Quick notes about category theory

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1. The category of categories Cat is co/complete.

proof: We will prove this interesting statment by the realization-nerv relation between Cat and sSet.

First note that:
- $\mathrm{h} : \mathrm{sSet} \to \mathrm{Cat}$ is the **left** adjoint, which preserves **colimit**.
- $\mathrm{Nerv} : \mathrm{Cat} \to \mathrm{sSet}$ is the **right** adjoint; and it is **fully faithful**, which implies the counit $\varepsilon : \mathrm{h} \circ \mathrm{Nerv} \Rightarrow \mathrm{id}_{\mathrm{Cat}}$ is an isomorphism.

Then we start the main proof:

- **Cocompleteness**: For all functors $F : J \to \mathrm{Cat}$, since the counit of realization-nerv adjunction is an isomorphism, we have $\mathrm{h} \circ \mathrm{Nerv} \circ F \simeq F$. The left-hand-side **does** have colimit since

  $$\mathrm{colim}(\mathrm{h} \circ \mathrm{Nerv} \circ F) \simeq \mathrm{h}(\mathrm{colim}(\mathrm{Nerv} \circ F)).$$

  And the latter colimit is in sSet, which is cocomplete since it is a presheaf category.

  Also note that there is another abstract proof based on theory of enriched category, see The answer of Keith Harbaugh.
- **Completeness**: There is a general theorem: Reflective full subcategory is closed under limit. (Reason: Every reflective subcategory inclusion is a monadic functor, and monadic functor creat limit. Reference: Prop 5.1 and Prop 3.1).

  However, the arbitrary products and equalizer in Cat is obvious. So we do not need the above abstract theorem (at least in this proof).

  Also see Riehl Example 4.5.14 (vi), Page 143.

2. Localization of categories and reflective full subcategories

There is a complex relation between localization of categories and reflective full subcategories.
- The localization can be done on any classes of morphisms W in category $\mathcal{C}$, but when the class W is a *weak equivalence* (includes all isomorphisms and satisfies the two-out-of-three rule) things become smoother.
  - Note that the general localization is also called *category of fractions*.
- All reflective subcategories are localizations, see Prop 3.1.

3. Right adjoints is fully faithful $\iff$ The counit is an isomorphism

proof: For counit $\varepsilon : FU \to \mathrm{Id}_D$, thanks to the triangle equallity we have $U\varepsilon_d \circ \eta_{Ud} = \mathrm{id}_{Ud}$, where η is the unit, we know that there exists a counterpart l of η_{Ud} in $D(x, y)$ such that $\varepsilon_d \circ l = \mathrm{id}_d$, since D is fullu faithful.

4. Properties of monadic functors

For monadic functor $U : \mathcal{D} \to \mathcal{C}$ with left adjoint $F : \mathcal{C} \to \mathcal{D}$ and induced monad $T := UF$, induced comparison functor $K : \mathcal{D} \to \mathcal{C}^T$, we claim that U:

1. … is faithful

   proof: In fact for a monad induced by a general adjunction where the right adjoint is U, there is a decomposition of U: $U = U^T \circ K$ where $U^T : C^T \to C$ is the forgetful functor and K the comparison functor.

   Note that U^T is always faithful, so when K is a equivalence their composition is also faithful.

   - *Remark:* A maybe useful observation: For functor $U : \mathcal{D} \to \mathcal{C}$, it is faithful if and only if the induced natural transformation $\mathcal{D}(x, y) \to \mathcal{C}(Ux, Uy)$ is a monomorphism.
2. … creats limit

   proof: Here I only make a short description, please draw a diagram by yourself, really, it is benifit for you.

   Suppose $H : J \to \mathcal{D}$ is a diagram, and lim UH exists in $\mathcal{C}$ with a bunch of projections p_i, we need to construct a limit in $\mathcal{D}$. Thanks to the equivalence between $\mathcal{D}$ and $\mathcal{C}^T$, we need only to construct limit in the later category.

   Define $V := \lim\ UH$ We claim that the object $(V, h : UFV \to V)$ is the limit we want, where h is induced by the universal property of V with a bunch of morphisms $U\varepsilon_{Hi} \circ UFp_i : UFV \to UHi$. One can verify that h is indeed satisfies the requirements of $\mathcal{C}^T$. (**Warning**: please note that the triangle equallity of T-module $(c, v_c : Tc \to c)$ is that $v_c \circ \eta_c = \mathrm{id}_c$! Caution that the right hand side is *not* id_{Tc}!).

   V is a universal cone in $\mathcal{C}^T$ can be infered from its universal property in $\mathcal{C}$.

   So lim KH exists and thus lim H does so.

   - *Remark:* The diagram chasing is cubersome, please also see Disccusion on math exchange.

     we need to prove that it both preserves and reflects limits.
     - Preserves limits: Since U is a right adjoint.
     - Reflects limits: Since all faithful functors reflects limits, and U is just faithful.

5. What if $\mathrm{Hom}(-, x)$ preserves filtered colimit?

Such hom functors are trivial in the following sense: $\mathrm{Hom}(y, x) = \emptyset$ or $\mathrm{Hom}(y, x) = \mathrm{Hom}(x, x)$.

- *proof:* Suppose $x \in C$, note that $\mathbf{2} := \{0 \to 1\}$ is a filtered category (in fact all posets are filtered). For $y \in C$ if there is no morphisms from y to x then we are done, otherwise there is at least one morphism $f : y \to x$ and we define $F : \mathbf{2} \to C$ sending $\{0 \to 1\}$ to $\{y \to x\}$.

  So by assumption $\mathrm{Hom}(\mathrm{colim}_{\mathbf{2}} F, x) = \mathrm{colim}_{\mathbf{2}}\ \mathrm{Hom}(F -, x) = \mathrm{Hom}(x, x)$, and note that $\mathrm{colim}_{\mathbf{2}} F = x$, we are done.

6. Filtered colimits commute with finite limits in Set

For a functor $F : I \times J \to \mathrm{Set}$ with I filtered and J finite, we claim that

$$\mathrm{colim}_I \lim_J F \simeq \lim_J \mathrm{colim}_I F.$$

- *proof sketch*: We do not present the full proof in detail (see Prop. 23.7 in Dmitri's lecture) but point out two important observations:
  - First, please note that $F(i, j)$ is **not** a hom set, and F is **not** a hom functor, which means it is **covariant in both i and j**!
  - Second, in the proof of surjectivity, the claim "If we increase i further, we can also assume that for any morphism $f : j \to j'$ in J we have $D(-, f)(y_j) = y_{j'}$" is the result of universal property of limit:

$$\begin{tikzcd}
& x \in \lim_J \mathrm{colim}_I F \arrow[dl] \arrow[dr] & \\
\left(i, y_j : D(i, j)\right) \in \mathrm{colim}_I D(-, j) & & \left(i, y_{j'} : D(i, j')\right) \in \mathrm{colim}_I D(-, j') \arrow[ll, "\mathrm{colim}_I D(-, f)"']
\end{tikzcd}$$

  For any $f : j \to j'$ the above diagram must commute, so by increasing i to a suitable i' (this process will halt since there are only finite such f) we can claim that result.

7. $\mathrm{Ran}_G G$ is a monad if it exists

This is the construction of socalled **codensity** monad of an arbitrary functor $G : A \to B$, and the monadness can be proved in a clever way:

Define a category r_G whose:
- Objects: $(X : B \to B, x : XG \Rightarrow G)$, i.e., right extensions of G;
- Morphisms between (X, x) and (Y, y): Natural transformations $\eta : X \Rightarrow Y$ which campatible with x and y.

And $(r_G, \mathrm{id}_B, \circ)$ is a (strict) monoidal category.

Then we find **$\mathbf{Ran}_G G$ is the terminal object in r_G**! So by common abstract nonsense argument, it has an unique monoid structure.

Ref. *CODENSITY AND THE ULTRAFILTER MONAD*, Section 5. Tom Leinster.