

Selected Minicourses in *Beyond Uniform Hyperbolicity* 2023

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1 Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori (Yi Shi)

§1.1 Local Rigidity (Apr 25)

Definition 1.1.1. $f \in \text{Diff}^1(M)$ is **Anosov** if there exists a continuous Df -invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

Example 1.1.2 (Arnold's cat map)

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

Question 1.1.3. Is every Anosov diffeomorphism transitive?

Question 1.1.4. Topological classification of Anosov diffeomorphism.

Theorem 1.1.5 (Franks-Manning)

Every Anosov diffeomorphism $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \rightarrow H_1(d, \mathbb{Z})$.

Theorem 1.1.6 (Franks-Newhouse)

Every codimension-1 Anosov diffeomorphism must be supported on \mathbb{T}^d .

Definition 1.1.7. $f \in \text{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df -invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \rightarrow (0, 1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0, 1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.1.10 $E^s \oplus E^u$ integrable \implies NOT accessible.

However, Dolgopyat-Wilkinson and Hertz-Hertz-Ures, etc. showed that “MOST” partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

Geometric Rigidity \iff Dynamic Spectral Rigidity

That is, $E^s \oplus E^u$ is integrable $\implies E^c$ has exponents rigidity.

Example 1.1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \rightarrow \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22} -close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \rightarrow \mathbb{T}^4$ which is C^1 -along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \rightarrow \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be C^1 -close to A . Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \rightarrow \mathbb{T}^4$$

is an Anosov diffeomorphism C^1 -close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

NO exponents rigidity.

Main theorem: local rigidity. Assume that $A \in \text{GL}(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K \rightarrow \infty} \frac{\#\{A \text{ is generic} : \|A\| \leq K\}}{\#\{A : \|A\| \leq K\}} = 1.$$

Denote

$$T\mathbb{T}^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$

the finest dominated splitting, then $\dim L_i^{s/u} \leq 2$.

Let $f \in \text{Diff}^2(\mathbb{T}^d)$ be C^1 -close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$

Assume that $l \geq 2$ and $1 \leq k < l$. Denote

$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$

Then

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$$

makes f be an absolutely partially hyperbolic system.

Theorem 1.1.13 (Local rigidity, Gogolev-Shi, [arXiv: 2207.00704](https://arxiv.org/abs/2207.00704))

Assume $A \in \text{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \text{Diff}^2(\mathbb{R}^d)$ be C^1 -close to A , the following are equivalent:

1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws} :

$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \dots, l.$$

3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws} .

Dimension 3 case.

Theorem 1.1.14 (Hammerlindl-Ures, 2014)

Let $f \in \text{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \text{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then

- either f is accessible, thus ergodic.
- or there exists an f -invariant minimal foliation \mathcal{F}^{su} such that $T\mathcal{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_* .

Theorem 1.1.15 (Gan-Shi, 2020)

Let $f \in \text{Diff}_m^{1+}(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:

- $E^s \oplus E^u$ is integrable;
- f has spectral rigidity in E^c : $\lambda^c(f) \equiv \lambda^c(f_*)$.

Both imply f is Anosov.

Corollary 1.1.16 Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.1.13 – spectral rigidity \implies joint integrability. The case of all E_i^s are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \text{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E_l^s \implies$ smooth conjugacy in $E_l^s \implies h(\mathcal{F}_{l-1}^s) = \mathcal{L}_{l-1}^s$ (+spectral rigidity in $E_{l-1}^s \implies$ smooth conjugacy in $E_{l-1}^s \implies \dots \implies h(\mathcal{F}_{k+1}^s) = \mathcal{L}_{k+1}^s$ (+spectral rigidity in $E_{k+1}^s \implies h(\mathcal{F}^{ss}) = \mathcal{L}^{ss} \implies \mathcal{F}^{ss} \oplus \mathcal{F}^u = h^{-1}(\mathcal{L}^{ss} \oplus \mathcal{L}^u)$ joint integrability.

Proof of Theorem 1.1.13 – joint integrability \implies spectral rigidity. Main ideas:

1. $E^{ss} \oplus E^u$ integrability $\implies h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathcal{F}^{ss} \implies$ spectral rigidity in E_{k+1}^s .
3. $E^{ss} \oplus E_{k+1}^s \oplus E^u$ is integrable, and play induction on E_{k+2}^s .

Lemma 1.1.17

For every $1 \leq i \leq l$, the conjugation h preserves the center foliation: $h(\mathcal{F}_{(i,l)}^s) = \mathcal{L}_{(i,l)}^s$. Here, $\mathcal{F}_{(i,l)}^s$ and $\mathcal{L}_{(i,l)}^s$ are the foliations tangent to $E_i^s \oplus \dots \oplus E_l^s$ and $L_i^s \oplus \dots \oplus L_l^s$, respectively.

Proof. Since f is C^1 -close to A , we have

$$\|A_{L_{i-1}^s}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E_i^s(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be lifts of f and h , then $y \in \tilde{\mathcal{F}}_{(i,l)}^s(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leq (\rho_i - \varepsilon)^{-n} \|x - y\| + C < (\|A_{L_{i-1}^s}\| + \varepsilon)^{-n} \|x - y\| + C,$$

iff $H(y) \in \tilde{\mathcal{L}}_{(i,l)}^s(H(x))$. □

Lemma 1.1.18

If \mathcal{F} is a C^0 -foliation sub-foliated by a minimal linear foliation \mathcal{L} on \mathbb{T}^d , then \mathcal{F} is minimal and linear.

Proof. **Minimal.** every leaf $\mathcal{F}(x) \supset \mathcal{L}(x)$ is dense.

Linear. We will show that, on universal cover, $\tilde{\mathcal{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \tilde{\mathcal{F}}(0)$, there exists $v_n \rightarrow \tilde{\mathcal{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \rightarrow x$. Since \mathcal{F} is sub-foliated by \mathcal{L} and \mathcal{L} is linear, we have

$$y + k_n + v_n \in \tilde{\mathcal{F}}(y + k_n) = \tilde{\mathcal{F}}(k_n) = \tilde{\mathcal{F}}(k_n + v_n).$$

Take $n \rightarrow \infty$, then $y + x \in \tilde{\mathcal{F}}(x) = \tilde{\mathcal{F}}(0)$. □

Lemma 1.1.19 If $E^{ss} \oplus E^u$ is integrable to \mathcal{F}^{su} , then $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ is linear.

Proof. Note that $h(\mathcal{F}^{su})$ is sub-foliated by $h(\mathcal{F}^u) = \mathcal{L}^u$, where \mathcal{L}^u is linear and minimal on \mathbb{T}^d . Hence $h(\mathcal{F}^{su})$ is linear, A -invariant and transverse to $\mathcal{L}^{ws} = h(\mathcal{F}^{ws})$. This implies $h(\mathcal{F}^{su}) = \mathcal{L}^{su}$. So

$$h(\mathcal{F}^{ss}) = h(\mathcal{F}^s \cap \mathcal{F}^{su}) = h(\mathcal{F}^s) \cap h(\mathcal{F}^{su}) = \mathcal{L}^s \cap \mathcal{L}^{su} = \mathcal{L}^{ss}.$$

□

Corollary 1.1.20

Recall that $T\mathcal{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$, then for $T\mathcal{F}_j^s = E_j^s$, we have

$$h(\mathcal{F}_j^s) = \mathcal{L}_j^s, \quad \forall j = k, k+1, \dots, l.$$

Lemma 1.1.21 (Diophantine approximation of \mathcal{F}^{ss})

There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathcal{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$ -dense in \mathbb{T}^d .

Proof. Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of \mathcal{L}^{ss} . Here A is irreducible and \mathcal{L}^{ss} is algebraic, hence Diophantine. \square

Proof of Theorem 1.1.13. We will first show that the Lyapunov exponent at every point is the same in the $\dim E_{k+1}^s = 1$ case. Take $p, q \in \text{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f .

Take

- $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \rightarrow \infty$ and $d(x_n, q) \leq C \cdot K_n^{-\alpha}$.
 - Segments $J \subset \mathcal{F}_{k+1}^s(p)$ and $J_n \subset \mathcal{F}_{k+1}^s(x_n)$ such that $J_n = \text{Hol}^{ss}(J)$ ($x_n = \text{Hol}^{ss}(p)$).
- Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{L}_{k+1}^s) = \mathcal{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \implies \exists C_0 > 0, |J_n| \geq C_0 |J|.$$

Now we choose m_n, k_n such that

- x_n and q are very close in first k_n -steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathcal{F}_1^{ss}(p)$.

Then

$$|f^{m_n}(J_n)| \geq \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n\lambda_{k+1}^s(q)] |J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geq \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \rightarrow \infty.$$

However, $J_n = \text{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \text{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathcal{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \text{Hol}^{ss}(p)$, this contradicts to \mathcal{F}^{ss} is C^1 -smooth in $\mathcal{F}^{ss} \oplus \mathcal{F}_{k+1}^s(p)$.

For the case of $\dim E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain

- For every periodic points p, q , we have $\min \lambda_{k+1}^s(p) = \min \lambda_{k+1}^s(q)$.
- Considering the growth of area of local disks, we have

$$\text{Jac}(Df, E_{k+1}^s(p)) = \text{Jac}(Df, E_{k+1}^s(q)), \quad \forall p, q \in \text{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda_{k+1}^s(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda_{k+1}^s(A)$. \square

§1.2 Global Rigidity (Apr 26)

In the last lecture, we have shown a local rigidity result. That is, we only consider diffeomorphisms f that is C^1 -close to A . Today we will consider the global rigidity, i.e., the relation between f and $f_* \in \text{GL}(d, \mathbb{Z})$.

Question 1.2.1. What happens if f is not close to $A = f_*$?

Theorem 1.2.2 (Gogolev-Farell)

For $d \geq 10$, let $A \in \text{GL}(d, \mathbb{Z})$ be a hyperbolic matrix. Then

$$\mathcal{A}_A^{1+}(\mathbb{T}^d) := \{ f \in \text{Diff}^{1+}(\mathbb{T}^d) : f \text{ is Anosov, } f_* = A \}$$

has infinitely many connected components.

Theorem 1.2.3 (Full leaf conjugacy, Gogolev-Shi, arXiv: 2207.00704)

Let $f \in \text{Diff}^1(\mathbb{T}^d)$ be Anosov with absolutely partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$:

$$\|Df|_{E^{ss}}\| < \mu < m(Df|_{E^{ws}}) < \|Df|_{E^{ws}}\| < 1 < m(Df|_{E^u}).$$

If $E^{ss} \oplus E^u$ is integrable, then

1. $A = f_* \in \text{GL}(d, \mathbb{Z})$ is partially hyperbolic:

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u, \quad \dim L^\sigma = \dim E^\sigma, \quad \sigma = ss, ws, u.$$

2. f is dynamically coherent and fully conjugate to A :

$$h(\mathcal{F}^\sigma) = \mathcal{L}^\sigma, \quad \sigma = ss, ws, u.$$

Here $h \circ f = A \circ h$.

Question 1.2.4. Let $f \in \text{Diff}^1(\mathbb{T}^d)$ be Anosov with partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$.

- Is $f_* \in \text{GL}(d, \mathbb{Z})$ partially hyperbolic?
- Is f dynamically coherent or not? If yes, does f leaf conjugate to A .

Lemma 1.2.5

Let \mathcal{F} be a C^0 -foliation on \mathbb{T}^d with C^1 -leaves. If there exists a homeomorphism $h : \mathbb{T}^d \rightarrow \mathbb{T}^d$ homotopic to $\text{Id}_{\mathbb{T}^d}$ such that $h(\mathcal{F}) = \mathcal{L}$ is a linear foliation, then \mathcal{F} is quasi-isometric:

$$d_{\tilde{\mathcal{F}}}(x, y) \leq a \cdot d(x, y) + b, \quad \forall x \in \mathbb{R}^d, y \in \tilde{\mathcal{F}}(x).$$

Here $a, b > 0$ and $\tilde{\mathcal{F}}$ is the lift of \mathcal{F} in \mathbb{R}^d .

Proof of Theorem 1.2.3. Since $h(\mathcal{F}^{ss} \oplus \mathcal{F}^u)$ is sub-foliated by minimal linear foliation $h(\mathcal{F}^u) = \mathcal{L}^u$ is linear. We have $\mathcal{L}^{ss} := h(\mathcal{F}^{ss}) = h(\mathcal{F}^s) \cap h(\mathcal{F}^{ss} \oplus \mathcal{F}^u)$ is linear.

Brin's argument shows that $E^{ws} \oplus E^u$ integrate to \mathcal{F}^{cu} and $h(\mathcal{F}^{cu})$ is linear and minimal. Then \mathcal{F}^{ws} integrate to \mathcal{F}^{ws} and $\mathcal{L}^{ws} := h(\mathcal{F}^{ws})$ is A -invariant and linear.

Note that \mathcal{L}^{ws} and \mathcal{L}^{ss} are transverse in \mathcal{L}^s , then A admits an invariant splitting $T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u$. We need to show this is a dominated splitting. This follows from the above lemma and the fact that h is homotopic to $\text{Id}_{\mathbb{T}^d}$. \square

Theorem 1.2.6 (Global rigidity, Gogolev-Shi, [arXiv: 2207.00704](#))

Let $f \in \text{Diff}^2(\mathbb{T}^d)$ be Anosov and irreducible. Assume that f is absolutely partially hyperbolic $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$ and center bunching. If $E^{ss} \oplus E^u$ is integrable, then

1. f has a finest dominated splitting on E^{ws} with the same dimensions for $A|_{L^{ws}}$:

$$E^{ws} = E_1^{ws} \oplus \dots \oplus E_k^{ws}, \quad \dim E_i^{ws} = \dim L_i^{ws}.$$

2. f is spectrally rigid along every E_i^{ws} :

$$\lambda(E_i^{ws}, f) \equiv \lambda(L_i^{ws}, A), \quad \forall i = 1, \dots, k.$$

Remark 1.2.7 • Here f need NOT to be C^1 -close to $A = f_*$.

- To get dominated splitting, we usually need some C^1 -robust property like: robustly transitive, far from homoclinic bifurcations.
- If $A = f_*$ satisfies the generic assumption in the last lecture, then the conjugacy h is C^{1+} -smooth along \mathcal{F}^{ws} .
- The center bunching condition

$$\|Df|_{E^{ws}(x)}\| < m(Df|_{E^{ws}(x)}) \cdot m(Df|_{E^u(x)})$$

is a technical condition, which guarantees C^{1+} -smoothness of \mathcal{F}^{su} .

Corollary 1.2.8

Let $A \in \text{GL}(d, \mathbb{Z})$ be codimension one with real simple spectrum. For every Anosov $f \in \text{Diff}_m^2(\mathbb{T}^d)$ with $f_* = A$ and

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u, \quad \dim E^{ss} = 1,$$

if

- $E^{ss} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then f is C^{1+} -conjugate to A .

Main idea for showing Theorem 1.2.6. Play the game similar to the last lecture. We will use the Diophantine approximation of \mathcal{F}^{ss} to show the rigidity of smallest exponent in E^{ws} :

$$\lambda_{\min}^{ws}(p) = \lambda_{\min}^{ws}(q), \quad \forall p, q \in \text{Per}(f).$$

Then we will show the dimension of λ_{\min}^{ws} for each periodic point is constant. Next, we define the Pesin stable foliation \mathcal{F}_{\min}^{ws} and show it is \mathcal{F}^{su} -holonomy invariant, that is $\text{Hol}^{su} : \mathcal{F}^{ws}(p) \rightarrow \mathcal{F}^{ws}(q)$ preserves \mathcal{F}_{\min}^{ws} , for every $p, q \in \text{Per}(f)$. Finally, we show a uniform spectral exponents gap and extract out \mathcal{F}_{\min}^{ws} .

Lemma 1.2.9

Let $\text{Hol}_{x,y}^{su} : \mathcal{F}(x) \rightarrow \mathcal{F}(y)$ be the holonomy map of \mathcal{F}^{su} with $\text{Hol}_{x,y}^{su}(x) = y$ for every $x \in \mathbb{T}^d$ and $y \in \mathcal{F}^{su}(x)$. Then

$$\text{Hol}_{x,y}^{su}(K) = h^{-1} \circ T_{h(x),h(y)} \circ h(K).$$

Here $T_{h(x),h(y)} : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is the linear translation send $h(x)$ to $h(y)$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $K \subset \mathcal{F}^{ws}(x)$ with $\text{diam}(K) > \varepsilon$, then

$$\text{diam}(\text{Hol}_{x,y}^{su}(K)) > \delta, \quad \forall y \in \mathcal{F}^{su}(x).$$

Remark 1.2.10 The same holds for $\text{Hol}_{x,y}^{ss} : \mathcal{F}^{ws}(x) \rightarrow \mathcal{F}^{ws}(y)$ where $y \in \mathcal{F}^{ss}(x)$.

Proof. It follows immediately from f is fully conjugate to A . \square

Proof of Theorem 1.2.6. We first show that

Claim 1.2.11. $\lambda_{\min}^{ws}(p) = \lambda_{\min}^{ws}(q)$, $\forall p, q \in \text{Per}(f)$.

Proof. Assume that $\lambda_{\min}^{ws}(p) < \lambda_{\min}^{ws}(q)$. Take $x_n \in \mathcal{F}^{ss}(p)$ such that $d^{ss}(x_n, p) = K_n \rightarrow \infty$ and $d(x_n, q) \leq C \cdot K_n^{-\alpha}$. Take disk $D \subset \mathcal{F}_{\min}^{ws}(p)$, the Pesin stable manifold associated to $\lambda_{\min}^{ws}(p)$. Let $D_n = \text{Hol}^{ss}(D) \subset \mathcal{F}^{ws}(x_n)$, then $\text{diam}(D_n) \gg \text{diam}(D)$. Applying a similar (k_n, m_n) -argument, we get a contradiction since \mathcal{F}^{ss} is C^1 -smooth in $\mathcal{F}^{ws}(p)$. \square

Now we have $\lambda_{\min}^{ws} := \lambda_{\min}^{ws}(p)$ for every $p \in \text{Per}(f)$. We define the Pesin stable foliation associated to λ_{\min}^{ws} for each periodic point.

Claim 1.2.12. \mathcal{F}_{\min}^{ws} is Hol^{su} -invariant.

Proof. Let $\mathcal{L}_{\min}^{ws}|_{\mathcal{L}^{ws}(p)} := h(\mathcal{F}_{\min}^{ws}|_{\mathcal{L}^{ws}(p)})$, it suffices to show

$$T_{h(p),h(x)}(\mathcal{L}_{\min}^{ws}(p)) \subset \mathcal{L}_{\min}^{ws}(x)$$

for every $p, q \in \text{Per}(f)$ and $x \in \mathcal{F}^{ws}(q)$. Otherwise, take a disk $D \subset \mathcal{F}_{\min}^{ws}(p)$, then $T_{h(p),h(x)}(h(D))$ is transverse to $\mathcal{L}_{\min}^{ws}|_{\mathcal{L}^{ws}(q)}$ at $h(x)$. Take $x_n \in \mathcal{F}^{ss}$ such that $d^{ss}(p, x_n) = K_n \rightarrow \infty$ and $d(x_n, x) \ll K_n^{-\alpha}$, then

$$D_n := \text{Hol}_{p,x_n}^{ss}(D) \rightarrow h^{-1} \circ T_{h(p),h(x)} \circ h(D).$$

It follows that $\text{Hol}_{\text{loc}}^u(D)$ is “uniformly transverse” (the angle will not tend to zero) to \mathcal{L}_{\min}^{ws} in $\mathcal{F}_{\text{loc}}^{ws}(q)$, where $\text{Hol}_{\text{loc}}^u(D) : \mathcal{F}^{ws}(x_n) \rightarrow \mathcal{F}^{ws}(q)$ is C^{1+} -smooth. Since the transverse direction has a weaker contracting rate, we play the (k_n, m_n) -game and get a contradiction. \square

Let $\mathcal{L}_{\min}^{ws} := h(\mathcal{L}_{\min}^{ws})$, then the density of $\text{Per}(f)$ and minimality of \mathcal{F}^{ws} imply $T_{x,y}(\mathcal{L}_{\min}^{ws}(x)) \subset \mathcal{L}_{\min}^{ws}(y)$. By the translation invariance and the A -invariance, we have

- \mathcal{L}_{\min}^{ws} is a linear foliation on \mathbb{T}^d , and
- $L_{\min}^{ws} := T\mathcal{L}_{\min}^{ws}$ associate to an eigenspace of A .

Also by an estimate of the growth, we get $\lambda(A, L_{\min}^{ws}) \equiv \lambda_{\min}^{ws}$.

Finally, we establish the induction step. Following the idea of [Bonatti-Díaz-Pujals, 2003], consider the quotient cocycle $D\tilde{f} : E^{ws}/E_{\min}^{ws} \rightarrow E^{ws}/E_{\min}^{ws}$ which is Hölder continuous over f . Again by a (k_n, m_n) -game, we can show that λ_2^{ws} is uniformly larger than λ_{\min}^{ws} . Then the splitting $T\mathbb{T}^d = (E^{ss} \oplus E_{\min}^{ws}) \oplus F \oplus E^u$ is an absolutely partially hyperbolic splitting. The joint integrability follows from $h(\mathcal{F}^{ss} \oplus \mathcal{F}_{\min}^{ws})$ is linear. \square

§1.3 Rigidity on \mathbb{T}^4 (Apr 27)

Let us recall some results shown in last two lectures. We remark that the key point is that

$$E^{ss} \oplus E^u \text{ is integrable} \implies h(\mathcal{F}^{ss} = \mathcal{L}^{ss}) \text{ is linear.}$$

Question 1.3.1. Let f be C^1 -close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_j^u \oplus \cdots \oplus E_m^u.$$

What happens if $E_k^s \oplus E_j^u$ is jointly integrable? Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$?

Theorem 1.3.2 (Gogolev-Kalinin-Sadovskya)

Spectral rigidity in $E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u$ implies $h(\mathcal{F}_k^s) = \mathcal{L}_k^s$ and $h(\mathcal{F}_j^u) = \mathcal{L}_j^u$ hence $E_k^s \oplus E_j^u$ is jointly integrable.

The work of Avila-Viana.

Theorem 1.3.3 (Avila-Viana, 2010)

For every symplectic f which is C^∞ -close to A with splitting

$$T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u,$$

then

- either f is accessible and non-uniformly hyperbolic;
- or $E^s \oplus E^u$ is integrable and $\exists h \in \text{Diff}_m^\infty(\mathbb{T}^4)$ such that

$$h \circ f = A \circ h.$$

In particular, f is Bernoulli.

Main theorem.

Theorem 1.3.4 (Gogolev-Shi, arXiv: 2207.00704)

Let $A \in \text{GL}(d, \mathbb{Z})$ be an irreducible Anosov automorphism with dominated splitting

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}, \quad \text{with} \quad \dim L^{ws} = \dim L^{wu} = 1.$$

For $f \in \text{Diff}^2(\mathbb{T}^d)$ be C^1 -close to A with splitting

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is spectral rigid along E^{ws} and E^{wu} .

Corollary 1.3.5

Let $A \in \text{Sp}(4, \mathbb{Z})$ be hyperbolic and irreducible with dominated splitting

$$T\mathbb{T}^4 = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}.$$

For symplectic $f \in \text{Diff}_\omega^2(\mathbb{T}^4)$ be C^1 -close to A with

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is C^{1+} -smoothly conjugate to A .

Proof of corollary. If $E^{ss} \oplus E^{uu}$ is integrable, then we have spectral rigidity in $E^{ws} \oplus E^{wu}$, h is smooth along $E^{ws} \oplus E^{wu}$ and $h(\mathcal{L}^{ss}) = \mathcal{L}^{ss}$, $h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$. Since h is smooth along \mathcal{F}^{ws} and \mathcal{F}^{wu} , the holonomy map $\text{Hol}_{\mathcal{F}}^{su}$ is C^{1+} . Then we use the symplectic structure that $E^c = E^{ws} \oplus E^{wu}$ is perpendicular to E^{su} (with respect to ω). Hence $\mathcal{F}^{ws} \oplus \mathcal{F}^{wu}$ is C^{1+} . Then we can show that h is absolutely continuous in \mathcal{F}^{su} and hence h is C^{1+} . \square

Proof of main theorem. Main problem is whether $h(\mathcal{F}^{su}) = \mathcal{L}^{su}$ is the linear one? Or equivalently, whether we have $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ or $h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$? This is nontrivial.

Lemma 1.3.6

If one of $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ is integrable, then f is spectral rigid in $E^{ws} \oplus E^{wu}$.

Proof. If $E^{ss} \oplus E^u$ is integrable, then $h(\mathcal{F}^{ss} \oplus \mathcal{F}^u)$ is linear and hence $h(\mathcal{F}^{ss}) = h(\mathcal{F}^{ss} \oplus \mathcal{F}^u) \cap \mathcal{L}^s = \mathcal{L}^{ss}$ is linear. Then both $h(\mathcal{F}^{su})$ and $h(\mathcal{F}^{uu})$ are linear. Then we obtain a spectral rigidity by Theorem 1.1.13. \square

The solvable action. Let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$ and $L^c(0) = L^{ws}(0) \oplus L^{wu}(0) \subset \mathbb{R}^d$. Define the linear action

$$\alpha_0 : \Gamma \times L^c(0) \rightarrow L^c(0), \quad \alpha_0(k, n)(x) = L^{su}(A^k(x) + n) \cap L^c(0).$$

If we write $n = n^s + n^c + n^u \in L^s \oplus L^c \oplus L^u$, then $\alpha_0(k, n)(x) = A^k x + n^c$.

For $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the lift of f and $F(0) = 0$, then

- $F^k(x + n) = F^k(x) + A^k n$, $\forall x \in \mathbb{R}^d$ and $\forall n \in \mathbb{Z}^d$.
- $F(\tilde{\mathcal{F}}^c(0)) = \tilde{\mathcal{F}}^c(0)$.

Then $\Gamma \curvearrowright \tilde{\mathcal{F}}^c(0)$ given by

$$\alpha(k, n)(x) = \tilde{\mathcal{F}}^{su}(F^k(x) + n) \cap \tilde{\mathcal{F}}^c(0), \quad \forall (k, n) \in \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d, x \in \tilde{\mathcal{F}}^c(0).$$

Lemma 1.3.7 This is a group action by the solvable group Γ .

Main idea. If both $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ are not integrable, then we can find a free subgroup by a pingpong argument, which contradicts Γ is solvable.

Lemma 1.3.8

If $\alpha(0, n)(\tilde{\mathcal{F}}^{ws}(0)) \subset \tilde{\mathcal{F}}^{ws}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$, then both $h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ and $h(\mathcal{F}^{uu}) = \mathcal{L}^{uu}$ are linear. The same holds if $\alpha(0, n)(\tilde{\mathcal{F}}^{wu}(0)) \subset \tilde{\mathcal{F}}^{wu}(\alpha(0, n)0)$ for all $n \in \mathbb{Z}^d$.

Proof. Note that $\bigcup_{n \in \mathbb{Z}^d} \tilde{\mathcal{F}}^{ws}(n)$ is dense in \mathbb{R}^d and hence $E^{ss} \oplus E^{ws} \oplus E^{uu}$ jointly integrates to $\mathcal{F}^{su} \oplus \mathcal{F}^{ws}$. Then we deduce the linearity. \square

Proof of Theorem 1.3.4. Assume for a contradiction that there exists $n_1, n_2 \in \mathbb{Z}^d$ such that

- $\alpha(0, n_1)(\tilde{\mathcal{F}}^{ws}(0))$ is transverse to $\tilde{\mathcal{F}}^{ws}(\alpha(0, n_1)(0))$;
- $\alpha(0, n_1)(\tilde{\mathcal{F}}^{wu}(0))$ is transverse to $\tilde{\mathcal{F}}^{wu}(\alpha(0, n_1)(0))$.

Lemma 1.3.9

There exists $m_1, m_2 \in \mathbb{Z}^d$ such that

- $\alpha(0, m_1)(\tilde{\mathcal{F}}^{ws}(0))$ is transverse to $\tilde{\mathcal{F}}^{ws}(0)$;
- $\alpha(0, m_1)(\tilde{\mathcal{F}}^{wu}(0))$ is transverse to $\tilde{\mathcal{F}}^{wu}(0)$.

Lemma 1.3.10

For l large enough, $n = A^l m_1 - A^{-l} m_2 \in \mathbb{Z}^d$ satisfies

- $\alpha(0, n)(\tilde{\mathcal{F}}^{ws}(0))$ is transverse to $\tilde{\mathcal{F}}^{ws}(0)$;
- $\alpha(0, n)(\tilde{\mathcal{F}}^{wu}(0))$ is transverse to $\tilde{\mathcal{F}}^{wu}(0)$.

Now we consider $F : \tilde{\mathcal{F}}(0) \rightarrow \tilde{\mathcal{F}}(0)$ and

$$G : \alpha(0, n) \circ \alpha(1, 0) \circ \alpha(0, -n) : \tilde{\mathcal{F}}(0) \rightarrow \tilde{\mathcal{F}}(0).$$

Then F is saddle-like dynamics at $\tilde{\mathcal{F}}^{ws}(0) \cup \tilde{\mathcal{F}}^{ws}(0)$ near 0. The map G is also saddle-like near $\alpha(0, n)0$. By a pingpong-argument, we can show that $\{F^k, G^k\}$ generates a free group for a sufficiently large k . This contradicts that Γ is solvable. \square

§1.4 Anosov Maps (Apr 28)

Cone-field. Let f be an Anosov diffeomorphism with splitting $TM = E^s \oplus E^u$. Then there are cone-fields C^s, C^u containing E^s, E^u such that

$$Df(\overline{C^u(x)}) \subset C^u(fx), \quad Df^{-1}(\overline{C^s(x)}) \subset C^s(f^{-1}x).$$

Then $E^s(x)$ is determined by $\text{Orb}^+(x)$ as

$$E^s(x) = \bigcap_{n \geq 0} Df^{-n}(C^s(f^n x)),$$

and $E^u(x)$ is determined by $\text{Orb}^-(x)$ as

$$E^u(x) = \bigcap_{n \geq 0} Df^n(C^u(f^{-n} x)).$$

Theorem 1.4.1 (Anosov, 1967)

The Arnold's cat map $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is structurally stable. That is, for every $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ C^1 -close to A , there exists a homeomorphism $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ close to $\text{Id}_{\mathbb{T}^2}$ such that $h \circ f = A \circ h$.

Remark 1.4.2 Every Anosov diffeomorphism is structurally stable.

Remark 1.4.3 If $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is continuous with $f_* = A$, then there exists a surjective $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $h \circ f = A \circ h$.

By a cone-argument, we can show that a small perturbation of an Anosov diffeomorphism is also Anosov. In general, we have Franks-Manning's global classification of Anosov diffeomorphisms.

Theorem 1.4.4 (Franks-Manning)

Every Anosov diffeomorphism $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \rightarrow H_1(d, \mathbb{Z})$.

Anosov maps.

Definition 1.4.5. A local diffeomorphism $f : M \rightarrow M$ is **Anosov**, if there exists a continuous, Df invariant subbundle $E^s \subset TM$ such that

- $\|Df(v^s)\| < 1$ for every $v^s \in E^s$ with $\|v^s\| = 1$;
- Df induces an expanding map $D\tilde{f} : TM/E^s \rightarrow TM/E^s$, that is

$$\|D\tilde{f}(\tilde{v}^u)\| > 1, \quad \forall \tilde{v}^u \in TM/E^s, \|\tilde{v}^u\| = 1.$$

In this lecture, the Anosov map always refers to the non-invertible Anosov map.

Remark 1.4.6 Since $\text{Orb}^-(x)$ is not unique, there may be no $E^u(x)$.

Theorem 1.4.7 (Mañe-Pugh, 1974)

$f : M \rightarrow M$ is an Anosov map iff $\tilde{f} : \tilde{M} \rightarrow \tilde{M}$ is an Anosov diffeomorphism.

Definition 1.4.8 (Przytycki, 1976). A local diffeomorphism $f : M \rightarrow M$ is an **Anosov map**, if in the orbit space

$$\tilde{x} = (x_i)_{i \in \mathbb{Z}} \in M_f := \{(x_i) : f(x_i) = x_{i+1}, \forall i \in \mathbb{Z}\},$$

there exists a splitting

$$T_{x_i}M = E^s(x_i) \oplus E^u(x_i), \quad \forall i \in \mathbb{Z}$$

which is Df -invariant

$$D_{x_i}f(E^s(x_i)) = E^s(x_{i+1}), \quad D_{x_i}f(E^u(x_i)) = E^u(x_{i+1}), \quad \forall i \in \mathbb{Z},$$

and for every $v^{s/u} \in E^{s/u}(x_i)$ with $\|v^{s/u}\| = 1$:

$$\|D_{x_i}f(v^s)\| < 1, \quad \|D_{x_i}f(v^u)\| > 1.$$

Example 1.4.9

For every $n \geq 3$, the map

$$A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$$

is an Anosov map.

Remark 1.4.10 Every Anosov map $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ has a hyperbolic linearization $f_* \in M(\mathbb{Z}, d)$.

Unlike the Anosov diffeomorphisms, the Anosov map is not structurally stable.

Theorem 1.4.11 (Mañe-Pugh, 1974; Przytycki, 1976)

Let $A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ with $n \geq 3$. Then A is **NOT** structurally stable. That is, for every $\varepsilon > 0$, there exists an Anosov map $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ with $d_{C^\infty}(f, A) < \varepsilon$ such that there is no $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ homotopic to $\text{Id}_{\mathbb{T}^2}$ with $h \circ f = A \circ h$.

Remark 1.4.12 Every non-invertible Anosov map is not structurally stable unless it is expanding.

Proof. Take $p \neq 0$ such that $A(p) = 0$. Let U, U' be disjoint neighborhoods of 0 and p . Let (x_i) be an A -orbit satisfying

$$x_0 = p, \quad x_i = 0, \forall i > 0, \quad \text{and} \quad x_i \notin U', \forall i < 0.$$

Take a C^∞ ε -perturbation of f on U' : push p along the stable leaf.

Then there exists an f -orbit $\{y_i\}$ satisfying

$$y_0 = p, \quad \text{and} \quad y_i = x_i, \forall i < 0.$$

Then $y_i \in \mathcal{F}_\varepsilon^s(0)$ for every $i > 0$, where \mathcal{F}^s is the stable leaf of A . Then the A -orbit x_i shadows the f -orbit y_i and hence the conjugacy $h(y_i) = 0$. But there is no homeomorphism $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that $h(y_i) = 0$ for every $i > 0$. \square

Theorem 1.4.13 (Przytycki, 1976)

An Anosov map $f : M \rightarrow M$ is structurally in the orbit space (M_f, σ_f) , where $\sigma_f : (x_i) \mapsto (x_{i+1})$. That is, for every $g : M \rightarrow M$ C^1 -close to f , there exists a homeomorphism $\bar{h} : M_g \rightarrow M_f$ such that $\bar{h} \circ \sigma_g = \sigma_f \circ \bar{h}$.

Question 1.4.14.

- Assume that $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is an Anosov map with $f_* = A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}, n \geq 3$. When f topologically conjugate to A ?
- Assume that $f, g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ are Anosov maps with $f_* = g_*$. When f topologically conjugates to g ?

Example 1.4.15 (Przytycki, 1976)

Let

$$A = \begin{bmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & n \end{bmatrix} : \mathbb{T}^3 \rightarrow \mathbb{T}^3, \quad n \geq 2$$

be a **special Anosov map** (E^u does not depend on the choice of the inverse orbit). When n is big enough, for every $x \in \mathbb{T}^3$, there exists an f C^1 -close to A such that

$$\left\{ D\pi(E^u(x_0)) : \tilde{x} = (x_i) \in M_f \text{ with } x_0 = x \right\} \subset \mathcal{G}^2(T_x \mathbb{T}^3)$$

contains a curve in the Grassmannian $\mathcal{G}^2(T_x \mathbb{T}^3)$.

Theorem 1.4.16 (Micena-Tahzibi, 2016)

Let $f : M \rightarrow M$ be a transitive Anosov map, then

- either f has an integrable E^u (f is special),
- or there exists a residue set $\mathcal{R} \subset M$ such that x has infinitely many unstable directions for every $x \in \mathcal{R}$.

Main theorems.**Theorem 1.4.17** (An-Gan-Gu-Shi, arXiv: 2205.13144)

Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a C^{1+} -Anosov map, then the following are equivalent:

- f topologically conjugate to $f_* = A$;
- f is spectral rigid in stable bundle:

$$\lambda^s(p, f) \equiv \log \|A|_{L^s}\|, \quad \forall p \in \text{Per}(f).$$

Remark 1.4.18 The same holds if $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is an irreducible Anosov map with $\dim E^s = 1$.

Theorem 1.4.19 (An-Gan-Gu-Shi, arXiv: 2205.13144)

Let $A \in M(d, \mathbb{Z})$ be Anosov, irreducible and $|\det(A)| > 1$. If A has real simple spectrum in the stable bundle:

$$T\mathbb{T}^d = L_1^s \oplus L_2^s \oplus \cdots \oplus L_k^s \oplus L^u, \quad \dim L_i^s = 1,$$

then for every f C^1 -close to A , the following are equivalent:

- f topologically conjugates to A ,
- f is spectral rigidity in stable bundle, i.e. f admits dominated splitting

$$T\mathbb{T}^d = E_1^s \oplus E_2^s \oplus \cdots \oplus E_k^s \oplus E^u$$

and

$$\lambda(E_i^s, f) \equiv \log \|A|_{L_i^s}\|, \quad \forall i = 1, \dots, k.$$

Main philosophy. For every $y, z \in \mathbb{T}^d$, they are in the same “strongest stable manifold” if

$$f^n(y) = f^n(z), \quad \text{for some } n > 0.$$

Then f topologically conjugates to $A \iff E^u$ does not depend on $\text{Orb}^-(x)$. Hence we have $E^u(x) = E^u(y)$ if $f^n(y) = f^n(z)$. This is equivalent to E^u is “jointly integrable” with

$$\mathcal{F}^{ss}(x) := \{z : f^n(x) = f^n(z), \text{ for some } n > 0\}.$$

This leads to a spectral rigidity in E^s , which is the weak stable direction in this view.

Topological classification.

Theorem 1.4.20 (Gu-Shi, arXiv: 2212.11457)

Let $f, g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be homotopic C^{1+} -Anosov maps, then the following are equivalent:

- f topologically conjugates to g ;
- for every $p \in \text{Per}(f)$ and corresponding $p' \in \text{Per}(g)$,

$$\lambda^s(p, f) \equiv \lambda^s(p', g).$$

Remark 1.4.21 Since there is no a priori conjugacy, we should explain the meaning of “corresponding point”. This can be given by a (stable) leaf conjugacy, which is defined a priori. Note that each periodic stable leaf admits a unique periodic point since f is uniformly contracting on the stable leaf. The corresponding point can be defined in this way.

Corollary 1.4.22 (Gu-Shi, arXiv: 2212.11457)

Let $f, g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h . Then h is C^r -smooth along the stable foliation.

Theorem 1.4.23 (Gu-Shi, arXiv: 2212.11457)

Let $f, g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h . If

$$\text{Jac}(f^{\pi(p)}(p)) = \text{Jac}(g^{\pi(p)}(h(p))), \quad \forall p \in \text{Per}(f),$$

then h is C^{r_*} -smooth. Here $r_* = \begin{cases} r - 1 + \text{Lip}, & r \in \mathbb{N} \\ r, & r \notin \mathbb{N} \end{cases}$.

2 **Methods for Studying Abelian Actions and Centralizers (Danijela Damjanović / Disheng Xu)**

3 **Dimension of Stationary Measures** **(Francios Ledrappier / Pablo Lessa)**