Some topics on homogeneous dynamics

Yuxiang Jiao

https://yuxiangjiao.github.io

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- **Measure rigidity for diagonalizable actions (Manfred Einsiedler)**
  Minicourse at AMSS, Beijing.
- **Lattices, submanifolds and diophantine approximations (Nicolas de Saxcé)**
  Winter school at Fudan University, Shanghai.
- **Totally geodesic submanifolds and arithmeticity (Manfred Einsiedler)**
  Winter school at Fudan University, Shanghai.

Contents

1 Measure rigidity for diagonalizable actions (Manfred Einsiedler)

§1.1 Lecture 1

Theorem 1.1.1 (Furstenberg)

Let $A \subset \mathbb{T}$ be a closed and $\times 2, \times 3$-invariant set. Then

- $\# A < \infty$ consisting of periodic points, or
- $A = \mathbb{T}$.

Conjecture 1.1.2 (Furstenberg)

Let μ be an invariant probability measure for the joint $\times 2, \times 3$-action that is ergodic. Then

- $\# \operatorname{supp} \mu < \infty$, or
- $\mu = m_{\mathbb{T}}$ the Lebesgue measure.

Theorem 1.1.3 (Rudolph)

Let μ be $\times 2, \times 3$-invariant ergodic probability measure. If $h_\mu(\times 2) > 0$ (or $h_\mu(\times 3) > 0$, or $\dim \mu > 0$), then $\mu = m_{\mathbb{T}}$.

Theorem 1.1.4 (Einsiedler-Katok-Lindenstrauss, 2005)

Let $A = \left\{ \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix} \right\} \subset \mathrm{SL}(3, \mathbb{R})$ act on $X_3 = \mathrm{SL}(3, \mathbb{R}) / \mathrm{SL}(3, \mathbb{Z})$. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then $\mu = m_{X_3}$ is the uniform measure.

Theorem 1.1.5 (Lindenstauss, 2003)

Let $A = \left\{ \begin{bmatrix} * & \\ & * \end{bmatrix} \times \begin{bmatrix} * & \\ & * \end{bmatrix} \right\} \subset \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ act on $X = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) / \Gamma$ with Γ irreducible. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then $\mu = m_X$.

Theorem 1.1.6 (Einsiedler-Lindenstrauss, 2023)

Let $A \subset \mathrm{SL}(2,\mathbb{R})^k$ be isomorphic to $\mathbb{R}^2$ and $\mathbb{R}$-diagonalizable. Let $\Gamma < \mathrm{SL}(2,\mathbb{R})^k$ be irreducible and $X = \mathrm{SL}(2,\mathbb{R})^k/\Gamma$. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$ for some $a \in A$. Then

- μ is homogeneous with semisimple stabilizer, or
- X is non-compact and μ is invariant under a unipotent flow, and supported on an orbit of a solvable group.

Example 1.1.7

Let $K = \mathbb{Q}(\sqrt{3}) \hookrightarrow \mathbb{R}\times\mathbb{R}$ and $\mathbb{Z}[\sqrt{3}] \hookrightarrow \mathbb{R}\times\mathbb{R}$ which gives an irreducible lattice. Then $\mathrm{SL}(2,\mathbb{Z}[\sqrt{3}])$ also gives an irreducible lattice in $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. We consider the unipotent subgroup $U = \left\{ \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \times \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \right\}$. Then $U\Gamma \cong \mathbb{R}^2/\mathrm{Galois}(\mathbb{Z}[\sqrt{3}]) \cong \mathbb{T}^2$. This gives an example for the second case in the theorem. To understand these cases, we should classify invariant measures on tori.

Theorem 1.1.8 (Einsiedler-Lindenstrauss, 2023)

Let $A = \left\{ \begin{bmatrix} h & \\ & h^{-1} \end{bmatrix} : h \in \mathbb{Q} \right\} < \mathrm{SL}(2,\mathbb{A})$ where $\mathbb{A} = \mathbb{R}\times\prod'_p \mathbb{Q}_p$ is the adel. Let μ be an A-invariant ergodic probability measure on $X_\mathbb{A} = \mathrm{SL}(2,\mathbb{A})/\mathrm{SL}(2,\mathbb{Q})$. Then

- $\mu = m_{X_\mathbb{A}}$, or
- μ is the uniform Haar measure on a periodic orbit of a unipotent subgroup, or
- μ is the Dirac measure on a fixed point.

§1.2 Lecture 2

Leafwise measures. We consider the leafwise measure on $X = G/\Gamma$ with respect to $H < G$: a measure μ_x^H on H for almost every $x \in X$ so that the conditional measure of $\mu|_{\mathrm{box}}$ on the local pieces of H-orbits can be obtained by

$$(\mu|_{\mathrm{box}})^{\mathcal{A}^H_{\mathrm{box}}}_{V_x\cdot x} = \frac{1}{\mu_x^H(V_x)}(\mu_x^H|_{V_x})\cdot x,$$

where box is a "rectangle" (product of H-direction and some transverse direction) on X, $\mathcal{A}^H_{\mathrm{box}}$ is the σ-algebra whose atoms are pieces of H-orbits, $h \mapsto h\cdot x$ gives the map from $V_x \subset H$ to the box.

Fubini-construction of leafwise measure. Define $\widetilde{X} = X \times H$ equipped with $\mu \times m_H$. Let $\mathcal{A}_H$ be the preimage of $\mathcal{B}_X$ under $(x_0,h_0) \mapsto h_0^{-1}x_0 \in H$. The atom $[(x_0,h_0)]_{\mathcal{A}_H} = \Delta_H(x_0,h_0)$ where $\Delta(h)(x_0,h_0) := (hx_0, hh_0)$.

Multiplying by a density function $f_0 \in L^1(H)$. Taking conditional measure and dividing by the density we create a Radon measure (somehow the conditional measure of the infinite measure $\mu\times m_H$) on the Δ_H-orbits

$$(\mu\times m_H)^{\mathcal{A}_H}_{(x_0,h_0)}.$$

Projected to H, we obtain μ_x^H. Moreover, the h_0-coordinate is only relevant for the position of $\Delta_H(x_0, h_0)$.

Compatibility of leafwise measures: If $x, h \cdot x \in X$ for some $h \in H$, then $\mu_{hx}^H h \propto \mu_x^H$.

Entropy. Let $a \in G$ be diagonalizable preserving μ. Let $U < G_a^+$ be normalized by a. Then we can look at μ_x^U and these relate to entropy:

$$h_\mu(a, U) = \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^n B_1^U a^{-n}).$$

On the other hand, the ergodic theory also gives

$$h_\mu(a, U) = \lim_{n\to\infty} -\frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n).$$

These two inequality tell us a phenomenon: the global growth rate of the measure of a U-ball equals the local dimension of μ.

There are also several properties:

- If $U = G_a^+$ then $h_\mu(a) = h_\mu(a, U)$.
- If $h_\mu(a, U) = 0$ then $\mu_x^U = \delta_e$.
- If $h_\mu(a, U) = h_{m_X}(a, U)$ is maximal, then μ is U-invariant.

Product structure of leafwise measures. If $G_a^+ = U_{\alpha_1} \cdots U_{\alpha_n}$ is a direct product of root groups, then

$$\mu_x^{G_a^+} \propto \mu_x^{\alpha_1} \times \cdots \times \mu_x^{\alpha_n} \quad \text{a.s.}.$$

In particular, $h_\mu(a) = \sum h_\mu(a, U_{\alpha_i})$.

Idea of the proof. Say $G_a^+ = U_\alpha U_\beta$. Assume that we can distinguish U_α, U_β by some $b \in A$: b commutes with U_α but $U_\beta \subset G_b^-$. Choose $x \in X$ and elements u_α, u_β. We aim to show that the conditional measure $\mu_x^{U_\alpha}$ is proportion to an appropriate translation of $\mu_{u_\alpha u_\beta x}^{U_\alpha}$.

We iteration them by b. We have $\mu_x^\alpha = \mu_{b^n x}^\alpha$. Assume $b^n x \to y$ as $n \to \infty$. Applying Luzin's theorem, we can assume the conditional measures are continuous on a large set. Then $\mu_{b^n x}^\alpha \to \mu_y^\alpha$, where $y \in U_\alpha x$ because of the choice of b. Then we get the product structure. □

§1.3 Lecture 3

Symmetry of entropy contributions. If α have $-\alpha$ have unequal entropy contributions, then μ is invariant under a nontrivial unipotent subgroup of U_α or $U_{-\alpha}$.

All statement made for entropy and contributions also work conditionally over a factor of the action (in another word, conditioned on an A-invariant σ-algebra). We use $\mathcal{A}_\alpha$ to denote the σ-algebra generated by $x \mapsto \mu_x^\alpha$.

What is the leafwise measure for U_β conditioned on $\mathcal{A}_\alpha$: $\mu_x^{\beta|\mathcal{A}_\alpha}$ describes $\mu_x^{\mathcal{A}_\alpha}$ along U_β-orbits. Then $\mu_x^{\beta|\mathcal{A}_\alpha} = \mu_x^\beta$ because of the product structure for $U_\alpha U_\beta$.

We consider the diagram with three roots α, β, γ on the plane. Recall the entropy contribution formula (assume that $a \in A$ is chosen that $h_\mu(a) > 0$ and α, β contributes to $h_\mu(a)$, γ contributes to $h_\mu(a^{-1})$)

$$\begin{aligned} h_\mu(a) &= h_\mu(a, U_\alpha) + h_\mu(a, U_\beta) \\ &= h_\mu(a^{-1}) = h_\mu(a^{-1}, U_\gamma). \end{aligned}$$

For conditional entropies,

$$\begin{aligned} & h_\mu(a|\mathcal{A}_\alpha) = h_\mu(a, U_\alpha|\mathcal{A}_\alpha) + h_\mu(a, U_\beta) \\ = \; & h_\mu(a^{-1}) = h_\mu(a^{-1}, U_\gamma). \end{aligned}$$

This tells us $h_\mu(a, U_\alpha) = h_\mu(a, U_\alpha|\mathcal{A}_\alpha)$. By the assumption, we have $h_\mu(a, U_\alpha) > 0$. Therefore, $h_\mu(a, U_\alpha|\mathcal{A}_\alpha) > 0$. This means that within the same $\mathcal{A}_\alpha$-atom, we can find pairs of different points on the same U_α-orbit: $x, u_\alpha x$, where $u_\alpha \neq e$. This gives $\mu_x^\alpha = \mu_{u_\alpha x}^\alpha$. Then we obtain some translation invariance of μ_x^α.

Non-maximal torus actions. Our next goal is to show the following:

Theorem 1.3.1 (Einsiedler-Lindenstrauss, 2023)
$X = \mathrm{SL}(2,\mathbb{R})^k/\Gamma$ and Γ is irreducible (arithmetic). Let $A \subset \mathrm{SL}(2,\mathbb{R})^k$ be isometric to $\mathbb{R}^2$ and diagonalizable. Let μ be an A-invariant ergodic probability measure with $h_\mu(a) > 0$, then μ has nontrivial unipotent invariance.

Let $\mathrm{SL}(2,\mathbb{R})^k = G_1 \times G_2 \times G_3$ satisfy that $a \neq e \in G_1, b \neq e \in G_2$ are contained in A. Let $U = U_\alpha = G_a^+$.

Recall that $h_\mu(a) > 0$ tells us μ_x^U is nontrivial with a growth rate. In Lindenstrauss's low entropy method, he used a fact that μ is U-recurrent iff μ_x^U is infinite. We now have a quantitative version of μ_x^U is infinite. So we expect to show that μ satisfies a quantitative recurrence statement for U.

The idea is the following. If cover the space by r^{-d} balls of radius r. By Kac's lemma, for each r-ball, the points that don't return within $r^{-d-\varepsilon}$ has the measure less than $r^{d+\varepsilon}$. So that the total measure of non-recurrent points in the r^{-d} ball's is at most r^ε. We take $r = e^{-n}$ and apply Borel-Cantelli lemma. We obtain a polynomial recurrence.

For the actual practice, we should combine this philosophy with the nontrivial growth of leafwise measures to obtain a similar polynomial recurrence statement. A precise statement is as the following: given $B \subset G/\Gamma$, we have

$$\mu\left\{ x \in B : \mu_x^U \text{ has nontrivial growth rate and does not return within } a^n B_2^{U_\alpha} a^{-n} \right\} \leqslant e^{-h_\mu(a,U_\alpha)n}.$$

Now we want to show $h_\mu(b) > 0$. We assume for the purpose of a contradiction that $h_\mu(b) = 0$. By Brin-Katok, the entropy is also

$$h_\mu(b) = \lim_{n\to\infty} \frac{1}{2n} \log \mu(\text{Bowen } n\text{-ball for two sided map defined by } b).$$

Here two sided Bowen ball at x is $D_n \cdot x := \left(\bigcap_{k=-n}^n b^k B_\varepsilon^G b^{-k}\right) \cdot x$. The zero entropy shows that the measures of Bowen balls are not decay so fast. We will combine this with the recurrence argument to obtain a contradiction.

Using these ideas we obtain: for μ-almost every x and all sufficiently large n (depending on x) we have $e^{\frac{1}{2}h_\mu(a,U_\alpha)n}$-many different returns within $a^n B_2^{U_\alpha} a^{-n}$ to $D_{100n} \cdot x$.

Write $x = g\Gamma$. Then we have $ug = hg\gamma$, where $u \in a^n B_2^{U_\alpha} a^{-n}$ and $h \in D_{100n}$. Now we need to use the arithmeticity of Γ. The heights of the γ responsible for the return is $\ll e^{2n}$.

Claim 1.3.2. All γ commute.

Proof. Because $[\gamma_1, \gamma_2]$ has height $\ll e^{8n}$ and $\|[\gamma_1, \gamma_2] - \mathrm{id}_{G_2}\| \ll e^{-200n}$. $\square$

There are two cases:

- γ's are unipotent, then γ must be identity. But we have several returns, we obtain a contradiction.
- γ's are diagonalizable: too many lattice elements, a contraction.

2 Lattices, submanifolds and diophantine approximations (Nicolas de Saxcé)

§2.1 Classical results and general settings

Theorem 2.1.1

For every $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there exists infinitely many $p/q \in \mathbb{Q}$ such that $|\theta - p/q| \leqslant 1/q^2$.

The first proof (continued fractions). Let $\theta_0 = \theta$ and $a_0 = \lfloor \theta_0 \rfloor$. For every $i \geqslant 1$, we define inductively that

$$\theta_i = \frac{1}{\theta_{i-1} - a_{i-1}}, \quad a_i = \lfloor \theta_i \rfloor.$$

We can check that

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{\theta_n}}}.$$

We have the following two facts:

1. $\begin{bmatrix} 1 \\ \theta \end{bmatrix} \mathbb{R} = \begin{bmatrix} 0 & 1 \\ 1 & a_0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \begin{bmatrix} 1 \\ \theta_n \end{bmatrix} \mathbb{R}$.
2. Let $p_n/q_n = a_0 + \cfrac{1}{\ddots + 1/a_n}$, then $\begin{bmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}$.

In particular, for every n, $\theta \in [p_n/q_n, p_{n+1}/q_{n+q}]$ (maybe reverse order). Then

$$\left| \theta - \frac{p_n}{q_n} \right| \leqslant \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} \leqslant \frac{1}{q_n^2}.$$

□

Exercise 2.1.2. (1) Show that $q_{n+1} = a_{n+1} q_n + q_{n-1}$, and deduce that there are infinitely many n such that $q_{n+1} \geqslant \phi \cdot q_n$, where $\phi = (1+\sqrt{5})/2$.
(2) Conclude that there are infinitely many p_n/q_n such that $|\theta - p_n/q_n| \leqslant 1/(\sqrt{5} q_n^2)$.
(3) Check that the constant $\sqrt{5}$ is optimal.

The second proof (using Dirichlet's theorem).

Theorem 2.1.3 (Dirichlet)

For every $\theta \in \mathbb{R}$ and $Q \geqslant 1$, there exists $q \in \{1, \cdots, Q\}$ and $p \in \mathbb{Z}$ such that

$$\left| \theta - \frac{p}{q} \right| \leqslant \frac{1}{qQ} \leqslant \frac{1}{q^2}.$$

□

Definition 2.1.4. For $\theta \in \mathbb{R}$, we define its Diophantine exponent as

$$\beta(\theta) := \sup \left\{ \beta > 0 : \exists p/q \text{ arbitrarily close to } \theta \text{ with } |\theta - p/q| \leqslant q^{-\beta} \right\}.$$

There are several basic properties:

(D) By Dirichlet's theorem, $\beta(\theta) \geqslant 2$ for every $\theta \in \mathbb{R}$.

(BC) By Borel-Cantelli lemma, $\beta(\theta) = 2$ for almost every $\theta \in \mathbb{R}$.

(R) Roth showed that $\beta(\theta) = 2$ for every $\theta \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

Exercise 2.1.5 (Liouville). Show that if $f(\theta) = 0$ for some $f \in \mathbb{Z}[X] \setminus \{0\}$, then $\beta(\theta) \leqslant (\deg f)$ if $f \notin \mathbb{Q}$.

Approximation in $\mathbb{R}^n$.

Let $\theta = [\theta_1 \ \cdots \ \theta_n]^t \in \mathbb{R}^n$. We can consider several types of approximations:

- **Simultaneous approximations**: $|\theta_i - p_i/q| \leqslant q^{-\beta}$ for $i = 1, \cdots, n$.
- **Linear form approximations**: $|q - p_1\theta_1 - \cdots - p_n\theta_n| \leqslant q^{-\beta+1}$.

Here, the simultaneous approximation can also be considered as a **projective approximations**. Let $x = \mathbb{R}\begin{bmatrix} 1 \\ \theta \end{bmatrix} \subset \mathbb{R}^d$, which is a point in $\mathbb{P}(\mathbb{R}^d)$. Let v be an element in $\mathbb{P}(\mathbb{Q}^d) \subset \mathbb{P}(\mathbb{R}^d)$. Then v is also a rational line in $\mathbb{R}^d$, which can be written as $\mathbb{R}\mathbf{v}$ for some primitive $\mathbf{v} = [q \ p_1 \ \cdots \ p_n]^t \in \mathbb{Z}^d$. The **height** of v is given by $H(v) := \|\mathbf{v}\|$. We want to study $d(x, v) \leqslant H(v)^{-\beta}$. Here the distance is understood in the projective space.

Theorem 2.1.6

(D) For every $x \in \mathbb{P}(\mathbb{R}^d)$, $\beta(x) \geqslant d/(d-1)$.

(BC) For almost every $x \in \mathbb{P}(\mathbb{R}^d)$, $\beta(x) = d/(d-1)$.

(R-S) For every $x \in \mathbb{P}(\overline{\mathbb{Q}}^d)$ not in any proper rational subspace, $\beta(x) = d/(d-1)$.

Exercise 2.1.7. Check (D) and (BC).

Theorem 2.1.8 (Subspace theorem, Schmidt, 1970s)

Let $L \in \mathrm{GL}(d, \overline{\mathbb{Q}})$ and write $L_1, \cdots, L_d$ for the rows of L. For every $\varepsilon > 0$, all solutions $\mathbf{v} \in \mathbb{Z}^d$ satisfying the inequality

$$|L_1(\mathbf{v}) \cdots L_d(\mathbf{v})| \leqslant \|\mathbf{v}\|^{-\varepsilon}$$

are contained in a finite union of $\mathbb{Q}$-hyperplanes.

Exercise 2.1.9. Check the theorem when $L \in \mathrm{GL}(d, \mathbb{Q})$.

Proof of (RS) assuming the subspace theorem. Write $x = \mathbb{R}[1 \ \theta_2 \ \cdots \ \theta_d]^t$ with $\theta_i \in \overline{\mathbb{Q}}$. Take

$$L = \begin{bmatrix} 1 & & & \\ -\theta_2 & 1 & & \\ \vdots & & \ddots & \\ -\theta_d & & \cdots & 1 \end{bmatrix}.$$

Assume that $d(x,v) \leqslant H(v)^{-\beta}$ for some $v \in \mathbb{P}(\mathbb{Q}^d)$. Take $\mathbf{v} \in \mathbb{Z}^d$ corresponding to v. Then $L_1(\mathbf{v}) = |q|$ and $L_i(\mathbf{v}) = |-q\theta_i + p_i|$ for $i \geqslant 2$. By the assumption, we have $L_i(\mathbf{v}) \leqslant \|\mathbf{v}\|H(v)^{-\beta}$ for every $i \geqslant 2$. Hence $|L_1(\mathbf{v}) \cdots L_d(\mathbf{v})| \leqslant \|v\|^{d-(d-1)\beta}$. If $d-(d-1)\beta > 0$ then $\mathbf{v}$ belongs to a finite union of $\mathbb{Q}$-hyperplanes $V_1 \cup \cdots \cup V_k$. But $x \notin \mathbb{P}(V_1 \cup \cdots \cup V_k)$, so $d(x,v)$ is bounded away from 0. There are only finitely many v with bounded height. A contradiction. □

Exercise 2.1.10. Prove (RS) for linear form approximations.

Approximation by linear subspaces.

Schmidt's question. Fix integers $q \leqslant k \leqslant \ell < d$. Given an ℓ-dimensional subspace $x \in \mathbb{R}^d$. Study k-dimensional rational subspace v lying close to x.

Definition 2.1.11 (distance). $d(v,x) := \max\{ d(\mathbf{u},x) : \mathbf{u} \in v, \|\mathbf{u}\| = 1 \}$.

Notation 2.1.12. Denote X_ℓ to be the grassmannian variety of ℓ-dimensional subspaces in $\mathbb{R}^d$. Let $X_k(\mathbb{Q})$ to be the $\mathbb{Q}$-points in X_k (corresponding to $\mathbb{Q}$-subspaces).

Definition 2.1.13 (height). For every $v \in X_k(\mathbb{Q})$, the intersection $v \cap \mathbb{Z}^d$ is a subgroup of $\mathbb{Z}^d$, which can be written as $\mathbb{Z}v_1 \oplus \cdots \mathbb{Z}v_k$. The **height** of v is defined to be

$$H(v) := \operatorname{vol}(v_1 \wedge \cdots \wedge v_k) = \operatorname{vol}(v/(v \cap \mathbb{Z}^d)).$$

Proposition 2.1.14

There exists $C = C(d)$ such that $N_d(H) := \#\{ v \in X_k(\mathbb{Q}) : H(v) \leqslant H \}$ satisfies

$$C^{-1}H^d \leqslant N_d(H) \leqslant CH^d.$$

Exercise 2.1.15. Check this for $k=1$ and $k=d-1$.

Theorem 2.1.16

(D) For every $x \in X_\ell(\mathbb{R})$, $\beta_k(x) \geqslant \frac{d}{k(d-\ell)}$.

(BC) For almost every $x \in X_\ell(\mathbb{R})$, $\beta_k(x) = \frac{d}{k(d-\ell)}$.

(R) For every $x \in X_\ell(\overline{\mathbb{Q}})$ not contained in any proper rational pencil, $\beta_k(x) = \frac{d}{k(d-\ell)}$.

Definition 2.1.17. A **pencil** in X_ℓ is the a subset

$$\mathscr{P}_{W,r} := \{ x \in X_\ell(\mathbb{R}) : \dim x \cap W \geqslant r \},$$

where $W \subset \mathbb{R}^d$ is a rational subspace and $r \geqslant 1$.

Now we explain the intuition of this theorem. For every $v \in X_k(\mathbb{Q})$ and $\varepsilon > 0$. The set $\{ x \in X_\ell(\mathbb{R}) : d(v,x) \leqslant \varepsilon \}$ is an ε-neighborhood of $E_v = \{ x \in X_\ell(\mathbb{R}) : v \subset x \}$. Here E_v is a submanifold of $X_\ell(\mathbb{R})$ and $\dim E_v = (d-\ell)(\ell-k)$. Then $\operatorname{codim} E_v = k(d-\ell)$ and hence $\operatorname{vol}\{ x : d(v,x) \leqslant \varepsilon \} \asymp \varepsilon^{k(d-\ell)}$.

On the other hand, the number of $v \in X_k(\mathbb{Q})$ with $H(v) \leqslant H$ is approximately H^d. So that expected value for ε satisfies $H^d \varepsilon^{k(d-\ell)} = 1$. This gives $\varepsilon = H^{-\frac{d}{k(d-\ell)}}$.

Exercise 2.1.18. Use this argument to show that $\beta(x) \leqslant \frac{d}{k(d-\ell)}$ for almost every x.

§2.2 The correspondence between lattices and subspaces

Lattices in $\mathbb{R}^d$

Proposition 2.2.1
If Λ is a discrete subgroup of $\mathbb{R}^d$, then there exists k linearly independent vectors $v_1, \cdots, v_k$ such that $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$

Proof. Take $v_1 \in \Lambda$ with minimal norm. Consider $P_{v_1^\perp}(\Lambda)$, which is a discrete subgroup of $v_1^\perp$ since v_1 is the shortest vector. By induction, we may write

$$P_{v_1^\perp}(\Lambda) = \mathbb{Z}P_{v_1^\perp}(v_2) \oplus \cdots \oplus \mathbb{Z}P_{v_1^\perp}(v_k).$$

Then $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$. □

Definition 2.2.2. A **lattice** in $\mathbb{R}^d$ is a discrete subgroup of rank d. We can write $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d$ with (v_i) a basis of $\mathbb{R}^d$ in this case.

Definition 2.2.3. The **first minimum** of a lattice is $\lambda_1(\Lambda) := \min\{\|v\| : v \in \Lambda \setminus \{0\}\}$. The **co-volume** of Λ is $\operatorname{covol}\Lambda = \operatorname{vol}(v_1 \wedge \cdots \wedge v_k)$, where $v_1, \cdots, v_k$ is given above.

Theorem 2.2.4 (Minkowski I)
Let Δ be a lattice in $\mathbb{R}^d$. If C is a convex symmetric set in $\mathbb{R}^d$ with $\operatorname{vol} C > 2^d \operatorname{covol}\Delta$, then $C \cap \Delta \neq \{0\}$. In particular, $\lambda_1(\Delta)^d \leqslant \frac{2^d}{\operatorname{vol} B(0,1)} \operatorname{covol}\Delta$.

Proof. Consider $\Delta_q = \frac{1}{q}\Delta$ for $q \in \mathbb{N}_+$. The number of points in $\Delta_q \cap \frac{C}{2}$ is approximately $q^d \frac{\operatorname{vol}(C)}{2^d \operatorname{covol}\Delta}$. If $\operatorname{vol} C > 2^d \operatorname{covol}\Delta$, for q large enough, there exists $v_1, v_2 \in \Delta_1 \cap \frac{C}{2}$ with the same image in $\Delta_q/\Delta \cong (\mathbb{Z}/q\mathbb{Z})^d$. Then $0 \neq v_1 - v_2 \in \Delta \cap C$. □

Definition 2.2.5. The **successive minima** of Δ is $\lambda_1(\Delta) \leqslant \cdots \leqslant \lambda_d(\Delta)$, where

$$\lambda_i(\Delta) := \inf\{\lambda > 0 : \Delta \cap B(0,\lambda) \text{ contains } i \text{ linearly independent vectors}\}.$$

Theorem 2.2.6 (Minkowski II)
$\operatorname{covol}\Delta \leqslant \lambda_1(\Delta) \cdots \lambda_d(\Delta) \leqslant \frac{2^d}{\operatorname{vol} B(0,1)} \operatorname{covol}\Delta.$

Proof. If $v_1, \cdots, v_d$ are linearly independent with $\|v_i\| = \lambda_i$, then $\Delta' = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d < \Delta$. Hence $\lambda_1 \cdots \lambda_d \geqslant \operatorname{covol}(\Delta') \geqslant \operatorname{covol}(\Delta)$.

For the converse, we first construct an orthogonal basis $u_1, \cdots, u_d$ satisfying

$$\operatorname{span}\{u_1, \cdots, u_i\} = \operatorname{span}\{v_1, \cdots, v_i\}, \quad \forall 1 \leqslant i \leqslant d.$$

Let $T : u_i \mapsto \lambda_i^{-1}(\Delta)u_i$. We denote $\Delta_T = T\Delta$.

Claim 2.2.7. $\lambda_1(\Delta_T) \geqslant 1$.

Proof. Indeed, for every $v \in \Delta$, write $v = \sum_{i=1}^{I} \alpha_i v_i$ with $\alpha_I \neq 0$. Since v is linearly independent with $(v_1, \cdots, v_{I-1})$, $\|v\| \geqslant \lambda_I(\Delta)$. Therefore,

$$\|Tv\| \geqslant \frac{\|v\|}{\|T^{-1}|_{\mathrm{span}\{v_1,\cdots,v_I\}}\|} = \frac{\|v\|}{\|T^{-1}|_{\mathrm{span}\{u_1,\cdots,u_I\}}\|} \geqslant \frac{\lambda_I(\Delta)}{\lambda_I(\Delta)} = 1.$$

□

Now we apply Minkowski I to Δ_T, we obtain

$$1 \leqslant \frac{2^d \operatorname{covol} \Delta_T}{\operatorname{vol} B(0,1)} = \frac{2^d \operatorname{covol} \Delta}{\lambda_1(\Delta) \cdots \lambda_d(\Delta) \operatorname{vol} B(0,1)}.$$

□

Remark 2.2.8 We proved this theorem for euclidean norm above. But it is true in general for any norm with

$$\frac{\operatorname{covol} \Delta}{d!} \leqslant \lambda_1(\Delta) \cdots \lambda_d(\Delta) \leqslant \frac{2^d}{\operatorname{vol} B(0,1)} \operatorname{covol} \Delta.$$

Dani's correspondence

Let $x \in X_\ell(\mathbb{R})$. We want to study the diophantine exponent $\beta_k(x)$. Let $G = \mathrm{SL}(d, \mathbb{R})$ and $P = \mathrm{Stab}_G(x_0)$ where $x_0 = \mathrm{span}\{e_1, \cdots, e_\ell\} \in X_\ell(X)$. Then G acts transitively on $X_\ell(\mathbb{R}) \cong P \backslash G$, here the isomorphism is given by $gx_0 \mapsto Pg^{-1}$.

Notation 2.2.9. For $x \in X_\ell(\mathbb{R})$, let $u_x \in G$ be such that $x = Pu_x$ (hence $u_x x = x_0$).

The **zooming flow** is given by

$$a_t = \begin{bmatrix} e^{-t/\ell} & & & & & \\ & \ddots & & & & \\ & & e^{-t/\ell} & & & \\ & & & e^{t/(d-\ell)} & & \\ & & & & \ddots & \\ & & & & & e^{t/(d-\ell)} \end{bmatrix}, \quad t \in \mathbb{R}.$$

Proposition 2.2.10 (Dani's correspondence, version 1)
For $x \in X_\ell(\mathbb{R})$, let $\Delta_x = u_x \mathbb{Z}^d$ be the lattice in $\mathbb{R}^d$. Let

$$\gamma_1(x) := \limsup_{t \in +\infty} -\frac{1}{t} \log \lambda_1(a_t \Delta_x).$$

Then

$$\beta_1(x) = \frac{d}{(d-\ell)(1-\ell\gamma_1(x))}.$$

Applications.

(1) Lower bound on β. Minkowski's first theorem shows that $\lambda_1(a_t\Delta_x) \lesssim 1$. Hence $\gamma_1(x) \geqslant 0$ and $\beta_1(x) \geqslant \frac{d}{d-\ell}$.

(2) Let Ω be the space of unimodular lattices in $\mathbb{R}^d$. Then $\Omega \cong \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z}) = G/\Gamma$ and it admits a finite G-invariant measure m_Ω. For $f \in C_c(\mathbb{R}^d)$, we define

$$\widetilde{f}(\Delta) := \sum_{\text{primitive } v\in\Delta} f(v).$$

Then $\int_\Omega \widetilde{f}\,\mathrm{d}m_\Omega = \int_{\mathbb{R}^d} f$. Take $f = \mathbb{1}_{B(0,\varepsilon)}$, then $\widetilde{f}(\Delta) \geqslant \mathbb{1}_{\lambda_1(\Delta)\leqslant\varepsilon}$. Therefore,

$$m_\Omega(\{\lambda_1 \leqslant \varepsilon\}) \leqslant \int \widetilde{f} = \int f \lesssim \varepsilon^d.$$

Claim 2.2.11. For almost every $\Delta \in \Omega$, $\lim_{t\to+\infty} \frac{1}{t}\log\lambda_1(a_t\Delta) = 0$.

Proof. For every $\varepsilon > 0$, we aim to show $\lambda_1(a_t\Delta) \geqslant e^{-\varepsilon t}$ for t large enough. It is enough to check for $t \in \mathbb{N}$. Note that

$$|\{\Delta : \lambda_1(a_t\Delta) \leqslant e^{-\varepsilon t}\}| = |\{\Delta : \lambda_1(\Delta) \leqslant e^{-\varepsilon t}\}| \lesssim \varepsilon^{-d\varepsilon t}.$$

By Borel-Cantelli lemma, we have $\limsup -\frac{1}{t}\log\lambda_1(a_t\Delta) \leqslant 0$. Hence the limit is 0 because $\lambda_1(a_t\Delta) \leqslant 1$ for every t. This implies that $\lambda_1(x) = 0$ for almost every x. □

Exterior powers. For $0 \leqslant k \leqslant d$, the exterior power $\wedge^k\mathbb{R}^d$ is a vector space with basis e_I where $I \subset \{1,\cdots,d\}$ and $\#I = k$. If $I = \{i_1 < \cdots < i_k\}$ then $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$.

> **Exercise 2.2.12.** If $\wedge^k\mathbb{R}^d$ is endowed with the euclidean structure making e_I an orthonormal basis, then, for $W = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$ a discrete subgroup of $\mathbb{R}^d$, we have $|W| = \|v_1 \wedge \cdots \wedge v_k\|$, where $|W|$ denotes the covolume of W in its real span.

Note that a_t acts on $\wedge^k\mathbb{R}^d$ with eigenvalues $\exp(-(\frac{k}{\ell} - i\frac{d}{\ell(d-\ell)})t)$ for $0 \leqslant i \leqslant k$. An element e_I is an eigenvector corresponding to the eigenvalue $\exp(-(\frac{k}{\ell} - i\frac{d}{\ell(d-\ell)})t)$ if and only if $\#(I \setminus \{1,\cdots,\ell\}) = i$. We write $\pi_+ : \wedge^k\mathbb{R}^d \to \wedge^k\mathbb{R}^d$ to be the projection to the eigenspace with the eigenvalue $e^{-kt/\ell}$ (parallel to other eigenspaces).

Proposition 2.2.13 (Dani's correspondence, version 2)
For $x \in X_\ell(\mathbb{R})$, let

$$\gamma_k(x) := \sup\left\{\gamma \in \mathbb{R} : \begin{array}{c} \exists t > 0 \text{ large },\ \exists w \in a_t \wedge^k u_x\mathbb{Z}^d \text{ with} \\ \|w\| \leqslant e^{-\gamma t}, \|\pi_+ w\| \geqslant \frac{1}{2}\|w\| \end{array}\right\}.$$

Then

$$\beta_k(x) = \frac{d}{(d-\ell)(k-\ell\gamma_k(x))}.$$

Proof. Assume $\beta < \beta_k(x)$, then there exists $v \in X_k(\mathbb{Q})$ close to x with $d(v,x) \leqslant H(v)^{-\beta}$. Take $\mathbf{v} \in \wedge^k\mathbb{Z}^d$ representing v. We want to make $\|a_t u_x \mathbf{v}\|$ small. We write $u_x\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \cdots$ such that $a_t\mathbf{v}^{(i)} = \exp(-(\frac{k}{\ell} - i\frac{d}{\ell(d-\ell)})t)\mathbf{v}^{(i)}$.

Lemma 2.2.14
If v is close to x, then $\|\mathbf{v}^{(0)}\| \asymp H(v)$, $\|\mathbf{v}^{(1)}\| \asymp H(v)d(v,x)$ and $\|\mathbf{v}^{(p)}\| \lesssim H(v)d(v,x)^p$ for every $p \geqslant 2$.

Proof. Fix x and so does u_x. Then $H(v) = \|\mathbf{v}\| \asymp \max_i \|\mathbf{v}^{(i)}\|$. Note that $d(v,x) = d_{X_k}(v, E_x)$ where $E_x = \{ y \in X_k : y \subset x \}$. We have

$$d(v,x) \asymp d(u_x v, u_x x) = d_{X_k}(u_x v, E_{x_0}) \asymp d(\frac{u_x \mathbf{v}}{\|u_x \mathbf{v}\|}, \wedge^k \operatorname{span}\{ e_1, \cdots, e_\ell \}).$$

Note that $\wedge^k E_\ell$ is exactly the eigenspace of a_t with the eigenvalue $e^{-kt/\ell}$. Therefore,

$$d(v,x) \asymp \frac{1}{\|u_x \mathbf{v}\|} \max_{i \geqslant 1} \|\mathbf{v}^{(i)}\| \asymp \frac{1}{H(v)} \max_{i \geqslant 1} \|\mathbf{v}^{(i)}\|.$$

If $d(v,x)$ is small enough, then $\max_{i \geqslant 1} \|\mathbf{v}^{(i)}\|$ is much smaller than $H(v) \asymp \max_{i \geqslant 0} \|\mathbf{v}^i\|$. Therefore, $\mathbf{v}^{(0)}$ is the main term and $\|\mathbf{v}^{(0)}\| \asymp H(v)$.

Besides, we also obtain $\max_{i \geqslant 1} \|\mathbf{v}^{(i)}\| \lesssim H(v)d(v,x)$. Now we demonstrate the remaining two estimates. For simplicity, we assume that $k = \ell$. After some appropriate rotations, we may assume that $\pi_+(u_x \mathbf{v})$ is parallel to $e_1 \wedge \cdots \wedge e_\ell$. Then we write (**we cheat here**)

$$\frac{u_x \mathbf{v}}{\|u_x \mathbf{v}\|} = \begin{bmatrix} \mathrm{id} & 0 \\ (u_{ij}) & \mathrm{id} \end{bmatrix} (e_1 \wedge \cdots \wedge e_\ell)$$

with $u_{ij} \in \mathbb{R}$ small. So we have $d(v,x) \asymp \max_{i,j} |u_{ij}|$. But then

$$\frac{\mathbf{v}^{(1)}}{\|\mathbf{v}^{(0)}\|} = \sum \pm u_{ij} \cdot e_{\{1, \cdots, \ell\} \setminus \{j\} \cup \{i\}}$$

is with norm $\asymp \max |u_{ij}| \asymp d(v,x)$. For $p \geqslant 2$, we can find that $\|\mathbf{v}^{(p)}\| / \|\mathbf{v}^{(0)}\|$ is a homogeneous polynomial of deg p, so we have $\|\mathbf{v}^{(p)}\| \lesssim \|\mathbf{v}^{(0)}\| (\max |u_{ij}|)^p \asymp H(v)d(v,x)^p$. $\square$

So we have

$$\|a_t u_x \mathbf{v}\| \asymp H(v) \max \left\{ e^{-\frac{kt}{\ell}}, e^{-(\frac{k}{\ell} - \frac{d}{\ell(d-\ell)})t} d(x,v), \cdots \right\}.$$

Take $t > 0$ so that $e^{\frac{dt}{\ell(d-\ell)}} = H(v)^\beta$. Then

$$\|a_t u_x \mathbf{v}\| \lesssim H(v) e^{-\frac{kt}{\ell}} = e^{-(\frac{k}{\ell} - \frac{1}{\beta}\frac{d}{\ell(d-\ell)})t}.$$

Thus $\gamma_k(x) \geqslant \frac{k}{\ell} - \frac{1}{\beta}\frac{d}{\ell(d-\ell)}$.

For the converse direction, assume that $\|a_t u_x \mathbf{v}\| \leqslant e^{-\gamma t}$ and $\|\pi_+(a_t u_x \mathbf{v})\| \gtrsim \|a_t u_x \mathbf{v}\|$. Using the above computation, this yields:

$$e^{-\gamma t} \gtrsim H(v) \max \left\{ e^{-\frac{kt}{\ell}}, e^{-(\frac{k}{\ell} - \frac{d}{\ell(d-\ell)})t} d(x,v) \right\} \quad \text{and} \quad e^{-(\frac{k}{\ell} - \frac{d}{\ell(d-\ell)})t} \|\mathbf{v}^{(1)}\| \lesssim e^{-\frac{kt}{\ell}} \|\mathbf{v}^{(0)}\|.$$

Therefore, $H(v) \lesssim e^{(\frac{k}{\ell} - \gamma)t}$ and $d(x,v) \lesssim H(v)^{-\frac{d}{(d-\ell)(k-\ell\gamma)}}$. $\square$

During the proof of Lemma 2.2.14, we assume implicitly that $\mathbf{v}$ was decomposable. That is $\mathbf{v} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k$ for some $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{R}^d$. This is always possible thanks to the following lemma:

Lemma 2.2.15 (Mahler)
If Δ is a lattice in $\mathbb{R}^d$, then the successive minima of $\wedge^k\Delta$ are essentially (up to a multiplicative constant) equal to the

$$\lambda_I(\Delta) = \lambda_{i_1}(\Delta)\cdots\lambda_{i_k}(\Delta), \quad I \subset \{1,\cdots,\ell\}, \#I = k,$$

and achieved by decomposable vectors.

Proof. Assume Δ is unimodular and hence so is $\wedge^k\Delta$. If $\|v_i\| = \lambda_i(\Delta)$ with $v_1,\cdots,v_d$ linearly independent, then $v_I = v_{i_1}\wedge\cdots\wedge v_{i_k}$ satisfies $\|v_I\| \leqslant \lambda_I(\Delta)$. But by Minkowski II, $\prod_I \lambda_I(\Delta) \lesssim 1$ and hence $\|v_I\| \asymp \lambda_I(\Delta)$ for each I. $\square$

Going back to the correspondence, if there exits $w \in \wedge^k a_t u_x \mathbb{Z}^d$ with $\|w\| \leqslant e^{-\gamma t}$ (i.e. $\lambda_1(\wedge^k a_t u_x\mathbb{Z}^d) \leqslant e^{-\gamma t}$) and $\|\pi_+(w)\| \gtrsim \|w\|$, then we can find such w with that is decomposable.

§2.3 Algebraic subspaces

Grayson polygon and Harder-Narasimhan filtration.

Let Δ be a lattice in $\mathbb{R}^d$, let $\mu_i(\Delta) = \min\{\,|V| : V < \Delta, V \cong \mathbb{Z}^i\,\}$ be the successive covolumes of Δ.

Definition 2.3.1. The **Grayson polygon** C_Δ is the maximal convex function on $[0,d]$ whose graph has below each point $(i, \log\mu_i(\Delta))$.

Proposition 2.3.2 (Harder-Narasimhan filtration)
If C_Δ has angle at the point i then there exists $V_i < \Delta$ of rank i with $|V_i| = \log\mu_i(\Delta)$. Moreover, if $I = \{\,i_1 < \cdots < i_k\,\}$ is the set of angle points then

$$\{\,0\,\} < V_{i_1} < \cdots < V_{i_k} < \Delta.$$

Definition 2.3.3. Let $\mathbb{K}$ be a field with characteristic 0. A map $\tau : \mathrm{Gr}(\mathbb{K}^d) \to \mathbb{R}$ is **submodular** if

$$\tau(V\cap W) + \tau(V+W) \leqslant \tau(V) + \tau(W), \quad \forall V, W \subset \mathbb{K}^d.$$

Example 2.3.4
If Δ is a lattice in $\mathbb{R}^d$ then $\{$ primitive subgroups of $\Delta\,\} \hookrightarrow \mathrm{Gr}(\mathbb{Q}^d)$. Then $\tau(V) = \log|V|$ is submodular, or equivalently $|V\cap W|\cdot|V+W| \leqslant |V|\cdot|W|$.

> **Exercise 2.3.5.** Check this inequality.

Lemma 2.3.6 (Submodularity)
Let $\tau : \mathrm{Gr}(\mathbb{K}^d) \to \mathbb{R}$ be submodular with $\tau(\{0\}) = 0$. Then there exists a unique maximal subspace with

$$\frac{\tau(V)}{\dim V} = \inf\left\{ \frac{\tau(W)}{\dim W} : W \subset \mathbb{K}^d \right\}.$$

Proof. Assume for simplicity that V, W both attain the infimum a. Then

$$\tau(V+W) \leqslant a(\dim V + \dim W) - a\dim(V \cap W) = a\dim(V+W).$$

This proves the lemma. ∎

Theorem 2.3.7
If $\tau : \mathrm{Gr}(\mathbb{K}^d) \to \mathbb{R}$ is submodular with $\tau(0) = 0$. Define its Grayson polygon C_τ as the maximal convex function on $[0, d]$ lying below all points $(\dim W, \tau(W))$. If C_τ has angle at i, then there is a unique V_i such that $\dim V_i = i$ and $C_\tau(i)$, and if $I = \{ i_1 < \cdots < i_k \}$ is the set of angle points for C_τ then we have a HN-filtration

$$\{0\} < V_{i_1} < \cdots < V_{i_k} < \mathbb{K}^d.$$

Remark 2.3.8 By Minkowski II, $\mu_i(\Delta) \asymp \lambda_1(\Delta) \cdots \lambda_i(\Delta)$. So the shapes of C_Δ are (up to a additive constant) equal to $(\log \lambda_1(\Delta), \cdots, \log \lambda_d(\Delta))$.

Parametric subspace theorem.

Aim 2.3.9. Given $\Delta \subset \mathbb{R}^d$ a lattice, describe $C_{a_t\Delta}$ for $t > 0$, where $a_t = \mathrm{diag}(e^{\alpha_1 t}, \cdots, e^{\alpha_d t})$.

Theorem 2.3.10 (Parametric subspace theorem)
Assume that $\Delta = L\mathbb{Z}^d$ with $L \in \mathrm{GL}(d, \overline{\mathbb{Q}})$. Then there exists C_∞ such that

$$\lim_{t \to +\infty} \frac{1}{t} C_{a_t\Delta} = C_\infty.$$

Moreover, if $I = \{ i_1 < \cdots < i_k \}$ are the angles of C_∞ then there exists a filtration $\{0\} < V_{i_1} < \cdots < V_{i_k} < \mathbb{R}^d$ such that for every $t > 0$ large enough and for every s, $a_t L V_{i_s}$ contains the first i_s successive minima of $a_t L\mathbb{Z}^d$.

§2.4 Rational approximation to linear subspaces

Definition 2.4.1. For $W < \mathbb{R}^d$, the **expansion rate** of W under the flow $a_t L$ is

$$\tau_L(W) := \lim_{t \to +\infty} \frac{1}{t} \log \|a_t L w\|,$$

where $w \in \wedge^{\dim W} \mathbb{R}^d$ represents W.

Remark 2.4.2 $\tau_L(W)$ is the logarithm of the largest eigenvalue occurring in the decomposition of Lw along the eigenspaces of a_t in $\wedge^{\dim W}\mathbb{R}^d$.

Remark 2.4.3 If Λ_W is a lattice in W, then $|a_t L\Lambda_W| \asymp e^{\tau_L(W)}|\Lambda_W|$.

Exercise 2.4.4. $\tau_L : \mathrm{Gr}(\mathbb{Q}^d) \to \mathbb{R}$ is submodular.

Theorem 2.4.5 (Precision on the parametric subspace theorem.)
C_∞ is the Grayson polygon associated to τ_L and the HN filtration also corresponds.

Proof. V_{i_1} minimizes the rate $\frac{\tau_L(V_{i_1})}{i_1} = \min_V \frac{\tau_L(V)}{\dim V}$ and any V satisfying $\frac{\tau_L(V_{i_1})}{i_1} = \frac{\tau_L(V)}{\dim V}$ is a subspace of V_{i_1}. Observe that $|a_t L V_{i_1}(\mathbb{Z})| \asymp e^{t\tau_L(V_{i_1})}|V_{i_1}(\mathbb{Z})|$. So by Minkowski's first theorem, there exists $v \in a_t L V_{i_1}(\mathbb{Z})$ with $\|v\| \lesssim e^{t\frac{\tau_L(V_{i_1})}{i_1}}$. This shows that for every $t > 0$ large, $\lambda_1(a_t L\mathbb{Z}^d) \lesssim e^{t\frac{\tau_L(V_{i_1})}{i_1}}$. So we have $\frac{1}{t}\log \mu_{i_1}(a_t L\mathbb{Z}^d) \leqslant \tau_L(V_{i_1}) + o(1)$.

To check that $\frac{1}{t}C_t \to C_\infty$ on $[0, i_1]$, all we need to show is that

$$\lambda_1(a_t L\mathbb{Z}^d) \geqslant e^{t(\frac{\tau_L(V_{i_1})}{i_1}-\varepsilon)}$$

for every $\varepsilon > 0$ and $t > 0$ large enough. Let $V \leqslant \mathbb{Q}^d$ of minimal dimension such that there exists arbitrarily large t with $v \in V(\mathbb{Z})$ satisfying $\|a_t L v\| \leqslant e^{t(\frac{\tau_L(V_{i_1})}{i_1}-\varepsilon)}$. Let $k = \dim V$. We apply the subspace theorem.

Let $L_1, \cdots, L_d$ be the rows of L. Let j_1 be minimal such that $L_{j_1}|_V \neq 0$. We then find $j_1, \cdots, j_k$ inductively such that $L_{j_1}|_V, \cdots, L_{j_k}|_V$ are linearly independent. Then $\tau_L(V) = A_{j_1} + \cdots + A_{j_k}$. We have

$$\begin{aligned}
\|L_{j_1}(v)\cdots L_{j_k}(v)\| \leqslant& e^{-\tau_L(V)t}\prod_{s=1}^{k}\left|e^{A_{j_s}}tL_{j_s}(v)\right| \\
\leqslant& e^{\tau_L(V)t}\prod_{s=1}^{k}\|a_t L v\| \leqslant e^{\tau_L(V)t}e^{kt(\frac{\tau_L(V_{i_1})}{i_1}-\varepsilon)} \\
\leqslant& e^{-kt(\varepsilon - o(1))} \leqslant \|v\|^{-\varepsilon'}.
\end{aligned}$$

So all such v must belong to a finite union of proper subspaces of V. By the minimality of V, there can be such solutions only for bounded t. Hence we obtain that $\frac{1}{t}C_{a_t\Delta} \to \tau_L$ on $[0, i_1]$. Then we apply an induction and we are done. $\square$

Application to rational approximation to linear subspaces.

Let $x \in X_\ell(\overline{\mathbb{Q}})$ and $u_x \in \mathrm{SL}(d, \overline{\mathbb{Q}})$ such that $x = Pu_x$ ($x = u_x^{-1}\operatorname{span}\{e_1, \cdots, e_\ell\}$). We want to understand the successive minima of $a_t u_x \mathbb{Z}^d$. For $W \leqslant \mathbb{Z}^d$, write $\tau_x(W) = \tau_{u_x}(W)$. Then

$$\tau_x(W) = -\frac{\dim x \cap W}{\ell} + \frac{\dim W - \dim x \cap W}{d-\ell}.$$

So

$$\frac{\tau_x(W)}{\dim W} = \frac{1}{d-\ell} - \frac{\dim x \cap W}{\dim W} \cdot \frac{d}{\ell(d-\ell)}.$$

To minimize this, one has to maximize $\frac{\dim x \cap W}{\dim W}$.

Example 2.4.6

V_{i_1} is the unique subspace such that $\frac{\dim x \cap V_{i_1}}{\dim V_{i_1}} = \max_{W \leqslant \mathbb{Q}^d} \frac{\dim x \cap W}{\dim W}$.

Recall that a pencil for $W \subset \mathbb{Q}^d$ and $r \geqslant 1$ is

$$\mathscr{P}_{W,r} = \{ x \in X_\ell(\mathbb{R}) : \dim x \cap W \geqslant r \}.$$

We say the pencil is **constraining** if $\frac{r}{\dim W} > \frac{\ell}{d}$.

Corollary 2.4.7

If $x \in X_\ell(\overline{\mathbb{Q}})$ is not in any constraining rational pencil, then $\beta_k(x) = \frac{d}{k(d-\ell)}$.

Proof. By the example above, $V_{i_1} = \mathbb{Q}^d$. So the filtration is trivial and $C_\infty = 0$. Hence for every $i = 1, \cdots, d$, $\lambda_i(a_t u_x \mathbb{Z}^d) = e^{o(t)}$. But recall that the successive minima of $\wedge^k a_t u_x \mathbb{Z}^d$ are essentially the $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_k} = e^{o(t)}$. So $\wedge^k a_t u_x \mathbb{Z}^d$ has a nice basis consisting of vectors of length $e^{o(t)}$. One of them must satisfy $\|\pi_+(w)\| \gtrsim \|w\|$ so $\gamma_k(x) \geqslant 0$. Hence we obtain $\beta_k(x) \geqslant \frac{d}{k(d-\ell)}$. But we also know that $\wedge^k a_t u_x \mathbb{Z}^d$ contains no vector of norm less than $e^{\varepsilon t}$, so $\gamma_k(x) \leqslant 0$ and hence $\beta_k(x) = \frac{d}{k(d-\ell)}$. □

For general cases, V_{i_1} is the maximal maximizing $\frac{\dim x \cap V_{i_1}}{i_1} = \frac{\ell_1}{i_1}$; V_{i_2} is maximal maximizing $\frac{\dim x \cap V_{i_2} - \dim x \cap V_{i_1}}{i_2 - i_1} = \frac{\ell_2 - \ell_1}{i_2 - i_1}, \cdots$. To understand the successive minimas of $\wedge^k a_t u_x \mathbb{Z}^d$, we decompose

$$\wedge^k \mathbb{Q} = \bigoplus_{k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s = k} \underbrace{\bigwedge^{k_1} V_{i_1} \wedge \bigwedge^{k_2 - k_1} (V_{i_2}/V_{i_1}) \wedge \cdots \wedge \bigwedge^{k_s - k_{s-1}} (V_{i_s}/V_{i_{s-1}})}_{\text{denoted by } W_{\underline{k}} = W_{k_1, \cdots, k_s}}.$$

The logarithm of the successive minmas in $a_t u_x W_{\underline{k}}$ are essentially equal to

$$\Lambda_{\underline{k}} = \frac{k}{d-\ell} - \frac{d}{\ell(d-\ell)} \left(\frac{k_1 \ell_1}{i_1} + \frac{(k_2 - k_1)(\ell_2 - \ell_1)}{i_2 - i_1} + \cdots + \frac{(k_s - k_{s-1})(\ell_s - \ell_{s-1})}{i_s - i_{s-1}} \right).$$

To minimizing $\Lambda_{\underline{k}}$, one should take $k_1 = i_1, k_2 = i_2, \cdots, k_s = \min\{ i_s, k \}$. But then, one might not have $\|\pi_+(w)\| \gtrsim \|w\|$. To ensure this, it is necessary to have $k_r \leqslant \ell_r$ for every r. Indeed, otherwise, $u_x W_{\underline{k}} \cap \wedge^k \operatorname{span}\{ e_1, \cdots, e_\ell \} = \{ 0 \}$. Then we have $\|u_x v - \pi_+ u_x v\| \geqslant c \|u_x v\|$. So

$$\|a_t u_x v\| \geqslant c^{-(\frac{k}{\ell} - \frac{d}{\ell(d-\ell)})t} \|u_x v\| \gtrsim e^{\frac{dt}{\ell(d-t)}} \|\pi_+(a_t u_x v)\|$$

for $v \in W_{\underline{k}}$. This is not as desired.

Best possible choice is therefore $k_r = \min\{\ell_r, k\}$ for every i. Then we get the correct value. For example,

$$\gamma_\ell(x) = -\frac{\ell}{d-\ell} + \frac{d}{\ell(d-\ell)} \sum_{r=1}^{s} \frac{(\ell_r - \ell_{r-1})^2}{i_r - i_{r-1}}.$$

Finally, we can prove the first item in Theorem 2.1.16. It suffices to show that $\gamma_k \geqslant 0$. We consider a simpler case that $k = \ell$.

Proof. For $k = \ell$, by Cauchy-Schwartz, we have

$$d \sum_{r=1}^{s} \frac{(\ell_r - \ell_{s-1})^2}{i_r - i_{r-1}} \geqslant \left(\sum(\ell_r - \ell_{r-1})\right) \geqslant \ell^2.$$

Hence $\gamma_\ell \geqslant 0$. $\square$

3 Totally geodesic submanifolds and arithmeticity (Manfred Einsiedler)

§3.1 Lecture 1

1. Arithmeticity.

This minicourse focus on two following theorems about the arithmeticity of lattices.

Theorem 3.1.1 (Margulis) A lattice $\Gamma < \mathrm{SL}(3,\mathbb{R})$ is arithmetic.

Theorem 3.1.2 (Bader-Fisher-Miller-Stover)
Let $\Gamma < \mathrm{SO}(d,1)(\mathbb{R})$ be a lattice. Suppose that $M = \Gamma\backslash\mathbb{H}^d$ contains infinitely many maximal proper totally geodesic closed submanifolds of dimension at least two. Then Γ is arithmetic.

Reminders on arithmetic lattices.

Example 3.1.3
Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{Q}$-subgroup of $\mathrm{SL}(d,\mathbb{C})$. Then $\Gamma = \mathbf{G}(\mathbb{Z}) = \mathbf{G}(\mathbb{R}) \cap \mathrm{SL}(d,\mathbb{Z})$ is a lattice in $G = \mathbf{G}(\mathbb{R})$. For instance, $\mathrm{SL}(d,\mathbb{Z}) < \mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(d,1)(\mathbb{Z}) < \mathrm{SO}(d,1)(\mathbb{R})$.

Example 3.1.4 (Restriction of scalar)
Let $F/\mathbb{Q}$ be a number field and fix a basis of F over $\mathbb{Q}$. For any $\lambda \in F$, we let A_λ be the representation of the $\mathbb{Q}$-linear map $\lambda\cdot : x \in F \mapsto \lambda x \in F$. Let $\mathscr{A}_F$ be the image of F under the map $\lambda \mapsto A_\lambda \in \mathscr{A}_F \subset \mathbb{Q}^{d\times d}$. Then $\mathscr{A}_F$ is a subalgebra defined over $\mathbb{Q}$. For example, $F = \mathbb{Q}(\sqrt{a})$ for some $a \in \mathbb{Q}$ not a square. Then $\{1, \sqrt{a}\}$ form a $\mathbb{Q}$-basis of F. We have

$$\mathscr{A}_F = \left\{ \begin{bmatrix} x & ya \\ y & x \end{bmatrix} : x, y \in \mathbb{Q} \right\}.$$

Now let $\mathbf{G}$ be an algebraic subgroup of $\mathrm{SL}(n,\mathbb{C})$ defined over F. The restriction of scalar $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ is the following algebraic subgroup of $\mathrm{SL}(nd,\mathbb{C})$ defined over $\mathbb{Q}$:

$$\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} = \left\{ \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} : A_{ij} \in \mathscr{A}_F \text{ satisfy as blocks all equations that } \mathbf{G} \text{ satisfies} \right\}.$$

For example,

$$\operatorname{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : A_{ij} = \begin{bmatrix} x_{ij} & y_{ij}a \\ y_{ij} & x_{ij} \end{bmatrix}, A_{11}A_{22} - A_{12}A_{21} = \mathrm{id} \right\}.$$

Claim 3.1.5. $(\operatorname{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{C}) \cong \mathbf{G}(\mathbb{C})^d$.

This claim follows from the following observation. Considering $\mathscr{A}_F$ as a linear variety in $\mathbb{C}^{d\times d}$. Then

(1) the $\mathbb{Q}$-points of $\mathscr{A}_F$ are isomorphic to F;
(2) the $\mathbb{R}$-points of $\mathscr{A}_F$ are isomorphic to $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$;
(3) the $\mathbb{C}$-points of $\mathscr{A}_F$ are isomorphic to $F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^d$.

Recall that $F = \mathbb{Q}(\lambda)$ for some $\lambda \in F$. Note that the characteristic polynomial of A_λ is the minimal polynomial of λ. Hence the eigenvalue of A_λ are the Galois conjugates of λ in $\mathbb{R}$ or in $\mathbb{C}$. We can diagonalize A_λ by some $g \in \mathrm{GL}(d,\mathbb{R})$ as

$$g^{-1}A_\lambda g = \operatorname{diag}(\varphi_1(\lambda), \cdots, \varphi_r(\lambda), \psi_1(\lambda), \cdots, \psi_s(\lambda)),$$

where $\varphi_i : F \to \mathbb{R}$ and $\psi_i : F \to \mathbb{C}$, $\psi_i(\lambda)$ can be viewed as 2×2-real matrix.

Now we conjugate $\operatorname{Res}_{F/\mathbb{Q}}(\mathbf{G})$ by $\operatorname{diag}(g, \cdots, g)$, we obtain the following.

Claim 3.1.6. $\operatorname{Res}_{F/\mathbb{Q}}(\mathbb{R}) \cong \prod_{\varphi:F\to\mathbb{R}} \mathbf{G}^\varphi(\mathbb{R}) \times \prod_{\text{pairs of } \varphi:F\to\mathbb{C}} \mathbf{G}^\varphi(\mathbb{C})$, where $\mathbf{G}^\varphi$ is the algebraic group defined by the polynomials f^φ for all relations f that $\mathbf{G}$ satisfies.

Example 3.1.7

Let F be a totally real number field and $\lambda \in F$ such that $\varphi(\lambda) > 0$ for precisely one Galois embedding. Let

$$Q(x_1, \cdots, x_n, y) = x_1^2 + \cdots + x_n^2 - \lambda y^2.$$

Then $\mathbf{G} = \mathrm{SO}(Q)$ is a semisimple algebraic group defined over F if $n \geqslant 2$. Hence

$$\operatorname{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbb{R}) \cong \mathrm{SO}(n,1)(\mathbb{R}) \times \mathrm{SO}(n+1,\mathbb{R})^{d-1},$$

which is also semisimple. Using the first example we know that $(\operatorname{Res}_{F/\mathbb{Q}}\mathbf{G})(\mathbb{Z})$ is a lattice and hence the projection to $\mathrm{SO}(n,1)(\mathbb{R})$ is also a lattice.

Definition 3.1.8. Let G be a Lie group and Γ be a lattice. We say that Γ is **arithmetic** if there exists an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R}) = G \times K$ for a compact group K, $\mathbb{G}(\mathbb{Z}) < \mathbb{G}(\mathbb{R})$ is a lattice, and Γ is commensurable to a conjugate of the projection of $\mathbf{G}(\mathbb{Z})$ module K to G.

It is also worth noting that $\mathrm{SO}(n,1)(\mathbb{R})$ contains some non-arithmeticity lattices. An approach to construct non-arithmetic lattices is the following. We begin with two non compact arithmetic hyperbolic spaces $M_i = \Gamma_i\backslash\mathbb{H}^n$ and assume that they contain a same hyperbolic submanifold N. We then divide these them along N respectively and glue them back with exchanged pieces such that the resulting hyperbolic manifold M is still non compact. The non arithmeticity of M can be deduced from the following: the trace field for non-cocompact arithmetic lattices is $\mathbb{Q}$ and hence the length of closed geodesics are in $\exp(\mathbb{Q})$, but this is not always true for some weird ways of gluing manifolds.

§3.2 Lecture 2

2. Finite generation.

Theorem 3.2.1 (Garland-Raghunathan)
If G is a semisimple Lie group and $\Gamma < G$ is a lattice, then Γ is finitely generated.

We do not prove this theorem in this lecture. We will show the following proposition instead, which is easier to establish.

Proposition 3.2.2
If G is compactly generated and $\Gamma < G$ is a cocompact lattice, then Γ is finitely generated.

Proof. Let $Q \subset G$ be a compact subset such that $G = \bigcup_{n=1}^{\infty} Q^n$. Let $B \subset G$ be compact such that $\Gamma B = G$. Define $S := \Gamma \cap (B \cup BQB^{-1})$, which is a finite set.

Claim 3.2.3. $BQ \subset SB$.

Indeed, let $b \in B, g \in Q$ then $bg = \gamma b_1$ with $\gamma \in \Gamma, b_1 \in B$. Then $\gamma = bgb_1^{-1} \in S$.

Therefore, $BQ^n \subset S^n B$ and hence $G \subset \langle S \rangle B$. For any $\gamma \in \Gamma$, there exists some $\eta \in \langle S \rangle$ and $b \in B$ with $\gamma = \eta b$. Note that $b = \eta^{-1}\gamma \in \Gamma \cap B \subset S$, hence $\gamma \in \langle S \rangle$. ☐

3. Trace fields.

Proposition 3.2.4
Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})$ has no compact factors. Let $\Gamma < G$ be a lattice. Then

$$F := \mathbb{Q}(\{\operatorname{tr}(\operatorname{Ad}_\gamma) : \gamma \in \Gamma\})$$

is a finitely generated field. Moreover, there exists an algebraic group $\mathbf{G}^{\mathrm{ad}}$ defined over F and an algebraic isogeny $\varphi : \mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ defined over $\mathbb{R}$ such that $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(F)$.

Proof. We define the map $T : h \in \mathbf{G} \mapsto \operatorname{tr}(\operatorname{Ad}_h)$, which is a polynomial function on $\mathbf{G}$. For every $g \in \mathbf{G}$, we have that $g.T : h \in \mathbf{G} \mapsto \operatorname{tr}(\operatorname{Ad}_{hg})$ is another polynomial of the same degree. Hence $V = \langle g.T : g \in \mathbf{G} \rangle$ is finite dimensional.

Claim 3.2.5. $V = \langle \gamma.T : \gamma \in \Gamma \rangle$.

Proof. Because the right hand side $W = \langle \gamma.T : \gamma \in \Gamma \rangle$ satisfies $\gamma.W = W$ for all $\gamma \in \Gamma$. By Borel density (Γ is Zariski dense in $\mathbf{G}$), this implies that W is invariant for every $g \in \mathbf{G}$ and hence $W = W$ by the definition of V. ☐

Let $\gamma_1, \cdots, \gamma_n \in \Gamma$ be such that $\{\gamma_i.T\}$ forms a basis of V. We define $\varphi(g)$ to be the matrix representation of g. on V with respect to the basis $\{\gamma_i.T\}$. Then $\varphi(g) \in \mathrm{GL}(n, \mathbb{C})$ and we take $\mathbf{G}^{\mathrm{ad}}$ to be the image of φ.

Exercise 3.2.6. (1) Use Borel density to show that $\{\gamma_i.T|_\Gamma\}$ is linearly independent.

> (2) Moreover, there exists $s_1, \cdots, s_n \in \Gamma$ such that $\{\gamma_i.T\}$ is linearly independent restricted to $\{s_1, \cdots, s_n\}$.

Consequently, $A = \left[\mathrm{tr}(\mathrm{Ad}_{s_i\gamma_j})\right]_{1\leqslant i,j\leqslant n} \in \mathrm{GL}(n, \mathbb{C})$. Fix j and conclude that

$$\gamma\gamma_j.T = \sum_i \varphi(\gamma)_{ij}\gamma_i.T.$$

Now we evaluate this polynomial on s_k, we obtain

$$\gamma\gamma_i.T(s_k) = \sum_i \varphi(\gamma)_{ij}\gamma_i.T(s_k) = \sum_i \varphi(\gamma)_{ij}A_{kj}.$$

On the other hand, $\gamma\gamma_i.T(s_k) = \mathrm{tr}(\mathrm{Ad}_{s_k\gamma\gamma_i}) \in F$. Hence $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(F)$. By Borel density,

$$\overline{\mathbf{G}^{\mathrm{ad}}(F)}^{\mathrm{Zar}} = \varphi(\overline{\Gamma}^{\mathrm{Zar}}) = \varphi(\mathbf{G}) = \mathbf{G}^{\mathrm{ad}}.$$

Hence $\mathbf{G}^{\mathrm{ad}}$ is defined over F.

Finally, recall that Γ is finitely generated by some $S \subset \Gamma$. Let $L \subset F$ be the field generated by the matrix entries of $\varphi(\gamma)$ for $\gamma \in S$. Then L is finitely generated and $\varphi(\Gamma) \subset \mathbf{G}^{\mathrm{ad}}(L)$. This implies that both $\mathbf{G}^{\mathrm{ad}}$ and its Lie algebra are defined over L. We conclude that $\mathrm{tr}(\mathrm{Ad}_\gamma)$ calculated after applying the derivative of φ inside the Lie algebra of $\mathbf{G}^{\mathrm{ad}}$ gives values in L. We obtain $F \subset L \subset F$ and hence $L = F$. □

4. Margulis's strategy for arithmeticity.

Suppose $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$ and $\Gamma \subset \mathbf{G}(F)$, F is finitely generated satisfying $F \subset \mathbb{R}$. Let $\Bbbk$ be a local field and $\varphi : F \to \Bbbk$ be a Galois embedding. Let $\mathbf{H} = \mathbf{G}^\varphi$ be the algebraic $\Bbbk$-group obtained by applying φ to the coefficient of the elements of $\mathbf{G}$. Then $\varphi(\Gamma) \subset \mathbf{H}(\Bbbk)$ is Zariski dense by Borel density theorem.

Claim 3.2.7. Suppose for any such $\Bbbk$ and any group homomorphism $\varphi_\Gamma : \Gamma \to H = \mathbf{H}(\Bbbk)$ one of the followings holds:

- φ_Γ has a continuous extension to G, or
- φ_Γ has bounded image, i.e. $\overline{\varphi_\Gamma(\Gamma)} \subset \mathbf{H}(\Gamma)$ is compact in H.

Then Γ is arithmetic.

Notation 3.2.8. $\overline{V}^{\mathrm{Zar}}$ is the closure in Zariski topology and $\overline{V}$ is the closure in Hausdorff topology induced by the local field.

§3.3 Lecture 3

This time, we aim to show the claim mentioned at the end of last course.

Step 1: F is a number field. Suppose for a contradiction that $\mathrm{tr}(\mathrm{Ad}_{\gamma_0}) = x_0 \in F$ is transcendental for some $\gamma_0 \in \Gamma$. Pick p to be a prime and some transcendental $x_0' \in \mathbb{Q}_p$ with $\|x_0'\|_p > 1$. We can find a finite field extension $\Bbbk/\mathbb{Q}_p$ and $\varphi : F \to \Bbbk$ with $x_0 \mapsto x_0'$.

We apply the assumption for this φ and $\Bbbk$:

- φ_Γ cannot have a continuous extension $\varphi_G : G \to H$. Notice that G° is connected and hence $\varphi_G(G^\circ) = \{\mathrm{id}\}$. This implies that $\varphi_G(G)$ is finite, which contradicts the Zariski density.
- $\overline{\varphi_\Gamma(\Gamma)}$ cannot be compact because $\|\mathrm{tr}\,\mathrm{Ad}_{\varphi_\Gamma(\gamma_0)}\|_p = \|x_0'\|_p > 1$ and hence $\mathrm{Ad}_{\varphi_\Gamma(\gamma_0)}$ has an eigenvalue larger than 1.

Step 2: Γ is "almost integral". For simplicity, we assume that $F = \mathbb{Q}$. As Γ is finitely generated, there exists primes $p_1, \cdots, p_\ell \in \mathbb{N}$ such that $\Gamma \subset \mathbf{G}(\mathbb{Z}[1/(p_1 \cdots p_\ell)])$. Applying the assumption for $\varphi : \mathbb{Q} \to \mathbb{Q}_p$ with $p = p_j$, we have that $\overline{\varphi_\Gamma(\Gamma)}$ is compact. This means that all $\gamma \in \Gamma$ have entries where the powers of p in the denominator is bounded. In other words, since $\mathbb{H}(\mathbb{Z}_p)$ is compact open and $\overline{\varphi_\Gamma(\Gamma)}$ is compact, we have $\overline{\varphi_\Gamma(\Gamma)} \cap \mathbb{H}(\mathbb{Z}_p)$ has finite index in $\overline{\varphi_\Gamma(\Gamma)}$.

Applying this for all primes $p = p_j$, we obtain that $[\Gamma : \Gamma \cap \mathbf{G}(\mathbb{Z})] < \infty$. For general fields F, this argument shows that $[\Gamma : \Gamma \cap \mathbf{G}(\mathcal{O}_F)] < \infty$.

Step 3: Informations from the real and complex φ's.
Case 1. φ_Γ has a continuous extension.

Claim 3.3.1. In this case $\varphi = \mathrm{id} : F \hookrightarrow \mathbb{R}$.

Proof. If $\Bbbk = \mathbb{R}$ then φ is clearly an isogeny. Hence calculating the trace in the Lie algebras of H and G gives the same. This gives $\varphi(\operatorname{tr} \mathrm{Ad}_\gamma) = \operatorname{tr} \mathrm{Ad}_\gamma$ for $\gamma \in \Gamma$ and hence $\varphi = \mathrm{id}$ as F is generated by traces.

Suppose $\Bbbk = \mathbb{C}$. Let $\mathfrak{m}$ be the image of the real Lie algebra of G under the derivative of φ_G. Let $\mathfrak{h}$ be the complex Lie algebra of H. Then $\mathfrak{m}$ is an $\mathbb{R}$ Lie subalgebra of $\mathfrak{h}$. Note that $\mathfrak{m}_\mathbb{C}$ is preserved by $\varphi_\Gamma(\Gamma)$ and hence preserved by $\mathbf{H}$. Therefore $\mathfrak{m}_\mathbb{C}$ is an ideal in $\mathfrak{h}$. Since $\varphi_\Gamma(\Gamma)$ is Zariski dense in $\mathbf{H}$, $\mathfrak{m}_\mathbb{C}$ must be $\mathfrak{h}$ itself. Now we take an $\mathbb{R}$-basis of the Lie algebra of G. The pushforward of this basis under the derivative φ_G is an $\mathbb{R}$-basis of $\mathfrak{m}$ and hence a $\mathbb{C}$-basis of $\mathfrak{m}_\mathbb{C} = \mathfrak{h}$. Applying the same argument with the case $\Bbbk = \mathbb{R}$, we obtain that $\varphi|_F = \mathrm{id}$. □

Case 2. $\overline{\varphi_\Gamma(\Gamma)}$ is compact in H.

Claim 3.3.2. $\varphi(F) \subset \mathbb{R}$ and H is compact.

Proof. Let $M = \overline{\varphi_\Gamma(\Gamma)} \subset H$, which is compact by the assumption. Let $\mathfrak{m}$ be the real Lie algebra of M. Then $\mathfrak{m}_\mathbb{C} = \mathfrak{h}$ by the same argument. Moreover, if $\Bbbk = \mathbb{C}$ then $\mathfrak{m} \cap (i\mathfrak{m}) = \varnothing$. This is because for every $v \in \mathfrak{m} \cap (i\mathfrak{m})$, the exponential map $\exp : t \mapsto \exp(tv) \in M$ is a bounded entire function over $\mathbb{C}$ and hence $v = 0$.

We obtain that if $\Bbbk = \mathbb{C}$ then $\mathfrak{h} = \mathfrak{m} \oplus i\mathfrak{m}$. Using a $\mathbb{R}$-basis of $\mathfrak{m}$ as a $\mathbb{C}$-basis of $\mathfrak{h}$, we see that $\varphi(F) \subset \mathbb{R} \subset \mathbb{C}$. So we can assume without loss of generality that $\Bbbk = \mathbb{R}$.

As in the real world, every compact subgroups are algebraic. Because M is compact and Zariski dense in $\mathbf{H}$, we obtain that H is compact. □

We conclude what we have obtained from different embeddings.
(1) F is a number field.
(2) $\Gamma \cap \mathbf{G}(\mathcal{O}_F)$ is finite index in Γ.
(3) For $\varphi : F \to \mathbb{R}/\mathbb{C}$ with continuous extensions: $\varphi|_F = \mathrm{id}$.
(4) For $\varphi : F \to \mathbb{R}/\mathbb{C}$ with compact closures, $\varphi(F) \subset \mathbb{R}$ and $\mathbf{G}^\varphi(\mathbb{R})$ is compact.

Now we apply $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ and obtain a new semisimple algebraic group $\mathbb{Q}$. Its group of $\mathbb{R}$-points is isomorphic to $\mathbf{G}^{\mathrm{id}}(\mathbb{R}) \times K$, where $K = \prod_{\varphi \neq \mathrm{id}, \varphi : F \to \mathbb{R}} \mathbf{G}^\varphi(\mathbb{R})$ is compact. Moreover (by choosing a $\mathbb{Z}$-basis of $\mathcal{O}_F$ in the construction of $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$) we can ensure that $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})(\mathbb{Z}) \cong \mathbf{G}(\mathcal{O}_K)$. Finally, projecting module K we obtain the arithmetic lattice $\mathbf{G}^{\mathrm{id}}(\mathcal{O}_F) \subset G$. As $\Gamma \cap \mathbf{G}^{\mathrm{id}}(\mathcal{O}_F)$ has finite index in Γ, we obtain that Γ is arithmetic. □

§3.4 Lecture 4

5. Superrigidity.

Theorem 3.4.1 (Margulis's Superrigidity)
Let $G = \mathrm{SL}(3, \mathbb{R})$ and Γ be a lattice. Let $\Bbbk$ be a local field and $\mathbf{H}$ be a simple adjoint algebraic group over $\Bbbk$. Let $\varphi : \Gamma \to H = \mathbf{H}(\Bbbk)$ a homomorphism with a Zariski dense image. Then one of the following must hold:
(1) φ has a continuous extension $\varphi_G : G \to H$, or
(2) $\overline{\varphi(\Gamma)}$ is compact in H.

This theorem implies the arithmeticity by Claim 3.2.7.

6. Getting started for $\mathrm{SL}(3, \mathbb{R})$.

Let $U < \mathrm{SL}(3, \mathbb{R})$ be a root subgroup, for example $\left\{ \begin{bmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{bmatrix} \right\}$. Then U acts ergodically on $X = \Gamma \backslash G$ by Moore's ergodic theorem. Let $x_0 \in X$ be a U-generic point for U, that is

$$\frac{1}{T} \int_0^T \delta_{x_0 u_t} \, \mathrm{d}t \xrightarrow{w*} m_X \quad \text{as } T \to \infty.$$

Let V be an irreducible representation of H over $\Bbbk$. Then H acts on $\mathbb{P}(V)$ without fixed points. Restricting on $\varphi(\Gamma)$ this remains true.

Although this superrigidity also states for $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, we can keep in mind that H is a p-adic Lie group but G is a real Lie group. In this case, φ is the only thing links these two group. So we may consider the space

$$\widetilde{X} = \Gamma \backslash (G \times \mathbb{P}(V)),$$

where γ acts on $G \times \mathbb{P}(V)$ as $(g, [v]) \mapsto (\gamma g, \varphi(\gamma)[v])$ diagonally. Note that the projection $\widetilde{X} \to X, \Gamma(g, [v]) \mapsto \Gamma g$ is a nice factor map (projecting to $\mathbb{P}(V)$ is not nice since Γ acting on $\mathbb{P}(V)$ is not properly discontinuously).

Let $\widetilde{x}_0$ be any point in $\widetilde{X}$ mapping to x_0. Let

$$\mu_T = \frac{1}{T} \int_0^T \delta_{\widetilde{x}_0 u_t} \, \mathrm{d}t.$$

Suppose $\mu_T \to \mu$ along a subset of T's, then μ satisifies
- μ is U-invariant, and
- μ is a probability measure projecting to m_X.

7. Getting started for $\mathrm{SO}(d, 1)(\mathbb{R})$.

We skip this part for the moment. This will be discussed in Lecture 8.

8. A measure-valued map.

We are given a subgroup $S < G$ (for example, $S = U$), an extension $\widetilde{X} = \Gamma \backslash (G \times \mathbb{P}(V))$ and an S-invariant measure μ on $\widetilde{X}$ projecting to m_X.

We unfold $\widetilde{X}$ to create an infinite Γ invariant measure $\widetilde{\mu}$ on $G \times \mathbb{P}(V)$. We want to use conditional measures for the σ-algebra $\mathscr{C} = \mathscr{B}_G \times \mathscr{W}_{\mathbb{P}(V)}$, where $\mathscr{W}_{\mathbb{P}(V)}$ is the trivial σ-algebra on $\mathbb{P}(V)$. This way we get a measurable map

$$g \times G \to \delta_g \times \nu_g,$$

where ν_g is a probability measure on $\mathbb{P}(V)$. Moreover, $\widetilde{\mu}$ is invariant under S. Hence the conditional measure satisfy a resulting compatibility. In this case, we obtain $\nu_{gs} = \nu_g$ for $s \in S$ and almost every g.

Also $\widetilde{\mu}$ is Γ invariant. Then the conditional measure also satisfies

$$\delta_{\gamma g} \times \nu_{\gamma g} = \gamma_*(\delta_g \times \nu_g) = \delta_{\gamma g} \times (\varphi(\gamma)_* \nu_g),$$

and hence $\nu_{\gamma g} = \varphi(\gamma)_* \nu_g$ for γ and almost every $g \in G$. We can interpret this as a measurable Γ-equivariant map

$$\phi : G/S \to \mathcal{M}^1(\mathbb{P}(V)), \quad gS \mapsto \nu_g.$$

9. Locally closed orbits.

Let $V = \mathbf{V}(\mathbb{k})$ be a variety over a local field $\mathbb{k}$. Let $H = \mathbf{H}(\mathbb{k})$ act algebraically on V. We want to understand H-orbits and H-ergodic measures on V.

Claim 3.4.2. H-orbits are locally close, i.e. for any $v \in V$ there exists a neighborhood B of v so that $B \cap \overline{Hv} = B \cap Hv$.

Proof. $h \in H \mapsto h.v \in V$ is an algebraic map (possibly with a non-trivial stabilizer). Using that a polynomial regular map will only miss points from a lower dimension subvariety of the Zariski closure of the image, one can choose B. □

Corollary 3.4.3

Let μ be a measure on V that is H-ergodic. Then there is some $v \in V$ such that μ gives full measure to $H.v$.

§3.5 Lecture 5

Proof. Let $B_1, B_2, \cdots$ be a basis of the topology of V. For any n we apply the assumed ergodicity to $H.B_n$. Hence we have $\mu(H.B_n) = \varnothing$ or $\mu(V \setminus H.B_n) = \varnothing$. We take the union of these null sets and suppose v_0, v_1 do not belong to these null sets.

Claim 3.5.1. $H.v_0 = H.v_1$.

Proof. By the local closeness, we can take $B_{n_0} \ni v_0$ such that $B_{n_0} \cap \overline{H.v_0} = B_{n_0} \cap H.v_0$. Since v_0 does not belong to these null set, we have $\mu(H.B_{n_0}) > 0$. Consequently, $\mu(V \setminus H.B_{n_0}) = 0$ and hence $v_1 \in H.B_{n_0}$. Then we can take some $h_1 \in H$ such that $h_1 v_1 \in B_{n_0}$.

Assume that $h_1 v_1 \notin H.v_0 \cap B_{n_0}$. By the local discreteness of the orbits, we can take some $B_{n_1} \ni h_1 v_1$ such that $B_{n_1} \subset B_{n_0}$ and $B_{n_1} \cap H.v_0 = \varnothing$. A same deduction as above, we have $\mu(H.B_{n_1}) > 0$ and $v_0 \in H.B_{n_1}$. This contradicts $B_{n_1} \cap H.v_0 = \varnothing$. □

□

Proposition 3.5.2

Let $H = \mathbf{H}(\Bbbk)$ act algebraically on $V = \mathbf{V}(\Bbbk)$. Let μ be a Borel probability measure on V. Then $\mathrm{Stab}_H(\mu) = \{ h \in H : h_*\mu = \mu \}$ is a compact extension of the $\Bbbk$-points of the algebraic group $\mathrm{Fix}_{\mathbf{H}}(\mu) = \{ h : h.v = v, \forall v \in \operatorname{supp} \mu \}$.

Sketch of the proof. Without loss of generality, we can assume that $\operatorname{supp} \mu$ is Zariski dense in $\mathbf{V}$. If $h \in \mathrm{Stab}_H(\mu)$ then h normalizes $\mathrm{Fix}_{\mathbf{H}}(\mu)$. By taking the quotient we can assume that $\mathrm{Fix}_{\mathbf{H}}(\mu) = \{ \mathrm{id} \}$. Then we can find a finite set $\{ v_1, \cdots, v_n \} \subset \operatorname{supp} \mu$ such that $\dim \mathrm{Fix}_{\mathbf{H}}(v_1, \cdots, v_n) = 0$.

> **Exercise 3.5.3.** For all $(v_1', \cdots, v_n')$ in a sufficiently small neighborhood of $(v_1, \cdots, v_n)$, we have $\dim \mathrm{Fix}_{\mathbf{H}}(v_1', \cdots, v_n') = 0$.

Assuming by contradiction that $\mathrm{Stab}_{\mathbf{H}}(\mu)$ is non compact, we can apply Poincaré recurrence. Choose $(v_1', \cdots, v_n')$ near $(v_1, \cdots, v_n)$ which is infinitely recurrent under the $\mathrm{Stab}_{\mathbf{H}}(\mu)$ action. But the orbits are locally closed. Therefore there are infinitely many $h \in \mathrm{Stab}_{\mathbf{H}}(\mu)$ fixing $(v_1', \cdots, v_n')$. This contradicts $\dim \mathrm{Fix}_{\mathbf{H}}(v_1', \cdots, v_n') = 0$. □

Proposition 3.5.4 (Zimmer)

The H-actions on $\mathcal{M}^1(\mathbb{P}(V))$ has locally closed orbits.

10. Creating a map with values in H/L.

Recall that we have a Γ-equivariant map

$$\phi : G/U \to \mathcal{M}^1(\mathbb{P}(V)).$$

Let $m_{G/U}$ be a smooth measure on G/U. By ergodicity of U on $\Gamma\backslash G$, we have by duality that the Γ-action on $(G/U, m_{G/U})$ is ergodic. Hence $\phi_*(m_{G/U})$ is a Γ-ergodic measure on $\mathcal{M}^1(\mathbb{P}(V))$. So it is also H-ergodic. Since H-orbits on $\mathcal{M}^1(\mathbb{P}(V))$ are locally closed, $\phi_*(m_{G/U})$ gives the full measure to a single H-orbit $H.\nu_0$. That is,

$$\phi : G/U \to H.\nu_0 \cong H/\operatorname{Stab}_H(\nu_0) \quad \text{a.s..}$$

We distinguish two cases:
1. $\mathrm{Stab}_H(\nu_0)$ is non-compact. Then we take $\mathbf{L}_0 = \mathrm{Fix}_{\mathbf{H}}(\operatorname{supp} \nu_0)$ satisfying that $\mathbf{L}(k)$ is non compact but not all of H. In this case $\mathrm{Stab}_H(\nu_0) \subset N_H(\mathbf{L}_0) = \mathbf{L}$, where $\mathbf{L}$ is a proper algebraic subgroup of $\mathbf{H}$.
2. $\mathrm{Stab}_H(\nu_0)$ is compact.

Hence these two cases come to be
1. There exists a Γ-equivariant $\phi : G/U \to H/L$ for $L = \mathbf{L}(\Bbbk)$ and $\mathbf{L} < \mathbf{H}$ a proper $\Bbbk$-subgroup.
2. There exists a Γ-equivariant $\phi : G/U \to H/L$ where $L < H$ is compact.

§3.6 Lecture 6

Let us recall our strategy to establish Margulis superrigidity:

(1) Consider a root group U acting ergodically on $X = \Gamma\backslash G$ with a generic point x_0.
(2) Using an $\widetilde{x}_0 \in \widetilde{X} = \Gamma\backslash(G \times \mathbb{P}(V))$ to construct a lifting measure $\widetilde{\mu}$ on $\widetilde{X}$ which is U-invariant and projects to m_X.
(3) Consider the conditional measure of m_X, which gives a U-invariant and Γ-equivariant map
$$\phi : G \to \mathcal{M}^1(\mathbb{P}(V)).$$
(4) By the ergodicity of $\Gamma \curvearrowright G/U$ and the local closeness of H-orbits on $\mathcal{M}^1(\mathbb{P}(V))$, we know that $\phi(G)$ falls in one H-orbit $H.\nu_0 \cong H/\operatorname{Stab}_H(\nu_0)$.
(5) By studying the algebraic structure of $\operatorname{Stab}_H(\nu_0)$, there are only two cases should be considered:
   - There exists a Γ-equivariant $\phi : G/U \to H/L$ for $L = \mathbf{L}(\mathbb{k})$ and $\mathbf{L} < \mathbf{H}$ a proper $\mathbb{k}$-subgroup.
   - There exists a Γ-equivariant $\phi : G/U \to H/L$ where $L < H$ is compact.

11. Metric ergodicity (Bader-Gelander)

We now consider the case $\phi : G/U \to H/L$ where L is compact. In this case, H/L has an H-invariant metric.

Lemma 3.6.1
Let S be a unbounded subgroup of a simple group G. Let $\phi : G/S \to Y$ be continuous and G-equivariant for an action of G on Y preserving a metric on Y. Then ϕ is constant.

Proof. **Case 1.** Assume that S contains some diagonalizable element a. Let $u \in U = G_a^+$ then $a^n u a^{-n} \to \mathrm{id}$. Then

$$a^n\phi(uS) = \phi(a^n uS) = \phi(a^n u a^{-n} S) \to \phi(\mathrm{id}S), \quad n \to +\infty.$$

Note that $d(a^n\phi(uS), \phi(\mathrm{id}S)) = d(\phi(uS), a^{-n}\phi(\mathrm{id}S))$, we also have

$$\phi(\mathrm{id}S) = a^{-n}\phi(\mathrm{id}S) \to \phi(uS).$$

Hence ϕ is u-invariant. Noting that G_a^+, G_a^- generates G, we obtain that ϕ is constant.

Case 2. Assume that $S \supset U$ a unipotent subgroup. We think the case that $G = \mathrm{SL}(2,\mathbb{R})$ and $U = \left\{ \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \right\}$. Let $v_n = \begin{bmatrix} 1 & \\ \frac{1}{n} & 1 \end{bmatrix}$ and then there exists $u_n, u_n' \in U$ such that $u_n v_n u_n' \to \begin{bmatrix} 2 & \\ & \frac{1}{2} \end{bmatrix} = a$. Then we have $u_n\phi(v_nS) \to \phi(aS)$ and $\phi(v_nS) \to \phi(\mathrm{id}S)$. Since the metric is G-invariant, we have $u_n\phi(v_nS) \to u_n\phi(\mathrm{id}S) = \phi(\mathrm{id}S)$ and hence $\phi(aS) = \phi(\mathrm{id}S)$. This argument works for every a so we obtain that ϕ is A-invariant. Then we can apply the result of the first case. ☐

But in our case, the map ϕ is only measurable and Γ-equivariant. The assumption of this lemma is too strong to apply. We need to apply the lemma to another map associated to ϕ.

Theorem 3.6.2
Let $G = \mathrm{GL}(3,\mathbb{R}) \supset U$ a root group, $\Gamma < G$ a lattice and $\phi : G/U \to H/L$ a measurable Γ-equivariant map, where $L < H$ is compact so that H/L has an H-invariant metric. Then ϕ is constant almost surely. In particular, $\overline{\varphi_\Gamma(\Gamma)} \subset H$ is compact.

Proof of "in particular". If $\phi(gU) \equiv h_0 L$ for m_G-almost every $g \in G$. Then

$$\varphi(\gamma) h_0 L \doteq \phi(\gamma g U) \doteq h_0 L, \quad \forall \gamma \in \Gamma,$$

here $\doteq$ denotes the almost surely equality. Therefore, $\overline{\varphi_\Gamma(\Gamma)} \in h_0 L h_0^{-1}$ which is compact. □

Proof. Replacing the metric on H/L be a Γ-equivariant one, we may assume that the metric is bounded. We define

$$Y = L(G, H/L)^\Gamma := \{\, \Gamma\text{-equivariant measurable maps from } G \text{ to } H \,\}.$$

We endow $Y = L(G, H/L)^\Gamma$ with the metric

$$d_Y(f_1, f_2) = \int_F d_{H/L}(f_1(g), f_2(g))\, \mathrm{d}m_G(g),$$

where F is a fundamental domain of Γ. The action of G on Y is given by

$$\forall g_0 \in G, f \in Y, \quad g_0.f := (g \in G \mapsto f(g g_0)) \in Y.$$

Exercise 3.6.3. Show that the G-action is continuous and isometric on Y.

Now we define a new map $\widetilde{\phi} : G/U \to Y$ given by

$$g_0 U \mapsto (g \in G \mapsto \phi(g g_0 U) \in H/L).$$

Note that this map is also G-invariant. By applying the lemma to $\widetilde{\phi}$, we know that ϕ is a constant almost surely. □

12. Algebraic T-shadows (Bader-Furman)

This concept occurs in the study of **algebraic representations of ergodic actions (AREA)**. Recall that we want to study the Γ-equivariant map $\phi : G/U \to H/L$ for $L = \mathbf{L}(\Bbbk)$ noncompact and $\mathbf{L} < \mathbf{H}$ a proper $\Bbbk$-subgroup.

Definition 3.6.4. Let $T < G$ be unbounded. A measurable map $\psi : G \to H/L$ is called **an algebraic T-shadow** (for the $(\Gamma \times T)$-space G) if

(1) $L = \mathbf{L}(\Bbbk)$ for an algebraic subgroup $\mathbf{L} < \mathbf{H}$ over $\Bbbk$.
(2) ψ is measurable and defined almost everywhere.
(3) For every $\gamma \in \Gamma$, $\psi(\gamma g) = \varphi(\gamma)\psi(g)$ almost everywhere.
(4) For every $t \in T$, there exists $\tau(t) \in N_H(L)/L$ so that

$$\psi(gt) = \psi(g)\tau(t), \quad \text{a.e..}$$

Lemma 3.6.5

τ is uniquely determined by the definition and τ is a measurable (hence continuous) homomorphism $\tau : T \to N_H(L)/L$.

Lemma 3.6.6

If $\psi : G \to H/L$ is a T_j-shadow for $j = 1, \cdots, \ell$ and $T = \langle T_1, \cdots, T_\ell \rangle$, then ψ is also a T-shadow.

§3.7 Lecture 7

Aim 3.7.1. To show ψ is a T-shadow for large T.

Lemma 3.7.2 (G-shadow)
Suppose $\psi : G \to H/L$ is a G-shadow. Then L is a normal subgroup of H and there exists an $h_0 \in H$ so that $\tau(\gamma) = h_0 \gamma h_0^{-1} \in H/L$. In particular, if H is simple, adjoint and $L \neq H$ then $L = \{\,\mathrm{id}\,\}$.

Proof. For every g, we have $\psi(g_0 g)L = \psi(g_0)\tau(g)L$ for almost every g_0. By a Fubini argument, for almost every g_0, we have (without loss of generality)

$$\psi(g_0 g)L = \psi(g_0)\tau(g)L, \quad \forall g \in G.$$

Let $\psi(g_0) = h_0 L$. Then for every $\gamma \in \Gamma$,

$$\psi(\gamma g_0 g) = \gamma \psi(g_0 g) \subset \gamma h_0 N_H(L)/L.$$

On the other hand, we have

$$\psi(\gamma g_0 g) = \psi(g_0(g_0^{-1}\gamma g_0)g) = \psi(g_0)\tau(g_0^{-1}\gamma g_0)\tau(g) \subset h_0 N_H(L)/L.$$

Hence we obtain $h_0^{-1}\Gamma h_0 \in N_H(L)$ and hence $N_H(L) = H$ by the Zariski density of Γ. □

Definition 3.7.3. Let $\psi_1 : G \to H/L_1$ and $\psi_2 : G \to H/L_2$ be two T-shadows. We say that ψ_2 is a **factor** of ψ_1 if there exists $p \in H$ with $L_1 p \subset pL_2$ and the following diagram commutes:

$$\begin{tikzcd}
 & G/L_1 \arrow[dd, "\cdot pL_2\,\cdot"] \\
G \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\
 & G/L_2
\end{tikzcd}$$

Lemma 3.7.4
Assuming ψ_2 is a factor of ψ_1 and $p \in H$ is as in the definition. Then

$$\tau_1(t)pL_2 = p\tau_2(t)L_2, \quad \forall t \in T.$$

Proposition 3.7.5 (Initial T-shadow)
Assume that T is unbounded. There exists a T-shadow $\psi : G \to H/L_{\min}$ so that any other T-shadow is a factor. In fact, every T-shadow with $\mathbf{L}_{\min} = \overline{L_{\min}}^{\mathrm{Zar}}$ minimal in the set of all such Zariski closures is an initial T-shadow as above.

Corollary 3.7.6 (Normalizer)
Let $\psi_{\min} : G \to H/L_{\min}$ be an initial T-shadow as in the proposition. Then $\psi_{\min}$ is also an initial $N_G(T)$-shadow.

Proof. Let $a \in N_G(T)$, we define a new T-shadow $\psi_a : G \to H/L_{\min}$ by $\psi_a(g) = \psi_{\min}(ga)$ and $\tau_a(t) = \tau_{\min}(a^{-1}ta)$. Then

$$\psi_a(gt) = \psi_{\min}(gta) = \psi_{\min}(ga)\tau_{\min}(a^{-1}ta) = \psi_a(g)\tau_a(t).$$

Noting that $\tau_{\min}$ is initial, there exists $p = p_a$ such that $\psi_a = (\cdot pL_{\min}) \circ \psi_{\min}$. That is, $\psi_{\min}(ga)L_{\min} = \psi_{\min}(g)p_aL_{\min}$. Therefore we obtain a $N_G(T)$-shadow by letting $\tau(a) = p_a$. □

13. Conclusion for $\mathrm{SL}(3, \mathbb{R})$.

Aim 3.7.7. Start with a U-shadow and end up with a G-shadow.

Proof. For the $\mathrm{SL}(3, \mathbb{R})$ case, the root space can be generated by $\alpha, \beta, \gamma, -\alpha, -\beta, -\gamma$. We start with $U = U_\alpha$. By the proposition and the corollary, there exists an initial U_α-shadow $\psi_{\min}$, which is also a U_β-shadow. Applying the corollary again, we can find an initial U_β-shadow $\psi'_{\min}$ (which a priori can be greater than $\psi_{\min}$). Then we have that $\psi'_{\min}$ is also U_γ-shadow. We continue this process and will turn back to get a U_α-shadow.

This means that $\psi'_{\min}$ can not be better than $\psi_{\min}$. Moreover, $\psi_{\min}$ is an initial T-shadow for $T = U_\alpha, U_\beta, U_\gamma, U_{-\alpha}, U_{-\beta}, U_{-\gamma}$. These root groups generate G and hence $\psi_{\min}$ is a G-shadow. Therefore L is trivial and we obtain a continuous extension of φ_Γ. □

Proof idea for Proposition 3.7.5. Given $\psi_{\min}$ and ψ, we define

$$\psi_V(g) = (\psi_{\min}(g), \psi(g)) \in V = H/L_{\min} \times H/L.$$

Then $\mathbf{M} = \overline{\{ (\tau_{\min}(t), \tau(t)) : t \in T \}}^{\mathrm{Zar}} \subset N_{\mathbf{H}}(\mathbf{L}_{\min}) \times N_{\mathbf{H}}(\mathbf{L})$ and $\mathbf{V} = \overline{V}^{\mathrm{Zar}}$ is homogeneous for $\mathbf{H}$ acting diagonally. By the ergodicity and locally-closed orbits we obtain that ψ_V takes values in only one $H \times M$-orbit. □

§3.8 Lecture 8

7. Getting started for $\mathrm{SO}(d, 1)(\mathbb{R})$.

Now we will discuss about this skipped part. First, we discuss about the totally geodesic submanifolds in $\Gamma\backslash\mathbb{H}^d$.

For the case of $d = 3$, $\mathbb{H}^3$ can be interpreted as $\mathrm{SO}(3,1)(\mathbb{R})^\circ/\mathrm{SO}(3, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2, \mathbb{R})$. In $\mathbb{H}^3$, there is a standard embedded $\mathbb{H}^2$ as

$$\mathbb{H}^2 \cong \mathrm{SO}(2,1)(\mathbb{R})^\circ \mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(3, \mathbb{R}) \subset \mathrm{SO}(3,1)(\mathbb{R})^\circ/\mathrm{SO}(3, \mathbb{R}),$$

or

$$\mathrm{SL}(2, \mathbb{R})/\mathrm{SU}(2, \mathbb{R}) \subset \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2, \mathbb{R}).$$

Then $g\mathrm{SO}(2,1)(\mathbb{R})^\circ \mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(3, \mathbb{R})$ for $g \in \mathrm{SO}(3,1)(\mathbb{R})^\circ$ or $g\mathrm{SL}(2, \mathbb{R})/\mathrm{SU}(2, \mathbb{R})$ for $g \in \mathrm{SL}(2, \mathbb{C})$ give the algebraic description of two-dimensional hyperbolic planes inside $\mathbb{H}^3$.

The totally geodesic (closed) 2-dimensional subspace of $M = \Gamma\backslash\mathbb{H}^3$ are precisely of the form

$$\Gamma g \mathrm{SL}(2,\mathbb{R})/\mathrm{SU}(2,\mathbb{R}) \subset M$$

if the set is closed. Any closed totally geodesic plane in M corresponds this way to a closed orbit

$$\Gamma g \mathrm{SL}(2,\mathbb{R}) \subset X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C}).$$

Lemma 3.8.1 (Dani's argument)
These closed orbits always have finite volume.

The proof uses a version of Margulis-Dani's nondivergence:

Theorem 3.8.2 (Nondivergence)
Given $\varepsilon > 0$ and a compact $A \subset X$, there exists a compact $B \subset X$ so that for all $x \in A$ and $T > 0$ we have

$$\frac{1}{T}\left|\{\, t \in [0,T] : xu_t \in B \,\}\right| > 1 - \varepsilon.$$

Proof. Consider the Haar measure μ on the closed $\mathrm{SL}(2,\mathbb{R})$-orbit. We apply the nondivergence for the unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then we can find a compact B contained in the closed orbit. Consider the function

$$f = \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}_B(\cdot u_t)\,\mathrm{d}t.$$

Exercise 3.8.3. Show that $f \in L^2(\mu)$.

Then f is a $\{\, u_t \,\}$-invariant function. By Mautner's phenomenon, f is $\mathrm{SL}(2,\mathbb{R})$-invariant. Hence f is constant and $f > 0$ on A. Therefore μ is a finite. □

Theorem 3.8.4 (Mozes-Shah)
For a sequence of probability measure μ_n on $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ corresponding to a sequence of pairwise distinct totally geodesic closed 2-dimensional submanifolds we have equidistribution to the Haar measure on X.

Proof. Assume that $\mu_n \to \mu$. As μ_n is $\mathrm{SL}(2,\mathbb{R})$-invariant, then
1. μ is $\mathrm{SL}(2,\mathbb{R})$ invariant, and
2. μ is a probability measure: because there exists a compact subset $A \subset \mathbb{C}$ such that any closed geodesic has to hit A. Then for every ε, let B be given by the nondivergence, we have $\mu_n(B) > 1-\varepsilon$ and hence $\mu(B) > 1-\varepsilon$.

Then we apply Ratner's theorem for $\mathrm{SL}(2,\mathbb{R})$-invariant measures:

Theorem 3.8.5 (Ratner)
$\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measures are homogeneous.

Finally we need a linearization argument. We explain the idea here. Assume that $\mu = \sum c_j \nu_j + c_0 m_X$ be the ergodic decomposition of μ, where ν_j supported on closed orbits. (We admit that there are only countably many ergodic components). For this we assume that $Y = \Gamma g_0 \mathrm{SL}(2,\mathbb{R})$ is a closed orbit. We want to show $\mu(Y) = 0$. We can choose an $\mathrm{SL}(2,\mathbb{R})$-invariant complement of Y and use this to construct a transversal neighborhood of Y. Let x be a generic point of μ_n. By the property of polynomials (the (C,α)-good property), the time of the orbit of x entering a super small transversal neighborhood of Y is a little. Hence we can conclude that $\mu(Y) = 0$. $\square$

Mozes-Shah's theorem plays a crucial role in the construction of a measure on the fiber bundle $\widetilde{X}$ as in Part 6 (Getting started for $\mathrm{SL}(3,\mathbb{R})$). This helps to establish the arithmeticity of certain lattices in $\mathrm{SO}(d,1)$.