Theorem Prover in Functional Analysis

small example, axiom of choice, and analysis

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why?

Story from Vladimir Voevodsky

1. "Cohomological Theory of Presheaves with Transfers" 1992/93
2. a key lemma in the paper 1999/2000
3. weaker and more complicated lemma 2006
4. 1993-2006?

intro example

```
/-- The set of upper bounds of a set of real numbers ℝ -/
def up_bounds (A : set ℝ) := { x : ℝ | ∀ a ∈ A, a ≤ x}

/-- Predicate `is_maximum a A` means `a` is a maximum of `A` -/
def is_maximum (a : ℝ) (A : set ℝ) := a ∈ A ∧ a ∈ up_bounds A
```

```
lemma unique_max (A : set ℝ) (x y : ℝ) (hx : is_maximum x A) (hy : is_maximum y A) : x = y :=
begin
  -- We first break our assumptions in their two constituent pieces.
  -- We are free to choose the name following `with`
  cases hx with x_in x_up,
  cases hy with y_in y_up,
  -- Assumption `x_up` means x isn't less than elements of A, let's apply this to y
  specialize x_up y,
  -- Assumption `x_up` now needs the information that `y` is indeed in `A`.
  specialize x_up y_in,
  -- Let's do this quicker with roles swapped
  specialize y_up x x_in,
  -- We explained to Lean the idea of this proof.
  -- Now we know `x ≤ y` and `y ≤ x`, and Lean shouldn't need more help.
  -- `linarith` proves equalities and inequalities that follow linearly from
  -- the assumption we have.
  linarith,
end
```

intro example

```lean
/-- The set of upper bounds of a set of real numbers ℝ -/
def up_bounds (A : set ℝ) := { x : ℝ | ∀ a ∈ A, a ≤ x}

/-- Predicate `is_maximum a A` means `a` is a maximum of `A` -/
def is_maximum (a : ℝ) (A : set ℝ) := a ∈ A ∧ a ∈ up_bounds A
```

```lean
example (A : set ℝ) (x y : ℝ) (hx : is_maximum x A) (hy : is_maximum y A) : x = y :=
begin
  have : x ≤ y, from hy.2 x hx.1,
  have : y ≤ x, from hx.2 y hy.1,
  linarith,
end
```

axiom of choice

Definition 1.1

Every family of nonempty sets has a choice function. If S is a family of sets and $\varnothing \notin S$, then a choice function for S is a function f on S s.t. $f(X) \in X$ for every $X \in S$.

```
axiom choice {α : Sort u} : nonempty α → α
```

```
theorem axiom_of_choice' {α : Sort u} {β : α → Sort v} (h : ∀ x, nonempty (β x)) :
  nonempty (Π x, β x) :=
⟨λ x, classical.choice (h x)⟩
```

Sort u : the universe of types at universe level u

Π x : α, β : the type of functions taking an element x of α to an element of β, where β is an expression whose type is a Sort

λ x : α, t : the function mapping any value x of type to t, where t is an expression

analysis

```
example (hu : seq_limit u l) (hv : seq_limit v l') :
seq_limit (u + v) (l + l') :=
begin
  intros ε ε_pos,
  cases hu (ε/2) (by linarith) with N₁ hN₁,
  cases hv (ε/2) (by linarith) with N₂ hN₂,
  use max N₁ N₂,
  intros n hn,
  cases ge_max_iff.mp hn with hn₁ hn₂,
  have fact₁ : |u n - l| ≤ ε/2,
    from hN₁ n (by linarith),  -- note the use of `from`.
                               -- This is an alias for `exact`,
                               -- but reads nicer in this context
  have fact₂ : |v n - l'| ≤ ε/2,
    from hN₂ n (by linarith),
  calc
  |(u + v) n - (l + l')| = |u n + v n - (l + l')|   : rfl
                     ... = |(u n - l) + (v n - l')| : by congr' 1 ; ring
                     ... ≤ |u n - l| + |v n - l'|   : by apply abs_add
                     ... ≤  ε                       : by linarith,
end
```

Theorem 1.2: $L^1(\mathbb{R})$ closed under addition

Given $f_1, f_2 \in L^1(\mathbb{R})$, then $f_1 + f_2 \in L^1(\mathbb{R})$, namely

$$\int_{-\infty}^{\infty} f_1(x) + f_2(x)dx = \int_{-\infty}^{\infty} f_1(x)dx + \int_{-\infty}^{\infty} f_2(x)dx$$

```
theorem lintegral_nnnorm_add
  {f : α → β} {g : α → γ} (hf : ae_strongly_measurable f μ) (hg : ae_strongly_measurable g μ) :
  ∫⁻ a, nnnorm (f a) + nnnorm (g a) ∂μ = ∫⁻ a, nnnorm (f a) ∂μ + ∫⁻ a, nnnorm (g a) ∂μ
```

Theorem 1.3: Monotone Convergence Theorem in L^0

Suppose we have a sequence $f_n \in L^0(\mathbb{R})$ with $f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$ for all $x \in \mathbb{R}$ and suppose there exists a constant M s.t. $\int_{-\infty}^{\infty} f_n(x)dx \leq M$ for all n.
Then there eixsts $f \in L^0(\mathbb{R})$ s.t. $f_n \to f$ except possibly on a set of measure zero and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

```
theorem lintegral_supr
  {f : ℕ → α → ℝ≥0∞} (hf : ∀n, measurable (f n)) (h_mono : monotone f) :
  (∫⁻ a, ⨆n, f n a ∂μ) = (⨆n, ∫⁻ a, f n a ∂μ)
```

end

But I think that the sense of urgency that pushed me to hurry with the program remains. Sooner or later computer proof assistants will become the norm, but the longer this process takes the more misery associated with mistakes and with unnecessary self-verification the practitioners of the field will have to endure.

Vladimir Voevodsky, Univalent Foundations, 2014

Bibliography

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