Group actions and rigidity: around Zimmer program, Part II

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These notes involve some minicourses of the research school of
Group actions and rigidity: around Zimmer program

Contents

1 Big mapping class groups (Kathryn Mann)

§1.1 Lecture 1 (Apr 29)

Problem 1.1.1 (C^0 Zimmer program)
High rank lattices should not act nontrivially on low dimensional manifolds by homeomorphisms.

Progress.
- $d = 1$: Morris 1994, Deroin-Hurtado 2020.
- $d \geqslant 2$: Still open.

Question 1.1.2
Fix M with $\dim M \geqslant 2$. Find a torsion-free, finitely generated (presented) Γ such that $\Gamma \not\curvearrowright \mathrm{Homeo}(M)$?

Remark 1.1.3 $\mathrm{MCG}(\Sigma) := \mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma)$, for finite type surfaces, is pretty well understood.

Theorem 1.1.4 (Farb-Masur, Kaimmovich-Masur 1996)
Any morphism from a higher rank lattice to $\mathrm{MCG}(\Sigma)$ has finite image.

Next step. understand $\mathrm{Homeo}_0(\Sigma)$.

Problem 1.1.5
Study $\mathrm{Homeo}_0(\mathbb{D}^2, \partial)$ (restrict to identity on boundary).

If $g(\Sigma) \geqslant 2$, that is, Σ admits a hyperbolic structure. Then

$$\mathrm{Homeo}_0(\Sigma) \hookrightarrow \mathrm{Homeo}_0(\mathbb{D}^2, \partial).$$

There are two **Approaches**:
- "ignore" $\mathrm{Homeo}_0(\mathbb{D}^2, \partial)$ when studying subgroups of $\mathrm{Homeo}_0(\Sigma)$, e.g. study **undistorted subgroups**, study actions up to **semi-conjugacy**.
- "add topology" to discs.

Calegari 2009. Suppose $\Gamma \curvearrowright \Sigma$ a surface. If there exists a invariant finite $X \subset \Sigma$, then we get $\Gamma \to \mathrm{MCG}(\Sigma \setminus X)$, where $\Sigma \setminus X$ is a punctured surface. Can we generalize this to arbitrary closed invariant set?

Suppose $\Sigma \neq \mathbb{S}^2$ and there exists a point $x \in \Sigma$ such that $\overline{\Gamma x} \subset$ a embedded proper disc of Σ. Consider the unique connected component of $\Sigma \setminus \overline{\Gamma.x}$, call this Σ', which is a surface probably not of finite type. Then $\Gamma \curvearrowright \Sigma'$ by homeomorphisms. We want to understand the induced map

$$\Gamma \xrightarrow{\Phi_x} \mathrm{Homeo}(\Sigma')/\mathrm{Homeo}_0(\Sigma') = \mathrm{MCG}(\Sigma').$$

This includes two ingredients:

- Understand $\mathrm{MCG}(\Sigma')$ for infinite type surfaces.
- Understand $\ker(\Phi_x)$.

Infinite type surfaces

How to build ∞-type surfaces? Attach pants or cap.

Theorem 1.1.6 This procedure produces all examples.

Theorem 1.1.7
$S_1 \cong S_2$ homeomorphically iff $\mathrm{Ends}(S_1) \cong \mathrm{Ends}(S_2)$ and $g(S_1) = g(S_2)$.

For such surfaces, we endow $\mathrm{Homeo}(\Sigma')$ with the compact-open topology.

Definition 1.1.8. The **mapping class group** is $\mathrm{MCG}(\Sigma') := \mathrm{Homeo}(\Sigma')/\mathrm{Homeo}_0(\Sigma')$.

Remark 1.1.9 $\mathrm{MCG}(\Sigma')$ is **NOT** discrete, unless Σ' is of finite type.

Remark 1.1.10 $\mathrm{MCG}(\Sigma')$ is completely metrizable (Polish).

Question 1.1.11 (Calegari)
Does $\mathrm{MCG}(\mathbb{R}^2 \setminus \mathrm{Cantor})$ (For which Σ' does $\mathrm{MCG}(\Sigma')$) have nontrivial quasimorphisms admit interesting actions (by isomorphisms) on Gromov-hyperbolic metric spaces?

Theorem 1.1.12 (Haettel 2016)
Higher rank lattices acting on Gromov hyperbolic spaces are always elementary.

Theorem 1.1.13 (Calegari) No, when $\Sigma' = \mathbb{S}^2 \setminus \mathrm{Cantor}$.

Theorem 1.1.14 (Bavard) Yes, when $\Sigma' = \mathbb{R}^2 \setminus \mathrm{Cantor}$.

§1.2 Lecture 2

Geometric group theory

Γ a locally compact, compactly generated group with a well-defined quasi-isometric type via the word metric.

Rosendal 2010: generalize to more topological groups.

Definition 1.2.1. G a topological group. A subset $A \subset G$ is **coarsely bounded (CB)** if for every left invariant metric compatible with topology, A has finite diameter.

Example 1.2.2 A compact in locally compact groups.

Proposition 1.2.3
$A \subset G$ is coarsely bounded

- iff for every continuous action of G on a metric space X by isometries, A-orbits are bounded
- iff for every neighborhood V of id_G, there exists a finite $F \subset G$ and $n \in \mathbb{N}$ such that $A \subset (F \cdot V)^n$

Theorem 1.2.4 (Rosendal)
Assume that G is locally CB (there exists a neighborhood of id_G which is CB) and generated by CB set. Then it has a "well-defined QI type" such that for every CB generated sets A, A' there exists K, C such that

$$d_A(g_1, g_2) \leqslant K d_{A'}(g_1, g_2) + C.$$

Moreover,

- there exists a metric quasi-isometric to the word metric and compatible with the topology on G,
- for every metric d compatible with the topology, there exists $K, C > 0$ such that

$$d(g_1, g_2) \leqslant K d_A(g_1, g_2) + C.$$

Theorem 1.2.5 (Mann-Rosendal)
If M a compact manifold then $\mathrm{Homeo}_0(M)$ is locally CB, CB generated. If $\pi_1(M)$ is infinity and $M \neq \mathbb{S}^1$ then $\mathrm{Homeo}_0(M)$ is not CB.

Remark 1.2.6 $\mathrm{Homeo}_0(\mathbb{S}^n)$ is CB.

Question 1.2.7 Is $\mathrm{Homeo}_0(\mathbb{RP}^2)$ CB?

$\mathrm{Diff}^r_0(M)$ also fits this framework for $r < \infty$.

Definition 1.2.8. Say a finitely generated subgroup $G < \mathrm{Diff}^r(M)$ is **distorted** if the inclusion is not QI-embedding.

Question 1.2.9

Is g distorted (there exists finitely generated $\Gamma \subset \mathrm{Diff}(M)$ such that $g \in \Gamma$ is distorted) equivalent to $\langle g \rangle$ distorted in $\mathrm{Diff}(M)$?

Problem 1.2.10

Find M, N such that $\mathrm{Homeo}_0(M) \not\simeq_{\mathrm{QI}} \mathrm{Homeo}_0(N)$ but both are nontrivial.

Bounded and unbounded geometries

Proposition 1.2.11 $G = \mathrm{MCG}(\mathbb{S}^2 \setminus \mathrm{Cantor})$ is CB.

Proof. Given a neighborhood V of the identity. WLOG, $V = \{ [f] : f|_K = \mathrm{id} \}$ for some compact subsurface K. Let f be such that $f(K) \cap K = \varnothing$ and every component of $f(K) \cup K$ contains some Cantor set. Let $F = \{ f, f^{-1}, \mathrm{id} \}$.

Claim 1.2.12. $G \subset (FV)^4$.

□

Proposition 1.2.13 (Mann-Rafi)

If Σ has genus 0 or ∞ and $\mathrm{Ends}(\Sigma)$ is self-similar then $\mathrm{MCG}(\Sigma)$ is CB.

Unbounded geometry

Proposition 1.2.14

If G has a unbounded continuous length function ($\ell : G \to \mathbb{R}_{\geqslant 0}$ with $\ell(\mathrm{id}) = 0$, $\ell(g) = \ell(g^{-1})$ and $\ell(gf) \leqslant \ell(g) + \ell(f)$) then G is not CB.

Definition 1.2.15. Say $S \subset \Sigma$ is nondisplaceable if for every $f \subset \mathrm{Homeo}(\Sigma)$, $f(S) \cap S = \varnothing$.

Theorem 1.2.16 (Mann-Rafi)

If Σ contains nondisplaceable finite type S, then $\mathrm{MCG}(\Sigma)$ has unbounded length function, so not CB.

Corollary 1.2.17

Many, but not all ∞-type surfaces have a well-defined and often nontrivial QI type for $\mathrm{MCG}(\Sigma)$.

2 Orderability of lattices in semisimple Lie groups (Bertrand Deroin)

§2.1 Lecture 1

Introduction

Theorem 2.1.1 (Deroin-Hurtado)
An irreducible lattice $\Gamma < G$ in a semisimple Lie group of real rank at least 2 acts (nontrivially) on $\mathbb{R}$ by homeomophisms iff (up to finite covering) there exists a morphism $G \to \widetilde{\mathrm{Aut}(\mathbb{RP}^1)}$ and the actions are semi-conjugated to actions of Γ on $\mathbb{R} \cong \widetilde{\mathbb{RP}^1}$ given by the composition

$$\Gamma \hookrightarrow G \to \widetilde{\mathrm{Aut}(\mathbb{RP}^1)} \hookrightarrow \mathrm{Homeo}(\mathbb{R}).$$

This was originally conjectured by Ghys and Witte.

- **(Witte)** Demonstrating the theorem for certain arithmetic lattices $\Gamma < G$ of $\mathbb{Q}$-rank at least 2, for example, finite index subgroups of $\mathrm{SL}_n(\mathbb{Z})$.
- **(Ghys)** If $\Gamma < G$ an irreducible lattice in a semisimple Lie group of real rank at least 2 acts on $\mathbb{S}^1$ by homeomophisms, then either there exists a finite orbit or there is

  $$\Gamma \hookrightarrow G \to \mathrm{Aut}(\mathbb{RP}^1) \hookrightarrow \mathrm{Homeo}(\mathbb{S}^1).$$

  Moreover, if the action is by diffeomorphisms then the action always semi-conjugate to an $\mathrm{Aut}(\mathbb{RP}^1)$-action.

The goal of this minicourse is to prove Theorem 2.1.1.

Contraction properties

Proposition 2.1.2 (Ghys)
Let $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ be action of a finitely generated group on $\mathbb{R}$. Then either

(1) there exists a Radon measure on $\mathbb{R}$ invariant under Γ-action, or
(2) the action is semi-conjugated to an action commuting with $x \mapsto x+1$, or
(3) we have the **global contraction property**: for every compact interval $I \subset \mathbb{R}$ there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that $\{\phi(\gamma_n)(I)\}$ converges to a point.

Idea of the proof. If there is no invariant measures then we can obtain a local contraction property: for every x, there is a neighborhood of x which is contracted by a sequence in Γ. To obtain the global one, we consider the map

$$\psi(x) := \sup\{y > x : [x, y] \text{ can be shrunk to a point}\}.$$

If $\psi(x)$ is finite for some x then we will obtain the second possible in the proposition. $\square$

Harmonic actions

Let Γ be a finitely generated group. Let μ_Γ be a finitely supported probability measure on Γ which is symmetric and $\langle \operatorname{supp} \mu_\Gamma \rangle = \Gamma$.

Definition 2.1.3. A **μ_Γ-harmonic action** $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is an action so that the Lebesgue measure is stationary, that is

$$\int_\Gamma (\phi(\gamma)(y) - \phi(\gamma)(x)) \mathrm{d}\mu_\Gamma(\gamma) = y - x, \quad \forall x, y \in \mathbb{R}.$$

Definition 2.1.4. A μ_Γ-harmonic action has the **Derriennic property** if the **drift**

$$\int_\Gamma (\phi(\gamma)(x) - x) \mathrm{d}\mu_\Gamma(\gamma)$$

is identically equal to zero.

Lemma 2.1.5 (Kleptsyn) Any μ_Γ-harmonic action has the Derriennic property.

The proof can be found in [the notes of the previous minicourse].

Theorem 2.1.6 (Deroin-Kleptsyn-Navas-Parwani)
Any action $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ that does not have a discrete orbit on $\mathbb{R}$ is semi-conjugated to a μ_Γ-harmonic action, which is unique up to conjugate by affine maps.

Outline of the proof of the existence.

Step 1. Every nontrivial stationary measure $\mu_\mathbb{R}$ is bi-infinite ($\mu_\mathbb{R}(]-\infty, x]) = \infty$ and $\mu_\mathbb{R}([x, +\infty[) = \infty$ for every $x \in \mathbb{R}$).

Step 2. For every $x \in \mathbb{R}$ and $\mu_\Gamma^\mathbb{N}$ almost every (γ_n), we have

$$\limsup_{n \to +\infty} \phi(\gamma_n \cdots \gamma_1)(x) = \infty, \quad \liminf_{n \to -\infty} \phi(\gamma_n \cdots \gamma_1)(x) = -\infty.$$

Step 3. There exists a Radon measure $\mu_\mathbb{R}$ which is μ_Γ-stationary.

Step 4. $\mu_\mathbb{R}$ has no atoms.

Proof of Step 1. Assume by contradiction that

$$\mu_\mathbb{R}(]-\infty, x]) = \infty, \quad \forall x \in \mathbb{R}.$$

We consider the function $\psi : x \mapsto \mu_\mathbb{R}(]-\infty, x]) \in \mathbb{R}$, which is a μ_Γ-harmonic function

$$\int_\gamma \psi(\phi(\gamma)(x)) \mathrm{d}\mu_\Gamma(x) = \psi(x), \quad \forall x \in \mathbb{R}.$$

Let $c \in \mathbb{Q}$ and consider

$$\psi_c(x) := \max\{c - \psi(x), 0\}.$$

Then ψ_c is an $L^1(\mu_\mathbb{R})$ function which is μ_Γ-subharmonic

$$\psi_c(x) \leqslant \int_\gamma \psi_c(\phi(\gamma)(x)) \mathrm{d}\mu_\Gamma(x), \quad \forall x \in \mathbb{R}.$$

Note that a μ_Γ-subharmonic and $L^1(\mu_\mathbb{R})$ function is μ_Γ-harmonic $\mu_\mathbb{R}$-almost everywhere. Then ψ_c is μ_Γ-harmonic $\mu_\mathbb{R}$-almost for any $c \in \mathbb{Q}$. Therefore ψ is a constant (ψ_c is not harmonic near x_c with $\psi(x_c) = c$), and hence $\mu_\mathbb{R} \equiv 0$ is the trivial one. □

Proof of Step 2. For $c \in \mathbb{R}$, let

$$p(x) := \mathbf{P}\left(\limsup_{n\to+\infty} \phi(\gamma_n \cdots \gamma_1)(x) \geqslant c\right).$$

We have

- $p(x)$ is non-decreasing.
- $p(x)$ is μ_Γ-harmonic because $\left\{ \limsup_{n\to+\infty} \phi(\gamma_n \cdots \gamma_1)(x) \geqslant c \right\}$ is a tail event.

We also consider

$$\overline{p}(x) := \lim_{y\to x^+} p(y).$$

Then there exists a measure $\mu_\mathbb{R}$ on $\mathbb{R}$ such that

$$\mu_\mathbb{R}(]x,y]) = \overline{p}(y) - \overline{p}(x), \quad \forall x < y \in \mathbb{R}.$$

Note that $\mu_\mathbb{R}$ is μ_Γ-stationary and $\mu_\mathbb{R}(]x,y]) \leqslant 1$ by definition. So we conclude that $\mu_\mathbb{R} \equiv 0$ and hence p is constant by the first step.

The 0-1 law shows that $p \equiv 0$ or $p \equiv 1$. Let us prove that $p \geqslant 1/2$. Because of the symmetry of μ_Γ, we always have

$$\mathbf{P}\left(\phi(\gamma_n \cdots \gamma_1)(c) \geqslant c\right) \geqslant \frac{1}{2}, \quad \forall n \geqslant 0.$$

Therefore $p(c) \geqslant 1/2$. □

Proof of Step 3. This step uses a general fact of the existence of stationary measures for non-compact spaces.

Fact 2.1.7. Assume a finitely generated Γ acts on a topological space X and μ_Γ is a finitely supported probability measure on Γ. Assume that there exists a compact subset $K \subset X$ such that for every $x \in X$ and $\mu_\Gamma^\mathbb{N}$ almost every (γ_n),

$$\phi(\gamma_n \cdots \gamma_1)(x) \in K, \quad \text{for infinitely many } n.$$

Then there exists a Radon μ_Γ-stationary measure μ_X on X.

We can take $K = [\inf_{\gamma\in\operatorname{supp}\mu_\Gamma} \phi(\gamma)(0), \sup_{\gamma\in\operatorname{supp}\mu_\Gamma} \phi(\gamma)(0)]$ in our case. □

The proof of Step 4 is omitted because it is complicated. To show the uniqueness, it suffices to establish the uniqueness of μ_Γ-stationary measures (up to a constant). The argument is also in the global contraction property regime.

The almost-periodic space.

For a homeomorphism $h : \mathbb{R} \to \mathbb{R}$, we consider the **Kleptsyn constant**

$$K(h,x) := \begin{cases} \displaystyle\int_{h^{-1}(x)}^{x} [h(s) - x]\mathrm{d}s, & h(x) \geqslant x; \\ \displaystyle\int_{h(x)}^{x} [h^{-1}(s) - x]\mathrm{d}s, & h(x) < x. \end{cases}$$

Fact 2.1.8. For a μ_Γ-harmonic function, the function

$$x \mapsto \int_\Gamma K(\phi(\gamma), x)\mathrm{d}\mu_\Gamma(x)$$

is constant. We denote this constant by $K(\phi)$.

A basic fact is that every μ_Γ-harmonic action is bi-Lipschitz. For a bi-Lipschitz map, we have

$$K(h,x) \asymp (h(x)-x)^2.$$

As a conclusion, there exists $c_1, c_2 > 0$ (depending only on μ_Γ) so that for any μ_Γ-harmonic action

$$c_1 K(\phi) \leqslant \sup_{\gamma \in \operatorname{supp} \mu_\Gamma} (\phi(\gamma)(x) - x) \leqslant c_2 K(\phi)$$

and

$$-c_2 K(\phi) \leqslant \inf_{\gamma \in \operatorname{supp} \mu_\Gamma} (\phi(\gamma)(x) - x) \leqslant -c_1 K(\phi).$$

Corollary 2.1.9

The space of normalized ($K(\phi) = 1$) μ_Γ-harmonic actions is compact (equivalent with the compact-open topology on generators of Γ).