Algebra 2 Draft

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1 Introduction

This paper aims to introduce the properties of irreducible representations of the symmetric group, an extension of the properties of irreducible representation in a boarder context: topological context, so that hopefully some geometric intuitions could be grasped. Besides abstract algebra, some knowledge about point-set topology is assumed.

The goal is to glance at the infinite-dimensional representations by studying Verma modules.

2 Lie Algebra

Definition 2.1 (bracket, commutator, and Lie algebra, Humphreys, 1972). A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [xy]$ and called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following axioms are satisfied:

1. The bracket operation is bilinear
   (a) $[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ and $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
   (b) $[\lambda x, y] = \lambda[x, y] = [x, \lambda y]$
2. $[xx] = 0$ for for all $x \in L$
3. $[x[yz]] + [y[zx]] + [z[xy]] = 0$, $x, y, z \in L$

usually, $[x, y] = xy - yx$

Notice that **1.** and **2.** together implies the **anti-commutativity** of Lie-algebra, namely

$$[x+y, x+y] = [x, x+y]+[y, x+y] = [x, x]+[x, y]+[y, x]+[y, y] = [x, y]+[y, x] = 0$$

so we have another version of **2.**

$$[x, y] = -[y, x]$$

Let's consider our Lie group to be $SL_2(\mathbb{C})$. Intuitively, the Lie algebra that corresponds to a given Lie group is the tangent space of the manifold at the identity element of the group.

To find the Lie algebra, $\mathfrak{sl}_2(\mathbb{C})$, of Lie group ,$SL_2(\mathbb{C})$, we need to use ϵ to find the tangent space. Denote ϵ as a first order infinitesimal, that is, it's closer to 0 than any other real number, $\epsilon \neq 0$ and $\epsilon^2 = 0$. Thus the tangent space at the identity is simply all matrices A s.t. $I + A\epsilon \in SL_2(\mathbb{C})$.

$$I + A\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \epsilon = \begin{bmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{bmatrix}$$

Since we want $I + A\epsilon \in SL_2(\mathbb{C})$, we need to calculate its determinant:

$$\det(I + A\epsilon) = (1 + a\epsilon)(1 + d\epsilon) - cb\epsilon^2 = (1 + (a + d)\epsilon + ad\epsilon^2) - bc\epsilon^2 = 1 + (a + d)\epsilon$$

Thus for the determinant to be zero, we need the trace of $\mathfrak{sl}_2(\mathbb{C})$ to be 0.

Example 2.2 (basis of $\mathfrak{sl}_2\mathbb{C}$, Fulton and Harris, 2004). The trace of $\mathfrak{sl}_2\mathbb{C}$ needs to be 0, and since they are 2×2 matrices, it means the number on the diagnoal must be the inverse of each other under addition. Explicitly, we could write $\mathfrak{sl}_2\mathbb{C}$ as:

$$\mathfrak{sl}_2\mathbb{C} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{C} \right\}.$$

We could see that each matrix in $\mathfrak{sl}_2\mathbb{C}$ is determined by 3 numbers (a, b, c), thus the basis has 3 elements. If we just let a, b, c to be 1 we have an intuitive basis composed by:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Example 2.3 (bracket of $\mathfrak{sl}_2\mathbb{C}$). Let's define the bracket/commutator of $\mathfrak{sl}_2\mathbb{C}$ as the following. Given any $A, B \in \mathfrak{sl}_2\mathbb{C}$, let:

$$[A, B] = AB - BA.$$

One could check this bracket is well-defined. i.e. it satisfies the property we defined in 2.1

3 Weyl group

Definition 3.1 (adjoint map). Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Then for each $A \in G$, define a linear map $Ad_A : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$Ad_A(X) = AXA^{-1}.$$

Definition 3.2 (Weyl group). Let $\mathfrak{h}$ be the two-dimensional subspace of $sl(3, \mathbb{C})$. Let N be the subgroup of $SU(3)$ consisting of those $A \in SU(3)$ s.t. $Ad_A(H)$ is an element of $\mathfrak{h}$ for all $H \in \mathfrak{h}$. Let Z be the subgroup of $SU(3)$ consisting of those $A \in SU(3)$ s.t. $Ad_A(H) = H$ for all $H \in \mathfrak{h}$.

The **Weyl group** of $SU(3)$, denoted W, is the quotent group N/Z.

References

N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*. Birkhäuser Boston, Boston, 2010. ISBN 978-0-8176-4937-1 978-0-8176-4938-8. doi: 10.1007/978-0-8176-4938-8. URL http://link.springer.com/10.1007/978-0-8176-4938-8.

J. M. Douglass and G. Roehrle. The Steinberg Variety and Representations of Reductive Groups. *arXiv:0802.0764 [math]*, Oct. 2008. URL http://arxiv.org/abs/0802.0764. arXiv: 0802.0764.

W. Fulton and J. Harris. *Representation Theory*, volume 129 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 2004. ISBN 978-3-540-00539-1 978-1-4612-0979-9. doi: 10.1007/978-1-4612-0979-9. URL http://link.springer.com/10.1007/978-1-4612-0979-9.

J. E. Humphreys. *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

M. Seitz-Mcleese. CLASSIFYING THE REPRESENTATIONS OF sl2(C). page 19.