NOTES ON THE DIMENSION OF STATIONARY MEASURES

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CONTENTS

1. INTRODUCTION

1.1. The philosophy.

Recall the celebrated entropy formula proved by Ledrappier-Young in 1985. Let $f : M \to M$ be a C^2 diffeomorphism on a closed Riemannian manifold M. Let m be an f-invariant Borel probability measure on M. Then by Oseledec's Theorem, for m-a.e. points $x \in M$, there exists an f-invariant measurable splitting

$$T_x M = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$$

and numbers $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ (called **Lyapunov exponents**) such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df_x^n v\| = \lambda_i(x), \quad \forall v \in E_i(x).$$

Let $E^s(x) = \bigoplus_{\lambda_i(x)<0} E_i(x)$ and $E^u(x) = \bigoplus_{\lambda_i(x)>0} E_i(x)$, let

$$W^s(x) = \left\{ y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\},$$

$$W^u(x) = \left\{ y \in M : \limsup_{n\to-\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\}.$$

Then $W^s(x), W^u(x)$ are immersed submanifolds tangent to $E^s(x), E^u(x)$, respectively. Which are called the **stable and unstable manifold** of x. Denote the **entropy** f with respect to m by $h_m(f)$.

Theorem 1.1 ([LY85a, Theorem A]). *Then*

$$h_m(f) = \int \sum_i \lambda_i^+(x) \dim E_i(x) \mathrm{d}m(x)$$

if and only if m has absolutely continuous conditional measures on unstable manifolds, where $a^+ = \max\{a, 0\}$.

In the second paper published by Lerappier-Young, they give a more precisely formula, which included the dimension of measures. We assume that m is ergodic for convenience. Then the fast unstable direction is integrable as

$$W^i(x) = \left\{ y \in M : \limsup_{n \to -\infty} \frac{1}{n} \log d(f^n x, f^n y) \leqslant -\lambda_i \right\},$$

where $W^i(x)$ is tangent to $\bigoplus_{j \leqslant i} E_j(x)$. We consider the conditional measure of m along W^i, denote by m_x. One can show that

$$\delta_i(x) = \lim_{\varepsilon \to 0^+} \frac{\log m_x B^i(x, \varepsilon)}{\log \varepsilon}$$

exists and is a constant almost every where, where $B^i(x, \varepsilon)$ is the ε-ball along $W^i(x)$. We call δ_i the **(exact) dimension** of m on W^i.

Theorem 1.2 ([LY85b, Theorem C′]). *Assume m is ergodic. Let $\lambda_1 > \cdots > \lambda_u$ denote the distinct positive Lyapunov exponents of f and let δ_i be the dimension of m on W^i. Write $\gamma_i = \delta_i - \delta_{i-1}$ (where $\delta_0 = 0$), then $0 \leqslant \gamma_i \leqslant \dim E_i$ and*

$$h_m(f) = \sum_{i \leqslant u} \lambda_i \gamma_i.$$

These theorems show a philosophy that

$$\textbf{Entropy} = \textbf{Lyapunov Exponent} \times \textbf{Dimension}.$$

1.2. **Hochman's result.**

Setting

- $G = \mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^2$ linearly and induces an action on $\mathbb{RP}^1$.
- μ a finitely supported probability measure on G.
- Assume that G_μ the semigroup generated by $\operatorname{supp} \mu$ is not compact and acts strongly irreducible on $\mathbb{R}^2$.

Recall the Lyapunov exponents of random product of matrices

$$\chi = \chi(\mu) = \lim_{n \to \infty} \frac{1}{n} \log \| X_n \cdots X_1 \|,$$

where X_i are independent identically distributed obeys the distribution μ. Furstenberg's theorem [Fur63] asserts that $\chi > 0$ under the assumptions above. (For notes about Furstenberg's theorem, see here.)

Remark 1.3. We will take the logarithm base 2 for the convenience of later discussion. In fact, we will do some dyadic decomposition on $\mathbb{RP}^1$.

Definition 1.4. A probability measure ν on $\mathbb{RP}^1$ is called **μ-stationary** if

$$\nu = \mu.\nu = \int_G g.\nu \, \mathrm{d}\mu(g),$$

where $g.\nu$ is the push forward of ν under g.

Under the same assumption (in the case of $\mathrm{SL}(2, \mathbb{R})$), it can be shown that μ admits a unique μ-stationary measure ν on $\mathbb{RP}^1$.

Definition 1.5. ν is called **exact dimensional** if there exists $\alpha \geqslant 0$ such that

$$\lim_{r\to 0^+} \frac{\log \nu(B(x,r))}{\log r} = \alpha \tag{1.1}$$

for ν-a.e. x. At this time, α is called the **dimension** of ν, denote by $\dim \nu$.

Remark 1.6. The limit (if exists) in (1.1) is called the **local dimension** of x.

Remark 1.7. Intuitively, ν is of (exact) dimension α means that $\nu(B(x,r)) = r^{\alpha+o(1)}$ for almost every x.

Example 1.8. A Dirac measure is of exact dimension 0.

Example 1.9. The Lebesgue measure on $\mathbb{R}^d$ is of exact dimension d.

Definition 1.10. The random walk entropy of μ is defined as

$$h_{\mathrm{RW}}(\mu) := \lim_{n\to\infty} \frac{1}{n} H(\mu^{\star n}),$$

where $H(\mu^{\star n})$ is the entropy of the n-th convolution of μ.

Example 1.11. Assume $g, h \in \mathrm{SL}(2,\mathbb{R})$ generates a free semigroup. Let

$$\mu = \frac{1}{2}\delta_g + \frac{1}{2}\delta_h,$$

then $H(\mu^{\star n}) = n$. Hence $h_{\mathrm{RW}}(\mu) = 1$.

Example 1.12. Assume $g, h \in \mathrm{SL}(2,\mathbb{R})$ generates a free group. Let

$$\mu = \frac{1}{4}\delta_g + \frac{1}{4}\delta_{g^{-1}} + \frac{1}{4}\delta_h + \frac{1}{4}\delta_{h^{-1}}.$$

Then $H(\mu^{\star n}) = n\log 3 + o(n)$, hence $h_{\mathrm{RW}} = \log 3$.

Fix a left invariant metric $d(\cdot,\cdot)$ on $G = \mathrm{SL}(2,\mathbb{R})$. For a subset $\mathcal{A} \subset \mathrm{SL}(2,\mathbb{R})$, we say $\mathcal{A}$ is **Diophantine** if there exists $c > 0$, such that for every $A_1, \cdots, A_n$ and $A_1', \cdots, A_n'$ in $\mathcal{A}$,

$$A_1 \cdots A_n \neq A_1' \cdots A_n' \implies d(A_1 \cdots A_n, A_1' \cdots A_n') > c^n.$$

Assumption. $\operatorname{supp}\mu$ *satisfies the Diophantine condition.*

Theorem 1 ([HS17, Theorem 1.1])**.** *Under the assumptions above, the unique μ-stationary measure ν is exact dimensional and*

$$\dim \nu = \min\left\{1, \frac{h_{\mathrm{RW}}(\mu)}{2\chi}\right\}.$$

Remark 1.13. Why 2χ at the denominator? This is because if we consider g acts on $\mathbb{RP}^1$, then derivate at $[v]$ approximates $(\|gv\| \,/\, \|v\|)^{-2}$. Or, the compression ratio of g approximates $\|g\|^{-2}$. Then regard the random walk on $\mathbb{RP}^1$ as a dissipative system, the Lyapunov exponent is -2χ.

Remark 1.14. The additional Diophantine condition is reasonable that h_{RW} does not distinguish two different elements closed enough. But if two elements are very closed, then the actions on $\mathbb{RP}^1$ are very similar, we need to exclude this case.

2. Exact dimensionality of the stationary measure

2.1. **A baby example.**

In this section, we will show the first key result that a stationary measure is exact dimensional. At first, we give an example of stationary measure on the unit interval $[0,1]$, which is called the self-similar measure. In this baby case, the dimension can be calculated precisely. It might gives more intuition to the dimension of measures.

Example 2.1. Let $F_0, F_1 : [0,1] \to [0,1]$ given by $F_0(x) = \frac{1}{2}x$ and $F_1(x) = \frac{1}{2}(x+1)$. For $0 \leqslant p \leqslant \frac{1}{2}$, consider the probability measure

$$\mu_p = p\delta_{F_0} + (1-p)\delta_{F_1}.$$

Then there exists a unique probability measure ν_p on $[0,1]$ such that $\nu_p = \mu_p.\nu_p$. For every $k > 0$, for $\boldsymbol{i} = (i_1, i_2, \cdots, i_k) \in \{0,1\}^k$, let $F_{\boldsymbol{i}} = F_{i_k} \cdots F_{i_1}$. Then $F_{\boldsymbol{i}}$ maps $[0,1)$ to an interval with length 2^{-k}, denoted by $I_{\boldsymbol{i}}$, which consists of numbers with binary expansion beginning with $0.i_1 \cdots i_k$. By the explicit expansion

$$\nu_p = \mu_p.\nu_p = \mu_p^{\star k}.\nu_p = \sum_{\boldsymbol{i} \in \{0,1\}^k} p_{\boldsymbol{i}} F_{\boldsymbol{i}}.\nu_p,$$

we have

$$\nu_p(I_{\boldsymbol{j}}) = \sum_{\boldsymbol{i} \in \{0,1\}^k} p_{\boldsymbol{i}} F_{\boldsymbol{i}}.\nu_p(I_{\boldsymbol{j}}) = p_{\boldsymbol{j}} = p^{n_0}(1-p)^{n_1},$$

where n_0 and n_1 denote the number of times that 0 and 1, respectively, occurs in $\boldsymbol{j}$. See the following figure, which shows the method of constructing the self-similar measure for $p = \frac{1}{3}$.

By the strong law of large number, we know that for ν_p almost every $x \in [0,1]$, the frequency of 0 occurs in the binary expansion of x at beginning tends to p. For a such typical x, let $I_k(x)$ be the interval $I_{\mathbf{i}}$ with length 2^{-k} containing x. Then

$$\frac{\log \nu_p(I_k(x))}{\log |I_k(x)|} \to -(p\log p + (1-p)\log(1-p)).$$

Hence $\dim \nu_p = -(p\log p + (1-p)\log(1-p))$, which coincides with the entropy of $h_{\mathrm{RW}}(\mu_p)$. (Note that the Lyapunov exponent is $\log\frac{1}{2} = -1$.) Besides, there is also an inequality between the dimension of measure and the dimension of the supports

$$\dim \nu_p \leqslant \dim_{\mathrm{H}} \operatorname{supp} \nu_p = 1.$$

For more discussions about basic properties of dimension of measures and fractal geometry, we refer to [Fal97, Chapter 10].

2.2. **Exact dimensionality of the stationary measure.**

Back to our setting. Let μ be a finitely supported probability measure on $G = \mathrm{SL}(2,\mathbb{R})$ satisfying some conditions and ν be the unique μ-stationary measure on $\mathbb{RP}^1$.

Definition 2.2. The **Furstenberg entropy** of ν is defined by

$$h_{\mathrm{F}}(\nu) = \int\int \log\frac{\mathrm{d}A.\nu}{\mathrm{d}\nu}(x)\mathrm{d}A.\nu(x)\mathrm{d}\mu(A).$$

Remark 2.3. By Jensen's inequality, the Furstenberg entropy is positive in our setting.

Theorem 2.4 ([HS17, Theorem 3.4])**.** *ν is exact dimensional and the dimension equals to $h_{\mathrm{F}}(\nu)/2\chi$.*

Let $\Omega = G^{\mathbb{N}}$ and $\mathbf{P} = \mu^{\mathbb{N}}$ be the probability on Ω. Let $\sigma : \Omega \to \Omega$ be the shift map, explicitly write as $\omega = (X_0, X_1, \cdots) \mapsto (X_1, X_2, \cdots)$. Recall the original proof of Furstenberg's theorem, in which he showed that

$$X_0(\omega)X_1(\omega)\cdots X_{n-1}(\omega).\nu \to \delta_{\pi(\omega)}$$

in the weak* topology. Where $\pi(\omega)$ is now known as the **Furstenberg boundary**. It has two properties that

(i) $X_0(\omega)^{-1}\pi(\omega) = \pi(\sigma\omega)$,
(ii) $\int \pi(\omega)\mathrm{d}\mathbf{P}(\omega) = \nu$.

For prove the exact dimensionality, it suffices to show that for typical ω, the local dimension of $\pi(\omega)$ equals to $h_{\mathrm{F}}(\nu)/2\log\chi$.

Idea of the Proof. Write $x = \pi(\omega)$ for some typical $\omega = (X_0, X_1, \cdots) \in \Omega$. Let $r = 2^{-2\chi N}$ for some N large enough. We want to calculate the measure $\nu(B(x,r))$. Since $X_0X_1\cdots X_n$ contracts the circle to $\pi(\omega)$, which means that $X_n^{-1}\cdots X_1^{-1}X_0^{-1}$ expands a neighborhood of $\pi(\omega)$. The idea is to expand $B(x,r)$ to an interval with a ν-measure bounded from below. We have

$$\nu(B(x,r)) = \left(\prod_{n=0}^{N-1}\frac{\nu((X_0\cdots X_{n-1})^{-1}B(x,r))}{\nu((X_0\cdots X_n)^{-1}B(x,r))}\right)\nu((X_0\cdots X_{N-1})^{-1}B(x,r)).$$

Roughly speaking, we can expect that $X_{n-1}^{-1}\cdots X_0^{-1}$ expands $B(x,r)$ with the rate $\approx 2^{2\chi n}$. And the last term has an order of constant. Taking logarithms and multiplying $1/N$, we have

$$-\frac{1}{N}\log\nu(B(x,r)) = \frac{1}{N}\sum_{n=0}^{N-1}\log\frac{\nu(X_n^{-1}B_n)}{\nu(B_n)} + O(\frac{1}{N}),$$

where $B_n = (X_0 \cdots X_{n-1})^{-1} B(x, r)$. Note that $\pi(\sigma^n \omega) \in B_n$, then

$$B_n \approx B(\pi(\sigma^n \omega), 2^{-2\chi(N-n)+o(N)}),$$

with length $o(1)$ for most n. Note that

$$\frac{\nu(X^{-1} I)}{\nu(I)} = \frac{X.\nu(I)}{\nu(I)} \to \frac{\mathrm{d}X.\nu}{\mathrm{d}\nu}(x) \quad \nu-\text{a.e.}$$

for $I \ni x, |I| \to 0$. By ergodic theorem, we can expect that

$$\frac{1}{N} \sum_{n=0}^{N-1} \log \frac{\nu(X_n^{-1} B_n)}{\nu(B_n)} \to h_{\mathrm{F}}(\nu).$$

Then the local dimension of x equals to $h_{\mathrm{F}}(\nu)/2\chi$. □

3. An inverse theorem

Now we know that ν is exact dimensional and $\dim \nu = \alpha$ for some $\alpha > 0$. In what follows, we will show the idea to demonstrate

$$\alpha = \min \left\{ 1, \frac{h_{\mathrm{RW}}(\mu)}{2\chi(\mu)} \right\}$$

under the Diophantine condition. By Jensen's inequality, we can show that

$$h_F(\nu) \leqslant H(\mu),$$

hence $h_F(\nu, \mu^{\star n}) \leqslant H(\mu^{\star n})$. Since $\chi(\mu^{\star n}) = n\chi(\mu)$, it follows that $\alpha \leqslant h_{\mathrm{RW}}(\mu)/2\chi(\mu)$.

The key point is that, if $\alpha < \min \{1, h_{\mathrm{RW}}(\mu)/2\chi(\mu)\}$, then there are room for growth of ν, which contradicts with the "stationary identity" $\mu.\nu = \nu$. The idea comes from additive combinatorics, we give some examples to show the philosophy.

Example 3.1. Let $A \subset \mathbb{R}$ be a finite subset, let $A + A := \{a + b : a, b \in A\}$. Then

$$\sharp(A + A) \geqslant 2\sharp A - 1,$$

the equality holds iff A is an arithmetic progression. This example shows that if the cardinality does not grow so much under the addition, then A must possess some arithmetic structure.

Example 3.2. Let A, B be finite subsets of $\mathbb{Z}/p\mathbb{Z}$, where p is a prime. If $|A| \leqslant p - 1$ and $|B| \geqslant 2$, then $|A + B| \geqslant |A| + 1$. (In fact, by a theorem of Cauchy-Davenport, $|A + B| \geqslant \min \{p, |A| + |B| - 1\}$.) Then

$$A + B + B + \cdots + B$$

will eventually cover $\mathbb{Z}/p\mathbb{Z}$.

Example 3.3. Let A, B be finite subsets of $\mathbb{R}$, if there exists $c > 0$ such that $|A + B| \leqslant c|A|$. Then $|A_0 + \sum_k B| \leqslant c^k |A_0|$ for some non empty $A_0 \subset A$, for any positive integer k, where $\sum_k B = B + B + \cdots + B$ for k-times. This is a famous result in additive combinatorics called the Plünnecke-Ruzsa inequality.

For a non discrete context (comparing with a finite subset of $\mathbb{R}$, the measure is a continuous object), we will consider a discretized version. That is, the entropy of ν under a dyadic decomposition. We regard $\mathbb{RP}^1$ as $[0, 1)$ and consider the dyadic decomposition

$$\mathcal{D}_n = \left\{ [k2^{-n}, (k+1)2^{-n}) : 0 \leqslant k \leqslant 2^n - 1 \right\}$$

on $[0,1)$. We consider the entropy $H(\nu, \mathcal{D}_n)$, then

$$H_n(\nu) := \frac{1}{n} H(\nu, \mathcal{D}_n) \to \alpha. \tag{3.1}$$

Remark 3.4. The entropy of a measure helps us to consider a measure as a discrete object. The dimension (entropy) plays the role of cardinality of a finite set. The following example is a similar result as example 3.3 in a measure theoretical setting.

Example 3.5 ([Hoc14a, Proposition 4.9]). Let θ, η be probability measures on $\mathbb{R}$ and $H_n(\theta), H_n(\eta) < \infty$, then

$$H_n((\theta^{*k}) * \eta) \leqslant H_n(\eta) + k(H_n(\theta * \eta) - H_n(\eta)) + O\left(\frac{k}{n}\right),$$

where $*$ is the additive convolution on $\mathbb{R}$.

Remark 3.6. Combining the philosophy of example 3.2, 3.3, 3.5. Here is a rough idea: for two probability measures θ, η on $\mathbb{R}$ with positive dimension and satisfying some suitable regularity condition. Then the dimension of $(\theta^{*k}) * \eta$ will eventually attain the full dimension. Besides, if $\dim \eta < 1$, then there must have some dimension (entropy) growth of $\theta * \eta$ comparing with η.

For every $x \in [0,1)$ and $n > 0$, let $I_n(x)$ be the component in $\mathcal{D}_n$ containing x, let

$$\nu_{x,n} = \frac{1}{\nu(I_n(x))} \nu|_{I_n(x)}$$

be the component measure. Then $\nu = \mathbf{E}_{i=n} \nu_{x,i}$, where $\mathbf{E}_{n_1 \leqslant i \leqslant n_2}$ denote taking the expectation for $(x,i) \in [0,1) \times \{n_1, \cdots, n_2\}$, obeys the distribution ($\nu \times$ uniform). Similarly, let $\mathbf{P}_{n_1 \leqslant i \leqslant n_2}$ be the probability with respect to the same distribution. Then the conditional entropy

$$H(\nu, \mathcal{D}_{n+m} | \mathcal{D}_n) = \mathbf{E}_{i=n} H(\nu_{x,i}, \mathcal{D}_{i+m}).$$

We decompose the measure into an expectation of component measures is for the sake of regard the measure as a discrete object. Each component measure can be regard as a unit. Comparing with (3.1), the stationary measure has a stronger regularity. Which is called the **porosity**.

Proposition 3.7 ([HS17, Proposition 5.5])**.** *ν is **α-porous**. That is, for every $\delta > 0$, for every $m > m(\delta)$ and $n > n(\delta, m)$,*

$$\mathbf{P}_{0 \leqslant i \leqslant n} \left(\frac{1}{m} H(\nu_{x,i}, \mathcal{D}_{i+m}) \leqslant \alpha + \delta \right) > 1 - \delta.$$

Remark 3.8. The intuition of this proposition is that: for m large enough, for most local parts, when we look down m-levels, the component measure ν represents as a fractal of dimension about α. In particular, the porosity shows a phenomenon that at each level, if we regard the dyadic partition $\mathcal{D}_n$ as $\mathbb{Z}/(2^n\mathbb{Z})$, then ν is regular (not too large) at most levels.

The porosity gives a regularity condition as we mentioned in remark 3.6. This is sufficient to give a growth of entropy (dimension).

Theorem 3.9 (The inverse theorem, [Hoc14a, Theorem 2.8], [HS17, Theorem 5.7])**.** *For every $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 = n_0(\varepsilon, \delta)$. For every probability measures θ, η on $[0,1)$ such that η is $(1-\varepsilon)$-porous, then for every $n > n_0$,*

$$H_n(\theta) > \varepsilon \implies H_n(\theta * \eta) > H_n(\eta) + \delta.$$

Another technique in [HS17] is to linearize the measure μ. Since μ is a probability measure on $G = \mathrm{SL}(2, \mathbb{R})$, we should push forward the measure to $\mathbb{RP}^1$ and consider the additive convolution on $\mathbb{RP}^1 = [0, 1)$.

For $g_0 \in \operatorname{supp} \mu^{\star n}$, such that $\mu^{\star n}$ has some entropy near g_0. This can be shown by the Diophantine condition and the assumption that $\alpha < \min \{1, h_{\mathrm{RW}}(\mu)/2\chi(\mu)\}$. For $x_0 \in [0, 1)$, we can write the action as

$$g.x = g_0.x_0 + (g - g_0).x_0 + g_0.(x - x_0) + \text{higher order terms}$$

around g_0 and x_0. By some distortion control and choosing some good positions (with large probability), the action of g_0 around x_0 can be viewed as affine maps. The main terms in the expression above are $(g - g_0).x_0$ and $g_0.(x - x_0)$, which carries the information of μ and ν, respectively. Roughly speaking,

$$\nu_{g_0.x_0,-} \approx (\mu_{g_0,-}).x_0 * g_0.\nu_{x_0,-},$$

where $*$ denotes the additive convolution on $[0, 1)$. Since μ has some entropy near g_0 and $\nu_{x_0,-}$ satisfies the porosity assumption. We can expect an entropy near $g_0.x_0$. This argument is valid on a large probability set. But when we sum up these measures, it will contradict with the assumption that $\mu.\nu = \nu$.

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