# Dimension of Stationary Measures

François Ledrappier and Pablo Lessa
Notes: Yuxiang Jiao

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## §1 Generalities about dimension and statement of results (Françios, May 1)

We will follow the paper [LL23].

Let $(X, d)$ be a separable metric space and $\mu$ be a Radon measure on $X$. The local dimension for $x \in X$ is defined as

$$\overline{\dim}_x(\mu) := \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \underline{\dim}_x(\mu) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

**Definition 1.1.** We say $\mu$ is **exact dimensional** if there is a constant $\delta$ such that for $\mu$ almost every $x$,

$$\overline{\dim}_x(\mu) = \underline{\dim}_x(\mu) = \delta.$$

This is also related to the Hausdorff dimension. For a subset $A \subset X$ and $\alpha > 0$, the Hausdorff outer measure

$$H_\alpha(A) := \lim_{\varepsilon\to 0} \inf\Big\{ \sum \varepsilon_i^\alpha \ :\ A \subset \bigcup B(x_i, \varepsilon_i), \varepsilon_i < \varepsilon \text{ for every } i \Big\}.$$

The Hausdorff dimension of $A$ is defined as

$$\dim_{\mathrm{H}} A := \inf\{ \alpha \geqslant 0 \ :\ H_\alpha(A) = 0 \}.$$

**Fact 1.2.** If $\mu$ is exact dimensional with dimension $\delta$, then

$$\delta = \inf\{ \dim_{\mathrm{H}}(A) \ :\ \mu(A) > 0 \} = \inf\{ \dim_{\mathrm{H}} \ :\ \mu(X \setminus A) = 0 \}.$$

**Example 1.3**

Graph of the Weierstrass function

$$\phi(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x)$$

where $b \in \mathbb{N}$ and $\lambda \in (\frac{1}{b}, 1)$.

- Besicovitch-Ursell (1937): $\dim_{\mathrm{H}} \{ (x, \phi(x)) \} \leqslant 2 + \log \lambda / \log b$.
- W. Shen (2018): $\dim_{\mathrm{H}} \{ (x, \phi(x)) \} = 2 + \log \lambda / \log b$.

Let $(X_1, d_1, \mu_1), (X_2, d_2, \mu_2)$ be two spaces with $\dim \mu_i = d_i$. Then $\mu_1 \otimes \mu_2$ is exact dimensional on $(X_1 \times X_2, \max\{d_1, d_2\})$ and $\dim(\mu_1 \otimes \mu_2) = \delta_1 + \delta_2$.

Let $(X, d_X, \mu)$ be a space and $\pi(X, d_X) \to (Y, d_Y)$ be a Lipschitz map. Then

$$\overline{\dim}_{\pi(x)}(\mu_* \mu) \leqslant \overline{\dim}_x(\mu), \quad \underline{\dim}_{\pi(x)}(\mu_* \mu) \leqslant \underline{\dim}_x(\mu).$$

Moreover, there exists a family of $y \mapsto \mu_y$ of disintegration, that is

$$\int f(x) \mathrm{d}\mu(x) = \int_Y \int_{\pi^{-1}(y)} f(x) \mathrm{d}\mu_y(x) \mathrm{d}\mu(y).$$

Assume that for $\mu$ almost every $y$, $\mu_y$ is exact dimensional with dimension $\delta$. If $(X, \delta)$ is Lipschitz equivalent to an Euclidean space, then

$$\underline{\dim}_x(\mu) \geqslant \underline{\dim}_{\pi(x)}(\mu_* \mu) + \delta.$$

**Example 1.4**

1. The Cantor measure is exact dimensional and with dimension $\log 2 / \log 3$.
2. Let $\mu_p$ be the Bernoulli measure with law $(p, 1-p)$ on $\{0, 1\}^{\mathbb{N}} \approx [0, 1]$, then $\dim \mu_p = -p \log p - (1-p) \log(1-p)$.
3. Consider $\mu_p$ on $\{0, 1\}^{\mathbb{N}}$ isomorphic to the Cantor set embedded into $[0, 1]$, then $\dim \mu_p = [-p \log p - (1-p) \log(1-p)] / \log 3$.
4. In general, push $\mu_p$ on $\{0, 1\}^{\mathbb{N}}$ to the $(\lambda, \rho)$-Cantor set (the limit set given by $(x \mapsto \lambda x)$ and $(x \mapsto \rho x + (1-\rho))$ on $[0, 1]$), also denoted by $\mu_p$. Then the dimension is

$$\dim \mu_p = \frac{-p \log p - (1-p) \log(1-p)}{-p \log \lambda - (1-p) \log \rho}.$$

**Random walk on matrices.** Let $\mu$ be a countably supported probability measure on $\mathrm{SL}(d, \mathbb{R})$. Let $(\Omega, m) := (\mathrm{SL}(d, \mathbb{Z}), \mu)^{\mathbb{Z}}$ and $\sigma$ be the left shift map on it. Let $g_n : \Omega \to \mathrm{SL}(d, \mathbb{R})$ be the projection onto its $n$-th coordinate. Let

$$X_n(\omega) = \begin{cases} g_{n-1}(\omega) \cdots g_0(\omega), & n \geqslant 0; \\ g_n^{-1}(\omega) \cdots g_{-1}^{-1}(\omega), & n < 0. \end{cases}$$

Then $X_{m+n}(\omega) = X_m(\sigma^n \omega) X_n(\omega)$.

Assume that $\int \log \|g\| \, \mathrm{d}\mu(g) < \infty$. By the Oseledets' theorem, there exists a splitting

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_N(\omega)$$

such that

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|X_n(\omega)v\| = \chi_i, \quad \forall v \neq 0 \in E_i(\omega),$$

where $\chi_1 > \chi_2 > \cdots > \chi_N$ are all the different Lyapunov exponents. Let $d_i = \dim E_i$, then

$$\sum_{i=1}^{N} d_i = d, \quad \sum_{i=1}^{N} d_i\chi_i = 0.$$

Let

$$\mathscr{X}(\omega) = (E_1(\omega), \cdots E_n(\omega)) \in \prod_{i=1}^{N}\mathscr{G}_{d_i}(\mathbb{R}^d) =: \mathscr{X},$$

where $\mathscr{G}_{d_i}(\mathbb{R}^d)$ is the Grassmannian.

**Theorem 1.5** (Main Theorem) The distribution of $\mathscr{X}(\omega)$ is exact dimensional.

## §2 Stationary measures and entropies (Françios, May 2)

More precisely, let $M$ be the distribution of $\mathscr{X}(\omega)$, that is

$$M(A) = m(\{\omega : \mathscr{X}(\omega) \in A\}), \quad \forall A \in \mathscr{B}(\mathscr{X}).$$

Then

$$\dim M = \Delta = \sum_{i\neq j}\gamma_{i,j}$$

where $0 \leqslant \gamma_{i,j} \leqslant d_i d_j$ will be explained later.

We also consider the flag variety on $\mathbb{R}^d$ as

$$\mathscr{F} = \left\{\{0\} \subset U_1 \subset U_2 \subset \cdots \subset U_N = \mathbb{R}^d : U_i \text{ are subspaces of } \mathbb{R}^d, \ \dim U_j = \sum_{i\leqslant j} d_i\right\}.$$

For every $\omega \in \Omega$, let

$$f(\omega) = \left\{\{U_j(\omega)\} : U_j(\omega) = \bigoplus_{i\leqslant j} E_i(\omega)\right\} \in \mathscr{F}.$$

Then

$$v \in U_j(\omega) \iff \limsup_{n\to-\infty}\frac{1}{|n|}\log\|X_n(\omega)v\| \leqslant -\chi_j.$$

Note that $f(\omega)$ only depends on the negative coordinates of $\omega$, or equivalently, $f(\omega)$ is $\sigma(g_n(\omega) : n < 0)$-measurable.

We also consider another flag variety

$$\mathscr{F}' = \left\{\{0\} \subset U_1' \subset U_2' \subset \cdots \subset U_N' = \mathbb{R}^d : U_i' \text{ are subspaces of } \mathbb{R}^d, \ \dim U_k' = \sum_{i>N-k} d_i\right\}.$$

Let

$$f'(\omega) = \left\{\{U_k'(\omega)\} : U_k'(\omega) = \bigoplus_{i>N-k} E_i(\omega)\right\} \in \mathscr{F}'.$$

Then

$$v \in U_k(\omega)' \iff \limsup_{n\to+\infty} \frac{1}{n} \log \|X_n(\omega)v\| \leqslant \chi_{N-k+1}.$$

Similarly, $f'(\omega)$ is $\sigma(g_n(\omega) : n \geqslant 0)$-measurable.

Let $\nu$ be the distribution of $f(\omega)$ and $\nu'$ be the distribution of $f'(\omega)$ on the flag varieties respectively.

**Theorem 2.1** (Ledrappier-Lessa) $(\mathscr{F}, \nu)$ is exact dimensional with $\dim \nu = \sum_{i<j} \gamma_{i,j}$.

We can show the cocycle invariance of $f(\omega)$ as $f(\sigma\omega) = g_0(\omega) f(\omega)$. It follows that $\nu$ is a $\mu$-stationary measure on $\mathscr{F}$.

**Remark 2.2** We have no further assumptions on $\mu$ (such as the usual Zariski dense condition). So the $\mu$-stationary measure on $\mathscr{F}$ might not be unique. But we only consider this specific stationary measure.

**Example 2.3**

For the case of $d = 3$ and $d_i = 1$, we have two projection $(f(\omega) \mapsto U_1(\omega))$ and $(f(\omega) \mapsto U_2(\omega))$. These projections give two stationary measures $(\mathscr{L}, \nu_{\mathscr{L}})$ and $(\mathscr{P}, \nu_{\mathscr{P}})$. Rapaport (2021) has show that these projection measures are exact dimensional.

**Definition 2.4.** Let $(Y, \nu)$ be a $(G, \mu)$-space with $\mu * \nu = \nu$, the **Furstenberg entropy** is

$$\kappa(\mu, \nu) := \int_{G\times Y} \log \frac{\mathrm{d}g_*\nu}{\mathrm{d}\nu}(gy)\mathrm{d}\mu(g)\mathrm{d}\nu(y).$$

**Observation 2.5.** $\kappa(\mu, \nu) = I(g_{-1}, f)$.

Here

$$I(\mathscr{A}, \mathscr{B}) = \sup_{A\in\mathscr{A}, B\in\mathscr{B}} H(A) + H(B) - H(A \vee B)$$

for two sub-algebras $\mathscr{A}, \mathscr{B}$.

**Example 2.6**

Back to Example 2.3, we have $\kappa(\mu, \nu_{\mathscr{F}}) - \kappa(\mu, \nu_{\mathscr{L}}) = I(g_{-1}, f|E_1)$. Note that the projections are indeed fiber bundles with disintegrations $\nu_{\mathscr{F}}^{\mathscr{L}}$ over $\nu^{\mathscr{L}}$ and disintegrations $\nu_{\mathscr{F}}^{\mathscr{P}}$ over $\nu_{\mathscr{P}}$.

**Theorem (Lessa).** *Both $\nu_{\mathscr{F}}^{\mathscr{L}}$ and $\nu_{\mathscr{F}}^{\mathscr{P}}$ are exact dimensional and*

$$\dim \nu_{\mathscr{F}}^{\mathscr{L}} = \frac{\kappa(\mu, \nu_{\mathscr{F}}) - \kappa(\mu, \nu_{\mathscr{L}})}{\chi_2 - \chi_3}.$$

## §3 (Françios, May 3)

Now we consider the case in $d = 3$. Let $\mu$ be a countably supported probability measure on $\mathrm{SL}(d, \mathbb{R})$ with $\int \|g\| \,\mathrm{d}\mu(g) < \infty$. Assume that there are three distinct Lyapunov exponents $\chi_1 > \chi_2 > \chi_3$. We have the Oseledts' splitting $\mathbb{R}^3 = E_1(\omega) \oplus E_2(\omega) \oplus E_3(\omega)$. Then we have

- the unstable flag $(E_1(\omega), E_1(\omega) \oplus E_2(\omega))$, and

- the stable flag $(E_3(\omega), E_3(\omega) \oplus E_2(\omega))$.

Other than the natural projections $(E_1, E_1 \oplus E_2) \mapsto E_1$ and $(E_1, E_1 \oplus E_2) \mapsto E_1 \oplus E_2$, we have another projection

$$(E_1, E_1 \oplus E_2) \mapsto (E_1 \oplus E_3, E_2).$$

This is a codimension one projection satisfying

- equivariance,
- contraction $e^{n(\chi_3 - \chi_1)}$ along the fiber,
- entropy $K_{1,3} = I(g_1, f | E_1 \oplus E_3, E_2)$.

We have the following two claims. The proofs will be left for later lectures.

**Claim 3.1.** Conditional measures are exact dimensional and the dimension is $\frac{K_{1,3}}{\chi_1 - \chi_3}$.

**Claim 3.2.** In the setting, if the contraction is stronger in the fiber than in the quotient, then dimensions add up.

To understand the distribution of $(E_1 \oplus E_3, E_2)$, we consider the projections

- $(E_1 \oplus E_3, E_2) \mapsto E_2$, with the contraction rate $\chi_1 - \chi_2$ along fibers, and
- $(E_1 \oplus E_3, E_2) \mapsto E_1 \oplus E_3$ with the contraction rate $\chi_2 - \chi_3$ along fibers.

Now we apply the claim. If $\chi_2 \leqslant 0$, then $\chi_1 - \chi_2 \geqslant \chi_2 - \chi_3$ and we use the first projection. Otherwise, $\chi_2 - \chi_3 \geqslant \chi_1 - \chi_2$, we use the second way of projection. This choice allows us to add the dimension.

Combing above two claims, we can show that $\nu_{\mathscr{F}}$ is exact dimension and

$$\dim \nu_{\mathscr{F}} = \begin{cases} \gamma_{1,3} + \gamma_{2,3} + \gamma'_{1,2} = \dfrac{K_{1,3}}{\chi_1 - \chi_3} + \dfrac{K_{2,3}}{\chi_2 - \chi_3} + \dfrac{K'_{1,2}}{\chi_1 - \chi_2}, & \chi_2 \geqslant 0; \\ \gamma_{1,3} + \gamma_{1,2} + \gamma'_{2,3} = \dfrac{K_{1,3}}{\chi_1 - \chi_3} + \dfrac{K_{1,2}}{\chi_1 - \chi_2} + \dfrac{K'_{2,3}}{\chi_2 - \chi_3}, & \chi_2 \leqslant 0. \end{cases}$$

It also shows a Ledrappier-Young formula as

$$\kappa(\mu, \nu_{\mathscr{F}}) = (\chi_1 - \chi_2)\gamma_{1,3} + (\chi_1 - \chi_2)\overline{\gamma}_{1,2} + (\chi_2 - \chi_3)\overline{\gamma}_{2,3},$$

each $\gamma \in [0, 1]$.

**Corollary 3.3** $\dim \nu_{\mathscr{F}} \leqslant \dim_{\mathrm{LY}} \nu_{\mathscr{F}}$.

**For general $d \geqslant 3$.** For the random walks on $\mathrm{SL}(d, \mathbb{R})$, let $E_1(\omega) \oplus \cdots \oplus E_N(\omega)$ be the splitting. Let $T$ be a topology on $\{1, 2, \cdots, N\}$ which is finer than $T_0 = \{\{1, \cdots, N\}, \{2, \cdots, N\}, \cdots, \{N\}\}$. In another word, let $T(i)$ denote the atom of $i$, then $T(i) \subset \{i, i+1, \cdots, N\}$. All such topologies correspond to all the ways to projection. See Intermediate bundles.

## §4 Applications of exact dimension to Anosov representations (Pablo, May 4)

**Example 4.1** (Schottky groups in $\mathrm{SL}(2, \mathbb{R})$)
Let $R_i, A_i$ be disjoint closed intervals in $X = \mathscr{P}(\mathbb{R}^2)$ and $\gamma_i \in \mathrm{SL}(2, \mathbb{R})$ such that

$$\gamma_i(X \setminus R_i) \subset A_i \quad \text{and} \quad \gamma_i^{-1}(X \setminus A_i) \subset R_i.$$

The generated group $\Gamma$ is free. Let $\Lambda_\Gamma$ be the limit set, i.e., the smallest closed $\Gamma$-invariant set.

**Theorem (Bowen, Patterson, Sullivan, 1970s)**
$\dim_{\mathrm{H}} \Lambda_\Gamma = \delta$, *the critical exponent of* $\sum_{\gamma\in\Gamma} \|\gamma\|^{-2s}$.

**Example 4.2** (Anosov representations in $\mathrm{SL}(3,\mathbb{R})$)

Letting $\gamma_i$ be as before, perturb $\begin{bmatrix} \gamma_i & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}(3,\mathbb{R})$ slightly, the generated group $\Gamma < \mathrm{SL}(3,\mathbb{R})$ is Anosov.

**Definition 4.3.** Let $\Gamma$ be a finitely generated group, a representation $\rho : \Gamma \to \mathrm{SL}(3,\mathbb{R})$ is **Anosov** if there exists $c > 0$ such that

$$\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} > c\exp(c|\gamma|),$$

where $\sigma_i$ are singular values and $|\gamma|$ denotes the word norm.

**Remark 4.4** It took long to get this definition: Hitchin 1990s, Labourie 2000s, Kapovich-Leeb-Porti, Gvenritard-Guchard-Kassel-Weinhard, Bochi-Potrie-Sambarino 2010s.

If $\sigma_1(\rho(\gamma)) > \sigma_2(\rho(\gamma)) > \sigma_3(\gamma)$, there are well defined
- $\xi^1(\gamma)$ the most contracted line.
- $\xi^2(\gamma)$ the most contracted plane.
- $\xi(\gamma)$ the most contracted flag.

Then we can define the limit set

$$\Lambda_\Gamma := \left\{ \text{all limits of } \lim_{|\gamma_n|\to\infty} \xi(\gamma_n) \right\}.$$

**Question 4.5.** What is $\dim_{\mathrm{H}}(\Lambda_\Gamma)$?

**Random walks on groups.** Let $\Gamma$ be a word hyperbolic group and $\xi$ can be extended to the Gromov boundary $\partial\Gamma$ which is Hölder continuous. Furthermore, if $x \neq y \in \partial\Gamma$, then $\xi(x)$ and $\xi(y)$ are in general positions.

Let $\mu$ be a probability measure on $\Gamma$ satisfying $\sum \mu(\gamma)|\gamma| < \infty$. Assume that $\Gamma_\mu$, the semigroup generated by supp $\mu$, is non-elementary.

**Theorem 4.6** (Furstenberg, Maher-Tiozzo)

There exists a unique $\mu$-stationary measure $\nu_\mu$ on the boundary $\partial\Gamma$.

Then we have $\xi_*\nu_\mu$ on $\Gamma_\Lambda$ and $\xi^1_*\nu_\mu, \xi^1_*\nu_\mu$ on its projections to $\mathscr{P}(\mathbb{R}^3)$ and $\mathscr{G}_2(\mathbb{R}^3)$.

**Fact 4.7.** $\kappa = \kappa(\mu, \mu_\nu) = \kappa(\mu, \xi_*\nu_\mu) = \kappa(\mu, \xi^1_*\nu_\mu) = \kappa(\mu, \xi^1_*\nu_\mu)$.

**Estimate the dimension.** We have

$$\dim(\xi_* \nu_\mu) = \frac{\kappa_{1,3}^{\mathcal{F}}}{\chi_1 - \chi_3} + \frac{\kappa_{1,2}^{\mathcal{F}}}{\chi_1 - \chi_2} + \frac{\kappa_{2,3}^{\mathcal{F}}}{\chi_2 - \chi_3},$$

$$\dim(\xi_*^1 \nu_\mu) = \frac{\kappa_{1,3}^{\mathcal{P}}}{\chi_1 - \chi_3} + \frac{\kappa_{1,2}^{\mathcal{P}}}{\chi_1 - \chi_2},$$

$$\dim(\xi_*^2 \nu_\mu) = \frac{\kappa_{1,3}^{\mathcal{L}}}{\chi_1 - \chi_3} + \frac{\kappa_{2,3}^{\mathcal{L}}}{\chi_2 - \chi_3}.$$

Although is is hard to understand each $\kappa_{i,j}^*$, but we have

$$0 \leqslant \kappa_{i,j}^* \leqslant \chi_i - \chi_j.$$

We also no that they sum up to the same $\kappa$. Hence

$$\kappa \leqslant \chi_1 - \chi_3 + \min\{\chi_1 - \chi_2, \chi_2 - \chi_3\},$$

it follows that

$$\dim(\xi_* \nu_\mu) \leqslant 2.5.$$

**Question 4.8.** Is $\sup_\mu \dim(\xi_* \nu_\mu) = \dim_{\mathrm{H}}(\Lambda_\Gamma)$?

**Theorem 4.9** (Li-Pan-Xu, in preparation) $\dim_{\mathrm{H}}(\Lambda_\Gamma) \leqslant \sup_\mu \dim_{\mathrm{LY}}(\xi_* \nu_\mu)$.

**Corollary 4.10** (Ledrappier-Lessa)
If $\rho : \Gamma \to \mathrm{SL}(3, \mathbb{R})$ is Anosov, then $\dim_{\mathrm{H}}(\Lambda_\Gamma) \leqslant 2.5$.

## §5 Examples and idea of the proof (Pablo, May 4)

**Example 5.1**
Let $X = \{(x_1, x_2) : x_i \text{ are 1-dimensional subspaces of } \mathbb{R}^2, \text{ and } x_1 \oplus x_2 = \mathbb{R}^2\}$. We consider the projection $\pi : X \to X' = \mathcal{P}(\mathbb{R}^2), (x_1, x_2) \mapsto x_2$.

**Aim 5.2.** Turn it into a vector bundle with a nice $\mathrm{SL}(2, \mathbb{R})$-action.

**Coordinates.** For a pair $V = (V_1, V_2)$ where $V_1 \oplus V_2 = \mathbb{R}^2$ and $V_2 = x' \in X'$. Let

$$\mathrm{Nil}(V) := \left\{ f : \mathbb{R}^2 \to \mathbb{R}^2 : V_1 \xrightarrow{f} V_2 \xrightarrow{f} 0 \right\}.$$

We define

$$\varphi_V : \mathrm{Nil}(V) \to \pi^{-1}(x'), \quad \varphi_V(f) = ((\mathrm{id} + f)V_1, (\mathrm{id} + f)V_2).$$

This gives the coordinates of the vector bundle. For every $x' \in X'$, let $V = ((x')^\perp, x')$ and set $\psi_{x'} = \varphi_V$. We obtain the vector bundle structure.

**The action of** $\mathrm{SL}(2,\mathbb{R})$. Note that $\varphi_{gV}^{-1} g \varphi_V(f) = gfg^{-1}$. Then we have

$$\psi_{gx'}^{-1} g \psi_{x'}(f) = \pi_{(gx')^\perp} - \pi_{g(x')^\perp} + gfg^{-1},$$

which is an affine map.

**Example 5.3**

Let $X = \left\{(x_1, x_2, x_3) \,:\, x_i \text{ are 1-dimensional subspaces of } \mathbb{R}^3, \text{ and } x_1 \oplus x_2 \oplus x_3 = \mathbb{R}^3\right\}$. We consider the map

$$\pi \,:\, X \to X' := \left\{x_3' \subset x_{23}' \,:\, \dim x_3' = 1 \text{ and } \dim x_{23}' = 2\right\}, \quad (x_1, x_2, x_3) \mapsto (x_3, x_2 \oplus x_3).$$

**Coordinates.** $V = (V_1, V_2, V_3)$ where $V_1 \oplus V_2 \oplus V_3 = \mathbb{R}^3, V_2 \oplus V_3 = x_{23}'$ and $V_3 = x_3'$. Let

$$\mathrm{Nil}(V) := \left\{ f \,:\, \mathbb{R}^3 \to \mathbb{R}^3 \,:\, V_1 \xrightarrow{f} V_2 \oplus V_3, \quad V_2 \xrightarrow{f} V_3 \xrightarrow{f} 0 \right\}.$$

Let $\varphi_V(f) = ((\mathrm{id} + f)V_1, (\mathrm{id} + f)V_2, (\mathrm{id} + f)V_3)$. Let $V = \left\{(x_{23}')^\perp, x_{23}' \cap (x_3')^\perp, x_3'\right\}$ and set $\psi_{x'} = \varphi_V$, which gives the bundle structure. We can also verify that the action of $\mathrm{SL}(3,\mathbb{R})$ is fiberwise affine.

**Intermediate bundles.** The following is a diagram of all intermediate bundles in the case of $d = 3$. The arrows denote a fiber bundle with one-dimensional fibers. Here $x_i$ denotes a one-dimensional subspace and $x_{ij}$ denotes a two-dimensional subspace.

$$\begin{tikzcd}
 & & \{(x_1,x_2,x_3)\} \arrow[dll, "\chi_1-\chi_2"'] \arrow[d] \arrow[drr, "\chi_2-\chi_3"] & (X_1,\nu_1) \arrow[l, dashed] & \\
\{(x_{12},x_{23},x_3)\} \arrow[d, "\chi_1-\chi_3"] & & \{(x_{13},x_2,x_3)\} \arrow[dll, "\chi_1-\chi_2"'] \arrow[drr, "\chi_2-\chi_3"] & & \{(x_1,x_{23},x_3)\} \arrow[d, "\chi_1-\chi_3"] \\
\{(x_2,x_3)\} \arrow[drr, "\chi_2-\chi_3"'] & & & & \{(x_{13},x_{23},x_3)\} \arrow[dll, "\chi_1-\chi_2"] \\
 & & \{(x_{23},x_3)\} & (X_0,\nu_0) \arrow[l, dashed] &
\end{tikzcd}$$

**How do we use this.** Let $\mu$ be a probability measure on $\mathrm{SL}(3,\mathbb{R})$ with $\chi_1 > \chi_2 > \chi_3$ and Oseledets' splitting $E_1(\omega) \oplus E_2(\omega) \oplus E_3(\omega)$. Consider its distribution on $X_1 \subset (\mathscr{P}(\mathbb{R}^3))^3$, denoted by $\nu_1$. Then we project it onto $X_0$, the flag space. The projection measure is denoted by $\nu_0$. Then we can show $\nu_0$ is exact dimensional by two steps:

- For each one-dimensional fibers, show the disintegration is exact dimensional.
- For a fiber bundle over a fiber bundle $X \to X' \to X''$, if $X \to X'$ contracts stronger than $X' \to X''$, then the dimensions add up.

# §6 (Pablo, May 5)