Selected Minicourses in *Beyond Uniform Hyperbolicity 2023*

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Contents

1 Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori (Yi Shi)

§1.1 Local Rigidity (Apr 25)

Definition 1.1.1. $f \in \mathrm{Diff}^1(M)$ is **Anosov** if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

Example 1.1.2 (Arnold's cat map)
$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

Question 1.1.3. Is every Anosov diffeomorphism transitive?

Question 1.1.4. Topological classification of Anosov diffeomorphism.

Theorem 1.1.5 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Theorem 1.1.6 (Franks-Newhouse)
Every codimension-1 Anosov diffeomorphism must be supported on $\mathbb{T}^d$.

Definition 1.1.7. $f \in \mathrm{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \to (0,1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0,1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \to M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.1.10 $E^s \oplus E^u$ integrable $\implies$ NOT accessible.

However, Dolgopyat-Wilkinsonm and Hertz-Hertz-Ures, etc. showed that "MOST" partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

$$\textbf{Geometric Rigidity} \iff \textbf{Dynamic Spectral Rigidity}$$

That is, $E^s \oplus E^u$ is integrable $\implies$ E^c has exponents rigidity.

Example 1.1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22}-close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \to \mathbb{T}^4$ which is C^1-along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \to \mathbb{T}^2$ be C^1-close to A. Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$$

is an Anosov diffeomorphism C^1-close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

NO exponents rigidity.

Main Theorem: Local Rigidity. Assume that $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K \to \infty} \frac{\#\{\, A \text{ is generic} : \|A\| \leqslant K \,\}}{\#\{\, A : \|A\| \leqslant K \,\}} = 1.$$

Denote
$$T\mathbb{T}^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$
the finest dominated splitting, then $\dim L_i^{s/u} \leqslant 2$.

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting
$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$
Assume that $l \geqslant 2$ and $1 \leqslant k < l$. Denote
$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$
Then
$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$$
makes f be an absolutely partially hyperbolic system.

Theorem 1.1.13 (Local rigidity, Gogolev-Shi, arXiv: 2207.00704)

Assume $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \mathrm{Diff}^2(\mathbb{R}^d)$ be C^1-close to A, the following are equivalent:

1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws}:
$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \cdots, l.$$
3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws}.

Dimension 3 case.

Theorem 1.1.14 (Hammerlindl-Ures, 2014)

Let $f \in \mathrm{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \mathrm{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then

- either f is accessible, thus ergodic.
- or there exists an f-invariant minimal foliation $\mathscr{F}^{su}$ such that $T\mathscr{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_*.

Theorem 1.1.15 (Gan-Shi, 2020)

Let $f \in \mathrm{Diff}_m^{1+}(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:

- $E^s \oplus E^u$ is integrable;
- f has spectral rigidity in E^c: $\lambda^c(f) \equiv \lambda^c(f_*)$.

Both imply f is Anosov.

Corollary 1.1.16 Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.1.13 — spectral rigidity $\Longrightarrow$ joint integrability. The case of all E_i^s are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \mathrm{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E_l^s \Longrightarrow$ smooth conjugacy in $E_l^s \Longrightarrow h(\mathscr{F}_{l-1}^s) = \mathscr{L}_{l-1}^s$ (+spectral rigidity in E_{l-1}^s) $\Longrightarrow$ smooth conjugacy in $E_{l-1}^s \Longrightarrow \cdots \Longrightarrow h(\mathscr{F}_{k+1}^s) = \mathscr{L}_{k+1}^s$ (+spectral rigidity in E_{k+1}^s) $\Longrightarrow h(\mathscr{F}^{ss}) = \mathscr{L}^{ss} \Longrightarrow \mathscr{F}^{ss} \oplus \mathscr{F}^u = h^{-1}(\mathscr{L}^{ss} \oplus \mathscr{L}^u)$ joint integrability.

Proof of Theorem 1.1.13 — joint integrability $\Longrightarrow$ spectral rigidity. Main ideas:

1. $E^{ss} \oplus E^u$ integrability $\Longrightarrow h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathscr{F}^{ss} \Longrightarrow$ spectral rigidity in E_{k+1}^s.
3. $E^{ss} \oplus E_{k+1}^s \oplus E^u$ is integrable, and play induction on E_{k+2}^s.

Lemma 1.1.17

For every $1 \leqslant i \leqslant l$, the conjugation h preserves the center foliation: $h(\mathscr{F}_{(i,l)}^s) = \mathscr{L}_{(i,l)}^s$. Here, $\mathscr{F}_{(i,l)}^s$ and $\mathscr{L}_{(i,l)}^s$ are the foliations tangent to $E_i^s \oplus \cdots \oplus E_l^s$ and $L_i^s \oplus \cdots \oplus L_l^s$, respectively.

Proof. Since f is C^1-close to A, we have

$$\|A_{L_{i-1}^s}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E_i^s(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \to \mathbb{R}^d$ be lifts of f and h, then $y \in \widetilde{\mathscr{F}}_{(i,l)}^s(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leqslant (\rho_i - \varepsilon)^{-n}\|x-y\| + C < (\|A_{L_{i-1}^s}\| + \varepsilon)^{-n}\|x-y\| + C,$$

iff $H(y) \in \widetilde{\mathscr{L}}_{(i,l)}^s(H(x))$. □

Lemma 1.1.18

If $\mathscr{F}$ is a C^0-foliation sub-foliated by a minimal linear foliation $\mathscr{L}$ on $\mathbb{T}^d$, then $\mathscr{F}$ is minimal and linear.

Proof. **Minimal.** every leaf $\mathscr{F}(x) \supset \mathscr{L}(x)$ is dense.
Linear. We will show that, on universal cover, $\widetilde{\mathscr{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \widetilde{\mathscr{F}}(0)$, there exists $v_n \to \widetilde{\mathscr{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \to x$. Since $\mathscr{F}$ is sub-foliated by $\mathscr{L}$ and $\mathscr{L}$ is linear, we have

$$y + k_n + v_n \in \widetilde{\mathscr{F}}(y + k_n) = \widetilde{\mathscr{F}}(k_n) = \widetilde{\mathscr{F}}(k_n + v_n).$$

Take $n \to \infty$, then $y + x \in \widetilde{\mathscr{F}}(x) = \widetilde{\mathscr{F}}(0)$. □

Lemma 1.1.19 If $E^{ss} \oplus E^u$ is integrable to $\mathscr{F}^{su}$, then $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.

Proof. Note that $h(\mathscr{F}^{su})$ is sub-foliated by $h(\mathscr{F}^u) = \mathscr{L}^u$, where $\mathscr{L}^u$ is linear and minimal on $\mathbb{T}^d$. Hence $h(\mathscr{F}^{su})$ is linear, A-invariant and transverse to $\mathscr{L}^{ws} = h(\mathscr{F}^{ws})$. This implies $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$. So

$$h(\mathscr{F}^{ss}) = h(\mathscr{F}^s \cap \mathscr{F}^{su}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{su}) = \mathscr{L}^s \cap \mathscr{L}^{su} = \mathscr{L}^{ss}.$$

□

Corollary 1.1.20

Recall that $T\mathcal{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$, then for $T\mathcal{F}_j^s = E_j^s$, we have

$$h(\mathcal{F}_j^s) = \mathcal{L}_j^s, \quad \forall j = k, k+1, \cdots, l.$$

Lemma 1.1.21 (Diophantine approximation of $\mathcal{F}^{ss}$)

There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathcal{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$-dense in $\mathbb{T}^d$.

Proof. Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of $\mathcal{L}^{ss}$. Here A is irreducible and $\mathcal{L}^{ss}$ is algebraic, hence Diophantine. □

Proof of Theorem 1.1.13. We will fist show that the Lyapunov exponent at every point is the same in the dim $E_{k+1}^s = 1$ case. Take $p, q \in \mathrm{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f.

Take

- $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$.
- Segments $J \subset \mathcal{F}_{k+1}^s(p)$ and $J_n \subset \mathcal{F}_{k+1}^s(x_n)$ such that $J_n = \mathrm{Hol}^{ss}(J)$ ($x_n = \mathrm{Hol}^{ss}(p)$). Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{L}_{k+1}^s) = \mathcal{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \quad \Longrightarrow \quad \exists C_0 > 0, |J_n| \geqslant C_0 |J|.$$

Now we choose m_n, k_n such that

- x_n and q are very close in first k_n-steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathcal{F}_1^{ss}(p)$.

Then

$$|f^{m_n}(J_n)| \gtrsim \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n \lambda_{k+1}^s(q)] |J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geqslant \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \to \infty.$$

However, $J_n = \mathrm{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \mathrm{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathcal{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \mathrm{Hol}^{ss}(p)$, this contradicts to $\mathcal{F}^{ss}$ is C^1-smooth in $\mathcal{F}^{ss} \oplus \mathcal{F}_{k+1}^s(p)$.

For the case of dim $E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain

- For every periodic points p, q, we have $\min \lambda_{k+1}^s(p) = \min \lambda_{k+1}^s(q)$.
- Considering the growth of area of local disks, we have

$$\mathrm{Jac}(Df, E_{k+1}^s(p)) = \mathrm{Jac}(Df, E_{k+1}^s(q)), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda_{k+1}^s(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda_{k+1}^s(A)$. □

§1.2 Global Rigidity (Apr 26)

In the last lecture, we have shown a local rigidity result. That is, we only consider diffeomorphisms f that is C^1-close to A. Today we will consider the global rigidity, i.e., the relation between f and $f_* \in \mathrm{GL}(d,\mathbb{Z})$.

Question 1.2.1. What happens if f is not close to $A = f_*$?

Theorem 1.2.2 (Gogolev-Farell)
For $d \geqslant 10$, let $A \in \mathrm{GL}(d,\mathbb{Z})$ be a hyperbolic matrix. Then

$$\mathscr{A}_A^{1+}(\mathbb{T}^d) := \left\{ f \in \mathrm{Diff}^{1+}(\mathbb{T}^d) \;:\; f \text{ is Anosov, } f_* = A \right\}$$

has infinitely many connected components.

Theorem 1.2.3 (Full leaf conjugacy, Gogolev-Shi, arXiv: 2207.00704)
Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with absolutely partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$:

$$\|Df_{E^{ss}}\| < \mu < m(Df|_{E^{ws}}) < \|Df|_{E^{ws}}\| < 1 < m(Df|_{E^u}).$$

If $E^{ss} \oplus E^u$ is integrable, then

1. $A = f_* \in \mathrm{GL}(d,\mathbb{Z})$ is partially hyperbolic:

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u, \quad \dim L^\sigma = \dim E^\sigma, \quad \sigma = ss, ws, u.$$

2. f is dynamically coherent and fully conjugate to A:

$$h(\mathscr{F}^\sigma) = \mathscr{L}^\sigma, \quad \sigma = ss, ws, u.$$

Here $h \circ f = A \circ h$.

Question 1.2.4. Let $f = \mathrm{Diff}^1(\mathbb{T}^d)$ be Anosov with partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$.

- Is $f_* \in \mathrm{GL}(d,\mathbb{Z})$ partially hyperbolic?
- Is f dynamically coherent or not? If yes, does f leaf conjugate to A.

Lemma 1.2.5
Let $\mathscr{F}$ be a C^0-foliation on $\mathbb{T}^d$ with C^1-leaves. If there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to $\mathrm{Id}_{\mathbb{T}^d}$ such that $h(\mathscr{F}) = \mathscr{L}$ is a linear foliation, then $\mathscr{F}$ is quasi-isometric:

$$d_{\widetilde{\mathscr{F}}}(x,y) \leqslant a \cdot d(x,y) + b, \quad \forall x \in \mathbb{R}^d, y \in \widetilde{\mathscr{F}}(x).$$

Here $a, b > 0$ and $\widetilde{\mathscr{F}}$ is the lift of $\mathscr{F}$ in $\mathbb{R}^d$.

Proof of Theorem 1.2.3. Since $h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is sub-foliated by minimal linear foliation $h(\mathscr{F}^u) = \mathscr{L}^u$ is linear. We have $\mathscr{L}^{ss} := h(\mathscr{F}^{ss}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear.

Brin's argument shows that $E^{ws} \oplus E^u$ integrate to $\mathscr{F}^{cu}$ and $h(\mathscr{F}^{cu})$ is linear and minimal. Then $\mathscr{F}^{ws}$ integrate to $\mathscr{F}^{ws}$ and $\mathscr{L}^{ws} := h(\mathscr{F}^{ws})$ is A-invariant and linear.

Note that $\mathscr{L}^{ws}$ and $\mathscr{L}^{ss}$ are transverse in $\mathscr{L}^{s}$, then A admits an invariant splitting $T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u$. We need to show this is a dominated splitting. This follows from the above lemma and the fact that h is homotopic to $\mathrm{Id}_{\mathbb{T}^d}$. □

Theorem 1.2.6 (Global rigidity, Gogolev-Shi, arXiv: 2207.00704)

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be Anosov and irreducible. Assume that f is absolutely partially hyperbolic $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$ and center bunching. If $E^{ss} \oplus E^u$ is integrable, then

1. f has a finest dominated splitting on E^{ws} with the same dimensions for $A|_{L^{ws}}$:

$$E^{ws} = E^{ws}_1 \oplus \cdots \oplus E^{ws}_k, \quad \dim E^{ws}_i = \dim L^{ws}_i.$$

2. f is spectrally rigid along every E^{ws}_i:

$$\lambda(E^{ws}_i, f) \equiv \lambda(L^{ws}_i, A), \quad \forall i = 1, \cdots, k.$$

Remark 1.2.7 • Here f need NOT to be C^1-close to $A = f_*$.

- To get dominated splitting, we usually need some C^1-robust property like: robustly transitive, far from homoclinic bifurcations.
- If $A = f_*$ satisfies the generic assumption in the last lecture, then the conjugacy h is C^{1+}-smooth along $\mathscr{F}^{ws}$.
- The center bunching condition

$$\|Df|_{E^{ws}(x)}\| < m(Df|_{E^{ws}(x)}) \cdot m(Df|_{E^u(x)})$$

is a technical condition, which guarantees C^{1+}-smoothness of $\mathscr{F}^{su}$.

Corollary 1.2.8

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be codimension one with real simple spectrum. For every Anosov $f \in \mathrm{Diff}^2_m(\mathbb{T}^d)$ with $f_* = A$ and

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u, \quad \dim E^{ss} = 1,$$

if

- $E^{ss} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then f is C^{1+}-conjugate to A.

Main idea for showing Theorem 1.2.6. Play the game similar to the last lecture. We will use the Diophantine approximation of $\mathscr{F}^{ss}$ to show the rigidity of smallest exponent in E^{ws}:

$$\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we will show the dimension of $\lambda^{ws}_{\min}$ for each periodic point is constant. Next, we define the Pesin stable foliation $\mathscr{F}^{ws}_{\min}$ and show it is $\mathscr{F}^{su}$-holonomy invariant, that is $\mathrm{Hol}^{su} : \mathscr{F}^{ws}(p) \to \mathscr{F}^{ws}(q)$ preserves $\mathscr{F}^{ws}_{\min}$, for every $p, q \in \mathrm{Per}(f)$. Finally, we show a uniform spectral exponents gap and extract out $\mathscr{F}^{ws}_{\min}$.

Lemma 1.2.9

Let $\mathrm{Hol}^{su}_{x,y} : \mathscr{F}(x) \to \mathscr{F}(y)$ be the holonomy map of $\mathscr{F}^{su}$ with $\mathrm{Hol}^{su}_{x,y}(x) = y$ for every $x \in \mathbb{T}^d$ and $y \in \mathscr{F}^{su}(x)$. Then

$$\mathrm{Hol}^{su}_{x,y}(K) = h^{-1} \circ T_{h(x),h(y)} \circ h(K).$$

Here $T_{h(x),h(y)} : \mathbb{T}^d \to \mathbb{T}^d$ is the linear translation send $h(x)$ to $h(y)$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $K \subset \mathscr{F}^{ws}(x)$ with $\mathrm{diam}(K) > \varepsilon$, then

$$\mathrm{diam}(\mathrm{Hol}^{su}_{x,y}(K)) > \delta, \quad \forall y \in \mathscr{F}^{su}(x).$$

Remark 1.2.10 The same holds for $\mathrm{Hol}^{ss}_{x,y} : \mathscr{F}^{ws}(x) \to \mathscr{F}^{ws}(y)$ where $y \in \mathscr{F}^{ss}(x)$.

Proof. It follows immediately from f is fully conjugate to A. □

Proof of Theorem 1.2.6. We fist show that

Claim 1.2.11. $\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f)$.

Proof. Assume that $\lambda^{ws}_{\min}(p) < \lambda^{ws}_{\min}(q)$. Take $x_n \in \mathscr{F}^{ss}(p)$ such that $d^{ss}(x_n, p) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$. Take disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, the Pesin stable manifold associated to $\lambda^{ws}_{\min}(p)$. Let $D_n = \mathrm{Hol}^{ss}(D) \subset \mathscr{F}^{ws}(x_n)$, then $\mathrm{diam}(D_n) \gg \mathrm{diam}(D)$. Applying a similar (k_n, m_n)-argument, we get a contradiction since $\mathscr{F}^{ss}$ is C^1-smooth in $\mathscr{F}^{ws}(p)$. □

Now we have $\lambda^{ws}_{\min} := \lambda^{ws}_{\min}(p)$ for every $p \in \mathrm{Per}(f)$. We define the Pesin stable foliation associated to $\lambda^{ws}_{\min}$ for each periodic point.

Claim 1.2.12. $\mathscr{F}^{ws}_{\min}$ is Hol^{su}-invariant.

Proof. Let $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)} := h(\mathscr{F}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)})$, it suffices to show

$$T_{h(p),h(x)}(\mathscr{L}^{ws}_{\min}(p)) \subset \mathscr{L}^{ws}_{\min}(x)$$

for every $p, q \in \mathrm{Per}(f)$ and $x \in \mathscr{F}^{ws}(q)$. Otherwise, take a disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, then $T_{h(p),h(x)}(h(D))$ is transverse to $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}_{\mathrm{loc}}(q)}$ at $h(x)$. Take $x_n \in \mathscr{F}^{ss}$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, x) \ll K_n^{-\alpha}$, then

$$D_n := \mathrm{Hol}^{ss}_{p,x_n}(D) \to h^{-1} \circ T_{h(p),h(x)} \circ h(D).$$

It follows that $\mathrm{Hol}^{u}_{\mathrm{loc}}(D)$ is "uniformly transverse" (the angle will not tend to zero) to $\mathscr{L}^{ws}_{\min}$ in $\mathscr{F}^{ws}_{\mathrm{loc}}(q)$, where $\mathrm{Hol}^{u}_{\mathrm{loc}}(D) : \mathscr{F}^{ws}(x_n) \to \mathscr{F}^{ws}(q)$ is C^{1+}-smooth. Since the transverse direction has a weaker contracting rate, we play the (k_n, m_n)-game and get a contradiction. □

Let $\mathscr{L}^{ws}_{\min} := h(\mathscr{L}^{ws}_{\min})$, then the density of $\mathrm{Per}(f)$ and minimality of $\mathscr{F}^{ws}$ imply $T_{x,y}(\mathscr{L}^{ws}_{\min}(x)) \subset \mathscr{L}^{ws}_{\min}(y)$. By the translation invariance and the A-invariance, we have

- $\mathscr{L}^{ws}_{\min}$ is a linear foliation on $\mathbb{T}^d$, and
- $L^{ws}_{\min} := T\mathscr{L}^{ws}_{\min}$ associate to an eigenspace of A.

Also by an estimate of the growth, we get $\lambda(A, L^{ws}_{\min}) \equiv \lambda^{ws}_{\min}$.

Finally, we establish the induction step. Following the idea of [Bonatti-Díaz-Pujals, 2003], consider the quotient cocycle $D\widetilde{f} : E^{ws}/E^{ws}_{\min} \to E^{ws}/E^{ws}_{\min}$ which is Hölder continuous over f. Again by a (k_n, m_n)-game, we can show that λ^{ws}_2 is uniformly larger than $\lambda^{ws}_{\min}$. Then the splitting $T\mathbb{T}^d = (E^{ss} \oplus E^{ws}_{\min}) \oplus F \oplus E^u$ is an absolutely partially hyperbolic splitting. The joint integrability follows from $h(\mathscr{F}^{ss} \oplus \mathscr{F}^{ws}_{\min})$ is linear. □

§1.3 Rigidity on $\mathbb{T}^4$ (Apr 27)

Let us recall some results shown in last two lectures. We remark that the key point is that

$$E^{ss} \oplus E^u \text{ is integrable } \implies h(\mathscr{F}^{ss} = \mathscr{L}^{ss}) \text{ is linear.}$$

Question 1.3.1. Let f be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_j^u \oplus \cdots \oplus E_m^u.$$

What happens if $E_k^s \oplus E_j^u$ is jointly integrable? Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$?

Theorem 1.3.2 (Gogolev-Kalinin-Sadovskya)

Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$ implies $h(\mathscr{F}_k^s) = \mathscr{L}_k^s$ and $h(\mathscr{F}_j^u) = \mathscr{L}_j^u$ hence $E_k^s \oplus E_j^u$ is jointly integrable.

The work of Avila-Viana.

Theorem 1.3.3 (Avila-Viana, 2010)

For every symplectic f which is C^∞-close to A with splitting

$$T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u,$$

then

- either f is accessible and non-uniformly hyperbolic;
- or $E^s \oplus E^u$ is integrable and $\exists h \in \mathrm{Diff}_m^\infty(\mathbb{T}^4)$ such that

$$h \circ f = A \circ h.$$

In particular, f is Bernoulli.

Main Theorem.

Theorem 1.3.4 (Gogolev-Shi, arXiv: 2207.00704)

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an irreducible Anosov automorphism with dominated splitting

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}, \quad \text{with} \quad \dim L^{ws} = \dim L^{wu} = 1.$$

For $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is spectral rigid along E^{ws} and E^{wu}.

Corollary 1.3.5

Let $A \in \mathrm{Sp}(4, \mathbb{Z})$ be hyperbolic and irreducible with dominated splitting

$$T\mathbb{T}^4 = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}.$$

For symplectic $f \in \mathrm{Diff}^2_\omega(\mathbb{T}^4)$ be C^1-close to A with

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is C^{1+}-smoothly conjugate to A.

Proof of corollary. If $E^{ss} \oplus E^{uu}$ is integrable, then we have spectral rigidity in $E^{ws} \oplus E^{wu}$, h is smooth along $E^{ws} \oplus E^{wu}$ and $h(\mathcal{L}^{ss}) = \mathcal{L}^{ss}, h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$. Since h is smooth along $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$, the holonomy map $\mathrm{Hol}^{su}_{\mathcal{F}}$ is C^{1+}. Then we use the symplectic structure that $E^c = E^{ws} \oplus E^{wu}$ is perpendicular to E^{su} (with respect to ω). Hence $\mathcal{F}^{ws} \oplus \mathcal{F}^{wu}$ is C^{1+}. Then we can show that h is absolutely continuous in $\mathcal{F}^{su}$ and hence h is C^{1+}. □

Proof of Main Theorem. Main problem is whether $h(\mathcal{F}^{su}) = \mathcal{L}^{su}$ is the linear one? Or equivalently, whether we have $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ or $h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$? This is nontrivial.

Lemma 1.3.6

If one of $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ is integrable, then f is spectral rigid in $E^{ws} \oplus E^{wu}$.

Proof. If $E^{ss} \oplus E^u$ is integrable, then $h(\mathcal{F}^{ss} \oplus \mathcal{F}^u)$ is linear and hence $h(\mathcal{F}^{ss}) = h(\mathcal{F}^{ss} \oplus \mathcal{F}^u) \cap \mathcal{L}^s = \mathcal{L}^{ss}$ is linear. Then both $h(\mathcal{F}^{su})$ and $h(\mathcal{F}^{uu})$ are linear. Then we obtain a spectral rigidity by Theorem 1.1.13. □

The solvable action. Let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$ and $L^c(0) = L^{ws}(0) \oplus L^{wu}(0) \subset \mathbb{R}^d$. Define the linear action

$$\alpha_0 : \Gamma \times L^c(0) \to L^c(0), \quad \alpha_0(k, n)(x) = L^{su}(A^k(x) + n) \cap L^c(0).$$

If we write $n = n^s + n^c + n^u \in L^s \oplus L^c \oplus L^u$, then $\alpha_0(k, n)(x) = A^k x + n^c$.

For $F : \mathbb{R}^4 \to \mathbb{R}^4$ be the lift of f and $F(0) = 0$, then

- $F^k(x + n) = F^k(x) + A^k n, \forall x \in \mathbb{R}^d$ and $\forall n \in \mathbb{Z}^d$.
- $F(\widetilde{\mathcal{F}}^c(0)) = \widetilde{\mathcal{F}}^c(0)$.

Then $\Gamma \curvearrowright \widetilde{\mathcal{F}}^c(0)$ given by

$$\alpha(k, n)(x) = \widetilde{\mathcal{F}}^{su}(F^k(x) + n) \cap \widetilde{\mathcal{F}}^c(0), \quad \forall (k, n) \in \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d, x \in \widetilde{\mathcal{F}}^c(0).$$

Lemma 1.3.7 This is a group action by the solvable group Γ.

Main idea. If both $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ are not integrable, then we can find a free subgroup by a pingpong argument, which contradicts Γ is solvable.

Lemma 1.3.8

If $\alpha(0, n)(\widetilde{\mathcal{F}}^{ws}(0)) \subset \widetilde{\mathcal{F}}^{ws}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$, then both $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$ are linear. The same holds if $\alpha(0, n)(\widetilde{\mathcal{F}}^{wu}(0)) \subset \widetilde{\mathcal{F}}^{wu}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$.

Proof. Note that $\bigcup_{n\in\mathbb{Z}^d}\widetilde{\mathscr{F}}^{ws}(n)$ is dense in $\mathbb{R}^d$ and hence $E^{ss}\oplus E^{ws}\oplus E^{uu}$ jointly integrates to $\mathscr{F}^{su}\oplus\mathscr{F}^{ws}$. Then we deduce the linearity. □

Proof of Theorem 1.3.4. Assume for a contradiction that there exists $n_1, n_2 \in \mathbb{Z}^d$ such that

- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(\alpha(0,n_1)(0))$;
- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(\alpha(0,n_1)(0))$.

Lemma 1.3.9

There exists $m_1, m_2 \in \mathbb{Z}^d$ such that

- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Lemma 1.3.10

For l large enough, $n = A^l m_1 - A^{-l} m_2 \in \mathbb{Z}^d$ satisfies

- $\alpha(0,n)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,n)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Now we consider $F : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0)$ and

$$G : \alpha(0,n)\circ\alpha(1,0)\circ\alpha(0,-n) : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0).$$

Then F is saddle-like dynamics at $\widetilde{\mathscr{F}}^{ws}(0)\cup\widetilde{\mathscr{F}}^{ws}(0)$ near 0. The map G is also saddle-like near $\alpha(0,n)0$. By a pingpong-argument, we can show that $\left\{F^k, G^k\right\}$ generates a free group for a sufficiently large k. This contradicts that Γ is solvable. □

2 Methods for Studying Abelian Actions and Centralizers (Danijela Damjanović / Disheng Xu)

3 Dimension of Stationary Measures (Francios Ledrappier / Pablo Lessa)