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HW5

1. (a)
   - The macro-explanatory variable is w_j, as it is indexed by j, meaning it varies between groups but is constant within a group.
   - The micro-explanatory variable is $x_{i,j}$, as it is indexed by both i and j, meaning it varies across individuals within a group.
   - The fixed effect parameters are $\beta_0, \beta_1, \beta_2$. β_0 is the fixed intercept, β_1 is the fixed slope for the micro-level variable $x_{i,j}$, and β_2 is the fixed slope for the macro-level variable w_j.
   - The random effect parameters are $\alpha_{0,j}$, and $\alpha_{1,j}$, where $\alpha_{0,j}$ is the random intercept, and $\alpha_{1,j}$ is the random slope.
   - σ^2 is the variance of the residual errors $\epsilon_{i,j}$, ψ_0^2 is the variance of the random intercept $\alpha_{0,j}$, ψ_1^2 is the variance of the random slope $\alpha_{1,j}$.
   - The covariance between $\alpha_{0,j}$ and $\alpha_{1,j}$ is ψ_{01}.

   (b) The variable w_j is a group-level random variable that does not vary within a group. Adding $a_{2,j}w_j$ is unnecessary as all effects should've been already covered by $\beta_j w_j$.

   (c) Notice that under M_0 assumption, since $\psi_1^2 = 0$, the random slope $a_{1,j} = 0$ since $\mathbb{E}[a_{1,j}] = 0$. Then, we have the following null and alternative hypotheses:

   - Null hypothesis: The random effect slope, $a_{1,j}$ is 0.
   - Alternative hypothesis: The random effect slope, $a_{1,j}$, is not zero.

   From the hypotheses stated above, M_0 has 1 random effect coefficient, while M_1 has 2 random effect coefficients. Therefore,

$$\lambda(y) = \begin{cases} X_1 \text{ with probability } 1/2 \\ X_2 \text{ with probability } 1/2 \end{cases}$$

   where $X_1 \sim \chi_1^2$ and $X_2 \sim \chi_2^2$. Hence, the distribution of the LRT statistic under the null hypothesis is

$$\lambda | M_0 \sim \frac{1}{2}\left(\chi_1^2 + \chi_2^2\right).$$

   To obtain a p-value, we will need the LRT statistic, which is

$$\lambda(y) = 2 \cdot (\log_{M_1}(y) - \log_{M_2}(y)).$$

   Then, the p-value is

$$\text{p-value} = \frac{1}{2}\Pr(\chi_1^2 \geq \lambda) + \frac{1}{2}\Pr(\chi_2^2 \geq \lambda).$$

   (d)