Notes on Algebraic Topology

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Contents

I plan to learn algebraic topology with Prof. Löh's notes, some other references:
- A Basic Course in Algebraic Topology (GTM 127), William S. Massey, Springer.
- A concise course in Algebraic Topology, Peter May.

1. Basic Definitions

1.1. Relative Homotopy

Two morphisms f and g from topology space X to Y is called *homotopic relative to* A for $A \subset X$, denoted as $f \simeq g$ rel A, if there exists a morphism $h : X \times I \to Y$ such that
- $h(x, 0) = f(x)$ for all $x \in X$;
- $h(x, 1) = g(x)$ for all $x \in X$;
- $h(a, t) = f(a) = g(a)$ for all $a \in A$, $t \in I$.

Remark. So the classical homotopic relation between two paths with same end points is exactly the relative homotopy when $A = \{\text{initial point}, \text{final point}\}$.

1.2. Retract and Deformation Retract

- A subspace $i : A \hookrightarrow X$ is called a *retract* of X if i admits a left inverse $r : X \twoheadrightarrow A$, i.e. $r \circ i = \mathrm{id}_A$;
- It is called a *deformation retract* of X if $i \circ r \simeq \mathrm{id}_X$ rel A.

Remark. Note that $r \circ i = \mathrm{id}_A$ is equivalent to $r \circ i \simeq \mathrm{id}_A$ rel A – so the condition of deformation retract is rather natural – indeed:
- If $r \circ i = \mathrm{id}_A$, then define the homotopy $h(x, t) = r \circ i(x) = \mathrm{id}_A(x) = x$, which is of course continuous in both x and t;
- If $r \circ i \simeq \mathrm{id}_A$ rel A, then by definition there exists a homotopy $h : A \times I \to A$ such that $h(a, t) = r \circ i(a) = \mathrm{id}_A(a)$, implies that $r \circ i = \mathrm{id}_A$.

The main importance of deformation retract is embodied in the following theorem:

> **Theorem 1.2.1 (A deformation retract shares the same fundamental group of the ambinent sapce)**
>
> For a deformation retract $i : A \hookrightarrow X$, we have $\pi_1(A, a) = \pi_1(X, a)$ for all $a \in A$.

Proof. Suppose the relative homotopy is witness by h.

Then by the proposition in Section 4, Chapter 1 of [May], we need only to prove that $\gamma[h(a, -)] = \mathrm{id}$, but by the definition of relative homotopy, $h(a, -) == a$ so the equation is tautology. □

> **Remark**
>
> We can prove it directly, first we need a lemma: if $f, g : X \to Y$ is relative homotopic respect to $x_0 \in X$, i.e. there exists a homotopy $h : f \simeq g \text{ rel } \{x_0\}$, we claim that $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 := f(x_0) = g(x_0)$.
>
> By this lemma, and note that $i \circ r \simeq \mathrm{id}_A \text{ rel } \{a\}$, we have $(i \circ r)_* = i_* \circ r_* = (\mathrm{id}_A)_*$. Furthur since A is a retract, $r \circ i = \mathrm{id}$ and thus $r_* \circ i_* = \mathrm{id}$. So we finish the proof of the theorem.
>
> For the lemma, suppose $[p] \in \pi_1(X, x_0)$ is a path, we prove that $f \circ p \simeq g \circ p$: the homotopy $\hat{h} : I \times I \to Y$ is given by
>
> $$\begin{tikzcd} I \times I \arrow[d, "p \times \mathrm{id}"'] \arrow[dr, dashed, "\hat{h}"] & \\ X \times I \arrow[r, "h"'] & Y \end{tikzcd}.$$
>
> Note that $\hat{h}(s, t) = h(p(s), t)$ is indeed a homotopy between f and g. We are finished.

This can be used to compute the fundamental group $\pi_1(\mathbb{R}^n, x_0)$ for all $x_0 \in \mathbb{R}^n$: we claim that $\{x_0\}$ is a deformation retract of $\mathbb{R}^n$, and one of the required homotopies is given by $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ which sends (x, t) to $tx + (1 - t)x_0$.

So $\pi_1(\mathbb{R}^n, x_0) = \pi_1(\{x_0\}, x_0) = \{*\}$.

2. Covering Space

There are (at least) two definitions of covering spaces over a given base topological space X:

- **The "new" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an open neihgbourhood $x \in U_x$ such that $p^-1(U_x) \cong U_x \times p^-1(x)$.
- **The "old" fashion**: A covering space over X is a morphism $p : E \to X$ such that: for all $x \in X$, there exists an *path-connected* open neihgbourhood U_x such that each path-connected component of $p^{-1}(U_x)$ is homeomorphic to U_x via p.

The new fashion can be found in [Wedhorn], and *covering space* on nLab while the old one can be found in [Massey] and [May].

> **Remark**
>
> The old fashion definition is in fact not consistent:
>
> - [Massey] requires that both covering spaces and base spaces are *path-connected* and *locally path-connected*.
> - [May] requires nothing.

TODO: Verify that nLab's definition of covering space coinsides with May's: the number of path-connected components of $p^{-1}(U_x)$ equals to $|p^{-1}(x)|$

In short: Covering space is a discrete bundle.

2.1. Lebesgue in Algebraic Topology

Yes, Lebesgue and algebraic topology. We should thanks to him for the following useful lemma:

Theorem 2.1.1 (Lebesgue number)

For an open covering $\{U_i\}$ of a *compact metric space* X, there exists $\delta > 0$, which is called a Lebesgue number, such that for all $x \in X$ the open ball centered x with radius δ is fully contained in one of those open sets, formally: $B(x, \delta) \subset U_i$ for some U_i.

Proof. Suppose not, that is, for all n, there exists x_n such that $B\left(x_n, \frac{1}{n}\right)$ does not fully contained any U_i.

Since X is compact, $\{x_n\}$ has a coverage subsequence $\{y_n\}$ tends to y_0. Let ε_n the associated radius of y_n. Since $\{U_i\}$ is an open cover of X, there exists $\varepsilon_0 > 0$ and N such that $B(y_0, \varepsilon) \subset U_i$ for some i, and for all $m > N$ we have $y_m \in B\left(y_0, \frac{\varepsilon_0}{2}\right)$.

Now make m so large that $\varepsilon_m < \frac{\varepsilon_0}{2}$ and $m > N$, so $B(y_m, \varepsilon_m) \subset B(y_0, \varepsilon_0) \subset U_i$, contradiction to our assumption! $\square$