

Reading Seminar on Homogeneous Dynamics (2023 Spring)

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Contents

1	Introduction (Pengyu Yang, Mar 17)	2
2	Ergodic theory (Yuxiang Jiao, Mar 31)	4
3	Preparation on algebraic groups I (Yuxiang Jiao, Mar 31)	7
4	Preparation on algebraic groups II (Yuxiang Jiao, Apr 7)	8
5	Margulis super-rigidity theorem I (Jiesong Zhang, Apr 7)	10
6	Margulis super-rigidity theorem II (Bohan Yang, Apr 14)	11
7	Margulis arithmeticity theorem (Apr 21)	13
	References	15

§1 Introduction (Pengyu Yang, Mar 17)

Arithmetic & Super-rigidity

Let \mathbb{G} be a connected semisimple algebraic \mathbb{Q} -group.

Theorem 1.1 (Borel-Harish-Chandra) $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$.

Definition 1.2. We say $\Gamma, \Gamma' \subset \mathbb{G}$ are **commensurable** if

$$[\Gamma : \Gamma \cap \Gamma'] < \infty, \quad [\Gamma' : \Gamma \cap \Gamma'] < \infty$$

Definition 1.3 (Restriction of scalar). Let $[k : \mathbb{Q}] = d$ and \mathbb{G} be a k -group. The restriction $R_{k/\mathbb{Q}}\mathbb{G}$ is a \mathbb{Q} -group such that for every $k \subset K$,

$$R_{k/\mathbb{Q}}\mathbb{G}(K) \cong \mathbb{G}^{\sigma_1}(K) \times \mathbb{G}^{\sigma_2}(K) \times \cdots \times \mathbb{G}^{\sigma_d}(K)$$

where $\sigma_i : k \hookrightarrow \mathbb{C}$ are embeddings.

Remark 1.4 — $R_{k/\mathbb{Q}}\mathbb{G}(\mathbb{Q}) \cong \mathbb{G}(k)$, $R_{k/\mathbb{Q}}\mathbb{G}(\mathbb{Z}) \cong \mathbb{G}(\mathcal{O}_k)$.

Definition 1.5. Let G be a connected semisimple real Lie group with trivial center and no compact factor. Let $\Gamma \subset G$ be a lattice. We say Γ is **arithmetic** if there exists a semisimple algebraic \mathbb{Q} -group \mathbb{H} such that there is a surjective $\varphi : \mathbb{H}(\mathbb{R})^0 \rightarrow G$ with compact kernel such that $\varphi(\mathbb{H}(\mathbb{Z}) \cap \mathbb{H}(\mathbb{R})^0)$ is commensurable with Γ .

Example 1.6

1. $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ or congruence subgroups.
2. $G = \mathrm{Sp}(2n, \mathbb{R})$ and $\Gamma = \mathrm{Sp}(2n, \mathbb{Z})$.
3. $B = \mathbb{Q}(2, 3) := \langle i, j \mid i^2 = 2, j^2 = 3, ij = -ji \rangle$. Then $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{Mat}(2, \mathbb{R})$. Let

$$\mathbb{G} = B^{(1)} := \{a + bi + cj + dij : a^2 - 2b^2 - 3c^2 + 6d^2 = 1\}.$$

Then $\mathbb{G}(\mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$ given by $i \mapsto \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix}$ and $j \mapsto \begin{bmatrix} & 1 \\ 3 & \end{bmatrix}$. Then $\mathbb{G}(\mathbb{Z})$ is a cocompact arithmetic lattice in $\mathrm{SL}(2, \mathbb{R})$, which is not commensurable with $\mathrm{SL}(2, \mathbb{Z})$.

4. $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$, we consider the embedding $\Gamma \hookrightarrow G$ given by $A \mapsto (A, {}^\sigma A)$. The restriction of scalar $\mathbb{G} = R_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SL}(2, \mathbb{Q}(\sqrt{2}))$.
5. $G = \mathrm{SL}(2, \mathbb{C})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{-1}])$, $\mathbb{G} = R_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\mathrm{SL}(2, \mathbb{Q}(\sqrt{-1}))$.
6. Let $J = x_1^2 + x_2^2 + (1 - \sqrt{2})x_3^2$. Let $G = \mathrm{SO}(J)(\mathbb{R})^0 \cong \mathrm{SO}(2, 1)^0$. Let $\mathbb{H} = R_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SO}(J)$. Then $\mathbb{H}(\mathbb{R}) \approx G \times \mathrm{SO}(x_1^2 + x_2^2 + (1 + \sqrt{2})x_3^2)(\mathbb{R}) \cong G \times \mathrm{SO}(3)$.

Theorem 1.7 (Margulis Arithmeticity)

Let G be a semisimple real Lie group with $\mathrm{rank}_{\mathbb{R}} G \geq 2$ without compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is a arithmetic.

Theorem 1.8 (Margulis Super-rigidity)

Let G be a semisimple real Lie group with $\text{rank}_{\mathbb{R}} G \geq 2$. Assume that G is with trivial center and no compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Let $H = \mathbb{H}(k)$ be a connected simple k -group where k is a local field $(\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \dots)$. Let $\varphi : \Gamma \rightarrow H$ be a homomorphism such that $\varphi(\Gamma)$ is Zariski dense and unbounded. Then φ extends to G , that is, $\exists \psi : G \rightarrow H$ continuous such that $\psi|_{\Gamma} = \varphi$.

Remark 1.9 — Margulis Super-rigidity \implies Margulis Arithmeticity.

Real rank one case

$X = G/K$	\mathbb{H}^n	$\mathbb{C}\mathbb{H}^n$	$\mathbb{H}\mathbb{H}^n$	$\mathbb{O}\mathbb{H}^n$
G	$\text{SO}(n, 1)^0$	$\text{SU}(n, 1)$	$\text{Sp}(n, 1)$	F_4^{-20}
K	$\text{SO}(n)$	$U(n)$	$\text{Sp}(n)$	$\text{Spin}(9)$

$\text{SO}(2, 1)$ case. $G = \text{PSL}(2, \mathbb{R}) \cong \text{SO}(2, 1)^0 \cong \text{Isom}(\mathbb{H}^2)^+$. Let $\Gamma = \pi_1(\Sigma_g)$. We consider

$$\mathcal{M}_g := \text{Hom}(\Gamma, G) / \sim = \{\text{hyperbolic structure on } S\} = \{\text{complex structure on } S\}.$$

\mathcal{M}_g is a complex orbifold of complex dimension $3g - 3$. There is no rigidity.

$\text{SO}(n, 1)$ case for $n \geq 3$. There is some rigidity.

Theorem 1.10 (Mostow strong rigidity)

Let M, N be compact hyperbolic n -manifolds. Let $\varphi : M \rightarrow N$ be a homotopy equivalence. Then there exists an isometry $\psi : M \rightarrow N$ which is homotopic to φ .

Theorem 1.11 (Gromov-Piatetski-Shapiro)

For every $n \geq 3$, $\text{SO}(n, 1)$ contains infinitely many commensurable classes of non-arithmetic lattices.

$\text{Sp}(n, 1)$ case and F_4^{-20} case.

Theorem 1.12 (Corlette, Gromov-Shoen)

Let $G = \text{Sp}(n, 1)$ or F_4^{-20} . Every lattice $\Gamma < G$ is arithmetic.

$\text{SU}(n, 1)$ case. The only known non-arithmetic lattices are for $n = 2, 3$. For the $\text{SU}(2, 1)$ case, Mostow constructed reflection groups which are non-arithmetic. For the $\text{SU}(3, 1)$ case, Deligne-Mostow constructed non-arithmetic lattices.

This semester**Theorem 1.13** (Bader-Fisher-Miller-Stover, [BFMS21, Theorem 1.1])

Let $\Gamma \subset \mathrm{SO}(n, 1)^0$ be a lattice. Suppose $K \backslash G/\Gamma$ contains infinitely many maximal totally geodesic subspace of $\dim \geq 2$. Then Γ is arithmetic.

Theorem 1.14 ([BFMS21, Theorem 1.5])

Let $W = \mathrm{SO}(m, 1)^0 < G = \mathrm{SO}(n, 1)^0$ where $1 < m < n$. If there exists $\{\mu_i\}$ a sequence of W -invariant ergodic probability measure on G/Γ such that $\mu_i \xrightarrow{w^*} \mu_{G/\Gamma}$. Then Γ is arithmetic.

Theorem 1.15 (Super-rigidity, [BFMS21, Theorem 1.6])

Let $W = \mathrm{SO}(m, 1)^0 < G = \mathrm{SO}(n, 1)^0$ where $1 < m < n$. Let k be a local field. Let \mathbb{H} be a connected k -algebraic group. Assume that (k, \mathbb{H}) is compatible with G . Let $\rho : \Gamma \rightarrow \mathbb{H}(k)$ be a homomorphism with unbounded and Zariski dense image. If there exists $\mathbb{H} \rightarrow \mathrm{SL}(V)$ a k -representation on a k -vector space V and a W -invariant probability measure ν on

$$(G \times \mathbb{P}(V))/\Gamma : \{(g, v) \sim (g\gamma, \rho(\gamma)^{-1}v)\}$$

such that ν projects to $\mu_{G/\Gamma}$. Then ρ extends to $G \rightarrow \mathbb{H}(k)$.

- There are two good surveys about rigidity theory [S04] and [F22].
- We will follow a textbook by Zimmer [Z13] at the beginning in this semester.

§2 Ergodic theory (Yuxiang Jiao, Mar 31)

Setting

- G locally compact second countable group.
- S a Borel space (isomorphic to a complete separable metric space with Borel σ -algebra).
- S is a G -space: G acts on S (measurably).
- A quasi-invariant measure μ on S , that is, for every $A \subset S, g \in G, \mu(Ag) = 0$ iff $\mu(A) = 0$.

Definition 2.1. The action is called **ergodic** if every G -invariant measurable subset of S is either null or conull.

Example 2.2

1. $S = M$ a smooth manifold, $G \subset \mathrm{Diff}(M)$, $\mu \approx \mathrm{Leb}$ which is quasi-invariant.
2. $H < G$ a closed subgroup, $X = G/H$. Then $G \curvearrowright (X, \mu_X)$ is ergodic (by transitivity).
3. $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$ is ergodic (by Fourier analysis).
4. $X = \prod_{\mathbb{Z}} \{\pm 1\}$ a compact abelian group. $H = \{x \in X : x_i = 1 \text{ for all but finitely many } i\}$. Then $H \curvearrowright X$ is ergodic (by Fourier analysis).
5. $\Gamma = \mathrm{SL}(2, \mathbb{Z}) \curvearrowright \mathbb{RP}^1$. Ergodic? Regard $\mathbb{RP}^1 \cong \mathrm{SL}(2, \mathbb{R})/P$ where $P = \{g : g \cdot \infty = \infty\}$. We remark that there is no Γ -invariant measure on \mathbb{RP}^1 . Proposition 2.3 helps to deal with this action.

Moore's ergodicity theorem

Proposition 2.3 ([Z13, Corollary 2.2.3])

Let H_1, H_2 be closed subgroups of G . Then $H_1 \curvearrowright G/H_2$ ergodic $\iff H_2 \curvearrowright G/H_1$ ergodic.

Proof. Let S be a G -space and $H \subset G$ a closed subgroup. Then $H \curvearrowright S$ is ergodic iff $G \curvearrowright (S \times G/H)$ is ergodic. \square

Definition 2.4. Let G be a connected semisimple Lie group with finite center and $\Gamma < G$ is a lattice. We say Γ **irreducible** if for every normal subgroup $H \subset G$, $\Gamma H/H$ is dense in G/H .

Theorem 2.5 (Moore's ergodicity theorem, [Z13, Theorem 2.2.6])

Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let $\Gamma < G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and H is not compact. Then H is ergodic on G/Γ .

Example 2.6 $\mathrm{SL}(n, \mathbb{Z})$ acts ergodically on \mathbb{RP}^{n-1} .

Example 2.7 $\mathrm{SL}(n, \mathbb{Z})$ acts ergodically on $(\mathbb{R}^n, \mathrm{Leb})$. Since $\mathbb{R}^n \setminus \{0\} \cong \mathrm{SL}(n, \mathbb{R})/H$.

Definition 2.8. Let G be a finite product of connected non-compact simple Lie groups with finite center. Let S be an ergodic G -space with finite invariant measure. We say the action is **irreducible** if for every non-central normal subgroup $N \subset G$, N is ergodic on S .

Proposition 2.9 $\Gamma < G$ is an irreducible lattice $\iff G \curvearrowright G/\Gamma$ is irreducible.

Example 2.10

G as above. Assume that $G \hookrightarrow H$ where H is a simple Lie group. Let $\Gamma < H$ be a lattice (hence irreducible). By Moore's ergodicity, H/Γ is an ergodic G -space. Furthermore, it is an irreducible G -space.

Theorem 2.11 (Moore's ergodicity theorem, general version, [Z13, Theorem 2.2.15])

Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let S be an irreducible ergodic G -space with finite invariant measure. If $H \subset G$ is a closed subgroup and H is not compact. Then H is ergodic on S .

Relation with unitary representations

Let us show the idea of proof of Moore's ergodicity theorem. Note that $G \curvearrowright S$ induces an action $G \curvearrowright L^2(S)$. Since we assume that μ is G -invariant, then G acts by unitary operators. Denote as $\pi : G \rightarrow \mathcal{U}(L^2(S))$. We equip $\mathcal{U}(L^2(S))$ with strong operator topology, then π is continuous.

Denote $L_0^2(S)$ to be the orthogonal complement of \mathbb{C} in $L^2(S)$ which is G -invariant.

Proposition 2.12 ([Z13, Corollary 2.2.17])

G acts ergodically on $S \iff$ there is no non-trivial G -invariant vectors in $L_0^2(S)$.

Combining this proposition, it suffices to show

Theorem 2.13 ([Z13, Theorem 2.2.19])

Let $G = \prod G_i$ be a finite product of connected non-compact simple Lie groups with finite center. Let π be a unitary representation of G such that $\pi|_{G_i}$ has no invariant vectors. If $H \subset G$ is a closed subgroup and $\pi|_H$ has non-trivial invariant vectors, then H is compact.

This theorem follows from the following result.

Theorem 2.14 (Vanishing of matrix coefficients, [Z13, Theorem 2.2.20])

Let G, G_i, π be as above. For every unit vectors $v, w \in \mathcal{H}$, the Hilbert space where G acts on. The matrix coefficient $f_{v,w}(g) = (\pi(g)v, w)$ tends to zero as g tending to infinity.

If there exists an H -invariant vector v , then H is compact since $(\pi(h)v, v) \equiv 1$ for $h \in H$.

Remark 2.15 — Vanishing of matrix coefficients can be viewed as “mixing”, which is stronger than ergodicity.

“Mixing” in $\mathrm{SL}(2, \mathbb{R})$ **Theorem 2.16**

Let $G = \mathrm{SL}(2, \mathbb{R})$ and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation without invariant vectors. Then for every $\varphi, \psi \in \mathcal{H}$ and (g_n) divergent in $\mathrm{SL}(2, \mathbb{R})$, we have $(g_n \cdot \varphi, \psi) \rightarrow 0$.

Proof. By KAK decomposition, it suffices to consider $g_n \in A$. Let $g_n = a_{t_n} = \mathrm{diag}(e^{t_n}, e^{-t_n})$ with $t_n \rightarrow \infty$. Assume for a contradiction that $(g_n \cdot \varphi, \psi) \not\rightarrow 0$, we can assume that $(g_n \cdot \varphi, \psi) \rightarrow c \neq 0$. Take a countable dense set $\mathcal{A} \subset \mathcal{H}$ containing φ, ψ above. Passing to a subsequence if necessary, we can assume that $(g_n \cdot \varphi, \psi)$ convergent for every $\varphi, \psi \in \mathcal{A}$. Define

$$f(\varphi, \psi) = \lim_{n \rightarrow \infty} (g_n \cdot \varphi, \psi),$$

which forms a nonzero sesquilinear form on \mathcal{H} . By Riesz representation theorem, there exists $E \in \mathcal{L}(\mathcal{H})$ such that $f(\varphi, \psi) = (E\varphi, \psi)$.

We want to show that every vector in $\mathrm{Im} E$ is fixed by $\mathrm{SL}(2, \mathbb{R})$. For every $u = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$, we have $g_n^{-1} u g_n \rightarrow \mathrm{id}$. Then

$$(u \cdot E\varphi, \psi) = \lim_{n \rightarrow \infty} (u g_n \cdot \varphi, \psi) = \lim_{n \rightarrow \infty} (g_n \cdot \varphi, \psi) = (E\varphi, \psi).$$

Hence $u \circ E = E$. It follows that $\mathrm{Im} E$ is fixed by U . Similarly, $E \circ v = E$ for every $v = \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix}$. This does not lead to $\mathrm{Im} E$ are fixed by V directly.

We use a trick of considering the adjoint operator. Note that $E^* = \lim g_n^{-1}$ in the weak sense. By the commutativity, we have

$$(E\varphi, E\varphi) = \lim_k \lim_l (g_k \cdot \varphi, g_l \cdot \varphi) = \lim_k \lim_l (g_l^{-1} \cdot \varphi, g_k^{-1} \cdot \varphi) = (E^* \varphi, E^* \varphi).$$

Then $\ker E^* = \ker E$. Hence $\text{Im}(\text{id} - v) \subset \ker E = \ker E^*$. It follows that $E^* \circ v = E^*$ and hence $v^* \circ E = E$. Since $v^* = v^{-1}$ run over V , we get the V -invariance.

Because U, V generates $\text{SL}(2, \mathbb{R})$, we have $\text{Im } E$ is fixed by $\text{SL}(2, \mathbb{R})$ and hence is trivial. We get a contradiction. \square

In the case of $\text{SL}(n, \mathbb{R})$, we can similarly define U^+, U^- as

$$U^+ = \{u : g_n^{-1} u g_n \rightarrow \text{id}\}, \quad U^- = \{u : g_n u g_n^{-1} \rightarrow \text{id}\}.$$

By some calculation on the Lie algebra, we can show that U^+ and U^- together generate $\text{SL}(n, \mathbb{R})$.

§3 Preparation on algebraic groups I (Yuxiang Jiao, Mar 31)

Setting

- G a locally compact second countable group and S a measurable G -space.
- $k \subset K$ where k is a local field (where $\text{char } k = 0$) and K is algebraic closed.
- \mathbb{G} a linear algebraic group defined over k , \mathbb{G}_k is its k -points.
- Regard $\mathbb{G} \subset \text{GL}(n, \mathbb{K})$, it then \mathbb{G}_k has a locally compact topology (the usual topology given by $\text{GL}(n, k)$). We call it the Hausdorff topology.

Theorem 3.1 (Chevalley, [Z13, Proposition 3.1.4])

If $\mathbb{H} \subset \mathbb{G}$ is a k -subgroup of \mathbb{G} , then there is a k -rational representation $\mathbb{G} \rightarrow \text{GL}(n, K)$ and a point $x \in \mathbb{P}^{n-1}(k)$ such that $\mathbb{H}_k = \text{Stab}_{\mathbb{G}_k}(x)$.

There are several definitions.

- A set is called **locally closed** if it is open in its closure.
- A Borel space is called **countably separated** if there exists a countable family of Borel sets $\{A_i\}$ which separate points.
- A Borel space is called **countably generated** if we additionally requires that $\{A_i\}$ generates the Borel σ -algebra.
- Let S be a Borel G -space which is countably separated, we call the action is smooth if S/G is countably separated.

Proposition 3.2

If G acts smoothly on S . Then every quasi-invariant measure on S is supported on an orbit (measurable support).

Theorem 3.3 ([Z13, Theorem 2.1.4])

Suppose G acts continuously on a complete separable metrizable space S . Then the following are equivalent

- (1) All orbits are locally closed.
- (2) The action is smooth.
- (3) For every $s \in S$, $G/\text{Stab}_G(s) \rightarrow \text{Orb}(s)$ is a homeomorphism.

Fact 3.4. Let V, W be varieties and $f : V \rightarrow W$ is a regular map. Then $f(V)$ contains an open set in its closure (in Zariski topology).

Now we consider an algebraic group \mathbb{G} acts algebraically on a variety V . Then for every $x \in V$, the orbit $\mathbb{G}.x$ contains an open subset $U \subset \overline{\mathbb{G}.x}^{\text{Zar}}$. Hence $\mathbb{G}.x = \mathbb{G}.U$ which is open in $\overline{\mathbb{G}.x}^{\text{Zar}}$. Since a Zariski topology is coarser than Hausdorff topology, we deduce (general version needs to show a certain Galois cohomology group is finite)

Theorem 3.5 (Borel-Serre, [Z13, Theorem 3.1.3])

If k is a local field of characteristic 0, and a k -group \mathbb{G} acts k -algebraically on a k -variety V . Then every \mathbb{G}_k -orbit in V_k is locally closed in the Hausdorff topology.

Group actions on the measure space

Let $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(k)$ be the projective space. Let $G = \text{PGL}(n, k)$ with a natural action on $\mathbb{P}^{n-1}(k)$. It induces an action on $\text{Prob}(\mathbb{P}^{n-1})$, the family of probability measures on \mathbb{P}^{n-1} . We equip $\text{Prob}(\mathbb{P}^{n-1})$ with the weak* topology, which makes it a compact metrizable space.

Theorem 3.6 ([Z13, Theorem 3.2.4])

For any $\mu \in \text{Prob}(\mathbb{P}^{n-1})$, the stabilizer $\text{Stab}_G(\mu)$ has a normal subgroup of finite index which is k -almost algebraic (a compact extension of the k -points of a k -group). In particular, if $k = \mathbb{R}$, $\text{Stab}_G(\mu)$ is the real points of an \mathbb{R} -group.

Theorem 3.7 ([Z13, Theorem 3.2.6])

Every G -orbit in $\text{Prob}(\mathbb{P}^{n-1})$ is locally closed, hence $G \curvearrowright \text{Prob}(\mathbb{P}^{n-1})$ is smooth.

§4 Preparation on algebraic groups II (Yuxiang Jiao, Apr 7)

Let us sketch the proof of Theorem 3.6 here.

Lemma 4.1 (Furstenberg)

Let $(g_n) \subset G$ such that $g_n.\mu \rightarrow \nu$ where $\mu, \nu \in \text{Prob}(\mathbb{P}^{n-1})$, then

- (1) either (g_n) is bounded in G ,
- (2) or there exists proper subspaces $V, W \subset k^n$ such that $\text{supp } \nu \subset [V] \cup [W]$.

Corollary 4.2 ([Z13, Corollary 3.2.2])

Let $\mu \in \text{Prob}(\mathbb{P}^{n-1})$, then

- (1) either $\text{Stab}_G(\mu)$ is compact,
- (2) or there exists a proper subspace $V_0 \subset k^n$ such that $\mu([V_0]) > 0$ and $\text{Stab}_G(\mu).[V_0] = [V_0] \cup [V_1] \cup \cdots \cup [V_r]$, a finite union of proper subspaces.

Proof of Theorem 3.6. Decompose μ into a sum of countably many $\mu_i \in \text{Prob}(\mathbb{P}^{n-1})$, such that for each μ_i :

- (i) μ_i is invariant under $\text{Stab}_G(\mu)$.
 - (ii) $\text{supp } \mu_i \subset [V_{i0}] \cup [V_{i1}] \cup \cdots \cup [V_{ir_i}]$, a finite union of subspaces with same dimension.
 - (iii) for each $V \subset k^n$ with $\dim V < \dim V_{i0}$, $\mu_i(V) = 0$.
- Then $\text{Stab}_G(\mu) = \bigcap_i \text{Stab}_G(\mu_i)$. For each i , we consider

$$H_i = \{g \in G : g \cdot [V_{i0}] \subset [V_{i1}] \cup \cdots \cup [V_{ir_i}]\}, \quad N_i = \{g \in G : g|_{V_{i0}} \text{ is a scalar}\}.$$

Then $\bigcap_i N_i \subset \text{Stab}_G(\mu) \subset \bigcap_i H_i$. Since H_i, N_i are algebraic, the intersection can be replaced by a finite intersection. By previous lemma, we have

$$\bigcap_{i \in F} N_i \subset_{\text{Cocompact}} \text{Stab}_G(\mu) \cap \bigcap_{i \in F} H'_i \subset_{\text{Finite index}} \text{Stab}_G(\mu) \subset \bigcap_{i \in F} H_i,$$

where $H'_i := \{g \in G : g \cdot [V_{ij}] = [V_{ij}], \forall j\}$. □

Theorem 4.3 (Borel density theorem)

Let \mathbb{G} be a connected semisimple \mathbb{R} -group, $G = \mathbb{G}_{\mathbb{R}}^0$ and assume that G has no compact factor. Let Γ be a closed subgroup such that G/Γ has a finite G -invariant measure. Then

1. Γ is Zariski dense in \mathbb{G} .
2. Γ^0 is normal in G . In particular, if G is simple and Γ is a proper subgroup, then Γ is discrete.

Proof. Let \mathbb{H} be the Zariski closure of Γ and $H = \mathbb{H} \cap G$. Since G is Zariski dense in \mathbb{G} [Z13, Theorem 3.1.9], it suffices to show $H = G$. By Chevalley's theorem (Theorem 3.1), there is a \mathbb{R} -regular homomorphism $\mathbb{G} \rightarrow \text{GL}(n, \mathbb{C})$ such that $H = \text{Stab}_G(x)$ for some $x \in \mathbb{P}^{n-1}(\mathbb{R})$. WLOG, we assume that $G \cdot x$ linearly spans $\mathbb{P}^{n-1}(\mathbb{R})$. The conclusion follows if $n = 1$.

Assume that $n \geq 2$. Since G/H has a finite G -invariant measure, there is also a G -invariant measure μ on $G \cdot x \subset \mathbb{P}^{n-1}(\mathbb{R})$. Note that G has no compact factor and hence there is a proper subspace V with $\mu([V]) > 0$ and $\mu([V']) = 0$ for every proper subspace $V' \subset V$. Then $G \cdot V$ is a finite union of proper subspaces, by connectedness, $G \cdot V = V$. But $G \cdot x \cap [V] \neq 0$ since $\mu(G \cdot x) = 1$ and $\mu([V]) > 0$, hence $G \cdot x \subset [V]$. We get a contradiction. □

Theorems 3.6 and 3.7 together gives a clear description of the action of $\text{PGL}(n, k)$ on $\text{Prob}(\mathbb{P}^{n-1}(k))$. There are several corollaries as below.

Corollary 4.4 ([Z13, Corollary 3.2.12])

If $\mathbb{G} \subset \text{PGL}(G, K)$ is a k -group, then the action of \mathbb{G}_k on $\text{Prob}(\mathbb{P}^{n-1}(k))$ is smooth.

Proof. It suffices to consider \mathbb{G}_k -orbits on $G \cdot \mu$ and note that $G \cdot \mu \cong G/\text{Stab}_G(\mu)$. □

Corollary 4.5 ([Z13, Corollary 3.2.17])

If $\mathbb{H} < \mathbb{G}$ are k -groups such that $\mathbb{G}_k/\mathbb{H}_k$ is compact, then \mathbb{G}_k acts smoothly on $\text{Prob}(\mathbb{G}_k/\mathbb{H}_k)$.

Corollary 4.6 ([Z13, Corollary 3.2.18])

If $\mathbb{H} < \mathbb{G}$ are \mathbb{R} -groups such that $\mathbb{G}_{\mathbb{R}}/\mathbb{H}_{\mathbb{R}}$ is compact, then for every $\mu \in \text{Prob}(\mathbb{G}_{\mathbb{R}}/\mathbb{H}_{\mathbb{R}})$, $\text{Stab}_{\mathbb{G}_{\mathbb{R}}}(\mu)$ is the real points of an \mathbb{R} -group.

Group actions on the function space

Let X be a σ -finite measure space and V be a locally compact space. Denote $F(X, V)$ be the space of measurable maps $f : X \rightarrow V$. We endow $F(X, V)$ with the topology in the sense of converging in measure. Then $F(X, V)$ is a complete separable metrizable space.

Proposition 4.7

Let \mathbb{G} be a k -group and V be a k -variety, \mathbb{G} acts k -regularly on V . Then the action of \mathbb{G}_k on $F(X, V_k)$ is smooth and the stabilizers are k -points of a k -group.

Let V be an \mathbb{R} -variety. Define

$$\text{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{C})) := \{f \text{ is the restriction to } V_{\mathbb{R}} \text{ of an } \mathbb{R}\text{-rational function } f : V \rightarrow \mathbb{P}^m(\mathbb{C})\}.$$

Proposition 4.8

Let \mathbb{G}, \mathbb{H} be \mathbb{R} -groups acting on $\mathbb{P}^n(\mathbb{C}), \mathbb{P}^m(\mathbb{C})$ respectively. Let $V \subset \mathbb{P}^n(\mathbb{C})$ be a closed \mathbb{G} -invariant \mathbb{R} -subvariety, such that $V_{\mathbb{R}}$ is Zariski dense in V . Then $\mathbb{G}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}$ induces an action on $\text{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{R}))$. We have

1. The $\mathbb{G}_{\mathbb{R}}, \mathbb{H}_{\mathbb{R}}$ and $\mathbb{G}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}$ actions on $\text{Rat}(V_{\mathbb{R}}, \mathbb{P}^m(\mathbb{R}))$ are smooth.
2. The stabilizers are real points of algebraic \mathbb{R} -groups.

§5 Margulis super-rigidity theorem I (Jiesong Zhang, Apr 7)

Let \mathbb{G} be a connected semisimple \mathbb{R} -group, $G = \mathbb{G}_{\mathbb{R}}^0$ and assume that G has trivial center and no compact factors. Let $\Gamma \subset G$ be an irreducible lattice. Let $H = \mathbb{H}_k$ be the k -points of a k -group (take $k = \mathbb{R}$ today), which is center-free. Let $\varphi : \Gamma \rightarrow H$ be a homomorphism such that

1. $\varphi(\Gamma)$ is Zariski dense and,
2. unbounded.

Today's main result is the following lemma.

Lemma 5.1

There are proper algebraic \mathbb{R} -subgroups $\mathbb{P} \subset \mathbb{G}, \mathbb{L} \subset \mathbb{H}$ and a Γ -map $\psi : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \rightarrow \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}$.

Definition 5.2. We say $\mathbb{H} \subset \mathbb{G}$ is **parabolic** if \mathbb{G}/\mathbb{H} is a projective variety.

Proposition 5.3

If \mathbb{G} is a k -group and \mathbb{H} is a parabolic subgroup of \mathbb{G} . Then $\mathbb{G}_k/\mathbb{H}_k$ is compact.

Definition 5.4. Let G be a topological group. We say G is **amenable**, if every continuous G -action on a compact metrizable space admits a G -invariant probability measure.

Proposition 5.5

Let \mathbb{P} be a minimal parabolic subgroup of \mathbb{G} and $\Gamma \subset G$ is a lattice. Then \mathbb{P} is an amenable group and Γ acts amenably on \mathbb{G}/\mathbb{P} .

The definition of amenable action, see

Proposition 5.6

Let S be an amenable Γ -space and X be a compact G -space. Then there is a measurable Γ -map $S \rightarrow \text{Prob}(X)$.

We will skip the definition of an amenable action. We proof the following result directly.

Proposition 5.7

If $\Gamma \curvearrowright X$ where X is a compact metrizable space. Then there exists a Γ -map $\omega : \mathbb{G}/\mathbb{P} \rightarrow \text{Prob}(X)$.

Proof. Let μ be the Haar measure on \mathbb{G} . Consider the action

$$(\Gamma \times \mathbb{G}) \curvearrowright (\mathbb{G} \times X), \quad (\gamma, g)(h, x) = (\gamma hg^{-1}, \gamma x).$$

Let $p : \mathbb{G} \times X \rightarrow \mathbb{G}$ be the projection. Let Q be the family of Borel measures τ on $\mathbb{G} \times X$ satisfying $p_*\tau = \mu$ and $(\gamma, 1)_*\tau = \tau$. We claim that Q is nonempty. In fact, let D be a fundamental domain of Γ and $x_0 \in X$, let $\phi : \mathbb{G} \rightarrow \mathbb{G} \times X$ given by $g \mapsto (g, \gamma_g x_0)$ where $\gamma_g \in \Gamma$ is the unique element such that $g \in \gamma_g D$. Then ϕ is Γ -equivalent and hence $\phi_*\mu \in Q$.

Note that Q is a compact and convex set and Q is $(\Gamma \times \mathbb{G})$ -invariant. Recall that \mathbb{P} is amenable, then there exists a $(1, \mathbb{P})$ -invariant element $\tau \in Q$. Write

$$\tau = \int_{\mathbb{G}} \delta_g \otimes \nu_g d\mu(g), \quad \nu_g \in \text{Prob}(X).$$

We can see that $\nu_g = \gamma_*\nu_{\gamma^{-1}gp} = \nu_{gp}$ for almost every g . It induces a Γ -map $\omega : gp \mapsto \nu_g$. \square

Proof of Lemma 5.1. Let $\mathbb{P} \subset \mathbb{G}$ be a minimal parabolic group and $\mathbb{P}' \subset \mathbb{H}$ be a parabolic subgroup. Then $\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}}$ is a compact space. Note that Γ acts amenably on $\mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}}$. Hence there is a Γ -map

$$\varphi : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \rightarrow \text{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}}).$$

It induces a map $\tilde{\varphi} : \mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}} \rightarrow \text{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$, which is Γ -invariant. Recall that the action $\mathbb{H}_{\mathbb{R}} \curvearrowright \text{Prob}(\mathbb{H}_{\mathbb{R}}/\mathbb{P}'_{\mathbb{R}})$ is smooth. Hence $\tilde{\varphi}$ is essential constant. Hence $\varphi(\mathbb{G}_{\mathbb{R}}/\mathbb{P}_{\mathbb{R}})$ falls in an orbit $\mathbb{G}.\mu$. Take $\mathbb{L}_{\mathbb{R}} = \text{Stab}_{\mathbb{H}_{\mathbb{R}}}(\mu)$, the conclusion follows. \square

§6 Margulis super-rigidity theorem II (Bohan Yang, Apr 14)

Let us recall the Margulis' superrigidity theorem.

Theorem 6.1 (Margulis' superrigidity)

Let \mathbb{G} be a connected semisimple algebraic \mathbb{R} -group with \mathbb{R} -rank at least 2. Assume that $\mathbb{G}_{\mathbb{R}}^0$ has no compact factors. Let $\Gamma \subset \mathbb{G}_{\mathbb{R}}^0$ be an irreducible lattice. Let \mathbb{H} be a connected simple algebraic \mathbb{R} -group and $\mathbb{H}_{\mathbb{R}}$ is not compact. Assume that $\pi : \Gamma \rightarrow \mathbb{H}_{\mathbb{R}}$ is a homomorphism with $\pi(\Gamma)$ Zariski dense. Then π extends to a rational homomorphism $\mathbb{G} \rightarrow \mathbb{H}$ defined over \mathbb{R} .

Throughout this section, we will use the notation in Zimmer's book [Z13], which is terrible. There, $G/\Gamma = \{\Gamma \cdot g : g \in G\}$ and the action $G \curvearrowright X$ is always an right action $(g, x) \mapsto xg$. This means that G has a natural (right) action on G/Γ .

Lemma 6.2 ([Z13, Lemma 5.1.3])

Suppose $\mathbb{P} \subset \mathbb{G}$ and $\mathbb{L} \subset \mathbb{H}$ are proper algebraic \mathbb{R} -subgroups, and $\varphi : \mathbb{G}/\mathbb{P} \rightarrow \mathbb{H}/\mathbb{L}$ is a rational Γ -map, then π extends to a rational homomorphism $\mathbb{G} \rightarrow \mathbb{H}$.

Hence it suffices to find such rational Γ -map φ . We will use the map constructed last time (Lemma 5.1). The aim is to show the constructed map is (essentially) rational (Step 2 in [Z13]).

Definition 6.3. Let V be a complex variety and W be an \mathbb{R} -variety. Let $A \subset V_{\mathbb{R}}$ be a set of positive measure. We say $f : A \rightarrow V$ is **essentially rational** if there exists a rational map $R : W \rightarrow V$ such that $R = f$ on A .

We want to show that φ is rational. On criterion for rationality is a unipotent representation of a unipotent group. So want to replace $\mathbb{G}_{\mathbb{R}}^0/P_0$ by a such group, where $P_0 = \mathbb{P} \cap \mathbb{G}_{\mathbb{R}}^0$.

Lemma 6.4 ([Z13, Lemma 5.1.4])

There exists a connected unipotent \mathbb{R} -subgroup $U \subset \mathbb{G}$ such that the product map $m : U \times \mathbb{P} \rightarrow \mathbb{G}$ is injective and the image is a Zariski dense \mathbb{R} -open set. Furthermore, it induces a map $U_{\mathbb{R}} \rightarrow \mathbb{G}_{\mathbb{R}}^0/P_0$ which is a measure space isomorphism.

In our case, $\mathbb{G} = \mathrm{SL}(n, \mathbb{C})$ and \mathbb{P} is the triangular matrices. Then we can take U to be the lower triangular matrices with diagonal entries equal to 1.

Lemma 6.5 ([Z13, Lemma 5.1.5])

It suffices to show for some $g \in \mathbb{G}_{\mathbb{R}}^0$, the map $u \mapsto \varphi(ug)$ is essentially rational on $U_{\mathbb{R}}$.

Definition 6.6. For every $t \in A \subset \mathbb{G}$, let C_t be the centralizer of t in \mathbb{G} . Let $C_t^u = C_t \cap U$.

Lemma 6.7 ([Z13, Lemma 5.1.6])

There exists $t_1, \dots, t_n \in A_{\mathbb{R}}^0$, $t_i \neq e$ and connected subgroups $U_i \subset C_{t_i}^u$ such that

- (1) $\prod_{i=1}^r U_i \rightarrow U$ is an \mathbb{R} -isomorphism.
- (2) For each r , $\prod_{i=1}^r U_i \subset U$ is a subgroup and $\prod_{i=r+1}^n U_i$ is normal in $\prod_{i=r}^n U_i$.

Lemma 6.8 ([Z13, Lemma 5.1.7])

To prove Step 2, it suffices to prove if $e \neq t \in A_{\mathbb{R}}^0$, $V \subset C_t^0$ is a connected algebraic \mathbb{R} -group, then for almost every $g \in \mathbb{G}_{\mathbb{R}}^0$, $u \mapsto \varphi(ug)$ is essentially rational on $V_{\mathbb{R}}$.

Proof. Induction on $n - r$, we prove that $\varphi : u \mapsto \varphi(ug)$ is essentially rational on $\prod_{i=r}^n (U_i)_{\mathbb{R}}$. If $r = n$, then this is the “suffices to show” part. Suppose we have $u \mapsto \varphi(ug)$ is essentially rational on $\prod_{i=r}^n (U_i)_{\mathbb{R}}$. We define $\varphi_g(c, u) = \varphi(cug)$, $c \in U_{r-1}$. It suffices to show φ_g is essentially rational for almost every g .

By the “suffices to show” part, for every u and almost every g , $c \mapsto \varphi(cug)$ is essentially rational. By Fubini, for almost every g , $c \mapsto \varphi_g(c, u)$ is essentially rational. On the other hand, $\varphi(cug) = \varphi((cuc^{-1})cg)$, then for every c , for almost every g , $u \mapsto \varphi(cug)$ is essentially rational. By another Fubini and Theorem [Z13, Theorem 3.4.4], we have for almost every g , φ_g is essentially rational. \square

Lemma 6.9 ([Z13, Lemma 5.1.8])

To prove Step 2, it suffices to prove if $e \neq t \in A_{\mathbb{R}}^0$, then for almost every g , there exists

- (1) an \mathbb{R} -subvariety $W_g \in \mathbb{H}/\mathbb{L}$ such that $\varphi_g : (C_t)_{\mathbb{R}}^0 \rightarrow \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}, c \mapsto cg$ satisfies $\varphi_g(c) \in W_g$ for almost all c ;
 - (2) an \mathbb{R} -algebraic group Q_g which acts \mathbb{R} -regularly on W_g ;
 - (3) a measurable homomorphism $h_g : (C_t)_{\mathbb{R}}^0 \rightarrow (Q_g)_{\mathbb{R}}$;
 - (4) a point $x_g \in W_g \cap \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}$;
- such that $\varphi_g(c) = x_g h_g(c)$ for almost all $c \in (C_t)_{\mathbb{R}}^0$.

Proof. Let $V \subset C_t^u$ be a connected algebraic \mathbb{R} -group. If $\varphi_g = x_g h_g$ holds for all c , then $h_g|_{V_{\mathbb{R}}}$ is unipotent by [Z13, Proposition 3.4.2] and hence $\varphi_g|_{V_{\mathbb{R}}}$ is rational. But $V_{\mathbb{R}}$ is of measure zero in $(C_t)_{\mathbb{R}}^0$, we need a further argument. For each $u \in V_{\mathbb{R}}$ and almost all $g \in \mathbb{G}_{\mathbb{R}}^0, c \in (C_t)_{\mathbb{R}}^0$, we have

$$\varphi(ucg) = x_g h_g(uc) = x_g h_g(u) h_g(c).$$

By Fubini, there exists a fixed c such that the equation holds for almost every g and almost every $u \in V_{\mathbb{R}}$. Therefore, $u \mapsto x_g h_g(u) h_g(c)$ is rational. Hence $u \mapsto \varphi(ug)$ is essentially rational. \square

Proposition 6.10 ([Z13, Proposition 3.5.2])

Let C be a locally compact group and $\varphi \in F(C, \mathbb{H}_k/\mathbb{L}_k)$. For every $g \in C$, let $\varphi_g \in F(C, \mathbb{H}_k/\mathbb{L}_k), \varphi_g(c) = \varphi(cg)$. Assume that almost every φ_g lie in a single \mathbb{H}_k -orbit of $F(C, \mathbb{H}_k/\mathbb{L}_k)$, then there exists (1)(2)(3)(4) as above.

Proof of Step 2. By the above proposition, we should check that for almost every $g \in \mathbb{G}_{\mathbb{R}}^0$, for almost every $c \in C = (C_t)_{\mathbb{R}}^0$, $(\varphi_g)_c$ lies in a common $\mathbb{H}_{\mathbb{R}}$ -orbit. By a Fubini argument, it suffices to show that almost every φ_g lies in a same $\mathbb{H}_{\mathbb{R}}$ -orbit.

Define $\Phi : \mathbb{G}_{\mathbb{R}}^0 \rightarrow F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}}), g \mapsto \varphi_g$. Let $T = \{t^n\} \subset A$, which is unbounded. Then

$$\varphi_{tg}(c) = \varphi(ctg) = \varphi(tcg) \stackrel{T \subseteq P_0}{=} \varphi(cg) = \varphi_g(c).$$

Hence Φ is a T -invariant measurable map, which induces $T : \mathbb{G}_{\mathbb{R}}^0/T \rightarrow F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})$. Recall φ is a Γ -map, then $\varphi_{g\gamma} = \varphi_g \pi(\gamma)$. Note that $\pi(\gamma) \in \mathbb{H}_{\mathbb{R}}$, consider the induced map

$$\bar{\Phi} : \mathbb{G}_{\mathbb{R}}^0/T \rightarrow F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$$

which is essentially Γ -invariant. Since T is unbounded, $\Gamma \curvearrowright \mathbb{G}_{\mathbb{R}}^0/T$ is ergodic. Combining with $F(C, \mathbb{H}_{\mathbb{R}}/\mathbb{L}_{\mathbb{R}})/\mathbb{H}_{\mathbb{R}}$ is countably separated, $\bar{\Phi}$ is essentially constant. This complete the proof. \square

§7 Margulis arithmeticity theorem (Apr 21)

Definition 7.1 (Restriction of scalar). Let $[k : \mathbb{Q}] = d$ and \mathbb{G} be a k -algebraic group. We define the \mathbb{Q} -algebraic group $R_{k/\mathbb{Q}}\mathbb{G}$ such that

$$R_{k/\mathbb{Q}}\mathbb{G} \cong \prod_{i=1}^d \mathbb{G}^{\sigma_i},$$

where $\sigma_1, \dots, \sigma_d$ are the \mathbb{Q} -embeddings of $k \hookrightarrow \mathbb{C}$.

Proposition 7.2 $(R_{k/\mathbb{Q}}(\mathbb{G}))_{\mathbb{Q}} \cong \mathbb{G}_k$ and $(R_{k/\mathbb{Q}}(\mathbb{G}))_{\mathbb{Z}} \cong \mathbb{G}_{\mathcal{O}_k}$.

Theorem 7.3 (Margulis Arithmeticity)

Let G be a semisimple real Lie group with $\text{rank}_{\mathbb{R}} G \geq 2$ without compact factor. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is arithmetic.

The aim is to put Γ into some $\mathbb{G}_k \cong (R_{k/\mathbb{Q}}\mathbb{G})_{\mathbb{Q}}$. Then we consider the Zariski closure $\overline{\alpha(\Gamma)} = \mathbb{H}$, the rational points $\mathbb{H}_{\mathcal{O}_k}$ will be a desired construction.

First we want to find an algebraic extension k/\mathbb{Q} such that $\Gamma \subset \mathbb{G}_k$. Note that G can be equipped with an algebraic structure.

Lemma 7.41. There exists an embedding $\pi : \Gamma \rightarrow \text{GL}(m, k)$.

Proof. For every $g \in G$, we define $T(g) = \text{tr}(\text{Ad}(G))$, then T is a polynomial. Let V be the linear space of $\{gT\}$ which is finite dimensional with an G -action on it. It induces a G representation which is faithful. Since Γ is Zariski dense in G , there is $\{\gamma_1, \dots, \gamma_m\} \subset \Gamma$ such that $\{\pi(\gamma_i)T\}$ is a basis of V . We need the following fact.

Fact 7.5. For every $\gamma \in \Gamma$, $\text{tr}(\text{Ad}(\gamma))$ is algebraic.

Proof. It suffices to show for every $\gamma \in \Gamma$, $\text{Aut}(\mathbb{C})(\text{tr}(\text{Ad}(\gamma)))$ is bounded. Note that for every σ , we have $\sigma(\text{tr}(\text{Ad}(\gamma))) = \text{tr}(\text{Ad}(\sigma(\gamma)))$. It suffices to show the following fact.

Fact 7.6. $\{\text{tr}(\text{Ad}(\sigma(\gamma))) : \sigma \in \text{Aut}(\mathbb{C})\}$ is bounded.

Proof. Let $G = \prod H_i$ and $\mathfrak{g} = \sum \mathfrak{h}_i$ be the Lie algebras. □

□

Then for every $\gamma, \pi(\gamma) \in \text{GL}(n, k)$. This can be shown by the following way. Let c_{ij} be coefficient of matrices. Then for every $\gamma \in \Gamma$, we have

$$\pi(\gamma)(\pi(\gamma_i)T) = \sum_{j=1}^m c_{ij}(\pi(\gamma_j)T),$$

expand T into $\text{tr}(\text{Ad})$, which is algebraic for every element in Γ . The conclusion follows. □

Indeed, Γ is finitely generated. Hence k is a finite algebraic extension.

Proof of Theorem 7.3. Let $[k : \mathbb{Q}] = d$. We take the restriction of scalar, let $\alpha : G_k \rightarrow (R_{k/\mathbb{Q}}G)_{\mathbb{Q}}$ be the isomorphism. Let $H = \overline{\alpha(\Gamma)}$, which is an algebraic \mathbb{Q} -group. Let $P : R_{k/\mathbb{Q}}(G) \rightarrow G$ such that $(p \circ \alpha)|_{\mathbb{G}_k} = \text{id}$. Note that Γ is Zariski dense in G , we have $P(H) = G$. Since G is semisimple and center-free, we have $P(\text{Rad}(H)) = \text{id}$ and $P(C(G)) = \text{id}$. Hence we can also assume that H is semisimple and center-free. H is also connected since G is connected.

Claim 7.7. $(\ker P)_{\mathbb{R}}$ is compact.

Proof. Let F be a simple factor of $\ker P$, it suffices to check $F_{\mathbb{R}}$ is compact. Assume that $F_{\mathbb{R}}$ is non-compact, by Margulis' super-rigidity theorem, the map $G \xrightarrow{\alpha} H \xrightarrow{\text{projection}} F$ extends to a rational homomorphism $h : G \rightarrow F$. Contradicts $\alpha(\Gamma)$ is Zariski dense in H . □

For a prime p , let \mathbb{Q}_p be the p -adic field. □

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