Analysis daily problems

Maxwell Gong

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Contents

Day 1

Q1 (Sum of Series)

If $\sum a_n$ and $\sum b_n$ are both convergent, conclude about $\sum(a_n + b_n)$

Proof : Because $\sum a_n$, $\sum b_n$ convergent, suppose $\sum a_n \to A$, $\sum b_n \to B$. By definition of convergence, we have $\forall \epsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$, $\forall n \geq N_1$, we have

$$\left| \sum_{k=1}^{n} a_k - A \right| < \epsilon$$

And this implies that

$$A - \epsilon < \sum_{k=1}^{n} a_k < A + \epsilon$$

Similarly, we have $\forall n \geq N_2$, we have

$$B - \epsilon < \sum_{k=1}^{n} b_k < B + \epsilon$$

Now I claim that $\sum(a_k + b_k) \to A + B$. We can take $N = max\{N_1, N_2\}$, hence $\forall n \geq N$,

$$A + B - 2\epsilon < \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k < A + B + 2\epsilon$$

Hence we can say that $\sum(a_k + b_k) \to A + B$

Extended question: What about $\sum a_k b_k$?

One can show that this can be either convergent or divergent.

Case 1, it is divergent:

Let $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = a_n$. By alternating series test, one can show that $\sum a_k b_k = \sum \frac{1}{n}$, which means that even if $\sum a_k$ and $\sum b_k$ are convergent, $\sum a_k b_k$ is divergent.

The divergent case is very obvious so we do not leave a construction here. But we notice that if $\{a_n\}$ and $\{b_n\}$ are all positive sequences, then $\sum a_k b_k$ must be convergent.

Q2 (Sum of neighbour terms)

If $\sum(a_n + a_{n+1})$ is convergent. Will $\sum a_n$ be convergent ?

Answer: No, we can actually give counterexample for this. Just let $a_n = (-1)^n$, then we have $a_n + a_{n+1} = 0$. So $\sum(a_n + a_{n+1})$ is convergent. However, if $\{a_n\}$ is positive sequence, then we are able to deduce that $\sum a_n$ must be convergent since

$$\sum a_n < \sum(a_n + a_{n+1}) \leq A$$

where A denotes the limit of $\sum(a_n + a_{n+1})$. As $\sum a_n$ is bounded, it is convegent.

Q3 (Exchange odd and even terms)

Suppose $\sum a_n$ is convergent to S. If for all n, we exchange the position of a_{2n-1} and a_{2n}, prove that the new series $\sum a'_n$ is convergent, and find its sum.

Proof: Define $s_n = \sum_{k=1}^{n} a'_k$. We will next consider $\{s_{2n}\}$ and $\{s_{2n+1}\}$. Since $\sum_{n=1}^{\infty} a_n$ is convergent, so we have two things. One thing is that $\sum_{n=1}^{\infty} a_n = A$ and $\lim_{n\to\infty} a_n = 0$

So $\forall \epsilon > 0$, $\exists N_1 \in \mathbb{N}$, $\forall n \geq N_1$,

$$\left|\sum_{k=1}^{n} a_k - A\right| < \epsilon$$

Take $N > \frac{N_1}{2}$. So we have $\forall n \geq N$, $2n \geq N_1$, hence

$$|s_{2n} - A| = \left|\sum_{k=1}^{2n} a_k - A\right| < \epsilon$$

Now we have $s_{2n} \to A$ as $n \to \infty$.

Since $\lim_{n\to\infty} a_n = 0$, $\exists N_2$, $\forall n \geq N_2$, $|a_n| < \epsilon$. So let's take $N = max\{N_2, \frac{N1}{2}\}$, $\forall n \geq N$, we have $2n+1 \geq N_1$ and $2n \geq N_2$. So

$$|s_{2n+1} - A| = \left|\sum_{k=1}^{2n} a_k + a_{2n+2} - A\right| < \left|\sum_{k=1}^{2n} a_k - A\right| + |a_{2n+2}| < \epsilon + \epsilon = 2\epsilon$$

Q4 (Can $a_n \neq 0$?) Suppose we have positive sequence $\{a_n\}$. $\sum a_n$ is convergent. Suppose $s_n = \sum_{k=1}^{n} a_k$, $s_n \to S$ as $n \to \infty$. Define $R_n = S - s_n$.

I leave this question as practice when I review analysis.

Prove that if $a_n \leq a_n R_n$, then $\sum a_n$ is a sum of finite terms.

Kummer test for convergence

Q5 (Kummer Test)

Prove that series of positive terms $\sum_{n=1}^{\infty} a_n$ convergent iff there exists positive sequence $\{b_n\}$ and positive number $\delta > 0$, for big enough n, we have

$$b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \geq \delta > 0$$

Proof:

($\Longrightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Then define $b_n = \frac{R_n}{a_n}$, so we have $b_n > 0$ since $a_n > 0$ and $R_n > 0$ as defined before. So,

$$b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} = \frac{R_n}{a_{n+1}} - \frac{R_{n+1}}{a_{n+1}} = 1 > 0$$

Notice that this is for all n.

($\Longleftarrow$) Consider we have

$$b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \geq \delta > 0$$

for $n \geq N$, we can actually discard the previous terms in $\sum a_n$. So wlog, we assume this inequality holds for all n. Since a_{n+1} is positive, so we multiply it both sides to obtain

$$a_n b_n - a_{n+1} b_{n+1} \geq \delta a_{n+1} > 0$$

Then we sum both sides, we obtain the inequality such that

$$\sum_{n=1}^{N} \delta a_{n+1} \leq \sum_{n=1}^{N} (a_n b_n - a_{n+1} b_{n+1}) = a_1 b_1 - a_{N+1} b_{N+1} < a_1 b_1$$

hence we can deduce that $\sum a_n$ is bounded above. So $\sum a_n$ is convergent. Done!

Day 2

Kummer test for divergence

Q1 (Kummer test for divergence)

$\sum_{n=1}^{\infty} a_n$ is divergent iff there exists divergent postive terms series $\sum_{n=1}^{\infty} \frac{1}{b_n}$, such that for big enough n, we have

$$b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \leq 0$$

We will come back to this proof later after we prove Sapagof test for convergence.

Q2 ($\sum a_n$ and $\sum \frac{1}{a_n}$)

Prove first that if $\sum a_n$ is convergent, then $\sum \frac{1}{a_n}$ is divergent, given that $a_n \neq 0$. Notice that this is true for all series, we are not only restrict in positive terms series. Can you show that even if $\sum \frac{1}{a_n}$ is divergent, but $\sum a_n$ might not be convergent.

Proof: Suppose $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Now let's take $\epsilon = 1$, by definition of the limit, $\exists N \in \mathbb{N}$, $\forall n \geq N$, we have $|a_n| < 1$. This implies that $\left|\frac{1}{a_n}\right| \geq 1$. So $\lim_{n\to\infty} \frac{1}{a_n} \neq 0$. As a result, it must be divergent.

Consider $a_n = 1$. Both $\sum \frac{1}{a_n}$ and $\sum a_n$ are divergent.

Sapagof test

Q3 (Sapagof test)

Suppose there is a positive decreasing sequence $\{a_n\}$, then $\lim\limits_{n\to\infty} a_n = 0$ iff postive terms series $\sum_{n=1}^{\infty}(1 - \frac{a_{n+1}}{a_n})$

is divergent.

Proof:

($\Longrightarrow$) Can we prove that $1-\frac{a_{n+1}}{a_n}$ does not tend to 0 as $n\to\infty$? Obviously we can't do this. Since we can simply by picking an counterexample to tell ourselves that there might be exceptional cases. For example, we can let $a_n=\frac{1}{n}$.

However, comparing to this test, proving whether a sequence is a Cauchy sequence is the stronges way we can use to prove whether a series is convergent.

Now, let

$$S_n=\sum_{k=1}^{n}(1-\frac{a_{k+1}}{a_k})$$

we can see that

$$S_{n+p}-S_n=\sum_{k=n+1}^{n+p}(1-\frac{a_{k+1}}{a_k})>\frac{a_{n+1}-a_{n+p}}{a_{n+1}}=1-\frac{a_{n+p}}{a_{n+1}}$$

Notice that we can take p to be large enough and since $a_n\to 0$, so we can make $\frac{a_{n+p}}{a_{n+1}}$ to be small enough. So we conclude that it is not a Cauchy sequence, so it does not convergent.

($\Longleftarrow$) We prove this side by contrapositive.

Suppose $\lim_{n\to\infty}a_n=a\neq 0$, then

$$\sum(1-\frac{a_{n+1}}{a_n})\leq\frac{\sum(a_k-a_{k+1})}{a}=\frac{a_1-a}{a}$$

Hence we can conclude that $\sum(1-\frac{a_{n+1}}{a_n})$ is convergent.

Q4 (Other forms of Sapagof test)

Prove first that if $\{a_n\}$ is positive monotone increasing sequence, then a_n is convergent iff $\sum(1-\frac{a_n}{a_{n+1}})$ is convergent.

Also show that if $S_n=\sum_{k=1}^{n}a_k$ for a positive terms series $\sum a_n$. Then $\sum a_n$ is convergent iff $\sum\frac{a_n}{S_n}$ is convergent.

Proof:

($\Longrightarrow$)Consider if $\{a_n\}$ is positive monotone increasing sequence. We have

$$1-\frac{a_n}{a_{n+1}}=\frac{a_{n+1}-a_n}{a_{n+1}}>0$$

And one can use similar method to show that this series is bounded. This is because

$$\sum(1-\frac{a_n}{a_{n+1}}) < \frac{\sum(a_{n+1}-a_n)}{a_1} = \frac{a-a_1}{a_1}$$

Hence this series is bounded above, so it is convergent as it is a positive terms series.

($\Longleftarrow$)Again we use contrapositive to prove this. Suppose $\{a_n\}$ is divergent, i.e $a_n \to \infty$ as $n \to \infty$. Then I will prove that $\sum_{k=1}^{n}(1-\frac{a_k}{a_{k+1}})$ is not Cauchy sequence.

Just consider

$$\sum_{k=n}^{n+p}\frac{a_{k+1}-a_k}{a_{k+1}} > \frac{a_{n+p+1}-a_n}{a_{n+p+1}} = 1-\frac{a_n}{a_{n+p+1}}$$

So that we can take p to large enough, we can always find such a p for $\sum_{k=n}^{n+p}\frac{a_{k+1}-a_k}{a_{k+1}} > \frac{1}{2}$.

The proof for **next one** just need us to notice that

$$\sum\frac{a_n}{S_n} = \sum\frac{S_n-S_{n-1}}{S_n} = \sum 1-\frac{S_{n-1}}{S_n}$$

where S_n is a monotone increasing positive sequence, it convergent iff $\sum(1-\frac{S_n}{S_{n+1}})$ is convergent by Sapagof test. So we proved such statement.

I will go back for Kummer test now.

Proof:

($\Longrightarrow$)If $\sum a_n$ is divergent, then just let $b_n = \frac{S_n}{a_n}$. Hence we have

$$\frac{S_n-S_{n+1}}{a_{n+1}} = \frac{-a_{n+1}}{a_{n+1}} = -1 < 0$$

Hence we have this inequality for all n, this side is done.

($\Longleftarrow$)I will now prove that $\sum a_n$ is divergent. Wlog, let's assume the inequailty above holds for all n. So we have

$$\frac{a_n}{a_{n+1}} \leq \frac{b_{n+1}}{b_n}$$

It is equivalent to

$$a_{n+1} \geq \frac{a_n b_n}{b_{n+1}} \geq \frac{b_n}{b_{n+1}}\cdot\frac{b_{n-1}}{b_n}\cdot a_{n-1}$$

Carrying on this inequailty, we have $a_{n+1} \geq \frac{b_1 \cdot a_1}{b_{n+1}}$ by induction. Hence use comparison test, we ca show that $\sum a_n$ is divergent.

Q5 (One Putnam question)

Show first that if $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$, then $\lim_{n\to\infty} \frac{x_n}{n} = 0$. Then given the same condition, show that

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0$$

(Just use this question as a review for sequence.)

Day 3

Q1 (The limit of a recursion sequence)

Suppose sequence $\{a_n\}$ satisfy $0 < a_1 < 1$ and $a_{n+1} = a_n(1 - a_n)$. Prove

$$\lim_{n\to\infty} n \cdot a_n = 1$$

Proof: Notice that

$$a_{n+1} - a_n = a_n - a_n^2 - a_n = -a_n^2 < 0$$

this means that the sequence is decreasing. Also, the sequence is bounded below by 0, which we can check by induction. Then we can assume $\lim_{n\to\infty} a_n = a$. Take limit both sides on the recursion formula, one can verify that

$$a = a - a^2$$

We solve the equation, so we get that $\lim_{n\to\infty} a_n = 0$.

Now consider $n \cdot a_n$, notice that $n \cdot a_n = \frac{n}{\frac{1}{a_n}}$. Additionally we can check the denominator is increasing and tends to infinity. Now we can apply Stolz in this limit, we get

$$\lim_{n\to\infty} n \cdot a_n = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n}} = \lim_{n\to\infty} \frac{n-(n-1)}{\frac{1}{a_n} - \frac{1}{a_{n-1}}} = \lim_{n\to\infty} \frac{a_n a_{n-1}}{a_{n-1} - a_n} = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$$

To compute the limit $\frac{a_n}{a_{n-1}}$, we notice that $\frac{a_{n+1}}{a_n} = 1 - a_n$. Take limit both sides, we can show that $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = 1$. We use the fact that $\lim_{n\to\infty} a_n = 0$ here.

Q2 (Divergent series's convergent Arithmetic mean)

Suppose $\{a_{2k-1}\}$ is convergent to a, $\{a_{2k}\}$ is convergent to b. Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a+b}{2}$$

Proof: Define $S_n = \frac{a_1+a_2+\cdots+a_n}{n}$. By considering $\{S_{2n}\}$ and $\{S_{2n+1}\}$, one can get remarkable result. I'll first prove that $S_{2n} \to \frac{a+b}{2}$ as $n \to \infty$. Notice that

$$S_{2n} = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = \frac{1}{2} \cdot \frac{a_1 + a_3 + \cdots + a_{2n-1}}{n} + \frac{1}{2} \cdot \frac{a_2 + a_4 + \cdots + a_{2n}}{n}$$

By Stolz,

$$\lim_{n\to\infty}\frac{a_1+a_3+\cdots+a_{2n-1}}{n}=\lim_{n\to\infty}\frac{a_{2n+1}}{1}=b$$

Similarly, we can prove that $\lim_{n\to\infty}\frac{a_2+a_4+\cdots+a_{2n}}{n}=a$. Hence we have $S_{2n}\to\frac{a+b}{2}$ as $n\to\infty$.

Next, I'll prove that S_{2n+1} also has a limit of $\frac{a+b}{2}$. Notice

$$S_{2n+1}=\frac{a_1+a_2+\cdots+a_{2n+1}}{2n+1}+\frac{a_2+a_4+\cdots+a_{2n}}{2n+1}$$

Using Stolz again, one can show that $S_{2n+1}\to\frac{a+b}{2}$. Done!

Inequailty about e

Q3 (An inequality about $n!$)

Prove that

$$\left(\frac{n+1}{e}\right)^n<n!<e\cdot\left(\frac{n+1}{e}\right)^{n+1}$$

Use this to deduce the limit of

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$$

Proof: If $n=1$, inequailty denotes that $2<e<4$, which is obviously true. By mathematical induction, we have

$$\left(\frac{n}{e}\right)^{n-1}<(n-1)!<e\cdot\left(\frac{n}{e}\right)^n$$

One can show that

$$n!=n\cdot(n-1)!>n\cdot\left(\frac{n}{e}\right)^{n-1}=\frac{n^n}{e^n}\cdot e>\left(\frac{n+1}{e}\right)^n$$

In the final inequailty, we use the fact that $e>\left(1+\frac{1}{n}\right)^n$. We are done on the left inequality, then we give insight into the right inequailty. Notice that

$$n!=n\cdot(n-1)!<n\cdot e\cdot\left(\frac{n}{e}\right)^n=\frac{n^{n+1}}{e^{n+1}}\cdot e^2<\frac{n^{n+1}}{e^{n+1}}\cdot\left(1+\frac{1}{n}\right)^{n+1}<e\cdot\left(\frac{n+1}{e}\right)^{n+1}$$

where we use the inequality $e < \left(1 + \frac{1}{n}\right)^{n+1}$.

We then use this fact to find an upper and lower bound of $\frac{n}{\sqrt[n]{n!}}$. Which is

$$\frac{e \cdot n}{(n+1) \cdot (n+1)^{\frac{1}{n}}} < \frac{n}{\sqrt[n]{n!}} < \frac{e \cdot n}{n+1}$$

Using sandwich's theorem, we can deduce that

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

Q4 If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove that

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = a \cdot b$$

I will leave this as an exercise for me to review since it includes very important method dealing with this kind of questions.

Q5 (Equivalent definition for limit of function)

Prove the following two definition are equivalent.

I $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$, $\forall x \in \mathbb{R}$, $0 < |x - x_0| < \delta \implies |f(x) - y| < \epsilon$

II $\forall J \subset \mathbb{R}$ containing y, $\exists I \subset \mathbb{R}$ containing x_0, such that $f\left(I \setminus \{x_0\}\right) \subset J$, where J and I are open intervals.

Proof:

(I $\implies$ II) Since J containing y is an open interval. Assume it is (a, b), take $\epsilon = min\{y - a, b - y\}$. By definition of I, we can always find $\delta > 0$, so that consider an open interval $(x_0 - \delta, x_0 + \delta)$. By I, any x in this interval except x_0 has the property that $f(x) \in (y - \epsilon, y + \epsilon) \subset J$. Done!

(II $\implies$ I) This one is quite obvious to see.

Day 4

Rabbe test

Q1 (Kummer test and Rabbe test)

Rabbe test for a positive terms series $\sum a_n$ is a stronger version of normal ratio test, which is stated as following:

$$\lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) = r$$

Then if $r > 1$, the series is convergent. If $r < 1$, the series is divergent.

Prove Rabbe test by using Kummer test.

Proof: let $b_n = n$ in the Kummer test. Then one can show that $\sum a_n$ is convergent if

$$n \cdot \frac{a_n}{a_{n+1}} - n > 1$$

for large enough n. So we can say that if $\lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) > 1$, then there must exist one N, such that $\forall n \geq N$, the inequality in Kummer test is satisfied. Hence we proved that $\sum a_n$ is convergent. Similarly, we can use the Kummer test for divegence to prove the other side for divergence of $\sum a_n$. Since $\sum \frac{1}{n}$ is divergent, so if we have $r < 1$, then we are able to deduce that for large enough n,

$$n \cdot \frac{a_n}{a_{n+1}} - (n+1) < 0$$

which satisfies Kummer test for divergence, hence $\sum a_n$ is divergent.

Bertrand test

Q2 (Kummer test and Bertrand test)

Bertrand test states that for a positive terms series $\sum a_n$, let

$$\lim_{n\to\infty} \ln n \cdot \left[n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right] = r$$

If $r > 1$, then the series is convergent. If $r < 1$, then the series is divergent.

(I will leave space here and this is a very good practice for analysis skills)

Q3 (Some applications of Sapagof test)

State first that what is Sapagof test. Prove using Sapagof test that whether the limit of the following sequence

$$\left\{ \frac{(2n)!}{4^n (n!)^2} \right\}$$

is convergent to 0.

Q4 (Proof from first principle of divergence)

Prove first that $\{\sin n\}$ is divergent by using prove by contradiction. Deduce that $\{\tan n\}$ is divergent.

Proof: Suppose $\{\sin n\}$ is convergent and suppose it's limit is a. By considering

$$\lim_{n\to\infty} \sin(n+1) - \sin(n-1) = \lim_{n\to\infty} 2\cos n \cdot \sin 1$$

Hence we can deduce that $\cos n \to 0$ as $n \to \infty$. Again consider

$$\lim_{n\to\infty} \cos(n-1) - \cos(n+1) = \lim_{n\to\infty} 2\sin 1 \sin n$$

This means $\sin n \to 0$ as $n \to \infty$. However if we consider $\sin^2 n + \cos^2 n = 1$, this will give a contradiction.

Then let's prove $\tan n$ is divergent. This one can be solved by similar method. Now let's assmue $\tan n$ is convergent to a. Consider

$$\tan(2n) = \frac{2\tan n}{1 - \tan^2 n}$$

Take limit both sides as $n \to \infty$. We have $a = \frac{2a}{1-a^2}$, but solving this, we can obtain that $\tan n \to 0$ as $n \to \infty$. Then consider $\tan n \cdot \cos n = \sin n$, one can take limit both sides and this implies that $\sin n \to 0$ as

$n \to \infty$, so this comes to a contradition, hence we proved that $\tan n$ is divergent.

Q5 (The relation between the sequence and the ratio)

Suppose $\{a_n\}$ is convergent to 0. Also

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists and convergent to a. Prove that $a \leq 1$.

Proof: I will prove this statement by contrapositive.

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ is convergent to $a > 1$. Then there must exist some $N \in \mathbb{N}$, such that $\forall n \geq N$, we have $\left| \frac{a_{n+1}}{a_n} \right| > 1$.(You can verify this by taking $\epsilon = \frac{|1-a|}{2}$) Hence one can show that $|a_{n+1}| > |a_n|$ for all $n \geq N$. Now take $\epsilon = |a_N|$, we can prove the negation of $\lim_{n\to\infty} a_n = 0$. That is $\forall N' \in \mathbb{N}$, $\exists n \geq N'$, take this n to be greater than N. Notice that by mathematical induction $|a_n| > |a_N|$, $\forall n > N$. Hence we proved that $a_n \not\to 0$ as $n \to \infty$.

Day 5

In the following days, we combine sequences and series and functions.

Q1

Define

$$a_n = \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right)$$

Find $\lim_{n\to\infty} a_n$.

Solution: One can apply sandwich theorem on this sequence. Notice that

$$\begin{aligned} \sqrt{1+\frac{k}{n^2}} - 1 &= \frac{1+\frac{k}{n^2}-1}{\sqrt{1+\frac{k}{n^2}}+1} \\ &= \frac{\frac{k}{n^2}}{\sqrt{1+\frac{k}{n^2}}+1} \end{aligned}$$

Notice that

$$\frac{1}{\sqrt{1+\frac{1}{n}}+1} < \frac{1}{\sqrt{1+\frac{k}{n^2}}+1} < \frac{1}{2}$$

If we sum over n, we will obtain that

$$\frac{1}{\sqrt{1+\frac{1}{n}}+1} \cdot \frac{n\cdot(n+1)}{2n^2} < \sum_{k=1}^{n} \left(\sqrt{1+\frac{k}{n^2}} - 1 \right) < \frac{n\cdot(n+1)}{4n^2}$$

Since the limits of left side and right side are both $\frac{1}{4}$, we can conclude that $\lim_{n\to\infty} a_n = \frac{1}{4}$. We can actually use Python to run a project in order to see where the limit goes.

```
s = 0
n = int(input("enter the number: "))
for i in range(1,n):
    s += (1+i/(n**2))**(1/2) - 1
print(s)
```

```
enter the number: 100000
0.24999708334113935
```

Figure 1: This figure means that if we take n = 100000, $a_n \approx 0.2499970$

Q2 (Prime numbers with limit of sequence)

Denote the number of prime numbers which can divde n as $p(n)$, prove that

$$\lim_{n\to\infty} \frac{p(n)}{n} = 0$$

Proof: Now consider, if p is a prime number who can divde n. Then there must exist one number k, such that $p \cdot k = n$. One can prove from this equality that one of them must be less than $\sqrt{n}$. This implies that all the prime numbers who can divde n, they can be either less than $\sqrt{n}$ or the paired number will be less than $\sqrt{n}$. So we split them into two cases, case one is that the prime number is less than $\sqrt{n}$, those numbers can be at most $\sqrt{n}$. And for those prime numbers who are factors of n, we can count them by using the paired number of them, this can be at most $2\sqrt{n}$. So we can get two inequalities for $\frac{p(n)}{n}$, i.e.

$$0 < \frac{p(n)}{n} < \frac{2\sqrt{n}}{n}$$

Again, we use sandwich theorem, $\lim_{n\to\infty} \frac{p(n)}{n} = 0$.

Q3

Let $a_0, a_1, \dots a_p$ be $p+1$ fixed numbers which satisfy

$$a_0 + a_1 + \cdots + a_p = 0$$

Find

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_p\sqrt{n+p})$$

Solution: Notice that $a_0 = -a_1 - a_2 - \cdots - a_p$, which helps us to simplify the equation into

$$a_1 \cdot (\sqrt{n+1} - \sqrt{n}) + a_2 \cdot (\sqrt{n+2} - \sqrt{n}) + \cdots + a_p \cdot (\sqrt{n+p} - \sqrt{n})$$

Since those numbers are fixed and p is just a constant, so we can deduce that

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_p\sqrt{n+p}) = 0$$

Done!

Q4 (Supremum and infimum)

Prove that for any sequences $\{x_n\}$ and $\{y_n\}$, following inequailties are true

$$\sup\{x_n + y_n\} \leq \sup\{x_n\} + \sup\{y_n\}$$
$$\inf\{x_n + y_n\} \geq \inf\{x_n\} + \inf\{y_n\}$$

Pick one example where the strict inequailty holds, and pick an example that equality holds.

Prove also that if A and B are sets of numbers which are bounded above, then

$$C \subset \{x + y \mid x \in A, y \in B\} \implies \sup C \leq \sup A + \sup B$$

and

$$\{x + y \mid x \in A, y \in B\} \subset C \implies \sup C \geq \sup A + \sup B$$

Hence deduce that if $C = \{x + y \mid x \in A, y \in B\}$, then $\sup C = \sup A + \sup B$.

(I leave this as an exercise)

Q5 (Bolzano-Weierstrass)

State Bolzano-Weierstrass theorem. Prove that a sequence has a convergent subsequence iff the sequence does not diverge to infinity. Use Bolzano-Weierstrass further deduce that a sequence is convergent iff it's a Cauchy sequence.

Day 6

Contraction mapping

Q1 (Contraction mapping)

Definition: Function f is called a contraction mapping if f is defined on $[a, b]$. $f([a, b]) \subset [a, b]$ and there exists a constant k, such that $0 < k < 1$. And $\forall x, y \in [a, b]$, $|f(x) - f(y)| \leq k|x - y|$.

Prove the following statements:

1. f has an unique invariant point ξ, i.e. a point which satisfies $\xi = f(\xi)$
2. For any initial value a_0, if the sequence is defined as $a_{n+1} = f(a_n)$, then this sequence must converge to ξ.
3. For every a_n,

$$|a_n - \xi| \leq \frac{k}{1-k}|a_n - a_{n-1}| \quad \text{and} \quad |a_n - \xi| \leq \frac{k^n}{1-k}|a_1 - a_0|$$

Proof:

First I am going to prove that such ξ exists for such function, which we can pick an arbitrary sequence $\{a_n\}$ with initial term a_0 within the domain of the function f. Here I claim that this sequence is convergent. We can test this claim by proving it is a Cauchy sequence. Consider

$$|a_{n+p} - a_n| = |f(a_{n+p-1}) - a_{n-1}| \leq k|a_{n+p-1} - a_{n-1}| \tag{1}$$

Then apply this procedure n times, we obtain that

$$|a_{n+p} - a_n| \leq k^n|a_p - a_0| \leq k^n|a - b|$$

Since $k^n|a - b| \to 0$ as $n \to \infty$, one can always find an N, such that $\forall n \geq N$, $|a_{N+p} - a_N| < \epsilon$. Hence, we proved that for any sequence with initial value within the domain $[a, b]$, it is convergent and now we can call this limit ξ. We can actually prove that $f(a_n) \to f(\xi)$, this is just because $|f(a_n) - f(\xi)| \leq k|a_n - \xi|$. As a result, one can show that $\xi = f(\xi)$.

Let us prove that this ξ is unique, suppose there is another invariant point which is called η. Since we want to prove they are unique, just consider the distance between them. Notice that

$$|\xi - \eta| = |f(\xi) - f(\eta)| \leq k|\xi - \eta| < |\xi - \eta|$$

if $|\xi - \eta| \neq 0$, so the only way to make sense is $\xi = \eta$. Now we are done on 1 and 2.

For 3, we just apply our properties again, one can show that

$$|a_n - \xi| = |f(a_{n-1}) - f(\xi)| \leq k|a_{n-1} - \xi| \leq k(|a_n - a_{n-1}| + |a_n - \xi|)$$

rearrange of this inequality leads to $|a_n - \xi| \leq \frac{k}{1-k}|a_n - a_{n-1}|$. Use (1), we obtain second inequality we want to prove.

Some important properties about e

Q2 (Formal definition of e)

Define $a_n = \left(1 + \frac{1}{n}\right)^n$ and $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$. Prove that a_n is simply increasing and b_n is simply decreasing. Prove also that a_n and b_n are bounded. Hence deduce that both a_n and b_n are convergent to same limit, define this limit as e. Prove further that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Proof:

Notice that

$$a_n = \left(1 + \frac{1}{n}\right)^n \cdot 1 \leq \left(\frac{n+2}{n+1}\right)^{n+1} = a_{n+1}$$

where we use the AM-GM inequality in "$\leq$". Again, one can see

$$\frac{1}{b_n} = \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{n}{n+1}\right)^{n+1} \cdot 1 \leq \left(\frac{n+1}{n+2}\right)^{n+2} = \frac{1}{b_{n+1}}$$

this implies that $b_{n+1} \leq b_n$, i.e. $\{b_n\}$ is a simply decreasing sequence. Then let's prove that a_n and b_n is bounded above and below, respectively. Notice

$$\begin{aligned} a_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \left(\frac{1}{n}\right)^{n-k} &= \sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} \\ &= 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \dots + \frac{1}{(n-1) \times n} \\ &= 1 + 1 + 1 - \frac{1}{n-1} \\ &\leq 3 \end{aligned}$$

So a_n is bounded above by 3. Obviously, b_n is bounde below by 0. And we can see that $\lim_{n\to\infty} a_n = e = \lim_{n\to\infty} a_n \cdot \left(1 + \frac{1}{n}\right)$, where the right hand side is just b_n.

Finally, let's prove that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. Let's define $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, obviously, s_n is simply increasing and bounded above.(One can easily check this) So we assume that $s_n \to s$ as $n \to \infty$. Since

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} \cdot \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \leq s_n$$

So we can deduce that $e \leq s$. Additionally, if we fix m as a constant, then

$$a_n \geq \sum_{k=0}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$$

take n to ∞, one can see that $e \geq s_m$. Hence one can deduce that $e \geq s$. Combine those two inequalities, we can obtain that $e = s$.

Q3 (How fact is the convergence of e)

Define $\epsilon_n = e - (1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!})$. Use Stolz theorem to prove that

$$\lim_{n\to\infty} \epsilon_n \cdot (n+1)! = 1$$

By considering $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, show that

$$\frac{1}{(n+1)!} < \epsilon_n < \frac{1}{n \cdot n!}$$

Q4 (Euler constant γ)

Prove first that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

Use inequality above to prove that $\{c_n\}$ is convergent, given that $c_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$. Normally, we call this limit $\lim_{n\to\infty} c_n = \gamma$ Euler constant. Hence, use c_n to find the following limit,

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$$

Q5 (Trigonometric and e sequence)

Find

$$\lim_{n\to\infty} n \sin(2\pi n! e)$$

Use Stolz theorem to find

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \ln \binom{n}{k}}{n^2}$$

Deduce that the geometric mean of binomial coefficients $\to \sqrt{e}$ as $n \to \infty$

Day 7

Q1 (A recursion sequence used in Kepler equation)

Kepler obtained the following equation

$$x - q\sin x = a$$

where a is an arbitrary constant and $0 < q < 1$. This equation can not be rearranged in order to solve x. But there is a method which can approximate the solution by using a recursion sequence. This method is to use $a_{n+1} = q\sin a_n + a$, where it states that we can use arbitrary initial value for the sequence. Prove that this is true.

Proof: Define function $f(x) = q\sin x + a$, we want to apply contraction mapping theorem here. For any $x \in \mathbb{R}$, one can show that $f(x) \in [a-q, a+q]$. So we4 can just define the function on $[a-q, a+q]$. Notice that $f([a-q, a+q]) \subset [a-q, a+q]$. And $\forall x, y \in [a-q, a+q]$, we have

$$\begin{aligned} |f(x) - f(y)| &= |q(\sin x - \sin y)| \\ &= \left| q \cdot 2 \cdot \cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right) \right| \\ &\leq \left| q \cdot \cos\left(\frac{x+y}{2}\right) \cdot |x-y| \right| \end{aligned}$$

here, k can be chosed to be q. This implies that $f(x)$ is a contraction mapping, one can use the theorem we proved before to deduce that there exists unique ξ, such that $\xi = f(\xi)$, which is just the x we want. Additionally, use the second criteria that we proved, one can showt that any sequence converges to this value. Done!

Heine-Borel Theorem

Q2 (Heine-Borel Theorem)

If $\{G_\alpha\}$ is an open cover of closed interval $[a, b]$, then there exists one finite subset $\{G_1, G_2, \ldots, G_n\}$ such that it is a finite open cover of $[a, b]$.

Proof: Now suppose $[a, b]$ has an open cover $\{G_\alpha\}$. Now define

$$A = \{x \geq a \mid [a, x] \text{ has a finite open cover}\}$$

Since $\{G_\alpha\}$ is an open cover of $[a, b]$, so there must exist such an open set which can contain point a, this is finite open cover since only one open set is involved in covering $[a, a]$. One can separate this into two cases now, one is A is not bounded above, and the other is that A is bounded above. For the first case, we have a finite open cover for $[a, \infty]$, so if we use the same open cover for $[a, b]$, $[a, b]$ must be covered. For the second case, suppose A is bounded by ξ. Now we want to prove that $\xi \geq b$, so the only case we need to consider is that $\xi < b$. Since $\xi \in [a, b]$, one can always find an open interval G_i which can cover ξ. By adding G_1 into the open cover which can cover $[a, \xi]$, it is still a finite open cover, but it must cover some number which is greater than ξ. This leads to contradition.

Q3 (Stronger Heine-Borel theorem)

Suppose $\{G_\alpha\}$ is an open cover of $[a, b]$. Prove that there exists a positive number $\delta > 0$, such that $\forall x', x'' \in [a, b]$, which satisfy $|x' - x''| < \delta$, we can always find an element of $\{G_\alpha\}$ which contains x' and x''.

Proof: Apply Heine-Borel theorem here, one can find a finite subset of $\{G_\alpha\}$ which covers $[a, b]$, assume it is $\{G_1, G_2, \ldots G_n\}$. Now for each of the G_i, we can find its two endpoints as a_i, a_{i+1}. Collecting all the endpoints of G_1 to G_n and arrange them in order, which is

$$x_1 < x_2 < x_3 < \cdots < x_m$$

Take $\delta = \min\{x_2 - x_1, x_3 - x_2, \ldots, x_m - x_{m-1}\}$. Now I claim that this is the δ we want to find. Here are several cases. Case one is that x' and x'' are both inside some interval $[x_i, x_{i+1}]$. Since x_i and x_{i+1} are both some endpoints of some open interval, such that there must exist some x_j where $j < i$ such that x' and x'' are contained in an open interval in the form of (x_j, x_k), where $k \geq i + 1$. The second case is that x' and x'' are at different side of an endpoint x_i. Since $|x' - x''| < \delta$, so they must be contained in (x_{i-1}, x_{i+1}). And since x_i is endpoint of an open interval, so there must be another open set which can contain it, this is at least (x_{i-1}, x_{i+1}). Done!

Q4 (equivalent definitions of upper limit)

Prove first that every sequence must have at least one limit point. Prove also that

$$\overline{\lim_{n\to\infty}} x_n = \lim_{n\to\infty} \sup_{k\geq n}\{x_k\}$$

Proof:

The first statement is very easy as an exercise.

First, let us develop several notation to make the question simplified. Denote

$$b_n = \sup_{k\geq n}\{x_k\} = \sup\{x_n, x_{n+1}, \dots\}$$

Hence the thing we want to prove becomes equivalent to $\lim\limits_{n\to\infty} b_n = \overline{\lim\limits_{n\to\infty}} x_n$. Here, we have several things to discuss with, one is that b_n might be infinity at some time for n. Notice that $b_n \geq x_n$, so b_n can only tend to $+\infty$. If b_n is infinity at some time, this can actually implies x_n is not bounded. Consider if x_n is bounded above. Then every b_n will be less than M, given that M is the upper bound of x_n. Impossible for b_n to be $+\infty$, so in this case x_n must be not bounded above, and this implies that $\overline{\lim\limits_{n\to\infty}} x_n = \lim\limits_{n\to\infty} \sup_{k\geq n}\{x_k\} = +\infty$. The remaining cases are those b_n are finite, $\forall n$. We have $b_{n+1} \leq b_n$, so $\{b_n\}$ is a decreasing sequence. Let $b = \lim\limits_{n\to\infty} b_n$. To be rigorous, one need to prove that a decreasing sequence must have only one limit point, which is denoted by b. This is easy to see, we can talk about it in two cases. One is that the b_n is bounded below, done. Another case is that b_n is not bounded below, which means $b_n \to -\infty$, at this time, we define $b = -\infty$. Here, it becomes clear that there are two cases we need to give a insight. One is that $b = -\infty$, noticing $x_n \leq b_n$ will solve this problem as it tells us $x_n \to -\infty$ as $n \to \infty$.

Final case it that b is a finite number. Claim: b is a limit point of $\{x_n\}$. One can choose $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$. In each case, we have

$$b \leq x_{k_i} \leq b_{k_i} < b + \frac{1}{i}$$

$x_{k_n} \to b$, b is a limit point of $\{x_n\}$. Suppose now c is another limit point of $\{x_n\}$. $\{x_{n'_k}\}$ has limit of c. However consider $\{b_{n'_k}\}$ We have $x_{n'_k} \leq b_{n'_k}$. So $c \leq b$, this proves that b is the upper limit of $\{x_n\}$.

Definitions of limit of functions

Q5 (Equivalent definition of limits of functions)

Prove that the following definitions are equivalent to the definition of the limit of function.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in O_\delta(a) - \{a\}, |f(x) - A| \leq \epsilon \quad \text{(Definition 1)}$$

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in O_\delta(a) - \{a\}, |f(x) - A| < k\epsilon \quad \text{(Definition 2)}$$

$$\forall n \in \mathbb{N}, \exists \delta > 0, \forall x \in O_\delta(a) - \{a\}, |f(x) - A| < \frac{1}{n} \quad \text{(Definition 3)}$$

$$\forall \epsilon > 0, \exists n, \forall x \in O_{\frac{1}{n}}(a) - \{a\}, |f(x) - A| < \epsilon \quad \text{(Definition 4)}$$

(Leave as an exercise)

Day 8

Cantor set

Q1 (Cantor set)

Denote $[0,1]$ as E_0. Now eliminate $\left(\frac{1}{3}, \frac{2}{3}\right)$ from E_0 and denote the remaining closed interval as E_1. Show that

$$E_1 \supset E_2 \supset E_3 \ldots$$

Call $P = \bigcap_{n=1}^{\infty} E_n$. Prove that there is no open intervals of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ in P. Hence deduce that there is no open intervals contained in P.

Q2

Prove that for $0 < k < 1$,

$$\lim_{n\to\infty} [(1+n)^k - n^k] = 0$$

Proof: Consider following inequality first,

$$(1+n)^k - n^k = n^k \cdot \left(\frac{(1+n)^k}{n^k} - 1\right) = n^k \cdot \left(\left(\frac{1+n}{n}\right)^k - 1\right) \leq n^k \cdot \left(1 + \frac{1}{n} - 1\right)$$

By sandwich's theorem, we can see that

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} [(1+n)^k - n^k] \leq \lim_{n\to\infty} n^{k-1}$$

And the left hand side and right hand side is both 0, so we obtain the limit is 0.

Q3

Suppose $\{x_n\}$ is convergent. Define

$$y_n = n(x_n - x_{n-1})$$

Will $\{y_n\}$ be convergent?

Proof: There are two ways of doing this, one is using an algebraic trick and another one is to prove it directly from definition. Let's do the tricky one, $y_n = n(x_n - x_{n-1}) = nx_n - (n-1)x_{n-1} - x_{n-1}$. Then consider

$$\frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{nx_n - (x_1 + x_2 + \cdots + x_{n-1})}{n}$$

Now use Stolz, one can show that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{y_1 + y_2 + \cdots + y_n}{n} = \lim_{n\to\infty} \frac{nx_n - (x_1 + x_2 + \cdots + x_{n-1})}{n} = \lim_{n\to\infty} x_n - \lim_{n\to\infty} x_{n-1} = 0$$

This implies that $\{y_n\}$ is not only convergent, but convergent to 0.

Q4 (Some properties about lower limit)

A finite number b is the lower limit of $\{x_n\}$ iff $\forall \epsilon > 0$, there exists infinite terms of $\{x_n\}$ in $(b-\epsilon, b+\epsilon)$ and $\exists N$, $\forall n \geq N$, $x_n > b - \epsilon$.

Proof: Let do $\Longrightarrow$ first. Since b is a limit of subsequence of $\{x_n\}$. Then there must exist infinite terms in $(b-\epsilon, b+\epsilon)$. Now we just want to show that $\exists N$, $\forall n \geq N$, $x_n > b - \epsilon$. Suppose $\forall N$, $\exists n \geq N$, such that $x_n \leq b - \epsilon$. We can do the following procedure, pick $N = 1$ first, then we can find a $x_{n_1} \leq b - \epsilon$, taking $N = n_1 + 1$ now. There is always x_{n_2}, such that $x_{n_2} \leq b - \epsilon$. Continue this procedure, one can show that there exists a sequence which has all the terms less than $b - \epsilon$. And this can imply that we must have a limit point less than b. Contradiction. Done!

Now let's do $\Longleftarrow$. b is obviously a limit point of $\{x_n\}$, which we can verify this by considering $\epsilon = \frac{1}{n}$ at each time. The only thing left is to prove that this is the lowest limit point of this sequence. Now suppose there is $p < b$, where p is a limit point of this sequence. So there must exist infinite terms between a small interval of p, this contradicts the second condition, since we can not find such a N which satisfies $\forall n \geq N$, $x_n > b - \epsilon$.

Q5 (Why upper limits and lower limits are so useful)

Use upper-lower limits method to prove that a Cauchy sequence must be convergent. Use upper-lower limits to prove that a sequence defined as below is covergent.

Define $a_1 > 2$, and

$$a_{n+1} = 2 + \frac{1}{a_n} \text{ for } n \geq 1$$

(this is from 2024 Jan analysis question)

Proof: I leave this one as an exercise.

Day 9

General continuity for a polynomial

Q1

Prove that polynomials $p_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is continuous at every point.

Proof: This is equivalent to prove following:

$$\lim_{x\to a} p_n(x) = p_n(a)$$

We just need to prove that $\lim_{x\to a} x^i = a^i$, where c, i are both constant. Then use the addition property of limit, one can then deduce what we want to prove. $\forall\epsilon > 0$, let's first take $\delta < 1$. Consider if $|x - a| < \delta$, one can say that $|x| < |a| + 1$, so now $|x|$ is bounded. Suppose it is bounded by m. Now take $M = \max\{|a|, m\}$. Let's then take $\delta < \frac{\epsilon}{iM^{i-1}}$. Then we have the following inequailties:

$$|x^i - a^i| = |x - a| \cdot |x^{i-1} + x^{i-2} \cdot a + \cdots + a^{i-1}| \leq |x - a| \cdot (i \cdot M^{i-1}) < \epsilon$$

Notice that the final inequality is obtained by $\delta < \frac{\epsilon}{iM^{i-1}}$. Final thing to do is to apply the addition property and scalar multi property. Done!

Composition of limits

Q2

Given that $F(x) = f(g(x))$, $\forall x$, with the conditions $\lim_{x\to a} g(x) = A$ and $\lim_{y\to A} f(y) = B$. Does this imply that

$$\lim_{x\to a} F(x) = \lim_{x\to a} f(g(x)) = \lim_{y\to A} f(y) = B?$$

Solution: This is impossible, the following is a quick thought. For $f(g(x))$, if we want the limit of this one is same as $f(y)$, we need to restrict the $g(x)$ input to be different from A. However, we only have $g(x) \to A$, we do not have a condition to let $g(x)$ to be different from A. This might be like $g(x)$ is A during small interval near a, and this would imply that the limit of $F(x)$ to be exactly the value at A for $f(y)$.

After saying those, let's just pick one example when this is not true.

$$f(y) = \begin{cases} 1 & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

And $g(x) = 0$. So we notice that $\lim_{x \to 0} f(g(x)) = 0$, but $\lim_{y \to 0} f(y) = 1$. They are not equal.

Q3 (Conditions which the composition of limits hold)

Suppose $\lim_{x \to a} g(x) = A$, $\lim_{y \to A} f(y) = B$. Then if one of the following conditions are satisfied,

$$\exists B = (a - \delta_0, a + \delta_0) \setminus \{a\}, \quad g(x) \neq A \text{ if } x \in B \tag{Condition 1}$$

$$\lim_{y \to A} f(y) = f(A) \tag{Condition 2}$$

$$A = \infty \text{ and } \lim_{y \to A} f(y) = B \tag{Condition 3}$$

Then

$$\lim_{x \to a} F(x) = \lim_{y \to A} f(y)$$

(Leave this as an exercise)

Q4 (Limited possibilities)

Prove that if $\lim_{x\to a} g(x) = A$ and $\lim_{y\to A} f(y)$, then $\lim_{x\to a} f(g(x))$ only has three possibilities.

1. $\lim_{x\to a} f(g(x)) = B$
2. $\lim_{x\to a} f(g(x)) = f(A)$
3. the limit does not exist

(Also leave this as an exercise)

Q5

Prove that $\lim_{x\to 0} f(x)$ exists iff $\lim_{x\to 0} f(x^3)$ exists and they are equal when both exist.

Day 10

Q1 (Limit only at one point)

Pick an example of a function where it is defined at $(-\infty, +\infty)$ with limit only at $x = 0$. That is, at any other point, the function does not have a limit.

Solution: Just consider the following function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Now, let's prove that it has limit at zero. $\forall \epsilon > 0$, just let $\delta = \epsilon$, notice that if $0 < |x| < \delta$, then $|f(x) - 0| = |x| < \epsilon$. Then the second thing we should do for this function is to prove it does not have limit at any other point. Suppose $x_0 \neq 0$, now let us take two sequences $\{a_n\}$ and $\{b_n\}$ both converges to x_0. But a_n are all rational and b_n are all irrational. We can see $f(a_n)$ tends to x_0, but $f(b_n)$ tends to $-x_0$.(We can carefully prove those two properties). Now by Heine's theorem(we will prove later), this is impossible for $f(x)$ to have a limit at x_0.

Q2 (Periodic function with its limit)

Prove that if f is periodic function and $\lim\limits_{x\to\infty} f(x) = 0$, then $f(x) \equiv 0$.

Proof: This one is an obvious one we should use contradiction to solve. Suppose $f(x_0) = a \neq 0$ for some a. Since f is periodic, there will be $T > 0$ as its period. So $\forall N \in \mathbb{N}$, we have $f(x_0 + NT) = f(x_0) \neq 0$.

Let's prove the negation of $\lim\limits_{x\to\infty} f(x) = 0$. Set $\epsilon = |a|$, $\forall M > 0$, there must be $x_0 + N'T > M$, this follows from Archimedean property. So $\exists x = x_0 + N'T > M$, such that $|f(x) - 0| = |a| \geq \epsilon$. This means that we proved the negation of this statement.

Q3 (Periodic rational functions)

Prove that all non-constant rational functions can not be periodic.

(Leave as an exercise)

Q4 (One-sided limit for functions)

Suppose a is a finite number, then

$$\lim_{x \to a} f(x) = A$$

if and only if $f(a^-) = f(a^+) = A$. A can either be finite or infinite.

Proof: ($\Longleftarrow$) By definitions of one-sided limit, $\forall \epsilon > 0$, we can always find δ_1 and δ_2, such that, $\forall x \in a - \delta_1 < x < a$ and $a < x < a + \delta_2$, $|f(x) - A| < \epsilon$. We then take $\delta = \min\{\delta_1, \delta_2\}$, for $0 < |x - a| < \delta$, we have $|f(x) - A| < \epsilon$.

($\Longrightarrow$) $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall x \in 0 < |x - a| < \delta$, $|f(x) - A| < \epsilon$. Now for each side, just take $\delta_1 = \delta_2 = \delta$, this works to show that both sides tends to A. Done!

Q5

Suppose f is monotone on (a, b), then $f(b^-) = \lim_{x \to b^-} f(x)$ must be meaningful.

Proof: Wlog, let's assume that $f(x)$ is increasing on (a, b). Now, let $A = \{a \mid \exists x \in (a, b), f(x) = a\}$. Since (a, b) is not empty, so we can say that A is not empty, it must have a $\sup A$ if finite. Now we are dealing with two cases. Case one is that A is bounded above. In this case, let $\xi = \sup A$. Using the property of supremum, $\forall \epsilon > 0$, there must exist one $x' \in A$, such that $f(x') > \xi - \epsilon$ and then we claim that $\forall x \in A > x'$, we have $\xi - \epsilon < f(x') \leq f(x) \leq \xi$. This actually implies that $\lim_{x \to b^-} f(x)$.

Then the case left is when A is not bounded. Now we claim that $\lim_{x \to b^-} f(x) = +\infty$. This is because $\forall M > 0$, there must be $x \in (a, b)$, such that $f(x) > M$, take $b - x$ as δ, one can show from the definition that $\lim_{x \to b^-} f(x) = +\infty$. Done!

Day 11

Alternative definition of closed sets

Q1

Show that a set B is closed if and only if $\forall$ convergent $\{x_n\} \subset B$, where

$$\lim_{n\to\infty} x_n = x$$

we have $x \in B$.

Proof:

($\Longrightarrow$) B is closed, B^c is open. We can actually show that $\forall b \in B^c$, it can not be the limit of any convergent sequence in B. There exists δ, such that $(b-\delta, b+\delta) \subset B^c$. Hence if one sequence tends to b, we can let $\epsilon = \delta$, this would result in there must be infinite terms of $\{x_n\}$ in B^c. Contradiction, so we must have all limit of convergent sequences in B to be one element in B.

($\Longleftarrow$) Suppose B^c is not an open interval, so exists $b \in B^c$, such that $\forall \epsilon > 0$, $\exists b' \in B$, such that $b' \in (b-\epsilon, b+\epsilon)$. Now take $\epsilon = \frac{1}{n}$ each time, so we can find a sequence that is belong to B but it tends to b. So we must have $b \in B$. Contradiction.

Q2 (Referenced from Rudin)

Defintion (Limit points)

A point p is said to be limit point of set E, if $\forall \delta > 0$, $\exists q \in E$, such that, $q \neq p$ and $q \in N_\delta(p)$.

This is how we define a limit point on a metric space. Now go back to $\mathbb{R}$ for a moment.

Prove that E is closed if and only if every limit point p of E belongs to E.

(Leave as an exercise)

Compact & Closed and Bounded

Q3

Definition (Compact sets)

A set $K \subset \mathbb{R}$ is compact if and only if every sequence $\{x_n\} \subset K$ contains a convergent subsequence $\{x_{n_i}\}$ and $\lim_{i\to\infty} x_{n_i} \in K$.

Now prove that

$$K \text{ is compact} \Leftrightarrow K \text{ is closed and bounded}$$

Proof:

($\Longrightarrow$) Let's use Q1 in this direction. Since $\forall$ convergent $\{x_n\} \subset K$. K is compact tells us that there is a subsequence converges to $k \in K$. From the property of connvergent sequences we know that $\{x_n\}$ must have same limit as its subsequence. So, $\{x_n\} \to k \in K$. Done!

($\Longleftarrow$) This side is very easy as we can use the Bolzano-Weierstrass theorem here to say that since the sequence is bounded, so it must have a subsequence which converges to some value in this K. Done!

Q4 (Intervals that are compact)

Prove that intervals in the form of $[a, b]$ are compact. Then prove that all closed subsets of a compact set is compact.

Proof:

Notice first that $[a, b]$ is closed and bounded, so it is compact. The second one is just some deductions of definitions. Let's call A as any closed subset of E, where E is the compact set. Suppose $\{x_n\} \subset A$. $\{x_n\} \subset K$, there exists $\{x_{n_i}\}$ converges to some value $k \in K$. We can actually argue that $k \in A$. Then the question is done!

Equivalent definitions for Compact sets

Q5

Prove that a set E is compact if and only if for every open cover of E, there will be a finite subsets $\{G_\alpha\}$ of open cover, such that

$$E \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$$

Proof: Let's denote the finite open cover property as P_F. A set satisfies this property is called P_F set.

($\Longrightarrow$) I will first prove that if a set is compact, then it must have property P_F. We proved before that if a set E is compact, then it must be closed and bounded. Therefore, there must exist one $[a, b]$, containing E. Then I will introduce a **theorem** first.

Theorem: A closed subset of P_F set must also be P_F set.

Denote the P_F set as K and the closed subset of it as E. Let $\{V_\alpha\}$ to be any open cover of E. Now add E^c into the $\{V_\alpha\}$. This means that the new open cover $\{U_\alpha\} = E^c \cup \{V_\alpha\}$ must be an open cover of K. Because K is a P_F set, then there must exist finite subset of $\{U_\alpha\}$ that can cover K. There are two cases remaining, one is that E^c is in this set. If this is true, then take out E^c from this subset, the remaining sets are a finite subset of $\{V_\alpha\}$ which also covers E. Another case is that E^c is not in the subset, this finite subset of $\{U_\alpha\}$ is also a finite subset of $\{V_\alpha\}$, hence this can cover E. Done!

Let's come back to our proof now, using Heine-Borel's theorem we stated in **Day 7**, one can show that $[a, b]$ is a P_F set. Then E must be a P_F set.

($\Longleftarrow$) Let's then prove that if E is a P_F set, then E is compact. Let's first prove that every infinite subset K of E must have a limit point $p \in E$. This can be done using contradiction. Suppose that there is no limit points of K in E. Hence $\forall q \in E$, we can always find $\delta_q > 0$, such that the open interval $(q - \delta_q, q + \delta_q)$ can contain at most 1 point of K. So take all such open interval for all points in E. This is an open cover of E. However, there can not be finite subset of it which can cover E since K is infinite. This contradicts, so every infinite subset of E must have a limit point in E. Then I am going to use this property to prove E is bounded and closed. Suppose E is not bounded, i.e. $\forall N \in \mathbb{N}$, we can find $|x_n| > N$ and $x_n \in E$. Now $\{x_n\}$ is an infinite subset of E, however it does not have any limit point in $\mathbb{R}$. So it does not have any limit point in E. Contradiction shows that E must be bounded. Suppose E is not closed, meaning that $\exists p \notin E$, such that p is a limit point of E (Notice that this comes out directly from the alternating definition of closed set). If each time I take $\delta = \frac{1}{n}$, I can find $\{y_n\}$ such that $y_n \in E$ and $y_n \in (p - \frac{1}{n}, p + \frac{1}{n})$. Since $\{y_n\}$ is an infinite subset of E, so it must have a limit point $y \in E$. However, the definition shows that the sequence must converge to only one limit, which is p.(One can show this by contradiction). After proving this by oneself, we have proved that E is bounded and closed, which is equivalent as E being compact. Done!

Several ways of proving Bolzano Theorem

Bolzano Theorem

Bolzano's theorem states that if f is continuous on $[a, b]$ and assume

$$f(a) < 0 < f(b)$$

then there exists $x \in (a, b)$, such that $f(x) = 0$.

This is an important theorem in continuity of a function, and there are a lot of ways to prove the theorem, which we will introduce later.

Lebesgue method

Let's use my favourite method to prove **Bolzano Theorem** first. Notice that since $f(a)$ is less than zero and f is continuous on $[a, b]$, $\exists \delta > 0$, $\forall x \in [a, a + \delta)$, $f(x) < 0$. Now let's apply Lebesgue method here, define $A = \{x \mid x \in [a, b],\ f(x) < 0\}$. We notice that A is non-empty and A is bounded above. So we can apply the supremum theorem to A, let $\alpha = \sup A$. And now we have several things to do, first of all, we notice that $\alpha < b$ this is because there also exists an interval on the left of b, such that $f(x) > 0$ in this interval. So we can pick any element in this interval, one can show that α is less than this element, and this element is less than b. There are three possibilities there for α. $f(\alpha) > 0$, $f(\alpha) < 0$ and $f(\alpha) = 0$. We only need to show that the first two possibilities can not happen here. Let's assume $f(\alpha) < 0$. Now since f is continuous at α, so we can find an element $\alpha' > \alpha$ and $f(\alpha') < 0$, this contradicts that α is the supremum of A. Suppose now $f(\alpha) > 0$, then there exists a small interval on the left of α, for all x in this interval, $f(x) > 0$. However, we can use property of supremum to contradict this. Done!

Nested Intervals method

Consider bisection of the interval of $[a, b]$. The first step is to consider $\frac{a+b}{2}$ and we test whether $f(\frac{a+b}{2})$ is less than zero or larger than zero. If $f(\frac{a+b}{2}) = 0$. we proved the theorem. If $f(\frac{a+b}{2}) < 0$, let $a_1 = \frac{a+b}{2}$ and $b_1 = b$, similar for $f(\frac{a+b}{2}) > 0$. Constructing those a_n and b_n. We have $|a_n - b_n| \leq \frac{|a-b|}{2^n}$, by nested intervals theorem, $\exists \xi$ such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$. Then we can claim that $f(\xi) = 0$. If $f(\xi) \neq 0$, wlog, I assume that $f(\xi) > 0$. By continuity of f, we can prove that there exists a small interval of ξ, where all $f(x)$ in this interval remains the same sign as $f(\xi)$. However, this is impossible because we can always find a_k in this interval such that $f(a_k) > 0$. Done!

Heine Borel method

This method uses Stronger Heine Borel Thoerem in order to prove the statement. f is continuous on $[a, b]$ suppose that we can not find a point $\xi \in (a, b)$, such that $f(\xi) = 0$. This implies that for all the value between $[a, b]$, we can always find a small open interval which all the $f(x)$ inside it preserves the same sign as the point, so this forms an open cover of the interval $[a, b]$, apply stronger Heine Borel Theorem here, we see that there exists a δ, which satisfies that $\forall x', x'' \in [a, b]$, if $|x' - x''| < \delta$, there exists an open interval from the set which can cover the points x' and x''. Now we can further claim that $f(x')$ and $f(x'')$ have the same sign since they are contained in same open interval we constructed earlier. Now consider we divide the interval $[a, b]$ into some pieces where each piece has a length of $\frac{\delta}{2}$, this implies that $f(a)$ has the same sign as $f(b)$, because that $f(a + \frac{\delta}{2})$ preserves the same sign as $f(a)$ which preserves the same sign as $f(a + \delta)$, continuing this process. This one is a contradiction, so we can now prove by contradiction that the bolzano theorem holds in this situation.

Day 12

Q1

Suppse f is continuous on $[a, b]$, $\forall$ rational numbers $r_1, r_2 \in [a, b]$, $r_1 < r_2 \implies f(r_1) \leq f(r_2)$. Prove that f is monotone increasing on $[a, b]$.

Proof:

Prove by contradiction, assume that there exists two points x_0 and x_1 such that $x_1 < x_0$ but $f(x_1) < f(x_0)$ (This is the negation of montone increasing). Since f is continuous, we can find δ_1 and δ_0 such that $\forall x \in (x_1 - \delta_1)$, $f(x)$ is greater than all $f(y)$ for $y \in (x_0, x_0 + \delta_0)$. To be more specific, we can let $\delta_0 + \delta_1 < |x_1 - x_0|$. Pick any rational number in those two intervals, we can show a contradiction to the condition.

Q2

Prove that the discontinuities of monotone functions can only be jump points.

Hint: Let's conside $f(x_0^-)$ and $f(x_0^+)$. Since the function is monotone, we can claim that those two limits exist and have an inequality

$$f(x_0^-) \leq f(x_0) \leq f(x_0^+)$$

However, it is discontinous at x_0, so one can actually show that the inequality should be strict, this is the definition of a jump point. Done!