Elementary Complex Functions

Yuxin Gong

April 2025

1 Introduction

In this note, we will introduce some basic functions $f : \mathbb{C} \to \mathbb{C}$. Things seem to become complicated when they are written complex numbers. However, it turns out many things in complex will become much simpler in real functions. For example, if we say some functions is holomorphic, it will then be infinitly differentiable. Anyway, those are strayed off topic we gonna discuss today. Before discussing the topic, let me quote a sentence that might be instructive for complex analysis.

Sometimes the quickest way forward is the long road around.

2 Problems

Consider the following function

$$f : \mathbb{C} \to \mathbb{C}\,,\ f(z) = z^{\alpha}$$

Think about the cases where z is a real number. If α is an integer, we have nothing to worry about cuz it is just some self-multiplication. In real analysis, we know that x^{α} is continuous for every real power on the region $(0, \infty)$. However, in complex analysis, we should really think about a general way to define such function.

3 Exponential Functions

In general, there are two ways to define e^z, one defines using the series

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Alternatively, one can define $e^z = e^x \cos(y) + e^x \sin(y)i$. They are equivalenet, having benefits for different purposes. Let's use the second one here. Using Cauchy-Riemann equations,

$$\frac{\partial}{\partial x} e^x \cos(y) = \frac{\partial}{\partial y} e^x \sin(y) \tag{1}$$

$$\frac{\partial}{\partial y} e^x \cos(y) = -\frac{\partial}{\partial x} e^x \sin(y) \tag{2}$$

e^z is holomorphic for all $z \in \mathbb{C}$. It has some property we expect

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$$