Measure rigidity for diagonalizable actions (Manfred Einsiedler, Winter 2024)

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Contents

§1 Lecture 1

Theorem 1.1 (Furstenberg)

Let $A \subset \mathbb{T}$ be a closed and $\times 2, \times 3$-invariant set. Then

- $\# A < \infty$ consisting of periodic points, or
- $A = \mathbb{T}$.

Conjecture 1.2 (Furstenberg)

Let μ be an invariant probability measure for the joint $\times 2, \times 3$-action that is ergodic. Then

- $\# \operatorname{supp} \mu < \infty$, or
- $\mu = m_{\mathbb{T}}$ the Lebesgue measure.

Theorem 1.3 (Rudolph)

Let μ be $\times 2, \times 3$-invariant ergodic probability measure. If $h_\mu(\times 2) > 0$ (or $h_\mu(\times 3) > 0$, or $\dim \mu > 0$), then $\mu = m_{\mathbb{T}}$.

Theorem 1.4 (Einsiedler-Katok-Lindenstrauss, 2005)

Let $A = \left\{ \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \right\} \subset \mathrm{SL}(3, \mathbb{R})$ act on $X_3 = \mathrm{SL}(3, \mathbb{R}) / \mathrm{SL}(3, \mathbb{Z})$. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then $\mu = m_{X_3}$ is the uniform measure.

Theorem 1.5 (Lindenstauss, 2003)

Let $A = \left\{ \begin{bmatrix} * & \\ & * \end{bmatrix} \times \begin{bmatrix} * & \\ & * \end{bmatrix} \right\} \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ act on $X = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Gamma$ with Γ irreducible. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then $\mu = m_X$.

Theorem 1.6 (Einsiedler-Lindenstrauss, 2023)

Let $A \subset \mathrm{SL}(2,\mathbb{R})^k$ be isomorphic to $\mathbb{R}^2$ and $\mathbb{R}$-diagonalizable. Let $\Gamma < \mathrm{SL}(2,\mathbb{R})^k$ be irreducible and $X = \mathrm{SL}(2,\mathbb{R})^k/\Gamma$. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then

- μ is homogeneous with semisimple stabilizer, or
- X is non-compact and μ is invariant under a unipotent flow, and supported on an orbit of a solvable group.

Example 1.7

Let $K = \mathbb{Q}(\sqrt{3}) \hookrightarrow \mathbb{R} \times \mathbb{R}$ and $\mathbb{Z}[\sqrt{3}] \hookrightarrow \mathbb{R} \times \mathbb{R}$ which gives an irreducible lattice. Then $\mathrm{SL}(2,\mathbb{Z}[\sqrt{3}])$ also gives an irreducible lattice in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We consider the unipotent subgroup $U = \left\{ \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \times \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \right\}$. Then $U\Gamma \cong \mathbb{R}^2/\mathrm{Galois}(\mathbb{Z}[\sqrt{3}]) \cong \mathbb{T}^2$. This gives an example for the second case in the theorem. To understand these cases, we should classify invariant measures on tori.

Theorem 1.8 (Einsiedler-Lindenstrauss, 2023)

Let $A = \left\{ \begin{bmatrix} h & \\ & h^{-1} \end{bmatrix} : h \in \mathbb{Q} \right\} < \mathrm{SL}(2,\mathbb{A})$ where $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$ is the adel. Let μ be an A-invariant ergodic probability measure on $X_{\mathbb{A}} = \mathrm{SL}(2,\mathbb{A})/\mathrm{SL}(2,\mathbb{Q})$. Then

- $\mu = m_{X_{\mathbb{A}}}$, or
- μ is the uniform Haar measure on a periodic orbit of a unipotent subgroup, or
- μ is the Dirac measure on a fixed point.

§2 Lecture 2

Leafwise measures. We consider the leafwise measure on $X = G/\Gamma$ with respect to $H < G$: a measure μ_x^H on H for almost every $x \in X$ so that the conditional measure of $\mu|_{\mathrm{box}}$ on the local pieces of H-orbits can be obtained by

$$(\mu|_{\mathrm{box}})^{\mathcal{A}^H_{\mathrm{box}}}_{V_x \cdot x} = \frac{1}{\mu_x^H(V_x)} (\mu_x^H|_{V_x}) \cdot x,$$

where box is a "rectangle" (product of H-direction and some transverse direction) on X, $\mathcal{A}^H_{\mathrm{box}}$ is the σ-algebra whose atoms are pieces of H-orbits, $h \mapsto h \cdot x$ gives the map from $V_x \subset H$ to the box.

Fubini-construction of leafwise measure. Define $\widetilde{X} = X \times H$ equipped with $\mu \times m_H$. Let $\mathcal{A}_H$ be the preimage of $\mathcal{B}_X$ under $(x_0, h_0) \mapsto h_0^{-1} x_0 \in H$. The atom $[(x_0, h_0)]_{\mathcal{A}_H} = \Delta_H(x_0, h_0)$ where $\Delta(h)(x_0, h_0) := (hx_0, hh_0)$.

Multiplying by a density function $f_0 \in L^1(H)$. Taking conditional measure and dividing by the density we create a Radon measure (somehow the conditional measure of the infinite measure $\mu \times m_H$) on the Δ_H-orbits

$$(\mu \times m_H)^{\mathcal{A}_H}_{(x_0, h_0)}.$$

Projected to H, we obtain μ_x^H. Moreover, the h_0-coordinate is only relevant for the position of $\Delta_H(x_0, h_0)$.

Compatibility of leafwise measures: If $x, h \cdot x \in X$ for some $h \in H$, then $\mu_{hx}^H h \propto \mu_x^H$.

Entropy. Let $a \in G$ be diagonalizable preserving μ. Let $U < G_a^+$ be normalized by a. Then we can look at μ_x^U and these relate to entropy:

$$h_\mu(a, U) = \lim_{n \to \infty} \frac{1}{n} \log \mu_x^U(a^n B_1^U a^{-n}).$$

On the other hand, the ergodic theory also gives

$$h_\mu(a, U) = \lim_{n \to \infty} -\frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n).$$

These two inequality tell us a phenomenon: the global growth rate of the measure of a U-ball equals the local dimension of μ.

There are also several properties:

- If $U = G_a^+$ then $h_\mu(a) = h_\mu(a, U)$.
- If $h_\mu(a, U) = 0$ then $\mu_x^U = \delta_e$.
- If $h_\mu(a, U) = h_{m_X}(a, U)$ is maximal, then μ is U-invariant.

Product structure of leafwise measures. If $G_a^+ = U_{\alpha_1} \cdots U_{\alpha_n}$ is a direct product of root groups, then

$$\mu_x^{G_a^+} \propto \mu_x^{\alpha_1} \times \cdots \times \mu_x^{\alpha_n} \quad \text{a.s.}.$$

In particular, $h_\mu(a) = \sum h_\mu(a, U_{\alpha_i})$.

Idea of the proof. Say $G_a^+ = U_\alpha U_\beta$. Assume that we can distinguish U_α, U_β by some $b \in A$: b commutes with U_α but $U_\beta \subset G_b^-$. Choose $x \in X$ and elements u_α, u_β. We aim to show that the conditional measure $\mu_x^{U_\alpha}$ is proportion to an appropriate translation of $\mu^{U_\alpha}_{u_\alpha u_\beta x}$.

We iteration them by b. We have $\mu_x^\alpha = \mu_{b^n x}^\alpha$. Assume $b^n x \to y$ as $n \to \infty$. Applying Luzin's theorem, we can assume the conditional measures are continuous on a large set. Then $\mu_{b^n x}^\alpha \to \mu_y^\alpha$, where $y \in U_\alpha x$ because of the choice of b. Then we get the product structure. □

§3 Lecture 3

Symmetry of entropy contributions. If α have $-\alpha$ have unequal entropy contributions, then μ is invariant under a nontrivial unipotent subgroup of U_α or $U_{-\alpha}$.

All statement made for entropy and contributions also work conditionally over a factor of the action (in another word, conditioned on an A-invariant σ-algebra). We use $\mathcal{A}_\alpha$ to denote the σ-algebra generated by $x \mapsto \mu_x^\alpha$.

What is the leafwise measure for U_β conditioned on $\mathcal{A}_\alpha$: $\mu_x^{\beta|\mathcal{A}_\alpha}$ describes $\mu_x^{\mathcal{A}_\alpha}$ along U_β-orbits. Then $\mu_x^{\beta|\mathcal{A}_\alpha} = \mu_x^\beta$ because of the product structure for $U_\alpha U_\beta$.

We consider the diagram with three roots α, β, γ on the plane. Recall the entropy contribution formula (assume that $a \in A$ is chosen that $h_\mu(a) > 0$ and α, β contributes to $h_\mu(a)$, γ contributes to $h_\mu(a^{-1})$)

$$\begin{aligned} h_\mu(a) &= h_\mu(a, U_\alpha) + h_\mu(a, U_\beta) \\ = \; h_\mu(a^{-1}) &= h_\mu(a^{-1}, U_\gamma). \end{aligned}$$

For conditional entropies,

$$\begin{aligned} h_\mu(a|\mathcal{A}_\alpha) &= h_\mu(a, U_\alpha|\mathcal{A}_\alpha) + h_\mu(a, U_\beta) \\ = \; h_\mu(a^{-1}) &= h_\mu(a^{-1}, U_\gamma). \end{aligned}$$

This tells us $h_\mu(a, U_\alpha) = h_\mu(a, U_\alpha|\mathcal{A}_\alpha)$. By the assumption, we have $h_\mu(a, U_\alpha) > 0$. Therefore, $h_\mu(a, U_\alpha|\mathcal{A}_\alpha) > 0$. This means that within the same $\mathcal{A}_\alpha$-atom, we can find pairs of different points on the same U_α-orbit: $x, u_\alpha x$, where $u_\alpha \neq e$. This gives $\mu_x^\alpha = \mu_{u_\alpha x}^\alpha$. Then we obtain some translation invariance of μ_x^α.

Non-maximal torus actions. Our next goal is to show the following:

Theorem 3.1 (Einsiedler-Lindenstrauss, 2023)
$X = \mathrm{SL}(2, \mathbb{R})^k/\Gamma$ and Γ is irreducible (arithmetic). Let $A \subset \mathrm{SL}(2, \mathbb{R})^k$ be isometric to $\mathbb{R}^2$ and diagonalizable. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$, then μ has nontrivial unipotent invariance.

Let $\mathrm{SL}(2, \mathbb{R})^k = G_1 \times G_2 \times G_3$ satisfy that $a \neq e \in G_1, b \neq e \in G_2$ are contained in A. Let $U = U_\alpha = G_a^+$.

Recall that $h_\mu(a) > 0$ tells us μ_x^U is nontrivial with a growth rate. In Lindenstrauss's low entropy method, he used a fact that μ is U-recurrent iff μ_x^U is infinite. We now have a quantitative version of μ_x^U is infinite. So we expect to show that μ satisfies a quantitative recurrence statement for U.

The idea is the following. If cover the space by r^{-d} balls of radius r. By Kac's lemma, for each r-ball, the points that don't return within $r^{-d-\varepsilon}$ has the measure less than $r^{d+\varepsilon}$. So that the total measure of non-recurrent points in the r^{-d} ball's is at most r^ε. We take $r = e^{-n}$ and apply Borel-Cantelli lemma. We obtain a polynomial recurrence.

For the actual practice, we should combine this philosophy with the nontrivial growth of leafwise measures to obtain a similar polynomial recurrence statement. A precise statement is as the following: given $B \subset G/\Gamma$, we have

$$\mu\left\{ x \in B : \mu_x^U \text{ has nontrivial growth rate and does not return within } a^n B_2^{U_\alpha} a^{-n} \right\} \leqslant e^{-h_\mu(a, U_\alpha)n}.$$

Now we want to show $h_\mu(b) > 0$. We assume for the purpose of a contradiction that $h_\mu(b) = 0$. By Brin-Katok, the entropy is also

$$h_\mu(b) = \lim_{n\to\infty} \frac{1}{2n} \log \mu(\text{Bowen } n\text{-ball for two sided map defined by } b).$$

Here two sided Bowen ball at x is $D_n \cdot x := \left(\bigcap_{k=-n}^n b^k B_\varepsilon^G b^{-k}\right) \cdot x$. The zero entropy shows that the measures of Bowen balls are not decay so fast. We will combine this with the recurrence argument to obtain a contradiction.

Using these ideas we obtain: for μ-almost every x and all sufficiently large n (depending on x) we have $e^{\frac{1}{2}h_\mu(a,U_\alpha)n}$-many different returns within $a^n B_2^{U_\alpha} a^{-n}$ to $D_{100n} \cdot x$.

Write $x = g\Gamma$. Then we have $ug = hg\gamma$, where $u \in a^n B_2^{U_\alpha} a^{-n}$ and $h \in D_{100n}$. Now we need to use the arithmeticity of Γ. The heights of the γ responsible for the return is $\ll e^{2n}$.

Claim 3.2. All γ commute.

Proof. Because $[\gamma_1, \gamma_2]$ has height $\ll e^{8n}$ and $\|[\gamma_1, \gamma_2] - \mathrm{id}_{G_2}\| \ll e^{-200n}$. □

There are two cases:

- γ's are unipotent, then γ must be identity. But we have several returns, we obtain a contradiction.
- γ's are diagonalizable: too many lattice elements, a contraction.