

# Variational Inference and Generative Models

CS 285

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UC Berkeley

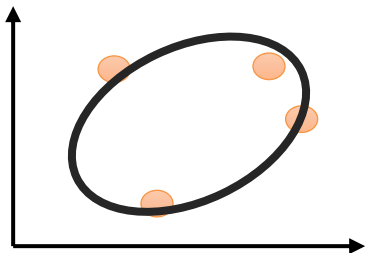


# Today's Lecture

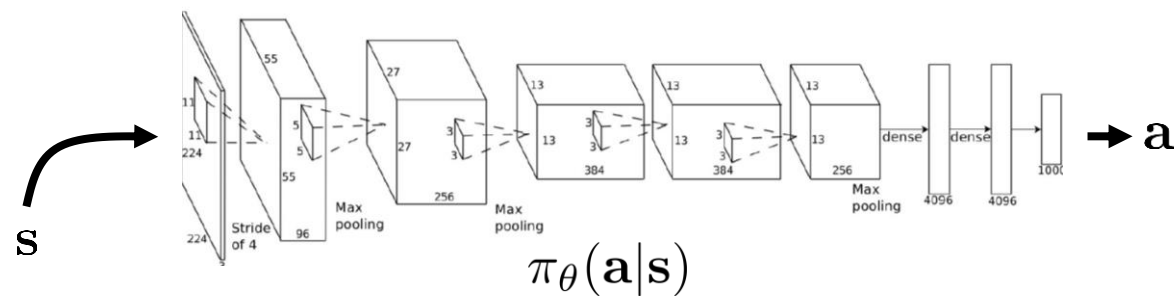
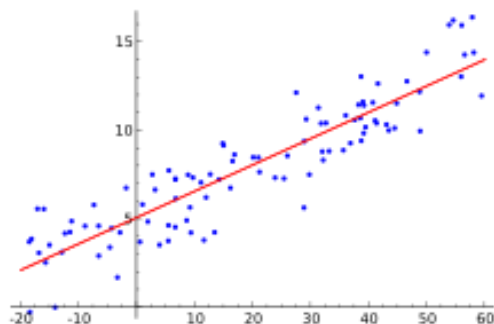
1. Probabilistic latent variable models
  2. Variational inference
  3. Amortized variational inference
  4. Generative models: variational autoencoders
- Goals
    - Understand latent variable models in deep learning
    - Understand how to use (amortized) variational inference

# Probabilistic models

$p(x)$



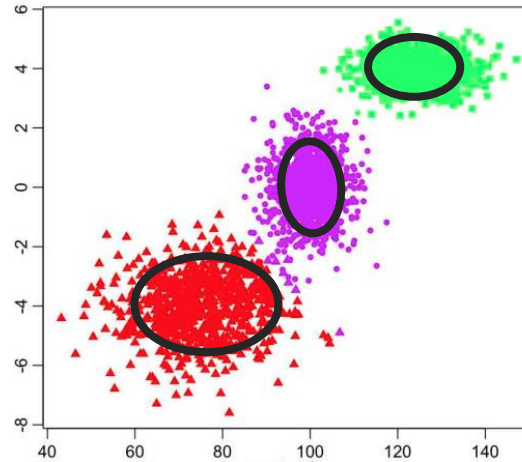
$p(y|x)$



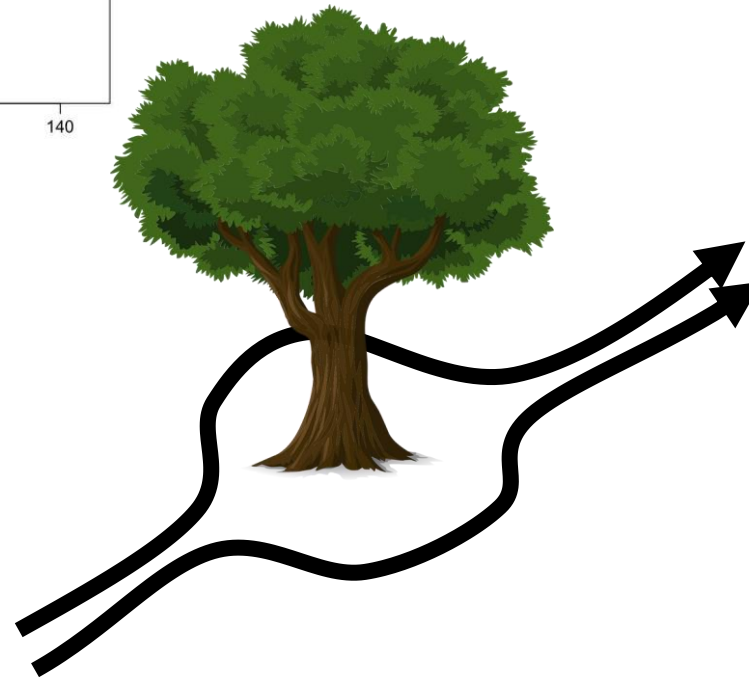
# Latent variable models

$$p(x) = \sum_z p(x|z)p(z)$$

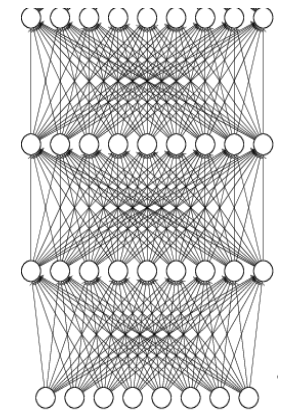
↑  
mixture  
element



$$p(y|x) = \sum_z p(y|x, z)p(z)$$



$$w_1, \mu_1, \Sigma_1, \dots, w_N, \mu_N, \sigma_N$$

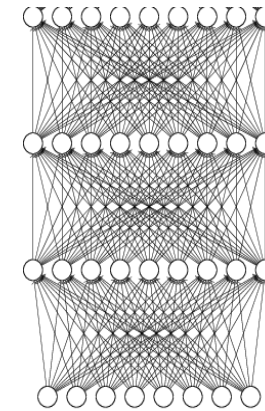
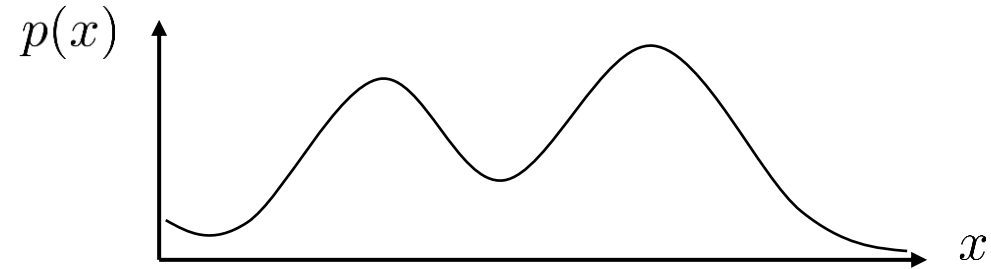


# Latent variable models in general

$$p(x) = \int p(x|z)p(z)dz$$

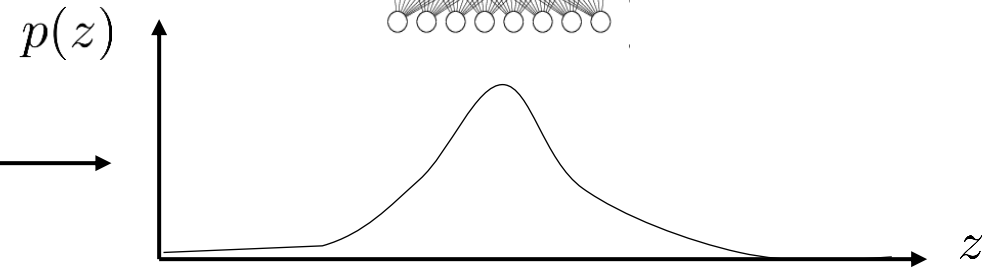
“easy” distribution  
(e.g., conditional Gaussian)

“easy” distribution  
(e.g., Gaussian)



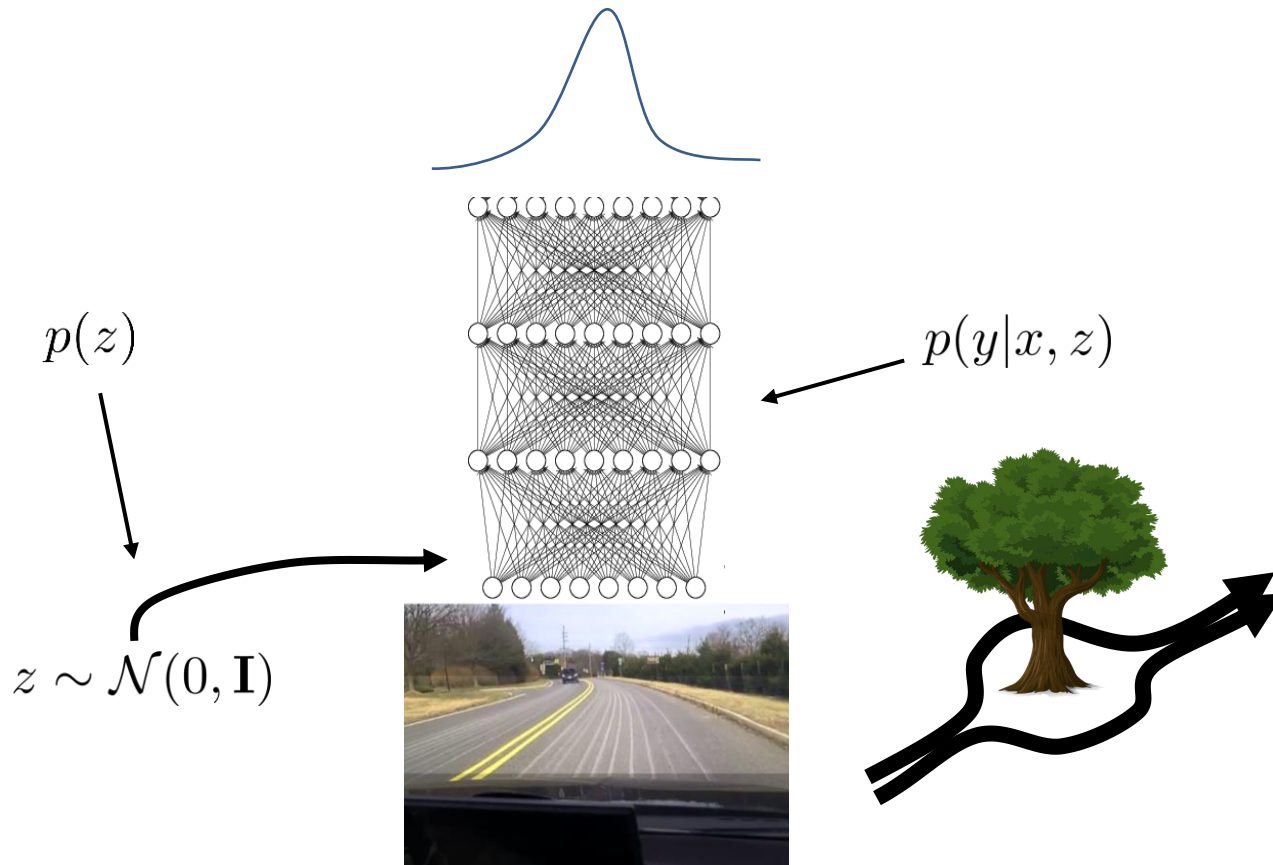
$$p(x|z) = \mathcal{N}(\mu_{\text{nn}}(z), \sigma_{\text{nn}}(z))$$

“easy” distribution  
(e.g., Gaussian)

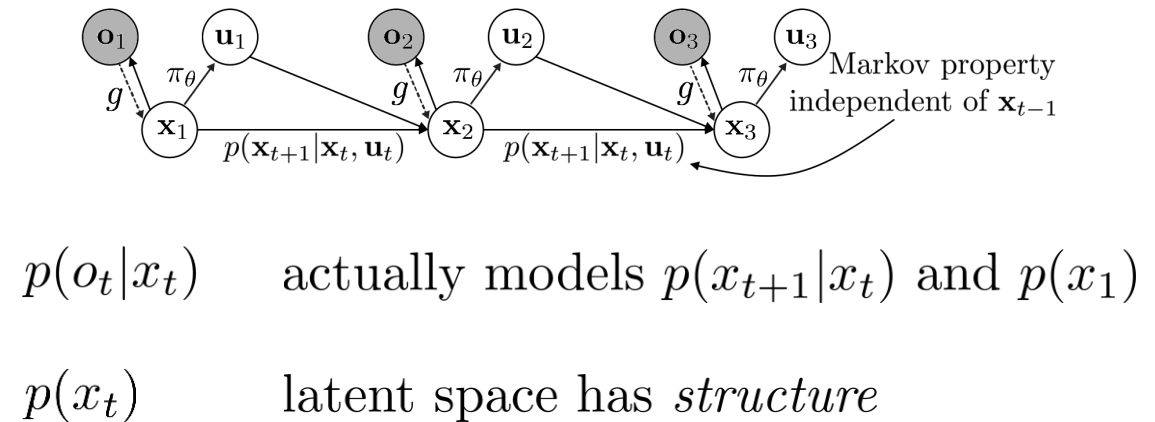


# Latent variable models in RL

conditional latent variable  
models for multi-modal policies

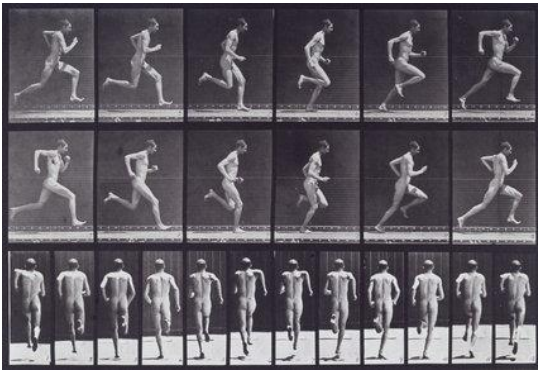


latent variable models for  
model-based RL



# Other places we'll see latent variable models

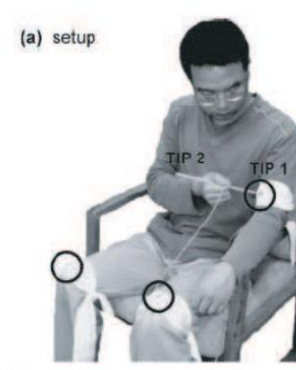
Using RL/control + variational inference to model human behavior



Muybridge (c. 1870)



Mombaur et al. '09

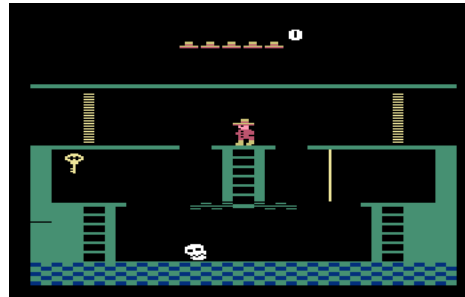


Li & Todorov '06



Ziebart '08

Using generative models and variational inference for exploration



# How do we train latent variable models?

the model:  $p_\theta(x)$

the data:  $\mathcal{D} = \{x_1, x_2, x_3, \dots, x_N\}$

maximum likelihood fit:

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log p_\theta(x_i)$$

$$p(x) = \int p(x|z)p(z)dz$$

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log \left( \int p_\theta(x_i|z)p(z)dz \right)$$



completely intractable



# Estimating the log-likelihood

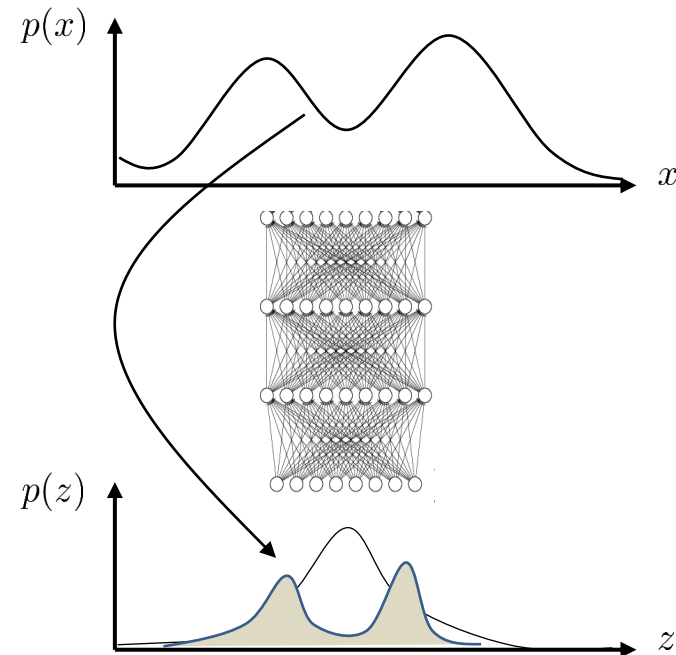
alternative: *expected* log-likelihood:

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i E_{z \sim p(z|x_i)} [\log p_{\theta}(x_i, z)]$$

but... how do we calculate  $p(z|x_i)$ ?

intuition: “guess” most likely  $z$  given  $x_i$ ,  
and pretend it’s the right one

...but there are many possible values of  $z$   
so use the distribution  $p(z|x_i)$



# Variational Inference

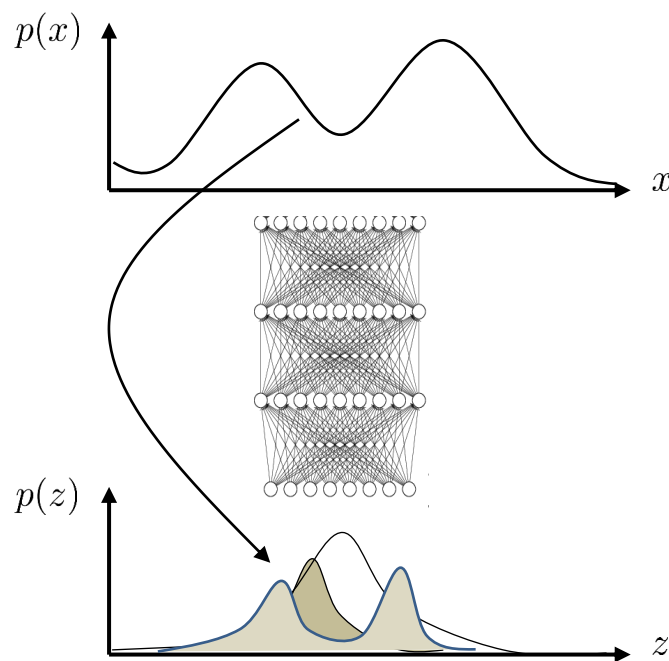
# The variational approximation

but... how do we calculate  $p(z|x_i)$ ?

what if we approximate with  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

can bound  $\log p(x_i)$ !

$$\begin{aligned}\log p(x_i) &= \log \int_z p(x_i|z)p(z) \\ &= \log \int_z p(x_i|z)p(z) \frac{q_i(z)}{q_i(z)} \\ &= \log E_{z \sim q_i(z)} \left[ \frac{p(x_i|z)p(z)}{q_i(z)} \right]\end{aligned}$$



# The variational approximation

but... how do we calculate  $p(z|x_i)$ ?

can bound  $\log p(x_i)$ !

$$\log p(x_i) = \log \int_z p(x_i|z)p(z)$$

$$= \log \int_z p(x_i|z)p(z) \frac{q_i(z)}{q_i(z)}$$

$$= \log E_{z \sim q_i(z)} \left[ \frac{p(x_i|z)p(z)}{q_i(z)} \right]$$

$$\geq E_{z \sim q_i(z)} \left[ \log \frac{p(x_i|z)p(z)}{q_i(z)} \right] = E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] + E_{z \sim q_i(z)} [\log q_i(z)]$$

Jensen's inequality

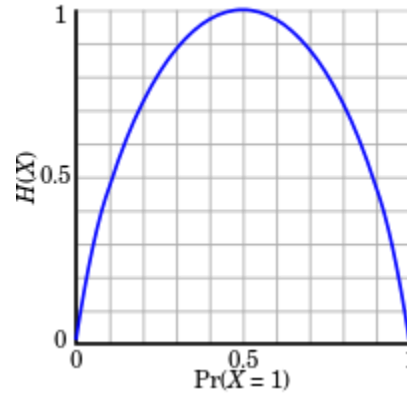
$$\log E[y] \geq E[\log y]$$

maximizing this maximizes  $\log p(x_i)$



# A brief aside...

## Entropy:



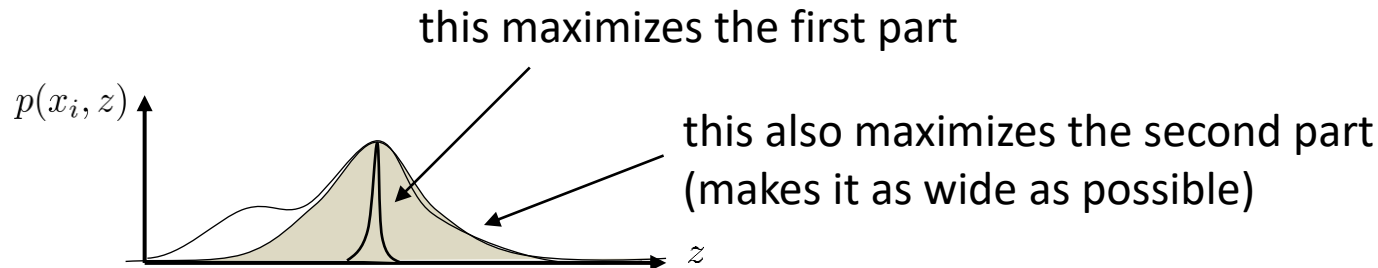
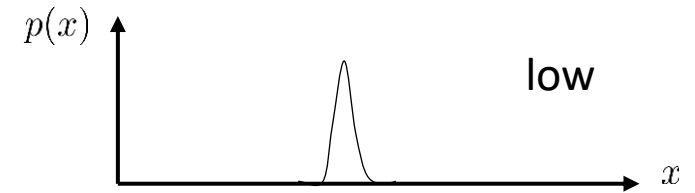
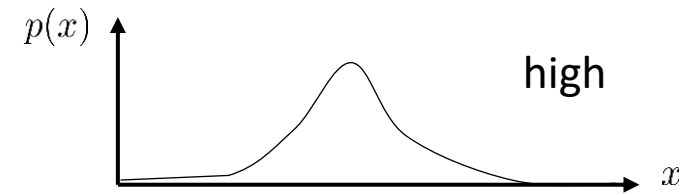
$$\mathcal{H}(p) = -E_{x \sim p(x)}[\log p(x)] = - \int_x p(x) \log p(x) dx$$

Intuition 1: how *random* is the random variable?

Intuition 2: how large is the log probability in expectation *under itself*

what do we expect this to do?

$$E_{z \sim q_i(z)}[\log p(x_i|z) + \log p(z)] + \mathcal{H}(q_i)$$



# A brief aside...

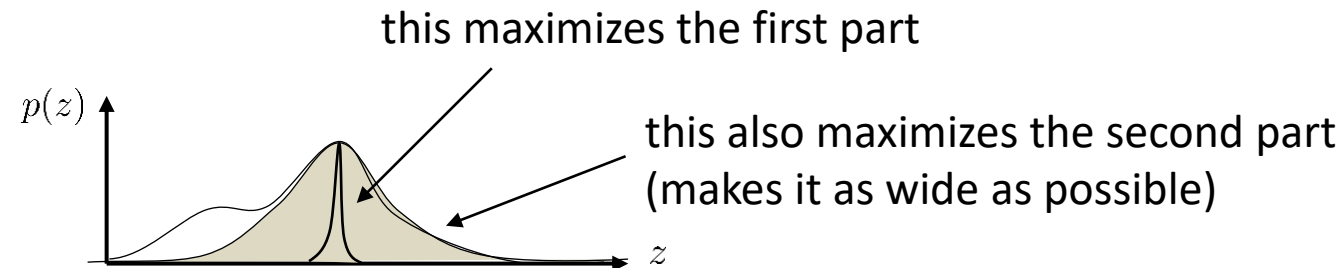
## KL-Divergence:

$$D_{\text{KL}}(q||p) = E_{x \sim q(x)} \left[ \log \frac{q(x)}{p(x)} \right] = E_{x \sim q(x)} [\log q(x)] - E_{x \sim q(x)} [\log p(x)] = -E_{x \sim q(x)} [\log p(x)] - \mathcal{H}(q)$$

Intuition 1: how *different* are two distributions?

Intuition 2: how small is the expected log probability of one distribution under another, minus entropy?

why entropy?



# The variational approximation

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

what makes a good  $q_i(z)$ ?

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$

approximate in what sense?

compare in terms of KL-divergence:  $D_{\text{KL}}(q_i(z) \| p(z|x))$

why?

$$\begin{aligned} D_{\text{KL}}(q_i(x_i) \| p(z|x_i)) &= E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)}{p(z|x_i)} \right] = E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)p(x_i)}{p(x_i, z)} \right] \\ &= -E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] + E_{z \sim q_i(z)} [\log q_i(z)] + E_{z \sim q_i(z)} [\log p(x_i)] \\ &= -E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)] - \mathcal{H}(q_i) + \log p(x_i) \\ &= -\mathcal{L}_i(p, q_i) + \log p(x_i) \end{aligned}$$

$$\log p(x_i) = D_{\text{KL}}(q_i(z) \| p(z|x_i)) + \mathcal{L}_i(p, q_i)$$

$$\log p(x_i) \geq \mathcal{L}_i(p, q_i)$$

# The variational approximation

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

$$\log p(x_i) = D_{\text{KL}}(q_i(z) \| p(z|x_i)) + \mathcal{L}_i(p, q_i)$$

$$\log p(x_i) \geq \mathcal{L}_i(p, q_i)$$

$$\begin{aligned} D_{\text{KL}}(q_i(z) \| p(z|x_i)) &= E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)}{p(z|x_i)} \right] = E_{z \sim q_i(z)} \left[ \log \frac{q_i(z)p(x_i)}{p(x_i, z)} \right] \\ &= \underbrace{-E_{z \sim q_i(z)} [\log p(x_i|z) + \log p(z)]}_{-\mathcal{L}_i(p, q_i)} - \mathcal{H}(q_i) + \log p(x_i) \end{aligned}$$

independent of  $q_i$ !

$\Rightarrow$  maximizing  $\mathcal{L}_i(p, q_i)$  w.r.t.  $q_i$  minimizes KL-divergence!



# How do we use this?

$$\log p(x_i) \geq \overbrace{E_{z \sim q_i(z)} [\log p_\theta(x_i|z) + \log p(z)]}^{\mathcal{L}_i(p, q_i)} + \mathcal{H}(q_i)$$

~~$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \log p_\theta(x_i)$$~~

$$\theta \leftarrow \arg \max_{\theta} \frac{1}{N} \sum_i \mathcal{L}_i(p, q_i)$$

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}_i(p, q_i)$$

update  $q_i$  to maximize  $\mathcal{L}_i(p, q_i)$

how?

let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

gradient ascent on  $\mu_i, \sigma_i$

# What's the problem?

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

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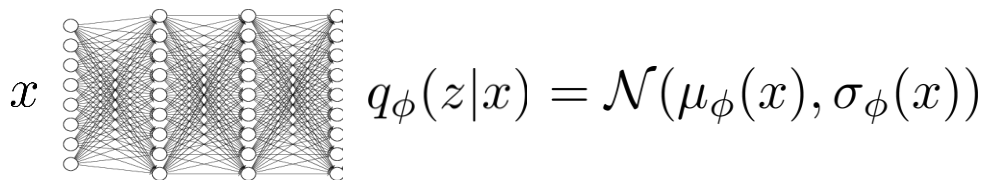
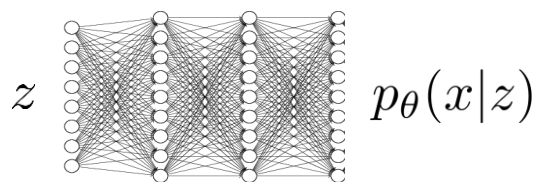
gradient ascent on  $\mu_i, \sigma_i$

How many parameters are there?

$$|\theta| + (|\mu_i| + |\sigma_i|) \times N$$

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$

what if we learn a *network*  $q_i(z) = q(z|x_i) \approx p(z|x_i)$ ?



# Amortized Variational Inference

# What's the problem?

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}_i(p, q_i)$ :

sample  $z \sim q_i(z)$

$$\nabla_{\theta} \mathcal{L}_i(p, q_i) \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$$

$$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}_i(p, q_i)$$

update  $q_i$  to maximize  $\mathcal{L}_i(p, q_i)$

let's say  $q_i(z) = \mathcal{N}(\mu_i, \sigma_i)$

use gradient  $\nabla_{\mu_i} \mathcal{L}_i(p, q_i)$  and  $\nabla_{\sigma_i} \mathcal{L}_i(p, q_i)$

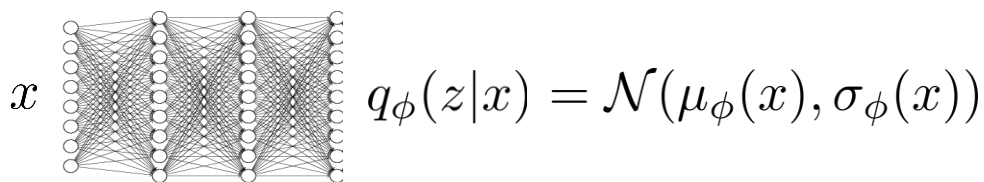
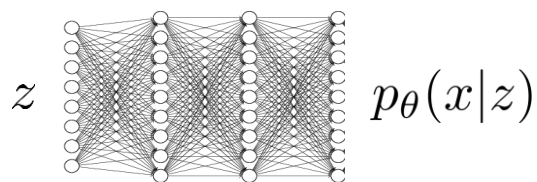
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How many parameters are there?

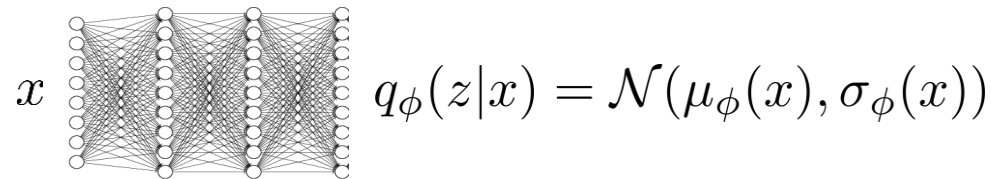
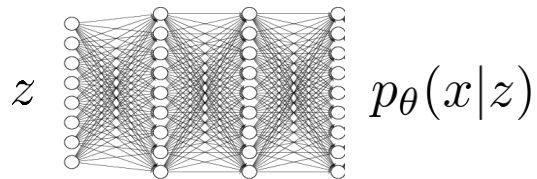
$$|\theta| + (|\mu_i| + |\sigma_i|) \times N$$

intuition:  $q_i(z)$  should approximate  $p(z|x_i)$

what if we learn a *network*  $q_i(z) = q(z|x_i) \approx p(z|x_i)$ ?



# Amortized variational inference



for each  $x_i$  (or mini-batch):

calculate  $\nabla_\theta \mathcal{L}(p_\theta(x_i|z), q_\phi(z|x_i))$ :

sample  $z \sim q_\phi(z|x_i)$

$\nabla_\theta \mathcal{L} \approx \nabla_\theta \log p_\theta(x_i|z)$

$\theta \leftarrow \theta + \alpha \nabla_\theta \mathcal{L}$

$\phi \leftarrow \phi + \alpha \nabla_\phi \mathcal{L}$

how do we calculate this?

$$\log p(x_i) \geq \overbrace{E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z) + \log p(z)]}^{\mathcal{L}(p_\theta(x_i|z), q_\phi(z|x_i))} + \mathcal{H}(q_\phi(z|x_i))$$

# Amortized variational inference

for each  $x_i$  (or mini-batch):

calculate  $\nabla_{\theta} \mathcal{L}(p_{\theta}(x_i|z), q_{\phi}(z|x_i))$ :

sample  $z \sim q_{\phi}(z|x_i)$

$\nabla_{\theta} \mathcal{L} \approx \nabla_{\theta} \log p_{\theta}(x_i|z)$

$\theta \leftarrow \theta + \alpha \nabla_{\theta} \mathcal{L}$

$\phi \leftarrow \phi + \alpha \nabla_{\phi} \mathcal{L}$

$$q_{\phi}(z|x) = \mathcal{N}(\mu_{\phi}(x), \sigma_{\phi}(x))$$

look up formula for  
entropy of a Gaussian



$$\mathcal{L}_i = \underbrace{E_{z \sim q_{\phi}(z|x_i)} [\log p_{\theta}(x_i|z) + \log p(z)]}_{J(\phi)} + \mathcal{H}(q_{\phi}(z|x_i))$$

$$J(\phi) = E_{z \sim q_{\phi}(z|x_i)} [r(x_i, z)]$$

can just use policy gradient!

What's wrong with this gradient?

$$\nabla J(\phi) \approx \frac{1}{M} \sum_j \nabla_{\phi} \log q_{\phi}(z_j|x_i) r(x_i, z_j)$$

# The reparameterization trick

Is there a better way?

$$\begin{aligned} J(\phi) &= E_{z \sim q_\phi(z|x_i)}[r(x_i, z)] \\ &= E_{\epsilon \sim \mathcal{N}(0,1)}[r(x_i, \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i))] \end{aligned}$$

estimating  $\nabla_\phi J(\phi)$ :

sample  $\epsilon_1, \dots, \epsilon_M$  from  $\mathcal{N}(0, 1)$  (a single sample works well!)

$$\nabla_\phi J(\phi) \approx \frac{1}{M} \sum_j \nabla_\phi r(x_i, \mu_\phi(x_i) + \epsilon_j \sigma_\phi(x_i))$$

most autodiff software (e.g., TensorFlow)  
will compute this for you!

$$q_\phi(z|x) = \mathcal{N}(\mu_\phi(x), \sigma_\phi(x))$$

$$z = \mu_\phi(x) + \epsilon \sigma_\phi(x)$$

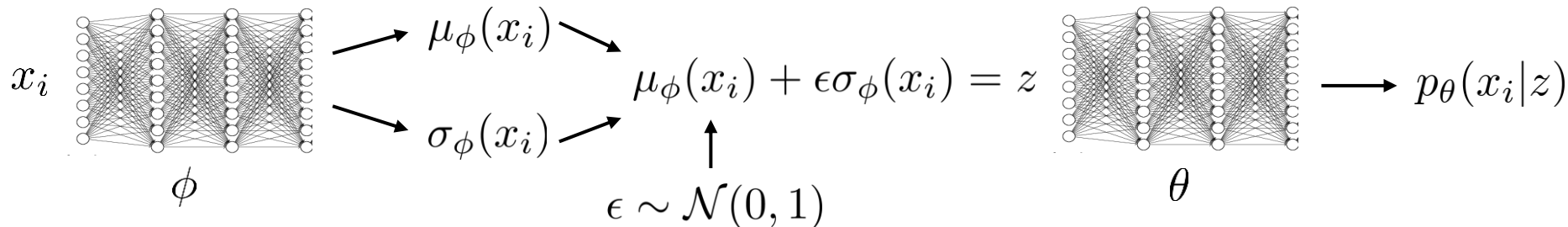


$$\epsilon \sim \mathcal{N}(0, 1)$$

independent of  $\phi$ !

# Another way to look at it...

$$\begin{aligned}\mathcal{L}_i &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z) + \log p(z)] + \mathcal{H}(q_\phi(z|x_i)) \\ &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z)] + \underbrace{E_{z \sim q_\phi(z|x_i)} [\log p(z)] + \mathcal{H}(q_\phi(z|x_i))}_{-D_{\text{KL}}(q_\phi(z|x_i) \| p(z))} \leftarrow \text{this often has a convenient analytical form (e.g., KL-divergence for Gaussians)} \\ &= E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z)] - D_{\text{KL}}(q_\phi(z|x_i) \| p(z)) \\ &= E_{\epsilon \sim \mathcal{N}(0,1)} [\log p_\theta(x_i | \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i))] - D_{\text{KL}}(q_\phi(z|x_i) \| p(z)) \\ &\approx \log p_\theta(x_i | \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i)) - D_{\text{KL}}(q_\phi(z|x_i) \| p(z))\end{aligned}$$





# Reparameterization trick vs. policy gradient

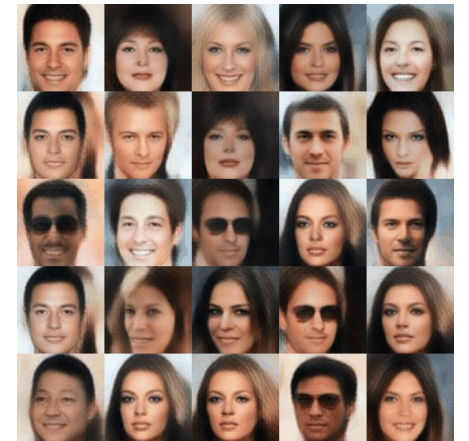
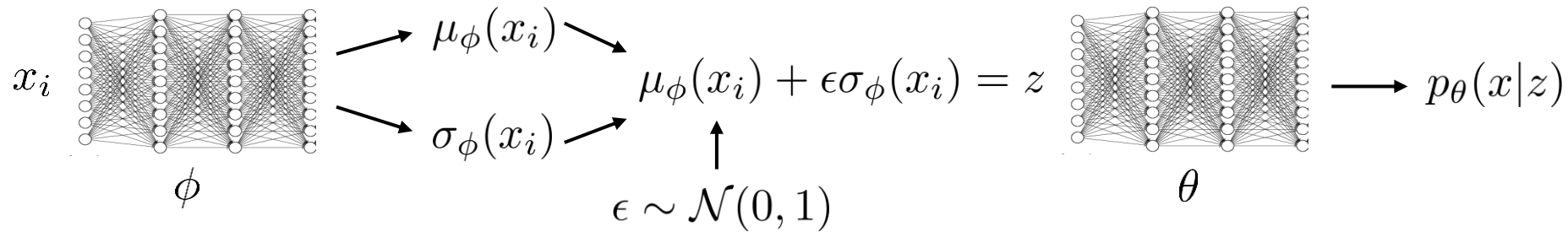
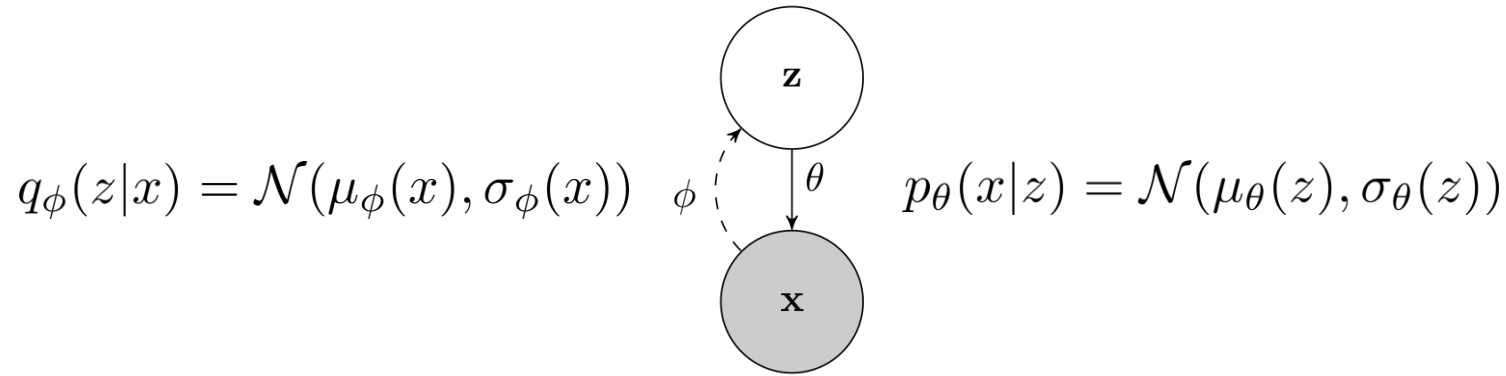
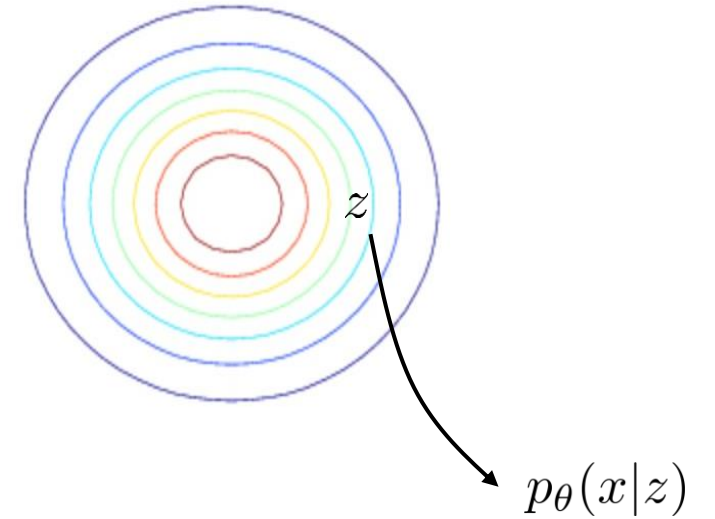
- Policy gradient
  - Can handle both discrete and continuous latent variables
  - High variance, requires multiple samples & small learning rates
- Reparameterization trick
  - Only continuous latent variables
  - Very simple to implement
  - Low variance

$$J(\phi) \approx \frac{1}{M} \sum_j \nabla_{\phi} \log q_{\phi}(z_j|x_i) r(x_i, z_j)$$

$$\nabla_{\phi} J(\phi) \approx \frac{1}{M} \sum_j \nabla_{\phi} r(x_i, \mu_{\phi}(x_i) + \epsilon_j \sigma_{\phi}(x_i))$$

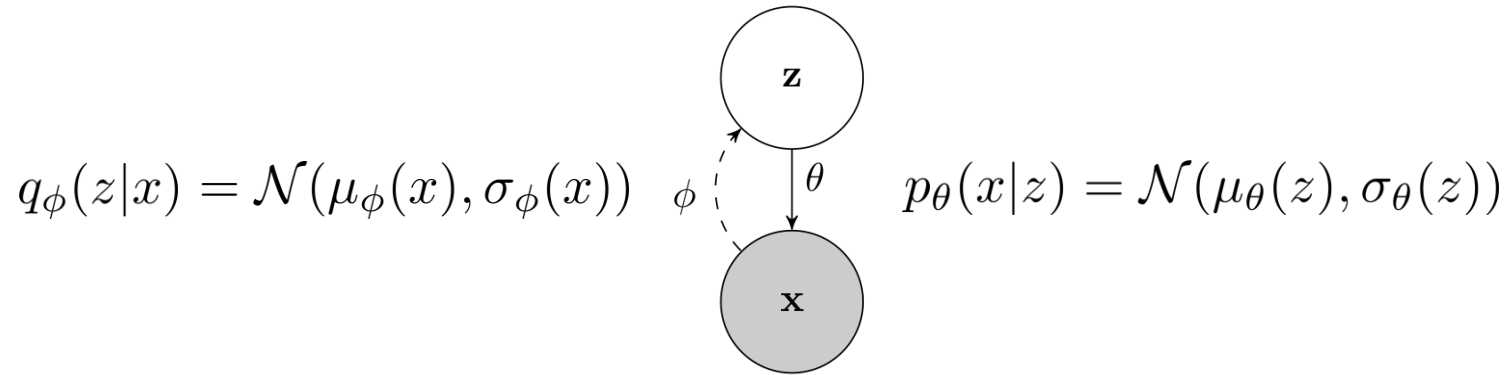
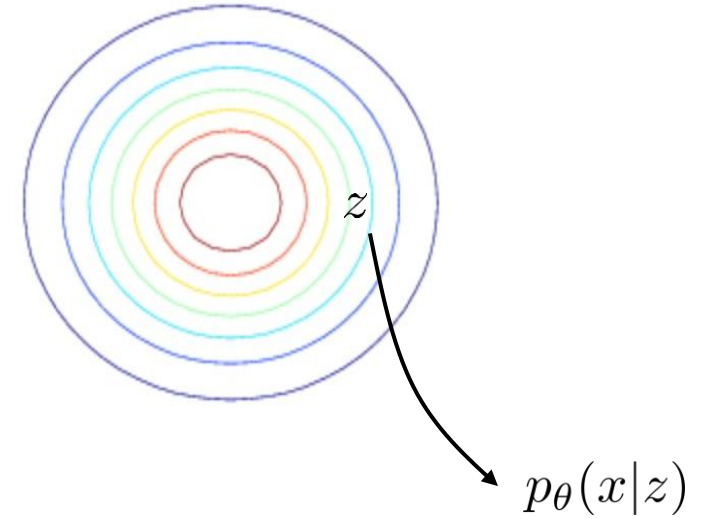
# Example Models

# The *variational* autoencoder



$$\max_{\theta, \phi} \frac{1}{N} \sum_i \log p_\theta(x_i | \mu_\phi(x_i) + \epsilon \sigma_\phi(x_i)) - D_{\text{KL}}(q_\phi(z|x_i) \| p(z))$$

# Using the variational autoencoder



$$p(x) = \int p(x|z)p(z)dz$$

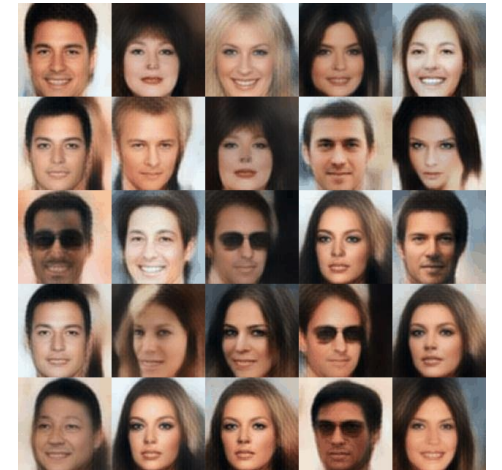
why does this work?

sampling:

$$z \sim p(z)$$

$$x \sim p(x|z)$$

$$\mathcal{L}_i = E_{z \sim q_\phi(z|x_i)} [\log p_\theta(x_i|z)] - D_{\text{KL}}(q_\phi(z|x_i) || p(z))$$



# Conditional models

$$\mathcal{L}_i = E_{z \sim q_\phi(z|x_i, y_i)} [\log p_\theta(y_i|x_i, z) + \log p(z|x_i)] + \mathcal{H}(q_\phi(z|x_i, y_i))$$

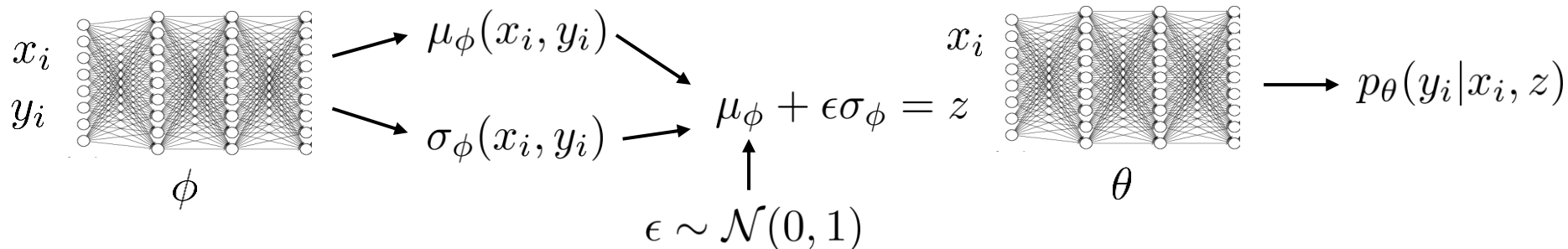
just like before, only now generating  $y_i$   
and *everything* is conditioned on  $x_i$

at test time:

$$z \sim p(z|x_i)$$

$$y \sim p(y|x_i, z)$$

can *optionally* depend on  $x$



$$z \sim \mathcal{N}(0, \mathbf{I})$$

$$p(z)$$

$$p(y|x, z)$$



# Examples

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## Embed to Control: A Locally Linear Latent Dynamics Model for Control from Raw Images

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**Manuel Watter\***

**Jost Tobias Springenberg\***

**Martin Riedmiller**

**Joschka Boedecker**

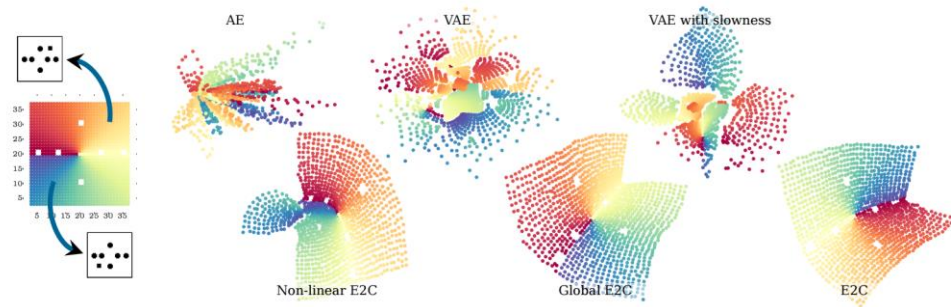
Google DeepMind

University of Freiburg, Germany

London, UK

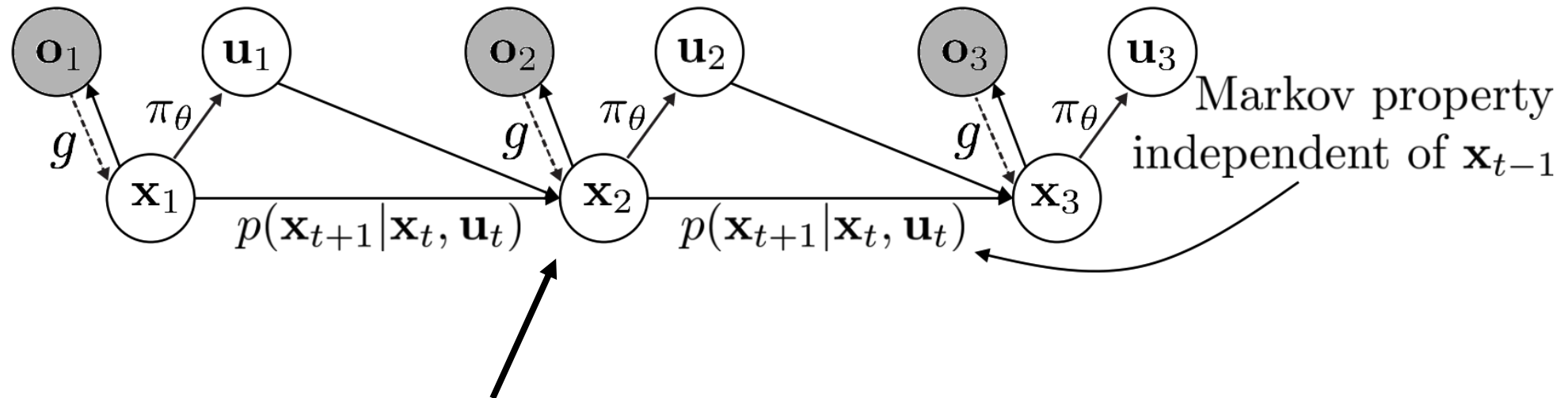
{watterm, springj, jboedeck}@cs.uni-freiburg.de

riedmiller@google.com



Swing-up with the E2C algorithm

1. collect data
2. learn embedding of image & dynamics model (**jointly**)
3. run iLQG to learn to reach image of goal



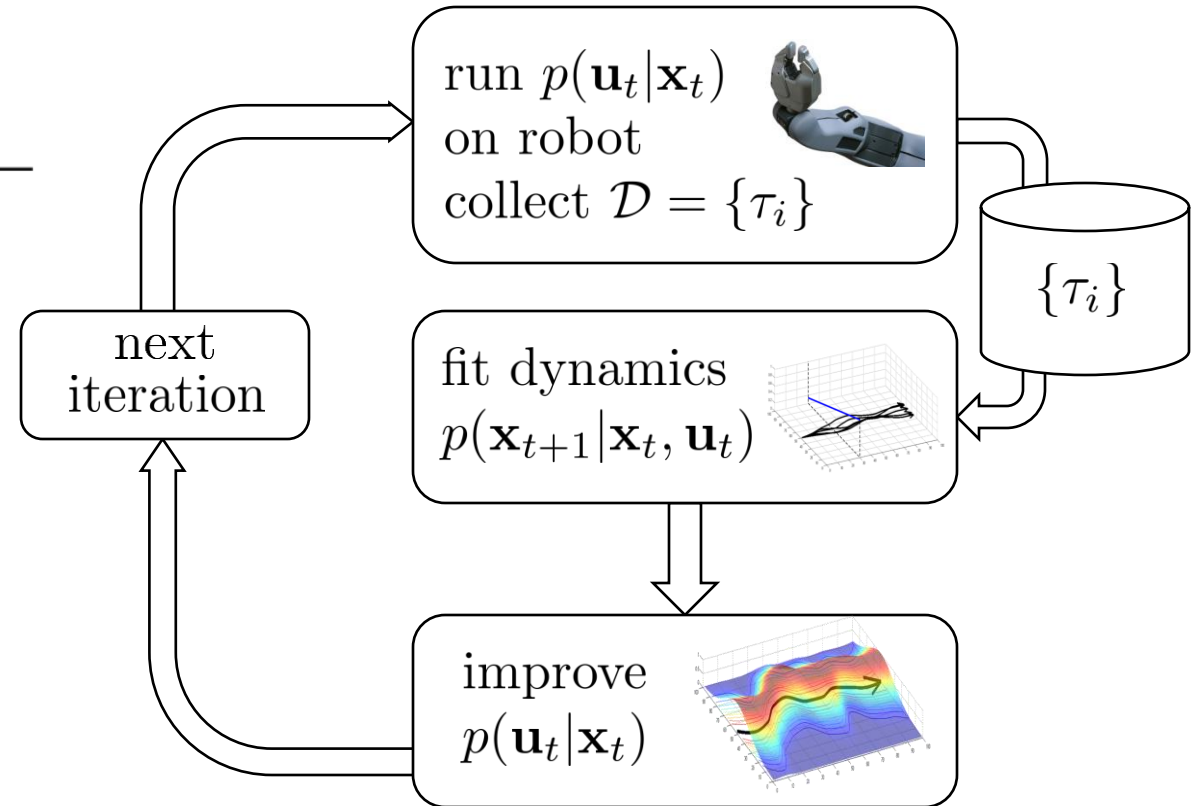
a type of variational autoencoder with temporally decomposed latent state!

# Local models with images

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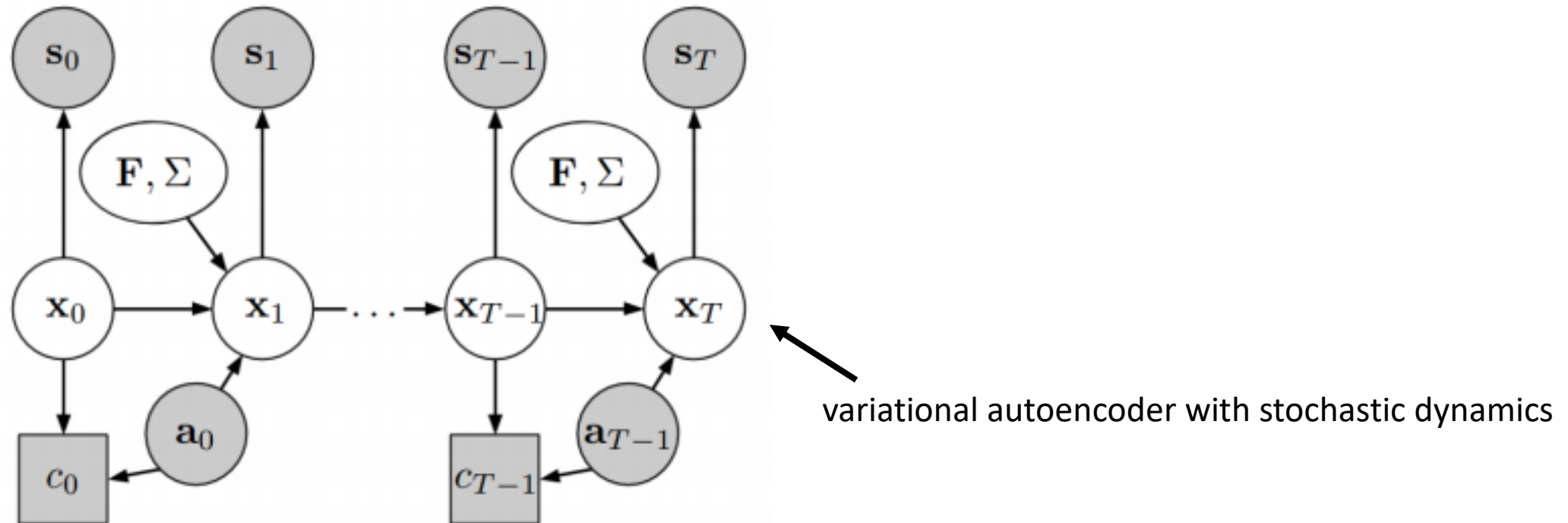
## SOLAR: Deep Structured Latent Representations for Model-Based Reinforcement Learning

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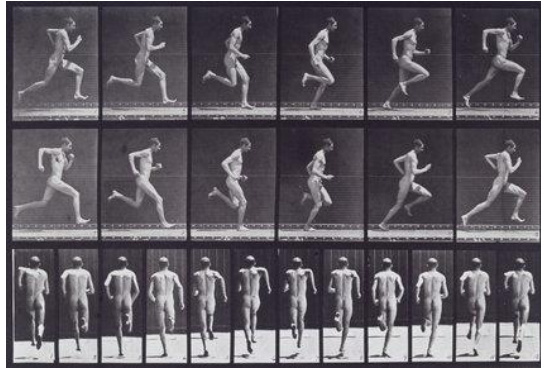


# Local models with images



# We'll see more of this for...

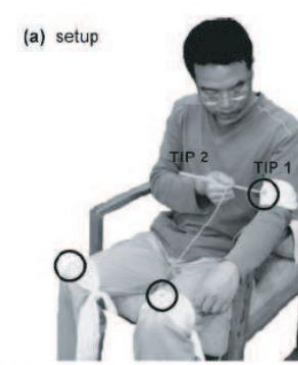
Using RL/control + variational inference to model human behavior



Muybridge (c. 1870)



Mombaur et al. '09



Li & Todorov '06



Ziebart '08

Using generative models and variational inference for exploration

