Notes on Frames and Locales

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This is my notes on Picardo and Pultr's book *Frames and Locales*, which is a good monograph of point-free topology. This notes will, at least, as my intent, *not* be something contain the full contents of the book, instead, it should serve as a helper to prevent me from fogetting everything. And I hope this notes can clarify as much misunderstoodings as posible.

Contents

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There are many preliminaries needed be established first: the category **Pos** which consists of all partial order sets[1], the category **Lac** of lattices, Galois adjunctions, Heyting and Boolean algebras. The story is not long, but give yourself some time to digest it and, *appreciate* it.

1.1. The category Pos and Loc

The essence of a category is its morphisms, which respect some specific structures, so what maps should be, or *can* be morphisms? Well, just look at the poset's structures – order.

[1]Ingore size issues, of course.

Definition 1.1.1 (Monotone functions): Those polite maps *respect* the structures of posets, i.e., a map $f : A \to B$ is called *monotone* if and only if

$$u \leq v \Rightarrow f(u) \leq f(v).$$

Of course, both A and B are posets.

There is nothing more interesting than its morphisms, in **Pos**. However, the funny fact is that any poset P is itslef a category, so the objects in the category **Pos** are also categories! We use $\mathbf{Pos}(P)$ or just P to denote the category generated by a single poset P, as follows:

Definition 1.1.2 (View posets as categories): Given any poset P, it can be viewd as a category, where

- Objects: elements in it;
- Morphisms: define morphisms between two objects (hence elements) $f : a \to b$ if and only if $a \leq b$;
- Identity morphisms exist: for all a, $a \leq a$ by the definition of partial order, which induce a morphism;
- Composition rules: by the transivity of partial order.

Remark: Further, a functor between two categories $\mathbf{Pos}(X)$ and $\mathbf{Pos}(Y)$ is exactly a monotone map m between X and Y, viewd as posets.

- Preserve morphisms: if $f : x_1 \to x_2$ is a morphism in $\mathbf{Pos}(X)$, then there exist a morphism $m(f) : m(x_1) \to m(x_2)$ in $\mathbf{Pos}(Y)$ as well. And this is just a fancy way to describe the monotonity;
- Preserve indentity: also trivial;
- Preserve composition: by monotonity and transivity.

In addition, one can find that the *suprema* of a poset A is just the *coproduct* (or *sum*) of the category, and *infima* is the *product*. And the bottom $\bot$ is exactly the initial object (hence we sometimes use 0 to denote it), while $\top$ is the terminal object (hence 1, sometimes). This observation is important, since we know left adjoint preserve coproduct[2] and right adjoint preverse product.

[2]Actually, co-limit. Also the "prodcut" later is "limit"

$$
\begin{tikzcd}
 & \text{lower bound} \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \inf\{a, b\} = a \wedge b \arrow[dl, bend right] \arrow[dr, bend left] & \\
a & & b
\end{tikzcd}
$$

Don't forget the morphism $a \to b$ is defined if and only if $a \leq b$.

1.2. Galois adjunctions

Galois adjunctions are just adjunctions between two posets A and B, viewd as categories, that's all:

$$\mathrm{Hom}_A(f(a), b) \cong \mathrm{Hom}_B(a, g(b)).$$

Unpack the definition of adjunctions, we immediately get the explicit version of the Galois adjunctions between two posets A and B:

Definition 1.2.1 (**Galois adjunctions**): Two monotone functions $f : A \to B$ and $g : B \to A$ are said to be *Galois adjoint* if for all $a \in A$ and $b \in B$, we have

$$f(a) \leq b \Longleftrightarrow a \leq g(b).$$

This is essencially a reformation of the abstract definition. Note that, if we are in the category setting, the left adjoint of g is *the* f, since each left adjoint is isomorphic to others, of course right adjoints share this note as well.

Example: The so-called "to-range" map $f : \mathcal{P}(A) \to \mathcal{P}(B)$ with its "inverse" f^{-1} are adjoint to each other, since

$$f(U) \subset V \Longleftrightarrow U \subset f^{-1}(V).$$

Here is a interesting proposition, where f and g are more closed to each other – they are on the same side:

Proposition 1.2.1: Two morphisms $f : X \to Y$ and $g : Y \to X$ in **Pos** are adjoint to each other, if and only if for all $x \in X$ and $y \in Y$, there holds

$$fg(y) \leq y \quad \text{and} \quad x \leq gf(x).$$

Proof: (Abstract nonsense version). As noted in Definition 1.1.2, f and g can be viewed as *functors* as well. Guess what? By category theory, two functors are adjoint if and only if there exists natural transformations "unit" $\eta : \mathrm{id}_X \Longrightarrow gf$ and "counit" $\varepsilon : fg \Longrightarrow \mathrm{id}_Y$, in diagram, it says, for all $x \in X$,

$$\begin{tikzcd} x \arrow[r, "\varepsilon_x"] \arrow[d, Rightarrow, "\mathrm{id}_x"'] & gf(x) \arrow[d, "gf(\mathrm{id}_x)"] \\ x \arrow[r, "\varepsilon_x"'] & gf(x) \end{tikzcd}$$

commutes. Actually the commutativity is a little bit too much, we only need the existence of ε_x, which indicate that $x \leq gf(x)$.

Man, what can I say? Category theory is baesd. ■

Proof: (Not based version). Note that the proposition has two sides:

- ($\Longrightarrow$) Move the outside f and g in both inequalities and immediately we find it obvious.
- ($\Longleftarrow$) This is more interesting. Keep in mind that now, we want to prove

$$f(a) \leq b \Longleftrightarrow a \leq g(b),$$

so by the introduction rule, we firstly obtain $f(a) \leq b$ and goal is $a \leq g(b)$. So by assumption,

$$a \leq gf(a) = g(f(a)) \leq g(b),$$

which is we wanted. ■

Theorem 1.2.2: Left Galois adjoint preserve suprema, while the right one preserve infima.

Proof: Guess what? By **Category theory**, left adjoint preserve not only coproduct (hence suprema), but also co-limit, and the situation of right adjoint is similar, qed. ■

In lattice theory, the converse statement is also true:

Theorem 1.2.3: If a monotone maps $f : X \to Y$ preverse suprema, then it is a left adjoint.

The goal is to construct, in a explicit way, a map $g : Y \to X$ satisfies the definition. That is to arrange each $y \in Y$ to some value in X. Let's first assume g is indeed athe right adjoint of f, and find some properties it **must** satisfy which guide our constructtion.

We claim that $g(y)$ **must** be the suprema of all x such that $f(x) \leq y$, i.e., this equation must holds:

$$g(y) = \sup\{x : f(x) \leq y\}.$$

There is no doubt $g(y)$ is a general upper bound of $\{x : f(x) \leq y\}$, also note this set contain $g(y)$ itself as well, since $g(y) \leq g(y)$ and then by the adjunction's property. Accordingly, our cliam is true.

So there is actually no other ways to define the behaviour of g, we are **forced** to arrange each $y \in Y$ to $\sup\{x : f(x) \leq y\}$, no other choices.

Proof ***of*** **Theorem 1.2.3**: Define $g : Y \to X, y \mapsto \sup\{x : f(x) \leq y\}$. And we verify this is indeed the right adjoint of f:

- $f(x) \leq y$ implies $x \leq g(y)$: this is trivial by the definition of $g(y)$.
- $x \leq g(y)$ implies $f(x) \leq y$: Please think about the assumption that f preverse arbitrary suprema, which hasn't been used yet. Apply f to both side of the inequality $x \leq g(y)$, and by the monotonity:

$$f(x) \leq f(g(y)) = f(\sup\{u : f(u) \leq y\});$$

since f preserve suprema, the last stuff equals to $\sup\{f(u) : f(u) \leq y\}$, which is aparently less than or equal to y. ■

1.3. Lattices

Lattices are just better posets, we concern thoes fancy sets since the topology of a sapce X, or ΩX, the set of all its open sets is indeed a lattice.[3]

[3]Why don't you use paratheness, namely use $\Omega(X)$? Lol, because the world is ruled by **Category theory**, Ω is a *functor*!

Definition 1.3.1 (**Lattices**): Lattices are just posets, which are close under *finite* meet and join.

Remark: In the book, the authors differ the notion of lattices and *bounded* lattices, while we **do not** – lattices *are* bounded lattices.

The notion of lattices is necessary, we show that there exists poset which is not a lattice: equip the two points set $\{*, \dagger\}$ with the partial order defined as exactly equal, i.e., $x \sqsubset y$ if and only if $x = y$. This dummy set is indeed a poset, but neither $(* \wedge \dagger)$ nor $(* \vee \dagger)$ exists.

The following definition is a rutine.

Definition 1.3.2 (***Complete*** **Lattices**): If a lattice admits arbitrary size of meet and join, then it is called *complete* lattice.

1.4. Distributive lattices

Why study distributive lattices? Well, since ΩX is a lattice, we'd better, then, clasify some specific lattices and capture as much properties as posible, and a satisfying thing is that

> - Any Boolean algebra, and even any Heyting algebra, is a distributive lattice.
> - Every frame and every σ-frame is a distributive lattice.
>
> — https://ncatlab.org/nlab/show/distributive+lattice

Definition 1.4.1 (**Distributive lattices**): A lattice is *distributive* if and only if this distributive law holds:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Remark: An immediate observation is that this law implies the dual one:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$$

which can be proved straightforward: $(a \vee b) \wedge (a \vee c) = ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$, which by the subsititution of the former one's a by $(a \vee b)$, and the right-hand-side can be computed, with the fact $(a \vee b) \wedge a = a$, as $\text{rhs} = a \vee ((c \wedge a) \vee (c \wedge b)) = a \vee (c \wedge b)$, where the last equality followed by that $\vee$ is asscociative.

By the remark above, we distil this conclusion:

A lattice L is distributive $\Longleftrightarrow$ its opposite lattice L^{op} is distributive.

There are more symmetric laws, but, please consult to ncatlab.

1.5. Filters and ideals

So you already known what is a

Definition 1.5.1 (**Filter**): A *filter* F on a distributive lattice L is a subset of L, such that

- $1 \in F$; (the top element is big)
- $a \in F$ and $b \in F \Longrightarrow a \wedge b \in F$; (two big sets' intersection is also big)
- $a \in F,$and $b \geq a \Longrightarrow b \in F$ (if you are bigger than a big guy, then you are big too)

Remark: Sometimes we talk about *prime* filters, those filters somehow act like prime numbers: a filter F is called *prime* if and only if

$$a \vee b \in F \Longrightarrow a \in F \text{ or } b \in F;$$

compare with prime numbers:

$$m \times n = p \Longrightarrow m \mid p \text{ or } n \mid p.$$

The definition of *ideal* is just a filter in the opposite lattice, or just modify the above definition to a "small guys" version.

The main result of this section is the so called “ultralfilters exists” theorem, stated as bellow:

Lemma 1.5.1: Let J be an ideal and F be a filter, which satisfy their intersection is empty. Then there exists a maximal *prime filter* $\overline{F}$ which respect to the property $\overline{F}$ does not interesects with J.

Proof: There are two points to prove:

- Maximum: when talk about maximum, one should always remember the Zorn lemma, which is exactly a proposition about maximal things. And, note that, there is not too much restrictions, being *envy* to construct what you want is welcome. (proof omitted, as exercise. Lol)
- Primeness: this is more chanllengable, give me some times to digest it.

■

1.6. Pseudocomplement

This is the last section of today (I hope), fuh!

Since defining Heyting algebra needs the preparation of pseudocomplement, we first define this slightly abstract notion:

Definition 1.6.1 (**Pseudocomplement**): In a $\wedge$-lattice, The pseudocomplement of a element $a \in F$ is the *largest* element x such that

$$x \wedge a = 0,$$

where “largest” means for all $b \in F$,

$$b \wedge a = 0 \Longleftrightarrow b \leq x.$$

We use the convenient notation a^* to denote this (unique) pseudocomplement.

Remark: Of course, this pseudocomplement need not to exist. For instance:

```
...           ...
  \           /
   4         3
    \       /
     2     1
      \   /
        0
```

1.7. Heyting algebras

Heyting algebras captrue the notion of "semantics implication" in some sense – those structures are equipped with a so-called *Heyting operation* "$\rightarrow$", which can be indentified with the logic implication. There is a interesting correspondence between implications and complements of elements: in (classic) logic, 0 is the complement of 1 and vice versa, with these equations:

$$(1 \rightarrow 0) = \text{the complement of } 1;$$

$$(0 \rightarrow 0) = \text{the complement of } 0;$$

by definition of implication.

A Heyting algebra extends this correspondence: change "complement" to "pseudocomplement" and add more elements other than 1 and 0, at the same time preserving the equations above, namely the pseudocomplement a^* of $a \in H$ is equal to $a \rightarrow 0$. And this Heyting operation can be extended, too, that is we can define more formulars than just $(\Box \rightarrow 0)$, so, finally, here is the definition:

Definition 1.7.1 (**Heyting algebras**): A lattice is called a *Heyting algebra* if it is equipped with a binary *Heyting operation* "$\rightarrow$" such that

$$c \leq a \rightarrow b \text{ if and only if } c \wedge a \leq b.$$

Remark:
- This law is indeed a Galois adjunction, where the left adjoint is the map $(\Box \mapsto a \wedge \Box)$ and the right one is $(\Box \mapsto (a \rightarrow \Box))$.
- This law ensures that the Heyting operation is indeed a extension of implication of classic logic, i.e.,

$$a^* = a \rightarrow 0,$$

where a^* is the pseudocomplement of a.

The equation in the definition is **like** the definition of Galois adjunction, but we need to justify that these maps are really *leagal morphisms* – monotone maps.

- The left adjoint is monotone: this is obvious, since by the definition of infima:

$$x \leq y \Longrightarrow u \wedge x \leq u \wedge y.$$

- The right guy: suppose $x \leq y$ and fix u, we want to prove $(u \rightarrow x) \leq (u \rightarrow y)$, which is equavalent to, by definition, $u \wedge (u \rightarrow x) \leq y$. The trick is:

$$u \wedge (u \rightarrow x) \leq x,$$

since $u \rightarrow x = u \rightarrow x$.

There is a striking result, of the specific left adjoint $(\square \mapsto (a \wedge \square))$:

Theorem 1.7.1: Every Heyting algebra is distributive.

Proof: A lattice is distributive if and only if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

And a left adjoint preverse suprema (the $\vee$ operation), apply this little lemma to our left adjoint, we immediately obtain the distributive law. ■

Thanks to Galois adjunction, we can obtain two more amazing theorems about Heyting algebra, which is of sattisfactory, in all sense:

Theorem 1.7.2: Heyting operation is unique

Proof: By the uniqueness of adjoint, and notice the left one $(\square \mapsto (a \wedge \square))$ is fixed. ■

Theorem 1.7.3: A lattice admits a Heyting algebra if this arbitrary distributive law holds:

$$b \wedge \left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} (b \wedge a_i).$$

Proof: By `Theorem 1.2.3`, plus the observation of monotonity of the map $(\square \mapsto (a \wedge \square))$, we obtain a right adjoint of this map, but don't forget the Heyting operation *is* a right adjoint of it and, adjointy *is* unique. We are done. ■

Some warm notices:
- All Heyting algebras are distributive, that's true.
- All *arbitrary distributive* lattices are equipped with Heyting operations, that's also true.
- General distributive lattices don't promise arbitrary distributive law.

1.8. TODO: Boolean algebras