Functional Analysis II
(Fall 2021, Zhifei Zhang)

Yuxiang Jiao

Contents

1 Distributions and Sobolev Spaces

Example 1.0.1

Consider the Heaviside function

$$Y(x) = \begin{cases} 1, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

How to define $Y'(x)$ properly ?

Example 1.0.2

Consider the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\mathrm{d}\xi = \delta_x.$$

Strictly speaking, this is not right. But it makes sense.

§1.1 Distributions

Basic space

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For $u \in C(\overline{\Omega})$, let $\operatorname{supp} u = \overline{\{x \in \Omega : u(x) \neq 0\}}$.

Notation 1.1.1. $C_c^k(\Omega), C_c^\infty(\Omega), \partial^\alpha \varphi$.

We consider an important smooth function

$$j(x) = \begin{cases} C_0 e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases}$$

where $C_0 > 0$ such that $\int_{\mathbb{R}^n} j(x)\mathrm{d}x = 1$. For every $\delta > 0$, let

$$j_\delta(x) = \delta^{-n} j\left(\frac{x}{\delta}\right),$$

then $\int j_\delta = 1$.

For every $u \in L^1(\Omega)$, let

$$u_\delta = u * j_\delta = \int_\Omega u(y) j_\delta(x-y)\mathrm{d}y,$$

then $u_\delta \in C^\infty(\Omega)$. Moreover, if $\operatorname{supp} u \subset K \subset \Omega$ for some compact K, then $u_\delta \in C_c^\infty(\Omega)$ when $\delta \ll 1$.

Definition 1.1.2. For $\{\varphi_j\} \subset C_c^\infty(\Omega), \varphi \in C_c^\infty(\Omega)$, we define the convergence $\varphi_j \to \varphi$ if

(i) $\exists K$ compact, $\operatorname{supp} \varphi_j \subset K$ for every j.

(ii) $\forall \alpha, \max_K |\partial^\alpha(\varphi_j - \varphi)| \to 0, j \to \infty$.

The set $C_c^\infty(\Omega)$ equipped with this topology is denoted by $\mathcal{D}(\Omega)$.

The definition of distributions

Definition 1.1.3. The family of continuous linear functionals on $\mathcal{D}(\Omega)$ is called the **distribution space**, denoted by $\mathcal{D}'(\Omega)$.

Example 1.1.4

1. For every $f \in L^1_{\text{loc}}(\Omega)$, there is an action

$$\langle f, \varphi \rangle := \int_\Omega f\varphi \mathrm{d}x, \quad \forall \varphi \in D(\Omega).$$

   This induces a natural embedding $L^1_{\text{loc}}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$.

2. Dirac measure $\delta_{x_0} : \langle \delta_{x_0}, \varphi \rangle := \varphi(x_0)$.

Theorem 1.1.5

$f \in \mathcal{D}'(\Omega)$ iff: for every $K \subset \Omega$ compact, there exists $C > 0$ and $m \in \mathbb{N}$ such that

$$|\langle f, \varphi \rangle| \leqslant C \sum_{|\alpha| \leqslant m} \max_{x \in K} |\partial^\alpha \varphi(x)|$$

holds for every $\varphi \in \mathcal{D}(\Omega), \operatorname{supp} \varphi \subset K$.

Proof. It suffices to verify the "only if" part. Otherwise, there exists $K \subset \Omega$ compact, such that for every $j \in \mathbb{Z}_+$, there exists φ_j,

$$\langle f, \varphi_j \rangle = 1 \quad \text{and} \quad \max_{x \in K} |\partial^\alpha \varphi_j| \leqslant \frac{1}{j}, \quad \forall |\alpha| \leqslant j.$$

Then $\varphi_j \to 0$ but $\langle f, \varphi_j \rangle \equiv 1$, a contradiction. $\square$

The convergence on $\mathcal{D}'(\Omega)$

Definition 1.1.6. For $\{f_j\} \subset \mathcal{D}'(\Omega)$, $f \in \mathcal{D}'(\Omega)$, we define the the convergence $f_j \to f$ in $\mathcal{D}'$ if

$$\langle f_j, \varphi \rangle \to \langle f, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example 1.1.7

1. Consider $f_j(x) = \frac{1}{\pi} \frac{\sin jx}{x} \in L^1_{\text{loc}}(\mathbb{R})$, then $f_j \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
2. Consider $j_\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$, then $j_\delta \to \delta_0(\delta \to 0^+)$ in $\mathcal{D}'(\mathbb{R})$.

§1.2 B_0 spaces

Definition 1.2.1. Let X be a linear space, it is called a **countably-normed space (B_0^* space)** if there exists a countable number of semi-norms $\{\|\cdot\|_m\}_{m=1}^\infty$ on X satisfying

$$\forall m, \ \|x\|_m = 0 \ \text{iff} \ x = 0.$$

Remark 1.2.2 — WLOG, we usually assume that $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \cdots$.

Remark 1.2.3 — A B_0^* space is a Frechet* space, for we can define a quasi-norm

$$\|x\| = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|x\|_m}{1+\|x\|_m}.$$

Example 1.2.4

There are several examples of B_0^* spaces:

1. For $K \subset \Omega$ compact, let $D_K(\Omega) := \{\varphi \in C^\infty(\Omega) : \operatorname{supp}\varphi \subset K\}$, let

$$\|\varphi\|_m := \sum_{|\alpha| \leqslant m} \max_{x \in K} |\partial^\alpha \varphi(x)|.$$

2. Let $K_1 \subset K_2 \subset \cdots \subset \Omega$ be a sequence of compact subsets such that $\Omega = \bigcup_{m=1}^{\infty} K_m$. Let $\mathcal{E}(\Omega)$ be the set $C^\infty(\Omega)$ with countable norms $\{\|\cdot\|_m\}$ where

$$\|\varphi\|_m := \sum_{|\alpha| \leqslant m} \max_{x \in K_m} |\partial^\alpha \varphi(x)|.$$

   The topology on $\mathcal{E}(\Omega)$ is independent with the choice of $\{K_m\}$.

3. The **Schwarz space**

$$\mathcal{S}(\mathbb{R}^n) := \left\{\varphi \in C^\infty(\mathbb{R}^n) : \sup |(1+|x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty,\ \forall k, \alpha\right\}.$$

   We can define the semi-norm as

$$\|\varphi\|_m := \sum_{|\alpha| \leqslant m} \sup_{x \in \mathbb{R}^n} \left|(1+|x|)^{\frac{m}{2}} \partial^\alpha \varphi(x)\right|.$$

A complete B_0^* space is called a **B_0 space**. $\mathcal{S}(\mathbb{R}^n)$ is a B_0 space.

Lemma 1.2.5

Let X be a B_0 space, then f is a continuous linear functional on X iff $\exists m, C_m > 0$ such that $|\langle f, \varphi\rangle| \leqslant C_m \|\varphi\|_m$ for every $\varphi \in X$.

Theorem 1.2.6

There are sufficiently many continuous linear functionals on every B_0 space, i.e. $\forall x_0 \neq 0$, there exists $f \in X'$, $\langle f, x\rangle = 1$.

Proof. $\forall x_0 \neq 0$, there exists m such that $\|x_0\|_m \neq 0$. Let $f_0(\lambda x_0) = \lambda$ and apply Hahn-Banach to get f such that $f|_{\{\lambda x_0\}} = f_0$ and $|f| \leqslant C_m \|\cdot\|_m$. □

Now, we focus on the space $\mathcal{S}'(\mathbb{R}^n)$, which is called the family of **tempered distributions**. Because $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{D}(\mathbb{R}^n)$, hence $\mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.2.7

$f \in \mathcal{S}'(\mathbb{R}^n)$ iff: $\exists m \in \mathbb{Z}_+$ and $\{u_\alpha : |\alpha| \leqslant m\} \subseteq L^2(\mathbb{R}^n)$ such that

$$\langle f, \varphi \rangle = \sum_{|\alpha| \leqslant m} \int_{\mathbb{R}^n} u_\alpha(x) \partial^\alpha \varphi(x) (1 + |x|^2)^{\frac{m}{2}} \mathrm{d}x, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. We can define countable norms $\{\|\cdot\|'_m\}$ as

$$\|\varphi\|'_m := \left(\sum_{|\alpha| \leqslant m} \int_{\mathbb{R}^n} (1 + |x|^2)^m |\partial^\alpha \varphi(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Then $\{\|\cdot\|'_m\}$ is equivalent to $\{\|\cdot\|_m\}$ because $\|\cdot\|'_m$ can be bounded by $\|\cdot\|_{m+n}$ and vice versa. By the previous lemma, f can be bounded by some $\|\cdot\|'_m$. We consider a Hilbert space

$$\mathcal{H}_m = \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|'_m}$$

with a inner product $(\varphi, \psi)_m$. Then by Riesz representation theorem, there exists $u \in \mathcal{H}_m$ such that $\langle f, \varphi \rangle = (u, \varphi)_m$. Let $u_\alpha(x) = \partial^\alpha \varphi(x)(1 + |x|^2)^{m/2}$, then $u_\alpha \in L^2(\mathbb{R}^n)$. □

§1.3 Operations on distributions

Definition 1.3.1. Let $A : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ be a linear operator, it is called continuous if

$$\varphi_j \to \varphi \text{ in } \mathcal{D}(\Omega) \implies A\varphi_j \to A\varphi \text{ in } \mathcal{D}(\Omega).$$

Example 1.3.2

The followings are continuous linear operators on $\mathcal{D}(\Omega)$.

1. $\partial^\alpha : \varphi \mapsto \partial^\alpha \varphi$.
2. $\tau_{x_0} : \varphi(x) \mapsto \varphi(x - x_0)$.
3. $\sigma : \varphi(x) \mapsto \varphi(-x)$.
4. $A_\psi : \varphi \mapsto \psi\varphi$, where $\psi \in C^\infty(\Omega)$.

For every continuous linear operator $A : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$, it induces a continuous linear operator $A^* : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle A^* f, \varphi \rangle = \langle f, A\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Derivations on distributions

Definition 1.3.3. For every $f \in \mathcal{D}'(\Omega)$, we can define the derivative $\partial^\alpha f$ of f as

$$\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

In particular, if $f \in C^1(\Omega)$, then ∂f is the usual derivative of f.

Example 1.3.4

Consider the Heaviside function

$$Y(x) = \begin{cases} 1, & x \geqslant 0; \\ 0, & x < 0. \end{cases}$$

Then $\langle Y', \varphi \rangle = -\langle Y, \varphi' \rangle = -\int_0^\infty \varphi(x)\mathrm{d}x = \varphi(0)$, hence $Y' = \delta_0$.

Another application of the derivations on distributions is in PDE. Recall the basic solution of Laplace's equation, we can show that

- $\Delta \ln |x| = 2\pi\delta_0$ when $n = 2$.
- $\Delta |x|^{2-n} = (2-n)\Omega_n \delta_0$ when $n \geqslant 3$, where Ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Multiplication, translation and reflection

- Given $\psi \in C^\infty(\Omega)$, for every $f \in \mathcal{D}'(\Omega)$, we can define the multiplication $\psi f \in \mathcal{D}'(\Omega)$ as

$$\langle \psi f, \varphi \rangle := \langle f, \psi\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

- Given $x_0 \in \mathbb{R}^n$, for every $f \in \mathcal{D}'(\Omega)$, we can the translation $\tau_{x_0} f \in \mathcal{D}'(\Omega)$ as

$$\langle \tau_{x_0} f, \varphi \rangle := \langle f, \tau_{-x_0}\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

- For every $f \in \mathcal{D}'(\Omega)$, we can the reflection $\sigma f \in \mathcal{D}'(\Omega)$ as

$$\langle \sigma f, \varphi \rangle := \langle f, \sigma\varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example 1.3.5

Let $\delta = \delta_0$ be the Dirac measure in $\mathbb{R}$. For every $m, n \in \mathbb{N}$, we calculate $x^n \partial^m \delta$. For every $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle x^n \partial^m \delta, \varphi \rangle = \langle \partial^m \delta, x^n \varphi \rangle = (-1)^m \langle \delta, \partial^m (x^n \varphi) \rangle.$$

If $m < n$, then $x^n \partial^m \delta = 0$. If $m \geqslant n$, then

$$\langle x^n \partial^m \delta, \varphi \rangle = \frac{(-1)^m m!}{(n-m)!} \partial^{m-n} \varphi(0).$$

§1.4 Fourier transform on $\mathcal{S}'$

For every $f \in L^1(\mathbb{R}^n)$, we can define the Fourier transform f as

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) \mathrm{d}x.$$

Question 1.4.1. How to generalize the definition of Fourier transform to $L^p(\mathbb{R}^n)$?

Proposition 1.4.2
$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous linear operator.

We can also define the inverse Fourier transform as

$$\overline{\mathcal{F}}\varphi(x) = \varphi^\vee(x) := \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}\varphi(\xi)\mathrm{d}\xi.$$

Then $\overline{\mathcal{F}} \in \mathscr{L}(\mathcal{S})$. For every $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\langle \mathcal{F}\varphi, \psi\rangle = \langle \varphi, \mathcal{F}\psi\rangle\,, \quad \langle \overline{\mathcal{F}}\varphi, \psi\rangle = \langle \varphi, \overline{\mathcal{F}}\psi\rangle\,.$$

Definition 1.4.3. For every $f \in \mathcal{S}'(\mathbb{R}^n)$, we can the **Fourier transform** $\mathcal{F}f \in \mathcal{S}'(\mathbb{R}^n)$ as

$$\langle \mathcal{F}f, \varphi\rangle := \langle f, \mathcal{F}\varphi\rangle\,, \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^n).$$

Similarly, we can define the inverse Fourier transform of tempered distributions. Then $\mathcal{F}, \overline{\mathcal{F}} \in \mathscr{L}(\mathcal{S}')$. In particular, for every $f \in L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform $\mathcal{F}f \in \mathcal{S}'(\mathbb{R}^n)$. We also have the identity

$$\mathcal{F}\overline{\mathcal{F}} = \overline{\mathcal{F}}\mathcal{F} = \mathrm{Id}.$$

Theorem 1.4.4 (Plancherel's Identity)
If $f \in L^2$, then $\mathcal{F}f \in L^2$ and $\|f\|_{L^2} = \|\mathcal{F}f\|_{L^2}$.

Corollary 1.4.5 For every $f, g \in L^2$, we have $(f, g)_{L^2} = (\widehat{f}, \widehat{g})_{L^2}$.

§1.5 Sobolev spaces

Definition 1.5.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. We define the **Sobolev space** as the space

$$W^{m,p} := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p, \forall|\alpha| \leqslant m\}$$

with the norm

$$\|u\|_{m,p} := \left(\sum_{|\alpha|\leqslant m} \|\partial^\alpha u\|_p^p\right)^{\frac{1}{p}}.$$

Theorem 1.5.2 $W^{m,p}(\Omega)$ is a Banach space.

Theorem 1.5.3 $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.

Definition 1.5.4. Let $S := \left\{u \in C^\infty(\Omega) : \|u\|_{m,p} < \infty\right\}$, let

$$H^{m,p}(\Omega) := \overline{S}^{\|\cdot\|_{m,p}}, \quad H_0^{m,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{m,p}}.$$

Remark 1.5.5 — We will omit p if $p = 2$.

By theorem 1.5.2, for every Ω, we have

$$H_0^{m,p}(\Omega) \hookrightarrow H^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega).$$

When $\Omega = \mathbb{R}^n$, by theorem 1.5.3, we have $H_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$. For general Ω, does it hold?

Theorem 1.5.6 (Meyers-Serrin)
For $1 \leqslant p < \infty$, we have $H^{m,p}(\Omega) = W^{m,p}(\Omega)$.

Definition 1.5.7. Open set $\Omega \subset \mathbb{R}^n$ is called **extensible** if for every $m \in \mathbb{N}, q \leqslant p < \infty$, there exists a continuous linear operator $T : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ such that $Tu|_\Omega = u$ for every $u \in W^{m,p}(\Omega)$.

Theorem 1.5.8 (Sobolev Embedding Theorem)
If $\Omega \subset \mathbb{R}^n$ is extensible, then for every $m > n/2$, $W^{m,2} \hookrightarrow C(\overline{\Omega})$.

Proof. It suffices to show for $\Omega = \mathbb{R}^n$. We know that

$$\|u\|'_m = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^m |\widehat{u}(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}$$

is a norm equivalent to the norm $\|\cdot\|_m$ on $W^{m,2}$. Then for $m > n/2$,

$$\|\widehat{u}\|_{L^1} \leqslant \|u\|'_m \left(\int_{\mathbb{R}^n} \frac{\mathrm{d}\xi}{(1 + |\xi|^2)^m}\right)^{\frac{1}{2}} \leqslant c_{n,m} \|u\|'_m .$$

Because $\|u\|_{C(\mathbb{R}^n)} \leqslant \|\widehat{u}\|_{L^1}$, hence $W^{m,2} \hookrightarrow C(\overline{\Omega})$ for $m > n/2$. □

General versions of Sobolev embedding theorem are stated as the following two facts.

Fact 1.5.9. $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$, where $m \leqslant \frac{n}{p}$.

Fact 1.5.10. $W^{m,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ when $m > \frac{n}{p}$.

Theorem 1.5.11 (Rellich)
Let Ω be a bounded extensible region, then the unit ball of $W^{1,2}(\Omega)$ is sequentially compact in $L^2(\Omega)$.

Proof. Take a sequence $\{u_m\} \subset W^{1,2}(\Omega)$ such that $\|u_m\|_1 \leqslant 1$. Let $T : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ be an extension operator, let $U_m = Tu_m$. We can assume that $\operatorname{supp} U_m \subset K$ for every m, where $K \subset \mathbb{R}^n$ compact. Then

$$\|u_m - u_l\|^2_{L^2(\Omega)} \leqslant C \|U_m - U_l\|^2_{L^2(\mathbb{R}^n)} \approx \int |\widehat{U}_m - \widehat{U}_l|^2 = \int_{|\xi| \geqslant r} + \int_{|\xi| < r}.$$

For the part of high frequency, we have the estimate

$$\int_{|\xi| \geqslant r} \leqslant Cr^{-2} \int |\xi|^2 (|\widehat{U}_m|^2 + |\widehat{U}_l|^2) \mathrm{d}\xi \leqslant Cr^{-2}.$$

Because $\{U_m\}$ is bounded in $L^2(\mathbb{R}^n)$, hence there exits a subsequence $\{U'_m\}$ such that

$$U_{m'} \xrightarrow{w} g \in L^2(\mathbb{R}^n).$$

Note that $\widehat{U}_m(\xi) = \langle U_m, e^{2\pi i \xi \cdot x} \alpha_K \rangle$ for some $\alpha_K \in C_c^\infty(\mathbb{R}^n)$, $\alpha_K|_K = 1$. Then $\widehat{U}_{m'} \to \widehat{g}$. Hence by dominated convergence theorem,

$$\int_{|\xi| < r} |\widehat{U}_{m'} - \widehat{U}_{l'}|^2 \to 0$$

for a given $r > 0$. Hence the sequence $\{U_{m'}\}$ is Cauchy in $L^2(\mathbb{R}^n)$. $\square$

Note that

$$\mathcal{D}(\Omega) \hookrightarrow H_0^m(\Omega) \hookrightarrow L^2(\Omega),$$

take a duality,

$$L^2(\Omega) \hookrightarrow H_0^m(\Omega)^* \hookrightarrow \mathcal{D}'(\Omega).$$

We usually denote $H_0^m(\Omega)^*$ by $H^{-m}(\Omega)$, for we have the following fact.

Fact 1.5.12. $f \in H^{-m}(\Omega)$ iff $\exists g_\alpha \in L^2(\Omega)$ for every $|\alpha| \leqslant m$ such that

$$\langle f, \varphi \rangle = \sum_{|\alpha| \leqslant m} \int g_\alpha(x) \partial^\alpha \varphi(x) \mathrm{d}x.$$

Hence we can write as

$$f = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \partial^\alpha g_\alpha.$$

2 Unbounded Operators

§2.1 Closed operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, then $(X \times Y, \|(x, y)\| := \|x\|_X + \|y\|_Y)$ is a Banach space.

Definition 2.1.1. Let $T : D(T) \subseteq X \to R(T) \subseteq Y$ be a linear operator. The **graph** of T is

$$\Gamma(T) := \{(x, Tx) : x \in D(T)\} \subseteq X \times Y.$$

Definition 2.1.2. A linear operator $T : D(T) \to R(T)$ is called **closed** if $\Gamma(T)$ is closed in $X \times Y$.

Definition 2.1.3. Let T_1, T_2 be two linear operators. T_2 is called an **extension** of T_1 if $\Gamma(T_1) \subset \Gamma(T_2)$, denoted by $T_1 \subset T_2$.

Definition 2.1.4. Let T be a linear operator, T is called **closable** if there exists a linear operator S such that $\overline{\Gamma(T)} = \Gamma(S)$. We denote $S = \overline{T}$.

Fact 2.1.5. A linear operator T is closable iff, for every $\{x_n\} \subset D(T)$ such that $x_n \to 0$ and $Tx_n \to y$, then $y = 0$.

We state several properties of closed operators here.

1. If T is a one-one closed operator, then T^{-1} is closed.
2. If T is closed, then $N(T) = \ker T$ is closed.
3. If T is closable, S is closed and $T \subset S$, then $\overline{T} \subset S$.
4. If T is closed and bounded, then $D(T)$ is closed.
5. (Closed Graph Theorem) If T is closed and $D(T) = X$, then T is bounded.

So when we study an unbounded closed operator, $D(T)$ must be a proper subset of X.

Definition 2.1.6. A linear operator T is called **densely defined** if $\overline{D(T)} = X$.

> **Remark 2.1.7** — WLOG, we can always assume that a closed operator is densely defined, because $X_0 = \overline{D(T)}$ is also a Banach space.

Example 2.1.8

Let $X = Y = L^2(\mathbb{R}^n)$.

1. $T_0 = -\Delta$, $D(T_0) = C_c^\infty(\mathbb{R}^n)$, then T_0 is not closed.
2. $T_1 = -\Delta$, $D(T_1) = H^2(\mathbb{R}^n)$, then T_1 is closed. Because if $u_n \to u, \Delta u_n \to v$ in $L^2(\mathbb{R}^n)$, then $v = \Delta u$. Moreover, we can show that $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for every $|\alpha| \leqslant 2$, hence T_1 is closed.
3. T_0 is closable. In fact, $\overline{T}_0 = T_1$.

Let X^*, Y^* be the dual spaces of X, Y, respectively.

Definition 2.1.9. Let T be a densely defined operator, let

$$D(T^*) = \{y^* \in Y^* : \exists x^* \in X^*, \text{ such that } \langle y^*, Tx\rangle = \langle x^*, x\rangle, \forall x \in D(T)\}.$$

Let $T^* : D(T^*) \to X^*$, $y^* \mapsto x^*$. Then T^* is called the **adjoint operator** of T.

Remark 2.1.10 — The density of $D(T)$ guarantees the uniqueness of x^*.

Let $X = Y = \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Then

$$D(T^*) = \{y \in \mathcal{H} : \exists M_y > 0, |(y, Tx)| \leqslant M_y \|x\|, \forall x \in \mathcal{H}\}.$$

Lemma 2.1.11
Let $V : X \times Y \to Y \times X$, $(x, y) \mapsto (-y, x)$. Then $\Gamma(T^*) = {}^{\perp}V(\Gamma(T))$.

Proof. $(y^*, x^*) \in \Gamma(T^*) \iff \langle y^*, Tx\rangle = \langle x^*, x\rangle \iff \langle (y^*, x^*), (-Tx, x)\rangle = 0.$ □

Theorem 2.1.12 T^* is closed. Besides, if $T_1 \subset T_2$, then $T_2^* \subset T_1^*$.

Theorem 2.1.13
Let $\mathcal{H}$ be a Hilbert space and T is a densely defined operator on $\mathcal{H}$. Then T is closable iff T^* is densely defined. Moreover, if T is closable, then $\overline{T} = (T^*)^*$.

Proof. "If": if T^* is densely defined, then T^{**} is well-defined. Moreover,

$$\Gamma(T^{**}) = {}^{\perp}V^{\perp}V\Gamma(T) = \overline{\Gamma(T)},$$

hence T is closable and $\overline{T} = T^{**}$.
"Only if": if T is closable, assume $\overline{D(T^*)} \neq \mathcal{H}$. Then there exists $y_0 \neq 0 \in D(T^*)^{\perp}$. Then $(0, y_0) \in {}^{\perp}V(\Gamma(T^*))$. Hence ${}^{\perp}V\Gamma(T^*)$ cannot be a graph of a linear operator. But it corresponds to $\overline{\Gamma(T)} = \Gamma(\overline{T})$, a contradiction. □

Example 2.1.14
Recall T_0, T_1 in the previous example. Then $T_0^* = T_1$ and $T_1^* = T_1$. Hence $\overline{T}_0 = T_1$.

Definition 2.1.15. Let $\mathcal{H}$ be a Hilbert space, T is a densely defined operator on $\mathcal{H}$. Then

- We call T is **symmetric** if $T \subset T^*$, i.e. $(Tx, y) = (x, Ty)$ for every $x, y \in D(T)$.
- We call T is **self-adjoint** if $T = T^*$. In particular, T is closed if T is self-adjoint.
- We call T is **essentially self-adjoint** if T is closable and $\overline{T}$ is self-adjoint.

Remark 2.1.16 — If T is self-adjoint and S is symmetric such that $T \subset S$. Then

$$T \subset S \subset S^* \subset T^* = T,$$

hence $T = S$. Every self-adjoint operator is the maximum symmetric extension of itself. In particular, an essentially self-adjoint operator T has a unique self-adjoint extension $\overline{T}$.

Example 2.1.17
T_0 is symmetric and T_1 is self-adjoint. By $\overline{T}_0 = T_1$, we know that T_0 is essentially self-adjoint.

Theorem 2.1.18 Assume T is an invertible self-adjoint operator, then T^{-1} is self-adjoint.

Proof. Let's first prove that $R(T)$ is dense. Take $u \in R(T)^{\perp}$, then

$$(Tu, v) = (u, Tv) = 0, \quad \forall v \in D(T).$$

Because $\overline{D(T)} = \mathcal{H}$, hence $Tu = 0$, and it follows $u = 0$. Then T^{-1} is densely-defined. By T is self-adjoint, we get

$$\Gamma(T) = \Gamma(T^*) = {}^{\perp}V\Gamma(T) \implies \Gamma((T^{-1})^*) = {}^{\perp}V\Gamma(T^{-1}) = \Gamma(T^{-1}).$$

Hence T^{-1} is self adjoint. $\square$

Example 2.1.19
Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{H} = L^2(\Omega)$. Let $P(x, D)$ be a C^∞-coefficient differential operator. Let $D(P^\Omega) = C_c^\infty(\Omega)$, $P^\Omega : u \mapsto Pu$. Then P^Ω is not closed but closable. Define

$$D(P_{\min}^\Omega) = \left\{u \in L^2(\Omega) : \exists u_n \in C_c^\infty(\Omega), u_n \to u \text{ and } \{Pu_n\} \text{ convergent}\right\},$$

let $P_{\min}^\Omega u := \lim_{n\to\infty} Pu_n$. Then $\overline{\Gamma(P^\Omega)} = \Gamma(P_{\min}^\Omega)$. But we can show that $\overline{P^\Omega} \subset (P^\Omega)^*$ and it is in general not equal. Hence P^Ω is symmetric, closable but not self-adjoint or essentially self-adjoint in some cases.

Aim 2.1.20. To construct a self-adjoint extension.

§2.2 Representation theorems

Theorem 2.2.1 (Riesz Representation Theorem)
Let $\mathcal{H}$ be a Hilbert space and $f \in \mathcal{H}^*$. Then there exists a unique element $w \in \mathcal{H}$ such that

$$\langle f, u\rangle = (u, w)_{\mathcal{H}}, \quad \forall u \in \mathcal{H}.$$

Besides, $\|u\|_{\mathcal{H}} = \|f\|_{\mathscr{L}(\mathcal{H},\mathbb{C})}$.

Remark 2.2.2 — If f is a skewlinear functional on $\mathcal{H}$, then $\langle f, \cdot \rangle = (w, \cdot)_{\mathcal{H}}$ for a $w \in \mathcal{H}$.

Let V be a Hilbert space, let $a : V \times V \to \mathbb{C}$ be a sesquilinear form satisfying

(i) Continuity: $\exists C > 0$ such that $|a(u, v)| \leqslant C \|u\|_V \|v\|_V$.

(ii) Positivity: $\exists \alpha > 0$ such that $|a(u, u)| \geqslant \alpha \|u\|_V^2$.

By Riesz representation theorem, there exists $A \in \mathscr{L}(V)$ such that $a(u, v) = (Au, v)_V$.

Theorem 2.2.3 (Lax-Milgram) $A : V \to V$ is an isomorphism.

Now, we consider two Hilbert spaces $V, \mathcal{H}$ such that $V \hookrightarrow \mathcal{H}$ and V is dense in $\mathcal{H}$. Then $\| \cdot \|_{\mathcal{H}} \leqslant C \| \cdot \|_V$. Let a be a sesquilinear form on V. Define

$$D(S) := \{u \in V : |a(u, v)| \leqslant C_u \|v\|_{\mathcal{H}}, \forall v \in V\}.$$

For every $u \in D(S)$, we can find $Su \in \mathcal{H}$ such that $a(u, \cdot) = (Su, \cdot)_{\mathcal{H}}$.

Theorem 2.2.4

$S : D(S) \to \mathcal{H}$ is a bijection, $S^{-1} \in \mathscr{L}(\mathcal{H})$ and $D(S)$ is dense in $\mathcal{H}$.

Proof. 1. S is injective:

$$\alpha \|u\|_{\mathcal{H}}^2 \leqslant \alpha C \|u\|_V^2 \leqslant C|a(u, u)| \leqslant C \|Su\|_{\mathcal{H}} \|u\|_{\mathcal{H}} \implies \alpha \|u\|_{\mathcal{H}} \leqslant C \|Su\|_{\mathcal{H}}.$$

2. S is surjective and $S^{-1} \in \mathscr{L}(\mathcal{H})$: take $h \in \mathcal{H}$, $(h, \cdot)_{\mathcal{H}}$ is a bounded skewlinear functional on $\mathcal{H}$ hence on V. Then there exists $u \in V$ such that

$$(h, \cdot)_{\mathcal{H}} = (u, \cdot)_V = a(A^{-1}u, \cdot).$$

Moreover, $S^{-1} : h \mapsto A^{-1}u$ is bounded.

3. $\overline{D(S)}^{\| \cdot \|_{\mathcal{H}}} = \mathcal{H}$: assume $h \in D(S)^{\perp}$, let $h = Sv$, then

$$0 = |(Sv, v)_{\mathcal{H}}| = |a(v, v)| \geqslant \alpha \|v\|_V^2.$$

Hence $h = Sv = 0$.

□

Theorem 2.2.5

If we further assume that a is symmetric, i.e. $a(u, v) = \overline{a(v, u)}$, then $S = S^*$ and $D(S)$ is dense in V.

Proof. S is symmetric because a is symmetric. It suffices to show $D(S^*) \subset D(S)$. For every $v \in D(S^*)$, there is $v_0 \in D(S)$ such that $Sv_0 = S^*v$. Hence

$$(Su, v_0)_{\mathcal{H}} = (u, Sv_0)_{\mathcal{H}} = (u, S^*v)_{\mathcal{H}} = (Su, v)_{\mathcal{H}}, \quad \forall u \in D(S),$$

then $v = v_0$. The conclusion follows. □

Annotation 2.2.6 Why we consider $V \hookrightarrow \mathcal{H}$? My comprehension is that: if we consider a densely defined closed operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, we can first consider the bounded operator

$$A : (D(A), [\![\cdot]\!]) \to (\mathcal{H}, \|\cdot\|),$$

where $[\![u]\!] := \|u\| + \|Au\|$ is the graph norm. Hence the embedding

$$\iota : (D(A), [\![\cdot]\!]) \hookrightarrow (\mathcal{H}, \|\cdot\|), \ \ u \mapsto u$$

is a natural study object. Note that $V = D(A)$ is complete with respect to a stronger norm $[\![\cdot]\!]$ and is dense in $\mathcal{H}$ with respect to $\|\cdot\|$. This is our $V \hookrightarrow \mathcal{H}$.

§2.3 The Friedrichs extension

Definition 2.3.1. Let T_0 be a symmetric operator, T_0 is **semi-bounded from below** if there exists a constant c,

$$(T_0 u, u)_{\mathcal{H}} \geqslant -c \|u\|_{\mathcal{H}}^2, \quad \forall u \in D(T_0).$$

Example 2.3.2

Consider $T_0 = -\Delta + V(x)$ where $V(x)$ is a real-valued function such that $V(x) \geqslant -c$. Let $D(T_0) = C_c^\infty(\mathbb{R}^n)$, then T_0 is semi-bounded from below.

WLOG, we can assume that $(T_0 u, u)_{\mathcal{H}} \geqslant \|u\|_{\mathcal{H}}$. This can be accomplished by replacing T_0 by $T_0 + \lambda \mathrm{Id}$. Let

$$a_0(u, v) = (T_0 u, v)_{\mathcal{H}}, \quad \forall u, v \in D(T_0),$$

then $a_0(u, u) \geqslant \|u\|_{\mathcal{H}}^2$. Let $p_0(u) = \sqrt{a_0(u, u)}$. Let

$$V = \{u \in \mathcal{H} : \exists u_n \in D(T_0), \ \ \|u_n - u\|_{\mathcal{H}} \to 0, \ \ p_0(u_n - u_m) \to 0\},$$

define the norm on V as $\|u\|_V = \lim_{n\to\infty} p_0(u_n) \geqslant \|u\|_{\mathcal{H}}$.

Fact 2.3.3. If $u_n \to 0$ in $\mathcal{H}$ and $p_0(u_n - u_m) \to 0$, then $p_0(u_n) \to 0$.

This fact shows that $\|\cdot\|_V$ is independent with the choice of $\{u_n\}$. Then $V \hookrightarrow \mathcal{H}$ and V is dense in $\mathcal{H}$. We can define the inner product

$$(u, v)_V := \lim_{n\to\infty} a_0(u_n, v_n), \quad \forall u, v \in V.$$

Which forms V a Hilbert space. Now, we let $a(u, v) = (u, v)_V$. Then there exists a self-adjoint operator S such that $a(u, \cdot) = (Su, \cdot)_{\mathcal{H}}$ for every $u \in D(S)$. Note that $a(u, \cdot) = (T_0 u, \cdot)_{\mathcal{H}}$ for every $u \in D(T_0)$ and $D(T_0)$ is dense in $\mathcal{H}$. Hence S is a self-adjoint extension of T_0.

Theorem 2.3.4 (Friedrichs Extension Theorem)

Let T_0 be a symmetric operator on $\mathcal{H}$ semi-bounded from below. Then there exists a self-adjoint extension of T_0, which is also semi-bounded from below by the same constant.

Applications

Dirichlet Laplacian Let $\Omega \subseteq \mathbb{R}^n$ be a bounded region with a smooth boundary. Let $\mathcal{H} = L^2(\Omega)$ and $T = -\Delta$, $D(T) = C_c^\infty(\Omega)$. Let

$$a_0(u, v) = (-\Delta u, v) = \int_\Omega \nabla u \cdot \nabla \overline{v} \mathrm{d}x.$$

Then by Poincaré inequality, there exists $c > 0$ such that

$$a_0(u, u) = \int_\Omega |\nabla u|^2 \mathrm{d}x \geqslant c \|u\|_{L^2}^2 .$$

Let $p_0(u) = \sqrt{a_0(u, u)}$, then $p_0(u) \approx \|u\|_{H^1}$. Hence

$$\overline{C_c^\infty(\Omega)}^{p_0(\cdot)} = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}} = H_0^1(\Omega).$$

Then $V = H_0^1(\Omega)$ and

$$D(S) = \left\{ u \in H_0^1(\Omega) : \left| \int_\Omega \nabla u \cdot \nabla \overline{v} \mathrm{d}x \right| \leqslant C_u \|v\|_{L^2}, \ \forall v \in H_0^1(\Omega) \right\}.$$

That is, for every $u \in D(S)$, there exists $f \in L^2(\Omega)$ such that

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla \overline{v} \mathrm{d}x = \int_\Omega f \cdot \overline{v} \mathrm{d}x.$$

It follows that $-\Delta u = f \in L^2(\Omega)$, hence $u \in H^2(\Omega)$.

We define the **Dirichlet Laplacian** $-\Delta_D := S$, where $D(S) = H^2(\Omega) \cap H_0^1(\Omega)$. Then the Dirichlet Laplacian is a self-adjoint extension of $T = -\Delta$.

Neumann Laplacian Let $\Omega \subseteq \mathbb{R}^n$ be a bounded region with a smooth boundary. Let $\mathcal{H} = L^2(\Omega)$ and $T = -\Delta$, $D(T) = C^\infty(\Omega)$. Let

$$a_0(u, v) = (-\Delta u + u, v) = \int_\Omega \nabla u \cdot \nabla \overline{v} \mathrm{d}x + \int_\Omega u \cdot \overline{v} \mathrm{d}x.$$

Then $a_0(u, u) = \|u\|_{H^1}^2$, hence $V = \overline{C^\infty(\Omega)}^{\|\cdot\|_{H^1}} = H^1(\Omega)$. For every $u \in D(S)$, we have

$$f = -\Delta u + u \in L^2(\Omega).$$

Then $D(S) \subseteq W = \left\{ u \in H^1(\Omega) : -\Delta u \in L^2(\Omega) \right\}$.

§2.4 The Cayley transform

Let A be a symmetric operator on Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$,

$$\|(A + i\mathrm{Id})x\|^2 = \|Ax\|^2 + \|x\|^2 .$$

There are several properties of a symmetric operator.

- $\ker(A \pm i\mathrm{Id}) = \{0\}$.
- If A is a closed symmetric operator, then $R(A \pm i\mathrm{Id})$ is closed.
- $\ker(A^* \pm i\mathrm{Id}) = R(A \mp i\mathrm{Id})^\perp$.

Theorem 2.4.1

Let A be a symmetric operator, then the following are equivalent:

(1) A is self-adjoint.

(2) A is closed and $\ker(A^* \pm i\mathrm{Id}) = \{0\}$.

(3) $R(A \mp i\mathrm{Id}) = \mathcal{H}$.

Proof. (1) $\Longrightarrow$ (2): $\ker(A^* \pm i\mathrm{Id}) = \ker(A \pm i\mathrm{Id}) = \{0\}$.
(2) $\Longrightarrow$ (3): $R(A \mp i\mathrm{Id})^{\perp} = \ker(A^* \pm i\mathrm{Id}) = \{0\}$ and $R(A \mp i\mathrm{Id})$ is closed by A closed.
(3) $\Longrightarrow$ (1): Hence $\ker(A^* \pm i\mathrm{Id}) = \{0\}$. Take $y \in D(A^*)$, then there exists $z \in D(A)$ such that

$$(A^* \mp i\mathrm{Id})y = (A \mp i\mathrm{Id})z.$$

Because $A \subset A^*$, we have $y - z \in \ker(A^* \mp i\mathrm{Id})$. Hence $y = z \in D(A)$. □

Corollary 2.4.2

Let A be a symmetric operator, then the following are equivalent:

(1) A is essentially self-adjoint.

(2) $\ker(A^* \pm i\mathrm{Id}) = \{0\}$.

(3) $\overline{R(A \mp i\mathrm{Id})} = \mathcal{H}$.

Theorem 2.4.3

Let A be a closed symmetric operator, let

$$U : R(A + i\mathrm{Id}) \to R(A - i\mathrm{Id}), \quad x \mapsto (A - i\mathrm{Id})(A + i\mathrm{Id})^{-1}x.$$

Then U is an onto isometric closed operator. In particular, U is unitary if A is self-adjoint.

Proof. For every $y \in R(A + i\mathrm{Id})$, there is a unique $x \in D(A)$ such that $y = (A + i\mathrm{Id})x$. Then

$$\|Uy\| = \|(A - i\mathrm{Id})x\| = \|Ax\| + \|x\| = \|(A + i\mathrm{Id})x\| = \|x\|.$$

Assume $y_n \in R(A + i\mathrm{Id})$ and $z_n \in R(A - i\mathrm{Id})$ such that $Uy_n = z_n$, $y_n \to y$, $z_n \to z$. Then

$$\begin{cases} (A + i\mathrm{Id})x_n = y_n, \\ (A - i\mathrm{Id})x_n = z_n, \end{cases} \implies \begin{cases} x_n = \dfrac{1}{2i}(y_n - z_n), \\ Ax_n = \dfrac{1}{2}(y_n + z_n). \end{cases}$$

Hence $x_n \to \frac{1}{2i}(y - z)$ and $Ax_n \to \frac{1}{2}(y + z)$. Because A closed, hence U closed. □

Definition 2.4.4. For a closed symmetric operator A, the operator $U := (A - i\mathrm{Id})(A + i\mathrm{Id})^{-1}$ is called the **Cayley transform** of A.

Furthermore, if U is a closed isometric operator and we assume that U is the Cayley transform of a closed symmetric operator A. Then we can solve A by

$$A = i(I + U)(I - U)^{-1},$$

which is called the **inverse Cayley transform**. That is, A is uniquely determined by its Cayley transform.

3 Spectra of Linear Operators

§3.1 The Spectrum

Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed operator. A is called **invertible** if there exists A^{-1} such that

$$AA^{-1} = \mathrm{Id}_X, \quad A^{-1}A = \mathrm{Id}_{D(A)}.$$

That is, A is invertible iff A maps one-to-one onto X. If A is invertible, then A^{-1} is closed and $D(A^{-1}) = X$ is closed, hence A^{-1} is bounded by CGT.

Definition 3.1.1. Let A be a closed operator, the **resolvent set** of A is denoted as

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda\mathrm{Id} - A \text{ is invertible}\}.$$

The **resolvent** is denoted as $R_A(\lambda) = (\lambda\mathrm{Id} - A)^{-1}$ for every $\lambda \in \rho(A)$. The **spectrum** of A is the complement of its resolvent set, denoted as $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

The classification of spectra

(1) The **point spectrum** $\sigma_p(A) := \{\lambda \in \mathbb{C} : \ker(\lambda - A) \neq \{0\}\}$. Every $\lambda \in \sigma_p(A)$ is called an **eigenvalue** and every $x \neq 0 \in \ker(\lambda - A)$ is called an **eigenvector**. The dimension of $\ker(\lambda - A)$ is called the **geometrically multiplicity** of A.

(2) The **continuous spectrum**

$$\sigma_c(A) := \left\{\lambda \in \mathbb{C} : \ker(\lambda - A) = 0, R(\lambda - A) \neq X, \overline{R(\lambda - A)} = X\right\}.$$

(3) The **residual spectrum**

$$\sigma_c(A) := \left\{\lambda \in \mathbb{C} : \ker(\lambda - A) = 0, \overline{R(\lambda - A)} \neq X\right\}.$$

(4) The **discrete spectrum**

$$\sigma_c(A) := \{\lambda \text{ is an isolated point in } \sigma(A) \text{ and the algebraic multiplicity of } \lambda \text{ is finite}\}.$$

The algebraic multiplicity will be defined later.

(5) The **essential spectrum** $\sigma_{ess}(A) := \sigma(A) \setminus \sigma_d(A)$.

Then $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A) = \sigma_d(A) \cup \sigma_{ess}(A)$.

Example 3.1.2

1. Let $X = L^2([0,1])$, $Au = -\frac{d^2}{dt^2}u(t)$, where $D(A) = \left\{u \in L^2 : \Delta u \in L^2\right\}$. Then

$$\sigma(A) = \sigma_p(A) = \sigma_d(A) = \left\{(2\pi n)^2 : n = 0, 1, 2, \cdots\right\}.$$

2. Let $X = C([0,1])$, $Au = tu$, then $\sigma(A) = \sigma_r(A) = [0,1]$.
3. Let $X = L^2([0,1])$, $Au = tu$, then $\sigma(A) = \sigma_c(A) = [0,1]$.

Lemma 3.1.3
If $T \in \mathscr{L}(X)$ and $\|T\| < 1$. Then $\mathrm{Id} - T$ is invertible and $\left\|(\mathrm{Id} - T)^{-1}\right\| \leqslant (1 - \|T\|)^{-1}$.

In fact, we have

$$(\mathrm{Id} - T)^{-1} = \sum_{k=0}^{\infty} T^k,$$

which is called the **Neumann series**.

Theorem 3.1.4

1. $\rho(A) \subset \mathbb{C}$ is open and $\sigma(A)$ is closed.
2. $R_A(\lambda)$ is analytic on $\rho(A)$.

Remark 3.1.5 — $\sigma(A)$ is compact for a bounded operator, but this in general is not true for an unbounded operator.

Theorem 3.1.6 (Resolvent Identity)

1. $R_A(\lambda) - R_A(\mu) = (\mu - \lambda) R_A(\lambda) R_A(\mu)$ for every $\lambda, \mu \in \rho(A)$.
2. $R_A(\lambda) - R_B(\lambda) = R_A(\lambda)(A - B) R_B(\lambda)$ for every $\lambda \in \rho(A) \cap \rho(B)$.

Theorem 3.1.7
If $A \in \mathscr{L}(X)$, then:

1. $\sigma(A) \neq \varnothing$.
2. The **spectral radius** $r_\sigma(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

§3.2 Spectra of self-adjoint operators

Lemma 3.2.1
Let A be a self-adjoint operator, $\lambda \in \mathbb{C}$. If there exists $M > 0$ such that

$$\|(\lambda - A)u\| \geqslant M \|u\|$$

for every $u \in D(A)$, then $\lambda \in \rho(A)$.

Proof. Obviously, $\ker(\lambda - A) = \{0\}$. For A is self-adjoint, we know that $\overline{R(\lambda - A)} = \mathcal{H}$. Besides, $R(\lambda - A)$ is closed because A is closed and $(\lambda - A)^{-1}$ is bounded on $R(\lambda - A)$. □

Theorem 3.2.2

Let A be a self-adjoint operator, then $\sigma(A) \subset \mathbb{R}$ and $\sigma_r(A) = \varnothing$.

Proof. $\sigma(A) \subset \mathbb{R}$ follows from the previous lemma. Take $\lambda \in \sigma(A) \subset \mathbb{R}$, if $\ker(\lambda - A) = \{0\}$, then $R(\lambda - A)^\perp = \ker(\lambda - A^*) = \ker(\lambda - A) = \{0\}$, hence $\lambda \in \sigma_c(A)$. □

Remark 3.2.3 — By the previous discussion, we can show that

$$\|R_A(z)\| \leqslant \frac{1}{|\operatorname{Im} z|}, \quad \forall z \in \rho(A).$$

In fact, we can prove a stronger result (easily follows from an operational calculus) that

$$\|R_A(z)\| \leqslant d(z, \sigma(A))^{-1}, \quad \forall z \in \rho(A).$$

Lemma 3.2.4

Let A be a self-adjoint operator, $\lambda \in \mathbb{R}$. If there exists $\varepsilon > 0$ such that $\|(\lambda - A)u\| \leqslant \varepsilon \|u\|$, then $\sigma(A) \cap [\lambda - \varepsilon, \lambda + \varepsilon] \neq \varnothing$.

Proof. By the remark. □

Theorem 3.2.5 (Weyl's Criterion)

Let A be a self-adjoint operator, then $\lambda \in \sigma(A)$ iff $\exists \{u_n\} \in D(A)$ such that $\|u_n\| = 1$ and $\|(\lambda - A)u_n\| \to 0$.

Proof. "If": Assume $\lambda \in \rho(A)$, then $(\lambda - A)^{-1}$ is bounded, a contradiction.
"Only if": If $\lambda \in \sigma_p(A)$, taking an eigenvalue is enough. Otherwise, $\lambda \in \sigma_c(A)$, then $(\lambda - A)^{-1}$ is unbounded. □

Definition 3.2.6. Let A be a closed operator, A is called **positive** if $(u, Au) \geqslant 0$ for every $u \in D(A)$, denoted by $A \geqslant 0$.

Proposition 3.2.7 Let A be a self-adjoint operator, then $A \geqslant 0$ iff $\sigma(A) \subset \mathbb{R}_{\geqslant 0}$.

Proof. If A is positive, then for every $\lambda > 0$, we have $\|(A + \lambda)u\| \geqslant \lambda \|u\|$, hence $-\lambda \in \rho(A)$. If $\sigma(A) \subset \mathbb{R}_{\geqslant 0}$, then for every $\lambda > 0$, we have $\|R_A(-\lambda)\| \leqslant \lambda^{-1}$. Hence for every $u \in D(A)$,

$$\|(A + \lambda)u\|^2 \geqslant \lambda^2 \|u\|^2 \implies 2\lambda(Au, u) \geqslant - \|Au\|^2.$$

Let $\lambda \to \infty$, we have $(Au, u) \geqslant 0$. □

Fact 3.2.8. If A is a bounded self-adjoint operator, then $\sup_{\|u\|=1} |(Au, u)| = \|A\|$.

Theorem 3.2.9 If A is a bounded self-adjoint operator, then $r_\sigma(A) = \|A\|$.

Proof. It suffices to show $r_\sigma(A) \geqslant \|A\|$. Take $u_n \in D(A)$ such that $\|u_n\| = 1$ and $(Au_n, u_n) \to \lambda_0$, where $\lambda_0 = \pm \|A\|$. Then $\|(A - \lambda_0)u_n\| \to 0$, hence $r_\sigma(A) = \|A\|$. □

§3.3 Riesz projections

Let A be a closed operator on X and let λ_0 be an isolated point in $\sigma(A)$. Define

$$P_{\lambda_0} := \frac{1}{2\pi i}\oint_{\Gamma_{\lambda_0}} R_A(\lambda)\mathrm{d}\lambda,$$

where Γ_{λ_0} is a closed contour around λ_0 such that the closure of the region bounded by Γ_{λ_0} intersects $\sigma(A)$ only at λ_0. By the Cauchy integral formula and the Hahn-Banach theorem, we can show the following fact.

Fact 3.3.1. The definition of P_{λ_0} does not depend on the choice of Γ_{λ_0}.

Theorem 3.3.2

1. P_{λ_0} is a projection.
2. $\ker(\lambda_0 - A) \subset R(P_{\lambda_0})$.
3. If $X = \mathcal{H}$ is a Hilbert space and A is self-adjoint, then $P_{\lambda_0} : \mathcal{H} \to \ker(\lambda_0 - A)$ is the orthogonal projection.

Proof. 1. We do some calculation

$$\begin{aligned}
P_{\lambda_0}^2 &= \frac{1}{(2\pi i)^2}\oint_{\Gamma_{\lambda_0}} \mathrm{d}\lambda \oint_{\widetilde{\Gamma}_{\lambda_0}} \mathrm{d}\mu\, R_A(\lambda)R_A(\mu) \\
&= \frac{1}{(2\pi i)^2}\oint_{\Gamma_{\lambda_0}} \mathrm{d}\lambda \oint_{\widetilde{\Gamma}_{\lambda_0}} \mathrm{d}\mu \left(\frac{R_A(\lambda)}{\mu-\lambda} - \frac{R_A(\mu)}{\mu-\lambda}\right) \\
&= P_{\lambda_0} - 0 = P_{\lambda_0}.
\end{aligned}$$

2. For every $u \in \ker(\lambda_0 - \lambda)$, we have $R_A(\lambda)u = (\lambda-\lambda_0)^{-1}u$ for every $\lambda \in \Gamma_{\lambda_0}$. Hence $P_{\lambda_0}u = u$. Then dimension of $\ker R(P_{\lambda_0})$ is called the **algebraic multiplicity** of λ_0.

3. Let $\Gamma_{\lambda_0} = \partial B(\lambda_0, r)$, then we can verify that P_{λ_0} is self-adjoint. It suffices to show that $R(P_{\lambda_0}) \subset \ker(\lambda_0 - A)$. Where we have

$$(\lambda_0 - A)P_{\lambda_0} = \frac{1}{2\pi i}\oint_{\Gamma_{\lambda_0}} (\lambda_0 - A)R_A(\lambda)\mathrm{d}\lambda = \frac{1}{2\pi i}\oint_{\Gamma_{\lambda_0}} (\lambda_0-\lambda)R_A(\lambda)\mathrm{d}\lambda.$$

Because we have the estimate $\|R_A(\lambda)\| \leqslant d(\lambda, \sigma(A))^{-1}$, take Γ_{λ_0} closed enough to λ_0, we have $\|(\lambda-\lambda_0)R_A(\lambda)\| \leqslant 1$ in the region bounded by Γ_{λ_0}. Hence $(\lambda-\lambda_0)R_A(\lambda)$ can be extended to λ_0 analytically. Then $(\lambda_0 - A)P_{\lambda_0} = 0$.

□

Remark 3.3.3 — The definition of algebraic multiplicity coincides with the concept in linear algebra when X is finite-dimensional and A is a linear map on X.

Definition 3.3.4. P_{λ_0} is called the **Riesz integral** for A and λ_0.

Remark 3.3.5 — For a more general concept, refer to the Riesz-Dunford functional calculus.

Lemma 3.3.6
If A is self-adjoint and λ_0 is an isolated point in $\sigma(A)$, then $\lambda_0 \in \sigma_p(A)$.

Proof. If $\lambda_0 \notin \sigma_p(A)$, then $R(P_{\lambda_0}) \equiv 0$. Because $\|R_A(\lambda)\| \leqslant |\lambda - \lambda_0|^{-1}$ on a neighborhood of λ_0, then λ_0 cannot be a pole of multiplicity $\geqslant 2$. Hence $R_A(\lambda)$ can be extended to λ_0 analytically. A contradiction. □

Lemma 3.3.7
If A is self-adjoint and λ_0 is an isolated point in $\sigma(A)$, then $\lambda_0 \notin \sigma(A|_{\ker(A-\lambda_0)^\perp})$.

Remark 3.3.8 — If A is self-adjoint, then $\ker A^\perp$ is A-invariant and $A|_{\ker A^\perp}$ is self-adjoint.

Proof. Let $A_1 = A|_{\ker(A-\lambda_0)^\perp}$, then $\lambda_0 \notin \sigma_p(A_1)$ hence $\lambda_0 \notin \sigma(A_1)$. □

Theorem 3.3.9
If A is self-adjoint, then $\lambda_0 \in \sigma_d(A)$ iff $\dim\ker(\lambda_0 - A) < \infty$ and $\lambda_0 \notin \sigma(A|_{\ker(A-\lambda_0)^\perp})$.

Proof. "Only if" follows from previous discussions. "If" part: let $P_{\lambda_0} : \mathcal{H} \to \ker(\lambda_0 - A)$ be the orthogonal projection. For $|\lambda - \lambda_0| \ll 1$, $(\lambda - A)^{-1}$ exists on $\ker(A-\lambda_0)^\perp$. We define

$$R(\lambda) = (\lambda - \lambda_0)^{-1} P_{\lambda_0} + (\lambda - A)^{-1}(\mathrm{Id} - P_{\lambda_0}), \quad \forall \lambda \neq \lambda_0, |\lambda - \lambda_0| \ll 1,$$

then $(\lambda - A)R(\lambda) = \mathrm{Id}$. Hence $\lambda \in \rho(A)$ and $\lambda_0 \in \sigma_d(A)$. □

Corollary 3.3.10
If A is self-adjoint and $\dim\ker(\lambda_0 - A) < \infty$, then $\lambda_0 \in \sigma_{ess}(A)$ iff $(A-\lambda_0)|_{\ker(A-\lambda_0)^\perp}$ has an unbounded inverse.

Now, we assume that σ_1 is a compact connected component of $\sigma(A)$. Then we can define

$$P_{\sigma_1} = \frac{1}{2\pi i} \oint_{\Gamma_1} R_A(\lambda) \mathrm{d}\lambda,$$

where Γ_1 is a simple closed curve winding σ_1 and the intersection of the region bounded by Γ_1 and $\sigma(A)$ is σ_1. We can show that

Fact 3.3.11. P_{σ_1} commutes with A on $D(A)$ and $\sigma(AP_{\sigma_1}) = \sigma_1$, $\sigma(A(\mathrm{Id} - P_{\sigma_1})) = \sigma(A) \setminus \sigma_1$.

Definition 3.3.12. Let A be a self-adjoint operator, an eigenvalue λ is called an **embedded eigenvalue** if λ is not isolated in $\sigma(A)$.

For an embedded eigenvalue λ, let $P_\lambda : \mathcal{H} \to \ker(A - \lambda)$ be the orthogonal projection.

Theorem 3.3.13
Let A be a self-adjoint operator and λ is an embedded eigenvalue of A. Then

$$P_\lambda = s - \lim_{\epsilon \to 0} (-i\epsilon)(A - \lambda - i\epsilon)^{-1}.$$

Proof. Let $P_\epsilon = (-i\epsilon)(A - \lambda - i\epsilon)^{-1}$, then $P_\epsilon P_\lambda = P_\lambda$. Let $Q_\lambda = \mathrm{Id} - P_\lambda$, it suffices to show

$$s - \lim_{\epsilon \to 0} P_\epsilon Q_\lambda = 0.$$

For every $u \in D(A)$, we can calculate that

$$\begin{aligned} \|P_\epsilon Q_\lambda u\|^2 &= \|Q_\lambda u\|^2 - (Q_\lambda u, (A-\lambda)^2[(A-\lambda)^2 + \epsilon^2]^{-1} Q_\lambda u) \\ &= \|Q_\lambda u\|^2 - \left\|(A - \lambda - i\epsilon)^{-1}(A-\lambda) Q_\lambda u\right\|^2 \to 0. \end{aligned}$$

□

§3.4 Weyl's criterion for the essential spectrum

Let A be a self-adjoint operator on Hilbert space $\mathcal{H}$.

Definition 3.4.1. A sequence $\{u_n\} \subset D(A)$ is called a **Weyl sequence** for A and λ if

$$\|u_n\| = 1, \ \ u_n \xrightarrow{w} 0, \ \ (A - \lambda) u_n \to 0.$$

Theorem 3.4.2 (Weyl's criterion for the essential spectrum)
If A is self-adjoint, then $\lambda \in \sigma_{ess}(A)$ iff there exists a Weyl sequence for A and λ.

Proof. "Only if": If $\dim \ker(A - \lambda) = \infty$, taking an orthogonal sequence in $\ker(A-\lambda)$ is enough. If $\dim \ker(A - \lambda) < \infty$, let $A_1 = (A - \lambda)|_{\ker(A-\lambda)^\perp}$, then A_1^{-1} is unbounded. Then we can find $u_n \in \ker(A - \lambda)^\perp$ such that $\|u_n\| = 1$ and $A_1 u_n \to 0$. It suffices to show $u_n \xrightarrow{w} 0$. For every $f \in D((A_1^{-1})^*)$, we have

$$(f, u_n) = ((A_1^{-1})^* f, A_1 u_n) \to 0.$$

It suffices to show $D((A_1^{-1})^*)$ is dense in $\ker(A - \lambda)^\perp$. Note that for every $f \in D(A_1)$, we have

$$(A_1 f, u) = (f, A_1 u), \quad \forall u \in D(A_1).$$

Hence $A_1 f \in D((A_1^{-1})^*)$, which follows $R(A_1) \subset D((A_1^{-1})^*)$. And $R(A_1)$ is dense because the self-adjointness of A_1, this part holds.

"If": Assume for a contradiction that $\lambda \in \sigma_d(A)$, then $\dim \ker(\lambda - A) < \infty$. Let P_λ be the orthogonal projection onto $\ker(\lambda - A)$, then

$$P_\lambda u_n \xrightarrow{w} 0.$$

Because the weak convergence is equivalent to the strong convergence on a finite-dimensional space, hence

$$P_\lambda u_n \to 0.$$

Then we can assume without lost of generality that the Weyl sequence takes value in $\ker(A-\lambda)^\perp$. But if $\lambda \in \sigma_d(A)$, we know that $(A - \lambda)|_{\ker(\lambda - A)^\perp}$ is invertible. This contradicts with $\|u_n\| = 1$ and $\|Au_n\| \to 0$. □

Application: the Laplacian on $\mathbb{R}^n$

Consider $\mathcal{H} = L^2(\mathbb{R}^n)$ and the operator $-\Delta, D(-\Delta) = H^2(\mathbb{R})$. At this time, $-\Delta$ is self-adjoint. Because $-\Delta \geqslant 0$, we know that $\sigma(-\Delta) \subseteq [0, +\infty)$. We will show that $\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0, +\infty)$.

For every $\lambda > 0$, we construct

$$u_m(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{u}_m(\xi) e^{i\xi x} \mathrm{d}\xi,$$

where

$$\widehat{u}_m := (2\pi m)^{\frac{n}{2}} e^{-m^2|\xi-\xi_0|^2}, \quad |\xi_0|^2 = \lambda.$$

Then $\|u_m\|_{L^2} = \|\widehat{u}_m\|_{L^2} = 1$. For every $f \in \mathcal{S}(\mathbb{R}^n)$, $(u_m, f) = (\widehat{u}_m, \widehat{f}) \to 0$. And

$$\|(-\Delta - \lambda)u_m\|_{L^2} = \left\|(|\xi|^2 - \lambda)^2 \widehat{u}_m\right\|_{L^2} \to 0.$$

Hence $\{u_m\}$ forms a Weyl sequence for $-\Delta$ and λ. Hence $(0, +\infty) \subseteq \sigma_{ess}(-\Delta)$. Then $\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0, +\infty)$.

> **Annotation 3.4.3** In this example, we do not need the Weyl sequence because every point in the spectrum is not isolated.

4 Perturbations of Self-Adjoint Operators

Consider the operator $H = -\Delta + V(x)$ where $V(x)$ is a real-valued perturbation.

> **Question 4.0.1.** Does H still self-adjoint? What about the spectrum of H?

§4.1 Perturbations of densely defined operators

Definition 4.1.1. Let A, B be densely defined operators on Hilbert space $\mathcal{H}$, endowing $D(A)$ with the **graph norm** $[\![x]\!] = \|x\| + \|Ax\|$. Then B is called **bounded relative to** A if

(i) $D(B) \supset D(A)$.

(ii) $B : (D(B), [\![\cdot]\!]) \to (\mathcal{H}, \|\cdot\|)$ is bounded.

Note that B is bounded relative to A iff there exists $a, b > 0$ such that

$$\|Bx\| \leqslant a \|Ax\| + b \|x\|, \quad \forall x \in \mathcal{H}.$$

The infimum of a is called the **bound of B relative to A**. In particular, if $B \in \mathscr{L}(\mathcal{H})$, then B is bounded relative to A and the bound is 0.

Theorem 4.1.2

Let A, B be densely defined operators, assume that B is bounded relative to A and the bound is smaller than 1. Then $A + B$ is closable iff A is closable. Furthermore, $D(\overline{A+B}) = D(\overline{A})$.

Proof. Recall that T is closable iff, $x_n \to 0, Tx_n \to y \implies y = 0$. By assumption,

$$(1-a)\|Ax\| - b\|x\| \leqslant \|(A+B)x\| \leqslant (1+a)\|Ax\| + b\|x\|.$$

Assume A is closable, then if $x_n \to 0$ and $(A+B)x_n \to y$, we know that $\{Ax_n\}$ is Cauchy and hence $Ax_n \to 0$. Then $(A+B)x_n \to 0$, which shows $A + B$ is closable. And vice versa. □

Corollary 4.1.3

Let A, B be densely defined operators, assume that B is bounded relative to A and the bound is smaller than 1. Then $A + B$ is closed iff A is closed.

Corollary 4.1.4

Let A, T be densely defined operators such that $D(A) = D(T)$. Assume that there exists $a', a'' < 1, b > 0$, such that

$$\|(T-A)x\| \leqslant a' \|Ax\| + a'' \|Tx\| + b\|x\|, \quad \forall x \in \mathrm{D}(A),$$

then A is closable iff T is closable.

Proof. Let $B = T - A$. For every $\lambda \in [0,1]$, let $T_\lambda = A + \lambda B$. Then $T_0 = A$ and $T_1 = T$. Let $a = \max\{a', a''\} < 1$, then for every $\lambda \in [0,1], x \in D(A)$,

$$\|Bx\| \leqslant a\|Ax\| + a\|Tx\| + b\|x\| \leqslant 2a\|T_\lambda x\| + a\|Bx\| + b\|x\|.$$

For every $0 < h < \frac{1-a}{2a}$, we have

$$h\|Bx\| \leqslant \frac{2ah}{1-a}\|T_\lambda x\| + \frac{bh}{1-a}\|x\|, \quad \forall x \in D(A).$$

Hence T_λ is closable iff $T_{\lambda+h}$ is closable. Then $A = T_0$ is closable iff $T = T_1$ is closable. □

Definition 4.1.5. Let A, B be densely defined operators on Hilbert space $\mathcal{H}$, endowing $D(A)$ with the graph norm $[\![x]\!] = \|x\| + \|Ax\|$. Then B is called **compact relative to A** (or **A-compact**) if

(i) $D(B) \supset D(A)$.

(ii) $B : (D(B), [\![\cdot]\!]) \to (\mathcal{H}, \|\cdot\|)$ is compact.

Proposition 4.1.6

If B is closable and B is compact relative to A. Then for every $\varepsilon > 0$, there exists $b_\varepsilon > 0$ such that

$$\|Bx\| \leqslant \varepsilon\|Ax\| + b_\varepsilon\|x\|, \quad \forall x \in D(A).$$

Proof. Assume for a contradiction, there exists $u_n \in D(A)$, such that

$$\|Bx_n\| = 1, \quad \varepsilon\|Ax_n\| + n\|x_n\| \leqslant 1.$$

Then $[\![x_n]\!] \leqslant 1 + \varepsilon$, there exists $Bx_{n_j} \to y$. But $x_{n_j} \to 0$, for B is closable, $Bx_{n_j} \to 0$. A contradiction. □

Theorem 4.1.7

Assume B is closable, then B is A-compact iff B is $(A+B)$-compact.

Proof. By the previous proposition, we can show that $[\![\cdot]\!]_A$ is equivalent to $[\![\cdot]\!]_{A+B}$. □

Annotation 4.1.8 The condition of B is closable in previous results can be replaced by A is closable. But there are some gaps in the proof on textbook. I complement the proof here.

A same argument in proposition 4.1.6, note that $\{Ax_n\}$ is bounded and $\mathcal{H}$ is reflexive. We can assume WLOG that

$$\|Bx_n\| = 1, \quad x_n \to 0, \quad Ax_n \xrightarrow{w} y.$$

Because A is closable, hence A^* is densely defined. For every $z \in D(A^*)$,

$$(y,z)_{\mathcal{H}} = \lim_{n\to\infty}(Ax_n, z)_{\mathcal{H}} = \lim_{n\to\infty}(x_n, A^*z)_{\mathcal{H}} = 0.$$

Hence $y = 0$, then x_n weakly convergent to 0 with respect to $[\![\cdot]\!]$. By the relative compactness of B, $Bx_n \to 0$. A contradiction.

§4.2 Perturbations of self-adjoint operators

Theorem 4.2.1 (Kato-Rellich)
Let A be a self-adjoint operator and B be a symmetric operator, assume that B is bounded relative to A and the bound is smaller than 1. Then $A+B$ is self-adjoint. In particular, $A+B$ is self-adjoint if B is a bounded self-adjoint operator.

Proof. It suffices to find $\mu_0 > 0$ such that $A + B \pm i\mu_0 \mathrm{Id}$ is invertible. We can write

$$A + B \pm i\mu_0 \mathrm{Id} = [\mathrm{Id} + B(A \pm i\mu_0)^{-1}](A \pm i\mu_0),$$

Note that

$$\|(A \pm i\mu \mathrm{Id})x\| = \|Ax\| + \mu \|x\|, \quad \forall x \in D(A).$$

Hence for every $x \in \mathcal{H}$,

$$\left\|B(A \pm i\mu)^{-1}x\right\| \leqslant a \left\|A(A \pm i\mu)^{-1}x\right\| + b \left\|(A \pm i\mu)^{-1}x\right\| \leqslant (a + \frac{b}{\mu}) \|x\|.$$

Take $\mu = \mu_0$ sufficiently large such that $\left\|B(A \pm i\mu)^{-1}x\right\| < 1$. The conclusion follows. □

The Schrödinger operator Let $H = -\Delta + V(x)$, where $V = V_1 + V_2$ such that V_1, V_2 are both real-valued and $V_1 \in L^2(\mathbb{R}^3), V_2 \in L^\infty(\mathbb{R}^3)$.

- Let $D(-\Delta) = H^2(\mathbb{R}^3)$, then $-\Delta$ is self-adjoint.
- Let $D(V) = \left\{u \in L^2(\mathbb{R}^3) : Vu \in L^2(\mathbb{R}^3)\right\}$, then V is symmetric.

We want to show H is self-adjoint. By Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, hence $D(-\Delta) \subset D(V)$. Because

$$\|Vu\|_{L^2} \leqslant \|V_1\|_{L^2} \|u\|_{L^\infty} + \|V_2\|_{L^\infty} \|u\|_{L^2},$$

it suffices to estimate $\|u\|_{L^\infty}$. For every $\lambda > 0$, we have

$$\begin{aligned}
\|u\|_{L^\infty} &\leqslant \int |\hat{u}(\xi)| \mathrm{d}\xi = \int_{|\xi| \leqslant \lambda} |\hat{u}(\xi)| \mathrm{d}\xi + \int_{|\xi| > \lambda} |\hat{u}(\xi)| \mathrm{d}\xi \\
&\leqslant \|u\|_{L^2} \left(\int_{|\xi| \leqslant \lambda} \mathrm{d}\xi\right)^{\frac{1}{2}} + \left(\int_{|\xi| > \lambda} |\widehat{\Delta u}(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{|\xi| > \lambda} |\xi|^{-4} \mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\leqslant c\lambda^{\frac{3}{2}} \|u\|_{L^2} + c\lambda^{-\frac{1}{2}} \|\Delta u\|_{L^2}.
\end{aligned}$$

Take λ sufficiently large such that $c\lambda^{-1/2} \|V_1\|_{L^2} < 1$, we know that the bound of V relative to $-\Delta$ is smaller than 1. Hence H is self-adjoint.

Corollary 4.2.2
Let A be an essentially self-adjoint operator and B be a symmetric operator, assume that B is bounded relative to A and the bound is smaller than 1. Then $A+B$ is essentially self-adjoint and $\overline{A+B} = \overline{A} + \overline{B}$.

Theorem 4.2.3

Let A be a self-adjoint operator and B be a symmetric operator, assume that B is bounded relative to A such that

$$\|Bx\| \leqslant \|Ax\| + b\|x\|, \quad \forall x \in D(A).$$

Then $A+B$ is essentially self-adjoint on $D(A)$.

Proof. It suffices to show $\ker((A+B)^* \pm i\mathrm{Id}) = \{0\}$, assume that $((A+B)^* - i\mathrm{Id})h = 0$. For every $s \in (0,1)$, let $T_s = A + sB$ which is self-adjoint. Then $R(T_s + i\mathrm{Id}) = \mathcal{H}$. Take $x_s \in D(A)$ such that $(T_s + i\mathrm{Id})x_s = h$. Let $y_s = (A + B + i\mathrm{Id})x_s$, then

$$(h, y_s)_{\mathcal{H}} = (((A+B)^* - i\mathrm{Id})h, x_s)_{\mathcal{H}} = 0.$$

We claim that $y_s \xrightarrow{w} h$, then $\|h\|^2 = \lim_{s\to1}(h, y_s) = 0$. Note that $y_s = h + (1-s)Bx_s$, it suffices to show $(1-s)Bx_s \xrightarrow{w} 0$. For every $z \in D(A)$, we have

$$((1-s)Bx_s, z) = ((1-s)x_s, Bz) \to 0 (s \to 1).$$

Because $D(A)$ is dense in $\mathcal{H}$, we only need to bound $\|(1-s)Bx_s\|$. Note that $\|h\|^2 = \|T_s x_s\|^2 + \|x_s\|^2$, hence $\|x_s\| \leqslant \|h\|$. Besides

$$\|Ax_s\| \leqslant \|(A+sB)x_s\| + s\|Bx_s\| \leqslant \|h\| + s\|Ax_s\| + sb\|x_s\|,$$

hence $(1-s)\|Ax_s\| \leqslant \|h\| + sb\|x_s\|$. Then

$$\|(1-s)Bx_s\| \leqslant (1-s)\|Ax_s\| + (1-s)b\|x_s\| \leqslant (1-s+b)\|h\|.$$

The statement follows. □

Theorem 4.2.4

Let A be a self-adjoint operator semi-bounded from below, let B be a symmetric operator, assume that B is bounded relative to A and the bound is smaller than 1. Then $A+B$ is self-adjoint and semi-bounded from below.

Proof. Note that a self-adjoint operator $T \geqslant C$ iff $(-\infty, C) \subset \rho(T)$. It's more convenient to use some result of spectral resolutions of self-adjoint operators. □

Theorem 4.2.5 (Kato-Lax-Milgram-Nelson)

Let A be a positive self-adjoint operator and B be a closed symmetric operator satisfying:

(i) $D(B) \supset D(A)$.

(ii) There exists $0 < a < 1, b > 0$ such that $|(Bu, u)_{\mathcal{H}}| \leqslant a(Au, u)_{\mathcal{H}} + b\|u\|_{\mathcal{H}}$ for every $u \in D(A)$.

Then there exists a unique self-adjoint operator C corresponding to a sesquilinear form $a(u, v)$ on $V = D(C) = D(A^{1/2})$ such that

(i) $a(u, v) = (Cu, v)_{\mathcal{H}}$, for every $u, v \in V$.

(ii) $a(u, v) = ((A+B)u, v)$, for every $u \in D(A), v \in V$.

§4.3 Spectra under perturbations

Let A be a self-adjoint operator and B be a symmetric operator, assume that B is bounded relative to A and the bound is smaller than 1. Then $A+B$ is self-adjoint. Now we study the changes of spectra.

Theorem 4.3.1 (Weyl)
If B is A-compact, then $\sigma_{ess}(A+B)=\sigma_{ess}(A)$.

Proof. For every $\lambda\in\sigma_{ess}(A)$, take a Weyl sequence for A and λ. Assume that

$$\|u_n\|=1,\quad u_n\xrightarrow{w}0,\quad (A-\lambda)u_n\to 0.$$

Then $Au_n\xrightarrow{w}0$, hence u_n weakly convergent to 0 with respect to $[\![\cdot]\!]$. Then $Bu_n\to 0$, hence $\{u_n\}$ is also a Weyl sequence for λ and $A+B$. Hence $\sigma_{ess}(A)\subset\sigma_{ess}(A+B)$ and vice versa. □

Now we focus on the change of discrete spectrum. Let $\lambda_0\in\sigma_p(A)$, then

- $\dim\ker(\lambda_0-A)=m<\infty$.
- There exists $r>0$ such that $[\lambda_0-r,\lambda_0+r]\cap\sigma(A)=\{\lambda_0\}$. Let

$$D:=\left\{z\in\mathbb{C}:|z-\lambda_0|<\frac{r}{2}\right\},\quad \Gamma:=\partial D.$$

We assume that

$$\|Bx\|\leqslant a\|Ax\|+b\|x\|,\quad \forall x\in D(A).$$

Theorem 4.3.2
If $2a(|\lambda_0|+r)+2b<r$, then $A+B$ has exactly m eigenvalues (counting the multiplicity) and no other spectrum points in D.

We need the following fact.

Fact 4.3.3. Let P,Q be two projection operators on $\mathcal{H}$ such that $\|P-Q\|<1$, then

$$\dim P\mathcal{H}=\dim Q\mathcal{H}.$$

Proof of theorem 4.3.2. Let $T_s=A+sB$, we first show that $\Gamma\in\rho(T_s)$ for every $s\in[0,1]$. Write

$$\lambda-T_s=[\mathrm{Id}-sB(\lambda-A)^{-1}](\lambda-A).$$

For every $x\in\mathcal{H}$, let $y=(\lambda-A)^{-1}x$, then $\|y\|\leqslant\frac{2}{r}\|x\|$. We have

$$\left\|sB(\lambda-A)^{-1}x\right\|\leqslant sa\|Ay\|+sb\|y\|\leqslant sa(1+\frac{2}{r}(|\lambda_0|+\frac{r}{2}))+sb\frac{2}{r}=sh<1,$$

where $h=\frac{2}{r}(a|\lambda_0|+ar+b)<1$. Hence $\lambda-T_s$ is invertible and $\|(\lambda-T_s)\|\leqslant\frac{2}{r(1-h)}$.

Consider the Riesz projection

$$P_s=\frac{1}{2\pi i}\oint_\Gamma\frac{\mathrm{d}\lambda}{\lambda-T_s},$$

it suffices to show that $\dim P_s\mathcal{H}$ is a constant. We have

$$P_s-P_{s'}=\frac{1}{2\pi i}\oint_\Gamma(R_{T_s}(\lambda)-R_{T_{s'}}(\lambda))\mathrm{d}\lambda=\frac{1}{2\pi i}\oint_\Gamma R_{T_s}(\lambda)(T_s-T_{s'})R_{T_{s'}}(\lambda)\mathrm{d}\lambda.$$

Same as former estimates, we can show that $\|P_s-P_{s'}\|<1$ for $|s-s'|$ small enough. Then the conclusion follows. □

5 Banach Algebras

§5.1 Preliminaries

Definition 5.1.1. $\mathcal{A}$ is called an **algebra** on $\mathbb{C}$ if

(i) $\mathcal{A}$ is a linear space on $\mathbb{C}$.

(ii) There is a multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is compatible with the scalar multiplication of the linear space.

Definition 5.1.2. There are several definitions:

1. The **unit** $e \in \mathcal{A}$ is the element such that $ea = ae = a$ for every $a \in \mathcal{A}$.
2. If $\mathcal{A}$ has a unit e, an element a is called **invertible** if $\exists b \in \mathcal{A}$, $ba = ab = e$. The element b is denoted by a^{-1}.
3. $\mathcal{A}$ is called a **division algebra** if a is invertible for every $a \neq 0 \in \mathcal{A}$.
4. $\mathcal{A}$ is called **commutative** if $ab = ba$ for every $a, b \in \mathcal{A}$.

Definition 5.1.3. Let $\mathcal{A}$ be an algebra, subspace $\mathcal{B} \subset \mathcal{A}$ is called a **sub-algebra** if $\mathcal{B}$ forms an algebra under the operations on $\mathcal{A}$. A sub-algebra $J \subsetneqq \mathcal{A}$ is called an **ideal** if $\forall a \in \mathcal{A}$, $aJ \subset J, Ja \subset J$.

Definition 5.1.4. Let $\mathcal{A}, \mathcal{B}$ be two algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ is called a **homomorphism** if φ is a linear map and preserves the multiplication. Moreover, φ is called an **isomorphism** if φ is a bijection.

Fact 5.1.5. Let $\mathcal{A}, \mathcal{B}$ be two algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism. Then $\ker \varphi = \varphi^{-1}(0)$ is $\mathcal{A}$ or an ideal.

Fact 5.1.6. Let $\mathcal{A}$ be an algebra with a unit, $J \subset \mathcal{A}$ is an ideal. Then for every $a \in \mathcal{A}$ invertible, $a \notin J$.

Let J be an ideal of $\mathcal{A}$, we consider the quotient space $\mathcal{B} = \mathcal{A}/J$. An element $[a] \in \mathcal{B}$ is of the form $[a] = \{b \in \mathcal{A} : b - a \in J\}$. Then $\mathcal{B}$ is also an algebra. The natural projection $\varphi : \mathcal{A} \to \mathcal{B}, a \mapsto [a]$ is a homomorphism such that $\ker \varphi = J$.

Definition 5.1.7. Let J be an ideal of $\mathcal{A}$, J is called a **maximal ideal** if there is no ideal contains J properly.

Fact 5.1.8. Let $\mathcal{A}$ be a commutative algebra with a unit. For every $a \in \mathcal{A}$, a is contained in some maximal algebra iff a is non-invertible.

Theorem 5.1.9

Let $\mathcal{A}$ be a commutative algebra with a unit, then J is a maximal ideal iff $\mathcal{A}/J$ is a division algebra.

§5.2 Banach algebras

Definition 5.2.1. $\mathcal{A}$ is called a **Banach algebra** if:

(i) $\mathcal{A}$ is an algebra on $\mathbb{C}$.

(ii) $(\mathcal{A}, \|\cdot\|)$ is a Banach space.

(iii) For every $a, b \in \mathcal{A}$, we have $\|ab\| \leqslant \|a\| \, \|b\|$.

Remark 5.2.2 — The multiplication on $\mathcal{A}$ is continuous with respect to $\|\cdot\|$.

Remark 5.2.3 — If $\mathcal{A}$ has a unit e, then $\|e\| \geqslant 1$. Besides, we can define another norm as

$$|a| := \sup_{b \in \mathcal{A}} \frac{\|ab\|}{\|b\|}.$$

Then $|e| = 1$ and $|\cdot|$ is equivalent to $\|\cdot\|$. Hence we can always assume that $\|e\| = 1$.

Example 5.2.4

Let M be a compact Hausdorff space, then $C(M)$ is a commutative algebra with a unit. Let $\|f\| = \max_{x \in M} |f(x)|$, then $C(M)$ is a Banach algebra.

Example 5.2.5

Let X be a Banach space, then $\mathscr{L}(X)$ is a non-commutative Banach algebra with a unit.

Example 5.2.6

Consider the convolution $*$ on $L^1(\mathbb{R}^n)$. By Young's inequality $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$, $(L^1(\mathbb{R}^n), *)$ forms a commutative Banach algebra with no unit.

Example 5.2.7 (Continuous functions with uniformly convergent Fourier series)

Let

$$\mathcal{A} := \left\{ u \in C(\mathbb{S}^1) : u(e^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}, \sum_{n \in \mathbb{Z}} |c_n| < \infty \right\}$$

with the norm $\|u\| := \sum_{n \in \mathbb{Z}} |c_n|$. Then for every $u, v \in \mathcal{A}$,

$$(uv)(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} u_{n-k} v_k \right) e^{in\theta} \in \mathcal{A}.$$

Hence $\mathcal{A}$ is a commutative Banach algebra with a unit.

The Gelfand representation

Theorem 5.2.8 (Gelfand-Mazur)

Let $\mathcal{A}$ be a Banach algebra with a unit that is a division algebra, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

Proof. Let $\mathcal{B} := \{ze : z \in \mathbb{C}\} \subset \mathcal{A}$, it suffices to show $\mathcal{B} = \mathcal{A}$. Otherwise, there exists $a \in \mathcal{A} \backslash \mathcal{B}$, for every $z \in \mathbb{C}$, $ze - a$ is invertible. Let

$$r(z) = (ze - a)^{-1}.$$

We can show that

1. r is weakly analytic, that is, for every $f \in \mathcal{A}^*$, $\langle f, r(z) \rangle$ is analytic.
2. $\|r(z)\|$ is bounded.

Combining these two facts, we know that $\langle f, r(z) \rangle \equiv 0$ for every $f \in \mathcal{A}^*$. Hence $r(z) = 0$, a contradiction. □

Lemma 5.2.9

Let $\mathcal{A}$ be a Banach algebra with a unit, then every maximal ideal J of $\mathcal{A}$ is closed.

Proof. Consider $\overline{J}$ which is also an ideal of $\mathcal{A}$. It suffices to show that $\overline{J} \subsetneqq \mathcal{A}$. We claim that $e \notin \overline{J}$. This is because $B(e, 1)$ is invertible in $\mathcal{A}$, hence $B(e, 1) \cap J = \varnothing$. □

By this lemma, for every maximal ideal J of $\mathcal{A}$, $\mathcal{A}/J$ is also a Banach space. Besides, we can define the quotient norm $[\![[a]]\!] = \inf_{x \in [a]} \|x\|$ on it. Then $\mathcal{A}/J$ forms a Banach algebra.

Theorem 5.2.10

Let $\mathcal{A}$ be a commutative Banach algebra with a unit, J is a maximal ideal of $\mathcal{A}$. Then $\mathcal{A}/J$ is isometrically isomorphic to $\mathbb{C}$, denotes by $\mathcal{A}/J \cong \mathbb{C}$.

Definition 5.2.11. A **multiplicative functional** φ in a Banach algebra $\mathcal{A}$ is an algebra homomorphism of $\mathcal{A}$ into $\mathbb{C}$.

Remark 5.2.12 — The definition of a multiplicative functional is purely algebraic. But it still has some analytic properties for a Banach algebra.

The previous theorem shows that every maximal ideal J of $\mathcal{A}$ corresponds to a multiplicative functional $\varphi_J : \mathcal{A} \to \mathbb{C}$ such that $\ker \varphi_J = J$. Besides, $|\varphi_J(a)| \leqslant \|a\|$.

Now, let's take another view. Let $\mathfrak{M}$ be the family of maximal ideals in $\mathcal{A}$. Then $\mathcal{A}$ can act on $\mathfrak{M}$ as

$$\widehat{a} : \mathfrak{M} \to \mathbb{C}, \quad J \mapsto \varphi_J(a), \qquad \forall a \in \mathcal{A}.$$

Then $\Gamma : a \mapsto \widehat{a} \in \{\text{complex functions on } \mathfrak{M}\}$ is a homomorphism.

Definition 5.2.13. Let $\mathcal{A}$ be a commutative Banach algebra with a unit, the homomorphism $\Gamma : a \mapsto \widehat{a}$ is called the **Gelfand representation** of $\mathcal{A}$.

Aim 5.2.14. To define a topology on $\mathfrak{M}$, such that $\mathfrak{M}$ is a compact Hausdorff space and Γ maps $\mathcal{A}$ to $C(\mathfrak{M})$.

Let $\mathcal{A}$ be a commutative Banach algebra with a unit, consider

$$\Delta = \{\varphi \in \mathcal{A}^* : \varphi \text{ is a non-zero multiplicative functional}\} .$$

Then we have the following fact.

Fact 5.2.15. $J \mapsto \varphi_J$ is a bijection between $\mathfrak{M}$ and Δ, the inverse is given by $\varphi \mapsto \ker \varphi$.

Theorem 5.2.16 (Alaoglu)

Let X be a Banach space, then the closed unit ball of X^* is weak* compact.

Hence $\Delta \hookrightarrow \mathcal{A}^*$ is a compact subspace with respect to the weak* topology. Endowing $\mathfrak{M}$ with this topology, then $\mathfrak{M}$ is a compact Hausdorff space. For every $J_0 \in \mathfrak{M}$, it has neighborhood bases as

$$\begin{aligned} N(J_0; \varepsilon, A) &= \{J \in \mathfrak{M} : |\varphi_J(a) - \varphi_{J_0}(a)| < \varepsilon, \forall a \in A\} \\ &= \{J \in \mathfrak{M} : |\widehat{a}(J) - \widehat{a}(J_0)| < \varepsilon, \forall a \in A\} , \end{aligned}$$

where $A \subset \mathcal{A}$ is a finite set. Then $C(\mathfrak{M})$ is a commutative Banach algebra with a unit and $\Gamma : \mathcal{A} \to C(\mathfrak{M})$ is a homomorphism.

Theorem 5.2.17

Let $\mathcal{A}$ be a commutative Banach algebra with a unit, then $\Gamma : \mathcal{A} \to C(\mathfrak{M})$ is a continuous homomorphism and $\|\Gamma a\|_{C(\mathfrak{M})} \leqslant \|a\|$.

Spectrum of an element

Definition 5.2.18. Let $\mathcal{A}$ be a Banach algebra with a unit, let $G(\mathcal{A})$ be the family of invertible elements in $\mathcal{A}$. For every $a \in \mathcal{A}$, the **resolvent set** of a is $\rho(a) := \{\lambda \in \mathbb{C} : \lambda e - a \in G(\mathcal{A})\}$ and the **spectrum** of a is $\sigma(a) := \mathbb{C} \setminus \rho(a)$.

Fact 5.2.19. $\forall a \in \mathcal{A}$, $\rho(a)$ is open and $\sigma(a)$ is compact.

Theorem 5.2.20

Let $\mathcal{A}$ be a Banach algebra with a unit, then for every $a \in \mathcal{A}$, we have

$$\sigma(a) = \{\widehat{a}(J) : J \in \mathfrak{M}\} .$$

Hence

$$\|\Gamma a\|_{C(\mathfrak{M})} = \sup \{|\lambda| : \lambda \in \sigma(a)\} .$$

Proof. $\lambda e - a \notin G(\mathcal{A})$ iff $\lambda e - a \in J$ for a maximal ideal J. This is equivalent to $\lambda = \widehat{a}(J)$. ☐

Lemma 5.2.21

Let $\mathcal{A}$ be a Banach algebra with a unit, then for every $a \in \mathcal{A}$, we have

$$\|\Gamma a\|_{C(\mathfrak{M})} = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

> **Question 5.2.22.** Γ is an isomorphism onto its image? Γ is isometric? Γ is onto?

Definition 5.2.23. Let $\mathcal{A}$ be an algebra, $\mathfrak{M}$ be the family of maximal ideals of $\mathcal{A}$. The **Jacobson radical** of $\mathcal{A}$ is $R := \bigcap_{J\in\mathfrak{M}} J$. The algebra $\mathcal{A}$ is called **semi-simple** if $R = \{0\}$.

Fact 5.2.24. Γ is isomorphic iff $\mathcal{A}$ is semi-simple.

This fact gives an algebraic description of when Γ is isomorphic. In the view of analytic, we can give another description. These results answer the first question.

Theorem 5.2.25

Let $\mathcal{A}$ be a Banach algebra with a unit, then $\mathcal{A}$ is semi-simple iff

$$\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0 \implies a = 0.$$

For the second question, we have the following result.

Theorem 5.2.26

Let $\mathcal{A}$ be a Banach algebra with a unit, then Γ is an isomeric isomorphism onto its image iff $\left\|a^2\right\| = \|a\|^2$ for every $a \in \mathcal{A}$.

We will discuss the third question in a C^* algebra setting.

Examples and applications

Algebra of continuous functions Let M be a compact Hausdorff space, then $C(M)$ is a Banach algebra. A natural question is whether the Gelfand representation $\Gamma : C(M) \to C(\mathfrak{M})$ corresponds to the natural isomorphism.

Theorem 5.2.27

Let $\mathcal{A} = C(M)$, then $\mathfrak{M}$ is homeomorphic to M.

Proof. There is a natural bijection between M and $\mathfrak{M}$ is given by

$$x_0 \in M \mapsto J_{x_0} := \{a \in C(M) : a(x_0) = 0\} = \ker(a \mapsto a(x_0)).$$

Notice that the topology on $\mathfrak{M}$ is the weakest topology such that every Γa is continuous. Hence the topology on M is stronger than the topology on $\mathfrak{M}$. But $\mathfrak{M}$ is Hausdorff and M is compact, hence $M \cong \mathfrak{M}$. $\square$

Continuous functions with uniformly convergent Fourier series Now, we consider $\mathcal{A}$ to be the Banach algebra in example 5.2.7. Then $\mathcal{A} \subset C(\mathbb{S}^1)$. A natural question is to ask what $\mathfrak{M}$ is. We can show a same result by a similar argument.

Theorem 5.2.28 $\mathfrak{M} \cong \mathbb{S}^1$.

Proof. The only difference with the previous example is to show that every maximal ideal J is of the form $J_{\theta_0} = \{a \in \mathcal{A} : a(e^{i\theta_0}) = 0\}$. Consider the homomorphism φ_J, because

$$\langle \varphi_J, e^{in\theta} \rangle = \langle \varphi_J, e^{i\theta} \rangle^n, \quad |\langle \varphi_J, e^{in\theta} \rangle| \leqslant 1,$$

we have $|\langle \varphi_J, e^{in\theta} \rangle| \equiv 1$. Hence $\langle \varphi_J, e^{in\theta} \rangle = e^{in\theta_0}$ for some θ_0. It follows $\varphi_J : a \mapsto a(e^{i\theta_0})$. □

As an application of this result, we have

Theorem 5.2.29 (Wiener)
Assume $f \in \mathcal{A}$ such that $f(e^{i\theta}) \neq 0$ for every $e^{i\theta} \in \mathbb{S}^1$, then $1/f \in \mathcal{A}$, i.e. the Fourier series of $1/f$ is uniformly convergent.

§5.3 C^* algebras

Definition 5.3.1. Let $\mathcal{A}$ be an algebra, an operation $* : \mathcal{A} \to \mathcal{A}$ is called an **involution** if

(i) $(a + b)^* = a^* + b^*, \forall a, b \in \mathcal{A}$.

(ii) $(\lambda a)^* = \overline{\lambda} a^*, \forall a \in \mathcal{A}, \lambda \in \mathbb{C}$.

(iii) $(ab)^* = b^* a^*, \forall a, b \in \mathcal{A}$.

(iv) $(a^*)^* = a, \forall a \in \mathcal{A}$.

Example 5.3.2

1. On $C(M)$, $* : \varphi \mapsto \overline{\varphi}$.
2. On $\mathscr{L}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, $* : A \mapsto A^*$.

Definition 5.3.3. Let $\mathcal{A}$ be a Banach algebra with a unit and an involution $*$ on it. $\mathcal{A}$ is called a **C^* algebra** if $\|a^* a\| = \|a\|^2$ for every $a \in \mathcal{A}$.

Example 5.3.4 $C(M), \mathscr{L}(\mathcal{H})$ are both C^* algebras.

Definition 5.3.5. An element $a \in \mathcal{A}$ is called **self-adjoint** if $a^* = a$.

Lemma 5.3.6
Let $\mathcal{A}$ be a C^* algebra, let $a \in \mathcal{A}$. Then:

1. $a + a^*, i(a - a^*), aa^*$ are self-adjoint.
2. a can be written as $a = u + iv$, where u, v are both self-adjoint. Moreover, this decomposition is unique.
3. e is self-adjoint.
4. $a \in G(\mathcal{A})$ iff $a^* \in G(\mathcal{A})$ and $(a^*)^{-1} = (a^{-1})^*$.
5. $\lambda \in \sigma(a)$ iff $\overline{\lambda} \in \sigma(a^*)$.

Proof. The key point is 3, which follows by $e^* = ee^*$ is self-adjoint. □

Lemma 5.3.7
Let $\mathcal{A}$ be a C^* algebra, let $a \in \mathcal{A}$. Then:

1. $\|a^*\| = \|a\|$.
2. If a is self-adjoint, then $\|a^2\| = \|a\|^2$.

Proof. 1 follows by $\|a\|^2 = \|a^* a\| \leqslant \|a^*\| \|a\|$, hence $\|a\| \leqslant \|a^*\|$ and vice versa. □

Theorem 5.3.8 (Gelfand-Naimark)
Let $\mathcal{A}$ be a commutative C^* algebra, then $\Gamma : \mathcal{A} \to C(\mathfrak{M})$ is an onto $*$ isometrical isomorphism, i.e.

(i) $\forall a \in \mathcal{A}, \Gamma a^* = \overline{\Gamma a}$.

(ii) Γ is onto.

(iii) $\forall a \in \mathcal{A}, \|\Gamma a\| = \|a\|$.

Proof. Γ is isometric follows by

$$\|a^2\|^2 = \|(a^2)^* a^2\| = \|(a^* a)^2\| = \|a\|^4.$$

For proving (i), it suffices to show that Γa is real-valued for every self-adjoint element a. Assume that $\Gamma a(J) = \alpha + i\beta$, then for every $t \in \mathbb{R}$,

$$\alpha^2 + (t + \beta)^2 = |\varphi_J(a + ite)|^2 \leqslant \|a + ite\|^2 \leqslant \|a\|^2 + t^2.$$

It is possible only if $\beta = 0$. And (ii) is an immediate corollary of Stone-Weierstrass's theorem. □

§5.4 Operational calculus of (bounded) normal operators

Definition 5.4.1. Let $\mathcal{H}$ be a Hilbert space, $N \in \mathscr{L}(\mathcal{H})$ is called **normal** if $NN^* = N^*N$.

Definition 5.4.2. Given a normal operator N, let $\mathcal{A}_N \subseteq \mathscr{L}(\mathcal{H})$ be the smallest closed C^* algebra containing N and Id.

Now, we fix a normal operator $N \in \mathscr{L}(\mathcal{H})$. Then

$$\mathcal{A}_N = \overline{\{P(N, N^*) : P(x, y) \in \mathbb{C}[x, y]\}}^{\|\cdot\|_{\mathscr{L}(\mathcal{H})}}.$$

The involution is $* : P(N, N^*) \mapsto \overline{P}(N^*, N)$. Hence $\mathcal{A}_N$ forms a commutative C^* algebra. There are several facts of $\mathcal{A}_N$.

Fact 5.4.3. The spectrum of N as an operator on $\mathcal{H}$ corresponds to the spectrum of N in $\mathcal{A}_N$.

Fact 5.4.4. Let $\mathfrak{M}$ be the family of maximal ideals of $\mathcal{A}_N$, then $\mathfrak{M} \cong \sigma(N)$. The continuous map $\psi_0 : J \mapsto \varphi_J(N)$ gives the homeomorphism.

Then $\mathcal{A}_N$ is $*$ isometrically isomorphic onto $C(\sigma(N))$. The isomorphisms can be written explicitly as

$$\widetilde{\Gamma} : \mathcal{A}_N \to C(\sigma(N)), \quad a \mapsto \Gamma a(\psi_0^{-1} z).$$

Note that $\widetilde{\Gamma}\mathrm{Id} = (z \mapsto 1), \widetilde{\Gamma}N = (z \mapsto z), \widetilde{\Gamma}N^* = (z \mapsto \overline{z})$.

Definition 5.4.5. For every $\varphi \in C(\sigma(N))$, we define $\varphi(N) := \widetilde{\Gamma}^{-1}\varphi \in \mathcal{A}_N$.

This definition allows us to consider a continuous function acts on a normal operator. Combining previous discussions, it's not hard to verify the following rules of operational calculus.

- $(\alpha\varphi + \beta\psi)(N) = \alpha\varphi(N) + \beta\psi(N)$.
- $(\varphi\psi)(N) = \varphi(N)\psi(N)$.
- $\varphi(N)^* = \overline{\varphi}(N)$.
- $1(N) = \mathrm{Id}$.
- $z(N) = N$.
- $\overline{z}(N) = N^*$.

Moreover, we can show that

$$\sigma(\varphi(N)) = \varphi(\sigma(N)), \quad (\varphi \circ \psi)(N) = \varphi(\psi(N)).$$

Annotation 5.4.6 A baby case: let N be a normal operator and $P \in \mathbb{C}[x]$ such that $P(\sigma(N)) = 0$. Then $P(N) = 0$. Is this an analogue of Hamilton-Cayley theorem?

Applications

Theorem 5.4.7

Let $N \in \mathscr{L}(\mathcal{H})$ be a normal operator, then

1. N is self-adjoint iff $\sigma(N) \subset \mathbb{R}$.
2. N is positive iff $\sigma(N) \subset \mathbb{R}_{\geqslant 0}$.

Proof. 1. If N is self-adjoint, then $\sigma_{\mathcal{A}_N}(N) \subset \mathbb{R}$. If $\sigma(N) \subset \mathbb{R}$, then $N^* = \overline{z}(N) = N$.

2. If $\sigma(N) \subset \mathbb{R}_{\geqslant 0}$, then $z^{1/2} \in C(\sigma(N))$. Write $N = N^{1/2}N^{1/2}$ and $N^{1/2}$ is self-adjoint. Hence $(Nx, x) = (N^{1/2}x, N^{1/2}x) \geqslant 0$.

   If N is positive and hence self-adjoint. We consider $\varphi_1, \varphi_2 \in C(\mathbb{R})$ where $\varphi_1 = \max\{t, 0\}$ and $\varphi_2 = \max\{-t, 0\}$. Then $N_1 = \varphi_1(N), N_2 = \varphi_2(N)$ are positive operators and $N = N_1 - N_2, N_1N_2 = 0$. We have

$$0 \leqslant (NN_2x, N_2x) = -(N_2^2x, N_2x) \leqslant 0,$$

   hence $(N_2^2x, N_2x) = (N_2^3x, x) \equiv 0$. Then $N_2^3 = 0$, hence

$$\|N\|_2 = \|\Gamma N_2\| = \lim_{n\to\infty} \|N_2^n\|^{\frac{1}{n}} = 0.$$

   That is, $\sigma(N) = \sigma(N_1) \subset \mathbb{R}_{\geqslant 0}$.

□

Corollary 5.4.8

Let N be a normal operator, $\varphi \in C(\sigma(N))$. Then $\varphi(N)$ is self-adjoint iff φ is real-valued, $\varphi(N)$ is positive iff $\varphi \geqslant 0$.

Corollary 5.4.9

Let P be a positive operator, then there exists a unique positive square root $Q \in \mathscr{L}(\mathcal{H})$ such that $Q^2 = P$. Moreover, for every $A \in \mathscr{L}(\mathcal{H})$ commutes with P, A also commutes with Q.

Proof. Because P is positive, hence P is self-adjoint and normal. Let $Q = z^{1/2}(P)$, then $Q \in \mathscr{L}(\mathcal{H})$ such that $Q^2 = P$. The positivity of Q follows from $\sigma(Q) = \left\{z^{1/2} : z \in \sigma(P)\right\}$. Because $Q \in \mathcal{A}_P$, then if A commutes with P, A also commutes with Q.

Now we show the uniqueness of Q. If Q_1 is another positive square root of P, then Q_1 commutes with P and Q. We consider the C^* algebra generated by Id, P, Q, Q_1, denoted by $\mathcal{A}$. Then $\mathcal{A}$ is a commutative C^* algebra. Then the Gelfand representation of $\mathcal{A}$ is an isomorphism. Note that

$$\Gamma(Q)^2 = \Gamma(Q_1)^2, \quad \Gamma(Q), \Gamma(Q_1) \geqslant 0,$$

hence $\Gamma(Q) = \Gamma(Q_1)$. It follows $Q = Q_1$. □

6 Spectral Resolution

The idea of spectral resolution is to generalize the concept of diagonalization in linear algebra to linear operators. In linear algebra, a normal operator A can be written as

$$A = \sum \lambda_i P_i.$$

Where $\{\lambda_i\}$ are eigenvalues of A and $\{P_i\}$ are projections to the corresponding eigenvectors. Naturally, for a normal operator $N \in \mathscr{L}(\mathcal{H})$, we want to write as

$$N = \sum_{\lambda \in \sigma(N)} \lambda P_\lambda.$$

But this in general does not make sense because $\sigma(N)$ might be very large (a non-countable set) and there might no projections in $\mathcal{A}_N$. Hence we expect to write N as an integral of λ with respect to a "projection-valued measure" on $\sigma(N)$, which is called the **spectral resolution**.

§6.1 Projection operators

Let $\mathcal{H}$ be a Hilbert space and $\mathscr{L}(\mathcal{H})$ be bounded linear operators on $\mathcal{H}$.

Definition 6.1.1. $P \in \mathscr{L}(\mathcal{H})$ is called a **projection operator** if P is self-adjoint and idempotent.

Fact 6.1.2. Assume P_1, P_2 are two projection operators. Then P_1P_2 is a projection operator iff P_1 commutes with P_2.

Fact 6.1.3. Assume P_1, P_2 are two projection operators. Then $P_1 + P_2$ is a projection operator iff $P_1P_2 = 0$ or $P_1\mathcal{H} \perp P_2\mathcal{H}$.

Definition 6.1.4. Assume P_1, P_2 are two projection operators. P_1 is called a **part operator** of P_2 if $P_1\mathcal{H} \subset P_2\mathcal{H}$.

Lemma 6.1.5

P_1 is a part operator of P_2 iff one of the following is true:

(1) $P_1P_2 = P_2P_1 = P_1$.

(2) $\forall x \in \mathcal{H}, \|P_1x\| \leqslant \|P_2x\|$.

Theorem 6.1.6

Assume P_1, P_2 are two projection operators. Then $P_2 - P_1$ is a projection operator iff P_1 is a part operator of P_2.

§6.2 Spectral resolution of (bounded) normal operators

Our first aim is to generalize operations on N. That is, not only consider the continuous functions acting on N, we also want to define a Borel measurable function acts on N. Let $B(\sigma(N))$ be the set of bounded Borel measurable functions on $\sigma(N)$, the norm on $B(\sigma(N))$ is defined as $\|\psi\| := \sup_{n\in\sigma(N)} |\psi(z)|$.

Given $x, y \in \mathcal{H}$, we consider the map $C(\sigma(N)) \to \mathbb{C}$, $\varphi \mapsto (\varphi(N)x, y)$. This is a bounded linear functional on $C(\sigma(N))$, because

$$(\varphi(N)x, y) \leqslant \|\varphi(N)\| \|x\| \|y\| = \|\varphi\| \|x\| \|y\| .$$

By Riesz representation theorem, there exists a unique complex Borel measure $m_{x,y}$ on $\sigma(N)$ such that

$$(\varphi(N)x, y) = \int_{\sigma(N)} \varphi(z) \mathrm{d}m_{x,y}(z).$$

Definition 6.2.1. $m_{x,y}$ is called the **spectral measure** of N corresponds to $x, y \in \mathcal{H}$.

Then,

1. The total variation of $m_{x,y}$ as

$$|m_{x,y}| = \int_{\sigma(N)} |\mathrm{d}m_{x,y}(z)| \leqslant \|x\| \|y\| .$$

2. For every Borel subset Ω of $\sigma(N)$, $m_{x,y}(\Omega)$ is sesquilinear with respect to x, y. That is
   - $m_{\alpha x_1 + \beta x_2, y}(\Omega) = \alpha m_{x_1,y}(\Omega) + \beta m_{x_2,y}(\Omega)$.
   - $m_{x, \alpha y_1 + \beta y_2}(\Omega) = \overline{\alpha} m_{x,y_1}(\Omega) + \overline{\beta} m_{x,y_2}(\Omega)$.

3. $m_{x,x}$ is real and non-negative. Because for every $\psi \geqslant 0$, $\psi(N)$ is a positive operator, hence $(\psi(N)x, x) \geqslant 0$. Which shows that $m_{x,x}$ is a non-negative measure.

Now we consider a measurable function $\psi \in B(\sigma(N))$, let

$$a_\psi(x, y) = \int_{\sigma(N)} \psi(z) \mathrm{d}m_{x,y}(z).$$

Then $|a_\psi(x, y)| \leqslant \|\psi\| \|x\| \|y\|$. By Riesz representation theorem, there is a unique linear operator $\psi(N) \in \mathscr{L}(\mathcal{H})$, such that

$$(\psi(N)x, y) = \int_{\sigma(N)} \psi(z) \mathrm{d}m_{x,y}(z).$$

This induces a map $\tau : B(\sigma(N)) \to \mathscr{L}(\mathcal{H})$, $\psi \mapsto \psi(N)$.

Theorem 6.2.2

$\tau : B(\sigma(N)) \to \mathscr{L}(\mathcal{H})$ is a $*$ homomorphism satisfying:

(i) $\tau|_{C(\sigma(N))} = \widetilde{\Gamma}^{-1}$.

(ii) $\|\tau\psi\|_{\mathscr{L}(\mathcal{H})} \leqslant \|\psi\|$ for every $\psi \in B(\sigma(N))$.

Fact 6.2.3. If $\psi_n, \psi \in B(\sigma(N))$, $\psi_n \to \psi$ $m_{x,x} - \text{a.e.}$ for every $x \in \mathcal{H}$, assume that there exists M, $\|\psi\|_n \leqslant M$. Then for every $x \in \mathcal{H}$, $\psi_n(N)x \to \psi(N)x$.

Fact 6.2.4. $C(\sigma(N))N$ is dense in $B(\sigma(N))N$ with respect to the strong operator topology.

Fact 6.2.5. If $A \in \mathscr{L}(\mathcal{H})$ commutes with N, then A commutes with $\psi(N)$, $\forall \psi \in B(\sigma(N))$.

Now, we give a general definition of spectral families. Let X be a locally compact topological space, $\mathscr{B}(X)$ is the Borel σ-algebra on X. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{P}(\mathcal{H})$ be the family of projection operators on $\mathcal{H}$.

Definition 6.2.6. A triple $(X, \mathscr{B}(X), E)$ is called a **spectral family** if the map $E : \mathscr{B}(X) \to \mathcal{P}(\mathcal{H})$ satisfying:

(i) $E(X) = \mathrm{Id}$.

(ii) For every pairwise disjoint Borel subsets $\{A_i\} \subset \mathscr{B}(X)$,

$$E\left(\bigcup_{i=1}^{\infty} A_i\right) = s - \lim_{n\to\infty} \sum_{i=1}^{n} E(A_i),$$

where $s - \lim$ means the strong limit of operators.

Back to our case. Let $N \in \mathscr{L}(\mathcal{H})$ be a normal operator, let $X = \mathbb{C}$ and $\mathscr{B} = \mathscr{B}(\mathbb{C})$ be the Borel σ-algebra on $\mathbb{C}$. We define

$$E : \mathscr{B} \to \mathscr{L}(\mathcal{H}), \quad \Omega \mapsto \tau \mathbb{1}_{\Omega\cap\sigma(N)}.$$

Note that a characteristic function $\mathbb{1}_{\square}$ is real-valued and idempotent, hence $\tau\mathbb{1}_{\square}$ is a projection operator. Indeed, $E : \mathscr{B} \to \mathcal{P}(\mathcal{H})$. Then $(C, \mathcal{B}, E)$ forms a spectral family. In particular, for every $\Omega \in \mathcal{B}(\mathbb{C})$

$$(E(\Omega)x, y) = \int \mathbb{1}_{\Omega\cap\sigma(N)} \mathrm{d}m_{x,y} = m_{x,y}(\Omega \cap \sigma(N)).$$

Besides, for every $z \in \mathbb{C}$, let

$$\Omega_z := \{\alpha + i\beta \in \sigma(N) : \alpha \leqslant \operatorname{Re} z, \beta \leqslant \operatorname{Im} z\}.$$

Let $E(z) := E(\Omega_z)$, then $E(z)$ is a distribution function on $\mathbb{C} \cong \mathbb{R}^2$. In this sense, we can rewrite the integral (with respect to the spectral measure) as a Lebesgue-Stieljes integral. We can reformulate our result by the following spectral resolution theorem.

Theorem 6.2.7 (Spectral resolution of bounded normal operators)

Let N be a normal operator on $\mathscr{L}(\mathcal{H})$. Let $(\mathbb{C}, \mathscr{B}, E)$ be the spectral family we defined before. Then for every $\psi \in B(\sigma(N))$, there exists a unique operator $\psi(N) \in \mathscr{L}(\mathcal{H})$ such that

$$(\psi(N)x, y) = \int_{\sigma(N)} \psi(z)\mathrm{d}(E(z)x, y), \quad \forall x, y \in \mathcal{H}.$$

Denotes by (the integral will be understood in the weak sense)

$$\psi(N) = \int_{\sigma(N)} \psi(z)\mathrm{d}E(z). \tag{6.2.1}$$

Fact 6.2.8. The integral (6.2.1) can also be understood in the uniform sense, i.e. the integral is convergent with respect to the operator norm in the sense of Lebesgue integral.

This fact is a stronger version of spectral resolution. In particular, for every $\psi \in B(\sigma(N))$ and $x \in \mathcal{H}$, we have

$$\psi(N)x = \int_{\sigma(N)} \psi(z)\mathrm{d}E(z)x.$$

Example 6.2.9

- $N = \int_{\sigma(N)} \lambda \mathrm{d}E.$
- $\|\psi(N)x\|^2 = \int |\psi|^2 \mathrm{d}m_{x,x} = \int |\psi(z)|^2 \mathrm{d}(E(z)x, x) = \int_{\sigma(N)} |\psi(z)|^2 \mathrm{d}\,\|E(z)x\|^2.$

Example 6.2.10

Let $A \in \mathscr{L}(\mathcal{H})$ be a self-adjoint operator, then $\sigma(A) \subset \mathbb{R}$. In this case,

$$A = \int_{\sigma(A)} \lambda \mathrm{d}E_\lambda,$$

where $E_\lambda := E((-\infty, \lambda] \cap \sigma(A))$. Then:

1. If $\lambda \leqslant \lambda'$, then $E_\lambda \leqslant E_{\lambda'}$.
2. $E_\lambda = s - \lim_{\lambda' \to \lambda+} E'_\lambda$.
3. $E_a = 0$ and $E_b = I$, where $a = \inf_{\lambda \in \sigma(A)} \lambda$ and $b = \sup_{\lambda \in \sigma(A)} \lambda$.

Example 6.2.11

Let U be a unitary operator, then $\sigma(U) \subset \mathbb{S}^1$. Let $F_\theta := E(e^{i[0,\theta]} \cap \sigma(U))$, then

$$U = \int_0^{2\pi} e^{i\theta} \mathrm{d}F_\theta.$$

§6.3 Spectra of (bounded) normal operators

Let $N \in \mathscr{L}(\mathcal{H})$ be a normal operator, then

$$\sigma(N) = \sigma_p(N) \cup \sigma_c(N) \cup \sigma_r(N) = \sigma_d(N) \cup \sigma_{ess}(N).$$

Let $(\mathbb{C}, \mathscr{B}, E)$ be the corresponding spectral family.

Theorem 6.3.1 $\lambda_0 \in \sigma_p(N)$ iff $E(\{\lambda_0\}) \neq 0$.

Proof. "If": take $x_0 \in E(\{\lambda_0\})\mathcal{H}$, then

$$Nx_0 = \int z \mathrm{d}E(z) x_0 = \lambda_0 x_0.$$

"Only if": take $f_\delta \in B(\sigma(N))$ such that $f_\delta|_{B(\lambda_0,\delta)} = 0$ and $f_\delta(z) = 1/(\lambda_0 - z)$ on $\mathbb{C} \setminus B(\lambda_0, \delta)$. Then

$$E(\mathbb{C} \setminus B(\lambda_0, \delta))x_0 = f_\delta(N)(\lambda_0 \mathrm{Id} - N)x_0.$$

Let $\delta \to 0^+$, we have $E(\mathbb{C} \setminus \{\lambda_0\})x_0 = 0$. Hence $E(\{\lambda_0\})x_0 = x_0$. □

Theorem 6.3.2 $\sigma_r(N) = \varnothing$.

Proof. Assume for a contradiction that $\lambda_0 \in \sigma_r(N)$. Then

$$\ker(\overline{\lambda}_0 \mathrm{Id} - N^*) = R(\lambda_0 \mathrm{Id} - N)^\perp \neq \{0\}.$$

Hence $\overline{\lambda}_0 \in \sigma_p(N^*)$ and by the previous theorem, $E_{N^*}(\{\overline{\lambda}_0\}) \neq 0$. But $E_N(\{\lambda_0\}) = E_{N^*}(\{\overline{\lambda}_0\})$, hence $\lambda_0 \in \sigma_p(N)$. □

Theorem 6.3.3 $\lambda_0 \in \sigma(N)$ iff $E(U) \neq 0$ for every neighborhood U of λ_0.

Proof. "If": by E "supports" on $\sigma(N)$.
"Only if": it suffices to show for $\lambda_0 \in \sigma_c(N)$. Then $(\lambda_0 \mathrm{Id} - N)^{-1}$ is unbounded, take $\|x_n\| = 1$ such that $(\lambda_0 \mathrm{Id} - N)x_n \to 0$. Assume that $E(B(\lambda_0, \delta)) = 0$, then

$$\|(\lambda_0 \mathrm{Id} - N)x_n\|^2 = \int_{\sigma(N)} |\lambda_0 - z|^2 \mathrm{d}\, \|E(z)x_n\|^2 \geqslant \delta^2 \|x_n\|^2 ,$$

a contradiction. □

Theorem 6.3.4
$\lambda_0 \in \sigma_d(N)$ iff λ_0 is an isolated eigenvalue and $\dim \ker(\lambda_0 \mathrm{Id} - N) < \infty$.

Proof. For every $\lambda \in \rho(N)$, let $\delta = d(\lambda, \sigma(N))$, let

$$f_\delta(z) = \begin{cases} 0, & z \in B(\lambda, \delta) \\ (\lambda - z)^{-1}, & z \notin B(\lambda, \delta). \end{cases}$$

Then for every $x \in \mathcal{H}$,

$$\|R_\lambda(N)x\|^2 = \int |f_\delta(z)|^2 \mathrm{d}\, \|E_z x\|^2 \geqslant \frac{1}{\delta^2} \|x\|^2 ,$$

hence $\|R_\lambda(N)\| \leqslant d(\lambda, \sigma(N))^{-1}$. A same argument as in theorem 3.3.2, we can show that $R(P_{\lambda_0}) = \ker(\lambda_0 \mathrm{Id} - N)$. Hence the "If" part follows. If $\lambda_0 \in \sigma_d(N)$, it suffices to show that $\lambda_0 \in \sigma_p(N)$. This follows by $E(U_{\lambda_0}) = E(\{\lambda_0\}) \neq 0$ for a neighborhood U_{λ_0} of λ_0. □

Annotation 6.3.5 I have lost the original proof and this is my own proof. There might be some gaps here.

§6.4 Spectral resolution of self-adjoint operators

Let A be a (maybe unbounded) self-adjoint operator. We consider the Cayley transform $U = (A - iI)(A + iI)^{-1}$, then U is a unitary operator. Let $\{F_\theta\}$ be the spectral family of U, see example 6.2.11. Let $E_\lambda = F_\theta$, where $\lambda = -\cot \frac{\theta}{2}$. Then $\{E_\lambda\}$ gives a spectral family on $\mathbb{R}$. Let $B(\mathbb{R})$ be the family of bounded Borel measurable functions on $\mathbb{R}$. For every $\phi \in B(\mathbb{R})$, consider

$$\phi(A)x := \int_{\mathbb{R}} \phi(\lambda) \mathrm{d}E_\lambda x = \lim_{n \to \infty} \int_{-n}^{n} \phi(\lambda) \mathrm{d}E_\lambda x.$$

This limit is convergent. Then we can define

$$\phi(A) = s - \lim_{n \to \infty} \int_{-n}^{n} \phi(\lambda) \mathrm{d}E_\lambda.$$

The map $\phi(\lambda) \mapsto \phi(A)$ gives a $*$ isometric homomorphism from $B(\mathbb{R})$ to $\mathscr{L}(\mathcal{H})$.

Fact 6.4.1. Assume $\phi_j \in B(\mathbb{R})$, $\|\phi_j\| \leqslant M < \infty$ and $\phi_j \to \phi$ a.e. . Then

$$\phi(A) = s - \lim_{j \to \infty} \phi_j(A).$$

Aim 6.4.2. To generalize the operational calculus to unbounded Borel measurable functions.

Let ϕ be a Borel measurable function on $\mathbb{R}$. A natural idea is truncating the function. Let

$$\phi_n = \phi \mathbb{1}_{\{|\phi| \leqslant n\}},$$

then $\phi_n \to \phi$ a.e. .

Lemma 6.4.3

Let

$$E_\phi := \left\{ x \in \mathcal{H} : \int_{-\infty}^{\infty} |\phi(\lambda)|^2 \mathrm{d} \|E_\lambda x\|^2 < \infty \right\},$$

then E_ϕ is dense in $\mathcal{H}$. Furthermore, for every $x \in E_\phi$, the limit $\lim_{n\to\infty} \phi_n(A)x$ exists.

Proof. Let $F_n = \{|\phi| \leqslant n\}$, then $\mathbb{1}_{F_n} \to \mathbb{1}_{\mathbb{R}}$ a.e. . By $\mathbb{1}_{F_n}(A) = E(F_n)$, we have

$$\int_{-\infty}^{\infty} |\phi(\lambda)|^2 \mathrm{d} \|E_\lambda x\|^2 \leqslant n^2 \|x\|^2, \quad \forall x \in \mathbb{1}_{F_n}(A)\mathcal{H}.$$

Then $\mathbb{1}_{F_n}(A)\mathcal{H} \subset E_\phi$. Besides, $\mathbb{1}_{F_n}(A)x \to x$, hence E_ϕ is dense. □

As a conclusion, we can define

$$\phi(A)x := \lim_{n\to\infty} \phi_n(A)x, \quad \forall x \in E_\phi.$$

There are several properties of operational calculus on Borel measurable functions.

- $\phi(A)$ is a closed densely defined linear operator.
- $a_1\phi_1(A) + a_2\phi_2(A) \subset (a_1\phi_1 + a_2\phi_2)(A)$.
- $\phi_1(A)\phi_2(A) \subset (\phi_1\phi_2)(A)$.
- $\overline{\phi}(A) = \phi(A)^*$.

Fact 6.4.4. If ϕ is a real-valued Borel measurable function, then $\phi(A)$ is self-adjoint.

Example 6.4.5

Consider the Laplacian $-\Delta$, then for every Borel measurable ϕ, we have

$$\phi(-\Delta)u = \int \phi(|\xi|^2)\widehat{u}(\xi)e^{ix\xi}\mathrm{d}\xi,$$

hence the truncation (of $\phi(z) = z$) in this case is indeed a truncation of low frequencies.

Theorem 6.4.6 (von Neumann)

Let A be a self-adjoint operator, then there exists a unique spectral family $(\mathbb{R}, \mathscr{B}(\mathbb{R}), E)$ such that

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda.$$

Proof. Let
$$B=\int_{\mathbb{R}}\lambda\mathrm{d}E_\lambda=\int_0^{2\pi}i\frac{1+e^{i\theta}}{1-e^{i\theta}}\mathrm{d}F_\theta=\int_0^{2\pi}\psi(\theta)\mathrm{d}F_\theta.$$
Then the Cayley transform of B is
$$\int_0^{2\pi}\frac{\psi(\theta)-i}{\psi(\theta)+i}\mathrm{d}F_\theta=\int_0^{2\pi}\frac{2ie^{i\theta}}{1-e^{i\theta}}\frac{1-e^{i\theta}}{2i}\mathrm{d}F_\theta=\int_0^{2\pi}e^{i\theta}\mathrm{d}F_\theta=U.$$
Hence $B=A$. The uniqueness follows from the uniqueness of spectral family of U. □

§6.5 Self-adjoint extensions

Theorem 6.5.1 (von Neumann)
Let A be a closed symmetric operator on $\mathcal{H}$. Then
$$D(A^*)=D(A)\oplus D_+\oplus D_-,$$
where $D_\pm(A):=\ker(A^*\mp i\mathrm{Id})$.

Definition 6.5.2. Let $n_\pm:=\dim D_\pm=\dim\ker(A^*\mp i\mathrm{Id})$, then (n_+,n_-) are called the **deficiency indices** of A, denoted by $\mathrm{def}(A)$.

Fact 6.5.3. A is self-adjoint iff $\mathrm{def}(A)=(0,0)$.

Fact 6.5.4. A has self-adjoint extensions iff $\mathrm{def}(A)=(n,n)$.

Definition 6.5.5. Let $\mathcal{H}$ be a Hilbert space, let $C:\mathcal{H}\to\mathcal{H}$ be a skew-linear map, i.e.
$$C(\alpha x+\beta y)=\overline{\alpha}C(x)+\overline{\beta}C(y).$$
C is called a **conjugate** on $\mathcal{H}$ if C is isometric and $C^2=\mathrm{Id}$.

Theorem 6.5.6 (Von Neumann Criterion)
Let A be a closed symmetric operator on $\mathcal{H}$. If there exists a conjugate C such that $C:D(A)\to D(A)$ and $AC=CA$, then A has self-adjoint extensions.

Proof. If $x\in D_\pm$, by $AC=CA$, we can show that $Cx\in D_\mp$. Hence $C:D_\pm\to D_\mp$ is a isometry. That is, $\mathrm{def}(A)=(n,n)$. □

Annotation 6.5.7 A general method to study a given unbounded operator T. Find a self-adjoint extension S of T. Then applying spectral resolution of S to do some estimate. For example, to show S is semi-bounded from below, to estimate the spectrum of S, etc.

7 Semigroups of Operators

§7.1 Semigroups of operators and generators

Definition 7.1.1. Let X be a Banach space, a family of linear operators $\{T(t) : t \geqslant 0\} \subset \mathscr{L}(X)$ is called a **strongly continuous semigroup (of operators on X)** if

(i) $T(0) = \mathrm{Id}$.

(ii) $T(t)T(s) = T(t+s)$ for every $s, t \geqslant 0$.

(iii) $T(\cdot)x \in C([0,\infty), X)$ for every $x \in X$.

Remark 7.1.2 — The third condition can reduce to $T(t)x \to x(t \to 0^+)$ for every $x \in X$.

Example 7.1.3 (Flow)
Consider the autonomous system in $\mathbb{R}^n$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t), \\ x(0) = x_0, \end{cases}$$

where $A \in M(n, \mathbb{R})$. Then $T(t) : x_0 \mapsto x(t)$ forms a strongly continuous semigroup.

Example 7.1.4
For every $A \in \mathscr{L}(X)$, let

$$e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!},$$

then $\{e^{tA} : t \geqslant 0\}$ forms a strongly continuous semigroup.

Let $\{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup on X. For every $t > 0$, let

$$A_t := \frac{T(t) - \mathrm{Id}}{t}.$$

Definition 7.1.5. Let $\{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup on X, let

$$D(A) := \left\{x \in X : \lim_{t \to 0^+} A_t x \text{ exists}\right\}.$$

For every $x \in D(A)$, let $Ax := \lim_{t \to 0^+} A_t x$. Then A is called the **infinitesimal generator** of $\{T(t) : t \geqslant 0\}$.

Fact 7.1.6. $D(A)$ is dense, because every element of the form

$$x_s = \frac{1}{s}\int_0^s T(t)x\mathrm{d}t$$

is in $D(A)$. [This is a very common method in operator semigroups.]

Theorem 7.1.7

Let $\{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup. Then the infinitesimal generator A is a densely defined closed operator. Furthermore, for every $x \in D(A)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = T(t)Ax = AT(t)x.$$

Remark 7.1.8 — For an infinitesimal generator A, we can denote as $T(t) = e^{tA}$ formally. In fact, $T(t)x = \lim_{\varepsilon \to 0+} \exp(tA_\varepsilon)x$ for every $x \in X$ (not only in $D(A)$).

§7.2 The Hille-Yosida theorem

Definition 7.2.1. Let $\{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup. It is called a **contraction semigroup** if $\|T(t)\| \leqslant 1$ for every $t \geqslant 0$.

If we write $T(t) = e^{tA}$, we can do some formal calculation as

$$\int_0^\infty e^{-\lambda t} T(t) \mathrm{d}t = \int_0^\infty e^{-\lambda t} e^{tA} \mathrm{d}t = (\lambda - A)^{-1} = R_A(\lambda).$$

This enlightens us to consider

$$R_\lambda := \int_0^\infty e^{-\lambda t} T(t) \mathrm{d}t.$$

If $\{T(t) : t \geqslant 0\}$ is a contraction semigroup, then this integral converges on $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$. Furthermore, $\|R_\lambda\| \leqslant (\operatorname{Re} \lambda)^{-1}$.

Lemma 7.2.2

Let A be the generator of a contraction semigroup $\{T(t) : t \geqslant 0\}$, then

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\} \subset \rho(A).$$

Besides, for every $\lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0$,

$$R_\lambda(A) = \int_0^\infty e^{-\lambda t} T(t) \mathrm{d}t.$$

Proof. For every $x \in \mathcal{H}$, we have (by a specific calculation)

$$(\lambda - A)R_\lambda x = \lim_{s \to 0^+} (\lambda - A_s) R_\lambda x = x.$$

When $x \in D(A)$, we have

$$R_\lambda A x = \int_0^\infty e^{-\lambda t} \frac{\mathrm{d}}{\mathrm{d}t} T(t) x \mathrm{d}t = -x + \lambda R_\lambda x.$$

Hence R_λ is the inverse of $\lambda - A$. □

Combining previous discussions, if A is a generator of a contraction semigroup, then

- A is a densely defined closed operator.
- $(0, \infty) \subset \rho(A)$.
- $\|R_A(\lambda)\| \leqslant \lambda^{-1}$, for every $\lambda > 0$.

Theorem 7.2.3 (Hille-Yosida)
A densely defined closed operator A is a generator of a contraction semigroup iff

(i) $(0, \infty) \subset \rho(A)$.

(ii) $\|R_A(\lambda)\| \leqslant \lambda^{-1}$, for every $\lambda > 0$.

Proof. For every $\lambda > 0$, we define $B_\lambda = \lambda^2(\lambda - A)^{-1} - \lambda$. Let $T_\lambda(t) = \exp(tB_\lambda)$, we can show that $T(t) = s - \lim_{\lambda\to\infty} T_\lambda(t)$ is the expected semigroup. □

Theorem 7.2.4
Let $\{T(t) : t \geqslant 0\}$ be a contraction semigroup and A be the generator. Then

$$T(t) = s - \lim_{n\to\infty} \left(\mathrm{Id} - \frac{t}{n}A\right)^{-n}.$$

For a general strongly continuous semigroup, by the resonance theorem,

$$\sup_{0\leqslant t\leqslant 1} \|M(t)\| < \infty.$$

Hence $\|T(t)\| \leqslant Me^{\omega t}$ for some $M, \omega > 0$.

Theorem 7.2.5
A densely defined closed operator A is a generator of a strongly continuous semigroup iff:

(i) There exists $\omega_0 > 0$, such that $(\omega_0, \infty) \subset \rho(A)$.

(ii) There exists $M > 0$, such that for every $\lambda > \omega > \omega_0$,

$$\left\|(\lambda - A)^{-n}\right\| \leqslant \frac{M}{(\lambda-\omega)^n}, \quad \forall n \in \mathbb{N}.$$

§7.3 Examples

Example 7.3.1 (The translation semigroup)
Let $X = C_0([0, +\infty))$ be the space of continuous functions on $[0, \infty)$ vanishing at infinity, let $\|u\| := \sup|u|$. Consider

$$T_1(t) : u(\,\cdot\,) \mapsto u(\,\cdot\, + t), \quad t \geqslant 0.$$

Then $\{T_1(t) : t \geqslant 0\}$ forms a contraction semigroup.

Theorem 7.3.2
Let A_1 be the generator of $\{T_1(t) : t \geqslant 0\}$, then

$$D(A_1) = \left\{u \in X : u' \in X\right\}, \quad A_1 u = u'.$$

Proof. Take $\lambda > 0$, then $R_{A_1}(\lambda)X = D(A_1)$. Besides,

$$R_{A_1}\lambda = \int e^{-\lambda t} T(t) \mathrm{d}t,$$

hence we can show that $D(A_1) \subset \{u \in X : u' \in X\}$. Then we can get the result. □

Example 7.3.3

Let A_2 be a positive self-adjoint operator, then $-A_2$ is a generator of a contraction semigroup. By the spectral resolution, write

$$A_2 = \int_0^\infty \lambda \mathrm{d}E_\lambda.$$

Let

$$T_2(t) = \int_0^\infty e^{-\lambda t} \mathrm{d}E_\lambda, \quad \forall t \geqslant 0,$$

then $\{T_2(t) : t \geqslant 0\}$ forms a contraction semigroup. We can verify that $-A_2$ is the generator of this semigroup.

Example 7.3.4 (The Gaussian semigroup)

Let $X = C_0(\mathbb{R}^n)$, equipped with the L^∞ norm. Then $C_0(\mathbb{R}^n) = \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_{L^\infty}}$. We define

$$T_3(t)u = G_t * u = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u(y) \mathrm{d}y, \quad \forall t > 0,$$

and $T_3(0) = \mathrm{Id}$. Then $\{T_3(t) : t \geqslant 0\}$ forms a contraction semigroup. This can be verified by note that

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \quad \text{and} \quad \widehat{G_t}(\xi) = e^{-4t\pi^2|\xi|^2}.$$

Our next aim is to figure out the infinitesimal generator A_3 of $\{T_3(t) : t \geqslant 0\}$. For every $u \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}u \in \mathcal{S}(\mathbb{R}^n)$. Hence for every $u \in \mathcal{S}(\mathbb{R}^n)$,

$$A_3 u = \mathcal{F}^{-1}(-4\pi^2|\xi|^2(\mathcal{F}u)(\xi)) = \Delta u.$$

In fact, we can show that

Theorem 7.3.5

In $C_0(\mathbb{R}^n)$, the generator of the Gaussian semigroup $\{T_3(t) : t \geqslant 0\}$ is Δ, where

$$D(\Delta) = \{u \in C_0(\mathbb{R}^n) : \Delta u \in C_0(\mathbb{R}^n)\}.$$

Definition 7.3.6. Let A be a densely defined closed operator on X, A is called **dissipative** if, for every $x \in D(A)$, there exists $x^* \in X^*$ such that

$$\|x^*\|^2 = \|x\|^2 = \langle x^*, x \rangle \quad \text{and} \quad \mathrm{Re}\,\langle x^*, Ax \rangle \leqslant 0.$$

Remark 7.3.7 — If $X = \mathcal{H}$ is a Hilbert space, then A is dissipative iff $\mathrm{Re}(x, Ax)_{\mathcal{H}} \leqslant 0$ for every $x \in D(A)$.

Example 7.3.8

Consider Δ on $C_0(\mathbb{R}^n)$, where $D(\Delta) = \{u \in C_0(\mathbb{R}^n) : \Delta u \in C_0(\mathbb{R}^n)\}$. For every $u \in D(\Delta)$, assume $|u(x_u)| = \|u\|$. Let $u^* \in X^*$

$$\langle u^*, v\rangle := u(x_u)\overline{v}(x_u), \quad \forall v \in X.$$

Then $\|u^*\|^2 = \|u\|^2 = \langle u^*, u\rangle$. Besides, note that $\mathrm{Re}\,\Delta|u(x_u)|^2 \leqslant 0$,

$$\mathrm{Re}\,\langle u^*, \Delta u\rangle = \frac{1}{2}\mathrm{Re}\,\Delta|u(x_u)|^2 \leqslant 0 - |\nabla u(x_u)|^2 \leqslant 0.$$

Then Δ is a dissipative operator.

Theorem 7.3.9

A densely defined closed operator A is a generator of a contraction semigroup iff A is dissipative and there exists $\lambda_0 > 0$ such that $R(\lambda_0 - A) = X$.

Proof. A generator of a contraction semigroup is always dissipative. It suffices to show the inverse direction. Assume A is dissipative, then for every $\lambda > 0, x \in D(A)$

$$\|(\lambda - A)u\| \geqslant \frac{1}{\|u^*\|}\mathrm{Re}\,\langle u^*, (\lambda - A)u\rangle \geqslant \lambda\,\|u\|.$$

Hence $R(\lambda - A)$ is closed in X. Besides $R(\lambda_0 - A) = X$, and we can show that for every $|\lambda - \lambda_0| < \lambda_0$, λ is invertible. Hence we can extends to every $\lambda \in \mathbb{R}$ that $R(\lambda - A) = X$. This satisfies the conditions in the Hille-Yosida theorem. □

§7.4 The Stone theorem and its applications

Definition 7.4.1. Let $\mathcal{H}$ be a Hilbert space, $\{U(t) : t \in \mathbb{R}\}$ is called a **(strongly continuous) unitary group** if

(i) $U(t)$ is a unitary operator on $\mathcal{H}$ for every $t \in \mathbb{R}$.

(ii) $U(t)U(s) = U(t+s)$ for every $s, t \in \mathbb{R}$.

(iii) $U(\cdot)x \in C(\mathbb{R}, \mathcal{H})$ for every $x \in \mathcal{H}$.

Theorem 7.4.2

Let $\mathcal{H}$ be a Hilbert space and $\{T(t) : t \geqslant 0\}$ is a contraction semigroup, let A be its generator. Then $\{T(t)^* : t \geqslant 0\}$ is also an contraction semigroup with the generator $B = A^*$.

Proof. It suffices to show $B = A^*$. For every $x \in D(A)$ and $y \in D(B)$, we have

$$(Ax, y) = \lim_{t\to 0}\left(\frac{T(t)x - x}{t}, y\right) = \lim_{t\to 0}\left(x, \frac{T(t)^*y - y}{t}\right) = (x, By),$$

hence $D(B) \subset D(A^*)$. Besides, for every $x \in D(A)$ and $y \in D(A^*)$, we have

$$(T(t)x - x, y) = \int_0^t (AT(s)x, y)\mathrm{d}s = \int_0^t (x, A^*T(s)^*y)\mathrm{d}s,$$

hence $D(A^*) \subset D(B)$. Which follows $A^* = B$. □

Remark 7.4.3 — Let $\{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup of unitary operators, let

$$U(t) = \begin{cases} T(t), & t \geqslant 0; \\ T(-t)^*, & t < 0. \end{cases}$$

Then $\{U(t) : t \in \mathbb{R}\}$ forms a strongly continuous unitary group.

Remark 7.4.4 — The condition of strongly continuous can be replaced by a weak continuity. That is, for every $x, y \in \mathcal{H}$, the map $t \mapsto (U(t)x, y)_{\mathcal{H}}$ is continuous. For a weak convergence and a norm convergence can imply the strong convergence in the Hilbert space.

Proof. Assume that

$$\|x_n\| \to \|x\|, \quad x_n \xrightarrow{w} x.$$

WOLG, assume that $\|x_n\| = \|x\| = 1$. If $x_n \not\to x$, by passing to a subsequence if necessary, assume that $\|x_n - x\| > \varepsilon > 0$. Then

$$2\operatorname{Re}(x_n, x)_{\mathcal{H}} = \|x_n\|^2 + \|x\|^2 - \|x_n - x\|^2 < 2 - \varepsilon^2,$$

which cannot converge to $2\operatorname{Re}(x, x)_{\mathcal{H}} = 2$. □

Remark 7.4.5 — If further assume that $\mathcal{H}$ is separable, then the condition can be reduced to a weakly measurable condition. That is, for every $x, y \in \mathcal{H}$, the map $t \mapsto (U(t)x, y)_{\mathcal{H}}$ is measurable.

Theorem 7.4.6 (Stone)

A densely defined closed operator B is a generator of a unitary group iff there exists a self-adjoint operator A such that $B = iA$. In this case,

$$U(t) = \exp(itA) = \int_{-\infty}^{\infty} e^{it\lambda}\mathrm{d}E_\lambda,$$

where $\{E_\lambda : \lambda \in \mathbb{R}\}$ is the spectral family of A.

Proof. If B is the generator, then

$$Bx = \lim_{t\to 0^+}\frac{U(t)x - x}{t} = \lim_{t\to 0^+} U(t)\frac{x - U(t)^*x}{t} = -B^*x.$$

Hence $B = iA$ where A is self-adjoint. The inverse direction can be verified by an explicit construction applying the operational calculus. □

Applications

The Schrödinger equation Consider the Schrödinger equation

$$i\hbar\partial_t\psi = H\psi, \quad H = -\frac{\hbar^2}{2m}\Delta + V.$$

Where $\psi(t,x)$ refers to a wave function, $|\psi(t,x)|^2$ means the probability density of the particle occurs at the position x at time t. Then

$$\int_{\mathbb{R}^3} |\psi(t,x)|^2 \mathrm{d}x = 1.$$

Define

$$U(t) : \psi(0,x) \mapsto \psi(t,x),$$

then $U(t)$ is isometric. Hence $\{U(t) : t \in \mathbb{R}\}$ forms a unitary group. We always assume that $U(t)$ is weakly measurable, then H must be self-adjoint and

$$U(t) = \exp\left(-i\frac{1}{\hbar}Ht\right).$$

The ergodic theorem

Theorem 7.4.7 (von Neumann)
Let $\{U(t) : t \in \mathbb{R}\}$ be a strongly continuous unitary group. Let

$$\mathcal{H}_0 := \{y \in \mathcal{H} : U(t)y = y, \forall t \in \mathbb{R}\}$$

be a closed subspace of $\mathcal{H}$. Let $P : \mathcal{H} \to \mathcal{H}_0$ be the orthogonal projection, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T U(t)x\mathrm{d}t = Px, \quad \forall x \in \mathcal{H}.$$

Proof. By the Stone theorem, we have

$$U(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\mathrm{d}E_\lambda.$$

Then

$$\frac{1}{T}\int_0^T e^{i\lambda t}\mathrm{d}t = \begin{cases} \dfrac{e^{i\lambda T}-1}{i\lambda T}, & \lambda \neq 0; \\ 1, & \lambda = 0. \end{cases}$$

Which follows

$$\left\|\frac{1}{T}\int_0^T U(t)x\mathrm{d}t - E_{\{0\}}x\right\|^2 \to 0.$$

It suffices to show that $P = E_{\{0\}}$. For every $x \in \mathcal{H}$, we have

$$U(t)E_{\{0\}}x = \int_{-\infty}^{\infty} e^{i\lambda t}\mathrm{d}E_\lambda E_{\{0\}}x = E_{\{0\}}x,$$

hence $R(E_{\{0\}}) \subset \mathcal{H}_0$. Besides, for every $x \in \mathcal{H}_0$, it is obvious that

$$x = \lim_{T\to\infty}\frac{1}{T}\int_0^T U(t)x\mathrm{d}t = E_{\{0\}}x.$$

Hence the statement follows. □