NINE INTRODUCTIVE QUESTIONS TO MATHEMATICS

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Abstract

This booklet is designed with nine questions for students who are just entering the Alevel-center for studying to test themselves. Math Club in JinLing High School Alevel Center started from 2023, this is intended to be a complementary material for the club. All exercises are *STEP-style*, which you may encounter later in A-level mathematics. This booklet is dedicated to my high-school teachers and classmates for their support.

1 Introduction

Before getting questions to do, we need to understand what is Mathematics.

When we are young, we picked apples one by one and teachers taught us that $1+1=2$.

Figure 1: A child counting apples.

Those seems to be natural for a human-being in earth as we are living inside a physical world. But have you ever thought about why we can do that? Why $1+1=2$? How many things can we assume to be the common sense like $1+1=2$?

Figure 2: Euclid, the father of geometry.

Here is something seems to be absurd. A very famous mathematics question is 'Fermat's Last Theorem', which states that there is no positive integer solution to the equation

$$x^n + y^n = z^n$$

for $n > 2$. It took mathematicians more than 350 years to prove it. The proof was finally completed by Andrew Wiles in 1994. The proposition was first stated as a theorem by Pierre de Fermat around 1637 in the margin of a copy of Arithmetica. Fermat added that he had a proof that was too large to fit in the margin. And nobody has ever found Fermat's proof, which becomes a mystery in the history of mathematics. However, if we don't specify what are the 'common sense' of mathematics, I can now prove Fermat's Last Theorem using one sentence:

Proof. Fermat's Last Theorem seems to be a very simple equation, so it is a common sense for me that it is true. □

Indeed, you can even generate this idea to prove any theorem in mathematics by just saying that it is a common sense although nobody is going to trust you on them. Maths should be rigorous by identifying the things we can use and we can't use. Those are called the axioms which you might not see before in your middle school. Why Elements of Geometry by Euclid is so famous? Because it is the first book that systematically and rigorously proves theorems in geometry using axioms instead of geometric intuitions. I believe most or majority of you are using the geometric intuitions to solve the questions in middle school, although some steps seem to be ok, but probably you are not getting the ideas of real mathematics. Those will be the things we are going to introduce in the Math club by weekly lectures. Think slowly and carefully! Let's enjoy the journey of mathematics together! – *Maxwell Gong 2025*.

Q1: Basic Inequalities, Functions, and Graphs

(i) Expand $(\sqrt{x} - \sqrt{y})^2$ to show

$$x + y \geq 2\sqrt{xy}, \quad x, y \geq 0,$$

and determine when equality holds.

(ii) Let $f(x) = x + \frac{1}{x}$. Find its domain and prove that its minimum for $x > 0$ is 2. Sketch the graph by examining behavior as x approaching to 0 and x being large enough.

(iii) If $a > 2$ and the line $y = a$ meets $y = x + 1/x$ at (x_1, a) and (x_2, a) with $x_2 > x_1$, by first finding a relation between x_1 and x_2, and use it to prove $|x_2 - 1| > |x_1 - 1|$.

Comment on the question:

By introducing the basic inequality, we can derive some extreme values of some functions. Then we examine a classical rational function by plotting its diagram. Note that some tricks can be applied to plot the function, note $f(x) = -f(-x)$, so the graph is symmetric about the origin.

Q2: Parabolas, Lines, and Geometry

On the Cartesian plane, let $A = (0, \frac{1}{4})$ lie on the y-axis and let $B = (t, -\frac{1}{4})$ move on the line $y = -\frac{1}{4}$.

(i) Show the midpoint C of AB lies on the x-axis and express its coordinate in terms of t.

(ii) Through B, draw the line perpendicular to $y = -\frac{1}{4}$, and through the midpoint D of AB, draw the line perpendicular to AB. Let these meet at M. Find $M(t)$.

(iii) Sketch the locus of M as t ranges over $\mathbb{R}$, and show CM is tangent to this curve.

Comment on the question:

> *This question is about the geometry of parabolas and lines. The first part is to find the midpoint of a line segment, which is a common task in geometry. The second part involves finding the intersection of two lines, which requires knowledge of slopes and perpendicularity. The final part is to sketch the locus of points, which involves understanding how the coordinates change as t varies. Indeed, you may want an additional question to tackle (ii) more smoothly, prove that two lines $y = k_1 x$ and $y = k_2 x$ are perpendicular iff $k_1 k_2 = -1$, and extend this to any pair of lines.*

Q3: Calculus and Coordinate Transformations

(i) For $x^2 + 2xy + y^2 = a$ $(a > 0)$, find $\frac{dy}{dx}$. Plot this for $a = 1$. Under the rotation

$$x' = \frac{1}{\sqrt{2}}(x - y), \quad y' = \frac{1}{\sqrt{2}}(x + y),$$

derive the new equation and plot it alongside.

(ii) Show that under the same rotation, the circle $x^2 + y^2 = a$ remains invariant. Interpret this geometrically, then consider $x^2 + 2xy + y^2 = \sqrt{2}(y - x)$: find a point (x_0, y_0) and line $x + y + k = 0$ so that each solution (x_1, y_1) satisfies

$$\mathrm{dist}((x_1, y_1), (x_0, y_0)) = \mathrm{dist}((x_1, y_1), \{x + y + k = 0\}).$$

(iii) Plot $\frac{1}{2}x^2 - xy + \frac{1}{2}y^2 = e^{x^2+2xy+y^2}$ and show its minimal distance to $(0, 0)$ is 1.

Comment on the question:

This question explores the relationship between calculus and coordinate transformations. The first part involves finding the derivative of a curve, which is a fundamental concept in calculus. The second part introduces a rotation transformation, which is a common technique in geometry to simplify problems. The final part combines these concepts to find the minimum distance from a point to a curve, which requires both calculus and geometric reasoning.

Q4: General Inequalities

(i) For $x_i > 0$ $(i = 1, 2, 3)$, prove

$$x_1 + x_2 + x_3 \geq 3\sqrt[3]{x_1 x_2 x_3}.$$

(ii) Show $x_2 > x_1 \Leftrightarrow e^{x_2} > e^{x_1}$, and that $e^x > x$ for all real x. Prove $y = \frac{e}{a}x$ is tangent to $y = e^{x/a}$, identifying the point of contact.

(iii) Deduce the general AM–GM inequality

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}.$$

(iv) If $x_i > 0$ and $\prod_i x_i = 1$, show

$$(n + x_1)(n + x_2)\cdots(n + x_n) \geq (n+1)^n.$$

Comment on the question:

The meaningful part of this question is (i) - (iii), where the forth question is some tricky question I picked from Olympiad style question to let you think deeper of when the equality holds and how to apply those inequalities.

Q5: Trigonometry, Integration, and Inequalities

(i) From $\sin^2\alpha + \cos^2\alpha = 1$ derive the half-angle forms

$$\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)}, \quad \cos\alpha = \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)},$$

and show

$$\int_0^{\pi/2} \tfrac{dx}{\sin x + \cos x} = \sqrt{2}\ln(\sqrt{2}+1).$$

(ii) If $x, y > 0$ and $x + y = 1$, find the minima of

$$\tfrac{1}{x} + \tfrac{1}{y} \quad \text{and} \quad \tfrac{1}{x+1} + \tfrac{1}{y+2}.$$

(iii) For $x_i > 0$ with $\sum x_i = 1$, use AM–GM to prove

$$1 + x_i \geq (n+1)(x_1 x_2 \cdots x_n x_i)^{1/(n+1)},$$

and a similar bound for $1 - x_i$. Deduce

$$\prod_{i=1}^{n}\left(\tfrac{1}{x_i^2} - 1\right) \geq (n^2-1)^n.$$

Comment on the question:

This is a bad question I have to admit. (i) is irrelevant to other parts. Hint of (iii), think of converting 1 *to some formula in* x_i*s. It aims to test your understanding of AM-GM inequality.*

Q6: Polynomials and Trigonometry

(i) By letting $x = a + y$, show $x^2 - 2ax + b = c$ becomes $y^2 = a^2 + c - b$, and solve $x^2 - 6x + 1 = 8$.

(ii) Solve $\cos\theta = \frac{\sqrt{2}}{2}$. Assume $\cos^{-1}(2)$ is well-defined, solve $\cos 2\theta = 2$, and show there are only two possible values for $\cos\theta$.

(iii) Using $e^{i\theta} = \cos\theta + i\sin\theta$ (without the double-angle formula), find all $\cos\theta$ satisfying $\cos 2\theta = 2$, and determine corresponding θ.

(iv) Find the solution to $2x^2 - 1 = 2$ and show that the solutions are $\cos\theta$ for some θ satisfying $\cos 2\theta = 2$.

Comment on the question:

> *This question explores the relationship between polynomials and trigonometric functions. The first part involves manipulating a polynomial equation, which is a common task in algebra. The second part introduces the concept of inverse trigonometric functions, which is a fundamental concept in trigonometry. The third part combines these concepts to find solutions to a trigonometric equation using complex numbers, which is a powerful technique in mathematics. The additional question encourages reflection on the connections between these concepts. If you are interested in this concept, try to generate this idea to polynomials with degree* 3 *and solve them.*

Q7: Number Theory and Euclid's Algorithm

(i) For integers a, b with $b > 0$, show there are unique q, r such that $a = qb + r$, $0 \leq r < b$. Find examples for $(a, b) = (133, 21)$ and $(-50, 8)$.

(ii) Show that the greatest common divisor of a and b is same as g.c.d of a and $a - b$ given $a \geq b$.

(iii) If $L = \{ua + vb : u, v \in \mathbb{Z}\}$ has minimal positive element d, prove every common divisor of a, b divides d, and conclude $d = \gcd(a, b)$.

Comment on the question:

This question introduces the concept of the greatest common divisor (gcd) and the Euclidean algorithm, which is a fundamental algorithm in number theory. The first part involves finding the unique representation of an integer in terms of another integer, which is a common task in number theory. The second part uses the concept of linear combinations to prove properties of the gcd, which is a powerful technique in mathematics. This question is essential for understanding the foundations of number theory and its applications.

Q8: Mechanics and Young's Modulus

(i) A uniform string of length L, cross-section A, modulus K, density ρ, hangs under gravity. Show the extension to point B at distance ℓ from the top and total extension $\rho g L^2/(2K)$.

(ii) If density varies as $f(\ell) = \sin(\ell)/\ell$, prove the force at B is $< Ag\ln(L/\ell)$, the total extension $< gL/K$, and hence maximized by $g\pi/K$ when $L \leq \pi$.

Comment on the question:

This is a question suggested by one of my classmates, designed for whom are interested in mechanics. It is a good question to test your understanding of Young's modulus and the mechanics of materials. The first part involves calculating the extension of a string under gravity, which is a common problem in mechanics. The second part introduces a variable density function and requires understanding of logarithmic functions and their properties.

Q9: Vector Identities and Einstein Summation

(i) Prove

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c},$$

and

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

then deduce the cyclic sum identity.

(ii) Let $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Show

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2.$$

Explain the Einstein summation convention, define δ_{ij} and ε_{ijk}, and verify

$$\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp},$$

showing how $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$ recovers Q9(i).

Comment on the question:

This question requires some knowlegde from vector calculus. Especially the Einstein summation convention, I would suggest you to leave this question to the point where you have some insights on them.

2 Conclusion

This concludes the nine questions to test your understanding of mathematics. Each question is designed to challenge your knowledge and skills in various areas of mathematics, including algebra, geometry, calculus, and number theory. I hope you find these questions engaging and thought-provoking. Remember, the journey of learning mathematics is continuous, and each question is a step towards deeper understanding and mastery. If you are not able to do all the questions, that's fine, you will have a totally different viewpoint if you take the lectures in the Math club. I hope you will enjoy the process of learning and exploring mathematics.