Lie Theory

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1 Background

1.1 Topology

Definition 1.1.1. A topological space is *locally connected* at point x if everyneighborhood of x contains a connected open neighborhood.

2 Topologcial Groups

2.1 Introduction

Definition 2.1.1. A *topological group* is a group such that

1. he product $p : G \times G \to G, p(g, h) = gh$, is a continuous map if $G \times G$ has the product topology;
2. The map $\iota : G \to G, \iota(g) = g^{-1}$, is continuous (hence, a homeomorphism, as $\iota^{-1} = \iota\big)$.

Each element $g \in G$ defines the following maps.

- *left translation*: $L_g : G \to G, L_g(h) = gh$;
- *right translation*: $R_g : G \to G, R_g(h) = hg$;
- *conjugation*: $C_g : G \to G, C_g(h) = ghg^{-1}$.

2.2 Neighborhoods of Identity

An (open) neighborhood of $x \in X$, where X is a topological space, is an open set U that coitains x.

Let G be a topological group, and $1 \in G$ is the identity. $V(1)$ refers to the set of all neighborhoods of 1.

Proposition 2.2.1 (Proposition 2.2)**.** *Let G be a t.g. (topological group), $V = V(1)$. Then we'll have*

1. *(T1)for all $u \in V, 1 \in u$;*
2. *(T2)$u, v \in V \implies u \cap v \in V$;*
3. *(TG1) for all $u \in V$, there exists $v \in V$ s.t. $v^2 \subseteq u$;*
4. *(TG2) $u \in V \implies u^{-1} \in V$;*
5. *(TG3) $u \in V, g \in G \implies gug^{-1} \in V$.*

Definition 2.2.2. Let G be a group, not necessarily topological group. A system of neighborhood of $1 \in G$ is a family of sets sastisfying (T1) to (TG3).

Definition 2.2.3. Let X be a topological space and $x \in X$. A fundamental system of neighborhoods of x is a family F of open sets containing x s.t. for all open u that contains x, there exists $v \in F$ s.t. $v \subseteq u$.

Theorem 2.2.4 (Proposition 2.5)**.** *Let G be an abstract group, V be a system of neighborhoods of 1. There exists a unique topology on G making G into a topological group and s.t. V is a fundamental system of neighborhoods of 1.*

idea of proof. □

Proposition 2.2.5. *Let G be a topological group. TFAE*

1. *topology of G is a Hausdorff*
2. *$\{1\}$ is closed in*
3. *$\bigcap_{U \in \mathcal{V}(1)} U = \{1\}$*

2.3 Metrizable Groups

Definition 2.3.1. Let G be a topological group. G is metrizable if it has a left-(or right-) invariatn distance which defines the tooplogy left-invariant for all $g \in G$ and $d(gx, gy) = d(x, y)$ for all $x, y \in G$.

Theorem 2.3.2. *A topological grouop G is metrizable iff it has a countable system of neighborhoods of 1.*

2.4 Homomorphisms

We need to talk about $G \to H$ continuous homomorphisms.

Example 2.4.1. *The determinant homomorphism* $\det : GL_n(\mathbb{R}) \to \mathbb{R}^* = GL(1, \mathbb{R})$ *is continuous.*

Theorem 2.4.2. *Let G, H be topological group. A group homomorphism $\phi : G \to H$ is continuous iff ϕ is continuous at $1 \in G$.*

Proof. $\implies$ is obvious. Let's look at the other direction.

Note that $\phi \circ L_g = L_{\phi(g)} \circ \phi$ as maps $G \to H$ because

$$(\phi \circ L_g)(g') = \phi(gg') = \phi(g)\phi(g') = (L_{\phi(g)} \circ \phi)(g').$$

Then

$$\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$$

is continuous at g, as $L_{g^{-1}}$ is continuous at g, ϕ continuous at 1, and $L_{\phi(g)}$ continuous everywhere. $\square$

Theorem 2.4.3. *A map $\phi : G \to H$ is a group homomorphism (G, H are just grouops) iff*

$$gr(\phi) := \{(g, \phi(g)) \mid g \in G\} \subseteq G \times H.$$

Proposition 2.4.4. *Let X and Y be topological spaces, such that Y is Hausdorff. A map $\phi : X \to Y$ is continuous if and only if its graph $gr(\phi)$ is closed and the projection $p(x, \phi(x)) = x$ is a homeomorphism.*

Proof. Suppose ϕ is continuous. Then

$$gr(\phi) = \theta^{-1}(\Delta y) \text{ w.r.t. } \theta : X \times Y \to Y \times Y$$

is closed, since *tehta* is continuous and Δy is closed. $\square$

Theorem 2.4.5. *Suppose G, H are topological groups, H is Hausdorff. The map $G \to H$ is a continuous homomorphism iff $gr(\phi)$ is a closed subgroup and $p : gr(\phi) \to G$ is a homeomorphism.*

2.5 Subgroups

Let G be a topological group. $H \subseteq G$ is a *topological subgroup* if H is a topological group w.r.t. the induced topology.

Proposition 2.5.1. *Let G be a topological group. If $H \subseteq G$ a subgroup, which is open. Then H is also closed.*

Proof. Consider

$$Y = \bigcup_{g \in G - H} gH.$$

Y is open, as it is a union of open sets. H is also closed, as $G - Y = H$. Hence, H is closed. $\square$

Proposition 2.5.2. *G a topological group, $H \subseteq G$ a subgroup. Then $\overline{H}$ is also a subgroup of G.*

Proof. Note that $A \subseteq X$ (subset of a topological space), $x \in \overline{A}$ iff for all open U that contains x, $U \cap A \neq \emptyset$. Then we check the followings.

1. $\overline{H}$ is closed under $m : G \times G \to G$.

$\square$