Notes on Algebraic Topology

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Contents

I plan to learn algebraic topology with Prof. Löh's notes, some other references:

- *A Basic Course in Algebraic Topology* (GTM 127), William S. Massey, Springer.
- *A Concise Course in Algebraic Topology*, Peter May, availabe at https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

We denote the concatenation of two path $g_{12} : x_1 \rightsquigarrow x_2$ and $g_{23} : x_2 \rightsquigarrow x_3$ by $g_{12} \cdot g_{23} : x_1 \rightsquigarrow x_2$, i.e., we apply g_{12} **first**, then g_{23}.

1. Basic Definitions

1.1. Relative Homotopy

Two morphisms f and g from topology space X to Y is called *homotopic relative to* A for $A \subset X$, denoted as $f \simeq g \text{ rel } A$, if there exists a morphism $h : X \times I \to Y$ such that

- $h(x, 0) = f(x)$ for all $x \in X$;
- $h(x, 1) = g(x)$ for all $x \in X$;
- $h(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

Remark. So the classical homotopic relation between two paths with same end points is exactly the relative homotopy when $A = \{\text{initial point}, \text{final point}\}$.

1.2. Retract and Deformation Retract

- A subspace $i : A \hookrightarrow X$ is called a *retract* of X if i admits a left inverse $r : X \twoheadrightarrow A$, i.e. $r \circ i = \mathrm{id}_A$;
- It is called a *deformation retract* of X if $i \circ r \simeq \mathrm{id}_X \text{ rel } A$.

Remark. Note that $r \circ i = \mathrm{id}_A$ is equivalent to $r \circ i \simeq \mathrm{id}_A \text{ rel } A$ – so the condition of deformation retract is rather natural – indeed:

- If $r \circ i = \mathrm{id}_A$, then define the homotopy $h(x, t) = r \circ i(x) = \mathrm{id}_A(x) = x$, which is of course continuous in both x and t;
- If $r \circ i \simeq \mathrm{id}_A \text{ rel } A$, then by definition there exists a homotopy $h : A \times I \to A$ such that $h(a, t) = r \circ i(a) = \mathrm{id}_A(a)$, implies that $r \circ i = \mathrm{id}_A$.

The main importance of deformation retract is embodied in the following theorem:

Theorem 1.2.1 (A deformation retract shares the same fundamental group of the ambient sapce)

For a deformation retract $i : A \hookrightarrow X$, we have $\pi_1(A, a) = \pi_1(X, a)$ for all $a \in A$.

Proof. Suppose the relative homotopy is witness by h.

Then by the proposition in Section 4, Chapter 1 of [May], we need only to prove that $\gamma[h(a, -)] = \mathrm{id}$, but by the definition of relative homotopy, $h(a, -) \equiv a$ so the equation is tautology. □

Remark

We can prove it directly, first we need a lemma: if $f, g : X \to Y$ is relative homotopic respect to $x_0 \in X$, i.e. there exists a homotopy $h : f \simeq g$ rel $\{x_0\}$, we claim that $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 := f(x_0) = g(x_0)$.

By this lemma, and note that $i \circ r \simeq \mathrm{id}_A$ rel $\{a\}$, we have $(i \circ r)_* = i_* \circ r_* = (\mathrm{id}_A)_*$. Furthur since A is a retract, $r \circ i = \mathrm{id}$ and thus $r_* \circ i_* = \mathrm{id}$. So we finish the proof of the theorem.

For the lemma, suppose $[p] \in \pi_1(X, x_0)$ is a path, we prove that $f \circ p \simeq g \circ p$: the homotopy $\hat{h} : I \times I \to Y$ is given by

$$\begin{tikzcd} I \times I \arrow[d, "p \times \mathrm{id}"'] \arrow[dr, dashed, "\hat{h}"] & \\ X \times I \arrow[r, "h"'] & Y \end{tikzcd}.$$

Note that $\hat{h}(s, t) = h(p(s), t)$ is indeed a homotopy between f and g. We are finished.

This can be used to compute the fundamental group $\pi_1(\mathbb{R}^n, x_0)$ for all $x_0 \in \mathbb{R}^n$: we claim that $\{x_0\}$ is a deformation retract of $\mathbb{R}^n$, and one of the required homotopies is given by $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ which sends (x, t) to $tx + (1 - t)x_0$.

So $\pi_1(\mathbb{R}^n, x_0) = \pi_1(\{x_0\}, x_0) = \{*\}$.

2. Covering Space

There are (at least) two definitions of covering spaces over a given base topological space X:

- **The "new" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an open neihgbourhood $x \in U_x$ such that $p^{-1}(U_x) \cong U_x \times p^{-1}(x)$, where $p^{-1}(x)$ equipped with discrete topology.
- **The "old" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an *path-connected* open neihgbourhood U_x such that each path-connected component of $p^{-1}(U_x)$ is homeomorphic to U_x via p.

The new fashion can be found in [Wedhorn], [Löh], and *covering space* on nLab while the old one can be found in [Massey] and [May].

Remark

The old fashion definition is in fact not consistent:

- [Massey] requires that both covering spaces and base spaces are *path-connected* and *locally path-connected*.
- [May] requires nothing.

In short: Covering space is a locally trivial bundle with discrete fiber.

2.1. Examples and non-exmaples

Examples:

1. **The trivial bundle**: Identity $\mathrm{id}_X : X \to X$ is a covering space, with fiber $\{*\}$.
2. **Global trivial bundle with discrete fiber**: For a discrete space D, the projection $X \times D \twoheadrightarrow X$ is of course a covering space, and D is the fiber.
3. **Exponential**: The exponential function $\exp : \mathbb{R} \to S^1$ sends θ to $(\cos\theta, \sin\theta)$ is a local trivial bundle with fiber $\mathbb{Z}$.

Note that to be a covering map, the fiber must be *discrete*. Here is one of non-exmaples:

1. The projection $S^1 \times [0,1] \twoheadrightarrow S^1$ is not a covering map, since the fiber $[0,1]$ is not discrete.

2.2. Basic Topological Properties

Theorem 2.2.1 (fiber-wise diagonal of covering space is open and closed)

For $p : E \to X$ a covering, the diagonal $\Delta(E) := \{(e,e) : e \in E\}$ is both closed and open in $E \times_X E$, the pullback of $p : E \to X$.

Proof. See nLab - *Covering spaces*, Lemma 3.2. □

This will be used to prove the result: two lifts in a covering space of a path are either same, or different everywhere.

2.3. Produce covering spaces from group actions

In this section, G is a group endowed with discrete topology and X is an arbitrary space. We fix a G-action on space X: $\alpha : G \to \mathrm{Aut}_{\mathrm{Top}}(X)$, then it induces a covering space if this action is good enough. We always write gx for $\alpha_g x$.

Denote the orbit of $x \in X$ as $Gx := \{gx : g \in G\}$.

Firstly we need some definitions:

Definition 2.3.1 (Orbit space)

The orbit space GX of X related to the action α is defined as:

$$GX := \{Gx : x \in X\}.$$

We endow this set with quotient topology induced from the projection $X \twoheadrightarrow GX$.

2.4. Lebesgue number

Yes, Lebesgue and algebraic topology. We should say “thank you” to him for the following useful lemma:

Theorem 2.4.1 (Lebesgue number)

For an open covering $\{U_i\}$ of a *compact metric space* X, there exists $\delta > 0$, which is called a Lebesgue number, such that for all $x \in X$ the open ball centered x with radius δ is fully contained in one of those open sets, formally: $B(x, \delta) \subset U_i$ for some U_i.

Proof. Suppose not, that is, for all n, there exists x_n such that $B\left(x_n, \frac{1}{n}\right)$ does not fully contained any U_i.

Since X is compact, $\{x_n\}$ has a covergence subsequence $\{y_n\}$ tends to y_0. Let ε_n the associated radius of y_n. Since $\{U_i\}$ is an open cover of X, there exists $\varepsilon_0 > 0$ and N such that $B(y_0, \varepsilon) \subset U_i$ for some i, and for all $m > N$ we have $y_m \in B(y_0, \frac{\varepsilon_0}{2})$.

Now make m so large that $\varepsilon_m < \frac{\varepsilon_0}{2}$ and $m > N$, so $B(y_m, \varepsilon_m) \subset B(y_0, \varepsilon_0) \subset U_i$, contradiction to our assumption! □

Remark

We need X to be compact, otherwise there may be no such δ, for example: $\left\{B\left(r, \frac{r}{2}\right)\right\}_{r \in (0,1)}$ covers $(0, 1)$.

Compactness ensures the existence of covergence subsequence of a infinite sequence: suppose $\{x_n\}$ does not admit a covergence subsequence, then for all $x \in X$, there exists a open neihgbourhood U_x of x that only contains finitely many x_n. Now note that $\{U_x\}$ is an open cover of a compact space so it admits a finite subcover $\{V_x\}$, and their union only contains finitely many x_n.

Since $\{V_x\}$ covers X, we know that there are only finitely many distinct points in $\{x_n\}$, which implies this sequence itself must coverage, contradiction!

This theorem is especially useful and when we want to prove something about covering spaces, it serves as a bridge from the trivial covering to general situations. We will see such examples in the next section, where we concern lifting properties.

2.5. Lifting Properties

Theorem 2.5.1 (Lifting of paths)

For a given covering space $p : \hat{X} \twoheadrightarrow X$ and a path $f : I \to X$ with the initial point $x_0 := f(0)$, then for any $\widehat{x_0} \in \hat{X}$ such that $p(\widehat{x_0}) = x_0$, there exists an unique path $\hat{f} : I \to E$ with initial point $\widehat{x_0}$ such that $p \circ \hat{f} = f$.

In diagram:

$$\begin{tikzcd}
\{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\
{[0,1]} \arrow[r, "f"'] & X
\end{tikzcd}$$

induces an unique $\hat{f}$ such that

$$\begin{tikzcd} \{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\ {[0,1]} \arrow[r, "f"'] \arrow[ur, "\exists! \hat{f}"] & X \end{tikzcd}$$

commutes.

Proof. The main idea is that we first prove the case $\hat{X}$ is *global trivial*, and then proceed to the general case.

- **Trivial Case**: Suppose $\hat{X} \cong X \times D$ for a discrete space D.
  - Existence: We define $\hat{f}(t) := (f(t), d_0)$, where $d_0 = \mathrm{pr}_2(\widehat{x_0})$. It is continuous and indeed a lifting.
  - Uniqueness: For another lift $\tilde{f} : [0,1] \to \hat{X}$, since the diagram commutes, we have $\tilde{f}(t) = (f(t), d(t))$ and $d(0) = d_0$.

    A continuous image of a path-connected space is again path-connected, so $\tilde{f}(I)$ is path-connected and we claim that $d(t) \equiv d_0$. Otherwise, because all discrete spaces are not path-connected, $\left(\tilde{f}(t_1), d(t_1)\right)$ can not be connected to $\left(\tilde{f}(t_2), d(t_2)\right)$ by path since D is discrete.

- **General Case**: Thanks to the local triviality of a covering space, for each $x \in X$ there exists a open neihgbourhood U_x such that $p^{-1}(U_x) \cong U_x \times D$ a trivial covering of U_x, where $D := p^{-1}(x)$ is equipped with discrete topology.

  Now we need only to divide f into pieces of sub-paths $\{f_i\}$ that each of them is fully contained in some trivial open neihgbourhoods. Then use the result from the trivial case we obtain sub-liftings $\left\{\hat{f}_i\right\}$, and finally glueing them!

  Since $[0,1]$ is compact and $\{f^{-1}(U_x)\}_{x \in X}$ is an open cover of it, we obtain a finite subcover $\{f^{-1}(V_x)\}$, now we let $\frac{1}{n}$ be a Lebesgue number of this cover where n is a integer, and divide $[0,1]$ into $\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], ..., \left[1 - \frac{1}{n}, 1\right]\right\}$. Let $f_k : \left[\frac{k-1}{n}, \frac{k}{n}\right] \to X$ be the restriction of f. By the definition of n, image of each f_k is contained in a trivial open neihgbourhood U_k, and thus the problem for each f_k reduces into the global trivial case, finally we obtain the sub-liftings $\left\{\hat{f}_i\right\}$ with the given initial point of $\widehat{f_{k+1}}$ defined to be the final point of $\hat{f}_k$.

  Now glued them, we obtain the lifting $\hat{f}$.

  For the uniqueness, if there are two such liftings, restrict them to $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ and by the uniqueness of trivial case, these two restriction are same. Thus the two liftings are same.

$\square$

Theorem 2.5.2 (lifts out of connected space into covering spaces are unique relative to any point)

For a commutative diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{f}_2"] \arrow[ur, shift right, "\tilde{f}_1"'] \arrow[r, "f"'] & X \end{tikzcd}$$

where Y is connected and $p : E \to X$ a covering, either $\tilde{f}_1$ and $\tilde{f}_2$ are same, or different everywhere.

Proof. These two lifts $\tilde{f}_1$ and $\tilde{f}_2$ induces a map $\langle \tilde{f}_1, \tilde{f}_2 \rangle : Y \to E \times_X E$.

Since $\Delta(E)$ is clopen in $E \times_X E$ (recall Theorem 2.2.1), the pre-image of it under $\langle \tilde{f}_1, \tilde{f}_2 \rangle$ is also clopen. But Y is connected, so the pre-image is either Y or empty. □

Theorem 2.5.3 (Lifting of homotopies)

TODO.

Proof. TODO. □

Theorem 2.5.4 (Lifting of arbitrary maps)

Let $\left(\tilde{X}, p\right)$ be a covering of X and $\tilde{x}_0 \in \tilde{X}$ be a pre-image of x_0. Suppose Y is **connected and locally path-connected**. For a continuous map $\varphi : (Y, y_0) \to (X, x_0)$, there exists a lifting $\tilde{\varphi} : (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ if and only if $\varphi_* \pi_1(Y, y_0) \subset p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right)$.

$$\begin{tikzcd} & \left(\tilde{X}, \tilde{x}_0\right) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\exists\tilde{\varphi}"] \arrow[r, "\varphi"] & (X, x_0) \end{tikzcd}$$

Proof. For simplicity we denote the fundamental group of a space W at point w as $\pi(W, w)$, i.e. omit the subscript 1.

- If such $\tilde{\varphi}$ exists, then since the fundamental group $\pi : \mathrm{Top}_* \to \mathrm{Grp}$ is a functor, the following diagram also commutes:

$$\begin{tikzcd} & \pi\left(\tilde{X}, \tilde{x}_0\right) \arrow[d, "p_*"] \\ \pi(Y, y_0) \arrow[ur, dashed, "\tilde{\varphi}_*"] \arrow[r, "\varphi_*"] & \pi(X, x_0) \end{tikzcd}$$

  So the desired property is immediate.
- If $\varphi_* \pi_1(Y, y_0) \subset p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right)$, we need to construct a $\tilde{\varphi}$ such that the diagram in theorem commutes.

  Note that Y is global path-connected since it is connected and locally path-connected, so there exists at least one path $g : y_0 \rightsquigarrow y$ for all point $y \in Y$. Then by compose φ we obtain a path $\varphi g : x_0 \rightsquigarrow x := \varphi g(y)$. By the lifting property of paths in X, we then obtain a path $\widetilde{\varphi g}$ in $\tilde{X}$ with the chosen initial point $\tilde{x}_0$, let $\tilde{x}$ be the final point of this lifting path.

So we start with an arbitrary point y in Y and end with a point $\tilde{x}$ in $\tilde{X}$, and we, bravely, define $\tilde{\varphi}(y) = \tilde{x}$.

Of course we need to verify this procedure is well-defined:

- **$\tilde{x}$ is independent with the connecting path:** Suppose we have two different paths $g_1, g_2 : y_0 \rightsquigarrow y$, and the resulting final points by using these paths in $\tilde{X}$ are $\tilde{x}_1$ and $\tilde{x}_2$, resp. We need to verify $\tilde{x}_1 = \tilde{x}_2$.

  Let $g_2^{-1} : y \rightsquigarrow y_0$ be the inverse path of g_2, then $g_1 \cdot g_2^{-1} : y_0 \rightsquigarrow y_0$ is an (representative) element in $\pi(Y, y_0)$. And thus $\varphi_*(g_1 \cdot g_2^{-1}) = (\varphi g_1) \cdot (\varphi g_2)^{-1} \in \pi(X, x_0)$. By assumption, there exists a path $\alpha \in \pi(\tilde{X}, \tilde{x}_0)$ such that $p_* \alpha = (\varphi g_1) \cdot (\varphi g_2)^{-1}$.

  So tired, for a complete proof see nLab - *Coverign space*, Proposition 3.9.
- **$\tilde{\varphi}$ makes the diagram commutative:** Because of the lifting property of $\widetilde{\varphi g}$.
- **$\tilde{\varphi}$ is continuous:** **TODO**.

□

3. Basic notions in Singular Homology

For motivations, see []