A rim hook rule for the equivariant quantum cohomology of the Grassmannian

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joint work with
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Cohomology: A First Example

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$$\ydiagram{1} \cdot \ydiagram{3} = 0$$

Cohomology: The Algebra

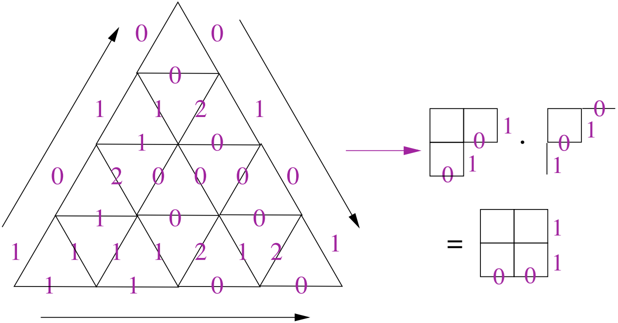
The cohomology of the Grassmannian has a nice algebraic structure. The *Borel isomorphism* says that

$$H^*(Gr(k,n)) \cong \mathbb{Z}[e_1,\ldots,e_k]/\langle h_{n-k+1},\ldots,h_n\rangle$$

- e_i elementary symmetric polynomials
- h_i homogeneous symmetric polynomials
- in variables $x_1,\ldots,x_k$.

$H^*(Gr(k,n))$ has a $\mathbb{Z}$-algebra basis of *Schubert classes* indexed by Young diagrams λ which fit inside a $k\times(n-k)$ box.

A completed puzzle with a unique filling:



In general, there may be either none or several. Each valid puzzle contributes a term to the product in $H^*(Gr(k,n))$.

The Rim Hook Rule

The Idea: Compute $QH^*(Gr(k,n))$ from $H^*(Gr(k, 2n-k))$, where all products of $k \times (n-k)$ boxes "fit", and then remove rim hooks in exchange for the quantum parameter.

Example

To compute $\sigma_{\square} \star \sigma_{\ydiagram{2,2}}$ in $QH^*(Gr(2,4))$, first compute the classical product in $H^*(Gr(2,6))$:

$$\ydiagram{1} \cdot \ydiagram{2,2} = \ydiagram{3,2} \mapsto \ytableau{ \, & \times & \times \\ \times & \times } = q\, \ydiagram{1}$$

Then remove all possible 4-rim hooks, picking up a (signed) power of q for each rim hook removed. This gives

$$\sigma_{\square} \star \sigma_{\ydiagram{2,2}} = q\sigma_{\square}$$

Equivariant Rim Hook Rule

Theorem (Bertiger, M–, Taipale)

The following algorithm gives quantum equivariant products in $QH_T^*(Gr(k,n))$:

- *Take classical product of factorial Schur functions (do equivariant Littlewood-Richardson in "large enough" Grassmannian)*
- *In the quantum ideal, $\sigma_\lambda = (-1)^\epsilon q^d \sigma_\nu$ if we can remove d n-rimhooks from λ to get ν and the n-core $c(\nu)$ fits in $k \times (n-k)$ rectangle.*
- *Reduce equivariant coefficents by $t_i \mapsto t_{i \bmod n}$*
- *Result gives quantum equivariant product of Schubert classes.*

Cyclic Factorial Schur Polynomials

Symmetric function versions of the Peterson isomorphism:

$QH^*(G/B)$	$H_*(Gr_G)$
Schubert polynomials	k-Schur polynomials
$QH^*_T(G/B)$	$H^T_*(Gr_G)$
double Schubert polynomials	double k-Schur polynomials
$QH^*_T(Gr(k,n))$	$H^T_*(Gr_G)/J$
cyclic factorial Schurs	???