Sparse Sensor Selection for Distributed Systems: An l_1-Relaxation Approach ⋆

Yuxing Zhong [a], Nachuan Yang [a], Lingying Huang [b], Guodong Shi [c], Ling Shi [a]

[a] *Department of Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*

[b] *School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore*

[c] *Australian Center for Field Robotics, School of Aerospace, Mechanical and Mechatronic Engineering, University of Sydney, New South Wales, Australia*

Abstract

We study the problem of sensor selection for distributed systems, where a large number of sensors are located spatially in many different locations. Specifically, we consider both perfect and packet-dropping communication channels. While the original problem is NP-hard, by adopting a sparse design, we can solve the problem via convex optimization and reduce the computation cost significantly. Our method not only handles correlated measurement noise but also can be easily extended to actuator selection or sensor-and-actuator (SaA) selection problems. Simulation shows that our sparsity-based approach performs similarly to the brute force optimal strategy while consuming significantly less computation time. Additionally, our method is shown to outperform the state-of-art method notably.

Key words: Sensor selection; Sparsity; Packet dropouts; l_1-relaxation; Convex optimization.

A Lemma 16

Lemma 16 ([1]) Consider the state estimation problem of a stable system with $x_{k+1} = Ax_k + w_k$. The estimation error is updated by $e_{k+1} = x_{k+1} - \hat{x}_{k+1} = A(x_k - \hat{x}_k) + w_k = Ae_k + w_k$. The steady state error covariance is $\bar{P} = P_{\text{opt}}$, where P_{opt} is the solution to the following optimization problem:

$$\begin{aligned} \min_{P} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & APA^T - P \preceq -Q, \\ & P \succ 0. \end{aligned}$$

B Proof of Lemma 6

It is easy to see that the estimation error is updated by

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= (A - L_k^\star CA)e_k + (I - L_k^\star C)w_k - L_k^\star v_k. \end{aligned}$$

According to Lemma 16, we obtain

$$\begin{aligned} \min_{P,L} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & (A - LCA)P(A - LCA)^T - P \\ & \quad \preceq -(I - LC)Q(I - LC)^T - LRL^T, \\ & P \succ 0. \end{aligned}$$

Introducing $\Phi = P^{-1}$ and $Y = \Phi L$ and pre- and post-multiplying the constraint by Φ, we obtain

$$\min_{\Phi,Y} \quad \mathrm{Tr}(\Phi^{-1}) \tag{B.1a}$$

$$\text{s.t.} \quad (\Phi A - YCA)\Phi^{-1}(\Phi A - YCA)^T - \Phi \preceq -(\Phi - YC)Q(\Phi - YC)^T - YRY^T, \tag{B.1b}$$

$$\Phi \succ 0. \tag{B.1c}$$

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Email addresses: yuxing.zhong@connect.ust.hk (Yuxing Zhong), nc.yang@connect.ust.hk (Nachuan Yang), lingying.huang@ntu.edu.sg (Lingying Huang), guodong.shi@sydney.edu.au (Guodong Shi), eesling@ust.hk (Ling Shi).

By using Schur complement decomposition, the constraint (B.1b) can be rewritten as

$$\begin{bmatrix} \Phi & \Phi A - YCA & Y & \Phi - YC \\ * & \Phi & 0 & 0 \\ * & * & R^{-1} & 0 \\ * & * & * & Q^{-1} \end{bmatrix} \succeq 0.$$

Therefore, (B.1) becomes an SDP, i.e.,

$$\begin{aligned} \min_{\Phi, Y} \quad & \mathrm{Tr}(\Phi^{-1}) \\ \text{s.t.} \quad & \text{(6c) and (6b).} \end{aligned} \tag{B.2}$$

The original variable can be uniquely recovered by $\bar{P}^\star = \Phi_{\mathrm{opt}}^{-1}$ and $\bar{L}^\star = \Phi_{\mathrm{opt}}^{-1} Y_{\mathrm{opt}}$, where Φ_{opt} and Y_{opt} are solutions to (B.2).

C Proof of Lemma 8

Define an operator:

$$\begin{aligned} \Psi(L, X, \Gamma) = {} & (A - AL\Gamma C)X(A - AL\Gamma C)^T + Q \\ & + AL\Gamma R \Gamma L^T A^T. \end{aligned}$$

With packet dropouts, the error is updated by

$$\begin{aligned} e_{k+1}^- &= x_{k+1} - \hat{x}_{k+1}^- \\ &= Ax_k + w_k - A\left[\hat{x}_k^- + L_k^\star\left(\Gamma_k y_k - \Gamma_k C \hat{x}_k^-\right)\right] \\ &= (A - AL_k^\star \Gamma_k C)e_k^- + w_k - AL_k^\star \Gamma_k v_k. \end{aligned} \tag{C.1}$$

Hence, the error covariance is computed by

$$P_{k+1}^{\star -} = \Psi(L_k^\star, P_k^{\star -}, \Gamma_k) = \min_L \Psi(L, P_k^{\star -}, \Gamma_k).$$

where

$$L_k^\star = P_k^{\star -} C^T \Gamma_k (\Gamma_k C P_k^{\star -} C^T \Gamma_k + \Gamma_k R \Gamma_k)^\dagger.$$

Define

$$K_k \triangleq P_k^{\star -} C^T \left[G \odot (CP_k^{\star -} C^T + R)\right]^{-1} \Lambda^{-1}.$$

We obtain $P_{k+1}^{\star -} \preceq \Psi(K_k, P_k^{\star -}, \Gamma_k)$. Therefore

$$\begin{aligned} \mathbb{E}[P_{k+1}^{\star -}] &\preceq \mathbb{E}\left[\Psi(K_k, P_k^{\star -}, \Gamma_k)\right] \\ &= \mathbb{E}\left[(A - AK_k \Lambda C)P_k^{\star -}(A - AK_k \Lambda C)^T + Q\right. \\ &\quad + AK_k \Lambda R \Lambda K_k^T A^T \\ &\quad \left.+ AK_k\left((\Sigma \mathcal{I}) \odot (CP_k^{\star -} C^T + R)\right) K_k^T A^T\right] \\ &= \mathbb{E}\left[AP_k^{\star -} A^T + Q - AP_k^{\star -} C^T\right. \\ &\quad \left.\times (G \odot (CP_k^{\star -} C^T + R))^{-1} C P_k^{\star -} A^T\right] \\ &= \mathbb{E}[g_\lambda(P_k^{\star -})], \end{aligned}$$

where the first equality holds due to the independence between $P_k^{\star -}$ and Γ_k, and the second one holds by substituting K_k.

Since $g_\lambda(X)$ is concave and non-decreasing, by the assumption $\mathbb{E}[P_t^{\star -}] \preceq V_t^{\star -}$, we obtain

$$\mathbb{E}[P_{t+1}^{\star -}] \preceq \mathbb{E}[g_\lambda(P_t^{\star -})] \preceq g_\lambda(\mathbb{E}[P_t^{\star -}]) \preceq g_\lambda(V_t^{\star -}) = V_{t+1}^{\star -}.$$

By induction, $\mathbb{E}[P_k^{\star -}] \preceq V_k^{\star -}$ for all k.

Analogously, by calculating the update rule of e_k in terms of e_k^-, we obtain $\mathbb{E}[P_k^\star] \preceq \mathbb{E}[\hat{g}_\lambda(P_k^{\star -})]$. Since $\hat{g}_\lambda(X)$ is concave and non-decreasing, we have

$$\mathbb{E}[P_k^\star] \preceq \mathbb{E}[\hat{g}_\lambda(P_k^{\star -})] \preceq \hat{g}_\lambda(\mathbb{E}[P_k^{\star -}]) \preceq \hat{g}_\lambda(V_k^{\star -}) = V_k^\star,$$

which completes the proof.

D Proof of Theorem 11

The update rule $V_{k+1}^{\star -} = \min_L f_\lambda(V_k^{\star -}, L)$ can be obtained by direct calculation of (9)(10) and $\min_L f_\lambda(V, L)$, and the detailed process is omitted here. Nevertheless, we can provide an alternative explanation of V_k^- by the update rule of e_{k+1}^- with packet dropouts as shown in (C.1). We obtain

$$\begin{aligned} V_{k+1}^{\star -} &= \mathbb{E}\left[(A - \hat{L}_k^\star \Gamma_k C)V_k^{\star -}(A - \hat{L}_k^\star \Gamma_k C)^T\right. \\ &\quad \left.+ Q + \hat{L}_k^\star \Gamma_k R \Gamma_k \hat{L}_k^{\star T}\right] \\ &= f_\lambda\left(V_k^{\star -}, \hat{L}_k^\star\right) = \min_L f_\lambda(V_k^{\star -}, L), \end{aligned}$$

with $\hat{L}_k^\star = AV_k^{\star -} C^T [G \odot (CV_k^{\star -} C^T + R)]^{-1} \Lambda^{-1}$.

Similar to Lemma 16, for steady state, it is equivalent to solve

$$\begin{aligned} \min_{P, L} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & f_\lambda(P, L) \preceq P, \\ & P \succ 0. \end{aligned}$$

Remark 1 Intuitively, $V_k^{\star -}$ is equal to $\mathbb{E}[P_k^{\star -}]$ if $g_\lambda(X)$ were linear in X, in which case we have $\mathbb{E}[g_\lambda(X)] = g_\lambda(\mathbb{E}[X])$.

Note that the last term of $f_\lambda(P, L)$ can be rewritten as

$$L \Lambda F \left(\sum_{i=1}^{M} \bar{C}_i X (\bar{C}_i)^T + \bar{R}\right) F \Lambda L^T.$$

Therefore, by introducing $\Phi = P^{-1}$ and $Y = \Phi L$ and pre- and post-multiplying the constraint by Φ, we obtain

$$
\begin{aligned}
\min_{\Phi, Y} \quad & \mathrm{Tr}(\Phi^{-1}) \\
\text{s.t.} \quad & (\Phi A - Y\Lambda C)\Phi^{-1}(\Phi A - Y\Lambda C)^T - \Phi + Q \\
& \preceq -Y\Lambda F \left(\sum_{i=1}^{M} \bar{C}_i \Phi^{-1} (\bar{C}_i)^T + \bar{R} \right) F \Lambda Y^T \\
& \quad - Y\Lambda R \Lambda Y^T, \\
& \Phi \succ 0.
\end{aligned}
\tag{D.1}
$$

By using Schur complement several times, (D.1) can be transformed into (11), which completes the proof.

E Proof of Proposition 13

Since $A\bar{P}^\star A^T + Q - \bar{P}^\star \succeq 0$, $\bar{S}^\star \succeq \eta S \succeq 0$, it is easy to verify that

$$
\begin{aligned}
\mathcal{C}^\star &= \mathrm{Tr}(W\bar{P}^\star) + \mathrm{Tr}\left(\bar{S}^\star(A\bar{P}^\star A^T + Q - \bar{P}^\star)\right) \\
&\geq \mathrm{Tr}(W\bar{P}^\star) + \eta \mathrm{Tr}\left(S(A\bar{P}^\star A^T + Q - \bar{P}^\star)\right) \\
&= \eta \mathrm{Tr}(SQ) + \mathrm{Tr}\left(h(\eta S)\bar{P}^\star\right) \\
&= \eta \mathrm{Tr}(SQ) + \eta \mathrm{Tr}\left(h(S)\bar{P}^\star\right) + (1-\eta)\mathrm{Tr}(W\bar{P}^\star),
\end{aligned}
$$

where the operate $h(X) \triangleq A^T X A + W - X$.

1) When A is stable, it is easy to verify that when $X_1 \succeq X_2$, $h(X_1) \preceq h(X_2)$. Therefore, we have

$$
0 \preceq h(\bar{S}^\star) \preceq h(\eta S) = \eta A^T S A + W - \eta S.
$$

Thus,

$$
\begin{aligned}
\mathcal{C}^\star &\geq \eta \mathrm{Tr}(SQ) + \mathrm{Tr}\left(h(\eta S)\bar{P}^\star\right) \\
&\geq \eta \mathrm{Tr}(SQ) + \eta \mathrm{Tr}\left(h(\eta S)P)\right) \\
&= \eta \mathrm{Tr}(WP) + \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(S(APA^T - P)\right).
\end{aligned}
$$

2) When A is not stable, it is easy to verify that $h(S) \succeq 0$. Thus,

$$
\begin{aligned}
\mathcal{C}^\star &\geq \eta \mathrm{Tr}(SQ) + \eta \mathrm{Tr}\left(h(S)\bar{P}^\star\right) + (1-\eta)\mathrm{Tr}(W\bar{P}^\star) \\
&\geq \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(h(S)P)\right) + (1-\eta)\eta \mathrm{Tr}(WP) \\
&= \eta \mathrm{Tr}(WP) + \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(S(APA^T - P)\right).
\end{aligned}
$$

Therefore, (13) holds for a general A. The proof is now complete.

References

[1] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, 1994.