Algebra 2 Term Paper Proposal

Yuxuan Sun

March 18, 2022

1 Introduction

This paper aims to offer sufficient introductions and examples to understand that given a finite solvable group G which acts faithfully, irreducibly and quasi-primitively on a finite vector space V, if ~~and~~ G is not metacylic, then G always has a regular orbit on V except for a few small cases (Yang et al., 2020).

Do you plan on listing the exceptions? Or is that too technical? You should at least explain something about the exceptions.

2 Terms and Theorems

The followings are from Artin, 1991.

Definition 1 (simple group)**.** A group G is *simple* if it is not the trivial group and if it contains no proper normal subgroup - no normal subgroup other than $\langle 1 \rangle$ and G.

Corollary 1. Cyclic groups of prime order are simple groups.

Definition 2 (solvable)**.** A finite group G is *solvable* if it contains a chain~~s~~ of subgroups

$$G = H_0 \subset H_1 \subset H_1 \subset \ldots \subset H_k = \{1\}$$

such that for every $i = 1, \ldots, k$, H_i is a normal subgroup of H_{i-1}, and the quotient group $H_i H_{i+1}$ is a cyclic group.

The followings are from Fawcett et al., 2016.

Definition 3 (base)**.** Let G be a finite group acting faithfully on a set Ω. A *base* $\mathscr{B}$ for G is a non-empty subset of Ω with the property that only the identity fixes every element of $\mathscr{B}$.

Definition 4 (regular orbit)**.** Let G be a finite group acting faithfully on a set Ω. If there is a base $\mathscr{B} = \{\omega\}$ for some $\omega \in \Omega$, the~~n~~ the orbit $\{\omega g : g \in G\}$ of G on Ω is regular.

(Also look for alternate forms of this definition...)

The followings are from Gelander and Glasner, 2008.

Definition 5. (primitive action) An action of a group G on a set X is *primitive* if $|X| > 1$ and there are no G-invariant equivalence relations on X apart from the two trivial ones.

The trivial equivalence relations are those with a unique equivalence class, or with singletons as equivalence classes. When $|X| = 2$, we require that the action is not trivial.

Definition 6. quasiprimitive action An action is called *quasiprimitive* if every normal subgroup acts either trivially or transitively.

Is there a connection to primitive action?

Definition 7. quasiprimitive group A group is *quasiprimitive* if it admits a faithful quasiprimitive action on a set.

The following is from Li and Liu, 2021.

Definition 8. metacyclic A group G is *metacyclic* if it has a cyclic normal subgroup N such that GN is cyclic.

The followings are notations from Yang et al., 2020.

Notation 1. Let G be a finite group, let S be a subset of G and let π be a set of different primes.

For each prime s, we denote

$$SP_s(S) = \{\langle x\rangle | o(x) = s, x \in S\} \quad \text{and} \quad EP_s(S) = \{x | o(x) = s, x \in S\}$$

also, we denote

\bigcup

$$SP(S) = \cup SP_s(S) \quad \text{and} \quad EP(S) = \cup EP_s(S)$$

$$EP_\pi(S) = \cup_{s\in\pi} EP_s(S)$$

also, we denote

$$NEP(S) = |EP(S)| \quad \text{and} \quad NEP_s(S) = |EP_s(S)|$$

$$NEP_\pi(S) = |EP_\pi(S)|$$

It's not clear to me that this notation is really needed. Why not write $|EP(S)|$, etc?

3 Outline

1. provide examples of:
   - a finite solvable group G acts faithfully on a finite vector space V
   - a finite solvable group G acts faithfully, quasi-primitively on a finite vector space V
   - a metacyclic group G

     perhaps also an example of an action that meets all the conditions, and for which you exhibit the regular orbit?
   - a finite solvable, non-metacyclic, group G acts faithfully on a finite vector space V

     presumably you want an example without a regular orbit in these cases..
   - a finite solvable, non-metacyclic, group G acts faithfully , quasi-primitively on a finite vector space V
   - the notations listed above

     Beyond an example, also give context for the notation – from the definitions alone it's hard for me to see what the point of the notation is.
2. Pick a theorem from the paper that could be expained regarding the limits of pages.

I suggest you actually start your work here. Before working on Section 1, try to map out the general outline of the theorem so that you can plan what you want to include in Section 2; then, go back to section 1, emphasizing examples that help explain what you've chosen to write about in section 2.

References

M. Artin. *Algebra*. Prentice Hall, Inc., Englewood Cliffs, NJ, 1991. ISBN 0-13-004763-5.

J. B. Fawcett, E. A. O'Brien, and J. Saxl. Regular orbits of symmetric and alternating groups. *Journal of Algebra*, 458:21–52, July 2016. ISSN 00218693. doi: 10.1016/j.jalgebra.2016.02.018. URL http://arxiv.org/abs/1812.05880. arXiv: 1812.05880.

T. Gelander and Y. Glasner. Countable Primitive Groups. *Geometric and Functional Analysis*, 17(5):1479–1523, Jan. 2008. ISSN 1016-443X, 1420-8970. doi: 10.1007/s00039-007-0630-y. URL http://link.springer.com/10.1007/s00039-007-0630-y.

P. Li and R. Liu. Finite p-groups all of whose proper subgroups of class 2 are metacyclic. *Comm. Algebra*, 49(4):1667–1675, 2021. ISSN 0092-7872. doi: 10.1080/00927872.2020.1843048. URL https://doi.org/10.1080/00927872.2020.1843048.

Y. Yang, A. Vasil'ev, and E. Vdovin. Regular orbits of finite primitive solvable groups, III. *arXiv:1612.05959 [math]*, Dec. 2020. URL http://arxiv.org/abs/1612.05959. arXiv: 1612.05959.

You should have annotated this bibliography, i.e., made some remarks about how you think each source will be useful $-\frac{1}{2}$

Are you only using these papers for definitions, or are they substantially related to Yang, et al? If the former, it is probably better to use a general group theory text for the definitions... see below.

Yuxuan,

This is a good proposal and you are ready to start working on the paper. See my remarks on page 2 about the strategy of starting with Section 2. You may need a group theory text as a reference; one I often recommend is

Rotman, *An Introduction to the theory of groups*
https://mathscinet.ams.org/mathscinet-getitem?mr=1307623

Have fun working on this, and let me know if you have any questions!

-David