Peano axioms for Natural Number

An interesting question: how to prove that $3 \neq 0$?

Today, we will use Peano axioms to construct the Natural Number and then use them to prove that the Natural Number is the same as what we deal with in our daily life.

Here are some common **sets**:

$-\mathbb{N}$ stands for natural numbers

$-\mathbb{N}_0$ stands for natural numbers with zero

$-\mathbb{Z}$ stands for integers

$-\mathbb{Q}$ stands for rational numbers

There are also many sets we have not mentioned above, they will be discussed later in analysis.

We can give our insights into construction of the number system.

Definition: "++" represents the number next to then number which is under operation. (Note, this symbol can be any symbol you want, but this note is referenced to a book written by Tao—An American mathematician, so I just keep it original)

Example: $n++$ represents the next number of n

We can now introduce our first axiom here, which is necessary, since it is the basic element of the $\mathbb{N}$:

Axiom I:

$$0 \in \mathbb{N}$$

To make the Natural Number more like what we actually have in our daily life, we give the second axiom:

Axiom II:

$$(n \in \mathbb{N}) \Longrightarrow (n++ \in \mathbb{N})$$

We are then able to define the number using our familiar symbols. $1 := 0++$,

$2 \coloneqq (0++)++$, $3 \coloneqq ((0++)++)++$ ......

Question I: Use axioms we have now, prove that 3 is a natural number

Proof:

$$(0 \in \mathbb{N}) \implies (1 \coloneqq 0++ \in \mathbb{N}) \implies (2 \coloneqq 1++ \in \mathbb{N}) \implies (3 \coloneqq 2++ \in \mathbb{N})$$

■

Question II: Is our system complete?

Answer: No, because there exists such a number system. We can consider the system below

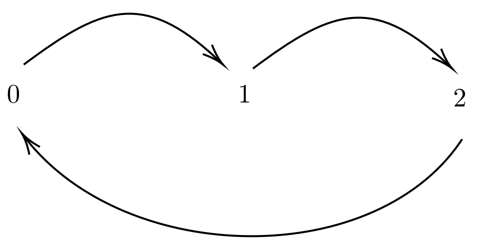

Figure 1: A circular system

It's noticeable that the system satisfy the first three axioms in an obvious way, however, the trick here is that $3 = 0$, $4 = 1$, $5 = 2$ and so on.

This implies our next axiom:

Axiom III:

$$(\forall n \in \mathbb{N}) n++ \neq 0$$

Now there will be an interesting proof which is obvious in our daily life, but now we are going to prove it.

Question III: Prove that $3 \neq 0$.

Proof: Since $3 \coloneqq 2++$(Definition) and $2 \in \mathbb{N}$, so $2++ \neq 0$. ■

However there are still some problems, so we are gonna have next axiom.

Axiom IV:

$$(\forall n, m \in \mathbb{N}), (n \neq m) \implies (n++ \neq m++)$$

Question IV: Prove that the system below does not satisfy Axiom IV

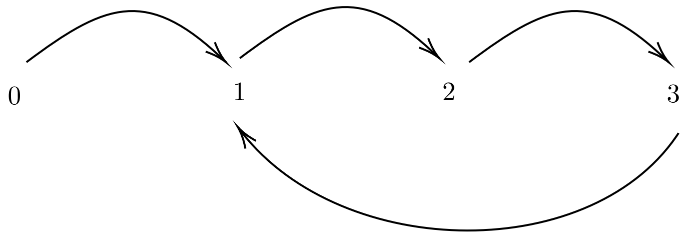

Figure 2: A special case

Notice that this system satisfy the first three Axioms, because of this system, the forth Axiom is given to avoid this system.

Proof:

Because $3 := 2{+}{+} \neq 0$(Axiom III)

Now suppose this system satisfies Axiom IV. According to Axiom IV, $3, 0 \in \mathbb{N}$, so we have $3{+}{+} \neq 0{+}{+}$. However $3{+}{+} = 0{+}{+} = 1$. Contradiction, so the system must not satisfy Axiom IV. $\blacksquare$

Question V: Can you pick a counterexample to illustrate that this "natural number set" can still be weird?

Answer:

Consider the set below:

$$\{0, \mathbf{0.5}, 1, \mathbf{1.5}, 2, \mathbf{2.5}, 3, \mathbf{3.5}, 4, \mathbf{4.5}...\}$$

To avoid the case like this, we publish axiom V

Axiom V:(**Mathematical Induction**)Let $P(n)$ be some property of natural number. Suppose $P(0)$ is true and if $(\forall n \in \mathbb{N})P(n) \implies P(n{+}{+})$. Then we can conduct that $P(n)$ is true for all natural numbers.

This axiom gives us the property regarded to "the smallest"

Question VI: Prove that the $\mathbb{N}$ in Question V is not satisfied with axiom V

Proof:

Suppose

$$\mathbb{N} = \{0, \mathbf{0.5}, 1, \mathbf{1.5}, 2, \mathbf{2.5}, 3, \mathbf{3.5}, 4, \mathbf{4.5}...\}$$

Take the subset $\mathbb{A} = \{0, 1, 2, 3...\}$ and $\mathbb{B} = \mathbb{N} \backslash \mathbb{A} = \{0.5, 1.5, 2.5, 3.5...\}$, in another word $\mathbb{A}$ has these properties:

1. all elements of $\mathbb{A}$ belong to $\mathbb{N}$

2. $0 \in \mathbb{A}$

3. $(n \in \mathbb{A}) \Longrightarrow (n{+}{+} \in \mathbb{A})$

Let $P(n)$ be the property that $n \in \mathbb{A}$. Then we use mathematical induction above.(Axiom V)

Obviously that $P(0)$ is true.

Now we suppose $P(n)$ is true, that is $n \in \mathbb{A}$, according to the third property, we know that $n{+}{+} \in \mathbb{A}$, meaning $P(n+1)$ is true. So $P(n)$ is true for all natural numbers, which implies that all natural numbers belong to $\mathbb{A}$.

By property 1, we can deduce that $\mathbb{A} = \mathbb{N}$. So $\mathbb{B} = \mathbb{N} \backslash \mathbb{A} = \{0.5, 1.5, 2.5, 3.5...\} = \varnothing$, which is a contradiction. So $\mathbb{N} \neq \{0, \mathbf{0.5}, 1, \mathbf{1.5}, 2, \mathbf{2.5}, 3, \mathbf{3.5}, 4, \mathbf{4.5}...\}$ ■

Notes taken by Maxwell Gong