Introduction

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Duke STA 610

Multilevel data

Multilevel data: Data for which there are

- multiple nested levels of sampling, and/or
- multiple nested sources of variability.

Such data are also often called *hierarchical data* or *clustered data*.

Examples:

Educational testing: students nested within classes;

Small area estimation: households nested within counties;

Agricultural experiments: subplots nested within whole plots;

Clinical trials: measurements nested within patients, patients within hospitals.

Terminology

observational unit: an object or condition for which data are measured.

macro-level unit: a unit within which other units are nested.

micro-level unit: a unit nested within another unit.

Synonyms:

- macro-level unit, top-level unit, clusters, groups;
- micro-level unit, bottom-level unit, units.

If there are only two levels, we will say *units* are nested within *groups*.

Notation: $y_{i,j}$ = measurement of ith unit in jth group.

Populations:

- The population: all possible units from all possible groups;
- A subpopulation: all possible units from a single group group.

Types of multilevel inference

Subpopulation inferences: Group-specific features are of primary interest.

- What is the mean within each group, based on a sample from each group?
- What is the treatment effect for each group?
- Do the groups differ? If so, how do they differ?

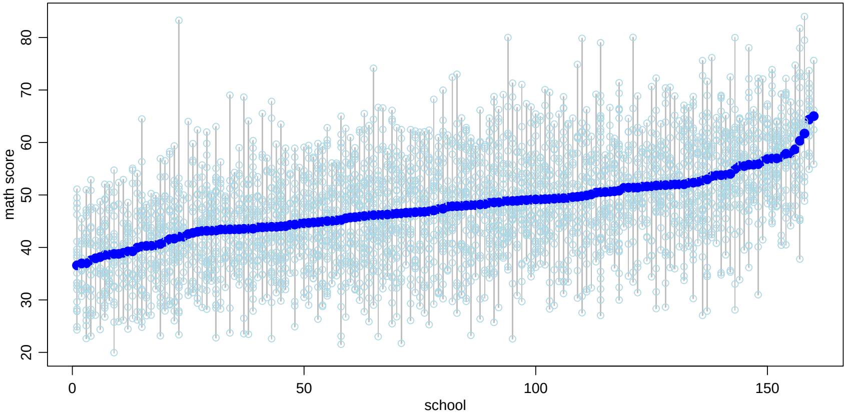
Population inferences: Across-group averages are of primary interest.

- What is the population mean, based on cluster sample?
- What is the population treatment effect?

Cross-level inferences: Both types of features are important.

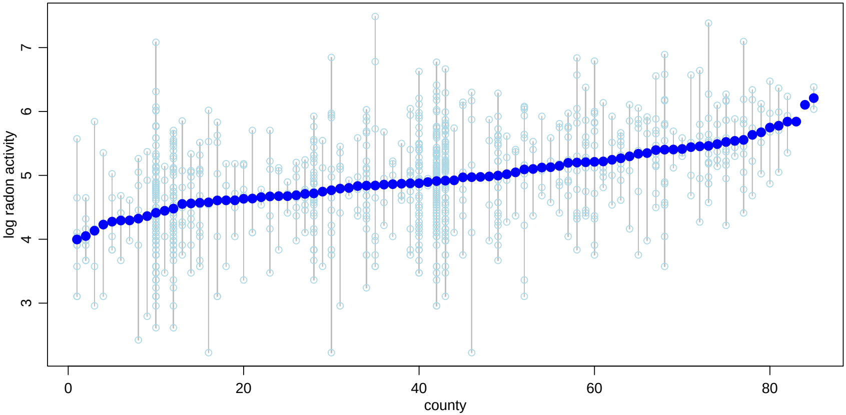
- What is the average treatment effect, adjusting for group differences?

Example: Educational testing data



Exercise: Identify the populations and subpopulations.

Example: Environmental monitoring data



Exercise: Identify the populations and subpopulations.

Group-specific inferences

Targets of inference: Subpopulation means $\theta_1, \ldots, \theta_p$.

Data: Subpopulation samples $\{y_{1,1}, \ldots, y_{1,n_1}\}, \ldots, \{y_{1,p}, \ldots, y_{1,n_p}\}$.

Statistical methods:

- Variance tests and estimation: What is $\text{Var}[\theta_1, \ldots, \theta_p]$? Is it zero?
- Estimates of θ_j: $\hat{\theta}_j = \bar{y}_{\cdot j}$ or $\hat{\theta}_j = w\bar{y}_{\cdot j} + (1-w)\bar{y}_{\cdot\cdot}$;
- Confidence intervals: $\Pr(\theta_j \in C(\mathbf{y})|\theta_j) = 1-\alpha$, or $\Pr(\theta^* \in C(\mathbf{y})) = 1-\alpha$;

Cluster sampling

Survey design: Consider the costs of obtaining soil samples from

- 100 randomly sampled locations in a city, versus
- 10 randomly sampled locations from 10 randomly sampled neighborhoods.

Cluster sampling:
The second sampling scheme is called *cluster sampling* or *two-stage sampling*.

Cluster sampling

- is often cheaper per sampled unit;
- often gives less reliable estimates of population means.

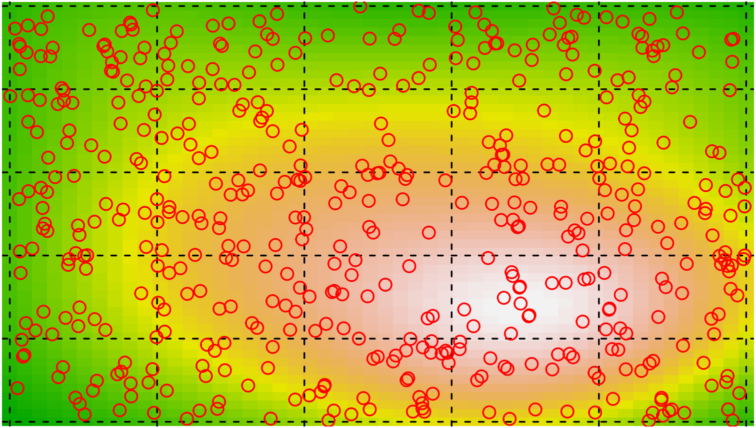
Estimation of a population mean

Task: Estimate the population mean μ from sample data.

Questions:

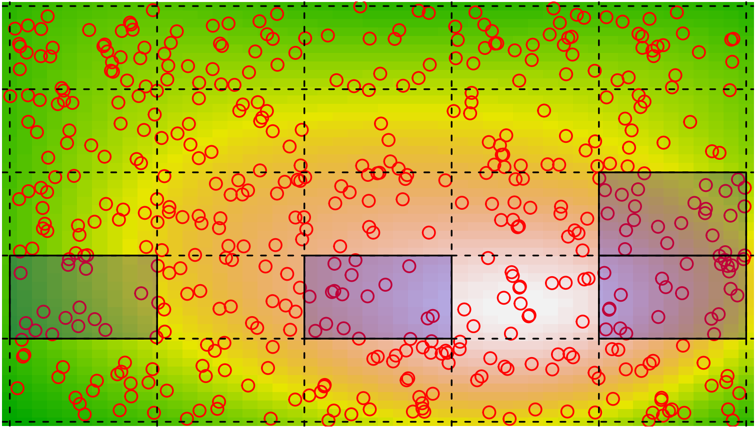
- How do cluster sampling and SRS compare?
- How do you infer μ from cluster sample data?

Two-stage sampling



μ=2.0494009

Two-stage sampling



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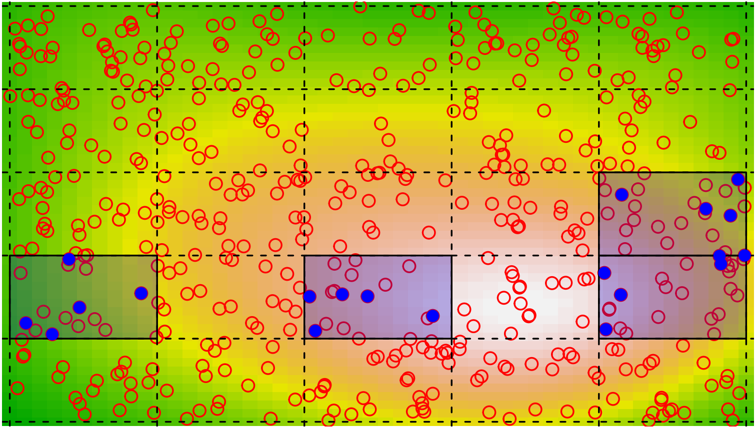
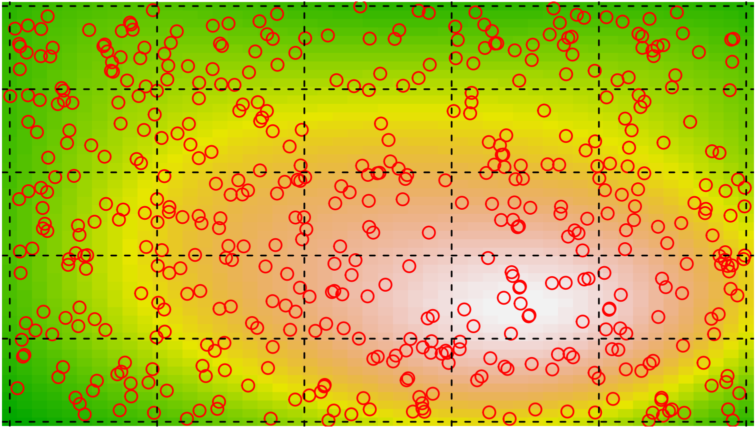
Two-stage sampling



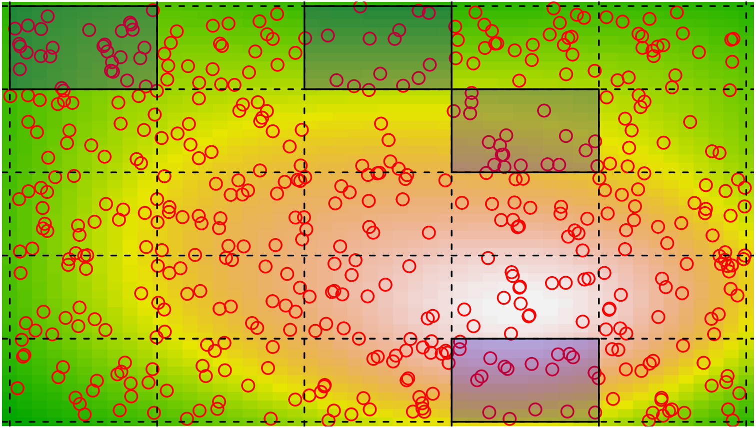
μ=2.0494009 , $\bar{y}$=2.3547727

Two-stage sampling



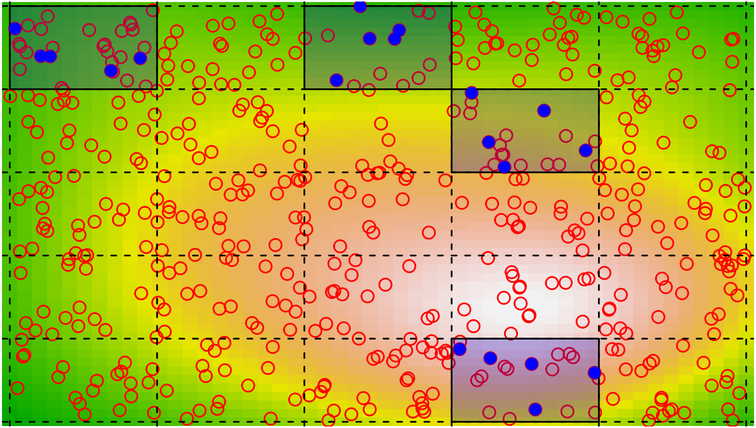
μ=2.0494009

Two-stage sampling



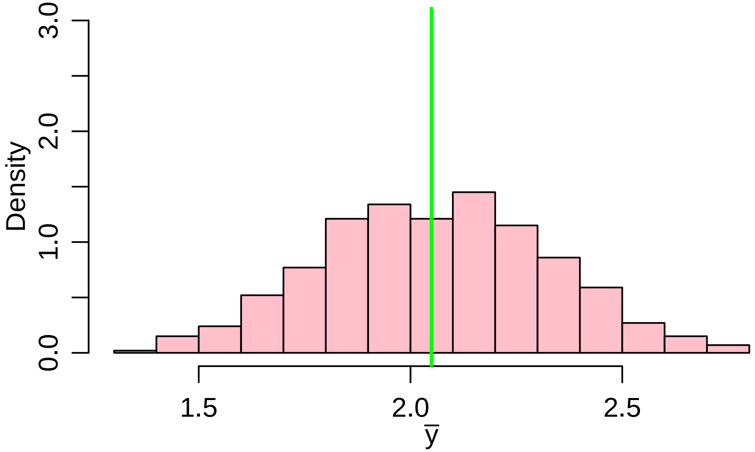
μ=2.0494009

Two-stage sampling

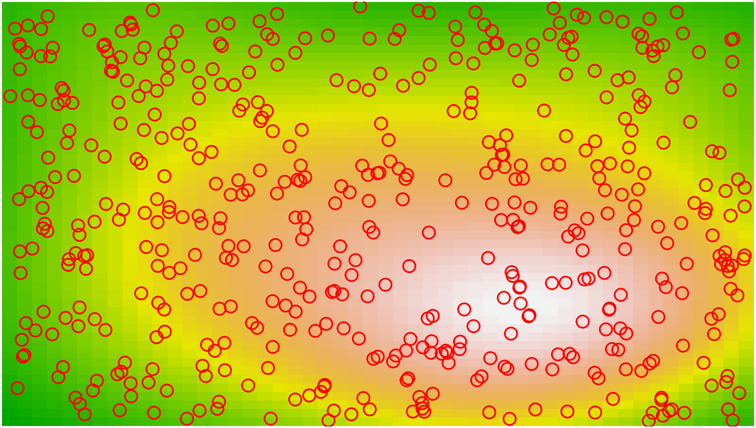


μ=2.0494009 , $\bar{y}$=1.896463

Variability of sample mean

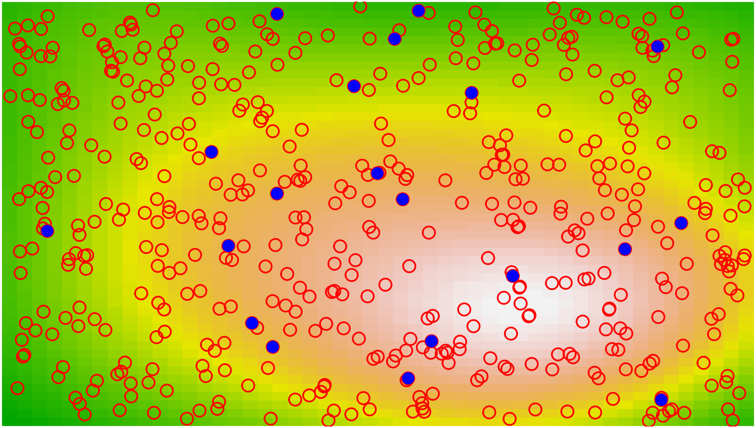


Comparison to SRS



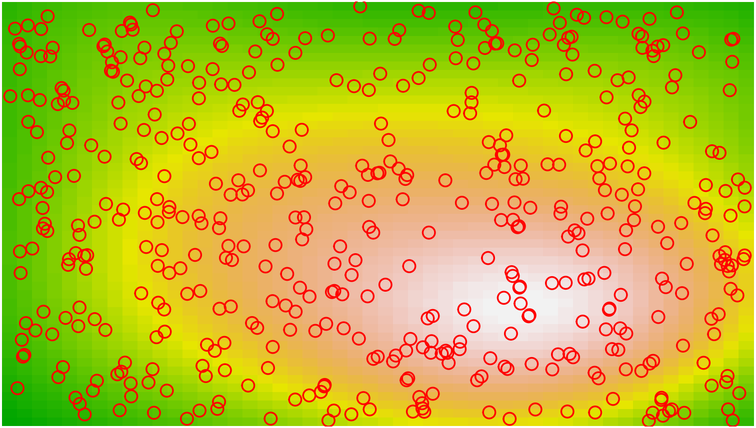
μ=2.0494009

Comparison to SRS



μ=2.0494009 , $\bar{y}$=2.1696295

Comparison to SRS



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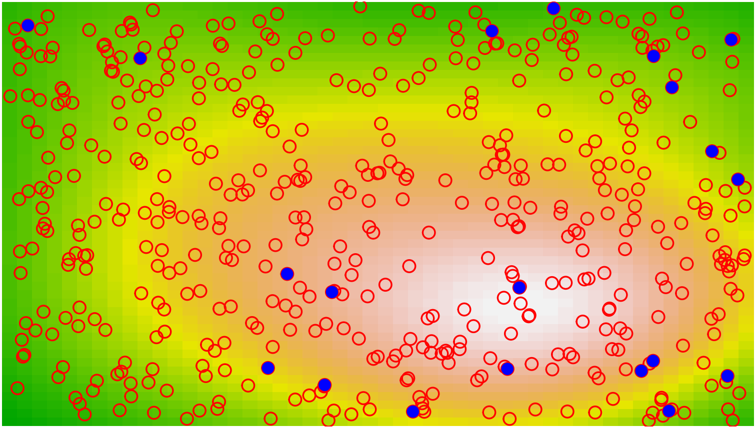
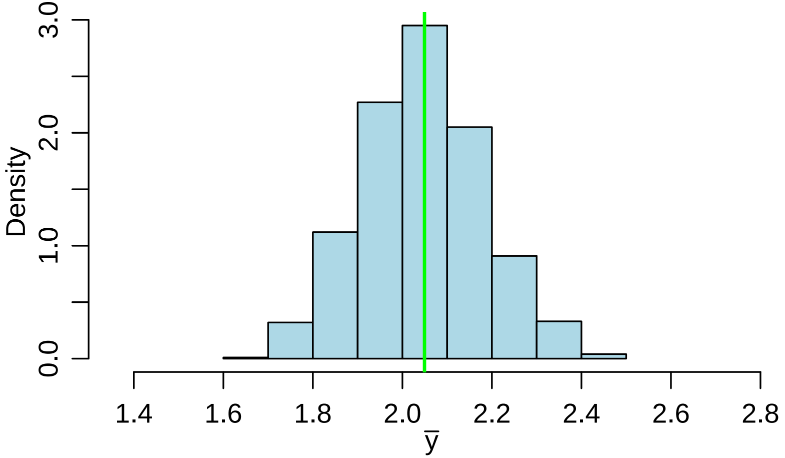
Comparison to SRS

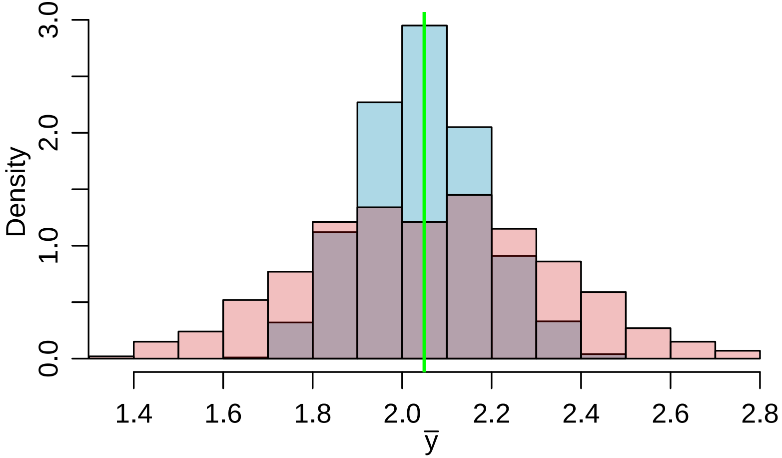


μ=2.0494009 , $\bar{y}$=1.9804926

Variability of sample mean



Comparison of sampling variability



Heterogeneity, homogeneity and dependence

As we will show mathematically,

$$\begin{aligned} \text{across-group heterogeneity} &\Leftrightarrow \text{within-group homogeneity} \\ &\Leftrightarrow \text{within-group correlation or dependence} \end{aligned}$$

Across-group heterogeneity increases the variance of the sample mean, and so

$$\text{Var}[\bar{y}_{tss}] \geq \text{Var}[\bar{y}_{srs}]$$

if the total samples sizes are the same.

Ignoring across-group heterogeneity

Task: Construct a 95% CI for the population mean.

t-interval for SRS:
If $y_1, \ldots, y_n$ is an iid sample with $\mathrm{E}[y_i] = \mu$ and $\mathrm{Var}[y_i] = \sigma^2$,

$$\mathrm{E}[\bar{y}] = \mu \ , \ \mathrm{Var}[\bar{y}] = \sigma^2/n.$$

By the central limit theorem,

$$\bar{y} \dot{\sim} N(\mu, \sigma^2/n) \ , \ \frac{\bar{y} - \mu}{\sigma/\sqrt{n}} \dot{\sim} N(0,1).$$

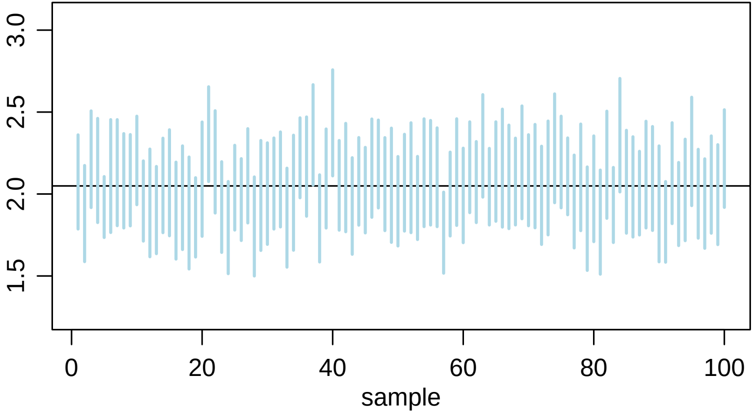
As σ^2 is generally unknown, we use

$$\frac{\bar{y} - \mu}{s/\sqrt{n}} \dot{\sim} t_{n-1}, \ \text{, where } s^2 = \frac{1}{n-1} \sum (y_i - \bar{y})^2.$$

From this, we have

$$\bar{y} \pm t_{n-1,.975} \times s/\sqrt{n} \ \text{ is a 95\% CI for } \mu.$$

Ignoring across-group heterogeneity



Ignoring across-group heterogeneity

$$\bar{y} \pm t_{n-1,.975} \times s/\sqrt{n}$$

What if we apply the formula to data from a cluster sample?

If $y_1, \ldots, y_n$ are from a SRS, then

$$\text{Var}[\bar{y}] = \sigma^2/n = \text{E}[s^2/n].$$

$s/\sqrt{n}$ provides a good estimate of the sd of $\bar{y}$.

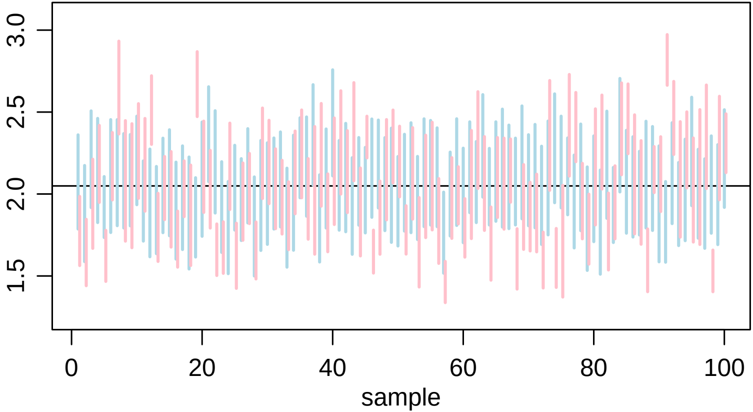
If $y_1, \ldots, y_n$ are from a cluster sample, then generally

$$\text{Var}[\bar{y}] > \sigma^2/n \approx \text{E}[s^2/n].$$

$s/\sqrt{n}$ is generally an underestimate of the sd of $\bar{y}$.

How will the resulting confidence interval behave if $\text{sd}(\bar{y}) > s/\sqrt{n}$?

Ignoring across-group heterogeneity



Ignoring across-group heterogeneity

Summary:

- Across-group heterogeneity = within-group similarity.
- Within-group similarity leads to positively correlated cluster sample data.
- The variance of the sample mean from (positively) correlated data is higher than that of the mean of uncorrelated data.
- Statistical inference ignoring such correlation will be inaccurate.

Remedy: We will develop techniques to

- evaluate within- and across-group heterogeneity;
- provide accurate statistical inference based on cluster samples.

Estimation of a treatment effect

Suppose

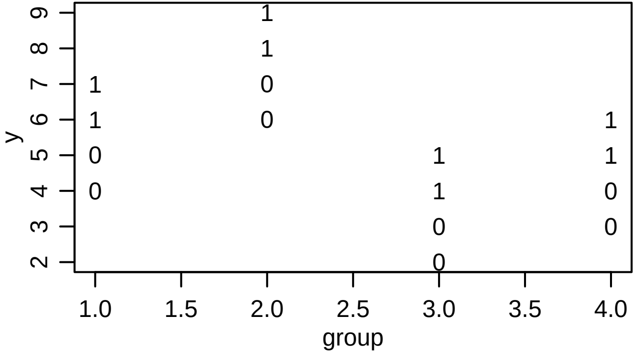
- $x \in \{0, 1\}$
- $\mu_1 = \mathrm{E}[y|x = 1]$
- $\mu_0 = \mathrm{E}[y|x = 0]$

Task: Estimate the difference $\delta = \mu_1 - \mu_0$ based on cluster sample data.

Data: For each group j, we have $(y_{1,j}, x_{1,j}), \ldots, (y_{n,j}, x_{n,j})$.

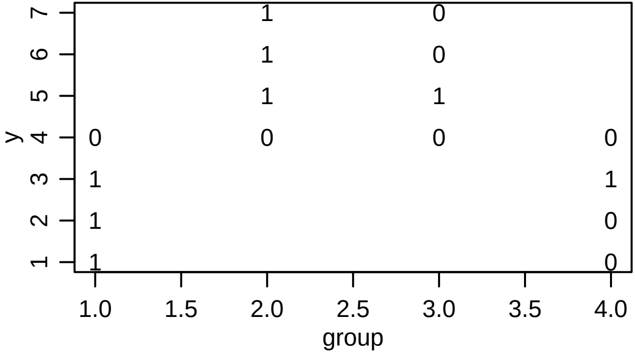
Question: What could go wrong by ignoring the multilevel nature of the data?

Overconservative analysis



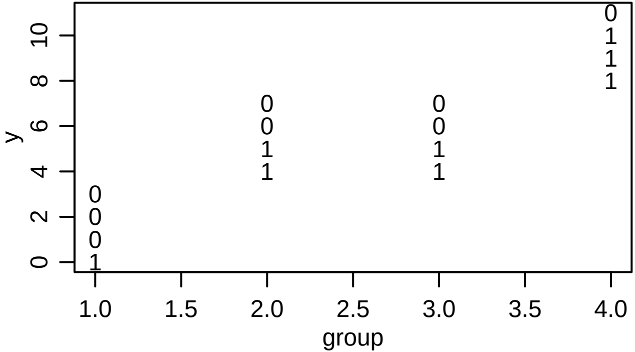
- Overlap across groups, no overlap within groups.
- Across-group variation is *large* compared to the treatment effect.
- Ignoring group differences can lead to *overconservative analysis*.

Underconservative analysis



- The population mean difference is zero.
- The sample mean difference based on pairs of two groups is not zero.
- Ignoring group differences can lead to *underconservative analysis.*

Effect reversal



- $\mu_1 - \mu_0 > 0$ in population, $\mu_{1,j} - \mu_{0,j} < 0$ in every group.
- Within-group effects may be different from population effects.
- This is sometimes called *Simpson's paradox*.

Consequences of across-group heterogeneity

Summary:

- Across-group heterogeneity can lead to *over or under* conservative analysis.
- Population-level effects may be different from group-level effects.
- Data analysis ignoring groups can be inaccurate in *unpredictable* ways.

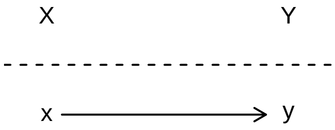
Remedy: We will develop techniques to

- differentiate between macro and micro level effects;
- appropriately control for within and between-group heterogeneity.

Macro and micro effects

X, x are macro and micro level explanatory variables

Y, y are macro and micro level outcome variables



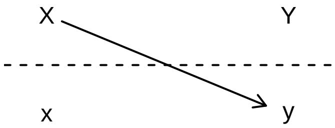
What are the effects of SES (x) on political opinion (y)?

(a *micro-micro effect*)

Macro, micro and cross-level effects

X, x are macro and micro level explanatory variables

Y, y are macro and micro level outcome variables



What are the effects of State GDP (X) on political opinion (y) ?

(a *macro-micro effect*)

Macro, micro and cross-level effects

X, x are macro and micro level explanatory variables

Y, y are macro and micro level outcome variables

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \hdashline x & & y \end{array}$$

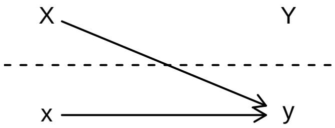
What are the effects of State GDP (X) on statewide political opinion (Y)?

(a *macro-macro effect*)

Macro, micro and cross-level effects

X, x are macro and micro level explanatory variables

Y, y are macro and micro level outcome variables



What are the effects of State GDP (X) and SES (x) on political opinion (y)?

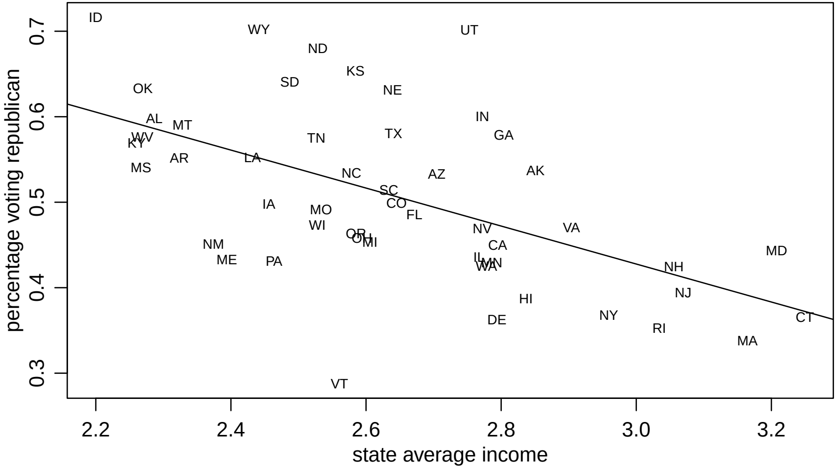
(*multilevel effects*)

Example: Income and voting patterns

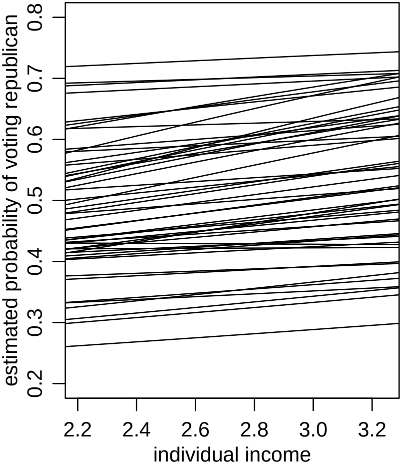
Exit poll data from 2004 presidential election

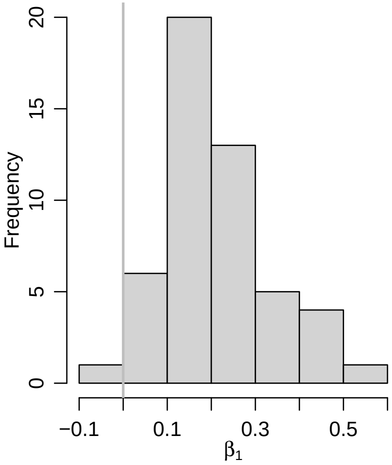
- $j \in \{1, \ldots, 50\}$ indexes the states,
- $y_{i,j}$ is the voting variable for person i in state j,
- $x_{i,j}$ is a measure of income for person (i, j).

Macro effects



Micro effects





Joint estimation of effects

In general we may be interested in understanding all of the following:

- macro level effects,
- micro level effects,
- macro effects on micro variables,
- heterogeneity of micro effects across groups.

Inference for these items can be made with LME and GLME models:

$$\begin{aligned} y_{i,j} &\sim \quad a_j \quad + \quad b_j x_{i,j} \quad + \epsilon_{i,j} \\ &= (\alpha_0 + \alpha_1 w_j + z_j) + (\beta_0 + \beta_1 w_j + e_j) x_{i,j} + \epsilon_{i,j}. \end{aligned}$$