Spectrum Rigidity and Joint Integrability for Anosov Systems on Tori

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§1 Local Rigidity (Apr 25)

Definition 1.1. $f \in \mathrm{Diff}^1(M)$ is **Anosov** if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^u$ such that for every unit vector $v^{s/u} \in E^{s/u}$:

$$\|Df(v^s)\| < 1, \quad \|Df(v^u)\| > 1.$$

Example 1.2 (Arnold's cat map)
$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is Anosov.

There are two main open problems in the study of Anosov diffeomorphisms.

Question 1.3. Is every Anosov diffeomorphism transitive?

Question 1.4. Topological classification of Anosov diffeomorphism.

Theorem 1.5 (Franks-Manning)
Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Theorem 1.6 (Franks-Newhouse)
Every codimension-1 Anosov diffeomorphism must be supported on $\mathbb{T}^d$.

Definition 1.7. $f \in \mathrm{Diff}^r(M)$ is **partially hyperbolic**, if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

and functions $\xi, \eta : M \to (0,1)$ such that for every $x \in M$ and unit vectors $v^{s/c/u} \in E^{s/c/u}$,

$$\|Df(v^s)\| < \xi(x) < \|Df(v^c)\| < \eta(x)^{-1} < \|Df(v^u)\|.$$

Definition 1.8. A partially hyperbolic diffeomorphism f is **absolutely partially hyperbolic** if $\xi = \xi_0, \eta = \eta_0 \in (0,1)$,

$$\|Df(v^s)\| < \xi_0 < \|Df(v^c)\| < \eta_0^{-1} < \|Df(v^u)\|.$$

Let $f : M \to M$ be a partially hyperbolic diffeomorphism, then

$$TM = E^s \oplus E^c \oplus E^u.$$

Question 1.9. What happens if $E^s \oplus E^u$ is integrable?

Remark 1.10 $E^s \oplus E^u$ integrable $\implies$ NOT accessible.

However, Dolgopyat-Wilkinsonm and Hertz-Hertz-Ures, etc. showed that "MOST" partially hyperbolic diffeomorphisms are accessible.

Main philosophy.

Geometric Rigidity $\iff$ **Dynamic Spectral Rigidity**

That is, $E^s \oplus E^u$ is integrable $\implies$ E^c has exponents rigidity.

Example 1.11

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4,$$

which is irreducible and partially hyperbolic

$$T\mathbb{T}^4 = L^s \oplus L^c \oplus L^u,$$

where $\dim L^c = 2$ and $\lambda^c(A) \equiv 0$.

Theorem (F. R. Hertz, 2005). For every f which is C^{22}-close to A with splitting $T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u$, if $E^s \oplus E^u$ is integrable, then there exists homeomorphism $h : \mathbb{T}^4 \to \mathbb{T}^4$ which is C^1-along E^c such that $h \circ f = A \circ h$. In particular, all center exponents $\lambda^c(f) \equiv 0$.

Example 1.12 (Reducible case)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $F_0 = \begin{bmatrix} A^2 & 0 \\ 0 & A \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$. Assume $f : \mathbb{T}^2 \to \mathbb{T}^2$ be C^1-close to A. Then

$$F = \begin{bmatrix} A^2 & 0 \\ 0 & f \end{bmatrix} : \mathbb{T}^4 \to \mathbb{T}^4$$

is an Anosov diffeomorphism C^1-close to F_0 with splitting

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}.$$

Here $E^{ss} \oplus E^{wu} \oplus E^{uu}$, $E^{ss} \oplus E^{ws} \oplus E^{uu}$, $E^{ss} \oplus E^{uu}$ are all integrable, but f is arbitrary:

NO exponents rigidity.

Main theorem: local rigidity. Assume that $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies *generic properties*:

- A is irreducible and hyperbolic;
- two eigenvalues of A have the same absolute value must be conjugate complex numbers.

Here the generic property means that

$$\lim_{K\to\infty} \frac{\#\{\, A \text{ is generic} \,:\, \|A\| \leqslant K \,\}}{\#\{\, A \,:\, \|A\| \leqslant K \,\}} = 1.$$

Denote

$$T\mathbb{T}^d = L_1^s \oplus \cdots \oplus L_l^s \oplus L_1^u \oplus \cdots \oplus L_m^u$$

the finest dominated splitting, then $\dim L_i^{s/u} \leqslant 2$.

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E_1^s \oplus \cdots \oplus E_k^s \oplus E_{k+1}^s \oplus \cdots \oplus E_l^s \oplus E_1^u \oplus \cdots \oplus E_m^u.$$

Assume that $l \geqslant 2$ and $1 \leqslant k < l$. Denote

$$E^{ss} = E_1^s \oplus \cdots \oplus E_k^s \text{ and } E^{ws} = E_{k+1}^s \oplus \cdots \oplus E_l^s.$$

Then

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$$

makes f be an absolutely partially hyperbolic system.

Theorem 1.13 (Local rigidity, Gogolev-Shi, arXiv: 2207.00704)
Assume $A \in \mathrm{GL}(d, \mathbb{Z})$ satisfies generic properties. For every $f \in \mathrm{Diff}^2(\mathbb{R}^d)$ be C^1-close to A, the following are equivalent:

1. $E^{ss} \oplus E^u$ is integrable.
2. f has spectral rigidity in E^{ws}:

$$\lambda(E_i^s, f) \equiv \lambda(L_i^s, A), \quad \forall i = k+1, \cdots, l.$$

3. The conjugacy h ($h \circ f = A \circ h$) is smooth along E^{ws}.

Dimension 3 case.

Theorem 1.14 (Hammerlindl-Ures, 2014)
Let $f \in \mathrm{Diff}_m^r(\mathbb{T}^3)$ be partially hyperbolic and $f_* \in \mathrm{GL}(3, \mathbb{Z})$ be hyperbolic (f is a DA-diffeo), then

- either f is accessible, thus ergodic.
- or there exists an f-invariant minimal foliation $\mathscr{F}^{su}$ such that $T\mathscr{F}^{su} = E^s \oplus E^u$ and f is topologically conjugate to f_*.

Theorem 1.15 (Gan-Shi, 2020)

Let $f \in \mathrm{Diff}^{1+}_m(\mathbb{T}^3)$ be a partially hyperbolic DA-diffeo. The following are equivalent:
- $E^s \oplus E^u$ is integrable;
- f has spectral rigidity in E^c: $\lambda^c(f) \equiv \lambda^c(f_*)$.

Both imply f is Anosov.

Corollary 1.16 Every C^{1+} partially hyperbolic DA-diffeo is ergodic.

Proof of Theorem 1.13 — spectral rigidity $\implies$ joint integrability. The case of all E^s_i are 1-dimensional is shown by [Gogolev, 2018]. For generic $A \in \mathrm{GL}(d, \mathbb{Z})$, the statement is shown by [Gogolev-Kalinin-Sadovskaya, 2011, 2020].

Spectral rigidity in $E^s_l \implies$ smooth conjugacy in $E^s_l \implies h(\mathcal{F}^s_{l-1}) = \mathcal{L}^s_{l-1}$ (+spectral rigidity in E^s_{l-1}) $\implies$ smooth conjugacy in $E^s_{l-1} \implies \cdots \implies h(\mathcal{F}^s_{k+1}) = \mathcal{L}^s_{k+1}$ (+spectral rigidity in E^s_{k+1}) $\implies h(\mathcal{F}^{ss}) = \mathcal{L}^{ss} \implies \mathcal{F}^{ss} \oplus \mathcal{F}^u = h^{-1}(\mathcal{L}^{ss} \oplus \mathcal{L}^u)$ joint integrability.

Proof of Theorem 1.13 — joint integrability $\implies$ spectral rigidity. Main ideas:
1. $E^{ss} \oplus E^u$ integrability $\implies h(\mathcal{F}^{ss}) = \mathcal{L}^{ss}$ is linear.
2. Diophantine approximation of $\mathcal{F}^{ss} \implies$ spectral rigidity in E^s_{k+1}.
3. $E^{ss} \oplus E^s_{k+1} \oplus E^u$ is integrable, and play induction on E^s_{k+2}.

Lemma 1.17

For every $1 \leqslant i \leqslant l$, the conjugation h preserves the center foliation: $h(\mathcal{F}^s_{(i,l)}) = \mathcal{L}^s_{(i,l)}$. Here, $\mathcal{F}^s_{(i,l)}$ and $\mathcal{L}^s_{(i,l)}$ are the foliations tangent to $E^s_i \oplus \cdots \oplus E^s_l$ and $L^s_i \oplus \cdots \oplus L^s_l$, respectively.

Proof. Since f is C^1-close to A, we have

$$\|A_{L^s_{i-1}}\| < \inf_{x \in \mathbb{T}^d} m(Df|_{E^s_i(x)}) =: \rho_i.$$

Let $F, H : \mathbb{R}^d \to \mathbb{R}^d$ be lifts of f and h, then $y \in \widetilde{\mathcal{F}}^s_{(i,l)}(x)$ iff

$$\|H^{-1} \circ A^{-n} \circ H(x) - H^{-1} \circ A^{-n} \circ H(y)\| \leqslant (\rho_i - \varepsilon)^{-n}\|x - y\| + C < (\|A_{L^s_{i-1}}\| + \varepsilon)^{-n}\|x - y\| + C,$$

iff $H(y) \in \widetilde{\mathcal{L}}^s_{(i,l)}(H(x))$. □

Lemma 1.18

If $\mathcal{F}$ is a C^0-foliation sub-foliated by a minimal linear foliation $\mathcal{L}$ on $\mathbb{T}^d$, then $\mathcal{F}$ is minimal and linear.

Proof. **Minimal.** every leaf $\mathcal{F}(x) \supset \mathcal{L}(x)$ is dense.
Linear. We will show that, on universal cover, $\widetilde{\mathcal{F}}(0) \subset \mathbb{R}^d$ is closed under addition. For every $x, y \in \widetilde{\mathcal{F}}(0)$, there exists $v_n \to \widetilde{\mathcal{L}}(0)$ and $k_n \in \mathbb{Z}^d$ such that $k_n + v_n \to x$. Since $\mathcal{F}$ is sub-foliated by $\mathcal{L}$ and $\mathcal{L}$ is linear, we have

$$y + k_n + v_n \in \widetilde{\mathcal{F}}(y + k_n) = \widetilde{\mathcal{F}}(k_n) = \widetilde{\mathcal{F}}(k_n + v_n).$$

Take $n \to \infty$, then $y + x \in \widetilde{\mathcal{F}}(x) = \widetilde{\mathcal{F}}(0)$. □

Lemma 1.19 If $E^{ss} \oplus E^u$ is integrable to $\mathscr{F}^{su}$, then $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ is linear.

Proof. Note that $h(\mathscr{F}^{su})$ is sub-foliated by $h(\mathscr{F}^u) = \mathscr{L}^u$, where $\mathscr{L}^u$ is linear and minimal on $\mathbb{T}^d$. Hence $h(\mathscr{F}^{su})$ is linear, A-invariant and transverse to $\mathscr{L}^{ws} = h(\mathscr{F}^{ws})$. This implies $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$. So

$$h(\mathscr{F}^{ss}) = h(\mathscr{F}^s \cap \mathscr{F}^{su}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{su}) = \mathscr{L}^s \cap \mathscr{L}^{su} = \mathscr{L}^{ss}.$$

□

Corollary 1.20

Recall that $T\mathscr{F}^{ss} = E_1^s \oplus \cdots \oplus E_k^s$. If $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$, then for $T\mathscr{F}_j^s = E_j^s$, we have

$$h(\mathscr{F}_j^s) = \mathscr{L}_j^s, \quad \forall j = k, k+1, \cdots, l.$$

Lemma 1.21 (Diophantine approximation of $\mathscr{F}^{ss}$)

There exists $C, \alpha > 0$ such that for every $x \in \mathbb{T}^d$ and $R > 0$, the disk $\mathscr{F}_R^{ss}(x)$ is $C \cdot R^{-\alpha}$-dense in $\mathbb{T}^d$.

Proof. Since $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and h is Hölder continuous, it suffices to show the Diophantine property of $\mathscr{L}^{ss}$. Here A is irreducible and $\mathscr{L}^{ss}$ is algebraic, hence Diophantine. □

Proof of Theorem 1.13. We will fist show that the Lyapunov exponent at every point is the same in the dim $E_{k+1}^s = 1$ case. Take $p, q \in \mathrm{Per}(f)$ such that

$$\min \lambda_{k+1}^s(f) \approx \lambda_{k+1}^s(p) < \lambda_{k+1}^s(q) \approx \lambda_{k+1}^s(f).$$

Without loss of generality, we assume that p, q are fixed by f.

Take
- $x_n \in \mathscr{F}^{ss}(p)$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$.
- Segments $J \subset \mathscr{F}_{k+1}^s(p)$ and $J_n \subset \mathscr{F}_{k+1}^s(x_n)$ such that $J_n = \mathrm{Hol}^{ss}(J)$ $(x_n = \mathrm{Hol}^{ss}(p))$. Besides, we have $|f^m(J)| \approx \exp[m \cdot \lambda_{k+1}^s(p)] \cdot |J|$.

Since $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and $h(\mathscr{L}_{k+1}^s) = \mathscr{L}_{k+1}^s$ both are linear, we have

$$|h(J_n)| \equiv |h(J)| \quad \Longrightarrow \quad \exists C_0 > 0, |J_n| \geqslant C_0 |J|.$$

Now we choose m_n, k_n such that
- x_n and q are very close in first k_n-steps;
- $f^{m_n}(x_n)$ is the first time entering $\mathscr{F}_1^{ss}(p)$.

Then

$$|f^{m_n}(J_n)| \gtrsim \exp[(m_n - k_n)\lambda_{k+1}^s(p) + k_n \lambda_{k+1}^s(q)] |J_n|.$$

From Diophantine estimation, $d(x_n, q) \ll [d^{ss}(p, x_n)]^{-\alpha}$, there exists $\delta > 0$ such that $k_n > \delta m_n$. It follows that

$$\frac{|f^{m_n}(J_n)|}{|f^{m_n}(J)|} \geqslant \frac{\exp[\delta(\lambda_{k+1}^s(q))]}{\exp[\delta(\lambda_{k+1}^s(p))]} \cdot \frac{|J_n|}{|J|} \to \infty.$$

However, $J_n = \mathrm{H}^{ss}(J)$ implies that $f^{m_n}(J_n) = \mathrm{Hol}^{ss}(f^{m_n}(J))$. Since $f^{m_n}(x_n) \in \mathscr{F}_1^{ss}(p)$ and $f^{m_n}(x_n) = \mathrm{Hol}^{ss}(p)$, this contradicts to $\mathscr{F}^{ss}$ is C^1-smooth in $\mathscr{F}^{ss} \oplus \mathscr{F}_{k+1}^s(p)$.

For the case of dim $E_{k+1}^s = 2$, we repeat the argument of 1-dim case. We can obtain

- For every periodic points p, q, we have $\min \lambda^s_{k+1}(p) = \min \lambda^s_{k+1}(q)$.
- Considering the growth of area of local disks, we have

$$\operatorname{Jac}(Df, E^s_{k+1}(p)) = \operatorname{Jac}(Df, E^s_{k+1}(q)), \quad \forall p, q \in \operatorname{Per}(f).$$

Then we estimate the growth on the universal cover, the Lyapunov exponents $\lambda^s_{k+1}(f)$ at periodic points are forced to coincide with the Lyapunov exponent $\lambda^s_{k+1}(A)$. $\square$

§2 Global Rigidity (Apr 26)

In the last lecture, we have shown a local rigidity result. That is, we only consider diffeomorphisms f that is C^1-close to A. Today we will consider the global rigidity, i.e., the relation between f and $f_* \in \mathrm{GL}(d, \mathbb{Z})$.

Question 2.1. What happens if f is not close to $A = f_*$?

Theorem 2.2 (Gogolev-Farell)
For $d \geqslant 10$, let $A \in \mathrm{GL}(d, \mathbb{Z})$ be a hyperbolic matrix. Then

$$\mathscr{A}^{1+}_A(\mathbb{T}^d) := \left\{ f \in \operatorname{Diff}^{1+}(\mathbb{T}^d) \ : \ f \text{ is Anosov, } f_* = A \right\}$$

has infinitely many connected components.

Theorem 2.3 (Full leaf conjugacy, Gogolev-Shi, arXiv: 2207.00704)
Let $f \in \operatorname{Diff}^1(\mathbb{T}^d)$ be Anosov with absolutely partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$:

$$\|Df_{E^{ss}}\| < \mu < m(Df|_{E^{ws}}) < \|Df|_{E^{ws}}\| < 1 < m(Df|_{E^u}).$$

If $E^{ss} \oplus E^u$ is integrable, then

1. $A = f_* \in \mathrm{GL}(d, \mathbb{Z})$ is partially hyperbolic:

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u, \quad \dim L^\sigma = \dim E^\sigma, \quad \sigma = ss, ws, u.$$

2. f is dynamically coherent and fully conjugate to A:

$$h(\mathscr{F}^\sigma) = \mathscr{L}^\sigma, \quad \sigma = ss, ws, u.$$

   Here $h \circ f = A \circ h$.

Question 2.4. Let $f = \operatorname{Diff}^1(\mathbb{T}^d)$ be Anosov with partially hyperbolic splitting $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$.
- Is $f_* \in \mathrm{GL}(d, \mathbb{Z})$ partially hyperbolic?
- Is f dynamically coherent or not? If yes, does f leaf conjugate to A.

Lemma 2.5

Let $\mathscr{F}$ be a C^0-foliation on $\mathbb{T}^d$ with C^1-leaves. If there exists a homeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to $\mathrm{id}_{\mathbb{T}^d}$ such that $h(\mathscr{F}) = \mathscr{L}$ is a linear foliation, then $\mathscr{F}$ is quasi-isometric:

$$d_{\widetilde{\mathscr{F}}}(x, y) \leqslant a \cdot d(x, y) + b, \quad \forall x \in \mathbb{R}^d, y \in \widetilde{\mathscr{F}}(x).$$

Here $a, b > 0$ and $\widetilde{\mathscr{F}}$ is the lift of $\mathscr{F}$ in $\mathbb{R}^d$.

Proof of Theorem 2.3. Since $h(\mathscr{F}^{ss} \oplus \mathscr{F}^{u})$ is sub-foliated by minimal linear foliation $h(\mathscr{F}^u) = \mathscr{L}^u$ is linear. We have $\mathscr{L}^{ss} := h(\mathscr{F}^{ss}) = h(\mathscr{F}^s) \cap h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear.

Brin's argument shows that $E^{ws} \oplus E^u$ integrate to $\mathscr{F}^{cu}$ and $h(\mathscr{F}^{cu})$ is linear and minimal. Then $\mathscr{F}^{ws}$ integrate to $\mathscr{F}^{ws}$ and $\mathscr{L}^{ws} := h(\mathscr{F}^{ws})$ is A-invariant and linear.

Note that $\mathscr{L}^{ws}$ and $\mathscr{L}^{ss}$ are transverse in $\mathscr{L}^s$, then A admits an invariant splitting $T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^u$. We need to show this is a dominated splitting. This follows from the above lemma and the fact that h is homotopic to $\mathrm{id}_{\mathbb{T}^d}$. □

Theorem 2.6 (Global rigidity, Gogolev-Shi, arXiv: 2207.00704)

Let $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be Anosov and irreducible. Assume that f is absolutely partially hyperbolic $T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u$ and center bunching. If $E^{ss} \oplus E^u$ is integrable, then

1. f has a finest dominated splitting on E^{ws} with the same dimensions for $A|_{L^{ws}}$:

$$E^{ws} = E_1^{ws} \oplus \cdots \oplus E_k^{ws}, \quad \dim E_i^{ws} = \dim L_i^{ws}.$$

2. f is spectrally rigid along every E_i^{ws}:

$$\lambda(E_i^{ws}, f) \equiv \lambda(L_i^{ws}, A), \quad \forall i = 1, \cdots, k.$$

Remark 2.7 • Here f need NOT to be C^1-close to $A = f_*$.
- To get dominated splitting, we usually need some C^1-robust property like: robustly transitive, far from homoclinic bifurcations.
- If $A = f_*$ satisfies the generic assumption in the last lecture, then the conjugacy h is C^{1+}-smooth along $\mathscr{F}^{ws}$.
- The center bunching condition

$$\|Df|_{E^{ws}(x)}\| < m(Df|_{E^{ws}(x)}) \cdot m(Df|_{E^u(x)})$$

is a technical condition, which guarantees C^{1+}-smoothness of $\mathscr{F}^{su}$.

Corollary 2.8

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be codimension one with real simple spectrum. For every Anosov $f \in \mathrm{Diff}^2_m(\mathbb{T}^d)$ with $f_* = A$ and

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^u, \quad \dim E^{ss} = 1,$$

if

- $E^{ss} \oplus E^u$ is integrable;
- the metric entropy $h_m(f) = h_m(A)$;

then f is C^{1+}-conjugate to A.

Main idea for showing Theorem 2.6. Play the game similar to the last lecture. We will use the Diophantine approximation of $\mathscr{F}^{ss}$ to show the rigidity of smallest exponent in E^{ws}:

$$\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f).$$

Then we will show the dimension of $\lambda^{ws}_{\min}$ for each periodic point is constant. Next, we define the Pesin stable foliation $\mathscr{F}^{ws}_{\min}$ and show it is $\mathscr{F}^{su}$-holonomy invariant, that is $\mathrm{Hol}^{su} : \mathscr{F}^{ws}(p) \to \mathscr{F}^{ws}(q)$ preserves $\mathscr{F}^{ws}_{\min}$, for every $p, q \in \mathrm{Per}(f)$. Finally, we show a uniform spectral exponents gap and extract out $\mathscr{F}^{ws}_{\min}$.

Lemma 2.9

Let $\mathrm{Hol}^{su}_{x,y} : \mathscr{F}(x) \to \mathscr{F}(y)$ be the holonomy map of $\mathscr{F}^{su}$ with $\mathrm{Hol}^{su}_{x,y}(x) = y$ for every $x \in \mathbb{T}^d$ and $y \in \mathscr{F}^{su}(x)$. Then

$$\mathrm{Hol}^{su}_{x,y}(K) = h^{-1} \circ T_{h(x),h(y)} \circ h(K).$$

Here $T_{h(x),h(y)} : \mathbb{T}^d \to \mathbb{T}^d$ is the linear translation send $h(x)$ to $h(y)$. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $K \subset \mathscr{F}^{ws}(x)$ with $\mathrm{diam}(K) > \varepsilon$, then

$$\mathrm{diam}(\mathrm{Hol}^{su}_{x,y}(K)) > \delta, \quad \forall y \in \mathscr{F}^{su}(x).$$

Remark 2.10 The same holds for $\mathrm{Hol}^{ss}_{x,y} : \mathscr{F}^{ws}(x) \to \mathscr{F}^{ws}(y)$ where $y \in \mathscr{F}^{ss}(x)$.

Proof. It follows immediately from f is fully conjugate to A. □

Proof of Theorem 2.6. We fist show that

Claim 2.11. $\lambda^{ws}_{\min}(p) = \lambda^{ws}_{\min}(q), \quad \forall p, q \in \mathrm{Per}(f)$.

Proof. Assume that $\lambda^{ws}_{\min}(p) < \lambda^{ws}_{\min}(q)$. Take $x_n \in \mathscr{F}^{ss}(p)$ such that $d^{ss}(x_n, p) = K_n \to \infty$ and $d(x_n, q) \leqslant C \cdot K_n^{-\alpha}$. Take disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, the Pesin stable manifold associated to $\lambda^{ws}_{\min}(p)$. Let $D_n = \mathrm{Hol}^{ss}(D) \subset \mathscr{F}^{ws}(x_n)$, then $\mathrm{diam}(D_n) \gg \mathrm{diam}(D)$. Applying a similar (k_n, m_n)-argument, we get a contradiction since $\mathscr{F}^{ss}$ is C^1-smooth in $\mathscr{F}^{ws}(p)$. □

Now we have $\lambda^{ws}_{\min} := \lambda^{ws}_{\min}(p)$ for every $p \in \mathrm{Per}(f)$. We define the Pesin stable foliation associated to $\lambda^{ws}_{\min}$ for each periodic point.

Claim 2.12. $\mathscr{F}^{ws}_{\min}$ is Hol^{su}-invariant.

Proof. Let $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)} := h(\mathscr{F}^{ws}_{\min}|_{\mathscr{L}^{ws}(p)})$, it suffices to show

$$T_{h(p),h(x)}(\mathscr{L}^{ws}_{\min}(p)) \subset \mathscr{L}^{ws}_{\min}(x)$$

for every $p, q \in \mathrm{Per}(f)$ and $x \in \mathscr{F}^{ws}(q)$. Otherwise, take a disk $D \subset \mathscr{F}^{ws}_{\min}(p)$, then $T_{h(p),h(x)}(h(D))$ is transverse to $\mathscr{L}^{ws}_{\min}|_{\mathscr{L}^{ws}_{\mathrm{loc}}(q)}$ at $h(x)$. Take $x_n \in \mathscr{F}^{ss}$ such that $d^{ss}(p, x_n) = K_n \to \infty$ and $d(x_n, x) \ll K_n^{-\alpha}$, then

$$D_n := \mathrm{Hol}^{ss}_{p,x_n}(D) \to h^{-1} \circ T_{h(p),h(x)} \circ h(D).$$

It follows that $\mathrm{Hol}^{u}_{\mathrm{loc}}(D)$ is "uniformly transverse" (the angle will not tend to zero) to $\mathscr{L}^{ws}_{\min}$ in $\mathscr{F}^{ws}_{\mathrm{loc}}(q)$, where $\mathrm{Hol}^{u}_{\mathrm{loc}}(D) : \mathscr{F}^{ws}(x_n) \to \mathscr{F}^{ws}(q)$ is C^{1+}-smooth. Since the transverse direction has a weaker contracting rate, we play the (k_n, m_n)-game and get a contradiction. □

Let $\mathscr{L}^{ws}_{\min} := h(\mathscr{L}^{ws}_{\min})$, then the density of $\mathrm{Per}(f)$ and minimality of $\mathscr{F}^{ws}$ imply $T_{x,y}(\mathscr{L}^{ws}_{\min}(x)) \subset \mathscr{L}^{ws}_{\min}(y)$. By the translation invariance and the A-invariance, we have

- $\mathscr{L}^{ws}_{\min}$ is a linear foliation on $\mathbb{T}^d$, and
- $L^{ws}_{\min} := T\mathscr{L}^{ws}_{\min}$ associate to an eigenspace of A.

Also by an estimate of the growth, we get $\lambda(A, L^{ws}_{\min}) \equiv \lambda^{ws}_{\min}$.

Finally, we establish the induction step. Following the idea of [Bonatti-Díaz-Pujals, 2003], consider the quotient cocycle $\widetilde{Df} : E^{ws}/E^{ws}_{\min} \to E^{ws}/E^{ws}_{\min}$ which is Hölder continuous over f. Again by a (k_n, m_n)-game, we can show that λ^{ws}_2 is uniformly larger than $\lambda^{ws}_{\min}$. Then the splitting $T\mathbb{T}^d = (E^{ss} \oplus E^{ws}_{\min}) \oplus F \oplus E^u$ is an absolutely partially hyperbolic splitting. The joint integrability follows from $h(\mathscr{F}^{ss} \oplus \mathscr{F}^{ws}_{\min})$ is linear. □

§3 Rigidity on $\mathbb{T}^4$ (Apr 27)

Let us recall some results shown in last two lectures. We remark that the key point is that

$$E^{ss} \oplus E^u \text{ is integrable} \implies h(\mathscr{F}^{ss} = \mathscr{L}^{ss}) \text{ is linear.}$$

> **Question 3.1.** Let f be C^1-close to A with splitting
>
> $$T\mathbb{T}^d = E^s_1 \oplus \cdots \oplus E^s_k \oplus \cdots \oplus E^s_l \oplus E^u_1 \oplus \cdots \oplus E^u_j \oplus \cdots \oplus E^u_m.$$
>
> What happens if $E^s_k \oplus E^u_j$ is jointly integrable? Spectral rigidity in $E^s_{k+1} \oplus \cdots \oplus E^s_l \oplus E^u_1 \oplus \cdots \oplus E^u_{j-1}$?

Theorem 3.2 (Gogolev-Kalinin-Sadovskya)

Spectral rigidity in $E^s_{k+1} \oplus \cdots \oplus E^s_l \oplus E^u_1 \oplus \cdots \oplus E^u_{j-1}$ implies $h(\mathscr{F}^s_k) = \mathscr{L}^s_k$ and $h(\mathscr{F}^u_j) = \mathscr{L}^u_j$ hence $E^s_k \oplus E^u_j$ is jointly integrable.

The work of Avila-Viana.

Theorem 3.3 (Avila-Viana, 2010)

For every symplectic f which is C^∞-close to A with splitting

$$T\mathbb{T}^4 = E^s \oplus E^c \oplus E^u,$$

then

- either f is accessible and non-uniformly hyperbolic;
- or $E^s \oplus E^u$ is integrable and $\exists h \in \mathrm{Diff}^\infty_m(\mathbb{T}^4)$ such that

$$h \circ f = A \circ h.$$

In particular, f is Bernoulli.

Main theorem.

Theorem 3.4 (Gogolev-Shi, arXiv: 2207.00704)

Let $A \in \mathrm{GL}(d, \mathbb{Z})$ be an irreducible Anosov automorphism with dominated splitting

$$T\mathbb{T}^d = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}, \quad \text{with} \quad \dim L^{ws} = \dim L^{wu} = 1.$$

For $f \in \mathrm{Diff}^2(\mathbb{T}^d)$ be C^1-close to A with splitting

$$T\mathbb{T}^d = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is spectral rigid along E^{ws} and E^{wu}.

Corollary 3.5

Let $A \in \mathrm{Sp}(4, \mathbb{Z})$ be hyperbolic and irreducible with dominated splitting

$$T\mathbb{T}^4 = L^{ss} \oplus L^{ws} \oplus L^{wu} \oplus L^{uu}.$$

For symplectic $f \in \mathrm{Diff}^2_\omega(\mathbb{T}^4)$ be C^1-close to A with

$$T\mathbb{T}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu},$$

the following are equivalent:

- $E^{ss} \oplus E^{uu}$ is integrable;
- f is C^{1+}-smoothly conjugate to A.

Proof of corollary. If $E^{ss} \oplus E^{uu}$ is integrable, then we have spectral rigidity in $E^{ws} \oplus E^{wu}$, h is smooth along $E^{ws} \oplus E^{wu}$ and $h(\mathscr{L}^{ss}) = \mathscr{L}^{ss}, h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$. Since h is smooth along $\mathscr{F}^{ws}$ and $\mathscr{F}^{wu}$, the holonomy map $\mathrm{Hol}^{su}_{\mathscr{F}}$ is C^{1+}. Then we use the symplectic structure that $E^c = E^{ws} \oplus E^{wu}$ is perpendicular to E^{su} (with respect to ω). Hence $\mathscr{F}^{ws} \oplus \mathscr{F}^{wu}$ is C^{1+}. Then we can show that h is absolutely continuous in $\mathscr{F}^{su}$ and hence h is C^{1+}. $\square$

Proof of main theorem. Main problem is whether $h(\mathscr{F}^{su}) = \mathscr{L}^{su}$ is the linear one? Or equivalently, whether we have $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ or $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$? This is nontrivial.

Lemma 3.6

If one of $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ is integrable, then f is spectral rigid in $E^{ws} \oplus E^{wu}$.

Proof. If $E^{ss} \oplus E^u$ is integrable, then $h(\mathscr{F}^{ss} \oplus \mathscr{F}^u)$ is linear and hence $h(\mathscr{F}^{ss}) = h(\mathscr{F}^{ss} \oplus \mathscr{F}^u) \cap \mathscr{L}^s = \mathscr{L}^{ss}$ is linear. Then both $h(\mathscr{F}^{su})$ and $h(\mathscr{F}^{uu})$ are linear. Then we obtain a spectral rigidity by Theorem 1.13. □

The solvable action. Let $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$ and $L^c(0) = L^{ws}(0) \oplus L^{wu}(0) \subset \mathbb{R}^d$. Define the linear action

$$\alpha_0 : \Gamma \times L^c(0) \to L^c(0), \quad \alpha_0(k,n)(x) = L^{su}(A^k(x) + n) \cap L^c(0).$$

If we write $n = n^s + n^c + n^u \in L^s \oplus L^c \oplus L^u$, then $\alpha_0(k,n)(x) = A^k x + n^c$.

For $F : \mathbb{R}^4 \to \mathbb{R}^4$ be the lift of f and $F(0) = 0$, then

- $F^k(x+n) = F^k(x) + A^k n$, $\forall x \in \mathbb{R}^d$ and $\forall n \in \mathbb{Z}^d$.
- $F(\widetilde{\mathscr{F}}^c(0)) = \widetilde{\mathscr{F}}^c(0)$.

Then $\Gamma \curvearrowright \widetilde{\mathscr{F}}^c(0)$ given by

$$\alpha(k,n)(x) = \widetilde{\mathscr{F}}^{su}(F^k(x) + n) \cap \widetilde{\mathscr{F}}^c(0), \quad \forall (k,n) \in \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d, x \in \widetilde{\mathscr{F}}^c(0).$$

Lemma 3.7 This is a group action by the solvable group Γ.

Main idea. If both $E^{ss} \oplus E^u$ and $E^s \oplus E^{uu}$ are not integrable, then we can find a free subgroup by a pingpong argument, which contradicts Γ is solvable.

Lemma 3.8

If $\alpha(0,n)(\widetilde{\mathscr{F}}^{ws}(0)) \subset \widetilde{\mathscr{F}}^{ws}(\alpha(0,n)0)$ for all $n \in \mathbb{Z}^d$, then both $h(\mathscr{F}^{ss}) = \mathscr{L}^{ss}$ and $h(\mathscr{F}^{uu}) = \mathscr{L}^{uu}$ are linear. The same holds if $\alpha(0,n)(\widetilde{\mathscr{F}}^{wu}(0)) \subset \widetilde{\mathscr{F}}^{wu}(\alpha(0,n)0)$ for all $n \in \mathbb{Z}^d$.

Proof. Note that $\bigcup_{n\in\mathbb{Z}^d} \widetilde{\mathscr{F}}^{ws}(n)$ is dense in $\mathbb{R}^d$ and hence $E^{ss} \oplus E^{ws} \oplus E^{uu}$ jointly integrates to $\mathscr{F}^{su} \oplus \mathscr{F}^{ws}$. Then we deduce the linearity. □

Proof of Theorem 3.4. Assume for a contradiction that there exists $n_1, n_2 \in \mathbb{Z}^d$ such that

- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(\alpha(0,n_1)(0))$;
- $\alpha(0,n_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(\alpha(0,n_1)(0))$.

Lemma 3.9

There exists $m_1, m_2 \in \mathbb{Z}^d$ such that

- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,m_1)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Lemma 3.10

For l large enough, $n = A^l m_1 - A^{-l} m_2 \in \mathbb{Z}^d$ satisfies

- $\alpha(0,n)(\widetilde{\mathscr{F}}^{ws}(0))$ is transverse to $\widetilde{\mathscr{F}}^{ws}(0)$;
- $\alpha(0,n)(\widetilde{\mathscr{F}}^{wu}(0))$ is transverse to $\widetilde{\mathscr{F}}^{wu}(0)$.

Now we consider $F : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0)$ and

$$G : \alpha(0,n) \circ \alpha(1,0) \circ \alpha(0,-n) : \widetilde{\mathscr{F}}(0) \to \widetilde{\mathscr{F}}(0).$$

Then F is saddle-like dynamics at $\widetilde{\mathscr{F}}^{ws}(0) \cup \widetilde{\mathscr{F}}^{ws}(0)$ near 0. The map G is also saddle-like near $\alpha(0,n)0$. By a pingpong-argument, we can show that $\left\{F^k, G^k\right\}$ generates a free group for a sufficiently large k. This contradicts that Γ is solvable. □

§4 Anosov Maps (Apr 28)

Cone-field. Let f be an Anosov diffeomorphism with splitting $TM = E^s \oplus E^u$. Then there are cone-fields C^s, C^u containing E^s, E^u such that

$$Df(\overline{C^u(x)}) \subset C^u(fx), \quad Df^{-1}(\overline{C^s(x)}) \subset C^s(f^{-1}x).$$

Then $E^s(x)$ is determined by $\mathrm{Orb}^+(x)$ as

$$E^s(x) = \bigcap_{n \geqslant 0} Df^{-n}(C^s(f^n x)),$$

and $E^u(x)$ is determined by $\mathrm{Orb}^-(x)$ as

$$E^u(x) = \bigcap_{n \geqslant 0} Df^{n}(C^u(f^{-n} x)).$$

Theorem 4.1 (Anosov, 1967)

The Arnold's cat map $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ is structurally stable. That is, for every $f : \mathbb{T}^2 \to \mathbb{T}^2$ C^1-close to A, there exists a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ close to $\mathrm{id}_{\mathbb{T}^2}$ such that $h \circ f = A \circ h$.

Remark 4.2 Every Anosov diffeomorphism is structurally stable.

Remark 4.3 If $f : \mathbb{T}^2 \to \mathbb{T}^2$ is continuous with $f_* = A$, then there exists a surjective $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h \circ f = A \circ h$.

By a cone-argument, we can show that a small perturbation of an Anosov diffeomorphism is also Anosov. In general, we have Franks-Manning's global classification of Anosov diffeomorphisms.

Theorem 4.4 (Franks-Manning)

Every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ conjugates to $f_* : H_1(d, \mathbb{Z}) \to H_1(d, \mathbb{Z})$.

Anosov maps.

Definition 4.5. A local diffeomorphsim $f : M \to M$ is **Anosov**, if there exists a continuous, Df invariant subbundle $E^s \subset TM$ such that

- $\|Df(v^s)\| < 1$ for every $v^s \in E^s$ with $\|v^s\| = 1$;
- Df induces an expanding map $\widetilde{Df} : TM/E^s \to TM/E^s$, that is

$$\|\widetilde{Df}(\tilde{v}^u)\| > 1, \quad \forall \tilde{v}^u \in TM/E^s, \|\tilde{v}^u\| = 1.$$

In this lecture, the Anosov map always refers to the non-invertible Anosov map.

Remark 4.6 Since $\mathrm{Orb}^-(x)$ is not unique, there may be no $E^u(x)$.

Theorem 4.7 (Mañe-Pugh, 1974)
$f : M \to M$ is an Anosov map iff $\widetilde{f} : \widetilde{M} \to \widetilde{M}$ is an Anosov diffeomorphism.

Definition 4.8 (Przytycki, 1976)**.** A local diffeomorphsim $f : M \to M$ is an **Anosov map**, if in the orbit space

$$\tilde{x} = (x_i)_{i\in\mathbb{Z}} \in M_f := \{(x_i) : f(x_i) = x_{i+1}, \ \ \forall x \in \mathbb{Z}\},$$

there exists a splitting

$$T_{x_i}M = E^s(x_i) \oplus E^u(x_i), \quad \forall i \in \mathbb{Z}$$

which is Df-invariant

$$D_{x_i}f(E^s(x_i)) = E^s(x_{i+1}), \quad D_{x_i}f(E^u(x_i)) = E^u(x_{i+1}), \quad \forall i \in \mathbb{Z},$$

and for every $v^{s/u} \in E^{s/u}(x_i)$ with $\|v^{s/u}\| = 1$:

$$\|D_{x_i}f(v^s)\| < 1, \quad \|D_{x_i}f(v^u)\| > 1.$$

Example 4.9
For every $n \geqslant 3$, the map

$$A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

is an Anosov map.

Remark 4.10 Every Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^d$ has a hyperbolic linearization $f_* \in M(\mathbb{Z}, d)$.

Unlike the Anosov diffeomorphisms, the Anosov map is not structurally stable.

Theorem 4.11 (Mañe-Pugh, 1974; Przytycki, 1976)
Let $A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$ with $n \geqslant 3$. Then A is **NOT** structurally stable. That is, for every $\varepsilon > 0$, there exists an Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ with $d_{C^\infty}(f, A) < \varepsilon$ such that there is no $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to $\mathrm{id}_{\mathbb{T}^2}$ with $h \circ f = A \circ h$.

Remark 4.12 Every non-invertible Anosov map is not structurally stable unless it is expanding.

Proof. Take $p \neq 0$ such that $A(p) = 0$. Let U, U' be disjoint neighborhoods of 0 and p. Let (x_i) be an A-orbit satisfying

$$x_0 = p, \quad x_i = 0, \forall i > 0, \quad \text{and} \quad x_i \notin U', \forall i < 0.$$

Take a C^∞ ε-perturbation of f on U': push p along the stable leaf.

Then there exists an f-orbit $\{ y_i \}$ satisfying

$$y_0 = p, \quad \text{and} \quad y_i = x_i, \forall i < 0.$$

Then $y_i \in \mathscr{F}^s_\varepsilon(0)$ for every $i > 0$, where $\mathscr{F}^s$ is the stable leaf of A. Then the A-orbit x_i shadows the f-orbit y_i and hence the conjugacy $h(y_i) = 0$. But there is no homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that $h(y_i) = 0$ for every $i > 0$. ☐

Theorem 4.13 (Przytycki, 1976)

An Anosov map $f : M \to M$ is structurally in the orbit space (M_f, σ_f), where $\sigma_f : (x_i) \mapsto (x_{i+1})$. That is, for every $g : M \to M$ C^1-close to f, there exists a homeomorphism $\overline{h} : M_g \to M_f$ such that $\overline{h} \circ \sigma_g = \sigma_f \circ \overline{h}$.

Question 4.14.

- Assume that $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an Anosov map with $f_* = A = \begin{bmatrix} n & 1 \\ 1 & 1 \end{bmatrix}, n \geqslant 3$. When f topologically conjugate to A?
- Assume that $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ are Anosov maps with $f_* = g_*$. When f topologically conjugates to g?

Example 4.15 (Przytycki, 1976)

Let

$$A = \begin{bmatrix} n & 1 & 0 \\ 1 & n & 0 \\ 0 & 0 & n \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3, \quad n \geqslant 2$$

be a **special Anosov map** (E^u does not depend on the choice of the inverse orbit). When n is big enough, for every $x \in \mathbb{T}^3$, there exists an f C^1-close to A such that

$$\left\{ D\pi(E^u(x_0)) : \tilde{x} = (x_i) \in M_f \text{ with } x_0 = x \right\} \subset \mathscr{G}^2(T_x\mathbb{T}^3)$$

contains a curve in the Grassmannian $\mathscr{G}^2(T_x\mathbb{T}^3)$.

Theorem 4.16 (Micena-Tahzibi, 2016)

Let $f : M \to M$ be a transitive Anosov map, then

- either f has an integrable E^u (f is special),
- or there exists a residue set $\mathscr{R} \subset M$ such that x has infinitely many unstable directions for every $x \in \mathscr{R}$.

Main theorems.

Theorem 4.17 (An-Gan-Gu-Shi, arXiv: 2205.13144)
Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a C^{1+}-Anosov map, then the following are equivalent:
- f topologically conjugate to $f_* = A$;
- f is spectral rigid in stable bundle:

$$\lambda^s(p, f) \equiv \log \|A|_{L^s}\|, \quad \forall p \in \mathrm{Per}(f).$$

Remark 4.18 The same holds if $f : \mathbb{T}^d \to \mathbb{T}^d$ is an irreducible Anosov map with $\dim E^s = 1$.

Theorem 4.19 (An-Gan-Gu-Shi, arXiv: 2205.13144)
Let $A \in M(d, \mathbb{Z})$ be Anosov, irreducible and $|\det(A)| > 1$. If A has real simple spectrum in the stable bundle:

$$T\mathbb{T}^d = L_1^s \oplus L_2^s \oplus \cdots \oplus L_k^s \oplus L^u, \quad \dim L_i^s = 1,$$

then for every f C^1-close to A, the following are equivalent:
- f topologically conjugates to A,
- f is spectral rigidity in stable bundle, i.e. f admits dominated splitting

$$T\mathbb{T}^d = E_1^s \oplus E_2^s \oplus \cdots \oplus E_k^s \oplus E^u$$

  and

$$\lambda(E_i^s, f) \equiv \log \|A|_{L_i^s}\|, \quad \forall i = 1, \cdots, k.$$

Main philosophy. For every $y, z \in \mathbb{T}^d$, they are in the same "strongest stable manifold" if

$$f^n(y) = f^n(z), \quad \text{for some } n > 0.$$

Then f topologically conjugates to $A \iff E^u$ does not depend on $\mathrm{Orb}^-(x)$. Hence we have $E^u(x) = E^u(y)$ if $f^n(y) = f^n(z)$. This is equivalent to E^u is "jointly integrable" with

$$\mathcal{F}^{ss}(x) := \{ z : f^n(x) = f^n(z), \text{ for some } n > 0 \}.$$

This leads to a spectral rigidity in E^s, which is the weak stable direction in this view.

Topological classification.

Theorem 4.20 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be homotopic C^{1+}-Anosov maps, then the following are equivalent:
- f topologically conjugates to g;
- for every $p \in \mathrm{Per}(f)$ and corresponding $p' \in \mathrm{Per}(g)$,

$$\lambda^s(p, f) \equiv \lambda^s(p', g).$$

Remark 4.21 Since there is no a priori conjugacy, we should explain the meaning of "corresponding point". This can be given by a (stable) leaf conjugacy, which is defined a priori. Note that each periodic stable leaf admits a unique periodic point since f is uniformly contracting on the stable leaf. The corresponding point can be defined in this way.

Corollary 4.22 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. Then h is C^r-smooth along the stable foliation.

Theorem 4.23 (Gu-Shi, arXiv: 2212.11457)
Let $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ be C^r Anosov maps ($r > 1$) topologically conjugated via h. If

$$\mathrm{Jac}(f^{\pi(p)}(p)) = \mathrm{Jac}(g^{\pi(p)}(h(p))), \quad \forall p \in \mathrm{Per}(f),$$

then h is C^{r_*}-smooth. Here $r_* = \begin{cases} r - 1 + \mathrm{Lip}, & r \in \mathbb{N} \\ r, & r \notin \mathbb{N} \end{cases}$.