Combinatorial Theory

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Contents

1 Chapter 1

1.1 Permutations, Subsets, Multisets

Example 1.1.1. *Suppose n people give their n hats to a hat check. Let $g(n)$ be the number ways hats could be given back so no person receives their own hat.*

Answer.

$$g(n) = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!}.$$

□

Example 1.1.2. *Let $h(n)$ be the number of domino tilings of a $2 \times n$ rectangle using 2×1 rectangles.*

Answer.

1. For all $n \geq 3$, $h(n) = h(n-1) + h(n_2)$.
2. Using rational generating function associated to linear recurrence relations:

$$h(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right).$$

□

Definition 1.1.3. Let S be a finite set. A k-permutation of S is a sequence $(s_1, s_2, \ldots, s_k)$ as long as $k \leq |s|$.

The number of k-permutation of $[n]$ is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}, \quad \text{denoted by } (n)_k \text{ or falling factorial.}$$

Definition 1.1.4. Let $\binom{n}{k}$ denote the number of subsets of $[n]$ of size k.

Theorem 1.1.5 (Sagan 1.3.2)**.**

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(n)_k}{k!}.$$

Theorem 1.1.6 (Sagan 1.3.3)**.** *We have*

1. $$\binom{0}{0} = 1 \quad \binom{0}{k} = 0.$$

2. $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

3. $$\binom{n}{k} = \binom{n}{n-k}.$$

4. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

5. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}.$$

1.2 Generating Functions

Given a numerical sequence

$$a_0, a_1, a_2, a_3, \ldots$$

The ordinary generating function is

$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Note: $k[[x]]$ is a local ring.

Claim: $A(x)$ is invertible if and only if $a_0 \neq 0$.

Let

$$A_m(x) = \sum_{n=0}^{m} x^n.$$

Then

$$A(x)(1-x) = \lim_{m \to \infty} A_m(x)(1-x) = 1.$$

Two generating functions are the same if they converge to each other.

Theorem 1.2.1 (Binomial Theorem)**.**

$$\sum_{k\geq 0}\binom{n}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n.$$

We first do some disambiguating. We use multivariables instead of just one.

$$\begin{aligned}(1+x_1)(1+x_2)\cdots(1+x_n) &= \sum_{1\leq i_1<i_2\cdots i_k\leq n} x_{i_1}x_{i_2}\cdots x_{i_k}\\ &= \sum_{T\subseteq[n]}\prod_{i\in T}x_i\\ &= \sum_{k=0}^{n}\binom{n}{k}x^k\end{aligned}$$

Definition 1.2.2. Let α be any complex number, k non-negative integer. We define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.$$

Consider the genreating function of $\binom{-3}{k}$.

$$\binom{-3}{0}=1, \binom{-3}{1}=-3, \binom{-3}{2}=6, \binom{-3}{3}=-10, \ldots.$$

First note that

$$\sum_{n\geq 0}\binom{-3}{n}x^n = \sum_{n\geq 0}(-1)^n\frac{(n+2)(n+1)}{2}x^n.$$

Then do some differenciation to $\frac{1}{1-x}$ we'll eventually be

$$(1+x)^{-3}.$$

Theorem 1.2.3 (Generalized Binomial Theorem)**.**

$$\sum_{k\geq 0}\binom{\alpha}{k}x^k = (1+x)^\alpha.$$

This could be proved/shown by doing taylor series expansions.

Definition 1.2.4. n multichoose k is the number of ways of choosing a multiset from $[n]$ of size k. Denoted by

$$\left(\!\!\binom{n}{k}\!\!\right).$$

Example 1.2.5.

$$\left(\!\!\binom{3}{2}\!\!\right) = \#\{11,12,13,22,23,33\} = 6.$$

Theorem 1.2.6.
$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Theorem 1.2.7.
$$\sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k = (1-x)^{-n} \quad or \quad \left(\frac{1}{1-x}\right)^n.$$

Recall $h(n)$ is the number of tilings of a $2 \times n$ rectangle.

$$h(n) = \sum_{k=0}^{\frac{n}{2}} \binom{n-k}{k}$$
$$H(x) = \sum_{n\geq 0} h(n)x^n$$
$$H(x) = \frac{1}{1-x-x^2}$$

Example 1.1.13, 1.1.15 from Stanely.

Definition 1.2.8. A *composition* of $[n]$ is an ordered sum of positive integers that sum to n. *k-composition* has exactly k parts.

The number of k-compositions of $[n]$ is $\binom{n-1}{k-1}$ and the number of compositions is 2^{n-1}.

Definition 1.2.9. *Multinomial coefficients* are

$$\binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1!a_2!\cdots a_m!} = \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{m-1}}{a_m}$$