

Notes on Frames and Locales

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This is my notes on Picardo and Pultr's book *Frames and Locales*, which is a good monograph of point-free topology. This notes will, at least, as my intent, *not* be something contain the full contents of the book, instead, it should serve as a helper to prevent me from fogetting everything. And I hope this notes can clarify as much misunderstandings as posible.

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1. June 1, 2024

There are many preliminaries needed be established first: the category **Pos** which consists of all partial order sets¹, the category **Lac** of lattices, Galois adjunctions, Heyting and Boolean algebras. The story is not long, but give yourself some time to digest it and, *appreciate* it.

1.1. The category **Pos** and **Loc**

The essence of a category is its morphisms, which respect some specific structures, so what maps should be, or *can* be morphisms? Well, just look at the poset's structures – order.

¹Ingore size issues, of course.

Definition 1.1.1 (Monotone functions): Those polite maps *respect* the structures of posets, i.e., a map $f : A \rightarrow B$ is called *monotone* if and only if

$$u \leq v \Rightarrow f(u) \leq f(v).$$

Of course, both A and B are posets.

There is nothing more interesting than its morphisms, in **Pos**. However, the funny fact is that any poset P is itself a category, so the objects in the category **Pos** are also categories! We use $\mathbf{Pos}(P)$ or just P to denote the category generated by a single poset P , as follows:

Definition 1.1.2 (View posets as categories): Given any poset P , it can be viewed as a category, where

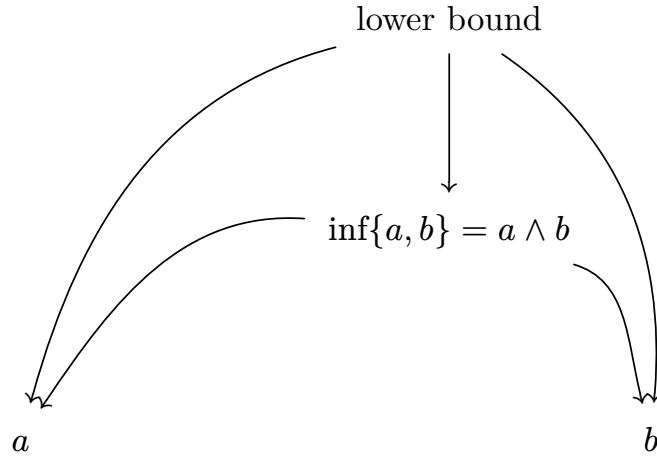
- Objects: elements in it;
- Morphisms: define morphisms between two objects (hence elements) $f : a \rightarrow b$ if and only if $a \leq b$;
- Identity morphisms exist: for all a , $a \leq a$ by the definition of partial order, which induce a morphism;
- Composition rules: by the transitivity of partial order.

Remark: Further, a functor between two categories $\mathbf{Pos}(X)$ and $\mathbf{Pos}(Y)$ is exactly a monotone map m between X and Y , viewed as posets.

- Preserve morphisms: if $f : x_1 \rightarrow x_2$ is a morphism in $\mathbf{Pos}(X)$, then there exist a morphism $m(f) : m(x_1) \rightarrow m(x_2)$ in $\mathbf{Pos}(Y)$ as well. And this is just a fancy way to describe the monotonicity;
- Preserve identity: also trivial;
- Preserve composition: by monotonicity and transitivity.

In addition, one can find that the *suprema* of a poset A is just the *coproduct* (or *sum*) of the category, and *infima* is the *product*. And the bottom \perp is exactly the initial object (hence we sometimes use 0 to denote it), while \top is the terminal object (hence 1 , sometimes). This observation is important, since we know left adjoint preserve coproduct² and right adjoint preserve product.

²Actually, co-limit. Also the “product” later is “limit”



Don't forget the morphism $a \rightarrow b$ is defined if and only if $a \leq b$.

1.2. Galois adjunctions

Galois adjunctions are just adjunctions between two posets A and B , viewed as categories, that's all:

$$\text{Hom}_A(f(a), b) \cong \text{Hom}_B(a, g(b)).$$

Unpack the definition of adjunctions, we immediately get the explicit version of the Galois adjunctions between two posets A and B :

Definition 1.2.1 (Galois adjunctions): Two monotone functions $f : A \rightarrow B$ and $g : B \rightarrow A$ are said to be *Galois adjoint* if for all $a \in A$ and $b \in B$, we have

$$f(a) \leq b \iff a \leq g(b).$$

This is essentially a reformation of the abstract definition. Note that, if we are in the category setting, the left adjoint of g is *the* f , since each left adjoint is isomorphic to others, of course right adjoints share this note as well.

Example: The so-called “to-range” map $f : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ with its “inverse” f^{-1} are adjoint to each other, since

$$f(U) \subset V \iff U \subset f^{-1}(V).$$

Here is an interesting proposition, where f and g are more closed to each other – they are on the same side:

Proposition 1.2.1: Two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ in **Pos** are adjoint to each other, if and only if for all $x \in X$ and $y \in Y$, there holds

$$fg(y) \leq y \text{ and } x \leq gf(x).$$

Proof: (Abstract nonsense version). As noted in **Definition 1.1.2**, f and g can be viewed as *functors* as well. Guess what? By category theory, two functors are adjoint if and only if there exists natural transformations “unit” $\eta : \text{id}_X \Rightarrow gf$ and “counit” $\varepsilon : fg \Rightarrow \text{id}_Y$, in diagram, it says, for all $x \in X$,

$$\begin{array}{ccc} x & \xrightarrow{\varepsilon_x} & gf(x) \\ \text{id}_x \parallel \downarrow & & \downarrow gf(\text{id}_x) \\ x & \xrightarrow{\varepsilon_x} & gf(x) \end{array}$$

commutes. Actually the commutativity is a little bit too much, we only need the existence of ε_x , which indicate that $x \leq gf(x)$.

Man, what can I say? Category theory is baesd. ■

Proof: (Not based version). Note that the proposition has two sides:

- (\Rightarrow) Move the outside f and g in both inequalities and immediately we find it obvious.
- (\Leftarrow) This is more interesting. Keep in mind that now, we want to prove

$$f(a) \leq b \iff a \leq g(b),$$

so by the introduction rule, we firstly obtain $f(a) \leq b$ and goal is $a \leq g(b)$. So by assumption,

$$a \leq gf(a) = g(f(a)) \leq g(b),$$

which is we wanted. ■

Theorem 1.2.2: Left Galois adjoint preserve suprema, while the right one preserve infima.

Proof: Guess what? By **Category theory**, left adjoint preserve not only coproduct (hence suprema), but also co-limit, and the situation of right adjoint is similar, qed. ■

In lattice theory, the converse statement is also true:

Theorem 1.2.3: If a monotone maps $f : X \rightarrow Y$ preverse suprema, then it is a left adjoint.

The goal is to construct, in a explicit way, a map $g : Y \rightarrow X$ satisfies the definition. That is to arrange each $y \in Y$ to some value in X . Let's first assume g is indeed at the right adjoint of f , and find some properties it **must** satisfy which guide our construction.

We claim that $g(y)$ **must** be the suprema of all x such that $f(x) \leq y$, i.e., this equation must holds:

$$g(y) = \sup\{x : f(x) \leq y\}.$$

There is no doubt $g(y)$ is a general upper bound of $\{x : f(x) \leq y\}$, also note this set contain $g(y)$ itself as well, since $g(y) \leq g(y)$ and then by the adjunction's property. Accordingly, our claim is true.

So there is actually no other ways to define the behaviour of g , we are **forced** to arrange each $y \in Y$ to $\sup\{x : f(x) \leq y\}$, no other choices.

Proof of Theorem 1.2.3: Define $g : Y \rightarrow X, y \mapsto \sup\{x : f(x) \leq y\}$. And we verify this is indeed the right adjoint of f :

- $f(x) \leq y$ implies $x \leq g(y)$: this is trivial by the definition of $g(y)$.
- $x \leq g(y)$ implies $f(x) \leq y$: Please think about the assumption that f preverse arbitrary suprema, which hasn't been used yet. Apply f to both side of the inequality $x \leq g(y)$, and by the monotonicity:

$$f(x) \leq f(g(y)) = f(\sup\{u : f(u) \leq y\});$$

since f preserve suprema, the last stuff equals to $\sup\{f(u) : f(u) \leq y\}$, which is apparently less than or equal to y . ■

1.3. Lattices

Lattices are just better posets, we concern those fancy sets since the topology of a space X , or ΩX , the set of all its open sets is indeed a lattice.³

³Why don't you use paratheness, namely use $\Omega(X)$? Lol, because the world is ruled by **Category theory**, Ω is a *functor*!

Definition 1.3.1 (Lattices): Lattices are just posets, which are close under *finite* meet and join.

Remark: In the book, the authors differ the notion of lattices and *bounded* lattices, while we **do not** – lattices *are* bounded lattices.

The notion of lattices is necessary, we show that there exists poset which is not a lattice: equip the two points set $\{*, \dagger\}$ with the partial order defined as exactly equal, i.e., $x \sqsubseteq y$ if and only if $x = y$. This dummy set is indeed a poset, but neither $(* \wedge \dagger)$ nor $(* \vee \dagger)$ exists.

The following definition is a routine.

Definition 1.3.2 (*Complete Lattices*): If a lattice admits arbitrary size of meet and join, then it is called *complete* lattice.

1.4. Distributive lattices

Why study distributive lattices? Well, since ΩX is a lattice, we'd better, then, classify some specific lattices and capture as much properties as possible, and a satisfying thing is that

- Any Boolean algebra, and even any Heyting algebra, is a distributive lattice.
- Every frame and every σ -frame is a distributive lattice.

— <https://ncatlab.org/nlab/show/distributive+lattice>

Definition 1.4.1 (Distributive lattices): A lattice is *distributive* if and only if this distributive law holds:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Remark: An immediate observation is that this law implies the dual one:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$$

which can be proved straightforward: $(a \vee b) \wedge (a \vee c) = ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$, which by the substitution of the former one's a by $(a \vee b)$, and the right-hand-side can be computed, with the fact $(a \vee b) \wedge a = a$, as $\text{rhs} = a \vee ((c \wedge a) \vee (c \wedge b)) = a \vee (c \wedge b)$, where the last equality followed by that \vee is associative.

By the remark above, we distil this conclusion:

A lattice L is distributive \iff its opposite lattice L^{op} is distributive.

There are more symmetric laws, but, please consult to ncatlab.

1.5. Filters and ideals

So you already know what is a

Definition 1.5.1 (Filter): A *filter* F on a distributive lattice L is a subset of L , such that

- $1 \in F$; (the top element is big)
- $a \in F$ and $b \in F \implies a \wedge b \in F$; (two big sets' intersection is also big)
- $a \in F$, and $b \geq a \implies b \in F$ (if you are bigger than a big guy, then you are big too)

Remark: Sometimes we talk about *prime* filters, those filters somehow act like prime numbers: a filter F is called *prime* if and only if

$$a \vee b \in F \implies a \in F \text{ or } b \in F;$$

compare with prime numbers:

$$m \times n = p \implies m \mid p \text{ or } n \mid p.$$

The definition of *ideal* is just a filter in the opposite lattice, or just modify the above definition to a “small guys” version.

The main result of this section is the so called “ultrafilters exists” theorem, stated as bellow:

Lemma 1.5.1: Let J be an ideal and F be a filter, which satisfy their intersection is empty. Then there exists a maximal *prime filter* \overline{F} which respect to the property \overline{F} does not interesects with J .

Proof: There are two points to prove:

- Maximum: when talk about maximum, one should always remember the Zorn lemma, which is exactly a proposition about maximal things. And, note that, there is not too much restrictions, being *envy* to construct what you want is welcome. (proof omitted, as exercise. Lol)
- Primeness: this is more chanllengable, give me some times to digest it.

■

1.6. Pseudocomplement

This is the last section of today (I hope), fuh!

Since defining Heyting algebra needs the preparation of pseudocomplement, we first define this slightly abstract notion:

Definition 1.6.1 (Pseudocomplement): In a \wedge -lattice, The pseudocomplement of a element $a \in F$ is the *largest* element x such that

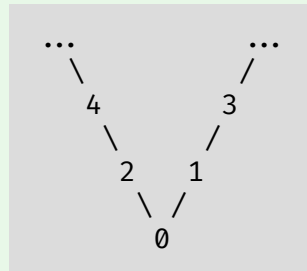
$$x \wedge a = 0,$$

where “largest” means for all $b \in F$,

$$b \wedge a = 0 \iff b \leq x.$$

We use the convenient notation a^* to denote this (unique) pseudocomplement.

Remark: Of course, this pseudocomplement need not to exist. For instance:



1.7. Heyting algebras

Heyting algebras capture the notion of “semantics implication” in some sense – those structures are equipped with a so-called *Heyting operation* “ \rightarrow ”, which can be identified with the logic implication. There is an interesting correspondence between implications and complements of elements: in (classic) logic, 0 is the complement of 1 and vice versa, with these equations:

$$(1 \rightarrow 0) = \text{the complement of } 1;$$

$$(0 \rightarrow 0) = \text{the complement of } 0;$$

by definition of implication.

A Heyting algebra extends this correspondence: change “complement” to “pseudocomplement” and add more elements other than 1 and 0, at the same time preserving the equations above, namely the pseudocomplement a^* of $a \in H$ is equal to $a \rightarrow 0$. And this Heyting operation can be extended, too, that is we can define more formulas than just $(\Box \rightarrow 0)$, so, finally, here is the definition:

Definition 1.7.1 (Heyting algebras): A lattice is called a *Heyting algebra* if it is equipped with a binary *Heyting operation* “ \rightarrow ” such that

$$c \leq a \rightarrow b \text{ if and only if } c \wedge a \leq b.$$

Remark:

- This law is indeed a Galois adjunction, where the left adjoint is the map $(\Box \mapsto a \wedge \Box)$ and the right one is $(\Box \mapsto (a \rightarrow \Box))$.
- This law ensures that the Heyting operation is indeed an extension of implication of classic logic, i.e.,

$$a^* = a \rightarrow 0,$$

where a^* is the pseudocomplement of a .

The equation in the definition is **like** the definition of Galois adjunction, but we need to justify that these maps are really *legal morphisms* – monotone maps.

- The left adjoint is monotone: this is obvious, since by the definition of infima:

$$x \leq y \implies u \wedge x \leq u \wedge y.$$

- The right guy: suppose $x \leq y$ and fix u , we want to prove $(u \rightarrow x) \leq (u \rightarrow y)$, which is equivalent to, by definition, $u \wedge (u \rightarrow x) \leq y$. The trick is:

$$u \wedge (u \rightarrow x) \leq x,$$

since $u \rightarrow x = u \rightarrow x$.

There is a striking result, of the specific left adjoint $(\Box \mapsto (a \wedge \Box))$:

Theorem 1.7.1: Every Heyting algebra is distributive.

Proof: A lattice is distributive if and only if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

And a left adjoint preverse suprema (the \vee operation), apply this little lemma to our left adjoint, we immediately obtain the distributive law. ■

Thanks to Galois adjunction, we can obtain two more amazing theorems about Heyting algebra, which is of satisfactory, in all sense:

Theorem 1.7.2: Heyting operation is unique

Proof: By the uniqueness of adjoint, and notice the left one $(\Box \mapsto (a \wedge \Box))$ is fixed. ■

Theorem 1.7.3: A lattice admits a Heyting algebra if this arbitrary distributive law holds:

$$b \wedge (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} (b \wedge a_i).$$

Proof: By [Theorem 1.2.3](#), plus the observation of monotonicity of the map $(\Box \mapsto (a \wedge \Box))$, we obtain a right adjoint of this map, but don't forget the Heyting operation *is* a right adjoint of it and, adjointy *is* unique. We are done. ■

Some warm notices:

- All Heyting algebras are distributive, that's true.
- All *arbitrary distributive* lattices are equipped with Heyting operations, that's also true.
- General distributive lattices don't promise arbitrary distributive law.

1.8. TODO: Boolean algebras

2. June 4, 2024

Today we play with category theory.

2.1. Orthogonal morphisms

2.2. Extremal, strong, and regular

Definition 2.2.1 (Extremal morphisms): A mono m is extremal, if for any factorisation $m = f \circ e$, where e is epi, the e must be an iso, too.

Example:

1. All injections in **Set**, **Grp** and so on.