Geometric Group Theory
(Spring 2023, Wenyuan Yang)

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Contents

Introduction

There are three main topics in this semester.

I. Groups acting on δ-hyperbolic spaces.
II. Boundary Theory for groups: dynamics.
III. Patterson-Sullivan measure on boundary.

§0.1 Groups acting on hyperbolic spaces

Hyperbolic groups. Gromov (1987), Rips, Cannon.
The fundamental group of a closed Riemannian manifold with negative curvature.

Relatively hyperbolic groups. Gromov, Farb (96), Bowditch, Osin

(1) The fundamental group of a finite volume Riemannian manifold with negative curvature.
(2) $H * K$, for example, $\mathbb{Z}^2 * \mathbb{Z}^3$.

Acylinderically hyperbolic groups. Osin (2015), Guirardel, Dahmani(2012)
$\Longleftrightarrow$ Groups with hyperbolic embed subgroups.

(1) Mapping class groups
(2) $\mathrm{Out}(F_n)$.
(3) Cremona groups $\mathrm{Aut}(\mathbb{P}^2(\mathbb{C}))$.
(4) Groups with contracting elements.

§0.2 Boundary theory

We focus on **Gromov boundary** of a δ-hyperbolic (geodesic) space. Let X and Y be two hyperbolic spaces associated with boundaries ∂X and ∂Y, respectively. Let $\psi : X \to Y$ be a QIE, then it induces a boundary map $\partial\psi : \partial X \to \partial Y$ which is continuous. The boundary is "better".

We will equip the boundary with a visual metric, then the boundary map $\partial\psi$ will be quasi-conformal.

Motivation. Mostow Rigidity Theorem.

Applications. Quasi-isometric rigidity.

1 Groups acting on hyperbolic spaces

§1.1 Feb 23

Let (X, d) be a **length space**, that is,

$$d(x, y) = \inf \{\mathrm{len}(\gamma) : \gamma \text{ is a path that connects } x \text{ and } y\}$$

where

$$\mathrm{len}(\gamma) := \sup_{t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})).$$

A path γ is called rectifiable if $\mathrm{len}(\gamma) < \infty$.

We say a metric space is **proper** if for every $o \in X$ and $r > 0$, $\overline{B(o, r)}$ is compact.

Definition 1.1.1. A path γ is called a **geodesic** if there is a parametrization of γ as

$$\gamma : [0, \mathrm{len}(\gamma)] \to X$$

which is a isometric embedding. Or equivalently, let $\gamma : [0, 1] \to X$, then for every $0 \leqslant s \leqslant t \leqslant 1$, $d(\gamma(s), \gamma(t)) = \mathrm{len}\, \gamma([s, t])$.

Theorem 1.1.2 (Arzela-Ascoli Lemma)

Let (M, d) be a compact metric space and $\gamma_n \in C((M, d) \to (M, d))$. Then there is a subsequence of γ_n converges in uniform convergence iff $\{\gamma_n\}$ is equi-continuous and uniformly bounded.

Let γ_n be a sequence of path uniformly converges to γ, then

$$\liminf_{n \to \infty} \mathrm{len}(\gamma_n) \geqslant \mathrm{len}(\gamma).$$

And if all of γ_n's are geodesics, then γ is a geodesic.

Theorem 1.1.3 (Hopf-Rinow)

Let X be a length space. Then X is proper if and only if

(i) X is locally compact, and

(ii) X is complete.

In particular, X is a **geodesic space** in this case.

Example 1.1.4

A connected graph with combinatorial metric is a geodesic space. But it may not be proper if the graph is not locally finite.

Definition 1.1.5. A geodesic metric space (X, d) is **δ-hyperbolic** for $\delta \geqslant 0$ if for every geodesic triangle $\Delta(x, y, z)$, every side is contained in the δ-neighborhood of the other two sides.

A δ-hyperbolic space satisfies a **Thin Triangle Property**: let $\Delta(x, y, z)$ be a geodesic triangle with three sides α, β, γ, then there exists $o \in \alpha$ such that $d(o, \beta) \leqslant \delta$ and $d(o, \gamma) \leqslant \delta$. Such point o is called a δ-center.

Proposition 1.1.6 (Exponential divergence)
Let p be a rectifiable path in a δ-hyperbolic space (X, d). Let α be a geodesic connecting extremal points of p. Then for every $x \in \alpha$,

$$d(x, p) \leqslant \delta \lceil \log \operatorname{len}(p) \rceil + 1.$$

Or equivalently,

$$\operatorname{len}(p) \geqslant 2^{(d(x,p)-1)/\delta}.$$

Definition 1.1.7. A path p is called a **(λ, c)-quasi geodesic** in (X, d) if for every rectifiable subpath $q \subseteq p$,

$$d(q_-, q_+) \leqslant \operatorname{len}(q) \leqslant \lambda d(q_-, q_+) + c$$

where q_- and q_+ are endpoints of q.

Theorem 1.1.8 (Morse Lemma: stability of quasi-geodesics)
Let p be a (λ, c)-quasi geodesic in a δ-hyperbolic space. Then there exists $D = D(\lambda, c, \delta)$ such that

$$p \subset \mathcal{N}_D([p_-, p_+]), \quad [p_-, p_+] \subset \mathcal{N}_D(p),$$

where $\mathcal{N}_D$ denotes D-neighborhood and $[p_-, p_+]$ denotes the geodesic.

Remark 1.1.9 Morse lemma does not hold in an Euclidean space. For example, let Δxyz be a right triangle with $xy \perp yz$. Then $[xy][yz]$ is a $(2, 0)$-quasi-geodesic. But the is no $D = D(2, 0)$ such that $[x, y][y, z] \subset \mathcal{N}_D([x, z])$.

Proof. It suffices to prove the second assertion. The first one follows from the second one by a connected argument.

Take $x \in [p_-, p_+]$ and let $R = d(x, p)$, then there exists $\theta = \theta(\delta) > 0$ such that $\operatorname{len}(p) \geqslant e^{\theta R}$. On the other hand,

$$\operatorname{len}(p) \leqslant \lambda d(p_-, p_+) + c.$$

It suffices to control $d(p_-, p_+)$ by a linear function of R. Then we can get a contradiction. $\square$

§1.2 Feb 28

Example 1.2.1 (Some examples of hyperbolic spaces)
1. Tree: $\delta = 0$.
2. $\mathbb{H}^2$ & Poincaré disk.

Continued proof of Theorem 1.1.8. Take $x \in [p_-, p_+]$ such that $d(x,p) = R$ is maximal. Take y_1, y_2 on $[p_-, p_+]$ such that $d(y_1, x) = d(x, y_2) = 2R$. Let z_1, z_2 be the projection of y_1, y_2 on p, respectively. We consider the path

$$\widetilde{p} \coloneqq y_1 \rightsquigarrow z_1 \rightsquigarrow z_2 \rightsquigarrow y_1.$$

Since $d(y_1, z_1), d(y_2, z_2) \leqslant R$, then $\widetilde{p}$ is disjoint with $B(x, R)$. Then we have

$$\mathrm{len}(\widetilde{p}) \leqslant 2R + \mathrm{len}(p[z_1, z_2]) \leqslant 2R + \lambda d(z_1, z_2) + c \leqslant 2R + 6\lambda R + c.$$

Combining with $\mathrm{len}(\widetilde{p}) \geqslant e^{\theta R}$, we get a uniform bound on R. □

Definition 1.2.2. Let $x, y, z \in (X, d)$ be three points, we define the **Gromov product** as

$$\langle x, y \rangle_z = \frac{1}{2} \left(d(x,z) + d(y,z) - d(x,y) \right).$$

Example 1.2.3

1. In $\mathbb{E}^2$, let $\Delta(x,y,z)$ be a triangle and $\odot i$ be the incircle which tangents $[yz]$ at a. Then $\langle x, y \rangle_z = d(z, a)$.
2. In a tree, we have $\langle x, y \rangle_z = d(z, [x,y])$. This identity is true for general spaces. But it always holds $\langle x, y \rangle_z \leqslant d(z, [x,y])$.

Definition 1.2.4. A point $x \in X$ is a **δ-center** for a triangle $\Delta(\alpha, \beta, \gamma)$ if

$$d(x, \alpha) \leqslant \delta, \quad d(x, \beta) \leqslant \delta, \quad d(x, \gamma) \leqslant \delta.$$

Lemma 1.2.5

If there exists $\delta > 0$ such that for every geodesic triangle $\Delta \subset (X, d)$, Δ has a δ-center, then for every $x, y, z \in X$,

$$d(z, [x,y]) \leqslant \langle x, y \rangle_z + 2\delta.$$

Proof. Consider a geodesic triangle $\Delta(x,y,z)$. By the condition, there exists $o \in [x,y]$ such that $d(o, [x,z]), d(o, [y,z]) \leqslant \delta$. By triangle inequality, the conclusion follows. □

Lemma 1.2.6

If there exists $\delta > 0$ such that for every geodesic triangle $\Delta \subset (X, d)$, Δ has a δ-center, then (X, d) is $\widetilde{\delta}$-hyperbolic for $\widetilde{\delta} = \widetilde{\delta}(\delta)$.

Proof. Let $\Delta(x,y,z)$ be geodesic triangle and o be a δ-center. Then $p = [x,o][o,y]$ is a $(1, 2\delta)$-geodesic. Hence for every $z \in p$, we have $\langle x, y \rangle_z \leqslant \delta$. Let α be the edge of Δ connecting x and y. By the lemma above, we have $d(z, \alpha) \leqslant \delta + 4\delta$ for every $z \in p$. Hence $p \subset \mathcal{N}_{5\delta}(\alpha)$. Also we have $\alpha \subset \mathcal{N}_{10\delta}(p)$, the conclusion follows. □

Tree approximation for hyperbolic spaces.

Let (X,d) be a δ-hyperbolic space and $F \subset (X,d)$ be a finite set with $\#F = n$. We construct an embedded tree T with leaves containing F as follows:

1) Let $F = \{x_0, \cdots, x_{n-1}\}$.
2) Let $T_1 = [x_0, x_1]$. Assume that T_i is constructed, we construct

$$T_{i+1} = T_i \cup [x_i, z_i]$$

where z_i is the shortest projection from x_i to T_i.

Then d induces a metric d_T on the tree T.

Proposition 1.2.7 (Tree approximation)
There exists $c = c(n,\delta)$ such that for every $x, y \in T$,

$$d(x,y) \leqslant d_T(x,y) \leqslant d(x,y) + c.$$

Corollary 1.2.8
There exists $\delta' = \delta'(\delta)$ such that for every $x, y, z, o \in (X,d)$,

$$\langle x, y\rangle_o \geqslant \min\{\langle x, z\rangle_o, \langle z, y\rangle_o\} - \delta'.$$

Remark 1.2.9 This is also a equivalent definition of a hyperbolic space.

Proof. This conclusion holds for a tree (0-hyperbolic space) with δ'. For general cases, it follows by the tree approximation. □

Let p be a path and $x \in X$, we define the projection

$$\pi_p(x) := \{y \in p : d(x,y) = d(x,p)\}.$$

Lemma 1.2.10 (Strong contractility of quasi-convex subsets)
Let α be a geodesic in a δ-hyperbolic space (X,d). Then for every metric ball B with $B \cap \alpha = \varnothing$, we have

$$\operatorname{diam} \pi_\alpha(B) \leqslant C$$

where $C = C(\delta)$ only depends on δ.

Remark 1.2.11 If X is a tree, then $\pi_\alpha(B)$ can only have one point if $B \cap \alpha = \varnothing$.

§1.3 Mar 2

Some properties of a Hyperbolic space:

- **Thin triangle property**.
- **Morse lemma**.
- **Contracting property**. This property can describe a "partially hyperbolic space" with some "hyperbolic direction": the geodesics satisfy the contracting property.

Lemma 1.3.1 (Bounded image property)
There exists $C = C(\delta) > 0$ such that for every geodesics $\alpha = [x, y]$ and γ, if

$$d(\pi_\gamma(x), \pi_\gamma(y)) \geqslant C,$$

then

$$d(\pi_\gamma(x), \alpha) \leqslant C, \quad d(\pi_\gamma(y), \alpha) \leqslant C.$$

Proof. Let $u \in \pi_\gamma(x)$ and $v \in \pi_\gamma(y)$, then $[x, u][u, v]$ is a $(3, 0)$-quasi-geodesic. Let $D = D(3, 0, \delta)$, then $d(u, [x, v]) \leqslant D$. Let $z \in \pi_{[x,v]}(u)$, we know that $z \in \mathcal{N}_\delta([x, y][y, v])$.

If $z \in \mathcal{N}_\delta([y, v])$, take $w \in \pi_{[y,v]}(z)$, then $d(w, \gamma) \leqslant d(w, z) + d(z, u) \leqslant \delta + D$. Hence $d(w, v) = d(w, \gamma) \leqslant D + \delta$. It follows that $d(u, v) \leqslant d(w, v) + d(w, u) \leqslant 2(D + \delta)$.

Otherwise $z \in \mathcal{N}_\delta([x, y])$, then $d(u, \alpha) \leqslant (D + \delta)$. Similarly, $d(v, \alpha) \leqslant (D + \delta)$. Take $C := 2(D + \delta)$ is enough. □

Proof of Lemma 1.2.10. Let $B = B(x, R)$. Take $C = 10C'$ where C' is the constant given by previous lemma. Take $y \in B(x, R)$ and let u, v be projections of x, y on α respectively. If $d(u, v) > 10C'$, then $d(u, [x, y]), d(v, [x, y]) \leqslant C'$. Let u_1, v_1 be the projections of u, v on $[x, y]$ respectively. Then $d(x, u_1), d(x, v_1) \geqslant R - C'$. On the other hand, $d(u_1, v_1) \geqslant d(u, v) - 2C'$. Then

$$R \geqslant d(x, y) \geqslant (R - C') + (d(u, v) - 2C') \geqslant R + 7C'.$$

We get a contradiction. □

Lemma 1.3.2 (Section 4.1, Exercise 1.5)
Let (X, d) be a general geodesic space. Let γ be a C-contracting geodesic. Then for every (λ, c)-quasi-geodesic p with endpoints on γ, we have $p \subset \mathcal{N}_D(\gamma)$ where $D = D(\lambda, c, C)$.

Centers. Let $\Delta(x, y, z)$ be a geodesic triangle in a δ-hyperbolic space. Then the projection point $\pi_{[y,z]}(x)$ is a $D(3, 0, \delta)$-center of $\Delta(x, y, z)$.

Now we consider the points $u \in [y, z]$ such that $d(u, z) = \langle x, y \rangle_z$. We construct $v \in [x, z]$ and $w \in [x, y]$ similarly. These points u, v, w are called **congruent points**. One can show that congruent points are uniform centers of $\Delta(x, y, z)$.

Lemma 1.3.3
For every $C > 0$, there exists $D > 0$ such that for every geodesic triangle Δ,

$$\operatorname{diam} \{C\text{ – centers of } \Delta\} \leqslant D.$$

Proof. The key point is that if o is a C–center of Δ then $|d(x, o) - \langle y, z \rangle_x| \leqslant 3C$. Then for two different centers o_1, o_2. Let w_1, w_2 be the projections of o_1, o_2 on $[x, y]$. Then $d(w_1, w_2) \leqslant 2(3C + C) = 8C$ and hence $d(o_1, o_2) \leqslant 8C + 2C = 10C$. □

Proof of tree approximation (Proposition 1.2.7). We want to show that any arc in T_i is a $(1, c_i)$-quasi-geodesic. When $i = 1$, we have $c_1 = 0$. When $i = 2$, we can choose $c_2 = 2D$ since the point $\pi_{[x_1,x_2]}(x_3)$ is a center. Then we can do the induction on i. □

§1.4 Mar 9

Definition 1.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called a **(λ, c)-quasi-isometric embedding (QIE)** where $\lambda \geqslant 1$ and $c \geqslant 0$, if for every $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - c \leqslant d_Y(fx_1, fx_2) \leqslant \lambda d_X(x_1, x_2) + c.$$

Remark 1.4.2 A QIE is not necessarily injective or continuous. But it is coarsely injective. Specifically, if $d_X(x_1, x_2) \geqslant \lambda c + 1$ then $fx_1 \neq fx_2$.

Definition 1.4.3. Let $f : X \to Y$ be a QIE, we say f is a **(λ, c)-quasi-isometry (QI)** if there exists $R \geqslant 0$ such that $\mathcal{N}_R(f(X)) \supset Y$. In this case, we say X and Y are **quasi-isometric**.

Example 1.4.4

1. Every bounded metric space is quasi-isometric with $\{\mathrm{pt}\}$.
2. The natural embedding $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a QI.

Definition 1.4.5. Let $f : X \to Y$ be a map. A map $g : Y \to X$ is called a **quasi-inverse** of f if there exists $R \geqslant 0$ such that for every $x \in X$, $d_X(gfx, x) \leqslant R$.

Remark 1.4.6 The map g is a quasi-inverse of f does not imply f is a quasi-inverse of g.

Lemma 1.4.7 A map f is QI $\iff$ f admits a quasi-inverse which is also a QIE.

Remark 1.4.8 During the proof, we can see that if f admits a quasi-inverse g which is a QIE then f is also a quasi-inverse of g. It asserts that the quasi-isometry between two spaces is a equivalence relation.

Given a metric space (X, d), we can consider the family of self-quasi-isometries on X. Define

$$\mathrm{QI}(X, d) := \{f : X \to X \text{ is a quasi-isometry}\} / \sim$$

where $f \sim g$ if $\sup_{x \in X} d(fx, gx) < \infty$. Then $\mathrm{QI}(X, d)$ is a group and $\mathrm{QI}(X, d_X) \cong \mathrm{QI}(Y, d_Y)$ if X and Y are quasi-isometric.

Program (Gromov). Classify the class of finitely generated groups up to quasi-isometry (between Cayley graphs).

The Cayley graph

Given a finitely generated group, we want to correspond it with a geometric object (a proper geodesic space). The Cayley graph.

Definition 1.4.9. Let G be a group generated by a finite, symmetric generating set S. The **Cayley graph** $\mathrm{Cay}(G, S)$ is a directed graph defined as below:

- The vertices $V := G$.

- The edges $E := G \times S$. An edge $e = (g, s)$ has two endpoints $e_- = g$ and $e_+ = gs$.

For example, if $g_1, g_2 \in G$ satisfying $g_2 = g_1 s$ with $s \in S$. Then there are two edges between g_1, g_2 (with a little abuse of notation): $g_1 \xrightarrow{s} g_2$ and $g_2 \xrightarrow{s^{-1}} g_1$.

Fact 1.4.10. A Cayley graph $\mathrm{Cay}(G, S)$ is connected and regular. The degree of every vertex equals to $\#S$.

The Cayley graph is equipped with a metric d induced by paths on the graph, defined as:

$$d(g_1, g_2) := \inf \{l(p) : p \text{ is a directed path connecting } g_1 \text{ and } g_2\}.$$

We refer to d as the **word metric**. Thus, $(\mathrm{Cay}(G, S), d)$ is a proper metric space. Note that G acts naturally on the Cayley graph $\mathrm{Cay}(G, S)$ by left multiplication, and this action is by isometries.

> Remark 1.4.11 Cayley graph is dual graph of tessellation of discrete groups on $\mathbb{H}^n$ or $\mathbb{E}^n$.

Example 1.4.12

1. $\mathbb{Z}^n$.
2. $\pi_1(\Sigma_g)$ with $g \geqslant 2$. The dual graph of tessellation of $\Sigma_g \curvearrowright \mathbb{H}^2$.
3. The Baumslag-Solitar group $\mathrm{BS}(m, n) = \langle a, t | t a^m t^{-1} = a^n \rangle$.
   - $\mathrm{BS}(1, 2) \curvearrowright \mathbb{H}^2$ by $a = (z \mapsto z + 1)$ and $t = (z \mapsto 2z)$. Note that there is a elementary cycle in the graph as a, a, t, a^{-1}, t^{-1}. Then we can draw the graph as a fractal of such rectangles (see wiki).

§1.5 Mar 14

Fundamental lemma

Let G be a group and (X, d) be a length space. We consider G acting on (X, d) isometrically, that is, a homomorphism $G \to \mathrm{Isom}(X, d)$.

Definition 1.5.1. Let G be a group and X be a set. An **action** $G \curvearrowright X$ means a map $G \times X \to X, (g, x) \mapsto g.x$ satisfying $1.x = x$ and $g_1.(g_2.x) = (g_1 g_2).x$. Equivalently, it is a homomorphism $G \to \mathrm{Sym}(X)$. Let us first recall some definition of group actions.

Definition 1.5.2. 1) We say the action is **effective** if $\ker(G \to \mathrm{Sym}(X))$ is finite.
2) We say the action is **faithful** if $\ker(G \to \mathrm{Sym}(X))$ is trivial.
3) If X is a topological space. We say the action is **proper** if for every compact subset $K \subset X$, we have
$$\# \{g \in G : g.K \cap K \neq \varnothing\} < \infty.$$

Fact 1.5.3. Let X be a locally compact space. It the action $G \curvearrowright X$ is proper, then $G_x = \{g \in G : g.x = x\}$ is finite and the orbit $Gx = \{g.x : g \in G\}$ is a discrete closed subset in X.

Fact 1.5.4. The converse is true. [Ratciffe, Found Hypermanifold]

Definition 1.5.5. We say $G \curvearrowright X$ is **cocompact** if $\exists K \subset X$ compact such that $G.K = X$.

Remark 1.5.6 If X is locally compact then $G \curvearrowright X$ is cocompact iff X/G is compact.

Remark 1.5.7 The action $G \curvearrowright X$ is proper $\iff$ $\Phi : G \times X \to X \times X$, $(g, x) \mapsto (g.x, x)$ is proper (preimage of compact is compact), where we equip G with the discrete topology.

Theorem 1.5.8 (Fundamental Lemma, Milnor-Svarc)

If G (isometrically) acts on a proper geodesic space (X, d) properly and cocompactly. Then

1. G is finitely generated by S.
2. Fix $o \in X$, then the map $(G, d_S) \to (X, d)$, $g \mapsto g.o$ is a quasi-isometry.

Remark 1.5.9 For every finite generating sets S, T, we have (G, d_S) and (G, d_T) are quasi-isometric.

Remark 1.5.10 If we do not assume that (X, d) is proper, then the second term still holds while S may be infinite. (Need the action is cobounded, see Section 4.2.)

Proof. Take compact $K \subset X$ such that $G.K = X$. Let $R = \operatorname{diam}(K)$, then $\mathcal{N}_R(Go) = X$ for some $o \in K$. For $g \in G$, assume that $n \leqslant d(o, go) < n + 1$. Let $x_0, x_1, \cdots, x_n, x_{n+1}$ be points on $[o, go]$ such that $o = x_0 < x_1 < \cdots < x_n \leqslant x_{n+1} = go$ with $d(x_{i-1}, x_i) = 1$ for $1 \leqslant i \leqslant n$. Then there exists $g_i \in G$ such that $d(g_i o, x_i) \leqslant R$. We have

$$d(o, g_i^{-1} g_{i+1} o) = d(g_i o, g_{i+1} o) \leqslant 2R + 1.$$

Let $S = \{s \in G : d(o, so) \leqslant 2R + 1\}$, which is a finite set. Then $\langle S \rangle = G$.

Now we verifies the second term. Since $\mathcal{N}_R(Go) = X$, it suffices to show $g \mapsto g.o$ is a QIE. For every $g \in G$, write $g = s_1 \cdots s_l$ a geodesic word. Then

$$\begin{aligned} d(o, go) &\leqslant d(o, s_1 o) + d(s_1 o, s_1 s_2 o) + \cdots + d(s_1 \cdots s_{l-1} o, s_1 \cdots s_l o) \\ &= d(o, s_1 o) + d(o, s_2 o) + \cdots + d(o, s_l o) \leqslant \lambda d_S(1, g). \end{aligned}$$

On the other hand, if $d(o, go) \geqslant n$, then g can be written into a multiplication of at most $n+1$ elements s_i with $d(o, s_i o) \leqslant 2R + 1$. Then $d(1, g) \leqslant Cn$. Hence $d(o, go) \geqslant C^{-1} d_S(1, g)$. $\square$

Corollary 1.5.11

Let $H < G$ be a finite index subgroup. If G is finitely generated then H is finitely generated.

Proof. Since H is a finite index subgroup, the action $H \curvearrowright \operatorname{Cay}(G, S)$ is cocompact. $\square$

Corollary 1.5.12

The fundamental group of a compact manifold is finitely generated.

Proof. Consider $\pi_1(M) \curvearrowright (\widetilde{M}, d)$ where $\widetilde{M}$ is the universal cover of M. $\square$

Corollary 1.5.13

Assume that $1 \to N \to G \to \Gamma \to 1$ and G is finitely generated, N is finite. Then G is quasi-isometric to Γ.

Some history

1. **Milnor, 1968.** Let M be a closed manifold with negative curvature, then $\pi_1(M)$ has exponential growth. Specifically, let $B(n)$ be the ball of radius n on $\mathrm{Cay}(G,S)$, then $\varphi : n \mapsto \#B(n)$ has exponential growth.
2. **Milnor-Wolf, 1968.** A finitely generated solvable group is of exponential growth or polynomial growth. Furthermore, if it has polynomial growth then it is nilpotent. Their work induced two questions:
   - Does there exists an intermediate growth?
   - Polynomial growth implies (virtually) nilpotent?
3. **Grigorchuk, 1980s.** Grigorchuk's group has an intermediate growth.
4. **Gromov, 1980.** Polynomial growth implies (virtually) nilpotent.

Remark 1.5.14 Growth function and hyperbolicity are QI-invariants.

§1.6 Mar 16

Definition 1.6.1. A finitely generated group G is **hyperbolic** if for some finite generating set S the Cayley graph $(\mathrm{Cay}(G,S), d_S)$ is δ-hyperbolic for some $\delta \geqslant 0$.

Example 1.6.2 (Hyperbolic groups)
1. Finite groups are hyperbolic.
2. Free groups $\mathbb{F}(S)$ are hyperbolic.
3. Let (X,d) be a proper δ-hyperbolic space and G act on (X,d) properly and cocompactly. Then $\mathrm{Cay}(G,S)$ is quasi-isometric with (X,d) hence G is hyperbolic.
4. $\pi_1(\Sigma_g) \curvearrowright \mathbb{H}^2$, hence $\pi_1(\Sigma_g)$ is hyperbolic.
5. π_1(compact Riemann manifold with negative curvature) is hyperbolic.

Lemma 1.6.3

Let $\varphi : X \to Y$ be a QIE between any two length spaces. If Y is hyperbolic then X is hyperbolic. In particular, hyperbolicity is QI-invariant.

Proof. First we consider that $\Phi : I = [a,b] \subset \mathbb{R} \to (X,b)$ is a (λ,c)-QIE. Then there exists a (λ',c')-quasi-geodesic from $\Phi(a)$ to $\Phi(b)$ has a (D,λ)-Hausdorff distance to $\Phi(I)$. The aim of this assertion is to modify $\Phi(I)$ (which can be a discrete set) a little to make it be a path.

The rest of the proof is a direct consequence of Morse lemma. □

Remark 1.6.4 Then G is hyperbolic iff $\mathrm{Cay}(G,S)$ is hyperbolic for a fixed S.

Example 1.6.5 $\mathbb{Z}^2$ is not hyperbolic.

Corollary 1.6.6
A finitely generated group is hyperbolic iff it admits a geometric (proper and cocompact) action on a proper δ-hyperbolic space.

Properties. [We will prove later]

1. Hyperbolic groups are finitely presentable. If it is torsion free, then it has a finite classifying space.
2. There exists only finitely many finite subgroups up to conjugacy class.
3. Word/conjugacy problem is solvable.
4. Hyperbolic groups are automatic groups.

Now we consider a finitely generated group G with a finite generating set S. Let $\mathbb{F}(S)$ be the free group generated by S. Then we have an exact sequence

$$1 \to N \to \mathbb{F}(S) \to G \to 1.$$

Hence $\operatorname{Cay}(\mathbb{F}(S), S)$ is a cover of $\operatorname{Cay}(G, S)$. Since $\operatorname{Cay}(\mathbb{F}(S), S)$ is simply connected, we have

$$\pi_1(\operatorname{Cay}(G, S)) = N.$$

Then $N \longleftrightarrow \{\text{word labeling loops at } 1 \in G\}$. Now we modify $\operatorname{Cay}(G, S)$ to a Cayley complex X given by attaching cells to loops such that X is simply connected and $\pi_1(X/G) = G$.

Definition 1.6.7. We say $G = \langle S|\mathcal{R}\rangle$ if $G \cong \mathbb{F}(S)/\langle\langle\mathcal{R}\rangle\rangle$.

Theorem 1.6.8
Hyperbolic groups are finitely presentable, that is,

$$G = \langle S|\mathcal{R}\rangle, \quad \#\mathcal{R} < \infty.$$

Proof. We have $N \hookrightarrow \mathbb{F}(S) \twoheadrightarrow G$. The aim is to find a finite set $\mathcal{R}$ such that

$$N = \langle\langle\mathcal{R}\rangle\rangle = \left\{\prod_{i=1}^{n} g_i r_i g_i^{-1} : g_i \in G, r_i \in \mathcal{R}\right\}.$$

For a loop $w = s_1 s_2 \cdots s_n$ in $\operatorname{Cay}(G, S)$, it suffices to write $\widetilde{w}_1 s_i \widetilde{w}_2^{-1}$ as a product of conjugate of $r \in \mathcal{R}$ where $\widetilde{w}_1, \widetilde{w}_2$ are geodesics from 1 to another point. The triangle $\widetilde{w}_1 s_i \widetilde{w}_2^{-1}$ has an edge with length 1. Hence for every point on edge $\widetilde{w}_1$, there exists a point in $\widetilde{w}_2^{-1}$ with distance at most $\delta+1$. We separate $\widetilde{w}_1$ into segments with length $2\delta+3$, then we can divide the triangle into small pieces with circumference at most $O_\delta(1)$. Take $\mathcal{R} = \{w \in N : |w| < O_\delta(1)\}$, the conclusion follows. $\square$

§1.7 Mar 23

Definition 1.7.1. Let $G = \langle S | \mathcal{R} \rangle$ be a finitely presentable group. We define the **Dehn function**

$$\Phi(n) := \sup_{W \in \langle\langle \mathcal{R} \rangle\rangle, |W| \leqslant n} \min \left\{ m : W = \prod_{i=1}^{m} g_i r_i g_i^{-1}, g_i \in \mathbb{F}(S), r_i \in \mathcal{R} \right\}.$$

Notation 1.7.2. Let $f, g : \mathbb{N} \to \mathbb{N}$ be monotone increasing functions. We denote $f \preccurlyeq g$ if there exists $a, b, c, d, e > 0$ such that

$$f(n) \leqslant ag(bn + c) + dn + e.$$

We denote $f \asymp g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Theorem 1.7.3

Any hyperbolic group G admits a **Dehn presentation** $G = \langle S | \mathcal{R} \rangle$. That is, for every word w with $w \equiv_G 1$, there exists a subword $u \subseteq w$ and $r \in \mathcal{R}$ with $r = uv$ and $|u| > |v|$.

Corollary 1.7.4 Any hyperbolic group G has linear Dehn function.

Remark 1.7.5 If $\Phi(n) \preccurlyeq n$, then G is hyperbolic.

Remark 1.7.6 (Gromov) If $\Phi(n) \precnsim n^2$, then $\Phi \asymp n$.

Remark 1.7.7 (Braidy-Bridson) There exists $A \subset [2, \infty)$ with $\overline{A} = [2, \infty)$ such that for every $d \in A$, there exists G with $\Phi(n) \asymp n^d$.

Remark 1.7.8 $BS(1, 2)$ has exponential Dehn function.

Proof of Theorem 1.7.3. Let $\mathcal{R} := \{w \in \mathbb{F}(S) : w \equiv_G 1, |w| \leqslant 100\delta\}$. We prove that $\langle S | \mathcal{R} \rangle$ is a Dehn presentation. Let γ be a path in $\mathrm{Cay}(G, S)$ corresponds to w. Let $v \in \gamma$ such that $d(1, v)$ is maximal.

Claim 1.7.9. The subpath $\alpha \subset \gamma$ of length 10δ with midpoint v must not be a geodesic.

Proof. Assume that α is a geodesic. Denote $\alpha = [x, v][v, y]$. Then $d(x, v) = d(v, y) = 5\delta$. Then $d(v, z) \leqslant \delta$ for some $z \in [x, 1]$. We have $d(x, 1) \geqslant d(x, v) + d(v, 1) - 2\delta \geqslant d(1, v) + 3\delta$. A contradiction. □

Case 1. There exists such α in the claim. Let $\alpha = \alpha(x, y)$. Let $[x, y]$ be the geodesic between x and y. Then let $r = \alpha[y, x]$ satisfying the condition.
Case 2. Let $w_1 = \gamma(1, v)$ and $w_2 = \gamma(v, 1)$. Then $d(1, v) < 5\delta$ and hence $\gamma \subset B(1, 5\delta)$. If $|\gamma| > 5\delta$, cut off a subpath with length $5\delta + 1$. Otherwise $|\gamma| \leqslant 5\delta$, then $\gamma \in \mathcal{R}$. □

Theorem 1.7.10

There are only finitely many conjugacy classes of finite subgroups in a hyperbolic group.

Fact 1.7.11. Let F be a bounded subset in a δ-hyperbolic space (X, d). Then there exists $D = D(\delta)$ such that

$$\operatorname{diam}\{\text{centers of } F\} \leqslant D.$$

Definition 1.7.12. Let F be a bounded set. For $x \in X$, we define $r_x = \inf\left\{r > 0 : \overline{B(x,r)} \supset F\right\}$. A point c is called a **center of F** if $r_c = \inf\{r_x : x \in X\}$.

Proof of Theorem 1.7.10. Let F be a finite subgroup. Then

- $f\mathrm{Center}(F) = \mathrm{Center}(F)$ for every $f \in F$.
- $gFg^{-1}(g\mathrm{Center}(F)) = g\mathrm{Center}(F)$, for every $g \in G$.

Claim 1.7.13. There exists $g \in G$ and $g\,\mathrm{Center}(F) \subset B(1, 2D)$. Then for every $f \in F$, $d(gfg^{-1}, 1) \leqslant 6D$.

Proof. Note that $gFg^{-1}(g\,\mathrm{Center}(F)) = g\,\mathrm{Center}(F)$, then there exists $o \in g\mathrm{Center}(F)$ with $d(o, 1) \leqslant 2D$. We have

$$d(gfg^{-1}, 1) \leqslant d(gfg^{-1}, gfg^{-1}o) + d(o, 1) + d(o, gfg^{-1}o) \leqslant 6D$$

since $o, gfg^{-1}o \in g\,\mathrm{Center}(F)$. □

Hence $G \curvearrowright \{F < G : \#F < \infty\}$ by conjugacy has only finite orbits. □

§1.8 Mar 28

Rips complex

Theorem 1.8.1

Let G be a hyperbolic group. Then G acts geometrically on a contractible simplicial complex. In particular, if G is torsion-free, then is has finite $K(G, 1)$.

Remark 1.8.2 G has finite $K(G, 1)$ means that there is a finite simplicial complex X with $\pi_1(X) \cong G$ and $\pi_n(X) \cong \{1\}$ for $n \geqslant 2$.

Let X be a metric space. Fix $R > 0$, we construct the **Victoris-Rips complex** $P_R(X)$ as below:

- The vertex set is X.
- For every $x_0, \cdots x_n \subset X$, we add an n-simplex iff $d(x_i, x_j) \leqslant R$ for every i, j.

If we assume that X is a discrete, proper space, then

1. $\mathrm{Isom}(X)$ acts on $P_R(X)$ cellularly.
2. $X \hookrightarrow P_R(X)$ is a QI.

Lemma 1.8.3

Let G be a hyperbolic group. Let $X = (G, d_S)$ where S is a fixed finite generating set, such that $\mathrm{Cay}(G, S)$ is δ-hyperbolic. Then for every $R \geqslant 10\delta$, $P_R(X)$ is contractible.

Proof. Note that G acts transitively on the vertexes of $P_R(X)$. Hence $\dim(P_R(X)) \leqslant \#B(1,R)$, which is finite. Besides, for every $d \leqslant \dim P_R(X)$, $G \curvearrowright P_R(X)$ has finitely many orbits of Δ^d. Then the action is geometric. It remains to show $P_R(X)$ is contractible. It suffices to show $\pi_n(P_R(X)) = \{1\}$, equivalently, to show every finite simplicial complex is homotopic to $\{\mathrm{pt}\}$. Let $L \subset P_R(X)$ be a simplicial complex and take $v \in L$ maximizing $d(o,v)$. Let $v' \in [o,v]$ such that $d(v',v) = R/2$. The key point is the following claim.

Claim 1.8.4. $\mathrm{St}(v) \subset \mathrm{St}(v')$, where $\mathrm{St}(v)$ is the star of v in L, that is the union of simplexes containing v.

Then we push $L = \{v, x_1, \cdots, x_r\}$ to $L' = \{v', x_1, \cdots, x_r\}$, where L' is closer to o. For complete proofs, see Bridson's textbook. □

Rips sequence

Theorem 1.8.5

Let Q be a finitely presentable group. Then there exists a hyperbolic group G and a finitely generated normal subgroup $N \lhd G$ such that

$$1 \to N \to G \to Q \to 1.$$

The proof uses small cancellation theory.

Definition 1.8.6. Let $\lambda \in (0,1)$, a finitely presentable group $\langle S|\mathcal{R}\rangle$ where $\mathcal{R}$ is closed under cyclic permutation and inverse, is called **$C'(\lambda)$-group (small cancellation group)** if for every $r \neq r' \in \mathcal{R}$, the maximal common prefix (a piece) of r and r' is with length less than $\lambda \min\{|r|,|r'|\}$.

Example 1.8.7

The surface group $\pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g|\mathcal{R}_g\rangle$ where $\mathcal{R}_g$ contains $[a_1,b_1]\cdots[a_g,b_g]$ and its cyclic permutations and their inverses. Then it is a $C'(1/(4g-1))$-group.

Theorem 1.8.8

If a finitely presentable group $\langle S|\mathcal{R}\rangle$ is a $C'(1/6)$-group, then it is a hyperbolic group.

Proof of Theorem 1.8.5 assuming Theorem 1.8.8. Let $Q = \langle S|\mathcal{R}\rangle$. We take

$$G = \langle S \cup \{a,b\} | \mathcal{R}'\rangle, \quad N = \langle a,b\rangle < G.$$

We $\mathcal{R}'$ is constructed as below

- For every $r \in \mathcal{R}$, choose $w_r \in W(\{a,b\})$.
- For every $s \in S$, we choose $u_{s^+}, u_{s^-}, v_{s^+}, v_{s^-} \in W(\{a,b\})$.
- Set $r = W_r$ in $\mathcal{R}$.
- Set $sas^{-1} = u_{s^+}$, $s^{-1}as = u_{s^-}$, $sbs^{-1} = v_{s^+}$, $s^{-1}bs = v_{s^-}$ in $\mathcal{R}$.

If the chosen words in $W(\{a,b\})$ are complicated enough, G is a $C'(1/6)$-group. □

§1.9 Mar 30

Theorem 1.9.1 (Tits alternative in hyperbolic groups)

A subgroup in a Gromov-hyperbolic group G is

(1) either virtually cyclic

(2) or contains a free subgroup $\mathbb{F}_2$ such that $(\mathbb{F}_2, d) \hookrightarrow (G, d)$ is a QIE.

Remark 1.9.2 Tits alternative is first proved in linear groups by Tits. He showed that such group is either virtually solvable or contains a free subgroup $\mathbb{F}_2$.

Corollary 1.9.3

A Gromov-hyperbolic hyperbolic cannot contain a $\mathbb{Z}^2$ or a non-virtually-cyclic solvable group as a subgroup.

Open Problem 1.9.4 (Gromov)

Does there exist a closed surface group in every one-ended hyperbolic group?

Remark 1.9.5 (Kahn-Markovich) For every hyperbolic three-manifold M^3, there exists a surface group in $\pi_1(M^3)$.

Definition 1.9.6. Let (X, d) be a geodesic space. A subset $S \subset (X, d)$ is called ***σ*-quasi-convex** for $\sigma > 0$ if for every $x, y \in S$, we have $[x, y] \in \mathcal{N}_\sigma(S)$.

Example 1.9.7

1. Bounded subsets are quasi-convex.
2. Convex subsets are quasi-convex.
3. A quasi-geodesic is a quasi-convex subset in a hyperbolic space.
4. If $H < G$ is quasi-convex with respect to $\mathrm{Cay}(G, S)$, then it is NOT necessarily quasi-convex for other generating set. For example, $\mathbb{Z}^2 = \langle a, b | ab = ba \rangle$, then $\langle ab \rangle$ is not quasi-convex. But $\langle a \rangle$ is not quasi-convex with respect to $\mathbb{Z}^2 = \langle a, b, ab | \rangle$, where $\langle ab \rangle$ is quasi-convex.
5. If $H < G$ is quasi-convex and G is a hyperbolic group, then H is quasi-convex for every generating set.

Lemma 1.9.8

Let $H < G$ be a σ-quasi-convex subgroup with respect to $\mathrm{Cay}(G, S)$, $\#S < \infty$. Then H is finitely generated by T and $(H, d_T) \hookrightarrow (G, d_S)$ is a QIE.

Proof. Let $h \in H$ and γ be a geodesic in $\mathrm{Cay}(G, S)$. Let $\gamma \subset \mathcal{N}_\sigma(H)$. We take $T = \{t \in H : d_S(1, t) \leqslant 2\sigma + 1\}$. The conclusion follows. □

Definition 1.9.9. Let $H < G$ be two finitely generated groups. We say H is **undistorted** if $H \hookrightarrow G$ is a QIE for some word metric.

Remark 1.9.10 This definition does not depend on the choice of generating set.

Remark 1.9.11 Every $H < \mathbb{Z}^n$ is undistorted.

Then quasi-convex subgroups are undistorted. Furthermore, we can show that every undistorted subgroup is quasi-convex in a Hyperbolic group.

Theorem 1.9.12

For every $g \in G$ where G is a hyperbolic group, the centralizer

$$C_G(g) := \{c \in G : cgc^{-1} = g\}$$

is quasi-convex in G.

Proof. It suffices to show there exists $\sigma > 0$ such that $\forall c \in C_G(g)$, $[1, c] \in \mathcal{N}_\sigma(C_G(g))$. We consider a quadrangle $[1, c][c, cg][cg, g][g, 1]$. We label the vertexes on $[1, c]$ by x_i's and the vertexes on $[g, cg]$ by y_i's.

Claim 1.9.13. If $d(x_i, 1), d(x_i, g) \geqslant d(1, g) + 2\delta$, then $d(x_i, y_i) \leqslant D(|g|, \delta)$.

Proof. Applying the thin-quadrangle property, we have $d(x_i, [g, cg]) \leqslant \delta$. Then there exists $z_i \in [g, cg]$ with $d(x_i, z_i) \leqslant \delta$. Note that $|d(g, z_i) - d(1, x_i)| \leqslant d(1, g) + \delta$ and $d(g, y_i) = d(1, x_i)$, hence $d(x_i, y_i) \leqslant 2\delta + d(1, g)$. Take $D = 2(\delta + d(1, g))$ as desired. □

By this claim, we have $N = \#\left\{x_i^{-1}y_i\right\} < \infty$. Hence if $d(1, c) \geqslant N + 1$, there are i, j such that $x_i^{-1}y_i = x_j^{-1}y_j$. Now we consider $c' = x_i x_j^{-1} c$, which is also contained in $C_G(g)$. It shortens $d(1, c)$. □

§1.10 Apr 6

Remark 1.10.1 $\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$ can be a QIE but not quasi-convex.

Lemma 1.10.2

Let H, K be quasi-convex subgroups in a group G. Then $H \cap K$ is also quasi-convex.

Proof. Let $g \in H \cap K$ be an element. Assume that the geodesic $[1, g]$ is $x_0 x_1 \cdots x_n$. Then there exists $y_i \in H$ and $z_i \in K$ such that $d(x_i, y_i) \leqslant \sigma$ and $d(x_i, z_i) \leqslant \sigma$. Then $d(y_i, z_i) \leqslant 2\sigma$ hence $y_i^{-1} z_i \in B(1, 2\sigma)$. If $d(1, g) > \#B(1, 2\sigma)$, then there exists $i \neq j$ with $\Delta(x_i, y_i, z_i) \cong \Delta(x_j, y_j, z_j)$. Besides $c = y_i y_j^{-1} = z_i z_j^{-1} \in H \cap K$. Now we replace g by cg , then $d(x, cg) < d(x, g)$ where x is a given point on $[1, g]$. Then the conclusion follows by an inductive argument. □

Theorem 1.10.3

If g is of infinite order in a hyperbolic group G. Then $\langle g\rangle$ is a quasi-convex subgroup in G. Equivalently, $n \mapsto g^n$ is a QIE.

Proof. Note that $\langle g\rangle \subset C_G(g)$. Here $H = C_G(g)$ is a quasi-convex subgroup of G and hence finitely generated by T. Besides

$$Z(H) = \cap_{t\in T} Z_H(t) < H,$$

hence $Z(H)$ is a quasi-convex subgroup of H. We have

$$\langle g\rangle \hookrightarrow Z(H) \hookrightarrow H \hookrightarrow G.$$

Note that $Z(H) \hookrightarrow G$ is a QIE and $Z(H)$ is a finitely generated abelian group. Hence $\langle g\rangle \hookrightarrow Z(H) \hookrightarrow G$ is a QIE. □

Lemma 1.10.4

Let H be a infinite quasi-convex subgroup in a hyperbolic group G. Then $[E(H) : H] < \infty$ where

$$E(H) := \{g \in G : d_{\mathrm{H}}(H, gH) < \infty\}.$$

Corollary 1.10.5

If g is of infinite order in a hyperbolic group G, then

$$\langle g\rangle \subset C_G(g) \subset N_G(g) \subset E(\langle g\rangle).$$

In particular, both $C_G(g)$ and $N_G(g)$ are virtually $\mathbb{Z}$.

Proof. It suffices to show that $N_G(g) \subset E(\langle g\rangle)$. Note that for every $f \in N_G(g)$, we have $f\langle g\rangle = \langle g\rangle f$. Then $d_{\mathrm{H}}(f\langle g\rangle, \langle g\rangle) = d_{\mathrm{H}}(\langle g\rangle f, \langle g\rangle) \leqslant d(1, f) < \infty$. □

Corollary 1.10.6

If H is a finitely generated normal subgroup of a hyperbolic group G and $\#H = \infty$, then $[G : H] < \infty$.

Corollary 1.10.7

If $G = \langle S|\mathcal{R}\rangle$ is a finitely presentable group and $\langle\langle\mathcal{R}\rangle\rangle$ is infinite, then G is finite.

§1.11 Apr 11

Proof of Lemma 1.10.4. Let $D = \delta + 2\sigma$, we will show that for every $g \in E(H)$, $d_{\mathrm{H}}(H, gH) \leqslant D$. Assume that $d_{\mathrm{H}}(H, gH) = r$. Note that gHg^{-1} acts transitively on gH. Then for every $x \in gH$, there exists $y, z \in gH$ such that

(i) $d(y,z) \geqslant 2r + 10\delta$,
(ii) x is σ-close to the mid point of $[y,z]$.

Recall that $y, z \in \mathcal{N}_r(H)$, by the thin-quadrangle property, if $r > D$ then $d(x,H) \leqslant D$. □

Definition 1.11.1. A group is called **elementary** if it is virtually cyclic.

> **Remark 1.11.2** Let g be a infinite order element in a hyperbolic group. Then $\langle g \rangle$ is contained in a maximal elementary subgroup $E(\langle g \rangle)$.

Recall Rips sequence

$$1 \to N \to G \to Q \to 1,$$

where $N \lhd G$ and G is a hyperbolic group. If $\#Q = \infty$, then we know that N is NOT quasi-convex in G.

Corollary 1.11.3

Let G be a hyperbolic group, then
1. $\mathbb{Z}^2 \not< G$.
2. $\mathrm{BS}(m,n) = \langle a, t | t a^m t^{-1} = a^n \rangle \not< G$.

Proof. 1 follows from $C_G(g) \geqslant \mathbb{Z}^2$ for an element $g \in \mathbb{Z}^2$, which leads to a contradiction. Now we show the second item. It suffices to show for the case $m > n$. Assume that $\mathrm{BS}(m,n) \hookrightarrow G$ a hyperbolic group. For every $l \geqslant 0$, we have $t^l a^{m^l} t^{-l} = a^{n^l}$. Recall that $n \mapsto a^n$ is a QIE. Then

$$2l|t| + m^l|a| \asymp n^l|a|.$$

We get a contradiction. □

2 Boundary Theory

§2.1 Apr 11

Let X be a δ-hyperbolic proper geodesic space. The **Gromov boundary** ∂X of X is defined by

$$\partial X \coloneqq \{\text{ geodesic rays }\} / \sim_{\text{asymptotic}},$$

where the asymptotic relation is given by $\alpha \sim \beta$ iff $d_{\mathrm{H}}(\alpha, \beta) < \infty$.

Fix a base point $o \in X$, we consider the set

$$\partial_o X \coloneqq \{\text{ geodesic rays from } o\,\} / \sim .$$

Then $\partial_o X \hookrightarrow \partial X$.

Lemma 2.1.1 $\partial_o X = \partial X$.

Proof. Let α be a geodesic ray from another point x. Let $\alpha(n)$ be the point on α with $d(x, \alpha(n)) = n$. Let $\beta_n = [o, \alpha(n)]$. By Arzela-Ascoli lemma, there exists a subsequence $\beta_{n_k} \to \beta_\infty$ where β_∞ is a geodesic ray from o. Since X is hyperbolic, we have $\beta_n \subset \mathcal{N}_D(\alpha)$ for $D = \delta + d(o, x)$. Then $\beta_\infty \subset \mathcal{N}_D(\alpha)$. We also have $\alpha \subset \mathcal{N}_{D'}(\beta_\infty)$ for some D' by the connectedness argument. $\square$

A bi-infinite geodesic $\gamma : (-\infty, +\infty) \to X$ connects two points $\gamma^+ \coloneqq \gamma([0, +\infty)) \in \partial X$ and $\gamma^- \coloneqq \gamma((-\infty, 0]) \in \partial X$.

Lemma 2.1.2 (∂X is visual) For every $p \neq q \in \partial X$, there exists γ connecting p to q.

Proof. Fix a base point o and assume that $p = [\alpha]$ and $q = [\beta]$ where α, β are geodesics from o. For every $n \in \mathbb{Z}_+$, there exists a geodesic $\gamma_n = [\alpha(n), \beta(n)]$. We want to show that $\gamma_n \to \gamma_\infty$ which is a desired bi-infinite geodesic. We need the following claim, which guarantees the condition of Arzela-Ascoli lemma.

Claim 2.1.3. There exists $D > 0$ such that $d(o, \gamma_n) \leqslant D$.

Proof. Assume that $d(o, \gamma_n) \to \infty$. Let $x_n = \pi_{\gamma_n}(o)$. Then there exists $y_n \in \alpha$ and $z_n \in \beta$ such that $d(y_n, z_n) \leqslant D$ for some $D = D(\delta)$ but $d(o, y_n) \to \infty$. This will lead to $d_{\mathrm{H}}(\alpha, \beta) < \infty$ by an application of Morse lemma. $\square$

$\square$

§2.2 Apr 13

Lemma 2.2.1 (Asymptotic rays are eventually uniform thin)
Let $\alpha \sim \beta$ be two asymptotic geodesic rays. Then there exists $s_0, t_0 > 0$ such that

$$\alpha[s_0, \infty) \subset \mathcal{N}_{6\delta}(\beta[t_0, \infty)), \quad \text{and} \quad \beta[t_0, \infty) \subset \mathcal{N}_{6\delta}(\alpha[s_0, \infty)).$$

Proof. Let $T = d(\alpha(0), \beta(0))$ and $D = d_{\mathrm{H}}(\alpha, \beta) < \infty$. Let $s_0 = D + L + 4\delta$. Consider the quadrangle $[\alpha(0), \alpha(2s_0), \beta(t_1), \beta(0)]$ where $\beta(t_1) = \pi_\beta(\alpha(2s_0))$. By the thin-quadrangle property, $d(\alpha(s_0), \beta) \leqslant 2\delta$. Let t_0 be the positive number such that $d(\beta(t_0), \alpha(s_0)) \leqslant 2\delta$. It suffices to show that for every $s \geqslant s_0$, $\alpha(s) \subset \mathcal{N}_{6\delta}(\beta[t_0, \infty))$. If $s \leqslant s_0 + 4\delta$, the conclusion is direct. If $s \geqslant s_0 + 4\delta$, let $s' = s + D + 2\delta$, by a thin-quadrangle argument, the conclusion follows. □

Remark 2.2.2 For every $x, y, z \in \partial X$, the triangle $\Delta(x, y, z)$ is uniform thin.

The topology on the boundary. Now we construct a cone topology on $\partial_o X \cong \partial X$, which compactifies X, that is

$$X \hookrightarrow \overline{X} := X \cup \partial_o X$$

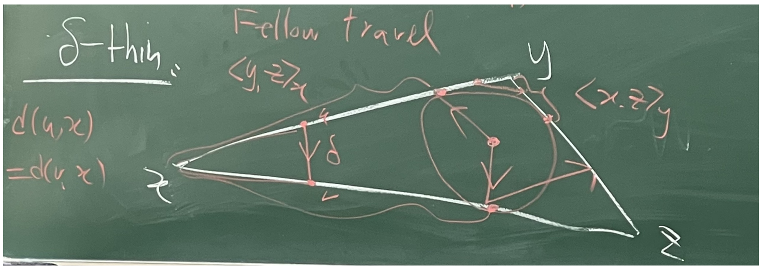
is an open dense subset. Let $k = 12\delta$, we construct the topology as follows.

(1) For every $x \in X$, let $U(x, n) := B(x, 1/n)$.

(2) For every $x \in \partial_o X$, let

$$U(x, n) := \left\{ y \in \overline{X} : \exists \alpha \in x, \beta \in y \text{ such that } d(\alpha(kn), \beta(kn)) < 4\delta \right\}.$$

For simplicity, we assume that δ satisfies the following "fellow travel" property. This guaran-



tees the following fact, which shows that $\{U(x, n)\}$ forms a topological basis.

Fact 2.2.3. $U(x, n) \supset U(x, n + 1)$.

Definition 2.2.4. A subset $S \subset \overline{X}$ is open if for every $x \in S$, there exists n such that $U(x, n) \subset S$.

This topology is equivalent with the following topology which is constructed by giving the convergence sequences in $\overline{X}$.

$$x_n \to x \text{ iff } \exists \alpha_n \in x, \alpha \in x \text{ such that } \alpha_n \to \alpha \text{ locally uniformly.}$$

The visual metric on the boundary. We present two examples to draw some inspiration.

Example 2.2.5

1. X is a tree. Then $\partial X = \{\text{ geodesic rays from } o\}$. For $x, y \in \partial\partial X$, let $\rho(x, y) = 2^{-n}$ where n is the length of the longest common subpath of x and y. Indeed, $n = \langle x, y \rangle_o$. Then $\rho(x, y)$ is an ultra-metric, i.e., $\rho(x, y) \leqslant \max\{\rho(x, z), \rho(y, z)\}$.
2. $X = \mathbb{H}^n$ and $\partial X = \mathbb{S}^{n-1}$. We equip $\mathbb{S}^{n-1}$ with the chord-metric. For every $x, y \in \mathbb{S}^{n-1}$, let $\alpha = [o, x]$ and $\beta = [o, y]$ then

$$\rho(x, y) = \frac{|x - y|}{2} = \lim_{n \to \infty} e^{-\langle \alpha(n), \beta(n) \rangle_o}.$$

Recall that for a hyperbolic space X and every $x, y, z, o \in X$, we have

$$\langle x, y \rangle_o \geqslant \min \{\langle x, z \rangle_o, \langle x, y \rangle_o\} - \delta.$$

This inequality can be extended to $\overline{X}$. Besides, we have $|\langle x, y \rangle_o - d(o, [x, y])| \leqslant O(\delta)$ for every $x, y \in X$. Then there exists $\delta' > 0$ such that for every $o \in X, x, y, z \in \overline{X}$,

$$d(o, [x, y]) \geqslant \min \{d(o, [x, z]), d(o, [y, z])\} - \delta'.$$

Fix a positive number a. We first define a quasi-metric on $\overline{X}$ as

$$\overline{\rho}_a(x, y) := e^{-a\langle x, y \rangle_o}$$

for every $x, y \in X$. For the points $x, y \in \partial X$, we define

$$\overline{\rho}_a(x, y) := e^{-ad(o,[x,y])}.$$

Fact 2.2.6. (1) $\overline{\rho}_a(x, y) = \overline{\rho}_a(y, x)$.
(2) $\overline{\rho}_a(x, y) \leqslant K \max \{\overline{\rho}_a(x, z), \overline{\rho}_a(y, z)\}$, where $K = e^{a\delta'} \in [1, \infty)$.

Lemma 2.2.7 (Frink)
If $1 \leqslant K \leqslant \sqrt{2}$, then there exists a metric ρ_a on ∂X such that

$$\frac{1}{K^2}\overline{\rho}_a(x, y) \leqslant \rho_a(x, y) \leqslant \overline{\rho}_a(x, y). \tag{2.2.1}$$

Definition 2.2.8. The metric ρ_a is called the **visual metric** on ∂X.

Remark 2.2.9 The metric ρ_a can also be defined on X, but the topology it induced is different with the original topology on X, since $\overline{\rho}_a(o, x) = 1$ for every $x \in X$.

Proof. For every $x, y \in \partial X$, we define

$$\rho_a(x, y) := \inf \left\{ \sum_{i=1}^{n} \overline{\rho}_a(x_{i-1}, x_i) : x = x_0 \to x_1 \to \cdots \to x_n = y \right\}.$$

It suffices to show (2.2.1). We induct on $n \geqslant 2$ to prove

$$\overline{\rho}_a(x, y) \leqslant K^2 \sum_{i=1}^{n} \overline{\rho}_a(x_{i-1}, x_i).$$

It is direct when $n = 2$. For the case of $n + 1$. Assume that $\sum_{i=1}^{n+1} \overline{\rho}_a(x_{i-1}, x_i) = R$, take the maximal p such that $\sum_{i=1}^{p} \overline{\rho}_a(x_{i-1}, x_i) < R/2$. By inductive hypothesis, we have

$$\begin{aligned}
\overline{\rho}_a(x, y) &\leqslant K \max \{\overline{\rho}_a(x, x_p), \overline{\rho}_a(x_p, y)\} \\
&\leqslant \max \left\{ K^3 \frac{R}{2}, K^2 \max \{\overline{\rho}_a(x_p, x_{p+1}), \overline{\rho}_a(x_{p+1}, y)\} \right\} \\
&\leqslant \max \left\{ K^3 \frac{R}{2}, K^2 R, K^4 \frac{R}{2} \right\} \leqslant K^2 R.
\end{aligned}$$

The conclusion follows. $\square$

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Let X be a hyperbolic space. We have constructed metrics ρ_a on the Gromov boundary ∂X.

Fact 2.3.1. If X is proper, then $(\partial X, \rho_a)$ is compact.

Fact 2.3.2. For different choices of the base point o, o' we have

$$\frac{1}{\lambda}\rho_a^{o'}(x,y) \leqslant \rho_a^o(x,y) \leqslant \lambda_a^{o'}(x,y),$$

where $\lambda = \lambda(d(o,o'))$. For different choices of a, we have a Hölder dependence between metrics as

$$\rho_a^o(x,y) \asymp \left[\rho_{a'}^o(x,y)\right]^{a/a'}.$$

Theorem 2.3.3

Let $\psi : X \to Y$ be a QI between hyperbolic spaces. Then ψ extends to a homeomorphism between Gromov boundaries. We denote by $\partial\psi : \partial X \to \partial Y$. The extension is continuous in the following sense, if $x_n \to x \in \partial X$ then $\psi(x_n) \to \partial\psi(x)$. Moreover, if we fix $o \in X$ and $o' = \psi(o)$ then $\partial\psi : (\partial X, \rho^o) \to (\partial Y, \rho^{o'})$ is **quasi-conformal**, i.e.

$$H_p := \limsup_{r\to 0^+} \frac{\sup\left\{\rho^{o'}(\partial\psi(p), \partial\psi(q)) : \rho^o(p,q) = r\right\}}{\inf\left\{\rho^{o'}(\partial\psi(p), \partial\psi(q)) : \rho^o(p,q) = r\right\}}$$

is uniformly bounded for $p \in \partial X$.

Proof. We define the boundary map as

$$\partial\psi : \alpha \in X \mapsto [\beta] \in \partial Y$$

where β is a geodesic ray in Y with finite Hausdorff distance to $\psi(\alpha)$.

Fact 2.3.4 (Section 4.4)**.** QIE coarsely commutes with the projection map.

$\partial\psi$ is injective. For $p \neq q$, we have

$$\rho^{o'}(p', q') \gg \overline{\rho}^{o'}(p', q') \gg e^{-ad(o', [p', q'])} \gg e^{-Cad(o,[p,q])}$$

is uniformly bounded away from 0.

$\partial\psi$ is quasi-conformal. Let $p \in \partial X$ and q_1, q_2 such that $\rho^o(p, q_1) = \rho^o(p, q_2) = r$. Then we have

$$|d(o, [p, q_1]) - d(o, [p, q_2])| \leqslant D_1.$$

It suffices to show that $|d(o', [p', q_1']) - d(o', [p', q_2'])| \leqslant D_2$ where D_2 is independent with the choice of r and p. Assume that r is small enough, then $d(\pi_{[p,q_1]}(o), \pi_{p,q_2})(o) \leqslant D_1 + \delta$. Then the conclusion follows by ψ is a QI and the previous fact. □

For a hyperbolic group G, the Gromov boundary of G is defined to be the Gromov boundary if its Cayley graph. Then for different generating sets S, S', we have

$$(\partial \operatorname{Cay}(G,S), \rho) \cong (\partial \operatorname{Cay}(G,S'), \rho)$$

which is a quasi-conformal isomorphism.

Conjecture 2.3.5 (Cannon)
If $\partial G \cong \mathbb{S}^2$, then there exists a finite index subgroup $\dot{G} < G$ such that $\dot{G}$ acts properly, cocompactly on $\mathbb{H}^3$.

One of consequences of this conjecture is Thurston's hyperbolization conjecture.

Theorem 2.3.6 (Bonk-Kleiner)
If

$$\inf \{\dim_{\mathrm{H}}(\partial G, \rho) : \rho \text{ is quasi-conformal to } \rho_a\} = 2,$$

then G is virtually a subgroup of $\mathrm{Isom}(\mathbb{H}^3)$.

3 The Patterson-Sullivan Measure

4 Homework

§4.1 Exercise 1

EXERCISE SHEET #1

Let (X, d) be a geodesic metric space. We denote by $[x, y]$ a choice of a geodesic between x and y. Here we collect a few elementary facts in general metric spaces.

Exercise 0.1. *Let γ be a geodesic in X. Let $x \in X$ and $y \in \pi_\gamma(x)$. Then for any point $z \in \gamma$, we have the path $[x, y][y, z]$ is a $(3, 0)$-quasi-geodesic.*

Could you propose a version of this statement if γ is a (λ, c)-quasi-geodesic.

Exercise 0.2. *Let p be a rectifiable path in X so that $Len(p) \leq d(p_-, p_+) + c$ for some $c > 0$. Then any subpath q of p satisfies $Len(q) \leq d(q_-, q_+) + c$.*

Exercise 0.3. *Let x, y, z be any points in X. Then $\langle x, y\rangle_z \leq d(z, [x, y])$.*

Exercise 0.4. *Let α, β be two (λ, c)-quasi-geodesics for $\lambda, c > 0$. If $\alpha \subset N_D(\beta)$ for some $D > 0$, then $\beta \subset N_{2\lambda D+c}(\alpha)$.*

A geodesic α is *C-contracting* for some $C \geq 0$ if for any metric ball B with $B \cap \alpha = \emptyset$, $diam(\pi_\alpha(B)) \leq C$.

Exercise 0.5 (Alternative proof of Morse Lemma)**.** *Let α be a C-contracting geodesic. Then for any $\lambda, c > 0$, there exists $D = D(\lambda, c, C) > 0$ with the following property. Let p be any (λ, c)-quasi-geodesic with two endpoints on α. Then $p \subset N_D(\alpha)$. (Tips: find an appropriate cover of p by balls and then project them to α.)*

We assume now that (X, d) is a δ-hyperbolic space.

Exercise 0.6 (Strengthened version of Morse Lemma)**.** *Let p be a path in (X, d). Given a non-decreasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, let p be a path such that $Len(q) \leq f(d(q_-, q_+))$ for any subpath q of p. Assume that f is sub-exponential, i.e.:*

$$\lim_{n\to\infty} \log f(n)/n = 0$$

Then p is a quasi-geodesic. (Tips: prove that p is contained in a uniform neighborhood of $[p_-, p_+]$.)

Answers

Exercise 1.1. For every $u \in [x, y], v \in [y, z]$, we have $d(u, v) \geqslant d(u, y)$. By triangle inequality, $d(y, v) \leqslant d(y, u) + d(u, v) \leqslant 2d(u, v)$. It follows that

$$\operatorname{len}([u, y][y, v]) \leqslant 3d(u, v),$$

hence $[x, y][y, z]$ is a $(3, 0)$-quasi-geodesic.

If γ is a (λ, c)-quasi-geodesic, then $p = [x, y]\gamma(y, z)$ is a $(2\lambda + 1, c)$-quasi-geodesic. $\square$

Exercise 1.2. Assume that $\mathrm{len}(q) > d(q_-, q_+) + c$, then we have

$$\mathrm{len}(p) \geqslant d(p_-, q_-) + \mathrm{len}(q) + d(q_-, p_-) > d(p_-, q_-) + d(q_-, q_+) + d(q_-, p_-) + c \geqslant d(p_-, p_+) + c.$$

We get a contradiction. □

Exercise 1.3. Let $o \in \pi_{[x,y]}(z)$, then $d(x, z) \leqslant d(x, o) + d(o, z)$ and $d(y, z) \leqslant d(y, o) + d(o, z)$. Hence we have $\langle x, y \rangle_z \leqslant d(o, z) = d(z, [x, y])$. □

Exercise 1.4. Let x, y be endpoints of α and β. For any $z \in \beta$, assume without loss of generality that $d(z, \alpha) > D$. Then we consider closed sets $\mathcal{N}_D(\beta(x, z))$ and $\mathcal{N}_D(\beta(z, y))$. Since they cover α, which is connected, then they have a nonempty intersection. Then we can take $w \in \alpha$ and $a \in \alpha(x, z), b \in \alpha(z, y)$ such that $d(w, a) \leqslant D$ and $d(w, b) \leqslant D$. Combining with α is a (λ, c)-quasi-geodesic, we have

$$\mathrm{len}(\alpha(a, z)\alpha(z, b)) \leqslant \lambda d(a, b) + c \leqslant 2\lambda D + c.$$

Then at least one of $\mathrm{len}(\alpha(a, z))$ and $\mathrm{len}(\alpha(a, z))$ is less than $\lambda D + c/2$. Which implies that

$$d(z, \alpha) \leqslant D + (\lambda D + \frac{c}{2}) = (\lambda + 1)D + \frac{c}{2} \leqslant 2\lambda D + c.$$

The conclusion follows. □

Exercise 1.5. Let $x \in p$ such that $d(x, \alpha) = D$. Assume that $D > 100\lambda^2 C$. We consider set

$$\mathcal{E} = \left\{ y \in p : d(y, \alpha) \geqslant \frac{D}{10\lambda} \right\}.$$

Let $p(x_1, x_2)$ be the connected component of $\mathcal{E}$ containing x. Then

$$L = \mathrm{len}(p(x_1, x_2)) \geqslant 2D - d(x_1, \alpha) - d(x_2, \alpha) \geqslant D.$$

On the other hand, for every $y \in p(x_1, x_2)$, $B(y, 10\lambda C) \cap \alpha = \varnothing$ by the assumption that $D/(10\lambda) > 10\lambda C$. Then we can cover $p(x_1, x_2)$ by at most $\lceil L/(10\lambda C) \rceil$ metric balls with radius $10\lambda C$. Take $y_1 \in \pi_\alpha(x_1)$ and $y_2 \in \pi_\alpha(x_2)$, then

$$d(y_1, y_2) \leqslant C \cdot \lceil L/(10\lambda C) \rceil \leqslant \frac{L}{10\lambda} + C.$$

Since $p(x_1, x_2)$ is a (λ, c)-quasi-geodesic and $d(x_1, y_1) = d(x_2, y_2) = D/(10\lambda)$. We have

$$L \leqslant \lambda d(x_1, x_2) + c \leqslant \lambda \left(\frac{3L}{10\lambda} + C \right) + c.$$

Which implies that $L \leqslant 2(\lambda C + c)$. Hence

$$D \leqslant \max \left\{ 100\lambda^2 C, 2(\lambda C + c) \right\},$$

only depends on (λ, c, C). □

Exercise 1.6. We first show the following claim.

Claim. There exists $D > 0$ such that for every subpath q of p, $[q_-, q_+] \subset \mathcal{N}_D(q)$.

Proof. We apply a similar argument with the proof of Morse lemma. There exists $\theta = \theta(\delta) > 0$ such that for every rectifiable path γ, if there exists $x \in \gamma$ satisfying $d(x, [\gamma_-, \gamma_+]) > R$, then $\mathrm{len}(\gamma) \geqslant 2^{\theta R - 1}$. Assume that there exists $x \in [q_-, q_+]$ such that $d(x, q) > D$ and maximizing $d(\,\cdot\,, q)$. Without loss of generality, we assume $d(x, q_-), d(x, q_+) \geqslant 2D$. Take x_1, x_2 on $[q_-, q_+]$ with $d(x_1, x) = d(x, x_2) = 2D$. Let $y_1 \in \pi_q(x_1)$ and $y_2 \in \pi_q(x_2)$, then $d(x_i, y_i) \leqslant D$ for $i = 1, 2$.

On one hand, $\widetilde{q} = [x_1, y_1]q(y_1, y_2)[y_2, x_2] \cap B(x, D) = \varnothing$, hence $\operatorname{len}(\widetilde{q}) \geqslant 2^{\theta D - 1}$. On the other hand, $\operatorname{len}(q(y_1, y_2)) \leqslant f(d(y_1, y_2)) \leqslant f(6D)$. Thus $2^{\theta D-1} \leqslant f(6D) + 2D$. Since f is sub-exponential, it follows that D is bounded. This claim holds. □

For every subpath $q \subset p$, we separate $[q_-, q_+]$ into segments with length 1 except the last segment. Denote the endpoints of these segments by $q_0, q_1, \cdots q_n$. Then $n = \lceil d(q_-, q_+) \rceil \leqslant d(q_-, q_+) + 1$. For each i, take $x_i \in \pi_q(q_i)$. In particular, $x_0 = q_0 = q_-$ and $x_n = q_n = q_+$. By the claim, we know that $d(x_i, q_i) \leqslant D$. Hence for every $0 \leqslant i \leqslant n-1$, we have $d(x_i, x_{i+1}) \leqslant 2D + 1$. Then

$$\operatorname{len}(q) \leqslant n f(2D+1) \leqslant f(2D+1)(d(q_-, q_+) + 1).$$

Take $\lambda = c = f(2D+1)$, p is a (λ, c)-quasi-geodesic. □

§4.2 Exercise 2

EXERCISE SHEET #2

We call an isometric action of a group G on a metric space X is *co-bounded* if there exists a bounded set K such that $G \cdot K = X$.

Exercise 0.1. *Suppose G acts by co-boundedly on a length space (X, d). Fix a basepoint $o \in X$. Then there exists a (possibly infinite) generating set S of G such that the map*

$$(G, d_S) \to (Go, d), \ g \mapsto go,$$

is a G-equivariant quasi-isometric map.

Exercise 0.2. *Let $d \geq 3$ be an integer. Prove that any two trees with vertices of degree between 3 and d are quasi-isometric.*

Exercise 0.3. *Prove that finite presentability is a quasi-isometric invariant: Assume that two finitely generated groups G and Γ are quasi-isometric. If G is finitely presentable, then Γ is finitely presentable.*

We consider the set of all quasi-isometries of X. Two quasi-isometries $\phi, \psi : X \to X$ are called *equivalent* if they differ by a bounded constant: $||\phi - \psi||_\infty < \infty$. Denote by $QI(X)$ the set of equivalent classes of quasi-isometries of X.

Exercise 0.4. *The set $QI(X)$ with the composition operation is a group. Moreover, there exists a homomorphism from the isometry group $Isom(X)$ of X into the group $QI(X)$.*

Exercise 0.5. *Suppose two metric spaces X, Y are quasi-isometric. Then $QI(X)$ is isomorphic to $QI(Y)$ (given by conjugating the isometric actions on X).*

Answers

Exercise 2.1. Take a bounded set $K \subset X$ such that $G.K = X$. Let $R = \operatorname{diam}(K)$, then $\mathcal{N}_R(Go) = X$ for some $o \in K$. For $g \in G$, assume that $n \leqslant d(o, go) < n+1$. Let $x_0, x_1, \cdots, x_n, x_{n+1}$ be points on $[o, go]$ such that $o = x_0 < x_1 < \cdots < x_n \leqslant x_{n+1} = go$

with $d(x_{i-1}, x_i) = 1$ for $1 \leqslant i \leqslant n$. Then there exists $g_i \in G$ such that $d(g_i o, x_i) \leqslant R$. We have

$$d(o, g_i^{-1} g_{i+1} o) = d(g_i o, g_{i+1} o) \leqslant 2R + 1.$$

Let $S = \{s \in G : d(o, so) \leqslant 2R + 1\}$, then $\langle S \rangle = G$.

Since $\mathcal{N}_R(Go) = X$, it suffices to show $g \mapsto g.o$ is a QIE. For every $g \in G$, write $g = s_1 \cdots s_l$ a geodesic word. Then

$$\begin{aligned} d(o, go) &\leqslant d(o, s_1 o) + d(s_1 o, s_1 s_2 o) + \cdots + d(s_1 \cdots s_{l-1} o, s_1 \cdots s_l o) \\ &= d(o, s_1 o) + d(o, s_2 o) + \cdots + d(o, s_l o) \leqslant \lambda d_S(1, g). \end{aligned}$$

On the other hand, if $d(o, go) \geqslant n$, then g can be written into a multiplication of at most $n + 1$ elements $s_i \in S$. Then $d_S(1, g) \leqslant n + 1$ and hence $d_S(1, g) \leqslant d(o, go) + 1$. □

Exercise 2.2. It suffices to show that any such tree is quasi-isometric to a 3-regular tree. Let T_1 be a 3-regular tree and T_2 be any such tree. We will construct a QI $f : T_1 \to T_2$. We assume that both T_1 and T_2 are trees with roots, denote r_1 and r_2 respectively. Then for every $i = 1, 2$, each vertex of T_i has one father and 2 to d children. We construct $f(r_1) = r_2$.

Assume that $f(u)$ is defined for some $u \in T_1$, then the number of children of $v = f(u)$ is more than the number of u's. Let u_1, u_2 (or maybe three children if u is the root) be all children of u and $v_1, \cdots, v_i$ be children of v with $2 \leqslant i \leqslant d - 1$. Let $u_2 u_3 \cdots u_{i-1}$ be the path such that u_{j+1} is a child of u_i. Let u_j' be the another child of u_j differs from u_{j+1}. Then we construct

$$f : u_1 \mapsto v_1, \quad u_j \mapsto v_1, u_j' \mapsto v_j (2 \leqslant j \leqslant i - 1), \quad u_i \mapsto v_i.$$

Each edge maps to the corresponding edge. This construction can ran over all points of T_1. Then f is a $(1, d)$-quasi-isometric embedding and indeed surjective. Hence a QI. □

Exercise 2.3. Let S, S' be generating sets of G and Γ respectively. We consider two Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(\Gamma, S')$. Since G is finite presentable by $\langle S | \mathcal{R} \rangle$, every cycle in $\mathrm{Cay}(G, S)$ can be divided into small cycles such that the edges of each cycle is labeled in $\mathcal{R}$. In particular, every cycle can be divided into small cycles with a uniform bound of circumferences. This property is invariant under quasi-isometry since the circumferences are expanded by at most a multiplicity of $(\lambda + c)$ where (λ, c) is the constant given by QIE. □

Exercise 2.4. If ϕ is QI, the it has a quasi-inverse ψ which is QI. Then $\|\phi \circ \psi - \mathrm{id}\|_\infty < \infty$ and $\|\psi \circ \phi - \mathrm{id}\|_\infty < \infty$. Hence $\mathrm{QI}(X)$ is closed under inversion. It is obvious that $\mathrm{QI}(X)$ is closed under convolution, hence $\mathrm{QI}(X)$ is a group. There exists a natural map $\iota : \mathrm{Isom}(X) \to \mathrm{QI}(X)$ given by $f \mapsto [f]$, which is a homomorphism. □

Exercise 2.5. Let $\phi : X \to Y$ be a QI and $\psi : Y \to X$ be a quasi-inverse of ϕ which is also a QI. Then for every $f \in \mathrm{QI}(X)$, the map $\widetilde{f} = \phi \circ f \circ \psi : Y \to Y$ is also a QI since ϕ, f, ψ are QI. Moreover, if $\|f - g\|_\infty < \infty$ then $\|\widetilde{f} - \widetilde{g}\| < \infty$ by the property of quasi-isometry. Then $\Phi : f \mapsto \phi \circ f \circ \psi$ gives a well-defined map between $\mathrm{QI}(X)$ and $\mathrm{QI}(Y)$. It preserves the group operation since

$$\|f \circ g - f \circ (\psi \circ \phi) \circ g\|_\infty < \infty.$$

Hence Φ is a group homomorphism. It has a inverse Φ^{-1} given by $\widetilde{f} \mapsto \psi \circ \widetilde{f} \circ \phi$ since

$$\|f - \psi \circ (\phi \circ f \circ \psi) \circ \phi\|_\infty < \infty.$$

□

§4.3 Exercise 3

EXERCISE SHEET #3

Prove that there are only finitely many conjugacy classes of finite subgroups in a hyperbolic group. You may proceed by the following steps:

Execise 0.1. *Assume that a group G acts geometrically on a proper hyperbolic space (X, d).*

(1) *Define a notion of the center for any bounded set B in a metric space X. Define first the radius of B:*
$$r_B := \inf\{r : B \subset B(x, r), r \geq 0, x \in X\}.$$
where $B(x, r)$ is the closed ball of radius r at x. The center of B is then defined to be set of points $o \in X$ such that
$$B \subset B(o, r_B + 1).$$
(2) *Prove that if X is δ-hyperbolic space, the center of any bounded set is bounded by a constant depending only on δ.*
(3) *Apply the assertion (2) to the orbit $B = F \cdot x$ of a finite group F of G, and conclude the proof that there are finitely many conjugacy classes of finite subgroups F.*

Here are two useful facts about quasiconvex subgroups.

Execise 0.2. *If H is a undistorted subgroup in a hyperbolic group G, then it is quasi-convex.*

Execise 0.3. *Prove that any finitely generated subgroup in a free group of finite rank is quasiconvex.*

The following fact allows to solve conjugacy problem for hyperbolic groups.

Execise 0.4. *Let g, h be two conjugate elements in a hyperbolic group G. Prove that there exists a short conjugator $f \in G$ of length at most $D = D(|g|, |h|)$ so that $g = fhf^{-1}$.*

Answers

Exercise 3.1. (2) Take $D = 4(\delta + 1)$. Let B be a bounded set. Assume there are two points x, y in the center of B with $d(x, y) > D$. Taking $o \in [x, y]$ with $d(x, o) = d(x, y)/2$, we will show that $B \subset B(o, r_B - 1)$, which leads to a contradiction. For every $z \in B$, note that $d(x, z), d(y, z) \leqslant r_B + 1$. By δ-thin triangle property, $d(o, [x, z]) \leqslant \delta$ or $d(o, [y, z]) \leqslant \delta$. Without loss of generality, there is $o_1 \in [x, z]$ such that $d(o, o_1) \leqslant \delta$. Note that $d(o, x) \geqslant D/2$, we have $d(o_1, x) \geqslant D/2 - \delta \geqslant \delta + 2$. Then
$$d(o, z) \leqslant d(o, o_1) + d(o_1, z) \leqslant \delta + (r_B + 1) - (\delta + 2) \leqslant r_B - 1.$$
Which contradicts the definition of r_B.

(3) Let C be the center of $B = F.x$, then C is an F-invariant set that $\operatorname{diam}(C)$ is uniformly bounded by $D = D(\delta)$. Let K be a fundamental domain of G which is compact, then

there exists $g \in G$ such that $g.C \cap K \neq \varnothing$. Hence $g.C \subset \widetilde{K}$ where $\widetilde{K} = \overline{\mathcal{N}_D(K)}$ is a fixed compact set. Note that $g.C$ is invariant under gFg^{-1}, hence

$$h.\widetilde{K} \cap \widetilde{K} \neq \varnothing, \quad \forall h \in gFg^{-1}.$$

Since the action is proper, we have gFg^{-1} is contained in a finite subset $A \subset G$. It follows that every finite group conjugates to a subgroup contained in A.

□

Exercise 3.2. Let $(H, d_T) \hookrightarrow (G, d_S)$ be a QIE. Let $x_0 x_1 \cdots x_n$ be a geodesic in $\mathrm{Cay}(H,T) \hookrightarrow \mathrm{Cay}(G,S)$, let $[x_i, x_j]$ be the geodesic in $\mathrm{Cay}(G,S)$. Then the path $[x_0,x_1][x_1,x_2]\cdots[x_{n-1},x_n]$ is a (λ, C)-quasi-geodesic in $\mathrm{Cay}(G,S)$ where (λ, C) only depends on the QIE $(H, d_T) \hookrightarrow (G, d_S)$ and $\sup_{t \in T} d_S(1,t)$. By Morse lemma, we have

$$[x_0, x_n] \subset \mathcal{N}_D([x_0,x_1][x_1,x_2]\cdots[x_{n-1},x_n])$$

for some $D = D(\lambda, C)$. Taking $\sigma = D + \sup_{t\in T} d_S(1,t)$, we have

$$[x_0, x_n] \subset \mathcal{N}_\sigma(\{x_0, x_1, \cdots, x_n\}) \subset \mathcal{N}_\sigma(H).$$

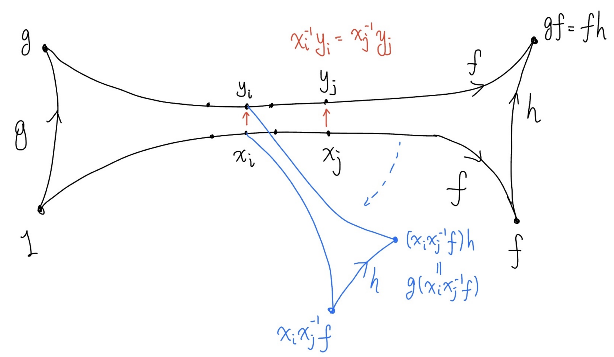
Hence H is σ-quasi-convex. □

Exercise 3.3. Let $G = \mathbb{F}(S)$ with $\#S < \infty$, then G is hyperbolic. Let $H = \langle T \rangle$ be a subgroup generating by a finite set T. Let $\lambda = \sup_{t\in T} d_S(1,t)$ and $\widetilde{T} = \{t \in H : d_S(t) \leqslant \lambda\}$. Note that $H = \langle \widetilde{T} \rangle$. Then for every $x, y \in H$, we have

$$d_{\widetilde{T}}(x,y) - 1 \leqslant d_S(x,y) \leqslant \lambda d_{\widetilde{T}}(x,y).$$

Hence $(H, d_{\widetilde{T}}) \hookrightarrow (G, d_S)$ is a QIE. The conclusion follows by the previous exercise. □

Exercise 3.4. Note that $g = fhf^{-1}$ iff there is a quadrangle formed by $[1,g]$, $[g,gf][gf,f][f,1]$ such that $[f,gf]$ is labeled by h. Denote the points on $[1,f]$ and $[g,gf]$ by x_i's and y_i's respectively, such that $[x_i,x_{i+1}]$ and $[y_i,y_{i+1}]$ is labeled by the same element in S, the generating set of G. Assume that $|f| > D$, which is a constant only depends on $|g|, |h|$ to be determined later. For every y_i with $d(y_i, g) > |g| + \delta$ and $d(y_i, gf) > |h| + \delta$, by the δ-thin-quadrangle property, $y_i \in \mathcal{N}_\delta([g,1][1,f][f,gf])$. Hence there exists $z_i \in [1,f]$ such that $d(y_i, z_i) \leqslant \delta$. Since $d(g, y_i) = d(1, x_i)$, we have $|d(1,z_i) - d(1,x_i)| \leqslant |g| + \delta$. Hence $d(x_i,y_i) \leqslant |g| + 2\delta$. Take $D = (|g| + |h| + 2\delta) + \#\{s \in G : |s| \leqslant |g| + 2\delta\}$. Then there are $i < j$, such that $x_i^{-1}y_i = x_j^{-1}y_j$. Now we consider another element $x_i x_j^{-1} f$, which has a shorter word norm than $|f|$ and also conjugates g and h. By an inductive method. We conclude that g and h can always be conjugated by an element of length at most $D = D(|g|,|h|)$.



□

§4.4 Exercise 4

EXERCISE SHEET #4

Let X be a proper δ-hyperbolic space with Gromov boundary ∂X.

Execise 0.1. *There exists a uniform constant C depending only on δ such that the following thin triangle property holds.*

Let $x, y, z \in X \cup \partial X$ be any triple of distinct points. Then any geodesic $[x, y]$ is contained in the C-neighborhood of $[x, z] \cup [y, z]$.

Let X be a metric complete space and A be a closed subset. Let $\pi_A : X \to A$ be the shortest projection (set-valued) map so that $\pi_A(x)$ is the set of points $a \in A$ satisfying $d(x, a) = d(x, A)$.

Execise 0.2. *Let $\phi : X \to Y$ be a (λ, c)-quasi-isometry between two proper geodesic δ-hyperbolic spaces X, Y. Let γ be a geodesic. Prove that there exists a constant $D = D(\lambda, c, \delta)$ such that for any point $x \in X$,*

$$d_H(\phi(\pi_\gamma(x)), \pi_{\phi\gamma}(\phi(x))) \leq D$$

where d_H denotes the Hausdorff distance.

We say that a (not necessarily geodesic) metric space X is δ-hyperbolic if for any four points x, y, z, w, we have

$$\langle x, y\rangle_w \geq \min\{\langle x, z\rangle_w, \langle z, y\rangle_w\} - \delta. \tag{1}$$

If X is a geodesic metric space, this is equivalent to the usual thin triangle property.

Definition 0.3 (Gromov boundary defined by sequences)**.** A sequence (x_n) in X *converges at infinity* if $(x_i, x_j)_o \to \infty$ as $i, j \to \infty$. Two such sequences $(x_n), (y_n)$ are called *equivalent* if $(x_i, y_j)_o \to \infty$ as $i, j \to \infty$. The *Gromov boundary* $\partial_s X$ of X is the set of all equivalent classes of sequences converging at infinity.

Execise 0.4. *If X is a proper geodesic hyperbolic space, there exists a natural bijection from $\partial_s X$ to ∂X.*

By using (1), we can prove the following.

Execise 0.5. *Consider $w, x, y, z \in X$, and $C \geq 0$. Assume $\langle w, y\rangle_x \leq C$ and $\langle x, z\rangle_y \leq C + \delta$ and $d(x, y) \geq 2C + 2\delta + 1$. Then $\langle w, z\rangle_x \leq C + \delta$.*

Definition 0.6. For $C, D \geq 0$, a sequence of points $x_0, \cdots, x_n$ is a (C, D)*-chain* if one has $\langle x_{i-1}, x_{i+1}\rangle_{x_i} \leq C$ for all $0 < i < n$, and $d(x_i, x_{i+1}) \geq D$ for all $0 \leq i < n$.

Using induction via the previous exercise, we can prove the following very useful fact, saying that a "long" local quasi-geodesic is a global quasi-geodesic.

Execise 0.7. *Let $x_0, \cdots, x_n$ be a (C, D)-chain with $D \geq 2C + 2\delta + 1$. Then $\langle x_0, x_n\rangle_{x_1} \leq C + \delta$, and*

$$d(x_0, x_n) \geq \sum_{i=0}^{n-1} (d(x_i, x_{i+1}) - (2C + 2\delta)) \geq n.$$

Corollary 0.8. *In particular, if $D > 2(2C+2\delta)$ then $2d(x_0, x_n) \geq \sum_{i=0}^{n-1} d(x_i, x_{i+1})$. This implies that the path $\cup_{i=0}^{n-1}[x_i, x_{i+1}]$ is a $(2, 4C + 4\delta + 2)$-quasi-geodesic.*

Answers

Exercise 4.1. There exists $x_1, y_1 \in [x, y]$ and $x_2 \in [z, x], y_2 \in [z, y]$ such that $[x_1, x] \subset \mathcal{N}_{6\delta}([x_2, x]), [y_1, y] \subset \mathcal{N}_{6\delta}([y_2, y])$. Moreover, we can choose x_1, x_2, y_1, y_2 satisfying $d(x_1, x_2) \leqslant 6\delta$ and $d(y_1, y_2) \leqslant 6\delta$. It suffices to show there exists C such that $[x_1, y_2] \subset \mathcal{N}_C([x, z] \cup [y, z])$. We can also take $z_1 \in [x, z]$ and $z_2 \in [y, z]$ such that $d(z_1, z_2) \leqslant 6\delta$. Consider the hexagon $(x_1, x_2, z_1, z_2, y_2, y_1)$, which is 4δ-thin. Then

$$[x_1, y_1] \subset \mathcal{N}_{4\delta}([x_1, x_2][x_2, z_1][z_1, z_2][z_2, y_2][y_2, y_1]) \subset \mathcal{N}_{10\delta}([x_2, z_1] \cup [z_2, y_2]).$$

The conclusion follows. □

Exercise 4.2. Let $\gamma = [y, z]$, then there exists $D_1 = D_1(\lambda, c, \delta)$ such that $d_{\mathrm{H}}(\phi\gamma, [\gamma y, \gamma z]) \leqslant D_1$. Then $|d(\phi x, \phi\gamma) - d(\phi x, [\phi y, \phi z])| \leqslant D_1$. It follows that $d_{\mathrm{H}}(\pi_{\phi\gamma}(\phi x), \phi_{[\phi y, \phi z]}(\phi x)) \leqslant D_1 + 10\delta$.

It suffices to show that $d_{\mathrm{H}}(\phi\pi_{[y,z]}(x), \pi_{[\phi y, \phi z]}(\phi x)) \leqslant D = D(\lambda, c, \delta)$ for every $x, y, z \in X$. For every $o \in \pi_{[y,z]}(x)$, o is a $D_2 = D_2(3, 0, \delta)$-center of the geodesic triangle $\Delta(x, y, z)$. Since ϕ is a QIE, combining with Morse lemma, $\phi(o)$ is a $D_3 = D_3(\lambda, c, \delta)$-center of the geodesic triangle $\Delta(\phi x, \phi y, \phi z)$. Recall that for every $C > 0$, every C-center of a geodesic triangle is uniformly bounded. Combining with both $\phi\pi_{[y,z]}(x)$ and $\pi_{[\phi y, \phi z]}(\phi x)$ are contained in the $\max\{D_2, D_3\}$-center of $\Delta(\phi x, \phi y, \phi z)$, we obtain the desired conclusion. □

Exercise 4.4. Fix a base point $o \in X$. It suffices to construct a natural bijection from $\partial_o X$ to $\partial_s X$. For every geodesic ray γ with $\gamma(0) = o$, we construct the sequence (x_n) with $x_n = \gamma(n)$, which is a sequence converges at infinity. For every $\gamma_1 \sim \gamma_2$, we have $d(\gamma_1(n), \gamma_2(n)) \leqslant 2d_{\mathrm{H}}(\gamma_1, \gamma_2) < \infty$. Then $(\gamma_1(n), \gamma_2(m))_o \geqslant \frac{1}{2}(n + m - |n - m|) - C$, where $C = d_{\mathrm{H}}(\gamma_1, \gamma_2)$. Hence $(\gamma_1(n))$ and $(\gamma_2(n))$ are equivalent. It shows that the map is well-defined.

Next we show that the following claim.

Claim 4.4.1. For every sequences $(x_n), (y_n) \subset X$ assume that $x_n \to x \in \partial X$ and $y_n \to y \in \partial Y$ and $(x_i, y_j)_o \to \infty$ as $i, j \to \infty$, then $x = y$.

Proof. It suffice to show that $\overline{\rho}(x, y) = 0$, where $\overline{\rho} = \overline{\rho}_a$ is a quasi-metric on X. Note that

$$\overline{\rho}(x, y) \asymp \lim_{n\to\infty} e^{-ad(o,[x_n,y_n])} = 0,$$

the conclusion follows. □

Now for every sequence (x_n) converges at infinity. We want to show that $x_n \to x$ for some $x \in \partial X$. This follows by a "sub-subsequence argument". By Arzela-Ascoli lemma, every subsequence of (x_n) has a further subsequence converges to some point in ∂X. By the claim above, every converging subsequence of (x_n) converges to the same point on ∂X. Hence (x_n) converges to some $x \in \partial X$. We maps such sequence (x_n) to the point $x \in \partial X$. Again by the claim above, two equivalent sequences map to the same point. This map is indeed the inverse of $\gamma \in \partial X \mapsto (\gamma(n))$. The conclusion follows. □

Exercise 4.5. Note that $\langle y, z\rangle_x = d(x, y) - \langle x, z\rangle_y \geqslant C + \delta + 1$. We have $\langle y, z\rangle_x - \delta \geqslant C + 1 > \langle w, y\rangle_x$. Hence $\langle w, z\rangle_x \leqslant \langle w, y\rangle_x + \delta \leqslant C + \delta$. □

Exercise 4.7. It suffices to show $\langle x_0, x_n\rangle_{x_{n-1}} \leqslant C + \delta$. We induct on k to show that $\langle x_0, x_k\rangle_{x_{k-1}} \leqslant C + \delta$. The case of $k = 1$ follows by the condition. By inductive hypothesis, $\langle x_0, x_k\rangle_{x_{k-1}} \leqslant C + \delta$. Combining with $\langle x_{k-1}, x_{k+1}\rangle_{x_k} \leqslant C$ and $d(x_{k-1}, x_k) \geqslant 2C + 2\delta + 1$, we obtain $\langle x_0, x_{k+1}\rangle_{x_k} \leqslant C + \delta$ by the previous exercise.

Since $\langle x_0, x_{i+1}\rangle_{x_i} \leqslant C + \delta$, we have

$$d(x_0, x_{i+1}) - d(x_0, x_i) = d(x_i, x_{i+1}) - 2\langle x_0, x_{i+1}\rangle_{x_i} \geqslant d(x_i, x_{i+1}) - (2C + 2\delta).$$

Summing i from 0 to $(n-1)$, we obtain the desired inequality. $\square$

Corollary 4.8. Denote this path by γ. For every $z_1, z_2 \in \gamma$, assume that $z_1 \in [x_i, x_{i+1}], z_2 \in [x_j, x_{j+1}]$ with $i \leqslant j$. If $d(x_i, z_i) \leqslant 4C + 4\delta$ for $i = 1, 2$ then

$$\operatorname{len}(\gamma(z_1, z_2)) \leqslant 8C+8\delta+\sum_{k=i+1}^{j-1} d(x_k, x_{k+1}) \leqslant 8C+8\delta+2d(x_{i+1}, x_j) \leqslant 2d(z_1, z_2)+24(C+\delta).$$

If $d(x_i, z_i) > 4C + 4\delta$, for $i = 1, 2$, we consider the path $[z_1, x_{i+1}] \cdots [x_{j-1}, x_j][x_j, z_2]$. Then conclusion follows directly. A similar argument works for one of $d(x_i, z_i)$ larger than $4C + 4\delta$. Hence γ is a $(2, 24(C + \delta))$-quasi-geodesic. $\square$