

Quick notes about category theory

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1. The category of categories \mathbf{Cat} is co/complete.

proof: We will prove this interesting statment by the realization-nerv relation between \mathbf{Cat} and \mathbf{sSet} .

First note that:

- $h : \mathbf{sSet} \rightarrow \mathbf{Cat}$ is the **left** adjoint, which preserves **colimit**.
- $\mathrm{Nerv} : \mathbf{Cat} \rightarrow \mathbf{sSet}$ is the **right** adjoint; and it is **fully faithful**, which implies the counit $\varepsilon : h \circ \mathrm{Nerv} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ is an isomorphism.

Then we start the main proof:

- **Cocompleteness:** For all functors $F : J \rightarrow \mathbf{Cat}$, since the counit of realization-nerv adjunction is an isomorphism, we have $h \circ \mathrm{Nerv} \circ F \simeq F$. The left-hand-side **does** have colimit since

$$\mathrm{colim}(h \circ \mathrm{Nerv} \circ F) \simeq h(\mathrm{colim}(\mathrm{Nerv} \circ F)).$$

And the latter colimit is in \mathbf{sSet} , which is cocomplete since it is a presheaf category.

Also note that there is another abstract proof based on theory of enriched category, see [The answer of Keith Harbaugh](#).

- **Completeness:** There is a general theorem: Reflective full subcategory is closed under limit. (Reason: Every reflective subcategory inclusion is a monadic functor, and monadic functor creat limit. Reference: [Prop 5.1](#) and [Prop 3.1](#)).

However, the arbitrary products and equalizer in \mathbf{Cat} is obvious. So we do not need the above abstract theorem (at least in this proof).

Also see [Riehl Example 4.5.14 \(vi\)](#), [Page 143](#).

2. Localization of categories and reflective full subcategories

There is a complex relation between localization of categories and reflective full subcategories.

- The localization can be done on any classes of morphisms W in category \mathcal{C} , but when the class W is a *weak equivalence* (includes all isomorphisms and satisfies the [two-out-of-three rule](#)) things become smoother.
 - Note that the general localization is also called *category of fractions*.
- All reflective subcategories are localizations, see [Prop 3.1](#).

3. Right adjoints is fully faithful \iff The counit is an isomorphism

proof: For counit $\varepsilon : FU \rightarrow \text{Id}_D$, thanks to the triangle equality we have $U\varepsilon_d \circ \eta_{Ud} = \text{id}_{Ud}$, where η is the unit, we know that there exists a counterpart l of η_{Ud} in $D(x, y)$ such that $\varepsilon_d \circ l = \text{id}_d$, since D is fullu faithful.

4. Properties of monadic functors

For monadic functor $U : \mathcal{D} \rightarrow \mathcal{C}$ with left adjoint $F : \mathcal{C} \rightarrow \mathcal{D}$ and induced monad $T := UF$, induced comparison functor $K : \mathcal{D} \rightarrow \mathcal{C}^T$, we claim that U :

1. ... is faithful

proof: In fact for a monad induced by a general adjunction where the right adjoint is U , there is a decomposition of U : $U = U^T \circ K$ where $U^T : \mathcal{C}^T \rightarrow \mathcal{C}$ is the forgetful functor and K the comparison functor.

Note that U^T is always faithful, so when K is a equivalence their composition is also faithful.

- *Remark:* A maybe useful observation: For functor $U : \mathcal{D} \rightarrow \mathcal{C}$, it is faithful if and only if the induced natural transformation $\mathcal{D}(x, y) \rightarrow \mathcal{C}(Ux, Uy)$ is a monomorphism.

2. ... creates limit

proof: Here I only make a short description, please draw a diagram by yourself, really, it is benifit for you.

Suppose $H : J \rightarrow \mathcal{D}$ is a diagram, and $\lim UH$ exists in \mathcal{C} with a bunch of projections p_i , we need to construct a limit in \mathcal{D} . Thanks to the equivalence between \mathcal{D} and \mathcal{C}^T , we need only to construct limit in the later category.

Define $V := \lim UH$ We claim that the object $(V, h : U F V \rightarrow V)$ is the limit we want, where h is induced by the universal property of V with a bunch of morphisms $U\varepsilon_{Hi} \circ U F p_i : U F V \rightarrow U H i$. One can verify that h is indeed satisfies the requirements of \mathcal{C}^T . (**Warning:** please note that the triangle equality of T -module $(c, v_c : Tc \rightarrow c)$ is that $v_c \circ \eta_c = \text{id}_c$! Caution that the right hand side is *not* id_{Tc} !).

V is a universal cone in \mathcal{C}^T can be infered from its universal property in \mathcal{C} .

So $\lim KH$ exists and thus $\lim H$ does so.

- *Remark:* The diagram chasing is cubersome, please also see [Discusion on math exchange](#).

we need to prove that it both preserves and reflects limits.

- Preserves limits: Since U is a right adjoint.
- Reflects limits: Since all faithful functors reflects limits, and U is just faithful.

5. What if $\text{Hom}(-, x)$ preserves filtered colimit?

Such hom functors are trivial in the following sense: $\text{Hom}(y, x) = \emptyset$ or $\text{Hom}(y, x) = \text{Hom}(x, x)$.

- *proof:* Suppose $x \in C$, note that $\mathbf{2} := \{0 \rightarrow 1\}$ is a filtered category (in fact all posets are filtered). For $y \in C$ if there is no morphisms from y to x then we are done, otherwise there is at least one morphism $f : y \rightarrow x$ and we define $F : \mathbf{2} \rightarrow C$ sending $\{0 \rightarrow 1\}$ to $\{y \rightarrow x\}$.

So by assumption $\text{Hom}(\text{colim}_2 F, x) = \text{colim}_2 \text{Hom}(F -, x) = \text{Hom}(y, x)$, and note that $\text{colim}_2 F = x$, we are done.

6. Filtered colimits commute with finite limits in Set

For a functor $F : I \times J \rightarrow \text{Set}$ with I filtered and J finite, we claim that

$$\text{colim}_I \lim_J F \simeq \lim_J \text{colim}_I F.$$

- *proof sketch:* We do not present the full proof in detail (see [Prop. 23.7 in Dmitri's lecture](#)) but point out two important observations:
 - First, please note that $F(i, j)$ is **not** a hom set, and F is **not** a hom functor, which means it is **covariant in both i and j !**
 - Second, in the proof of surjectivity, the claim “If we increase i further, we can also assume that for any morphism $f : j \rightarrow j'$ in J we have $D(-, f)(y_j) = y_{j'}$,” is the result of universal property of limit:

$$\begin{array}{ccc}
 & x \in \lim_J \text{colim}_I F & \\
 \swarrow & & \searrow \\
 (i, y_j : D(i, j)) \in \text{colim}_I D(-, j) & \xleftarrow{\text{colim}_I D(-, f)} & (i, y_{j'} : D(i, j')) \in \text{colim}_I D(-, j')
 \end{array}$$

For any $f : j \rightarrow j'$ the above diagram must commute, so by increasing i to a suitable i' (this process will halt since there are only finite such f) we can claim that result.

7. $\text{Ran}_G G$ is a monad if it exists

This is the construction of so-called **codensity** monad of an arbitrary functor $G : A \rightarrow B$, and the monadness can be proved in a clever way:

Define a category r_G whose:

- Objects: $(X : B \rightarrow B, x : XG \Rightarrow G)$, i.e., right extensions of G ;
- Morphisms between (X, x) and (Y, y) : Natural transformations $\eta : X \Rightarrow Y$ which compatible with x and y .

And $(r_G, \text{id}_B, \circ)$ is a (strict) monoidal category.

Then we find **$\text{Ran}_G G$ is the terminal object in r_G !** So by common abstract nonsense argument, it has an unique monoid structure.

Ref. [CODENSITY AND THE ULTRAFILTER MONAD](#), Section 5. Tom Leinster.