Combinatorial Theory

November 6, 2023

Contents

1 Chapter 1

1.1 Permutations, Subsets, Multisets

Example 1.1.1. *Suppose n people give their n hats to a hat check. Let $g(n)$ be the number ways hats could be given back so no person receives their own hat.*

Answer.

$$g(n) = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!}.$$

□

Example 1.1.2. *Let $h(n)$ be the number of domino tilings of a $2 \times n$ rectangle using 2×1 rectangles.*

Answer. 1. For all $n \geq 3$, $h(n) = h(n-1) + h(n_2)$.

2. Using rational generating function associated to linear recurrence relations:

$$h(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

□

Definition 1.1.3. Let S be a finite set. A k-permutation of S is a sequence $(s_1, s_2, \ldots, s_k)$ as long as $k \le |s|$.

The number of k-permutation of $[n]$ is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}, \quad \text{denoted by } (n)_k \text{ or falling factorial.}$$

Definition 1.1.4. Let $\binom{n}{k}$ denote the number of subsets of $[n]$ of size k.

Theorem 1.1.5 (Sagan 1.3.2)**.**

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(n)_k}{k!}.$$

Theorem 1.1.6 (Sagan 1.3.3)**.** *We have*

1. $$\binom{0}{0} = 1 \quad \binom{0}{k} = 0.$$

2. $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

3. $$\binom{n}{k} = \binom{n}{n-k}.$$

4. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

5. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1, & n = 0 \\ 0, & n \ge 1 \end{cases}.$$

1.2 Generating Functions

Given a numerical sequence

$$a_0, a_1, a_2, a_3, \ldots$$

The ordinary generating function is

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Note: $k[[x]]$ is a local ring.

Claim: $A(x)$ is invertible if and only if $a_0 \neq 0$.

Let

$$A_m(x) = \sum_{n=0}^{m} x^n.$$

Then

$$A(x)(1-x) = \lim_{m\to\infty} A_m(x)(1-x) = 1.$$

Two generating functions are the same if they converge to each other.

Theorem 1.2.1 (Binomial Theorem)**.**

$$\sum_{k\geq 0} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

We first do some disambiguating. We use multivariables instead of just one.

$$\begin{aligned}(1+x_1)(1+x_2)\cdots(1+x_n) &= \sum_{1\leq i_1<i_2\cdots i_k\leq n} x_{i_1}x_{i_2}\cdots x_{i_k} \\ &= \sum_{T\subseteq[n]} \prod_{i\in T} x_i \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k\end{aligned}$$

Definition 1.2.2. Let α be any complex number, k non-negative integer. We define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.$$

Consider the genreating function of $\binom{-3}{k}$.

$$\binom{-3}{0} = 1, \binom{-3}{1} = -3, \binom{-3}{2} = 6, \binom{-3}{3} = -10, \ldots.$$

First note that

$$\sum_{n\geq 0} \binom{-3}{n} x^n = \sum_{n\geq 0} (-1)^n \frac{(n+2)(n+1)}{2} x^n.$$

Then do some differenciation to $\frac{1}{1-x}$ we'll eventually be

$$(1+x)^{-3}.$$

Theorem 1.2.3 (Generalized Binomial Theorem)**.**

$$\sum_{k\geq 0}\binom{\alpha}{k}x^k = (1+x)^\alpha.$$

This could be proved/shown by doing taylor series expansions.

Definition 1.2.4. n multichoose k is the number of ways of choosing a multiset from $[n]$ of size k. Denoted by

$$\left(\binom{n}{k}\right).$$

Example 1.2.5.

$$\left(\binom{3}{2}\right) = \#\{11, 12, 13, 22, 23, 33\} = 6.$$

Theorem 1.2.6.

$$\left(\binom{n}{k}\right) = \binom{n+k-1}{k}.$$

Theorem 1.2.7.

$$\sum_{k\geq 0}\left(\binom{n}{k}\right)x^k = (1-x)^{-n} \quad or \quad \left(\frac{1}{1-x}\right)^n.$$

Recall $h(n)$ is the number of tilings of a $2\times n$ rectangle.

$$h(n) = \sum_{k=0}^{\frac{n}{2}}\binom{n-k}{k}$$
$$H(x) = \sum_{n\geq 0}h(n)x^n$$
$$H(x) = \frac{1}{1-x-x^2}$$

Example 1.1.13, 1.1.15 from Stanely.

Definition 1.2.8. A *composition* of $[n]$ is an ordered sum of positive integers that sum to n. *k-composition* has exactly k parts.

The number of k-compositions of $[n]$ is $\binom{n-1}{k-1}$ and the number of compositions is 2^{n-1}.

Definition 1.2.9. *Multinomial coefficients* are

$$\binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1! a_2! \cdots a_m!} = \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_1-\cdots-a_{m-1}}{a_m}$$

Definition 1.2.10. A permutation written in *cycle notation*:

1. each cycle has the largest element first
2. cycles arranged in increasing order by 1-st element.

Definition 1.2.11. Given $w \in S_n$, let $c_i(w)$ be the number of i-cycles in w. We define *cycle type* of w to be $(c_1, c_2, \ldots, c_n)$.

Proposition 1.2.12. *The number of permutations in S_n with cycle type $(c_1, c_2, \ldots, c_n)$ is edqual to*

$$\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}.$$

Definition 1.2.13. We define *cycle index polynomial* of S_n to be

$$Z_n(t_1, \ldots, t_n) := \frac{1}{n!} \sum_{w \in S_n} t^{\mathrm{type}(w)}$$

Theorem 1.2.14.

$$\sum_{n \geq 0} z_n x^n = \exp(t_1 x + t_2 \frac{x^2}{2} + \cdots) = \exp\left(\sum_{n \geq 1} t_n \frac{x^n}{n}\right).$$

1.3 Stirling Numbers

Stanely 1.3, 1.9

Segan 1.4, 1.5

Recall

$$z_n(t_1, t_2, \ldots, t_n) = \frac{1}{n!} \sum_{w \in S_n} t^{\mathrm{type}(w)}.$$

Definition 1.3.1. Let $c(n,k)$ be the number of permutations w of S_n with exactly k cycles.

Proposition 1.3.2 (Prop 1.3.7)**.**

$$\sum_{k=0}^{n} c(n,k) t^k = t(t+1)(t+2)\cdots(t+n-1).$$

Proof.

$$\begin{aligned}
\sum_{n=0} \left(\sum_{k=0}^{n} c(n,k) t^k \right) \frac{x^n}{n!} &= \exp(t \sum_{n \geq 1} \frac{x^n}{n}) \\
&= \exp\left(t \log \left(\frac{1}{1-x} \right) \right) \\
&= \exp\left(\log(1-x)^{-t} \right) \\
&= (1-x)^{-t} \\
&= \sum_{n \geq 0} (-1)^n \binom{-t}{n} x^n \\
&- \sum_{n \geq 0} \frac{t(t+1)(t+2)\cdots(t+n-1)x^n}{n!}
\end{aligned}$$

□

Lemma 1.3.3 (Lem 1.3.6). *The $c(n,k)$'s staisfy the recurrence*

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$$

for $n, k \geq 1$.

Proof. Building up an permutation. Build one in S_n using one in S_{n-1}.

1. Our perm $w \in S_n$ has n as a fixed point: has (n) as a 1-cycle. Build the rest of w by any p ermutation of S_{n-1} with $(k-1)$ cycles.
2. Our permutation $w \in S_n$ has element n in a cycle of length ≥ 2. Build by drawing diagraph of a perm on S_{n-1} and changing one arrow.

□

Definition 1.3.4. We define the *stirling number of first kind* to be

$$s(n,k) = (-1)^{n-k} c(n,k).$$

Definition 1.3.5. We define the *stirling number of second kind* to be

$$s(n,k) = \text{number of set of partition of } [n] \text{ into } k \text{blocks}.$$

Theorem 1.3.6 (Thm 1.4.2 Segan).

$$s(0,k) = \delta_{0,k} = \begin{cases} 1, & k = 0 \\ 0, & \textit{otherwise} \end{cases}.$$

and

$$s(n,k) = s(n-1,k) + ks(n-1,k-1) \textit{ for } n, k \geq 1.$$

Definition 1.3.7. Let $B(n)$ be the number of set partitions of $[n]$ regardless of the number of blocks.

$$B(n) = \sum_{k=1}^{n} s(n,k).$$

Theorem 1.3.8 (Theorem 1.4.1)**.** *$B(n)$ is defined by $B(0) = 1, B(n) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} B(n-k)$ for $n \geq 1$.*

1.4 Twelve Fold Way

Stanely 1.9

1.5 Integer Partitions

Let lowercase $p(n)$ equals the number of Partitions of size n. Let $p(n,k)$ be the number of partitions of n with $\leq k$ parts, which Stanley denotes as $p_k(n)$.

Theorem 1.5.1 (Theorem 1.6.2)**.** *$p(n,k)$ defined by*

$$p(0,k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases} \quad \textit{and} \quad p(n,k) = p(n-k,k) + p(n,k-1).$$

1.6 Permutation Statistics

Stanley 1.3-1.4 Sagan 3.2

Theorem 1.6.1 (Sagan Theorem 3.2.1)**.**

$$\sum_{w \in S_n} q^{inv(w)} = (1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}) = [n]_q!.$$

Definition 1.6.2. The inversion table $I(w)$ for a permutation $W \in S_n$ is

$$I(w) = (b_1, b_2, \ldots, b_n),$$

such that b_i is the number of (j,i) such that $i < j$, $w^{-1}(j) < w^{-1}(i)$.

Proposition 1.6.3 (Cor 1.3.13)**.**

$$\sum_{w \in S_n} q^{inv(w)} = \sum_{b_1=0}^{n-1} \sum_{b_2=0}^{n-2} \cdots \sum_{b_{n-1}=0}^{1} \sum_{b_n=0}^{0} q^{b_1+b_2+\cdots+b_n}.$$

This also equivalent to

$$\sum_{w \in S_n} q^{inv(w)} = \left(\sum_{b_1=0}^{n-1} q^{b_1}\right)\left(\sum_{b_2=0}^{n-2} q^{b_2}\right)\cdots\left(\sum_{b_n=0}^{0} q^{b_n}\right) = [n]_q[n-1]_q\cdots[2]_q[1]_q.$$

Definition 1.6.4. We say *descents* of w as i such that $w_i > w_{i+1}$.

Definition 1.6.5. We say *major index* of w as

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

Theorem 1.6.6 (Sagan Thm 3.2.2)**.**

$$\sum_{w \in S_n} q^{\mathrm{maj}(w)} = [n]_q!.$$

Definition 1.6.7. Given a permutation w, we define *$des(w)$* to be the number of descents of w. The generating function is

$$A_n(x) := \sum_{w \in S_n} x^{1+des(w)}$$

Definition 1.6.8. *exceedance* of a permutation is

$$exc(w) := \{i \mid i < w(i)\}.$$

and *weak exceedance* is

$$wexc(w) := \{i \mid i \leq w(i)\}.$$

Proposition 1.6.9 (Sagan 4.2.3)**.**

$$A_n(x) = \sum_{w \in S_n} x^{1+exc(w)} = \sum_{w \in S_n} x^{wexc(w)}.$$

Theorem 1.6.10 (3.2.6 Sagan)**.** *If V is a vector space over $\mathbb{F}_q$ where $q = p^k$ for a prime p, of dimension n, then then number of k-dimensional subspaces of V is $\binom{n}{k}_{q=p^k}$.*

1.7 Euler's Pentagonal Number Theorem

Theorem 1.7.1.

$$\prod_{k \geq 1}(1 - x^k) = 1 + \sum_{n \geq 1}(-1)^n x^{\frac{n(3n-1)}{2}} + \sum_{n \geq 1}(-1)^n x^{\frac{n(3n+1)}{2}}.$$

Proof. See Stanley Page 76. □

2

2.1 Ferrers Boards

Theorem 2.1.1 (Stanley Thm 2.4.1)**.** *Let $\sum r_k x^k$ be the rook polynomial of the Ferrers board B of shape $(b_1, \ldots, b_m)$. Set $s_i = b_i - i + 1$. Then*

$$\sum_k r_k \cdot (x)_{m-k} = \prod_{i=1}^{m} (x + s_i).$$

Corollary 2.1.2 (Stanley Cor 2.4.2)**.** *Let B be the triangular board (or staircase) of shape $(0, 1, 2, \ldots, m-1)$. Then $r_k = S(m, m-k)$, the 2nd stirling number, the number of set partitions of $[m]$ into $(m-k)$ blocks.*

Sagan sec 2.2 and 2.4

Definition 2.1.3. Given a set S, a function $f \to f$ is an *involution* iff $f \circ f = \mathrm{id} : S \to S$.

Definition 2.1.4. $f : S \to S$ is *sign-reversing involution* if $\operatorname{sgn}(f(s)) = -\operatorname{sgn}(s)$ unless s is a fixed point.

Then

$$\sum_{s\in S} \operatorname{sgn}(s) = \sum_{s\in Fix(f)} \operatorname{sgn}(s).$$

Section 2.4, Andre's reflection principle.

2.2 Lindström–Gessel–Viennot lemma

Given an $n \times n$ matrix $M = (m_{ij})$. We can represent it in a directed weighted and bipartite graph with vertices $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, and edges $A_i \to B_j$ with weight m_{ij}.

Definition 2.2.1. A *path* in a graph is a sequence $v_1e_1v_2e_2\cdots e_nv_n$.

The goal is to give a combinatorial interpretation for matrix determinant in terms of these graphs.

Definition 2.2.2. The *determinant* of a matrix M is

$$\det(M) = \sum_{\sigma\in S_n} sgn(\sigma)m_{1\sigma(1)}\cdots m_{n\sigma(n)}.$$

Recall $sgn(\sigma) = (-1)^{\#inv(\sigma)}$.

A *path system* $\mathcal{P}$ with permutation σ in a graph G is a collection of paths

$$P_i : A_i \to B_{\sigma(i)}.$$

We say $\mathcal{P}$ is *vertex disjoint* if distinct paths don't share vertices.

A path system $\mathcal{P}$ has weights

$$w(\mathcal{P}) = \prod w(P_i).$$

Now we have an alternative definition for determinant:

$$\det(M) = \sum_{\sigma\in S_n} sgn(\sigma)w(\mathcal{P}_\sigma).$$

Proposition 2.2.3.

$$\det(M) = \det(M^T).$$

We could have a graph-based proof for this familiar statement from linear algebra.

Proof. Notice that $sgn(\sigma) = sgn(\sigma^{-1})$. □

Let $G = (V, E)$ be a finite acyclic directed graph. Note that G is acyclic means that there are finitely many directed paths between any 2 vertices.

We'll give each edge e a weight $w(e)$. Let P be a directed path from A to B, then the weight of P is the product of weights of edges in P.

Suppose $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ are two subsets of V. They don't have to be disjoint.

To $\mathcal{A}, \mathcal{B}$, there is an associated path matrix $M = (m_{ij})$ where

$$m_{ij} = \sum_{P: A_i \to B_j} w(P).$$

We denote VD as the family of vertex disjoint path systems.

Lemma 2.2.4 (LGV Lemma)**.**

$$\det M = \sum_{\mathcal{P} \in VD} sgn(\mathcal{P}) w(\mathcal{P}).$$

A *spanning tree* in G is a connected acyclic subgraph using all vertices in G.

We define *Laplacian matrix* of a graph G as a matrix $L(G)$ whose i-row j-column element is negative number of edges from v_i to v_j if $i \neq j$, the degree v_i if otherwise.

Theorem 2.2.5 (Matrix Tree Theorem (Kirchoff's))**.** *Let $G = (V, E)$ be an undirected graph first. The absolute value of the determinant of the reduced Laplacian matrix (crossed out one row / one column) $L_0(G)$ equals to the number of spanning trees in G, which equals to connected acyclic subgraphs touching every vertex of G.*

Claim

$$\det L(G) = 0.$$

where it's unreduced.

2.3 Matrix Tree Theorem for Directed Graphs

When G is directed, the definition of Laplacian matrix turns into: a *Laplacian matrix* of a directed graph G as a matrix $L(G)$ whose i-row j-column element is negative number of edges from $v_i \to v_j$ if $i \neq j$, the out-degree v_i if otherwise.

First notice that $L(G)$ is not symmetric. More importantly, $\det L_0(G)$ depends on vertex index of row and column deleted.

Now, we have $\det(L_0(G))$ w.r.t. vertex v_i equals to the number of rooted directed spanning trees into v_i.

Now we will prove the Matrix Tree Theorem using Cauchy-Binet Theorem.

Theorem 2.3.1 (Cauchy-Binet THeorem)**.** *FOr $m \leq n$, Q a $m \times n$ matrix, and R a $n \times m$ matrix, then*

$$\det(QR) = \sum_{S \in \binom{[n]}{m}} \det Q_{[m],s} \cdot \det R_{s,[m]}.$$

Theorem 2.3.2 (Directed Matrix Tree Theorem)**.**

$$\det L_0^{out}(G) = \sum_T \mathrm{wt}(T).$$

where L_0 deletes the n-th row, n-th column, and T's are in-tree rooted at vertex n.

2.4 Rational Generating Functions and Linear Recursions

Example 2.4.1. *Let $a_0 = 1, a_1 = -4$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Define*

$$f(x) = \sum_{n\geq 0} a_n x^n.$$

Then we have

$$\begin{aligned} f(x) - a_0 - a_1 x &= \sum_{n\geq 2} a_n x^n \\ f(x) - 1 + 4x &= 4x(f(x) - 1) - 4x^2 f(x) \\ f(x) &= \frac{1-8x}{(1-2x)^2} = \frac{4}{1-2x} - \frac{3}{(1-2x)^2} \end{aligned}$$

Theorem 2.4.2.

$$\frac{1}{(1-rx)^a} = \sum_{n\geq 0} \binom{n+a-1}{a-1} r^n x^n.$$

Definition 2.4.3 (Segan 3.6)**.** Let $(a_n), n \geq 0$ be a sequence of complex numbers. We say that the sequence satisfies a *homogeneous linear recursion of degree d with constant coefficients* if there is $d \in \mathbb{Z}_+$ and constants $c_1, \dots, c_d \in \mathbb{C}$ with $c_d \neq 0$ such that

$$a_{n+d} + c_1 a_{n+d-1} + c_2 a_{n+d-2} + \cdots + c_d a_n = 0.$$

Theorem 2.4.4. *Given a sequence (a_n) satisfied the definition above, and $d \in \mathbb{Z}_+$. Let $q(x) = 1 + c_1 x + c_2 x^2 + \cdots + c_d x^d$. TFAE*

1. *The sequence is homogenous with linear recursion of degree d with constant coefficients*
2. *The generating function $f(x) = \sum_{n\geq 0} a_n x^n$ has the form*

   $$f(x) = \frac{p(x)}{q(x)}$$

   and degree $p(x) < d$.
3. *We can write $a_n = \sum_{i=1}^k p_i(n) r_i^n$ where r_i are distinct non zero complex numbers satisfying*

   $$q(x) = \prod_{i=1}^k (1 - r_i x)^{d_i}.$$

   And $p_i(n)$ is a polynomial with degree $p_i(n) < d_i$ for all i.

2.5

Theorem 2.5.1 (the BEST theorem)**.** *If G is a digraph that satisfies indeg = outdeg at every vertex, then the number of Eulerian cycles equals to the number of intress rooted at v times $\prod_{w \in V}(outdeg(w) - 1)!$.*

Definition 2.5.2. A *binary de bruijn sequences* of degree n is a sequence of 0's and 1's of length 2^n:

$$a_1 a_2 \cdots a_{2^n}.$$

Looking at circular windows of length n, we see all possible binary sequences of length n.

Notice that BDBS are really Eulaerian cycles! Denote the corresponding graph as D_n.

Claim: eigenvalues of $L(D_n)$ are $0, 2, 2, 2, \ldots, 2$ for $2^{n-1} - 1$ times.

The number of binary de Bruijn sequences of degree n is

$$\frac{1}{2^{n-1}} 2^{2^{n-1}-1} = 2^{2^{n-1}-n}.$$

Similarly, th number of k-ary dBS of degree n is

$$k^{k^{n-1}-n} \cdot (k-1)!^{k^{n-1}}.$$

2.6 Chromatic polynomials

Let G be an undirected simple graphs (no multiple edges, no loops) Denote $G = (V, E)$.

Definition 2.6.1. A *coloring* of G is a map $c : V \to S$ where S is the set of colors. A coloring is *proper* if $c(u) \neq c(v)$ when $(u, v) \in E$.

Definition 2.6.2. The *chromatic number* of G, denoted by $\chi(G)$, is the minimal cardinality of S such that there's a proper coloring.

An edgeless graph has $\chi(G) = 1$ and a bipartite graph has $\chi(G) = 2$.