Alternative definitions of determinant

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1 Introduction

This note will investigate several definitions of the determinant of a matrix in order to give a conclusion regarding benefits and drawbacks of each definition. We will always consider a matrix over a field F. All $n \times n$ matrices are denoted by $F^{n\times n}$

2 Classical Definition

To give a definition that every one can understand, you should use a recursive definition.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

The matrix above is a normal matrix with each entry belongs to F. Instead of writing a big 'guy' like this, we usually write it like $(a_{ij})_{1\leq i,j\leq n}$. Further simplify it, we write $A = (a_{ij})$ without specifying its dimension. We call the matrix which is obtained by deleting i-th row and j-th column ij minor of A.

If A is a 1×1 matrix, just define its det to be a_{11}. Suppose $A \in F^{n \times n}$, define

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det(A_{1i})$$

This is all about the classical definition.

3 Permutation Definition

To define the determinant in this way, one needs to introduce the idea of group. S_n denotes the symmetric group of $\{1, \ldots, n\}$. The general idea is to define a homomorphism $\mathrm{sgn} : S_n \to \{-1, 1\}$.

Claim 1. *There exists a unique homomorphism sgn* $: S_n \to \{-1, 1\}$ *sends all transpositions to* -1.

I will not offer the proof at this point but rather a procedure to achieve this well-defined definition for determinant. To give each permutation σ a unique sign, it will be sufficient to prove that

Claim 2. S_n *is generated by transpositions.*

After doing this, one should wonder why the homomorphism is well-defined, i.e., given $\sigma \in S_n$, $\sigma = \eta_1 \cdots \eta_m = \eta'_1 \cdots \eta'_n$. That is to say, the way of writing σ as transpositions is not unique. Hence we are not able to say sgn is now a well-defined homomorphism. However an amazing proof of

Claim 3. *Suppose*

$$\sigma = \eta_1 \cdots \eta_m = \eta'_1 \cdots \eta'_n$$

then $m \equiv n$ *(mod 2).*

Then $\mathrm{sgn}(\sigma) = (-1)^m = (-1)^n$ will always be well-defined. I refer this proof to Cambridge Math Notes - Groups Page 19. **Note this is not the only path to define sgn**. I will state other sources where you can find other approaches to this homomorphism from Algebra 2 by Kevin Buzzard - Chapter 3 Parity of Permutations. He uses polynomials to define sgn. This approach has an advantage that it owns a well-defined formula at beginning, so you don't need to worry about the well-defined problem. Another method to define this problem is given by Michael Artin in his book 'Algebra'. He uses classical definition for determinant and assign sgn a value by using determinant. However, this method might deviate from what we want to do.

After the terminology we introduced just now, we are able to define the determinant for $A \in F^{n \times n}$ through sgn. Define

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Equivalence of those definitions can be proved by showing

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \det(A_{1i})$$

4 Summary

These two methods to calculate the determinant are both important in different situations. In some proofs like proving

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \cdot \det(C)$$

the sgn definition leads to a simpler proof, giving amazing simplicity. However one should note that in daily applications, classical definition gives a more intuitive and easier method to calculate determinant.