Algebra 2 Draft

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April 12, 2022

1 Introduction

This paper aims to introduce the properties of irreducible representations of the symmetric group, an extension of the properties of irreducible representation in a boarder context: topological context, so that hopefully some geometric intuitions could be grasped. Besides abstract algebra, some knowledge about point-set topology is assumed.

2 Lie Algebra

Definition 2.1 (matrix Lie group)**.** *Seitz-Mcleese*

*A **matrix Lie group** over a filed $\mathbb{F}$ is a subgroup G of $GL_n(\mathbb{F})$, such that the group multiplication and iversion are smooth maps. need to figure out what smooth maps mean here later*

Definition 2.2 (bracket, commutator, Lie algebra)**.** *Humphreys, 1972*

*A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [xy]$ and called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following axioms are satisfied:*

1. *The bracket operation is bilinear*
   (a) $[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ *and* $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
   (b) $[\lambda x, y] = \lambda[x, y] = [x, \lambda y]$
2. $[xx] = 0$ *for for all* $x \in L$
3. $[x[yz]] + [y[zx]] + [z[xy]] = 0,\ x, y, z \in L$

usually, $[x, y] = xy - yx$

Notice that **1.** and **2.** together implies the **anti-commutativity** of Lie-algebra, namely

$$[x+y, x+y] = [x, x+y]+[y, x+y] = [x, x]+[x, y]+[y, x]+[y, y] = [x, y]+[y, x] = 0$$

so we have another version of **2.**

$$[x, y] = -[y, x]$$

Let's consider our Lie group to be $SL_2(\mathbb{C})$. Intuitively, the Lie algebra that corresponds to a given Lie group is the tangent space of the manifold at the identity element of the group.

To find the Lie algebra, $\mathfrak{sl}_2(\mathbb{C})$, of Lie group ,$SL_2(\mathbb{C})$, we need to use ϵ to find the tangent space. Denote ϵ as a first order infinitesimal, that is, it's closer to 0 than any other real number, $\epsilon \neq 0$ and $\epsilon^2 = 0$. Thus the tangent space at the identity is simply all matrices A s.t. $I + A\epsilon \in SL_2(\mathbb{C})$.

$$I + A\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \epsilon = \begin{bmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{bmatrix}$$

Since we want $I + A\epsilon \in SL_2(\mathbb{C})$, we need to calculate its determinant:

$$\det(I + A\epsilon) = (1 + a\epsilon)(1 + d\epsilon) - cb\epsilon^2 = (1 + (a + d)\epsilon + ad\epsilon^2) - bc\epsilon^2 = 1 + (a + d)\epsilon$$

Thus for the determinant to be zero, we need the trace of $\mathfrak{sl}_2(\mathbb{C})$ to be 0.

3 Steinberg variety

Definition 3.1 (Steinberg variety)**.** *Douglass and Roehrle, 2008*

Let G be a connected, reductive algebraic group defined over an algebraically closed field k, $\mathcal{B}$ is the variety of Borel subgroups of G, and u is a unipotent element in G.

Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{N}$ denote the variety of nilpotent elements in $\mathfrak{g}$. The Steinberg variety of G is

$$Z = \{(x, B, B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} | x \in Lie(B) \cap Lie(B')\}$$

4 Irreducible Representation

Theorem 4.1 (Irreducible Rperesentation of the Symmetric Group.)**.** *Chriss and Ginzburg, 2010*

Let $G = SL_n(\mathbb{C})$, $H(Z)$ stands for the top homology of the Steinberg variety Z.

For any $x \in \mathcal{N}$, let $d(x) = dim_{\mathbb{R}}\mathcal{B}_x$. Then

1. *The $H_m(Z)$-module $H_{d(x)}(\mathcal{B}_x)$ is simple;*
2. *The modules $H_{d(x)}(\mathcal{B}_x)$ and $H_{d(y)}(\mathcal{B}_y)$ are isomorphic if and only if x is conjugate by G to y.*
3. *The collection $\left\{H_{d(x)}(\mathcal{B}_x)\right\}$*

References

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