ODE: Qualitative Theory
(Spring 2022, Shaobo Gan)

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Contents

1 Basic Concepts

§1.1 Basic notions and results

Assume $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, x) \mapsto f(t, x)$ continuous, consider the **equation** (or **system**)

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x).$$

A differentiable function $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ is said to be a **solution** (or **solution curve**), if for every $t \in (a, b)$,

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = f(t, \gamma(t)).$$

The **graph** of γ is

$$\{(t, \gamma(t)) : t \in (a, b)\} \subset \mathbb{R} \times \mathbb{R}^n.$$

For $t_0 \in (a, b)$, let $x_0 = \gamma(t_0)$, then γ is called the solution of the **initial value problem**

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

The initial value problem has a unique solution: Let $\gamma_i : (a_i, b_i) \to \mathbb{R}^n$ be two solutions of the initial value problem. Then there exists $\delta > 0$, $(t_0 - \delta, t_0 + \delta) \subset (a_1, b_1) \cap (a_2, b_2)$, such that $\gamma_1(t) = \gamma_2(t), \forall t \in (t_0 - \delta, t_0 + \delta)$,

Theorem 1.1.1 (Existence and Uniqueness Theorem)
$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, x)$ continuous, given $t_0 \in \mathbb{R}^n, x_0 \in \mathbb{R}^n, a > 0, b > 0$, consider the region

$$R = R(t_0, x_0, a, b) = \{(t, x) : |t - t_0| \leqslant a, |x - x_0| \leqslant b\}.$$

If f is Lipchitz in x on R, i.e. $\exists L > 0, \forall (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leqslant L|x_1 - x_2|,$$

then the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

has a unique solution on $[t_0 - h, t_0 + h]$, where $h = \min\left\{a, \frac{b}{M}\right\}$, $M = \max_{(t,x)\in R} |f(t, x)|$.

Remark 1.1.2 — The solution is denoted as $\varphi(t; t_0, x_0)$.

Corollary 1.1.3
When $f \in C^1$, the existence and uniqueness theorem holds.

Denotes the **maximal interval** of $\begin{cases}\dot{x}=f(t,x)\\ x(t_0)=x_0\end{cases}$ as $I(t_0,x_0)$, it is an open interval.

Corollary 1.1.4
Assume $f\in C^1$ and $|f(t,x)|\leqslant A(t)|x|+B(t)$, then the maximal interval of the initial value problem is $(-\infty,+\infty)$.

§1.2 Flows

Now we consider the **autonomous equation**

$$\dot{x}=f(x).$$

$\mathbb{R}^n$ is called the **phase space** and $\mathbb{R}\times\mathbb{R}^n$ is called the **generalized phase space**.

The solution of the initial value problem $\begin{cases}\dot{x}=f(x)\\ x(0)=x_0\end{cases}$ is denoted as $\varphi(t,x_0)$, the set

$$\mathrm{Orb}(x_0):=\{\varphi(t,x_0):t\in I(x_0)\}\subset\mathbb{R}^n$$

is called the **orbit** pass by x_0.

Corollary 1.2.1 (Continuous Dependence on the Initial Value)
Assume $f\in C^1$, then $U=\{(t,x):t\in I(x)\}$ is open and $\varphi:U\to\mathbb{R}^n,(t,x)\mapsto\varphi(t,x)$ is continuous.

Theorem 1.2.2
$f(x)\in C^1$, then:

1. $\varphi_0(x)=x$ for every $x\in\mathbb{R}^n$.
2. $\varphi(t,\varphi(s,x))=\varphi(t+s,x)$ for every $s\in I(x),t\in I(\varphi(s,x))$.

Definition 1.2.3. $\psi:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, continuous, is said to be a **(continuous) flow** if:

(i) $\psi(0,x)=x$,

(ii) $\psi(t,\psi(s,x))=\psi(t+s,x)$.

Remark 1.2.4 — The solution of an autonomous equation is a **local flow.**

Corollary 1.2.5
Let $\psi:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be a flow, then $\psi_t:=\psi(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ are homeomorphisms.

Remark 1.2.6 — Consider the group of self-homeomorphisms of $\mathbb{R}^n$, denotes as $\mathrm{Homeo}(\mathbb{R}^n)$, then $\psi : \mathbb{R} \to \mathbb{R}^n$ is a group homomorphism. More generally, we can consider $G \to \mathrm{Homeo}(\mathbb{R}^n)$ for some group G.

Proposition 1.2.7

Assume f is a C^1 vector field, then the orbits of the flow generated by f are either coincide or disjoint.

$\bigcup_{x\in\mathbb{R}^n} \mathrm{Orb}(x)$ forms a partition of $\mathbb{R}^n$, is called the **orbit space**. For each orbit, orient it to indicate the direction of motion, the family of the oriented orbits $\varphi(t,x)/f(x)$ is called the **phase portrait**.

A point $x_0 \in \mathbb{R}^n$ with $f(x_0) = 0$ is called a **critical point** (or a **singularity**, **equilibrium**). The orbit $\mathrm{Orb}(x_0)$ is a single point $\{x_0\}$.

Example 1.2.8

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t, x_0) = x_0 e^t$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$.

Example 1.2.9

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2 \\ x(0) = x_0 \end{cases},$$

the solutions are $\varphi(t, x_0) = \frac{x_0}{1-x_0 t}$. There are three orbits $\mathbb{R}_+, \mathbb{R}_-, \{0\}$. But the phase portrait is different from the former examples, because the orientations on $\mathbb{R}_-$ are different.

Theorem 1.2.10

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, $\beta(x) : \mathbb{R}^n \to \mathbb{R} \in C^1$ and $\beta(x) > 0$. Then the equations $\dot{x} = f(x)$ and $\dot{x} = \beta(x) f(x)$ have the same phase portraits.

Proof. $\varphi : I \to \mathbb{R}^n$ a solution of f. Find a C^1 diffeomorphism $h : J \to I$ such that $\varphi \circ h$ is the solution of $\dot{x} = \beta(x) f(x)$. It suffices that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=h(s)} \varphi(t) \cdot \frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) f(\varphi \circ h(s)),$$

i.e. $\frac{\mathrm{d}h(s)}{\mathrm{d}s} = \beta(\varphi \circ h(s)) > 0$, it is an initial value problem. It shows that the maximal solution curve of f is contain in some solution curve of βf. □

Theorem 1.2.11 (Differentiable Dependence on the Initial Value)
Assume $f \in C^1$, it generates the flow ϕ_t, then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

Exercise 1.2.12.
$$\frac{\partial}{\partial t}\frac{\partial \phi(t,x)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi(t,x)}{\partial t}.$$

Let $\Phi(t,x) = \Phi_t(x) = \frac{\partial \phi(t,x)}{\partial x}$, then Φ is the solution of the equation

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = A(t)y(t), A(t) = \mathrm{D}f(\phi_t(x)) \\ y(0) = \mathrm{Id} \end{cases}.$$

The equation is called the **variation equation** of $f(x)$ along $\phi_t(x)$.

Lemma 1.2.13
$f \in C^1$, $\Phi(t,x)$, then
$$\Phi_t(\phi_s(x))\Phi_s(x) = \Phi_{t+s}(x).$$

Remark 1.2.14 — This property is called the **cocycle** condition.

We already know that ϕ_t are self-homeomorphisms of $\mathbb{R}^n$, and lemma 1.2.13 shows that the differential is invertible, hence ϕ_t are diffeomorphisms. Define

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R}^n \times \mathbb{R}^n \\ (t,x,v) &\mapsto (\phi_t(x), \Phi_t(x)v). \end{aligned}$$

Proposition 1.2.15
$\Phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a flow.

Remark 1.2.16 — We call Φ_t a skew product flow of ϕ_t.

Theorem 1.2.17
$$\Phi_t(x)f(x) = f(\phi_t(x)).$$

If ψ is a C^1 flow, let
$$g(x) = \left.\frac{\partial \psi(t,x)}{\partial t}\right|_{t=0},$$
then $\psi(t,x_0)$ solve the initial value problem $\begin{cases} \dot{x} = g(x) \\ x(0) = x_0 \end{cases}$. Because

$$\frac{\partial \psi(t,x_0)}{\partial t} = \left.\frac{\partial \psi(t+s,x_0)}{\partial s}\right|_{s=0} = \left.\frac{\partial \psi(s,\psi(t,x_0))}{\partial s}\right|_{s=0} = g(\psi(t,x_0)).$$

§1.3 Equations on manifolds

Let M be a closed smooth manifold, X is a C^1 vector field on M. Then X is bounded, hence the maximal intervals are $(-\infty, +\infty)$. Consider the equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = X(x) \\ x(0) = x_0 \end{cases},$$

then the solution $\varphi(t, x)$ generates a flow.

2 Linear Systems

§2.1 Plane linear sigularities

Consider the equation

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

It is said to be a **plane linear system** if f, g both linear functions of x, y, i.e.

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad a,b,c,d \in \mathbb{R}.$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then $(0,0)$ is the only singularity. In this case, we call O an **elementary singularity**.

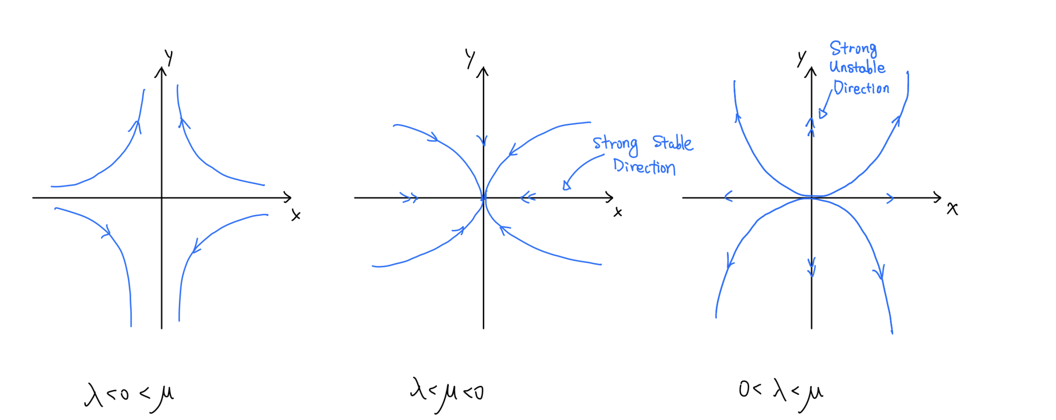
Consider the Jordan form of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, there are four cases:

I. Two different real eigenvalues: $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\mu t} \end{bmatrix}$.

   i. $\lambda < 0 < \mu$: the origin is called a **saddle point**.

   ii. $\lambda < \mu < 0$: the origin is called a **stable node**.

   iii. $0 < \lambda < \mu$: the origin is called an **unstable node**.



II. Conjugated imaginary eigenvalues: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\beta > 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

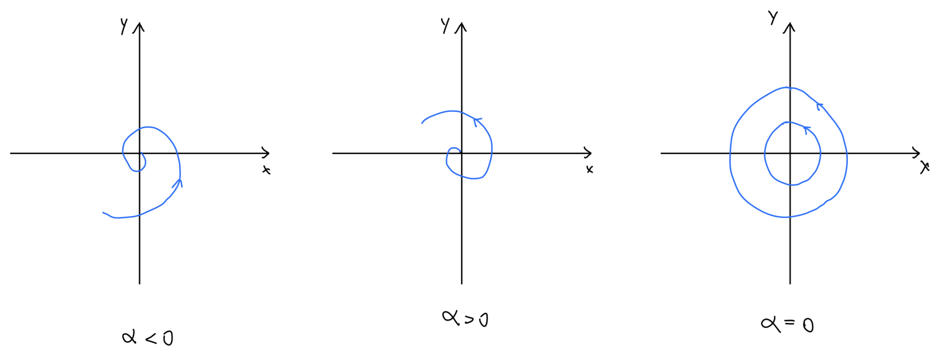
   If we consider this equation in the polar coordinates, it turns $\begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta \end{cases}$.

   i. $\alpha < 0$, the origin is called a **stable focus**.

   ii. $\alpha > 0$, the origin is called an **unstable focus**.

   iii. $\alpha = 0$, the origin is called a **center**.

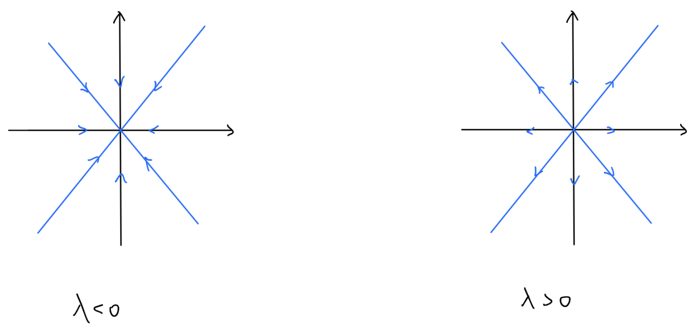
Definition 2.1.1. φ_t a flow. If p is not a singularity and $\exists T > 0$, such that $\varphi_T(p) = p$. Then p is called a **periodic point**, $\mathrm{Orb}(p)$ is called a **periodic orbit**. If p is a periodic point, the smallest $T > 0$ is called the **minimum positive period**.



III. Two same real eigenvalues, diagonalizable: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{\lambda t} \\ y_0 e^{\lambda t} \end{bmatrix}$.

i. $\lambda < 0$, the origin is called a **stable critical node**.

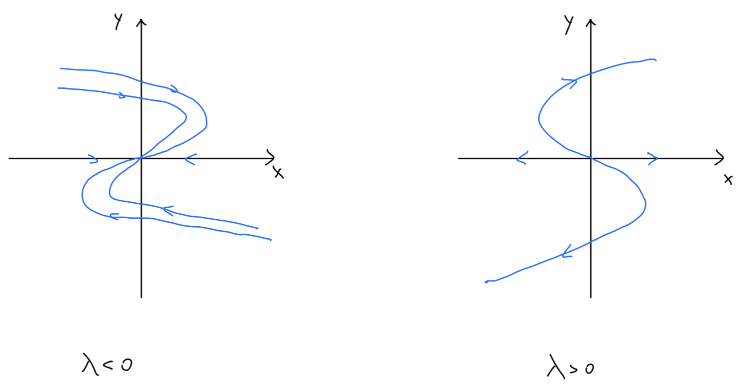
ii. $\lambda > 0$, the origin is called an **unstable critical node**.



IV. Two same real eigenvalues, not diagonalizable: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t}(x_0 + t y_0) \\ e^{\lambda t} \end{bmatrix} y_0$, or $x(t) = \frac{x_0}{y_0} y(t) + \frac{y(t)}{\lambda} \ln \frac{y(t)}{y_0}$.

i. $\lambda < 0$, the origin is called a **stable unidirectional node**.

ii. $\lambda > 0$, the origin is called an **unstable unidirectional node**.



Exercise 2.1.2. Draw the phase portraits of non-elementary plane systems (i.e. the determinant is 0).

§2.2 Topological conjugacies between linear systems

Definition 2.2.1. Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphisms. f and g are said to be **topologically conjugate** if there exists $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ f = g \circ h$.

Remark 2.2.2 — Conjugacy is a equivalence relation.

Definition 2.2.3. Let $\varphi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ be two flows, we call φ_t and ψ_t are **conjugate** if there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ \varphi_t = \psi_t \circ h$. Let X, Y be two C^1 vector fields on $\mathbb{R}^n$, we call X, Y are **conjugate** if the flows generated by them, respectively, are conjugate.

Example 2.2.4

$A, B \in M(n, \mathbb{R})$ are similar, then $\dot{x} = Ax$ and $\dot{y} = By$ are conjugate.

$f, g : \mathbb{R}^n \to \mathbb{R}^n$ C^1 vector fields, generate flows ϕ_t, ψ_t. Let $x = h(y) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 diffeomorphism gives the conjugate, i.e., $h\psi_t(y) = \phi_t h(y)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} h\psi_t(y)\bigg|_{t=0} = f(h(y)) \implies \mathrm{D}h(y)g(y) = f(h(y)).$$

If there exists a C^1 diffeomorphism conjugate $e^{Bt}y$ to $e^{At}x$ via $x = h(y)$, i.e. $h(e^{Bt}y) = e^{At}h(y)$. Then $\mathrm{D}h_0 e^{Bt} = e^{At}\mathrm{D}h_0$, hence $\mathrm{D}h_0 B = A\mathrm{D}h_0$. It shows that C^1 conjugate generically not hold even if topologically conjugate.

Proposition 2.2.5

Assume f, g C^1 vector fields generate ϕ_t, ψ_t, let h be a conjugate between ϕ_t and ψ_t. Then:

1. $h(\mathrm{Orb}(x, \phi)) = \mathrm{Orb}(hx, \psi)$.
2. h maps the singularities of f to the singularities of g.
3. h maps the periodic orbits of f to the periodic orbits of g. Moreover, it preserves the minimum positive period.

Example 2.2.6

$\dot{x} = -2x$ and $\dot{y} = -4y$ are conjugate.

Let $h : \mathbb{R} \to \mathbb{R}$, $h(0) = 0$. Take $x_0, y_0 > 0$, let $h(x_0) = y_0$, then $h(e^{-2t}x_0) = e^{-4t}y_0$ or $h(x) = \left(\frac{x}{x_0}\right)^2 y_0$. The construction for the negative part is similar.

Exercise 2.2.7. $\lambda\mu \neq 0$, show that $\dot{x} = \lambda x$ is conjugate to $\dot{y} = \mu y$ if and only if $\lambda\mu > 0$.

Proposition 2.2.8

$\phi_t^i, \psi_t^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ are topologically conjugate by h_i, $i = 1, 2$. Then $\phi_t^1 \times \phi_t^2$ and $\psi_t^1 \times \psi_t^2$ are topologically conjugate by $h_1 \times h_2$.

Example 2.2.9

$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$ are conjugate.

Proof. $\phi_t(x,y) = e^{-t}(x,y)$ and $\psi_t(x,y) = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. For every $(x,y) \neq (0,0)$, there exists unique $t = t(x,y)$ such that $\phi_t(x,y) \in \mathbb{S}^1$. Let $h(x,y) := \psi_{-t}\phi_t(x,y)$, where $t = t(x,y)$, then h gives the conjugate. □

Exercise 2.2.10. Show that $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ and $\begin{cases} \dot{x} = -x - y \\ \dot{y} = -y \end{cases}$ are conjugate.

Classification of elementary plane linear systems (under conjugacies):

(I) Stable: node, critical node, unidirectional node, focus.

(II) Unstable: node, critical node, unidirectional node, focus.

(III) Saddle point.

(IV) Center.

Definition 2.2.11. The linear system $\dot{x} = Ax$ in $\mathbb{R}^n$ is called **hyperbolic** if the real parts of eigenvalues of A are nonzero. The **(stable) index** of A is the number of eigenvalues with negative real parts, denoted by $\operatorname{Ind} A$.

Theorem 2.2.12

Two plane hyperbolic linear system $\dot{x} = Ax, \dot{y} = By$ are topologically conjugate if and only if $\operatorname{Ind} A = \operatorname{Ind} B$.

Proof. "$\Longrightarrow$": Let $W_A^s = \{x : e^{tA}x \to 0, t \to \infty\}$, $W_B^s = \{x : e^{tB}x \to 0, t \to \infty\}$, then h and h^{-1} preserves the stable manifolds. Then $\operatorname{Ind} A = \dim W_A^s = \dim W_B^s = \operatorname{Ind} B$. □

Example 2.2.13

Consider $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ and $\begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$ with the same phase portraits are not topologically conjugate. Because the topologically conjugate preserves the minimum positive orbits.

Definition 2.2.14. $\phi_t, \psi_t : \mathbb{R}^n \to \mathbb{R}^n$ flows, h is a homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ maps the orbit of ϕ to the orbit of ψ preserves the orientation. Then ϕ and ψ are called **topologically equivalent** or **flow equivalent**.

Theorem 2.2.15 (Grobman-Hartman)

If x_0 is a hyperbolic singularity of $f(x)$, then the flows generated by $\dot{x} = f(x)$ and $\dot{y} = Ay$ where $y = \mathrm{D}f(x_0)$ are topologically conjugate near 0.

§2.3 Non-autonomous linear systems

$A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, consider the equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n$$

a non-autonomous linear system.

Theorem 2.3.1

The followings hold:

1. The initial problem of the equation exist the unique solution.
2. The maximal interval of any solution is $(-\infty, \infty)$.
3. All solutions of the equation form an n-dimensional linear space S.

Theorem 2.3.2 (Liouville's Formular)

Assume $X(t)$ is a solution of $\dot{x} = A(t)x$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \det X(t) = \operatorname{tr} A(t) \det X(t),$$

hence $\det X(t) = \det X(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s) \mathrm{d}s$.

Let $X_1(t), X_2(t), \cdots, X_n(t)$ be a basis of S, let

$$X(t) := [X_1(t), X_2(t), \cdots, X_n(t)] \in \mathrm{GL}(n, \mathbb{R}),$$

it called a **fundamental solution** of the equation. The fundamental solution of

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = A(t)X \\ X(t_0) = I_n \in \mathrm{GL}(n, \mathbb{R}) \end{cases}$$

is called the **standard fundamental solution**.

If $X(t), Y(t)$ are two fundamental solutions, suppose $Y(0) = X(0)C$, then

$$\frac{\mathrm{d}X(t)C}{\mathrm{d}t} = \frac{\mathrm{d}X(t)}{\mathrm{d}t}C = A(t)X(t)C,$$

is a non-degenerate solution of $\frac{\mathrm{d}X}{\mathrm{d}t} = AX$. By the uniqueness, we get $Y(t) = X(t)C$.

Example 2.3.3

$A(t) \equiv A$, the fundamental solution of $\dot{x} = Ax$ is

$$e^{tA} = \mathrm{Id} + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{k!}t^kA^k + \cdots.$$

Example 2.3.4

$\dot{x} = f(x), x \in \mathbb{R}^n$, where $f \in C^1$, generates the flow $\varphi_t(x)$. Consider $\Phi_t(x) = \frac{\partial}{\partial t}\varphi_t(x)$ and the variation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x) = \mathrm{D}f(\varphi_t(x))\Phi_t(x).$$

Given $x \in \mathbb{R}^n$, let $A(t) := \mathrm{D}f(\varphi_t(x))$, then $\Phi_t(x)$ is the standard fundamental solution ($t_0 = 0$) of $\dot{x} = A(t)x$. Consider two special types of orbits:

- x is a singularity, denoted by σ. Then $\varphi_t(\sigma) = \sigma$, $\dot{x} = Ax$ where $A = Df(\sigma)$.
- x is a periodic point, denoted by p, the minimum period $T > 0$. Then A is T-periodic.

§2.4 Periodic linear systems

Definition 2.4.1. The equation $\dot{x} = A(t)x$ satisfies $A(t+T) = A(t)$ for some $T > 0$ is called a **periodic linear systems**.

Theorem 2.4.2 (Floquet)

Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution, then $X(t+T)$ is a fundamental solution, i.e. $\exists C \in \mathrm{GL}(n, \mathbb{R})$ such that $X(t+T) = X(t)C$. Moreover, there exists a T-periodic map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ and a constant matrix $B \in M(n, \mathbb{C})$ such that $X(t) = P(t)e^{tB}$.

Lemma 2.4.3

$\forall C \in \mathrm{GL}(n, \mathbb{R}), \exists B \in M(n, \mathbb{C})$ such that $C = e^B$.

Proof. It suffices to show for Jordan block. This follows by the matrix series

$$\ln(I + N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N^k$$

is convergence for nilpotent matrix N. ☐

Lemma 2.4.4

$\forall C \in \mathrm{GL}(n, \mathbb{R}), \exists B \in M(n, \mathbb{R})$ such that $C^2 = e^B$.

Proof. Note that the Jordan block of C^2 is either:

(i) $\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & & \lambda \end{bmatrix}$, where $\lambda > 0$, or

(ii) $\begin{bmatrix} J & I_2 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & J & I_2 \\ 0 & \cdots & & J \end{bmatrix}$, where $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, a, b \in \mathbb{R}, b > 0.$

And $J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ have a real matrix logarithm because $\left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\} \cong \mathbb{C} = \{a + bi\}$. □

Theorem 2.4.5 (Real Form of Floquet Theorem)
Assume $\dot{x} = A(t)x$ is a T-periodic linear system, if X is a fundamental solution. Then there exists a **$2T$-periodic** map $P : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ and a constant matrix $B \in M(n, \mathbb{R})$ such that $X(t) = P(t)e^{tB}$.

Example 2.4.6 ($2T$ is necessary)
Let $\Phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t \right) \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t \right)$. Let

$$A(t) = \dot{\Phi}(t)\Phi(t)^{-1} = \begin{bmatrix} -\cos t \sin t & -\sin^2 t \\ \cos^2 t & \cos t \sin t \end{bmatrix},$$

then $A(t)$ is π-periodic. Then $\Phi(t)$ is a standard fundamental solution of $\dot{x} = A(t)x$, hence $\exists \pi$-periodic $P(t)$ and B such that $\Phi(t) = P(t)e^{tB}$. Then $e^{\pi B} = \begin{bmatrix} -1 & -\pi \\ 0 & -1 \end{bmatrix}$, there is no real matrix B satisfying this equation.

Definition 2.4.7. In Floquet theorem, $X(t+T) = X(t)C$. We call C is a **monodromy matrix**. The eigenvalues of C are called **Floquet multipliers**. If ρ is a Floquet multiplier with $\rho = e^{\lambda T}$, then λ is called a **Floquet exponent**.

Corollary 2.4.8
Consider a T-periodic linear system $\dot{x} = A(t)x$. Then there exists a linear transformation (non-autonomous) $x = P(t)y$ such that $\dot{y} = By$.

Proof. Let $X(t) = P(t)e^{tB}$ be a fundamental solution, then

$$AX = \dot{X} \implies \dot{P}e^{tB} + PBe^{tB} = APe^{tB},$$

hence $\dot{P} + PB = AP$. Then $APy = \frac{\mathrm{d}}{\mathrm{d}t}(Py) = \dot{P}y + P\dot{y}$, hence $\dot{y} = By$. □

Remark 2.4.9 — This type of equation is called reducible, which means after some reduction, the equation can become independent with time t.

Corollary 2.4.10
Let λ be a Floquet multiplier of $\dot{x} = A(t)x$. Then there exists a T-periodic function $p(t)$ such that $e^{\lambda t}p(t)$ is a solution of the equation $\dot{x} = A(t)x$.

Proof. $e^{\lambda T}$ is an eigenvalue of C, then $\exists x_0$ such that $Cx_0 = e^{\lambda T}x_0$. Then $X(t)x_0$ is a solution. Let $p(t) = e^{-\lambda t}X(t)x_0$ is T-periodic and $e^{\lambda t}p(t)$ is a solution. □

Corollary 2.4.11

The equation admits a nonzero T-periodic solution if and only if 1 is a Floquet multiplier.

Corollary 2.4.12

Assume $\rho_1, \rho_2, \cdots, \rho_n$ are all Floquet multipliers of $\dot{x} = A(t)x$, then

$$\rho_1\rho_2\cdots\rho_n = \det\Phi(T) = \exp\int_0^T \operatorname{tr} A(t)\,\mathrm{d}t.$$

Example 2.4.13

The equation $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos^2 t & \frac{1}{2}\sin 2t - 1 \\ \frac{1}{2}\sin 2t + 1 & \sin^2 t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ has an unbounded solution. Because the product of two multipliers is $\exp\int_0^\pi 1\,\mathrm{d}t = e^\pi > 1$.

Consider **Hill equation**

$$\ddot{x} + p(t)x = 0,$$

where $p(t)$ is π-periodic. This is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x \end{cases},$$

then $\rho_1\rho_2 = \exp\int_0^\pi \operatorname{tr} A(t)\,\mathrm{d}t = 0$, where $A(t) = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}$.

Lemma 2.4.14

If ρ_1, ρ_2 both are imaginary numbers, then every solution of Hill equation is bounded.

Proof. Because ρ_1, ρ_2 are conjugate imaginary numbers, hence $\Phi(\pi)$ is similar to a rotation. Then $\Phi(\pi)^n$ is bounded independent of n and $\Phi(s)$ is bounded for $s \in [0, \pi]$. □

Definition 2.4.15. A particular Hill equation with $p(t) = a + \varepsilon\cos 2t$ is called **Mathieu equation**.

Exercise 2.4.16. Consider Mathieu equation

$$\ddot{x} + (a + \varepsilon\cos 2t)x = 0.$$

(1) $U = \{(a, \varepsilon) \in [0, 10] \times [-1, 1] : \text{ every solution is bounded}\}$. Draw the figure of U by some calculation.

(2) Guess some conclusions by the figure of U.

Example 2.4.17

Let $p(t)$ be a π-periodic continuous function satisfying

(i) $p(t) \not\equiv 0$.

(ii) $\int_0^\pi p(t)\mathrm{d}t \geqslant 0$.

(iii) $\pi \int_0^\pi |p(t)|\mathrm{d}t \leqslant 4$.

Then every solution of $\ddot{x} + p(t)x = 0$ is bounded.

Proof. If Floquet multipliers are conjugate imaginary numbers, the statement follows. Otherwise there is a real Floquet multiplier $\rho \neq 0$. There is a solution $x(t) \not\equiv 0$ such that $x(t+T) = \rho x(t)$. If $x(t)$ has no zeros, assume $x(t) > 0$, we have $\frac{\dot{x}}{x}(\pi) = \frac{\dot{x}}{x}(0)$. Note that

$$0 = \frac{\ddot{x}}{x} + p(t) = \left(\frac{\dot{x}}{x}\right)' + \left(\frac{\dot{x}}{x}\right)^2 + p(t) = 0,$$

take the integral and we get a contradiction. Then there must be some zeros, let a, b be two successive zeros, WLOG, $0 < a < b < \pi$. Assume $x(t) > 0$ in (a, b) and $x(c)$ takes the maximum. Then $\exists \alpha \in (a, c), \beta \in (c, b)$ such that $\dot{x}(\alpha) = \frac{x(c)}{c-a}, \dot{x}(\beta) = \frac{-x(c)}{b-c}$. We have

$$\frac{4}{\pi} \geqslant \int_0^\pi |p(t)|\mathrm{d}t > \int_a^b \left|\frac{\ddot{x}}{x}(t)\right| \mathrm{d}t \geqslant \frac{\int_\alpha^\beta |\ddot{x}(t)|\mathrm{d}t}{x(c)} \geqslant \frac{1}{c-a} + \frac{1}{b-c} \geqslant \frac{4}{a-b},$$

the identity holds if and only if $x \equiv 0$, contradiction. $\square$

Back to Mathieu equation, consider

$$\ddot{x} + (\omega^2 + \varepsilon \cos 2t)x = 0, \quad \omega > 0, \varepsilon < \omega^2.$$

We apply the conclusion of the example, for $\omega < \frac{2}{\pi}$,

$$\int_0^\pi (\omega^2 + \varepsilon \cos 2t)\mathrm{d}t = \omega^2 \pi \leqslant \frac{4}{\pi}.$$

Consider $\varepsilon = 0$, then

$$\Phi(t) = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}$$

is a standard fundamental solution. The monodromy matrix for (ω, ε) where $\omega > 0$ is a perturbation of

$$C = \Phi(\pi) = \begin{bmatrix} \cos \omega\pi & \frac{1}{\omega} \sin \omega\pi \\ -\omega \sin \omega\pi & \cos \omega\pi \end{bmatrix}.$$

Note that $|\operatorname{tr} \Phi(\pi)| = |2 \cos \omega\pi| < 2$ for $\omega \notin \mathbb{Z}$. Then there is a small neighborhood U of $(\omega, 0)$ such that every solution is bounded.

Definition 2.4.18. Let $A : \mathbb{R} \to M(n, \mathbb{R})$ continuous, bounded, assume that

$$\sup \{|A(t)| : t \in \mathbb{R}\} < \infty.$$

Let $\Phi(t)$ be a standard fundamental solution of the equation $\dot{x} = A(t)x$. For every $v \neq 0 \in \mathbb{R}^n$, define **Lyapunov exponent** of v

$$\chi(v) := \limsup_{t \to \infty} \frac{\ln \|\Phi(t)v\|}{t}.$$

Exercise 2.4.19. For every $v \neq 0$, show that $\chi(v) \neq \pm\infty$.

Then $\chi : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties

1. $\chi(\alpha v) = \chi(v)$ for every $\alpha \neq 0$.
2. $\chi(v + w) \leqslant \max\{\chi(v), \chi(w)\}$.
3. If $\chi(v) < \chi(w)$, then $\chi(v + w) = \chi(w)$.

Fact 2.4.20. The number of different Lyapunov exponents $\leqslant n$.

Example 2.4.21

$\dot{X} = AX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a constant matrix. Regard as a T-periodic system, then the eigenvalues λ_1, λ_2 of A are Floquet exponents. Lyapunov exponents are

(1) λ_1, λ_2, if $\lambda_1 \neq \lambda_2$ real.

(2) $\lambda = \lambda_1 = \lambda_2$, if $\lambda_1 = \lambda_2$.

(3) α, if $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$.

For the T-periodic system, assume that λ is a Floquet exponent, then $\chi = \mathrm{Re}(\lambda)$ is a Lyapunov exponent. For $n = 2$, T-periodic system, we always have

$$\chi_1 + \chi_2 = \mathrm{Re}(\lambda_1 + \lambda_2) = \frac{1}{T}\int_0^T \mathrm{tr}\, A(t)\mathrm{d}t.$$

Example 2.4.22

Consider

$$\begin{cases} \dot{x} = (-\mu - (\sin\ln t + \cos\ln t))x \\ \dot{y} = (-\mu + (\sin\ln t + \cos\ln t))y \end{cases},$$

then the solution

$$\begin{cases} x = C_1 e^{-\mu t - t\sin\ln t} \\ y = C_2 e^{-\mu t + t\sin\ln t} \end{cases}.$$

Then $\chi(v) = -\mu + 1$ for every $v \neq 0$. But $\chi_1 + \chi_2 = -2\mu + 2 \neq \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{tr}\, A(t)\mathrm{d}t = -2\mu$. This example is called non-regular.

3 Stability

§3.1 Lyapunov stability

Let $f : \mathbb{R}^n \to \mathbb{R}^n, 0 \in \mathbb{R}^n, f(0) = 0$, generates a (local) flow $\varphi_t(x)$.

Definition 3.1.1. 1. σ is called **(forward Lyapunov) stable**, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| < \varepsilon$ for $t \geqslant 0$. Otherwise, we call σ is **unstable**.

2. σ is called **asymptotically stable**, if
   (i) σ is stable,
   (ii) there exists $\delta_0 > 0$, such that if $|x - \sigma| < \delta$, then $\lim_{t\to\infty} \varphi_t(x) = \sigma$.

3. σ is called **exponentially stable**, if exists $\delta_0 > 0, C \geqslant 1, \lambda > 0$, such that if $|x - \sigma| < \delta$, then $|\varphi_t(x) - \sigma| \leqslant Ce^{-\lambda t}|x - \sigma|$ for $t \geqslant 0$.

Similarly, we can define backward stable, backward asymptotically stable, backward exponentially stable.

Remark 3.1.2 — If we replace the condition of stability by **given** $t \geqslant 0$, then it always holds by the continuous independence of solutions with respect to initial value.

Example 3.1.3

For the equation in polar coordinates

$$\begin{cases} \dot{r} = r(1 - r) \\ \dot{\theta} = \sin^2(\theta/2) \end{cases}.$$

Then the fixed point $(1, 0)$ satisfy the second condition of asymptotically stable but it is **not** stable.

In general, we can prove that if $\varphi_t(x) \not\equiv \sigma$ and $\lim_{t\to\pm\infty} \varphi_t(x) = \sigma$, then σ is not stable.

Example 3.1.4

Consider the linear elementary singularities, recall the classification, then

1. Stable type: forward stable.
2. Unstable type: unstable, bet backward stable.
3. Saddle point: unstable.
4. Center: forward and backward stable.

Theorem 3.1.5

Let $A \in M(n, \mathbb{R})$, consider the equation $\dot{X} = AX$, 0 is a singularity, then

1. 0 is stable iff each eigenvalue of A is with non-positive real part and Jordan block are trivial for every eigenvalue with zero real part.
2. 0 is asymptotically stable iff 0 is exponentially stable iff every eigenvalue of A is with negative real part.

Lemma 3.1.6 (Gronwall's Inequality)

Let $u : [0, T] \to \mathbb{R}$ non-negative, continuous. If $C \geqslant 0, K > 0$ such that for every $t \in [0, T]$,

$$u(t) \leqslant C + K \int_0^t u(s)\mathrm{d}s,$$

then $u(t) \leqslant Ce^{Kt}$ for $t \in [0, T]$.

Theorem 3.1.7

$f : \mathbb{R}^n \to \mathbb{R}^n$, C^1, $f(\sigma) = 0$. Assume that every eigenvalue of $A = Df(0)$ is with negative real part, then σ is exponentially stable.

Proof. There $\exists C \geqslant 1, \mu > 0$, such that $|e^{At}| \leqslant Ce^{-\mu t}$ for $t \geqslant 0$. WOLG, $\sigma = 0$. Let $f(x) = Ax + g(x)$ where $g(x) = o(|x|)$, let $\varphi_t(x)$ be a maximal solution of the initial value problem. Then

$$e^{-tA}(\dot{\varphi}_t(x) - A\varphi_t(x)) = e^{-tA}g(\varphi_t(x)),$$

hence

$$\varphi_t(x) = e^{tA}x + \int_0^t e^{(t-s)A}g(\varphi_s(x))\mathrm{d}s.$$

Fix $\varepsilon_0 > 0$ to be determined later, $\exists \delta_0 > 0$ such that $|g(x)| \leqslant \varepsilon_0|x|$ if $|x| \leqslant \delta_0$. Assume the right maximal interval of φ_t is $[0, \beta), \beta > 0$. Let

$$T^* = T^*(x) = \sup\left\{t < \beta : \varphi_{[0,t]}(x) \subseteq \overline{B(\sigma, \delta_0)}\right\}.$$

Then, for every $|x| \leqslant \delta_0, 0 \leqslant t \leqslant T^*$, we have

$$e^{\mu t}|\varphi_t(x)| \leqslant C|x| + C\varepsilon_0 \int_0^t e^{s\mu}|\varphi_s(x)|\mathrm{d}s.$$

By Gronwall's inequality, we have $|\varphi_t(x)| \leqslant C|x|e^{-(\mu - C\varepsilon_0)t}, \forall t < T^*$. Let $C\varepsilon_0 = \frac{\mu}{2}$ is enough. For all $|x| \leqslant \frac{\delta_0}{2C}$, then $|\varphi_t(x)| \leqslant \frac{\delta_0}{2}e^{-\mu t}$ for every $t < T^*$. Then we can show that $T^* = \beta = \infty$ and φ_t is exponentially stable. $\square$

Proposition 3.1.8

f, g, C^1 vector fields. Assume f, g are topologically conjugate, i.e. , $h \circ \varphi_t = \psi_t \circ h$ where φ_t, ψ_t are flows generated by f, g, respectively. Let $\sigma, h\sigma$ be singularities of f, g, respectively, then σ is stable if and only if $h\sigma$ is stable.

Now, we state a celebrated theorem, Hartman-Grobman Theorem. But we will not give a proof here.

Theorem 3.1.9 (Hartman-Grobman)

Let σ be a hyperbolic singularity of f. Then there exists a neighborhood $V \ni \sigma$ and a homeomorphism $h : V \to \mathbb{R}^n$ onto its image, $h(\sigma) = 0$, such that $h \circ \varphi_t(x) = Df(\sigma) \circ h(x)$ for every $x, \varphi_t(x) \in V$.

§3.2 Lyapunov functions

Definition 3.2.1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$, be a C^1 vector field, $f(0) = 0$. A C^1 function $V : D \to \mathbb{R}$ where D is a neighborhood of σ is called a **Lyapunov function** of f (for σ) if

(i) $V(\sigma) = 0, V(x) > 0, \forall x \in D \setminus \{\sigma\}$.

(ii) $\forall x \in D \setminus \{\sigma\}, \dot{V}(x) \leqslant 0$, where

$$\dot{V}(x) = \frac{\partial}{\partial t}\bigg|_{t=0} V(\varphi_t(x)) = DV(x)f(x).$$

V is called a **strict Lyapunov function** if $\dot{V}(x) \leqslant 0$ is replaced by $\dot{V}(x) < 0$.

Theorem 3.2.2

Assume σ is a singularity of f, if there is a Lyapunov function for σ, then σ is stable. If there is a strict Lyapunov function for σ, then σ is asymptotically stable.

Proof. Let V be a Lyapunov function, for every $\varepsilon > 0$, assume $B_\varepsilon(\sigma) = \{x : |x - \sigma| \leqslant \varepsilon\} \subseteq D$. Let $m = \min\{V(x) : x \in \partial B_\varepsilon(\sigma)\} > 0$, take $\delta > 0$ such that $V(x) < m, \forall x \in B_\delta(\sigma)$. By $\dot{V}(x) \leqslant 0$, we have that every solution curve start at $x \in B_\delta(\sigma)$ can not reach $\partial B_\varepsilon(\sigma)$.

If $\dot{V}(x) < 0$ for every $x \in D \setminus \{\sigma\}$, it suffices to show that each convergent subsequence of $\varphi_t(x)$ converges to σ. Otherwise, assume converges to $y \neq \sigma$, but $\dot{V}(y) < 0$, there is some $s > 0$ such that $V(\varphi_s(y)) < V(y)$. Contradiction. □

Example 3.2.3

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = 0$, hence 0 is stable.

Example 3.2.4

Consider the equation

$$\begin{cases} \dot{x} = -x + y \\ \dot{y} = -x - y^3 \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2 - 2y^4 < 0$, hence 0 is asymptotically stable.

Example 3.2.5

Consider the equation

$$\begin{cases} \dot{x} = -x - y + x^2 \\ \dot{y} = x \end{cases}.$$

Let $V(x, y) = x^2 + y^2$, then $\dot{V}(x, y) = -2x^2(1 - x) \leqslant 0$, hence 0 is stable. In fact, 0 is asymptotically stable, but we need to consider another Lyapunov function $Q(x, y) = x^2 + y^2 + xy$.

Theorem 3.2.6

If V is a Lyapunov function of f, assume

$$K = \left\{x \in D \setminus \{\sigma\}, \dot{V}(x) = 0\right\}$$

does not contain any complete positive orbit $\varphi_{[0,\infty)}(x)$, then σ is asymptotically stable.

Example 3.2.7

Let $f : \mathbb{R} \to \mathbb{R}, C^1, f(0) = 0$, satisfying $xf(x) > 0, \forall x \neq 0$. Consider the stability of $\ddot{x} + f(x) = 0$, or

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \end{cases}.$$

Let

$$E(x, y) = \frac{1}{2}y^2 + \int_0^x f(z)\mathrm{d}z$$

be an energy function, then $\dot{E}(x, y) \equiv 0$.

Example 3.2.8

Let $V : \mathbb{R}^n \to \mathbb{R}, C^2$, the gradient of V is

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

The system $\dot{x} = -V(x)$ is called the **gradient system** generated by V. Then,

1. $\dot{V}(x) \leqslant 0$.
2. σ is a singularity if and only if $\dot{V}(\sigma) = 0$.
3. If σ is a minimum point of $V(x)$, then σ is stable.

Theorem 3.2.9

Let σ be a singularity of C^1 vector field f, a C^1 function $V : D \to \mathbb{R}$ satisfies

(i) $V(\sigma) = 0$, and V can take positive value on any neighborhood of σ.

(ii) $\dot{V}(x) > 0, \forall x \in D \setminus \{0\}$.

Then σ is unstable.

Example 3.2.10

Consider the equation

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}.$$

Let $V(x, y) = x^2 - y^2$, then $\dot{V}(x, y) = 2x^2 + 2y^2 > 0$, hence 0 is unstable.

Theorem 3.2.11

Let f be a C^1 vector field, $f(\sigma) = 0$. If σ is stable, then every eigenvalue of $Df(\sigma)$ is with non-positive real part.

Proof. Prove for $n = 2$. Assume $\sigma = 0$, the equation is

$$\begin{cases} \dot{x} = \lambda x + \alpha(x, y) \\ \dot{y} = \mu y + \beta(x, y) \end{cases},$$

where $\lambda < \mu, \mu > 0, |\alpha|, |\beta| = o(r)$. Let $V(x, y) = -x^2 + y^2$, then

$$\dot{V}(x, y) = -2\lambda x^2 + 2\mu y^2 - 2x\alpha + 2y\beta.$$

If $\lambda < 0$, then $\dot{V} > 0$ in a neighborhood of 0, then 0 is unstable. If $\lambda \geqslant 0$, consider

$$C = \{(x, y) : V(x, y) \geqslant 0\}.$$

We can show that for some $\varepsilon_0 \geqslant 0$, $\dot{V}(x, y) > 0$ on $C \cap B(0, \varepsilon_0) \setminus \{0\}$. Let $H(x, y) = x^2 + y^2$, then $\dot{H}(x, y) \geqslant \frac{\mu}{2} H(x, y)$ on some neighborhood of 0. Hence

$$H(\varphi_t(x, y)) \geqslant H(x, y) e^{\frac{\mu}{2} t}$$

will be out of $C \cap B_\varepsilon(x, y)$. □

Remark 3.2.12 — In fact, there exists $(x, y) \in B(0, \varepsilon_0) \setminus \{0\}$, such that

$$\lim_{t \to -\infty} \varphi_t(x, y) = 0, \quad \frac{f(\varphi_t(x, y))}{|f(\varphi_t(x, y))|} \to (0, 1).$$

$\varphi_t(x, y)$ is called the unstable manifold.

Exercise 3.2.13. Prove the theorem for general dimension n.

Now, we consider a perturbation of a singularity of center type. Consider the system

$$\begin{cases} \dot{x} = -y + \alpha(x, y) \\ \dot{y} = x + \beta(x, y) \end{cases},$$

then

$$\dot{\theta} = 1 + \frac{x\beta - y\alpha}{x^2 + y^2},$$

$$\dot{r} = \frac{x\alpha + y\beta}{r} = \alpha\cos\theta + \beta\sin\theta = R_2(\theta)r^2 + R_3(\theta)r^3 + \cdots.$$

Example 3.2.14

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2y + x^3) \end{cases}.$$

Then

$$\dot{r} = \sin\theta(x^2y + x^3) = r^3(\cos^2\theta\sin^2\theta + \cos^3\theta\sin\theta),$$

we calculate

$$\overline{R}_3 = \int_0^{2\pi} R_3(\theta)\mathrm{d}\theta = \frac{\pi}{4} > 0.$$

Let $g(\theta) = \int_0^\theta R_3(\theta)\mathrm{d}\theta$, then

$$\varphi_3(\theta) = g(\theta) - \frac{\theta}{2\pi}\int_0^{2\pi} R_3(\theta)\mathrm{d}\theta$$

is 2π-periodic. Let $r = \rho + \varphi_3(\theta)\rho^3$, then

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = \overline{R}_3\rho^3 + \cdots,$$

hence ρ is increasing. Therefore, 0 is unstable.

Example 3.2.15

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We want to construct a Lyapunov function of the form $V(x, y) = x^2 + y^2 + F(x, y)$, where $F(x, y)$ is a homogeneous polynomial of $\deg = 3$. Then

$$\dot{V}(x, y) = -yF_x + xF_y + 2y^3 + y^2 F_y,$$

we want $-yF_x + xF_y + 2y^3 = 0$. Consider $L : H_k \to H_k$, where H_3 is the family of homogeneous polynomials of $\deg = k$, $L(F) = -yF_x + xF_y$. After repetition, we can let

$$V(x, y) = \lambda(x^2 + y^2)^k + \cdots.$$

Then 0 is stable if $\lambda < 0$, 0 is unstable if $\lambda > 0$. Or we can find V such that $\dot{V}(x, y) = 0$, then 0 is still a center.

Example 3.2.16

Consider the equation

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + y^2 \end{cases}.$$

We can solve this equation,

$$y^2 = -x + \frac{1}{2}(1 - e^{-2x}) + Ce^{-2x},$$

hence $e^{2x}(x^2 + y^2) = C + \cdots$. 0 is still a center.

Example 3.2.17

Consider the equation

$$\begin{cases} \dot{x} = -y & = X(x, y) \\ \dot{y} = x + y^2 & = Y(x, y) \end{cases}.$$

Notice that $X(x, -y) = -X(x, y), Y(x, -y) = Y(x, y)$, hence the solution curve is symmetric with respect to x-axis. We can prove this fact by showing $(x(-t), -y(-t))$ is a solution if $(x(t), y(t))$ is a solution. Then we can show 0 is a center.

§3.3 Stability under perturbations

Definition 3.3.1. Consider an autonomous system $\dot{x} = f(x)$, generating a flow φ_t. For every $x_0 \in \mathbb{R}^n$, the orbit $\varphi_t(x_0)$ is said to be **stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi_t(x) - \varphi_t(x_0)| < \varepsilon, \quad \forall t \geqslant 0, x \in B(x_0, \delta).$$

Example 3.3.2

Consider the equation

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r^2 \end{cases}.$$

Then the orbit of $(r_0, \theta_0) = (1, 0)$ is **not** stable.

Definition 3.3.3. Consider a non-autonomous system $\dot{x} = f(x, t)$, let $\varphi(t; t_0, x_0)$ be the solution of the initial value problem $x(t_0) = x_0$. The orbit $x(t; t_0, x_0)$ is said to be **stable**, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\varphi(t; t_0, x) - \varphi_t(t; t_0, x_0)| < \varepsilon, \quad \forall t \geqslant t_0, x \in B(x_0, \delta).$$

Similarly, we can define the asymptotically stable and the exponentially stable for general orbits of autonomous or non-autonomous systems.

Theorem 3.3.4

$A : \mathbb{R} \to M(n, \mathbb{R})$, consider a non-autonomous system $\dot{x} = A(t)x$. Then

1. Every solution is stable iff 0 is stable.
2. 0 is stable iff $\sup_{t \geqslant 0} |X(t)| < \infty$, where $X(t)$ is a fundamental solution.
3. 0 is asymptotically stable iff $\lim_{t \to \infty} |X(t)| = 0$.

Theorem 3.3.5

Consider a T-periodic system $\dot{x} = A(t)x$. Then

2. 0 is stable iff the Floquet exponents are of non-positive real parts and Jordan block are trivial for every Floquet exponent with zero real part.
2. 0 is asymptotically stable iff Floquet exponents are of negative real parts iff 0 is exponentially stable.

For an autonomous system, let $f(0) = 0, f(x) = Ax + \varphi(x)$, where $\varphi(0) = 0, D\varphi(0) = 0$. Rewrite the system as $\dot{x} = Ax + \varphi(x)$, if every eigenvalue of A is with negative real parts, then 0 is stable.

For a non-autonomous system, assume

$$\dot{x} = Ax + \varphi(t, x), \quad \varphi(t, 0) = 0, D\varphi(t, 0) = 0,$$

if every eigenvalue of A is with negative real parts, then 0 is stable. In general,

$$\dot{x} = A(t)x + \varphi(t, x),$$

but the negativeness of Lyapunov exponents do **not** imply the stableness. See the following example.

Example 3.3.6

Consider

$$\begin{cases}\dot{x} = (-\mu - (\sin\ln t + \cos\ln t))x \\ \dot{y} = (-\mu + (\sin\ln t + \cos\ln t))y + x^2\end{cases},$$

let $a(t) = t\sin\ln t$, the solutions are

$$\begin{cases}x = C_1 e^{-\mu t - a(t)} \\ y = C_2 e^{-\mu t + a(t)} + C_1^2 e^{-\mu t + a(t)} \displaystyle\int_1^t e^{-\mu s - 3a(s)} \mathrm{d}s\end{cases}.$$

Consider $\mu = 1 + \sigma, \sigma$ is sufficiently small, then 0 is not stable.

For this case, we need a stronger condition. Let $\Phi(t)$ be a fundamental solution of the linear part, if $\exists \mu > 0$,

$$|\Phi(t)\Phi(-s)| \leqslant Ce^{-\mu(t-s)}, \quad \forall t \geqslant s \geqslant 0,$$

then 0 is also stable under the perturbation .

4 Poincaré-Bendixson Theory

§4.1 Basic notions

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, generating a flow $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4.1.1. $A \subseteq \mathbb{R}^n$ is said to be f (or φ_t) **invariant** if for every $t \in \mathbb{R}$, $\varphi_t(A) = A$.

For every $x \in \mathbb{R}^n$, the orbit $\mathrm{Orb}(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$ is an invariant set. In general, if A is invariant, then

$$A = \bigcup_{x \in A} \mathrm{Orb}(x).$$

Definition 4.1.2. Let A be a compact invariant set, A is said to be **Lyapunov orbit stable** if for every neighborhood $U \supseteq A$, there exists a neighborhood $V \supseteq A$ such that

$$\varphi_t(x) \in U, \quad \forall x \in V, t \geqslant 0.$$

Let

$$\mathrm{Orb}^+ := \{\varphi_t(x) : t \geqslant 0\}, \quad \mathrm{Orb}^- := \{\varphi_t(x) : t \leqslant 0\}$$

be the **positive semi-orbit** and the **negative semi-orbit**.

Definition 4.1.3. Given $p \in \mathbb{R}^n$, x is called a **positive limit point** if $\exists t_n \to +\infty, \varphi_{t_n} \to x$. The set of all positive limit points is called the **α-limit set** of p, denoted by $\alpha(p)$. Similarly, we can define the **negative limit points**, they form a set is called **ω-limit set**, denoted by $\omega(p)$.

> **Remark 4.1.4** — In the Greek alphabet, α is the first letter and ω is the last letter, it is very graphic that the orbit of p ran from α to ω.

Example 4.1.5

Consider the equation

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}.$$

Then $\omega(0) = \alpha(0) = 0$. For every $p \in \mathbb{S}^1$, we have $\omega(p) = \alpha(p) = \mathbb{S}^1$. Otherwise, let $p = (x, y)$, we have

(1) If $0 < x^2 + y^2 < 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \{0\}$.

(2) If $x^2 + y^2 > 1$, then $\omega(p) = \mathbb{S}^1, \alpha(p) = \varnothing$.

Proposition 4.1.6

$\forall p \in \mathbb{R}^n$, we have

$$\omega(p) = \bigcap_{t \geqslant 0} \overline{\mathrm{Orb}^+(\varphi_t(p))} = \bigcap_{k \in \mathbb{Z}_+} \overline{\mathrm{Orb}^+(\varphi_k(p))}.$$

Proposition 4.1.7

Assume $\mathrm{Orb}^+(p)$ is bounded, then

1. $\omega(p)$ is non-empty, compact, invariant, connected.
2. $\lim_{t\to\infty} d(\varphi_t(p), \omega(p)) = 0$.

Proof. 1. Non-empty, compact, invariant are trivial. The connected follows by the fact that $A_k = \overline{\mathrm{Orb}^+(\varphi_k(p))}$ are connected and $A_k \supseteq A_{k+1} \supseteq \cdots$.

2. For every $\varepsilon > 0$, $A_k \subseteq B(\omega(p), \varepsilon)$ for every k sufficiently large.

□

Definition 4.1.8. p is said to be **positively recurrent** if $p \in \omega(p)$,

The singularities and periodic points are called trivial recurrent points, other recurrent points are said to be non-trivial.

Definition 4.1.9. Let Λ be a non-empty, compact, invariant set. Λ is called a **minimal set** of φ_t if it does not contain a proper, nonempty, compact invariant set.

Theorem 4.1.10 (Flow Box Theorem)

Let f be a C^1 vector field, $p \in \mathbb{R}^n$, $f(p) \neq 0$. Then there is a neighborhood $U \ni p$ and a C^1 diffeomorphism $h : U \to h(U)$ on to its image, such that $\mathrm{D}h(x)f(x) = (1, 0, \cdots, 0)^t$.

Proof. WLOG, $p = 0, f(p) = (1, 0, \cdots, 0)^t$. We construct $g : (-\varepsilon_0, \varepsilon_0) \times L \to U$ some neighborhood of p. Let

$$x = g(y) = g(y_1, y_2, \cdots, y_n) := \varphi_{y_1}(0, y_2, y_3, \cdots, y_n).$$

Then

$$\frac{\partial}{\partial t}\varphi_t(x)\bigg|_{(t,x)=(y_1,0,y_2,\cdots,y_n)} = f(\varphi_{y_1}(0, y_2, \cdots, y_n)) = f(g(y)),$$

let $(y_1, y_2, \cdots, y_n) = (0, 0, \cdots, 0)$, then $\frac{\partial g}{\partial y_1}(y) = f(g(y))$. Moreover,

$$\mathrm{Id} = \frac{\partial \varphi_t(x_1, \cdots, x_n)}{\partial(x_1, \cdots, x_n)}\bigg|_{t=0} \implies \frac{\partial g}{\partial y}\bigg|_{y=0} = \mathrm{Id}.$$

Hence, g gives a local diffeomorphism. Let $h = g^{-1}$, the statement follows. □

Remark 4.1.11 — Let $L_{\varepsilon_0} = \left\{(y_2, \cdots, y_n) : y_2^2 + \cdots + y_n^2 \leqslant \varepsilon_0^2\right\}$, let

$$U = h^{-1}((-\varepsilon_0, \varepsilon_0) \times L_{\varepsilon_0}),$$

then U is called a **tubular neighborhood** near p, or a **flow box** near p.

§4.2 The Poincaré-Bendixson Theorem

Definition 4.2.1. $C \subseteq \mathbb{R}^2$ is called a **Jordan curve** if it is homeomorphism to $\mathbb{S}^1$.

Theorem 4.2.2 (Jordan Seperation Theorem)
Let $C \subseteq \mathbb{R}^2$ be a Jordan curve. Then $\mathbb{R}^2 \setminus C$ has exactly two connected components. One of them is bounded, which is called the interior of C. Another one is bounded, which is called the exterior of C. Both of them are with bound C.

Theorem 4.2.3 (Jordan-Schoenflies)
Let C be a Jordan curve, then there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, such that $h(C) = \mathbb{S}^1$.

Definition 4.2.4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1, $L \subseteq \mathbb{R}^2$ is a line segment. L is called **transverse** to f if $\forall x \in L$, $f(x)$ and the direction of L generates $\mathbb{R}^2$. We then say L is a **transversal** to f.

Lemma 4.2.5
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1, L is a transversal to f. Assume there are three points $P_1, P_2, P_3 \in L$ and $x \in \mathbb{R}^2$ such that

$$\varphi_{t_i}(x) = P_i, \quad t_1 < t_2 < t_3,$$

$$\varphi_{(t_1,t_2)}(x) \cap L = \varnothing, \quad \varphi_{(t_2,t_3)}(x) \cap L = \varnothing,$$

then $P_2 \in (P_1, P_3)$.

Proof. Assume A, B are extreme points of L. Consider a Jordan curve

$$C = \varphi_{[t_1,t_2]}(x) \cup (P_1, P_2),$$

let D be the interior of C. Assume $B \in D$, we show that $P_3 \in D$. By the Flow Box Theorem, there exists $\varepsilon > 0$ such that $\varphi_{(t_2,t_2+\varepsilon]}(x) \subseteq D$. Let $\tau = \inf\{t > t_2 : \varphi_t(x) \notin D\} > t_2 + \varepsilon$ if exists. Then $\varphi_\tau(x) \in C$, but it can not on $\varphi_{(t_1,t_2)}(x)$ or P_1, P_2. So $\varphi_\tau(x) \in (P_1, P_2)$, but this contradict with L is a transversal to f. □

Remark 4.2.6 — Assume $\varphi_t(x)$ intersect with a transversal L at $P_i = \varphi_{t_i}(x), i = 1, 2, \cdots$ in chronological order, i.e., $0 < t_1 < t_2 < \cdots$, then

$$P_1 < P_2 < \cdot \quad \text{or} \quad P_1 > P_2 > \cdots \quad \text{or} \quad P_1 = P_2 = \cdots.$$

Proposition 4.2.7
Assume L is a transversal of f, then for every $x \in \mathbb{R}^2$,

$$\sharp(\omega(x) \cap L) \leqslant 1.$$

Proof. Assume for a contradiction. Let $q \neq q' \in \omega(x) \cap L$, then $\exists t_n \to \infty, t_n' \to \infty$ such that $\varphi_{t_n}(x) \to q, \varphi_{t_n'}(x) \to q'$. WLOG, assume $t_1 < t_1' < t_2 < t_2' < \cdots$. By the Flow Box Theorem, for k sufficiently large, there exists τ_k, τ_k' such that

$$|\tau_k - t_k| \to 0, |\tau_k' - t_k'| \to 0, \quad \varphi_{\tau_k}(x), \varphi_{\tau_k'}(x) \in L, \quad \varphi_{\tau_k}(x) \to q, \varphi_{\tau_k'} \to q'.$$

We can also assume that $\tau_k < \tau_k' < \tau_{k+1} < \cdots$, then this contradicts with the monotonicity of $\varphi_t(x)$ intersecting the transversal. □

Theorem 4.2.8 (Poincaré-Bendixson Theorem)

Assume $\mathrm{Orb}^+(x)$ is bounded and $\omega(x)$ contains no singularities, then $\omega(x)$ is a periodic orbit.

Proof. Because $\mathrm{Orb}^+(x)$ is bounded, $\omega(x) \neq \varnothing$. For every $p \in \omega(x)$, take $q \in \omega(p) \subseteq \omega(x)$ arbitrarily. Take a transversal L_q of f through q, then $\exists t_n \to \infty, \varphi_{t_n}(p) \to q$. WLOG, $\varphi_{t_n}(x) \in L_q$. Because $\varphi_{t_n}(p) \in \omega(x)$ and $\sharp\omega(x) \cap L_q = 1$, then $\varphi_{t_n}(p) = \varphi_{t_{n+1}}(p)$, hence p is a periodic point.

Take $p \in \omega(x)$, it is a periodic point. If $\omega(x) \neq \mathrm{Orb}(x)$, take a transversal L_p of f through p. Because $\omega(x)$ is connected, hence $\mathrm{Orb}(p)$ is not isolated in $\omega(x)$. Take $q_n \in \omega(x) \setminus \mathrm{Orb}(p), q_n \to \mathrm{Orb}(p)$. WLOG, $q_n \to p$ and $q_n \in L_p$, this contradicts with $\sharp\omega(x) \cap L_p \leqslant 1$. □

Theorem 4.2.9 (P-B Annular Region Theorem)

Assume A is an annular region and ∂A is two C^1 curves. If for every $x \in \partial A$, $f(x)$ is pointing inside of A, and A contains no singularities. Then there is a periodic orbit in A.

Example 4.2.10

The system

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 3x - 1) - y \\ \dot{y} = y(x^2 + y^2 - 3x - 1) + x \end{cases}$$

contains a non-trivial periodic orbit.

Proof. 0 is the only singularity. Let

$$A = \left\{(x, y) \in \mathbb{R}^2, r^2 \leqslant x^2 + y^2 \leqslant R^2\right\}, \quad r < R,$$

let $V(x, y) = x^2 + y^2$. Then for r small enough $\dot{V} < 0$, for R large enough $\dot{V} > 0$. Hence $f(x)$ is pointing outside of A on ∂A, consider the α-limit set. □

The **Liénard equation**

$$\ddot{x} + f(x)\dot{x} + g(x) = 0.$$

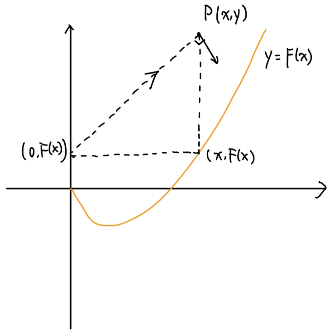
Let $F(x) = \int_0^x f(t)\mathrm{d}t$, then the equation is equivalent to

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}.$$

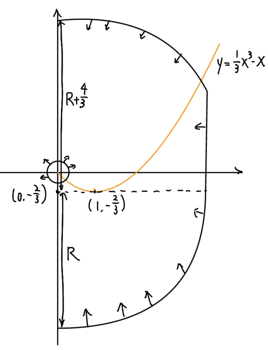
Consider a particular Liénard equation, which is called **van der Pol equation**: $f(x) = x^2-1, g(x) = x$. Then this equation is equivalent to

$$\begin{cases} \dot{x} = y - (\frac{1}{3}x^3 - x) \\ \dot{y} = -x \end{cases}.$$

We introduce the **Liénard graphing method.** Draw the graph of $y = F(x)$, for $P = (x, y) \in \mathbb{R}^2$, draw $(x, F(x))$ and $Q = (0, F(x))$. Hence the slope of $QP = \frac{F(x)-y}{-x}$, then rotate it $90°$ clockwise, we get $f(x, y)$.



Consider $V = x^2 + y^2$, then $\dot{V} = 2x^2(1 - \frac{1}{3}x^2) \geqslant 0$ when $|x| < 1$. Take $x = 1$, then $F(x) = -\frac{2}{3}$ attain the minimum. For R sufficiently large, we can construct a curve such that the poin, see the figure. For r small enough, vectors on B_r are pointing outside. This gives an annular region and hence there is a periodic orbit in it. Furthermore, we can prove that the periodic orbit is unique.



Theorem 4.2.11 (P-B)

f is a C^1 vector field, assume there are only finite singularities. If Orb^+ is bounded, then there is a trichotomy:

(1) $\omega(x)$ is a periodic orbit.

(2) $\omega(x)$ is a singularity.

(3) $\omega(x)$ contains regular points and singularities and $\forall y \in \omega(x)$, $\alpha(y), \omega(y)$ are both a singularity.

Proof. If $\omega(x)$ contains no singularities, by P-B, $\omega(x)$ is a periodic point. If $\omega(x)$ contains no regular points, then $\omega(x)$ is finite singularities, but $\omega(x)$ is connected, hence $\omega(x)$ is a singularity.

If $\omega(x)$ contains regular points and singularities both, for $y \in \omega(x)$, assume for a contradiction that $\omega(y)$ contains some regular points. Take $z \in \omega(y)$ regular and a transversal L_z contains z. Take $t_n \to +\infty, \varphi_{t_n}(y) \to z, \varphi_{t_n}(y) \in L_z$. Same argument shows that y is a periodic point.

Take a transversal L_y contains y, take $\tau_n \to +\infty, \varphi_{\tau_n}(x) \to y, \varphi_{\tau_n}(x) \in L_y$. Because $\omega(x)$ contains some singularities, then x is not a periodic point, hence $\varphi_{\tau_n}(x) \neq y$. Let

$$P_n = \varphi_{\tau_n}(x), \quad C_n = \varphi_{[\tau_n, \tau_{n+1}]}(x) \cap [P_n, P_{n+1}],$$

then $C_n \to \mathrm{Orb}(y)$, hence we can show that $\omega(x) = \mathrm{Orb}(y)$. A contradiction. □

§4.3 Poincaré recurrence and limit cycle

Lemma 4.3.1

Let f be a C^1 vector field, generating a flow φ_t. Let p be a regular point, L_t be a transversal of f at $\varphi_t(p)$. Then there is a neighborhood $U \ni p$ and a C^1 map: $U \to \mathbb{R}$ such that

$$\tau(p) = t, \quad \varphi_{\tau(x)} \in L_t, \forall x \in U.$$

Proof. WLOG, assume $L_t \perp f(\varphi_t(x))$. Let $Q = \varphi_t(p)$, consider

$$F(\tau, x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (\tau, x) \mapsto (\varphi_t(x) - Q) \cdot f(Q).$$

Then $\varphi_\tau(x) \in L_t$ iff $F(\tau, x) = 0$. We have

$$\left.\frac{\partial F}{\partial \tau}\right|_{(\tau,x)=(t,p)} = f(\varphi_\tau(x)) f(Q)|_{(\tau,x)=(t,p)} = |f(Q)|^2 > 0,$$

The conclusion follows by implicit function theorem. □

Corollary 4.3.2

Let f be a C^1 vector field, generating a flow φ_t. For every regular point p and two transversals L_0, L_t at p and $\varphi_t(p)$, respectively. There exists an neighborhood I_0 of p in L_0 and a C^1 map $\tau : I_0 \to \mathbb{R}$ such that

$$\tau(p) = t, \quad \varphi_{\tau(x)}(x) \in L_t, \forall x \in I_0.$$

Furthermore, let $P : I_0 \to L_t, x \mapsto \varphi_{\tau(x)}(x)$, then P is a diffeomorphism onto image.

Proof. Assume $L_t : (X - Q) \cdot f(Q) = 0$, then τ is given by $f(Q)^t(\varphi_\tau(x) - Q)$. Hence

$$D\tau = -\frac{f(Q)^t D\varphi_\tau(x)}{f(Q)^t f(\varphi_t(x))}.$$

Let I_0 be a connected component of $U \cap L_0$ containing x, define the **Poincaré map**

$$P : I_0 \to L_t, \quad x \mapsto \varphi_{\tau(x)}(x).$$

Then, for $v \in T_p L_0$.

$$DP(p)v = \frac{\partial \varphi}{\partial \tau}\frac{\partial \tau}{\partial x}v + D\varphi_{\tau(x)}(x)v|_{x=p} = -f(Q)\frac{f(Q)^t D\varphi_t(x)}{f(Q)^t f(Q)} + D\varphi_t(p)v,$$

it is the orthogonal projection of $D\varphi_t(x)v$ to L_t. It suffices to show $DP(p) \neq 0$. Otherwise, $D\varphi_t(p)v = \eta f(Q)$ for some v, but $D\varphi_t(p)f(p) = f(Q)$, a contradiction with $D\varphi_t(p)$ is an isomorphism. □

Let p be a T-periodic point, then $D\varphi_T f(p) = f(p)$. Assume $f(p) \parallel \mathbf{e_1}$, then

$$D\varphi_T(p) = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}.$$

Take $L_0 = L_T \parallel \mathbf{e_2}$, then $DP(p)e_2 = d \cdot \mathbf{e_2}$. We choose a coordinate on L_0, assume $p = 0$, then $P : I_0 \to L_0$ is a real-valued function on $\mathbb{R}$ such that $P'(0) = d$.

Proposition 4.3.3

Assume p is a T-periodic point of C^1 vector field, then $P'(0) = \det D\varphi_T(p) > 0$.

Proof. $t \mapsto \det D\varphi_t(p)$ is continuous, nonzero and $0 \mapsto 1$, hence is positive. □

Definition 4.3.4. An isolated closed orbit is called a **limit cycle**.

Lemma 4.3.5

Let γ be a periodic point, L is a transversal of some point on γ. Let $P : I \to L$ be the Poincaré recurrent map, $0 \in I$ correspond to the given periodic point. Then $x \in I$ is a fixed point of P if and only if x is a periodic point.

Proof. Let x be a periodic point of φ. If $P(x) \neq x$, then $P(x) \in \mathrm{Orb}(x) = \omega(x)$, but $\sharp(\omega(x) \cap L_0) \leqslant 1$, a contradiction. □

Definition 4.3.6. Let γ be a limit cycle. Then γ is said:

(1) **stable**, if there exists a neighborhood U of γ, such that $\forall x \in U, \omega(x) = \gamma$.

(2) **unstable**, if there exists a neighborhood U of γ, such that $\forall x \in U, \alpha(x) = \gamma$.

(3) **semi-stable**, if there exists a neighborhood U of γ and a splitting $U \setminus \gamma = U_+ \cup U_-$, such that $\forall x \in U_+, \omega(x) = \gamma$ and $\forall x \in U_-, \alpha(x) = \gamma$.

Proposition 4.3.7

Let γ be a limit cycle and $p \in \gamma$. If $P'(p) < 1$, then γ is stable. If $P'(p) > 1$, then γ is unstable.

Compare with the definition of Lyapunov asymptotically stable, this definition does not guarantee the Lyapunov stability. But under the condition $P'(p) < 1$, we can show that $\forall \varepsilon > 0, \exists \delta > 0$ such that if $|x - p| < \delta$, then $|\varphi_t(x) - \varphi_t(p)| < \varepsilon$ for every $t \geqslant 0$. It suffices to show for $x \in L$. Note that

$$P^n(x) = \varphi_{\tau(x)+\tau(P(x))+\cdots+\tau(P^{n-1}(x))}(x),$$

because $|P^n(x) - P^n(p)| \leqslant \lambda^n|x-p|$ for some $\lambda < 1$, we can bound the time $|\tau(x) + \tau(P(x)) + \cdots + \tau(P^{n-1}(x)) - nT|$ by some $\frac{C}{1-\lambda}|x-p|$.

Definition 4.3.8. Let p be a periodic point, if the every eigenvalue of $DP(p)$ is not on the unit circle, then p is called **hyperbolic**.

Exercise 4.3.9. $n \geqslant 2$, let p be a T-periodic point, then p is hyperbolic if and only if there are $n-1$ eigenvalues of $D\varphi_T(p)$ are with absolute value $\neq 1$.

Example 4.3.10

Let $H : \mathbb{R}^2 \to \mathbb{R}$, be a C^2 function. Consider the **plane Hamilton system**

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases},$$

then there is not limit cycle in this system.

Proof. Let $\Phi_t(x, y) = \mathrm{D}\varphi_t(x, y)$, then $\det \Phi_t(x, y) = 1$. Hence φ_t is area preserving. This contradicts with the existence of limit cycle. $\square$

Hilbert's 16-th problem Consider the plane polynomial system

$$\begin{cases} \dot{x} = P_n(x, y) = \displaystyle\sum_{i+j=0}^{n} a_{ij} x^i y^j, \\ \dot{y} = Q_n(x, y) = \displaystyle\sum_{i+j=0}^{n} b_{ij} x^i y^j. \end{cases}$$

The second part of this problem is considering about the number of limit cycles in this system. For a given system, Dulac-Ilyashenko-Écalle proved that the number of limit cycles is finite.

Furthermore, let $H(n)$ be the upper bound of all systems with degree at most n. A further problem is whether $H(n)$ is finite. It trivial that $H(1) = 0$, but the existence of $H(2)$ is still unknown.

5 Bifurcations

§5.1 Structural stability

Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be two C^1 vector fields. The C^1 distance of f, g is defined as

$$|f - g|_{C^1} := \sup \{|f(x) - g(x)|, |Df(x) - Dg(x)| : x \in \mathbb{R}^n\}.$$

Definition 5.1.1. Given a C^1 vector field f on $\mathbb{R}^n$, f is called C^1 **structurally stable** if there exists $\varepsilon > 0$ such that for every C^1 vector field g, $|g - f|_{C^1} < \varepsilon$, then g is topologically equivalent with f.

Remark 5.1.2 — The condition is just "topologically equivalent" is reasonable. A C^1-conjugate preserves the derivation at a singularity. A C^0-conjugate preserves the time on a periodic orbit. Both conditions are too much for a general vector field.

Example 5.1.3
$f(x) = x, x \in \mathbb{R}$ is C^1 structurally stable.

Exercise 5.1.4. Show that the plane system

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2) \\ \dot{y} = x + y(1 - x^2 - y^2) \end{cases}$$

is C^1 structurally stable.

Exercise 5.1.5. Show that a center is not structurally stable.

Let $\overline{\mathbb{D}}$ be the closed unit disk $\{x^2 + y^2 \leqslant 1\}$, let $\mathcal{X}^1(\overline{\mathbb{D}})$ be the set of C^1 vector fields on $\overline{\mathbb{D}}$ which are not tangent to $\partial\mathbb{D}$.

Theorem 5.1.6 (Andronov-Pontryagin)
Given $f \in \mathcal{X}^1(\overline{\mathbb{D}})$, then f is C^1 structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.

Corollary 5.1.7
Under the same condition, given $x \in \overline{\mathbb{D}}$, if $\mathrm{Orb}(x) \subset \overline{\mathbb{D}}$, then $\omega(x)$ is a periodic point or a singularity.

Theorem 5.1.8 (Peixoto-Pugh)
Let M be a 2-dimensional closed Riemannian manifold. Given $f \in \mathcal{X}^1(M)$, then f is C^1 structurally stable if and only if the followings hold:

- all singularities and periodic orbits are hyperbolic,
- there are no saddle connections.
- for every $x \in M$, $\omega(x)$ is a periodic point or a singularity.

Example 5.1.9
Consider a system on $\mathbb{T}^2$,

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha, \end{cases}$$

if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then this system satisfying the first two conditions. But this system is not C^1 structurally stable. Hence for a general Riemannian manifold, the third condition is necessary.

Remark 5.1.10 — Peixoto has proved C^r structural stability for orientable M. For general surface M, Pugh applied C^1-closed theorem to show the statement. But C^r-closed theorem is still a big open problem.

§5.2 Bifurcations

Definition 5.2.1. Let $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be a C^1 map, which is a family of vector fields with k-parameters. The parameter $\epsilon_0 \in \mathbb{R}^k$ is called a **bifurcation** if $\forall \delta > 0, \exists \epsilon_1 \in \mathbb{R}^k, |\epsilon_1 - \epsilon_0| < \delta$, such that f_{ϵ_1} is not topologically equivalent with f_{ϵ_0}.

Example 5.2.2
Consider $f_\beta(x, y) = (-\beta y, \beta x)$, then f_β is not C^1 structurally stable, but is also not a bifurcation.

Remark 5.2.3 — A general problem is how to embed a non structurally stable vector field into a vector field family to be a bifurcation. These embeddings are called an **unfolding**. Some topics are to find a **universal unfolding**.

Example 5.2.4
Consider $f_\epsilon(x) = x(\epsilon - x)$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **transcritical bifurcation**: there are two hyperbolic singularities before and after the collision, and after the collision, the stability of two singularities exchange.

Example 5.2.5

Consider $f_\epsilon(x) = \epsilon - x^2$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **saddle-node bifurcation**: it can be regarded as a collision of a saddle and a node, there is a non-hyperbolic singularity at the bifurcation which is called a saddle-node.

It comes from the plane vector fields $f_\epsilon(x, y) = (\epsilon - x^2, y)$, when $\epsilon = 0$, the singularity seems like a node on one side and saddle on another side.

Example 5.2.6

Consider $f_\epsilon(x) = x(\epsilon - x^2)$, then $\epsilon = 0$ is a bifurcation. This bifurcation is called a **pitchfork bifurcation**: $x = 0$ is always a singularity, it is non-hyperbolic only if $\epsilon = 0$, after that, it create two hyperbolic singularities, one stable and one unstable.

Example 5.2.7

Consider the system

$$\begin{cases} \dot{x} = \epsilon x - y - x(x^2 + y^2) \\ \dot{y} = x + \epsilon y - y(x^2 + y^2) \end{cases} \quad \begin{cases} \dot{r} = \epsilon r - r^3 \\ \dot{\theta} = 1 \end{cases}.$$

It is called a **Hopf bifurcation** and corresponds to a creation of periodic orbit (at $r = \sqrt{\epsilon}$ for $\epsilon > 0$).

6 Index Theory and Dynamics on $\mathbb{S}^1$

§6.1 Index

Definition 6.1.1. A **path** in $\mathbb{R}^2$ is a continuous map $\gamma : [0,1] \to \mathbb{R}^2$.

Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the covering map, $\pi(\theta) = (\cos\theta, \sin\theta)$.

Theorem 6.1.2

Let $\gamma : [a,b] \to \mathbb{S}^1$ be a continuous map, then for every $\theta_0 \in \pi^{-1}(\gamma(a))$, there exists a unique continuous map $\theta : [a,b] \to \mathbb{R}$, $\theta(a) = \theta_0$ such that $\gamma = \pi \circ \theta$.

Lemma 6.1.3

Given a path $\gamma : [a,b] \to \mathbb{R}^2$ and a continuous vector field $f = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$. If $f|_\gamma \neq 0$, then there exists $\theta : [a,b] \to \mathbb{R}$, such that

$$\begin{cases} \cos\theta(t) = \dfrac{P(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \\ \sin\theta(t) = \dfrac{Q(\gamma(t))}{\sqrt{P^2(\gamma(t)) + Q^2(\gamma(t))}} \end{cases},$$

i.e. $\pi \circ \theta = \frac{f\circ\gamma}{|f\circ\gamma|}$. If there exists another continuous map $\widetilde{\theta}$ such that $\pi \circ \widetilde{\theta} = \frac{f\circ\gamma}{|f\circ\gamma|}$, then $\exists k \in \mathbb{Z}$ such that $\forall t \in [a,b], \widetilde{\theta}(t) - \theta(t) = 2k\pi$. In particular, $\theta(b) - \theta(a) = \widetilde{\theta}(b) - \widetilde{\theta}(a)$.

Definition 6.1.4. A path $\gamma : [a,b]$ is called **closed** if $\gamma(a) = \gamma(b)$.

Definition 6.1.5. Let $\gamma : [a,b] \to \mathbb{R}^2$ be a closed path, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field, $f|_\gamma \neq 0$. Then

$$\frac{\theta(b) - \theta(a)}{2\pi}$$

is called the **index** of γ with respect to f, denoted by $\mathrm{Ind}_f(\gamma)$. Where $\theta : [a,b] \to \mathbb{R}$ satisfying $\pi \circ \theta = \frac{f\circ\gamma}{|f\circ\gamma|}$.

Lemma 6.1.6

$\mathrm{Ind}_f(\gamma) \in \mathbb{Z}$ and the index is independent with the choice of θ.

Theorem 6.1.7 (Hopf)

Let f be a C^1 vector field, let $\gamma : [0,T] \to \mathbb{R}^2$ be a periodic orbit of f, i.e., $\dot\gamma(t) = f(\gamma(t))$, $\gamma(0) = \gamma(T)$, then $\mathrm{Ind}_f(\gamma) = \pm 1$.

Proof. WLOG, we can assume that $\gamma(0) = 0$ is the lowest point on $\gamma([0,T])$, and $\dot\gamma(t)$ is pointing the positive direction of x-axis (this might take the index to its opposite). Let $\Delta = \{(s,t) : 0 \leqslant s \leqslant t \leqslant T\}$, we define the map $\eta : \Delta \to \mathbb{S}^1$ as

$$\eta(s,t) = \begin{cases} \dfrac{f(\gamma(t))}{|f(\gamma(t))|}, & s = t, \\ -\eta(0,0), & (s,t) = (0,T), \\ \dfrac{\gamma(t)-\gamma(s)}{|\gamma(t)-\gamma(s)|}, & s \neq t, (s,t) \neq (0,T). \end{cases}$$

Then $\eta : \Delta \to \mathbb{S}^1$ is continuous. There exists a unique continuous map $\phi : \Delta \to \mathbb{R}$, $\phi(0,0) = 0$ and $\pi \circ \phi = \eta$. By definition, we have

$$\mathrm{Ind}_f(\gamma) = \frac{\phi(T,T) - \phi(0,0)}{2\pi}.$$

On the other hand, note that $\phi(0,t) \in [0,\pi]$ and $\phi(t,T) \in [\pi, 2\pi]$, we have

$$\phi(0,T) - \phi(0,0) = \pi - 0 = \pi \quad \text{and} \quad \phi(T,T) - \phi(0,T) = 2\pi - \pi = \pi.$$

Hence $\mathrm{Ind}_f(\gamma) = 1$. □

Definition 6.1.8. Let $\gamma_0, \gamma_1 : [a,b] \to \mathbb{R}^2$ be two paths, γ_0 and γ_1 are called **homotopic** if $\exists H : [a,b] \times [0,1] \to \mathbb{R}^2$ continuous such that $H(\cdot,0) = \gamma_0, H(\cdot,1) = \gamma_1$. H is called a **homotopy** of γ_0 and γ_1.

Theorem 6.1.9

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field, let $H : [a,b] \times [0,1] \to \mathbb{R}^2$ be a homotopy of closed paths γ_0 and γ_1. If $f \circ H \neq 0$, then $\mathrm{Ind}_f(\gamma_0) = \mathrm{Ind}_f(\gamma_1)$.

Theorem 6.1.10

Let $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be a continuous family of vector field, $\gamma : [a,b] \to \mathbb{R}^2$ be a closed path such that $f_s(\gamma(t)) \neq 0$ for every s, t. Then $\mathrm{Ind}_{f_s}(\gamma)$ is a constant with respect to s.

Corollary 6.1.11

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field, if $\gamma : [a,b] \to \mathbb{R}^2$ is a Jordan curve and there is no singularity of f on γ or inside γ, then $\mathrm{Ind}_f(\gamma) = 0$.

Corollary 6.1.12

Let f be a C^1 vector field, if $\gamma : [0,T] \to \mathbb{R}^2$ is a periodic orbit, then there exists a singularity inside γ.

Theorem 6.1.13

Let f be a continuous vector field, γ be a C^1 Jordan curve. If f is transverse to γ, then $\mathrm{Ind}_f(\gamma) = \pm 1$.

Proof. Let
$$f_s(\gamma(t)) = sf(\gamma(t)) + (1-s)\dot{\gamma}(t),$$
then $f_s(\gamma(t)) \neq 0$, hence
$$\mathrm{Ind}_f(\gamma) = \mathrm{Ind}_{f_1}(\gamma) = \mathrm{Ind}_{f_0}(\gamma) = \mathrm{Ind}_{\dot{\gamma}}(\gamma) = \pm 1.$$
□

Corollary 6.1.14

Let f be a continuous vector field, γ be a C^1 Jordan curve. If f is transverse to γ, then there exists a singularity inside γ.

§6.2 Index of a singularity

Definition 6.2.1. Let f be a continuous vector field, σ is a singularity. σ is called an **isolated singularity** if $\exists \varepsilon_0 > 0$ such that σ is the unique singularity in $B(\sigma, \varepsilon_0)$.

Lemma 6.2.2

Let σ be an isolated singularity, then for every $\varepsilon \in (0, \varepsilon_0]$, $\mathrm{Ind}_f(\gamma_\varepsilon) = \mathrm{Ind}_f(\gamma_{\varepsilon_0})$, where
$$\gamma_\varepsilon : [0, 2\pi] \to \mathbb{R}^2, \quad \gamma_\varepsilon(t) = \sigma + \varepsilon(\cos t, \sin t).$$

Definition 6.2.3. Let f be a continuous vector field and σ is an isolated singularity of f. There exists $\varepsilon_0 > 0$ such that σ is the unique singularity in $B(\sigma, \varepsilon_0)$. Then $\mathrm{Ind}_f(\gamma_{\varepsilon_0})$ is called the **index of isolated singularity** σ, denoted by $\mathrm{Ind}_f(\sigma)$.

Lemma 6.2.4

Let $f = (P, Q)$ be a C^1 vector field, $\gamma : [a, b] \to \mathbb{R}^2$ be a path such that $f|_\gamma \neq 0$. Let $\theta : [a, b] \to \mathbb{R}$ be a lifting of $\frac{f\circ\gamma}{|f\circ\gamma|}$, then
$$\theta(b) - \theta(a) = \int_\gamma \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2}.$$
In particular, if γ is a closed path, then
$$\mathrm{Ind}_f(\gamma) = \frac{1}{2\pi} \oint_\gamma \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2}.$$

Theorem 6.2.5

Let f be a C^2 vector field and $\gamma : [a,b] \to \mathbb{R}^2$ be a C^1 Jordan curve with positive direction, $f \circ \gamma \neq 0$. If there are only finite singularities inside γ, denoted by $\sigma_1, \cdots, \sigma_n$, then

$$\mathrm{Ind}_f(\gamma) = \mathrm{Ind}_f(\sigma_1) + \mathrm{Ind}_f(\sigma_2) + \cdots + \mathrm{Ind}_f(\sigma_n).$$

Remark 6.2.6 — This theorem also is also valid for just continuous settings.

Example 6.2.7

Consider

$$f_{\alpha,\beta} = (\alpha x - \beta y, \beta x + \alpha y), \quad (\alpha, \beta) \neq (0,0).$$

Then O is the unique singularity, consider $\gamma = \gamma(t) = (\cos t, \sin t)$. Then $\mathrm{Ind}_{f_{\alpha,\beta}}(\gamma)$ is the constant 1.

Example 6.2.8

Consider

$$g_{\lambda,\mu}(x,y) = (\lambda x, \mu y), \quad \lambda\mu \neq 0.$$

Then for $\lambda\mu > 0$, $\mathrm{Ind}_{g_{\lambda,\mu}} = 1$; for $\lambda\mu < 0$, $\mathrm{Ind}_{g_{\lambda,\mu}} = -1$.

Exercise 6.2.9. Let $z = (x,y) = x + iy$, $f_n(z) = z^n, n \in \mathbb{N}$. Regard f_n as a vector field on $\mathbb{R}^2$, then O is the unique singularity. Calculate $\mathrm{Ind}_{f_n}(O)$.

Exercise 6.2.10. Let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ be two bounded continuous functions. Show that there exists singularities in the vector field $f(x,y) = (P(x,y) - y, x + Q(x,y))$.

§6.3 Rotation numbers on the torus

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus, then $\mathbb{T}^2$ is an abelian group and a Riemannian manifold with a flat metric. Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the covering map, then for every map $f : \mathbb{T}^2 \to \mathbb{R}^2$, the map $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{f} = f \circ \pi$ is a doubly periodic function.

Lemma 6.3.1

Let $f : \mathbb{R}^2 \to X$ be a map. Then f is doubly periodic if and only if there exists $\overline{f} : \mathbb{T}^2 \to X$ such that $f = \overline{f} \circ \pi$.

In particular, a doubly periodic vector field on $\mathbb{R}^2$ can be regarded as a vector field on $\mathbb{T}^2$.

Lemma 6.3.2

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 doubly periodic vector field, let ϕ_t be the flow generated by f, then

$$\phi_t(p + k) = \phi_t(p) + k, \quad \forall k \in \mathbb{Z}^2, p \in \mathbb{R}^2.$$

Therefore, $\pi\circ : \mathbb{R}^2 \to \mathbb{T}^2$ is a doubly periodic map. There exists $\overline{\phi}_t : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\pi \circ \phi_t = \overline{\phi}_t \circ \pi$. Then $\overline{\phi}_0 = \mathrm{Id}, \overline{\phi}_{t+s} = \overline{\phi}_t \circ \overline{\phi}_s$, hence $\{\overline{\phi}_t\}$ is a flow on $\mathbb{T}^2$, which is generated by $\overline{f}$.

Example 6.3.3

Consider the constant vector field $f(x, y) = (1, \alpha)$ on $\mathbb{R}^2$. It induces a flow on $\mathbb{T}^2$.

(1) If $\alpha = \frac{p}{q}, p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1$. Then $\pi(x(q), y(q)) = \pi(x(0), y(0))$, hence every orbit is a periodic orbit with a common period q.

(2) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $K = \{(\overline{x}, \overline{y}) \in \mathbb{T}^2 : \overline{x} = 0\}$. Then the orbit intersects with K at

$$\{(0, \overline{n\alpha + y_0 - x_0\alpha}) : n \in \mathbb{Z}\}.$$

Theorem 6.3.4

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then every orbit of the flow generated by $(1, \alpha)$ is dense on $\mathbb{T}^2$.

The flow generated by $(1, \alpha)$ on $\mathbb{T}^2$ is called the **irrational flow** on $\mathbb{T}^2$.

Definition 6.3.5. Let X, be a compact metric space, a flow on X is called **minimal** if every orbit is dense in X.

$K = \{(\overline{x}, \overline{y}) \in \mathbb{T}^2 : \overline{x} = 0\}$ is a transversal of $f = (1, \alpha)$, then the orbit recurs to K at times one. It induces a map $K \to K$, which is also called **Poincaré recurrent map**.

Lemma 6.3.6

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1, doubly periodic vector field without singularities. Then there exists an closed curve Γ on $\mathbb{T}^2$ such that Γ is a transversal of $\overline{f}$ and Γ is not null-homotopic.

Proof. Let g be the vector orthogonal complement vector field of f. If Γ has some periodic orbit, take it. Otherwise, take some $q \in \omega(p, g)$, take a flow box near q. the the orbit of p will recurs to the flow box. We can closed the orbit by a C^1 curve transverse to f in the flow box.

For every transversal closed curve Γ of f, if Γ is null-homotopic, then Γ can be lifted to a closed curve $\widetilde{\Gamma}$ in $\mathbb{R}^2$. By index theory, $\mathrm{Ind}_f(\widetilde{\Gamma}) \neq 0$, which contradicts with f is without singularities. $\square$

Exercise 6.3.7. Let f be a C^1, non-singularity vector field on $\mathbb{T}^2$. $K \subset \mathbb{T}^2$ be a C^1 Jordan curve, transverse to f. If there exists $p_0 \in K$ and $t > 0$ such that $\phi_t(p_0) \in K$, then for every $p \in K$, there exists $t > 0$ such that $\phi_t(p) \in K$.

Under the condition of the exercise, we can define the Poincaré recurrent map $P : K \to K$ as $P(x) = \phi_{t(x)}(x)$. Then P is a orientation preserving diffeomorphism of K.

Exercise 6.3.8. Let (P, Q) be a C^1, doubly periodic vector field on $\mathbb{R}^2$, $P > 0$. Let $\phi(x; x_0, y_0)$ be the solution of the following initial problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{Q(x, y)}{P(x, y)}, \\ y(x_0) = y_0. \end{cases}$$

Show that:

1. $\phi(x; x_0, y_0 + 1) = \phi(x; x_0, y_0) + 1$;
2. $\phi(x + 1; x_0 + 1, y_0) = \phi(x; x_0, y_0)$;
3. $f(y_0) = \phi(1; 0, y_0)$.

Lemma 6.3.9

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function, assume $f(y + 1) = f(y) + 1$. Then there exists $\alpha \in \mathbb{R}$ and $C > 0$ such that

$$|f^n(y) - y - n\alpha| < C, \quad \forall y \in \mathbb{R}, n \in \mathbb{Z}. \tag{$*$}$$

In particular,

$$\alpha = \lim_{n\to\infty} \frac{f^n(y) - y}{n} = \alpha, \quad \forall y \in \mathbb{R}.$$

Definition 6.3.10. The constant α is called the **rotation number** of f, denoted by $\rho(f)$. We call f a **bounded mean motion** if f satisfies $(*)$.

Proof. Let $\phi(y) = f(y) - y$, then $\phi(y + 1) = \phi(y)$. For every $0 \leqslant y \leqslant z \leqslant 1$, we have

$$\phi(y) - \phi(z) = f(y) - f(z) - y + z \geqslant f(y) - f(z) \geqslant f(y) - f(z - 1) - 1 \geqslant -1,$$

$$\phi(y) - \phi(z) = f(y) - f(z) - y + z \leqslant f(y) - f(z) + 1 \leqslant 1,$$

hence $|\phi(y) - \phi(z)| = 1$. Moreover, $|\phi(y) - \phi(z)| \leqslant 1$ for every $y, z \in \mathbb{R}$. Given $n \in \mathbb{Z}_+$, f^n is also a strictly increasing function satisfying $f^n(y + 1) = f^n(y) + 1$, hence for every $y, z \in \mathbb{R}$, $|f^n(y) - y - f^n(z) + z| \leqslant 1$.

Now, we take $y = x$ and $z = f^m(x)$, we have $|f^n(x) - x - f^{n+m}(x) + f^m(x)| \leqslant 1$. Fix $x \in \mathbb{R}$, let $a_n = f^n(x) - x$. Then we have

$$|a_{m+n} - a_m - a_n| = |f^{m+n}(x) - f^m(x) - f^n(x) + x| \leqslant 1.$$

For every positive integer k, n, we have $|a_{kn} - ka_n| \leqslant k - 1$. Hence,

$$\frac{a_n}{n} - \frac{1}{n} \leqslant \frac{a_{kn}}{kn} - \frac{1}{kn} \leqslant \frac{a_{kn}}{kn} \leqslant \frac{a_{kn}}{kn} + \frac{1}{kn} \leqslant \frac{a_n}{n} + \frac{1}{n}.$$

Now we consider the intervals $I_n = [\frac{a_n}{n} - \frac{1}{n}, \frac{a_n}{n} + \frac{a_n}{n}]$. Then for every $n \geqslant 0$, we have $\frac{a_{n!}}{n!} \in \bigcap_{k=1}^n I_k$. Because $|I_n| \to 0$, hence $J = \cap_{n\geqslant 1} I_n$ is a single-point set $\{\alpha\}$. That is,

$$\lim_{n\to\infty} \frac{f^n(x) - x}{n} = \alpha.$$

Because for every $y, z \in \mathbb{R}$, $|f^n(y) - y - f^n(z) + z| \leqslant 1$, hence this limit is independent with the choice of x. $\square$

Exercise 6.3.11. If $f : \mathbb{R} \to \mathbb{R}$ is increasing but not strictly increasing, $f(y + 1) = f(y) + 1$, does the conclusion still hold?

Proposition 6.3.12

$\rho(f)$ is continuous with respect to f, i.e. for every $\varepsilon > 0, \exists \delta > 0$, such that for every g satisfying the same condition, $\|g - f\|_\infty < \delta$, we have $|\rho(g) - \rho(f)| < \varepsilon$.

Proof. By the previous lemma, we have $|f^n(x) - x - n\alpha| < 1$, take N such that $\frac{2}{N} < \varepsilon$. Take $\delta > 0$, such that for $\|g - f\|_\infty < \delta$, we have $|g^N(x) - x - (f^N(x) - x)| < 1$. □

Theorem 6.3.13 (Poincaré)

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. Then $\rho(f) \in \mathbb{Q}$ if and only if $\overline{f} : \mathbb{T} \to \mathbb{T}$ has some periodic points.

Proof. We prove that if $\rho(f) = p/q$, then there exists $x \in \mathbb{R}$ such that $f^q(x) = x + p$. Let $\psi(y) = f^q(y) - y - p$, let $\alpha = \min \psi(y)$ and $\beta = \max \psi(y)$, we want to show $\alpha \leqslant 0$ and $\beta \geqslant 0$. Otherwise, if $\alpha > 0$, then

$$f^{nq}(y) - y = \sum_{k=0}^{n-1} (f^{(k+1)q}(y) - f^{kq}(y)) \geqslant n(p + \alpha).$$

Hence $\rho(f) \geqslant \frac{p+\alpha}{q}$, a contradiction. Then we get $\alpha \leqslant 0$ and $\beta \geqslant 0$, hence there exists x such that $f^q(x) = x + p$. □

§6.4 Denjoy's theorem

Definition 6.4.1. Let (X, d) be a metric space, $f : X \to X$ continuous. For $x \in X$, $y \in X$ is called a **positive limit point** of x if $\exists n_i \to \infty$, $f^{n_i}x \to y$. The **positive limit set** $\omega(x)$ is the set of all positive limit points of x.

Lemma 6.4.2

Let (X, d) be a metric space, $f : X \to X$ continuous. Then

1. $\forall x \in X, \omega(x) = \bigcap_{n=0}^\infty \overline{\mathrm{Orb}^+(f^n x)}$.
2. $\forall x \in X, \omega(x)$ is closed and $f(\omega(x)) = \omega(x)$.
3. If X is compact, then $\omega(x) \neq \varnothing$.

If $f : X \to X$ is a homeomorphism, then we can define negative limit points and the negative limit set $\alpha(x)$ similarly.

Definition 6.4.3. Let (X, d) be a metric space, $f : X \to X$ continuous. f is called **minimal** if $\forall x \in X$, $\overline{\mathrm{Orb}^+(x)} = X$. Let $\Lambda \subset X$ be a closed subset such that $f(\Lambda) = \Lambda$ and $\forall x \in \Lambda$, $\overline{\mathrm{Orb}^+(x)} = \Lambda$, then Λ is called a **minimal set**.

Theorem 6.4.4 (Poincaré)

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. If $\rho(f) \notin \mathbb{Q}$, then for every $\overline{x} \in \mathbb{T}, \omega(\overline{x})$ is a minimal set of $\overline{f}$. Moreover, $\forall \overline{y} \in \mathbb{T}, \omega(\overline{y}) = \omega(\overline{x})$.

Proof. It suffices to show $\omega(\overline{x}) = \omega(\overline{y})$ for every $\overline{x}, \overline{y} \in \mathbb{T}$. If $\omega(\overline{x}) \neq \omega(\overline{y})$, WLOG, assume that $\omega(\overline{x}) \neq \mathbb{T}$, we will show that $\omega(\overline{y}) \subseteq \omega(\overline{x})$. Otherwise, there must be $\overline{y} \notin \omega(\overline{x})$. Let $(\overline{a}, \overline{b}) \ni \overline{y}$ be a connected component of $\mathbb{T} \setminus \omega(\overline{x})$. Then $\overline{f}^n(\overline{a}, \overline{b})$ is a connected component of $\mathbb{T} \setminus \omega(\overline{x})$. Because $\rho(f) \notin \mathbb{Q}$, these intervals are disjoint. Hence $\operatorname{diam} \overline{f}^n(\overline{a}, \overline{b}) \to 0$. Hence $\omega(\overline{y}) \subseteq \omega(\overline{a}) \subseteq \omega(\overline{x})$. □

Remark 6.4.5 — Note that $\partial\omega(\overline{x})$ is closed and invariant, hence $\partial\omega(\overline{x}) = \varnothing$ or $\omega(\overline{x})$, which shows that $\omega(\overline{x})$ is either $\mathbb{T}$ or a Cantor set.

Definition 6.4.6. $\overline{f}$ is called **ergodic** if $\omega(\overline{x}) = \mathbb{T}$. Otherwise, $\overline{f}$ is said to be **exceptional**.

Theorem 6.4.7

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. If $\overline{y}$ is ergodic, then there exists a strictly increasing function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + 1$, $h \circ f = R_\alpha \circ h$ where $\alpha = \rho(f)$ and $R_\alpha(x) = x + \alpha$.

Proof. Let $h(f^n 0 + k) = n\alpha + k$ for every $n, k \in \mathbb{Z}$, then h is well-defined and strictly increasing. Also, h can be uniquely extended to $\mathbb{R}$ continuously. □

Theorem 6.4.8

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function, assume $f(y+1) = f(y) + 1$. Then there exists an increasing function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + 1$, $h \circ f = R_\alpha \circ h$ where $\alpha = \rho(f)$ and $R_\alpha(x) = x + \alpha$.

Remark 6.4.9 — In this case, h is called a **semi-conjugate**.

Definition 6.4.10. Let $f : \mathbb{T} \to \mathbb{T}$ be continuous, if exists a C^k lifting $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, then f is called C^k. Besides, f is called a **C^k diffeomorphism** if f is invertible and f, f^{-1} are both C^k.

Remark 6.4.11 — If $f : \mathbb{T} \to \mathbb{T}$ is C^1, then $f'(\overline{x}) := \widetilde{f}'(x)$ is well-defined. f is an orientation-preserving diffeomorphism iff $f' > 0$.

Definition 6.4.12. A function $g : \mathbb{T} \to \mathbb{R}$ is called of bounded variation if $\exists C > 0$ such that for every disjoint intervals $I_k = (x_k, y_k)$,

$$\sum_{k=1}^{n} |g(x_k) - g(y_k)| \leqslant C.$$

The smallest C satisfying the inequality is called the **total variation** of g, denoted by $\operatorname{Var}(g)$.

Lemma 6.4.13

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving diffeomorphism, then f' is of bounded variation iff $\log f'$ is of bounded variation.

Lemma 6.4.14 (distortion estimate)

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving diffeomorphism, assume f' is of bounded variation. If an interval $I \subset \mathbb{T}$ satisfies $I, fI, \cdots, f^{n-1}I$ are disjoint. Then for every $x, y \in I$,

$$e^{-V} \leqslant \frac{(f^n)'(x)}{(f^n)'(y)} \leqslant e^V, \quad V = \mathrm{Var}(\log f'). \tag{$*$}$$

Remark 6.4.15 — Note that the bound in $(*)$ is independent with n, this type of estimates are called **bounded distortion**.

Definition 6.4.16. Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $p, q \in \mathbb{Z}, q > 0$. The rational number p/q is called a **best rational approximation** of α if

$$|q\alpha - p| < |q'\alpha - p'|, \quad \forall p', q' \in \mathbb{Z}, 0 < q' \leqslant q, \frac{p'}{q'} \neq \frac{p}{q}.$$

Lemma 6.4.17

Every irrational number has infinite many best rational approximations.

Lemma 6.4.18

Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let p_n/q_n be the n-th best approximation of α. Then

$$(p_n/q_n - \alpha)(p_{n+1}/q_{n+1} - \alpha) < 0.$$

Lemma 6.4.19

Assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $R_\alpha : \mathbb{T} \to \mathbb{T}$ be the irrational rotation. Let p_n/q_n be the n-th best approximation of α, let I be the minor arc with endpoints x and $R_\alpha^{q_n}(x)$, then $I, R_\alpha I, \cdots, R_\alpha^{q_{n+1}-1} I$ are disjoint.

Corollary 6.4.20

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving diffeomorphism, $\rho(f)$ is irrational. Then there exists infinite many positive integer n such that for every $y \in \mathbb{T}$, the intervals $I, fI, \cdots, f^n I$ are disjoint, where I is an open interval with endpoints y and $f^{-n}y$.

Theorem 6.4.21 (Denjoy)

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving diffeomorphism, assume f' is of bounded variation. If $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$, then f is ergodic.

Proof. Otherwise, let Λ be the unique minimal set, let I be a connected component of $\mathbb{T} \setminus \Lambda$. There are infinite many n, such that for every $x \in \mathbb{T}$, the intervals $J, fJ, \cdots, f^n J$ are disjoint, where J is an open interval with endpoints x and $f^{-n}x$. By the distortion estimate, we have

$$(f^n)'(x)(f^{-n})'(x) = \frac{(f^n)'(x)}{(f^n)'(f^{-n}x)} \in [e^{-V}, e^V], \quad V = \mathrm{Var}(\log f').$$

We apply this estimate on I, we have

$$|f^n I| + |f^{-n} I| \geqslant \int_I (f^n)'(x) + (f^{-n})'(x)\mathrm{d}x \geqslant 2e^{-V/2}|I|.$$

Note that $\{f^n I\}_{n\in\mathbb{Z}}$ are disjoint, this estimate contradicts with $\sum_{n\in\mathbb{Z}} |f^n I| < \infty$. $\square$

Exercise 6.4.22. Consider the one-parameter diffeomorphisms $f_c : \mathbb{R} \to \mathbb{R}, x \mapsto x + \frac{1}{8}\sin 2\pi x + c$. Define $\rho : \mathbb{R} \to \mathbb{R}$ as $c \mapsto \rho(f_c)$. Show that:

1. ρ is increasing.
2. ρ is continuous and $\rho(\mathbb{R}) = \mathbb{R}$.
3. ρ is not strictly increasing.

§6.5 Dynamics on $\mathbb{T}^2$

Let (P, Q) be a doubly periodic vector field on $\mathbb{R}^2$, where $P > 0$. Let $\phi(x, y_0)$ be the solution of the initial problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{Q(x,y)}{P(x,y)}, \\ y(0) = y_0. \end{cases}$$

Let $f(y_0) = \phi(1, y_0)$, and $\overline{f} : \mathbb{T} \to \mathbb{T}$ be the corresponding map.

Theorem 6.5.1

If $\overline{f}$ is ergodic, then there exists a double periodic function $w : \mathbb{R}^2 \to \mathbb{R}^2$. Such that for every solution $y = g(x)$, there exists $c \in \mathbb{R}$,

$$g(x) = \alpha x + c + w(x, \alpha x + c),$$

where $\alpha = \rho(f)$.