Notes on *Measure, Integration and Real Analysis*

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1. Measure

2. Integration

The main idea of integration is that *every function can be approximated by simple function.* And, we can almost replace every function concerned with its approximation sequence (Examples are the proof of additivity of integration.).

2.1. Integration of simple funcitons

Definition 2.1.1 (Simple Function) Those functions are precisely whose range is finite.

Lemma 2.1.2 Every simple function admits a normal form as:

$$\sum_{E \in \mathcal{E}} c_E \chi_E.$$

where $\mathcal{E}$ is a finite family of sets, c_E are real number and χ is the characteristic function.

2.2. Integration plays well with limit, absolute value and ordering

Theorem 2.2.1 (Monotone Coverage) For measurable space (X, S, μ) and a sequence of increasing measurable funcitons f_i, if $f(x) := \lim_k f_k(x)$ is defined for all x, then we have:

$$\lim_k \int f_k d\mu = \int f d\mu.$$

Proof. Omitted □

Theorem 2.2.2 (Absolute Value Inequality) For measurable function f we have

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

Proof. By triangle inequality. □

To prove integration preserve ordering we will first introduce a lemma:

Lemma 2.2.3 (Positive function admit positive integration) If $f \geq 0$ then

$$\int f d\mu \geq 0.$$

Theorem 2.2.4 (Ordering are preserved) For $f \geq g$ we have

$$\int f \geq \int g.$$

Proof. Consider $\int (f - g)$. □

2.3. Integrations on small sets are small

Theorem 2.3.1 (You can not be powerful every where) For measurable function g, if it is not so extraordinary (that is, $\int g < \infty$), then for any $\varepsilon > 0$ there exists a number δ such that

$$\int_B g < \varepsilon,$$

for all B with $\mu B < \delta$.

Proof. Consider easy cases first: if g is a simple function, then just let $\delta = \frac{1}{2} \max(g) \cdot \varepsilon$, since $\int_B g \leq \mu B \cdot \max(g)$.

If g is not simple, then we just simply force it be simple. □

2.4. Domination Theorem

The powerful dominate the weak, thought I don't like the fact but, a fact is a fact.