Notes on Geometric Group Theory (and Model Theory)

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Contents

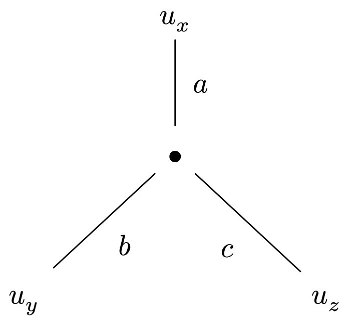
This note is taken on the Nankai Logic Summer School 2025: *Gromov's Randomness & Model Theory of Groups*, lectured by Prof. Rizos Sklinos.

I must admit that I almost know nothing about serious geometric group theory and large-scale geometry, and this note should not be used as a study matetial. I just written down things interested me.

Reference: Loeh's Notes.

1. Day 1

1. Dehn *solved* the word problem for fundamental groups of surfaces.
2. Hyperbolic groups: Those that satisfy Dehn's algorithm and admit a finitely presentation.
3. (**Gromov**) If we pick a finite presented group *at random*, then it is almost hyperbolic.
4. (**Geodesic Path**) For a metric space (X, d), a geodesic path from x to y in X is an isometry map $c : [0, L] \to X$ for some $L \geq 0$ such that:
$$\begin{cases} c(0) = x \\ c(L) = y \end{cases}.$$
In paticular, since c is a isometry, we know that $L = d(x, y)$.
5. (**Geodesic Space**) A metric space (X, d) is called a geodesic space if it satisfies: for all $x, y \in X$ there exists a geodesic path connecting x and y.
6. $(\mathbb{R}^2, d_E)$ is *uniquely* geodesic.
7. (**δ-slim**) Let $\delta \geq 0$, a geodesic triangle $[x, y, z]$ is called δ-slim if for any $u \in [x, y]$, there exists a point $v \in [y, z] \cup [z, x]$ such that $d(u, v) \leq \delta$.
   - *Remark:* By "geodesic triangle", we mean a triple (x, y, z) of points in a metric space, with a set of *chosen* geodesic paths between them.
8. (**δ-hyperbolic**) A geodesic metric space is called δ-*hyperbolic* if all geodesic triangles are δ-slim.
9. (**Hyperbolic**) A geodesic metric space is *hyperbolic* if it is δ-hyperbolic *for some* δ.
10. (**Real Tree**) A 0-hyperbolic space is called a *real tree*.
11. (**Tripod**) Given a geodesic triangle $[x, y, z]$, a tripod of it is a real tree with three vertex u_x, u_y, u_z:



such that

$$\begin{cases} a+b = d(x,y) \\ b+c = d(y,z) \\ c+a = d(z,x) \end{cases}$$

In short: a tripod of a geodesic triangle $[x,y,z]$ is a real tree $\left(u_x, u_y, u_z\right)$ isometric to it.

12. (**δ-thin**) For a geodesic triangle $\Delta \coloneqq [x,y,z]$ with a tripod isometry $X_\Delta : \Delta \to T(a,b,c)$, we say Δ is δ-thin if for any $p, q \in X_\Delta^{-1}(t)$ where $t \in T(a,b,c)$, we always have $d(p,q) \leq \delta$.
    - *Remark*: δ-thin is proper stronger than δ-slim.
13. (**(δ)-hyperbolic**) A metric space is (δ)-hyperbolic if for any $w, x, y, z \in X$ we have:

$$d(x,y) + d(y,z) \leq \max\{d(x,w) + d(y,z), d(x,z) + d(y,w)\} + \delta.$$

    - *Problem*: Does the class of (1)-hyperbolic finite graphs have the *amalgmation property*?
        - *Remark*: A theory admit the amalgmation property if for any extensions B, C of a model A, there exists an extension of both B and C such that

$$\begin{tikzcd} & D & \\ B \arrow[ur, hook] & & C \arrow[ul, hook'] \\ & A \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

        commutes.
    - *Problem*: What about (δ)-hyperbolic finite graphs for general positive real δ?
14. (**Quasi-isometry**) Definition of quasi-isometry was given here (but due to my laziness, ommitted).
    - *Remark*: This is indeed an equivalent relation.
    - *Examples*:
        - Any finite diameter space $\overset{\text{qi}}{\sim} \{\text{point}\}$.
        - $(\mathbb{Z} \times \mathbb{Z}, \text{taxi metric}) \overset{\text{qi}}{\sim} (\mathbb{R}^2, d_E))$ where d_E is the common Euclid's metric.

2. Day 2

1. "The" Cayley graph of a finitely generated group is unique (up to quasi-isometry).
    - *Remark:* For group $G = \langle S \mid R \rangle$ finitely generated, we can define the length for any element $g \in G$ as $|g|_G \coloneqq \min_{g=h} |h|_{\text{Cayley}}$.

2. (**Hyperbolic Group**) A group is called hyperbolic if its Cayley graph is hyperbolic as a metric spacce.
   - *Remark:* This is just one of the many definitions of hyperbolic groups.
3. Some facts:
   - $\mathrm{Cay}(G, A)$ for A a generating set of G, is a proper space with all geodesics.
   - G acts on $\mathrm{Cay}(G, A)$ by sendind (r, g) to $r \cdot g$, which is an isometry, i.e. $d(rg, rh) = d(g, h)$.
   - The above action is properly-discontinuous, i.e., for compact $X \subset \mathrm{Cay}(G, A)$, the set $\{g : gX \cap X \neq \emptyset\}$ is finite.
   - The action is co-compact, i.e., $\mathrm{Cay}(G, A)/G$ is compact.
4. (**Svarč-Milnar Theorem**) See Loeh's book.
5. For surface with genus $g \geq 2$, its fundamental group $\pi_1(\Sigma_g)$ is hyperbolic, since
$$\pi_1(\Sigma_g) \underset{\text{nicely}}{\circlearrowleft} \mathbb{H}^2 (\text{the hyperbolic plane}).$$
   It acts so nicely that the S-M theorem ensures $\pi_1(\Sigma_g)$ is hyperbolic.
6. Hyperbolicity is stable under finite index.
7. Finite groups and free groups are all hyperbolic.
8. A group G is free $\Longleftrightarrow$ It acts freely without inversions on a simplicial tree.
   - ($\Longrightarrow$) If G is free, then (assume $G = \boldsymbol{F}S$)
$$G \circlearrowleft \mathrm{Cay}(G, S),$$
     where $\mathrm{Cay}(G, S)$ is a simplicial tree.
   - ($\Longleftarrow$) Use covering space theory. If $G \circlearrowleft T$ nicely then we obtain a covering map $p : T \to T/G$, which is universal, so $\pi_1(T/G) \cong G$. But T/G is a graph so $\pi_1(T/G)$ is free.