$$\phi : G/U \to \mathcal{M}^1(\mathbb{P}(V)).$$

Let $m_{G/U}$ be a smooth measure on G/U. By ergodicity of U on $\Gamma\backslash G$, we have by duality that the Γ-action on $(G/U, m_{G/U})$ is ergodic. Hence $\phi_*(m_{G/U})$ is a Γ-ergodic measure on $\mathcal{M}^1(\mathbb{P}(V))$. So it is also H-ergodic. Since H-orbits on $\mathcal{M}^1(\mathbb{P}(V))$ are locally closed, $\phi_*(m_{G/U})$ gives the full measure to a single H-orbit $H.\nu_0$. That is,

$$\phi : G/U \to H.\nu_0 \cong H/\operatorname{Stab}_H(\nu_0) \quad \text{a.s..}$$

We distinguish two cases:

(1) $\operatorname{Stab}_H(\nu_0)$ is non-compact. Then we take $\mathbf{L}_0 = \operatorname{Fix}_{\mathbf{H}}(\operatorname{supp}\nu_0)$ satisfying that $\mathbf{L}(k)$ is non compact but not all of H. In this case $\operatorname{Stab}_H(\nu_0) \subset N_H(\mathbf{L}_0) = \mathbf{L}$, where $\mathbf{L}$ is a proper algebraic subgroup of $\mathbf{H}$.

(2) $\operatorname{Stab}_H(\nu_0)$ is compact.

Hence these two cases come to be

(1) There exists a Γ-equivariant $\phi : G/U \to H/L$ for $L = \mathbf{L}(\Bbbk)$ and $\mathbf{L} < \mathbf{H}$ a proper $\Bbbk$-subgroup.

(2) There exists a Γ-equivariant $\phi : G/U \to H/L$ where $L < H$ is compact.

§6 Lecture 6

Let us recall our strategy to establish Margulis superrigidity:

(1) Consider a root group U acting ergodically on $X = \Gamma\backslash G$ with a generic point x_0.

(2) Using an $\widetilde{x}_0 \in \widetilde{X} = \Gamma\backslash(G \times \mathbb{P}(V))$ to construct a lifting measure $\widetilde{\mu}$ on $\widetilde{X}$ which is U-invariant and projects to m_X.

(3) Consider the conditional measure of m_X, which gives a U-invariant and Γ-equivariant map

$$\phi : G \to \mathcal{M}^1(\mathbb{P}(V)).$$

(4) By the ergodicity of $\Gamma \curvearrowright G/U$ and the local closeness of H-orbits on $\mathcal{M}^1(\mathbb{P}(V))$, we know that $\phi(G)$ falls in one H-orbit $H.\nu_0 \cong H/\operatorname{Stab}_H(\nu_0)$.

(5) By studying the algebraic structure of $\operatorname{Stab}_H(\nu_0)$, there are only two cases should be considered:
   - There exists a Γ-equivariant $\phi : G/U \to H/L$ for $L = \mathbf{L}(\Bbbk)$ and $\mathbf{L} < \mathbf{H}$ a proper $\Bbbk$-subgroup.
   - There exists a Γ-equivariant $\phi : G/U \to H/L$ where $L < H$ is compact.

11. Metric ergodicity (Bader-Gelander)

We now consider the case $\phi : G/U \to H/L$ where L is compact. In this case, H/L has an H-invariant metric.

Lemma 6.1

Let S be a unbounded subgroup of a simple group G. Let $\phi : G/S \to Y$ be continuous and G-equivariant for an action of G on Y preserving a metric on Y. Then ϕ is constant.

Proof. **Case 1.** Assume that S contains some diagonalizable element a. Let $u \in U = G_a^+$ then $a^n u a^{-n} \to \mathrm{id}$. Then

$$a^n \phi(uS) = \phi(a^n uS) = \phi(a^n u a^{-n} S) \to \phi(\mathrm{id}S), \quad n \to +\infty.$$

Note that $d(a^n\phi(uS), \phi(\mathrm{id}S)) = d(\phi(uS), a^{-n}\phi(\mathrm{id}S))$, we also have

$$\phi(\mathrm{id}S) = a^{-n}\phi(\mathrm{id}S) \to \phi(uS).$$

Hence ϕ is u-invariant. Noting that G_a^+, G_a^- generates G, we obtain that ϕ is constant.

Case 2. Assume that $S \supset U$ a unipotent subgroup. We think the case that $G = \mathrm{SL}(2, \mathbb{R})$ and $U = \left\{ \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \right\}$. Let $v_n = \begin{bmatrix} 1 & \\ \frac{1}{n} & 1 \end{bmatrix}$ and then there exists $u_n, u_n' \in U$ such that $u_n v_n u_n' \to \begin{bmatrix} 2 & \\ & \frac{1}{2} \end{bmatrix} = a$. Then we have $u_n \phi(v_n S) \to \phi(aS)$ and $\phi(v_n S) \to \phi(\mathrm{id}S)$. Since the metric is G-invariant, we have $u_n\phi(v_n S) \to u_n \phi(\mathrm{id}S) = \phi(\mathrm{id}S)$ and hence $\phi(aS) = \phi(\mathrm{id}S)$. This argument works for every a so we obtain that ϕ is A-invariant. Then we can apply the result of the first case. □

But in our case, the map ϕ is only measurable and Γ-equivariant. The assumption of this lemma is too strong to apply. We need to apply the lemma to another map associated to ϕ.

Theorem 6.2

Let $G = \mathrm{GL}(3, \mathbb{R}) \supset U$ a root group, $\Gamma < G$ a lattice and $\phi : G/U \to H/L$ a measurable Γ-equivariant map, where $L < H$ is compact so that H/L has an H-invariant metric. Then ϕ is constant almost surely. In particular, $\overline{\varphi_\Gamma(\Gamma)} \subset H$ is compact.

Proof of "in particular". If $\phi(gU) \equiv h_0 L$ for m_G-almost every $g \in G$. Then

$$\varphi(\gamma) h_0 L \doteq \phi(\gamma g U) \doteq h_0 L, \quad \forall \gamma \in \Gamma,$$

here $\doteq$ denotes the almost surely equality. Therefore, $\overline{\varphi_\Gamma(\Gamma)} \in h_0 L h_0^{-1}$ which is compact. □

Proof. Replacing the metric on H/L be a Γ-equivariant one, we may assume that the metric is bounded. We define

$$Y = L(G, H/L)^\Gamma := \{ \Gamma\text{-equivariant measurable maps from } G \text{ to } H \}.$$

We endow $Y = L(G, H/L)^\Gamma$ with the metric

$$d_Y(f_1, f_2) = \int_F d_{H/L}(f_1(g), f_2(g)) \, \mathrm{d}m_G(g),$$

where F is a fundamental domain of Γ. The action of G on Y is given by

$$\forall g_0 \in G, f \in Y, \quad g_0.f := (g \in G \mapsto f(gg_0)) \in Y.$$

Exercise 6.3. Show that the G-action is continuous and isometric on Y.

Now we define a new map $\widetilde{\phi} : G/U \to Y$ given by

$$g_0 U \mapsto (g \in G \mapsto \phi(gg_0U) \in H/L).$$

Note that this map is also G-invariant. By applying the lemma to $\widetilde{\phi}$, we know that ϕ is a constant almost surely. □

12. Algebraic T-shadows (Bader-Furman)

This concept occurs in the study of **algebraic representations of ergodic actions (AREA)**. Recall that we want to study the Γ-equivariant map $\phi : G/U \to H/L$ for $L = \mathbf{L}(\Bbbk)$ noncompact and $\mathbf{L} < \mathbf{H}$ a proper $\Bbbk$-subgroup.

Definition 6.4. Let $T < G$ be unbounded. A measurable map $\psi : G \to H/L$ is called **an algebraic T-shadow** (for the ($\Gamma \times T$)-space G) if

1. $L = \mathbf{L}(\Bbbk)$ for an algebraic subgroup $\mathbf{L} < \mathbf{H}$ over $\Bbbk$.
2. ψ is measurable and defined almost everywhere.
3. For every $\gamma \in \Gamma$, $\psi(\gamma g) = \varphi(\gamma)\psi(g)$ almost everywhere.
4. For every $t \in T$, there exists $\tau(t) \in N_H(L)/L$ so that

$$\psi(gt) = \psi(g)\tau(t), \quad \text{a.e..}$$

Lemma 6.5

τ is uniquely determined by the definition and τ is a measurable (hence continuous) homomorphism $\tau : T \to N_H(L)/L$.

Lemma 6.6

If $\psi : G \to H/L$ is a T_j-shadow for $j = 1, \cdots, \ell$ and $T = \langle T_1, \cdots, T_\ell \rangle$, then ψ is also a T-shadow.

§7 Lecture 7

Aim 7.1. To show ψ is a T-shadow for large T.

Lemma 7.2 (G-shadow)

Suppose $\psi : G \to H/L$ is a G-shadow. Then L is a normal subgroup of H and there exists an $h_0 \in H$ so that $\tau(\gamma) = h_0\gamma h_0^{-1} \in H/L$. In particular, if H is simple, adjoint and $L \neq H$ then $L = \{ \mathrm{id} \}$.

Proof. For every g, we have $\psi(g_0 g)L = \psi(g_0)\tau(g)L$ for almost every g_0. By a Fubini argument, for almost every g_0, we have (without loss of generality)

$$\psi(g_0 g)L = \psi(g_0)\tau(g)L, \quad \forall g \in G.$$

Let $\psi(g_0) = h_0 L$. Then for every $\gamma \in \Gamma$,

$$\psi(\gamma g_0 g) = \gamma\psi(g_0 g) \subset \gamma h_0 N_H(L)/L.$$

On the other hand, we have

$$\psi(\gamma g_0 g) = \psi(g_0(g_0^{-1}\gamma g_0)g) = \psi(g_0)\tau(g_0^{-1}\gamma g_0)\tau(g) \subset h_0 N_H(L)/L.$$

Hence we obtain $h_0^{-1}\Gamma h_0 \in N_H(L)$ and hence $N_H(L) = H$ by the Zariski density of Γ. $\square$

Definition 7.3. Let $\psi_1 : G \to H/L_1$ and $\psi_2 : G \to H/L_2$ be two T-shadows. We say that ψ_2 is a **factor** of ψ_1 if there exists $p \in H$ with $L_1 p \subset pL_2$ and the following diagram commutes:

$$\begin{tikzcd} & G/L_1 \arrow[dd, "\cdot pL_2 \cdot"] \\ G \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\ & G/L_2 \end{tikzcd}$$

Lemma 7.4

Assuming ψ_2 is a factor of ψ_1 and $p \in H$ is as in the definition. Then

$$\tau_1(t)pL_2 = p\tau_2(t)L_2, \quad \forall t \in T.$$

Proposition 7.5 (Initial T-shadow)

Assume that T is unbounded. There exists a T-shadow $\psi : G \to H/L_{\min}$ so that any other T-shadow is a factor. In fact, every T-shadow with $\mathbf{L}_{\min} = \overline{L_{\min}}^{\text{Zar}}$ minimal in the set of all such Zariski closures is an initial T-shadow as above.

Corollary 7.6 (Normalizer)

Let $\psi_{\min} : G \to H/L_{\min}$ be an initial T-shadow as in the proposition. Then $\psi_{\min}$ is also an initial $N_G(T)$-shadow.

Proof. Let $a \in N_G(T)$, we define a new T-shadow $\psi_a : G \to H/L_{\min}$ by $\psi_a(g) = \psi_{\min}(ga)$ and $\tau_a(t) = \tau_{\min}(a^{-1}ta)$. Then

$$\psi_a(gt) = \psi_{\min}(gta) = \psi_{\min}(ga)\tau_{\min}(a^{-1}ta) = \psi_a(g)\tau_a(t).$$

Noting that $\tau_{\min}$ is initial, there exists $p = p_a$ such that $\psi_a = (\cdot pL_{\min}) \circ \psi_{\min}$. That is, $\psi_{\min}(ga)L_{\min} = \psi_{\min}(g)p_aL_{\min}$. Therefore we obtain a $N_G(T)$-shadow by letting $\tau(a) = p_a$. □

13. Conclusion for $\mathrm{SL}(3, \mathbb{R})$.

Aim 7.7. Start with a U-shadow and end up with a G-shadow.

Proof. For the $\mathrm{SL}(3,\mathbb{R})$ case, the root space can be generated by $\alpha, \beta, \gamma, -\alpha, -\beta, -\gamma$. We start with $U = U_\alpha$. By the proposition and the corollary, there exists an initial U_α-shadow $\psi_{\min}$, which is also a U_β-shadow. Applying the corollary again, we can find an initial U_β-shadow $\psi'_{\min}$ (which a priori can be greater than $\psi_{\min}$). Then we have that $\psi'_{\min}$ is also U_γ-shadow. We continue this process and will turn back to get a U_α-shadow.

This means that $\psi'_{\min}$ can not be better than $\psi_{\min}$. Moreover, $\psi_{\min}$ is an initial T-shadow for $T = U_\alpha, U_\beta, U_\gamma, U_{-\alpha}, U_{-\beta}, U_{-\gamma}$. These root groups generate G and hence $\psi_{\min}$ is a G-shadow. Therefore L is trivial and we obtain a continuous extension of φ_Γ. □

Proof idea for Proposition 7.5. Given $\psi_{\min}$ and ψ, we define

$$\psi_V(g) = (\psi_{\min}(g), \psi(g)) \in V = H/L_{\min} \times H/L.$$

Then $\mathbf{M} = \overline{\{(\tau_{\min}(t), \tau(t)) : t \in T\}}^{\mathrm{Zar}} \subset N_{\mathbf{H}}(\mathbf{L}_{\min}) \times N_{\mathbf{H}}(\mathbf{L})$ and $\mathbf{V} = \overline{V}^{\mathrm{Zar}}$ is homogeneous for $\mathbf{H}$ acting diagonally. By the ergodicity and locally-closed orbits we obtain that ψ_V takes values in only one $H \times M$-orbit. □

§8 Lecture 8

7. Getting started for $\mathrm{SO}(d,1)(\mathbb{R})$.

Now we will discuss about this skipped part. First, we discuss about the totally geodesic submanifolds in $\Gamma\backslash\mathbb{H}^d$.

For the case of $d = 3$, $\mathbb{H}^3$ can be interpreted as $\mathrm{SO}(3,1)(\mathbb{R})^\circ/\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{R})$. In $\mathbb{H}^3$, there is a standard embedded $\mathbb{H}^2$ as

$$\mathbb{H}^2 \cong \mathrm{SO}(2,1)(\mathbb{R})^\circ \mathrm{SO}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R}) \subset \mathrm{SO}(3,1)(\mathbb{R})^\circ/\mathrm{SO}(3,\mathbb{R}),$$

or

$$\mathrm{SL}(2,\mathbb{R})/\mathrm{SU}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2,\mathbb{R}).$$

Then $g\mathrm{SO}(2,1)(\mathbb{R})^\circ \mathrm{SO}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$ for $g \in \mathrm{SO}(3,1)(\mathbb{R})^\circ$ or $g\mathrm{SL}(2,\mathbb{R})/\mathrm{SU}(2,\mathbb{R})$ for $g \in \mathrm{SL}(2,\mathbb{C})$ give the algebraic description of two-dimensional hyperbolic planes inside $\mathbb{H}^3$.

The totally geodesic (closed) 2-dimensional subspace of $M = \Gamma\backslash\mathbb{H}^3$ are precisely of the form

$$\Gamma g \mathrm{SL}(2,\mathbb{R})/\mathrm{SU}(2,\mathbb{R}) \subset M$$

if the set is closed. Any closed totally geodesic plane in M corresponds this way to a closed orbit

$$\Gamma g \mathrm{SL}(2,\mathbb{R}) \subset X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C}).$$

Lemma 8.1 (Dani's argument)
These closed orbits always have finite volume.

The proof uses a version of Margulis-Dani's nondivergence:

Theorem 8.2 (Nondivergence)
Given $\varepsilon > 0$ and a compact $A \subset X$, there exists a compact $B \subset X$ so that for all $x \in A$ and $T > 0$ we have

$$\frac{1}{T}|\{t \in [0,T] : xu_t \in B\}| > 1 - \varepsilon.$$

Proof. Consider the Haar measure μ on the closed $\mathrm{SL}(2,\mathbb{R})$-orbit. We apply the nondivergence for the unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then we can find a compact B contained in the closed orbit. Consider the function

$$f = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}_B(\cdot u_t)\,\mathrm{d}t.$$

Exercise 8.3. Show that $f \in L^2(\mu)$.

Then f is a $\{u_t\}$-invariant function. By Mautner's phenomenon, f is $\mathrm{SL}(2,\mathbb{R})$-invariant. Hence f is constant and $f > 0$ on A. Therefore μ is a finite. □

Theorem 8.4 (Mozes-Shah)

For a sequence of probability measure μ_n on $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ corresponding to a sequence of pairwise distinct totally geodesic closed 2-dimensional submanifolds we have equidistribution to the Haar measure on X.

Proof. Assume that $\mu_n \to \mu$. As μ_n is $\mathrm{SL}(2,\mathbb{R})$-invariant, then

1. μ is $\mathrm{SL}(2,\mathbb{R})$ invariant, and
2. μ is a probability measure: because there exists a compact subset $A \subset \mathbb{C}$ such that any closed geodesic has to hit A. Then for every ε, let B be given by the nondivergence, we have $\mu_n(B) > 1-\varepsilon$ and hence $\mu(B) > 1-\varepsilon$.

Then we apply Ratner's theorem for $\mathrm{SL}(2,\mathbb{R})$-invariant measures:

Theorem 8.5 (Ratner)

$\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measures are homogeneous.

Finally we need a linearization argument. We explain the idea here. Assume that $\mu = \sum c_j\nu_j + c_0 m_X$ be the ergodic decomposition of μ, where ν_j supported on closed orbits. (We admit that there are only countably many ergodic components). For this we assume that $Y = \Gamma g_0\mathrm{SL}(2,\mathbb{R})$ is a closed orbit. We want to show $\mu(Y) = 0$. We can choose an $\mathrm{SL}(2,\mathbb{R})$-invariant complement of Y and use this to construct a transversal neighborhood of Y. Let x be a generic point of μ_n. By the property of polynomials (the (C,α)-good property), the time of the orbit of x entering a super small transversal neighborhood of Y is a little. Hence we can conclude that $\mu(Y) = 0$. □

Mozes-Shah's theorem plays a crucial role in the construction of a measure on the fiber bundle $\widetilde{X}$ as in Part 6 (Getting started for $\mathrm{SL}(3,\mathbb{R})$). This helps to establish the arithmeticity of certain lattices in $\mathrm{SO}(d,1)$.