

ODE: Qualitative Theory (Spring 2022, Shaobo Gan)

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Contents

| | | |
|----------|---|-----------|
| 1 | Basic Concepts | 3 |
| 1.1 | Basic notions and results | 3 |
| 1.2 | Flows | 4 |
| 1.3 | Equations on manifolds | 7 |
| 2 | Linear Systems | 8 |
| 2.1 | Plane linear singularities | 8 |
| 2.2 | Topological conjugacies between linear systems | 10 |
| 2.3 | Non-autonomous linear systems | 12 |
| 2.4 | Periodic linear systems | 13 |
| 3 | Stability | 18 |
| 3.1 | Lyapunov stability | 18 |
| 3.2 | Lyapunov functions | 20 |
| 3.3 | Stability under perturbations | 24 |
| 4 | Poincaré-Bendixson Theory | 27 |
| 4.1 | Basic notions | 27 |
| 4.2 | The Poincaré-Bendixson Theorem | 29 |
| 4.3 | Poincaré recurrence and limit cycle | 32 |
| 5 | Bifurcations | 35 |
| 5.1 | Structural stability | 35 |
| 5.2 | Bifurcations | 36 |
| 6 | Index Theory and Dynamics on \mathbb{S}^1 | 38 |
| 6.1 | Index | 38 |
| 6.2 | Index of a singularity | 40 |
| 6.3 | Rotation numbers on the torus | 41 |
| 6.4 | Denjoy's theorem | 44 |
| 6.5 | Dynamics on \mathbb{T}^2 | 47 |

1 Basic Concepts

§1.1 Basic notions and results

Assume $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(t, x) \mapsto f(t, x)$ continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{dx}{dt} = f(t, x).$$

A differentiable function $\gamma : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be a **solution** (or **solution curve**), if for every $t \in (a, b)$,

$$\frac{d\gamma(t)}{dt} = f(t, \gamma(t)).$$

The **graph** of γ is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For $t_0 \in (a, b)$, let $x_0 = \gamma(t_0)$, then γ is called the solution of the **initial value problem**

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

The initial value problem has a unique solution: Let $\gamma_i : (a_i, b_i) \rightarrow \mathbb{R}^n$ be two solutions of the initial value problem. Then there exists $\delta > 0$, $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$, such that $\gamma_1(t) = \gamma_2(t)$, $\forall t \in (t_0 - \delta, t_0 + \delta)$,

Theorem 1.1.1 (Existence and Uniqueness Theorem)

$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(t, x)$ continuous, given $t_0 \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $a > 0$, $b > 0$, consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}.$$

If f is Lipchitz in x on R , i.e. $\exists L > 0$, $\forall (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on $[t_0 - h, t_0 + h]$, where $h = \min \left\{ a, \frac{b}{M} \right\}$, $M = \max_{(t, x) \in R} |f(t, x)|$.

Remark 1.1.2 — The solution is denoted as $\varphi(t; t_0, x_0)$.

Corollary 1.1.3

When $f \in C^1$, the existence and uniqueness theorem holds.

Denotes the **maximal interval** of $\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$ as $I(t_0, x_0)$, it is an open interval.

Corollary 1.1.4

Assume $f \in C^1$ and $|f(t, x)| \leq A(t)|x| + B(t)$, then the maximal interval of the initial value problem is $(-\infty, +\infty)$.

§1.2 Flows

Now we consider the **autonomous equation**

$$\dot{x} = f(x).$$

\mathbb{R}^n is called the **phase space** and $\mathbb{R} \times \mathbb{R}^n$ is called the **generalized phase space**.

The solution of the initial value problem $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ is denoted as $\varphi(t, x_0)$, the set

$$\text{Orb}(x_0) := \{\varphi(t, x_0) : t \in I(x_0)\} \subset \mathbb{R}^n$$

is called the **orbit** pass by x_0 .

Corollary 1.2.1 (Continuous Dependence on the Initial Value)

Assume $f \in C^1$, then $U = \{(t, x) : t \in I(x)\}$ is open and $\varphi : U \rightarrow \mathbb{R}^n, (t, x) \mapsto \varphi(t, x)$ is continuous.

Theorem 1.2.2

$f(x) \in C^1$, then:

1. $\varphi_0(x) = x$ for every $x \in \mathbb{R}^n$.
2. $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for every $s \in I(x), t \in I(\varphi(s, x))$.

Definition 1.2.3. $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, continuous, is said to be a **(continuous) flow** if:

- (i) $\psi(0, x) = x$,
- (ii) $\psi(t, \psi(s, x)) = \psi(t + s, x)$.

Remark 1.2.4 — The solution of an autonomous equation is a **local flow**.

Corollary 1.2.5

Let $\psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a flow, then $\psi_t := \psi(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are homeomorphisms.

Remark 1.2.6 — Consider the group of self-homeomorphisms of \mathbb{R}^n , denoted as $\text{Homeo}(\mathbb{R}^n)$, then $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ is a group homomorphism. More generally, we can consider $G \rightarrow \text{Homeo}(\mathbb{R}^n)$ for some group G .

Proposition 1.2.7

Assume f is a C^1 vector field, then the orbits of the flow generated by f are either coincide or disjoint.

$\bigcup_{x \in \mathbb{R}^n} \text{Orb}(x)$ forms a partition of \mathbb{R}^n , is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbits $\varphi(t, x)/f(x)$ is called the **phase portrait**.

A point $x_0 \in \mathbb{R}^n$ with $f(x_0) = 0$ is called a **critical point** (or a **singularity, equilibrium**). The orbit $\text{Orb}(x_0)$ is a single point $\{x_0\}$.

Example 1.2.8

$$\begin{cases} \frac{dx}{dt} = x \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t, x_0) = x_0 e^t$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$.

Example 1.2.9

$$\begin{cases} \frac{dx}{dt} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t, x_0) = \frac{x_0}{1 - x_0 t}$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$. But the phase portrait is different from the former examples, because the orientations on \mathbb{R}_- are different.

Theorem 1.2.10

Assume $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 vector field, $\beta(x) : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1$ and $\beta(x) > 0$. Then the equations $\dot{x} = f(x)$ and $\dot{x} = \beta(x)f(x)$ have the same phase portraits.

Proof. $\varphi : I \rightarrow \mathbb{R}^n$ a solution of f . Find a C^1 diffeomorphism $h : J \rightarrow I$ such that $\varphi \circ h$ is the solution of $\dot{x} = \beta(x)f(x)$. It suffices that

$$\frac{d}{dt} \Big|_{t=h(s)} \varphi(t) \cdot \frac{dh(s)}{ds} = \beta(\varphi \circ h(s)) f(\varphi \circ h(s)),$$

i.e. $\frac{dh(s)}{ds} = \beta(\varphi \circ h(s)) > 0$, it is an initial value problem. It shows that the maximal solution curve of f is contained in some solution curve of βf . \square

Theorem 1.2.11 (Differentiable Dependence on the Initial Value)

Assume $f \in C^1$, it generates the flow ϕ_t , then $\phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^1 .

Exercise 1.2.12.

$$\frac{\partial}{\partial t} \frac{\partial \phi(t, x)}{\partial x} = \frac{\partial}{\partial x} \frac{\partial \phi(t, x)}{\partial t}.$$

Let $\Phi(t, x) = \Phi_t(x) = \frac{\partial \phi(t, x)}{\partial x}$, then Φ is the solution of the equation

$$\begin{cases} \frac{dy(t)}{dt} = A(t)y(t), A(t) = Df(\phi_t(x)) \\ y(0) = \text{Id} \end{cases}.$$

The equation is called the **variation equation** of $f(x)$ along $\phi_t(x)$.

Lemma 1.2.13

$f \in C^1$, $\Phi(t, x)$, then

$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

Remark 1.2.14 — This property is called the **cocycle** condition.

We already know that ϕ_t are self-homeomorphisms of \mathbb{R}^n , and lemma 1.2.13 shows that the differential is invertible, hence ϕ_t are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (t, x, v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

Proposition 1.2.15

$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ is a flow.

Remark 1.2.16 — We call Φ_t a skew product flow of ϕ_t .

Theorem 1.2.17

$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If ψ is a C^1 flow, let

$$g(x) = \left. \frac{\partial \psi(t, x)}{\partial t} \right|_{t=0},$$

then $\psi(t, x_0)$ solve the initial value problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$. Because

$$\frac{\partial \psi(t, x_0)}{\partial t} = \left. \frac{\partial \psi(t+s, x_0)}{\partial s} \right|_{s=0} = \left. \frac{\partial \psi(s, \psi(t, x_0))}{\partial s} \right|_{s=0} = g(\psi(t, x_0)).$$

§1.3 Equations on manifolds

Let M be a closed smooth manifold, X is a C^1 vector field on M . Then X is bounded, hence the maximal intervals are $(-\infty, +\infty)$. Consider the equation

$$\begin{cases} \frac{dx}{dt} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution $\varphi(t, x)$ generates a flow.

2 Linear Systems

§2.1 Plane linear singularities

Consider the equation

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}, \quad (x, y) \in \mathbb{R}^2.$$

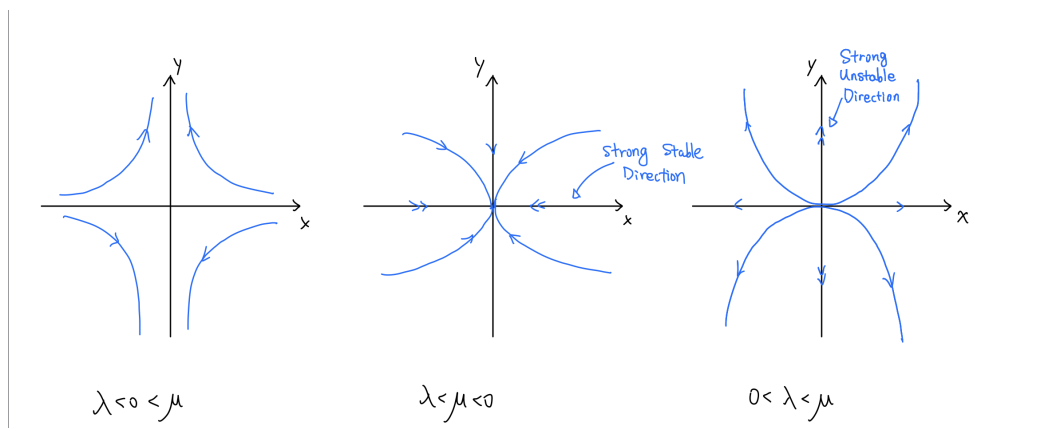
It is said to be a **plane linear system** if f, g both linear functions of x, y , i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a, b, c, d \in \mathbb{R}.$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $(0, 0)$ is the only singularity. In this case, we call O an **elementary singularity**.

Consider the Jordan form of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, there are four cases:

- I. Two different real eigenvalues: $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$.
 - i. $\lambda < 0 < \mu$: the origin is called a **saddle point**.
 - ii. $\lambda < \mu < 0$: the origin is called a **stable node**.
 - iii. $0 < \lambda < \mu$: the origin is called an **unstable node**.

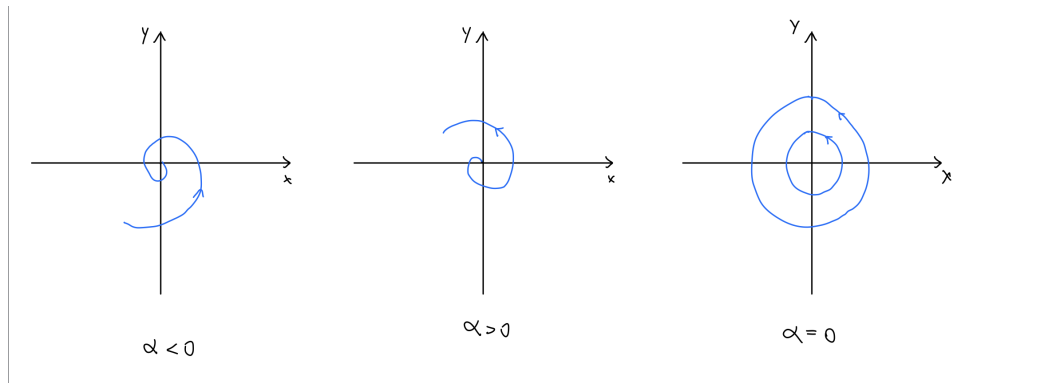


- II. Conjugated imaginary eigenvalues: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \beta > 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

If we consider this equation in the polar coordinates, it turns $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$.

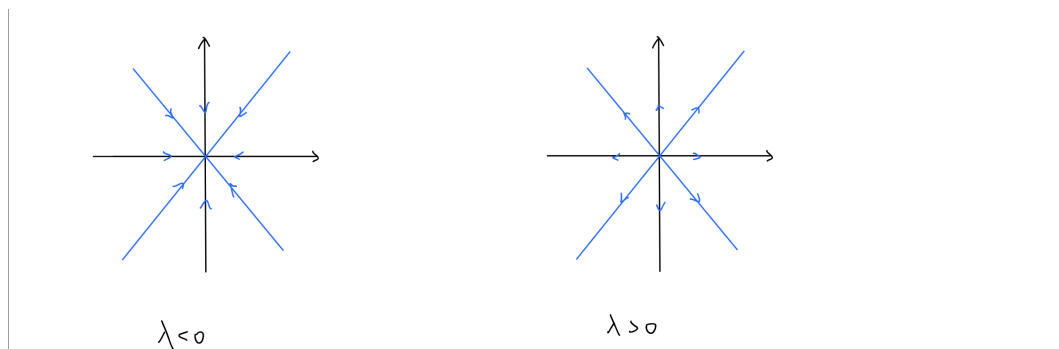
- i. $\alpha < 0$, the origin is called a **stable focus**.
- ii. $\alpha > 0$, the origin is called an **unstable focus**.
- iii. $\alpha = 0$, the origin is called a **center**.

Definition 2.1.1. φ_t a flow. If p is not a singularity and $\exists T > 0$, such that $\varphi_T(p) = p$. Then p is called a **periodic point**, $\text{Orb}(p)$ is called a **periodic orbit**. If p is a periodic point, the smallest $T > 0$ is called the **minimum positive period**.



III. Two same real eigenvalues, diagonalizable: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$.

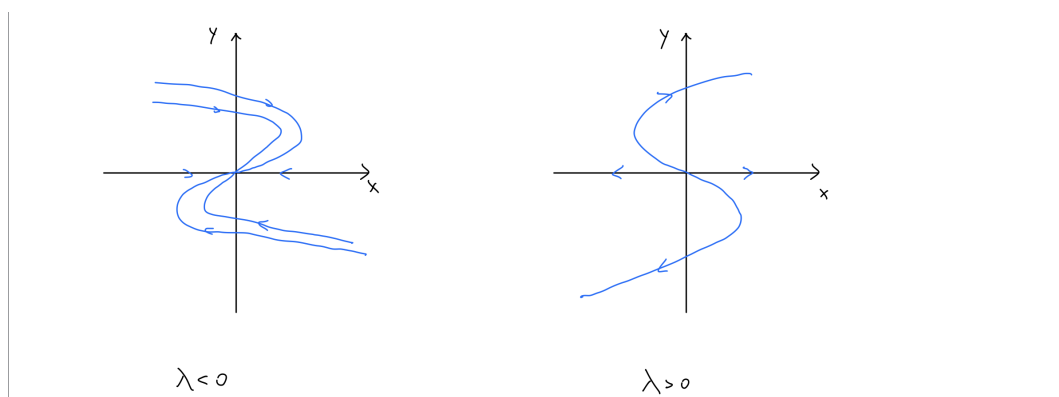
- i. $\lambda < 0$, the origin is called a **stable critical node**.
- ii. $\lambda > 0$, the origin is called an **unstable critical node**.



IV. Two same real eigenvalues, not diagonalizable: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + ty_0) \\ e^{\lambda t}y_0 \end{bmatrix}$, or

$$x(t) = \frac{x_0}{y_0}y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}.$$

- i. $\lambda < 0$, the origin is called a **stable unidirectional node**.
- ii. $\lambda > 0$, the origin is called an **unstable unidirectional node**.



Exercise 2.1.2. Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

§2.2 Topological conjugacies between linear systems

Definition 2.2.1. Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ homeomorphisms. f and g are said to be **topologically conjugate** if there exists $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $h \circ f = g \circ h$.

Remark 2.2.2 — Conjugacy is a equivalence relation.

Definition 2.2.3. Let $\varphi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two flows, we call φ_t and ψ_t are **conjugate** if there is a homeomorphism $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $h \circ \varphi_t = \psi_t \circ h$. Let X, Y be two C^1 vector fields on \mathbb{R}^n , we call X, Y are **conjugate** if the flows generated by them, respectively, are conjugate.

Example 2.2.4

$A, B \in M(n, \mathbb{R})$ are similar, then $\dot{x} = Ax$ and $\dot{y} = By$ are conjugate.

$f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ C^1 vector fields, generate flows ϕ_t, ψ_t . Let $x = h(y) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 diffeomorphism gives the conjugate, i.e., $h\psi_t(y) = \phi_t h(y)$. Then

$$\left. \frac{d}{dt} h\psi_t(y) \right|_{t=0} = f(h(y)) \implies Dh(y)g(y) = f(h(y)).$$

If there exists a C^1 diffeomorphism conjugate $e^{Bt}y$ to $e^{At}x$ via $x = h(y)$, i.e. $h(e^{Bt}y) = e^{At}h(y)$. Then $Dh_0 e^{Bt} = e^{At} Dh_0$, hence $Dh_0 B = A Dh_0$. It shows that C^1 conjugate generically not hold even if topologically conjugate.

Proposition 2.2.5

Assume f, g C^1 vector fields generate ϕ_t, ψ_t , let h be a conjugate between ϕ_t and ψ_t . Then:

1. $h(\text{Orb}(x, \phi)) = \text{Orb}(hx, \psi)$.
2. h maps the singularities of f to the singularities of g .
3. h maps the periodic orbits of f to the periodic orbits of g . Moreover, it preserves the minimum positive period.

Example 2.2.6

$\dot{x} = -2x$ and $\dot{y} = -4y$ are conjugate.

Let $h : \mathbb{R} \rightarrow \mathbb{R}$, $h(0) = 0$. Take $x_0, y_0 > 0$, let $h(x_0) = y_0$, then $h(e^{-2t}x_0) = e^{-4t}y_0$ or $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$. The construction for the negative part is similar.

Exercise 2.2.7. $\lambda\mu \neq 0$, show that $\dot{x} = \lambda x$ is conjugate to $\dot{y} = \mu y$ if and only if $\lambda\mu > 0$.

Proposition 2.2.8

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i}$ are topologically conjugate by $h_i, i = 1, 2$. Then $\phi_t^1 \times \phi_t^2$ and $\psi_t^1 \times \psi_t^2$ are topologically conjugate by $h_1 \times h_2$.

Example 2.2.9

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$ are conjugate.

Proof. $\phi_t(x, y) = e^{-t}(x, y)$ and $\psi_t(x, y) = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For every $(x, y) \neq (0, 0)$, there exists unique $t = t(x, y)$ such that $\phi_t(x, y) \in \mathbb{S}^1$. Let $h(x, y) := \psi_{-t}\phi_t(x, y)$, where $t = t(x, y)$, then h gives the conjugate. \square

Exercise 2.2.10. Show that $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$ are conjugate.

Classification of elementary plane linear systems (under conjugacies):

- (I) Stable: node, critical node, unidirectional node, focus.
- (II) Unstable: node, critical node, unidirectional node, focus.
- (III) Saddle point.
- (IV) Center.

Definition 2.2.11. The linear system $\dot{x} = Ax$ in \mathbb{R}^n is called **hyperbolic** if the real parts of eigenvalues of A are nonzero. The **(stable) index** of A is the number of eigenvalues with negative real parts, denoted by $\text{Ind } A$.

Theorem 2.2.12

Two plane hyperbolic linear system $\dot{x} = Ax, \dot{y} = By$ are topologically conjugate if and only if $\text{Ind } A = \text{Ind } B$.

Proof. “ \implies ”: Let $W_A^s = \{x : e^{tA}x \rightarrow 0, t \rightarrow \infty\}$, $W_B^s = \{x : e^{tB}x \rightarrow 0, t \rightarrow \infty\}$, then h and h^{-1} preserves the stable manifolds. Then $\text{Ind } A = \dim W_A^s = \dim W_B^s = \text{Ind } B$. \square

Example 2.2.13

Consider $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ and $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$ with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

Definition 2.2.14. $\phi_t, \psi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ flows, h is a homeomorphism $\mathbb{R}^n \rightarrow \mathbb{R}^n$ maps the orbit of ϕ to the orbit of ψ preserves the orientation. Then ϕ and ψ are called **topologically equivalent** or **flow equivalent**.

Theorem 2.2.15 (Grobman-Hartman)

If x_0 is a hyperbolic singularity of $f(x)$, then the flows generated by $\dot{x} = f(x)$ and $\dot{y} = Ay$ where $y = Df(x_0)$ are topologically conjugate near 0.

§2.3 Non-autonomous linear systems

$A : \mathbb{R} \rightarrow M(n, \mathbb{R})$ continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

Theorem 2.3.1

The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is $(-\infty, \infty)$.
3. All solutions of the equation form an n -dimensional linear space S .

Theorem 2.3.2 (Liouville's Formular)

Assume $X(t)$ is a solution of $\dot{x} = A(t)x$, then

$$\frac{d}{dt} \det X(t) = \operatorname{tr} A(t) \det X(t),$$

hence $\det X(t) = \det X(t_0) \exp \int_{t_0}^t \operatorname{tr} A(s) ds$.

Let $X_1(t), X_2(t), \dots, X_n(t)$ be a basis of S , let

$$X(t) := [X_1(t), X_2(t), \dots, X_n(t)] \in \operatorname{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \frac{dX}{dt} = A(t)X \\ X(t_0) = I_n \in \operatorname{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If $X(t), Y(t)$ are two fundamental solutions, suppose $Y(0) = X(0)C$, then

$$\frac{dX(t)C}{dt} = \frac{dX(t)}{dt}C = A(t)X(t)C,$$

is a non-degenerate solution of $\frac{dX}{dt} = AX$. By the uniqueness, we get $Y(t) = X(t)C$.

Example 2.3.3

$A(t) \equiv A$, the fundamental solution of $\dot{x} = Ax$ is

$$e^{tA} = \operatorname{Id} + tA + \frac{1}{2!}t^2A^2 + \dots + \frac{1}{k!}t^kA^k + \dots$$

Example 2.3.4

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$, where $f \in C^1$, generates the flow $\varphi_t(x)$. Consider $\Phi_t(x) = \frac{\partial}{\partial t} \varphi_t(x)$ and the variation equation

$$\frac{d}{dt} \Phi_t(x) = Df(\varphi_t(x)) \Phi_t(x).$$

Given $x \in \mathbb{R}^n$, let $A(t) := Df(\varphi_t(x))$, then $\Phi_t(x)$ is the standard fundamental solution ($t_0 = 0$) of $\dot{x} = A(t)x$. Consider two special types of orbits:

- x is a singularity, denoted by σ . Then $\varphi_t(\sigma) = \sigma$, $\dot{x} = Ax$ where $A = Df(\sigma)$.
- x is a periodic point, denoted by p , the minimum period $T > 0$. Then A is T -periodic.

§2.4 Periodic linear systems

Definition 2.4.1. The equation $\dot{x} = A(t)x$ satisfies $A(t+T) = A(t)$ for some $T > 0$ is called a **periodic linear systems**.

Theorem 2.4.2 (Floquet)

Assume $\dot{x} = A(t)x$ is a T -periodic linear system, if X is a fundamental solution, then $X(t+T)$ is a fundamental solution, i.e. $\exists C \in \text{GL}(n, \mathbb{R})$ such that $X(t+T) = X(t)C$. Moreover, there exists a T -periodic map $P : \mathbb{R} \rightarrow \text{GL}(n, \mathbb{C})$ and a constant matrix $B \in M(n, \mathbb{C})$ such that $X(t) = P(t)e^{tB}$.

Lemma 2.4.3

$\forall C \in \text{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{C})$ such that $C = e^B$.

Proof. It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N^k$$

is convergence for nilpotent matrix N . □

Lemma 2.4.4

$\forall C \in \text{GL}(n, \mathbb{R})$, $\exists B \in M(n, \mathbb{R})$ such that $C^2 = e^B$.

Proof. Note that the Jordan block of C^2 is either:

$$(i) \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}, \text{ where } \lambda > 0, \text{ or}$$

$$(ii) \begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}, \text{ where } J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0.$$

And $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have a real matrix logarithm because $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$. \square

Theorem 2.4.5 (Real Form of Floquet Theorem)

Assume $\dot{x} = A(t)x$ is a T -periodic linear system, if X is a fundamental solution. Then there exists a **2T-periodic** map $P : \mathbb{R} \rightarrow \text{GL}(n, \mathbb{R})$ and a constant matrix $B \in M(n, \mathbb{R})$ such that $X(t) = P(t)e^{tB}$.

Example 2.4.6 ($2T$ is necessary)

Let $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$. Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then $A(t)$ is π -periodic. Then $\Phi(t)$ is a standard fundamental solution of $\dot{x} = A(t)x$, hence $\exists \pi$ -periodic $P(t)$ and B such that $\Phi(t) = P(t)e^{tB}$. Then $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$, there is no real matrix B satisfying this equation.

Definition 2.4.7. In Floquet theorem, $X(t+T) = X(t)C$. We call C is a **monodromy matrix**. The eigenvalues of C are called **Floquet multipliers**. If ρ is a Floquet multiplier with $\rho = e^{\lambda T}$, then λ is called a **Floquet exponent**.

Corollary 2.4.8

Consider a T -periodic linear system $\dot{x} = A(t)x$. Then there exists a linear transformation (non-autonomous) $x = P(t)y$ such that $\dot{y} = By$.

Proof. Let $X(t) = P(t)e^{tB}$ be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PB e^{tB} = AP e^{tB},$$

hence $\dot{P} + PB = AP$. Then $APy = \frac{d}{dt}(Py) = \dot{P}y + P\dot{y}$, hence $\dot{y} = By$. \square

Remark 2.4.9 — This type of equation is called reducible, which means after some reduction, the equation can become independent with time t .

Corollary 2.4.10

Let λ be a Floquet multiplier of $\dot{x} = A(t)x$. Then there exists a T -periodic function $p(t)$ such that $e^{\lambda t}p(t)$ is a solution of the equation $\dot{x} = A(t)x$.

Proof. $e^{\lambda T}$ is an eigenvalue of C , then $\exists x_0$ such that $Cx_0 = e^{\lambda T}x_0$. Then $X(t)x_0$ is a solution. Let $p(t) = e^{-\lambda t}X(t)x_0$ is T -periodic and $e^{\lambda t}p(t)$ is a solution. \square

Corollary 2.4.11

The equation admits a nonzero T -periodic solution if and only if 1 is a Floquet multiplier.

Corollary 2.4.12

Assume $\rho_1, \rho_2, \dots, \rho_n$ are all Floquet multipliers of $\dot{x} = A(t)x$, then

$$\rho_1 \rho_2 \cdots \rho_n = \det \Phi(T) = \exp \int_0^T \operatorname{tr} A(t) dt.$$

Example 2.4.13

The equation $\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2} \sin 2t - 1 \\ \frac{1}{2} \sin 2t + 1 & \sin^2 t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has an unbounded solution. Because the product of two multipliers is $\exp \int_0^\pi 1 dt = e^\pi > 1$.

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where $p(t)$ is π -periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then $\rho_1 \rho_2 = \exp \int_0^\pi \operatorname{tr} A(t) dt = 0$, where $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$.

Lemma 2.4.14

If ρ_1, ρ_2 both are imaginary numbers, then every solution of Hill equation is bounded.

Proof. Because ρ_1, ρ_2 are conjugate imaginary numbers, hence $\Phi(\pi)$ is similar to a rotation. Then $\Phi(\pi)^n$ is bounded independent of n and $\Phi(s)$ is bounded for $s \in [0, \pi]$. \square

Definition 2.4.15. A particular Hill equation with $p(t) = a + \varepsilon \cos 2t$ is called **Mathieu equation**.

Exercise 2.4.16. Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon \cos 2t)x = 0.$$

- (1) $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] : \text{every solution is bounded}\}$. Draw the figure of U by some calculation.
- (2) Guess some conclusions by the figure of U .

Example 2.4.17

Let $p(t)$ be a π -periodic continuous function satisfying

- (i) $p(t) \not\equiv 0$.
- (ii) $\int_0^\pi p(t)dt \geq 0$.
- (iii) $\pi \int_0^\pi |p(t)|dt \leq 4$.

Then every solution of $\ddot{x} + p(t)x = 0$ is bounded.

Proof. If Floquet multipliers are conjugate imaginary numbers, the statement follows. Otherwise there is a real Floquet multiplier $\rho \neq 0$. There is a solution $x(t) \not\equiv 0$ such that $x(t+T) = \rho x(t)$. If $x(t)$ has no zeros, assume $x(t) > 0$, we have $\frac{\dot{x}}{x}(\pi) = \frac{\dot{x}}{x}(0)$. Note that

$$0 = \frac{\ddot{x}}{x} + p(t) = \left(\frac{\dot{x}}{x}\right)' + \left(\frac{\dot{x}}{x}\right)^2 + p(t) = 0,$$

take the integral and we get a contradiction. Then there must be some zeros, let a, b be two successive zeros, WLOG, $0 < a < b < \pi$. Assume $x(t) > 0$ in (a, b) and $x(c)$ takes the maximum. Then $\exists \alpha \in (a, c), \beta \in (c, b)$ such that $\dot{x}(\alpha) = \frac{x(c)}{c-a}, \dot{x}(\beta) = \frac{-x(c)}{b-c}$. We have

$$\frac{4}{\pi} \geq \int_0^\pi |p(t)|dt > \int_a^b \left| \frac{\ddot{x}}{x}(t) \right| dt \geq \frac{\int_\alpha^\beta |\ddot{x}(t)|dt}{x(c)} \geq \frac{1}{c-a} + \frac{1}{b-c} \geq \frac{4}{a-b},$$

the identity holds if and only if $x \equiv 0$, contradiction. \square

Back to Mathieu equation, consider

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \varepsilon < \omega^2.$$

We apply the conclusion of the example, for $\omega < \frac{2}{\pi}$,

$$\int_0^\pi (\omega^2 + \varepsilon \cos 2t)dt = \omega^2 \pi \leq \frac{4}{\pi}.$$

Consider $\varepsilon = 0$, then

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

is a standard fundamental solution. The monodromy matrix for (ω, ε) where $\omega > 0$ is a perturbation of

$$C = \Phi(\pi) = \begin{bmatrix} \cos \omega \pi & \frac{1}{\omega} \sin \omega \pi \\ -\omega \sin \omega \pi & \cos \omega \pi \end{bmatrix}.$$

Note that $|\text{tr } \Phi(\pi)| = |2 \cos \omega \pi| < 2$ for $\omega \notin \mathbb{Z}$. Then there is a small neighborhood U of $(\omega, 0)$ such that every solution is bounded.

Definition 2.4.18. Let $A : \mathbb{R} \rightarrow M(n, \mathbb{R})$ continuous, bounded, assume that

$$\sup \{|A(t)| : t \in \mathbb{R}\} < \infty.$$

Let $\Phi(t)$ be a standard fundamental solution of the equation $\dot{x} = A(t)x$. For every $v \neq 0 \in \mathbb{R}^n$, define **Lyapunov exponent** of v

$$\chi(v) := \limsup_{t \rightarrow \infty} \frac{\ln \|\Phi(t)v\|}{t}.$$

Exercise 2.4.19. For every $v \neq 0$, show that $\chi(v) \neq \pm\infty$.

Then $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the following properties

1. $\chi(\alpha v) = \chi(v)$ for every $\alpha \neq 0$.
2. $\chi(v + w) \leq \max \{\chi(v), \chi(w)\}$.
3. If $\chi(v) < \chi(w)$, then $\chi(v + w) = \chi(w)$.

Fact 2.4.20. The number of different Lyapunov exponents $\leq n$.

Example 2.4.21

$\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a constant matrix. Regard as a T -periodic system, then the eigenvalues λ_1, λ_2 of A are Floquet exponents. Lyapunov exponents are

- (1) λ_1, λ_2 , if $\lambda_1 \neq \lambda_2$ real.
- (2) $\lambda = \lambda_1 = \lambda_2$, if $\lambda_1 = \lambda_2$.
- (3) α , if $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$.

For the T -periodic system, assume that λ is a Floquet exponent, then $\chi = \text{Re}(\lambda)$ is a Lyapunov exponent. For $n = 2$, T -periodic system, we always have

$$\chi_1 + \chi_2 = \text{Re}(\lambda_1 + \lambda_2) = \frac{1}{T} \int_0^T \text{tr } A(t) dt.$$

Example 2.4.22

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y \end{cases},$$

then the solution

$$\begin{cases} x = C_1 e^{-\mu t - t \sin \ln t} \\ y = C_2 e^{-\mu t + t \sin \ln t} \end{cases}.$$

Then $\chi(v) = -\mu + 1$ for every $v \neq 0$. But $\chi_1 + \chi_2 = -2\mu + 2 \neq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{tr } A(t) dt = -2\mu$. This example is called non-regular.

3 Stability

§3.1 Lyapunov stability

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $0 \in \mathbb{R}^n$, $f(0) = 0$, generates a (local) flow $\varphi_t(x)$.

Definition 3.1.1. 1. σ is called **(forward Lyapunov) stable**, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| < \varepsilon$ for $t \geq 0$. Otherwise, we call σ is **unstable**.

2. σ is called **asymptotically stable**, if

(i) σ is stable,

(ii) there exists $\delta_0 > 0$, such that if $|x - \sigma| < \delta_0$, then $\lim_{t \rightarrow \infty} \varphi_t(x) = \sigma$.

3. σ is called **exponentially stable**, if exists $\delta_0 > 0$, $C \geq 1$, $\lambda > 0$, such that if $|x - \sigma| < \delta_0$, then $|\varphi_t(x) - \sigma| \leq Ce^{-\lambda t}|x - \sigma|$ for $t \geq 0$.

Similarly, we can define backward stable, backward asymptotically stable, backward exponentially stable.

Remark 3.1.2 — If we replace the condition of stability by **given $t \geq 0$** , then it always holds by the continuous independence of solutions with respect to initial value.

Example 3.1.3

For the equation in polar coordinates

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases}$$

Then the fixed point $(1, 0)$ satisfy the second condition of asymptotically stable but it is **not** stable.

In general, we can prove that if $\varphi_t(x) \not\equiv \sigma$ and $\lim_{t \rightarrow \pm\infty} \varphi_t(x) = \sigma$, then σ is not stable.

Example 3.1.4

Consider the linear elementary singularities, recall the classification, then

1. Stable type: forward stable.
2. Unstable type: unstable, but backward stable.
3. Saddle point: unstable.
4. Center: forward and backward stable.

Theorem 3.1.5

Let $A \in M(n, \mathbb{R})$, consider the equation $\dot{X} = AX$, 0 is a singularity, then

1. 0 is stable iff each eigenvalue of A is with non-positive real part and Jordan block are trivial for every eigenvalue with zero real part.
2. 0 is asymptotically stable iff 0 is exponentially stable iff every eigenvalue of A is with negative real part.

Lemma 3.1.6 (Gronwall's Inequality)

Let $u : [0, T] \rightarrow \mathbb{R}$ non-negative, continuous. If $C \geq 0, K > 0$ such that for every $t \in [0, T]$,

$$u(t) \leq C + K \int_0^t u(s) ds,$$

then $u(t) \leq Ce^{Kt}$ for $t \in [0, T]$.

Theorem 3.1.7

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n, C^1, f(\sigma) = 0$. Assume that every eigenvalue of $A = Df(0)$ is with negative real part, then σ is exponentially stable.

Proof. There $\exists C \geq 1, \mu > 0$, such that $|e^{At}| \leq Ce^{-\mu t}$ for $t \geq 0$. WOLG, $\sigma = 0$. Let $f(x) = Ax + g(x)$ where $g(x) = o(|x|)$, let $\varphi_t(x)$ be a maximal solution of the initial value problem. Then

$$e^{-tA}(\dot{\varphi}_t(x) - A\varphi_t(x)) = e^{-tA}g(\varphi_t(x)),$$

hence

$$\varphi_t(x) = e^{tA}x + \int_0^t e^{(t-s)A}g(\varphi_s(x))ds.$$

Fix $\varepsilon_0 > 0$ to be determined later, $\exists \delta_0 > 0$ such that $|g(x)| \leq \varepsilon_0|x|$ if $|x| \leq \delta_0$. Assume the right maximal interval of φ_t is $[0, \beta), \beta > 0$. Let

$$T^* = T^*(x) = \sup \left\{ t < \beta : \varphi_{[0,t]}(x) \subseteq \overline{B(\sigma, \delta_0)} \right\}.$$

Then, for every $|x| \leq \delta_0, 0 \leq t \leq T^*$, we have

$$e^{\mu t}|\varphi_t(x)| \leq C|x| + C\varepsilon_0 \int_0^t e^{s\mu}|\varphi_s(x)|ds.$$

By Gronwall's inequality, we have $|\varphi_t(x)| \leq C|x|e^{-(\mu - C\varepsilon_0)t}, \forall t < T^*$. Let $C\varepsilon_0 = \frac{\mu}{2}$ is enough. For all $|x| \leq \frac{\delta_0}{2C}$, then $|\varphi_t(x)| \leq \frac{\delta_0}{2}e^{-\mu t}$ for every $t < T^*$. Then we can show that $T^* = \beta = \infty$ and φ_t is exponentially stable. \square

Proposition 3.1.8

f, g, C^1 vector fields. Assume f, g are topologically conjugate, i.e., $h \circ \varphi_t = \psi_t \circ h$ where φ_t, ψ_t are flows generated by f, g , respectively. Let $\sigma, h\sigma$ be singularities of f, g , respectively, then σ is stable if and only if $h\sigma$ is stable.

Now, we state a celebrated theorem, Hartman-Grobman Theorem. But we will not give a proof here.

Theorem 3.1.9 (Hartman-Grobman)

Let σ be a hyperbolic singularity of f . Then there exists a neighborhood $V \ni \sigma$ and a homeomorphism $h : V \rightarrow \mathbb{R}^n$ onto its image, $h(\sigma) = 0$, such that $h \circ \varphi_t(x) = Df(\sigma) \circ h(x)$ for every $x, \varphi_t(x) \in V$.

§3.2 Lyapunov functions

Definition 3.2.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, be a C^1 vector field, $f(0) = 0$. A C^1 function $V : D \rightarrow \mathbb{R}$ where D is a neighborhood of σ is called a **Lyapunov function** of f (for σ) if

(i) $V(\sigma) = 0, V(x) > 0, \forall x \in D \setminus \{\sigma\}$.

(ii) $\forall x \in D \setminus \{\sigma\}, \dot{V}(x) \leq 0$, where

$$\dot{V}(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(\varphi_t(x)) = DV(x)f(x).$$

V is called a **strict Lyapunov function** if $\dot{V}(x) \leq 0$ is replaced by $\dot{V}(x) < 0$.

Theorem 3.2.2

Assume σ is a singularity of f , if there is a Lyapunov function for σ , then σ is stable. If there is a strict Lyapunov function for σ , then σ is asymptotically stable.

Proof. Let V be a Lyapunov function, for every $\varepsilon > 0$, assume $B_\varepsilon(\sigma) = \{x : |x - \sigma| \leq \varepsilon\} \subseteq D$. Let $m = \min \{V(x) : x \in \partial B_\varepsilon(\sigma)\} > 0$, take $\delta > 0$ such that $V(x) < m, \forall x \in B_\delta(\sigma)$. By $\dot{V}(x) \leq 0$, we have that every solution curve start at $x \in B_\delta(\sigma)$ can not reach $\partial B_\varepsilon(\sigma)$.

If $\dot{V}(x) < 0$ for every $x \in D \setminus \{\sigma\}$, it suffices to show that each convergent subsequence of $\varphi_t(x)$ converges to σ . Otherwise, assume converges to $y \neq \sigma$, but $\dot{V}(y) < 0$, there is some $s > 0$ such that $V(\varphi_s(y)) < V(y)$. Contradiction. \square

Example 3.2.3

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = 0$, hence 0 is stable.

Example 3.2.4

Consider the equation

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y^3 \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2 - 2y^4 < 0$, hence 0 is asymptotically stable.

Example 3.2.5

Consider the equation

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2(1 - x) \leq 0$, hence 0 is stable. In fact, 0 is asymptotically stable, but we need to consider another Lyapunov function $Q(x, y) = x^2 + y^2 + xy$.

Theorem 3.2.6

If V is a Lyapunov function of f , assume

$$K = \{x \in D \setminus \{\sigma\}, \dot{V}(x) = 0\}$$

does not contain any complete positive orbit $\varphi_{[0, \infty)}(x)$, then σ is asymptotically stable.

Example 3.2.7

Let $f : \mathbb{R} \rightarrow \mathbb{R}, C^1, f(0) = 0$, satisfying $xf(x) > 0, \forall x \neq 0$. Consider the stability of $\ddot{x} + f(x) = 0$, or

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}.$$

Let

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x f(z)dz$$

be an energy function, then $\dot{E}(x, y) \equiv 0$.

Example 3.2.8

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}, C^2$, the gradient of V is

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

The system $\dot{x} = -\nabla V(x)$ is called the **gradient system** generated by V . Then,

1. $\dot{V}(x) \leq 0$.
2. σ is a singularity if and only if $\dot{V}(\sigma) = 0$.
3. If σ is a minimum point of $V(x)$, then σ is stable.

Theorem 3.2.9

Let σ be a singularity of C^1 vector field f , a C^1 function $V : D \rightarrow \mathbb{R}$ satisfies

- (i) $V(\sigma) = 0$, and V can take positive value on any neighborhood of σ .
- (ii) $\dot{V}(x) > 0, \forall x \in D \setminus \{0\}$.

Then σ is unstable.

Example 3.2.10

Consider the equation

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

Let $V(x, y) = x^2 - y^2$, then $\dot{V}(x, y) = 2x^2 + 2y^2 > 0$, hence 0 is unstable.

Theorem 3.2.11

Let f be a C^1 vector field, $f(\sigma) = 0$. If σ is stable, then every eigenvalue of $Df(\sigma)$ is with non-positive real part.

Proof. Prove for $n = 2$. Assume $\sigma = 0$, the equation is

$$\begin{cases} \dot{x} = \lambda x + \alpha(x, y) \\ \dot{y} = \mu y + \beta(x, y) \end{cases},$$

where $\lambda < \mu, \mu > 0, |\alpha|, |\beta| = o(r)$. Let $V(x, y) = -x^2 + y^2$, then

$$\dot{V}(x, y) = -2\lambda x^2 + 2\mu y^2 - 2x\alpha + 2y\beta.$$

If $\lambda < 0$, then $\dot{V} > 0$ in a neighborhood of 0, then 0 is unstable. If $\lambda \geq 0$, consider

$$C = \{(x, y) : V(x, y) \geq 0\}.$$

We can show that for some $\varepsilon_0 \geq 0$, $\dot{V}(x, y) > 0$ on $C \cap B(0, \varepsilon_0) \setminus \{0\}$. Let $H(x, y) = x^2 + y^2$, then $\dot{H}(x, y) \geq \frac{\mu}{2}H(x, y)$ on some neighborhood of 0. Hence

$$H(\varphi_t(x, y)) \geq H(x, y)e^{\frac{\mu}{2}t}$$

will be out of $C \cap B_\varepsilon(x, y)$. □

Remark 3.2.12 — In fact, there exists $(x, y) \in B(0, \varepsilon_0) \setminus \{0\}$, such that

$$\lim_{t \rightarrow -\infty} \varphi_t(x, y) = 0, \quad \frac{f(\varphi_t(x, y))}{|f(\varphi_t(x, y))|} \rightarrow (0, 1).$$

$\varphi_t(x, y)$ is called the unstable manifold.

Exercise 3.2.13. Prove the theorem for general dimension n .

Now, we consider a perturbation of a singularity of center type. Consider the system

$$\begin{cases} \dot{x} = -y + \alpha(x, y) \\ \dot{y} = x + \beta(x, y) \end{cases},$$

then

$$\dot{\theta} = 1 + \frac{x\beta - y\alpha}{x^2 + y^2},$$

$$\dot{r} = \frac{x\alpha + y\beta}{r} = \alpha \cos \theta + \beta \sin \theta = R_2(\theta)r^2 + R_3(\theta)r^3 + \dots.$$

Example 3.2.14

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2y + x^3) \end{cases}.$$

Then

$$\dot{r} = \sin \theta (x^2y + x^3) = r^3 (\cos^2 \theta \sin^2 \theta + \cos^3 \theta \sin \theta),$$

we calculate

$$\overline{R}_3 = \int_0^{2\pi} R_3(\theta) d\theta = \frac{\pi}{4} > 0.$$

Let $g(\theta) = \int_0^\theta R_3(\theta) d\theta$, then

$$\varphi_3(\theta) = g(\theta) - \frac{\theta}{2\pi} \int_0^{2\pi} R_3(\theta) d\theta$$

is 2π -periodic. Let $r = \rho + \varphi_3(\theta)\rho^3$, then

$$\frac{d\rho}{d\theta} = \overline{R}_3\rho^3 + \dots,$$

hence ρ is increasing. Therefore, 0 is unstable.

Example 3.2.15

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We want to construct a Lyapunov function of the form $V(x, y) = x^2 + y^2 + F(x, y)$, where $F(x, y)$ is a homogeneous polynomial of $\deg = 3$. Then

$$\dot{V}(x, y) = -yF_x + xF_y + 2y^3 + y^2F_y,$$

we want $-yF_x + xF_y + 2y^3 = 0$. Consider $L : H_k \rightarrow H_k$, where H_k is the family of homogeneous polynomials of $\deg = k$, $L(F) = -yF_x + xF_y$. After repetition, we can let

$$V(x, y) = \lambda(x^2 + y^2)^k + \dots.$$

Then 0 is stable if $\lambda < 0$, 0 is unstable if $\lambda > 0$. Or we can find V such that $\dot{V}(x, y) = 0$, then 0 is still a center.

Example 3.2.16

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We can solve this equation,

$$y^2 = -x + \frac{1}{2}(1 - e^{-2x}) + Ce^{-2x},$$

hence $e^{2x}(x^2 + y^2) = C + \dots$. 0 is still a center.

Example 3.2.17

Consider the equation

$$\begin{cases} \dot{x} = -y & = X(x, y) \\ \dot{y} = x + y^2 & = Y(x, y) \end{cases}.$$

Notice that $X(x, -y) = -X(x, y)$, $Y(x, -y) = Y(x, y)$, hence the solution curve is symmetric with respect to x -axis. We can prove this fact by showing $(x(-t), -y(-t))$ is a solution if $(x(t), y(t))$ is a solution. Then we can show 0 is a center.

§3.3 Stability under perturbations

Definition 3.3.1. Consider an autonomous system $\dot{x} = f(x)$, generating a flow φ_t . For every $x_0 \in \mathbb{R}^n$, the orbit $\varphi_t(x_0)$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi_t(x) - \varphi_t(x_0)| < \varepsilon, \quad \forall t \geq 0, x \in B(x_0, \delta).$$

Example 3.3.2

Consider the equation

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2 \end{cases}.$$

Then the orbit of $(r_0, \theta_0) = (1, 0)$ is **not** stable.

Definition 3.3.3. Consider a non-autonomous system $\dot{x} = f(x, t)$, let $\varphi(t; t_0, x_0)$ be the solution of the initial value problem $x(t_0) = x_0$. The orbit $x(t; t_0, x_0)$ is said to be **stable**, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi(t; t_0, x) - \varphi(t; t_0, x_0)| < \varepsilon, \quad \forall t \geq t_0, x \in B(x_0, \delta).$$

Similarly, we can define the asymptotically stable and the exponentially stable for general orbits of autonomous or non-autonomous systems.

Theorem 3.3.4

$A : \mathbb{R} \rightarrow M(n, \mathbb{R})$, consider a non-autonomous system $\dot{x} = A(t)x$. Then

1. Every solution is stable iff 0 is stable.
2. 0 is stable iff $\sup_{t \geq 0} |X(t)| < \infty$, where $X(t)$ is a fundamental solution.
3. 0 is asymptotically stable iff $\lim_{t \rightarrow \infty} |X(t)| = 0$.

Theorem 3.3.5

Consider a T -periodic system $\dot{x} = A(t)x$. Then

2. 0 is stable iff the Floquet exponents are of non-positive real parts and Jordan block are trivial for every Floquet exponent with zero real part.
2. 0 is asymptotically stable iff Floquet exponents are of negative real parts iff 0 is exponentially stable.

For an autonomous system, let $f(0) = 0$, $f(x) = Ax + \varphi(x)$, where $\varphi(0) = 0$, $D\varphi(0) = 0$. Rewrite the system as $\dot{x} = Ax + \varphi(x)$, if every eigenvalue of A is with negative real parts, then 0 is stable.

For a non-autonomous system, assume

$$\dot{x} = Ax + \varphi(t, x), \quad \varphi(t, 0) = 0, D\varphi(t, 0) = 0,$$

if every eigenvalue of A is with negative real parts, then 0 is stable. In general,

$$\dot{x} = A(t)x + \varphi(t, x),$$

but the negativeness of Lyapunov exponents do **not** imply the stableness. See the following example.

Example 3.3.6

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin \ln t + \cos \ln t))x \\ \dot{y} = (-\mu + (\sin \ln t + \cos \ln t))y + x^2 \end{cases},$$

let $a(t) = t \sin \ln t$, the solutions are

$$\begin{cases} x = C_1 e^{-\mu t - a(t)} \\ y = C_2 e^{-\mu t + a(t)} + C_1^2 e^{-\mu t + a(t)} \int_1^t e^{-\mu s - 3a(s)} ds \end{cases}.$$

Consider $\mu = 1 + \sigma$, σ is sufficiently small, then 0 is not stable.

For this case, we need a stronger condition. Let $\Phi(t)$ be a fundamental solution of the linear part, if $\exists \mu > 0$,

$$|\Phi(t)\Phi(-s)| \leq C e^{-\mu(t-s)}, \quad \forall t \geq s \geq 0,$$

then 0 is also stable under the perturbation .

4 Poincaré-Bendixson Theory

§4.1 Basic notions

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 vector field, generating a flow $\varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 4.1.1. $A \subseteq \mathbb{R}^n$ is said to be f (or φ_t) **invariant** if for every $t \in \mathbb{R}$, $\varphi_t(A) = A$.

For every $x \in \mathbb{R}^n$, the orbit $\text{Orb}(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$ is an invariant set. In general, if A is invariant, then

$$A = \bigcup_{x \in A} \text{Orb}(x).$$

Definition 4.1.2. Let A be a compact invariant set, A is said to be **Lyapunov orbit stable** if for every neighborhood $U \supseteq A$, there exists a neighborhood $V \supseteq A$ such that

$$\varphi_t(x) \in U, \quad \forall x \in V, t \geq 0.$$

Let

$$\text{Orb}^+ := \{\varphi_t(x) : t \geq 0\}, \quad \text{Orb}^- := \{\varphi_t(x) : t \leq 0\}$$

be the **positive semi-orbit** and the **negative semi-orbit**.

Definition 4.1.3. Given $p \in \mathbb{R}^n$, x is called a **positive limit point** if $\exists t_n \rightarrow +\infty, \varphi_{t_n} \rightarrow x$. The set of all positive limit points is called the **α -limit set** of p , denoted by $\alpha(p)$. Similarly, we can define the **negative limit points**, they form a set is called **ω -limit set**, denoted by $\omega(p)$.

Remark 4.1.4 — In the Greek alphabet, α is the first letter and ω is the last letter, it is very graphic that the orbit of p ran from α to ω .

Example 4.1.5

Consider the equation

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

Then $\omega(0) = \alpha(0) = 0$. For every $p \in \mathbb{S}^1$, we have $\omega(p) = \alpha(p) = \mathbb{S}^1$. Otherwise, let $p = (x, y)$, we have

- (1) If $0 < x^2 + y^2 < 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \{0\}$.
- (2) If $x^2 + y^2 > 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \emptyset$.

Proposition 4.1.6

$\forall p \in \mathbb{R}^n$, we have

$$\omega(p) = \bigcap_{t \geq 0} \overline{\text{Orb}^+(\varphi_t(p))} = \bigcap_{k \in \mathbb{Z}_+} \overline{\text{Orb}^+(\varphi_k(p))}.$$

Proposition 4.1.7

Assume $\text{Orb}^+(p)$ is bounded, then

1. $\omega(p)$ is non-empty, compact, invariant, connected.
2. $\lim_{t \rightarrow \infty} d(\varphi_t(p), \omega(p)) = 0$.

Proof. 1. Non-empty, compact, invariant are trivial. The connected follows by the fact that $A_k = \text{Orb}^+(\varphi_k(p))$ are connected and $A_k \supseteq A_{k+1} \supseteq \dots$.

2. For every $\varepsilon > 0$, $A_k \subseteq B(\omega(p), \varepsilon)$ for every k sufficiently large. □

Definition 4.1.8. p is said to be **positively recurrent** if $p \in \omega(p)$,

The singularities and periodic points are called trivial recurrent points, other recurrent points are said to be non-trivial.

Definition 4.1.9. Let Λ be a non-empty, compact, invariant set. Λ is called a **minimal set** of φ_t if it does not contain a proper, nonempty, compact invariant set.

Theorem 4.1.10 (Flow Box Theorem)

Let f be a C^1 vector field, $p \in \mathbb{R}^n$, $f(p) \neq 0$. Then there is a neighborhood $U \ni p$ and a C^1 diffeomorphism $h : U \rightarrow h(U)$ on to its image, such that $Dh(x)f(x) = (1, 0, \dots, 0)^t$.

Proof. WLOG, $p = 0$, $f(p) = (1, 0, \dots, 0)^t$. We construct $g : (-\varepsilon_0, \varepsilon_0) \times L \rightarrow U$ some neighborhood of p . Let

$$x = g(y) = g(y_1, y_2, \dots, y_n) := \varphi_{y_1}(0, y_2, y_3, \dots, y_n).$$

Then

$$\left. \frac{\partial}{\partial t} \varphi_t(x) \right|_{(t,x)=(y_1,0,y_2,\dots,y_n)} = f(\varphi_{y_1}(0, y_2, \dots, y_n)) = f(g(y)),$$

let $(y_1, y_2, \dots, y_n) = (0, 0, \dots, 0)$, then $\frac{\partial g}{\partial y_1}(y) = f(g(y))$. Moreover,

$$\text{Id} = \left. \frac{\partial \varphi_t(x_1, \dots, x_n)}{\partial (x_1, \dots, x_n)} \right|_{t=0} \implies \left. \frac{\partial g}{\partial y} \right|_{y=0} = \text{Id}.$$

Hence, g gives a local diffeomorphism. Let $h = g^{-1}$, the statement follows. □

Remark 4.1.11 — Let $L_{\varepsilon_0} = \{(y_2, \dots, y_n) : y_2^2 + \dots + y_n^2 \leq \varepsilon_0^2\}$, let

$$U = h^{-1}((-\varepsilon_0, \varepsilon_0) \times L_{\varepsilon_0}),$$

then U is called a **tubular neighborhood** near p , or a **flow box** near p .

§4.2 The Poincaré-Bendixson Theorem

Definition 4.2.1. $C \subseteq \mathbb{R}^2$ is called a **Jordan curve** if it is homeomorphism to \mathbb{S}^1 .

Theorem 4.2.2 (Jordan Separation Theorem)

Let $C \subseteq \mathbb{R}^2$ be a Jordan curve. Then $\mathbb{R}^2 \setminus C$ has exactly two connected components. One of them is bounded, which is called the interior of C . Another one is bounded, which is called the exterior of C . Both of them are with bound C .

Theorem 4.2.3 (Jordan-Schoenflies)

Let C be a Jordan curve, then there is a homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that $h(C) = \mathbb{S}^1$.

Definition 4.2.4. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be C^1 , $L \subseteq \mathbb{R}^2$ is a line segment. L is called **transverse** to f if $\forall x \in L$, $f(x)$ and the direction of L generates \mathbb{R}^2 . We then say L is a **transversal** to f .

Lemma 4.2.5

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be C^1 , L is a transversal to f . Assume there are three points $P_1, P_2, P_3 \in L$ and $x \in \mathbb{R}^2$ such that

$$\varphi_{t_i}(x) = P_i, \quad t_1 < t_2 < t_3,$$

$$\varphi_{(t_1, t_2)}(x) \cap L = \emptyset, \quad \varphi_{(t_2, t_3)}(x) \cap L = \emptyset,$$

then $P_2 \in (P_1, P_3)$.

Proof. Assume A, B are extreme points of L . Consider a Jordan curve

$$C = \varphi_{[t_1, t_2]}(x) \cup (P_1, P_2),$$

let D be the interior of C . Assume $B \in D$, we show that $P_3 \in D$. By the Flow Box Theorem, there exists $\varepsilon > 0$ such that $\varphi_{(t_2, t_2+\varepsilon]}(x) \subseteq D$. Let $\tau = \inf \{t > t_2 : \varphi_t(x) \notin D\} > t_2 + \varepsilon$ if exists. Then $\varphi_\tau(x) \in C$, but it can not on $\varphi_{(t_1, t_2)}(x)$ or P_1, P_2 . So $\varphi_\tau(x) \in (P_1, P_2)$, but this contradict with L is a transversal to f . \square

Remark 4.2.6 — Assume $\varphi_t(x)$ intersect with a transversal L at $P_i = \varphi_{t_i}(x), i = 1, 2, \dots$ in chronological order, i.e., $0 < t_1 < t_2 < \dots$, then

$$P_1 < P_2 < \dots \quad \text{or} \quad P_1 > P_2 > \dots \quad \text{or} \quad P_1 = P_2 = \dots.$$

Proposition 4.2.7

Assume L is a transversal of f , then for every $x \in \mathbb{R}^2$,

$$\sharp(\omega(x) \cap L) \leq 1.$$

Proof. Assume for a contradiction. Let $q \neq q' \in \omega(x) \cap L$, then $\exists t_n \rightarrow \infty, t'_n \rightarrow \infty$ such that $\varphi_{t_n}(x) \rightarrow q, \varphi_{t'_n}(x) \rightarrow q'$. WLOG, assume $t_1 < t'_1 < t_2 < t'_2 < \dots$. By the Flow Box Theorem, for k sufficiently large, there exists τ_k, τ'_k such that

$$|\tau_k - t_k| \rightarrow 0, |\tau'_k - t'_k| \rightarrow 0, \quad \varphi_{\tau_k}(x), \varphi_{\tau'_k}(x) \in L, \quad \varphi_{\tau_k}(x) \rightarrow q, \varphi_{\tau'_k}(x) \rightarrow q'.$$

We can also assume that $\tau_k < \tau'_k < \tau_{k+1} < \dots$, then this contradicts with the monotonicity of $\varphi_t(x)$ intersecting the transversal. \square

Theorem 4.2.8 (Poincaré-Bendixson Theorem)

Assume $\text{Orb}^+(x)$ is bounded and $\omega(x)$ contains no singularities, then $\omega(x)$ is a periodic orbit.

Proof. Because $\text{Orb}^+(x)$ is bounded, $\omega(x) \neq \emptyset$. For every $p \in \omega(x)$, take $q \in \omega(p) \subseteq \omega(x)$ arbitrarily. Take a transversal L_q of f through q , then $\exists t_n \rightarrow \infty, \varphi_{t_n}(p) \rightarrow q$. WLOG, $\varphi_{t_n}(x) \in L_q$. Because $\varphi_{t_n}(p) \in \omega(x)$ and $\sharp \omega(x) \cap L_q = 1$, then $\varphi_{t_n}(p) = \varphi_{t_{n+1}}(p)$, hence p is a periodic point.

Take $p \in \omega(x)$, it is a periodic point. If $\omega(x) \neq \text{Orb}(p)$, take a transversal L_p of f through p . Because $\omega(x)$ is connected, hence $\text{Orb}(p)$ is not isolated in $\omega(x)$. Take $q_n \in \omega(x) \setminus \text{Orb}(p)$, $q_n \rightarrow \text{Orb}(p)$. WLOG, $q_n \rightarrow p$ and $q_n \in L_p$, this contradicts with $\sharp \omega(x) \cap L_p \leq 1$. \square

Theorem 4.2.9 (P-B Annular Region Theorem)

Assume A is an annular region and ∂A is two C^1 curves. If for every $x \in \partial A$, $f(x)$ is pointing inside of A , and A contains no singularities. Then there is a periodic orbit in A .

Example 4.2.10

The system

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 3x - 1) - y \\ \dot{y} = y(x^2 + y^2 - 3x - 1) + x \end{cases}$$

contains a non-trivial periodic orbit.

Proof. 0 is the only singularity. Let

$$A = \{(x, y) \in \mathbb{R}^2, r^2 \leq x^2 + y^2 \leq R^2\}, \quad r < R,$$

let $V(x, y) = x^2 + y^2$. Then for r small enough $\dot{V} < 0$, for R large enough $\dot{V} > 0$. Hence $f(x)$ is pointing outside of A on ∂A , consider the α -limit set. \square

The **Liénard equation**

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

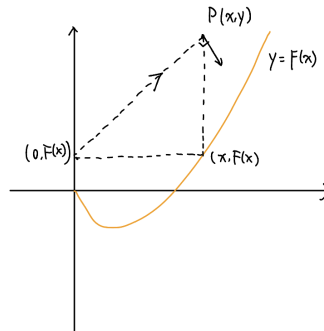
Let $F(x) = \int_0^x f(t)dt$, then the equation is equivalent to

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}.$$

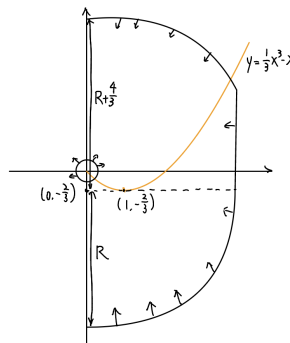
Consider a particular Liénard equation, which is called **van der Pol equation**: $f(x) = x^2 - 1$, $g(x) = x$. Then this equation is equivalent to

$$\begin{cases} \dot{x} = y - \left(\frac{1}{3}x^3 - x\right) \\ \dot{y} = -x \end{cases}.$$

We introduce the **Liénard graphing method**. Draw the graph of $y = F(x)$, for $P = (x, y) \in \mathbb{R}^2$, draw $(x, F(x))$ and $Q = (0, F(x))$. Hence the slope of $QP = \frac{F(x)-y}{-x}$, then rotate it 90° clockwise, we get $f(x, y)$.



Consider $V = x^2 + y^2$, then $\dot{V} = 2x^2(1 - \frac{1}{3}x^2) \geq 0$ when $|x| < 1$. Take $x = 1$, then $F(x) = -\frac{2}{3}$ attain the minimum. For R sufficiently large, we can construct a curve such that the poin, see the figure. For r small enough, vectors on B_r are pointing outside. This gives an annular region and hence there is a periodic orbit in it. Furthermore, we can prove that the periodic orbit is unique.



Theorem 4.2.11 (P-B)

f is a C^1 vector field, assume there are only finite singularities. If Orb^+ is bounded, then there is a trichotomy:

- (1) $\omega(x)$ is a periodic orbit.
- (2) $\omega(x)$ is a singularity.
- (3) $\omega(x)$ contains regular points and singularities and $\forall y \in \omega(x)$, $\alpha(y), \omega(y)$ are both a singularity.

Proof. If $\omega(x)$ contains no singularities, by P-B, $\omega(x)$ is a periodic point. If $\omega(x)$ contains no regular points, then $\omega(x)$ is finite singularities, but $\omega(x)$ is connected, hence $\omega(x)$ is a singularity.

If $\omega(x)$ contains regular points and singularities both, for $y \in \omega(x)$, assume for a contradiction that $\omega(y)$ contains some regular points. Take $z \in \omega(y)$ regular and a transversal L_z contains z . Take $t_n \rightarrow +\infty$, $\varphi_{t_n}(y) \rightarrow z$, $\varphi_{t_n}(y) \in L_z$. Same argument shows that y is a periodic point.

Take a transversal L_y contains y , take $\tau_n \rightarrow +\infty$, $\varphi_{\tau_n}(x) \rightarrow y$, $\varphi_{\tau_n}(x) \in L_y$. Because $\omega(x)$ contains some singularities, then x is not a periodic point, hence $\varphi_{\tau_n}(x) \neq y$. Let

$$P_n = \varphi_{\tau_n}(x), \quad C_n = \varphi_{[\tau_n, \tau_{n+1}]}(x) \cap [P_n, P_{n+1}],$$

then $C_n \rightarrow \text{Orb}(y)$, hence we can show that $\omega(x) = \text{Orb}(y)$. A contradiction. \square

§4.3 Poincaré recurrence and limit cycle

Lemma 4.3.1

Let f be a C^1 vector field, generating a flow φ_t . Let p be a regular point, L_t be a transversal of f at $\varphi_t(p)$. Then there is a neighborhood $U \ni p$ and a C^1 map: $U \rightarrow \mathbb{R}$ such that

$$\tau(p) = t, \quad \varphi_{\tau(x)} \in L_t, \forall x \in U.$$

Proof. WLOG, assume $L_t \perp f(\varphi_t(x))$. Let $Q = \varphi_t(p)$, consider

$$F(\tau, x) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (\tau, x) \mapsto (\varphi_t(x) - Q) \cdot f(Q).$$

Then $\varphi_{\tau}(x) \in L_t$ iff $F(\tau, x) = 0$. We have

$$\left. \frac{\partial F}{\partial \tau} \right|_{(\tau, x)=(t, p)} = f(\varphi_{\tau}(x))f(Q)|_{(\tau, x)=(t, p)} = |f(Q)|^2 > 0,$$

The conclusion follows by implicit function theorem. \square

Corollary 4.3.2

Let f be a C^1 vector field, generating a flow φ_t . For every regular point p and two transversals L_0, L_t at p and $\varphi_t(p)$, respectively. There exists an neighborhood I_0 of p in L_0 and a C^1 map $\tau : I_0 \rightarrow \mathbb{R}$ such that

$$\tau(p) = t, \quad \varphi_{\tau(x)}(x) \in L_t, \forall x \in I_0.$$

Furthermore, let $P : I_0 \rightarrow L_t$, $x \mapsto \varphi_{\tau(x)}(x)$, then P is a diffeomorphism onto image.

Proof. Assume $L_t : (X - Q) \cdot f(Q) = 0$, then τ is given by $f(Q)^t(\varphi_{\tau}(x) - Q)$. Hence

$$D\tau = -\frac{f(Q)^t D\varphi_{\tau}(x)}{f(Q)^t f(\varphi_t(x))}.$$

Let I_0 be a connected component of $U \cap L_0$ containing x , define the **Poincaré map**

$$P : I_0 \rightarrow L_t, \quad x \mapsto \varphi_{\tau(x)}(x).$$

Then, for $v \in T_p L_0$.

$$DP(p)v = \frac{\partial \varphi}{\partial \tau} \frac{\partial \tau}{\partial x} v + D\varphi_{\tau(x)}(x)v|_{x=p} = -f(Q) \frac{f(Q)^t D\varphi_t(x)}{f(Q)^t f(Q)} + D\varphi_t(p)v,$$

it is the orthogonal projection of $D\varphi_t(x)v$ to L_t . It suffices to show $DP(p) \neq 0$. Otherwise, $D\varphi_t(p)v = \eta f(Q)$ for some v , but $D\varphi_t(p)f(p) = f(Q)$, a contradiction with $D\varphi_t(p)$ is an isomorphism. \square

Let p be a T -periodic point, then $D\varphi_T f(p) = f(p)$. Assume $f(p) \parallel \mathbf{e}_1$, then

$$D\varphi_T(p) = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}.$$

Take $L_0 = L_T \parallel \mathbf{e}_2$, then $DP(p)e_2 = d \cdot \mathbf{e}_2$. We choose a coordinate on L_0 , assume $p = 0$, then $P : I_0 \rightarrow L_0$ is a real-valued function on \mathbb{R} such that $P'(0) = d$.

Proposition 4.3.3

Assume p is a T -periodic point of C^1 vector field, then $P'(0) = \det D\varphi_T(p) > 0$.

Proof. $t \mapsto \det D\varphi_t(p)$ is continuous, nonzero and $0 \mapsto 1$, hence is positive. \square

Definition 4.3.4. An isolated closed orbit is called a **limit cycle**.

Lemma 4.3.5

Let γ be a periodic point, L is a transversal of some point on γ . Let $P : I \rightarrow L$ be the Poincaré recurrent map, $0 \in I$ correspond to the given periodic point. Then $x \in I$ is a fixed point of P if and only if x is a periodic point.

Proof. Let x be a periodic point of φ . If $P(x) \neq x$, then $P(x) \in \text{Orb}(x) = \omega(x)$, but $\sharp(\omega(x) \cap L_0) \leq 1$, a contradiction. \square

Definition 4.3.6. Let γ be a limit cycle. Then γ is said:

- (1) **stable**, if there exists a neighborhood U of γ , such that $\forall x \in U, \omega(x) = \gamma$.
- (2) **unstable**, if there exists a neighborhood U of γ , such that $\forall x \in U, \alpha(x) = \gamma$.
- (3) **semi-stable**, if there exists a neighborhood U of γ and a splitting $U \setminus \gamma = U_+ \cup U_-$, such that $\forall x \in U_+, \omega(x) = \gamma$ and $\forall x \in U_-, \alpha(x) = \gamma$.

Proposition 4.3.7

Let γ be a limit cycle and $p \in \gamma$. If $P'(p) < 1$, then γ is stable. If $P'(p) > 1$, then γ is unstable.

Compare with the definition of Lyapunov asymptotically stable, this definition does not guarantee the Lyapunov stability. But under the condition $P'(p) < 1$, we can show that $\forall \varepsilon > 0, \exists \delta > 0$ such that if $|x - p| < \delta$, then $|\varphi_t(x) - \varphi_t(p)| < \varepsilon$ for every $t \geq 0$. It suffices to show for $x \in L$. Note that

$$P^n(x) = \varphi_{\tau(x)+\tau(P(x))+\dots+\tau(P^{n-1}(x))}(x),$$

because $|P^n(x) - P^n(p)| \leq \lambda^n |x - p|$ for some $\lambda < 1$, we can bound the time $|\tau(x) + \tau(P(x)) + \dots + \tau(P^{n-1}(x)) - nT|$ by some $\frac{C}{1-\lambda} |x - p|$.

Definition 4.3.8. Let p be a periodic point, if the every eigenvalue of $DP(p)$ is not on the unit circle, then p is called **hyperbolic**.

Exercise 4.3.9. $n \geq 2$, let p be a T -periodic point, then p is hyperbolic if and only if there are $n - 1$ eigenvalues of $D\varphi_T(p)$ are with absolute value $\neq 1$.

Example 4.3.10

Let $H : \mathbb{R}^2 \rightarrow \mathbb{R}$, be a C^2 function. Consider the **plane Hamilton system**

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases},$$

then there is not limit cycle in this system.

Proof. Let $\Phi_t(x, y) = D\varphi_t(x, y)$, then $\det \Phi_t(x, y) = 1$. Hence φ_t is area preserving. This contradicts with the existence of limit cycle. \square

Hilbert's 16-th problem Consider the plane polynomial system

$$\begin{cases} \dot{x} = P_n(x, y) = \sum_{i+j=0}^n a_{ij}x^i y^j, \\ \dot{y} = Q_n(x, y) = \sum_{i+j=0}^n b_{ij}x^i y^j. \end{cases}$$

The second part of this problem is considering about the number of limit cycles in this system. For a given system, Dulac-Ilyashenko-Écalle proved that the number of limit cycles is finite.

Furthermore, let $H(n)$ be the upper bound of all systems with degree at most n . A further problem is whether $H(n)$ is finite. It trivial that $H(1) = 0$, but the existence of $H(2)$ is still unknown.

5 Bifurcations

§5.1 Structural stability

Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two C^1 vector fields. The C^1 distance of f, g is defined as

$$|f - g|_{C^1} := \sup \{ |f(x) - g(x)|, |Df(x) - Dg(x)| : x \in \mathbb{R}^n \}.$$

Definition 5.1.1. Given a C^1 vector field f on \mathbb{R}^n , f is called C^1 **structurally stable** if there exists $\varepsilon > 0$ such that for every C^1 vector field g , $|g - f|_{C^1} < \varepsilon$, then g is topologically equivalent with f .

Remark 5.1.2 — The condition is just “topologically equivalent” is reasonable. A C^1 -conjugate preserves the derivation at a singularity. A C^0 -conjugate preserves the time on a periodic orbit. Both conditions are too much for a general vector field.

Example 5.1.3

$f(x) = x, x \in \mathbb{R}$ is C^1 structurally stable.

Exercise 5.1.4. Show that the plane system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

is C^1 structurally stable.

Exercise 5.1.5. Show that a center is not structurally stable.

Let \mathbb{D} be the closed unit disk $\{x^2 + y^2 \leq 1\}$, let $\mathcal{X}^1(\mathbb{D})$ be the set of C^1 vector fields on \mathbb{D} which are not tangent to $\partial\mathbb{D}$.

Theorem 5.1.6 (Andronov-Pontryagin)

Given $f \in \mathcal{X}^1(\mathbb{D})$, then f is C^1 structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.

Corollary 5.1.7

Under the same condition, given $x \in \mathbb{D}$, if $\text{Orb}(x) \subset \mathbb{D}$, then $\omega(x)$ is a periodic point or a singularity.

Theorem 5.1.8 (Peixoto-Pugh)

Let M be a 2-dimensional closed Riemannian manifold. Given $f \in \mathcal{X}^1(M)$, then f is C^1 structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.
- for every $x \in M$, $\omega(x)$ is a periodic point or a singularity.

Example 5.1.9

Consider a system on \mathbb{T}^2 ,

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha, \end{cases}$$

if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then this system satisfying the first two conditions. But this system is not C^1 structurally stable. Hence for a general Riemannian manifold, the third condition is necessary.

Remark 5.1.10 — Peixoto has proved C^r structural stability for orientable M . For general surface M , Pugh applied C^1 -closed theorem to show the statement. But C^r -closed theorem is still a big open problem.

§5.2 Bifurcations

Definition 5.2.1. Let $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a C^1 map, which is a family of vector fields with k -parameters. The parameter $\epsilon_0 \in \mathbb{R}^k$ is called a **bifurcation** if $\forall \delta > 0, \exists \epsilon_1 \in \mathbb{R}^k, |\epsilon_1 - \epsilon_0| < \delta$, such that f_{ϵ_1} is not topologically equivalent with f_{ϵ_0} .

Example 5.2.2

Consider $f_\beta(x, y) = (-\beta y, \beta x)$, then f_β is not C^1 structurally stable, but is also not a bifurcation.

Remark 5.2.3 — A general problem is how to embed a non structurally stable vector field into a vector field family to be a bifurcation. These embeddings are called an **unfolding**. Some topics are to find a **universal unfolding**.

Example 5.2.4

Consider $f_\epsilon(x) = x(\epsilon - x)$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **transcritical bifurcation**: there are two hyperbolic singularities before and after the collision, and after the collision, the stability of two singularities exchange.

Example 5.2.5

Consider $f_\epsilon(x) = \epsilon - x^2$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **saddle-node bifurcation**: it can be regarded as a collision of a saddle and a node, there is a non-hyperbolic singularity at the bifurcation which is called a saddle-node.

It comes from the plane vector fields $f_\epsilon(x, y) = (\epsilon - x^2, y)$, when $\epsilon = 0$, the singularity seems like a node on one side and saddle on another side.

Example 5.2.6

Consider $f_\epsilon(x) = x(\epsilon - x^2)$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **pitchfork bifurcation**: $x = 0$ is always a singularity, it is non-hyperbolic only if $\epsilon = 0$, after that, it create two hyperbolic singularities, one stable and one unstable.

Example 5.2.7

Consider the system

$$\begin{cases} \dot{x} = \epsilon x - y - x(x^2 + y^2) \\ \dot{y} = x + \epsilon y - y(x^2 + y^2) \end{cases} \quad \begin{cases} \dot{r} = \epsilon r - r^3 \\ \dot{\theta} = 1 \end{cases}.$$

It is called a **Hopf bifurcation** and corresponds to a creation of periodic orbit (at $r = \sqrt{\epsilon}$ for $\epsilon > 0$).

6 Index Theory and Dynamics on \mathbb{S}^1

§6.1 Index

Definition 6.1.1. A **path** in \mathbb{R}^2 is a continuous map $\gamma : [0, 1] \rightarrow \mathbb{R}^2$.

Let $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ be the covering map, $\pi(\theta) = (\cos \theta, \sin \theta)$.

Theorem 6.1.2

Let $\gamma : [a, b] \rightarrow \mathbb{S}^1$ be a continuous map, then for every $\theta_0 \in \pi^{-1}(\gamma(a))$, there exists a unique continuous map $\theta : [a, b] \rightarrow \mathbb{R}$, $\theta(a) = \theta_0$ such that $\gamma = \pi \circ \theta$.

Lemma 6.1.3

Given a path $\gamma : [a, b] \rightarrow \mathbb{R}^2$ and a continuous vector field $f = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. If $f|_\gamma \neq 0$, then there exists $\theta : [a, b] \rightarrow \mathbb{R}$, such that

$$\begin{cases} \cos \theta(t) = \frac{P(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \\ \sin \theta(t) = \frac{Q(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \end{cases},$$

i.e. $\pi \circ \theta = \frac{f \circ \gamma}{|f \circ \gamma|}$. If there exists another continuous map $\tilde{\theta}$ such that $\pi \circ \tilde{\theta} = \frac{f \circ \gamma}{|f \circ \gamma|}$, then $\exists k \in \mathbb{Z}$ such that $\forall t \in [a, b], \tilde{\theta}(t) - \theta(t) = 2k\pi$. In particular, $\theta(b) - \theta(a) = \tilde{\theta}(b) - \tilde{\theta}(a)$.

Definition 6.1.4. A path $\gamma : [a, b]$ is called **closed** if $\gamma(a) = \gamma(b)$.

Definition 6.1.5. Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a closed path, $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous vector field, $f|_\gamma \neq 0$. Then

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

is called the **index** of γ with respect to f , denoted by $\text{Ind}_f(\gamma)$. Where $\theta : [a, b] \rightarrow \mathbb{R}$ satisfying $\pi \circ \theta = \frac{f \circ \gamma}{|f \circ \gamma|}$.

Lemma 6.1.6

$\text{Ind}_f(\gamma) \in \mathbb{Z}$ and the index is independent with the choice of θ .

Theorem 6.1.7 (Hopf)

Let f be a C^1 vector field, let $\gamma : [0, T] \rightarrow \mathbb{R}^2$ be a periodic orbit of f , i.e., $\dot{\gamma}(t) = f(\gamma(t))$, $\gamma(0) = \gamma(T)$, then $\text{Ind}_f(\gamma) = \pm 1$.

Proof. WLOG, we can assume that $\gamma(0) = 0$ is the lowest point on $\gamma([0, T])$, and $\dot{\gamma}(t)$ is pointing the positive direction of x -axis (this might take the index to its opposite). Let $\Delta = \{(s, t) : 0 \leq s \leq t \leq T\}$, we define the map $\eta : \Delta \rightarrow \mathbb{S}^1$ as

$$\eta(s, t) = \begin{cases} \frac{f(\gamma(t))}{|f(\gamma(t))|}, & s = t, \\ -\eta(0, 0), & (s, t) = (0, T), \\ \frac{\gamma(t) - \gamma(s)}{|\gamma(t) - \gamma(s)|}, & s \neq t, (s, t) \neq (0, T). \end{cases}$$

Then $\eta : \Delta \rightarrow \mathbb{S}^1$ is continuous. There exists a unique continuous map $\phi : \Delta \rightarrow \mathbb{R}$, $\phi(0, 0) = 0$ and $\pi \circ \phi = \eta$. By definition, we have

$$\text{Ind}_f(\gamma) = \frac{\phi(T, T) - \phi(0, 0)}{2\pi}.$$

On the other hand, note that $\phi(0, t) \in [0, \pi]$ and $\phi(t, T) \in [\pi, 2\pi]$, we have

$$\phi(0, T) - \phi(0, 0) = \pi - 0 = \pi \quad \text{and} \quad \phi(T, T) - \phi(0, T) = 2\pi - \pi = \pi.$$

Hence $\text{Ind}_f(\gamma) = 1$. □

Definition 6.1.8. Let $\gamma_0, \gamma_1 : [a, b] \rightarrow \mathbb{R}^2$ be two paths, γ_0 and γ_1 are called **homotopic** if $\exists H : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$ continuous such that $H(\cdot, 0) = \gamma_0$, $H(\cdot, 1) = \gamma_1$. H is called a **homotopy** of γ_0 and γ_1 .

Theorem 6.1.9

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous vector field, let $H : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$ be a homotopy of closed paths γ_0 and γ_1 . If $f \circ H \neq 0$, then $\text{Ind}_f(\gamma_0) = \text{Ind}_f(\gamma_1)$.

Theorem 6.1.10

Let $f : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$ be a continuous family of vector field, $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a closed path such that $f_s(\gamma(t)) \neq 0$ for every s, t . Then $\text{Ind}_{f_s}(\gamma)$ is a constant with respect to s .

Corollary 6.1.11

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous vector field, if $\gamma : [a, b] \rightarrow \mathbb{R}^2$ is a Jordan curve and there is no singularity of f on γ or inside γ , then $\text{Ind}_f(\gamma) = 0$.

Corollary 6.1.12

Let f be a C^1 vector field, if $\gamma : [0, T] \rightarrow \mathbb{R}^2$ is a periodic orbit, then there exists a singularity inside γ .

Theorem 6.1.13

Let f be a continuous vector field, γ be a C^1 Jordan curve. If f is transverse to γ , then $\text{Ind}_f(\gamma) = \pm 1$.

Proof. Let

$$f_s(\gamma(t)) = sf(\gamma(t)) + (1-s)\dot{\gamma}(t),$$

then $f_s(\gamma(t)) \neq 0$, hence

$$\text{Ind}_f(\gamma) = \text{Ind}_{f_1}(\gamma) = \text{Ind}_{f_0}(\gamma) = \text{Ind}_{\dot{\gamma}}(\gamma) = \pm 1.$$

□

Corollary 6.1.14

Let f be a continuous vector field, γ be a C^1 Jordan curve. If f is transverse to γ , then there exists a singularity inside γ .

§6.2 Index of a singularity

Definition 6.2.1. Let f be a continuous vector field, σ is a singularity. σ is called an **isolated singularity** if $\exists \varepsilon_0 > 0$ such that σ is the unique singularity in $B(\sigma, \varepsilon_0)$.

Lemma 6.2.2

Let σ be an isolated singularity, then for every $\varepsilon \in (0, \varepsilon_0]$, $\text{Ind}_f(\gamma_\varepsilon) = \text{Ind}_f(\gamma_{\varepsilon_0})$, where

$$\gamma_\varepsilon : [0, 2\pi] \rightarrow \mathbb{R}^2, \quad \gamma_\varepsilon(t) = \sigma + \varepsilon(\cos t, \sin t).$$

Definition 6.2.3. Let f be a continuous vector field and σ is an isolated singularity of f . There exists $\varepsilon_0 > 0$ such that σ is the unique singularity in $B(\sigma, \varepsilon_0)$. Then $\text{Ind}_f(\gamma_{\varepsilon_0})$ is called the **index of isolated singularity** σ , denoted by $\text{Ind}_f(\sigma)$.

Lemma 6.2.4

Let $f = (P, Q)$ be a C^1 vector field, $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a path such that $f|_\gamma \neq 0$. Let $\theta : [a, b] \rightarrow \mathbb{R}$ be a lifting of $\frac{f \circ \gamma}{|f \circ \gamma|}$, then

$$\theta(b) - \theta(a) = \int_\gamma \frac{PdQ - QdP}{P^2 + Q^2}.$$

In particular, if γ is a closed path, then

$$\text{Ind}_f(\gamma) = \frac{1}{2\pi} \oint_\gamma \frac{PdQ - QdP}{P^2 + Q^2}.$$

Theorem 6.2.5

Let f be a C^2 vector field and $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a C^1 Jordan curve with positive direction, $f \circ \gamma \neq 0$. If there are only finite singularities inside γ , denoted by $\sigma_1, \dots, \sigma_n$, then

$$\text{Ind}_f(\gamma) = \text{Ind}_f(\sigma_1) + \text{Ind}_f(\sigma_2) + \dots + \text{Ind}_f(\sigma_n).$$

Remark 6.2.6 — This theorem also is also valid for just continuous settings.

Example 6.2.7

Consider

$$f_{\alpha, \beta} = (\alpha x - \beta y, \beta x + \alpha y), \quad (\alpha, \beta) \neq (0, 0).$$

Then O is the unique singularity, consider $\gamma = \gamma(t) = (\cos t, \sin t)$. Then $\text{Ind}_{f_{\alpha, \beta}}(\gamma)$ is the constant 1.

Example 6.2.8

Consider

$$g_{\lambda, \mu}(x, y) = (\lambda x, \mu y), \quad \lambda \mu \neq 0.$$

Then for $\lambda \mu > 0$, $\text{Ind}_{g_{\lambda, \mu}} = 1$; for $\lambda \mu < 0$, $\text{Ind}_{g_{\lambda, \mu}} = -1$.

Exercise 6.2.9. Let $z = (x, y) = x + iy$, $f_n(z) = z^n$, $n \in \mathbb{N}$. Regard f_n as a vector field on \mathbb{R}^2 , then O is the unique singularity. Calculate $\text{Ind}_{f_n}(O)$.

Exercise 6.2.10. Let $P, Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ be two bounded continuous functions. Show that there exists singularities in the vector field $f(x, y) = (P(x, y) - y, x + Q(x, y))$.

§6.3 Rotation numbers on the torus

Let $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ be the torus, then \mathbb{T}^2 is an abelian group and a Riemannian manifold with a flat metric. Let $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ be the covering map, then for every map $f : \mathbb{T}^2 \rightarrow \mathbb{R}^2$, the map $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\tilde{f} = f \circ \pi$ is a doubly periodic function.

Lemma 6.3.1

Let $f : \mathbb{R}^2 \rightarrow X$ be a map. Then f is doubly periodic if and only if there exists $\bar{f} : \mathbb{T}^2 \rightarrow X$ such that $f = \bar{f} \circ \pi$.

In particular, a doubly periodic vector field on \mathbb{R}^2 can be regarded as a vector field on \mathbb{T}^2 .

Lemma 6.3.2

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 doubly periodic vector field, let ϕ_t be the flow generated by f , then

$$\phi_t(p + k) = \phi_t(p) + k, \quad \forall k \in \mathbb{Z}^2, p \in \mathbb{R}^2.$$

Therefore, $\pi \circ : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is a doubly periodic map. There exists $\bar{\phi}_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $\pi \circ \phi_t = \bar{\phi}_t \circ \pi$. Then $\bar{\phi}_0 = \text{Id}$, $\bar{\phi}_{t+s} = \bar{\phi}_t \circ \bar{\phi}_s$, hence $\{\bar{\phi}_t\}$ is a flow on \mathbb{T}^2 , which is generated by \bar{f} .

Example 6.3.3

Consider the constant vector field $f(x, y) = (1, \alpha)$ on \mathbb{R}^2 . It induces a flow on \mathbb{T}^2 .

- (1) If $\alpha = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q > 0$, $\gcd(p, q) = 1$. Then $\pi(x(q), y(q)) = \pi(x(0), y(0))$, hence every orbit is a periodic orbit with a common period q .
- (2) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $K = \{(\bar{x}, \bar{y}) \in \mathbb{T}^2 : \bar{x} = 0\}$. Then the orbit intersects with K at

$$\{(0, \overline{n\alpha + y_0 - x_0\alpha}) : n \in \mathbb{Z}\}.$$

Theorem 6.3.4

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then every orbit of the flow generated by $(1, \alpha)$ is dense on \mathbb{T}^2 .

The flow generated by $(1, \alpha)$ on \mathbb{T}^2 is called the **irrational flow** on \mathbb{T}^2 .

Definition 6.3.5. Let X , be a compact metric space, a flow on X is called **minimal** if every orbit is dense in X .

$K = \{(\bar{x}, \bar{y}) \in \mathbb{T}^2 : \bar{x} = 0\}$ is a transversal of $f = (1, \alpha)$, then the orbit recurs to K at times one. It induces a map $K \rightarrow K$, which is also called **Poincaré recurrent map**.

Lemma 6.3.6

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 , doubly periodic vector field without singularities. Then there exists an closed curve Γ on \mathbb{T}^2 such that Γ is a transversal of \bar{f} and Γ is not null-homotopic.

Proof. Let g be the vector orthogonal complement vector field of f . If Γ has some periodic orbit, take it. Otherwise, take some $q \in \omega(p, g)$, take a flow box near q . the the orbit of p will recurs to the flow box. We can closed the orbit by a C^1 curve transverse to f in the flow box.

For every transversal closed curve Γ of f , if Γ is null-homotopic, then Γ can be lifted to a closed curve $\tilde{\Gamma}$ in \mathbb{R}^2 . By index theory, $\text{Ind}_f(\tilde{\Gamma}) \neq 0$, which contradicts with f is without singularities. \square

Exercise 6.3.7. Let f be a C^1 , non-singularity vector field on \mathbb{T}^2 . $K \subset \mathbb{T}^2$ be a C^1 Jordan curve, transverse to f . If there exists $p_0 \in K$ and $t > 0$ such that $\phi_t(p_0) \in K$, then for every $p \in K$, there exists $t > 0$ such that $\phi_t(p) \in K$.

Under the condition of the exercise, we can define the Poincaré recurrent map $P : K \rightarrow K$ as $P(x) = \phi_{t(x)}(x)$. Then P is a orientation preserving diffeomorphism of K .

Exercise 6.3.8. Let (P, Q) be a C^1 , doubly periodic vector field on \mathbb{R}^2 , $P > 0$. Let $\phi(x; x_0, y_0)$ be the solution of the following initial problem

$$\begin{cases} \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \\ y(x_0) = y_0. \end{cases}$$

Show that:

1. $\phi(x; x_0, y_0 + 1) = \phi(x; x_0, y_0) + 1$;
2. $\phi(x + 1; x_0 + 1, y_0) = \phi(x; x_0, y_0)$;
3. $f(y_0) = \phi(1; 0, y_0)$.

Lemma 6.3.9

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function, assume $f(y + 1) = f(y) + 1$. Then there exists $\alpha \in \mathbb{R}$ and $C > 0$ such that

$$|f^n(y) - y - n\alpha| < C, \quad \forall y \in \mathbb{R}, n \in \mathbb{Z}. \quad (*)$$

In particular,

$$\alpha = \lim_{n \rightarrow \infty} \frac{f^n(y) - y}{n} = \alpha, \quad \forall y \in \mathbb{R}.$$

Definition 6.3.10. The constant α is called the **rotation number** of f , denoted by $\rho(f)$. We call f a **bounded mean motion** if f satisfies $(*)$.

Proof. Let $\phi(y) = f(y) - y$, then $\phi(y + 1) = \phi(y)$. For every $0 \leq y \leq z \leq 1$, we have

$$\phi(y) - \phi(z) = f(y) - f(z) - y + z \geq f(y) - f(z) \geq f(y) - f(z - 1) - 1 \geq -1,$$

$$\phi(y) - \phi(z) = f(y) - f(z) - y + z \leq f(y) - f(z) + 1 \leq 1,$$

hence $|\phi(y) - \phi(z)| \leq 1$. Moreover, $|\phi(y) - \phi(z)| \leq 1$ for every $y, z \in \mathbb{R}$. Given $n \in \mathbb{Z}_+$, f^n is also a strictly increasing function satisfying $f^n(y + 1) = f^n(y) + 1$, hence for every $y, z \in \mathbb{R}$, $|f^n(y) - y - f^n(z) + z| \leq 1$.

Now, we take $y = x$ and $z = f^m(x)$, we have $|f^n(x) - x - f^{n+m}(x) + f^m(x)| \leq 1$. Fix $x \in \mathbb{R}$, let $a_n = f^n(x) - x$. Then we have

$$|a_{m+n} - a_m - a_n| = |f^{m+n}(x) - f^m(x) - f^n(x) + x| \leq 1.$$

For every positive integer k, n , we have $|a_{kn} - ka_n| \leq k - 1$. Hence,

$$\frac{a_n}{n} - \frac{1}{n} \leq \frac{a_{kn}}{kn} - \frac{1}{kn} \leq \frac{a_{kn}}{kn} \leq \frac{a_{kn}}{kn} + \frac{1}{kn} \leq \frac{a_n}{n} + \frac{1}{n}.$$

Now we consider the intervals $I_n = [\frac{a_n}{n} - \frac{1}{n}, \frac{a_n}{n} + \frac{1}{n}]$. Then for every $n \geq 0$, we have $\frac{a_n!}{n!} \in \bigcap_{k=1}^n I_k$. Because $|I_n| \rightarrow 0$, hence $J = \bigcap_{n \geq 1} I_n$ is a single-point set $\{\alpha\}$. That is,

$$\lim_{n \rightarrow \infty} \frac{f^n(x) - x}{n} = \alpha.$$

Because for every $y, z \in \mathbb{R}$, $|f^n(y) - y - f^n(z) + z| \leq 1$, hence this limit is independent with the choice of x . \square

Exercise 6.3.11. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing but not strictly increasing, $f(y + 1) = f(y) + 1$, does the conclusion still hold?

Proposition 6.3.12

$\rho(f)$ is continuous with respect to f , i.e. for every $\varepsilon > 0$, $\exists \delta > 0$, such that for every g satisfying the same condition, $\|g - f\|_\infty < \delta$, we have $|\rho(g) - \rho(f)| < \varepsilon$.

Proof. By the previous lemma, we have $|f^n(x) - x - n\alpha| < 1$, take N such that $\frac{2}{N} < \varepsilon$. Take $\delta > 0$, such that for $\|g - f\|_\infty < \delta$, we have $|g^N(x) - x - (f^N(x) - x)| < 1$. \square

Theorem 6.3.13 (Poincaré)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function, assume $f(y + 1) = f(y) + 1$. Then $\rho(f) \in \mathbb{Q}$ if and only if $\bar{f} : \mathbb{T} \rightarrow \mathbb{T}$ has some periodic points.

Proof. We prove that if $\rho(f) = p/q$, then there exists $x \in \mathbb{R}$ such that $f^q(x) = x + p$. Let $\psi(y) = f^q(y) - y - p$, let $\alpha = \min \psi(y)$ and $\beta = \max \psi(y)$, we want to show $\alpha \leq 0$ and $\beta \geq 0$. Otherwise, if $\alpha > 0$, then

$$f^{nq}(y) - y = \sum_{k=0}^{n-1} (f^{(k+1)q}(y) - f^{kq}(y)) \geq n(p + \alpha).$$

Hence $\rho(f) \geq \frac{p+\alpha}{q}$, a contradiction. Then we get $\alpha \leq 0$ and $\beta \geq 0$, hence there exists x such that $f^q(x) = x + p$. \square

§6.4 Denjoy's theorem

Definition 6.4.1. Let (X, d) be a metric space, $f : X \rightarrow X$ continuous. For $x \in X$, $y \in X$ is called a **positive limit point** of x if $\exists n_i \rightarrow \infty$, $f^{n_i}x \rightarrow y$. The **positive limit set** $\omega(x)$ is the set of all positive limit points of x .

Lemma 6.4.2

Let (X, d) be a metric space, $f : X \rightarrow X$ continuous. Then

1. $\forall x \in X$, $\omega(x) = \bigcap_{n=0}^{\infty} \overline{\text{Orb}^+(f^n x)}$.
2. $\forall x \in X$, $\omega(x)$ is closed and $f(\omega(x)) = \omega(x)$.
3. If X is compact, then $\omega(x) \neq \emptyset$.

If $f : X \rightarrow X$ is a homeomorphism, then we can define negative limit points and the negative limit set $\alpha(x)$ similarly.

Definition 6.4.3. Let (X, d) be a metric space, $f : X \rightarrow X$ continuous. f is called **minimal** if $\forall x \in X$, $\overline{\text{Orb}^+(x)} = X$. Let $\Lambda \subset X$ be a closed subset such that $f(\Lambda) = \Lambda$ and $\forall x \in \Lambda$, $\overline{\text{Orb}^+(x)} = \Lambda$, then Λ is called a **minimal set**.

Theorem 6.4.4 (Poincaré)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function, assume $f(y + 1) = f(y) + 1$. If $\rho(f) \notin \mathbb{Q}$, then for every $\bar{x} \in \mathbb{T}$, $\omega(\bar{x})$ is a minimal set of \bar{f} . Moreover, $\forall \bar{y} \in \mathbb{T}$, $\omega(\bar{y}) = \omega(\bar{x})$.

Proof. It suffices to show $\omega(\bar{x}) = \omega(\bar{y})$ for every $\bar{x}, \bar{y} \in \mathbb{T}$. If $\omega(\bar{x}) \neq \omega(\bar{y})$, WLOG, assume that $\omega(\bar{x}) \neq \mathbb{T}$, we will show that $\omega(\bar{y}) \subseteq \omega(\bar{x})$. Otherwise, there must be $\bar{y} \notin \omega(\bar{x})$. Let $(\bar{a}, \bar{b}) \ni \bar{y}$ be a connected component of $\mathbb{T} \setminus \omega(\bar{x})$. Then $\bar{f}^n(\bar{a}, \bar{b})$ is a connected component of $\mathbb{T} \setminus \omega(\bar{x})$. Because $\rho(f) \notin \mathbb{Q}$, these intervals are disjoint. Hence $\text{diam} \bar{f}^n(\bar{a}, \bar{b}) \rightarrow 0$. Hence $\omega(\bar{y}) \subseteq \omega(\bar{a}) \subseteq \omega(\bar{x})$. \square

Remark 6.4.5 — Note that $\partial\omega(\bar{x})$ is closed and invariant, hence $\partial\omega(\bar{x}) = \emptyset$ or $\omega(\bar{x})$, which shows that $\omega(\bar{x})$ is either \mathbb{T} or a Cantor set.

Definition 6.4.6. \bar{f} is called **ergodic** if $\omega(\bar{x}) = \mathbb{T}$. Otherwise, \bar{f} is said to be **exceptional**.

Theorem 6.4.7

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. If \bar{y} is ergodic, then there exists a strictly increasing function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x+1) = h(x) + 1$, $h \circ f = R_\alpha \circ h$ where $\alpha = \rho(f)$ and $R_\alpha(x) = x + \alpha$.

Proof. Let $h(f^n 0 + k) = n\alpha + k$ for every $n, k \in \mathbb{Z}$, then h is well-defined and strictly increasing. Also, h can be uniquely extended to \mathbb{R} continuously. \square

Theorem 6.4.8

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. Then there exists an increasing function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x+1) = h(x) + 1$, $h \circ f = R_\alpha \circ h$ where $\alpha = \rho(f)$ and $R_\alpha(x) = x + \alpha$.

Remark 6.4.9 — In this case, h is called a **semi-conjugate**.

Definition 6.4.10. Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be continuous, if exists a C^k lifting $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$, then f is called C^k . Besides, f is called a C^k **diffeomorphism** if f is invertible and f, f^{-1} are both C^k .

Remark 6.4.11 — If $f : \mathbb{T} \rightarrow \mathbb{T}$ is C^1 , then $f'(\bar{x}) := \tilde{f}'(x)$ is well-defined. f is an orientation-preserving diffeomorphism iff $f' > 0$.

Definition 6.4.12. A function $g : \mathbb{T} \rightarrow \mathbb{R}$ is called of bounded variation if $\exists C > 0$ such that for every disjoint intervals $I_k = (x_k, y_k)$,

$$\sum_{k=1}^n |g(x_k) - g(y_k)| \leq C.$$

The smallest C satisfying the inequality is called the **total variation** of g , denoted by $\text{Var}(g)$.

Lemma 6.4.13

Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be an orientation-preserving diffeomorphism, then f' is of bounded variation iff $\log f'$ is of bounded variation.

Lemma 6.4.14 (distortion estimate)

Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be an orientation-preserving diffeomorphism, assume f' is of bounded variation. If an interval $I \subset \mathbb{T}$ satisfies $I, fI, \dots, f^{n-1}I$ are disjoint. Then for every $x, y \in I$,

$$e^{-V} \leq \frac{(f^n)'(x)}{(f^n)'(y)} \leq e^V, \quad V = \text{Var}(\log f'). \quad (*)$$

Remark 6.4.15 — Note that the bound in $(*)$ is independent with n , this type of estimates are called **bounded distortion**.

Definition 6.4.16. Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $p, q \in \mathbb{Z}$, $q > 0$. The rational number p/q is called a **best rational approximation** of α if

$$|q\alpha - p| < |q'\alpha - p'|, \quad \forall p', q' \in \mathbb{Z}, 0 < q' \leq q, \frac{p'}{q'} \neq \frac{p}{q}.$$

Lemma 6.4.17

Every irrational number has infinite many best rational approximations.

Lemma 6.4.18

Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let p_n/q_n be the n -th best approximation of α . Then

$$(p_n/q_n - \alpha)(p_{n+1}/q_{n+1} - \alpha) < 0.$$

Lemma 6.4.19

Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $R_\alpha : \mathbb{T} \rightarrow \mathbb{T}$ be the irrational rotation. Let p_n/q_n be the n -th best approximation of α , let I be the minor arc with endpoints x and $R_\alpha^{q_n}(x)$, then $I, R_\alpha I, \dots, R_\alpha^{q_{n+1}-1}I$ are disjoint.

Corollary 6.4.20

Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be an orientation-preserving diffeomorphism, $\rho(f)$ is irrational. Then there exists infinite many positive integer n such that for every $y \in \mathbb{T}$, the intervals $I, fI, \dots, f^n I$ are disjoint, where I is an open interval with endpoints y and $f^{-n}y$.

Theorem 6.4.21 (Denjoy)

Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be an orientation-preserving diffeomorphism, assume f' is of bounded variation. If $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$, then f is ergodic.

Proof. Otherwise, let Λ be the unique minimal set, let I be a connected component of $\mathbb{T} \setminus \Lambda$. There are infinite many n , such that for every $x \in \mathbb{T}$, the intervals $J, fJ, \dots, f^n J$ are disjoint,

where J is an open interval with endpoints x and $f^{-n}x$. By the distortion estimate, we have

$$(f^n)'(x)(f^{-n})'(x) = \frac{(f^n)'(x)}{(f^n)'(f^{-n}x)} \in [e^{-V}, e^V], \quad V = \text{Var}(\log f').$$

We apply this estimate on I , we have

$$|f^n I| + |f^{-n} I| \geq \int_I (f^n)'(x) + (f^{-n})'(x) dx \geq 2e^{-V/2} |I|.$$

Note that $\{f^n I\}_{n \in \mathbb{Z}}$ are disjoint, this estimate contradicts with $\sum_{n \in \mathbb{Z}} |f^n I| < \infty$. \square

Exercise 6.4.22. Consider the one-parameter diffeomorphisms $f_c : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x + \frac{1}{8} \sin 2\pi x + c$. Define $\rho : \mathbb{R} \rightarrow \mathbb{R}$ as $c \mapsto \rho(f_c)$. Show that:

1. ρ is increasing.
2. ρ is continuous and $\rho(\mathbb{R}) = \mathbb{R}$.
3. ρ is not strictly increasing.

§6.5 Dynamics on \mathbb{T}^2

Let (P, Q) be a doubly periodic vector field on \mathbb{R}^2 , where $P > 0$. Let $\phi(x, y_0)$ be the solution of the initial problem

$$\begin{cases} \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \\ y(0) = y_0. \end{cases}$$

Let $f(y_0) = \phi(1, y_0)$, and $\bar{f} : \mathbb{T} \rightarrow \mathbb{T}$ be the corresponding map.

Theorem 6.5.1

If \bar{f} is ergodic, then there exists a double periodic function $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Such that for every solution $y = g(x)$, there exists $c \in \mathbb{R}$,

$$g(x) = \alpha x + c + w(x, \alpha x + c),$$

where $\alpha = \rho(f)$.