Algebra 2 Term Paper Proposal

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1 Introduction

This paper aims to offer sufficient introductions and examples to understand that given a finite solvable group G which acts faithfully, irreducibly and quasi-primitively on a finite vector space V and G is not metacylic, G always has a regular orbit on V except for a few small cases (Yang et al., 2020).

2 Terms and Theorems

The followings are from Artin, 1991.

Definition 1 (simple group)**.** A group G is *simple* if it is not the trivial group and if it contains no proper normal subgroup - no normal subgroup other than $\langle 1 \rangle$ and G.

Corollary 1. Cyclic groups of prime order are simple groups.

Definition 2 (solvable)**.** A finite group G is *solvable* if it contains a chaine of subgroups

$$G = H_0 \subset H_1 \subset H_1 \subset \ldots \subset H_k = \{1\}$$

such that for every $i = 1, \ldots, k$, H_i is a normal subgroup of H_{i-1}, and the quotient group $H_i H_{i+1}$ is a cyclic group.

The followings are from Fawcett et al., 2016.

Definition 3 (base)**.** Let G be a finite group acting faithfully on a set Ω. A *base* $\mathscr{B}$ for G is a non-empty subset of Ω with the property that only the identity fixes every element of $\mathscr{B}$.

Definition 4 (regular orbit)**.** Let G be a finite group acting faithfully on a set Ω. If a base $\mathscr{B} = \{\omega\}$ for some $\omega \in \Omega$, the the orbit $\{\omega g : g \in G\}$ of G on Ω is regular.

The followings are from Gelander and Glasner, 2008.

Definition 5. primitive action An action of a group G on a set X is *primitive* if $|X| > 1$ and there are no G-invariant equivalence relations on X apart from the two trivial ones.

The trivial equivalence relations are those with a unique equivalence class, or with singletons as equivalence classes. When $|X| = 2$, we require that the action is not trivial.

Definition 6. quasiprimitive action An action is called *quasiprimitive* if every normal subgroup acts either trivially or transitively.

Definition 7. quasiprimitive group A group is *quasiprimitive* if it admits a faithful quasiprimitive action on a set.

The following is from Li and Liu, 2021.

Definition 8. metacyclic A group G is *metacyclic* if it has a cyclic normal subgroup N such that GN is cyclic.

The followings are notations from Yang et al., 2020.

Notation 1. Let G be a finite group, let S be a subset of G and let π be a set of different primes.

For each prime s, we denote

$$SP_s(S) = \{\langle x\rangle | o(x) = s, x \in S\} \quad \text{and} \quad EP_s(S) = \{x | o(x) = s, x \in S\}$$

also, we denote

$$SP(S) = \cup SP_s(S) \quad \text{and} \quad EP(S) = \cup EP_s(S)$$

$$EP_\pi(S) = \cup_{s\in\pi} EP_s(S)$$

also, we denote

$$NEP(S) = |EP(S)| \quad \text{and} \quad NEP_s(S) = |EP_s(S)|$$

$$NEP_\pi(S) = |EP_\pi(S)|$$

3 Outline

1. provide examples of:
   - a finite solvable group G acts faithfully on a finite vector space V
   - a finite solvable group G acts faithfully, quasi-primitively on a finite vector space V
   - a metacyclic group G
   - a finite solvable, non-metacyclic, group G acts faithfully on a finite vector space V
   - a finite solvable, non-metacyclic, group G acts faithfully , quasi-primitively on a finite vector space V
   - the notations listed above
2. Pick a theorem from the paper that could be expained regarding the limits of pages.

References

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