Proposition 2.7. *Let $T: {}_R\mathbf{Mod} \to \mathbf{Ab}$ be an additive functor of either variance.*

(i) *If $0: A \to B$ is the **zero map**, that is, the map $a \mapsto 0$ for all $a \in A$, then $T(0) = 0$.*

(ii) $T(\{0\}) = \{0\}$.

Proof.

(i) Since T is additive, the function T_{AB} between Hom sets is a homomorphism, and so it preserves identity elements; that is, $T(0) = 0$.

(ii) If A is a left R-module, then $0 = 1_A$ if and only if $A = \{0\}$ [sufficiency is obvious; for necessity, if $1_A = 0$, then for all $a \in A$, we have $a = 1_A(a) = 0(a) = 0$, and so $A = \{0\}$]. By part (i), we have $T(1_{\{0\}}) = T(0) = 0$, and so $T(\{0\}) = \{0\}$. •

We now show that many constructions made for abelian groups and for vector spaces can be generalized to left modules over any ring. A *submodule* S is a left R-module contained in a larger left R-module M such that if $s, s' \in S$ and $r \in R$, then $s + s'$ and rs have the same meaning in S as in M.

Definition. If M is a left R-module, then a ***submodule*** N of M, denoted by $N \subseteq M$, is an additive subgroup N of M closed under scalar multiplication: $rn \in N$ whenever $n \in N$ and $r \in R$. A similar definition holds for right modules.

Example 2.8.

(i) A submodule of a $\mathbb{Z}$-module (i.e., of an abelian group) is a subgroup, and a submodule of a vector space is a subspace.

(ii) Both $\{0\}$ and M are submodules of a module M. A ***proper submodule*** of M is a submodule $N \subseteq M$ with $N \neq M$. In this case, we may write $N \subsetneq M$.

(iii) If a ring R is viewed as a left module over itself, then a submodule of R is a left ideal; I is a proper submodule when it is a proper left ideal. Similarly, if R is viewed as a right module over itself, then its submodules are its right ideals.

(iv) If M is an R-module and $r \in R$, where R is a commutative ring, then

$$rM = \{rm : m \in M\}$$

is a submodule of M. Here is a generalization. If J is an ideal in R and M is an R-module, then

$$JM = \Big\{ \sum_i j_i m_i \colon j_i \in J \text{ and } m_i \in M \Big\}$$

is a submodule of M.

(v) If S and T are submodules of a left module M, then

$$S + T = \{s + t \colon s \in S \text{ and } t \in T\}$$

is a submodule of M that contains S and T.

(vi) If $(S_i)_{i\in I}$ is a family of submodules of a left R-module M, then $\bigcap_{i\in I} S_i$ is a submodule of M.

(vii) A left R-module S is ***cyclic*** if there exists $s \in S$ with $S = \{rs : r \in R\}$. If M is an R-module and $m \in M$, then the ***cyclic submodule generated by*** m, denoted by $\langle m \rangle$, is

$$\langle m \rangle = \{rm \colon r \in R\}.$$

More generally, if X is a subset of an R-module M, then

$$\langle X \rangle = \Big\{ \sum_{\text{finite}} r_i x_i \colon r_i \in R \text{ and } x_i \in X \Big\},$$

the set of all ***R-linear combinations*** of elements in X. We call $\langle X \rangle$ the ***submodule generated by*** X. Exercise 2.10 on page 66 states that $\langle X \rangle = \bigcap_{X \subseteq S} S$. ◀

Definition. A left R-module M is ***finitely generated*** if M is generated by a finite set; that is, if there is a finite subset $X = \{x_1, \ldots, x_n\}$ with $M = \langle X \rangle$.

For example, a vector space V over a field k is a finitely generated k-module if and only if V is finite-dimensional.

Definition. If N is a submodule of a left R-module M, then the ***quotient module*** is the quotient group M/N (remember that M is an abelian group and N is a subgroup) equipped with the scalar multiplication

$$r(m + N) = rm + N.$$

The ***natural map*** $\pi \colon M \to M/N$, given by $m \mapsto m + N$, is easily seen to be an R-map.

Scalar multiplication in the definition of quotient module is well-defined: if $m + N = m' + N$, then $m - m' \in N$. Hence, $r(m - m') \in N$ (because N is a submodule), $rm - rm' \in N$, and $rm + N = rm' + N$.

Example 2.9. If $N \subseteq M$ is merely an additive subgroup of M but not a submodule, then the abelian group M/N is not an R-module. For example, let V be a vector space over a field k. If $a \in k$ and $v \in V$, then $av = 0$ if and only if $a = 0$ or $v = 0$ [if $a \neq 0$, then $0 = a^{-1}(av) = (a^{-1}a)v = v$]. Now $\mathbb{Q}$ is a vector space over itself, but $\mathbb{Q}/\mathbb{Z}$ is not a vector space over $\mathbb{Q}$ [we have $2(\frac{1}{2} + \mathbb{Z}) = \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$, and neither factor is zero]. ◀

Example 2.10.

(i) Recall that an additive subgroup $J \subseteq R$ of a ring R is a ***two-sided ideal*** if $x \in J$ and $r \in R$ imply $rx \in J$ and $xr \in J$. If $R = \mathrm{Mat}_2(k)$, the ring of all 2×2 matrices over a field k, then $I = \left\{\left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right] : a, b \in k\right\}$ is a left ideal and $I' = \left\{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] : a, b \in k\right\}$ is a right ideal, but neither is a two-sided ideal.

(ii) If J is a left (or right) ideal in R, then R/J is a left (or right) R-module. If J is a two-sided ideal, then R/J is a ring with multiplication

$$(r + J)(s + J) = rs + J.$$

This multiplication is well-defined, for if $r + J = r' + J$ and $s + J = s' + J$, then $rs + J = r's' + J$, because

$$rs - r's' = rs - r's + r's - r's' = (r - r')s + r'(s - s') \in J. \quad ◀$$

We continue extending definitions from abelian groups and vector spaces to modules.

Definition. If $f \colon M \to N$ is an R-map between left R-modules, then

$$\textbf{\textit{kernel} } \boldsymbol{f} = \ker f = \{m \in M \colon f(m) = 0\},$$
$$\textbf{\textit{image} } \boldsymbol{f} = \operatorname{im} f = \{n \in N \colon \text{there exists } m \in M \text{ with } n = f(m)\},$$
$$\textbf{\textit{cokernel} } \boldsymbol{f} = \operatorname{coker} f = N/\operatorname{im} f.$$

It is routine to check that $\ker f$ is a submodule of M and that $\operatorname{im} f$ is a submodule of N.

Theorem 2.11 (First Isomorphism Theorem). *If $f \colon M \to N$ is an R-map of left R-modules, then there is an R-isomorphism*

$$\varphi \colon M/\ker f \to \operatorname{im} f$$

given by

$$\varphi \colon m + \ker f \mapsto f(m).$$

Proof. If we view M and N only as abelian groups, then the first isomorphism theorem for groups says that $\varphi\colon M/\ker f \to \operatorname{im} f$ is a well-defined

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\text{nat}"'] \arrow[dr] & N \\ M/\ker f \arrow[r, dashed, "\varphi"'] & \operatorname{im} f \arrow[u, "\text{inc}"'] \end{tikzcd}$$

isomorphism of abelian groups. But φ is an R-map: if $r \in R$ and $m \in M$, then $\varphi(r(m+N)) = \varphi(rm+N) = f(rm)$; since f is an R-map, however, $f(rm) = rf(m) = r\varphi(m+N)$, as desired. •

The second and third isomorphism theorems are corollaries of the first.

Theorem 2.12 (Second Isomorphism Theorem). *If S and T are submodules of a left R-module M, then there is an R-isomorphism*

$$S/(S \cap T) \to (S+T)/T.$$

Proof. If $\pi\colon M \to M/T$ is the natural map, then $\ker \pi = T$; define $f = \pi|S$, so that $f\colon S \to M/T$. Now

$$\ker f = S \cap T \quad \text{and} \quad \operatorname{im} f = (S+T)/T,$$

for $(S+T)/T$ consists of all those cosets in M/T having a representative in S. The first isomorphism theorem now applies. •

Definition. If $T \subseteq S \subseteq M$ is a tower of submodules of a left R-module M, then ***enlargement of coset*** $e\colon M/T \to M/S$ is defined by

$$e\colon m+T \mapsto m+S$$

(e is well-defined, for if $m+T = m'+T$, then $m-m' \in T \subseteq S$ and $m+S = m'+S$).

Theorem 2.13 (Third Isomorphism Theorem). *If $T \subseteq S \subseteq M$ is a tower of submodules of a left R-module M, then enlargement of coset $e\colon M/T \to M/S$ induces an R-isomorphism*

$$(M/T)/(S/T) \to M/S.$$

Proof. The reader may check that $\ker e = S/T$ and $\operatorname{im} e = M/S$, so that the first isomorphism theorem applies at once. •

If $f\colon M \to N$ is a map of left R-modules and $S \subseteq N$, then the reader may check that $f^{-1}(S) = \{m \in M\colon f(m) \in S\}$ is a submodule of M containing $f^{-1}(\{0\}) = \ker f$.

Theorem 2.14 (Correspondence Theorem). *If T is a submodule of a left R-module M, then $\varphi\colon S \mapsto S/T$ is a bijection:*

$$\varphi\colon \{\textit{intermediate submodules } T \subseteq S \subseteq M\} \to \{\textit{submodules of } M/T\}.$$

Moreover, $T \subseteq S \subseteq S'$ in M if and only if $S/T \subseteq S'/T$ in M/T.

Proof. Since every module is an additive abelian group, every submodule is a subgroup, and so the usual correspondence theorem for groups shows that φ is an injection that preserves inclusions: $S \subseteq S'$ in M if and only if $S/T \subseteq S'/T$ in M/T. Moreover, φ is surjective: if $S^* \subseteq M/T$, then there is a unique submodule $S \supseteq T$ with $S^* = S/T$. The remainder of this proof is a repetition of the usual proof for groups, checking only that images and inverse images of submodules are submodules. •

The correspondence theorem is usually invoked tacitly: a submodule S^* of M/T is equal to $S^* = S/T$ for some unique intermediate submodule S.

Here is a ring-theoretic version.

Theorem 2.15 (Correspondence Theorem for Rings). *If I is a two-sided ideal of a ring R, then $\varphi\colon J \mapsto J/I$ is a bijection:*

$$\varphi\colon \{\textit{intermediate left ideals } I \subseteq J \subseteq R\} \to \{\textit{left ideals of } R/I\}.$$

Moreover, $I \subseteq J \subseteq J'$ in R if and only if $J/I \subseteq J'/I$ in R/I.

Proof. The reader may supply a variant of the proof of Theorem 2.14. •

Proposition 2.16. *A left R-module M is cyclic if and only if $M \cong R/I$ for some left ideal I.*

Proof. If M is cyclic, then $M = \langle m \rangle$ for some $m \in M$. Define $f\colon R \to M$ by $f(r) = rm$. Now f is surjective, since M is cyclic, and its kernel is a submodule of R; that is, $\ker f$ is a left ideal I. The first isomorphism theorem gives $R/I \cong M$.

Conversely, R/I is cyclic with generator $m = 1 + I$. •

Definition. A left R-module M is ***simple*** (or ***irreducible***) if $M \neq \{0\}$ and M has no proper nonzero submodules; that is, $\{0\}$ and M are the only submodules of M.

Corollary 2.17. *A left R-module M is simple if and only if $M \cong R/I$, where I is a maximal left ideal.*

Proof. This follows from the correspondence theorem. •

For example, an abelian group G is simple if and only if G is cyclic of order p for some prime p. The existence of maximal left ideals guarantees the existence of simple modules.