Lie Theory

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Contents

1 Background

1.1 Topology

Definition 1.1.1. A topological space is *locally connected* at point x if everyneighborhood of x contains a connected open neighborhood.

2 Topologcial Groups

2.1 Introduction

Definition 2.1.1. A *topological group* is a group such that

1. he product $p : G \times G \to G, p(g, h) = gh$, is a continuous map if $G \times G$ has the product topology;

2. The map $\iota : G \to G, \iota(g) = g^{-1}$, is continuous (hence, a homeomorphism, as $\iota^{-1} = \iota$).

Each element $g \in G$ defines the following maps.

- *left translation*: $L_g : G \to G, L_g(h) = gh$;
- *right translation*: $R_g : G \to G, R_g(h) = hg$;
- *conjugation*: $C_g : G \to G, C_g(h) = ghg^{-1}$.

2.2 Neighborhoods of Identity

An (open) neighborhood of $x \in X$, where X is a topological space, is an open set U that cointains x.

Let G be a topological group, and $1 \in G$ is the identity. $V(1)$ refers to the set of all neighborhoods of 1.

Proposition 2.2.1 (Proposition 2.2)**.** *Let G be a t.g. (topological group), $V = V(1)$. Then we'll have*

1. *(T1)for all $u \in V, 1 \in u$;*
2. *(T2)$u, v \in V \implies u \cap v \in V$;*
3. *(TG1) for all $u \in V$, there exists $v \in V$ s.t. $v^2 \subseteq u$;*
4. *(TG2) $u \in V \implies u^{-1} \in V$;*
5. *(TG3) $u \in V, g \in G \implies gug^{-1} \in V$.*

Definition 2.2.2. Let G be a group, not necessarily topological group. A system of neighborhood of $1 \in G$ is a family of sets sastisfying (T1) to (TG3).

Definition 2.2.3. Let X be a topological space and $x \in X$. A fundamental system of neighborhoods of x is a family F of open sets containing x s.t. for all open u that contains x, there exists $v \in F$ s.t. $v \subseteq u$.

Theorem 2.2.4 (Proposition 2.5)**.** *Let G be an abstract group, V be a system of neighborhoods of 1. There exists a unique topology on G making G into a topological group and s.t. V is a fundamental system of neighborhoods of 1.*

idea of proof. □

Proposition 2.2.5. *Let G be a topological group. TFAE*

1. *topology of G is a Hausdorff*
2. *$\{1\}$ is closed in*
3. *$\bigcap_{U \in \mathcal{V}(1)} U = \{1\}$*

2.3 Metrizable Groups

Definition 2.3.1. Let G be a topological group. G is metrizable if it has a left-(or right-) invariatn distance which defines the tooplogy left-invariant for all $g \in G$ and $d(gx, gy) = d(x, y)$ for all $x, y \in G$.

Theorem 2.3.2. *A topological grouop G is metrizable iff it has a countable system of neighborhoods of* 1.

2.4 Homomorphisms

We need to talk about $G \to H$ continuous homomorphisms.

Example 2.4.1. *The determinant homomorphism* $\det : GL_n(\mathbb{R}) \to \mathbb{R}^* = GL(1, \mathbb{R})$ *is continuous.*

Theorem 2.4.2. *Let G, H be topological group. A group homomorphism $\phi : G \to H$ is continuous iff ϕ is continuous at $1 \in G$.*

Proof. $\implies$ is obvious. Let's look at the other direction.

Note that $\phi \circ L_g = L_{\phi(g)} \circ \phi$ as maps $G \to H$ because

$$(\phi \circ L_g)(g') = \phi(gg') = \phi(g)\phi(g') = (L_{\phi(g)} \circ \phi)(g').$$

Then

$$\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$$

is continuous at g, as $L_{g^{-1}}$ is continuous at g, ϕ continuous at 1, and $L_{\phi(g)}$ continuous everywhere. $\square$

Theorem 2.4.3. *A map $\phi : G \to H$ is a group homomorphism (G, H are just grouops) iff*

$$gr(\phi) := \{(g, \phi(g)) \mid g \in G\} \subseteq G \times H.$$

Proposition 2.4.4. *Let X and Y be topological spaces, such that Y is Hausdorff. A map $\phi : X \to Y$ is continuous if and only if its graph $gr(\phi)$ is closed and the projection $p(x, \phi(x)) = x$ is a homeomorphism.*

Proof. Suppose ϕ is continuous. Then

$$gr(\phi) = \theta^{-1}(\Delta y) \text{ w.r.t. } \theta : X \times Y \to Y \times Y$$

is closed, since *tehta* is continuous and Δy is closed. $\square$

Theorem 2.4.5. *Suppose G, H are topological groups, H is Hausdorff. The map $G \to H$ is a continuous homomorphism iff $gr(\phi)$ is a closed subgroup and $p : gr(\phi) \to G$ is a homeomorphism.*

2.5 Subgroups

Let G be a topological group. $H \subseteq G$ is a *topological subgroup* if H is a topological group w.r.t. the induced topology.

Proposition 2.5.1. *Let G be a topological group. If $H \subseteq G$ a subgroup, which is open. Then H is also closed.*

Proof. Consider

$$Y = \bigcup_{g \in G-H} gH.$$

Y is open, as it is a union of open sets. H is also closed, as $G - Y = H$. Hence, H is closed. □

Proposition 2.5.2. *G a topological group, $H \subseteq G$ a subgroup. Then $\overline{H}$ is also a subgroup of G.*

Proof. Note that $A \subseteq X$ (subset of a topological space), $x \in \overline{A}$ iff for all open U that contains x, $U \cap A \neq \emptyset$. Then we check the followings.

1. $\overline{H}$ is closed under $m : G \times G \to G$.

□

2.6 Connected Components of Topological Groups

A *connected space* cannot be written as the union of two disjoint open sets.

A *connected component* of a point $x \in X$ is the union of all connected sets containing x, which is also the maximal connected set containing x.

A *connected component* of X is a maximal connected subset.

If $A \subseteq X$ is connected, then the closure $\overline{A}$ is connected. Thus, every connected component is closed.

Let G be a topological group, G_0 is the connected component of $1 \in G$.

Proposition 2.6.1. *G_0 is a closed normal subgroup of G. The connected components of G are exactly gG_0 for $g \in G$.*

A *neighborhood* N of $x \in X$ is a subset $N \subseteq X$, $x \in N$ and there exists an open $U \subseteq X$ s.t $x \in U \subseteq N$.

A space is *locally connected* if for every open neighborhood of every point contains a connected open neighborhood.

Proposition 2.6.2. *If G is locally conencted, then G_0 is open.*

Proposition 2.6.3. *If G connected, $U \in \mathcal{V}(1)$, then $G = \cup_{n \geq 1} U^n$.*

2.7 Group Action

Suppose G a group, X a set.

Definition 2.7.1. A *left action* of a group G on a set X is a function that associates to $g \in G$ a map $a(g) : X \to X$ which satisfies the properties: 1. $a(1) = \mathrm{id}_X$, that is, $a(1)(x) = x$, for every $x \in X$; 2. $a(gh) = a(g) \circ a(h)$.

Definition 2.7.2. Let $\phi_x : G \times X \to X, \phi_y : G \times Y \to Y$. A map $f : X \to Y$ is *G-equivariant* if

$$\phi_y(g, f(x)) = f(\phi_x(g, x)).$$

Same stroy for topological groups.

Definition 2.7.3. Let G be a topological group, X a topological space, an *action* G on X should be continuous. In other words, G acts on X by homeomorphisms ϕ_g.

Action is *transitive* if $X = Gx$ for some $x \in X$. We define the *orbit* of x to be $Gx = \{gx \mid g \in G\}$. A *stabilizer* or *isotropy subgroup* of x is $G_x = \{g \in G \mid gx = x\}$.

An action is an *effective action* or *faithful* if $gx = x, \forall x \in X \implies g = 1$, equivalently, $\cap_{x\in X} G_x = \{1\}$.

Proposition 2.7.4.

$$G/G_x \to X \quad \textit{where} \quad gG_x \mapsto gx.$$

This map is equivariant.

Proposition 2.7.5. *Suppose that the action of G on X is continuous and that X is a Hausdorff space. Then, any isotropy subgroup $G_x, x \in X$, is closed.*

2.8 Homogeneous Spaces

Let G be a topological group.

Definition 2.8.1. A *homogeneous G-space* is just G/H for a subgroup H of G.

Definition 2.8.2. A topological space X without regards to group is *homogeneous* if for all $x, y \in X$, there exists a homeomorphism $\phi : X \to X$ s.t. $\phi(x) = y$.

Topology on G/H is that of a quotient: $\pi : G \to G/H$. In other words, $U \subseteq G/H$ open if $\pi^{-1}(U) \subseteq G$ open.

Note: action of G on G/H is continuous:

$$G \times G/H \to G/H \quad \text{where} \quad (x, gH) \mapsto xgH.$$

Proposition 2.8.3. *We have the following facts.*

1. *G/H is a homogeneous space in the sense of topology.*

2. *$\pi : G \to G/H$ is an open map (it takes open sets to open sets).*
3. *H compact implies that π is a closed map.*
4. *G/H is Hausdorff iff H is closed.*
5. *G/H discrete iff H open. (HW2)*
6. *If G is compact, G/H discrete and finite iff H is open.*
7. *$H \lhd G$ implies G/H is a topological group.*
8. *$H := \overline{\{1\}}$. THen H is a normal subgroup of G, and G/H is Hausdorff topological group.*

Proof of 1. Consider left translation

$$L_x : gH \mapsto xgH.$$

This is a homeomorphism since $L_{x^{-1}}$ is an inverse and both are continuous. □

Proof of 2. We need to show that $\pi^{-1}\pi(U)$ is open. (Omitted, just do image preimage and write it as union of right cosets). □

Proof of 3. Take $F \subseteq G$ closed, if H is a compact subset, then $FH \subseteq G$ is closed. (From a proposition from textbook).

Notice that $\pi(F)$ closed iff $\pi^{-1}\pi(F)$ closed, and the latter equals to FH. □

Proof of 4. We first show $\implies$. Note that $H = \pi^{-1}(H)$, which is a point of G/H, so it's closed. Thus H is closed.

Then we show $\Leftarrow$. Consider the homeomorphism

$$f : G/H \times G/H \to G \times G/H \times H \quad \text{where} \quad (g_1 H, g_2 H) \mapsto (g_1, g_2)H \times H.$$

Denote $\Delta = \{(gH, gH)\}$. Then $f(\Delta) = \{(g, g)H \times H\}$ is closed iff $\pi^{-1}_{G\times G} f(\Delta)$ is closed, which equals to $\{(g_1, g_2) \mid g_1 H = g_2 H\} = \{(g_1, g_2) \mid g_1^{-1} g_2 \in H\}$. □

Let G be a topological group, $H \subseteq G$ a subgroup.

Proposition 2.8.4. *If H and G/H are compact, then so is G.*

Proof.

$$\pi : G \to G/H$$

is a *perfect map*, i.e., a continuous subjective closed map with compact fibers $\pi^{-1}(x), \forall x \in G/H$. □

Proposition 2.8.5. *If G/H and H are connected, then so is G.*

Proof. Suppose G is not connected, then there exists $A \bigsqcup B = G$, $A, B \neq \emptyset$ open, disjoin $\subseteq G$. Then $\pi(A), \pi(B) \neq \emptyset$, open because π is always open, $\pi(A) \cup \pi(B) = G/H$, which is connected. Therefore $\pi(A) \cap \pi(B) \neq \emptyset$. Thus there exsits $gH \in G/H$ s.t. $gH \cap A \neq \emptyset$ and $gH \cap B \neq \emptyset$. □

2.9 Orbits and Homogeneous Spaces

Homogeneous space G/G_x, we hav ea bijection:

$$G/G_x \to G \cdot x \quad \text{where} \quad gG_x \mapsto gx.$$

Proposition 2.9.1. *Let $G \times X \to X$ be a continuous and transitive action of G on X. Fix $x \in X$ and consider the bijection*

$$\xi_x : G/G_x \to X \text{ given by } \xi_x(gG_X) = gx.$$

Then ξ_x is continuous with respect to the quotient topology in G/G_x.

Proposition 2.9.2. *Let $G \times X \to X$ be a topological transitive group action. Suppose G is locally compact and spearable (i.e., has a countable dense subset) and X is Hausdorff and locally compact, Then*

$$\xi_x : G/G_x \to X = G \cdot x \quad \forall x \in X$$

is a homeomorphism.

2.10 Examples

We have

$$O(N) = \{g \in GL(n, \mathbb{R}) \mid gg^T = I_n(\det g = 1)\}.$$

$O(n)$ acts on $\mathbb{R}^n$ with orbits being $S_r^{n-1} - \{x \in \mathbb{R}^n \mid |x| = r\}, r \geq 0$.

Induction implies that $O(n), SO(n)$ are compact, $SO(n)$ connected.

Also $SL(n, \mathbb{R})$ is connected, as it has for $n > -2$ has 2 orbits on $\mathbb{R}^n : \{0\}, \mathbb{R}^n - \{0\}$. Also $SL(n, \mathbb{C})$ is connected.

Consider unitary groups

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid gg^{-T} - I_n(\det g = 1)\}.$$

$GL(n, \mathbb{F})$ acts on $\mathbb{P}^{n-1}$, which is the set of lines through 0 in $\mathbb{F}^n$.

$Gr_k(n, \mathbb{F})$ is the set of k-dimensional subspaces of $\mathbb{F}^n$, which is the quotient of the set of $n \times k$-matrices of rank k by $GL(k, \mathbb{F})$ acting on the right.

3 Lie Group

3.1 Basics

Definition 3.1.1. A *Lie group* G is a group and a manifold such that

$$m : G \times G \to G$$

is smooth.

The composition of two smooth maps is smooth.

Proposition 3.1.2. *The inverse map $\iota : G \to G$ is a diffeomorphism with*

$$d\iota_g = -(dL_{g^{-1}})_1 \circ (dR_{g^{-1}})_g.$$

Particularly, $\iota_1 = -\mathrm{id}$.

3.2 Tangent Bundle to a Manifold

A fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition outlined below. The space B is called the base space of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle projection). We shall assume in what follows that the base space B is connected.

We require that for every $x \in B$, there is an open neighborhood $U \subseteq B$ of x (which will be called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ (where $\pi^{-1}(U)$ is given the subspace topology, and $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:

ADD THIS!

Denote the *tangent bundle*

$$TM = \cup_{x \in M} T_x M \qquad T_x M = \{m(t) \mid m(0) = x\}/\sim .$$

3.3 Lie Groups

Let TG be the tangent bundle to a Lie group G. We define

$$d(L_g)_h : T_h G \to T_{gh} G \quad \text{where} \quad h'(t) \mapsto (gh)'(t).$$

Notice that then

$$d(L_g)_1 : T_1 G \simeq T_g G.$$

Moreover,

$$G \times T_1 G \simeq TG \quad \text{where} \quad (g, v) \mapsto (g, d(L_g)_1 v).$$

Thus, TG is trivial as a vector bundle for a Lie group G. i.e. G is *parallelizable*.

3.4 Lie Algebra

Proposition 3.4.1.

$$[\phi * X, \phi * Y] = \phi * ([X, Y]).$$

Definition 3.4.2. Let G be a Lie group. A vector field X on G is said to be

- *right invariant* if, for every $g \in G$, $(R_g)_* X = X$. In detail,

  $$d\left(R_g\right)_h (X(h)) = X(hg)$$

  for every $g, h \in G$;

- *left invariant* if, for every $g \in G$, $(L_g)_* X = X$, that is,

  $$d\left(L_g\right)_h (X(h)) = X(gh).$$

Definition 3.4.3. We define *Maurer-Cartan forms*, which are differential 1forms on G with values in $T_1 G$. They are defined by right or left translations by

$$\omega_g^r(v) = d\left(R_{g^{-1}}\right)_g (v) \quad \text{and} \quad \omega_g^l(v) = d\left(L_{g^{-1}}\right)_g (v)$$

for $g \in G$ and $v \in T_g G$.

Proposition 3.4.4. *If $X \in Vect(G)$ is right-invariant, then $\omega^r(X) = X(1)$, the constant T_1G-valued function. Similarly, if X is left-invariant, then $\omega^l(X) = X(1)$.*

Definition 3.4.5. We define the set of right invariant fields as

$$Inv_r = \cap_{g \in G} ker\left((R_g)_* - Id_{vect(G)}\right) \subseteq Vect(G).$$

Theorem 3.4.6. *Let $Inv_r \cong T_1G \cong Inv_e$*