Group actions and rigidity: around Zimmer program

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These notes involve four minicourses of the research school
Group actions and rigidity: around Zimmer program

- **Random processes on symmetric spaces and discrete groups of semisimple Lie groups**
  by Mikolaj Fraczyk
- **Basics on measure rigidity**
  by Aaron Brown
- **Margulis-Zimmer's super-rigidity**
  by Homin Lee
- **Space of actions of groups on the real line**
  by Bertrand Deroin

Contents

1 Random processes on symmetric spaces and discrete groups of semisimple Lie groups (Mikolaj Fraczyk)

Notation 1.0.1. • G real semisimple Lie group with the trivial center.
- K maximal compact subgroup.
- P minimal paraboli.
- A maximal split torus.
- N unipotent radical of P.
- $X = G/K$ the symmetric space.
- d left invariant metric on X (and G with $d(g,h) = d(gK, gK)$).

§1.1 Confined subgroups in higher rank

Definition 1.1.1. A discrete subgroup $\Lambda \subset G$ is **confined** if there exists a bounded set $W \supset \{1\}$ such that $\Lambda^g \cap W \supsetneq \{1\}$ for every $g \in G$, where $\Lambda^g := g^{-1}\Lambda g$.

Remark 1.1.2 Λ is confined iff $\Lambda \backslash X$ has uniformly bounded injective radius.

Exercise 1.1.3. (1) Lattices in G are always confined.
(2) Non-trivial normal subgroups of lattices are confined.
(3) If G is of real rank 1 then there are plenty of other examples.

In the higher rank case, one of Margulis's results state that every normal subgroup of a lattice is either of finite index (hence also a lattice) or contained in the center (hence trivial in our case).

Conjecture 1.1.4 (Margulis)
If G is higher rank simple and Λ is confined then it is a lattice.

Theorem 1.1.5 (Fraczyk-Gelander, 2023) The conjecture is true.

Space of subgroups of G. Let

$$\mathrm{Sub}(G) := \{ H \leqslant G : H \text{ is a closed subgroup of } G \},$$

equipped with the topology induced by Hausdorff convergence on bounded subsets. Then $\mathrm{Sub}(G)$ is a compact metrizable space. G acts continuously on $\mathrm{Sub}(G)$ by conjugations $\Lambda^g := g^{-1}\Lambda g$ for every $g \in G$. We also consider

$$\mathrm{Sub}_{\mathrm{d}}(G) := \{ \Lambda \in \mathrm{Sub}(G) : \Lambda \text{ is discrete} \}.$$

Fact 1.1.6. $\mathrm{Sub}_{\mathrm{d}}(G)$ is not closed.

Exercise 1.1.7. Consider $\Lambda_t = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix}\begin{bmatrix} 1 & \mathbb{Z} \\ & 1 \end{bmatrix}\begin{bmatrix} e & \\ & e^{-t} \end{bmatrix}$, then $\Lambda_t \to \begin{bmatrix} 1 & \mathbb{R} \\ & 1 \end{bmatrix}$.

Exercise 1.1.8. (1) Show that if $\Lambda_n \to H$ with $\Lambda_n \in \mathrm{Sub}_{\mathrm{d}}(G)$ then H^0 (the identity compenent) is solvable.
(2) Show that Λ is confined iff $\{1\} \notin \overline{\{\Lambda^g : g \in G\}}$.

Lemma 1.1.9 (Local rigidity lemma)

If G is simple higher rank and $\Gamma \subset G$ is a lattice then any $\Lambda \in \mathrm{Sub}(G)$ close enough to Γ is also a lattice.

Proof. Let S be a finite generating set of Γ and $\mathcal{R}$ be a finite set of relations such that $\Gamma = \langle S | \mathcal{R} \rangle$. Write $S = \{s_1, \cdots, s_k\}$. Choose $r > 0$ such that $S \subset B(r)$. Let $\delta > 0$ such that $\Gamma \cap B(\delta) = \varnothing$. Let $\varepsilon > 0$ such that for every $s_i' \in G$ with $d(s_i', s_i) < \varepsilon$ we have $w(s_1', \cdots, s_k') \in B(\delta/2)$ for every $w \in \mathcal{R}$. If Λ is close enough to Γ, then

(1) there exists $S' \subset \Lambda$ with $S' = \{s_1', \cdots, s_k'\}$ and $d(s_i', s_i) < \varepsilon$, and
(2) $\Lambda \cap (B(\delta) \setminus B(\delta/2)) = \varnothing$.

We can take δ small enough such that $B(\delta)$ contains no compact subgroup. Then $\Lambda \cap (B(\delta) \setminus B(\delta/2)) = \varnothing$ implies that $\Lambda \cap B(\delta/2) = \{1\}$. Hence Λ contains a copy of a quotient of Γ. By Margulis's super-rigidity and Margulis's normal subgroup theorem, Λ is a lattice. $\square$

§1.2 Stationary random subgroups (I)

Let μ be a probability measure on G. We assume that μ is bi-K-invariant and absolutely continuous with respect to Haar. Let $G \curvearrowright Z$ be a continuous action.

Definition 1.2.1. A measure ν on Z is **μ-stationary** if $\nu = \mu * \nu$. The action $G \curvearrowright (Z, \nu)$ is stationary if ν is.

Definition 1.2.2. A **stationary random subgroup** (resp. **discrete stationary random subgroup**) of G is a stationary probability measure on $\mathrm{Sub}(G)$ (resp. $\mathrm{Sub}_{\mathrm{d}}(G)$).

Example 1.2.3

1. $\nu = \delta_{\{1\}}$.
2. If $\Gamma < G$ is a lattice, let $\nu_\Gamma := \frac{1}{\mathrm{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \delta_{\{\Gamma^g\}} \mathrm{d}g$, which is G-invariant.
3. If Q is a parabolic subgroup (contains a conjugate of P) then there exists a unique K-invariant probability measure ν_Q on G/Q. Let $\widetilde{\nu}_Q := \int_{G/Q} \delta_{Q^{g^{-1}}} \mathrm{d}\nu_Q(gQ)$, which is μ-stationary.

Theorem 1.2.4 (Fraczyk-Gelander, 2023)

An ergodic discrete μ-stationary random subgroup of G is either the trivial one or the ν_Γ induced by some lattice Γ.

How to turn Λ into a stationary random subgroup? For a discrete subgoup Λ, consider

$$\nu_n = \frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_\Lambda.$$

Let ν be a weak* limit of $\{\nu_n\}$. Then ν is μ-stationary.

> **Question 1.2.5.** Can ν be a non-discrete stationary random subgroup?

We define the function

$$I : \mathrm{Sub}(G) \to [0, +\infty],\ H \mapsto \sup\{r \geqslant 0 : B(r) \cap H = \{1\}\}.$$

> **Exercise 1.2.6.** If $I(\Lambda) > 0$ then Λ is discrete.

To show ν is supported on $\mathrm{Sub}_{\mathrm{d}}(G)$, we make use of **Margulis function**.

Theorem 1.2.7 (Gelender-Levit-Margulis)
For a specific μ, there exists $\delta > 0$ such that $I^{-\delta}$ satisfies

$$\int_G I(\Lambda^g)^{-\delta} \mathrm{d}\mu(g) \leqslant c \cdot I(\Lambda)^{-\delta} + C$$

for some $0 < c < 1$ and $C > 0$, for every $\Lambda \in \mathrm{Sub}_{\mathrm{d}}(G)$.

By this inequality, we have

$$\int_G I(\Lambda^g)^{-\delta} \mathrm{d}\mu^{*k}(g) \leqslant c^k I(\Lambda)^{-\delta} + c^{k-1}C + \cdots + C \leqslant O(1).$$

By Markov's inequality,

$$\nu_n(\{\Lambda' \in \mathrm{Sub}(G) : I(\Lambda') < \varepsilon\}) \ll \varepsilon^\delta.$$

Taking the limit, we obtain that $\nu(\mathrm{Sub}(G) \setminus \mathrm{Sub}_{\mathrm{d}}(G)) = 0$. Therefore we obtain

Theorem 1.2.8
If $\Lambda \subset G$ is discrete, then any weak* limit of $\left\{\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_\Lambda\right\}$ is supported on $\mathrm{Sub}_{\mathrm{d}}(G)$.

Now we show that the classification of discrete stationary random subgroups (Theorem 1.2.4) implies Margulis's conjecture (Theorem 1.1.5). Start with a discrete confined subgroup $\Lambda \subset G$. Turn it into a discrete stationary random subgroup ν supported on $\overline{\{\Lambda^g : g \in G\}}$. Since Λ is confined, the ergodic decomposition of ν is of the form

$$\nu = \sum_{\Gamma \text{ lattice}} \alpha_\Gamma \cdot \nu_\Gamma.$$

Then there exists a lattice Γ with $\alpha_\Gamma \neq 0$. Therefore, $\Gamma \in \overline{\{\Lambda^g : g \in G\}}$ and hence Λ is a lattice by the local rigidity lemma (Lemma 1.1.9). □

Proof of Theorem 1.2.4. The key ingredient to show this classification is the following theorem.

Theorem 1.2.9 (Nevo-Zimmer)
Suppose (Y, ν) is an ergodic μ-stationary G-action. Then either
(1) ν is G-invariant, or
(2) there exists a G-equivariant and measure preserving $\pi : (Y, \nu) \to (G/Q, \nu_Q)$ with $Q \neq G$ a parabolic subgroup, or
(3) (if G is semisimple) the action factors through a rank-1 factor of G, that is, $G = G_1 \times G_2$ with rank $G_1 = 1$ and G_2 acts trivially.

Assume ν is an ergodic discrete stationary random subgroup. By Nevo-Zimmer's theorem, we have either
(1) ν is G-invariant, or
(2) $\pi : (\mathrm{Sub}_{\mathrm{d}}(G), \nu) \to (G/Q, \nu_Q)$.

Case (1). We use

Theorem 1.2.10 (Stuck-Zimmer)
If (Y, ν) is an ergodic probability measure preserving action of a higher rank simple G, then either
(1) the action is essentially free, i.e. $\mathrm{Stab}_G(y) = \{1\}$ for almost every $y \in Y$, or
(2) $(Y, \nu) \cong (G/\Gamma, \mathrm{Haar})$ for some lattice Γ.

But any $H \in \mathrm{Sub}(G)$ is stabilized by $N(H) \supset H$. So it can't be essentially free. Therefore $\nu = \nu_\Gamma$ for some Γ.

§1.3 Stationary random subgroups (II)

Case (2). There is $\pi : (\mathrm{Sub}_{\mathrm{d}}(G), \nu) \to (G/Q, \nu_Q)$ for some paraboli $Q \neq G$.

Furstenberg-Poisson boundary & decomposition. Let X_n be a random walk on G driven by μ with X_0. That is,

$$\mathbf{P}(X_{n+1} \in A | X_n) = \mu(A X_n^{-1}).$$

It induces a probability measure on $G^{\mathbb{N}}$ (with the initial law $X_0 \sim \mu$), also denoted by $\mathbf{P}$. For two elements $\xi, \xi' \in G^{\mathbb{N}}$, we define the equivalence relation $\xi \sim \xi'$ if there exists $n, m \geqslant 0$ such that $X_{n+k} = X'_{m+k}$ for every $k \geqslant 0$.

Definition 1.3.1. **Poisson boundary** for (G, μ) is probability space $(B, \tau) := (G^{\mathbb{N}}, \mathbf{P}) / \sim$.

Definition 1.3.2. For a given probability measure μ on G, a bounded function $f \in L^\infty(G)$ is **harmonic** if

$$\int f(gu) \mathrm{d}\mu(g) = f(u), \quad \forall u \in G.$$

The set of bounded harmonic function is denoted by $\mathcal{H}^\infty(G, \mu)$.

Using martingale convergence theorem, for every $f \in \mathcal{H}^\infty(G, \mu)$,

$$f(X_n) \to F(\xi), \quad \text{almost every } \xi = (X_0, X_1, \cdots).$$

The assignment

$$f \in \mathcal{H}^\infty(G, \mu) \to F \in L^\infty(B, \tau)$$

is a G-equivariant isomorphism.

Theorem 1.3.3 (Furstenberg) For μ as above, $(B, \tau) \cong (G/P, \nu_P)$.

For every μ-stationary probability action $G \curvearrowright (Y, \nu)$, there is a measurable map $\kappa : G/P \to \operatorname{Prob}(Y)$ satisfying

(1) κ is G-equivariant;
(2) $\kappa(gP)$ is a probability measure almost surely;
(3) $\nu = \int_{G/P} \kappa(gP) \mathrm{d}\nu_p(gP)$.

Now we have

- $\pi : (\operatorname{Sub}(G), \nu) \to (G/Q, \nu_Q)$;
- $\kappa : (G/P, \nu_P) \to \operatorname{Prob}(\operatorname{Sub}(G))$;
- $\kappa' : (G/P, \nu_P) \to \operatorname{Prob}(G/Q)$.

We can check that $\kappa'(gP) = \delta_{gQ}$ works. So by the uniqueness, κ' has to be this one.

Similarly $\pi_* \kappa : G/P \to \operatorname{Prob}(G/Q)$ also satisfies (1)(2)(3). So that $\pi_*\kappa(gP) = \delta_{gQ}$ for ν_P almost every gP. This means that $\kappa(gP)$ is a gPg^{-1}-invariant probability measure on $\operatorname{Sub}_{\mathrm{d}}(gQg^{-1})$.

It is enough to classify P-invariant discrete random subgroups of Q. Write τ for some discrete random subgroup of Q, that is, $\tau \in \operatorname{Prob}(\operatorname{Sub}_{\mathrm{d}}(Q))$.

Lemma 1.3.4 If τ is P-invariant then $\tau = \delta_{\{1\}}$.

Proof. Let A be a maximal split torus of P. Let L_Q be a Levi of Q containing A. Let N_Q be the unipotent radical of Q. Let A_Q be a maximal split torus in $Z(L_Q)$.

Example 1.3.5

$Q = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix} \right\}$ and $P = \left\{ \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix} \right\}$. Then

$$L_Q = \left\{ \begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \right\}, \quad N_Q = \left\{ \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} \right\}, \quad A_Q = \left\{ \begin{bmatrix} s & & \\ & s & \\ & & t \end{bmatrix} \right\}.$$

Exercise 1.3.6. There is no nontrivial discrete random subgroup of $\mathbb{R}$ which is invariant under dilations.

Step 1. A_Q-invariant random discrete subgroups are contained in L_Q.
Write $Q = L_Q N_Q$. Let $U \subset L_Q, V \subset N_Q$ be open neighborhoods of $\{1\}$. Consider

$$F_{U,V}(\Lambda) := \# \{ \Lambda \cap (UV \setminus L_Q) \}.$$

Then $F_{U,V}$ is finite almost surely. Moreover, we have

$$F_{U,V}(a^{-1}\Lambda a) = F_{U,aVa^{-1}}(\Lambda), \quad \forall a \in A_Q.$$

Use this we can show that

$$F_{U,N_Q} = 0, \quad \text{almost surely.}$$

Step 2. Show that $\bigcap_{p\in P} L_Q^p = \{1\}$. $\square$

By this lemma, we have

$$\nu = \int_{G/P} \kappa(gP)\mathrm{d}\nu_P(gP) = \int_{G/P} \delta_{\{1\}}\mathrm{d}\nu_P(gP) = \delta_{\{1\}}.$$

We complete the proof of Theorem 1.2.4.

2 Basics on measure rigidity (Aaron Brown)

§2.1 Lecture 1

Consider a group action $G \curvearrowright X$. There are two philosophies:

- **Extrainvariance.** $H < G$, μ is H-invariant and some additional assumptions $\implies \mu$ is G-invariant.
- **Stiffness.** ν a measure on G, some assumptions on ν, then the ν-stationary measure μ is G-invariant.

Higher rank rigidity. Consider $X = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $f : X \to X, x \mapsto 3x \bmod 1$.

Fact 2.1.1. For every $0 \leqslant \gamma \leqslant 1$, there exists a closed f-invariant set $\Lambda \subset \mathbb{S}^1$ and an ergodic f-invariant probability measure μ on $\mathbb{S}^1$ with $\dim_{\mathrm{H}} \Lambda = \gamma$ and $\dim_{\mathrm{H}} \mu = \gamma$.

Now we add another element $g : x \mapsto 2x \bmod 1$.

> **Question 2.1.2.** What are the $\langle f, g\rangle$ joint invariant closed set / measures.

Theorem 2.1.3 (Furstenberg)
Let $\Lambda \subset \mathbb{S}^1$ be closed sets which is $\langle f, g\rangle$-invariant. Then either
- $\Lambda = \mathbb{S}^1, \varnothing$, or
- Λ is a finite set.

Theorem 2.1.4 (Rudolph)
Let μ be an ergodic $\langle f, g\rangle$-invariant probability measure. Then either
(1) μ is Lebesgue, or
(2) $\dim_{\mathrm{H}} \mu = 0$.

We will use an example to illustrate the main theorem. Consider two matrices

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Claim 2.1.5. (1) $\det A = \det B = 1$.
(2) A has three real eigenvalues

$$\chi_A^1 > 1 > \chi_A^2 > \chi_A^3 > 0.$$

(3) B has three real eigenvalues

$$\chi_B^1 > \chi_B^3 > 1 > \chi_B^2 > 0.$$

(4) $AB = BA$, therefore they have 3 joint eigenspaces E_1, E_2, E_3.
(5) $A^\ell B^m = \mathrm{id}$ iff $\ell = m = 0$.

Then A, B induce a $\mathbb{Z}^2$-action on $\mathbb{T}^3$ by $\alpha(n,m)(x+\mathbb{Z}^3) = A^n B^m x + \mathbb{Z}^3$. Moreover, α leaves no rational sub-torus invariant (irreducible).

Theorem 2.1.6 (Simple case of Katok-Spatzier)
The only ergodic α-invariant measure on $\mathbb{T}^3$ are
(1) Lebesgue,
(2) $\dim \mu = 0$.

Structure in $\mathbb{T}^3$ for α-action. There exists $Q \in \mathrm{SL}(3, \overline{\mathbb{Q}})$ diagonalized both A and B. The **Lyapunov functionals** are $\lambda^j : \mathbb{Z}^2 \to \mathbb{R}$ with $\lambda^j(n,m) = \log((\chi_A^i)^n (\chi_B^j)^m) = \log \chi_A^j + m \log \chi_B^j$. The "**Lyapunov manifolds**" W^j are $W^j(x) := x + E^j$. Then $\alpha(n,m)$ expands or contracts W^j with the ratio $\lambda^j(n,m)$.

For each $1 \leqslant j \leqslant 3$, there exists (n,m) such that

$$\lambda^j(n,m) > 0 \quad \text{and} \quad \lambda^k(n,m) < 0 \text{ for } k \neq j.$$

Then W^j is the unstable manifold for $\alpha(n,m)$. The stable manifold of $\alpha(n,m)$ is $W^{k_1} \oplus W^{k_2}$.

Let μ be an α-invariant, ergodic measure on $\mathbb{T}^3$. We aim to study the behavior of μ along W^j. Let ξ^j be a measurable partition subordinate to W^j such that the boundary is a null set. Let $\widetilde{\mu}_x^j$ be conditional measures associate to ξ^i at x. The problem is that ξ^j and $\widetilde{\mu}_x^j$ are not α-equivariant.

Leafwise measures. For $x \in \mathbb{T}^3$, we build (μ-a.e.) measures ν_x^j on $E^j \cong \mathbb{R}$ such that
(1) ν_x^j is locally finite (but probably infinite),
(2) $\Phi_x^j : E^j \to W^j(x), v \mapsto x + v$ satisfying $(\Phi_x^j)_* \nu_x^j = \mu_x^j$,
(3) ν_x^j is normalized on $I = [-1, 1] \subset E^j$,
(4) $\mu_x^j \propto \mu_y^j$ for every $y \in W^j(x)$,
(5) $\alpha(n,m)_* \mu_x^j \propto \mu^j_{\alpha(n,m)(x)}$, equivalently $(m_{e^{\lambda^j(n,m)}})_* \nu_x^j = \nu^j_{\alpha(n,m)(x)}$ where $m_\lambda : \mathbb{R} \to \mathbb{R}, t \mapsto \lambda t$.
(6) can recover conditional measures from μ_x^j:

$$\widetilde{\mu}_x^j = \frac{\mu_x^j|_{\xi^j(x)}}{\mu_x^j(\xi^j(x))}.$$

Aim 2.1.7. Assuming $h_\mu(\alpha(n,m)) > 0$, show that ν_x^j is the Lebesgue measure on $\mathbb{R}$ for almost every x.

Remark 2.1.8 For $(n,m) \neq (0,0)$, $\alpha(n,m)$ is Anosov. Replacing (n,m) with $(-n,-m)$ if needed, we can pick $1 \leqslant j \leqslant 3$ such that

$$\lambda^j(n,m) > 0 \quad \text{and} \quad \lambda^k(n,m) < 0 \text{ for } k \neq j.$$

§2.2 Lecture 2

How does entropy help? Set $f = \alpha(n,m)$, $W^u_f = W^j$ and let μ be an ergodic f-invariant probability measure.

Proposition 2.2.1

The following are equivalent:

(1) $h_\mu(f) = 0$,

(2) for μ-almost every x, ν^j_x has at least one atom,

(3) for μ-almost every x, $\nu^j_x = \delta_0$ (i.e. $\mu^j_x = \delta_x$),

(4) the partition of $(\mathbb{T}^3, \mu)$ into full W^s-leaves is measurable,

(5) ν^j_x and μ^j_x are finite measures.

Measure preserving translations. Let ν be a locally finite measure on $\mathbb{R}$. Consider

$$\mathscr{G}(\nu) := \{ \text{ translations of } \mathbb{R}\ T_v : x \mapsto x + v \text{ such that } (T_v)_*\nu \propto \nu \} .$$

Exercise 2.2.2. $\mathscr{G}(v)$ is a closed subgroup of $\mathbb{R}$.

Exercise 2.2.3. Suppose $\mathscr{G}(\nu)$ has a dense orbit in $\operatorname{supp}\nu$, then either

(1) $\operatorname{supp}\nu$ is countable, $\mathscr{G}(\nu)$ is discrete, or

(2) $\mathscr{G}(\nu) = \mathbb{R}$ and ν is absolutely continuous with respect to Haar.

Exercise 2.2.4. Suppose $\mathscr{G}(\nu) = \mathbb{R}$, then there exists $C > 0$ and $\kappa \in \mathbb{R}$ such that $\mathrm{d}\nu = C \cdot e^{\kappa x}\mathrm{dLeb}$.

Proof of Katok-Spatzier's theorem (Theorem 2.1.6). (1) The entropy assumption $\Longrightarrow$ ν^j_x are "thick" ($\operatorname{supp}(\nu^j_x)$ is not countable).

(2) Isometry and recurrence $\Longrightarrow$ $\mathscr{G}(\nu^j_x)$ is big for almost every x $\Longrightarrow$ $\nu^j_x \approx$ Leb.

(3) Use dynamics on curvature κ $\Longrightarrow$ $\nu^j_x =$ Leb.

There is another point of view to the third step using the entropy and the Ledrappier-Young formula. Let $f = \alpha(n,m)$ and W^j be the unstable foliation.

Proposition 2.2.5 (Ledrappier-Young formula)

The following are equivalent.

(1) $h_\mu(f) = \lambda^j(n,m)$,

(2) $\nu^j_x \ll$ Leb,

(3) $\nu^j_x \approx$ Leb,

(4) $\nu^j_x =$ Leb.

In what follows, we will explain more carefully on the second step, which contains the most crucial argument. We already know that ν^j_x has an uncountable support. We have the equivariant relation

$$(m_{e^{\lambda^j(n,m)}})_*\nu^j_x = \nu^j_{\alpha(n,m)(x)}.$$

Heuristic argument. Suppose $(n,m) \in \ker \lambda^j$, we have $\nu_x^j = (m_1)_* \nu_x^j = \nu^j_{\alpha(n,m)(x)}$, which gives an extra invariance.

Now we construct an $\mathbb{R}^2$-action from the $\mathbb{Z}^2$-action. Consider $\mathbb{R}^2 \times \mathbb{T}^3$, it admits a left $\mathbb{R}^2$-action $s \cdot (t,x) = (s+t,x)$ and a right $\mathbb{Z}^2$ action $(t,x)\cdot n = (t+n, \alpha(-n)x)$. Let $N = \mathbb{R}^2 \times \mathbb{T}^3/\mathbb{Z}^2$, which admits an $\mathbb{R}^2$-action and a fiber bundle structure over $\mathbb{R}^2/\mathbb{Z}^2$ (fibers are $\mathbb{T}^3$). We can equip fibers of N with good metrics such that E^j acts by translations on each fiber. For every $x \in N, v \in E^j, t \in \mathbb{R}^2$,

$$\widetilde{\alpha}(t)(x+v) = \widetilde{\alpha}(t)(x) + e^{\lambda^j(t)}v.$$

Given an $\alpha(\mathbb{Z}^2)$-invariant ergodic measure μ, we can obtain an $\widetilde{\alpha}(\mathbb{R}^2)$-invariant ergodic measure $\widetilde{\mu}$ on N. We can build $\widetilde{\mu}_x^j, \widetilde{\nu}_x^j$ similarly for $\widetilde{\mu}$ on N.

Then for $s \in \ker \lambda^j$, we have

$$\widetilde{\nu}_x^j = \widetilde{\nu}^j_{\widetilde{\alpha}(s)(x)}.$$

Therefore $x \mapsto \widetilde{\nu}_x^j$ is constant along orbits of $\ker \lambda^j$. **Why is this useful?** **Suppose** $\widetilde{\alpha}(s)$ is $\widetilde{\mu}$-ergodic, then $x \mapsto \nu_x^j$ is a.s. constant. Then there exists ν on E^s such that $\widetilde{\nu}_x^j = \nu$. For another element $y = x+v$, this will gives an extra translation invariance $T_v \nu \propto \nu$. □

§2.3 Lecture 3

Let $y = x+v \in x + E^j$. Recall that

$$(\Phi_y^j)_* \widetilde{\nu}_y^j = \widetilde{\mu}_y^j \propto \widetilde{\mu}_x^j = (\Phi_x^j)_* \widetilde{\nu}_x^j.$$

Taking the inverse, we obtain

$$(T_v)_* \widetilde{\nu}_y^j = ((\Phi_x^j)^{-1} \circ \Phi_y^j)_* \widetilde{\nu}_y^j \propto \widetilde{\nu}_x^j.$$

As we discussed before, we hope that $\ker \lambda^j$ acts on $(N, \widetilde{\mu})$ ergodically. In fact, we don't need $x \mapsto \widetilde{\nu}_x^t$ is constant. We only need $x \mapsto \widetilde{\nu}_x^j$ is constant on each W^j-leaf.

Fix $s_0 \neq (0,0) \in \ker \lambda^j$. Let $\mathscr{E}_{s_0}$ be the ergodic decomposition of $\widetilde{\mu}$ with respect to $\widetilde{\alpha}(s_0)$. Then $x \widetilde{\nu}_x^j$ is $\mathscr{E}_{s_0}$-measurable. We want to show $x \mapsto \widetilde{\nu}_x^j$ is constant along W^j-leaves. Take the measurable hull Ξ^j of the partitions of $(N, \widetilde{\mu})$ into full W^j-leaves (each element in Ξ^j is W^j-saturated).

> **Aim 2.3.1.** To show $\mathscr{E}_{s_0} \prec \Xi^j$.

If this holds, using the fact that $x \mapsto \widetilde{\nu}_x^j$ is constant on $\mathscr{E}_{s_0}$-atoms, we obtain the conclusion.

To show the aim, we have some preparations.

1. **Ledrappier-Young.** Pick any $t \in \mathbb{R}^2$. Let

$$\Xi_t^s = \text{the measurable hull of partition into full stable manifolds for } \widetilde{\alpha}(t),$$

$$\Xi_t^u = \text{the measurable hull of partition into full unstable manifolds for } \widetilde{\alpha}(t).$$

   Ledrappier-Young I Theorem B states that $\Xi_t^s = \Xi_t^u = \Pi$, where Π is the Pinsker partition.
2. **Pointwise ergodic theorem.** $\mathscr{E}_{s_0} \prec \Xi_{s_0}^s$.
3. **Totally non-symplectic assumption.** $\lambda^j \neq -c\lambda^i$ for every $i \neq j$.

Proof. We will use the **Π-partition trick**. First we have

$$\mathscr{E}_{s_0} \prec \Xi^s_{s_0} = \Pi_{s_0} = \Xi^u_{s_0}.$$

Assume that $\lambda^j(s_0) = 0, \lambda^i(s_0) < 0$ and $\lambda^{i'}(s_0) > 0$. Using the totally non-symplectic assumption. We can find another $t \in \mathbb{R}^2$ such that $\lambda^j(t) < 0, \lambda^i(t) < 0$ and $\lambda^{i'}(t) > 0$. Therefore,

$$\Xi^u_{s_0} = \Xi^u_t = \Pi_t = \Xi^s_t \prec \Xi^j.$$

The last inequality follows from the fact that $W^j(x) \subset W^s_t(x)$. $\square$

Why care?

- **Orbit closures.** A group action Γ on a compact metric space. We want to classify the orbit closures. We always study the Γ-invariant / Γ-stationary measures on the orbit closures.

  > **Theorem 2.3.2** (Einsiedler-Katok-Lindenstrauss)
  > Let μ be an ergodic A-invariant probability measure on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ with $h_\mu(a) > 0$ for some $a \in A$. Then μ is Haar.

- **Joinings / Measurable factors.** By classifying the joinings, show that all measurable factors are smooth / homogeneous.

3 Margulis-Zimmer's super-rigidity (Homin Lee)

§3.1 Cocycles

Notation 3.1.1. • G, H nice topological groups.
• (S, μ) a Lebesgue space.
• $G \curvearrowright (S, \mu)$ and μ is G-quasi-invariant.

Definition 3.1.2. A measurable map $\alpha : G \times S \to H$ is called a **measurable cocycle** if for every $g_1, g_2 \in G$,

$$\alpha(g_1 g_2, s) = \alpha(g_1, g_2 \cdot s)\alpha(g_2, s), \quad \text{a.e.} s \in S.$$

Remark 3.1.3 We may assume that the equality holds for every $g_1, g_2 \in G$ and $s \in S$.

Example 3.1.4

0. $\pi : G \to H$ homomorphism. $\alpha_\pi(g, s) = \pi(g)$ gives a cocycle.
1. $G \curvearrowright (M, \mu)$ by ρ where M is a compact smooth manifold and μ is G-invariant. Then TM is a measurably trivialized by $\left\{ \psi_x : T_x M \to \mathbb{R}^d \right\}$. Therefore

$$D\rho : G \times M \to \mathrm{GL}(d, \mathbb{R}), \quad (g, x) \mapsto \psi_{g.x} \circ D_x \rho(g) \circ \psi_x^{-1}.$$

gives a cocycle.

Definition 3.1.5. Two cocycles $\alpha, \beta : G \times S \to H$ are **cohomologous** to each other if there exists a measurable $\phi : S \to H$ such that

$$\alpha(g, s) = \phi(g.s)^{-1} \beta(g, s)\phi(s), \quad \forall g \in G, s \in S.$$

Example 3.1.6

2. Let $\Gamma < G$ be a closed subgroup with a measurable fundamental domain $X \subset G$.
Definition 3.1.7. The **return cocycle** is defined to be

$$\mathcal{R} : G \times G/\Gamma \to \Gamma, \quad \mathcal{R}(g, x) = \gamma,$$

where γ is the unique element such that $gx\gamma^{-1} \in X$.

Proposition 3.1.8

Let $\alpha : G \times G/L \to H$ be a cocycle, where L is a closed subgroup of G. Then there exists a homomorphism $\rho_\alpha : L \to H$ such that $\rho_\alpha(\ell) = \alpha(\ell, [e])$.

Conversely, for every group homomorphism $\rho : L \to H$, there exists a cocycle $\alpha_\rho : G \times G/L \to H$ so that ρ_{α_ρ} is conjugate to ρ.

This proposition gives a 1-1 correspondence

$$\{\text{ cocycles } G \times G/L \to H\,\}\,/\text{cohomologous} \longleftrightarrow \mathrm{Hom}(L, H)/\text{conjugacy}.$$

Let $\alpha : G \times S \to H$ be a cocycle.

Question 3.1.9. Does there exist a minimal subgroup $L < H$ with a cocycle β cohomologous to α with $\beta(G \times S) \subset L$.

The answer in general is **NO**.

Proposition 3.1.10
Let $\alpha : G \times S \to H$ be a cocycle and $H < \mathrm{SL}_m(\mathbb{R})$ be a Zariski closed subgroup. Then there exists a Zariski closed subgroup $L \subset H$ with a cocycle β cohomologous to α taking values in L such that α is not cohomologous to a cocycle taking values in in any proper Zariski-closed subgroup of L. Moreover, L is unique up to conjugacy. Such L is called the **algebraic hull** of α.

Theorem 3.1.11 (Zimmer's cocycle super-rigidity)
Let $G = \mathrm{SL}_n(\mathbb{R})$ with $n \geqslant 3$. Consider $G \curvearrowright (S, \mu)$ where μ is an ergodic G-invariant probability measure. Let $\alpha : G \times S \to \mathrm{SL}_m(\mathbb{R}) = H$ and assume that H is the algebraic hull of α. Then $\alpha(g, x) = \phi(g.x)^{-1}\pi(g)\phi(x)$ for some homomorphism $\pi : G \to H$ and measurable map $\phi : S \to H$.

Theorem 3.1.12 (Margulis's super-rigidity)
Let $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geqslant 3$. Let $\Gamma < G$ be a lattice and $\pi : \Gamma \to \mathrm{SL}_m(\mathbb{R}) = H$ be a homomorphism. If $\pi(\Gamma)$ is Zariski dense in H then π extends to a homomorphism $\tilde{\pi} : G \to H$.

Exercise 3.1.13. Show Margulis's super-rigidity by Zimmer's cocycle super-rigidity.
Hint. Consider the cocycle $G \times G/\Gamma \to \Gamma \to H$, where the first map is given by the return cocycle.

§3.2 Rigidity theorems, relation with Zimmer's program

Theorem 3.2.1 (Zimmer, Fisher-Margulis)
Let $G \curvearrowright (S, \mu)$ and μ be an ergodic G-invariant measures. Let $\alpha : G \times S \to \mathrm{GL}_m(\mathbb{R})$ be a cocycle. Assume that $\log \|\alpha(g, -)\|_{\mathrm{op}} \in L^1(S, \mu)$ for every $g \in G$. Then there exists a measurable map $\phi : S \to \mathrm{GL}_m(\mathbb{R})$, a homomorphism $\pi : G \to \mathrm{GL}_m(\mathbb{R})$ and a cocycle $\mathcal{K} : G \times S \to K$ with $K < \mathrm{GL}_m(\mathbb{R})$ a compact subgroup, such that

$$\alpha(g, x) = \phi(g.x)^{-1}\pi(g)\mathcal{K}(g, x)\phi(x), \quad \forall g \in G, x \in S.$$

Fact 3.2.2. 1. Higher rank Lie groups and their lattices satisfy Property (T).
2. H Property (T) and amenable $\implies$ H is compact.
3. H Property (T) and $H \curvearrowright (S,\mu)$ with an ergodic H-invariant measure μ. Then for every cocycle $\alpha : H \times S \to F$ with an amenable F, α is conjugate to a compact group valued cocycle.
4. F amenable group and $F \curvearrowright (S,\mu)$ with an ergodic F-invariant measure μ. Then for every cocycle $\alpha : F \times S \to \mathrm{GL}_m(\mathbb{R})$, the algebraic hull of α is amenable.

Proof. Let H be the algebraic hull of α. By Levi decomposition, $H = F \ltimes U$ where F is reductive and U is unipotent. □

Theorem 3.2.3 (Margulis)
Let $\Gamma < G$ be a lattice and $\pi : \Gamma \to \mathrm{GL}_m(\mathbb{R})$ be a homomorphism. Then there exist homomorphisms $\tilde{\pi} : G \to \mathrm{GL}_m(\mathbb{R})$ and $\kappa : \Gamma \to K$ with a compact K, such that $\pi(\gamma) = \tilde{\pi}(\gamma)\kappa(\gamma)$ for every $\gamma \in \Gamma$.

Relation with the Zimmer program. This is somehow in the same flavor as the Zimmer program:

> **Question 3.2.4.** Consider a smooth action ρ of Γ on a closed manifold M. Can we classify ρ? Does ρ come from "algebraic actions" and "isometric actions"?

Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ act on $(\mathbb{T}^3, \mathrm{Vol})$ by volume-preserving diffeomorphisms. It induces the derivative cocycle $D : \Gamma \times \mathbb{T}^3 \to \mathrm{SL}_3(\mathbb{R})$. Zimmer's cocycle super-rigidity also holds: there exists a measurable map $\phi : \mathbb{T}^3 \to \mathrm{SL}_3(\mathbb{R})$, a homomorphism $\pi : G = \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})$ and a cocycle $\mathcal{K} : G \times \mathbb{T}^3 \to K$ with $K < \mathrm{SL}_3(\mathbb{R})$ a compact subgroup, such that

$$D(\gamma, x) = \phi(\gamma.x)^{-1}\pi(\gamma)\mathcal{K}(\gamma, x)\phi(x), \quad \forall \gamma \in G, x \in \mathbb{T}^3.$$

Here the homomorphism π is either trivial, defining representation, contragredient ($B \mapsto (B^{-1})^t$). In fact, if π is not trivial then K is trivial.

Lemma 3.2.5
(1) If there exists $\gamma \in \Gamma$ such that $h_{\mathrm{Vol}}(\rho(\gamma)) > 0$ then π is non-trivial.
(2) For every $\gamma \in \Gamma$, the Lyapunov exponent of $\rho(\gamma)$ is the logarithm of an algebraic number.

§3.3 Zimmer program

Let $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geqslant 3$ and $\Gamma < G$ be a lattice. We consider a smooth action $\alpha : \Gamma \to \mathrm{Diff}(M)$ where M is a smooth closed manifold.

Critical dimension.

Example 3.3.1
Consider the action $G \curvearrowright \mathbb{R}^n$ by linear transformations. This induces a smooth action $G \curvearrowright \mathbb{RP}^{n-1} = G/Q$.

Recall that the Zimmer's cocycle super-rigidity $D\alpha(\gamma, x) = \phi(\gamma, x)^{-1}\pi(\gamma)\mathcal{K}(\gamma, x)\phi(x)$. If α preserves the volume, then $\pi : G \to \mathrm{SL}_d(\mathbb{R})$. If $d \leqslant n-1$, then there is no nontrivial homomorphism $\pi : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$.

Theorem 3.3.2 (Zimmer's Conjecture, Brown-Fisher-Hurtado)
Let $\alpha : \Gamma \to \mathrm{Diff}^\infty(M)$ be a smooth action. Then
1. If $\dim M = n-1$ and α is volume preserving, then α is isometric and hence $\alpha(\Gamma)$ is finite.
2. If $\dim M = n-2$ then $\alpha(\Gamma)$ is finite.

Theorem 3.3.3 (Brown-Rodriguez Hertz-Wang)
Let $\alpha : \Gamma \to \mathrm{Diff}^\infty(M)$ be a smooth action. If $\dim M = n-1$ then $M \cong \mathbb{RP}^{n-1}$ or $\mathbb{S}^{n-1}$ and α is conjugate to the projective action.

Question 3.3.4. How about $\dim M = n$?

- For G-actions, we can classify the C^2-actions $\alpha : G \to \mathrm{Diff}^2(M)$.
- For Γ-actions, it is conjectured that either α extends to G-actions or α is (blow up + toral automorphisms).

Uniform / non-uniform hyperbolic systems.

Theorem 3.3.5 (Brown-Rodriguez Hertz-Wang)
Let $\alpha : \Gamma \to \mathrm{Diff}^\infty(\mathbb{T}^n)$ be a smooth action. If there exists $\gamma \in \Gamma$ such that $\alpha(\gamma)$ is an Anosov diffeomorphism then α smoothly conjugates to an affine action.

Theorem 3.3.6 (Katok-Lewis-Zimmer, Lee)
Let $\alpha : \Gamma \to \mathrm{Diff}^\infty_{\mathrm{Vol}}(M)$ be a volume-preserving smooth action. Assume that $\dim M = n$ and there exists $\gamma \in \Gamma$ such that $\alpha(\gamma)$ is Anosov. Then $M \cong \mathbb{T}^n$ and α is smoothly affine.

Question 3.3.7. Let $\alpha : \Gamma \to \mathrm{Diff}^\infty(M)$ be a smooth action. Assume that there exists $\gamma \in \Gamma$ such that $\alpha(\gamma)$ is partially hyperbolic (or Anosov). Is M a bi-homogeneous space? Is α smoothly algebraic?

Theorem 3.3.8 (Damjanovic-Spatzier-Vinhage-Xu)
Let $\alpha : \Gamma \to \mathrm{Diff}^\infty_{\mathrm{Vol}}(M)$ be a volume-preserving smooth action. Assume that it is "totally Anosov". Then M is bi-homogeneous and α is smoothly algebraic.

Theorem 3.3.9

Let $\alpha : \Gamma \to \mathrm{Diff}^{\infty}(M)$ be a smooth action. Assume that $\dim M = n$ and there exists a $\gamma \in \Gamma$ such that $h_{\mathrm{top}}(\alpha(\gamma)) > 0$. Then there exists an $\alpha(\Gamma)$-invariant absolutely continuous measure and hence M is measurably conjugate to $\mathbb{T}^n$ and $\Gamma \cong \mathrm{SL}_n(\mathbb{Z})$.

Low-regularity / low-dimension.

Question 3.3.10. Let $\Gamma < \mathrm{SL}_n(\mathbb{R})$ and $\alpha : \Gamma \to \mathrm{QC}_{\mathrm{Vol}}(\Sigma)$ be a volume-preserving quasi-conformal action on a closed surface. Is $\alpha(\Gamma)$ finite?

$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Theorem 3.3.11 (Margulis's normal subgroup theorem)

Let $\Gamma < G$ be a lattice, where G is a semisimple Lie group with finite center and real rank at least 2. Then every normal subgroup of Γ is either finite or of finite index.

Now we consider $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ is an irreducible lattice in this part. Note that G does not have Property (T) so that there is no Zimmer's cocycle super-rigidity.

Theorem 3.3.12 (Franks-Handel)

Let $\Gamma < \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ be a non-cocompact irreducible lattice. Then for every volume preserving smooth action $\alpha : \Gamma \to \mathrm{Diff}^{\infty}_{\mathrm{Vol}}(\Sigma)$ on a closed surface, $\alpha(\Gamma)$ is finite.

Question 3.3.13. Let $\Gamma < G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ be an irreducible lattice. Is it true that every volume preserving smooth action $\alpha : \Gamma \to \mathrm{Diff}^{\infty}_{\mathrm{Vol}}(\Sigma)$ on a closed surface has a finite image or it is essentially $\Gamma \hookrightarrow \mathrm{SO}(3) \curvearrowright \mathbb{S}^2$ by isometries?

4 Space of actions of groups on the real line (Bertrand Deroin)

§4.1 Lecture 1

Proposition 4.1.1
A countable group Γ embeds in $\mathrm{Homeo}^+(\mathbb{R})$ iff it is left-orderable.

Definition 4.1.2. A group Γ is **left-orderable** if there exists a total order $\prec$ on Γ which is left-invariant: if $g \prec h$ then $kg \prec kh$, for every $g, h, k \in \Gamma$.

Proof. Assume that $\Gamma < \mathrm{Homeo}^+(\mathbb{R})$. Let $(x_n)_n$ be a dense sequence of real numbers. For two different elements $g, h \in \Gamma$, letting $n_0 = \inf\{\, n : g(x_n) \neq h(x_n) \,\}$, we take $g \prec h$ if $g(x_{n_0}) < h(x_{n_0})$. Hence Γ is left-orderable.

Suppose now that Γ is countable and have a leaf-invariant total order $\prec$. We pick a numbering $(g_n)_{n \geqslant 0}$ of the elements of Γ. We will construct an embedding $t : \Gamma \to \mathbb{R}$ which preserves the order. The map t can be defined inductively on $g_0, g_1, \cdots$. Then Γ acts on $t(\Gamma)$ by $g.t(g_n) := t(gg_n)$. We can prove that the Γ-action extends to a C^0-action on $\overline{t(\Gamma)}$. We then extend the Γ-action on $\overline{t(\Gamma)}$ on the components of $\mathbb{R} \setminus \overline{t(\Gamma)}$ by affine maps. $\square$

Such constructions are called **dynamical realization of the order**.

Question 4.1.3 (Zimmer program)**.** Which lattices $\Gamma < G$ of semisimple Lie groups are left-orderable?

Let G be the isometry group of a symmetric space. Its real rank $\mathrm{rank}_{\mathbb{R}} G$ is the maximal dimension of a totally geodesic flat.

First we consider the case for the hyperbolic plane $\mathbb{H}^2$.

Lemma 4.1.4 Any torsion free lattice of $\mathrm{Isom}^+(\mathbb{H}^2)$ is left-orderable.

Proof. Note that $\mathrm{Isom}^+(\mathbb{H}^2)$ acts by diffeomorphisms on $\partial\mathbb{H}^2 \cong \mathbb{R}/\mathbb{Z}$.

Question 4.1.5. Given $\Gamma < \mathrm{Isom}^+(\mathbb{H}^2)$, is it possible to lift the action of Γ on $\partial\mathbb{H}^2$ to an action on $\widetilde{\partial\mathbb{H}^2} \cong \mathbb{R}$.

Case 1. If $\Gamma\backslash\mathbb{H}^2$ is non-compact, then Γ is freely generated by a finite set $S < \Gamma$. Therefore we can lift each element in S to $\mathbb{R}$ individually.
Case 2. Γ is a surface group:

$$\Gamma = \left\langle a_1, b_1, \cdots, a_g, b_g \middle| [a_1, b_1] \cdots [a_g, b_g] = 1 \right\rangle.$$

We can lift the a_i's, b_i's to homeomorphisms $\widetilde{a}_i, \widetilde{b}_i$'s on $\mathbb{R} = \widetilde{\partial\mathbb{H}^2}$. Then $[\widetilde{a}_1, \widetilde{b}_1] \cdots [\widetilde{a}_g, \widetilde{b}_g]$ is a deck transformation of $\widetilde{\partial\mathbb{H}^2} \to \partial\mathbb{H}^2$, which identifies with an integer. **BAD NEWS**: this integer is equal to $\pm(2 - 2g)$, which is not vanish in general.

So that we should choose another representation of Γ in $\mathrm{Isom}^+(\mathbb{H}^2)$. We look at the component of $\mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{H}^2))$ that contains the trivial representation. Every representation in this component can be lifted to $\mathbb{R}$ successfully. In fact, there exists a faithful representation in this component, which gives us a desired action. $\square$

Question 4.1.6. Is it true that a lattice in $\mathrm{Isom}(\text{real / complex hyperbolic space})$ is virtually left-orderable?

Question 4.1.7. Does there exists a left-orderable group with Kazhdan property (T)?

§4.2 Lecture 2

Theorem 4.2.1

If $\Gamma < G$ is an irreducible lattice in a semi-simple Lie group G of rank at least 2, and trivial center, then Γ is not left-orderable.

Definition 4.2.2. An element $h \in \mathrm{Homeo}^+(\mathbb{R})$ is **almost-periodic** if the set

$$\{ \tau_{-s} \circ h \circ \tau_s : s \in \mathbb{R} \}$$

is relatively compact, where τ_s is the translation $t \mapsto t + s$ and $\mathrm{Homeo}^+(\mathbb{R})$ is equipped with the compact-open topology on both g and g^{-1}.

Example 4.2.3

For some quasi-periodic function $f(t) = \sum_{k=0}^{n} [a_k \cos(\alpha_k t) + b_k \sin(\beta_k t)]$, the homeomorphism $h(t) = t + f(t)$ is almost-periodic.

Fact 4.2.4. The subset $\mathrm{APH}^+(\mathbb{R}) \subset \mathrm{Homeo}^+(\mathbb{R})$ of almost-periodic homeomorphisms is a subgroup.

Proposition 4.2.5

Let Γ be a finitely generated group and $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ be a homomorphism. Then ϕ is almost periodic ($\phi(\Gamma) \subset \mathrm{APH}^+(\mathbb{R})$) iff there exists a compact space Z, a free flow $\mathcal{T} = \{ T^t \}_{t \in \mathbb{R}}$ acting on Z, an action of Γ on Z and a point $z_0 \in Z$ such that the Γ-action preserves each $\mathcal{T}$-orbits and act on it by orientation-preserving maps and

$$g(T_t(z_0)) = T_{\phi(g)(t)}(z_0), \quad \forall g \in \Gamma, t \in \mathbb{R}.$$

Proof. Assume first that there is such space Z, flow $\mathcal{T}$ and point z_0 satisfying properties given in the proposition. For every $z \in Z$, there exists a Γ-action on $\mathbb{R}$ given by

$$\phi^z : \Gamma \to \mathrm{Homeo}^+(\mathbb{R}), \quad g(T_t(z)) = T_{\phi^z(g)(t)}(z).$$

Pick $g \in \Gamma$, the map $z \mapsto \phi^z(g) \in \mathrm{Homeo}^+(\mathbb{R})$ is continuous. By construction, we have the formula

$$\phi^{T_s(z)}(g) = \tau_{-s} \circ \phi^z(g) \circ \tau_s, \quad \forall s \in \mathbb{R}, z \in Z, g \in \Gamma.$$

By the compactness of Z, this shows that $\phi^z(g)$ stay in a compact set for each fixed g. Therefore $\phi(g) = \phi^{z_0}(g)$ is almost-periodic for every g.

Assume now that ϕ is almost-periodic. Consider the space $Z' = \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(\mathbb{R}))$, endowed with the subspace topology from $(\mathrm{Homeo}^+(\mathbb{R}))^S$ where S is a finite generating set of Γ. The map $T^t\psi := \tau_{-t} \circ \psi \circ \tau_t$ defines a flow on Z'. We define the Γ-action on Z' given by

$$g(\psi) := \tau_{-\psi(g)(0)} \circ \psi \circ \tau_{\psi(g)(0)} = T^{\psi(g)(0)}(\psi), \quad \forall g \in \Gamma, \psi \in Z'.$$

Let $Z = \overline{T^t(\phi)}$, which is preserved by T and by Γ. Let $z_0 = \phi \in Z$. □

Definition 4.2.6. Z is called the **almost-periodic space** and $\mathcal{T}$ is called the **translation flow**.

Theorem 4.2.7

Any Γ-action $\phi_0 : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is topologically conjugated to an almost-periodic action ϕ. Moreover, if the Γ-action ϕ_0 does not have any fixed point then ϕ does not "almost have fixed point".

Definition 4.2.8. An action $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is said to **almost have a fixed point** if

$$\inf_{t \in \mathbb{R}} \sup_{g \in S} |\phi(g)(t) - t| = 0,$$

where S is a finite generating set of Γ.

Remark 4.2.9 Let Z be the almost-periodic space, then there exists a sequence $\{ t_n \}$ such that $\phi(g)(t_n) - t_n \to 0$ for every $g \in S$. Let $z_n = T^{t_n}(z_0)$. Then any limit of z_n in Z is fixed by Γ.

Conjecture 4.2.10 (Linnell)

A finitely generated left-oderable group is either contains a non-abelian free group or it has a homomorphism onto $\mathbb{Z}$.

Theorem 4.2.11 (Witte)

Any finitely generated amenable left-orderable group has a homomorphism onto $\mathbb{Z}$.

§4.3 Lecture 3

Let Γ be a finitely generated group and μ be a symmetric finitely supported probability measure on Γ with the support S satisfying $\langle S \rangle = \Gamma$.

Definition 4.3.1. An action $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$ is **μ-harmonic** if the Lebesgue measure is μ-stationary.

Definition 4.3.2. The random walk has the Derriennic property if

$$x = \int g(x) \mathrm{d}\mu(g), \quad \forall x \in \mathbb{R}.$$

Proposition 4.3.3
A μ-stationary action is almost-periodic, does not have an almost periodic point, and it has the Derriennic property.

Proof (by Victor Kleptsyn). For $h \in \mathrm{Homeo}^+(\mathbb{R})$ and $c \in \mathbb{R}$, we define

$$\Delta^h(c) := \begin{cases} \int_{h^{-1}(c)}^{c} [h(s) - c]\mathrm{d}s, & h(c) \geqslant c; \\ \int_{h(c)}^{c} [h^{-1}(s) - c]\mathrm{d}s, & h(c) < c. \end{cases}$$

Lemma 4.3.4 $\int_a^b [h(s) - s] + [h^{-1}(s) - s]\mathrm{d}s = \Delta^h(b) - \Delta^h(a)$.

From μ-harmonicity, we have that the drift

$$\mathrm{Dr}(\phi, \mu) = \int [\phi(g)(x) - x]\mathrm{d}\mu(g)$$

does not depend on choice of $x \in \mathbb{R}$. Integral the equality of the lemma over Γ, we have

$$\int_a^b \left\{ \int_\Gamma [\phi(g)(s) - s]\mathrm{d}\mu(g) + \int_\Gamma [\phi(g)^{-1}(s) - s]\mathrm{d}\mu(g) \right\} \mathrm{d}s = \int_\Gamma [\Delta^{\phi(g)}(b) - \Delta^{\phi(g)}(a)]\mathrm{d}\mu(g).$$

Note that the left hand side equals to $2\mathrm{Dr}(\phi, \mu)(b - a)$. We have that the function

$$c \in \mathbb{R} \mapsto \int_\Gamma \Delta^{\phi(g)}(c)\mathrm{d}\mu(g)$$

is an affine function with the derivative $2\mathrm{Dr}(\phi, \mu)$. But $\Delta^h(c)$ is non-negative by our definition. So that the drift vanishes and the Derriennic property holds. Besides, $\int_\Gamma \Delta^{\phi(g)}(c)\mathrm{d}\mu(g)$ is the constant, which we denote by Δ.

To prove almost-periodicity, we will prove that the action is bi-Lipschitz and that the displacement $\sup_{s\in\mathbb{R}} |\phi(g)(s) - s| < \infty$ for every $g \in \Gamma$. The Lipschitz property is easy to establish. Note that for every $x < y$, we have

$$y - x = \int [\phi(g)(y) - \phi(g)(x)]\mathrm{d}\mu(g) \geqslant \mu(g)[\phi(g)(y) - \phi(g)(x)].$$

Recall that $\int_\Gamma \Delta^{\phi(g)}(c)\mathrm{d}\mu(g) = \Delta$. We have

$$\Delta^{\phi(g)}(c) \leqslant \frac{\Delta}{\mu(g)}, \quad \forall g \in S, c \in \mathbb{R}.$$

By the bi-Lipschitz property, we have

$$\Delta^{\phi(g)}(c) \geqslant \frac{\mu(g)}{2} |\phi(g)(c) - c||\phi(g)^{-1}(c) - c| \geqslant \frac{|\phi(g)(c) - c|^2}{2}.$$

Therefore, we obtain a uniform boundedness of the displacement of $\phi(g)$. □

Theorem 4.3.5

Any $\phi_0 : \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ without a discrete orbit is semi-conjugate to a μ-harmonic action ϕ. Moreover, ϕ is unique up to conjugacy by an affine map.

Definition 4.3.6. ϕ_0 is **semi-conjugate** to ϕ if there is a nondecreasing proper map $k : \mathbb{R} \to \mathbb{R}$ such that $\phi \circ k = k \circ \phi_0$.

The principle of proof is the existence of a Radon stationary measure ν which is bi-infinite ($\nu([c, +\infty[) = \infty, \nu(] - \infty, c]) = \infty$) and atomless. Then $k(c) := ([0, c])$ gives a semi-conjugacy from ϕ_0 to a μ-stationary action $\phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$. The unicity part of the theorem is the consequence of the uniqueness of the a Radon μ-stationary measure up to a multiplicative constant.

To show the existence of a such Radon μ-stationary measure, we consider the random sequence (g_n) which is an i.i.d. Γ-valued random variables obeying the law μ. Let $x_0 = x$ and $x_n = \phi(g_n) \cdots \phi(g_1)(x)$. There is an oscillation property:

$$\limsup_{n \to +\infty} x_n = \infty, \quad \liminf_{n \to -\infty} x_n = -\infty, \quad \text{almost surely.}$$

This oscillation property leads to the existence of a such stationary measure.