

Lie Theory

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1 Background

1.1 Topology

Definition 1.1.1. A topological space is *locally connected* at point x if every neighborhood of x contains a connected open neighborhood.

2 Topological Groups

2.1 Introduction

Definition 2.1.1. A *topological group* is a group such that

1. the product $p : G \times G \rightarrow G, p(g, h) = gh$, is a continuous map if $G \times G$ has the product topology;
2. The map $\iota : G \rightarrow G, \iota(g) = g^{-1}$, is continuous (hence, a homeomorphism, as $\iota^{-1} = \iota$).

Each element $g \in G$ defines the following maps.

- *left translation*: $L_g : G \rightarrow G, L_g(h) = gh$;
- *right translation*: $R_g : G \rightarrow G, R_g(h) = hg$;
- *conjugation*: $C_g : G \rightarrow G, C_g(h) = ghg^{-1}$.

2.2 Neighborhoods of Identity

An (open) neighborhood of $x \in X$, where X is a topological space, is an open set U that contains x .

Let G be a topological group, and $1 \in G$ is the identity. $V(1)$ refers to the set of all neighborhoods of 1.

Proposition 2.2.1 (Proposition 2.2). *Let G be a t.g. (topological group), $V = V(1)$. Then we'll have*

1. (T1) for all $u \in V, 1 \in u$;
2. (T2) $u, v \in V \implies u \cap v \in V$;
3. (TG1) for all $u \in V$, there exists $v \in V$ s.t. $v^2 \subseteq u$;
4. (TG2) $u \in V \implies u^{-1} \in V$;
5. (TG3) $u \in V, g \in G \implies gug^{-1} \in V$.

Definition 2.2.2. Let G be a group, not necessarily topological group. A system of neighborhood of $1 \in G$ is a family of sets satisfying (T1) to (TG3).

Definition 2.2.3. Let X be a topological space and $x \in X$. A fundamental system of neighborhoods of x is a family F of open sets containing x s.t. for all open u that contains x , there exists $v \in F$ s.t. $v \subseteq u$.

Theorem 2.2.4 (Proposition 2.5). *Let G be an abstract group, V be a system of neighborhoods of 1. There exists a unique topology on G making G into a topological group and s.t. V is a fundamental system of neighborhoods of 1.*

idea of proof.

□

Proposition 2.2.5. *Let G be a topological group. TFAE*

1. topology of G is a Hausdorff
2. $\{1\}$ is closed in
3. $\bigcap_{U \in V(1)} U = \{1\}$

2.3 Metrizable Groups

Definition 2.3.1. Let G be a topological group. G is metrizable if it has a left-(or right-) invariant distance which defines the topology left-invariant for all $g \in G$ and $d(gx, gy) = d(x, y)$ for all $x, y \in G$.

Theorem 2.3.2. A topological group G is metrizable iff it has a countable system of neighborhoods of 1.

2.4 Homomorphisms

We need to talk about $G \rightarrow H$ continuous homomorphisms.

Example 2.4.1. The determinant homomorphism $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^* = GL(1, \mathbb{R})$ is continuous.

Theorem 2.4.2. Let G, H be topological groups. A group homomorphism $\phi : G \rightarrow H$ is continuous iff ϕ is continuous at $1 \in G$.

Proof. \Rightarrow is obvious. Let's look at the other direction.

Note that $\phi \circ L_g = L_{\phi(g)} \circ \phi$ as maps $G \rightarrow H$ because

$$(\phi \circ L_g)(g') = \phi(gg') = \phi(g)\phi(g') = (L_{\phi(g)} \circ \phi)(g').$$

Then

$$\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$$

is continuous at g , as $L_{g^{-1}}$ is continuous at g , ϕ continuous at 1, and $L_{\phi(g)}$ continuous everywhere. \square

Theorem 2.4.3. A map $\phi : G \rightarrow H$ is a group homomorphism (G, H are just groups) iff

$$gr(\phi) := \{(g, \phi(g)) \mid g \in G\} \subseteq G \times H.$$

Proposition 2.4.4. Let X and Y be topological spaces, such that Y is Hausdorff. A map $\phi : X \rightarrow Y$ is continuous if and only if its graph $gr(\phi)$ is closed and the projection $p(x, \phi(x)) = x$ is a homeomorphism.

Proof. Suppose ϕ is continuous. Then

$$gr(\phi) = \theta^{-1}(\Delta Y) \text{ w.r.t. } \theta : X \times Y \rightarrow Y \times Y$$

is closed, since θ is continuous and ΔY is closed. \square

Theorem 2.4.5. Suppose G, H are topological groups, H is Hausdorff. The map $G \rightarrow H$ is a continuous homomorphism iff $gr(\phi)$ is a closed subgroup and $p : gr(\phi) \rightarrow G$ is a homeomorphism.

2.5 Subgroups

Let G be a topological group. $H \subseteq G$ is a *topological subgroup* if H is a topological group w.r.t. the induced topology.

Proposition 2.5.1. *Let G be a topological group. If $H \subseteq G$ a subgroup, which is open. Then H is also closed.*

Proof. Consider

$$Y = \bigcup_{g \in G-H} gH.$$

Y is open, as it is a union of open sets. H is also closed, as $G - Y = H$. Hence, H is closed. \square

Proposition 2.5.2. *G a topological group, $H \subseteq G$ a subgroup. Then \overline{H} is also a subgroup of G .*

Proof. Note that $A \subseteq X$ (subset of a topological space), $x \in \overline{A}$ iff for all open U that contains x , $U \cap A \neq \emptyset$. Then we check the followings.

1. \overline{H} is closed under $m : G \times G \rightarrow G$.

\square

2.6 Connected Components of Topological Groups

A *connected space* cannot be written as the union of two disjoint open sets.

A *connected component* of a point $x \in X$ is the union of all connected sets containing x , which is also the maximal connected set containing x .

A *connected component* of X is a maximal connected subset.

If $A \subseteq X$ is connected, then the closure \overline{A} is connected. Thus, every connected component is closed.

Let G be a topological group, G_0 is the connected component of $1 \in G$.

Proposition 2.6.1. *G_0 is a closed normal subgroup of G . The connected components of G are exactly gG_0 for $g \in G$.*

A *neighborhood* N of $x \in X$ is a subset $N \subseteq X$, $x \in N$ and there exists an open $U \subseteq X$ s.t $x \in U \subseteq N$.

A space is *locally connected* if for every open neighborhood of every point contains a connected open neighborhood.

Proposition 2.6.2. *If G is locally connected, then G_0 is open.*

Proposition 2.6.3. *If G connected, $U \in \mathcal{V}(1)$, then $G = \bigcup_{n \geq 1} U^n$.*

2.7 Group Action

Suppose G a group, X a set.

Definition 2.7.1. A *left action* of a group G on a set X is a function that associates to $g \in G$ a map $a(g) : X \rightarrow X$ which satisfies the properties: 1. $a(1) = \text{id}_X$, that is, $a(1)(x) = x$, for every $x \in X$; 2. $a(gh) = a(g) \circ a(h)$.

Definition 2.7.2. Let $\phi_x : G \times X \rightarrow X, \phi_y : G \times Y \rightarrow Y$. A map $f : X \rightarrow Y$ is *G -equivariant* if

$$\phi_y(g, f(x)) = f(\phi_x(g, x)).$$

Same story for topological groups.

Definition 2.7.3. Let G be a topological group, X a topological space, an *action* G on X should be continuous. In other words, G acts on X by homeomorphisms ϕ_g .

Action is *transitive* if $X = Gx$ for some $x \in X$. We define the *orbit* of x to be $Gx = \{gx \mid g \in G\}$. A *stabilizer* or *isotropy subgroup* of x is $G_x = \{g \in G \mid gx = x\}$.

An action is an *effective action* or *faithful* if $gx = x, \forall x \in X \implies g = 1$, equivalently, $\cap_{x \in X} G_x = \{1\}$.

Proposition 2.7.4.

$$G/G_x \rightarrow X \quad \text{where} \quad gG_x \mapsto gx.$$

This map is equivariant.

Proposition 2.7.5. Suppose that the action of G on X is continuous and that X is a Hausdorff space. Then, any isotropy subgroup $G_x, x \in X$, is closed.

2.8 Homogeneous Spaces

Let G be a topological group.

Definition 2.8.1. A *homogeneous G -space* is just G/H for a subgroup H of G .

Definition 2.8.2. A topological space X without regards to group is *homogeneous* if for all $x, y \in X$, there exists a homeomorphism $\phi : X \rightarrow X$ s.t. $\phi(x) = y$.

Topology on G/H is that of a quotient: $\pi : G \rightarrow G/H$. In other words, $U \subseteq G/H$ open if $\pi^{-1}(U) \subseteq G$ open.

Note: action of G on G/H is continuous:

$$G \times G/H \rightarrow G/H \quad \text{where} \quad (x, gH) \mapsto xgH.$$

Proposition 2.8.3. We have the following facts.

1. G/H is a homogeneous space in the sense of topology.

2. $\pi : G \rightarrow G/H$ is an open map (it takes open sets to open sets).
3. H compact implies that π is a closed map.
4. G/H is Hausdorff iff H is closed.
5. G/H discrete iff H open. (HW2)
6. If G is compact, G/H discrete and finite iff H is open.
7. $H \triangleleft G$ implies G/H is a topological group.
8. $H := \overline{\{1\}}$. Then H is a normal subgroup of G , and G/H is Hausdorff topological group.

Proof of 1. Consider left translation

$$L_x : gH \mapsto xgH.$$

This is a homeomorphism since $L_{x^{-1}}$ is an inverse and both are continuous. \square

Proof of 2. We need to show that $\pi^{-1}\pi(U)$ is open. (Omitted, just do image preimage and write it as union of right cosets). \square

Proof of 3. Take $F \subseteq G$ closed, if H is a compact subset, then $FH \subseteq G$ is closed. (From a proposition from textbook).

Notice that $\pi(F)$ closed iff $\pi^{-1}\pi(F)$ closed, and the latter equals to FH . \square

Proof of 4. We first show \implies . Note that $H = \pi^{-1}(\pi(H))$, which is a point of G/H , so it's closed. Thus H is closed.

Then we show \impliedby . Consider the homeomorphism

$$f : G/H \times G/H \rightarrow G \times G/H \times H \quad \text{where} \quad (g_1H, g_2H) \mapsto (g_1, g_2)H \times H.$$

Denote $\Delta = \{(gH, gH)\}$. Then $f(\Delta) = \{(g, g)H \times H\}$ is closed iff $\pi_{G \times G}^{-1}f(\Delta)$ is closed, which equals to $\{(g_1, g_2) \mid g_1H = g_2H\} = \{(g_1, g_2) \mid g_1^{-1}g_2 \in H\}$. \square

Let G be a topological group, $H \subseteq G$ a subgroup.

Proposition 2.8.4. *If H and G/H are compact, then so is G .*

Proof.

$$\pi : G \rightarrow G/H$$

is a *perfect map*, i.e., a continuous surjective closed map with compact fibers $\pi^{-1}(x), \forall x \in G/H$. \square

Proposition 2.8.5. *If G/H and H are connected, then so is G .*

Proof. Suppose G is not connected, then there exists $A \sqcup B = G$, $A, B \neq \emptyset$ open, disjoint $\subseteq G$. Then $\pi(A), \pi(B) \neq \emptyset$, open because π is always open, $\pi(A) \cup \pi(B) = G/H$, which is connected. Therefore $\pi(A) \cap \pi(B) \neq \emptyset$. Thus there exists $gH \in G/H$ s.t. $gH \cap A \neq \emptyset$ and $gH \cap B \neq \emptyset$. \square

2.9 Orbits and Homogeneous Spaces

Homogeneous space G/G_x , we have a bijection:

$$G/G_x \rightarrow G \cdot x \quad \text{where} \quad gG_x \mapsto gx.$$

Proposition 2.9.1. *Let $G \times X \rightarrow X$ be a continuous and transitive action of G on X . Fix $x \in X$ and consider the bijection*

$$\xi_x : G/G_x \rightarrow X \quad \text{given by} \quad \xi_x(gG_x) = gx.$$

Then ξ_x is continuous with respect to the quotient topology in G/G_x .

Proposition 2.9.2. *Let $G \times X \rightarrow X$ be a topological transitive group action. Suppose G is locally compact and separable (i.e., has a countable dense subset) and X is Hausdorff and locally compact, Then*

$$\xi_x : G/G_x \rightarrow X = G \cdot x \quad \forall x \in X$$

is a homeomorphism.

2.10 Examples

We have

$$O(N) = \{g \in GL(n, \mathbb{R}) \mid gg^T = I_n(\det g = 1)\}.$$

$O(n)$ acts on \mathbb{R}^n with orbits being $S_r^{n-1} = \{x \in \mathbb{R}^n \mid |x| = r\}, r \geq 0$.

Induction implies that $O(n), SO(n)$ are compact, $SO(n)$ connected.

Also $SL(n, \mathbb{R})$ is connected, as it has for $n > -2$ has 2 orbits on $\mathbb{R}^n : \{0\}, \mathbb{R}^n - \{0\}$. Also $SL(n, \mathbb{C})$ is connected.

Consider unitary groups

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid gg^{-T} = I_n(\det g = 1)\}.$$

$GL(n, \mathbb{F})$ acts on \mathbb{P}^{n-1} , which is the set of lines through 0 in \mathbb{F}^n .

$Gr_k(n, \mathbb{F})$ is the set of k -dimensional subspaces of \mathbb{F}^n , which is the quotient of the set of $n \times k$ -matrices of rank k by $GL(k, \mathbb{F})$ acting on the right.

3 Lie Group

3.1 Basics

Definition 3.1.1. A *Lie group* G is a group and a manifold such that

$$m : G \times G \rightarrow G$$

is smooth.

The composition of two smooth maps is smooth.

Proposition 3.1.2. *The inverse map $\iota : G \rightarrow G$ is a diffeomorphism with*

$$d\iota_g = -(dL_{g^{-1}})_1 \circ (dR_{g^{-1}})_g.$$

Particularly, $\iota_1 = -\text{id}$.

3.2 Tangent Bundle to a Manifold

A fiber bundle is a structure (E, B, π, F) , where E, B , and F are topological spaces and $\pi : E \rightarrow B$ is a continuous surjection satisfying a local triviality condition outlined below. The space B is called the base space of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle projection). We shall assume in what follows that the base space B is connected.

We require that for every $x \in B$, there is an open neighborhood $U \subseteq B$ of x (which will be called a trivializing neighborhood) such that there is a homeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ (where $\pi^{-1}(U)$ is given the subspace topology, and $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:

ADD THIS!

Denote the *tangent bundle*

$$TM = \cup_{x \in M} T_x M \quad T_x M = \{m(t) \mid m(0) = x\} / \sim.$$

3.3 Lie Groups

Let TG be the tangent bundle to a Lie group G . We define

$$d(L_g)_h : T_h G \rightarrow T_{gh} G \quad \text{where} \quad h'(t) \mapsto (gh)'(t).$$

Notice that then

$$d(L_g)_1 : T_1 G \simeq T_g G.$$

Moreover,

$$G \times T_1 G \simeq TG \quad \text{where} \quad (g, v) \mapsto (g, d(L_g)_1 v).$$

Thus, TG is trivial as a vector bundle for a Lie group G . i.e. G is *parallelizable*.

3.4 Lie Algebra

Proposition 3.4.1.

$$[\phi * X, \phi * Y] = \phi * ([X, Y]).$$

Definition 3.4.2. Let G be a Lie group. A vector field X on G is said to be

- *right invariant* if, for every $g \in G$, $(R_g)_* X = X$. In detail,

$$d(R_g)_h (X(h)) = X(hg)$$

for every $g, h \in G$;

- *left invariant* if, for every $g \in G$, $(L_g)_* X = X$, that is,

$$d(L_g)_h (X(h)) = X(gh).$$

Definition 3.4.3. We define *Maurer-Cartan forms*, which are differential 1-forms on G with values in $T_1 G$. They are defined by right or left translations by

$$\omega_g^r(v) = d(R_{g^{-1}})_g(v) \quad \text{and} \quad \omega_g^l(v) = d(L_{g^{-1}})_g(v)$$

for $g \in G$ and $v \in T_g G$.

Proposition 3.4.4. *If $X \in \text{Vect}(G)$ is right-invariant, then $\omega^r(X) = X(1)$, the constant T_1G -valued function. Similarly, if X is left-invariant, then $\omega^l(X) = X(1)$.*

Definition 3.4.5. We define the set of right invariant fields as

$$\text{Inv}_r = \cap_{g \in G} \ker((R_g)_* - \text{Id}_{\text{Vect}(G)}) \subseteq \text{Vect}(G).$$

Theorem 3.4.6. *Let $\text{Inv}_r \cong T_1G \cong \text{Inv}_e$*

Definition 3.4.7. $\mathfrak{g} = (\text{Inv}_r, [,])$ is the *Lie algebra* of a Lie group G .

Proposition 3.4.8. *This bracket gives the following bracket on T_1G :*

$$A \in T_1G \rightarrow A^r(g) = d(R_g)_1 A.$$

Moreover

$$[A, B] := [A, B]_r = [A^r, B^r](1).$$

Proposition 3.4.9. *Let $A, B \in T_1G$. Then, $[A, B]_r = -[A, B]_l$.*

$$[A, B] = -[A, B]_e = BA - AB.$$

3.5 Exponential Map

Remarks on flows on manifolds.

Let X be a vector field on manifold M , $X \in C^\infty(M, TM)$. A *flow* ϕ_t^x defined by $\phi_t^x(x) = x(t)$, $t \in (-\epsilon, \epsilon)$, and $\frac{dx}{dt} = X(x)$, $x(0) = x$.

Another notation is $X_t = \phi_t^x$.

WTS

$$X_{s+t} = X_s \circ X_t = X_t \circ X_s.$$

Take $X \in \mathfrak{g} = \text{Inv}_r$ right invariant vector field

Then $X_t(g)$ the flow equals to $g(t)$ and is given by

$$\frac{dg}{dt} = X(g), \quad g(0) = g.$$

For $g \in G$, $g(t) : (-\epsilon, \epsilon) \rightarrow G$.

Lemma 3.5.1. *For $X \in \text{Inv}_r$, we have*

$$X_t(gh) = X_t(g)h \quad \forall g, h \in G.$$

Theorem 3.5.2. *A right-invariant vector field X is complete, i.e., defined for all $t \in \mathbb{R}$.*

G a lie group, $\mathfrak{g} = T_1G$ its lie algebra.

Definition 3.5.3. The *exponential map*

$$\exp : \mathfrak{g} \rightarrow G$$

is defined by $X \in \mathfrak{g}$ generates the right invariant vector field $X^r(g) = d(R_g)_1 X, g \in G$.

Then we create a flow, denoted by $X_t^r = g(t)$, for $\frac{dg(t)}{dt} = X^r(g(t)), g(0) = g$, which gives that $X_t^r(1) |_{t=1} = \exp(X)$.

Proposition 3.5.4. *By doing the same procedure using left-invariant vector field X^l gives the same result:*

$$X_t^l(1) |_{t=1} = X_t^r(1) |_{t=1} = \exp(X).$$

Moreover,

$$X_t^l(1) = X_t^r(1) \quad \forall t \in \mathbb{R}.$$

Proof. Denote $g(t_0) = X_t^r(1), g(0) = 1$. It's sufficient to show that $\frac{dg}{dt} = X^l(g)$.

We know that

$$\begin{aligned} \frac{dg}{dt} &= \frac{d}{dt} (X_t^r(1)) = \frac{d}{ds} (X_{s+t}^r(1)) |_{s=0} \\ &= \frac{d}{ds} (X_t^r (X_s^r(1))) |_{s=0} \\ &= \frac{d}{ds} (X_t^r(1) X_s^r(1)) |_{s=0} \\ &= \frac{d}{ds} (L_{X_t^r(1)} X_s^r(1)) |_{s=0} \\ &= d(L_{X_t^r(1)})_1 \frac{d}{ds} (X_s^r(1)) |_{s=0} && \text{chain rule} \\ &= d(L_{X_t^r(1)})_1 X^r(1) \\ &= d(L_{X_t^r(1)})_1 X \\ &= X^l(X_t^r(1)) \\ &= X^l(g(t)) \end{aligned}$$

□

We have

$$X_t(1) : (\mathbb{R}, t) \rightarrow G.$$

a homomorphism, sometimes we call it a *one-parametric* subgroup of G generated by a right invariant vector field X^r .

Q: What is $X_t^r(1)$ and $X_t^l(1)$ via \exp ?

A: Suppose Y a vector field on M . Suppose we run a corresponding flow Y_t on M . Let $a \in \mathbb{R}$, then $(aY)_t = Y_{at}$ whenever flow Y_{at} and Y_t are defined.

$$(tY)_s |_{s=1} = Y_t.$$

Applying this to $M = G, Y = X^r$ at $g = 1, tX^r = (tX)^r$, we have

$$\exp(tX) = (tX)_s^r(1)|_{s=1} = (tX^r)_s(1)|_{s=1} = X_t^r(1).$$

Then

$$X_t^r(1) = \exp(tX) \quad X_t^l(1) = \exp(tX).$$

$$(\phi_*X)(y) = (d\phi)_{\phi^{-1}(y)}X(\phi^{-1}(y)) \text{ pushforward}$$