

# Comm & Hom

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## 1 Intro

**Theorem 1.0.1.** *radical ideal is generated by a polynomial  $f$  with no multiple roots.*

Suppose  $J \subset \mathbb{C}[x_1, \dots, x_n]$  is an ideal. Then  $I(Z(J)) = \text{rad}(J) = \{f \mid f^n \in J\}$

**Definition 1.0.2.** *radical ideal is generated by a polynomial  $f$  with no multiple roots. cokernel: take the image of  $f$  and mod out by image of  $f$ .*

### 1.1 Modules

Let  $R$  be a commutative ring. An  $R$ -module  $M$  is an abelian group  $(+)$  with a map  $R \times M \rightarrow M$  written  $(r, m) \mapsto rm$ . Satisfying

1. associativity:  $r(sm) = (rs)m$  for all  $r, s \in R, m \in M$ .
2. distributivity:  $r(m + m') = rm + rm'$  and  $(r + r')m = rm + r'm$  for all  $r, r' \in R, m, m' \in M$ .
3. unitality:  $1m = m$  for all  $m \in M$ .

Several things you could derive from the definition:  $0m = 0, (-1)m = -m$ , etc.

**Example 1.1.1.** *Let  $R = k[x]$ . A  $k[x]$ -module is*

- *a  $k$ -vector space  $M$*

- with a map  $xM \rightarrow M$ , where  $m \mapsto xm$ , a  $k$  linear transformation.

**Example 1.1.2.** What is an  $R$ -submodule of  $R$ ? It's

1.  $J \subseteq R$ ;
2. closed under addition, 0, negatives;
3. for any  $r \in R, j \in J, rj \in J$ .

an ideal.

**Definition 1.1.3.** Let  $M$  be an  $R$ -module,  $N$  a subgroup of  $M$ .  $N$  is a *submodule* if for any  $n \in N$  and  $r \in R$ , the product  $rn$  is in  $N$ .

**Definition 1.1.4.** If  $M$  is an  $R$ -module, we shall write  $\text{ann } M$  for the annihilator of  $M$ ; that is,

$$\text{ann } M = \{r \in R \mid rM = 0\},$$

which is an ideal.

**Definition 1.1.5.** Let  $I \subseteq R$  an ideal,  $M$  an  $R$ -module. We denote

$$IM = \left\{ \sum a_i m_i \mid a_i \in I, m_i \in M \right\} \subseteq M$$

the smallest  $R$ -submodule of  $M$  containing all elements of the form  $am$ , where  $a \in I, m \in M$ .

**Example 1.1.6.** Suppose  $M$  is an  $R$ -module. For  $N, N' \subseteq M$  submodules,

$$[N : N'] \subseteq R \quad x \in [N : N'] \iff xN' \subseteq N.$$

For  $N$  a submodule,  $I$  an ideal

$$[N : I] \subseteq M. \quad y \in [N : I] \iff Iy \subseteq N.$$

The point of having the above is to generalize the annihilator.

**Example 1.1.7.**  $\text{ann } M = [0 : M]$ .

Some operations we could do. Given a sequence of modules  $M_1, M_2, \dots$

**Definition 1.1.8.** We denote

$$\prod_{i \in I} M_i = \{(m_1, m_2, \dots) \mid m_i \in M_i\}.$$

$\prod M_i$  is an  $R$ -module with componentwise addition and scalar multiplication.

Note that  $\oplus M_i \subseteq \prod M_i$ , a sub- $R$ -module.

Also,  $\oplus M_i = \{(m_i)_{i \in I} \mid \text{only finitely many } m_i \text{ are zero}\}$

Suppose we have an  $R$ -module homomorphism

$$f : M \rightarrow N.$$

We could construct 3 modules:  $\ker(f) \subseteq M, \text{Im}(f) \subseteq N, \text{coker}(f) = N/\text{Im}(f)$

**Definition 1.1.9.** Suppose we have  $R$ -module homomorphism

$$f : M \rightarrow N \quad g : N \rightarrow P.$$

This is exact if  $\text{Im } f = \ker g$ .

**Definition 1.1.10.** If we have a sequence of maps

$$\cdots \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \cdots .$$

then we say it's exact iff each 2-term sequence is exact.

Saying  $0 \rightarrow M \rightarrow N$  is exact is saying  $f$  is injective. And  $M \rightarrow N \rightarrow 0$  is exact is saying  $f$  is surjective.

**Definition 1.1.11.** A short exact sequence is an exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0 \quad f : M \rightarrow N, g : N \rightarrow P.$$

This tells us that

1.  $M$  iso to  $\text{Im}(f)$
2.  $P$  iso to  $N/\ker(g)$
3.  $\ker(g) = \text{Im}(f)$ ,  $P$  iso  $\text{coker}(f)$

**Definition 1.1.12.** A free  $R$ -module is an  $R$ -module isomorphic to  $\oplus_{i \in I} R$ . In particular,  $R^n$  are the finitely generated free modules.

**Definition 1.1.13.** A module  $M$  is finitely generated if there exists  $m_1, \dots, m_n \in M$  such that every element of  $M$  is of the form  $\sum_{i=1}^n a_i m_i$  for some  $a_i \in R$ .

**Definition 1.1.14.** A module  $M$  is finitely presented if there exists an exact sequence

$$R^n \rightarrow R^m \rightarrow M \rightarrow 0.$$

## 2 Localization

### 2.1 Fractions

Suppose  $R$  is a ring,  $U \subseteq R$  is a subset that is closed under multiplication, and contains the unit  $1 \in R$ .

**Definition 2.1.1.** We can form the localization  $R[U^{-1}]$ , whose elements are

$$\{(r, s) \mid r \in R, s \in U\}.$$

We also put an equivalence relation on the elements.

$$(r, s) \equiv (r', s') \iff \exists u, v \in U, (ur, us) = (vr', vs').$$

Note that the equivalence relation is different from cross-multiplication as what we do in fractions.

**Example 2.1.2.** Let  $R = \mathbb{Z}, U = \{1, 2, 4, 6, 16, \dots\}$ . Then

$$R[U^{-1}] = \mathbb{Z}[\frac{1}{2}] = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q = 2^k \right\}.$$

**Definition 2.1.3.** In  $R[U^{-1}]$ , we have a ring.

$$\begin{aligned} (r, s) + (r', s') &= (rs' + r's, ss') \\ (r, s) \cdot (r', s') &= (rr', ss') \\ 0 &= (0, 1) \\ 1 &= (1, 1) \end{aligned}$$

**Example 2.1.4.** Let  $R = \mathbb{Z}/6, U = \{1, 3\}$ . Then localization is smaller:

$$R[U^{-1}] = \mathbb{Z}/2.$$

**Example 2.1.5.** Let  $R = \mathbb{C}[x], U = \{1, x, x^2, x^3, \dots\}$ . Then

$$R[U^{-1}] = \mathbb{C}[x, x^{-1}] = \{f(x)/x^n \mid f(x) \text{ poly}, n \in \mathbb{N}\} = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \mid a_n \in \mathbb{C} \right\}.$$

Note that in the summation there should only be finitely many  $n$ . The ring is also called Laurent polynomials.

This ring is isomorphic to

$$\mathbb{C}[x, y]/(yx - 1).$$

Note that there is always a ring homomorphism

$$\phi : R \rightarrow R[U^{-1}] \quad \phi(r) = \frac{r}{1}.$$

**Example 2.1.6.**  $R = \mathbb{C}[x_1, \dots, x_n], U = R - \{0\}$ . Note  $U$  is closed under multiplication because  $R$  is an integral domain.

$$R[U^{-1}] = \{f(\vec{x})/g(\vec{x}) \mid f, g \in \mathbb{C}[x_1, \dots, x_n], g \neq 0\}.$$

**Proposition 2.1.7.** The theory of ideals in  $R[U^{-1}]$  is closely related to the theory of ideals in  $R$ . Given an ideal  $J$  in  $R$ , we could have  $J \cdot R[U^{-1}]$ , which is an ideal in  $R[U^{-1}]$ .

The map from ideals of  $R[U^{-1}]$  to ideals of  $R$  is an injection. They are sort of “ideals that don’t meet the set  $U$ ”.

An ideal  $J$  is of the form  $\phi^{-1}(L)$  iff for any  $a, b$  s.t.  $a \in R, b \in U, ab \in J \implies a \in J$ .

There is a correspondence between prime ideals of  $R[U^{-1}]$  and prime ideals of  $R$  that don’t contain any elements of  $U$ .

**Example 2.1.8.** *prime ideals of  $\mathbb{Q}$ ; prime ideals of  $\mathbb{Z}$  that don't contain any elements of the set  $\{1, 2, 3, 4, 5, \dots\}$ ;  $\{(0)\}$ .*

**Definition 2.1.9.** Suppose  $R$  is a ring.  $P \subseteq R$  is a prime ideal. We define  $R_P$  to be the localization of the set  $U = R - P$ .

Note that  $U$  is closed under multiplication because  $P$  is prime.

Also,  $R_P$  has one maximal ideal:  $PR_P$ .

There is a correspondance between prime ideals of  $R_P$ ; prime ideals of  $R$  that don't contain any elements of  $U$ ; prime ideals of  $R$  contained in  $P$ .

**Example 2.1.10.**

$$\mathbb{Z}_{(2)} = \left\{ \frac{n}{m} \mid m \text{ odd} \right\}.$$

*This has 2 ideals:  $(0), (2)$ .*

**Definition 2.1.11.** A ring  $R$  is *local* if it has a unique maximal ideal.

$R_P$  is always local if  $P$  is prime.

If  $R$  is a ring,  $M$  is an  $R$ -module,  $U \subseteq R$  is a subset closed under multiplication and 1. We can construct

$$M[U^{-1}] = \left\{ \frac{m}{s} \mid m \in M, s \in U \right\}.$$

$M[U^{-1}]$  is an abelian group and a module on  $R[U^{-1}]$ .

**Example 2.1.12.**  $R = \mathbb{Z}, U = \{1, 3, 9, 27, \dots\}, M = \mathbb{Z}/10$ . Check that  $M[U^{-1}] \cong \{0\}$ .

## 2.2 Hom

For  $R$ -modules  $M, N$ . There is a new  $R$ -module  $\text{Hom}_R(M, N)$

$$\text{Hom}_R(M, N) \subseteq \{f : M \rightarrow N\}.$$

Functions that are

1. group homomorphisms
2.  $R$ -linear:  $f(rx) = rf(x)$

**Definition 2.2.1.**  $\text{Hom}_R(M, N)$  is an  $R$ -module in the following way.

- $f + g : (f + g)(m) = f(m) + g(m)$
- $rf : (rf)(m) = rf(m)$

There are some properties of  $\text{Hom}$ .

1.  $\text{Hom}_R(R, N) \cong N$ , where  $f \mapsto f(1), n \in N \mapsto f(r) = rn$ . Basically the same as picking an element from  $N$ .

2.

$$\text{Hom}_R(\oplus_{i \in I} M_i, N) \cong \prod_{i \in I} \text{Hom}_R(M_i, N).$$

The RHS is choosing for each  $i \in I$ , a homomorphism  $M_i \rightarrow N$ . There's also

$$\text{Hom}_R(M, \prod_{i \in I} N_i) \cong \prod_{i \in I} \text{Hom}_R(M, N_i).$$

3. If  $I$  have  $R$ -module homomorphisms

$$\alpha : M \rightarrow M' \quad \beta : N \rightarrow N'.$$

I get a map

$$\text{Hom}_R(\alpha, \beta) : \text{Hom}_R(M', N) \rightarrow \text{Hom}_R(M, N') \quad \text{where} \quad f \mapsto \beta f \alpha.$$

This respects identity functions and function composition. Functorial.

4. Exactness.  $\text{Hom}_R$  is *left-exact*:

(a) If  $M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence, then for any  $N$ ,

$$0 \rightarrow \text{Hom}_R(M'', N) \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M', N)$$

is also exact.

(b) If  $0 \rightarrow N' \rightarrow N \rightarrow N''$  is exact then

$$0 \rightarrow \text{Hom}_R(M, N') \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N'')$$

is exact.

For  $R$ -module  $M, N$  there is a tensor product  $M \otimes_R N$ , which we get by taking all formal sums of symbols  $m \otimes n, m \in M, n \in N$ , mod out by subgroup generated by elements of the form

- $(m + m') \otimes n - m \otimes n - m' \otimes n$ ;
- $m \otimes (n + n') - m \otimes n - m \otimes n'$ ;
- $(rm) \otimes n - m \otimes (rn)$ .

**Example 2.2.2.**

$$R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_n] \cong R[x_1, \dots, x_n, y_1, \dots, y_n].$$

Properties of  $\otimes_R$ .

1.

$$R \otimes_R M \cong M \quad \sum r_i \otimes m_i \mapsto \sum r_i m \quad 1 \otimes m \mapsto m.$$

2.

$$(\oplus M_i) \otimes_R N \cong \oplus (M_i \otimes_R N).$$

3. Functoriality. For  $R$ -module homomorphisms  $\alpha : M \rightarrow M'$ ,  $\beta : N \rightarrow N'$ , we get an  $R$ -module homomorphism

$$\alpha \otimes \beta : M \otimes_R N \rightarrow M' \otimes_R N' \quad \sum m_i \otimes n_i \mapsto \sum \alpha(m_i) \otimes \beta(n_i).$$

4. Right exactness. If  $M' \rightarrow M \rightarrow M''$  is exact, then

$$M' \otimes_R N \rightarrow M \otimes_R N \rightarrow M'' \otimes_R N \rightarrow 0.$$

is exact.

5. Symmetry.

$$M \otimes_R N \cong N \otimes_R M \quad \sum m_i \otimes n_i \mapsto \sum n_i \otimes m_i.$$

**Proposition 2.2.3.**

$$M[U^{-1}] \cong R[U^{-1}] \otimes_R M.$$

*Proof sketch.* The procedure we could do is

$$\frac{m}{u} \mapsto \frac{1}{u} \otimes m \quad \frac{rm}{u} \mapsto \frac{r}{u} \otimes m.$$

□

**Definition 2.2.4.** An  $R$ -module  $F$  is *flat* whenever

$$f : M \rightarrow N \quad \text{is injective,}$$

and the map

$$F \otimes_R M \rightarrow F \otimes_R N \quad \text{is injective.}$$

Alternatively,

$$0 \rightarrow M \rightarrow N \text{ exact} \implies 0 \rightarrow F \otimes_R M \rightarrow F \otimes_R N \text{ exact}.$$

**Theorem 2.2.5.**  $R[U^{-1}]$  is always a flat module over  $R$ .

*Proof.* Suppose  $f : M \rightarrow N$  is injective. We need to check  $M[U^{-1}] \rightarrow N[U^{-1}]$  is also injective.

Suppose  $\frac{m}{u} \in M[U^{-1}]$  which goes to 0 in  $N[U^{-1}]$ , then  $\frac{f(m)}{u} = \frac{0}{1}$  in  $N[U^{-1}]$ . This means there exists  $v \in U$  s.t.  $vf(m) = 0$ , which leads to  $vf(m) = f(vm) = 0$ . Since  $f$  is injective,  $vm = 0$  in  $M$ . Then  $\frac{m}{u} = \frac{vm}{vu} = \frac{0}{m} = \frac{0}{1}$ . □

**Example 2.2.6.**  $\mathbb{Q}$  is a flat module over  $\mathbb{Z}$ .  $\mathbb{Z}/2$  is a flat module over  $\mathbb{Z}/6$ . Both  $\mathbb{C}(x)$  and  $\mathbb{C}[x, x^{-1}]$  are flat over  $\mathbb{C}[x]$ .

**Theorem 2.2.7.** A module  $M$  over  $R$  is zero iff for every maximal ideal  $m$ , the localization  $M_m$  is zero.

**Definition 2.2.8.** An  $R$ -module  $M$  is *Noetherian* if every submodule of  $M$  is finitely generated.

**Theorem 2.2.9.** *If*

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

*is exact, then  $M$  is Noetherian (resp. Artinian) iff  $M'$  and  $M''$  are both Noetherian (resp. Artinian).*

### 2.3 Rings and Modules of Finite Length

**Definition 2.3.1.** An  $R$ -module  $M$  is *simple* iff  $M$  has exactly 2  $R$ -submodule: 0 and  $M$ .

**Definition 2.3.2.** A *composition series* for a module  $M$  is a chain

$$0 = M_0 < M_1 < M_2 < \cdots < M_{n-1} < M_n = M$$

of submodules (with strict inclusion) such that for all  $1 \leq i \leq n$ ,  $M_i/M_{i-1}$  is simple.

**Proposition 2.3.3.** *A  $\mathbb{Z}$ -module  $M$  is simple iff it's of the form  $\mathbb{Z}/p$  where  $p$  is prime.*

**Proposition 2.3.4.** *Let  $R$  be an  $R$ -module,  $J$  be an ideal.  $R/J$  is simple iff  $J$  is maximal.*

**Example 2.3.5.** *Let  $R = \mathbb{C}[x, y]$ ,  $M = \mathbb{C}_z[x, y]/(x^2, xy, y^2)$ .*

**Proposition 2.3.6.** *If  $k$  is a field, a compositions series for a vectors space  $V$  exists iff  $V$  is finite dimensional, the sequence always goes from 0 dimension to 1, 2, and grows to the entire thing  $V$ .*

**Definition 2.3.7.** The *length* of an  $R$ -module  $M$  is the minimal length of a compositions series, if one exists, or  $\infty$ . We denote it as  $l(M)$ .

**Theorem 2.3.8.** *Every composition series for  $M$  has the same length.*

Throughout the following, we assume all modules we work with have finite length.

**Proposition 2.3.9.** *If  $N < M \implies l(N) < l(M)$ .*

*Proof.* Choose a composition series for  $N$ :

$$0 < N_1 < N_2 < \cdots < N.$$

We start with a composition series for  $M$  of minimal length:

$$0 < M_0 < M_1 < \cdots < M_n = M.$$

We intersect it with  $N$ :  $N_k = M_k \cap N$ . Then we don't necessarily have strict containment.

$$0 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_n = N.$$

□



**Theorem 2.3.10.** *Every composition series of  $M$  has the same length.*

*Proof.* Suppose we have a composition series

$$0 = M_0 < M_1 < \cdots < M_n = M,$$

with the assumption that  $M_k/M_{k-1}$  simple. Then

$$0 \leq l(M_0) < l(M_1) < \cdots < l(M_n) = l(M).$$

Thus  $n \leq l(M)$ . From the definition of length, we know  $n = l(M)$ .  $\square$

**Proposition 2.3.11.**  $l(M) < \infty$  iff  $M$  satisfies ACC and DCC.

**Definition 2.3.12.** A *finite filtration* of a module  $M$  is a sequence

$$0 = M_0 \leq M_1 \leq M_2 \leq \cdots \leq M_n = M.$$

Associated to a filtration, we have *subquotients*

$$M_k/M_{k-1} = \text{gr}^k(M) \quad \text{grading } k.$$

**Proposition 2.3.13.** Suppose  $M, N$  are modules with filtrations  $\{M_k\}, \{N_k\}$  and a function  $f : M \rightarrow N$  s.t.  $f(M_k) \subseteq N_k$ . Then we get induced module maps

$$\text{gr}^k(M) \rightarrow \text{gr}^k(N) \quad \text{where } [x] \mapsto [f(x)].$$

Also, if all of these are isomorphisms, then so is  $f$ .

**Example 2.3.14.**  $\text{gr}^k(M) \cong 0 \iff M_k/M_{k-1} = 0 \iff M_{k-1} = M_k$ .

**Theorem 2.3.15 (Snake Lemma).** *If (DO THE TIKZ)*

**Proposition 2.3.16.** Suppose  $M$  has a composition series and  $N$  is a submodule with quotient  $M/N$ . We have

$$l(M) = l(N) + l(M/N).$$

Suppose  $M$  has a filtration, we could take the entire sequence and localize it.

**Proposition 2.3.17.** If  $R$  is Noetherian, then so is any quotient  $R/J$  and any localization  $S^{-1}R$ .

Subrings of Noetherian ring are not necessarily Noetherian.

**Theorem 2.3.18.** If  $R$  is Noetherian, then so is  $R[x]$ .

**Definition 2.3.19.** If we have a polynomial  $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ ,  $a_n \neq 0$ ,  $a_i \in R$ , we say  $f$  has *degree*  $n$  and  $a_n$  is its *lead coefficient*

**Theorem 2.3.20 (Hilbert basis theorem).** If  $R$  is Noetherian, then so is  $R[x]$ .

A natural proposition of the theorem will be the following.

**Proposition 2.3.21.** If  $R$  is Noetherian, so is  $R[x_1, \dots, x_n]$ .

**Theorem 2.3.22.** If  $R$  is Noetherian, so is the power series ring  $R[[x]]$ .

## 3 Primary Decomposition

### 3.1 Associated Primes

**Definition 3.1.1.** Suppose  $M$  is an  $R$ -module. The set of *associated primes* is the collection

$$\text{Ass}(M) = \{P \mid P \leq R \text{ is prime } P = a_n(x) \mid x \in M\}.$$

**Proposition 3.1.2.** *If  $M \neq 0$ , then  $\text{Ass}(M)$  is not empty. In fact, any ideal which is maximal in the set  $\{\text{ann}(x) \mid x \in M, x \neq 0\}$  is prime.*