Notes on Representation Theory

Author: 秦宇轩 (QIN Yuxuan)
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Contents

This is the note on representation theory.

Main references:

1. *Introduction to representation theory* (arXiv 0901.0827), Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. Accessed at 2025-10-22.

Abberiviations, default settings, terminologies:
- Intertwining maps: morphisms between representations (see Larsh's Note, Definition 3);
- ACF: Algebraic Closed Field;
- $\mathbb{F}$: an arbitrary field;
- k: an algebraic closed field;
- A: an algebra over an algebraic closed field k;

1. Basic notions of representation theory

Definition 1.1 (Irreducible and indecomposable representation)

A **nonzero** representation V of algebra A is called:
- **Irreducible** if the only sub-representations are 0 and V;
- **Indecomposable** if it is not isomorphic to a direct sum of two **nonzero** representations.

Theorem 1.2 (Schur's Lemma)

Let V_1, V_2 be representations of an algebra A over an arbitrary field $\mathbb{F}$, and let $\varphi : V_1 \to V_2$ be a nontrivial morphism between reps. Then:
- If V_1 is irreducible, φ is injective;
- If V_2 is irreducible, φ is surjective.

Proof. Since both $\ker \varphi$ and $\operatorname{im} \varphi$ are sub-representations of V_1 and V_2, resp. □

Remark

If both V_1 and V_2 are irreducible, then φ must be an isomorphism, or trivial.

Corollary 1.2.1 (Schur's Lemma for algebras over a ACF)

If we further assume $\mathbb{F} = k$ is an ACF and V is finite dimensional in Theorem 1.2, then we have this result:

For any endomorphism of an irreducible **finite dimensional** representation over an ACF $\varphi : V \to V$, we claim that $\varphi = \lambda \cdot \mathrm{id}$ for some $\lambda \in k$, i.e. φ is a scalar operator.

Proof. Thanks to your linear algebra course, we know that any endomorphism $\varphi : V \to V$ of a *finite dimensional* vector space B over an ACF admits a eigenvalue $\lambda \in k$ and, $(\varphi - \lambda \cdot \mathrm{id})$ is not a isomorphism since its determinant equals 0.

So $(\varphi - \lambda \cdot \mathrm{id}) = 0$ must be the trivial map since V is irreducible. □

By this corollary, we obtain a good description of irreducible representations of commutative algebras. The key is that if $\rho : A \to \mathrm{End}(V)$ is a representation and A is commutative, then $\rho_a : V \to V$ is automatically a intertwining operator for all $a \in A$.

Theorem 1.3 (A commutative: Irreducible $\iff$ 1-dimensional)

If A is commutative and $\rho : A \to \mathrm{End}(V)$ is an representation of A, then V is irreducible $\iff$ V is 1-dimensional.

Proof.
- ($\Longleftarrow$): Trivial;
- ($\Longrightarrow$): Since $\rho_a : V \to V$ is an intertwining map and V is irreducible, we know that ρ_a is an isomorphism. By Corollary 1.2.1, ρ_a is a scalar operator, since $V \neq 0$ by the definition of irreducible representation, V must be 1-dimensional.

□