Homogeneous Dynamics and Applications (Manfred Einsiedler)

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There notes are based on the course [1][2] at ETH in 2025 – 2026, given by Manfred Einsiedler.
See also the official lecture notes published on his website.

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1 Lattices and the space of lattices (2025 Autumn)

§1.1 Sep 18: Review on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$

Recall the hyperbolic plane $\mathbb{H}$ is the set $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ endowed with the hyperbolic metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. The Mobius transformation is induced by $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ given as

$$z \in \mathbb{H} \mapsto g.z = \frac{az+b}{cz+d}.$$

Note that $\begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} : z \mapsto z+s$ and $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} : z \mapsto -1/z$. We have $\operatorname{Im}(g.z) = \frac{\operatorname{Im} z}{|cz+d|^2}$. Using this one can show that Mobius transformations preserve the hyperbolic metric.

There are several important subgroups of $\mathrm{SL}_2(\mathbb{R})$ we may consider:

- Unipotent subgroup $N = U = \left\{ \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} : s \in \mathbb{R} \right\}$;
- Diagonal / Cartan subgroup $A = \left\{ \begin{bmatrix} e^{-t/2} & \\ & e^{t/2} \end{bmatrix} : t \in \mathbb{R} \right\}$;
- Borel subgroup $B = AN = \left\{ \begin{bmatrix} e^{-t/2} & s \\ & e^{t/2} \end{bmatrix} : t, s \in \mathbb{R} \right\}$;
- Maximal compact subgroup $K = \mathrm{SO}_2(\mathbb{R})$.

Then $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, but not simply transitive. The subgroup K fixes i and in fact $\mathrm{PSO}_2(\mathbb{R}) = \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(i)$.

We consider the tangent bundle $T\mathbb{H}$ and the unit tangent bundle

$$T^1\mathbb{H} = \left\{ (z,v) \in T\mathbb{H} : \|v\|_{\text{hyperbolic metric at } z} = \frac{\|v\|_{\text{Euclidean}}}{\operatorname{Im} z} = 1 \right\}.$$

The derivative of the Mobius transformations for $g \in \mathrm{PSL}_2(\mathbb{R})$ act on $T^1\mathbb{H}$ simply transitively: for any $(z_1, v_1), (z_2, v_2) \in T^1\mathbb{H}$ there exists a unique $g \in \mathrm{PSL}_2(\mathbb{R})$ mapping (z_1, v_1) to (z_2, v_2). Therefore we have the isomorphism

$$T^1\mathbb{H} \cong \mathrm{PSL}_2(\mathbb{R}).$$

The left hand side is a torsor: we forget the information of the identity. Usually we will choose $(i, \uparrow)$ as the element corresponding to the identity.

On $T^1\mathbb{H}$ there is a geodesic flow: simply following the geodesic determined by tangency by $(z,v) \in T^1\mathbb{H}$. For $(i,\uparrow)$, the geodesic orbit is $(e^t i, e^t \uparrow)$ and t is the time parameter for moving at unit speed. For a general starting point (z,v), assuming $(z,v) = Dg(i,\uparrow)$ for some $g \in \mathrm{PSL}_2(\mathbb{R})$, the geodesic flow is $(Dg)(e^t i, e^t \uparrow)$.

Claim 1.1.1. In $\mathrm{PSL}_2(\mathbb{R})$, the geodesic flow corresponds to the right multiplication by $a_t = \begin{bmatrix} e^{t/2} & \\ & e^{-t/2} \end{bmatrix}$ for $t \in \mathbb{R}$.

Proof. Note that $\mathrm{id} \cdot a_t = a_t \in \mathrm{PSL}_2(\mathbb{R})$ corresponds to $a_t.(i, \uparrow) = (e^t z, e^t \uparrow) \in T^1\mathbb{H}$. Hence the claim is true for $(i, \uparrow)$. Then the claim is true for general elements since the left and right multiplications commute. $\square$

For the group $\mathrm{PSL}_2(\mathbb{R})$ ($\cong T^1\mathbb{H}$), we have

- The eft-multiplications correspond to (the derivatives of) Mobius transformations.
- The right-multiplication by K fixes base-points and rotates vectors.
- The right-multiplication by $N = U = U^- = \left\{ \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \right\}$ creates stable horocycles; and the right-multiplication by $U^+ = \left\{ \begin{bmatrix} 1 & \\ s & 1 \end{bmatrix} \right\}$ creates unstable horocycles.

To get interesting dynamics, we need to fold up the space $T^1\mathbb{H}$. We can do this by a discrete subgroup of $\mathrm{Isom}(T^1\mathbb{H})^\circ = \mathrm{PSL}_2(\mathbb{R})$. Our example is $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Let M be the moduli surface, whose unit tangent bundle T^1M corresponds to

$$\mathbf{X}_2 := \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R}) = \{\Gamma g : g \in \mathrm{PSL}_2(\mathbb{R})\} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}).$$

On $\mathbf{X}_2$, right multiplications have still the same meaning. An fundamental domain of Γ can be given by

$$F = \left\{ -\frac{1}{2} \leqslant \operatorname{Re} z < \frac{1}{2} \right\} \cap \{|z| \geqslant 1\} \setminus \left\{ |z| = 1 \text{ and } 0 \leqslant \operatorname{Re} z < \frac{1}{2} \right\}.$$

A interesting question is for which $(z, |z| \cdot \uparrow) \in F$, the geodesic passing through this point is periodic. The answer is, the geodesic is periodic iff $\operatorname{Re} z$ is rational.

Another application of the geodesic flows comes from an equidistribution result by Sarnak. We consider two sets

$$Y = \left\{ \mathrm{SL}_2(\mathbb{Z}) u_s^- a_\varepsilon : s, \varepsilon \in (0,1) \right\} \text{ and } L = \left\{ \mathrm{SL}_2(\mathbb{Z}) u_r^+ : r \in (0,1) \right\},$$

where $a_t = \begin{bmatrix} e^{t/2} & \\ & e^{-t/2} \end{bmatrix}$, $u_s^- = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix}$ and $u_s^+ = \begin{bmatrix} 1 & \\ s & 1 \end{bmatrix}$. A theorem of Sarnak tells us that Y and La_t^{-1} will intersect often for large t. The intersection point represents the some elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$ with a fixed a. This relates to the study of parabola $\{bc \equiv a : b, c \in \mathbb{Z}\}$ for certain $a \in \mathbb{Z}$.

§1.2 Sep 19: Discrete subgroups and lattices

Let G be a locally compact, σ-compact group (this is assumed to be true all the time). We assume that $d = d_G$ is a left-invariant metric on G. That is $d(gg_1, gg_2) = d(g_1, g_2)$. The following exercise shows the existence of such metric for $G = \mathrm{GL}_d(\mathbb{R})$. In fact, such metric always exists.

> **Exercise 1.2.1.** Let $G = \mathrm{GL}_d(\mathbb{R})^\circ$. Let $\|\cdot\|$ be a norm on $\mathrm{Mat}_d(\mathbb{R})$. Let $p : [0,1] \to G$ be a continuous and piecewise differentiable map. We define
>
> $$L(p) := \int_0^1 \|p(t)^{-1} p'(t)\| \, \mathrm{d}t.$$
>
> Let d be defined as
>
> $$d(g_1, g_2) = \inf \{L(p) : p \text{ is a path with } p(0) = g_1, p(1) = g_2\}.$$

> Show that d is a left invariant metric on G.

We define $B_r^G = B_r^G(\mathrm{id})$. Note that

$$(B_r^G)^{-1} = B_r^G \text{ and } B_{r_1}^G B_{r_2}^G \subset B_{r_1+r_2}^G$$

by the left-invariance of d.

Let $\Gamma < G$ be a discrete subgroup. We define the quotient $X = \Gamma\backslash G := \{\Gamma g : g \in G\}$ be the space of right cosets. The metric d_X on X is given by

$$d_X(\Gamma g_1, \Gamma g_2) := \inf_{\gamma_1,\gamma_2\in\Gamma} d_G(\gamma_1 g_1, \gamma_2 g_2) = \inf_{\gamma\in\Gamma} d_G(\gamma g_1, g_2).$$

Lemma 1.2.2 (Injectivity radius)
For any compact subset $K \subset X = \Gamma\backslash G$, there exists a uniform **injectivity radius** $r > 0$ so that for any $x_0 \in K$ the map

$$g \in B_r^G \mapsto x_0 g \in B_r^X(x_0)$$

is an isometry. Moreover, for $K = \{\Gamma g_0\}$, we can take

$$r = \frac{1}{4} \inf_{\gamma\in\Gamma\backslash\{\mathrm{id}\}} d(g_0^{-1}\gamma g_0, \mathrm{id}).$$

Proof. We first show the case for $K = \{\Gamma g_0\}$. Let $r > 0$ be given by the formula above. Suppose $g_1, g_2 \in B_r^G$ and $\gamma \in \Gamma$ is such that $d(\gamma g_0 g_1, g_0 g_2) < 2r$. We will show that $\gamma = \mathrm{id}$. By the assumption, we have $d(g_0^{-1}\gamma g_0 g_1, g_2) < 2r$. Since $g_1, g_2 \in B_r^G$, by the triangle inequality,

$$d(\mathrm{id}, g_0^{-1}\gamma g_0) \leqslant d(\mathrm{id}, g_2) + d(g_2, g_0^{-1}\gamma g_0 g_1) + d(g_0^{-1}\gamma g_0 g_1, g_0^{-1}\gamma g_0) < 4r.$$

By the definition of r, this forces $\gamma = \mathrm{id}$. Therefore,

$$d_X(\Gamma g_0 g_1, \Gamma g_0 g_2) = \inf_{\gamma\in\Gamma} d_G(\gamma g_0 g_1, \gamma g_0 g_2) = d(g_1, g_2).$$

For general compact K, for any $x_0 \in K$ we can find r_0 as above. Note that for every $x \in B_{r_0/2}^X(x_0)$, we can use $r_0/2$ as an injectivity radius. Then the lemma follows by using the Lebesgue number of the cover given by $B_{r_0/2}^X(x_0)$'s. $\square$

Definition 1.2.3. A mesurable subset $F \subset G$ is called
- a **fundamental domain** if $G = \bigsqcup_{\gamma\in\Gamma} \gamma F$;
- **injective** if $\gamma_1 F \cap \gamma_2 F = \varnothing$ for every $\gamma_1 \neq \gamma_2 \in \Gamma$;
- **surjective** if $G = \bigcup_{\gamma\in\Gamma} \gamma G$.

Note that for the canonical projection map $\pi_X : G \to X = \Gamma\backslash G$, we have
- F is injective iff $\pi_X|_F$ is injective;
- F is surjective iff $\pi_X|_F$ is surjective;
- F is a fundamental domain iff $\pi_X|_F$ is bijective.

Lemma 1.2.4 (The existence of a fundamental domain)

Let $B_{\text{inj}} \subset B_{\text{suj}}$ be injective (resp. surjective) sets in G. Then there exists a fundamental domain F with $B_{\text{inj}} \subset F \subset B_{\text{suj}}$.

Proof. Applying the previous lemma, we find a sequence of sets $B_n \subset G$ such that $\pi_X|_{B_n}$ is bijective and $G = \bigcup B_n$. We inductively define the sets $F_0, F_1, \cdots$. Let $F_0 = B_{\text{inj}}$. For every $n \geqslant 1$, we define

$$F_n := B_{\text{suj}} \cap B_n \setminus \pi_X^{-1}(\pi_X(F_0 \cup F_1 \cup \cdots \cup F_{n-1})).$$

Let $F = \bigcup_{n=0}^{\infty} F_n$, which is a desired fundamental domain. □

Definition 1.2.5. Γ is called a **uniform lattice** if $X \backslash G$ is compact.

Note that by Lemma 1.2.2, we can find in this case a finite union of balls of compact closure in G whose images cover X. In particular, we can find a fundamental domain with compact closure.

Now we aim to give the general definition of lattices. For this purpose, we need Haar measures. There exists a left Haar measure m_G on G satisfying $m_G(gB) = m_G(B)$, $m(U) > 0$ and $m(K) < \infty$ where U is a nonempty open set and K is a compact set. This measure is unique up to a multiplicative constant. Also there exists a right Haar measure $m_G^{(\mathrm{r})}$ on G, which is also unique up to a multiplicative constant. The group G is called **unimodular** if m_G itself is right invariant,

Lemma 1.2.6

If B_1, B_2 are injective sets with $\pi_X(B_1) = \pi_X(B_2)$ then $m_G(B_1) = m_G(B_2)$.

Proof. We have

$$B_1 = \bigsqcup_{\gamma \in \Gamma} B_1 \cap \gamma B_2, \quad B_2 = \bigsqcup_{\gamma \in \Gamma} \gamma^{-1} B_1 \cap B_2.$$

Hence $m_G(B_1) = m_G(B_2)$. □

Given $X = \Gamma \backslash G$ and a fundamental domain $F \subset G$, we can define the measure m_X on X as $m_X(B) = m_G(F \cap \pi_X^{-1} B)$. The lemma shows that m_X does not depend on the choice of F.

Definition 1.2.7. Γ is called a **lattice** if $X = \Gamma \backslash G$ supports a right G-invariant finite measure.

Example 1.2.8 $\mathbb{Z}^d < \mathbb{R}^d$; uniform lattices; $\mathrm{SL}_d(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{R})$.

Starting with m_G, we define the measure $m_G^{(g)}(B) := m_G(Bg)$ for $B \subset G$. Note that $m_G^{(g)}$ is still a left Haar measure. By the uniqueness, there exists a positive constant $\mathrm{mod}(g)$ such that

$$m_G^{(g)} = \mathrm{mod}(g) m_G.$$

The map $\mathrm{mod} : G \to \mathbb{R}_{>0}$, which is obvious a group homomorphism, is called the **modular character**.

§1.3 Sep 25: Lattices & Orbits of closed subgroups

Proposition 1.3.1

The following are equivalent:

(a) Γ is a lattice.

(ã) There exists a fundamental domain F with $m_G^{(r)}(F) < \infty$ and $m_G^{(r)}$ is left Γ-invariant.

(b) There exists a fundamental domain F with $m_G(F) < \infty$.

If these hold, then G is unimodular (m_G is bi-invariant).

To show this proposition, we recall that $\mathrm{mod} : G \to \mathbb{R}_{>0}$ is the modular character: it is a homomorphism, continuous and satisfies

$$m_G(Bg) = \mathrm{mod}(g) m_G(B).$$

We will also use the Poincaré recurrence theorem from ergodic theory. In a way it is the ergodic pigeonhole principle as the following.

Theorem 1.3.2 (Poincaré recurrence)

Let X be a locally compact, σ-compact metric space. Let μ be a probability measure. Let $T : X \to X$ be a continuous measure-preserving map. Then for μ-almost every $x \in X$ there exists a sequence $n_k \to \infty$ with $T^{n_k}x \to x$ as $k \to \infty$.

Proof of Proposition 1.3.1. (b) $\Longrightarrow$ (ã). If $m_G(F) < \infty$ then $\mathrm{mod}(g) = m_G(Fg)/m_G(F) = 1$ for every $g \in G$. Hence G is unimodular and (ã) follows.

(a) $\Longrightarrow$ (ã). Let m_X be a right G-invariant probability measure on X. For $f \geqslant 0$ on G we define

$$\int_G f \, \mathrm{d}\mu := \int_X \sum_{\Gamma g = x} f(g) \, \mathrm{d}m_X(x).$$

Then for μ, we have

- μ is a right Haar measure on G.
- $\mu(F) = \int \mathbb{1} \, \mathrm{d}\mu_X = 1$.
- $\mu(\gamma_0 B) = \mu(B)$.

Hence we obtain (ã).

(ã) $\Longrightarrow$ (a). By the assumption, any two injective sets in G with the same image on X have the same $m_G^{(r)}$-measure. We use this to define m_X as

$$m_X(B) := m_G^{(r)}(F \cap \pi_X^{-1}(B)).$$

Then m_X is finite and independent of the choice of F. For $B \subset X$ and $g \in G$, we have

$$m_X(Bg) = m_G^{(r)}(F \cap \pi_X^{-1}(B)g) = m_G^{(r)}(Fg^{-1} \cap \pi_X^{-1}(B)) = m_X(B).$$

(ã) $\Longrightarrow$ (b). We aim to show that G is unimodular. Assume that m_X is a G-invariant probability measure on X. Then $\mathrm{supp}\, m_X = X$. Hence for every compact neighborhood $B \ni \mathrm{id}$ in G, we have $m_X(\Gamma B) > 0$.

Let $g \in G$. We define $T(x) = xg$. Then by Poincaré recurrence theorem, there exists Γb_0 and a sequence $n_k \to \infty$ such that $\Gamma b_0 g^{n_k} = \Gamma b_k$ where $b_k \in B$. By the definition of

X, there exists a sequence $\gamma_k \in \Gamma$ such that $b_0 g^{n_k} = \gamma_k b_k$. By the assumption, we have $\operatorname{mod}(\Gamma) = 1$. Hence

$$\operatorname{mod}(g)^{n_k} = \frac{\operatorname{mod}(\gamma_k b_k)}{\operatorname{mod}(b_0)} = \frac{\operatorname{mod}(b_k)}{\operatorname{mod}(b_0)},$$

which is contained in a compact subset of $(0, \infty)$. This forces $\operatorname{mod}(g) = 1$.

□

The following is the "folding" or "unfolding" of Haar measures. We will omit the proof, which is essentially the same to that of Proposition 1.3.1.

Proposition 1.3.3

Suppose G is unimodular and $\Gamma < G$ is discrete. Then there exists a locally finite right G-invariant measure m_X on $X \backslash G$ satisfying

$$\int_G f \, \mathrm{d}m_G = \int_X \sum_{\gamma \in \Gamma} f(\gamma g) \, \mathrm{d}m_X(\Gamma g)$$

for every $f \in L^1(G, m_G)$.

Proposition 1.3.4 (Abstract divergence criterion)

Let $\Gamma < G$ be a lattice and $x_n = \Gamma g_n \in X = \Gamma \backslash G$. The the following are equivalent:

(1) $x_n \to \infty$ as $n \to \infty$.

(2) The maximal injectivity radius r_{x_n} at x_n satisfies $r_{x_n} \to 0$ as $n \to \infty$.

Proof. (2) $\implies$ (1). If $x_n \not\to \infty$ then x_n visits a fixed compact subset of X infinitely often. This contradicts Lemma 1.2.2.

(1) $\implies$ (2). Suppose $x_n \to \infty$ but $r_{x_n} \geqslant \varepsilon > 0$ for all $n \geqslant 1$. Without loss of generality, we assume that $\overline{B_\varepsilon^G}$ is compact. By an inductive argument, we can find $n_1, n_2, \cdots$ such that

$$x_{n_k} \notin x_{n_1} \overline{B_\varepsilon^G} \cup \cdots \cup x_{n_{k-1}} \overline{B_\varepsilon^G}.$$

Hence the sets $x_{n_k} B^G_{\varepsilon/2}$ are pairwise disjoint. Let $g_n \in G$ such that $x_n = \Gamma g_n$. Then the set

$$B = \bigsqcup_{k=1}^{\infty} g_{n_k} B^G_{\varepsilon/2}$$

is a injective subset of G. But $m_G(B) = \infty$, which contradicts that Γ is a lattice.

□

Let $H < G$ be a closed subgroup. Then d_G also induces a metric on $H \backslash G$. Then $H \backslash G$ is also locally compact, σ-compact and complete. H also acts on $X = \Gamma \backslash G$. For any $x \in X$, the orbit is

$$H.x = xH \cong H / \operatorname{Stab}_H(x) \cong \operatorname{Stab}_H(x) \backslash H,$$

where $h.x = xh^{-1}$.

Lemma 1.3.5 The map $\mathrm{Stab}_H(x)\backslash H \to xH$ is continuous.

Proof. Suppose $\Lambda = \mathrm{Stab}_H(x) = H \cap g^{-1}\Gamma g$, where $g \in G$ satisfies $x = \Gamma g$. Let $h_n, h \in H$ be elements with $\Lambda h_n \to \Lambda h$. Hence there exist elements $\lambda_n \in \Lambda$ such that $\lambda_n h_n \to h$. Recall $\lambda_n = g^{-1}\gamma_n g$ for some $\gamma_n \in \Gamma$. This gives $\gamma_n g h_n \to gh$ and hence $\Gamma g h_n \to \Gamma gh$. □

Definition 1.3.6. We say xH has **finite volume** or is a **periodic orbit** if $\mathrm{Stab}_H(x) < H$ is a lattice.

Corollary 1.3.7
If xH has finite volume then xH is closed and the map $\mathrm{Stab}_H(x)\backslash H \to xH$ is proper.

Proof. Suppose $\Lambda h_n \to \infty$ in $\Lambda\backslash H$. By the previous proposition, the injectivity radius of Λh_n goes to zero. This means that there exists a sequence $\lambda_n \in \Lambda \setminus \{\mathrm{id}\}$ so that $h_n^{-1}\lambda_n h_n \to \mathrm{id}$ (see the proof of Lemma 1.2.2). Recall that we have $x = \Gamma g$ and $\Lambda = H \cap g^{-1}\Gamma g$. Hence $\lambda_n = g^{-1}\gamma_n g$ and so $h_n^{-1} g^{-1}\gamma_n g h_n \to \mathrm{id}$. Therefore, the injectivity of $\Gamma g h_n$ goes to zero and hence $\Gamma g h_n \to \infty$ in X.

Suppose now $xh_n \to z \in X$. In particular, $xh_n \not\to \infty$. By the argument above, $\Lambda h_n \not\to \infty$. Then there exists a subsequence such that $\Lambda h_{n_k} \to \Lambda h$. Hence $z = xh \in xH$. □

Proposition 1.3.8
If xH is a closed orbit then the map $\mathrm{Stab}_H(x)\backslash H \to xH$ is a homeomorphism. In particular, the Haar measure on $\mathrm{Stab}_H(x)\backslash H$ give rise to a locally finite measure on X with support H.

The proof of this proposition will be given later, see Proposition 1.4.3.

§1.4 Sep 26: Duality of orbits & Successive minimas

Proposition 1.4.1 (Topological duality)
Let $\Gamma, H < G$ be closed subgroups. The following are equivalent for $g_0 \in G$:
(1) $(\Gamma g_0)H \subset X = \Gamma\backslash G$ is closed.
(2) $\Gamma g_0 H \subset G$ is closed.
(3) $\Gamma(g_0 H) \subset Y = G/H$ is closed.
If Γ is discrete and these conditions hold, then the orbit $\Gamma(g_0 H) \subset Y$ is also discrete.

Proof. We suppose $B = BH \subset G$ that is invariant under H on the right.

Claim 1.4.2. B is closed in G iff $\pi_Y(B) \subset Y$ is closed.

This gives (2) $\iff$ (3) but also (1) $\iff$ (2) by switching sides.

Proof of the claim. The map $\pi_Y : G \to G/H$ is continuous. Hence if $\pi_Y(B)$ is closed then $B = \pi_Y^{-1}(\pi_Y(B))$ is also closed. We now assume that B is closed. Let $b_n H \to gH \in Y$, where $b_n H \in \pi_Y(B)$. Then there exists $h_n \in H$ such that $b_n h_n \to g \in G$. Note that $b_n h_n \in B$. We have $g \in B$ and hence $gH \in \pi_Y(B)$. □

Now we assume that Γ is discrete and $Y_0 = \Gamma(g_0 H) \subset Y$ is closed. Then Y_0 is complete. Suppose for the purpose that Y_0 is not discrete. Then there exists an accumulation point $\eta(g_0 H) \in Y_0$ if Y_0. Assume that $\gamma_n(g_0 H) \to \eta(g_0 H)$, where $\gamma_n(g_0 H) \subset Y_0 \setminus \eta(g_0 H)$. Then for every $\gamma \in \Gamma$, we have $\gamma(gH) = \lim_{n\to\infty} \eta^{-1}\gamma_n(g_0 H)$. Hence Y_0 is a perfect complete metric space. Then Y_0 is uncountable by the Baire category theorem, which contradicts that Γ is countable. □

Proposition 1.4.3

Let $\Gamma < G$ be discrete, $H < G$ be closed and $x_0 \in X = \Gamma \backslash G$ have a closed orbit $x_0 H$. Then $\mathrm{Stab}_H(x_0)\backslash H \to x_0 H \subset X$ is a homeomorphism. The volume measure on $\mathrm{Stab}_H(x_0)\backslash H$ give rise to a locally finite H-invariant measure on $x_0 H \subset X$.

Proof. We suppose $x_0 = \Gamma g_0$ and $x_0 h_n \to x_0 h$. Then there exists γ_n so that $\gamma_n g_0 h_n \to g_0 h$. We apply π_Y for $Y = G/H$ and get $\gamma_n g_0 H \to g_0 H$. By the proposition above, we have $\gamma_n g_0 H = g_0 H$ for large enough n. Equivalently, we have

$$g_0^{-1}\gamma_n g_0 \in H \cap g_0^{-1}\Gamma g_0 = \mathrm{Stab}_H(x_0).$$

We obtain $g_0^{-1}\gamma_n g_0 h_n \to h$. Hence $\mathrm{Stab}_H(x_0)h_n \to \mathrm{Stab}_H(x_0)h$. □

Lattices in $\mathbb{R}^d$.

Claim 1.4.4. A lattice in $\mathbb{R}^d$ always has the form $\Lambda = g\mathbb{Z}^d$ for $g \in \mathrm{GL}_d(\mathbb{R})$.

For a lattice $\Lambda = g\mathbb{Z}^d \subset \mathbb{R}^d$, the covolume is $|\det g|$. We call it **unimodular** if $\mathrm{covol}(\Lambda) = 1$. The space of all unimodular lattices is

$$\mathbf{X}_d := \left\{ g\mathbb{Z}^d : g \in \mathrm{SL}_d(\mathbb{R}) \right\} \cong \mathrm{SL}_d(\mathbb{R}) / \mathrm{Stab}_{\mathrm{SL}_d(\mathbb{R})}(\mathbb{Z}^d) = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}).$$

We will show that $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$. Moreover, the space $\mathbf{X}_d$ is of finite volume but not compact. We will also give a concrete criterion for sequence divergent to infinity in $\mathbf{X}_d$, which is called Mahler's criterion.

Theorem 1.4.5 (Minkowski's first theorem)

Let $\Lambda \subset \mathbb{R}^d$ be a lattice. Then there exists a nonzero vector in Λ of norm $\ll_d \sqrt[d]{\mathrm{covol}(\Lambda)}$.

Proof. Let $r_d > 0$ be such that $\mathrm{Vol}(B_{r_d}^{\mathbb{R}^d}) > 1$. Then $\sqrt[d]{\mathrm{covol}(\Lambda)} B_{r_d}^{\mathbb{R}^d}$ has volume strictly larger than $\mathrm{covol}(\Lambda)$. Then this set cannot be injective. So there exists $v_1 \neq v_2 \in \sqrt[d]{\mathrm{covol}(\Lambda)} B_{r_d}^{\mathbb{R}^d}$. Therefore, $v_1 - v_2 \in \Lambda \setminus \{0\}$ and $\|v_1 - v_2\| \leqslant 2r_d \sqrt[d]{\mathrm{covol}(\Lambda)}$. □

Theorem 1.4.6 (Minkowski's successive minimas)
Let $\Lambda \subset \mathbb{R}^d$ be a lattice. We define for $k = 1, \cdots, d$, the **successive minimas**

$$\lambda_k = \min \left\{ r : \Lambda \cap B_r^{\mathbb{R}^d} \text{ contains } k \text{ linearly independent vectors} \right\}.$$

Then we have $\lambda_1(\Lambda) \cdots \lambda_d(\Lambda) \asymp_d \operatorname{covol}(\Lambda)$.

Moreover, we define

$$\alpha_k(\Lambda) = \min \left\{ \operatorname{covol}(\Lambda \cap V \text{ inside V}) : V \subset \mathbb{R}^d \text{ is a linear subspace of dimension } k \right\}.$$

Then $\lambda_1(\Lambda) \cdots \lambda_k(\Lambda) \asymp_d \alpha_k(\Lambda)$ for $k = 1, \cdots, d$.

Proof. We begin with demonstrating the first claim by an induction on d. It is trivial for $d = 1$. Assume now the claim holds for $d - 1$. Let $v_1 \in \Lambda \setminus \{0\}$ of minmal norm $\|v_1\| = \lambda_1(\Lambda)$. We define $W = (\mathbb{R}v_1)^\perp$ and $\pi_W : \mathbb{R}^d \to W \cong \mathbb{R}^{d-1}$ the orthogonal projection. Let $\Lambda_W = \pi_W(\Lambda) \subset W$.

Claim 1.4.7. Λ_W is discrete and in fact $\lambda_1(\Lambda_W) \geqslant \frac{\sqrt{3}}{2}\lambda_1(\Lambda)$.

Proof. Suppose $w \in \Lambda_W \setminus \{0\}$ and has norm $< \frac{\sqrt{3}}{2}\lambda_1(\Lambda)$. Then there exists $v \in \Lambda$ with $v = w + tv_1$ such that $|t| \leqslant 1/2$. This implies that $\|v\| < \|v_1\|$, a contradiction. ☐

Claim 1.4.8. Λ_W is a lattice and $\operatorname{covol}(\Lambda) = \lambda_1(\Lambda) \cdot \operatorname{covol}(\Lambda_W)$.

Proof. Let F_W be a fundamental domain of Λ_W in W. Then $F = [-\frac{1}{2}, \frac{1}{2})v_1 + F_W$ is a fundamental domain of Λ. This shows the claim. ☐

Claim 1.4.9. $\lambda_k(\Lambda_W) \asymp \lambda_{k+1}(\Lambda)$ for $k = 1, \cdots, d - 1$.

Proof. Suppose $v_1, \cdots, v_{k+1} \in \Lambda$ are linearly independent and of norm $\lambda_1, \cdots, \lambda_{k+1}$ respectively. We apply π_W and obtain $w_2 = \pi_W(v_2), \cdots, w_{k+1} = \pi_W(w_{k+1})$ which are linearly independent and of norm at most $\lambda_{k+1}(\Lambda)$. Hence $\lambda_k(\Lambda_W) \leqslant \lambda_{k+1}(\Lambda)$.

Suppose $w_2, \cdots, w_{k+1} \in \Lambda_W$ be linearly independent and of norm $\leqslant \lambda_k(\Lambda_W)$. Then for each i, there exists $v_i = w_i + tv_1 \in V$ with $|t| \leqslant 1/2$. Then for every $i = 2, \cdots, k+1$,

$$\|v_i\| \leqslant \lambda_k(\Lambda_W) + \frac{1}{2}\lambda_1(\Lambda) \leqslant \lambda_k(\Lambda_W) + \frac{1}{\sqrt{3}}\lambda_k(\Lambda_W) \leqslant 2\lambda_k(\Lambda_W),$$

where we use the inequality from Claim 1.4.7. Since $v_1, \cdots, v_{k+1}$ are linearly independent, we have $\lambda_{k+1}(\Lambda) \leqslant 2\lambda_k(\Lambda_W)$. ☐

Therefore, we have

$$\operatorname{covol}(\Lambda) = \lambda_1(\Lambda) \cdot \operatorname{covol}(\Lambda_W) \asymp_d \lambda_1(\Lambda)\lambda_1(\Lambda_W) \cdots \lambda_{d-1}(\Lambda_W) \asymp_d \lambda_1(\Lambda) \cdots \lambda_d(\Lambda).$$

It remains to prove $\lambda_1(\Lambda) \cdots \lambda_k(\Lambda) \asymp_d \alpha_k(\Lambda)$. Let $v_1, \cdots, v_k \in \Lambda$ of norms $\lambda_1, \cdots, \lambda_k$ and linearly independent. Let $V = \mathbb{R}v_1 + \cdots + \mathbb{R}v_k$. Then

$$\alpha_k(\Lambda) \leqslant \operatorname{covol}(\Lambda \cap V \text{ in } V) \leqslant \|v_1\| \cdots \|v_k\| = \lambda_1 \cdots \lambda_k.$$

Now let V be an arbitrary subspace of dimension k. We only need to consider the case that $\Lambda \cap V$ is a lattice V. We apply the first statement of the theorem to $\Lambda \cap V$ in V. Then

$$\operatorname{covol}(\Lambda \cap V \text{ in } V) \asymp_d \lambda_1(\Lambda \cap V) \cdots \lambda_k(\Lambda \cap V) \geqslant \lambda_1(\Lambda) \cdots \lambda_k(\Lambda).$$

Hence $\alpha_k(\Lambda) \gg_d \lambda_1(\Lambda) \cdots \lambda_k(\Lambda)$. ☐

§1.5 Oct 2: Mahler's criterion & $\mathbf{X}_d$ is finite volume

Corollary 1.5.1
Let $\Lambda \subset \mathbb{R}^d$ be a lattice. Then there exists a $\mathbb{Z}$-basis $v_1, \cdots, v_d \in \Lambda$ with

$$\|v_1\| = \lambda_1(\Lambda),\ \|v_2\| \asymp_d \lambda_2(\Lambda),\ \cdots,\ \|v_d\| \asymp_d (\Lambda).$$

Proof. By an induction on d, we may assume that the corollary already holds for $d-1$. We define $v_1 \in \Lambda \setminus \{0\}$ as the shortest element. Then $v_1 = \lambda_1(\Lambda)$. We then define $W = (\mathbb{R}v_1)^{\perp}$ and π_W, Λ_W as in the previous theorem. By the inductive assumption, Λ_W has a $\mathbb{Z}$-basis $w_1, \cdots, w_{d-1}$ with $\|w_k\| \asymp_{d-1} \lambda_k(\Lambda_W)$ for $k = 1, \cdots, d-1$. By a similar argument as before, we can find $v_{k+1} \in \Lambda$ projecting to w_k such that

$$\|w_k\| \leqslant \|v_{k+1}\| \leqslant \|w_k\| + \frac{1}{2}\lambda_1(\Lambda) \leqslant \|w_k\| + \lambda_1\|\Lambda_w\| \ll \lambda_k(\Lambda_w) \ll \lambda_{k+1}(\Lambda).$$

By linear algebra, $v_1, \cdots, v_d$ is a $\mathbb{Z}$-basis. This shows the corollary. □

Theorem 1.5.2 (Mahler's compactness criterion)
A subset $B \subset \mathbf{X}_d = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ has compact closure if and only if there exists some $\delta > 0$ such that for every $\Lambda \in B$ we have $\lambda_1(\Lambda) \geqslant \delta$.

Proof. $\Longrightarrow$. This follows directly from the fact that $\lambda_1 : \mathbf{X}_d \to (0, \infty)$ is continuous.
$\Longleftarrow$. We assume that $\lambda_1(\Lambda) \geqslant \delta$ for all $\Lambda \in B$. Fix some $\Lambda \in B$ and apply the previous corollary. We find a $\mathbb{Z}$-basis $v_1, \cdots, v_d$ of Λ with $\|v_k\| \asymp_d \lambda_k(\Lambda) \geqslant \delta$ for every $k = 1, \cdots, d$. Moreover, we have $\|v_k\| \ll_d \delta^{-(d-1)}$ by Theorem 1.4.6. Let $g = [v_1, v_2, \cdots, v_d] \in \mathrm{Mat}_d(\mathbb{R})$. Then $\Lambda = g.\mathbb{Z}^d$ corresponds to $g\mathrm{SL}_d(\mathbb{Z}) \in \mathbf{X}_d$ and $\|g\| \leqslant c\delta^{-(d-1)}$ for some $c > 0$ depending only on d. Therefore, we obtain that

$$B \subset \left(\overline{B^{\mathrm{Mat}_d(\mathbb{R})}_{c\delta^{-(d-1)}}} \cap \mathrm{SL}_d(\mathbb{R})\right)\mathrm{SL}_d(\mathbb{R}),$$

where the right hand side is a compact subset of $\mathbf{X}_d$.

□

Our next aim is to show that $\mathbf{X}_d$ is of finite volume:

Theorem 1.5.3 $\mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SL}_d(\mathbb{R})$.

Lemma 1.5.4 $\mathrm{SL}_d(\mathbb{R})$ is unimodular.

Sketch proof I.. The d^2-dimension Lebesgue measure on $\mathrm{Mat}_d(\mathbb{R})$ is invariant under the left and right linear action by $\mathrm{SL}_d(\mathbb{R})$. For a Borel measurable subset B, we define the measure

$$m_{\mathrm{SL}_d(\mathbb{R})}(B) := m_{\mathrm{Mat}_d(\mathbb{R})}(\mathbb{R})([0,1]B),$$

where λg is the scaling of g in $\mathrm{Mat}_d(\mathbb{R})$ for $\lambda \in [0,1]$ and $g \in \mathrm{SL}_d(\mathbb{R})$. One can check that $m_{\mathrm{SL}_d(\mathbb{R})}$ is a bi-invariant Haar measure (see Exercise 1.7.1). □

Sketch proof II.. Recall the modular character mod : $\mathrm{SL}_d(\mathbb{R}) \to \mathbb{R}_{>0}$, which is a group homomorphism. The following lemma derives that $\mathrm{mod}(\mathrm{SL}_d(\mathbb{R})) = \{1\}$ and hence $\mathrm{SL}_d(\mathbb{R})$ is unimodular.

Lemma 1.5.5 $[\mathrm{SL}_d(\mathbb{R}), \mathrm{SL}_d(\mathbb{R})] = \mathrm{SL}_d(\mathbb{R})$.

This lemma follows from the following basic linear algebra fact.

Lemma 1.5.6
For any field $\mathbb{K}$, we have $\mathrm{SL}_d(\mathbb{K})$ is generated by the elementary unipotent subgroups: for every $1 \leqslant i \neq j \leqslant d$, we let U_{ij} be the subgroup of all matrices with 1's on the diagonal, all other entries 0 except for the (i,j)-th entry.

This gives the second sketch of the proof. □

Lemma 1.5.7
Let G be unimodular and $S, T < G$ be two closed subgroups with $S \cap T = \{\mathrm{id}\}$. Assume that $m_G(ST) > 0$. Then $m_G|_{ST}$ is (up to a scalar) the push forward of $m_S \times m_T^{(\mathrm{r})}$ under the map $(s,t) \in S \times T \mapsto st$.

Proof. Define the map $\psi : ST \to S \times T$, $st \mapsto (s, t^{-1})$, which is well-defined by the assumption $S \cap T = \{\mathrm{id}\}$. Let $\mu = \psi_*(m_G|_{ST})$. Since G is unimodular, μ is a left Haar measure of $S \times T$. Taking into account the inverse in the definition of ψ, we get the claim. □

For $G = \mathrm{SL}_d(\mathbb{R})$, we will use
- $K = \mathrm{SO}_d(\mathbb{R})$ the compact subgroup, and
- $B = AN = \left\{ \begin{bmatrix} a_1 & \cdots & * \\ & \ddots & \vdots \\ & & a_d \end{bmatrix} : a_1, \cdots a_d > 0, a_1 \cdots a_d = 1 \right\}$ the Borel subgroup.

Proposition 1.5.8 (Iwasawa decomposition) $\mathrm{SL}_d(\mathbb{R}) = KB$.

Proof. This follows from the Gram-Schmidt process for $g = [v_1, \cdots, v_d] \in \mathrm{SL}_d(\mathbb{R})$. □

Lemma 1.5.9
We have $B = AU$ for $A = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{bmatrix} : a_1, \cdots a_d > 0, a_1 \cdots a_d = 1 \right\}$ and $U = \left\{ \begin{bmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{bmatrix} \right\}$. Moreover, in this coordinate system we have

$$\mathrm{d}m_B^{(\mathrm{r})}(a,u) \propto \rho(a)\, \mathrm{d}m_A(a)\, \mathrm{d}m_U(u), \tag{1.5.1}$$

where m_A, m_U are bi-invariant measures and $\rho(a) = \prod_{i<j}(a_i/a_j)$.

Sketch of proof. In fact, m_U is the Lebesgume measure on the entries. We define a measure on AU by the right hand side in (1.5.1). We aim to show this measure is right invariant. Let $\phi \geqslant 0$ be a measurable function on AU. Then for $\widetilde{u} \in U$, we have

$$\int_{AU} f(au)\rho(a)\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u) = \int_{AU} f(au)\rho(a)\rho(a)\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u),$$

since m_U is the Haar measure. For $\widetilde{a} \in A$, we have

$$\int_{AU} f(au\widetilde{a})\rho(a)\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u) = \int_{AU} f(a\widetilde{a}\widetilde{a}^{-1}u\widetilde{a})\rho(a)\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u)$$
$$= \int_{AU} f(a'u')\rho(a')\prod_{i<j}\frac{\widetilde{a}_j}{\widetilde{a}_i}\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u), \qquad (a' := a\widetilde{a},\ u' := \widetilde{a}^{-1}u\widetilde{a}).$$

Noting that $u'_{ij} = \widetilde{a}_i^{-1}u_{ij}\widetilde{a}_j$ and m_U is Lebesgue on the entries, we obtain that this measure is also right A-invariant. $\square$

Applying Lemma 1.5.7 to $\mathrm{SL}_d(\mathbb{R}) = KB$ and the previous lemma, we obtain that

$$\mathrm{d}m_g = \mathrm{d}m_{kau} \propto \mathrm{d}m_K(k)\rho(a)\,\mathrm{d}m_A(a)\,\mathrm{d}m_U(u).$$

Definition 1.5.10. A **Siegel domain** is defined as $\Sigma_{s,t} := KA_tU_s$, where

$$A_t := \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{bmatrix} \in A : \frac{a_2}{a_1}, \cdots, \frac{a_d}{a_{d-1}} \geqslant t \right\} \text{ and } U_s = \left\{ u \in U : |u_{ij}| \leqslant s, \text{ for all } i < j \right\}.$$

§1.6 Oct 3: Siegel domain & Siegel transformation

Corollary 1.6.1 If $t \leqslant t_0$ and $s \geqslant 1/2$ then the Siegel domain $\Sigma_{s,t}$ is surjective.

Proof. Let $\Lambda \in \mathbf{X}_d$ be a unimodular lattice of $\mathbb{R}^d$. We apply Corollary 1.5.1 and find a $\mathbb{Z}$-basis $w_1, \cdots, w_d$ with $\|w_k\| \asymp_d \lambda_k(\Lambda)$. Let $g = [v_1, \cdots, v_d] \in \mathrm{SL}_d(\mathbb{R})$ representing Λ and $g = kau$ be the Iwasawa decomposition. By the Gram-Schmidt process, we can find that $a_k = \|\widetilde{v}_k\|$ where $\widetilde{v}_k$ is the orthogonal projection of v_k to $(\mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_{k-1})^\perp$ for $k = 1, \cdots, d$. By a slightly stronger version of Corollary 1.5.1, we also have $\|\widetilde{v}_k\| \asymp_d \lambda_k(\Lambda)$. Therefore,

$$\frac{a_{k+1}}{a_k} \gg_d \frac{\lambda_{k+1}(\Lambda)}{\lambda_k(\Lambda)} \geqslant 1.$$

Hence if we take $t_0 > 0$ small enough then $a \in A_t$ for every $t \leqslant t_0$.

For the unipotent part, we modify g by a $u_\mathbb{Z} \in U(\mathbb{Z}) = U \cap \mathrm{SL}_d(\mathbb{Z})$ on the right to ensure $u \in U_{1/2}$. Firstly we replace g by $gu_\mathbb{Z}$ for some $u_\mathbb{Z} \in U(\mathbb{Z}) = U \cap \mathrm{SL}_d(\mathbb{Z})$ to ensure that $u_{i(i+1)} \in [-\frac{1}{2}, \frac{1}{2})$. Then using another $u_\mathbb{Z} \in U(\mathbb{Z})$ with $(u_\mathbb{Z})_{i(i+1)} = 0$, we can replace g with another element representing Λ and satisfy $u_{i(i+1)}, u_{i(i+2)} \in [-\frac{1}{2}, \frac{1}{2})$. Proceeding by induction, we can find $g \in \mathrm{SL}_d(\mathbb{R})$ such that $\Lambda = g.\mathbb{Z}^d$ and $g = kau$ with $a \in A_t$ and $u \in U_{1/2}$. $\square$

Lemma 1.6.2 $m_{\mathrm{SL}_d(\mathbb{R})}(\Sigma_{s,t}) < \infty$ for all $s, t > 0$.

Proof. We clearly have $m_K(K) < \infty$ and $U_s < \infty$. It remains to show that $\int_{A_t} \rho(a)\, \mathrm{d}m_A(a) < \infty$. We use the isomorphism

$$a \in A \mapsto \left(\log \frac{a_2}{a_1}, \cdots, \log \frac{a_d}{a_{d-1}}\right) \in \mathbb{R}^{d-1}.$$

Let $y_i = \log(a_{i+1}/a_i)$. By the definition of $\rho(a)$, we have

$$\rho(a) = \prod_{i<j} \frac{a_i}{a} = \prod_{k=1}^{d-1} \left(\frac{a_k}{a_{k+1}}\right)^{r_k} = \prod_{k=1}^{d-1} e^{-r_k y_k}, \tag{1.6.1}$$

where $r_k = k(d-k) > 0$. In the linear coordinates A_t corresponds to $[\log t, +\infty)^{d-1} \in \mathbb{R}^{d-1}$. Taking the integral of (1.6.1), we obtain the lemma. $\square$

Theorem 1.5.3 then follows from the previous corollary and the previous lemma.

Siegel transform. Given $f \in C_c(\mathbb{R}^d)$ we define the **Siegel transform** by

$$\widetilde{f}(\Lambda) = \sum_{v \in \Lambda \setminus \{0\}} f(v), \quad \forall \Lambda \in \mathbf{X}_d = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}).$$

The map $\widetilde{f} : \mathbf{X}_d \to \mathbb{R}$ is well-defined by usually not compactly supported. Therefore, a priori $\widetilde{f}$ is not integrable.

Theorem 1.6.3 (Siegel formula) $\frac{1}{m_{\mathbf{X}_d}} \int_{\mathbf{X}_d} \widetilde{f}\, \mathrm{d}m_{\mathbf{X}_d} = \int_{\mathbb{R}^d} f\, \mathrm{d}v.$

Lemma 1.6.4 (Upper bound)
Assume that $\operatorname{supp} f \subset B_r(0)$. Then

$$|\widetilde{f}| \ll_d \max_{1,\cdots,d} \frac{r^k}{\lambda_1 \cdots \lambda_k} \cdot \|f\|_\infty.$$

Proof. Let V be the linear hull of $\Lambda \cap B_r(o)$ and $k = \dim V$. We apply Corollary 1.5.1 to $\Lambda \cap V$ and obtain $v_1, \cdots, v_k \in \Lambda \cap V$ such that $\|v_j\| \asymp \lambda_j(\Lambda \cap V) = \lambda_j(\Lambda)$. We define $F = \sum_{j=1}^k [0,1)v_j$, which is a fundamental domain for $\Lambda \cap V$ in V. Moreover, we have $\mathrm{Vol}_V(F) \asymp_d \lambda_1(\Lambda) \cdots \lambda_k(\Lambda)$. We now consider

$$\sum_{v \in \Lambda \cap B_r(o)} (F + v) \subset B_R^V(o),$$

where $R \asymp_d r$. Hence the volume on the left hand side is $\ll R^k \ll_d r^k$. This implies that $\#(\Lambda \cap B_r(0)) \ll_d r^k / \mathrm{Vol}_V(F)$, which gives the lemma. $\square$

Lemma 1.6.5 $(\lambda_1 \cdots \lambda_k)^{-1} : \mathbf{X}_d \to (0, \infty)$ is integrable.

Exercise 1.6.6. Show this lemma by a similar argument as the proof of Theorem 1.5.3.

Proof of Theorem 1.6.3. We define a new measure μ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f \, d\mu = \int_{\mathbf{X}_d} \widetilde{f} \, dm_{\mathbf{X}_d}.$$

Then μ is a locally finite finite measure on $\mathbb{R}^d$. As $m_{\mathbf{X}_d}$ is $\mathrm{SL}_d(\mathbb{R})$-invariant, μ is also. The action of $\mathrm{SL}_d(\mathbb{R})$ on $\mathbb{R}^d$ only has two orbits: $\{0\}$ and $\mathbb{R}^d \setminus \{0\}$. Hence $\mu = c_0\delta_0 + c m_{\mathbb{R}^d}$ by the uniqueness of invariant measure on G/H. We aim to show that $c_0 = 0$ and $c = m_{\mathbf{X}_d}(\mathbf{X}_d)$.

We define $f_r = \mathbb{1}_{B_r^{\mathbb{R}^d}}$. Note that $\widetilde{f_r} \to 0$ as $r \to 0$. By the dominated convergence theorem, we have

$$c_0 = \lim_{r\to 0} \int f_r \, d\mu = \lim_{r\to 0} \int_{\mathbf{X}_d} \widetilde{f_r} \, dm_{\mathbf{X}_d} = 0.$$

Moreover

$$c = \frac{1}{m_{\mathbb{R}^d}(B_r)} \int f_r \, d\mu = \int_{\mathbf{X}_d} \frac{1}{m_{\mathbb{R}^d}(B_r)} \widetilde{\mathbb{1}_{B_r}} \, dm_{\mathbf{X}_d}.$$

Letting $r \to \infty$, we have the pointwise convergence $\frac{1}{m_{\mathbb{R}^d}(B_r)} \widetilde{\mathbb{1}_{B_r}} \to 1$ by a counting argument of points in lattices. Hence the left hand side tends to $m_{\mathbf{X}_d}(\mathbf{X}_d)$ as $r \to \infty$. □

§1.7 Selected exercises of Chapter 1 in Manfred's notes

Chapter 1 of Manfred's official lecture notes can be found here. We select some of the exercises and provide their proofs. For each exercise, the corresponding number in Manfred's notes is indicated in parentheses after its label.

Exercise 1.7.1 (Exercise 1.6). For $d \geqslant 2$, let

$$m_{\mathrm{SL}_d(\mathbb{R})}(B) := m_{\mathrm{Mat}_d(\mathbb{R})}([0,1]B) = m_{\mathrm{Mat}_d(\mathbb{R})}\{tb : t \in [0,1], b \in B\},$$

for every measurable $B \subset \mathrm{SL}_d(\mathbb{R})$, where $m_{\mathrm{Mat}_d(\mathbb{R})}$ is the Lebesgue measure on $\mathbb{R}^{d^2}$. Show that $m_{\mathrm{SL}_d(\mathbb{R})}$ defines a bi-invariant Haar measure on $\mathrm{SL}_d(\mathbb{R})$.

Proof. If $B \subset \mathrm{SL}_d(\mathbb{R})$ is an open subset then $(0,1)B$ is an open subset of $\mathrm{Mat}_d(\mathbb{R})$. Hence $[0,1]B$ contains an open subset and $m_{\mathrm{SL}_d(\mathbb{R})}(B) > 0$. If B is compact then $[0,1]B$ is compact and hence $m_{\mathrm{SL}_d(\mathbb{R})}(B) < \infty$.

We now verify the bi-invariance. This follows by the following two facts:

- $[0,1](gB) = g([0,1]B)$ and $[0,1](Bg) = ([0,1]B)g$ for every g, B.
- The measure $m_{\mathrm{Mat}_d(\mathbb{R})}$ is bi-invariant for $\mathrm{SL}_d(\mathbb{R})$: Decomposing the elements of $\mathrm{Mat}_d(\mathbb{R})$ into column vectors yields

$$\mathrm{Mat}_d(\mathbb{R}) = \mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d, \text{ and } m_{\mathrm{Mat}_d(\mathbb{R})} = \mathrm{Leb}_{\mathbb{R}^d} \otimes \cdots \otimes \mathrm{Leb}_{\mathbb{R}^d}.$$

  Since the left multiplication by $\mathrm{SL}_d(\mathbb{R})$ preserves each copy of $\mathbb{R}^d$ and its Lebesgue measure, it also preserves $m_{\mathrm{Mat}_d(\mathbb{R})}$. The same holds for the right multiplication, by decomposing $\mathrm{Mat}_d(\mathbb{R})$ into row vectors instead.

□

Exercise 1.7.2 (Exercise 1.9)**.** Show that $\mathrm{SL}_2(\mathbb{Z})gA$ is a divergent trajectory ($A \ni a \mapsto \mathrm{SL}_2(\mathbb{Z})ga$ is proper) if and only if $ga \in \mathrm{SL}_2(\mathbb{Q})$ for some $a \in A$.

Proof. $\Longleftarrow$. Assume without loss of generality that $g \in \mathrm{SL}_2(\mathbb{Q})$. If $\mathrm{SL}_2(\mathbb{Z})gA$ is not divergent then there exists $a_n \to \infty$ such that $\mathrm{SL}_2(\mathbb{Z})ga_n \to \mathrm{SL}_2(\mathbb{Z})g'$ for some $g' \in \mathrm{SL}_2(\mathbb{R})$. Then there exist $\gamma_n \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_n g a_n \to g'$. Since $g \in \mathrm{SL}_2(\mathbb{Q})$, there exists a positive integer $q > 0$ such that $\mathrm{SL}_2(\mathbb{Z})g \in \mathrm{SL}_2(\mathbb{R}) \cap \mathrm{Mat}_2(\frac{1}{q}\mathbb{Z})$. Therefore, we have $\|\gamma_n g a_n\| \gg \frac{1}{q}\|a_n\|$, which contradicts $\gamma_n g a_n \to g'$.

$\Longrightarrow$. [The proof follows Tomanov-Weiss-Witte's notes using Kazhdan-Margulis's approach for general dimension d cases).] We show that if $ga \notin \mathrm{SL}_2(\mathbb{Q})$ for all $a \in A$ then $\mathrm{SL}_2(\mathbb{R})gA$ is not divergent. By Mahler's criterion, it suffices to show that there exists a neighborhood Ω_0 of 0 in $\mathbb{R}^2$ such that for every compact subset $C \subset A$ there exists $a \in A \setminus C$ satisfying $ag.\mathbb{Z}^2 \cap \Omega_0 = \{0\}$.

Lemma 1.7.3

There exists a neighborhood $\Omega \ni 0$ in $\mathbb{R}^2$ and a finite subset $F \subset A, c > 1$ such that for every $g \in \mathrm{SL}_2(\mathbb{R})$, there exists $f \in F$ satisfying

$$\|fv\| \geqslant c\|v\|, \quad \forall v \in g.\mathbb{Z}^2 \cap \Omega.$$

Proof. Since g is unimodular, $\lambda_2(g.\mathbb{Z}^2) \geqslant r$ for some uniform $r > 0$. Let $\Omega = B_r^{\mathbb{R}^2}(0)$. Then $g.\mathbb{Z}^2 \cap \Omega \subset \mathbb{R}v$ for some $v \in \mathbb{R}^2$ with $\|v\|$. For every $v \in \mathbb{R}^2$, there exists $a \in A$ such that $\|av\| > \|v\|$ and hence $\|av'\| > \|v'\|$ for every $v' \approx v$. By the compactness of $\mathbb{S}^1 \subset \mathbb{R}^2$, we can find a such finite subset $F \subset A$ and a uniform $c > 1$. ☐

Let $\Omega_0 \subset \mathbb{R}^2$ be a small ball such that if $v \in \Omega_0$ then $f^{\pm}v \in \Omega$ for every $f \in F$.
By the assumption, we have either $g.\mathbb{Z}^2 \cap \{(*,0)\} = \{0\}$ or $g.\mathbb{Z}^2 \cap \{(0,*)\} = \{0\}$. Therefore, for every compact subsets $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ and $C \subset A$, there exists $a \in A \setminus C$ such that $a.(\Omega_1 \cap g.\mathbb{Z}^2 \setminus \{0\}) \cap \Omega_2 = \varnothing$. We now take $a_0 \in A \setminus C$ which works for $\Omega_1 = C^{-1}.\Omega$ and $\Omega_2 = \Omega_0$. That is, if $v \neq 0 \in C^{-1}.\Omega \cap g.\mathbb{Z}^2$ then $a_0 v \notin \Omega_0$.
If $a_0 g.\mathbb{Z}^2 \cap \Omega_0 \neq \{0\}$ then we are done. Otherwise, we can find the sequence inductively $a_1, a_2 \cdots, \in F \subset A$ such that a_k stretches vectors in $a_{k-1} \cdots a_0 g.\mathbb{Z}^2 \cap \Omega_0$. Let $m > 0$ be the smallest integer such that $a_m \cdots a_1 a_0 g.\mathbb{Z}^2 \cap \Omega_0 = \{0\}$. Then $a_{m-1} \cdots a_1 a_0 g.\mathbb{Z}^2 \cap \Omega_0 \neq \{0\}$ and hence $a_m \cdots a_1 a_0 g.\mathbb{Z}^2 \cap \Omega \neq \{0\}$. We claim that $a_m \cdots a_1 a_0 \notin C$ and we complete the proof. Otherwise, since Ω_0 is a ball and m is the least, we can find $v \neq 0 \in g.\mathbb{Z}^2$ such that $a_k \cdots a_1 a_0 v \in \Omega_0$ for every $k = 0, \cdots, m-1$ and $a_m \cdots a_0 v \in \Omega$. That is, $v \neq 0 \in C^{-1}.\Omega \cap g.\mathbb{Z}^2$ and $a_0 v \in \Omega_0$, which contradicts our choice of a_0.
☐

Exercise 1.7.4 (Exercise 1.10)**.** Show that to any compact A-orbit in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ one can attach a real quadratic number field K such that the length of the orbit is $\log|\xi|$ where $\xi \in \mathcal{O}_K^*$ is a unit in the order $\mathcal{O}_K$ of K. Prove that there are only countably many such orbits.

Proof. If $\mathrm{SL}_2(\mathbb{Z})g$ has a compact A-orbit then there exists $a \in A$ such that $\mathrm{SL}_2(\mathbb{Z})ga = \mathrm{SL}_2(\mathbb{Z})g$. Hence there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $a = g^{-1}\gamma g$. For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the element g is uniquely determined up to the right multiplication by $C_{\mathrm{SL}_d(\mathbb{R})}(A) = \left\langle A, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$. Therefore, there are only countably many such orbits.

For a compact A-orbit, we have

$$\mathrm{SL}_2(\mathbb{Z})gA \cong \mathrm{Stab}_A(\mathrm{SL}_2(\mathbb{Z})g)\backslash A = \langle a \rangle \backslash A,$$

where $a \in A$ satisfying $gag^{-1} = \gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then the length of $\mathrm{SL}_2(\mathbb{Z})gA$ coincides with the covolume of $\langle a \rangle$ in A. Let K be the splitting field of the characteristic polynomial of γ. Then $a = \begin{bmatrix} \xi & \\ & \xi^{-1} \end{bmatrix}$, where $\xi \in \mathcal{O}_K^*$. Hence the length equals $\log|\xi|$. □

Exercise 1.7.5 (Exercise 1.12)**.** Show that $\mathrm{SL}_2(\mathbb{Z})gU^-$ is compact if and only if $g(\infty) \subset \mathbb{Q} \cup \{\infty\}$, where $U^- = \left\{ \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \right\}$. Moreover, show that if $\mathrm{SL}_2(\mathbb{Z})gU^-$ is compact then $\mathrm{SL}_2(\mathbb{Z})gU^- = \mathrm{SL}_2(\mathbb{Z})aU^-$ for some $a \in A$.

Proof. By the decomposition $\mathrm{SL}_2(\mathbb{R}) = PU^+AU^-$ and $U^- = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\infty)$, where P is a permutation matrix, we can assume without loss of generality that $g = \begin{bmatrix} x & \\ y & x^{-1} \end{bmatrix}$. If $\mathrm{SL}_2(\mathbb{Z})gU^-$ is compact then there exists $u = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \in U^-$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma = gug^{-1}$. Hence

$$gug^{-1} = \begin{bmatrix} x & xs \\ y & ys + x^{-1} \end{bmatrix} \begin{bmatrix} x^{-1} & \\ -y & x \end{bmatrix} = \begin{bmatrix} 1 - xys & x^2 s \\ -y^2 s & 1 + xys \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Letting $p = x^2 s$ and $q = \frac{y}{x}$, we have $p \in \mathbb{Z}$ and $1 - pq \in \mathbb{Z}$. Therefore, $q \in \mathbb{Q}$ and hence $g(\infty) = q^{-1} \in \mathbb{Q} \cup \{\infty\}$.

In oder to show that $SL_2(\mathbb{Z})gU^- = \mathrm{SL}_2(\mathbb{Z})aU^-$, we aim to find $u = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix}$ and $a = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix}$ such that $gua \in \mathrm{SL}_2(\mathbb{Z})$. We have

$$gua = \begin{bmatrix} tx & t^{-1}xs \\ ty & t^{-1}(ys + x^{-1}) \end{bmatrix}.$$

Since $x/y \in \mathbb{Q}$, there exists $t \in \mathbb{R}$ such that both tx, ty are integers. Moreover, we can take t such that tx, ty are coprime integers. Then there exists integers m, n such that $(tx)m - tyn = 1$. Let $s \in \mathbb{R}$ such that $t^{-1}xs = n$. Then $gua \in \mathrm{SL}_2(\mathbb{Z})$. □

Exercise 1.7.6 (Exercise 1.36)**.** Let $G \subset \mathrm{SL}_d(\mathbb{R})$ be a closed subgroup. Assume that $\Gamma = G \cap \mathrm{SL}_d(\mathbb{Z})$ is a non-uniform lattice in G. Show that Γ contains a unipotent element.

Proof. By the abstract divergence criterion, there exists $\gamma_n \in \Gamma$ and $g_n \in G$ such that $g_n \gamma_n g_n^{-1} \to \mathrm{id}$. Since $\gamma_n \in \mathrm{SL}_d(\mathbb{Z})$, the characteristic polynomial P_n of $g_n \gamma_n g_n^{-1}$ is defined over $\mathbb{Z}$. Note that the coefficients of the characteristic polynomial vary continuously with the matrix. Therefore, for large enough n, we have $P_n = (x-1)^d$ and hence $\gamma_n \in \Gamma$ is unipotent. □

> **Exercise 1.7.7** (Exercise 1.37)**.** Let $\Gamma < G$ be a uniform lattice in a connected σ-compact locally compact group G equipped with a proper left-invariant metric. Show that Γ is finitely generated.

Proof. Let F be a relatively compact fundamental domain of $\Gamma\backslash G$ containing id. Let $W \subset G$ be a compact subset that generates G as a semigroup. Let

$$S := \Gamma \cap FWF^{-1},$$

which is finite since $FWF^{-1} \subset G$ is compact. Then for every $w \in W, f \in F$, there exists $\gamma \in S$ and $f' \in F$ such that $fw = \gamma f'$. Now we verify that S is a generating set of Γ. For every $\gamma \in \Gamma \subset G$, write $\gamma = w_1 \cdots w_n$ where $w_1, \cdots, w_n \in W$. We then inductively define $\gamma_1, \cdots, \gamma_n \in S$ and $f_1, \cdots, f_n \in F$. Let γ_1, f_1 be elements such that $w_1 = \gamma_1 f_1$. For every $k \geqslant 2$, let γ_k, f_k be elements such that $f_{k-1} w_k = \gamma_k f_k$. Then we have

$$\gamma = w_1 \cdots w_n = \gamma_1 f_1 w_2 \cdots w_n = \gamma_1 \gamma_2 f_2 w_3 \cdots w_n = \cdots = \gamma_1 \cdots \gamma_n f_n.$$

Note that $f_n \in \Gamma \cap F$ and id $\in F$. We have $f_n = \mathrm{id}$ and $\gamma = \gamma_1 \cdots \gamma_n$. Therefore, Γ is finitely generated by S. $\square$

2 Ergodicity and mixing (2025 Autumn)

§2.1 Oct 3: Unitary representations & Lie groups and Lie algebras

In general, mixing property of a group action by G will imply the ergodicity of all $g \in G$ that generate unbounded subgroups. There are two main goals for the ergodic theory of group actions in homogeneous settings:

- Understand ergodicity of subgroups $H < G$ acting on X/Γ with respect to m_X;
- Ergodicity implies mixing in certain cases.

Let X be a locally compact and σ-compact metric space. Let G be a locally compact and σ-compact group. Suppose G acts continuously on X preserving a locally finite measure. Then the map

$$\pi_g : L^2(X, \mu) \to L^2(X, \mu), \quad f \mapsto \left(x \mapsto f(g^{-1})x)\right)$$

defines a unitary representation of G on $L^2(X, \mu)$. Here, a **unitary representation** of G on a Hilbert space $\mathcal{H}$ is a homomorphism π satisfying:

- $\pi : G \to \mathcal{U}(\mathcal{H})$, the space of unitary operators on $\mathcal{H}$;
- $\|\pi_g v\| = \|v\|$;
- strong continuity: for every $v \in \mathcal{H}$ the map $g \in G \to \pi_g v \in \mathcal{H}$ is continuous.

Now we verify that the action of G on $L^2(X, \mu)$ defined above gives a unitary representation.

Proof. We have

$$\|\pi_g f\|_2^2 = \int |f(g^{-1}x)|^2 \, \mathrm{d}\mu = \int |f|^2 \, \mathrm{d}\mu = \|f\|_2^2,$$

$$\pi_{g_1}(\pi_{g_2} f)(x) = (\pi_{g_2} f)(g_1^{-1}x) = f(g_2^{-1}g_1^{-1}x) = (\pi_{g_1 g_2} f)(x).$$

Now we establish the strong continuity. Let us consider first $f \in C_c(X)$ and a sequence $g_n \to g$. Then $f(g_n^{-1}x) \to f(g^{-1}x)$. By the dominated convergence theorem,

$$\|\pi_{g_n} f - \pi_g f\|_2^2 = \int |f(g_n^{-1}x) - f(g^{-1}x)|^2 \, \mathrm{d}\mu(x) \to 0.$$

Let $f \in L^2(X, \mu)$ be a general measurable function. Let $\varepsilon > 0$. Then there exists $f_0 \in C_c(X)$ such that $\|f - f_0\| < \varepsilon$. It follows that

$$\|\pi_{g_n} f - \pi_g f\|_2 \leqslant \|\pi_{g_n} f - \pi_{g_n} f_0\|_2 + \|\pi_{g_n} f_0 - \pi_g f_0\|_2 + \|\pi_g f_0 - \pi_g f\|_2 \leqslant 3\varepsilon$$

for n large enough. □

Lie groups and Lie algebras. For us it is enough to consider closed linear subgroups $G \subset \mathrm{SL}_d(\mathbb{R})$. In this context, the exponential map is

$$\exp : m \in \mathrm{Mat}_d(\mathbb{R}) \mapsto \sum_{i=0}^{\infty} \frac{1}{i!} m^i \in \mathrm{GL}_d(\mathbb{R}).$$

The adjoint representation is

$$\mathrm{Ad}_g : m \mapsto gmg^{-1}.$$

Therefore we have $\exp(\mathrm{Ad}_g m) = g\exp(m)g^{-1}$. By the Jordan normal form of a matrix, we can show that $\det\exp(m) = \exp(\operatorname{tr} m)$. This is the reason why $\mathfrak{gl}_d(\mathbb{R}) = \mathrm{Mat}_d(\mathbb{R})$ is the Lie algebra of $\mathrm{GL}_d(\mathbb{R})$ and $\mathfrak{sl}_d(\mathbb{R}) = \{m \in \mathrm{Mat}_d(\mathbb{R}) : \operatorname{tr} m = 0\}$ is the Lie algebra of $\mathrm{SL}_d(\mathbb{R})$.

In the matrix case, the Lie bracket is given by

$$[u, v] = uv - vu, \quad \forall u, v \in \mathfrak{sl}_d(\mathbb{R}).$$

We have the Jacobi identity

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

The reason to study Lie bracket is to consider the derivative of conjugations and adjoint representations. The conjugation by g gives the map $h \in H \mapsto ghg^{-1}$. Taking the derivative of h, we obtain the adjoint representation $\mathrm{Ad}_g : v \mapsto gvg^{-1}$. Then taking the derivative on g, we obtain the Lie bracket $[u, v] = uv - vu$.