Rigidity of (partially) hyperbolic abelian actions

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History and Motivation

The goal of these notes is to give an introduction to actions of higher-rank abelian and semisimple Lie groups. Examples of such groups include $\mathbb{R}^k \times \mathbb{Z}^\ell$ and $\mathrm{SL}_d(\mathbb{R})$, respectively.

We begin with a technical definition. Factors of dynamical systems are well-studied, even in rank 1. However, a special type of factor can occur when part of the group acts trivially on the factor.

Definition 0.1. Let $G \curvearrowright X$ be a C^∞ group action on a smooth manifold X. We say that an action $H \curvearrowright Y$ is a **C^∞ factor** of $G \curvearrowright X$ if there exists a submersion $\pi : X \to Y$ and a surjective homomorphism $\sigma : G \to H$ such that

$$\pi(g.x) = \sigma(g).\pi(x)$$

for all $g \in G$ and $x \in X$.

- If every factor of $G \curvearrowright X$ is finite-to-one, we say that the action is **irreducible**.
- If every factor of $G \curvearrowright X$ has $\ker \sigma = G$ or $\{e\}$, we say that the action is **group-irreducible**.
- If $G/\ker\sigma$ is a compact extension of $\mathbb{Z}$ or a reductive Lie group of real rank 1 (eg, $\mathrm{SL}_2(\mathbb{R})$ or $\mathbb{R}$), then we say that the factor has **rank one**.

When seeking new or "genuine" actions of groups which are not built from combining flows or diffeomorphisms, it is natural to ask for the action to not have rank one factors, or more strongly, for the action to be group-irreducible if the action shouldn't be built from a factor group.

Example 0.2

Consider an action by $\mathbb{Z}^k$ on a torus $\mathbb{T}^d$ generated by pairwise commuting matrices $A_1, \cdots, A_k \in \mathrm{SL}_d(\mathbb{Z})$. Then the action

- is *irreducible* when there does not exist a nontrivial common rational invariant subspace $V \subset \mathbb{R}^d$.
- is *group irreducible* if whenever $V \subset \mathbb{R}^d$ is a common rational invariant subspace, the action of A_i on $\mathbb{R}^d/V$ is either faithful or trivial
- *has a rank one factor* if and only if there exists a common invariant rational subspace $V \subset \mathbb{R}^d$ such that the induced action on $\mathbb{R}^d/V$ is generated by at most one infinite order matrix.

Some concrete examples are that

- $\mathbb{Z}^2 \curvearrowright \mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ defined by $(m,n).(x,y) = (A^m x, A^n y)$ is not irreducible or group irreducible, and has two rank one factors
- If $\alpha : \mathbb{Z}^2 \curvearrowright \mathbb{T}^3$ is generated by matrices A_1 and A_2 which commute and have three distinct real, irrational eigenspaces, then the action is both irreducible and group irreducible. A specific example is

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -4 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

- The action $\beta : \mathbb{Z}^2 \curvearrowright \mathbb{T}^3 \times \mathbb{T}^3$ which is the product action of the previous example with itself $\beta(m,n).(x,y) = (\alpha(m,n).x, \alpha(m,n).y)$ is group irreducible, but not irreducible and has no rank one factors
- The action $\gamma : \mathbb{Z}^4 \curvearrowright \mathbb{T}^3 \times \mathbb{T}^3$ which is the product action defined by

$$\gamma(m_1,n_1,m_2,n_2).(x,y) = (\alpha(m_1,n_1).x, \alpha(m_2,n_2).y)$$

  is neither irreducible, nor group irreducible, but has no rank one factors.
- The action $\delta : \mathbb{Z}^2 \curvearrowright \mathbb{T}^3 \times \mathbb{T}^2$ defined by

$$\delta(m,n).(x,y) = (\alpha(m,n)x, A^{m+n}y)$$

  has only one rank one factor.

The guiding philosophy of the higher-rank rigidity program is to run away from the headaches of classical hyperbolic theory: low regularity of foliations and conjugacies, orbit equivalences which are not conjugacies, a complicated cohomology theory with many nontrivial invariants and the corresponding thermodynamical formalism. All of these nuances (usually) disappear when the action is sufficiently far away from rank one actions (ie, has no rank one factors). We do not provide comprehensive references on the study of rigidity for (partially) hyperbolic group actions, but refer to David Fisher's survey on the Zimmer program and Ralf Spatzier's summary of the history and recent progress on the work of Zhiren Wang. The introductions of the following papers also summarize recent progress in the setting of abelian and semisimple group actions, respectively: Cartan actions of abelian groups and their classification and The Zimmer program for partially hyperbolic actions (the latter paper is what Lecture 2 is based on).

The context of these lectures is a continuation of theorems in these traditions: analyzing the structure and rigidity phenomena for actions without rank one factors.

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§1 Lecture 1: Fundamentals of partially hyperbolic abelian group actions

Let $\alpha : \mathbb{R}^k \times \mathbb{Z}^\ell \curvearrowright X$ be an action by C^∞ diffeomorphisms which is locally free. Fix a distribution E^c on X. An element $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ is **E^c-partially hyperbolic** if there exist two distributions $E_a^s, E_a^u \subset TX$ such that

$$TX = E_a^s \oplus E^c \oplus E_a^u$$

is an a-invariant dominated splitting with $a|_{E_a^s}$ contacting and $a|_{E_a^u}$ expanding. We let $\mathcal{PH}(E^c)$ denote the set of E^c-partially hyperbolic elements. If $E^c = T\mathcal{O}$, where $\mathcal{O}$ is the $\mathbb{R}^k$-orbit foliation, we call a **Anosov**.

Example 1.1

1. $(k,\ell) = (1,0)$ corresponds to Anosov / partially hyperbolic flows.
2. $(k,\ell) = (0,1)$ corresponds to Anosov / partially hyperbolic diffeomorphisms.

Lemma 1.2

Let $t \in \mathbb{R} \setminus \{0\}$. If $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ is Anosov or E^c-partially hyperbolic and $ta \in \mathbb{R}^k \times \mathbb{Z}^\ell$, then ta is Anosov or E^c-partially hyperbolic respectively.

Common stable foliations.

Theorem 1.3 (Hirsch-Pugh-Shub)

If $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ is Anosov or E^c-partially hyperbolic, then E_a^* integrates to a Hölder foliation with C^∞-leaves for $* = s, u$.

Remark 1.4 If $b \in \mathbb{R}^k \times \mathbb{Z}^\ell$ then $db(E_a*) = E_a^*$. This is because $db(E^*)$ is also an invariant distribution for a forming a new dominated splitting by the commutativity of a and b.

Now we look at the sequence of maps

$$b : \mathcal{W}_a^s(x) \to \mathcal{W}_a^s(b(x)) \to \mathcal{W}_a^s(b^2(x)) \to \cdots.$$

Theorem 1.5

If $a_1, \cdots, a_m$ are E^c-partially hyperbolic or Anosov, then

$$\mathcal{W}^s_{a_1,\cdots,a_n}(x) := \mathcal{W}^s_{a_1}(x) \cap \mathcal{W}^s_{a_2}(x) \cap \cdots \cap \mathcal{W}^s_{a_n}(x)$$

is a Hölder foliation with C^∞ leaves.

Definition 1.6. A foliation $\mathcal{W}$ is a **coarse Lyapunov foliation** if there exist $a_1, \cdots, a_n \in \mathbb{R}^k \times \mathbb{Z}^\ell$ such that $\mathcal{W} = \mathcal{W}^s_{a_1,\cdots,a_n}$, $\dim(\mathcal{W}) \geqslant 1$ and for every E^c-partially hyperbolic element $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$, $\mathcal{W}$ subfoliates either $\mathcal{W}_a^s$ or $\mathcal{W}_a^u$. We let Δ denote an indexing set for the coarse Lyapunov foliations and if $\chi \in \Delta$, let $\mathcal{W}^\chi$ denote the foliation corresponding to χ.

Corollary 1.7

For every $a \in \mathbb{R}^k \times \mathbb{Z}^\ell$ which is E^c-partially hyperbolic, there exists a decomposition $\Delta = \Delta^-(a) \sqcup \Delta^+(a)$ such that

$$E_a^s = \bigoplus_{\chi \in \Delta^-(a)} T\mathcal{W}^\chi \qquad \text{and} \qquad E_a^u = \bigoplus_{\chi \in \Delta^+(a)} T\mathcal{W}^\chi.$$

Example 1.8

Let $\psi_t \curvearrowright M$ and $\varphi_s \curvearrowright N$ be Anosov flows. Let α be the $\mathbb{R}^2$-action induces by $\psi_t \times \varphi_s$ acting on $M \times N$. The coarse Lyapunov foliations are $\mathcal{W}_\psi^{s/u}$ and $\mathcal{W}_\varphi^{s/u}$ on $M \times N$.

Example 1.9

$\alpha : \mathbb{R}^2 \curvearrowright \mathrm{SL}_3(\mathbb{R})/\Gamma$ by

$$\alpha(t_1, t_2) g\Gamma = \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} g\Gamma.$$

The coarse Lyapunov foliations are $\mathcal{W}^{ij}(x) = (\mathrm{id} + E_{ij}(\mathbb{R}))(x)$, where $E_{ij}(s)$ denotes the matrix with s in the (i,j)-entry and zeros elsewhere.

Focus on $\mathbb{R}^k$. How does $\mathcal{PH}(E^c)$, the set of E^c-partially hyperbolic elements, look?

Definition 1.10. An **open convex cone** in $\mathbb{R}^k$ is an open subset $\mathcal{C}$ such that for every $t > 0$, $t\mathcal{C} = \mathcal{C}$ and $\mathcal{C}$ is convex.

If $f : V \to W$ is an invertible map between normed vector spaces, let

$$m(f) = \min_{v \in V} \|f(v)\|_W / \|v\|_V = \|f^{-1}\|^{-1}$$

be the **conorm** of f. Then let

$$\mathcal{C}^-(\mathcal{W}) := \Bigg\{ a \in \mathbb{R}^k : \frac{\|d\alpha(na)|_{E^c}(x)\|}{m(d\alpha(na)|_{T\mathcal{W}}(x))} \leqslant e^{-n\epsilon} \text{ for some } \epsilon > 0, \text{ every } x \in X,$$
$$\text{and sufficiently large } n \in \mathbb{N} \Bigg\}.$$

Lemma 1.11 $\mathcal{C}^-(\mathcal{W})$ is an open convex cone.

Proof. By Lemma 1.2, it follows that $\mathcal{C}^-(\mathcal{W})$ is invariant under positive rescaling. It therefore suffices to show that if $a, b \in \mathcal{C}^-(\mathcal{W})$, then $a + b \in \mathcal{C}^-(\mathcal{W})$. Indeed, notice that

$$\begin{aligned}
\frac{\|d\alpha(n(a+b))|_{E^c}(x)\|}{m(d\alpha(n(a+b))|_{T\mathcal{W}}(x))} &= \frac{\|d\alpha(na)|_{E^c}(nb.x)\cdot d\alpha(nb)|_{E^c}(x)\|}{m(d\alpha(na)|_{T\mathcal{W}}(nb.x)\cdot d\alpha(nb)|_{T\mathcal{W}}(x))} \\
&\leqslant \frac{\|d\alpha(na)|_{E^c}(nb.x)\|}{m(d\alpha(na)|_{T\mathcal{W}}(nb.x))}\cdot\frac{\|d\alpha(nb)|_{E^c}(x)\|}{m(d\alpha(nb)|_{T\mathcal{W}}(x))} \\
&\leqslant e^{-n\epsilon_1}\cdot e^{-n\epsilon_2} \\
&= e^{-n(\epsilon_1+\epsilon_2)}
\end{aligned}$$

□

Recall that Δ denotes an indexing set for the coarse Lyapunov foliations, and we let $\mathcal{W}^\chi$ denote the foliation corresponding to $\chi \in \Delta$.

Corollary 1.12
$\mathcal{PH}(E^c) = \bigcap_{\chi\in\Delta}(\mathcal{C}^-(\mathcal{W}^\chi)\cup -\mathcal{C}^-(\mathcal{W}^\chi))$, where $\mathcal{PH}(E^c) \subset \mathbb{R}^k$ is the subset of E^c-partially hyperbolic elements.

Definition 1.13. α is called **totally partially hyperbolic** or **totally Anosov** if $\mathcal{PH}(E^c)$ is dense in $\mathbb{R}^k$. (For the $\mathbb{Z}^\ell$-case, we replace this with a projectively dense condition.)

> **Exercise 1.14.** Show that α is totally partially hyperbolic (or totally Anosov) if and only if $\mathcal{C}^-(\mathcal{W})$ is an open half space for every coarse Lyapunov foliation $\mathcal{W}$

Oseledets theorem for $\mathbb{R}^k\times\mathbb{Z}^\ell$-actions. Assume that α preserves an ergodic measure μ. There exists $\Delta \subset (\mathbb{R}^k\times\mathbb{Z}^\ell)^*$ and a splitting (recalling $\mathcal{O}$ is the foliation of $\mathbb{R}^k$-orbits)

$$TX = T\mathcal{O}\oplus\bigoplus_{\chi\in\Delta}E^\chi_\mu$$

such that for μ-almost every x, if $v \neq 0 \in E^\chi_\mu(x)$ then

$$\lim_{|a|\to\infty}\frac{\log\|d\alpha(a)v\|-\chi(a)}{|a|} = 0.$$

(See [Brown-Rodriguez Hertz-Wang] for some detailed discussions on the topic of Oseledets splittings for abelian actions.)

Remark 1.15 For every ergodic μ and coarse Lyapunov foliation $\mathcal{W} = \mathcal{W}^s_{a_1}\cap\cdots\mathcal{W}^s_{a_n}$, we have

$$T\mathcal{W}\subset\bigoplus_{\chi(a_i)<0}E^\chi_\mu.$$

If $a_1,\cdots,a_n$ are Anosov then $T\mathcal{W} = \bigoplus_{\chi(a_i)<0}E^\chi_\mu$.

Let $\mathcal{C}^-(\chi) := \{\, a : \chi(a) < 0 \,\}$. Then

$$\mathcal{C}^-(\mathcal{W}) \subset \bigcap_{\mu\in\mathcal{M}_{\alpha,\mathrm{erg}}}\ \bigcap_{E^\chi_\mu\subset T\mathcal{W}}\mathcal{C}^-(\chi).$$

If α is totally E^c-partially hyperbolic then $\mathcal{C}^-(\mathcal{W}) = \mathcal{C}^-(\chi)$ whenever χ is a Lyapunov exponent associate to $\mathcal{W}$. Since $\mathcal{C}^-(\chi_1) = \mathcal{C}^-(\chi_2)$ if and only if $\chi_1 = c\chi_2$ for some $c > 0$, if α is totally E^c-partially hyperbolic then for every $\mathcal{W}$ there exists a unique χ up to a positive scalar multiple. In this case, we call the projectivized functional $\mathbb{R}_+\chi$ the **exponent associated to $\mathcal{W}$**, and denote $\mathcal{W} = \mathcal{W}^\chi$.

Exercise 1.16. Show that if α is totally E^c-partially hyperbolic and there exist at least two coarse Lyapunov functionals $\chi_1 \neq -\chi_2 \in (\mathbb{R}^k)^*/\mathbb{R}_+$, then $d\alpha|_{E^c}$ has 0 Lyapunov exponents with respect to any invariant measure.
[*Hint*: Show that for every coarse exponent χ, $\ker \chi$ acts with 0 exponents on E^c.]

§2 Lecture 2: Rigidity

Rigidity results, by Damjanovic-Spatzier-Vinhage-Xu. Assuming that:

- $\alpha : \mathbb{R}^k \curvearrowright X$ is totally E^c-partially hyperbolic and volume-preserving ergodic.
- α is **super-accessible**: for every χ and x, y there exists a path connecting x, y by finitely many $\mathcal{W}^{\chi_i}$-segments where $\chi_i \neq c\chi$ for every i. This is equivalent to that there exists $a \in \ker \chi$ such that a is accessible.

> **Exercise 2.1.** Show that if α is super-accessible, then the action has no rank one factors.

- α is **measurably Oseledets conformal**: For every $\mathcal{W}^\chi$, $\mathrm{d}\alpha|_{T\mathcal{W}^\chi}$ is measurably conjugated to

$$\mathrm{d}\alpha(a)|_{T\mathcal{W}^\chi} = \begin{bmatrix} e^{c_1\chi(a)}H_1 & & & \\ & e^{c_2\chi(a)}H_2 & & \\ & & \ddots & \\ & & & e^{c_n\chi(a)}H_n \end{bmatrix},$$

where $H_i \in \mathrm{SO}_{d_i}(\mathbb{R})$, $c_i \in \mathbb{R}$.

Theorem 2.2 (Damjanovic-Spatzier-Vinhage-Xu)
If α satisfying the assumptions above then α is C^∞-conjugated to a translation action of the form

$$\beta : \mathbb{R}^k \curvearrowright K\backslash H/\Gamma$$

by $a.Kh\Gamma = K(ah)\Gamma$ and $\mathbb{R}^k \hookrightarrow Z_H(K)$ up to a finite cover.

The main application of Theorem 2.2 is to the setting of semisimple group actions. If G is a semisimple Lie group, we say that G is geniunely higher-rank if every simple factor of G has rank at least two. When G is genuinely higher-rank, and $G \curvearrowright X$ is a C^∞-action, we can restrict the action to an abelian subgroup known as a Cartan subgroup. When $G = \mathrm{SL}_d(\mathbb{R})$ with $d \geqslant 3$, the "canonical" Cartan subgroup is the group of diagonal matrices.

In this setting of genuinely higher-rank Lie group actions, super-accessibility is free, since accessibility of one element implies the super-accessibility (exercise), and Zimmer's cocycle superrigidity theorem implies the action is measurably Oseledets conformal.

Corollary 2.3 (Damjanovic-Spatzier-Vinhage-Xu: G-actions)
Assume that G is a genuinely higher-rank semisimple Lie group, and $\alpha : G \curvearrowright X$ is a C^∞-action such that

1. α preserves a volume;
2. $\alpha|_A$ is totally partially hyperbolic, where A is the Cartan subgroup;
3. $\alpha(a)$ is accessible for some partially hyperbolic $a \in A$.

Then there exists $\beta : G \curvearrowright K\backslash H/\Gamma$ a translation action such that α is a finite factor of β.

Step 1. Invariance principle. (Ledrappier, $\cdots$, Avila-Santamaria-Viana) Philosophy: over an accessible partially hyperbolic system, invariant measurable objects are Hölder continuous along stable / unstable manifolds. By the version of ASV (see also Kalinin-Sadovskaya), this holds for matrix cocycles with 1-exponent. This implies that the derivatives are C^0-trivializable (in the sense of modulo a rotation).

From now on, for simplicity we assume that $\mathcal{W}^\chi$ is equipped with a C^0-metric for which the dynamics is conformal, and the derivative of a restricted to each Lyapunov subspace $E^{c_i\chi}$ is $e^{c_i\chi(a)}R_{\theta_i}$ for some rotation matrix R_{θ_i}

Step 2. Homogeneous structure on each $\mathcal{W}^\chi$-leaf. We consider the action of $\ker\chi$ on $\mathcal{W}^\chi(x) \to \mathcal{W}^\chi(a.x)$. Since the derivative is C^0-trivilizable and $\ker\chi$ has zero exponents on $\mathcal{W}^\chi(x)$, we obtain an isometric action on $\mathcal{W}^\chi(x) \to \mathcal{W}^\chi(a.x)$. By the super-accessibility and totally partially hyperbolic assumption, any generic element in $\ker\chi$ is a volume preserving accessible, center bunched partially hyperbolic diffeomorphism (exercise), hence by Burns-Wilkinson's theorem, $\ker\chi$'s action is transitive.

Given $y \in W^\chi(x)$, we can find $a_k \in \ker\chi$ such that $a_k.x \to y$. Each such sequence of maps will converge to a single map $f = \lim_{k\to\infty} a_k : \mathcal{W}^\chi(x) \to \mathcal{W}^\chi(x)$ such that $f(x) = y$. This implies that $\mathrm{Isom}(\mathcal{W}^\chi(x))$ acts transitively on $\mathcal{W}^\chi(x)$ an isometric structure that compatible with the action α and $\mathrm{Isom}(\mathcal{W}^\chi(x))$ acts transitively on $\mathcal{W}^\chi(x)$ (by a result of Wilson 1982).

Moreover, there is a uniform contracting diffeomorphism on $\mathcal{W}^\chi(x)$ that normalizes $\mathrm{Isom}(\mathcal{W}^\chi(x))$ (some element $a \in \mathbb{R}^n$ with $\chi(a) < 0$). This implies that the nilpotent radical of $\mathrm{Isom}(\mathcal{W}^\chi(x))$, denoted by N_x^χ, is a nilpotent group, and N_x^χ that acts simply transitively on $\mathcal{W}^\chi(x)$.

Remark 2.4 While the N_x^χ-actions *seem* like the actions we want, in fact they need to be dualized. Indeed, if N_x^χ is nilpotent but not abelian, in the models, translation on nilpotent groups are *not* isometries. However, they are generated by vector fields dual to isometries. Thus, we should pass to the group generated by vector fields invariant under the action of N_x^χ.

Step 3. Constructing an extension. In the previous step, we have constructed a transitive N_x^χ-action on each leaf $\mathcal{W}^\chi(x)$. The goal is to glue the N_x^χ-actions into a single action. The problem is that there is no canonical way to choose the action of N^χ on different leaves. The following example illustrates this obstruction for the model.

Example 2.5 (Baby case: the geodesic flow on $\mathbb{H}^3/\Gamma$)
The action is
$$\begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \curvearrowright \left\{ \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} \right\} \backslash \mathrm{SL}_2(\mathbb{C})/\Gamma.$$
When we want to parametrize the horospheres, we want to use
$$\begin{bmatrix} 1 & z \\ & 1 \end{bmatrix} \curvearrowright \left\{ \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} \right\} \backslash \mathrm{SL}_2(\mathbb{C})/\Gamma,$$
which does not exist (even though it does on the homogeneous space $\mathrm{SL}_2(\mathbb{C})/\Gamma$). This problem persists in other translation models on $K\backslash G/\Gamma$.

This model also reveals the underlying cause of the problem: the action of $K = \left\{ \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} \right\}$. The unit tangent bundle to $\mathbb{H}^3/\Gamma$ is a bi-homogeneous space with this appearing as a left quotient. Note that K commutes with the action of the diagonal acting group, but not with the upper (or lower) triangular group. To build a horospherical action on the space, it is therefore necessary to first recover the K-component. In the model case, this can be achieved by lifting the action to the action on the frame bundle.

In general case, we also consider a fiber bundle over X. To simplify the construction, we assume that there is only one exponent in each coarse Lyapunov distribution (i.e., α acts conformally on each E^{χ}). This is not needed, but simplifies the construction of the extension.

For each $x \in X$, we build a fiber

$$\widetilde{K}_x = \left\{ \varphi : \mathbb{R}^d \to T_x\mathcal{W}^{\chi} \text{ such that } \{ \varphi(e_i) \} \text{ is an oriented orthonormal frame of } T_x\mathcal{W}^{\chi} \right\}.$$

Then SO_d acts transitively on each $\widetilde{K}_x$ by $k.\varphi = \varphi \circ k^{-1}$. Let $\widetilde{X}$ to be the associated C^0-fiber bundle, which is a principle SO_d-bundle. We define $\mathbb{R}^k \curvearrowright \widetilde{X}$ by

$$[a.\varphi](v) = \frac{\mathrm{d}\alpha(a)(\varphi(v))}{e^{\chi(a)}}.$$

> **Exercise 2.6.** Show that this gives a well-defined action on $\mathbb{R}^k$-action on $\widetilde{X}$ which lifts the action on X and commutes with the SO_d-action (i.e., the action is by bundle automorphisms).

Step 4. Path groups. In the final step, we need to build an action of a larger group H from generating subgroups $\hat{K}$, $\mathbb{R}^k$ and N^{χ}, $\chi \in \Delta$. Recall that we have three actions on the bundle $\widetilde{X}$ we built last time:

- $\widehat{K} \curvearrowright \widetilde{X}$, the structure group of principle bundle;
- $\mathbb{R}^k \curvearrowright \widetilde{X}$, the lifted action by pushing framings;
- $N^{\chi} \curvearrowright \widetilde{X}$, the nilpotent actions parametrizing $\mathcal{W}^{\chi}$-leaves.

Goal. "Glue" these actions into a single Lie group action $H \curvearrowright \widetilde{X}$.

To achieve this, we consider the free product

$$\mathcal{P} = \widehat{K} * \mathbb{R}^k * \mathop{\ast}_{\chi \in \Delta} N^{\chi}.$$

Free products of topological groups carry a unique topology defined through a universal property, namely that each component group in the product embeds continuously, and any family of continuous homomorphisms from the component group extends to a continuous homomorphism from the free product. In particular, since group actions can be thought of as continuous homomorphisms to diffeomorphism or homeomorphism groups, the actions of $\hat{K}$, $\mathbb{R}^k$ and N^{χ} induce an action of $\mathcal{P}$.

Proposition 2.7

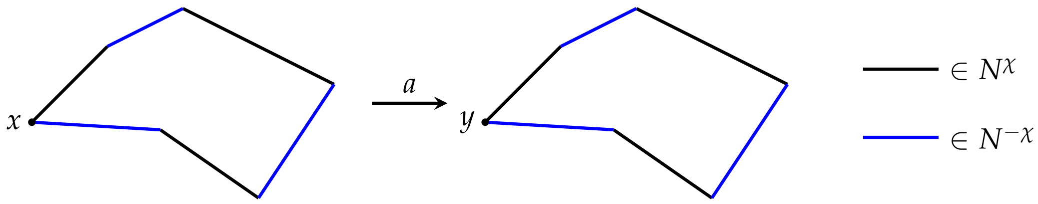
If $\mathrm{Stab}_{\mathcal{P}}(x)^{\circ} \lhd \mathcal{P}$ then $\mathcal{P}$ factors through a Lie group action.

Observation 2.8. The set $\mathrm{Stab}_{\mathcal{P}}(x)^{\circ}$ is exactly the set of "contractible cycles".

Proposition 2.9 $\mathrm{Stab}_{\mathcal{P}}(x)^{\circ} \lhd \mathcal{P}$ if and only if $\mathrm{Stab}_{\mathcal{P}}(x)^{\circ} = \mathrm{Stab}_{\mathcal{P}}(y)^{\circ}$ for every x, y.

To show constant cycle structure, we find "enough" constant cycles to generate $\mathrm{Stab}_{\mathcal{P}}(x)^{\circ}$ at every point. In fact, one can show that there are two classes of cycles which are sufficient:

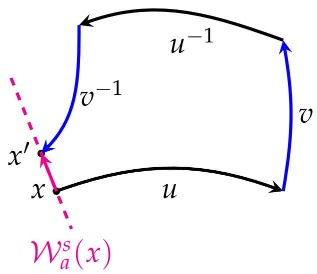
(1) $N^{\chi} * N^{-\chi}$-cycles: We consider a $N^{\chi} * N^{-\chi}$-cycle with the initial point x. Let $y = a.x$ for some $a \in \ker \chi$. Note that both $\mathcal{N}^{\chi}$ and $\mathcal{N}^{-\chi}$ actions commute with $\ker \chi$-actions. The image of the cycle starting at x under the action of a forms the same cycle starting at y (see the figure below). Consequently, cycles of this type are constant because of the $\ker \chi$-transitivity.



(2) Geometric commutators: Showing that the geometric commutators is the last step of the proof and one of the most technical. Indeed, if $u \in N^{\chi_1}$ and $v \in N^{\chi_2}$, then whenever $\chi_1(a), \chi_2(a) \leqslant 0$, and at least one is stricly negative,

$$d(na.v^{-1}u^{-1}vu \cdot x, na.x) \to 0,$$

since at least one of u and v contract under a, and both are non-expanding. If follows that $v^{-1}u^{-1}vu \cdot x$ is in the common stable manifold of x for such elements of $\mathbb{R}^k$.



> **Exercise 2.10.** $\{\lambda \in \Delta : \lambda(a) < 0$ whenever $\chi_1(a), \chi_2(a) \leqslant 0$ and $\chi_1(a) + \chi_2(a) < 0\}$ is the same as the set $\Sigma(\chi_1, \chi_2) = \{t\chi_1 + s\chi_2 : t, s > 0\} \cap \Delta$ (in particular, the second set is well-defined even though χ_1 and χ_2 are defined only up to positive scalar).

One must then make a careful analysis of the terms which appear in the "commutator," finding a unique way to express them in terms of the exponents in $\Sigma(\chi_1, \chi_2)$, and leveraging that the commutators end up satisfying a type of cocycle equation over the N^{χ_i}-actions, $i = 1, 2$. Under the simplifying conformality assumption made after Example 2.5, this becomes much simpler and follows a scheme in a precursor to this proof by Spatzier and Vinhage when the foliations are one-dimensional. Otherwise, several inductions need to be set up to establish polynomial forms of the geometric commutators.

We make a brief outline of the scheme with the simplifying conformality assumption. This in particular implies that each group N^{χ} is abelian, so we write them additively. The trick is to use induction on $\#\Sigma(\chi_1, \chi_2)$. Indeed, when $\Sigma(\chi_1, \chi_2) = \varnothing$, the groups N^{χ_1} and N^{χ_2} commute. Let us move one step into the induction, and assume that there exists a unique λ between χ_1 and χ_2, so $\lambda = t\chi_1 + s\chi_2$. In this case, if we let $\rho(u, v, x)$ be element of N^{λ} required to complete the geometric commutator shown above, one may show that ρ satisfies a cocycle property:

$$\rho(u_1 + u_2, v, x) = \rho(u_1, v, x) + \rho(u_2, v, u.x). \tag{2.1}$$

Furthermore, since the dynamics of $\mathbb{R}^k$ interacts with the nilpotent group actions in an exact way, we also have that

$$e^{t\chi_1(a) + s\chi_2(a)}\rho(u, v, x) = \rho(e^{\chi_1(a)}u, e^{\chi_2(a)}v, a.x) \tag{2.2}$$

A third more subtle property is crucial to the proof, namely that if $\rho \not\equiv 0$, then $t, s \geqslant 1$. This property is called *integral Lyapunov coefficients*, since t and s must be at least an integer (and in the end, turn out to be integers exactly). This property will allow us to rearrange (2.1) and rescale u_2 to a unit vector with (2.2) with $a \in \ker \chi_2$ to get a local Lipschitz property for ρ. In particular, it will be differentiable almost everyone, and the derivative will be $\mathbb{R}^k$-invariant. It will follow that it is constant almost everywhere, and this is enough to get that ρ is linear.

The next inductive step follows similarly, with an extra twist. When there are multiple exponents between χ_1 and χ_2, when defining the completion one must specify an order in which they appear. But when establishing something like (2.1), one must then rearrange the weights inside $\Sigma(\chi_1, \chi_2)$. These rearrangements cause commutators from prevous inductive steps appear. The resulting *cocycle-like equation* becomes

$$\rho(u_1 + u_2, v, x) = \rho(u_1, v, x) + \rho(u_2, v, u_1.x) + q(u_1, u_2, v) \tag{2.3}$$

for some polynomial $q : N^{\chi_1} \times N^{\chi_2} \to N^{\lambda}$. The proof is completed as in the cocycle equation, but ρ is no longer just a linear map, but polynomial. For example, the function $\rho(u, v, x) = u^2 v$ satisfies (2.3) with $q(u_1, u_2, v) = 2u_1 u_2 v$.

§3 Lecture 3: Non-rigidity and partial rigidity

Consider the acton $\alpha : \mathbb{R}^2 \curvearrowright X \times Y$, $\alpha = \psi_t \times \varphi_s$, where ψ_t, φ_s are volume-preserving Anosov flows. We define an action β which shares the same orbits as α by:

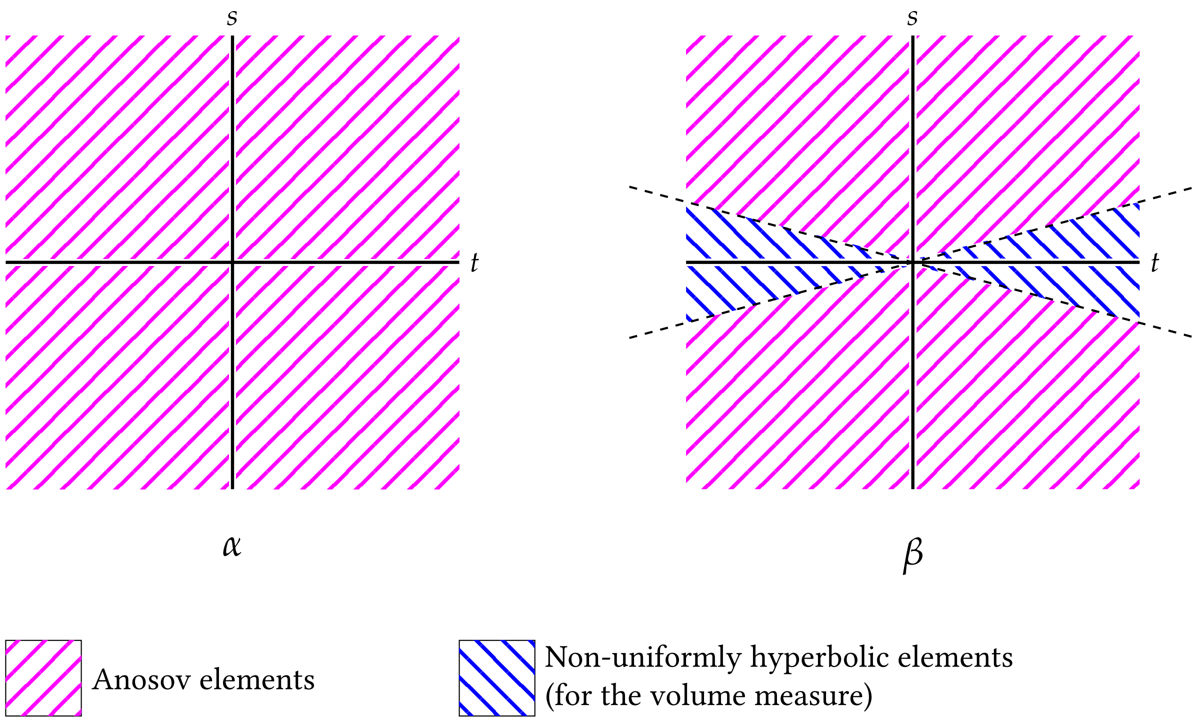
$$\beta(t,s)(x,y) = (\psi_t(x), \varphi_{s+\sigma(t,x)}(y)),$$

where $\sigma : \mathbb{R} \times X \to \mathbb{R}$ is a C^∞-cocycle such that

(1) $\int \sigma(t,x)\,\mathrm{dLeb}(x) = 0$ for every t;

(2) σ is not a coboundary (this is equivalent to $\int \sigma(t,x)\,\mathrm{d}\mu(x) \neq 0$ for some μ).

For the action α, every $(t,s) \in \mathbb{R}^2$ not on the axes is an Anosov element.



On the other hand, every $(t,s) \in \mathbb{R}^2$, we have $\beta(t,s)$ is uniformly hyperbolic if and only if $t \neq 0$ and

$$|s| > \int \sigma(t,x)\,\mathrm{d}\mu(x)$$

for every invariant measure μ. Thus, the action has Anosov elements, but is not totally Anosov.

Remark 3.1 The action of the horizontal axis $\mathbb{R} \times \{0\}$ is a classical example of a smooth system which is K but not Bernoulli. This is an adaptation of the (T, T^{-1}) Kalikow example adapted to the smooth setting by Rudolph. It is worth noting that every C^∞-example of K non-Bernoulli are elements of partially hyperbolic $\mathbb{Z} \times \mathbb{R}$ or $\mathbb{R}^2$-actions.

Question 3.2

Fix $r \geqslant 0$. If $f : X \to X$ is C^∞, volume preserving, K, but not Bernoulli, does $Z^r(f)/\langle f \rangle$ contain a copy of $\mathbb{R}$? If $f : X \to X$ is C^∞, volume preserving, K, non-Bernoulli, and partially hyperbolic, can f also be accessible?

Theorem 3.3 (Vinhage)

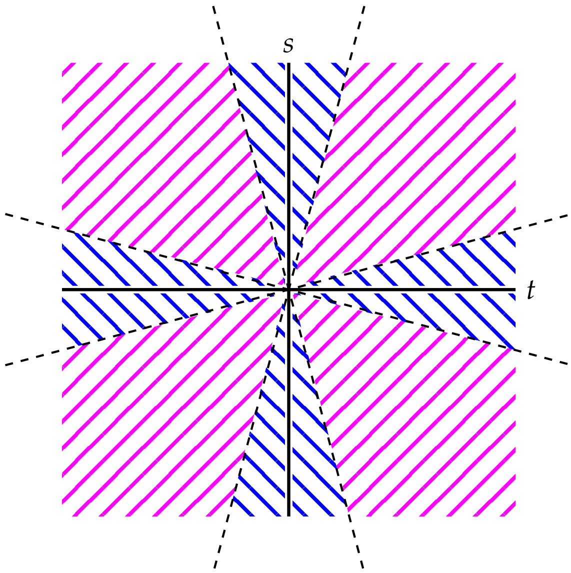
There exist "genuine" $\mathbb{R}^2$-time changes of product actions such that for every coarse $\mathcal{W}, \mathcal{C}^-(\mathcal{W})$ are properly contained in a half space. Consequently, this action is group-irreducible (i.e. has no factors).

Sketch of Proof. The idea is to imitate Rudolph's example, but shrink both Lyapunov half-spaces simultaneously. Note that we can't do this iteratively: once we change from a direct product to a skew product, we can't consider a cocycle depending only on the vertical coordinate. Instead, one must define reparameterization of $\mathbb{R}^2$-orbits all at once. Indeed, one may check that if $\tau : \mathbb{R}^2 \times X \to \mathbb{R}^2$ is a smooth function, when the family of transformations

$$\alpha_\tau(a).x = \alpha(\tau(a,x)).x$$

defines an action, τ is a cocycle over α_τ. This presents a complication when defining actions using cocycles, but notice that if $\tau_x(a) = \tau(a,x)$, then $\tau^{-1}(a,x) := (\tau_x^{-1}(a), x)$ is a cocycle over α. In fact, given any cocycle β over α such that β_x is a diffeomorphism of $\mathbb{R}^2$ for every x, one may perform this inversion operation to make a time change.

The resulting set of uniformly and non-uniformly hyperbolic elements for perturbations can be made to look as following:



□

Remark 3.4 As discussed at the beginning of the notes, it was believed that group-irreducible Anosov higher rank actions were all homogeneous. This theorem states that an group-irreducible Anosov higher rank action can be obtained by time changing a reducible action.

Question 3.5

Is every group-irreducible Anosov $\mathbb{R}^2$-action either a time change of a product or homogeneous?