# Notes on Frames and Locales

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This is my notes on Picardo and Pultr's book *Frames and Locales*, which is a good monograph of point-free topology. This notes will, at least, as my intent, *not* be something contain the full contents of the book, instead, it should serve as a helper to prevent me from fogetting everything. And I hope this notes can clarify as much misunderstoodings as posible.

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# 1. June 1, 2024

There are many preliminaries needed be established first: the category **Pos** which consists of all partial order sets<sup>1</sup>, the category **Lac** of lattices, Galois adjunctions, Heyting and Boolean algebras. The story is not long, but give yourself some time to digest it and, *appreciate* it.

# 1.1. The category Pos and Loc

The essence of a category is its morphisms, which respect some specific structures, so what maps should be, or can be morphisms? Well, just look at the poset's structures – order.

<sup>&</sup>lt;sup>1</sup>Ingore size issues, of course.

**Definition 1.1.1** (Monotone functions): Those polite maps respect the structures of posets, i.e., a map  $f: A \to B$  is called monotone if and only if

$$u \le v \Rightarrow f(u) \le f(v)$$
.

Of course, both A and B are posets.

There is nothing more interesting than its morphisms, in **Pos**. However, the funny fact is that any poset P is itslef a category, so the objects in the category **Pos** are also categories! We use **Pos**(P) or just P to denote the category generated by a single poset P, as follows:

**Definition 1.1.2** (View posets as categories): Given any poset P, it can be viewd as a category, where

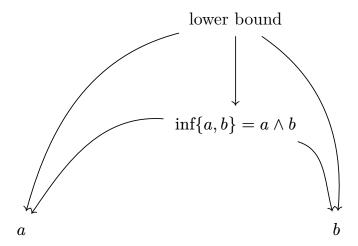
- Objects: elements in it;
- Morphisms: define morphisms between two objects (hence elements) f:  $a \to b$  if and only if  $a \le b$ ;
- Identity morphisms exist: for all a,  $a \le a$  by the definition of partial order, which induce a morphism;
- Composition rules: by the transivity of partial order.

Remark: Further, a functor between two categories  $\mathbf{Pos}(X)$  and  $\mathbf{Pos}(Y)$  is exactly a monotone map m between X and Y, viewd as posets.

- Preserve morphisms: if  $f: x_1 \to x_2$  is a morphism in  $\mathbf{Pos}(X)$ , then there exist a morphism  $m(f): m(x_1) \to m(x_2)$  in  $\mathbf{Pos}(Y)$  as well. And this is just a fancy way to describe the monotonity;
- Preserve indentity: also trivial;
- Preserve composition: by monotonity and transivity.

In addition, one can find that the *suprema* of a poset A is just the *coproduct* (or sum) of the categry, and infima is the product. And the bottom  $\bot$  is exactly the initial object (hence we sometimes use 0 to denote it), while  $\top$  is the terminal object (hence 1, sometimes). This observation is important, since we know left adjoint preserve coproduct<sup>2</sup> and right adjoint preverse product.

<sup>&</sup>lt;sup>2</sup>Actually, co-limit. Also the "prodcut" later is "limit"



Don't forget the morphism  $a \to b$  is defined if and only if  $a \le b$ .

### 1.2. Galois adjunctions

Galois adjunctions are just adjunctions between two posets A and B, viewd as categories, that's all:

$$\operatorname{Hom}_A(f(a), b) \cong \operatorname{Hom}_B(a, g(b)).$$

Unpack the definition of adjunctions, we immediately get the explicit version of the Galois adjunctions between two posets A and B:

**Definition 1.2.1** (Galois adjunctions): Two monotone functions  $f: A \to B$  and  $g: B \to A$  are said to be *Galois adjoint* if for all  $a \in A$  and  $b \in B$ , we have

$$f(a) \leq b \Longleftrightarrow a \leq g(b).$$

This is essencially a reformation of the abstract definition. Note that, if we are in the category setting, the left adjoint of g is the f, since each left adjoint is isomorphic to others, of course right adjoints share this note as well.

Example: The so-called "to-range" map  $f: \mathcal{P}(A) \to \mathcal{P}(B)$  with its "inverse"  $f^{-1}$  are adjoint to each other, since

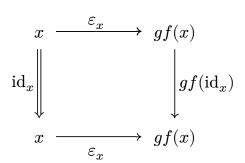
$$f(U)\subset V \Longleftrightarrow U\subset f^{-1}(V).$$

Here is a interesting proposition, where f and g are more closed to each other – they are on the same side:

**Proposition 1.2.1**: Two morphisms  $f: X \to Y$  and  $g: Y \to X$  in **Pos** are adjoint to each other, if and only if for all  $x \in X$  and  $y \in Y$ , there holds

$$fg(y) \le y$$
 and  $x \le gf(x)$ .

*Proof*: (Abstract nonsense version). As noted in Definition 1.1.2, f and g can be viewed as functors as well. Guess what? By category theory, two functors are adjoint if and only if there exists natural transformations "unit"  $\eta: \mathrm{id}_X \Longrightarrow gf$  and "counit"  $\varepsilon: fg \Longrightarrow \mathrm{id}_Y$ , in diagram, it says, for all  $x \in X$ ,



commutes. Actually the commutativity is a little bit too much, we only need the existence of  $\varepsilon_x$ , which indicate that  $x \leq gf(x)$ .

Man, what can I say? Category theory is based.

*Proof*: (Not based version). Note that the proposition has two sides:

- $(\Longrightarrow)$  Move the outside f and g in both inequalities and immediately we find it obvious.
- ( $\Leftarrow$ ) This is more interesting. Keep in mind that now, we want to prove

$$f(a) \leq b \Longleftrightarrow a \leq g(b),$$

so by the introduction rule, we firstly obtain  $f(a) \leq b$  and goal is  $a \leq g(b)$ . So by assumption,

$$a \leq gf(a) = g(f(a)) \leq g(b),$$

which is we wanted.

**Theorem 1.2.2**: Left Galois adjoint preserve suprema, while the right one preserve infima.

*Proof*: Guess what? By **Category theory**, left adjoint preserve not only coproduct (hence suprema), but also co-limit, and the situation of right adjoint is similar, qed.

In lattice theory, the converse statement is also true:

**Theorem 1.2.3**: If a monotone maps  $f: X \to Y$  preverse suprema, then it is a left adjoint.

The goal is to construct, in a explicit way, a map  $g: Y \to X$  satisfies the definition. That is to arrange each  $y \in Y$  to some value in X. Let's first assume g is indeed athe right adjoint of f, and find some properties it **must** satisfy which guide our construction.

We claim that g(y) must be the suprema of all x such that  $f(x) \leq y$ , i.e., this equation must holds:

$$g(y) = \sup\{x : f(x) \le y\}.$$

There is no doubt g(y) is a general upper bound of  $\{x: f(x) \leq y\}$ , also note this set contain g(y) itself as well, since  $g(y) \leq g(y)$  and then by the adjunction's property. Accordingly, our cliam is true.

So there is actually no other ways to define the behaviour of g, we are **forced** to arrange each  $y \in Y$  to  $\sup\{x : f(x) \le y\}$ , no other choices.

*Proof of Theorem 1.2.3*: Define  $g: Y \to X, y \mapsto \sup\{x: f(x) \le y\}$ . And we verify this is indeed the right adjoint of f:

- $f(x) \leq y$  implies  $x \leq g(y)$ : this is trivial by the definition of g(y).
- $x \leq g(y)$  implies  $f(x) \leq y$ : Please think about the assumption that f preverse arbitrary suprema, which hasn't been used yet. Apply f to both side of the inequality  $x \leq g(y)$ , and by the monotonity:

$$f(x) \leq f(g(y)) = f(\sup\{u: f(u) \leq y\});$$

since f preserve suprema, the last stuff equals to  $\sup\{f(u): f(u) \leq y\}$ , which is aparently less than or equal to y.

#### 1.3. Lattices

Lattices are just better posets, we concern those fancy sets since the topology of a sapce X, or  $\Omega X$ , the set of all its open sets is indeed a lattice.<sup>3</sup>

<sup>&</sup>lt;sup>3</sup>Why don't you use paratheness, namely use  $\Omega(X)$ ? Lol, because the world is ruled by **Category theory**,  $\Omega$  is a *functor*!

**Definition 1.3.1** (Lattices): Lattices are just posets, which are close under *finite* meet and join.

Remark: In the book, the authors differ the notion of lattices and bounded lattices, while we do not – lattices are bounded lattices.

The notion of lattices is necessary, we show that there exists poset which is not a lattice: equip the two points set  $\{*, \dagger\}$  with the partial order defined as exactly equal, i.e.,  $x \sqsubset y$  if and only if x = y. This dummy set is indeed a poset, but neither  $(* \land \dagger)$  nor  $(* \lor \dagger)$  exists.

The following definition is a rutine.

**Definition 1.3.2** (*Complete* Lattices): If a lattice admits arbitrary size of meet and join, then it is called *complete* lattice.

#### 1.4. Distributive lattices

Why study distributive lattices? Well, since  $\Omega X$  is a lattice, we'd better, then, clasify some specific lattices and capture as much properties as possible, and a satisfying thing is that

- Any Boolean algebra, and even any Heyting algebra, is a distributive lattice.
- Every frame and every  $\sigma$ -frame is a distributive lattice.

— https://ncatlab.org/nlab/show/distributive+lattice

**Definition 1.4.1** (**Distributive lattices**): A lattice is *distributive* if and only if this distributive law holds:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Remark: An immediate observation is that this law implies the dual one:

$$a \lor (b \land c) = (a \lor b) \land (a \lor c);$$

which can be proved straightforward:  $(a \lor b) \land (a \lor c) = ((a \lor b) \land a) \lor ((a \lor b) \land c)$ , which by the substitution of the former one's a by  $(a \lor b)$ , and the right-hand-side can be computed, with the fact  $(a \lor b) \land a = a$ , as  $c \lor b \lor (c \lor b) \lor (c \lor b) \lor (c \lor b) \lor (c \lor b)$ , where the last equality followed by that  $\lor$  is associative.

By the remark above, we distil this conclusion:

A lattice L is distributive  $\iff$  its opposite lattice  $L^{\text{op}}$  is distributive.

There are more symmetric laws, but, please consult to neatlab.

#### 1.5. Filters and ideals

So you already known what is a

**Definition 1.5.1** (Filter): A filter F on a distributive lattice L is a subset of L, such that

- $1 \in F$ ; (the top element is big)
- $a \in F$  and  $b \in F \Longrightarrow a \land b \in F$ ; (two big sets' intersection is also big)
- $a \in F$ , and  $b \ge a \Longrightarrow b \in F$  (if you are bigger than a big guy, then you are big too)

Remark: Sometimes we talk about prime filters, those filters somehow act like prime numbers: a filter F is called prime if and only if

$$a \lor b \in F \Longrightarrow a \in F \text{ or } b \in F$$
;

compare with prime numbers:

$$m \times n = p \Longrightarrow m \mid p \text{ or } n \mid p$$
.

The definition of *ideal* is just a filter in the opposite lattice, or just modify the above definition to a "small guys" version.

The main result of this section is the so called "ultralfilters exists" theorem, stated as bellow:

**Lemma 1.5.1**: Let J be an ideal and F be a filter, which satisfy their intersection is empty. Then there exists a maximal prime filter  $\overline{F}$  which respect to the property  $\overline{F}$  does not intersects with J.

*Proof*: There are two points to prove:

- Maximum: when talk about maximum, one should always remember the Zorn lemma, which is exactly a proposition about maximal things. And, note that, there is not too much restrictions, being *envy* to construct what you want is welcome. (proof omitted, as exercise. Lol)
- Primeness: this is more chanllengable, give me some times to digest it.

# 1.6. Pseudocomplement

This is the last section of today (I hope), fuh!

Since defining Heyting algebra needs the preparation of pseudocomplement, we first define this slightly abstract notion:

**Definition 1.6.1** (Pseudocomplement): In a  $\land$ -lattice, The pseudocomplement of a element  $a \in F$  is the *largest* element x such that

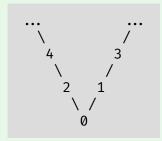
$$x \wedge a = 0$$
,

where "largest" means for all  $b \in F$ ,

$$b \wedge a = 0 \iff b \leq x$$
.

We use the convenient notation  $a^*$  to denote this (unique) pseudocomplement.

*Remark*: Of course, this pseudocomplement need not to exist. For instance:



### 1.7. Heyting algebras

Heyting algebras captrue the notion of "semantics implication" in some sense – those structures are equipped with a so-called *Heyting operation* " $\rightarrow$ ", which can be indentified with the logic implication. There is a interesting correspondence between implications and complements of elements: in (classic) logic, 0 is the complement of 1 and vice versa, with these equations:

$$(1 \rightarrow 0)$$
 = the complement of 1;

$$(0 \rightarrow 0)$$
 = the complement of 0;

by definition of implication.

A Heyting algebra extends this correspondence: change "complement" to "pseudocomplement" and add more elements other than 1 and 0, at the same time preserving the equations above, namely the pseudocomplement  $a^*$  of  $a \in H$  is equal to  $a \to 0$ . And this Heyting operation can be extended, too, that is we can define more formulars than just  $(\Box \to 0)$ , so, finally, here is the definition:

**Definition 1.7.1** (Heyting algebras): A lattice is called a *Heyting algebra* if it is equipped with a binary *Heyting operation* " $\rightarrow$ " such that

$$c \leq a \rightarrow b$$
 if and only if  $c \wedge a \leq b$ .

Remark:

- This law is indeed a Galois adjunction, where the left adjoint is the map  $(\square \mapsto a \wedge \square)$  and the right one is  $(\square \mapsto (a \to \square))$ .
- This law ensures that the Heyting operation is indeed a extension of implication of classic logic, i.e.,

$$a^* = a \to 0$$
,

where  $a^*$  is the pseudocomplement of a.

The equation in the definition is **like** the definition of Galois adjunction, but we need to justify that these maps are really *leagal morphisms* – monotone maps.

• The left adjoint is monotone: this is obvious, since by the definition of infima:

$$x \le y \Longrightarrow u \land x \le u \land y$$
.

• The right guy: suppose  $x \leq y$  and fix u, we want to prove  $(u \to x) \leq (u \to y)$ , which is equavalent to, by definition,  $u \wedge (u \to x) \leq y$ . The trick is:

$$u \wedge (u \to x) \leq x$$

since  $u \to x = u \to x$ .

There is a striking result, of the specific left adjoint  $(\Box \mapsto (a \land \Box))$ :

**Theorem 1.7.1**: Every Heyting algebra is distributive.

*Proof*: A lattice is distributive if and only if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

And a left adjoint preverse suprema (the  $\vee$  operation), apply this little lemma to our left adjoint, we immediately obtain the distributive law.

Thanks to Galois adjunction, we can obtain two more amazing theorems about Heyting algebra, which is of sattisfactory, in all sense:

### **Theorem 1.7.2**: Heyting operation is unique

*Proof*: By the uniqueness of adjoint, and notice the left one  $(\Box \mapsto (a \land \Box))$  is fixed.

**Theorem 1.7.3**: A lattice admits a Heyting algebra if this arbitrary distributive law holds:

$$b \wedge (\vee_{i \in I} \; a_i) = \vee_{i \in I} \; (b \wedge a_i).$$

*Proof*: By Theorem 1.2.3, plus the observation of monotonity of the map  $(\Box \mapsto (a \land \Box))$ , we obtain a right adjoint of this map, but don't forget the Heyting operation *is* a right adjoint of it and, adjointy *is* unique. We are done.

Some warm notices:

- All Heyting algebras are distributive, that's true.
- All arbitrary distributive lattices are equipped with Heyting operations, that's also true.
- General distributive lattices don't promise arbitrary distributive law.

# 1.8. TODO: Boolean algebras