Notes on simplicial stuffs

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1. The canonical category of *finite ordered sets* with *order preserving maps* is denoted by Δ, with a bunch of handy morphisms between neighboors:
   - *co-face* map $\delta_i^n : [n-1] \to [n]$, whose image is $[0, 1, ..., (\text{without } i), ..., n]$;
   - *co-degeneracy* map $\sigma_i^n : [n] \to [n-1]$, which is almost indentity but send both i and $i+1$ in $[n]$ to i in $[n-1]$.
2. A **simplicial object** in category C is just a functor $F : \Delta^{\text{op}} \to C$. We denote $F([k])$ by F_k, which is more convenient.
3. Since simplicial objects are contra-variant, we obtain:
   - *face* map $d_i^n := F(\delta_i^n) : F_n \to F_{n-1}$;
   - *degeneracy* map: $s_i^n := F(\sigma_i^n) : F_{n-1} \to F_n$.
4. An n-**simplex** of simplicial set F is just an element of set F_n.
5. A simplex is called **degenerate** if it is the degeneracy of another simplex. For example, we say an k-simplex x (i.e., $x \in X_k$) is degenerate if there exists $(k-1)$-simplex y (i.e., $y \in X_{k-1}$) and some number i such that $x = s_i(y)$.
   - Sometimes we use $D(X_k) := \cup_i \left(s_i^k(X_{i-1})\right)$ to denote all the degenerate k-simplex in X_k, for simplicial set X.
6. **Moore complex**: Given a simplicial object U in an abelian category $\mathcal{A}$, its associated *Moore complex* is a chain complex in $\mathcal{A}$, which looks like:

   $$... \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to ...$$

   where the boundary map $\partial_n : U_n \to U_n - 1$ is defined by the alternating sum of face maps:

   $$\partial_n := \sum_{i=0}^{n} (-1)^i d_i^n.$$

   Note that $d_k^n : U_n \to U_{n-1}$ for all k.
   - Also named **the alternating face map chain complex**.
7. **Singular simplicial complex**: For a topological space X, its *singular simplicial compelx* is indeed the simplicial set induced by the nerv-realazation relation of the canonical inclusion $\Delta \to \text{Top}$.
   - So a **singular n-simplex** for such a space X is just a morphism from Δ^n to X in Top, by the Yoneda lemma.
   - The **singular homology** of a space X is just the homology of the moore complex of its singular simplicial compelx. One usually writes $H_n(X, \mathbb{Z})$ or just $H_n(X)$ for the singular homology of X in degree n.