Lie algebra $\mathfrak{sl}_3$ and representation

eigenspaces and representation decomposition for $\mathfrak{sl}_3$

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Definition 1.1: bracket, commutator, and Lie algebra

A vector space L over a field F, with an operation $[\cdot,\cdot]$ from $L \times L \to L$, called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following axioms are satisfied:

1. The bracket operation is bilinear
   1. $[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ and $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
   2. $[\lambda x, y] = \lambda[x, y] = [x, \lambda y]$
2. $[x, x] = 0$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Theorem 1.2: anti-commutativity of Lie algebra

$$[x, y] = -[y, x]$$

Lie algebra

Example 1.3: $\mathfrak{sl}_3(\mathbb{R})$

$\mathfrak{sl}_3$ is a vector space of 3×3 matrices with trace 0,in other words:

$$\mathfrak{sl}_3 = \left\{ \begin{bmatrix} a & c & d \\ e & -a+b & f \\ g & h & -b \end{bmatrix} \mid a, b, c, d, e, f, g, h \in \mathbb{R} \right\}$$

equppied with $[A, B] = AB - BA$

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the basis is:

$$\mathcal{B}_{\mathfrak{sl}_3} = \left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \bigcup \{E_{i,j} \mid i \neq j\}.$$

$E_{i,j}$ is a matrix where ${E_{i,j}}_{ij} = 1$ and other places are 0.

Definition 1.4: adjoint representation of $\mathfrak{sl}_3(\mathbb{R})$

Give $X \in \mathfrak{sl}_3(\mathbb{R})$, define a linear map $ad_X : \mathfrak{sl}_3(\mathbb{R}) \to \mathfrak{sl}_3(\mathbb{R})$ as:

$$ad_X(Y) = [X, Y] = XY - YX$$

the map $X \mapsto ad_X$ is called an adjoint representation of $\mathfrak{sl}_3(\mathbb{R})$.

Definition 1.5: subspace $\mathfrak{h} \subseteq \mathfrak{sl}_3(\mathbb{R})$

Let $\mathfrak{h}$ be the two-dimensional subspace of all diagonal matrices, namely:

$$\mathfrak{h} = \operatorname{span}\left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

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What does it mean to be an eigenspace for $\mathfrak{h}$?

For all $H \in \mathfrak{h}$, an eigenspace $\mathfrak{g}_\lambda$ has all $M \in \mathfrak{sl}_3(\mathbb{R})$ s.t. $ad_H(M) = [H, M] = HM - MH = \lambda M$?

Let's pick an arbitrary $H \in \mathfrak{h}$ and $M \in \mathfrak{sl}_3(\mathbb{R})$, namely:

$$H = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

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Let's see the result under adjoint action, i.e.
$ad_H(M) = [H, M] = HM - MH = ?$

$$[H, M] = \begin{bmatrix} (a_1 - a_1)m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & (a_2 - a_2)m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & (a_3 - a_3)m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & 0 \cdot m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & 0 \cdot m_{33} \end{bmatrix}$$

some linear algebra

$$[H, M] = \begin{bmatrix} 0 & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & 0 & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & 0 \end{bmatrix}$$

observation: recall our only constrain on H: $a_1 + a_2 + a_3 = 0$, for M to be an eigenvector under bracket H, i.e. $[H, M] = \lambda M$ for some λ, M must be all 0 on the diagonal and only one place not equal to 0.

some linear algebra

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the eigenspace is generated by the other basis element $E_{i,j}$ where $i \neq j$ and $E_{i,j}$ has all but one entry ij zero, for example:

$$E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

linear functional

$$[H, M] = \begin{bmatrix} 0 & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & 0 & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & 0 \end{bmatrix}$$

one could see we have potentially 6 eigenvalues for each $H \in \mathfrak{h}$: $a_1 - a_2, a_1 - a_3, a_2 - a_3, a_2 - a_1, a_3 - a_1, a_3 - a_2$.

linear functional

Since we are asking for eigenspaces of $\mathfrak{h}$, we need a way to get $a_i - a_j$ from any $H \in \mathfrak{h}$ for us, thus we could define a linear functional:

Definition 1.6: L_i

L_i is a linear functional s.t.

$$L_i \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = a_i$$

it satisfies the requirement of a linear functional:
$L_i(A + B) = L_i(A) + L_i(B), L_i(aA) = aL_i(A)$.

eigenvalues and eigenspaces

in other words, the set of all eigenvalues of $\mathfrak{h}$ is

$$\mathfrak{h}^* = \mathbb{R}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0).$$

given the adjoint representation of $\mathfrak{sl}_3(\mathbb{R})$, the eigenspace, denoted $\mathfrak{g}_\alpha$, associated to the eigenvalue $\alpha \in \mathfrak{h}^*$ is the subspace of all $M \in \mathfrak{sl}_3(\mathbb{R})$ s.t.

$$H(M) = ad_H(M) = [H, M] = \alpha(H) \cdot M$$

for all $H \in \mathfrak{h}$.

Notice that for the adjoint representation of $\mathfrak{sl}_3(\mathbb{R})$, we have the following decomposition:

$$\mathfrak{sl}_3(\mathbb{R}) = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)$$

where α ranges over $\mathfrak{h}^*$ and $\mathfrak{g}_\alpha$ is the eigenspace w.r.t α.

why?

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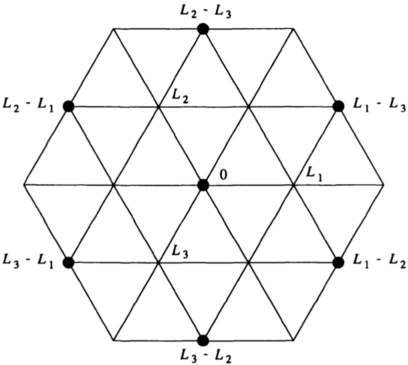
this is a special case of a general statement:

Theorem 2.1

For any semisimple Lie algebra $\mathfrak{g}$ (it is a direct sum of simple Lie algebras (non-abelian Lie algebras without any non-zero proper ideals), we could find an abelian subalgebra(closed under bracket) $\mathfrak{h} \subset \mathfrak{g}$ s.t. the action of $\mathfrak{h}$ on any representation V could be written as a direct sum decomposition of V into eigenspaces V_α for $\mathfrak{h}$.

graph of eigenspaces



Given any $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, where would ad_X sends Y to? i.e. $ad_X(Y) = [X, Y] =?$

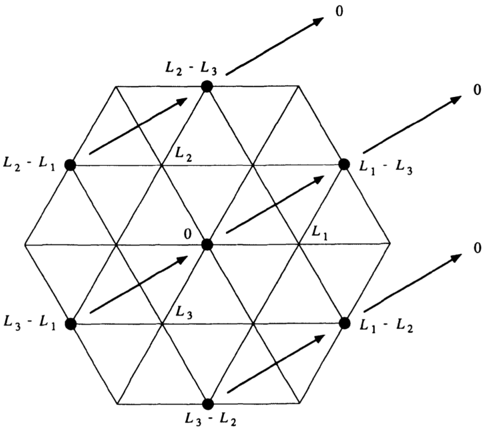
$\mathfrak{h}$ could help us, namely:

$$[H, [X, Y]] = ((\alpha + \beta)(H)) \cdot [X, Y]$$

in other words, $ad_X(Y)$ is an eigenvctor for $\mathfrak{h}$ with eigenvalue $\alpha + \beta$.

$$ad_{\mathfrak{g}_\alpha} : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$$

The action of $\mathfrak{g}_{L_1-L_3}$ is:



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