Notes on Algebra

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1. The algebraic closure of finite field

Fix a prime p, we claim that the algebraic closure of finite field $\mathbb{F}_{p^n}$ (for all $n \geq 1$) is:

$$\overline{\mathbb{F}}_p := \bigcup_{k \in \mathbb{N}} \mathbb{F}_{p^{k!}}.$$

Reasons:

1. We know that $\mathbb{F}_{p^k}$ is the splitting field of $x^{p^k} - x$ on $\mathbb{F}_p$ for all naturals $k \geq 1$;
2. For $k \mid m$ we have $\left(x^{p^k} - x\right) \mid \left(x^{p^m} - x\right)$, since if $x^{p^k} = x$ then $x^{p^m} = \left(x^{p^n}\right)^{p^{m-n}} = x^{p^{m-n}}$ and so on and so on, terminates at $x^{p^0} = x$ since $k \mid m$. So we always have $\mathbb{F}_{p^{k!}} \subset \mathbb{F}_{p^{(k+1)!}}$;
3. For any non-constant polynomial $f \in \overline{\mathbb{F}}_p$, there exist a natural number k such that all coefficients of it are in $\mathbb{F}_{p^{k!}}$, then the splitting field of f is a finite extension of $\mathbb{F}_{p^{k!}}$ and thus is also finite with characteristic p in form $\mathbb{F}_{p^m}$ for some naturals m, so its splitting field is contained in $(\mathbb{F})_{p^{m!}}$ by point 2, thus finally its splitting field is contained in $\overline{\mathbb{F}}_p$.

Ref. *algebraic closure of a finite field.* Planetmath. Ver. 2025-07-09.

2. The free functor from $\mathbf{Set}$ to $\mathbf{Mod}_R$

For a ring R, the demanded functor is

$$R \otimes_{\mathbb{Z}} \mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Mod}_{\boldsymbol{R}},$$

where

- $\mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Ab} := \mathbf{Mod}_{\mathbb{Z}}$ is the free abelian group functor;
- $\otimes_{\mathbb{Z}}$ is the common tensor product on $\mathbf{Ab}$.

This is because adjoint functors are stable under composition, and $R \otimes_{\mathbb{Z}} - : \mathbf{Ab} \to \mathbf{Mod}_R$ is the free functor from abelian groups to R-modules.

Further, note that $R \otimes_R M \cong M$ for any R-module M (this may be unrelated to the main property we want, but it justify it when $R = \mathbb{Z}$).

This result is used to define homology theories with coefficients different from $\mathbb{Z}$, see, eg. Loeh's note.