Reading Seminar on Homogeneous Dynamics (2023 Fall)

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§1 Introduction (Weikun He, Sep 22)

Let M be a hyperbolic surface, that is $M = \Gamma\backslash\mathbb{H}$ where Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ (**Fuchsian group**) and $\mathbb{H}$ is the hyperbolic space with the constant curvature -1. Let Δ be the **Laplace-Beltrami operator** on $\mathbb{H}$ given by (using the upper half plane model of $\mathbb{H}$)

$$\Delta f(x+iy) = -y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x+iy), \quad \forall f \in C^\infty(\mathbb{H}).$$

Then Δ induces a Laplace-Beltrami operator Δ_M on M with $\Delta_M : L^2(M,\mathrm{Vol}) \to L^2(M,\mathrm{Vol})$. Then Δ_M satisfies

$$\langle \Delta_M f, f\rangle = \int \|\nabla f\|^2 \,\mathrm{d}\,\mathrm{Vol}, \quad \forall f \in L^2(M).$$

Consider eigenvalues $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots$ and eigenfunctions $f_i \in L^2(M)$ of Δ_M with

$$\Delta_M f_i = \lambda_i f_i, \quad \|f_i\|_{L^2(M,\mathrm{Vol})} = 1.$$

Theorem 1.1 (Quantum ergodicity, Šnirel'man, Zelditch, Colin de Verdière)
Along a subsequence of density 1

$$|f_i|^2 \mathrm{d}\,\mathrm{Vol} \xrightarrow{\text{weak}^*} \mathrm{Vol},$$

provided that the geodesic flow (g^t) on (T^1M, μ) is ergodic, where μ is the Liouville measure on T^1M.

In our case, the geodesic flow on $T^1M = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$ is given by

$$g^t(\Gamma x) = \Gamma x a^t, \quad a^t = \begin{bmatrix} e^t & \\ & e^{-t}\end{bmatrix},$$

and μ is induced by Haar. The ergodicity follows from Howe-Moore. Actually for QE, working on T^1M, $|f_i|^2 \mathrm{d}\,\mathrm{Vol}$ can be lifted to a measure μ_i on T^1M (this operation is called a **microlocal lift**). A weak * limit of a subsequence (μ_i) is called a **quantum limit**.

Theorem 1.2 (Šnirel'man) A quantum limit is (g^t)-invariant.

Conjecture 1.3 (Quantum unique ergodicity, Rudnick-Sarnak)
For compact Riemannian manifold M with negative curvature, the Liouville measure is the unique quantum limit.

Remark 1.4 QUE fails for the billiard model of some regions $\Omega \subset \mathbb{R}^2$, which satisfies QE.

Arithmetic QUE. Let $M = \Gamma\backslash\mathbb{H}$ and Γ is arithmetic. Then Δ_M commutes with Hecke operators $T_n, n \in \mathbb{Z}$. For $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, Hecke operators are given by

$$T_n\psi(z) = \sum_{ad=n,\ b\in\mathbb{Z}/d\mathbb{Z}} \psi\left(\frac{az+b}{d}\right), \quad \forall \psi : \Gamma\backslash\mathbb{H} \to \mathbb{R}.$$

A Hecke eigenform is an eigenfunction joint for Δ_M and $T_n, n \in \mathbb{Z}$. From now on, a **quantum limit** means a limit Hecke eigenforms.

Theorem 1.5 (Lindenstrauss[Lin06]) QUE holds for compact arithmetic $\Gamma\backslash\mathbb{H}$.

He classified $(a_t, 1)$-invariant measures on $\mathrm{SL}(2,\mathbb{Z}[1/p])\backslash(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{Q}_p))/\mathrm{PSL}(2,\mathbb{Z}_p)$ for prime numbers p. A QUE for compact case follows from this and

Fact 1.6 (Rudnick-Sarnak). A closed geodesic cannot be a quantum limit.

For non compact cases, the proof needs additionally "non escape of mass". This is shown by Soundararajan.

Some important results.

1. QUE for continuous spectrum (Eisenstein series). Jakobson: subconvexity bounds for some L-functions
$$|\zeta(\frac{1}{2}+it)| \ll (1+|t|)^{\frac{1}{4}-\delta}.$$
2. Entropy bound. Anantharaman: If β is a quantum limit of a compact negatively curved manifold, then
$$h(T^1M, \beta, g^t) > 0.$$
3. Control of eigenfunctions. Dyatlov-Jin, Dyatlov-Jin-Nonnenmacher: Let M be an Anosov surface ((T^1M, g^t) is an Anosov flow). Let $\Omega \subset M$ be an open subset, then there exists $C_\Omega > 0$ such that for every eigenfunction f,
$$\int_\Omega |f|^2 \mathrm{d\,Vol} \geqslant C_\Omega |f|^2_{L(M)}.$$
This uses a **fractal uncertainty principle**.

Essential spectral gap. The limit set of Γ is $\Lambda_\Gamma := \overline{\Gamma x} \cap \partial\mathbb{H}$, where $\partial\mathbb{H} = \mathbb{S}^1$ (using the disk model). Let $\mathrm{Conv}(\Lambda_\Gamma)$ be the convex hull of $\Lambda_\Gamma \subset \mathbb{H}$, which is the union of geodesics connecting two points in Λ_Γ. We say $\Gamma\backslash\mathbb{H}$ is **convex cocompact** if $\Gamma\backslash\mathrm{Conv}(\Lambda_\Gamma)$ is compact.

Example 1.7 Schottky surfaces are convex cocompact.

Selberg zeta function: for $s \in \mathbb{C}$, let
$$Z_M(s) := \prod_{\ell}\prod_{k=1}^{\infty}(1-e^{-(\ell+k)s}),$$
where ℓ takes over lengths of primitive closed geodesics. Then Z_M extends to $\mathbb{C}$ meromorfically. In fact, $\#\{Z_M = 0\} \cap \{\mathrm{Re} > 1/2\} < \infty$, which corresponds to small eigenvalues of Δ_M.

We say M has an **essential spectral gap** if there exists $\beta > 0$ such that
$$\#\{Z_M = 0\} \cap \left\{\mathrm{Re} > \frac{1}{2} - \beta\right\} < \infty.$$
Patterson and Sullivan showed that M has an essential spectral gap with $\beta = \frac{1}{2} - \delta$ provided $0 < \delta(\Gamma) < \frac{1}{2}$, where $\delta(\Gamma)$ is the critical exponent of Γ.

Naud showed that β can be $\frac{1}{2} - \beta + \varepsilon$ for some $\varepsilon > 0$. This implies a similar version of the prime number theorem. Let $\mathcal{N}(L) := \#\{\text{closed primitive geodesics of length } \leqslant L\}$, then
$$\mathcal{N}(L) = \mathrm{Li}(e^{\delta T}) + O(e^{(\delta-\varepsilon)T}),$$

where $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$. Jakobson-Naud conjectured that the optimal β is $(1-\delta)/2$.

Another result is due to Dyatlov-Zahl. They showed that $\beta > 0$ for $\frac{1}{2}$. They established

$$\text{Fractal uncertainty principle} \implies \text{Essential spectral gap.}$$

Then Bourgain-Dyatlov showed the fractal uncertainty principle for every $\delta \in (0,1)$ and hence gave an essential spectral gap.

Fractal uncertainty principle. Let $h > 0$ be a small number. Consider the semiclassical Fourier transform

$$\mathscr{F}_h f(\xi) = (2\pi h)^{-\frac{1}{2}} \int e^{-i\xi x/h} f(x)\mathrm{d}x, \quad \forall f : \mathbb{R} \to \mathbb{C}.$$

For every $X \subset \mathbb{R}$, let $X^{(h)}$ denotes the h-neighborhood of X. We say (X,Y) satisfies **uncertainty principle** if

$$\|\mathbb{1}_{X^{(h)}} \mathscr{F}_h \mathbb{1}_{Y^{(h)}}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leqslant O(h^\beta)$$

for some $\beta > 0$.

Assume that X, Y are δ-Ahlfors-David regular, that is there exists ν supported on X and $C_R > 1$ such that

- $\forall x \geqslant \mathbb{R}$ and $r > 0$, $\nu(B(x,r)) \leqslant C_R r^\delta$.
- $\forall x \in X$ and $r > 0$, $\nu(B(x,r)) \geqslant C_R^{-1} r^\delta$.

Example 1.8

$X = \Lambda_\Gamma$ is δ-Ahlfors-David regular by taking ν to be Patterson-Sullivan measure.

Then we have a trivial bound

$$\|\mathbb{1}_{X^{(h)}} \mathscr{F}_h \mathbb{1}_{Y^{(h)}}\|_{L^2 \to L^2} \leqslant h^{\frac{1}{2}-\delta}.$$

- $\|\mathbb{1}_{Y^{(h)}}\|_{L^2 \to L^1} \leqslant |Y^{(h)}|^{\frac{1}{2}} \ll h^{(1-\delta)/2}$, by δ-Ahlfors-David regularity.
- $\|\mathscr{F}_h\|_{L^1 \to L^\infty} \ll h^{-\frac{1}{2}}$.
- $\|\mathbb{1}_{Y^{(h)}}\|_{L^\infty \to L^2} \leqslant |Y^{(h)}|^{\frac{1}{2}} \ll h^{(1-\delta)/2}$.

Dyatlov-Jin uses Dolgopyat method showed an FUP. Jin-Zhang gives an effective bound for $\beta = \beta(\delta)$. Backus-Leng-Tao showed for higher dimension.

§2 Preliminaries on spectral theories (Yao Ma, Oct 13)

Theorem 2.1

Let M be a compact Riemannian manifold and Δ is the Laplacian. Then the spectrum of Δ has the form

$$\sigma(\Delta) = \{0 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots, \quad \lambda_n \to \infty\}.$$

Theorem 2.2

Suppose A is a self-adjoint compact operator on a Hilbert space $\mathscr{H}$. Then there exists an orthonormal basis $\{\phi_i\}$ such that $A\phi_j = \lambda_i \phi_j, \lambda_j \in \mathbb{R}$, and the only limit point of $\{\lambda_j\}$ is 0.

Definition 2.3. A **finite meromorphic family** is a function $E : U \subset \mathbb{C} \to \mathscr{B}(\mathscr{H}, \mathscr{H})$ such that for every $a \in U$, E can be expressed as $E(s) = \sum_{k=-m}^{\infty}(s-a)^k A_k$ in a neighborhood of a, where A_k has finite rank for $k < 0$ and is compact for $k \geqslant 0$.

Theorem 2.4 (Analytic Fredholm)

Suppose $E(s)$ is a finite meromorphic family of compact operators. If $I - E(s)$ is invertible for at least one $s \in U$ then $(I - E(s))^{-1}$ exists as a finite meromorphic family.

Proof. It suffices to prove the result in a neighborhood of $s_0 \in U$. Decompose $E(s) = A(s) + F(s)$ where $F(s)$ is a meromorphic family of finite-rank operators and $A(s)$ is a holomorphic family of compact operators. Assume U is small enough, we can find a finite-rank operator R such that $\|A(s) - R\| \leqslant 1$. Let $G(s) = (F(s) + R)(I - A(s) + R)^{-1}$, then $I - E(s) = (I - G(s))(I - A(s) + R)$. Since we can write $G(s)$ as

$$G(s) = \sum_{j,k=1}^{N} \gamma_{jk}(s)\psi_j \langle \phi_j, \cdot \rangle ,$$

where γ_{jk}'s are meromorphic, we obtain the conclusion. □

Theorem 2.5 (Weyl's law)

Suppose $\{ \lambda_0 \leqslant \cdots \}$ is the spectrum of a self-adjoint positive elliptic operator A on M^n with the expression $A = a(x, D) = \sum_{k=0}^{m} a_m(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha \in C^\infty(M)[D]$ and $a_m \neq 0$. Then

$$N(t) := \#\{ \lambda_j \leqslant t \} \sim \frac{1}{(2\pi)^n} \int_M \int_{a_m(x,\xi)<t} \mathrm{d}\xi \mathrm{d}x = \frac{1}{n} \left(\int_M \mathrm{d}x \int_{|\xi'|=1} a_m^{-n/m} \mathrm{d}\xi' \right) t^{n/m}.$$

Proof. Define $A^z = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma^z}{\gamma I - A} \mathrm{d}\gamma$. Let $A_z(x, y)$, we consider $\xi(z) = \int_M A_z(x, x) \mathrm{d}x$.

Theorem 2.6 (Trace formula) $\xi(z) = \sum_{k=0}^{\infty} \lambda_k^z$.

Write $\xi(z) = \int_0^\infty t^z N(t)$, assume that $N(t) = ct^\alpha + o(t^\alpha)$. Then $\xi(z) = \frac{c\alpha}{z+\alpha} + f(z)$.

Theorem 2.7

$A_z(x, x)$ could be extended to a meromorphic function on $\mathbb{C}$ with poles only at $z_j = (j - n)/m, j = 0, 1, \cdots$ with residues γ_j. Here

$$\gamma_0 = -\frac{1}{m} \int_{|\xi|=1} a_m^{-n/m}(x, \xi) \mathrm{d}\xi'.$$

Then we obtain $c = n/m$ and $\alpha = \gamma_0/c$. □

§3 Micro-local lifts and quantum ergodicity (Yuxiang Jiao, Oct 20)

This lecture is based on two lecture notes on this topic: [Gorodnik] and [Einsiedler-Ward].

Setting

- $G = \mathrm{SL}(2,\mathbb{R})$ and $K = \mathrm{SO}(2,\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of G.
- $\mathbb{H}$ the real hyperbolic plane, $\mathbb{H} \cong G/K$, $T^1\mathbb{H} \cong \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{id}\}$.
- Γ a uniform lattice or a congruence lattice over $\mathbb{Q}$ (e.g. $\mathrm{SL}(2,\mathbb{Z})$) in G.
- $M = \Gamma\backslash\mathbb{H} = \Gamma\backslash G/K$ a hyperbolic surface with the volume form Vol_M.
- $X = \Gamma\backslash G$ with a right G-action, m_X the Haar measure on X.
- $u_x = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \in U, a_y = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix} \in A$, where $x + iy \in \mathbb{H}$. Let $k_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \in K$.

Theorem 3.1 (Micro-local lifts)

Let (ϕ_i) be an L^2-normalized eigenfunctions of Δ_M in $C^\infty(M) \cap L^2(M)$ with corresponding eigenvalues $\lambda_i \to \infty$. Assume that $|\phi_i|^2 \mathrm{d}\,\mathrm{Vol}_M \to \mu$ in the weak* sense. Then there exists lifted functions $\tilde{\phi}_i$ on X with a weak* limit $\tilde{\mu}$ of $(|\tilde{\phi}_i|^2 \mathrm{d}m_X)$ satisfying:

(1) $\tilde{\mu}$ projects to μ on M.
(2) $\tilde{\mu}$ is A-invariant.

The measure $\tilde{\mu}$ is called a **micro-local lift** of μ, or a **quantum limit** of (ϕ_i).

Remark 3.2 There is a natural way to lift ϕ to $\tilde{\phi}$ as a K-invariant functions. But it is hard to show the A-invariance of $\tilde{\mu}$ using these lifts.

Remark 3.3 A subtle point of this theorem is that the construction of $\tilde{\phi}_i$ only depends on ϕ_i but not on the entire sequence (ϕ_i). So we can focus on studying these $\tilde{\phi}$'s on $\Gamma\backslash G$.

Differential operators.

Definition 3.4. For every $v \in \mathfrak{g}$, we define the differential operator $D_v : C^\infty(X) \to C^\infty(X)$ as

$$D_v f(x) = \left[\frac{\partial}{\partial t} f(x\exp(tv))\right]_{t=0}.$$

Lemma 3.5 $D_v D_w - D_w D_v = D_{[v,w]}$ for every $v, w \in \mathfrak{g}$.

Proof. We have

$$\begin{aligned} D_v D_w f(x) &= \left[\frac{\partial^2}{\partial t_1 \partial t_2} f(x\exp(t_2 v)\exp(t_1 w))\right]_{t_1=t_2=0} \\ &= \left[\frac{\partial^2}{\partial t_1 \partial t_2} f(x\exp(t_1 w)\exp(t_2 \mathrm{Ad}_{\exp(-t_1 w)}(v))\right]_{t_1=t_2=0} \\ &= \left[\frac{\partial}{\partial t_1} D_{v_{t_1}} f(x\exp(t_1 w))\right]_{t_1=0} \qquad \text{where } v_{t_1} = \mathrm{Ad}_{\exp(-t_1 w)}(v) \\ &= D_w D_v f(x) + D_? f(x). \end{aligned}$$

Here $? = (\partial/\partial t)(\mathrm{Ad}_{\exp(-tw)} v)|_{t=0} = [v, w]$. □

By this identity, the differential operator can be extended to the universal enveloping algebra of $\mathfrak{g}$. There is a very special element in the universal enveloping algebra, the Casimir element, which is fixed by the adjoint representation. In fact, it induces the Laplacian on M. We consider

- $H = \begin{bmatrix} 1/2 & \\ & -1/2 \end{bmatrix}$, the direction of geodesic flow.
- $U^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, U^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, the directions of horocycle flow.
- $W = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = U^+ - U^-$, the direction of $K = \mathrm{SO}(2, \mathbb{R})$.

Definition 3.6. $\Omega = D_H D_H + \frac{1}{2} D_{U^+} D_{U^-} + \frac{1}{2} D_{U^-} D_{U^+}$ is called the **Casimir operator**.

Fact 3.7. Ω commutes with D_v for every $v \in \mathfrak{g}$.

Proposition 3.8

For every $f \in C^\infty(M)$, we regard f as a K-invariant smooth function on X. Then we have

$$\Omega f = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = -\Delta f.$$

Fourier analysis. First we recall the Fourier expansion on the torus $K = \mathrm{SO}(2, \mathbb{R})$. For $f \in C^\infty(K)$ and $x = k_\theta \in K$, we have

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{in\theta} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(k_\eta) e^{in(\theta - \eta)} \frac{\mathrm{d}\eta}{2\pi} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x k_\eta) e^{-in\eta} \frac{\mathrm{d}\eta}{2\pi}.$$

For $f \in C^\infty(X)$, we may define the Fourier expansion of f along K-direction. Let

$$f_n(x) = \int_0^{2\pi} f(x k_\eta) e^{-in\eta} \frac{\mathrm{d}\eta}{2\pi}.$$

Note that f_n satisfying $f_n(x k_\theta) = e^{in\theta} f_n(x)$.

We consider the subspace of ***K*-eigenfunctions of weight *n*** as

$$\mathscr{A}_n := \{ f \in C^\infty(X) : D_W f = inf \} = \left\{ f \in C^\infty(X) : f(x k_\theta) = e^{in\theta} f(x) \right\}.$$

We have $\mathscr{A}_n \mathscr{A}_m \subset \mathscr{A}_{n+m}$ and $\mathscr{A}_n \perp \mathscr{A}_m$ in the $L^2(X)$ sense for every $n \neq m$.

A function is said to be ***K*-finite** if it lies in a span of finitely many $\mathscr{A}_n$'s. For every $f \in C^\infty(X)$ and $L \geqslant 0$, let

$$f_{[-L,L]} = \sum_{n=-L}^{L} f_n \in \bigoplus_{n=-L}^{L} \mathscr{A}_n,$$

which is a K-finite function.

Lemma 3.9 (K-finite approximation)

Let $f \in C_c^\infty(\Gamma \backslash \mathrm{SL}(2, \mathbb{R}))$. Then $f_{[-L,L]} \to f$ uniformly as $L \to \infty$. Moreover, $D_H f_{[-L,L]}$ converges uniformly to $D_H f$.

The adjoint representation of W on $\mathfrak{g}$ can be diagonalized in $\mathfrak{g}(\mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})$. Letting

$$E^+ := \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad \text{and} \quad E^- := \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix},$$

we have $[W, E^\pm] = \pm 2i E^\pm$. Therefore (extending $D_\cdot$ to $\mathfrak{g}(\mathbb{C})$ complex linearly),

$$D_W D_{E^\pm} f = D_{E^\pm} D_W f + D_{[W, E^\pm]} f = i(n \pm 2) D_{E^\pm} f.$$

This implies $D_{E^\pm} \mathscr{A}_n = \mathscr{A}_{n \pm 2}$. Note that $\{ E^+, E^-, W \}$ forms a basis of $\mathfrak{g}(\mathbb{C})$, and

$$\Omega = D_{E^+} D_{E^-} - \frac{1}{4} D_W D_W + \frac{i}{2} D_W = D_{E^-} D_{E^+} - \frac{1}{4} D_W D_W - \frac{i}{2} D_W$$

Micro-local lifts. Now we are at the stage to construct the micro-local lift. Let $\phi \in C^\infty(M)$ be an eigenfunction of Δ with eigenvalue $\lambda = \frac{1}{4}+r^2$, $\|\phi\|_2 = 1$. We aim to construct a probability measure on X which projects to $|\phi|^2 \mathrm{d}\,\mathrm{Vol}_M$ and is asymptotically A-invariant as $\lambda \to \infty$.

We define inductively by $\phi_0 = \phi(xK) \in \mathscr{A}_0$, and

$$\phi_{2n+2} = \frac{1}{ir+\frac{1}{2}+n} D_{E^+}\phi_{2n} \in \mathscr{A}_{2n+2}, \quad n \geqslant 0, \tag{3.1}$$

$$\phi_{2n-2} = \frac{1}{ir+\frac{1}{2}-n} D_{E^-}\phi_{2n} \in \mathscr{A}_{2n-2}, \quad n \leqslant 0. \tag{3.2}$$

Since Ω commutes with $D_{E^\pm}$, $\Omega\phi_{2n} = \lambda\phi_{2n}$. Besides,

$$\begin{aligned}\|D_{E^+}\phi_{2n}\|^2 &= \left\langle D_{E^+}^* D_{E^+}\phi_{2n}, \phi_{2n}\right\rangle = -\left\langle D_{E^-} D_{E^+}\phi_{2n}, \phi_{2n}\right\rangle \\ &= -\left\langle (\Omega + \frac{1}{4}D_W^2 + \frac{i}{2}D_W)\phi_{2n}, \phi_{2n}\right\rangle = \left| ir + \frac{1}{2} + \frac{1}{2}n\right|^2 \|\phi_{2n}\|^2.\end{aligned}$$

Hence ϕ_n is L^2-normalized. We also mention that (3.1) and (3.2) hold for every $n \in \mathbb{Z}$.

Now we consider the L^2-normalized function

$$\psi = \psi_\lambda = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \phi_{2n},$$

where $N = N(\lambda)$ to be chosen later.

Lemma 3.10 (Almost lifts)
If $f \in C_c^\infty(M)$ (regarding f as a function in $\mathscr{A}_0$), then

$$\int f|\psi|^2 \mathrm{d}m_X = \int f|\phi|^2 \mathrm{d}\,\mathrm{Vol}_M + O_f(Nr^{-1}).$$

More generally, if f is a K-finite function in $C_c^\infty(X)$, then

$$\int f|\psi|^2 \mathrm{d}m_X = \left\langle f\sum_{n=-N}^{N}\phi_{2n}, \phi_0\right\rangle_{L^2(X)} + O_f(\max\{N^{-1}, Nr^{-1}\}).$$

Proof. Assume that $f \in \bigoplus_{\ell=-2L}^{2L}\mathscr{A}_n$ is a K-finite function. Then

$$\int f|\psi|^2 \mathrm{d}m_X = \langle f\psi, \psi\rangle_{L^2(X)} = \frac{1}{2N+1}\sum_{m,n=-N}^{N} \langle f\phi_{2m}, \phi_{2n}\rangle.$$

For $|m-n| > 2L$, $\langle f\phi_{2m}, \phi_{2n}\rangle = 0$. For $|m-n| \leqslant 2L$, we have

$$\begin{aligned}\langle f\phi_{2m}, \phi_{2n}\rangle &= \frac{1}{ir+m-\frac{1}{2}}\langle fD_{E^+}\phi_{2m-2}, \phi_{2n}\rangle \\ &= \frac{1}{ir+m-\frac{1}{2}}\left[\langle D_{E^+}(f\phi_{2m-2}), \phi_{2n}\rangle - \underbrace{\langle \phi_{2m-2}D_{E^+}f, \phi_{2n}\rangle}_{O_f(1)}\right] \\ &= \frac{1}{ir+m-\frac{1}{2}}\langle f\phi_{2m-2}, E_+^*\phi_{2n}\rangle + O_f(r^{-1}) = -\frac{-ir-n+\frac{1}{2}}{ir+m-\frac{1}{2}}\langle f\phi_{2m-2}, \phi_{2n-2}\rangle + O_f(r^{-1}) \\ &= \langle f\phi_{2m-2}, \phi_{2n-2}\rangle + O_f(r^{-1}) = \cdots = \left\langle f\phi_{2(n-m)}, \phi_0\right\rangle + O_f(Nr^{-1}).\end{aligned}$$

Hence,

$$\langle f\psi,\psi\rangle = \sum_{\ell=-L}^{L} \frac{2N+1-|\ell|}{2N+1} \langle f\phi_{2\ell},\phi_0\rangle + O_f(Nr^{-1}) = \left\langle f \sum_{\ell=-N}^{N} \phi_{2\ell},\phi_0 \right\rangle + O_f(N^{-1}) + O_f(Nr^{-1}).$$

□

Theorem 3.11 (Almost A-invariance, Zelditch)
If $f \in C_c^\infty(X)$ is a K-finite function and N is sufficiently large (depending on f), then

$$\left\langle (rD_H + \mathscr{L})(f) \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = 0,$$

where $\mathscr{L}$ is a fixed degree-two differential operator. In particular,

$$\left\langle D_H(f) \sum_{n=-N}^{N} \phi_{2n}, \phi_0 \right\rangle = O_f(N^{1/2}r^{-1}).$$

Proof. Assume that $f \in \bigoplus_{\ell=-2L}^{2L} \mathscr{A}_n$. Let $\tilde{\phi} = \sum_{n=-N}^{N} \phi_{2n}$. Recall that $D_{E^-}D_{E^+}\phi_0 = \Omega\phi_0 = \lambda\phi_0$. We have

$$\begin{aligned}\lambda\left\langle f\tilde{\phi},\phi_0\right\rangle &= \left\langle f\tilde{\phi}, D_{E^-}D_{E^+}\phi_0\right\rangle = \left\langle D_{E^-}D_{E^+}(f\tilde{\phi}),\phi_0\right\rangle \\ &= \left\langle D_{E^-}D_{E^+}(f)\tilde{\phi},\phi_0\right\rangle + \left\langle D_{E^+}(f)D_{E^-}(\tilde{\phi}),\phi_0\right\rangle + \left\langle D_{E^-}(f)D_{E^+}(\tilde{\phi}),\phi_0\right\rangle + \left\langle fD_{E^-}D_{E^+}(\tilde{\phi}),\phi_0\right\rangle.\end{aligned}$$

For the last term of the right hand side, we have (recalling $\Omega\phi_{2n} = \lambda\phi_{2n}$)

$$\left\langle fD_{E^-}D_{E^+}(\tilde{\phi}),\phi_0\right\rangle = \left\langle f\cdot(\Omega + \frac{1}{4}D_W^2 + \frac{i}{2}D_W)(\tilde{\phi}),\phi_0\right\rangle = \lambda(f\tilde{\phi},\phi_0) + \frac{1}{4}\left\langle f\cdot(D_W^2 + 2iD_W)(\tilde{\phi}),\phi_0\right\rangle.$$

Then two terms $\lambda(f\tilde{\phi},\phi_0)$ cancel out. For dealing with other terms, we use the fact that $D_{E^\pm}\tilde{\phi} \approx (ir \mp \frac{1}{2}D_W - \frac{1}{2})\tilde{\phi}$. The difference is a sum of K-eigenfunctions of weight about $\pm 2N$. Assuming N large enough comparing to L, we have

$$\left\langle D_{E^\pm}(f)D_{E^\mp}(\tilde{\phi}),\phi_0\right\rangle = \left\langle D_{E^\pm}(f)\cdot(ir \pm \frac{1}{2}D_W - \frac{1}{2})\tilde{\phi},\phi_0\right\rangle.$$

Recalling $E^+ + E^- = 2H$, we obtain

$$2ir\left\langle D_H(f)\tilde{\phi},\phi_0\right\rangle + \langle\square,\phi_0\rangle = 0,$$

where $\square$ is independent with r. To show that $\langle\square,\phi_0\rangle$ is indeed of the form $\left\langle \mathscr{L}(f)\tilde{\phi},\phi_0\right\rangle$, we note that it is of a special form: the only differential operator acting on $\tilde{\phi}$ is D_W. Recall $D_W\phi_0 = 0$. Therefore, for every f_1, f_2, we have

$$0 = \langle f_1f_2, D_W(\phi_0)\rangle = -\langle D_W(f_1f_2),\phi_0\rangle = -\langle D_W(f_1)f_2,\phi_0\rangle - \langle f_1D_W(f_2),\phi_0\rangle.$$

We obtain $\langle\square,\phi_0\rangle = \left\langle \mathscr{L}(f)\tilde{\phi},\phi_0\right\rangle$, where $\mathscr{L}$ is an explicit second order differential operator. □

Finally, we take $N = \lceil r^{1/2}\rceil \approx \lambda^{1/4}$, which guarantees $N^{-1}, Nr^{-1}, N^{1/2}r^{-1} \to 0$. Then the weak* limit $\tilde{\mu}$ (passing to a subsequence if necessary) projects to μ by Lemma 3.10 and A-invariant by Theorem 3.11. It is worth noting that we rely on Lemma 3.9 to verify this for not only K-finite function but also for every $f \in C_c^\infty(X)$. With this, we conclude the proof of Theorem 3.1.

Quantum ergodicity. Now we will show some idea of the proof of the quantum ergodicity, Theorem 1.1.

Theorem 3.12

For every K-finite function $f \in C^\infty(X)$, we have

$$\frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} \left| \int f |\phi_\lambda|^2 \mathrm{d}m_X - \int f \mathrm{d}m_X \right| \to 0, \tag{3.3}$$

where $N(L) = \#\{\lambda \in \mathrm{Spec}(\Delta) : \lambda \leqslant L\}$.

Lemma 3.13 (General Weyl law / Trace formula)

For every K-finite f, we have

$$\frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} \int f |\psi_\lambda|^2 \mathrm{d}m_X \to \int f \mathrm{d}m_X.$$

Proof of Theorem 3.12. For every K-finite f, let

$$A_T(f)(x) = \frac{1}{T} \int_0^T f(x \exp(tH)) \mathrm{d}t.$$

By Lemma 3.10 and Theorem 3.11,

$$|\langle f, |\psi_\lambda|^2 \rangle| = |\langle A_T(f), |\psi_\lambda|^2 \rangle| + O_f(T\lambda^{-1/4}).$$

Assuming $\int f \mathrm{d}m_X = 0$. For every $T > 0$, we have

$$\limsup \frac{1}{N(L)} \sum_{\lambda \in \mathrm{Spec}(\Delta), \lambda \leqslant L} |\langle f, |\phi_\lambda|^2 \rangle| \leqslant \limsup \frac{1}{N(L)} \sum \langle |A_T(f)|, |\phi_\lambda|^2 \rangle \leqslant \int |A_T(f)| \mathrm{d}m_X.$$

By the ergodicity of geodesic flow, we have $\int |A_T(f)| \mathrm{d}m_X \to 0$ as $T \to \infty$. □

To show Theorem 1.1, it remains two steps.

- First, for each f, extract a density 1 subsequence converging to $\int f \mathrm{d}m_X$ using (3.3). This needs an estimate on the spectral density, see Section 5 in [Zel87].
- Secondly, there is a density-1 subsequence independent with the choice of f, see Section 6 in [Zel87].

§4 Statement of main theorems and basic definitions (Disheng Xu, Oct 27)

Setting

- L is an S-algebraic group, $S \subset \{\mathbb{R}, \mathbb{C}, \mathbb{Q}_p\}$.
- $G = \mathrm{SL}(2, \mathbb{R}) \times L$, H the $\mathrm{SL}(2, \mathbb{R})$ factor of G.
- K a compact subgroup of L, Γ a discrete subgroup of G.
- $X = \Gamma \backslash G / K$.
- $A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\}$ the diagonal subgroup.

Example 4.1

1. $L = \mathrm{SL}(2,\mathbb{Q}_p)$, $K = \mathrm{SL}(2,\mathbb{Z}_p)$ and Γ is the diagonal embedding of $\mathrm{SL}(2,\mathbb{Z}[1/p])$ in G. Then $\Gamma\backslash G/K \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$.
2. $L = \mathrm{SL}(2,\mathbb{R})$ and $K = \{e\}$ and Γ is an irreducible lattice.

Invariant measures are impossible to classify in these two cases. The cases are similar to "rank one" hyperbolic dynamics, which do not have measure rigidity.

Theorem 4.2

Assume that $\Gamma \cap L$ is finite. Let μ be an A-invariant probability measure on Γ. Assume that

(1) All ergodic components of μ has positive entropy.

(2) μ is L/K recurrent.

Then μ is a combination of H-invariant algebraic measure.

We consider Γ as the following two cases:

(1) Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$;

(2) Γ derived from Eichler orders in an $\mathbb{R}$-split quaternion algebra over $\mathbb{Q}$. In this case, Γ is cocompact.

We call these lattices congruence lattices over $\mathbb{Q}$.

Theorem 4.3

Let $M = \Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ where Γ is a congruence lattice over $\mathbb{Q}$. Then every "arithmetic quantum limit" is $c\,\mathrm{Vol}_{\Gamma\backslash\mathrm{SL}(2,\mathbb{R})}$, where $c = 1$ for the cocompact case and $0 \leqslant c \leqslant 1$ for general cases.

Here "arithmetic quantum limit" requires each ϕ_i to be eigenfunctions for both Δ and Hecke operators. It is conjectured Δ has simple spectrums and hence the "arithmetic quantum limit" coincides the "quantum limit".

Theorem 4.3 follows from Theorem 4.2 by the following results:

(1) Any arithmetic quantum limit μ has positive entropy: every A-ergodic component has entropy $\geqslant 2/9$ [BL03].

(2) $\mathrm{SL}(2,\mathbb{Q}_p)/\mathrm{SL}(2,\mathbb{Z}_p)$-recurrence.

Two other consequences of Theorem 4.2 are the following.

Theorem 4.4

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. Let $A(\mathbb{A})$ be the diagonal group of $\mathrm{SL}(2,\mathbb{A})$ and let μ be an $A(\mathbb{A})$-invariant probability measure on $\mathrm{SL}(2,\mathbb{Q})\backslash\mathrm{SL}(2,\mathbb{A})$ then μ is $\mathrm{SL}(2,\mathbb{A})$-invariant.

Theorem 4.5

Let $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and H as before. Let Γ be a discrete subgroup of G satisfying its projection to each $\mathrm{SL}(2,\mathbb{R})$ factor is finite. Let μ be an ergodic invariant measure under the action of $B = \left\{\begin{bmatrix} * & \\ & * \end{bmatrix} \times \begin{bmatrix} * & \\ & * \end{bmatrix}\right\}$. Then

- either μ is an algebraic measure,
- or μ has zero entropy with respect to every one-parameter subgroup of B.

(G, T)**-spaces.**

- X locally compact separable metric space.
- T locally compact separable metric space with a distinguished point $e \in T$.
- G a locally compact second countable group.
- A continuous transitive action $G \curvearrowright T$.

Definition 4.6. X is called a **(G,T)-(foliated) space** if there is an open cover $\mathfrak{T}$ of X by relatively compact sets, and for every $U \in \mathfrak{T}$ a continuous map $t_U : U \times T \to X$ satisfying:

(1) For every $x \in U$, $t_U(x, e) = x$.

(2) For every $x \in U$ and $y \in t_U(x, T)$, and $V \in \mathfrak{T}$, there exists $\theta \in G$ such that $t_V(y, \cdot) \circ \theta = t_U(x, \cdot)$. In particular, $t_U(x, T) = t_V(y, T)$.

(3) There is some $r_U > 0$ so that for every $x \in U$, $t_U(x, \cdot)$ is injective on $\overline{B^T_{r_U}(e)}$.

Definition 4.7. We say a Radon measure μ on a (G, T) space is **recurrent**, if for every measurable set B with $\mu(B) > 0$, for every $x \in B, x \in U \in \mathfrak{T}$ and for every compact $K \subset T$, there is $t \in T \setminus K$ such that $t_U(x, t) \in B$.

§5 Hecke-Maass forms (Pengyu Yang, Nov 3)

Recall

$$\mathbb{Q}_p = \left\{ \sum_{i=\ell}^{\infty} a_i p^i \,:\, 0 \leqslant a_i \leqslant p-1 \right\}, \quad \mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i \,:\, 0 \leqslant a_i \leqslant p-1 \right\}.$$

For $a = a_\ell p^\ell + \cdots$ where $a_\ell \neq 0$, the p-adic norm is $|a| = p^{-\ell}$.

Consider $\mathbb{Z}[1/p] = \{ n/p^m \,:\, n \in \mathbb{Z}, m \geqslant 0 \}$. Then $\mathbb{Q}_p = \mathbb{Z}_p + \mathbb{Z}[1/p]$ and $\mathbb{Z}_p \cap \mathbb{Z}[1/p] = \mathbb{Z}$. Now we show the isomorphism

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{Z}[1/p])\backslash\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{Q}_p)/\mathrm{SL}(2,\mathbb{Z}_p).$$

We consider the map $[g_\infty] \mapsto [(g_\infty, 1)]$. This is well-defined a map.

Injectivity. If $[(g_\infty, 1)] = [(g'_\infty, 1)]$, then there exists $\gamma_p \in \mathrm{SL}(2,\mathbb{Z}[1/p])$ and $k_p \in \mathrm{SL}(2,\mathbb{Z}_p)$ such that $(g_\infty, 1) = (\gamma_p g'_\infty, \gamma_p k_p)$. Therefore, $\gamma_p = k_p^{-1} \in \mathrm{SL}(2,\mathbb{Z}[1/p]) \cap \mathrm{SL}(2,\mathbb{Z}_p) = \mathrm{SL}(2,\mathbb{Z})$. Hence $[g_\infty] = [g'_\infty]$.

Surjectivity. It suffices to show $\mathrm{SL}(2,\mathbb{Q}_p) = \mathrm{SL}(2,\mathbb{Z}[1/p])\mathrm{SL}(2,\mathbb{Z}_p)$. Note that $\mathrm{SL}(2,\mathbb{Q}_p)$ can be decomposed as a finite product of unipotent subgroups. Therefore $\mathrm{SL}(2,\mathbb{Z}[1/p])$ is dense in $\mathrm{SL}(2,\mathbb{Q}_p)$. Combining with $\mathrm{SL}(2,\mathbb{Z}_p)$ is open in $\mathrm{SL}(2,\mathbb{Q}_p)$, we obtain the desired conclusion. □

Maass forms. Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$.

Definition 5.1. $f \in C^\infty(\mathbb{H})$ is a **Maass form** for Γ if

(i) $f(\gamma z) = f(z)$, for every $\gamma \in \Gamma$.

(ii) $\Delta f = \lambda f$.

(iii) $f(x + iy) = O(y^N)$ for some $N > 0$.

We call f a **Maass cusp form** if $\int_0^1 f(z + x)\mathrm{d}x = 0$ for every z.

Fourier expansion. Since $f(x + 1) = f(x)$, we have

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(y) e^{2\pi i r x}.$$

Write $a_r(y) = \sqrt{y}k(2\pi|r|y)$. Assume that $\lambda = \frac{1}{4} - \nu^2$. Then we have

$$\left(y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - (y^2+\nu^2)\right)k = 0.$$

This ODE has two fundamental solutions I_ν, K_ν where I_ν is exponentially growth and K_ν is rapid decay. By the definition of Maass forms, k is a multiple of K_ν, where

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y\frac{t+t^{-1}}{2}} t^\nu \frac{\mathrm{d}t}{t}.$$

Therefore,

$$f(z) = \sum_r a(r)\sqrt{y}K_\nu(2\pi|r|y)e^{2\pi i r x}.$$

It is a cusp form iff $a(0) = 0$.

Hecke correspondence. Let

$$X_2 = \mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R}) = \mathrm{PGL}(2,\mathbb{Z}[1/p])\backslash\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{Q}_p)/\mathrm{PGL}(2,\mathbb{Z}_p).$$

There are four equivalent definitions of Hecke correspondence:

1. For every $\mathbb{H} \in z$, let $T_p z = \{\, pz, z/p, (z+1)/p, \cdots, (z+p-1)/p \,\}$.
2. Let $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$ and $\gamma_p = \mathrm{diag}(p,1)$. We have
$$\Gamma\gamma_p\Gamma = \Gamma\begin{bmatrix} p & \\ & 1\end{bmatrix} \sqcup \bigsqcup_{i=0}^{p-1}\Gamma\begin{bmatrix} 1 & i \\ & p\end{bmatrix}.$$
Let $T_p : \Gamma g \mapsto \Gamma\gamma_p\Gamma g$.
3. Using the fact $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{PGL}(2,\mathbb{R})$ is $\{\,\text{lattices in } \mathbb{R}^2\,\}/(\Lambda \sim \lambda\Lambda : \lambda \in \mathbb{R}^\times)$, then $T_p(\Lambda) = \{\, \Lambda' : [\Lambda : \Lambda'] = p \,\}$.
4. We have
$$\mathrm{PGL}(2,\mathbb{Z}_p)\begin{bmatrix} p & \\ & 1\end{bmatrix}\mathrm{PGL}(2,\mathbb{Z}_p) = \begin{bmatrix} p & \\ & 1\end{bmatrix} \sqcup \bigsqcup_{i=0}^{p-1}\begin{bmatrix} 1 & i \\ & p\end{bmatrix}\mathrm{PGL}(2,\mathbb{Z}_p).$$
Let $T_p([(g_\infty, g_p)]) = [(g_\infty, g_p\mathrm{diag}(p,1))]$.

Bruhat-Tits building. (gluing infinitely many euclidean spaces). Recall that for a real Lie group $\mathbb{G}(\mathbb{R})$, it acts on the symmetric space $\mathbb{G}/\mathbb{K}$ by isomorphisms. We want to define this notion similarly for p-adic groups $\mathbb{G}(\mathbb{Q}_p)$.

We here only give the example for $\mathrm{SL}(2,\mathbb{Q}_p)$ and list some properties. $\mathrm{SL}(2,\mathbb{Q}_p)$ acts on the $(p+1)$-regular tree T. The stabilizer of each vertex is a maximal compact subgroup. Let

$$\pi : \mathrm{SL}(2,\mathbb{Z}_p) \to \mathrm{SL}(2,\mathbb{F}_p), \quad \begin{bmatrix} a & b \\ c & d\end{bmatrix} \mapsto \begin{bmatrix} a \bmod p & b \bmod p \\ c \bmod p & d \bmod p\end{bmatrix}.$$

Then $\ker\pi = \begin{bmatrix} 1+\mathscr{P} & \mathscr{P} \\ \mathscr{P} & 1+\mathscr{P}\end{bmatrix}$ is the stabilizer of $\{\, v \in T : d(v, v_0) \leqslant 1 \,\}$ for some $v_0 \in T$. An **apartment** is a maximal flat geodesic subspace in the BT-tree. Then we have a 1-1 correspondence

$$\{\,\text{apartments}\,\} \longleftrightarrow \{\,\text{maximal split tori}\,\},$$

denoted by $\mathscr{A} \mapsto T(\mathscr{A})$. There are two properties of apartments:

(i) For every apartments $\mathscr{A}, \mathscr{A}'$, there exits $g \in G$ such that $g\mathscr{A} = \mathscr{A}'$ and $g|_{\mathscr{A}\cap\mathscr{A}'} = \mathrm{id}$.
(ii) For every distinct vertices x, x', there exits an apartment $\mathscr{A}$ such that $x, x' \in \mathscr{A}$. Moreover, if $x' = gx$ then there exists $a \in T(\mathscr{A})$ such that $x' = ax$.

Use these properties, we can prove Cartan decomposition $G = KAK$, where $K = G_o$ the stabilizer of o and $A = T(\mathscr{A})$ where $o \in \mathscr{A}$.

Proof. For every $g \in G$, there exists $\mathscr{A}'$ such that $o, go \in \mathscr{A}'$. Then there exists $g_1 \in G$ such that $\mathscr{A} = g_1\mathscr{A}'$ and $g_1 o = o$, hence $g_1 \in G_o$. Note that $g_1^{-1} go, o \in \mathscr{A}$, there exists $a \in T(\mathscr{A})$ such that $g_1^{-1} go = ao$. Therefore, $a^{-1} g_1^{-1} g \in G_o$. □

Hecke operators. For $N \in \mathbb{Z}_+$, let

$$T_N f(\Lambda) = \frac{1}{\sqrt{N}} \sum_{[\Lambda:\Lambda']=N} f(\Lambda').$$

We have $T_M T_N = T_N T_M$ and $T_N \Delta = \Delta T_N$. We say f is a **Hecke-Maass form** if f is a Maass cusp form and is an eigenform for all T_n. Write

$$f = \sum_n a(n)\sqrt{y} K_\nu(2\pi|n|y) e^{2\pi i n x}.$$

Assume that $a(1) = 1$ then $T_n(f) = a(n) f$.

§6 Positivity of the entropy of quantum limits (Jiesong Zhang, Nov 10)

This lecture is devoted to prove the positivity of entropies for arithmetic quantum limits, which is based on [BL03].

Theorem 6.1

Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a congruence lattice and μ be a quantum limit on $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. There exists $\tau_0 > 0 (\tau_0 = 1/50)$ and $\kappa' > 0 (\kappa' = 2/9)$ such that the following holds. For every compact subset K of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ and every $x \in K$, we have

$$\mu(xB(\varepsilon,\tau_0)) \ll_K \varepsilon^{\kappa'}.$$

Here, $B(\varepsilon,\tau) = a((-\tau,\tau))u^-((-\varepsilon,\varepsilon))u^+((-\varepsilon,\varepsilon))$, where $u^+ = \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}, u^- = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, a = \begin{bmatrix} a^t & \\ & a^{-t} \end{bmatrix}$.

Corollary 6.2

For every almost every ergodic component μ_0 of μ, we have $h(\mu_0) \geqslant \frac{\kappa'}{2} h(\mathrm{Haar})$.

Theorem 6.1 is a direct consequence of the following theorem.

Theorem 6.3

Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a cocompact lattice or a congruence lattice, $\Phi \in L^2(\Gamma\backslash\mathrm{SL}(2,\mathbb{R}))$ be an L^2-normalized eigenfunction of all Hecke operators. Then for every compact subset Ω and $x \in \Omega, r > 0$,

$$\int_{xB(r,\tau_0)} |\Phi(y)|^2 \,\mathrm{d}\,\mathrm{Vol}(y) \ll r^{\kappa'}.$$

To show this theorem, we need two following results.

Corollary 6.4 ([BL03, Corollary 3.7])

Let Φ be as above. Let $n = p_1 p_2 \cdots p_k$ be a square free positive integer. Take $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $\alpha_i = 1$ if $T_{p_i}\Phi = \lambda_{p_i}\Phi$ with $\lambda_{p_i} > \sqrt{p_i}/10$ and $\alpha_i = 2$ otherwise. Then for all $x \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, we have

$$|\Phi(x)|^2 \ll_k \sum_{y \in T_m(x)} |\Phi(y)|^2.$$

Theorem 6.5 ([BL03, Theorem 3.5])

For any set of prime numbers $\mathscr{P}$, $x \in \Lambda\backslash \mathrm{SL}(2,\mathbb{R})$ and $\varepsilon > 0$, there is a set W of cube free integers with the following properties:

(1) $n \in W$ has bounded number of prime factors (uniformly in x, ε).
(2) For every $n \in W$, $p^2 | n$ iff $p | n$ and $p > \mathscr{P}$.
(3) $\{ yB(\varepsilon, \tau_0) : y \in T_n(x), n \in W \}$ are pairwise disjoint.
(4) $\#W \gg \varepsilon^{-\kappa'/4}$.

Proof of Theorem 6.3. Let $\mathscr{P}$ be the set of all primes for which $\lambda_p < \sqrt{p}/10$. Let $x \in \Omega$. By the theorem above, there exists W satisfying the conditions. Then we have

$$(\#W) \int_{xB(\varepsilon,\tau_0)} |\Phi(y)|^2 \mathrm{d}\,\mathrm{Vol}(y) \ll \sum_{n \in W} \sum_{z \in T_n(x)} \int_{zB(\varepsilon,\tau_0)} |\Phi(y)|^2 \mathrm{d}\,\mathrm{Vol}(y) \leqslant -\!\!\!\!\!\!\int |\Phi(y)|^2 = 1.$$

Using $\#W \gg \varepsilon^{-\kappa'/4}$ we obtain the desired conclusion. □

Now we show the idea to prove Theorem 6.5. Let A be a finite set of integers and $\mathscr{P}$ be a set of primes. For every $y \geqslant 0$, we let $P(y) := \prod_{p \leqslant y, p \in \mathscr{P}} p$. Let

$$A_d := \#\{ a \in A : a \equiv 0 \bmod d \}.$$

$$S(A, \mathscr{P}, y) := \{ a \in A : \gcd(a, P(y)) = 1 \}.$$

Proposition 6.6

Let $\omega : \mathbb{Z} \to \mathbb{C}$ be a multiplicative function. Assume that

(1) $A_d = X\omega(d)/d + R(d)$.
(2) $\sum_{p \leqslant y} \omega(p)/p \ll \log\log y$.

Then there exists $\alpha > 0$ and for every $M > 0$, we have

$$S(A, \mathscr{P}, y) = X \prod_{p \leqslant y} (1 - \frac{\omega(p)}{p}) \left(1 + O\left(\frac{1}{\log^M y}\right)\right) + O(y^{\alpha \log\log y}).$$

Example 6.7

We estimate the number of prime numbers in $[Y, X + Y]$, denoted by $\pi(X, Y)$. Let $A = [Y, X+Y] \cap \mathbb{Z}$ and $\mathscr{P}$ be the set of all prime numbers. Take $\omega(p) = 1$. Then $A_d = X/d + O(1)$

and $\sum_{p\leqslant y} 1/p \ll \log\log y$. Therefore

$$\pi(X,Y) \leqslant S(A,\mathscr{P},y) + y = X\prod_{p\leqslant y}(1-\frac{1}{p})\left(1+O\left(\frac{1}{\log^M y}\right)\right) + O(y^{\alpha\log\log y}).$$

Taking $\log y = \log X/(\log\log X)$, we obtain $\pi(X,Y) \ll \frac{X(\log\log X)}{\log X}$.

Example 6.8 (Upper bound of twin prime numbers)

Let $\pi_2(X)$ be the number of twin prime numbers at most X. Let $A = \{x \leqslant X : x(x+2)\}$ and $\mathscr{P}$ be the set of all prime numbers. Let $\omega(p) := \begin{cases} 1, & p = 1 \\ 2, & p \geqslant 2 \end{cases}$. We can show that $A_d = X\omega(d)/d + O(1)$. We have $\pi_2(X) \leqslant S(A,\mathscr{P},y) + y$. Taking $\log y = \log X/(\log\log X)$, we obtain

$$S(A,\mathscr{P},y) \ll \frac{X(\log\log X)^2}{(\log X)^2}.$$

§7 Leafwise measures (Weikun He, Nov 17)

Recall the notion of (G,T)-space:

- G a locally compact topological space,
- T a locally compact separable metric space with a distinguished point e,
- $G \curvearrowright T$ transitive.

We further assume that G acts on T by isometries. Recall that a (G,T)-structure on X is a collection $(U,t_U)_{u\in\mathfrak{T}}$ where $\{U\}_{U\in\mathfrak{T}}$ is an open cover of X and $t_U : U\times T \to X$ continuous, satisfying:

(1) For every $x\in U$, $t_U(x,e) = x$.
(2) For every $x\in U$ and $y = t_U(x,t_0)$, and $V\in\mathfrak{T}(y)$, there exists $\theta\in G$ such that $t_V(y,\theta\cdot) = t_U(x,t)$. In particular, $t_U(x,T) = t_V(y,T)$. We also assume that $\theta t_0 = e$.
(3) There is some $r_U > 0$ so that for every $x\in U$, $t_U(x,\cdot)$ is injective on $\overline{B^T_{r_U}(e)}$.

For every $x\in X$, we let $B^T_r(x) := t_U(x,B^T_r) \subset X$, which is independent with the choice of $U\in\mathfrak{T}(x)$. The "T-leaf" of x is $T_x(x,T) = B^T_\infty(x)$.

Example 7.1

Let X with a right G-action. Let $T = G, e = 1_G$. Let $t_U(x,g) = xg$. For every $y = xg_0$, we have $xg = y(g_0^{-1}g)$.

In our case, we take

- $H = \mathrm{SL}(2,\mathbb{R})$, $L = \mathrm{SL}(2,\mathbb{Q}_p)$, $K = \mathrm{SL}(2,\mathbb{Z}_p) < L$.
- $T = L/K$, $e = K \in L/K$.
- $X = \Gamma\backslash H\times L/(1\times K)$, where Γ is a discrete subgroup of $H\times L$.

We have a (L,T)-structure on X. For every $x\in U\subset X$ where $x = \Gamma(h,\ell)K$, we want to take $t_U(x,t) = \Gamma(h,\ell g)K$ where $t = gK$. But this is not well-defined. We need to fix $\ell_U : U\to L$ and $h_U : U\to H$ such that for every $x\in U$ with $x = \Gamma(h_U(x),\ell_U(x))K$ and $t = gK$, we take $t_U(x,t) = \Gamma(h_U(x),\ell_U(x)gK)$.

Conditional measures. Recall $(X, \mathscr{B}, \mu)$ is a standard probability space. Let $\mathscr{A} \subset \mathscr{B}$ be a sub-σ-algebra. There exists $(\mu_x^{\mathscr{A}})_{x\in X}$, where $\mu_x^{\mathscr{A}} \in \mathscr{P}(X, \mathscr{B})$ such that for every $f \in L^1(X, \mu)$,

$$\int f \mathrm{d}\mu_x^{\mathscr{A}} = \mathbb{E}[f|\mathscr{A}](x),$$

where $\mathbb{E}[f|\mathscr{A}]$ is the conditional expectation.

Remark 7.2 For every $f \in L^1(X, \mu)$, $x \mapsto \int f \mathrm{d}\mu_x^{\mathscr{A}}$ is $\mathscr{A}$-measurable.

Definition 7.3. We say $\mathscr{A}$ is **countably generated** if it is generated as a σ-algebra by a countable set $\{ A_i \}_{i\in\mathbb{N}} \subset \mathscr{A}$.

Definition 7.4. If $\mathscr{A}$ is countably generated. For every $x \in X$, the **$\mathscr{A}$-atom of x** is

$$[x]_{\mathscr{A}} = \bigcap_{A\in\mathscr{A} : x\in A} A = \left(\bigcap_{i : x\in A_i} A_i \right) \cap \left(\bigcap_{i : x\notin A_i} (x \setminus A_i) \right),$$

which is measurable.

Remark 7.5 If $\mathscr{A}$ is countably generated then $\mu_x^{\mathscr{A}}([x]_{\mathscr{A}}) = 1$ for almost every x.

Example 7.6

X is a separable metric space and $\mathscr{B}$ is the Borel σ-algebra, then $\mathscr{B}$ is compactly generated.

Example 7.7

$\varphi : (Y, \mathscr{C}) \to (X, \mathscr{B})$ is measurable. If $\mathscr{B}$ is countably generated then so is $\mathscr{C}$.

Example 7.8

$X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $(X, \mathscr{B})$ is Borel. Let $\varphi_t : \mathbb{T}^2 \to \mathbb{T}^2$ be the irrational flow. Let $\mathscr{E} = \{ A \in \mathscr{B} : \forall t, \varphi_t A = A \}$. Then $\mathscr{E}$ is **NOT** compactly generated.

We can use conditional measures to show this fact. Let μ be the Lebesgue measure on $\mathbb{T}^2$. Then $\mu_x^{\mathscr{E}}([x]_{\mathscr{E}}) = 1$ by discussions above. But we can show that $\mu_x^{\mathscr{E}}$ is φ_t-invariant. Therefore $\mu_x^{\mathscr{E}}$ is Lebesgue by the unique ergodicity, which contradicts $\mu_x^{\mathscr{E}}([x]_{\mathscr{E}}) = 1$.

Remark 7.9 If μ is f-invariant. Let $\mathscr{I} = \left\{ A \in \mathscr{B} : f^{-1}A = A \right\}$. Then $\int \mu_x^{\mathscr{I}} \mathrm{d}\mu(x) = \mu$ is the ergodic decomposition.

Remark 7.10 If $\mathscr{A} \doteq \mathscr{A}'$ mod μ countably generated, then $\mu_x^{\mathscr{A}}([x]_{\mathscr{A}}) = \mu_x^{\mathscr{A}'}([x]_{\mathscr{A}'})$.

In Example 7.8, $\mathscr{E} = \{ \varnothing, X \}$ mod Leb. But we do not have $\mu^{\mathscr{E}} = \mu^{\{ \varnothing, X \}}$. This shows some limitation of the conditional measure. To deal with these cases, we consider the leafwise measure.

Leafwise measures. We illustrate the construction of leafwise measures using Example 7.8. We regard φ_t as an $T = \mathbb{R}$ action on $X = \mathbb{T}^2$. Consider the probability measure $\mu = \mathrm{Leb}|_Q/\mathrm{Leb}(Q)$ where Q is a region in X. The leafwise measure is $(\mu_x^T)_{x\in X}$, a collection of Radon measures on $T = \mathbb{R}$. It is given by

$$\mu_x^T = \mathbb{1}_{\{t\in T:\varphi_t(x)\in Q\}} \cdot \mathrm{Leb}.$$

Notation 7.11. $\mu \propto \nu$ if there exists $c > 0$ such that $\mu = c\nu$.

We need to mention that $\mu_x^T(T) = \infty$ and there is no canonical way to normalize μ_x^T. Therefore, $(\mu_x^T)_{x\in X}$ is defined up to a proportion and up to a null set.

Now we construct leafwise measures for general spaces. Let X be a (G,T)-space.

Definition 7.12. $A \subset X$ is an **open T-plaque** if for every $x \in A$,

(1) $\exists r > 0$, $A \subset B_r^T(x)$,

(2) $t_U(x,\cdot)^{-1}A$ is open on T for every $U \in \mathfrak{T}(x)$.

Definition 7.13. $(\mathscr{A},U)$ is an **(r,T)-flower with center $B \subset U$** if $U \subset X$ and $\mathscr{A}$ is a countably generated σ-algebra on U satisfying

($\clubsuit$-1) $B \subset U$ and $\overline{U}$ is compact.

($\clubsuit$-2) For every $y \in U$, $[y]_{\mathscr{A}} = U \cap B_{4r}^T(y)$.

($\clubsuit$-3) For every $y \in B$, $B_r^t(y) \subset [y]_{\mathscr{A}}$.

Theorem 7.14 (The existence of leafwise measures)

Let $\mu \in \mathscr{P}(X)$ where X is a (G,T)-space. Assume for μ-almost every $x \in X$, $B_\infty^T(x)$ is embedded (here "embedded" means $t \mapsto t_U(x,t)$ is injective). Then for every $V \in \mathfrak{T}$, there exists $(\mu_{x,T}^V)_{x\in V}$ where $\mu_{x,T}^V$ are Radon measures on T such that

(1) For almost every $x \in V$, $\mu_{x,T}^V(B_1^T) = 1$.

(2) If $(\mathscr{A},U)$ is an (r,T)-flower then for μ-almost every $x \in U$ and every $V \in \mathfrak{T}(x)$, we have

$$t_V(x,\cdot)_*^{-1}\mu_x^{\mathscr{A}} \propto \mu_{x,T}^V|_{t_V(x,\cdot)^{-1}[x]_{\mathscr{A}}}.$$

These two conditions determine $(\mu_{x,T}^V)_{x\in V}$ up to a null set.

Moreover, for every $x, U \in \mathfrak{T}(x)$ and $y \in B_\infty^T(x)$, $V \in \mathfrak{T}(y)$, we have

$$\theta_*\mu_{x,T}^U \propto \mu_{y,T}^V,$$

where $\theta \in G$ satisfying $t_U(y,\theta t) = t_U(x,t)$.

§8 Host's proof of Rudolph's theorem (Weikun He, Nov 24)

Setting

- μ a probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $F_p : \mathbb{T} \to \mathbb{T}, [x] \mapsto [px]$, where p is a prime number.
- $T_p^N = \frac{1}{p^N}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$, $N \geqslant 1$, are finite subgroups of $\mathbb{T}$.
- $T_p = \bigcup_{N\geqslant 1} T_p^N \subset \mathbb{R}/\mathbb{Z} = \mathbb{Z}[1/p]/\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$ is a dense subgroup of $\mathbb{T}$.
- $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{Z}[1/p]\backslash\mathbb{R} \times \mathbb{Q}_p/\mathbb{Z}_p$.

We want to consider "T_p-foliations". Let

$$\mathscr{A}^N = \left\{T_p^N\text{-invariant Borel subsets}\right\} = (F_p^N)^{-1}\mathscr{B},$$

which is countably generated. But

$$\mathscr{A}^\infty := \left\{ T_p\text{-invariant Borel subsets} \right\} = \bigcup_{N \geqslant 1} \mathscr{A}^N$$

is not countably generated.

We can disintegrate μ with respect to $\mathscr{A}^N$ as

$$\mu = \int \mu_x^{\mathscr{A}^N} \mathrm{d}\mu(x),$$

where $\mu_x^{\mathscr{A}^N}([x]_{\mathscr{A}^N}) = 1$. Note that

$$[x]_{\mathscr{A}^N} = x + T_p^N$$

with the cardinality p^N. We define $\mu_x^N := (-x)_* \mu_x^{\mathscr{A}^N}$ on T_p^N. Then μ_x^N satisfies the compatibility condition

$$\mu_x^N|_{T_p^M} \propto \mu_x^M, \quad \forall M \leqslant N.$$

Thus

$$\mu_x^\infty(\omega) := \frac{\mu_x^N(\{\omega\})}{\mu_x^N(\{0\})}, \quad \forall \omega \in T_p^N$$

is well-defined on T_p. It can be compared with the "leafwise measure".

Properties 8.1. (1) $\mu_x^\infty(\{0\}) = 1$.
(2) $\mu_x^\infty|_{T_p^N} \propto \mu_x^N$.
(3) $\forall y = x + \omega \in x + T_p$, $\mu_x^\infty \propto (+\omega)_* \mu_y^\infty$.

Lemma 8.2

Assume for μ-a.e. $x \in \mathbb{R}/\mathbb{Z}$ μ_x^∞ is T_p-invariant (i.e. $\mu_x^\infty(\{\omega\}) = 1$ for every $\omega \in T_p$), then μ is T_p-invariant hence $\mu = \text{Leb}$.

Proof. For every interval $I = [a/p^N, (a+1)/p^N[$, we have

$$\mu(I) = \iint \mathbb{1}_I \mathrm{d}\mu_x^{\mathscr{A}^N} \mathrm{d}\mu(x) = \iint \frac{1}{p^N} \mathrm{d}\mu(x) = \frac{1}{p^N}.$$

since $\mu_x^{\mathscr{A}^N} \propto (+x)_* \mu_x^\infty|_{T_p^N}$. Hence μ is Lebesgue. □

Recurrence. We say μ is **T_p-recurrent** if for every $B \subset \mathbb{R}/\mathbb{Z}$ with $\mu(B) > 0$, for μ-a.e. $x \in B$ and every compact $K \subset T_p$, there exists $\omega \in T_p \setminus K$ such that $x + \omega \in B$ [equivalently, for every $N > 1$, there exists $a \in \mathbb{Z}$ coprime with p such that $x + \frac{a}{p^N} \in B$]. This is equivalent to for every $B \subset \mathbb{R}/\mathbb{Z}$, $\mu(B) > 0$, there exists $\omega \in T_p \setminus \{0\}$ such that $\mu(B \cap (B + \omega)) > 0$.

Lemma 8.3 μ is T_p recurrent iff for μ-a.e. x, $\mu_x^\infty(T_p) = \infty$.

Proof. **The "only if" part.** Assume that $\mu(\{x : \mu_x^\infty(T_p) < \infty\}) > 0$. Then there exists N such that

$$B = B_N := \left\{ x : \mu_x^\infty(T_p^N) > 0.9\mu_x^\infty(T_p) \right\}$$

has the positive μ-measure. By recurrence, there exists $\omega \in T_p \setminus T_p^N$ such that $\mu(B \cap (B+\omega)) > 0$. For almost every $x \in B \cap (B+\omega)$, we have

$$\mu_x^\infty(T_p^N) + \mu_x^\infty(T_p^N + \omega) = \mu_x^\infty(T_p^N) + \mu_{x-\omega}^\infty(T_p^N) > \mu_x^\infty(T_p).$$

This contradicts $T_p^N \cap (T_p^N + \omega) = \emptyset$.

The "if" part. Assume that there exists B such that $\mu(B) > 0$ and $\mu(B \cap (B+\omega)) = 0$ for $\omega \in T_p \setminus \{0\}$. Replace B by $B \setminus \bigcup_{\omega \neq 0}(B+\omega)$, we can assume that $B \cap (B+\omega) = \emptyset$ for every $\omega \in T_p$. For every $x \in B + T_p$, there exists a unique $s(x) \in T_p$ such that $x + s(x) \in B$. We have

$$\mu(B) = \int \mathbb{1}_B \mathrm{d}\mu = \iint \mathbb{1}_B(y) \mathrm{d}\mu_x^{\mathscr{A}^N}(y) \mathrm{d}\mu(x) \tag{8.1}$$

$$= \int \mu_x^{\mathscr{A}^N}(x + s(x)) \mathrm{d}\mu(x) = \int \mu_{x+s(x)}^\infty (T_p^N)^{-1} \mathrm{d}\mu(x). \tag{8.2}$$

Here we use the fact that

$$\mu_x^{\mathscr{A}^N}(\{x\}) = \mu_x^N(\{0\}) = \frac{\mu_x^N(\{0\})}{\mu_x^N(T_p^N)} = \frac{\mu_x^\infty(\{0\})}{\mu_x^\infty(T_p^N)} = \mu_x^\infty(T_p^N)^{-1}.$$

Since $\mu_{x+s(x)}^\infty(T_p^N) \to \infty$, using the dominated convergence theorem, the right hand side of (8.1) tends to 0. This contradicts $\mu(B) > 0$. □

Theorem 8.4 (Host)

Assume $p > 2$, and

(1) μ is F_2-invariant.

(2) μ is T_p-recurrent.

Then μ is Lebesgue.

Proposition 8.5 (Host)

Assume μ is F_p-invariant. Then μ is T_p-recurrent iff for almost every F_p-ergodic component ν, $h(\nu, F_p) > 0$.

Proof of Rudolph's theorem. Let μ be an (F_2, F_3)-invariant ergodic probability measure. Assume that $h(\mu, F_3) > 0$. Decompose μ into F_3-ergodic measures

$$\mu = \int \mu_\alpha \mathrm{d}\eta(\alpha) = \int_{\mathrm{I}} + \int_{\mathrm{II}},$$

where $\mathrm{I} = \{\alpha : h(\mu_\alpha, F_3) > 0\}$. Then $\eta(\mathrm{I}) > 0$. Since F_2, F_3 are commuting,

$$\mu' = \eta(\mathrm{I})^{-1} \int_{\mathrm{I}} \mu_\alpha \mathrm{d}\eta(\alpha)$$

is an F_2-invariant probability measure. Then $\mu' = \mu$ since μ is (F_2, F_3)-ergodic. By the proposition above, μ is T_3-recurrent. Hence $\mu = \mathrm{Leb}$ by Host's theorem. □

Proof of Host's theorem. We aim to show that for every $\nu \in \mathbb{Z} \setminus \{0\}$, $\hat{\mu}(\nu) = 0$. We consider

$$\mathscr{G}_m(\cdot) = \frac{1}{m} \sum_{k=0}^{m-1} e(\nu 2^k \cdot),$$

then $\hat{\mu}(v) = \int \mathscr{G}_m \mathrm{d}\mu$ by the F_2-invariance of μ. The idea is to use the L^2-cancellation among $e(c2^k\cdot)$. Then by Cauchy-Schwartz, we have

$$\left|\int \mathscr{G}_m \mathrm{d}\mu\right| \leqslant \int \frac{|\mathscr{G}_m|^2}{\mu_x^N(\{0\})}\mathrm{d}\mu \int \mu_x^N(\{0\})\mathrm{d}\mu.$$

Recall that

$$\mu_x^{\mathscr{A}^N}(\{x\}) = \mu_x^N(\{0\}) = \frac{\mu_x^N(\{0\})}{\mu_x^N(T_p^N)} = \frac{\mu_x^\infty(\{0\})}{\mu_x^\infty(T_p^N)} = \mu_x^\infty(T_p^N)^{-1}.$$

By the T_p-recurrence, we have $\mu_x^N(\{0\}) \to 0$ for μ-a.e. x. By the dominated convergence theorem, $\int \mu_x^N(\{0\})\mathrm{d}\mu \to 0$.

It suffice to find (m, N) such that

$$\int \frac{|\mathscr{G}_m(x)|^2}{\mu_x^N(\{0\})}\mathrm{d}\mu$$

is uniformly bounded with respect to N. Denote the function in the integral by h. We have

$$\begin{aligned}\int h\mathrm{d}\mu &= \iint h\mathrm{d}\mu_x^{\mathscr{A}^N}\mathrm{d}\mu(x) = \iint h(x+\omega)\mathrm{d}\mu_x^N(\omega)\mathrm{d}\mu(x)\\ &= \int \sum_{\omega\in T_p^N} h(x+\omega)\mu_{x+\omega}^N(\{0\})\mathrm{d}\mu(x) = \int \sum_{\omega\in T_p^N} |\mathscr{G}_m(x+\omega)|^2\mathrm{d}\mu(x).\end{aligned}$$

Expand $\mathscr{G}_m$, we have

$$\begin{aligned}\sum_{\omega\in T_p^N} |\mathscr{G}_m(x+\omega)|^2 =& \frac{1}{m^2}\sum_{a=0}^{p^N-1}\sum_{k,\ell=0}^{m-1} e(v(2^k-2^\ell)(x+a/p^N))\\ =& \frac{p^N}{m^2}\#\left\{(k,\ell) : v(2^k-2^\ell) \equiv 0 \bmod p^N\right\}.\end{aligned}$$

Lemma 8.6

For every prime $p \neq 2$ and $v \in \mathbb{Z}_+$, there exists $c > 0$ such that the following holds. For every $0 \leqslant k \neq \ell \leqslant c\cdot p^N - 1$, $v(2^k - 2^\ell) \not\equiv 0 \bmod p^N$.

We take $m \approx cp^N$ and let $N \to \infty$. Then the above summation is uniformly bounded by $p^N/m \ll 1/c$ with respect to N. ☐

Proof of Proposition. Proof of the "if" part. Let μ be an F_p-invariant ergodic probability measure with $h(\mu, F_p) > 0$. We are going to show μ is T_p-recurrent. Let $\varphi(x) = \mu_x^{\mathscr{A}^1}(y) = \mu_x^1(\{0\})$. We have $\sum_{y\in F_p^{-1}(x)} F_p(x) = 1$. Note that

(1) $h(\mu, F_p) = h(\mu, \mathscr{B}|F_p^{-1}\mathscr{B}) = -\int \log\varphi\mathrm{d}\mu$.

(2) $\mu_x^N(\{0\}) = \varphi(x)\varphi(F_px)\cdots\varphi(F_p^{N-1}x)$.

By Birkhoff's ergodic theorem,

$$\frac{1}{N}\log\mu_x^N(\{0\}) \to \int \log\varphi\mathrm{d}\mu.$$

Therefore, $\mu_x^\infty(T_p^N) = \mu_x^N(\{0\})^{-1} \to 0$ and hence $\mu_x^\infty(T_p) = \infty$. By Lemma 8.3, we obtain the T_p-recurrence. ☐

Remark 8.7 The positivity of entropy directly implies $\mu_x^N(\{0\}) \to 0$ or equivalently $\mu_x^\infty(T_p)$ is infinite. This is exactly what we need in the proof of Host's theorem. Lemma 8.3 and the definition of recurrence is not needed in this sense. But the concept of T_p-recurrence gives some intuition on "μ_x^∞ is infinite for almost every x".

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