Fundamentals of Artificial Intelligence Programme (2024/25 Q1)

Multiobjective optimization for decision support

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In this assignment, we use two python libraries, one for exploratory modeling analysis The EMA workbench and one for multiobjective optimization Project Platypus, you can install both with pip as follows:

```
!pip install ema_workbench platypus-opt
```

```
Requirement already satisfied: ema_workbench in /usr/local/lib/python3.10/dist-packages (2.5.2)
Requirement already satisfied: platypus-opt in /usr/local/lib/python3.10/dist-packages (1.3.1)
Requirement already satisfied: numpy in /usr/local/lib/python3.10/dist-packages (from ema_workbench) (1.26.4)
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Requirement already satisfied: pyparsing>=2.3.1 in /usr/local/lib/python3.10/dist-packages (from matplotlib->ema_workbe
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Requirement already satisfied: threadpoolctl>=2.0.0 in /usr/local/lib/python3.10/dist-packages (from scikit-learn->ema_
Requirement already satisfied: patsy>=0.5.6 in /usr/local/lib/python3.10/dist-packages (from statsmodels->ema_workbench
Requirement already satisfied: six in /usr/local/lib/python3.10/dist-packages (from patsy>=0.5.6->statsmodels->ema_work
Requirement already satisfied: dill>=0.3.8 in /usr/local/lib/python3.10/dist-packages (from multiprocess->salib>=1.4.6-
```

The Lake model

The goal of this assignment is to demonstrate the use of multiobjective evolutionary optimization, to learn how to visualize and interpret the optimization results, and to use performance metrics to assess the results obtained via multiobjective evolutionary optimization. We will use the lake problem as a test case, this is a classic problem initially developed by Carpenter et al. (1999) where the population of a city has to decide the amount of annual pollution it will release into a lake. In this exercise, we will use the adapted version in Quinn et al. 2017 were the problem is defined as a state-based control problem (where actions are a function of the state of the system). In this case, the 'action' is the Phosphorous (P) emissions which are optimized to balance the economic benefits and the quality of the lake. Since this is a multi-objective problem, we need a flexible function to map the states to actions, so we use radial basis functions to parameterize the emission control policies. In fact, the MOEA will search for the optimal radii, centers and weights that yield good performance for the objectives of the lake model described below. See the paper for more details about the problem formulation.

The model is defined by the following equation:

$$X_{(t+1)} = X_t + a_t + \frac{(X_t^q)}{(1+X_t^q)} - bX_t + \epsilon_t$$

where X_t is the pollution at time t, a_t is the rate of anthropogenic pollution at time t, b is the lake's natural removal rate, q is the lake's natural recycling rate, ϵ_t is the rate of natural pollution at time t. The rate of anthropogenic pollution a_t is the decision variable where $a_t \in [0, 0.1]$.

There are four outcomes of interest. The first is the average concentration of phosphor in the lake.

$$f_{phosphorus} = \frac{1}{|T|} \sum_{t \in T} X_t$$

where $|T|$ is the cardinality of the set of points in time. The second objective is the economic benefit derived from polluting the lake defined as the discounted benefit of pollution minus the costs of having a polluted lake.

$$f_{economic} = \sum_{t \in T} \alpha a_t \delta^t$$

where α is the utility derived from polluting and δ is the discount rate. By default, α is 0.04. The third objective is related to the year over year change in the anthropogenic pollution rate.

$$f_{inertia} = \frac{1}{|T|-1} \sum_{t=1}^{|T|} I(|a_t - a_{t-1}| > \tau)$$

where I is an indicator function that is 0 if the statement is false, and 1 if the statement is true, τ is the threshold that is deemed undesirable, and is for illustrative purposes et to 0.2. Effectively, f_{inertia} is the fraction of years where the absolute value of the change in anthropogenic pollution is larger then τ. The fourth objective is the fraction of years where the pollution in the lake is below the critical threshold.

$$f_{reliability} = \frac{1}{|T|} \sum_{t \in T} I(X_t < X_{crit})$$

where I is an indicator function that is 0 if the statement is false, and 1 if the statement is true, X_{crit} is the critical threshold of pollution and is a function of both b and q.

The lake problem is characterized by both stochastic uncertainty and deep uncertainty. The stochastic uncertainty arises from the natural inflow. To reduce this stochastic uncertainty, multiple replications are performed and the average over the replication is taken. Deep uncertainty is presented by uncertainty about the mean μ and standard deviation $sigma$ of the lognormal distribution characterizing the natural inflow, the natural removal rate of the lake β, the natural recycling rate of the lake q, and the discount rate δ. The table below specifies the ranges for the deeply uncertain factors, as well as their best estimate or default values.

Lake model implementation in python

```python
import math
import numpy as np
from scipy.optimize import brentq


def get_antropogenic_release(xt, c1, c2, r1, r2, w1):
    '''

    Parameters
    ----------
    xt : float
         polution in lake at time t
    c1 : float
         center rbf 1
    c2 : float
         center rbf 2
    r1 : float
         ratius rbf 1
    r2 : float
         ratius rbf 2
    w1 : float
         weight of rbf 1

    Returns
    -------
    float

    note:: w2 = 1 - w1

    '''

    rule = w1 * (abs(xt - c1) / r1) ** 3 + (1 - w1) * (abs(xt - c2) / r2) ** 3
    at1 = max(rule, 0.01)
    at = min(at1, 0.1)

    return at


# def lake_model(b=0.42, q=2.0, mean=0.02,
#                              stdev=0.001, delta=0.98, alpha=0.4,
#                              nsamples=100, myears=100, c1=0.25,
#                              c2=0.25, r1=0.5, r2=0.5,
#                              w1=0.5, seed=123):

# adapted from the following request
def lake_model(b=0.42, q=2.0, mean=0.02,
                             stdev=0.001, delta=0.98, alpha=0.41,
                             nsamples=150, myears=100,
                             c1=0.25, c2=0.25, r1=0.5, r2=0.5,
                             w1=0.5, seed=123):
    '''runs the lake model for nsamples stochastic realisation using
    specified random seed.

    Parameters
    ----------
    b : float
            decay rate for P in lake (0.42 = irreversible)
    q : float
            recycling exponent
    mean : float
                    mean of natural inflows
    stdev : float
                    standard deviation of natural inflows
    delta : float
                    future utility discount rate
    alpha : float
                    utility from pollution
    nsamples : int, optional
    myears : int, optional
    c1 : float
    c2 : float
    r1 : float
    r2 : float
    w1 : float
    seed : int, optional
                  seed for the random number generator

    Returns
    -------
    tuple

    '''
    np.random.seed(seed)
    Pcrit = brentq(lambda x: x ** q / (1 + x ** q) - b * x, 0.01, 1.5)

    X = np.zeros((myears,))
    average_daily_P = np.zeros((myears,))
    reliability = 0.0
    inertia = 0
    utility = 0

    for _ in range(nsamples):
        X[0] = 0.0
        decision = 0.1

        decisions = np.zeros(myears, )
        decisions[0] = decision

        natural_inflows = np.random.lognormal(
            math.log(mean ** 2 / math.sqrt(stdev ** 2 + mean ** 2)),
            math.sqrt(math.log(1.0 + stdev ** 2 / mean ** 2)),
            size=myears)

        for t in range(1, myears):
            # here we use the decision rule
            decision = max(w1 * (abs(X[t - 1] - c1) / r1) ** 3 + (1 - w1) * (abs(X[t - 1]
            decision = min(decision, 0.1)
            decisions[t] = decision

            X[t] = (1 - b) * X[t - 1] + X[t - 1] ** q / (1 + X[t - 1] ** q) + decision
            average_daily_P[t] += X[t] / nsamples
            if isinstance(average_daily_P[t], (bool, np.bool_)):
                average_daily_P[t] = float(average_daily_P[t])
            # print(f"average_daily_P[{t}]: {average_daily_P[t]}, type: {type(average_daily_P[t])}")


        reliability += np.sum(X < Pcrit) / (nsamples * myears)
        inertia += np.sum(np.absolute(np.diff(decisions)
                                  < 0.02)) / (nsamples * myears)
        utility += np.sum(alpha * decisions * np.power(delta,
                                                             np.arange(my
    max_P = np.max(average_daily_P)
    # if isinstance(max_P, (bool, np.bool_)):
    #     max_P = 0.0  #

    # print(f"max_P: {max_P}, type: {type(max_P)}")


    return max_P, utility, inertia, reliability
```

1. Connecting the lake model with the EMA workbench.

Given the Python implementation of the lake problem above, adapt the code and connect it to the EMA workbench using the following lever ranges and uncertainty ranges:

Levers	Range	Default value
$r1$	0.0 – 2.0	0.5
$r2$	0.0 – 2.0	0.5.
$c1$	-2 – 2	0.25
$c2$	-2 – 2	0.25
$w1$	0.0-1.0	0.5

Uncertainties	Range	Default value
μ	0.01 – 0.05	0.02
σ	0.001 – 0.005	0.0017
b	0.1 – 0.45	0.42
q	2 – 4.5	2
δ	0.93 – 0.99	0.98

You can follow this tutorialfor guidance.

The outcomes in the EMA workbench refers to the objectives of the problem, in this case we have four. 1) maximum Phosphorous (to be minimized) 2) utility (to be maximized) 3) intertia (to be maximized), and reliability (to be maximized). Use an alpha value of 0.41, with number of samples= 150, and number of years = 100.

```
#  target:
#  Maximum  phosphorus  concentration:  should  be  minimized,  because  higher  phosphorus  concentration  means  deteri
#  Economic  benefits:  should  be  maximized  to  achieve  long-term  economic  benefits  through  pollution  control  s
#  Inertia:  should  be  maximized  to  maintain  consistency  and  stability  of  pollution  policies  between  years.
#  Reliability:  should  be  maximized  to  ensure  that  lake  pollution  is  below  the  critical  value  in  more  yea


from  ema_workbench  import  (Model,  RealParameter,  ScalarOutcome,  MultiprocessingEvaluator)

lakemodel  =  Model('lakemodel',  function=lake_model)

#  Levers
lakemodel.levers  =  [RealParameter('r1',  0.0,  2.0),
                              RealParameter('r2',  0.0,  2.0),
                              RealParameter('c1',  -2.0,  2.0),
                              RealParameter('c2',  -2.0,  2.0),
                              RealParameter('w1',  0.0,  1.0)]

#  Uncertainties
lakemodel.uncertainties  =  [RealParameter('mean',  0.01,  0.05),
                                      RealParameter('stdev',  0.001,  0.005),
                                      RealParameter('b',  0.1,  0.45),
                                      RealParameter('q',  2,  4.5),
                                      RealParameter('delta',  0.93,  0.99)]

#  Outcomes
lakemodel.outcomes  =  [ScalarOutcome('max_P',  kind=ScalarOutcome.MINIMIZE),
                                ScalarOutcome('utility',  kind=ScalarOutcome.MAXIMIZE),
                                ScalarOutcome('inertia',  kind=ScalarOutcome.MAXIMIZE),
                                ScalarOutcome('reliability',  kind=ScalarOutcome.MAXIMIZE)]
```

```
开始借助 AI 编写或生成代码。
```

2. How would you introduce a constrain within the optimization to reflect a desired performance threshold for a given objective?

You don't actually have to perform this step, simply specify how you would go about establishing a constraint in the optimization formulation, in such a way that it only finds solutions with maximum pollution (max phosphorous) of 0.85.

```
#  add  a  constraint  to  the  optimization  problem
from  ema_workbench  import  Constraint
constraints  =  [Constraint("max  pollution",  outcome_names="max_P",  function=lambda  x:  max(0.85,  x))]

```

3. Run the optimization and track the performance metrics

Tip: the EMA Workbench uses [Platypus](#) to run the optimization via the evaluator class, you can also collect metrics during runtime specifying the convergence option. Below is a sample snippet on how to run the optimization and collect performance metrics during runtime.

```
from  ema_workbench  import  MultiprocessingEvaluator,  ema_logging
from  ema_workbench.em_framework.optimization  import  (HyperVolume,
                                                                                          EpsilonProgress


convergence_metrics  =  [HyperVolume(minimum=[0,0,0,0],  maximum=[1,1.01,1.01,1.01]),
                                    EpsilonProgress()]
```

```


ema_logging.log_to_stderr(ema_logging.INFO)


with  MultiprocessingEvaluator(lakemodel)  as  evaluator:
        results,  convergence  =  evaluator.optimize(
                nfe=10000,
                searchover='levers',
                epsilons=[0.1,  0.1,  0.01,  0.1],
                convergence=convergence_metrics,
                constraints=constraints)


```

```
[MainProcess/INFO] pool started with 2 workers

   0%|                                                | 0/10000 [00:00<?, ?it/s]

   1%|▌                                       | 100/10000 [00:09<15:17, 10.79it/s]

   2%|▊                                       | 200/10000 [00:17<14:17, 11.43it/s]

   3%|█▏                                      | 300/10000 [00:25<13:33, 11.93it/s]

   4%|█▌                                      | 400/10000 [00:34<13:53, 11.52it/s]

   5%|██                                      | 500/10000 [00:43<13:48, 11.46it/s]

   6%|██▍                                     | 600/10000 [00:51<13:25, 11.66it/s]

   7%|██▊                                     | 700/10000 [01:00<13:32, 11.44it/s]

   8%|███▏                                    | 800/10000 [01:10<13:50, 11.07it/s]

   9%|███▋                                    | 900/10000 [01:17<12:41, 11.95it/s]

   9%|███▋                                    | 900/10000 [01:28<12:41, 11.95it/s]

  10%|███▉                                   | 1000/10000 [01:28<13:56, 10.76it/s]

  11%|████▎                                  | 1100/10000 [01:36<13:04, 11.35it/s]

  12%|████▋                                  | 1200/10000 [01:45<12:49, 11.44it/s]

  13%|█████                                  | 1300/10000 [01:52<12:16, 11.82it/s]

  14%|█████▍                                 | 1400/10000 [02:01<11:57, 11.99it/s]

  15%|█████▊                                 | 1500/10000 [02:09<11:50, 11.96it/s]

  16%|██████▏                                | 1600/10000 [02:16<11:14, 12.45it/s]

  17%|██████▋                                | 1700/10000 [02:26<11:46, 11.75it/s]

  18%|███████                                | 1800/10000 [02:33<10:56, 12.49it/s]

  19%|███████▍                               | 1900/10000 [02:42<11:15, 11.99it/s]

  20%|███████▊                               | 2000/10000 [02:49<10:31, 12.67it/s]

  21%|████████▏                              | 2100/10000 [02:58<10:51, 12.13it/s]

  22%|████████▌                              | 2200/10000 [03:04<10:04, 12.91it/s]

  23%|████████▉                              | 2300/10000 [03:14<10:35, 12.12it/s]

  24%|█████████▎                             | 2400/10000 [03:22<10:37, 11.93it/s]

  25%|█████████▊                             | 2500/10000 [03:31<10:23, 12.03it/s]
```

双击（或按回车键）即可修改

4. Selecting the objectives.

The outputs from the optimization runs will contain the decision variables (i.e. the parameters of the radial basis functions) and objectives combined. Create a data structure that only contains the objective values without the decision variables.

Tip: each row in the output matrix represents a different solution with it's obective values, the first columns are the decision variables and the last columns are the objective values.

```
results
```

	r1	r2	c1	c2	w1	max_P	utility	inertia	reliability
0	0.475550	0.976891	0.206675	0.183765	0.942480	0.097738	0.267078	0.990000	1.000000
1	0.085196	1.238544	0.240845	-0.261639	0.773910	0.522222	0.729131	0.987867	0.866733
2	0.085196	1.499870	0.240801	-0.405092	0.825415	0.838874	0.910375	0.985933	0.725267
3	0.085196	0.804180	0.242007	-0.012470	0.847954	0.481739	0.707849	0.743333	0.883867
4	0.086045	1.238544	0.240845	-0.261639	0.773910	0.356860	0.637278	0.989067	0.939400
5	0.456780	1.499182	0.327283	-0.298367	0.944601	0.194646	0.534778	0.990000	1.000000
6	0.085196	0.709425	0.240825	-0.062171	0.820034	0.659329	0.807812	0.986800	0.805200

```
convergence
```

	hypervolume	epsilon_progress	nfe
0	0.0	0	0
1	0.0	5	100
2	0.0	6	1100
3	0.0	6	2100
4	0.0	6	3100
5	0.0	6	4100
6	0.0	6	5100
7	0.0	10	6100
8	0.0	16	7100
9	0.0	24	8100
10	0.0	29	9100
11	0.0	31	10021

```
import pandas as pd

objective_columns = ['max_P', 'utility', 'inertia', 'reliability']


objectives_df = results[objective_columns]


```

```
print(objectives_df)

```

```
      max_P   utility   inertia  reliability
0  0.097738  0.267078  0.990000     1.000000
1  0.522222  0.729131  0.987867     0.866733
2  0.838874  0.910375  0.985933     0.725267
3  0.481739  0.707849  0.743333     0.883867
4  0.356860  0.637278  0.989067     0.939400
5  0.194646  0.534778  0.990000     1.000000
6  0.659329  0.807812  0.986800     0.805200
```

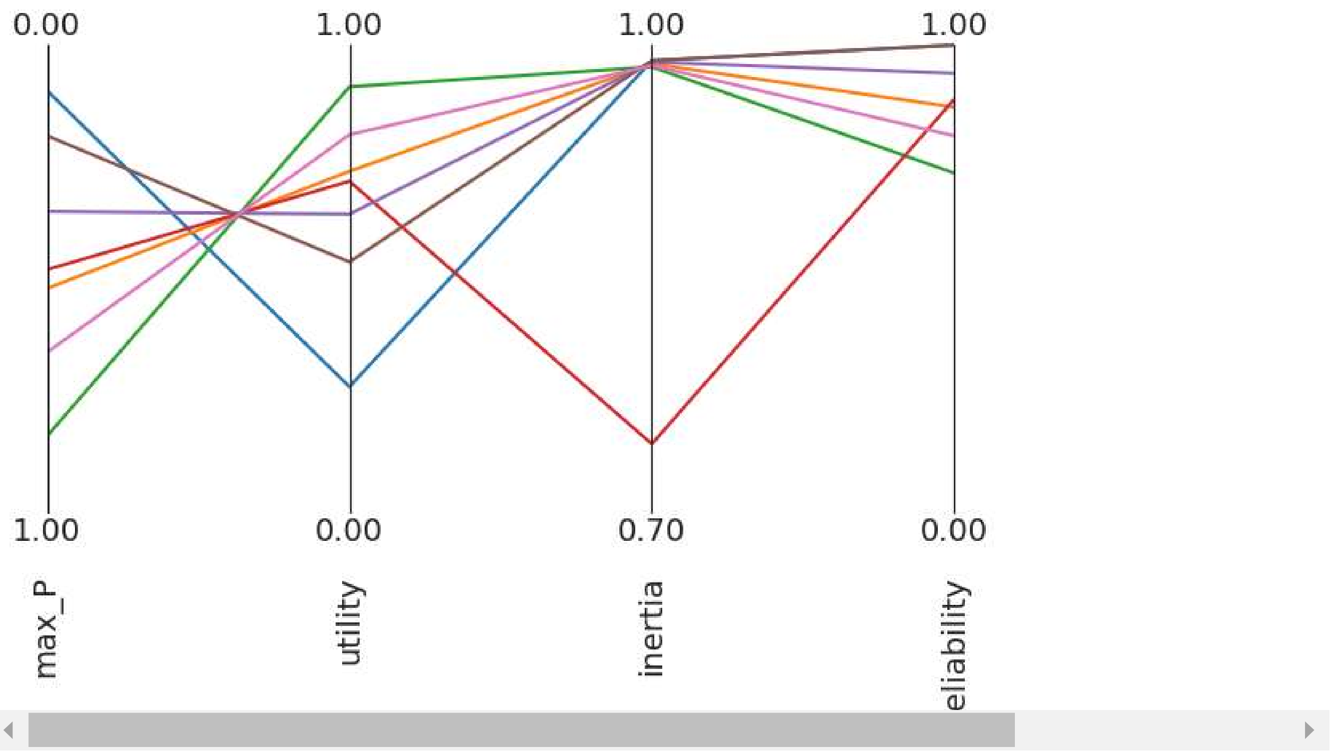
```
objectives_df.to_csv('objectives_only.csv', index=False)
results.to_csv('results.csv', index=False)
convergence.to_csv('convergence.csv', index=False)
```

5. Visualizing the results.

Present visually the results of the Pareto optimal solutions (in the objective space), feel free to be creative! Provide a brief discussion the results, are there any tradeoffs observed?

Tip: If you need inspiration check out the EMA workbench parcoords.

```
from ema_workbench.analysis import parcoords
import matplotlib.pyplot as plt

data = objectives_df


limits = parcoords.get_limits(data)
limits.loc[0, ['max_P', 'utility', 'reliability']] = 0
limits.loc[0, ['inertia']] = 0.7
limits.loc[1, ['max_P', 'utility', 'inertia', 'reliability']] = 1


paraxes = parcoords.ParallelAxes(limits)
paraxes.plot(data)
paraxes.invert_axis('max_P')


plt.show()
```

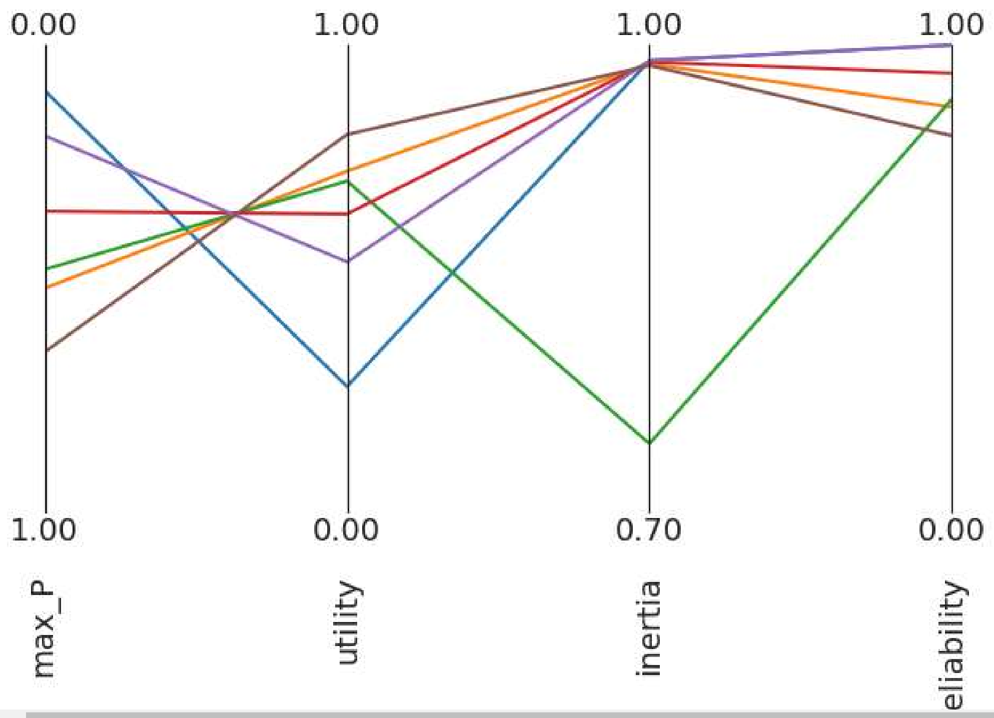


Yes, the parallel coordinates plot reveals several trade-offs among the objectives in the multi-objective optimization problem. Specifically, it shows that reducing maximum phosphorus levels (Max_P) often compromises utility and reliability. Additionally, there's an observable trade-off between inertia and reliability, suggesting that efforts to maintain the status quo (inertia) may adversely affect the reliability of the system.

6. Establishing a performance threshold.

Show visually only the solutions from the Pareto set that yield a reliability above 80%, and briefly discuss the results.

```
import plotly.express as px


filtered_data = objectives_df[objectives_df['reliability'] > 0.8]


limits = parcoords.get_limits(data)
limits.loc[0, ['max_P', 'utility', 'reliability']] = 0
limits.loc[0, ['inertia']] = 0.7
limits.loc[1, ['max_P', 'utility', 'inertia', 'reliability']] = 1


paraxes = parcoords.ParallelAxes(limits)
paraxes.plot(filtered_data)
paraxes.invert_axis('max_P')


plt.show()

```



The plot illustrates the balance between Max Phosphorus, Utility, Inertia, and Reliability across different solutions. A trade-off exists between Max Phosphorus and Reliability, where lower phosphorus doesn't always ensure higher reliability. Similarly, higher utility often corresponds with lower inertia, meaning more flexible, dynamic solutions tend to perform better.

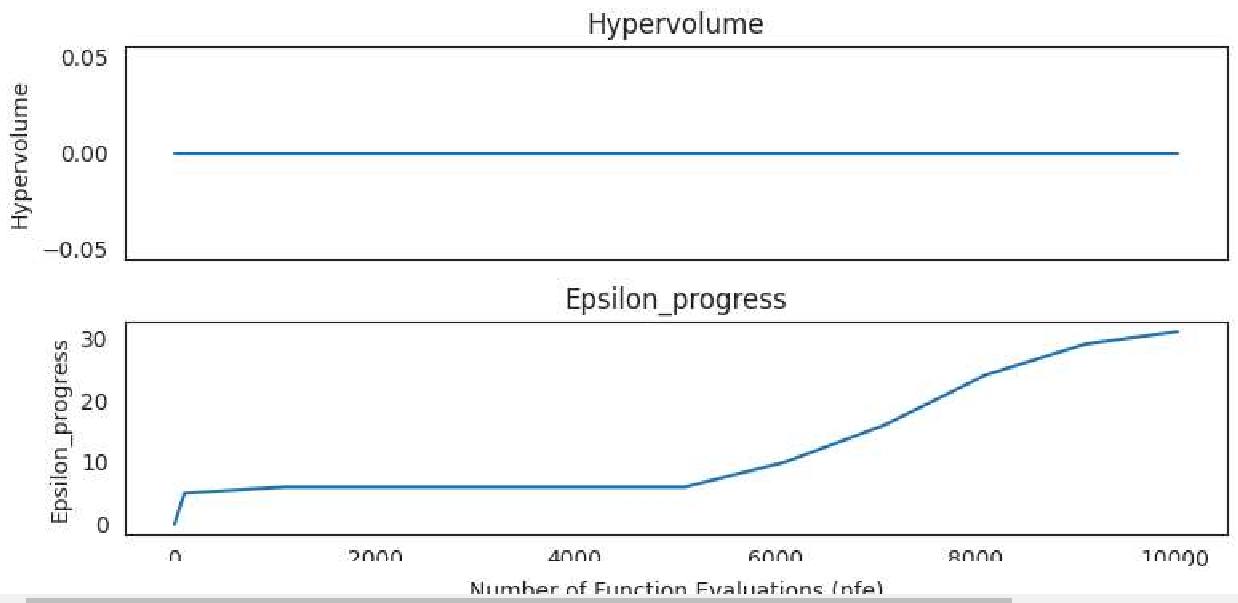
Solutions that achieve high reliability vary in their performance regarding phosphorus and utility, showing multiple paths to reliability. A cluster of solutions also balances high utility and reliability, suggesting an optimal compromise. This visualization helps decision-makers weigh trade-offs based on whether environmental impact or operational performance is prioritized.

7. Performance metrics

Show in a dataframe the results from the metrics (convergence) collected during the optimization. Plot the metrics (e.g hypervolume or epsilon progress) as a function of the number of function evaluations (nfe). Provide a brief discussion of the results.

```
import matplotlib.pyplot as plt
import seaborn as sns

# Set the style
sns.set_style("white")

# List of possible metrics we want to check if they exist in the DataFrame
possible_metrics = ['hypervolume', 'epsilon_progress', 'generational_distance', 'epsilon_indicator', 'inverted_gd',

# Filter out the metrics that exist in the DataFrame
available_metrics = [metric for metric in possible_metrics if metric in convergence.columns]

# Determine the number of plots based on available metrics
num_plots = len(available_metrics)

# Create subplots dynamically based on the number of available metrics
fig, axes = plt.subplots(nrows=num_plots, figsize=(8, 2 * num_plots), sharex=True)

# If only one metric available, wrap the axes object in a list
```

```
if  num_plots  ==  1:
        axes  =  [axes]

#  Plotting  each  available  metric
for  ax,  metric  in  zip(axes,  available_metrics):
        ax.plot(convergence['nfe'],  convergence[metric],  label=metric)
        ax.set_title(metric.capitalize())
        ax.set_ylabel(metric.capitalize())

#  Set  common  x-axis  label
axes[-1].set_xlabel('Number  of  Function  Evaluations  (nfe)')

#  Adjust  layout
plt.tight_layout()

#  Show  the  plot
plt.show()

```

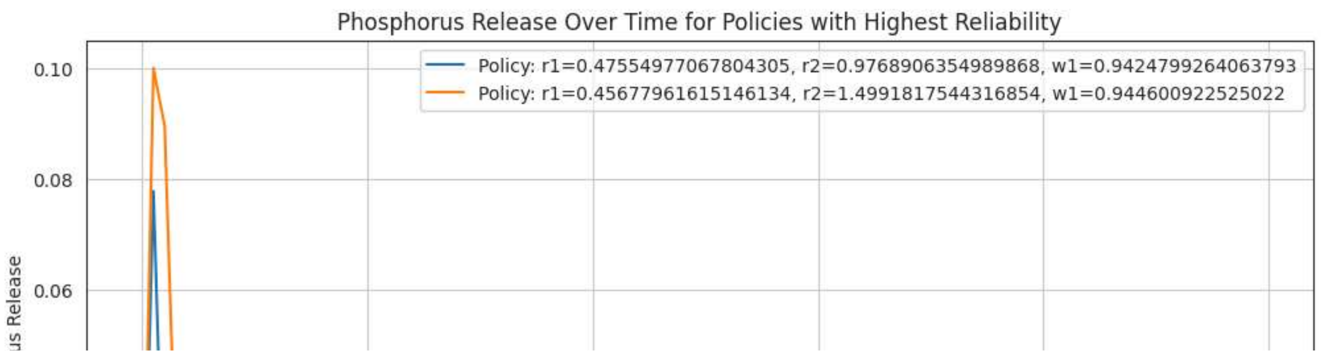


Extra credit: Visualize the phosphorous release from the policy with the highest reliability. If you plot phosphorous release as a function of time, what do you observe?

```
def  lake_model(b=0.42,  q=2.0,  mean=0.02,  stdev=0.001,  delta=0.98,  alpha=0.41,
                              nsamples=150,  myears=100,  c1=0.25,  c2=0.25,  r1=0.5,  r2=0.5,  w1=0.5,  seed=123):
        """
        Runs  the  lake  model  simulation  for  a  given  set  of  parameters  and  policies.

        Parameters:
        -  b,  q,  mean,  stdev,  delta,  alpha  :  float,  environmental  and  economic  parameters.
        -  nsamples  :  int,  number  of  samples  for  stochastic  realization.
        -  myears  :  int,  number  of  years  to  simulate.
        -  c1,  c2,  r1,  r2,  w1  :  float,  parameters  for  the  decision-making  policy.
        -  seed  :  int,  random  seed  for  reproducibility.

        Returns:
        -  tuple  (float,  float,  float,  float,  np.array),  containing  max  P,  utility,  inertia,  reliability,
           and  phosphorous  release  over  time  for  visualization.
        """
        np.random.seed(seed)
        Pcrit  =  brentq(lambda  x:  x  **  q  /  (1  +  x  **  q)  -  b  *  x,  0.01,  1.5)

        X  =  np.zeros((myears,))   #  Pollution  levels
        phosphorous_release_over_time  =  np.zeros((myears,))   #  Store  phosphorous  release  values  for  each  year
        reliability  =  0.0
```

```
    inertia = 0
    utility = 0

    for _ in range(nsamples):
        X[0] = 0.0  # Start with no pollution
        decision = 0.1  # Initial decision

        decisions = np.zeros(myears,)
        decisions[0] = decision
        natural_inflows = np.random.lognormal(math.log(mean ** 2 / math.sqrt(stdev ** 2 + mean ** 2))
                                                                math.sqrt(math.log(1.0 + st
                                                                size=myears)

        for t in range(1, myears):
            decision = get_antropogenic_release(X[t - 1], c1, c2, r1, r2, w1)
            decisions[t] = decision
            X[t] = (1 - b) * X[t - 1] + X[t - 1] ** q / (1 + X[t - 1] ** q) + decisic
            phosphorous_release_over_time[t] += decision / nsamples  # Accumulate average release

        reliability += np.sum(X < Pcrit) / (nsamples * myears)
        inertia += np.sum(np.absolute(np.diff(decisions) < 0.02)) / (nsamples * myears)
        utility += np.sum(alpha * decisions * np.power(delta, np.arange(myears))) / nsamples

    max_P = np.max(X)

    return max_P, utility, inertia, reliability, phosphorous_release_over_time

```

```
# Identify the policy with the highest reliability, and it's not only one
max_reliability = results['reliability'].max()
highest_reliability_policies = results[results['reliability'] == max_reliability]

plt.figure(figsize=(12, 6))

for _, policy in highest_reliability_policies.iterrows():
    max_P, utility, inertia, reliability, phosphorous_release = lake_model(
        c1=policy['c1'],
        c2=policy['c2'],
        r1=policy['r1'],
        r2=policy['r2'],
        w1=policy['w1']
    )

    plt.plot(np.arange(100), phosphorous_release, label=f'Policy: r1={policy["r1"]}, r2={policy["r2"]}, w1={poli

plt.xlabel('Year')
plt.ylabel('Phosphorus Release')
plt.title('Phosphorus Release Over Time for Policies with Highest Reliability')
plt.legend()
plt.grid(True)
plt.show()
```


Phosphorus Release Over Time for Policies with Highest Reliability
Policy: r1=0.47554977067804305, r2=0.9768906354989868, w1=0.9424799264063793
Policy: r1=0.45677961615146134, r2=1.4991817544316854, w1=0.944600922525022
0.10
0.08
0.06
us Release