Analysis 2 Functional Analysis

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Contents

1 Basics

Definition 1.1: characteristic function

Given $S \subset \mathbb{R}$, the corresponding characteristic function is

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \in S^C \end{cases}$$

Definition 1.2: Lebesgue integral of step function

Given a step function

$$f(x) = c_1\chi_{I_1}(x) + c_2\chi_{I_2}(x) + \ldots + c_n\chi_{I_n}(x)$$

We define its Lebesgue integral to be

$$\int_{-m}^{\infty} f(x)dx = c_1 m(I_1) + c_2 m(I_2) + \cdots + c_n m(I_n)$$

2 L1

Definition 2.1: $L^1\mathbb{R}$

We say $f : \mathbb{R} \to \mathbb{R}$ is in $L^2(\mathbb{R})$ if there are functions $g, h \in L^0(\mathbb{R})$ s.t. $f = g - h$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} g(x)dx - \int_{-\infty}^{\infty} h(x)dx$$

Theorem 2.2: $L^1(\mathbb{R})$ is a vector space

If $f_1, f_2 \in L^1(\mathbb{R})$ and c_1, c_2 are real constancts, then $c_1 f_1 + c_2 f_2 \in L^1(\mathbb{R})$

Theorem 2.3: $L^1(\mathbb{R})$ order integral

If $f_1, f_2 \in L^1(\mathbb{R})$ and $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f_1(x)dx \geq \int_{-\infty}^{\infty} f_2(x)dx.$$

Definition 2.4: $L^1_{nvs}(\mathbb{R})$

An element of $L^1_{nvs}(\mathbb{R})$ is a collection of functions in $L^1(\mathbb{R})$: specifically, two functions are in the same collection if they are equal except on a set of measure zero.
Given a collection S in $L^1_{nvs}(\mathbb{R})$, define $\|S\|_1$ by choosing any $f \in S$ and defining $\|S\|_1 = \|f\|_1$

Theorem 2.5: Monotone Convergence Theorem in L^1

Let $f_n \in L^1(\mathbb{R})$ which monotone increases for all $x \in \mathbb{R}$.
Suppose $\left\{\int_{-\infty}^{\infty} f_n(x)dx \; n \in \mathbb{N}\right\}$ is bounded.
Then there exists $f \in L^1(\mathbb{R})$ s.t. $f_n \to f$ pointwise except possibly on a set of measure zero, and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$$

Theorem 2.6: $L^1([a,b])$ is complete

If $f_n \in L^1([a,b])$ is a Cauchy sequence (with resepct to $\|\cdot\|_1$). Then there exists $f \in L^1([a,b])$ s.t. $f_n \to f$ in L^1

Definition 2.7: the spaces $L^p(\mathbb{R})$

For $p > 1$, we say that $f \in L^p(\mathbb{R})$ if f is a measurable function and $\int_{-\infty}^{\infty} |f(x)|^p dx$ is a finite number.

Theorem 2.8: $L^p(\mathbb{R})$ is a vector space

$L^p(\mathbb{R})$ is a vector space.

3 $L^1([a,b])$ and Fundamental THeorems of Calculus

Definition 3.1

if $f : [a,b] \to \mathbb{R}$, we say that $f \in L^1([a,b])$ if the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

is in $L^1(\mathbb{R})$. In that case, we write $\int_a^b f(x)dx = \int_{-\infty}^{\infty} g(x)dx$

4 L^2

Theorem 4.1: Inner Product on $L^2(\mathbb{R})$

if $f, g \in L^2$, then $fg \in L^1$ with:

$$\int_{-\infty}^{\infty} |f(x)g(x)| \leq \|f\|_2 \|g\|_2$$

Definition 4.2: Inner product on L^2

If $f, g \in L^2$, let:

$$\langle f, g \rangle - \int_{-\infty}^{\infty} f(x)g(x)dx$$

5 Geometry Recap

Theorem 5.1: Parallel-gram Law

d_1 and d_2 being the diagnoal an

$$s^2 + s^2 + t^2 + t^2 = d_1^2 + d_2^2$$

Corollary 5.2: inner product

$\|\cdot\|_2$ has an inner product that's like the dot product.
$\|\cdot\|_p$ and $\|\cdot\|_\infty$ don't have inner product.

6 Fourier

Example 6.1: orthonormal set

i.e. the inner product is 0

$$\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{2\pi}}, \frac{\cos 2x}{\sqrt{\pi}}$$

restricted to domain $[-\pi, \pi]$

Definition 6.2: Fourier senes

Given $f \in L^2$, define its Fourier senes as:

$$FS_f(x) = \langle f, f_0 \rangle f_0(x) + \langle f, f_1 \rangle f_1(x) + \langle f, f_2 \rangle f_2(x) + \langle f, f_3 \rangle f_3(x) + \dots$$

Comments
values of f outside $[-\pi, \pi]$ have no impact on FS_f
we could assume $f = 0$ outside $[-\pi, \pi]$, i.e. $f \in L^2([-\pi, \pi))$

Definition 6.3: inner product in L^2

Given $f, g \in L^2(\mathbb{R})$, we define there inner product by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

Theorem 6.4

If $f \in L^2([-\pi, \pi])$, $FS_f \to f$ in L^2

Theorem 6.5

If $f \in C([-\pi, \pi])$ and $f(\pi) = f(-\pi)$, then $FS_f \to f$ uniformly on $[-\pi, \pi]$

7 Fourier transform

Definition 7.1: rapidly decresing

f is rapidly decreasing if, for any $n \in \mathbb{N}$, there exists M_n, C_n s.t. $|f(x)| \leq C_n/x^n$ for all x with $|x| > M_n$. An alternative perspective: f is rapidly decreasing if and only if for any polynomial $p(x)$, we have $\lim_{x\to\infty} p(x)f(x) = \lim_{x\to-\infty} p(x)f(x) = 0$.

Definition 7.2: $C^\infty(\mathbb{R})$

$f \in C^\infty(\mathbb{R})$ is f has infinitely many derivatives at all $x \in \mathbb{R}$

Definition 7.3: $S(\mathbb{R})$

$f \in S(\mathbb{R})$ (Schwartz-class) if f is rapidly decreasing and in $C^\infty(\mathbb{R})$

$e^{-x^2} \in S(\mathbb{R})$

Definition 7.4: $\mathcal{D}([a, b])$

$f \in \mathcal{D}([a, b])$ if $C \in C^{\infty}(\mathbb{R})$ and $f = 0$ outside of $[a, b]$

Theorem 7.5: L^2 **and fourier**

if $f \in L^2([-T, T])$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2T} C_n e^{-in\pi x/T}$$

for

$$C_n = \int_{-T}^{T} f(x) e^{in\pi x/T}$$