

Notes on *Manifolds, sheaves and cohomology*

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Contents

1. Notations	1
2. Presheaf of (E -valued) functions	1
3. Morphisms between stalks are alyways more convenient	1
4. Constant sheaves	2
5. Restriction of presheaves	2
6. Locally constant sheaves	2
7. Constant sheaves and trivial etale spaces	2
8. Ringed spaces	2
9. Locally ringed spaces	2
10. Properties of morphisms between \mathbb{K}^n subspaces with C^α sheaves (Eg. 4.5)	3
11. Restriction is the left Kan extension (Def. 4.7)	3
12. Open subspaces of locally ringed space are also locally ringed (Def. 4.7)	3
13. C^α -premanifolds	3
14. Complex premanifolds	3
15. Manifolds	3

1. Notations

- X, Y, Z : topological spaces.
 - τ_X : the topology of space X , i.e., the set of open sets.
- F, G : general (pre)sheaves.
- \mathcal{O}_X : structure sheaf associated with ringed space X .
 - For open set $U \subset X$, we use \mathcal{O}_U to denote $\mathcal{O}_X|_U$.
- \mathbb{K} : used for denoting \mathbb{R} or \mathbb{C} .
- α : a generalized natural number, i.e., $\alpha \in \mathbb{N}$ or $\alpha = \infty$.
- C_X^α : Sheaf of C^α functions, on a suitable space X .

Warning: All “(pre)manifolds” are either C^α real-(pre)manifolds for some α , or C^ω complex-(pre)manifolds.

To be more informative we may use:

- (M, C_M^α) : for real-manifolds;
 - (M, hol_M) : for complex-manifolds;
 - (M, \mathcal{O}_M) : for general manifolds;
-

2. Presheaf of (E -valued) functions

Given a bare set E , those presheaves P such that $P(U)$ is some kinds of subset of $\text{Set}(U, E)$ for all opens U , where the restriction map is the obvious one.

3. Morphisms between stalks are alyways more convenient

- The possible reason of this phenomena is that, in general you can't go back from a set of local data to global data in presheaves, and *sheaves* are about local data, so it is difficult to define a morphism

from $\text{Sh}(F)$ to F for presheaves F . But the stalk F_x is about local data and permit you to construct an element by pure local data.

- For example, in the proof of **Example 3.30**, one can easily define a morphism $\alpha_x : \tilde{F}_x := \text{Sh}(F)_x \rightarrow F_x$ by specify that, for an element $(g, \{U_i\})$ in $\tilde{F}U$, the equivalence class defined by $(g|_{U_x}, U_x)$ where $x \in U_x$. And this is indeed a well morphism, which induces a map $\tilde{F}_x \rightarrow F_x$.

4. Constant sheaves

Unfortunately, the name is a little bit confusing (at least in some kind), since for such a sheaf F and an open U , the elements of FU is *not* constant functions but *locally constant* function.

- But it can be fixed by viewing that such sheaves are the sheafification of *constant presheaves*. The proof is straightforward by **Example 3.30**.

5. Restriction of presheaves

For a presheaf $F : \tau_X^{\text{op}} \rightarrow \text{Set}$, we can restrict it to a open $U \subset X$ by pre-composing the inclusion functor $\text{inclusion} : \tau_U^{\text{op}} \rightarrow \tau_X^{\text{op}}$. This is a presheaf by definition.

6. Locally constant sheaves

Those are sheaves F such that for any open U , there exist an open cover $\{U_i\}$ on where F is a constant sheaf, i.e., FU_i is a constant sheaf for all i . This name should be interpreted as “Locally (constant sheaf)”.

7. Constant sheaves and trivial etale spaces

We all agree something to be “constant” is some kinds of triviality, and under the equivalence of categories, this triviality should be preserved. And it is the fact that **constant sheaves correspond to trivial etale spaces**! To prove so, one need to prove that $\text{Et}(F)$ for some constant sheaf F is a trivial etale space, or $\text{Sh}(S, \pi)$ is a constant sheaf. And after trials one find that the latter is much easier.

- Remark of “trivial etale spaces”: The most trivial etale space of a topological space X is ... X it self with the identity map. The second class of trivial etale spaces is the class of $X \times E$ where E is a set equipped with discrete topology, with the projection map.
- Add the prefix “locally” we immediately obtain the finer correspondence between *locally constant sheaves* and *locally trivial etale space*, i.e., *covering spaces*.

8. Ringed spaces

- Objects: A ringed space is a topological space X with a R -algebra sheaf \mathcal{O}_X , where R is a ring.
- Morphisms: A morphism between two ringed spaces (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) is a pair (f, f^\flat) :
 - $f : X \rightarrow Y$ is a continuous map;
 - $f^\flat : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism between sheaves. (pronounced as “f flat”)
- Remark: By the adjunction of $f^{-1} \dashv f_*$, to specify the f^\flat is equivalent to specify $f^\sharp : f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$.

9. Locally ringed spaces

- Objects: Those ringed spaces such that $\mathcal{O}_{X,x}$ the stalk at x is a *local ring*.
- Morphisms: Those (f, f^\flat, f^\sharp) such that the morphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ send the maximal ideal into maximal ideal.

10. Properties of morphisms between \mathbb{K}^n subspaces with C^α sheaves (Eg. 4.5)

For open subsets $X \subset V$ and $Y \subset W$, where both V and W are **finite dimension** \mathbb{K} -vector spaces and \mathbb{K} -ringed C^α -map $(f, f^\flat) : (X, C_X^\alpha) \rightarrow (Y, C_Y^\alpha)$:

- (f, f^\flat) is automatically a **locally** ringed morphism.
- For all open $U \subset Y$, the morphism between structure sheaves $f_U^\flat : C_Y^\alpha U \rightarrow C_X^\alpha(f^{-1}U)$ is given by precomposition, i.e., $f^\flat = - \circ f$.

11. Restriction is the left Kan extension (Def. 4.7)

For an open subspace $i : U \rightarrow X$ and a sheaf \mathcal{O}_X on X , we claim that the restriction of \mathcal{O}_X on U , i.e., $\mathcal{O}_U := \mathcal{O}_X|_U$, is isomorphic to $i^{-1}\mathcal{O}_X$.

- *proof*: It's straightforward to verify that \mathcal{O}_U is exactly the left Kan extension of \mathcal{O}_X along i .
 - *Remark*: The canonical isomorphism is actually $i^\sharp : i^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_U$, the reason is that i^\sharp is the correspondence of i^\flat through the adjunction, but since this adjunction is about left Kan extension and precomposition, i^\sharp is induced by the universal property of $i^{-1}\mathcal{O}_X$, since there is a morphism $\mathcal{O}_X \rightarrow \mathcal{O}_U = \mathcal{O}_X|_U = \mathcal{O}_X \circ \text{bi}$, where $\text{bi} : \tau_X^{\text{op}} \rightarrow \tau_U^{\text{op}}$ induced by i .

12. Open subspaces of locally ringed space are also locally ringed (Def. 4.7)

By the above property, the stalk at $u \in U$ of \mathcal{O}_U is isomorphic to $\text{colim}_u i^{-1}\mathcal{O}_X = \mathcal{O}_{X,u}$, which by assumption is a local ring.

13. C^α -premanifolds

C^α -Premanifolds are **locally \mathbb{K} -ringed spaces** that are **locally isomorphic to open subspace of \mathbb{K}^n** , which endowed with C^α sheaf, for some n .

- Objects: A triple $(M, \mathcal{O}_M, \{U_i\})$ where
 - (M, \mathcal{O}_M) is a locally \mathbb{K} -ringed space;
 - $\{U_i\}$ is an open cover of M equipped with a bunch of **isomorphisms** between locally \mathbb{K} -ringed spaces: $w_i : (U_i, \mathcal{O}_{U_i}) \rightarrow (Y, C_Y^\alpha)$, where $Y \subset \mathbb{K}^n$ open.
 - *Remarks*: Y and n may depend on i , while α fixed.
- Morphisms: Plain morphisms between locally ringed space. For such morphisms between C^α premanifolds, we usually call them **C^α -maps**.
 - *Remarks 1*: When $\alpha = \infty$, we call them **smooth** maps. When $\alpha = \omega$, we call them **analytic** maps.
 - *Remarks 2*: Isomorphisms between C^α -premanifolds are called **C^α -diffeomorphisms**.

14. Complex premanifolds

When $\mathbb{K} = \mathbb{C}$, all functions that are C^α for any α are automatically C^ω , so it is dumb to differ them.

Therefore we call them just **complex premanifolds**.

The fancy name of morphisms between complex premanifolds is “**holomorphic maps**”, for isomorphisms, we use “**bi-holomorphic maps**”.

15. Manifolds

- Objects: Those premanifolds whose underlying spaces are **Hausdorff** and **second countable** (admit a countable bunch of basis).
- Morphisms: Inherits from premanifolds.