mot1442 handout 1 - The barest basics of deductive logic

(0) An argument is:
a set of ***premises*** plus a ***conclusion*** which is *claimed* to follow from these premises.

A (logically or deductively) ***valid*** argument is:
an argument where the conclusion indeed follows from the premises, meaning that the truth of the premises guarantees the truth of the conclusion or, equivalently, meaning that *if* the premises are true, then (you can be sure that) the conclusion is true.

A (logically or deductively) ***sound*** argument is:
a *valid* argument in which the *premises* are indeed *true*, guaranteeing the truth of the conclusion.

Notation:
$P_1, ..., P_n \models C$ means: C logically follows from $P_1, ..., P_n$, or: concluding C from $P_1, ..., P_n$ is valid.

NB: An argument is valid on the basis of its *logical form* alone What the premises actually say or what sentences actually serve as premises is not important, as long as, always when these sentences are true sentences, the conclusion is also true. This can only come about if the premises and the conclusion each have a certain *structure* and if these structures are connected.

(1) The basics of propositional logic

$\neg p$	**not** *p*
$p \vee q$	*p* **or** *q* (inclusive, not exclusive)
$p \wedge q$	*p* **and** *q*
$p \to q$	**if** *p* **then** *q*
$p \leftrightarrow q$	*p* **if and only if** *q*

The letters *p*, *q*, etc. stand for statements or sentences or, as they are called in logic, *propositions*. They are either true or false. The symbols $\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$ are the so-called *logical connectives*. They correspond to the conjunctions (‘and’, ‘or’, ‘but’, ‘if’, ‘unless’, ‘nor’, etc.) that connect sentences in our ordinary language.
In ordinary language some conjunctions are ambiguous (esp. ‘or’, ‘if’, ‘unless’). In logic, however, the connectives are defined exactly, with elimination of all ambiguity. They are defined by way of the following ‘truth-table’ (1 stands for ‘true’ and 0 stands for ‘false’):

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Note that this definition closely follows the everyday meaning of the conjunctions. If someone says ‘*p* is the case AND *q* is the case’, and it turns out that *q* is not the case, then what this person says cannot be true, even if *p* is the case. This is independent of the precise sentences that might take the place of *p* and *q* in ordinary talk. On the other hand, if someone says ‘*p* is the case OR *q* is the case’ and it turns out that *q* is not the case, then we consider what this persons says to be true as long as *p* is still the case. We consider what he/she says to be false only when NEITHER *p* NOR *q* is the case.

For the case of $p \rightarrow q$ it needs perhaps a little thought to see that the definition in the table matches our use of 'if... then...'. Think of it in this way: The *only* occasion where you have reason to say that someone who claims 'if p the q' is saying something false is when it turns out that p is the case but q is not the case.
Note that a formula like $p \wedge q$ is itself again a proposition. It can, therefore, take the place of, for example, r in the formula $r \rightarrow s$. A proposition that cannot be further analyzed into smaller ones linked by one or more logical connectives, e.g., p, is called an *atomic proposition*.
Sometimes Greek letters φ, ψ, ... are used to emphasize that a proposition could be non-atomic.

* **Equivalence**
  Two propositions are equivalent if the former can be derived from the latter and the latter can be derived from the former. We write $p \models q$ to indicate that q can be derived from p, or that q is a logical consequence of p, or that reasoning from p as a premise to q as a conclusion is valid. As a consequence, the equivalence of p and q can be written as $p \dashv\models q$, which merely abbreviates 'both $p \models q$ and $p \dashv q$'. If two propositions are equivalent, then if one is true, the other is true, and if one is false, the other is false.

* **Tautology** (also called: logical truth)
  A tautology is a proposition that is always true, independently of which atomic propositions are true or false. A simple atomic proposition p can therefore never be a tautology, since an atomic proposition is supposed to report a simple fact about the world, which may either be the case (in which case p is true) or not be the case (in which case p is false).
  Tautologies must therefore have a particular form in terms of atomic propositions. Examples are (note the use of parentheses to resolve ambiguity in a logical formula):

  $p \vee \neg p$
  $((p \rightarrow q) \wedge p) \rightarrow q$
  $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
  $(p \rightarrow q) \vee (q \rightarrow p)$

  You can prove that these expressions are tautologies by the truth-table method explained on the next page, by showing that, for all possible combinations of truth values of p and q, the whole expression always has the value 'true'.

  In deductive logic, all tautologies are of course equivalent. 'The' tautology is indicated by the symbol $\top$.
  A way to express that, for instance, $p \vee \neg p$ is a tautology is to write $\models p \vee \neg p$. This makes sense because the truth of $p \vee \neg p$ is guaranteed by any premise, even by no premise at all.

  Note the following: $\varphi \models \psi$ if and only if $\models \varphi \rightarrow \psi$, where φ and ψ can be any proposition.
  This means that if a conclusion C logically follows from premises P_1 to P_n, then the single proposition $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C$ is a tautology, i.e. true no matter what.

* **Contradiction** (more accurately: logical contradiction)
  A contradiction is a proposition that is always false, independently of which atomic propositions are true or false. Just as in the case of the tautology, a simple atomic proposition p can never be a contradiction; a contradiction must have a structure in atomic propositions.
  Examples are:

  $p \wedge \neg p$
  $(p \rightarrow q) \wedge p \wedge \neg q$

Proofs that these are contradictions can be obtained in exactly the same way as is done for tautologies.

Again, all contradictions are equivalent. The symbol for 'the' contradiction is $\perp$.

* **Useful relations**

(Note again the use of parentheses to resolve ambiguity. Without parentheses, $\neg$ only applies to the proposition immediately following it.)

$$\begin{aligned}
\neg\neg p &\;\Dashv\vDash\; p \\
p\rightarrow q &\;\Dashv\vDash\; \neg p \vee q \\
p\rightarrow q &\;\Dashv\vDash\; \neg q \rightarrow \neg p \\
p\leftrightarrow q &\;\Dashv\vDash\; (p\rightarrow q)\wedge(q\rightarrow p) \\
\neg(p\vee q) &\;\Dashv\vDash\; \neg p \wedge \neg q \\
\neg(p\wedge q) &\;\Dashv\vDash\; \neg p \vee \neg q
\end{aligned}$$

* How do you check that these relations are correct? By 'calculating', in the form of a truth-table, the truth-values of the left and the right term for all truth-values of the atomic propositions, using the definitions of the connectives, and checking that they are identical. This is done in the following table for the second and fifth relations.

p	q	$p\rightarrow q$	$\neg p$	$\neg p\vee q$	$p\vee q$	$\neg(p\vee q)$	$\neg p$	$\neg q$	$\neg p\wedge\neg q$
1	1	1	0	1	1	0	0	0	0
1	0	0	0	0	1	0	0	1	0
0	1	1	1	1	1	0	1	0	0
0	0	1	1	1	0	1	1	1	1

* How do you establish whether a particular argument is logically valid? Take for example: 'If cigarettes become more expensive because of raised taxes, less cigarettes will be sold. If less cigarettes will be sold, we must raise their price in order to stabilize our returns. Taxes on cigarettes have just been raised. Therefore, we must raise their price.'
Step 1: Start by bringing out the structure of the argument. Replace all atomic sentences by proposition letters and all conjunctions by logical connectives. It is important to bring out all logical structure present in the premises. If you replace each premise by a single proposition letter, you cannot see how the premises are related to the conclusion and consequently you cannot see what work they do for the truth of the conclusion. It is only on the basis of a specific relation that the conclusion bears to the premises that an argument is valid.
Step 2: The structure of the argument is seen to be: $p\rightarrow q$, $q\rightarrow r$, $p \mid r$. Write down a truth-table, containing all premises and the conclusion, for all possible combinations of truth values for the atomic propositions occurring in the premises and conclusion. In this case we have three atomic propositions and we get a truth-table with $2^3=8$ rows. In the table below, the green columns represent the three premises and the orange column represents the conclusion. Each row corresponds with a possible distribution of 'truth' and 'false' over the three atomic propositions. Only the rows where all premises receive the value 'true' need to be considered to see whether the argument is valid. (In our example this comes down to a single row.). In *all* cases where *all* premises are true, the conclusion should also receive the value 'true'. If that is so, the argument is valid. The table shows our example to be a valid argument.

p	q	r	$p\rightarrow q$	$q\rightarrow r$
1	1	1	1	1
1	1	0	1	0
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	1	0
0	0	1	1	1
0	0	0	1	1

If it is not so, the argument is invalid. A combination of truth-values for the atomic propositions for which the premises are all true but the conclusion false is called a *counterexample* to the argument.
The table below shows that the following argument is logically invalid: 'Computers are intelligent if and only if they can reason. If computers can reason they can apply logical rules. Computers can apply logical rules. So, computers are intelligent.' The structure of this argument is: $p\leftrightarrow q$, $q\rightarrow r$, $r \mid p$.

p	q	r	$p\leftrightarrow q$	$q\rightarrow r$	
1	1	1	1	1	
1	1	0	1	0	
1	0	1	0	1	
1	0	0	0	1	
0	1	1	0	1	
0	1	0	0	0	
0	0	1	1	1	counterexample! True premises, false conclusion.
0	0	0	1	1	

* Two basic **valid** arguments:

$$\begin{array}{l} p\rightarrow q \\ p \\ \hline q \end{array}$$

called **MODUS PONENS**

$$\begin{array}{l} p\rightarrow q \\ \neg q \\ \hline \neg p \end{array}$$

called **MODUS TOLLENS**

* Two basic **invalid** arguments:

$$\begin{array}{l} p\rightarrow q \\ q \\ \hline p \end{array}$$

called CONFIRMATION OF THE CONSEQUENT

$$\begin{array}{l} p\rightarrow q \\ \neg p \\ \hline \neg q \end{array}$$

called NEGATION OF THE ANTECEDENT

From the two valid arguments it can be seen that $p\rightarrow q$ expresses:
(1) that p is a **sufficient condition** for q : Given $p\rightarrow q$, the truth of p guarantees the truth of q
(2) that q is a **necessary condition** for p : Given $p\rightarrow q$, the falsity of q blocks the truth of p

(2) Some very brief remarks on predicate logic

In predicate logic propositions are given an internal structure, if only of a very limited sort: propositions are understood as stating that particular entities have particular properties.

* In order to make this possible, three new language elements are introduced:

Small-case letters x, y,..., standing for object variables, i.e. as yet unspecified objects;
Small-case letters a, b, ..., referring to *concrete* objects; I also use e_1, e_2, ... for this;
Capital letters A, B, ..., called ***predicates***, representing the *properties* of objects.
The symbols $\exists$ and $\forall$, called ***quantifiers***, which stand for ‘some...’ and ‘every...’, resp.

Thus Ba means: ‘object a has property B’; it could stand for, for instance, ‘Object a is red’.
Bx is not a complete formula and so has no meaning. ‘Some (unspecified) object has property B’ is $\exists x\, \mathrm{B}x$. Variables must always be ‘governed’ by quantifiers: one quantifier per variable.

* Any formula of the form Ax or Ba is treated as a proposition and can be combined with the logical connectives in the same way as ‘ordinary’ propositions like p and q can. Further, the quantifiers can be combined with such predicate propositions, again giving propositions.

$\forall x\,(\mathrm{A}x \wedge \mathrm{B}x)$ means ‘All objects (from the domain) have both property A and property B’.
$\forall y\,(\mathrm{R}y \rightarrow \mathrm{S}y)$ means ‘All objects (from the domain) that have property R also have property S’.
$\exists z\,(\mathrm{P}z \wedge \neg \mathrm{Q}z)$ means ‘At least one object (in the domain) has property P but lacks property Q’.

* The domain of objects is sometimes understood to be restricted to objects having particular properties. It is good practice, however, to make every relevant property explicit through a corresponding predicate: starting with $\forall y\,(\mathrm{R}y \rightarrow ...)$ you say ‘all R-objects are/have ...’.

* The quantifiers are not independent: $\forall x\,(\varphi)$ is equivalent to $\neg\exists x\,(\neg\varphi)$, and $\exists x\,(\varphi)$ to $\neg\forall x\,(\neg\varphi)$.

* The above framework can be extended by introducing not just predicates that denote the properties of single objects (*monadic* predicates), but also *dyadic*, *triadic*, etc., predicates that denote the relations between objects. The order in which object letters are attached to such *dyadic*, *triadic*, etc., predicates is all-important! Relations have a *direction*.
Think of ‘Object a is more expensive than object b’ (which is something totally different from ‘Object b is more expensive than object a’) or ‘An object is to the left of another object’.

Ae_1e_2 then means ‘Object e_1 stands in relation A to object e_2’.
$\exists y\,(\mathrm{P}ya)$ means ‘There is at least one object that stands in relation P to object a’.
$\forall x \exists y\,(y{\neq}x \wedge \mathrm{R}y \wedge \mathrm{S}xy)$ means ‘For each object there is at least one other object such that this second object has property R and such that the first object stands in relation S to the second object’. Note that = is considered a special cases, not an ordinary dyadic predicate; $a{\neq}b$ is an abbreviation of $\neg a{=}b$. (By the way, why is it not necessary to write this as $\neg(a{=}b)$?)

* As already stated, formulas like Ax and Ba and formulas beginning with $\forall$ or $\exists$ are themselves propositions, and the whole apparatus of propositional logic applies to them. So we can say something like $\models \forall y\,(\mathrm{R}y \rightarrow \mathrm{S}y) \leftrightarrow \neg\exists \mathrm{y}\,(\mathrm{R}y \wedge \neg \mathrm{S}y)$. (Can you see that this is a true statement?)
From the point of view of propositional logic, this statement is of the form $\models \varphi \leftrightarrow \psi$.

* Note, finally, that $\forall x\,(\mathrm{R}x \rightarrow \mathrm{S}x)$ is the customary way to represent scientific laws, in this case ‘All R-things are S-things’. Given the rules of propositional logic, there are two ways in which this expression can be true: either because all R-things in the universe are indeed S-things, or because the universe does not contain any R-things at all. So given the absence of unicorns from the universe, the statement ‘All unicorns are white’ is true, which is perhaps a bit counter-intuitive. But it is simply a consequence of the definition of $\rightarrow$ and this definition is the best we can do for ‘if... then...’. Any other choice would create far greater problems.