Notes on Algebraic Topology

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Last compiled at 2025-09-04

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I plan to learn algebraic topology with Prof. Löh's notes, some other references:
- A Basic Course in Algebraic Topology (GTM 127), William S. Massey, Springer.
- A concise course in Algebraic Topology, Peter May.

1. Basic Definitions

1.1. Relative Homotopy

Two morphisms f and g from topology space X to Y is called *homotopic relative to* A for $A \subset X$, denoted as $f \simeq g \text{ rel } A$, if there exists a morphism $h : X \times I \to Y$ such that
- $h(x, 0) = f(x)$ for all $x \in X$;
- $h(x, 1) = g(x)$ for all $x \in X$;
- $h(a, t) = f(a) = g(a)$ for all $a \in A$, $t \in I$.

Remark. So the classical homotopic relation between two paths with same end points is exactly the relative homotopy when $A = \{\text{initial point}, \text{final point}\}$.

1.2. Retract and Deformation Retract

- A subspace $i : A \hookrightarrow X$ is called a *retract* of X if i admits a left inverse $r : X \twoheadrightarrow A$, i.e. $r \circ i = \mathrm{id}_A$;
- It is called a *deformation retract* of X if $i \circ r \simeq \mathrm{id}_X \text{ rel } A$.

Remark. Note that $r \circ i = \mathrm{id}_A$ is equivalent to $r \circ i \simeq \mathrm{id}_A \text{ rel } A$ – so the condition of deformation retract is rather natural – indeed:
- If $r \circ i = \mathrm{id}_A$, then define the homotopy $h(x, t) = r \circ i(x) = \mathrm{id}_A(x) = x$, which is of course continuous in both x and t;
- If $r \circ i \simeq \mathrm{id}_A \text{ rel } A$, then by definition there exists a homotopy $h : A \times I \to A$ such that $h(a, t) = r \circ i(a) = \mathrm{id}_A(a)$, implies that $r \circ i = \mathrm{id}_A$.

The main importance of deformation retract is embodied in the following theorem:

> **Theorem 1.2.1 (A deformation retract shares the same fundamental group of the ambinent sapce)**
>
> For a deformation retract $i : A \hookrightarrow X$, we have $\pi_1(A, a) = \pi_1(X, a)$ for all $a \in A$.

Proof. Suppose the relative homotopy is witness by h.

Then by the proposition in Section 4, Chapter 1 of [May], we need only to prove that $\gamma[h(a, -)] = \mathrm{id}$, but by the definition of relative homotopy, $h(a, -) \equiv a$ so the equation is tautology. $\square$

Remark

We can prove it directly, first we need a lemma: if $f, g : X \to Y$ is relative homotopic respect to $x_0 \in X$, i.e. there exists a homotopy $h : f \simeq g \text{ rel } \{x_0\}$, we claim that $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 := f(x_0) = g(x_0)$.

By this lemma, and note that $i \circ r \simeq \mathrm{id}_A \text{ rel } \{a\}$, we have $(i \circ r)_* = i_* \circ r_* = (\mathrm{id}_A)_*$. Furthur since A is a retract, $r \circ i = \mathrm{id}$ and thus $r_* \circ i_* = \mathrm{id}$. So we finish the proof of the theorem.

For the lemma, suppose $[p] \in \pi_1(X, x_0)$ is a path, we prove that $f \circ p \simeq g \circ p$: the homotopy $\hat{h} : I \times I \to Y$ is given by

$$\begin{tikzcd}
I \times I \arrow[d, "p \times \mathrm{id}"'] \arrow[dr, dashed, "\hat{h}"] & \\
X \times I \arrow[r, "h"'] & Y
\end{tikzcd}.$$

Note that $\hat{h}(s, t) = h(p(s), t)$ is indeed a homotopy between f and g. We are finished.

This can be used to compute the fundamental group $\pi_1(\mathbb{R}^n, x_0)$ for all $x_0 \in \mathbb{R}^n$: we claim that $\{x_0\}$ is a deformation retract of $\mathbb{R}^n$, and one of the required homotopies is given by $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ which sends (x, t) to $tx + (1 - t)x_0$.

So $\pi_1(\mathbb{R}^n, x_0) = \pi_1(\{x_0\}, x_0) = \{*\}$.

2. Covering Space

There are (at least) two definitions of covering spaces over a given base topological space X:

- **The "new" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an open neihgbourhood $x \in U_x$ such that $p^{-1}(U_x) \cong U_x \times p^{-1}(x)$, where $p^{-1}(x)$ equipped with discrete topology.
- **The "old" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an *path-connected* open neihgbourhood U_x such that each path-connected component of $p^{-1}(U_x)$ is homeomorphic to U_x via p.

The new fashion can be found in [Wedhorn], [Löh], and *covering space* on nLab while the old one can be found in [Massey] and [May].

Remark

The old fashion definition is in fact not consistent:

- [Massey] requires that both covering spaces and base spaces are *path-connected* and *locally path-connected*.
- [May] requires nothing.

TODO: Verify that nLab's definition of covering space coinsides with May's: the number of path-connected components of $p^{-1}(U_x)$ equals to $|p^{-1}(x)|$

In short: Covering space is a locally trivial bundle with discrete fiber.

2.1. Examples and non-exmaples

Examples:

1. **The trivial bundle**: Identity $\mathrm{id}_X : X \to X$ is a covering space, with fiber $\{*\}$.
2. **Global trivial bundle with discrete fiber**: For a discrete space D, the projection $X \times D \twoheadrightarrow X$ is of course a covering space, and D is the fiber.
3. **Exponential**: The exponential function $\exp : \mathbb{R} \to S^1$ sends θ to $(\cos\theta, \sin\theta)$ is a local trivial bundle with fiber $\mathbb{Z}$.

Note that to be a covering map, the fiber must be *discrete*. Here is one of non-exmaples:

1. The projection $S^1 \times [0, 1] \twoheadrightarrow S^1$ is not a covering map, since the fiber $[0, 1]$ is not discrete.

2.2. Produce covering spaces from group actions

In this section, G is a group endowed with discrete topology and X is an arbitrary space. We fix a G-action on space X: $\alpha : G \to \mathrm{Aut}_{\mathrm{Top}}(X)$, then it induces a covering space if this action is good enough. We always write gx for $\alpha_g x$.

Denote the orbit of $x \in X$ as $Gx := \{gx : g \in G\}$.

Firstly we need some definitions:

Definition 2.2.1 (Orbit space)

The orbit space GX of X related to the action α is defined as:

$$GX := \{Gx : x \in X\}.$$

We endow this set with quotient topology induced from the projection $X \twoheadrightarrow GX$.

2.3. Lebesgue number

Yes, Lebesgue and algebraic topology. We should say "thank you" to him for the following useful lemma:

Theorem 2.3.1 (Lebesgue number)

For an open covering $\{U_i\}$ of a *compact metric space* X, there exists $\delta > 0$, which is called a Lebesgue number, such that for all $x \in X$ the open ball centered x with radius δ is fully contained in one of those open sets, formally: $B(x, \delta) \subset U_i$ for some U_i.

Proof. Suppose not, that is, for all n, there exists x_n such that $B\left(x_n, \frac{1}{n}\right)$ does not fully contained any U_i.

Since X is compact, $\{x_n\}$ has a covergence subsequence $\{y_n\}$ tends to y_0. Let ε_n the associated radius of y_n. Since $\{U_i\}$ is an open cover of X, there exists $\varepsilon_0 > 0$ and N such that $B(y_0, \varepsilon) \subset U_i$ for some i, and for all $m > N$ we have $y_m \in B(y_0, \frac{\varepsilon_0}{2})$.

Now make m so large that $\varepsilon_m < \frac{\varepsilon_0}{2}$ and $m > N$, so $B(y_m, \varepsilon_m) \subset B(y_0, \varepsilon_0) \subset U_i$, contradiction to our assumption! □

Remark

We need X to be compact, otherwise there may be no such δ, for example: $\left\{B\left(r, \frac{r}{2}\right)\right\}_{r \in (0,1)}$ covers $(0, 1)$.

Compactness ensures the existence of covergence subsequence of a infinite sequence: suppose $\{x_n\}$ does not admit a covergence subsequence, then for all $x \in X$, there exists a open neihgbourhood U_x of x that only contains finitely many x_n. Now note that $\{U_x\}$ is an open cover of a compact space so it admits a finite subcover $\{V_x\}$, and their union only contains finitely many x_n.

Since $\{V_x\}$ covers X, we know that there are only finitely many distinct points in $\{x_n\}$, which implies this sequence itself must coverage, contradiction!

This theorem is especially useful and when we want to prove something about covering spaces, it serves as a bridge from the trivial covering to general situations. We will see such examples in the next section, where we concern lifting properties.

2.4. Lifting Properties

Theorem 2.4.1 (Lifting of paths)

For a given covering space $p : \hat{X} \twoheadrightarrow X$ and a path $f : I \to X$ with the initial point $x_0 := f(0)$, then for any $\widehat{x_0} \in \hat{X}$ such that $p(\widehat{x_0}) = x_0$, there exists an unique path $\hat{f} : I \to E$ with initial point $\widehat{x_0}$ such that $p \circ \hat{f} = f$.

In diagram:

$$\begin{tikzcd}
\{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\
{[0,1]} \arrow[r, "f"'] & X
\end{tikzcd}$$

induces an unique $\hat{f}$ such that

$$\begin{tikzcd}
\{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\
{[0,1]} \arrow[r, "f"'] \arrow[ur, "\exists! \hat{f}"] & X
\end{tikzcd}$$

commutes.

Proof. The main idea is that we first prove the case $\hat{X}$ is *global trivial*, and then proceed to the general case.

- **Trivial Case**: Suppose $\hat{X} \cong X \times D$ for a discrete space D.
  - Existence: We define $\hat{f}(t) := (f(t), d_0)$, where $d_0 = \mathrm{pr}_2(\widehat{x_0})$. It is continuous and indeed a lifting.
  - Uniqueness: For another lift $\tilde{f} : [0, 1] \to \hat{X}$, since the diagram commutes, we have $\tilde{f}(t) = (f(t), d(t))$ and $d(0) = d_0$.

    A continuous image of a path-connected space is again path-connected, so $\tilde{f}(I)$ is path-connected and we claim that $d(t) \equiv d_0$. Otherwise, because all discrete spaces are not path-connected, $\left(\tilde{f}(t_1), d(t_1)\right)$ can not be connected to $\left(\tilde{f}(t_2), d(t_2)\right)$ by path since D is discrete.

- **General Case**: Thanks to the local triviality of a covering space, for each $x \in X$ there exists a open neihgbourhood U_x such that $p^{-1}(U_x) \cong U_x \times D$ a trivial covering of U_x, where $D := p^{-1}(x)$ is equipped with discrete topology.

  Now we need only to divide f into pieces of sub-paths $\{f_i\}$ that each of them is fully contained in some trivial open neihgbourhoods. Then use the result from the trivial case we obtain sub-liftings $\left\{\hat{f}_i\right\}$, and finally glueing them!

  Since $[0,1]$ is compact and $\{f^{-1}(U_x)\}_{x \in X}$ is an open cover of it, we obtain a finite subcover $\{f^{-1}(V_x)\}$, now we let $\frac{1}{n}$ be a Lebesgue number of this cover where n is a integer, and divide $[0,1]$ into $\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], ..., \left[1 - \frac{1}{n}, 1\right]\right\}$. Let $f_k : \left[\frac{k-1}{n}, \frac{k}{n}\right] \to X$ be the restriction of f. By the definition of n, image of each f_k is contained in a trivial open neihgbourhood U_k, and thus the problem for each f_k reduces into the global trivial case, finally we obtain the sub-liftings $\left\{\hat{f}_i\right\}$ with the given initial point of $\widehat{f_{k+1}}$ defined to be the final point of $\hat{f}_k$.

  Now glued them, we obtain the lifting $\hat{f}$.

  For the uniqueness, if there are two such liftings, restrict them to $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ and by the uniqueness of trivial case, these two restriction are same. Thus the two liftings are same.

$\square$