Sparse Sensor Selection for Distributed Systems: An l_1-Relaxation Approach

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A Lemma 16

Lemma 16 ([1]) Consider the state estimation problem of a stable system with $x_{k+1} = Ax_k + w_k$. The estimation error is updated by $e_{k+1} = x_{k+1} - \hat{x}_{k+1} = A(x_k - \hat{x}_k) + w_k = Ae_k + w_k$. The steady state error covariance is $\bar{P} = P_{\text{opt}}$, where P_{opt} is the solution to the following optimization problem:

$$\begin{aligned} \min_{P} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & APA^T - P \preceq -Q, \\ & P \succ 0. \end{aligned}$$

B Proof of Lemma 6

It is easy to see that the estimation error is updated by

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= (A - L_k^\star CA)e_k + (I - L_k^\star C)w_k - L_k^\star v_k. \end{aligned}$$

According to Lemma 16, we obtain

$$\begin{aligned} \min_{P,L} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & (A - LCA)P(A - LCA)^T - P \\ & \quad \preceq -(I - LC)Q(I - LC)^T - LRL^T, \\ & P \succ 0. \end{aligned}$$

Introducing $\Phi = P^{-1}$ and $Y = \Phi L$ and pre- and post-multiplying the constraint by Φ, we obtain

$$\min_{\Phi,Y} \quad \mathrm{Tr}(\Phi^{-1}) \tag{B.1a}$$

$$\text{s.t.} \quad (\Phi A - YCA)\Phi^{-1}(\Phi A - YCA)^T - \Phi \preceq -(\Phi - YC)Q(\Phi - YC)^T - YRY^T, \tag{B.1b}$$

$$\Phi \succ 0. \tag{B.1c}$$

By using Schur complement decomposition, the constraint (B.1b) can be rewritten as

$$\begin{bmatrix} \Phi & \Phi A - YCA & Y & \Phi - YC \\ * & \Phi & 0 & 0 \\ * & * & R^{-1} & 0 \\ * & * & * & Q^{-1} \end{bmatrix} \succeq 0.$$

Therefore, (B.1) becomes an SDP, i.e.,

$$\begin{aligned} \min_{\Phi,Y} \quad & \mathrm{Tr}(\Phi^{-1}) \\ \text{s.t.} \quad & \text{(6c) and (6b)}. \end{aligned} \tag{B.2}$$

The original variable can be uniquely recovered by $\bar{P}^\star = \Phi_{\text{opt}}^{-1}$ and $\bar{L}^\star = \Phi_{\text{opt}}^{-1} Y_{\text{opt}}$, where Φ_{opt} and Y_{opt} are solutions to (B.2).

C Proof of Lemma 8

Define an operator:

$$\begin{aligned} \Psi(L, X, \Gamma) = {} & (A - AL\Gamma C)X(A - AL\Gamma C)^T + Q \\ & + AL\Gamma R \Gamma L^T A^T. \end{aligned}$$

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With packet dropouts, the error is updated by

$$\begin{aligned}e_{k+1}^{-} &= x_{k+1} - \hat{x}_{k+1}^{-} \\ &= Ax_k + w_k - A\left[\hat{x}_k^{-} + L_k^{\star}\left(\Gamma_k y_k - \Gamma_k C\hat{x}_k^{-}\right)\right] \\ &= (A - AL_k^{\star}\Gamma_k C)e_k^{-} + w_k - AL_k^{\star}\Gamma_k v_k.\end{aligned} \tag{C.1}$$

Hence, the error covariance is computed by

$$P_{k+1}^{\star -} = \Psi(L_k^{\star}, P_k^{\star -}, \Gamma_k) = \min_L \Psi(L, P_k^{\star -}, \Gamma_k).$$

where

$$L_k^{\star} = P_k^{\star -} C^T \Gamma_k (\Gamma_k C P_k^{\star -} C^T \Gamma_k + \Gamma_k R \Gamma_k)^{\dagger}.$$

Define

$$K_k \triangleq P_k^{\star -} C^T \left[G \odot (CP_k^{\star -} C^T + R)\right]^{-1} \Lambda^{-1}.$$

We obtain $P_{k+1}^{\star -} \preceq \Psi(K_k, P_k^{\star -}, \Gamma_k)$. Therefore

$$\begin{aligned}\mathbb{E}[P_{k+1}^{\star -}] &\preceq \mathbb{E}\left[\Psi(K_k, P_k^{\star -}, \Gamma_k)\right] \\ &= \mathbb{E}\left[(A - AK_k\Lambda C)P_k^{\star -}(A - AK_k\Lambda C)^T + Q\right. \\ &\quad + AK_k\Lambda R\Lambda K_k^T A^T \\ &\quad \left.+AK_k\left((\Sigma\mathcal{I}) \odot (CP_k^{\star -}C^T + R)\right)K_k^T A^T\right] \\ &= \mathbb{E}\left[AP_k^{\star -}A^T + Q - AP_k^{\star -}C^T\right. \\ &\quad \left.\times (G \odot (CP_k^{\star -}C^T + R))^{-1} CP_k^{\star -}A^T\right] \\ &= \mathbb{E}[g_\lambda(P_k^{\star -})],\end{aligned}$$

where the first equality holds due to the independence between $P_k^{\star -}$ and Γ_k, and the second one holds by substituting K_k.

Since $g_\lambda(X)$ is concave and non-decreasing, by the assumption $\mathbb{E}[P_t^{\star -}] \preceq V_t^{\star -}$, we obtain

$$\mathbb{E}[P_{t+1}^{\star -}] \preceq \mathbb{E}[g_\lambda(P_t^{\star -})] \preceq g_\lambda(\mathbb{E}[P_t^{\star -}]) \preceq g_\lambda(V_t^{\star -}) = V_{t+1}^{\star -}.$$

By induction, $\mathbb{E}[P_k^{\star -}] \preceq V_k^{\star -}$ for all k.

Analogously, by calculating the update rule of e_k in terms of e_k^{-}, we obtain $\mathbb{E}[P_k^{\star}] \preceq \mathbb{E}[\hat{g}_\lambda(P_k^{\star -})]$. Since $\hat{g}_\lambda(X)$ is concave and non-decreasing, we have

$$\mathbb{E}[P_k^{\star}] \preceq \mathbb{E}[\hat{g}_\lambda(P_k^{\star -})] \preceq \hat{g}_\lambda(\mathbb{E}[P_k^{\star -}]) \preceq \hat{g}_\lambda(V_k^{\star -}) = V_k^{\star},$$

which completes the proof.

D Proof of Theorem 11

The update rule $V_{k+1}^{\star -} = \min_L f_\lambda(V_k^{\star -}, L)$ can be obtained by direct calculation of (9)(10) and $\min_L f_\lambda(V, L)$, and the detailed process is omitted here. Nevertheless, we can provide an alternative explanation of V_k^{-} by the update rule of e_{k+1}^{-} with packet dropouts as shown in (C.1). We obtain

$$\begin{aligned}V_{k+1}^{\star -} &= \mathbb{E}\left[(A - \hat{L}_k^{\star}\Gamma_k C)V_k^{\star -}(A - \hat{L}_k^{\star}\Gamma_k C)^T\right. \\ &\quad \left.+Q + \hat{L}_k^{\star}\Gamma_k R \Gamma_k \hat{L}_k^{\star T}\right] \\ &= f_\lambda\left(V_k^{\star -}, \hat{L}_k^{\star}\right) = \min_L f_\lambda(V_k^{\star -}, L),\end{aligned}$$

with $\hat{L}_k^{\star} = AV_k^{\star -}C^T[G \odot (CV_k^{\star -}C^T + R)]^{-1}\Lambda^{-1}$.

Similar to Lemma 16, for steady state, it is equivalent to solve

$$\begin{aligned}\min_{P,L} \quad & \mathrm{Tr}(P) \\ \text{s.t.} \quad & f_\lambda(P, L) \preceq P, \\ & P \succ 0.\end{aligned}$$

Remark 1 Intuitively, $V_k^{\star -}$ is equal to $\mathbb{E}[P_k^{\star -}]$ if $g_\lambda(X)$ were linear in X, in which case we have $\mathbb{E}[g_\lambda(X)] = g_\lambda(\mathbb{E}[X])$.

Note that the last term of $f_\lambda(P, L)$ can be rewritten as

$$L\Lambda F\left(\sum_{i=1}^{M} \bar{C}_i X (\bar{C}_i)^T + \bar{R}\right) F\Lambda L^T.$$

Therefore, by introducing $\Phi = P^{-1}$ and $Y = \Phi L$ and pre- and post-multiplying the constraint by Φ, we obtain

$$\begin{aligned}\min_{\Phi,Y} \quad & \mathrm{Tr}(\Phi^{-1}) \\ \text{s.t.} \quad & (\Phi A - Y\Lambda C)\Phi^{-1}(\Phi A - Y\Lambda C)^T - \Phi + Q \\ & \preceq -Y\Lambda F\left(\sum_{i=1}^{M} \bar{C}_i \Phi^{-1} (\bar{C}_i)^T + \bar{R}\right) F\Lambda Y^T \\ & \quad - Y\Lambda R\Lambda Y^T, \\ & \Phi \succ 0.\end{aligned} \tag{D.1}$$

By using Schur complement several times, (D.1) can be transformed into (11), which completes the proof.

E Proof of Proposition 13

Since $A\bar{P}^{\star}A^T + Q - \bar{P}^{\star} \succeq 0$, $\bar{S}^{\star} \succeq \eta S \succeq 0$, it is easy to verify that

$$\begin{aligned}\mathcal{C}^{\star} &= \mathrm{Tr}(W\bar{P}^{\star}) + \mathrm{Tr}\left(\bar{S}^{\star}(A\bar{P}^{\star}A^T + Q - \bar{P}^{\star})\right) \\ &\geq \mathrm{Tr}(W\bar{P}^{\star}) + \eta\mathrm{Tr}\left(S(A\bar{P}^{\star}A^T + Q - \bar{P}^{\star})\right) \\ &= \eta\mathrm{Tr}(SQ) + \mathrm{Tr}\left(h(\eta S)\bar{P}^{\star}\right) \\ &= \eta\mathrm{Tr}(SQ) + \eta\mathrm{Tr}\left(h(S)\bar{P}^{\star}\right) + (1-\eta)\mathrm{Tr}(W\bar{P}^{\star}),\end{aligned}$$

where the operate $h(X) \triangleq A^T X A + W - X$.

1) When A is stable, it is easy to verify that when $X_1 \succeq X_2$, $h(X_1) \preceq h(X_2)$. Therefore, we have

$$0 \preceq h(\bar{S}^\star) \preceq h(\eta S) = \eta A^T S A + W - \eta S.$$

Thus,

$$\begin{aligned}
\mathcal{C}^\star &\geq \eta \mathrm{Tr}(SQ) + \mathrm{Tr}\left(h(\eta S)\bar{P}^\star\right) \\
&\geq \eta \mathrm{Tr}(SQ) + \eta \mathrm{Tr}\left(h(\eta S)P\right) \\
&= \eta \mathrm{Tr}(WP) + \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(S(APA^T - P)\right).
\end{aligned}$$

2) When A is not stable, it is easy to verify that $h(S) \succeq 0$. Thus,

$$\begin{aligned}
\mathcal{C}^\star &\geq \eta \mathrm{Tr}(SQ) + \eta \mathrm{Tr}\left(h(S)\bar{P}^\star\right) + (1-\eta)\mathrm{Tr}(W\bar{P}^\star) \\
&\geq \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(h(S)P\right) + (1-\eta)\eta \mathrm{Tr}(WP) \\
&= \eta \mathrm{Tr}(WP) + \eta \mathrm{Tr}(SQ) + \eta^2 \mathrm{Tr}\left(S(APA^T - P)\right).
\end{aligned}$$

Therefore, (13) holds for a general A. The proof is now complete.

References

[1] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.