THE BOX DIMENSION FOR MINIMAL SETS OF CIRCLE DIFFEOMORPHISM GROUP ACTIONS

YUXIANG JIAO

Abstract. We show that the exceptional minimal set of a finitely generated subgroup of $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$ always has the same box dimension and Hausdorff dimension. In particular, according to [HJX23], its box dimension is strictly less than one. Our proof is based on the examination of conformal measures on the minimal set, extending the Patterson-Sullivan theory from Fuchsian groups to groups of circle diffeomorphisms.

Contents

1. Introduction

The investigation of dimensional properties in dynamical limits constitutes a fundamental question in the study of dynamical systems. Particularly in certain conformal systems, the theory of dimensions for these fractals is well-developed. This encompasses several notable results concerning important objects, such as Julia sets for one variable complex dynamics [ADU93; AL08; AL22; DU92; McM00; Sul83]; the limit sets for Fuchsian and Kleinian groups [Bea68; BJ97; Pat76; SU96; Sul79]; self-similar measures on $\mathbb{R}$ [Hoc14; Var19]; Furstenberg measure on $\mathbb{RP}^1$ [HS17; Led83].

In this article, we focus on studying the dimensional properties for another fundamental class of conformal systems: smooth group actions on the circle. Let G be a group of circle homeomorphisms without finite orbits on the circle. Then there is a unique G-invariant minimal set, which is either the entire circle or a Cantor set (see for example [Nav11, Theorem 2.1.1]). The later case is recognized as G preserving an *exceptional minimal set* Λ. We are interested in exploring the dimensional properties of Λ.

Under a smoothness condition for G, the dimension theory for Λ is established in [HJX23] as the following, which can be regarded as a generalization of a classic theory for second kind Fuchsian groups.

Theorem 1.1 ([HJX23, Main Theorem]). *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Then*

(1) $0 < \dim_{\mathrm{H}} \Lambda < 1$,
(2) $\dim_{\mathrm{H}} \Lambda = \delta(G)$,
(3) if there exists a $(k+1)$-multiple parabolic fixed point on Λ then $\dim_{\mathrm{H}} \Lambda > k/(k+1)$,

wherein $\delta(G) := \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \#\{ g \in G : \exists x \in \Lambda,\, g'|_{B(x,\varepsilon)} \geqslant e^{-n} \}$ *is the dynamical critical exponent of G.*

In addition to the Hausdorff dimension, the box dimension is another commonly considered measure of dimension (the precise definitions of these two notions will be provided in Section 2.4). For the limit sets of finitely generated Fuchsian groups, these two dimensions coincide by the classic theory. In this article, we extend this result for general real analytic smooth group actions on the circle.

Theorem 1.2. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Then the box dimension of Λ exists and* $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$. *In particular,* $\dim_{\mathrm{B}} \Lambda < 1$.

Our study of the property of Λ is closely related to investigations into the regularity of Λ. Motivated by a conjecture on codimension-one foliations proposed by Ghys-Hector-Sullivan, there is a question of whether every exceptional minimal set Λ is a Lebesgue null set for every finitely generated subgroup $G \subset \mathrm{Diff}^2_{+}(\mathbb{S}^1)$. Deroin-Kleptsyn-Navas developed a theory, extending Sullivan's expanding strategy, to study C^2-actions on the circle by diffeomorphisms [DKN09]. They also confirmed that $\mathrm{Leb}(\Lambda) = 0$ for the real analytic case [DKN18].

Their theory also provides useful tools for studying the dimension theory of Λ (see [HJX23]). Therefore, we are also interested in whether Λ has good dimensional properties for general Λ preserved by a finitely generated $G \subset \mathrm{Diff}^2_{+}(\mathbb{S}^1)$. It is worth noting that Λ could has different box dimension and Hausdorff dimension whenever $G \subset \mathrm{Diff}^{1+\alpha}(\mathbb{S}^1)$, as demonstrated by Denjoy examples, see for example [KS02].

Question 1.3. *Let $G \subset \mathrm{Diff}^2_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Does* $\dim_{\mathrm{B}} \Lambda$ *exist and coincide with* $\dim_{\mathrm{H}} \Lambda$*?*

The core of the proof is to extend the Patterson-Sullivan theory to smooth group actions on the circle. In the context of conformal dynamics, Sullivan introduced the concept of *conformal measures* [Sul83], which serve as a powerful tool for studying the dimension theory of dynamical limits.

Definition 1.4. For a subgroup $G \subset \mathrm{Diff}(\mathbb{S}^1)$, a Borel probability measure μ on $\mathbb{S}^1$ is a *conformal measure with exponent δ (δ-conformal measure)* for G, if for every Borel subset $E \subset \mathbb{S}^1$ and $g \in G$, we have

$$\mu(gE) = \int_E |g'(s)|^{\delta} \, \mathrm{d}\mu(s).$$

For every group G of circle diffeomorphisms, Lebesgue measure is a 1-conformal measure. However, in the case where G preserves an exceptional minimal set Λ, the conformal measure supported on Λ, particularly the atomless one, becomes the crucial object carrying the dimensional information of Λ. This conformal measure can be seen as an analogue of the Patterson-Sullivan measure for Fuchsian groups, and several of its properties were established in [DKN09; HJX23], which we summarize here.

Theorem 1.5. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ.*

(1) [DKN09, Theorem F] There exists at most one atomless conformal measure on Λ. If such measure exists then the exponent belongs to $(0, 1)$.

(2) [HJX23, Theorem 2.23] If there exists an atomless δ-conformal measure on Λ then $\delta = \dim_{\mathrm{H}} \Lambda$.

Unfortunately, the existence of an atomless conformal measure on Λ remains unclear. In [HJX23], the authors constructed an atomless conformal measure μ on Λ under a specific technical condition for G. However, this condition does not hold for general groups. In this article, we can overcome this condition and construct such measures in general cases.

Theorem 1.6. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Then there exists a unique atomless δ-conformal measure on Λ.*

A direct application of this theorem is $0 < \dim_{\mathrm{H}} \Lambda < 1$ by combining Theorem 1.5, which was also shown in [HJX23]. In order to use the atomless conformal measure μ to study the box dimension of Λ, we require some local dimension estimate for μ itself. We have the following uniform estimate of the upper local dimension of μ. Such estimate draws inspiration from the study of conformal measures for parabolic rational maps on $\overline{\mathbb{C}}$ [DU92].

Theorem 1.7. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Let $\Lambda' := \Lambda \setminus G(\mathrm{NE}(G))$, where $\mathrm{NE}(G) \cap \Lambda$ is a finite set of non-expandable points on Λ (see Section 2.3). Then there exists $c_0 > 0$ such that for every $r > 0$ small enough and $x \in \Lambda'$, we have $\mu(B(x, r)) \geqslant c_0 \cdot r^{\delta}$.*

Remark 1.8. Here Λ' corresponds to the conical limit set for Fuchsian groups. In fact, it can be shown that $G(\mathrm{NE}(G)) \cap \Lambda$ is exactly the set of points in Λ which are fixed by some element $f \in G$ as a parabolic fixed point.

Proof of Theorem 1.2. By Theorem 1.6, there exists an atomless δ-conformal measure of G on Λ. Combining the previous theorem with Lemma 2.10, we have $\overline{\dim}_{\mathrm{B}}\Lambda' \leqslant \delta$. By Theorem 1.5 (2), $\delta = \dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} \Lambda'$. Therefore,

$$\dim_{\mathrm{H}} \Lambda' \leqslant \underline{\dim}_{\mathrm{B}}\Lambda' \leqslant \overline{\dim}_{\mathrm{B}}\Lambda' \leqslant \delta = \dim_{\mathrm{H}} \Lambda'.$$

Hence the box dimension of Λ' exists and corresponds to $\dim_{\mathrm{H}} \Lambda'$. Taking into account that $\Lambda = \overline{\Lambda'}$, we have $\dim_{\mathrm{B}} \Lambda = \dim_{\mathrm{B}} \Lambda' = \dim_{\mathrm{H}} \Lambda$. □

Notation. In this article, we will use Vinogradov notations $A \ll B$ (A is dominated by B) and $A \ll_C B$ if the implicit constant depending only on C. We also denote $A \asymp B$ (resp. $A \asymp_C B$) if both $A \ll B$ and $A \ll B$ (resp. $A \ll_C B$ and $B \ll_C A$).

For a Borel subset $E \subset \mathbb{S}^1$, we use $|E|$ to denote the Lebesgue measure of E. To distinguish notation, we use $\#F$ to denote the cardinality of a finite set F.

Organization. In Section 2, we recall some preliminaries on distortion estimates, dimensional properties and group actions on the circle. Then we construct atomless conformal measures in Section 3 (the proof of Theorem 1.6), wherein a useful lemma of group theory is proved in Appendix A. Section 4 is devoted to study the dimensional properties of conformal measures (proof of Theorem 1.5).

Acknowledgement. The author would like to thank Weikun He, Disheng Xu, and Wenyuan Yang for their valuable discussions. Part of this work was completed during the author's visit to Great Bay University, where the author gratefully acknowledges their kind hospitality.

2. Preliminaries

2.1. **Distortion controls.** Let $f \in \mathrm{Diff}_+^2(\mathbb{S}^1)$ and $I \subset \mathbb{S}^1$. We denote

$$\varkappa(f, I) := \sup_{x,y \in I} |\log f'(x) - \log f'(y)|$$

to be the *distortion coefficient* of f on I. Then

(1) $\varkappa(f, I) \leqslant \|(\log f')'|_I\| \cdot |I|$, where $\|\cdot\|$ denotes the L^∞-norm.
(2) $\varkappa(f_1 f_2, I) \leqslant \varkappa(f_1, f_2 I) + \varkappa(f_2, I)$.

Now we fix a finite subset $S \subset \mathrm{Diff}_+^2(\mathbb{S}^1)$. Let $L = L(S) := \max_{g \in S} \|(\log g')'\|$ For an element $f = g_n \circ \cdots \circ g_1$ with $g_i \in S$, we denote $f_0 = \mathrm{id}$ and $f_i = g_i \circ \cdots \circ g_1$ for every $1 \leqslant i \leqslant n-1$. For an interval $I \subset \mathbb{S}^1$, we denote $I_i = f_i I$. Then

$$\varkappa(f, I) \leqslant \sum_{i=0}^{n-1} \varkappa(g_{i+1}, f_i I) \leqslant \sum_{i=0}^{n-1} \|(\log g_{i+1}')'|_{I_i}\| \cdot |I_i| \leqslant L \sum_{i=0}^{n-1} |I_i|.$$

Moreover, we have the following proposition of distortion control. Such kind of estimates date back to [Den32; Sul83]. The following version can be found in [DKN18].

Proposition 2.1. *Let $x_0 \in \mathbb{S}^1$ be a point and $f, (f_i)_{0 \leqslant i \leqslant n-1}$ be as above. Let $D = \sum_{i=0}^{n-1} f_i'(x_0)$. For every positive $\delta \leqslant (2LD)^{-1}$, we have $\varkappa(f, B(x_0, \delta)) \leqslant 2LD\delta$.*

2.2. **The behavior near parabolic fixed points.** Let $f \in \mathrm{Diff}_+^1(\mathbb{S}^1)$ and x_0 be a fixed point of f. We say that x_0 is a *parabolic fixed point* of f if $f'(x_0) = 1$. Now we describe the behavior of f near the parabolic fixed point. To simplify the statements, we assume additionally that f is real analytic on a neighborhood of x_0. For every such $f \neq \mathrm{id}$, there exists a unique positive integer k such that

$$f'(x_0) = 1, \quad f^{(2)}(x_0) = \cdots = f^{(k)}(x_0) = 0, \quad f^{(k+1)}(x_0) \neq 0.$$

Then we say that x_0 is of *multiplicity* $(k+1)$.

We consider a subinterval $[x_0, y_0] \subset \mathbb{S}^1$. Assume that f is real analytic and strictly contracting on $[x_0, y_0]$ for some $y_0 \neq x_0$. Let $y_n := f^n y_0$ for every $n \geqslant 0$. Then $y_n \to x_0$ as $n \to +\infty$. We denote $I_n := [y_n, y_{n-1}]$ for every $n \geqslant 1$, then $\{I_n : n \geqslant 1\}$ gives a close cover of $]x_0, y_0]$. We collect some useful estimates (the proof can be found in [HJX23]).

Lemma 2.2. *There exists $C_1 > 1$ such that for every $m \geqslant 0$ and $n \geqslant 1$, we have*

(1) $|[x_0, y_{n-1}]| \asymp_{C_1} n^{-1/k}$;
(2) $|I_n| \asymp_{C_1} n^{-(k+1)/k}$;
(3) $\varkappa(f^m, I_n) \ll_{C_1} 1$;
(4) $(f^m)'|_{I_n} \asymp_{C_1} \left(\frac{n}{n+m}\right)^{(k+1)/k}$, in particular, $(f^m)'|_{[x_0,y_0]} \gg_{C_1} (m+1)^{-(k+1)/k}$.

Corollary 2.3. *There exists $C_2 > 1$ such that for every $m \geqslant 0$, $n \geqslant 1$ and $z \in I_n$, we have*

(1) $\varkappa(f^m, [fz, z]) \ll_{C_2} 1$,
(2) $|[fz, z]| \asymp_{C_2} n^{-(k+1)/k}$.

Proof. (1) Note that $[fz, z] \subset I_n \cup I_{n+1} = [x_{n+1}, x_{n-1}]$, we have

$$\varkappa(f^m, [fz, z]) \leqslant \varkappa(f^m, I_n) + \varkappa(f^m, I_{n+1}) \ll_{C_1} 1.$$

(2) Consider the Taylor expansion of f around x_0, there is $C > 0$ such that

$$f(z) - z = (f(z) - z) - (f(x_0) - x_0) \asymp_C (z - x_0)^{1/(k+1)}.$$

Since $z \in I_n$, we have $z - x_0 \asymp_{C_1} n^{-1/k}$. Hence $|[fz, z]| \asymp_{C_2} n^{-(k+1)/k}$ for some $C_2 > 1$. $\square$

2.3. **Group actions on the circle.** In this subsection, we collect some useful facts about group actions on the circle.

Lemma 2.4 ([Alo+24, Lemma 7.12]). *Let $G \subset \mathrm{Homeo}(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ and $G_1 < G$ be a subgroup of finite index. Then Λ is also the G_1-minimal.*

Lemma 2.5 ([HJX23, Proposition 7.8]). *Let $G \subset \mathrm{Diff}^2_+(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ. Then there exists $f \in G$ has only finitely many fixed points, all of which are hyperbolic. Moreover, for every open set U intersecting Λ, the element f can be chosen such that at least one of its fixed point falls in U.*

Now we recall some notions and results introduced in [DKN09] to study smooth group actions on the circle. Property $(\Lambda\star)$ below leads to several consequences on the property of the minimal set Λ.

Definition 2.6. A point $x \in \mathbb{S}^1$ is *non-expandable* for the action of $G \subset \mathrm{Diff}^1_+(\mathbb{S}^1)$ if $g'(x) \leqslant 1$ for every $g \in G$. We denote by $\mathrm{NE} = \mathrm{NE}(G)$ the set of non-expandable points.

Definition 2.7. Let $G \subset \mathrm{Diff}^1(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ. We say $G \subset \mathrm{Diff}^1(\mathbb{S}^1)$ satisfies *property* $(\Lambda\star)$ if for every $x \in \mathrm{NE} \cap \Lambda$, there exists $g_+, g_- \in G$ such that $g_+(x) = g_-(x) = x$ and x is an isolated-from-the-right (resp. isolated-from-the-left) point of the set of fixed points $\mathrm{Fix}(g_+)$ (resp. $\mathrm{Fix}(g_-)$).

Theorem 2.8 ([DKN09, Theorem D]). *Let $G \subset \mathrm{Diff}^2_+(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Assume that G satisfies property $(\Lambda\star)$. Then $\mathrm{NE}(G) \cap \Lambda$ is a finite set.*

Deroin-Kleptsyn-Navas also made a breakthrough on the verification of the property fo ther real analytic case [DKN18].

Theorem 2.9. *Let $G \subset \mathrm{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ, then G satisfies property $(\Lambda\star)$.*

2.4. **Different notions of dimensions.** For a subset A in a metric space, there are two notions of the dimension of A usually took into account, the box dimension (or Minkowski dimension) and the Hausdorff dimension. The *upper and lower box dimension* of A are defined to be

$$\overline{\dim}_{\mathrm{B}} A := \limsup_{r \to 0^+} -\frac{\log \mathrm{Cov}(A, r)}{\log r} \quad \text{and} \quad \underline{\dim}_{\mathrm{B}} A := \liminf_{r \to 0^+} -\frac{\log \mathrm{Cov}(A, r)}{\log r},$$

where $\mathrm{Cov}(A, r)$ is the minimal number of r-balls covering A. If the upper and lower box dimensions of A coincide, the quantity is said to be the *box dimension* of A and denoted by $\dim_{\mathrm{B}} A$. A basic property of the box dimension is that $\dim_{\mathrm{B}} A = \dim_{\mathrm{B}} \overline{A}$, where $\overline{A}$ is the closure of A in the ambient metric space.

For every $s > 0$, the *s-dimensional Hausdorff outer measure* of A is defined to be

$$\mathcal{H}^s(A) := \lim_{r \to 0^+} \mathcal{H}^s_r(A),$$

where

$$\mathcal{H}^s_r(A) := \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : A \subset \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam} U_i \leqslant r \right\}.$$

The *Hausdorff dimension* of A is the unique $\alpha \geqslant 0$ such that $\mathcal{H}^s(A) = 0$ for $s > \alpha$ and $\mathcal{H}^s(A) = \infty$ for $s < \alpha$. We denote $\dim_{\mathrm{H}} A = \alpha$. It follows directly by definition that

$$\overline{\dim}_{\mathrm{B}} A \geqslant \underline{\dim}_{\mathrm{B}} A \geqslant \dim_{\mathrm{H}} A.$$

The dimension properties of sets are related to that of measures by the following lemma.

Lemma 2.10. *Let μ be a Borel probability measure on $\mathbb{S}^1$ and $A \subset \mathbb{S}^1$ be a Borel subset. If there exists $c > 0, \alpha > 0$ such that for every $r > 0$ small enough, $\mu(B(x,r)) \geqslant c \cdot r^{\alpha}$ for every $x \in A$, then $\overline{\dim}_{\mathrm{B}} A \leqslant \alpha$.*

Proof. For every $r > 0$ small enough, let $\mathscr{A}(r)$ be a cover of A by r-balls with the minimal cardinality. For each $B \in \mathscr{A}(r)$, we fix a point $x \in A \cap B$ and let $B' = B(x, 2r) \supset B$. Let $\mathscr{A}'(r)$ be the collection of all such B''s. Then $\mathscr{A}'(r)$ gives a cover of A by $2r$-balls, each ball is centered at a point in A and $\#\mathscr{A}'(r) = \#\mathscr{A}(r) = \operatorname{Cov}(A, r)$. We require the following two lemmas on the multiplicity of the cover $\mathscr{A}'(r)$.

Lemma 2.11. *For every $x \in \mathbb{S}^1$, there exists at most 2 balls $B \in \mathscr{A}(r)$ satisfying $x \in B$.*

Proof. Assume for a contradiction that there are $\{B_i\}_{i=1}^3 \subset \mathscr{A}(r)$ with $B_i \ni x$. For $r \leqslant 1/10$, $B(x, 2r)$ is a subinterval of $\mathbb{S}^1$ and $B_i \subset B(x, 2r)$. Assume without loss of generality that the centers of B_1, B_2, B_3 are arranged in order on the interval $B(x, 2r)$. Then $B_2 \subset B_1 \cup B_3$. Hence $\mathscr{A}(r) \setminus \{B_2\}$ also forms a cover of A, which contradicts that $\mathscr{A}(r)$ is of minimal cardinality. □

Lemma 2.12. *For every $x \in \mathbb{S}^1$, there exists at most 8 balls $B' \in \mathscr{A}'(r)$ satisfying $x \in B'$.*

Proof. Assume for a contradiction that there are $\{B'_i\}_{i=1}^9 \subset \mathscr{A}(r)$ with $B'_i \ni x$. Let $x_i \in A$ be the center of B'_i. Therefore $x_i \in B(x, 2r)$ for every i. Let $B_i \in \mathscr{A}(r)$ be the r-ball corresponding to B'_i. Note that $x_i \in B_i$ and $|B_i| = 2r$. We have $B_i \subset B(x, 4r)$. However,

$$\sum_{i=1}^{9} |B_i| = 18r > 2 \cdot |B(x, 4r)|.$$

Then there exists $y \in B(x, 4r)$ which belongs to at least three $B_i \in \mathscr{A}(r)$ by the pigeonhole principle. This contradicts the former lemma. □

Since every point is covered by at most 8 times of some ball in $\mathscr{A}'(r)$, we have

$$\sum_{B' \in \mathscr{A}'(r)} \mu(B') \leqslant 8 \cdot \mu(\mathbb{S}^1) = 8.$$

On the other hand, we have $\mu(B') \geqslant c \cdot (2r)^{\alpha}$ for every $B' \in \mathscr{A}'(r)$. Therefore, $\operatorname{Cov}(A, r) = \#\mathscr{A}'(r) \leqslant 8c^{-1} \cdot r^{-\alpha}$ for every $r > 0$ small enough. Hence $\overline{\dim}_{\mathrm{B}} A \leqslant \alpha$. □

3. Construction of atomless conformal measures

This section is devoted to show the existence of atomless conformal measures, i.e. the proof of Theorem 1.6. The strategy of construction is similar to the strategy of proof of [HJX23, Theorem 2.24]. In order to overcome the technical condition, we need to modify the behavior of G outside the minimal set. However, the rigidity of real analytic maps does not allow us to modify the G-action. So we will first construct the conformal measure for some C^{∞} cases.

3.1. The construction of conformal measures for certain C^∞ cases.

Definition 3.1. Let $G \subset \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ. For an element $f \in G$ and a connected component J of $\mathbb{S}^1 \setminus \Lambda$, we say (f, J) is a *fundamental pair* of G if

(1) the subgroup $\mathrm{Stab}_G(J) := \{g \in G : g(J) = J\}$ is exactly the cyclic group $\langle f \rangle$, and
(2) if $f \neq \mathrm{id}$ then f has at most one fixed point in J and the fixed point is hyperbolic if exists.

Remark 3.2. We allow f to be the identity map in this definition. In particular, (id, J) is a fundamental pair of G if $\mathrm{Stab}_G(J)$ is trivial.

Recall that the pointwise dynamical critical exponent introduced in [HJX23] is given by

$$\delta(G, x) := \lim_{\varepsilon\to\infty} \limsup_{n\to+\infty} \frac{1}{n} \log \# \left\{g \in G : g'|_{B(x,\varepsilon)} \geqslant e^{-n}\right\}.$$

Lemma 3.3. *Let $G \subset \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ be a finitely generated subgroup with a fundamental pair (f, J). Then for every $x \in J$ which is not fixed by $\langle f \rangle \setminus \{\mathrm{id}\}$, we have*

$$\delta(G, x) = \limsup_{n\to\infty} \frac{1}{n} \log \# \left\{g \in G : g'(x) \geqslant e^{-n}\right\}.$$

Proof. For every such point x, there exists $\varepsilon_0 > 0$ such that $f^m B(x, \varepsilon_0) \cap f^n B(x, \varepsilon_0) = \varnothing$ for every $f^m \neq f^n \in \langle f \rangle$. Since J is a connected component of $\mathbb{S}^1 \setminus \Lambda$, for every element $g \in G$, $g(J) \cap J \neq \varnothing$ if and only if $g(J) = J$. Because of $\mathrm{Stab}_G(J) = \langle f \rangle$, we have

$$g_1 B(x, \varepsilon_0) \cap g_2 B(x, \varepsilon_0) = \varnothing, \quad \forall g_1 \neq g_2 \in G.$$

In particular, this shows that $\sum_{g\in G} |gB(x, \varepsilon_0)| \leqslant 1$.

Let S be a finite symmetric generator of G. For an element $g \in G$, write $g = \gamma_m \cdots \gamma_1$ with $\gamma_i \in S$ in the simplest way. Let $g_i = \gamma_i \cdots \gamma_1$. Then g_i's are pairwise distinct. We have

$$\varkappa(g, B(x, \varepsilon_0)) \leqslant \sum_{i=1}^{m} \varkappa(\gamma_i, g_{i-1}B(x, \varepsilon_0)) \leqslant L \sum_{i=1}^{m} |g_{i-1}B(x, \varepsilon_0)| \leqslant L,$$

where $L = \max_{\gamma\in S} \|(\log \gamma')'\|$. Hence for every $0 < \varepsilon < \varepsilon_0$ and $g \in G$, we have a uniform control $\varkappa(g, B(x, \varepsilon)) \leqslant L$. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log \# \left\{g \in G : g'|_{B(x,\varepsilon)} \geqslant e^{-n}\right\} = \frac{1}{n} \log \# \left\{g \in G : g'(x) \geqslant e^{-n}\right\}$$

for every $0 < \varepsilon < \varepsilon_0$. We obtain the lemma. □

For the convenience of later considerations, we limit our discussions on the subgroups of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ which are good enough. This is enough for our constructions of conformal measures in real-analytic settings.

Definition 3.4. Let $G \subset \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ. We say G is *good* if it satisfies the following assumptions:

(1) G is finitely generated and satisfies property $(\Lambda\star)$,
(2) for every $g \in G$, g is real analytic on a neighborhood of Λ, and
(3) for every $x \in \Lambda$, the stabilizer $\mathrm{Stab}_G(x) := \{g \in G : g(x) = x\}$ is a cyclic group.

The main proposition of this subsection is the following one. This proposition is essentially established in [HJX23, Section 11.4]. We generalize it to certain C^∞-cases for the later use. We will omit some details during the proof. Further discussions can be found in [HJX23, Section 11].

Proposition 3.5. *Let G be a good subgroup of $\mathrm{Diff}_+^\infty(\mathbb{S}^1)$ with an exceptional minimal set Λ and a fundamental pair (f, J). Fixing a point $x_0 \in J$ which is not fixed by $\langle f\rangle \setminus \{\mathrm{id}\}$, let $\delta = \delta(G, x_0)$. Then there exists an atomless δ-conformal measure on Λ.*

Proof. For $s > 0$, we consider the series $P(s) = \sum_{g\in G} g'(x_0)^s$. By Lemma 3.3, δ corresponds to the critical exponent of the convergence of these series. We hope that the series is also divergent at $s = \delta$. So we recall a lemma of Patterson [Pat76, Lemma 3.1]

Lemma 3.6. *If $P(\delta)$ converges then there exists a decreasing function $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that*

(1) for every $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon) > 0$ such that $\phi(\lambda t) \leqslant \lambda^{-\varepsilon} h(t)$ for every $0 < \lambda < 1$ and $0 < t \leqslant t_0$.
(2) $\widetilde{P}(s) = \sum_{g\in G} \phi(g'(x_0))g'(x_0)$ diverges for $s \leqslant \delta$ and converges for $s > \delta$.

We take $\phi \equiv 1$ if $P(\delta)$ diverges and let $\widetilde{P}(s) = \sum_{g\in G} \phi(g'(x_0))g'(x_0)$. Now we consider the probability measures on $\mathbb{S}^1$ given by

$$\mu_s := \frac{1}{\widetilde{P}(s)} \sum \phi(g'(x_0))g'(x_0)\delta_{g(x_0)}$$

for every $s > \delta$. By the weak compactness, there exists $s_n \downarrow \delta$ such that $\mu_{s_n} \to \mu$ in the weak $*$ topology. We claim that μ is a desired conformal measure. The following lemmas are shown in [HJX23].

Lemma 3.7. *μ is δ-conformal.*

Lemma 3.8. $\operatorname{supp}\mu \subset \omega(x_0) := \{x : \exists (g_n) \subset G$ *pairwise distinct such that* $g_n x_0 \to x\}$.

Now we should show that μ is atomless. To establish this property, we first give an estimate of the critical exponent. The following lemma is essentially established in [HJX23, Proposition 10.4]. We only sketch the proof here.

Lemma 3.9. *Assume that there exists $h \in G$ with a parabolic fixed point $x \in \Lambda$ of multiplicity $(k+1)$. Then $\delta > \frac{k}{k+1}$.*

Proof. By the basic properties of dynamical critical exponent, we have $\delta(G, x_0) = \delta(G, Gx_0) = \delta(G, \overline{Gx_0}) \geqslant \delta(G, \Lambda)$ (see [HJX23, Section 11.2] for more discussions). So it suffices to show that $\delta(G, \Lambda) > \frac{k}{k+1}$. We assume without loss of generality that Λ is dense at the right of x and h is strictly contracting and real analytic on $[x, y]$ for some $y \neq x$. Recall the estimate provided in Lemma 2.2. Then there exists $C_1 > 0$ such that for every $n \geqslant 1$,

$$(h^m)'|_{[x,y]} \geqslant e^{-n}, \quad \forall m \leqslant C_1^{-1} e^{\frac{k}{k+1}n}.$$

Therefore $\delta(G, \Lambda) \geqslant \frac{k}{k+1}$.

The strict inequality can be deduced from a free product argument. We apply Lemma 2.5 to find an element $h_1 \in G$ that only has finitely many fixed points and at least one of them is contained in the open interval $]x, y[$. Take $z \in]x, y[$ so that h_1 has no fixed points on $[x, z[$. Assume that h_1 is also convergent on $[x, z]$. By replacing by a power of h_1, we can assume that $h_1[x, z] \cap h[x, z] = \varnothing$. Then the semigroup generated by h, h_1, denoted by T, is a free product of $\langle h\rangle$ and $\langle h_1\rangle$. This is enough to ensure that

$$\delta(G, \Lambda) \geqslant \delta(T, \Lambda) > \delta(\langle h\rangle, \Lambda) \geqslant \frac{k}{k+1}.$$

Here, a more precise argument can be found in [HJX23, Section 10.2]. □

Note that μ is δ-conformal. Therefore, if μ has an atom at $x \in \omega(x_0)$ then $\{g'(x) : g \in G\}$ must be bounded. Recall that G admits a fundamental pair (f, J). Hence $\omega(x_0) \setminus \Lambda$ is either empty or a single G-orbit Gy where y is the unique hyperbolic fixed point of f on J. Combined with the property $(\Lambda\star)$, every possible atom must be either in the G-orbit of $\mathrm{NE}(G)$ or in Gy. In particular, if μ has an atom at x then there exists some $h \neq \mathrm{id} \in G$ fixing x and $\mathrm{Stab}_G(x) = \langle h \rangle$. Moreover, if the multiplicity of g at x is $(k+1)$ then $\delta > \frac{k}{k+1}$ by Lemma 3.9 (here we assume the multiplicity at a hyperbolic fixed point is 1). Finally we apply [HJX23, Lemma 11.14] to conclude that μ has no atoms at these points. In particular, since Gy is isolated in $\Lambda \cup Gy$, we conclude that μ is supported on Λ. □

3.2. The C^∞-model.

Definition 3.10. Let $G \subset \mathrm{Diff}^\omega_+(\mathbb{S}^1)$ be a subgroup with an exceptional minimal set Λ. A *C^∞-model* of G is a subgroup G_0 of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$ associated with a group homomorphism $\rho : G \to G_0$ satisfying

(1) ρ is a group isomorphism, and
(2) for every $f \in G$, $\rho(f)$ agree with f on a neighborhood of Λ.

Lemma 3.11. *Let $G \subset \mathrm{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ and G_0 be a C^∞-model of G. Then G_0 shares the same minimal set to G and is a good subgroup of $\mathrm{Diff}^\infty_+(\mathbb{S}^1)$.*

Proof. Let $\rho : G \to G_0$ be the group isomorphism associated to the C^∞-model. By the requirements of C^∞-models, $G_0x = \rho(G)x = Gx$ which is dense in Λ for every $x \in \Lambda$. Hence Λ is also minimal for G_0. Moreover, for every $x \in \Lambda$, $\mathrm{Stab}_{G_0}(x) = \rho(\mathrm{Stab}_G(x))$ is a cyclic group by Hector's theorem. To show G_0 is good, it suffices to check G_0 satisfying property $(\Lambda\star)$.

Noting that Λ is a perfect set, we have $\rho(f)'(x) = f'(x)$ for every $x \in \Lambda$. In particular, $\mathrm{NE}(G_0) = \mathrm{NE}(G)$. By Deroin-Kleptsyn-Navas's result, every element $x \in \mathrm{NE}(G)$ is fixed by some nontrivial element $f \in G$. Then $\rho(f)$ fixes x and x is a isolated fixed point of $\rho(f)$ since $\rho(f)$ is real analytic on a neighborhood of Λ. Hence G_0 satisfies property $(\Lambda\star)$. □

Lemma 3.12. *Let $G \subset \mathrm{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated subgroup. Then there exists a finite index subgroup $H < G$ such that H admits a C^∞-model H_0 with a fundamental pair (h_0, J).*

Proof. By Ghys's theorem, G is a virtually free group. By passing to a finite index subgroup if necessary, we can assume without loss of generality that G is a free group. Now we take J to be an arbitrary connected component of $\mathbb{S}^1 \setminus \Lambda$. If $\mathrm{Stab}_G(J)$ is trivial, then (id, J) is a fundamental pair of G as we have mentioned in Remark 3.2.

Now we assume that $\mathrm{Stab}_G(J)$ is nontrivial. Let x be an endpoint of J. Then by Ghys's theorem, $\mathrm{Stab}_G(x)$ is a cyclic group. Hence there exists $h \neq \mathrm{id} \in G$ such that $\mathrm{Stab}_G(J) = \mathrm{Stab}_G(x) = \langle h \rangle$. To construct a C^∞-model, we need the following lemma in the group theory, whose proof will be given in Appendix A.

Lemma 3.13. *Let G be a finitely generated free group and $h \in G$ be a nontrivial element. Then there exists a subgroup $H < G$ of finite index (then H is also a finitely generated free group) containing h such that h is a free generator of H.*

Applying this lemma, we can find a finite index subgroup H containing h such that h is a free generator of H and $\mathrm{Stab}_H(J) = \langle h \rangle$. Note that h fixes the endpoints of J and $h \neq \mathrm{id}$ is real-analytic. We can take a closed subinterval $K \subset J$ such that the fix points of h on J are contained in the interior of K. Now we can take $h_0 \in \mathrm{Diff}^\infty_+(\mathbb{S}^1)$ such that

(1) $h_0|_{\mathbb{S}^1 \setminus K} = h|_{\mathbb{S}^1 \setminus K}$, and

(2) $h_0|_K$ has at most one fixed point which is hyperbolic if exists. The existence of the hyperbolic fixed point only depends on the multiplicities of h at the endpoints of J.

Let $S = \{\gamma_1, \cdots, \gamma_k\}$ be a free generating set of H with $\gamma_1 = h$. Let $\widetilde{\gamma}_1 = h_0$ and $\widetilde{\gamma}_i = \gamma_i$ for $i \geqslant 2$. Limiting on the H-invariant set Λ, elements $\gamma_i|_\Lambda$ freely generate a free group. Therefore $\widetilde{S} = \{\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_k\}$ also freely generates a free group because of $\widetilde{\gamma}_i|_\Lambda = \gamma_i|_\Lambda$. Let H_0 be the group generated by $\widetilde{S}$ and $\rho : H \to H_0$ be the group homomorphism induced by $\gamma_i \mapsto \widetilde{\gamma}_i$. Then ρ is an isomorphism by the freeness and $\rho(g)$ agrees with g on a neighborhood of Λ for every $g \in H$. Moreover, since $\partial J \subset \Lambda$, we have

$$\mathrm{Stab}_{H_0}(J) = \mathrm{Stab}_{H_0}(\partial J) = \rho(\mathrm{Stab}_H(\partial J)) = \rho(\langle h \rangle) = \langle h_0 \rangle .$$

Hence H_0 is a C^∞-model of H and H_0 has a fundamental pair (h_0, J). □

3.3. **Atomless conformal measures for real analytic cases.** Combined Proposition 3.5 with Lemma 3.12, we can obtain an atomless conformal measure supported on Λ for some finite index subgroup of G. This is enough for studying the dimension properties of Λ. But the existence of atomless conformal measures is also a question of independent interests. The following lemma reveals the fact that our construction actually gives the atomless conformal measure for the original group.

Lemma 3.14. *Let $G \subset \mathrm{Diff}_+^\omega(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ and $H < G$ be a subgroup of finite index. Assume that μ is an atomless conformal measure for H supported on Λ. Then μ is also a G-conformal measure.*

Proof. Assume that μ is δ-conformal. For every Borel set $E \subset \mathbb{S}^1$, we have

$$((h^{-1})_*\mu)(E) = \mu(hE) = \int_E |h'(x)|^\delta \, \mathrm{d}\mu(x), \quad \forall h \in H.$$

In another word, μ satisfies the identity

$$\mu = h_*(|h'|^\delta \mu), \quad \forall h \in H.$$

Let $\mathcal{M}(\mathbb{S}^1)$ be the space of Radon measures on $\mathbb{S}^1$. We consider the map

$$\varphi : G \times \mathcal{M}(\mathbb{S}^1) \to \mathcal{M}(\mathbb{S}^1), \quad (g, \xi) \mapsto g_*(|g'|^\delta \mu).$$

Then $\varphi(\mathrm{id}, \xi) = \xi$. Moreover, for every continuous function ψ on $\mathbb{S}^1$ and $g_1, g_2 \in G, \xi \in \mathcal{M}(\mathbb{S}^1)$, we have

$$\begin{aligned}
\int \psi \, \mathrm{d}\varphi(g_1 g_2, \xi) &= \int (\psi \circ g_1 g_2)(x) \cdot [(g_1 g_2)'(x)]^\delta \, \mathrm{d}\xi(x) \\
&= \int (\psi \circ g_1)(g_2 x) \cdot [g_1'(g_2 x)]^\delta \cdot [g_2'(x)]^\delta \, \mathrm{d}\xi(x) \\
&= \int (\psi \circ g_1)(y) \cdot [g_1'(y)]^\delta \, \mathrm{d}\varphi(g_2, \xi)(y) \\
&= \int \psi \, \mathrm{d}\varphi(g_1, \varphi(g_2, \xi)).
\end{aligned}$$

Therefore, φ induces a G-action on $\mathcal{M}(\mathbb{S}^1)$. As we have mentioned, μ is an H-fixed point of this action. Hence for every $g \in G$, $\varphi(g, \mu)$ is the gHg^{-1}-fixed point of this action.

For a fixed element $g \in G$, let $H_g = gHg^{-1} \cap H$, which is also a finite index subgroup of G and hence has the same exceptional minimal set Λ. Notice that both μ and $\varphi(g, \mu)$ are atomless δ-conformal measures on Λ for the group H_g. By the uniqueness of such measure (Theorem 1.5 (1)), there exists a positive constant $\lambda(g)$ such that $\varphi(g, \mu) = \lambda(g) \cdot \mu$.

Note that $\lambda : G \to \mathbb{R}_{>0}$ is a group homomorphism and $H \subset \ker \lambda$ is a finite index subgroup. Hence $\operatorname{Im} \lambda = \{1\}$, which is the unique finite subgroup of $\mathbb{R}_{>0}$. Therefore, $\varphi(g, \mu) = \mu$ for every $g \in G$, or equivalently, μ is a δ-conformal measure for G. □

Corollary 3.15. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with an exceptional minimal set Λ. Then G admits an atomless conformal measure supported on Λ.*

Proof. By Lemma 3.12, there exists a subgroup $H < G$ of finite index which has a C^{∞}-model H_0 with a fundamental pair (h_0, J). Since H is also a finitely generated subgroup of $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ with the exceptional minimal set Λ, H_0 is a good subgroup of $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ by Lemma 3.11. Therefore, H_0 has a atomless conformal measure μ supported on Λ by Proposition 3.5. Let $\rho : H \to H_0$ be the group isomorphism associated to the C^{∞}-model. Then $\rho(h)'(x) = h'(x)$ for every $h \in H$ and $x \in \Lambda$ due to the perfectness of Λ. Hence μ is also an atomless H-conformal measure. By Lemma 3.14, μ is also G-conformal. □

4. Dimensional properties of conformal measures

This subsection is devoted to prove Theorem 1.7. We first explain the idea. Let μ be an atomless δ-conformal measure on Λ. In order to study $\mu(B(z, r))$, we want to find some element $f \in G$ which has a bounded distortion on $B(z, r)$ and expands $B(z, r)$ to an interval of constant length. Then by the conformality, we have

$$1 \asymp \mu(fB(z,r)) = \int_{B(z,r)} [f'(s)]^{\delta} \,\mathrm{d}\mu(s) \asymp \mu(B(z,r)) \left(\frac{|fB(z,r)|}{|B(z,r)|} \right)^{\delta} \asymp \mu(B(z,r)) \cdot r^{-\delta}.$$

Therefore, we have $\mu(B(z,r)) \asymp r^{\delta}$.

In the case that $\mathrm{NE}(G) = \varnothing$, we can always achieve this and μ is equivalent to the δ-dimensional Hausdorff measure on Λ, which is essentially shown in [Sul83, Theorem 4]. For general case, we require the expanding argument introduced in [DKN09] and study more carefully the behavior of the conformal measure near non-expandable points.

4.1. Expanding strategy.

Such expanding strategy initially used in [DKN09] to derive the ergodicity of G by property $(\Lambda\star)$ or property $(\star)$. We will use it carefully to study the dimension property of conformal measures.

Since G satisfies property $(\Lambda\star)$ by Theorem 2.9, the set $\mathrm{NE}(G) \cap \Lambda$ is finite by Theorem 2.8. For every $x \in \mathrm{NE}(G) \cap \Lambda$, there exists disjoint intervals $I_x^+ =]x, x + \varepsilon_x^+]$ and $I_x^- = [x - \varepsilon_x^-, x[$ with elements $g_{x,+}, g_{x,-} \in G$ fixing x such that $g'_{x,+}|_{I_x^+} > 1$ and $g'_{x,-}|_{I_x^-} > 1$. We consider the closed intervals

$$J_x^+ := [g_{x,+}^{-1}(x + \varepsilon_x^+), x + \varepsilon_x^+] \subset I_x^+, \quad J_x^- := [x - \varepsilon_x^-, g_{x,-}^{-1}(x - \varepsilon_x^-)] \subset I_x^-.$$

Then $\inf_{J_x^+} g'_{x,+} > 1$, and $\inf_{J_x^-} g'_{x,-} > 1$. Let U_x be the interior of $\{x\} \cup I_x^- \cup I_x^+$, which is an open neighborhood of x. Let $U_{\mathrm{NE}} := \bigcup_{x \in \Lambda \cap \mathrm{NE}(G)} U_x$. By the finiteness of $\mathrm{NE}(G) \cap \Lambda$, shrinking ε_x^{+} if necessary, we can also assume that for every $x \neq x' \in \mathrm{NE}(G) \cap \Lambda$,

$$g_{x,+} I_x^+ \cap \overline{U_{x'}} = \varnothing, \quad g_{x,-} I_x^- \cap \overline{U_{x'}} = \varnothing, \quad g_{x,+} I_x^+ \cap g_{x,-} I_x^- = \varnothing. \tag{4.1}$$

For every $y \in \Lambda \setminus U_{\mathrm{NE}}$, there exists an element $f_y \in G$ satisfying $f_y'(y) > 1$. Therefore, for some open neighborhood $V_y \ni y$, we have $\inf_{V_y} f_y' > 1$. By the compactness of $\Lambda \setminus U_{\mathrm{NE}}$, we can find a finite cover $\Lambda \setminus U_{\mathrm{NE}} \subset \bigcup_{i=1}^{m} V_{y_i}$. By the finiteness of $\mathrm{NE}(G) \cap \Lambda$, the constant

$$\lambda := \min \left\{ \min_{x \in \mathrm{NE}(G) \cap \Lambda} \left\{ \inf_{J_x^+} g'_{x,+}, \inf_{J_x^-} g'_{x,-} \right\}, \min_{i=1,\cdots,m} \inf_{V_{y_i}} f'_{y_i} \right\}, \tag{4.2}$$

is strictly larger than 1. Let

$$L := \max \left\{ \max_{x \in \mathrm{NE}(G) \cap \Lambda} \|g'_{x,+}\|, \max_{x \in \mathrm{NE}(G) \cap \Lambda} \|g'_{x,-}\|, \max_{i=1,\cdots,m} \|f'_{y_i}\| \right\}. \tag{4.3}$$

Note that $\Lambda' = \Lambda \setminus G(\mathrm{NE}(G))$ is G invariant. We define maps $t, T : \mathbb{Z}_+ \times \Lambda' \to G$ as below. For every $z \in \Lambda'$, we define

$$T(1,z) := t(1,z) := \begin{cases} g_{x,+}, & z \in I_x^+; \\ g_{x,-}, & z \in I_x^-; \\ f_{y_i}, & z \notin \overline{U_{\mathrm{NE}}} \text{ and } z \in V_{y_i}. \end{cases}$$

Here we remark that if z belongs to at least two V_{y_i} then we let $t(1,z)$ be an arbitrary one corresponding f_{y_i}. For every $n \geqslant 2$, let

$$t(n,z) := t(1, T(n-1,z)z) \quad \text{and} \quad T(n,z) := t(n,z) \circ T(n-1,z).$$

The maps $t(\cdot,z), T(\cdot,z)$ make a rule to expand neighborhoods of z by elements in G. For every $z \in \Lambda'$, we define $n(z)$ to be the smallest positive integer such that $T(n(z)-1,z)z \notin \overline{U_{\mathrm{NE}}}$. Here, we regard $T(0,z)$ as the identity map. In other words, $n(z) = 1$ for every $z \in \Lambda' \setminus \overline{U_{\mathrm{NE}}}$.

Definition 4.1. We define the *good steps* for each $z \in \Lambda'$ as

$$\mathrm{GS}(z) := \{n_1, n_2, \cdots\}$$

where $n_1 = n_1(z) := n(z)$ and $n_{m+1} = n_{m+1}(z) := n_m + n(T(n_m,z)z)$.

The following is direct by the definition.

Lemma 4.2. *Let $z \in \Lambda'$ and $n' < n$ be two consecutive elements in $\mathrm{GS}(z)$. If $n \geqslant n'+2$ then there exists some $x \in \mathrm{NE}(\Lambda)$ and sign $\sigma \in \{+,-\}$ such that for every $m = n, \cdots, n'-2$, $T(m,z)z \in I_x^\sigma$ and $t(m+1,z) = g_{x,\sigma}$.*

Lemma 4.3. *Let λ, L be constants given in (4.2) and (4.3). For every $z \in \Lambda'$ and $n \in \mathbb{Z}_+$, we have*

(1) $1 < t(1,z)'(z) \leqslant L$;
(2) $n(z)$ is well-defined (finite);
(3) $T(n(z),z)'(z) \geqslant \lambda$.

Proof. (1) follows by the definition. To show (2), assume that $z \in I_x^+$ for some $x \in \Lambda \cap \mathrm{NE}(G)$. Since $g_{x,+}$ expands $I_x^+ =]x, x+\varepsilon_x^+]$ with a fixed point x, then there exists a smallest integer $n \geqslant 2$ such that $g_{x,+}^{n-2}(z) \in]g_{x,+}^{-1}(x+\varepsilon_x^+), x+\varepsilon_x^+]$. By our assumption (4.1),

$$]x+\varepsilon_x^+, g_{x,+}(x+\varepsilon_x^+)] \cap U_{\mathrm{NE}} = (g_{x,+}I_x^+ \setminus I_x^+) \cap U_{\mathrm{NE}} = (g_{x,+}I_x^+ \setminus I_x^+) \cap I_x^+ = \varnothing.$$

Hence $T(n-1,z)z = g_{x,+}^{n-1}(z) \notin \overline{U_{\mathrm{NE}}}$, that is $n(z) = n$.

(3) follows from that $t(n,z)'(T(n-1,z)z) \geqslant \lambda$ and $T(n-1,z)'(z) > 1$ by (1). □

A direct consequence of (3) in the lemma is

$$T(n_m,z)'(z) \geqslant \lambda^m.$$

Combined with (1) in the lemma, we obtain

Lemma 4.4. *For every $x \in \Lambda'$, $T(n,z)'(z)$ increases to infinity as n tending to infinity.*

The following one is [DKN09, Proposition 6.4]. We state it in our language. The proof relies on the estimate given above and a control of distortion (Proposition 2.1).

Proposition 4.5. *There exists constants $\varepsilon_1 > 0, C_3 > 0$ such that for every point $z \in \Lambda'$ and every good step $n \in \mathrm{GS}(z)$, we have*

$$\varkappa(T(n,z), B(z, \varepsilon_1 \cdot [T(n,z)'(z)]^{-1})) \leqslant C_3.$$

4.2. **The conformal measure around a parabolic fixed point.** The obstruction of the good steps being $\mathbb{Z}_{>0}$ is the existence of non-expandable points. So we should understand the behavior of the conformal measure near non-expandable points, or more general, all parabolic fixed points.

Let μ be an atomless δ-conformal measure on Λ. Let $x_0 \in \Lambda$ be a $(k+1)$-order parabolic fixed point of a nontrivial element $f \in G$. Assume that f is strictly contracting on the subinterval $]x_0, y_0]$ and $]x_0, y_0] \cap \Lambda \neq \varnothing$. We use the notation from Section 2.2. Let $y_n := f^n y$ and $I_n := [y_n, y_{n-1}]$ for $n \geqslant 1$.

Lemma 4.6. *There exists $c_1 > 0$ depending only on μ, f and $]x_0, y_0]$, such that for every $n \geqslant 1$ and $z \in I_n$, $\mu([fz, z]) \geqslant c_1 \cdot n^{-\frac{k+1}{k}\delta}$.*

Proof. Since $]x_0, y_0] \cap \Lambda \neq \varnothing$ and Λ is f-invariant, we have $I_1 \cap \Lambda \neq \varnothing$. Moreover, since Λ is a perfect set, $I_1^\circ \cap \Lambda \neq \varnothing$. Hence $\mu(I_1) > 0$. By Lemma 2.2, we have

$$\mu(I_n) = \mu(f^{n-1} I_1) = \int_{I_1} [(f^{n-1})'(s)]^\delta \, \mathrm{d}\mu(s) \asymp_{C_1} \mu(I_1) \cdot n^{-\frac{k+1}{k}\delta}.$$

Hence for some $c_1' > 0$, $\mu(I_n) \geqslant c_1' \cdot n^{-\frac{k+1}{k}\delta}$. Note that for every $z \in I_n$, $[f^2 z, z] \supset I_{n+1}$, we obtain $\mu([f^2 z, z]) \geqslant c_1' \cdot (n+1)^{-\frac{k+1}{k}\delta}$.

Besides,

$$\mu([f^2 z, fz]) = \int_{[fz,z]} [f'(s)]^\delta \, \mathrm{d}\mu(s) \asymp \mu([fz, z]),$$

where the implicit constant depending only on f. Hence $\mu([fz, z]) \gg \mu([f^2 z, z]) \geqslant c \cdot (n+1)^{-\frac{k+1}{k}\delta} \gg c_1' \cdot n^{-\frac{k+1}{k}\delta}$. That is, there exists $c_1 > 0$ such that $\mu([fz, z]) \geqslant c_1 \cdot n^{-\frac{k+1}{k}\delta}$. □

Lemma 4.7. *There exists $c_2, C_4 > 0$ depending only on μ, f and $]x_0, y_0]$, such that for every point $z \in I_n$ with $n \geqslant 2$ and the radius $r \in [C_4 \cdot n^{-(k+1)/k}, 1]$, we have $\mu(B(z, r)) \geqslant c_2 \cdot r^\delta$.*

Proof. We distinguish two cases to show this lemma.

Case 1. $r > d(z, y_0)$.

Then $B(z, r) \supset I_1$ since $z \in I_n$ with $n \geqslant 2$. Hence $\mu(B(z, r)) \geqslant \mu(I_1) \geqslant c_1 \geqslant c_1 \cdot r^\delta$, where $c_1 > 0$ is given by the previous lemma.

Case 2. $r \leqslant d(z, y_0)$.

By Lemma 2.2 and Corollary 2.3, $d(z, y_{n-2}) \ll_{C_1, C_2} n^{-(k+1)/k}$. By choosing a large enough C_4, we can assume that $B(z, r) \supset I_{n-1}$. We denote $z' = z + r$. Then $[z, z'] \subset \overline{B(x, r)}$ and $[z, z'] \subset]x_0, y_0]$. Moreover, since $[z, z']$ containing $I_n = [fy_{n-1}, y_{n-1}]$, we have $fz' \in [z, z']$. Then we can take m to be the largest positive integer such that $f^m z' \in [z, z']$. Therefore $[f^m z', z'] \subset B(z, r)$. Since μ is atomless, combining the previous lemma, we have

$$\mu(B(z, r)) \geqslant \sum_{\ell=0}^{m-1} \mu([f^{\ell+1} z', f^\ell z']) \geqslant c_1 \sum_{\ell=0}^{m-1} (n' + \ell)^{-\frac{k+1}{k}\delta},$$

where $n' \geqslant 1$ satisfying $z' \in I_{n'}$. Recall that $\delta \leqslant 1$. By Minkowski's inequality ($(a+b)^\delta \leqslant a^\delta + b^\delta$), we have

$$\mu(B(x, r)) \geqslant c_1 \left(\sum_{\ell=0}^{m-1} (n' + \ell)^{-\frac{k+1}{k}} \right)^\delta.$$

Applying Corollary 2.3 to $f^\ell z'$, we have $|[f^{\ell+1}z', f^\ell z']| \asymp_{C_2} (n'+\ell)^{-(k+1)/k}$. By the definition of m, we have

$$r = d(z,z') \leqslant |[f^{m+1}z', z']| \ll_{C_2} \sum_{\ell=0}^{m}(n'+\ell)^{-\frac{k+1}{k}} \ll \sum_{\ell=0}^{m-1}(n'+\ell)^{-\frac{k+1}{k}}.$$

Combining with the estimate above, we have $\mu(B(x,r)) \gg_{C_2} c_1 \cdot r^\delta$. That is, there exists $c_2 > 0$ such that $\mu(B(x,r)) \geqslant c_2 \cdot r^\delta$. ☐

Corollary 4.8. *For every $c > 0$ there exists $c_3 > 0$ depending only on c, μ, f and $]x_0, y_0]$, such that for every point $z \in I_n \cap \Lambda$ with $n \geqslant 2$ and the radius $r \in [c \cdot n^{-(k+1)/k}, 1]$, we have $\mu(B(z,r)) \geqslant c_3 \cdot r^\delta$.*

Proof. It suffices to show the lemma for $c > 0$ small enough. By Corollary 2.3, for $c > 0$ small enough, $B(z, c\cdot n^{-(k+1)/k}) \subset]fz, f^{-1}z[\subset I_{n+1} \cup I_n \cup I_{n-1}$. By Lemma 2.2,

$$(f^{n-2})'|_{I_1\cup I_2\cup I_3} \asymp_{C_1} n^{-\frac{k+1}{k}}.$$

Therefore, for some $\varepsilon > 0$ depending only on c, C_1 such that

$$f^{-(n-2)}B(z, c\cdot n^{-\frac{k+1}{k}}) \supset B(f^{-(n-2)}z, \varepsilon),$$

where $f^{-(n-2)}z \in \Lambda$ because $z \in \Lambda$. Recall that $\operatorname{supp}\mu = \Lambda$. Therefore, there exists $c' = c'(\mu,\varepsilon) > 0$ such that for every $z' \in \Lambda$, $\mu(B(z',\varepsilon)) > c_3'$. Hence

$$\mu(B(z, c\cdot n^{-\frac{k+1}{k}})) \geqslant \int_{B(f^{-(n-2)}z,\varepsilon)} [(f^{n-2})'(s)]^\delta \, d\mu(s) \gg_{C_1} c_3' \cdot n^{-\frac{k+1}{k}\delta}.$$

Let c_2, C_4 be the constants given by the previous lemma. Then there exists $0 < c_3 \leqslant c_2$ depending only on c, μ, f and $]x_0, y_0]$, such that

$$\mu(B(z,r)) \geqslant c_3 \cdot r^\delta, \quad \forall r \in [c\cdot n^{-(k+1)/k}, C_4 \cdot n^{-(k+1)/k}].$$

For $r \geqslant C_4 \cdot n^{-(k+1)/k}$, we obtain the estimate be the previous lemma. ☐

4.3. **Local dimension estimates.** Let μ be an atomless δ-conformal measure supported on Λ. Recall that $\Lambda' = \Lambda \setminus G(\mathrm{NE}(G))$. Now we are at the stage to prove Theorem 1.7. We aim to give a lower bound of $\mu(B(z,r))$ for every $z \in \Lambda'$ and $r > 0$ small enough. In fact, for the scale $r \asymp [T(n,z)'(z)]^{-1}$ with some good step $n \in \mathrm{GS}(z)$, we can expand $B(x,r)$ to a ball of the constant radius by some element in G with a controlled distortion (Proposition 4.5).

However, there is another case that r is far from $[T(n,z)'(z)]^{-1}$ for $n \in \mathrm{GS}(z)$. In this case, we can expand $B(x,r)$ with the controlled distortion to a ball near a non-expandable point and the radius is not very small. Therefore we can apply 4.8 to obtain the estimate of measures.

Let λ, L be constants given by Lemma 4.3. Let $\varepsilon_1, C_3 > 0$ be constants given by Proposition 4.5. For every $z \in \Lambda'$ and every $r > 0$, we take $n = n(z,r)$ to be the smallest positive integer satisfying

$$T(n,z)'(z) \geqslant \varepsilon_1 \cdot (Lr)^{-1}. \tag{4.4}$$

The existence of such n is guaranteed by Lemma 4.4. Recall that $T(n,z) = t(1, T(n-1,z)z) \circ T(n-1,z)$. Combined with Lemma 4.3 (1), we have

$$T(n,z)'(z) \leqslant \varepsilon_1 \cdot r^{-1}. \tag{4.5}$$

We take $n' \leqslant n$ to be the maximal integer in $\mathrm{GS}(z)$. Then

$$r \leqslant \varepsilon_1 \cdot [T(n,z)'(z)]^{-1} \leqslant \varepsilon_1 \cdot [T(n',z)'(z)]^{-1}$$

By Proposition 4.5, we have

$$\varkappa(T(n',z),B(z,r)) \leqslant C_3 \tag{4.6}$$

for some constant $C_3 > 0$ depending only on the group G.

Since μ is δ-conformal for G, we have

$$\mu(T(n',z)B(z,r)) = \int_{B(z,r)} [T(n',z)'(s)]^{\delta}\,\mathrm{d}\mu(s) \asymp_{C_3} [T(n',z)'(z)]^{\delta}\cdot\mu(B(z,r)). \tag{4.7}$$

This distinguishes into two cases:

Case 1. $n' = n$.

In this case, we know that

$$T(n',z)B(z,r) \supset B(T(n',z)z, e^{-C_3}\varepsilon_1 L^{-1})$$

by the estimate of the derivative (4.4) and the distortion control (4.6). Noting that $T(n',z)z \in \Lambda = \operatorname{supp}\mu$, we have

$$\mu(T(n',z)z, e^{-C_3}\varepsilon_1 L^{-1}) \geqslant c_4$$

for some positive constant $c_4 = c_4(\mu, e^{-C_3}\varepsilon_1 L^{-1}) > 0$. Taking into account (4.7) and (4.5), we have

$$\mu(B(z,r)) \gg_{C_3} c_4\cdot[T(n',z)'(z)]^{-\delta} \geqslant (c_4\varepsilon_1^{-\delta})\cdot r^{\delta}, \tag{4.8}$$

where the constant $c_4\varepsilon_1^{-\delta}$ is independent of $z \in \Lambda'$ and $r > 0$.

Case 2. $n' < n$.

Let n'' be the next element of n' in $\mathrm{GS}(x)$. Then $n'' > n$. By Lemma 4.2, there exists $x \in \Lambda \cap \mathrm{NE}(G)$ and sign $\sigma \in \{+,-\}$ such that for every $m = n', \cdots, n-1$, $T(m,z)z \in I_x^{\sigma}$ and $t(m+1,z) = g_{x,\sigma}$. Without loss of generality, we assume that the sign σ is $+$.

Let $z' = T(n',z)z$. Let $r' := e^{-C_3}T(n',z)'(z)\cdot r$. By the distortion control (4.6), we have

$$T(n',z)B(z,r) \supset B(z',r').$$

By the choice of n, we have

$$T(n,z)'(z) = T(n-n',z)'(z')\cdot T(n',z)'(z) \geqslant \varepsilon_1\cdot(Lr)^{-1}.$$

Hence

$$r' = e^{-C_3}T(n',z)'(z)\cdot r \geqslant (e^{-C_3}\varepsilon_1 L^{-1})\cdot[(g_{x,\sigma}^{n-n'})'(z')]^{-1}. \tag{4.9}$$

Lemma 4.9. *There exists $c_5 > 0$ depending only on $\mu, g_{x,+}, I_x^+$ and the constant $e^{-C_3}\varepsilon_1 L^{-1}$, such that $\mu(B(z',r')) \geqslant c_5\cdot(r')^{\delta}$.*

Proof. We assume that x is a $(k+1)$-multiple fixed point of $g_{x,+}$. We use the notations from Section 2.2 with $x_0 = x$, y_0 be the another endpoint of I_x^+ and $f = g_{x,+}^{-1}$ which is strictly contracting on $]x_0,y_0]$. Assume that $z' \in I_\ell = [f^{\ell}y_0, f^{\ell-1}y_0]$ for some $\ell \geqslant 1$. Since $f^{-(n-n')}z \in]x_0,y_0[$, we have $\ell \geqslant n-n'+1$. By Lemma 2.2 (4), there exists $C_1 > 1$ such that

$$[(g_{x,\sigma}^{n-n'})'(z')]^{-1} = [(f^{-(n-n')})'(z')]^{-1} = (f^{n-n'})'(f^{-(n-n')}z) \gg_{C_1} \left(\frac{\ell-(n-n')}{\ell}\right)^{\frac{k+1}{k}}.$$

Combining with (4.9), there exists $c > 0$ depending only on $g_{x,+}$, I_x^+ and $e^{-C_3}\varepsilon_1 L^{-1}$ such that

$$r' \geqslant c\cdot\left(\frac{\ell-(n-n')}{\ell}\right)^{\frac{k+1}{k}} \geqslant c\cdot\ell^{-\frac{k+1}{k}}.$$

Recall that $z' = T(n',z)z$ where $z \in \Lambda$ and $T(n',z) \in G$. We have $z' \in \Lambda$. Applying Corollary 4.8, there exists $c_5 = c_3(c,\mu,g_{x,+},I_x^+) > 0$ such that $\mu(B(z',r')) \geqslant c_5 \cdot (r')^\delta$. $\square$

Note that there are only finitely many pairs $(g_{x,\sigma}, I_x^\sigma)$ where $x \in \mathrm{NE}(G) \cap \Lambda$ and $\sigma \in \{+,-\}$. We can find a uniform $c_5 > 0$ depending only on μ and G such that $\mu(B(z',r')) \geqslant c_5 \cdot (r')^\delta$ for every such ball $B(z',r')$. Finally, we have

$$\mu(B(z,r)) \geqslant \int_{B(z',r')} [(T(n',z)^{-1})'(s)]^\delta \, \mathrm{d}\mu(s) \geqslant e^{-C_3\delta} \cdot [T(n',z)'(z)]^{-\delta} \cdot \mu(B(z',r')) \tag{4.10}$$

$$\geqslant c_5 e^{-C_3\delta} \cdot [T(n',z)'(z)]^{-\delta} \cdot (r')^\delta \geqslant c_5 e^{-2C_3\delta} \cdot r^\delta, \tag{4.11}$$

where the constant $c_5 e^{-2C_3\delta}$ depending only on μ and G.

Combining (4.8) and (4.11), let $c_0 := \min\left\{c_4\varepsilon_1^{-\delta}, c_5 e^{-2C_3\delta}\right\}$. We obtain Theorem 1.7. $\square$

Appendix A. A lemma of combinatorial group theory (Proof of Lemma 3.13)

If G is of rank 1 then the statement is trivial. Assume that G is a rank k free group with $k \geqslant 2$. We consider the topological space $X = \bigvee^k \mathbb{S}^1$, a union of k circles meeting at a point o. We denote these circles by $C_1, \cdots, C_k$. For each i, we use p_i to denote a fixed path $[0,1] \to C_i$ with $p_i(\{0,1\}) = o$ and p_i maps $(0,1)$ to $C_i \setminus \{o\}$ homeomorphically. Let $\iota : [0,1] \to [0,1], t \mapsto 1-t$ be the time reverse map. Let $\gamma_i = [p_i] \in \pi_1(X,o)$ and hence $\gamma_i^{-1} = [(p_i \circ \iota)]$. Then $S = \{\gamma_1, \cdots, \gamma_k\}$ is a free generator of $\pi_1(X,o) \cong G$.

We aim to construct another space $\widetilde{X}$ with an appropriate covering map $\rho : \widetilde{X} \to X$ of finite sheets such that $\pi_1(\widetilde{X}, \widetilde{o})$ is a freely generated by a finite set $\widetilde{S}$ and $\rho_*(\widetilde{h}) = h$ for some $\widetilde{h} \in \widetilde{S}$. Then $\rho_*\pi_1(\widetilde{X},\widetilde{o})$ is a desired free subgroup of $G = \pi_1(X,o)$.

Write $h = g_0 \cdots g_{n-1} \in G$ in the *reduced form* with respect to the generator S. That is, $g_j \in S \cup S^{-1}$ and $g_{j-1}g_j \neq \mathrm{id}$. We can assume without loss of generality that h is *cyclically reduced*, i.e. $g_{n-1}g_0 \neq \mathrm{id}$. This is because every element can be written in the form tgt^{-1} with $t \in G$ and a cyclically reduced $g \in G$ and the conclusion can be derived from the cyclically reduced case after an appropriate inner automorphism.

Convention. We use i to denote an element in $\{1, \cdots, k\}$ and j to denote an element in $\{0, \cdots, n-1\}$. We always understand the index j in module n.

We will construct $\widetilde{X}$ as well as a covering map $\rho : \widetilde{X} \to X$. The space $\widetilde{X}$ will be a graph, also be viewed as a 1-dimensional CW-complex. We first add n vertices $v_0, \cdots, v_{n-1}$ with $\rho(v_j) = o$ for every j. The edges we construct are distinguished in two kinds:

(1) For each j, we add an edge $e_j \cong]0,1[$ with $\partial e_j = \{v_j, v_{j+1}\}$. Let $\beta_j : [0,1] \to \overline{e_j}$ be a homeomophism with $\beta_j(0) = v_j$. Now we define

$$\rho|_{e_j} = \begin{cases} p_i \circ \beta_j^{-1}, & \text{if } g_j = \gamma_i; \\ p_i \circ \iota \circ \beta_j^{-1}, & \text{if } g_j = \gamma_i^{-1}. \end{cases}$$

(2) For each i, let

$$E_i^+ := \left\{ j : g_j = \gamma_i \text{ or } g_{j-1} = \gamma_i^{-1} \right\}, \quad E_i^- := \left\{ j : g_j = \gamma_i^{-1} \text{ or } g_{j-1} = \gamma_i \right\}.$$

Since $h = g_0 \cdots g_{n-1}$ is written in the reduced form and is cyclically reduced, we have $\#E_i^+ = \#E_i^-$. We take a bijection

$$b_i : \{0, \cdots, n-1\} \setminus E_i^+ \to \{0, \cdots, n-1\} \setminus E_i^-.$$

For each $j \in \{0, \cdots, n-1\} \setminus E_i^+$, we add an edge $e \cong]0,1[$ with $\partial e = \left\{v_j, v_{b_i(j)}\right\}$. Let $\beta : [0,1] \to \bar{e}$ be a continuous map with $\beta(0) = v_j, \beta(1) = v_{b_i(j)}$ and $\beta|_e$ is a homeomoprhism. The map ρ on e is given by $\rho|_e = p_i \circ \beta^{-1}$. Then ρ is continuous on $\bar{e}$.

Example A.1. Let G be a free group of rank $k = 2$. The space X is illustrated as below. Two circles correspond to the generators γ_1, γ_2 of G.

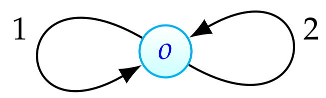

FIGURE A.2. The space X.

For an element $h = \gamma_1\gamma_2^{-1}\gamma_1^2 \in G$, the following graph illustrate a way to construct $\widetilde{X}$ for this h. First kind edges are colored in black and second kind edges are colored in blue. The label on each edge denotes which circle it maps under the map ρ.

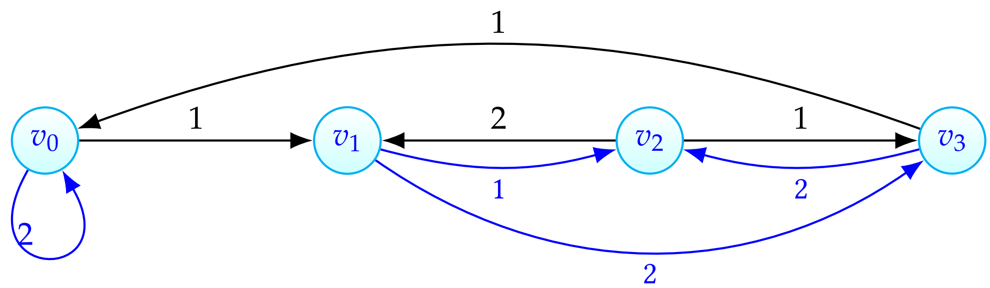

FIGURE A.3. The space $\widetilde{X}$ with the covering map ρ.

Now we obtain a space $\widetilde{X}$ with a continuous map $\rho : \widetilde{X} \to X$. The second kind of edges ensures that ρ is a local homeomorphism at vertices v_i's. Hence ρ is a covering map of n-sheets. Let $\widetilde{o} = v_0$ to be the base point. Then ρ_* induces an embedding $\pi_1(\widetilde{X}, \widetilde{o}) \hookrightarrow \pi_1(X, o)$ with an image of finite index.

Now we consider a spanning tree T of $\widetilde{X}$, whose edges are $\{e_0, \cdots, e_{n-2}\}$. Every edge e in $\widetilde{X} \setminus T$ determines a loop ℓ_e in $\widetilde{X}$ with the base point $\widetilde{o}$: goes from $\widetilde{o}$ to one endpoint of e by a path in T, crosses e and then back to $\widetilde{o}$ by a path in T. By considering the quotient space $\widetilde{X}/T$, we can find that $\left\{[\ell_e] : e \in \widetilde{X} \setminus T\right\}$ is a free generating set of $\pi_1(\widetilde{X}, \widetilde{o})$.

Therefore, the elements $\left\{\rho_*[\ell_e] = [\rho \circ \ell_e] : e \in \widetilde{X} \setminus T\right\}$ freely generates a finite index subgroup of $\pi_1(X, o) \cong G$. This subgroup will be a desired construction of H. In fact, we can take $e = e_{n-1}$, the last edge constructed in the first kind. Then $[\rho \circ \ell_{e_{n-1}}] = g_0 g_1 \cdots g_{n-1} = h$ is a free generator of H. $\square$

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Yuxiang Jiao. *Peking University, No.5 Yiheyuan Road, Haidian District, Beijing, China.*
email: ajorda@pku.edu.cn