Notes on Algebraic Topology

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Contents

I plan to learn algebraic topology with Prof. Löh's notes, some other references:

- *A Basic Course in Algebraic Topology* (GTM 127), William S. Massey, Springer.
- *A Concise Course in Algebraic Topology*, Peter May, availabe at https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

We denote the concatenation of two path $g_{12} : x_1 \rightsquigarrow x_2$ and $g_{23} : x_2 \rightsquigarrow x_3$ by $g_{12} \cdot g_{23} : x_1 \rightsquigarrow x_2$, i.e., we apply g_{12} **first**, then g_{23}.

1. Basic Definitions

1.1. Relative Homotopy

Two morphisms f and g from topology space X to Y is called *homotopic relative to* A for $A \subset X$, denoted as $f \simeq g \text{ rel } A$, if there exists a morphism $h : X \times I \to Y$ such that

- $h(x, 0) = f(x)$ for all $x \in X$;
- $h(x, 1) = g(x)$ for all $x \in X$;
- $h(a, t) = f(a) = g(a)$ for all $a \in A$, $t \in I$.

Remark. So the classical homotopic relation between two paths with same end points is exactly the relative homotopy when $A = \{\text{initial point}, \text{final point}\}$.

1.2. Retract and Deformation Retract

- A subspace $i : A \hookrightarrow X$ is called a *retract* of X if i admits a left inverse $r : X \twoheadrightarrow A$, i.e. $r \circ i = \mathrm{id}_A$;
- It is called a *deformation retract* of X if $i \circ r \simeq \mathrm{id}_X \text{ rel } A$.

Remark. Note that $r \circ i = \mathrm{id}_A$ is equivalent to $r \circ i \simeq \mathrm{id}_A \text{ rel } A$ – so the condition of deformation retract is rather natural – indeed:

- If $r \circ i = \mathrm{id}_A$, then define the homotopy $h(x,t) = r \circ i(x) = \mathrm{id}_A(x) = x$, which is of course continuous in both x and t;
- If $r \circ i \simeq \mathrm{id}_A \text{ rel } A$, then by definition there exists a homotopy $h : A \times I \to A$ such that $h(a,t) = r \circ i(a) = \mathrm{id}_A(a)$, implies that $r \circ i = \mathrm{id}_A$.

The main importance of deformation retract is embodied in the following theorem:

Theorem 1.2.1 (A deformation retract shares the same fundamental group of the ambient sapce)

For a deformation retract $i : A \hookrightarrow X$, we have $\pi_1(A,a) = \pi_1(X,a)$ for all $a \in A$.

Proof. Suppose the relative homotopy is witness by h.

Then by the proposition in Section 4, Chapter 1 of [1], we need only to prove that $\gamma[h(a,-)] = \mathrm{id}$, but by the definition of relative homotopy, $h(a,-) \equiv a$ so the equation is tautology. □

Remark

We can prove it directly, first we need a lemma: if $f, g : X \to Y$ is relative homotopic respect to $x_0 \in X$, i.e. there exists a homotopy $h : f \simeq g \text{ rel } \{x_0\}$, we claim that $f_* = g_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ where $y_0 := f(x_0) = g(x_0)$.

By this lemma, and note that $i \circ r \simeq \mathrm{id}_A \text{ rel } \{a\}$, we have $(i \circ r)_* = i_* \circ r_* = (\mathrm{id}_A)_*$. Furthur since A is a retract, $r \circ i = \mathrm{id}$ and thus $r_* \circ i_* = \mathrm{id}$. So we finish the proof of the theorem.

For the lemma, suppose $[p] \in \pi_1(X,x_0)$ is a path, we prove that $f \circ p \simeq g \circ p$: the homotopy $\hat{h} : I \times I \to Y$ is given by

$$\begin{tikzcd} I \times I \arrow[d, "p \times \mathrm{id}"'] \arrow[dr, dashed, "\hat{h}"] & \\ X \times I \arrow[r, "h"'] & Y \end{tikzcd}.$$

Note that $\hat{h}(s,t) = h(p(s),t)$ is indeed a homotopy between f and g. We are finished.

This can be used to compute the fundamental group $\pi_1(\mathbb{R}^n, x_0)$ for all $x_0 \in \mathbb{R}^n$: we claim that $\{x_0\}$ is a deformation retract of $\mathbb{R}^n$, and one of the required homotopies is given by $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ which sends (x,t) to $tx + (1-t)x_0$.

So $\pi_1(\mathbb{R}^n, x_0) = \pi_1(\{x_0\}, x_0) = \{*\}$.

2. Covering Space

There are (at least) two definitions of covering spaces over a given base topological space X:

- **The “new” fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an open neihgbourhood $x \in U_x$ such that $p^{-1}(U_x) \cong U_x \times p^{-1}(x)$, where $p^{-1}(x)$ equipped with discrete topology.
- **The “old” fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an *path-connected* open neihgbourhood U_x such that each path-connected component of $p^{-1}(U_x)$ is homeomorphic to U_x via p.

The new fashion can be found in [2], [3], and *covering space* on nLab while the old one can be found in [4] and [1].

In short: Covering space is a locally trivial bundle with discrete fiber.

This whole section can be rewritten in groupoid-theoretic language, as done in [May, Chapter 3].

2.1. Examples and non-exmaples

Examples:
1. **The trivial bundle**: Identity $\mathrm{id}_X : X \to X$ is a covering space, with fiber $\{*\}$.
2. **Global trivial bundle with discrete fiber**: For a discrete space D, the projection $X \times D \twoheadrightarrow X$ is of course a covering space, and D is the fiber.
3. **Exponential**: The exponential function $\exp : \mathbb{R} \to S^1$ sends θ to $(\cos\theta, \sin\theta)$ is a local trivial bundle with fiber $\mathbb{Z}$.

Note that to be a covering map, the fiber must be *discrete*. Here is one of non-exmaples:
1. The projection $S^1 \times [0, 1] \twoheadrightarrow S^1$ is not a covering map, since the fiber $[0, 1]$ is not discrete.

2.2. Basic Topological Properties

Theorem 2.2.1 (fiber-wise diagonal of covering space is open and closed)

For $p : E \to X$ a covering, the diagonal $\Delta(E) := \{(e, e) : e \in E\}$ is both closed and open in $E \times_X E$, the pullback of $p : E \to X$.

Proof. See nLab - *Covering spaces*, Lemma 3.2. □

This will be used to prove the result: two lifts in a covering space of a path are either same, or different everywhere.

2.3. Produce covering spaces from group actions

In this section, G is a group endowed with discrete topology and X is an arbitrary space. We fix a G-action $\alpha : G \to \mathrm{Aut}_{\mathrm{Top}}(X)$, then it induces a covering space if this action is good enough. We always write gx for $\alpha_g x$.

The investigations of group actions on spaces will help us to prove the correspondence between $\mathrm{Sub}(\pi_1(X))$ and $\mathrm{Cov}(X)$ for a good enough X, such as spaces which admit a universal covering space.

Denote the orbit of $x \in X$ as $Gx := \{gx : g \in G\}$.

Firstly we need some definitions:

Definition 2.3.1 (Orbit space)

The orbit space GX of X related to the action α is defined as:

$$GX := \{Gx : x \in X\}.$$

We endow this set with quotient topology induced from the projection $X \twoheadrightarrow GX$.

2.4. Lebesgue number

Yes, Lebesgue and algebraic topology. We should say "thank you" to him for the following useful lemma:

Theorem 2.4.1 (Lebesgue number)

For an open covering $\{U_i\}$ of a *compact metric space* X, there exists $\delta > 0$, which is called a Lebesgue number, such that for all $x \in X$ the open ball centered x with radius δ is fully contained in one of those open sets, formally: $B(x, \delta) \subset U_i$ for some U_i.

Proof. Suppose not, that is, for all n, there exists x_n such that $B\left(x_n, \frac{1}{n}\right)$ does not fully contained any U_i.

Since X is compact, $\{x_n\}$ has a covergence subsequence $\{y_n\}$ tends to y_0. Let ε_n the associated radius of y_n. Since $\{U_i\}$ is an open cover of X, there exists $\varepsilon_0 > 0$ and N such that $B(y_0, \varepsilon) \subset U_i$ for some i, and for all $m > N$ we have $y_m \in B(y_0, \frac{\varepsilon_0}{2})$.

Now make m so large that $\varepsilon_m < \frac{\varepsilon_0}{2}$ and $m > N$, so $B(y_m, \varepsilon_m) \subset B(y_0, \varepsilon_0) \subset U_i$, contradiction to our assumption! $\square$

Remark

We need X to be compact, otherwise there may be no such δ, for example: $\left\{B\left(r, \frac{r}{2}\right)\right\}_{r \in (0,1)}$ covers $(0, 1)$.

Compactness ensures the existence of covergence subsequence of a infinite sequence: suppose $\{x_n\}$ does not admit a covergence subsequence, then for all $x \in X$, there exists a open neihgbourhood U_x of x that only contains finitely many x_n. Now note that $\{U_x\}$ is an open cover of a compact space so it admits a finite subcover $\{V_x\}$, and their union only contains finitely many x_n.

Since $\{V_x\}$ covers X, we know that there are only finitely many distinct points in $\{x_n\}$, which implies this sequence itself must coverage, contradiction!

This theorem is especially useful and when we want to prove something about covering spaces, it serves as a bridge from the trivial covering to general situations. We will see such examples in the next section, where we concern lifting properties.

2.5. Lifting Properties

Theorem 2.5.1 (Lifting of paths)

For a given covering space $p : \hat{X} \twoheadrightarrow X$ and a path $f : I \to X$ with the initial point $x_0 := f(0)$, then for any $\widehat{x_0} \in \hat{X}$ such that $p(\widehat{x_0}) = x_0$, there exists an unique path $\hat{f} : I \to E$ with initial point $\widehat{x_0}$ such that $p \circ \hat{f} = f$.

In diagram:

$$\begin{tikzcd} \{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\ {[0,1]} \arrow[r, "f"'] & X \end{tikzcd}$$

induces an unique $\hat{f}$ such that

$$\begin{tikzcd} \{0\} \arrow[r, hook, "\widehat{x_0}"] \arrow[d, hook] & \hat{X} \arrow[d, two heads, "p"] \\ {[0,1]} \arrow[r, "f"'] \arrow[ur, "\exists! \hat{f}"] & X \end{tikzcd}$$

commutes.

Proof. The main idea is that we first prove the case $\hat{X}$ is *global trivial*, and then proceed to the general case.

- **Trivial Case**: Suppose $\hat{X} \cong X \times D$ for a discrete space D.
  - Existence: We define $\hat{f}(t) := (f(t), d_0)$, where $d_0 = \mathrm{pr}_2(\widehat{x_0})$. It is continuous and indeed a lifting.
  - Uniqueness: For another lift $\tilde{f} : [0,1] \to \hat{X}$, since the diagram commutes, we have $\tilde{f}(t) = (f(t), d(t))$ and $d(0) = d_0$.

  A continuous image of a path-connected space is again path-connected, so $\tilde{f}(I)$ is path-connected and we claim that $d(t) \equiv d_0$. Otherwise, because all discrete spaces are not path-connected, $\left(\tilde{f}(t_1), d(t_1)\right)$ can not be connected to $\left(\tilde{f}(t_2), d(t_2)\right)$ by path since D is discrete.

- **General Case**: Thanks to the local triviality of a covering space, for each $x \in X$ there exists a open neihgbourhood U_x such that $p^{-1}(U_x) \cong U_x \times D$ a trivial covering of U_x, where $D := p^{-1}(x)$ is equipped with discrete topology.

  Now we need only to divide f into pieces of sub-paths $\{f_i\}$ that each of them is fully contained in some trivial open neihgbourhoods. Then use the result from the trivial case we obtain sub-liftings $\left\{\hat{f}_i\right\}$, and finally glueing them!

  Since $[0,1]$ is compact and $\{f^{-1}(U_x)\}_{x \in X}$ is an open cover of it, we obtain a finite subcover $\{f^{-1}(V_x)\}$, now we let $\frac{1}{n}$ be a Lebesgue number of this cover where n is a integer, and divide $[0,1]$ into $\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], ..., \left[1 - \frac{1}{n}, 1\right]\right\}$. Let $f_k : \left[\frac{k-1}{n}, \frac{k}{n}\right] \to X$ be the restriction of f. By the definition of n, image of each f_k is contained in a trivial open neihgbourhood U_k, and thus the problem for each f_k reduces into the global trivial case, finally we obtain the sub-liftings $\left\{\hat{f}_i\right\}$ with the given initial point of $\widehat{f_{k+1}}$ defined to be the final point of $\hat{f}_k$.

  Now glued them, we obtain the lifting $\hat{f}$.

  For the uniqueness, if there are two such liftings, restrict them to $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ and by the uniqueness of trivial case, these two restriction are same. Thus the two liftings are same.

$\square$

Theorem 2.5.2 (lifts out of connected space into covering spaces are unique relative to any point)

For a commutative diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}_2", shift left] \arrow[ur, "\tilde{f}_1"', shift right] \arrow[r, "f"'] & X \end{tikzcd}$$

where Y is connected and $p : E \to X$ a covering, either $\tilde{f}_1$ and $\tilde{f}_2$ are same, or different everywhere.

Proof. These two lifts $\tilde{f}_1$ and $\tilde{f}_2$ induces a map $\langle \tilde{f}_1, \tilde{f}_2 \rangle : Y \to E \times_X E$.

Since $\Delta(E)$ is clopen in $E \times_X E$ (recall Theorem 2.2.1), the pre-image of it under $\langle \tilde{f}_1, \tilde{f}_2 \rangle$ is also clopen. But Y is connected, so the pre-image is either Y or empty. □

There is another proof does not use Theorem 2.2.1 from [Massey, Chapter V, Lemma 3.2] (and thus we need not to prove those tedious point-set topological results):

Proof. For simplicity we write $\tilde{f}_i$ as f_i.

We prove that, $K := \{y \in Y : f_1 y = f_2 y\}$ is clopen.

- K is closed: We prove that its closure $\overline{K} = K$ itself. For $a \in \overline{K} - K$, i.e. $f_1 a \neq f_2 a$, but $p f_1 a = p f_2 a$. Now let $V_1, V_2 \subset E$ be the elementary neihgbourhoods of these two point $p f_1 a$ and $p f_2 a$ resp. Since E is a covering space, by definition V_1 and V_2 are disjoint.

  Now consider $W := f_1^{-1} V_1 \cap f_2^{-1} V_2$ which is an open neihgbourhood of a, and $f_1 W \cap f_2 W = \emptyset$ by the definition of V_1 and V_2, this implies $W \cap K = \emptyset$, contradicts to the assumption that $a \in \overline{K}$.

- K is open: We prove that for any $y \in K$, there exists an open neihgbourhood of y which contained in K.

  If $y \in K$, i.e. $f_1 y = f_2 y =: \tilde{y}$, let V be the component in E contains $\tilde{y}$. Now consider $W := f_1^{-1} V \cap f_2^{-1} V$ which is an open neihgbourhood of y, we claim that $W \subset K$.

  For $a \in W$, $f_1 a$ and $f_2 a$ are both in V, an elementary neihgbourhood of $\tilde{y}$ in E, which is homeomorphic to an open set around $p\tilde{f}$. Note that both $f_1 a, f_2 a \in V$ are pre-images of $p f_1 a = p f_2 a$, since p restricted to V is a homeomorphism, it must be injective and thus $f_1 a = f_2 a$, so $a \in K$.

□

This result will be used to provide a neccessary condition of TODO: (a G-space Y is a covering space of Y/G via the canonical projection with $\mathrm{Aut}(Y) = G$)

Theorem 2.5.3 (Lifting of homotopies)

TODO.

Proof. TODO. □

Theorem 2.5.4 (Lifting of arbitrary maps)

Let $(\tilde{X}, p)$ be a covering of X and $\tilde{x}_0 \in \tilde{X}$ be a pre-image of x_0. Suppose Y is **connected and locally path-connected**. For a continuous map $\varphi : (Y, y_0) \to (X, x_0)$, there exists a lifting $\tilde{\varphi} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $\varphi_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[r, "\varphi"] \arrow[ur, dashed, "\exists \tilde{\varphi}"] & (X, x_0)
\end{tikzcd}$$

Proof. For simplicity we denote the fundamental group of a space W at point w as $\pi(W, w)$, i.e. omit the subscript 1.

- If such $\tilde{\varphi}$ exists, then since the fundamental group $\pi : \mathrm{Top}_* \to \mathrm{Grp}$ is a functor, the following diagram also commutes:

$$\begin{tikzcd}
 & \pi(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\
\pi(Y, y_0) \arrow[r, "\varphi_*"] \arrow[ur, dashed, "\tilde{\varphi}_*"] & \pi(X, x_0)
\end{tikzcd}$$

  So the desired property is immediate.
- If $\varphi_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$, we need to construct a $\tilde{\varphi}$ such that the diagram in theorem commutes.

  Note that Y is global path-connected since it is connected and locally path-connected, so there exists at least one path $g : y_0 \rightsquigarrow y$ for all point $y \in Y$. Then by compose φ we obtain a path $\varphi g : x_0 \rightsquigarrow x := \varphi g(y)$. By the lifting property of paths in X, we then obtain a path $\widetilde{\varphi g}$ in $\tilde{X}$ with the chosen initial point $\tilde{x}_0$, let $\tilde{x}$ be the final point of this lifting path.

  So we start with an arbitrary point y in Y and end with a point $\tilde{x}$ in $\tilde{X}$, and we, bravely, define $\tilde{\varphi}(y) = \tilde{x}$.

  Of course we need to verify this procedure is well-defined:
  - **$\tilde{x}$ is independent with the connecting path:** Suppose we have two different paths $g_1, g_2 : y_0 \rightsquigarrow y$, and the resulting final points by using these paths in $\tilde{X}$ are $\tilde{x}_1$ and $\tilde{x}_2$, resp. We need to verify $\tilde{x}_1 = \tilde{x}_2$.

    Let $g_2^{-1} : y \rightsquigarrow y_0$ be the inverse path of g_2, then $g_1 \cdot g_2^{-1} : y_0 \rightsquigarrow y_0$ is an (representative) element in $\pi(Y, y_0)$. And thus $\varphi_*(g_1 \cdot g_2^{-1}) = (\varphi g_1) \cdot (\varphi g_2)^{-1} \in \pi(X, x_0)$. By assumption, there exists a path $\alpha \in \pi(\tilde{X}, \tilde{x}_0)$ such that $p_* \alpha = (\varphi g_1) \cdot (\varphi g_2)^{-1}$.

    So tired, for a complete proof see nLab - *Coverign space*, Proposition 3.9.
  - **$\tilde{\varphi}$ makes the diagram commutative:** Because of the lifting property of $\widetilde{\varphi g}$.
  - **$\tilde{\varphi}$ is continuous:** TODO.

$\square$

Wow, a bunch of theorems, arguments, we need an application!

Universal Covering Space. If $(\tilde{X}, p)$ is a covering space of X, and $\tilde{X}$ is connected, **path-connected and simply-connected**; then it is unique up to isomorphisms.

Furthur, $\tilde{X}$ is the initial object in the category $\mathrm{Cov}(X)$, which consists of coverings of X.

Proof. For the first part, note that for two such coverings $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$. Choose $\tilde{x}_i \in p_i^{-1}(x_0)$, and by the lifting theorem we can lift p_1 to an **unique** $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ since $\pi(\tilde{X}_1)$ is trivial. By the same argument we obtain $\tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$. Now consider $\tilde{p}_1 \circ \tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_2, \tilde{x}_2)$, which must be the identity on $(\tilde{X}_2, \tilde{x}_2)$ since the both this map and the identity map are liftings of p_2, and by the lifting theorem, such lifting is **unique**, so the result is immediate. (See also nLab - *Universal covering space*, Proposition 1.1). □

2.6. Covering spaces from group actions

If a group G acts well on a good space, then we obtain a covering map for free, in detail:

> **Theorem 2.6.1 (Covering space from group action)**
>
> Let Y be a connected, locally path-connected space and let G be a **properly discontinuous** group of homeomorphisms of Y, then the projection $p : Y \to Y/G$ is a regular covering with the deck group $\operatorname{Deck}(p) := \operatorname{Aut}(Y, p) = G$.

When Y is simply-connected, it is further the universal covering of Y/G, thus we know the fundamental group of Y/G, as $\pi_1(Y/G) = \operatorname{Deck}(p) = G$.

We can use this theorem to compute fundamental groups of some good spaces.

1. **The fundamental group of $\mathbb{R}P^n$**: We claim

$$\pi_1(\mathbb{R}P^n) := \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z}/2 & \text{if otherwise.} \end{cases}$$

Proof. The case of $n = 1$ is left as an excersice.

When $n \geq 2$, consider the antipodal map $T : S^n \to S^n$, which is continuous and nilpotent, i.e. $T^2 = \mathrm{id}$. Note that $\{T, T^2\}$ is a group consists of discontinuous homeomorphisms of S^2, and S^n is a good space, the theorem tell us $p : S^n \to S^n/\{T, T^2\}$ is a covering map with deck group $\operatorname{Deck}(p) = \{T, T^2\}$. □

3. Basic notions in Singular Homology

For motivations, see [4].

3.1. Why are there so many different "homology"?

I have met so many "homology" so far, such as **singular** homology and **simplicial** homology, but what are they indeed?

Well, the fact is: **there are many different (some in form, some in essential) homology theories**, invented for different purposes.

The singular one is applied to any spaces, while the simplicial one is only applied to "a few" spaces, details below.

3.2. Simplicial Homology

The simplicial homology is literally the theory of somplices in a space,

> **Definition 3.2.1 (Standard simplex)**
>
> We denote the standard n-simplices by Δ_n, defined in the Euclidean space $\mathbb{R}^{n+1}$ by the formular:
>
> $$\Delta_n := \{\boldsymbol{x} : x_0 + \ldots + x_n = 1, x_i > 0\},$$

> **Definition 3.2.2 (Simplicial complex)**
>
> A simplicial complex is just a compound or combination of standard simplices, in a nice way.

The formal definition is wordy and omitted.

Definition 3.2.3 (Simplicial homology)

For K a simplicial complex, we can define its simplicial homology as follows:

1. $C_n^{\mathrm{Simp}}(K)$ is defined to be the free abelian group generated by all n-simplices in K;
2. $\mathrm{Simp}(K)$ is defined to be the chain complex consists of all $C_n^{\mathrm{Simp}}(K)$, (boundary maps omitted since it's wordy);
3. The simplicial n-homology group $H_n^{\mathrm{Simp}}(X)$ is defined as the n-homology group of the chain complex $\mathrm{Simp}(K)$.

Remark

So, at least for now, the simplicial homology is only defined for **simplicial complex**. Since we can not talk about "simplices" in a general space.

Based people can also define the simplicial homology not only for simplicial complex, but also **delta complex** (I don't know what is it).

But, I think there is some way to extend this definition to all spaces which admits a triangulation.

3.3. Singular Homology

As stated before, the simplicial one is only defined for a (rather small) class of spaces, but the singular one is defined for all spaces.

The main idea is copying the definition of the simplicial one in a celever way and modify it so we can talk about the modified "simplices" in a general space. And this is done by the concept of "shape" or "xxx object" in category theory.

Definition 3.3.1 (Singular homology)

Let X be general space. By "an n- simplicial map φ", we mean a continuous map $\varphi : \Delta_n \to X$ where Δ_n is the standard n-simplex;

1. $C_n^{\mathrm{Sing}}(X)$ is defined to be the free abelian group generated by all n-simplical maps to X;
2. $\mathrm{Sing}(X)$ is definned to be the chain complex consists of all $C_n^{\mathrm{Sing}}(X)$;
3. The singular n-homology group $H_n^{\mathrm{Sing}}(X)$ is defined as the n-homology group of $\mathrm{Sing}(X)$.

3.4. Scheme of different homology theory

We find the scheme of definitions of different homology theories can be summed up as follows:

1. First define a functor $\alpha : \mathrm{Space} \to \mathrm{Ch}_\bullet(\mathcal{C})$, where Space is a category consists of some kinds of spaces and $\mathcal{C}$ is an additive category;
2. Next, apply the n-homology functor defined on any chain complex category $H_n : \mathrm{Ch}_\bullet(\mathcal{C}) \to \mathrm{Alg}$, where Alg is a category consists some kind of algebras.

And finally we obtain a homology theory of spaces, denoted by H^α.

3.5. $H^{\mathrm{Simp}} = H^{\mathrm{Sing}}$

Yes, they are the same. **TODO**: proof.

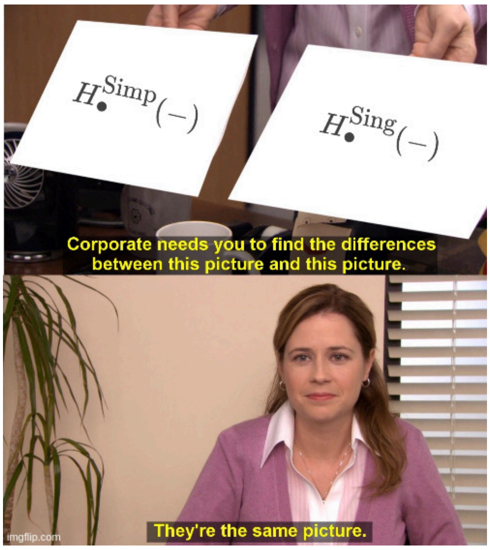

Figure 1: They are the same!

But why do people invent different-in-form but same-in-essential theories?

The reason is all theories sucks in some aspects, but for different purpose one can use appropriate theories. And that's it! The simplicial homology is rather easy for computation while very difficult for proof, and the singular one is exactly in the opposite.

3.6. Digression on Terminology

Let $(C_\bullet, \partial_\bullet)$ be a general chain complex, we know that the elements in $\ker \partial$ is called **cycles** and those in $\operatorname{im} \partial$ is called **boundaries**, but why?

Well, those wierd terms all come from algebraic topology.

Here is an exmaple: Suppose we have a standard 2-simplex Δ_2:

$$\begin{tikzcd} & v_2 & \\ v_0 \arrow[ur, "e_{02}"] \arrow[rr, "e_{01}"] & & v_1 \arrow[ul, "e_{12}"'] \end{tikzcd}$$

By simple computation we find: $\partial_1(e_{01} + e_{12} - e_{02}) = 0 \in C_0^{\mathrm{Simp}}(\Delta_2)$. And $e_{01} + e_{12} - e_{02}$ is literally a cycle!

Furthur, computation also shows $\ker \partial_1 = \langle e_{01} + e_{12} - e_{02} \rangle \cong \mathbb{Z}$, which means all the elements in $\ker \partial_1$ are just mutiples of a cycle!

As for "boundaries", note that $\partial_1 e_{01} = v_1 - v_0$ which is just a combination of boundaries of e_{01}.

I think that's the etymology of "cycle" and "boundary".

3.7. The long exact sequence lemma

Let X be a space A be a subspace of X. Denote the quotient map by $q : X \to X/A$ and the inclusion by $\iota : A \to X$, then we have the following long exact sequence:

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{H}_{n+1}(A) \arrow[r, "\iota_*"] & \tilde{H}_{n+1}(X) \arrow[r, "q_*"] & \tilde{H}_{n+1}(X/A) \arrow[dll, "\partial"] \\
& \tilde{H}_n(A) \arrow[r, "\iota_*"] & \tilde{H}_n(X) \arrow[r, "q_*"] & \tilde{H}_n(X/A) \arrow[dll, "\partial"] \\
& \tilde{H}_{n-1}(A) \arrow[r, "\iota_*"] & \tilde{H}_{n-1}(X) \arrow[r, "q_*"] & \tilde{H}_{n-1}(X/A) \arrow[r] & \ldots
\end{tikzcd}$$

where $\tilde{H}$ stands for the reduced singular homology.

Proof. We first obtain a short exact sequence of chain complex: $0 \to \mathrm{Sing}_\bullet(A) \hookrightarrow \mathrm{Sing}_\bullet(X) \twoheadrightarrow \mathrm{Sing}_\bullet(X/A)$, which can be verified quickly.

The rest of the proof can be found in [Miller] Theorem 9.1. □

TODO: what is ∂?

3.8. Compute $\tilde{H}_n(S^m)$

In this section we use the long exact sequence lemma for reduced singular homology groups to determine the homology groups of spheres:

$$\tilde{H}_n(S^m) = \begin{cases} \mathbb{Z} \ , \quad n = m \\ 0 \ , \ \text{otherwise} \end{cases}.$$

Our map chain in **Top** is $\mathbb{S}^{m-1} \hookrightarrow \mathbb{D}^m \twoheadrightarrow \mathbb{S}^m = \mathbb{D}^m / \mathbb{S}^{m-1}$, by the lemma, the following sequence is exact:

$$\begin{tikzcd}
& \tilde{H}_n(\mathbb{D}^m) \arrow[r] & \tilde{H}_n(\mathbb{S}^m) \arrow[dll, "\partial"] \\
\tilde{H}_{n-1}(\mathbb{S}^{m-1}) \arrow[r] & \tilde{H}_{n-1}(\mathbb{D}^m) &
\end{tikzcd}$$

Note that $H_n(\mathbb{D}^m) = 0$ for all $n, m \in \mathbb{Z}$ since $\mathbb{D}^m$ is contractible, and thus we know the ∂ here is actually an isomorphism, i.e. $\tilde{H}_n(S^m) \cong \tilde{H}_{n-1}(S^{m-1})$.

And we know the reduced singular homology groups $\tilde{H}_n(\mathbb{S}^0)$ of $\mathbb{S}^0 = \{\bullet \quad \bullet\}$ is $\mathbb{Z}$ when $n = 0$ and 0 otherwise. So by induction, we are done.

[4] provides another methods to determine $\tilde{H}_n(\mathbb{S}^n)$ by the Excision lemma below. (It is cubersome though)

3.9. Excision Lemma

The excision lemma is a powerful tool for computing homology groups.

> **Theorem 3.9.1 (Homology is invariant under excision)**
>
> For a space X, the homology group is invariant under tame excision, formally, if $A \subset X$ is a subspace and $W \subset A$, then
>
> $$H_n(X, A) \simeq H_n(X - W, A - W),$$
>
> provided that W is "totally" contained in A, to be precise, $\mathrm{cl}_X(W) \subset \mathrm{int}_X(A)$.

We can use this theorem to compute $H_n(S^k)$. (TODO: digest the computation in [4])

To prove this result, we need a construction called "subdivision", which is used to restrict our attention of singular cubes to a "smaller" ones without losing any information.

For example, if we are working in a metric space, then subdivision enables us to just consider cubes with diameter less than a given positive number, and the homology groups of those small cubes are isomorphic to the homology groups of all cubes.

To formalize what is "small" in an arbitrary topological space, we need the following definition:

TODO: Digest the more morden definition of small, i.e. locaility, see, eg. Haynes Miller's lecture 11.

Definition 3.9.2 (Small cube)

Let $\mathcal{U} := \{U_\lambda\}$ be a family of subsets in X such that $\{\mathrm{int}_X(U_\lambda)\}$ covers X.

A singular n-cube $T : I^n \to X$ is said to be small of order $\mathcal{U}$ if there exists an index λ such that $\mathrm{im}\ T \subset U_\lambda$.

Trivial small cube. The trivial n-cube T which sends all things to a single point is small of any order.

3.9.1. Application

Excision is useful to compute homology groups with the long exact sequence.

Ref. Anthony Bosman's lectures on Algebraic Topology, lecture 15, `47:04`, availabe on Youtube and 哔哩哔哩.

Theorem 3.9.3 ($\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$)

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be two opens, if they are isomorphic, then $n = m$.

Proof. This is really a good application of the Excision lemma. The main idea is use the homology groups of spheres. Since if there the homology groups of two spheres coincides on every level, these two spheres are forced to equal, which means their dimensions are the same.

We agree: in the following proof, $H_n(-)$ stands for the *reduced* homology group, for short. We prove it in x steps:

Step 1 (Excise what?) To make use of $H_k(\mathbb{S}^p)$, we need first make something homotopic to $\mathbb{S}^p$, and $\mathbb{R}^{p+1} - \{\mathrm{pt}\}$ comes into mind. By excision lemma $H_n(U, U - \{\mathrm{pt}\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathrm{pt}\})$ where we take $X = \mathbb{R}^n$, $A = \mathbb{R}^n - \{\mathrm{pt}\}$ and $W = U$ in Theorem 3.9.1.

Step 2 (Use the long exact sequence) By the long exact sequence lemma, the following sequence is exact for all $k \in \mathbb{Z}$:

$$
\begin{tikzcd}
 & 0 \arrow[d, equal] & H_k(U, U - \{\mathrm{pt}\}) \arrow[d, equal, "\text{Excision}"] \\
H_k(\mathbb{R}^n - \{\mathrm{pt}\}) \arrow[r] & H_k(\mathbb{R}^n) \arrow[r] & H_k(\mathbb{R}^n, \mathbb{R}^n - \{\mathrm{pt}\}) \arrow[dll, "\partial"] \\
H_{k-1}(\mathbb{R}^n - \{\mathrm{pt}\}) \arrow[r] \arrow[d, equal, "\mathbb{R}^n - \{\mathrm{pt}\} \simeq \mathbb{S}^{n-1}"'] & H_{k-1}(\mathbb{R}^n) \arrow[r] \arrow[d, equal] & H_{k-1}(\mathbb{R}^n, \mathbb{R}^n - \{\mathrm{pt}\}) \\
H_{k-1}(\mathbb{S}^{n-1}) & 0 &
\end{tikzcd}
$$

From the diagram, we observe that ∂ is actually an isomorphism. So $H_k(U, U - \{\mathrm{pt}\}) \cong H_{k-1}(\mathbb{S}^{n-1})$, similarly we have $H_k(V, V - \{\mathrm{pt}\}) \cong H_{k-1}(\mathbb{S}^{m-1})$.

Step 3 (Homology of spheres) By assumption, $U \cong V$, so the two relative homology groups $H_k(U, U - \{\mathrm{pt}\})$ and $H_k(V, V - \{\mathrm{pt}\})$ are isomorphic, which means $H_k(\mathbb{S}^{n-1}) \cong H_k(\mathbb{S}^{m-1})$ for all k by works in step 2. Finally, thanks to our computation on the homology groups of spheres, this immediately implies $n = m$, we are done.

□

4. (Co)fibration

TODO: Check the proofs in [1].

Cofibrations are good inclusions, in the sense that if $i : A \hookrightarrow X$ is a cofibration and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence. (TODO: counter-examples when i is not a cofibration)

With this property, we can show that all subgroups of a free group is free again: the main part using cofibration is that, for a maximal tree T of a finite connected plane graph X, the inclusion $i : T \hookrightarrow X$ is a cofibration. And note that T is contractible, X and X/A (a wedge product of circles) share the smae homotopy type, so the fundamental group of any finite connected plane graph is free.

Definition 4.1 (Cofibration)

A map $i : A \to X$ is called a cofibration if it admits the **homotopy extension property** (HEP): for any homotopy $h : A \times I \to Y$ makes the upper triangle commutes, it admits an extension $\tilde{h} : X \times I \to Y$, in diagram:

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{inl}_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i \times \mathrm{id}_I"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "\mathrm{inl}_0"'] & & X \times I \arrow[ul, dashed, "\exists \tilde{h}"']
\end{tikzcd}$$

> ⚠ **Warning**
>
> In the definition we do not require $\tilde{h}$ to be unique! (And it usually isn't)

There is an universal test for a map to be a cofibration: the mapping cylinder.

Definition 4.2 (Mapping Cylinder)

For a map $f : X \to Y$ between spaces, we define the mapping cylinder Mf of f to be the pushout:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A \times I \arrow[d] \\
X \arrow[r] & Mf := X \coprod_f (A \times I)
\end{tikzcd}$$

The category of compactly generated spaces admits pushout so it is well defined.

Proposition 4.3 (Mapping cylinder as universal test)

A map $i : A \to X$ is a cofibration if and only if HEP is true for Mi.

Proof. One direction is obvious. The other direction can be proved by the universal property of pushout. □

Proposition 4.4 (Cofibrations are injective with closed image)

As the name suggests.

Proof.

- For injectivity, use the universal property of Mi, and note that the inclusions into pushout are always injective.
- For the rest part, TODO.

□

Any map can be decomposed into a cofibration and a homotopy equivalence through its mapping cylinder.

Theorem 4.5 (Replacing maps by cofibrations)

For any map $f : X \to Y$, it admits a decomposition $f = r \circ (\iota \circ \mathrm{inl}_0)$:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{inl}_0"] \arrow[d, "f"'] & X \times I \arrow[d, hook, "\iota"'] \arrow[ddr, bend left, "(x,-) \mapsto f(x)"] & \\
Y \arrow[r, hook, "i"] \arrow[drr, bend right, equal] & Mf \arrow[dr, dashed, "\exists! r"] & \\
 & & Y
\end{tikzcd}$$

And r is a homotopy equivalence, $(\iota \circ \mathrm{inl}_0)$ is a cofibration.

Proof. TODO

May claim that it is not hard to directly show that $(\iota \circ \mathrm{inl}_0)$ is a cofibration, but there seems to be a more systematic way: the criteria of a map to be a cofibration. □

Theorem 4.6 (Criteria of cofibration)

For a **closed** subspace $i : A \hookrightarrow X$, the following are equivalent:
1. (X, A) is an NDR-pair;
2. $(X \times I, X \times \{0\} \cup A \times I)$ is a DR-pair;
3. $X \times \{0\} \cup A \times I$ is a retract of $X \times I$;
4. The inclusion $i : A \hookrightarrow X$ is a cofibration.

Proof. TODO □

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