Algebra 2 Draft

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1 Introduction

This paper aims to introduce the properties of irreducible representations of the symmetric group, an extension of the properties of irreducible representation in a boarder context: topological context, so that hopefully some geometric intuitions could be grasped. Besides abstract algebra, some knowledge about point-set topology is assumed.

The goal is to glance at the infinite-dimensional representations by studying Verma modules.

2 Lie Algebra

Definition 2.1 (bracket and Lie algebra, Hall, 2015). A vector space $\mathfrak{g}$ over a field F, with an operation $[\cdot,\cdot]$ from $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following properties are satisfied:

1. The bracket operation is bilinear
   (a) $[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ and $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
   (b) $[\lambda x, y] = \lambda[x, y] = [x, \lambda y]$
2. $[x, x] = 0$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Theorem 2.2 (anti-commutativity of bracket). Given a Lie algebra $\mathfrak{g}$ equipped with a bracket operation,for all $x, y \in \mathfrak{g}$, we always have $[x, y] = -[y, x]$, indicating the **anti-commutativity of bracket** on $\mathfrak{g}$.

proof. Given any $x, y \in \mathfrak{g}$, using the properties we defined above, we could have:

$$\begin{aligned}
[x+y, x+y] &= [x, x+y] + [y, x+y] && \textbf{property 1a}\\
&= [x,x] + [x,y] + [y,x] + [y,y] && \textbf{property 1a}\\
&= [x,y] + [y,x] && \textbf{property 2}\\
[x,y] &= -[y,x]
\end{aligned}$$

Definition 2.3 (adjoint representation of $\mathfrak{sl}_3\mathbb{C}$). Give $X \in \mathfrak{sl}_3\mathbb{C}$, define a linear map $\text{ad}_X : \mathfrak{sl}_3\mathbb{C} \to \mathfrak{sl}_3\mathbb{C}$ as:

$$\text{ad}_X(Y) = [X, Y]$$

the map $X \mapsto \text{ad}_X$ is called an **adjoint representation** of $\mathfrak{sl}_3\mathbb{C}$.

3 Representations of $\mathfrak{sl}_3\mathbb{C}$

In this section, we are going to look at the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ and see how it could be decomposed into eigenspaces of a special subspace of $\mathfrak{sl}_3\mathbb{C}$. This is a special case of a more general statement which would be mentioned at the end of the paper.

Before everything, let's look at what $\mathfrak{sl}_3\mathbb{C}$ is.

Example 3.1 ($\mathfrak{sl}_3\mathbb{C}$, Hall, 2015). The Lie algebra $\mathfrak{sl}_3\mathbb{C}$ is a vector space of 3×3 matrices with trace 0 over $\mathbb{C}$, in other words

$$\mathfrak{sl}_3\mathbb{C} = \left\{ \begin{bmatrix} a & c & d \\ e & -a+b & f \\ g & h & -b \end{bmatrix} \mid a, b, c, d, e, f, g, h \in \mathbb{C} \right\}$$

We could see that $\mathfrak{sl}_3\mathbb{C}$ is an 8-dimensional vector space with the following basis:

$$\mathcal{B}_{\mathfrak{sl}_3} = \left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\} \bigcup \{E_{i,j} \mid i \neq j, 1 \leq i, j \leq 3\}.$$

$E_{i,j}$ is a matrix where ${E_{i,j}}_{ij} = 1$ and other places are 0, for example:

$$E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark 3.2 ($\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$). Denote $\mathfrak{h}$ as the two dimensional subspace of all diagonal matrices in $\mathfrak{sl}_3\mathbb{C}$ (trace is 0). Namely:

$$\mathfrak{h} = \text{Span} \left\{ H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

Now we've chosen our special subspace $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$, let's look at what it means to be an eigenvector, eigenvalue, and eigenspace for $\mathfrak{h}$.

Definition 3.3 (linear functional). A linear functional T on a complex vector space V is a function $T : V \to \mathbb{C}$ which satisfies the following properties:

1. $T(v + w) = T(v) + T(w)$
2. $T(\alpha v) = \alpha T(v)$

Definition 3.4 (eigenvector and eigenvalue for a vector space). Let $\mathfrak{h}$ be a subspace of a vector space V, an **eigenvector** for $\mathfrak{h}$ refers to a vector v that is an eigenvector for all $H \in \mathfrak{h}$, in other words:

$$Hv = \alpha(H)v \quad \text{for all } H \in \mathfrak{h},$$

where α is a linear functional on H, and we call α as an **eigenvalue**.

Definition 3.5 (eigenspace for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3\mathbb{C}$). Give $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$ as we defined above, an eigenspace of $\mathfrak{h}$ with a linear functionl α as an eigenvalue, denoted as $\mathfrak{g}_\alpha$, contains all $M \in \mathfrak{sl}_3\mathbb{C}$ s.t.

$$[H, M] = \mathrm{ad}_H(M) = HM - MH = \alpha(H)\dot{M}.$$

It's still a bit fuzzy: we don't know what our linear functionals (eigenvalues) are and we also don't know what our M (eigenvectors) look at. Thus let's spell everything out explicitly.

Take any $H \in \mathfrak{h}$ and any $M \in \mathfrak{sl}_3\mathbb{C}$, namely:

$$H = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}, M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Let's see what the adjoint $\mathrm{ad}_H(M)$ give us.

$$\begin{aligned} [H, M] &= HM - MH \\ &= \begin{bmatrix} (a_1 - a_1)m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & (a_2 - a_2)m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & (a_3 - a_3)m_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot m_{11} & (a_1 - a_2)m_{12} & (a_1 - a_3)m_{13} \\ (a_2 - a_1)m_{21} & 0 \cdot m_{22} & (a_2 - a_3)m_{23} \\ (a_3 - a_1)m_{31} & (a_3 - a_2)m_{32} & 0 \cdot m_{33} \end{bmatrix} \end{aligned}$$

An immediate observation is that all diagnoal matrices are eigenvectors of $\mathfrak{h}$ with eigenvalue 0. In other words, $\mathfrak{h}$ is an eigenspace of itself with eigenvalue 0.

(If one wants to be more rigorous, by eigenvalue 0, we mean a linear functional that sends everything to 0, so it's a $T : \mathfrak{h} \to \mathbb{C}$ s.t. $H \mapsto 0$.)

Another important observation is that, we could only have 6 eigenvalues for each $H \in \mathfrak{h}$: $a_1 - a_2, a_1 - a_3, a_2 - a_3, a_2 - a_1, a_3 - a_1, a_3 - a_2$. Because we want our eigenvalue to be applicable for all $H \in \mathfrak{h}$, whose only constraint is $a_1 + a_2 + a_3 = 0$, we could only afford to hold them separately as eigenvalue.

Let's them formally define our linear functional:

Definition 3.6 (L_i). Let $L_i : \mathfrak{h} \to \mathbb{C}$ be a linear functional s.t.

$$L_i\left(\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}\right) = a_i$$

it satisfies the requirement of a linear functional:

$L_i(A + B) = L_i(A) + L_i(B), L_i(aA) = aL_i(A).$

4 Weyl group

Definition 4.1 (adjoint map). Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Then for each $A \in G$, define a linear map $Ad_A : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$Ad_A(X) = AXA^{-1}.$$

Definition 4.2 (Weyl group). Let $\mathfrak{h}$ be the two-dimensional subspace of $sl(3, \mathbb{C})$. Let N be the subgroup of $SU(3)$ consisting of those $A \in SU(3)$ s.t. $Ad_A(H)$ is an element of $\mathfrak{h}$ for all $H \in \mathfrak{h}$. Let Z be the subgroup of $SU(3)$ consisting of those $A \in SU(3)$ s.t. $Ad_A(H) = H$ for all $H \in \mathfrak{h}$.

The **Weyl group** of $SU(3)$, denoted W, is the quotent group N/Z.

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