ANCOVA

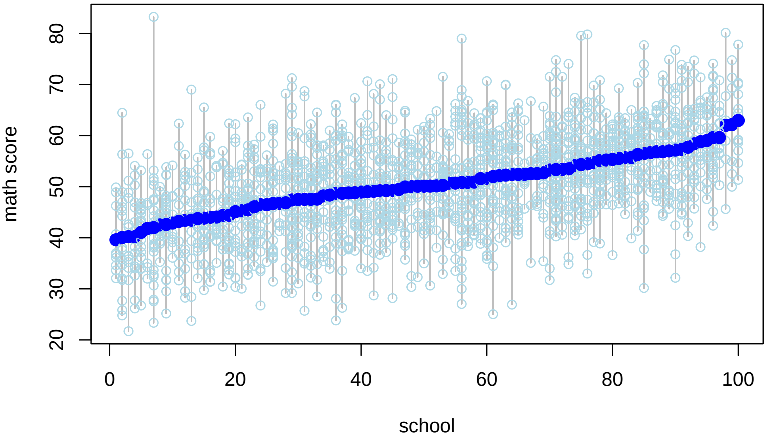
Peter Hoff
Duke STA 610

Motivating example

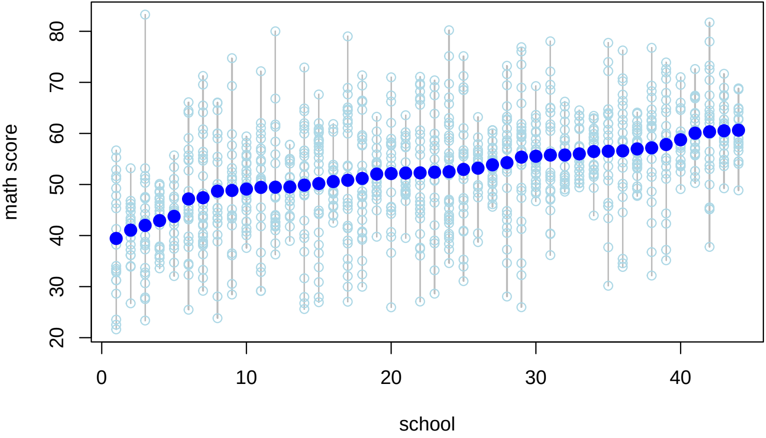
ANCOVA

NELS analysis

NELS data



Heteroscedasticity



Heteroscedasticity

Levene's test: If σ_j^2 is large, then $|y_{i,j} - \bar{y}_j| = |\hat{\epsilon}_{i,j}|$ should be large.

- Let $z_{i,j} = |\hat{\epsilon}_{i,j}|$
- Use the ANOVA F-test for across-group differences *in the $z_{i,j}$'s*

```
fit.nels<-lm(y.nels~as.factor(g.nels))
z.nels<-abs( fit.nels$res )
anova(lm(z.nels~as.factor(g.nels)) )
```

```
## Analysis of Variance Table
##
## Response: z.nels
##                       Df Sum Sq Mean Sq F value    Pr(>F)
## as.factor(g.nels)    683  27078  39.645  1.6092 < 2.2e-16 ***
## Residuals          12290 302776  24.636
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

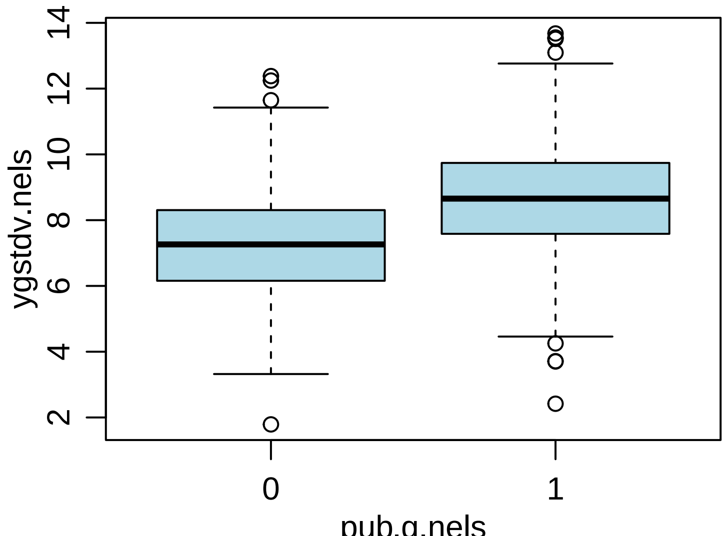
Sources of variation

```
nels_mathdat[1:5,]

##   school enroll flp public urbanicity hwh   ses mscore
## 1   1011      5   3      1      urban   2 -0.23  52.11
## 2   1011      5   3      1      urban   0  0.69  57.65
## 3   1011      5   3      1      urban   4 -0.68  66.44
## 4   1011      5   3      1      urban   5 -0.89  44.68
## 5   1011      5   3      1      urban   3 -1.28  40.57
```

What kind of schools might have higher variation?

What kind of schools have the highest variance?

```
ygstdv.nels<-c(tapply(y.nels,g.nels,sd))
boxplot(ygstdv.nels~pub.g.nels,col="lightblue")
```



Within-group variance models

Homoscedastic model: $y_{i,j} \sim N(\theta_j, \sigma^2)$.

- Simple to implement;
- The estimate of σ^2 will be precise if assumption is correct;
- The assumption could be wrong!

Heteroscedastic hierarchical normal model:

- Use $\hat{\sigma}_j^2 = \sum(y_{i,j} - \bar{y}_j)^2/(n_j - 1)$ if n_j's are large.
- Alternatively, use a hierarchical model for the variances.
- More appropriate inferences if variances are truly different.

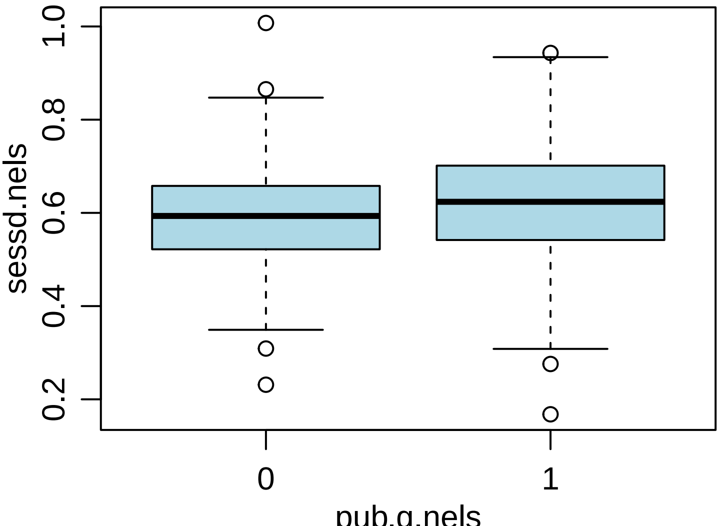
But this doesn't explain *why* variances are different.

Variance due to observable factors:

- Outcome could be related to unit-level characteristics $x_{i,j}$;
- Within-group variance can be partitioned:
  - variance explainable by observable unit-level characteristics;
  - unexplained variation.

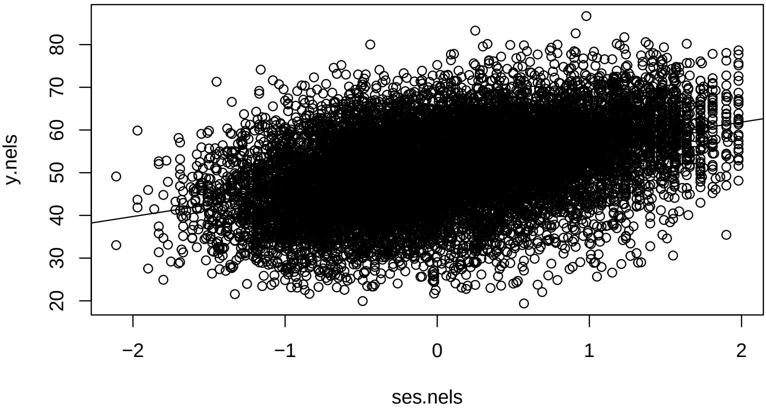
Heterogeneity attributable to observed covariates

```
sessd.nels<-tapply(ses.nels,g.nels,sd)
boxplot(sessd.nels~pub.g.nels,col="lightblue")
```

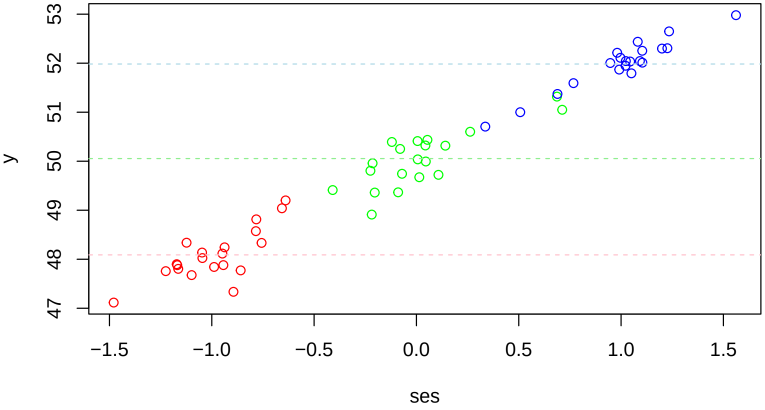


Marginal relationship

```
plot(y.nels~ses.nels)
abline(lm(y.nels~ses.nels))
```

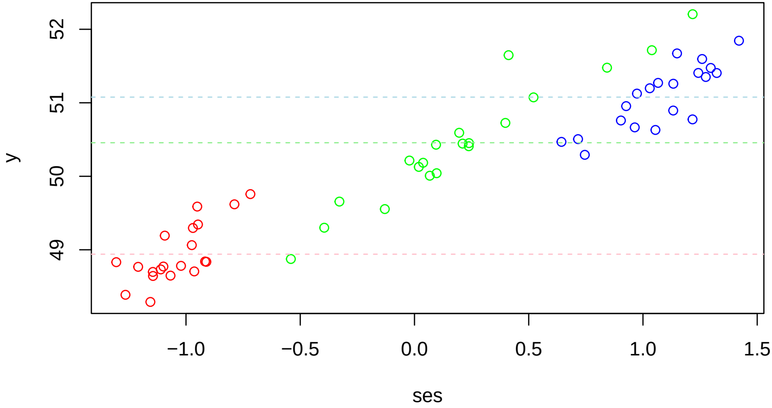


Possible explanations



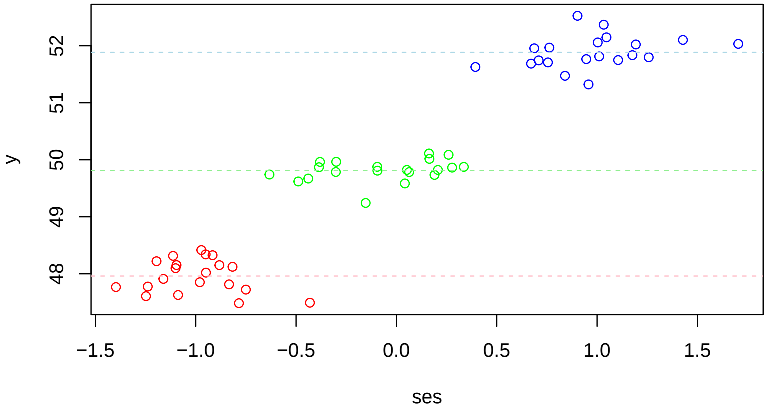
Variation across schools attributable to student-level variation in SES

Possible explanations



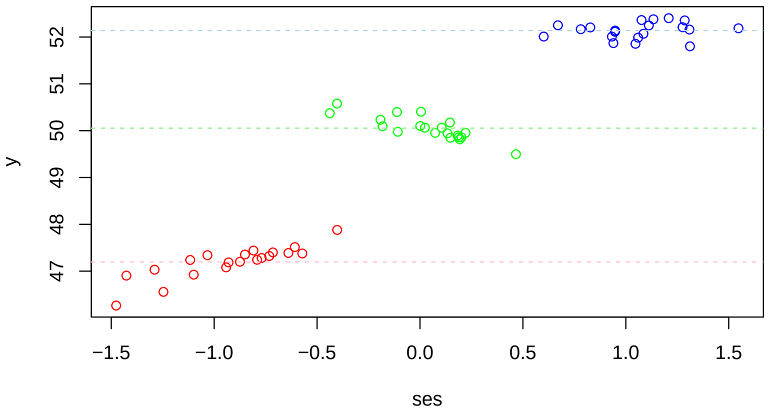
Variance across schools partially attributable to student-level variantion in SES

Possible explanations



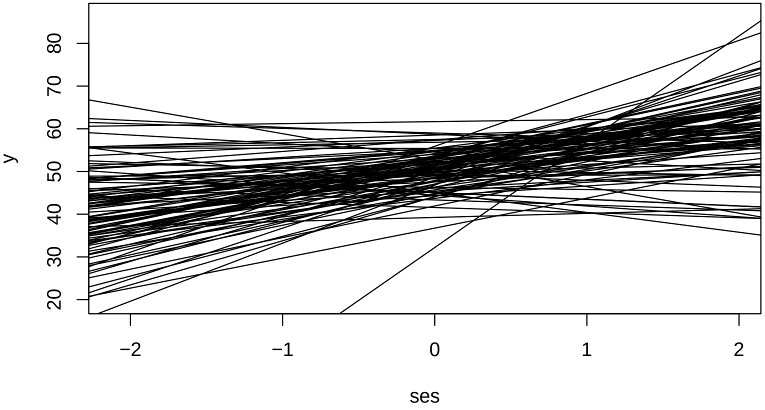
Variance across schools not attributable to student-level variantion in SES

Possible explanations



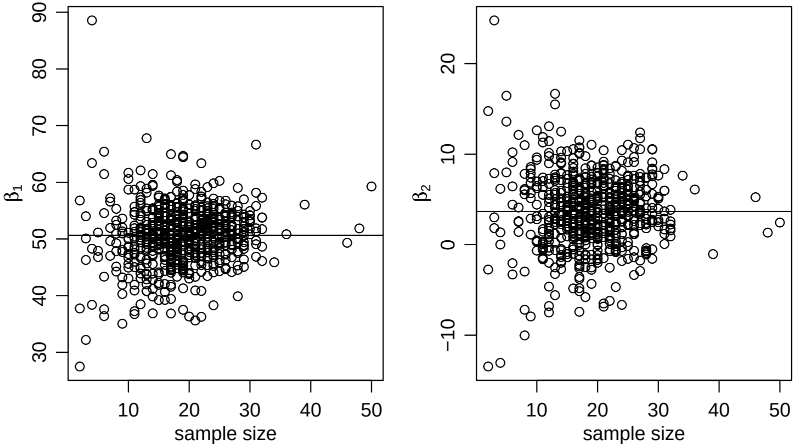
School specific OLS estimates

$$y_{i,j} = \hat{\beta}_{1,j} + \hat{\beta}_{2,j} x_{i,j} + \epsilon_{i,j}$$



School specific OLS estimates

$$y_{i,j} = \beta_{1,j} + \beta_{2,j} x_{i,j} + \epsilon_{i,j}$$



Estimation and testing

Hierarchical approach:

$$\begin{aligned} y_{i,j} &= \quad \beta_{1,j} \quad + \quad \beta_{2,j} x_{i,j} \quad + \ \epsilon_{i,j} \\ &= (\beta_1 + a_{1,j}) + (\beta_2 + a_{2,j}) x_{i,j} + \epsilon_{i,j}, \end{aligned}$$

Testing:

- Do the $a_{1,j}$'s vary across groups? $H_0 : a_{1,j} = 0$ for all j.
- Do the $a_{2,j}$'s vary across groups? $H_0 : a_{2,j} = 0$ for all j.

Note if $a_{1,j} = a_{1,j'} = 0$ for all j, then

- There still may be real heterogeneity in *mean* test scores, but
- all heterogeneity is attributable to heterogeneity in $x_{i,j}$.

Estimation: If H_0 is rejected, how do we estimate $\beta_{1,j}, \beta_{2,j}$?

- Unbiased OLS estimates?
- Biased shrinkage estimates?

Review of linear regression

Question:

- How does an outcome y vary with $\mathbf{x} = (x_1, \ldots, x_p)$ in a population?
- What is $p(y|\mathbf{x})$?

Data: A random sample of $(y, \mathbf{x})$ pairs from the population.

$$(y_1, \mathbf{x}_1), \ldots, (y_n, \mathbf{x}_n)$$

Task: Estimate $p(y|\mathbf{x})$ from the data.

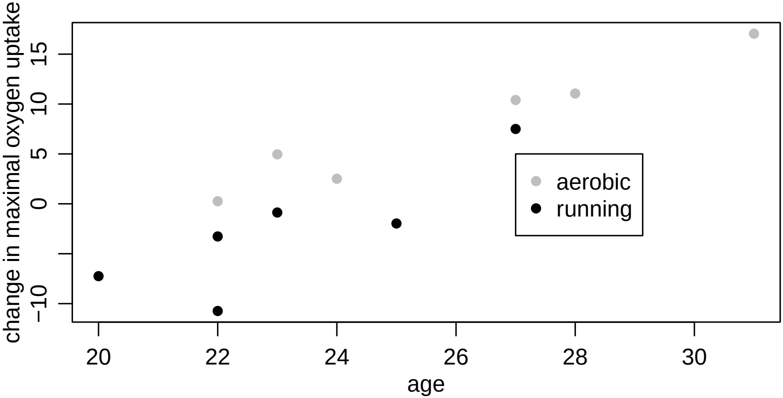
Example: O_2 uptake

Study design: 12 men randomly assigned to one of two regimens:

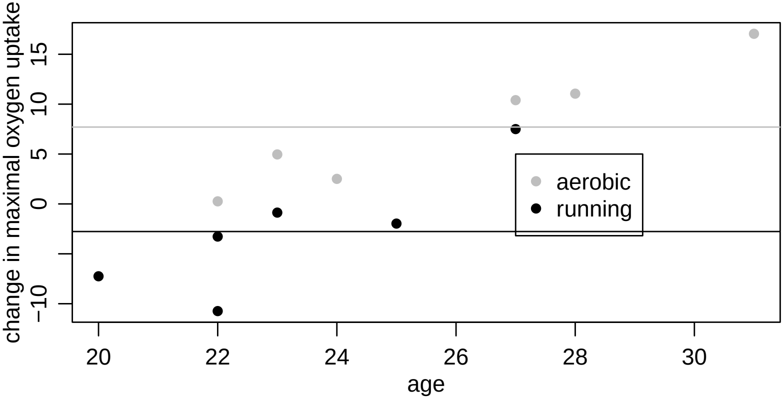
- flat terrain running;
- step aerobics.

The maximal O_2 uptake of each participant was measured after 3 months.

Age data is also available.



Example: O_2 uptake




```
mean(y[aerobic==1])

## [1] 7.705

mean(y[aerobic==0])

## [1] -2.766667
```

Regression and linear regression

How to estimate $p(y|\mathbf{x})$?

Unconstrained regression: Separately estimate the distribution of y for each age$\times$treatment combination.

- "unbiased"
- inefficient use of information;.

Constrained regression: Assume $p(y|\mathbf{x})$ has a simple form.

- biased, unless assumptions are correct;
- efficient use of information;
- interpretable parameters.

Linear regression: Assume $\mathrm{E}[y|\mathbf{x}]$ is linear in some unknown parameters:

$$\mathrm{E}[y|\mathbf{x}] = \int y p(y|\mathbf{x})\, dy = \beta_1 x_1 + \cdots + \beta_p x_p = \boldsymbol{\beta}^T \mathbf{x}$$

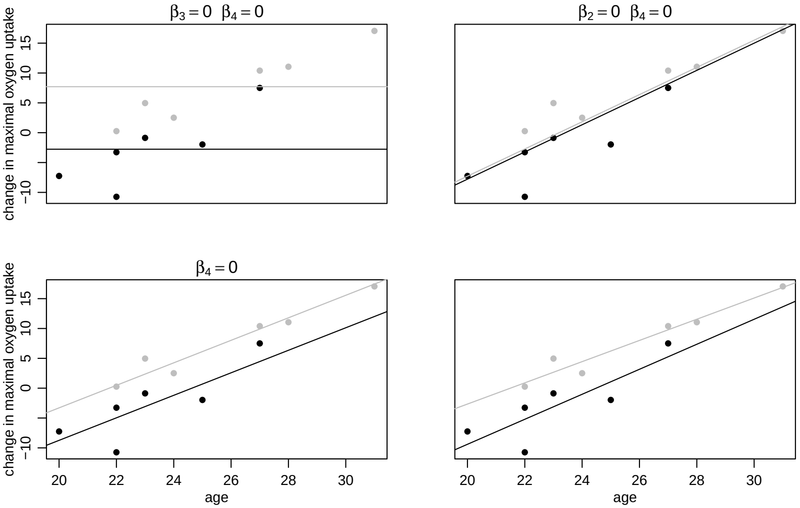
Linear regression for O_2 uptake

$$
\begin{aligned}
y_i &= \beta_1 x_{i,1} + \beta_2 x_{i,2} + \beta_3 x_{i,3} + \beta_4 x_{i,4} + \epsilon_i \ , \ \text{where} \\
x_{i,1} &= 1 \text{ for each subject } i \\
x_{i,2} &= 0 \text{ if subject } i \text{ is on the running program, 1 if on aerobic} \\
x_{i,3} &= \text{age of subject } i \\
x_{i,4} &= x_{i,2} \times x_{i,3}
\end{aligned}
$$

The conditional expectations of y for the two levels of $x_{i,2}$ are models as

$$
\begin{aligned}
\mathrm{E}[y|\boldsymbol{x}] &= \beta_1 + \beta_3 \times \ \text{age} && \text{if on running program} \\
\mathrm{E}[y|\boldsymbol{x}] &= (\beta_1 + \beta_2) + (\beta_3 + \beta_4) \times \ \text{age} && \text{if on aerobic program}
\end{aligned}
$$

Submodels


$\beta_3 = 0 \quad \beta_4 = 0$
$\beta_2 = 0 \quad \beta_4 = 0$
$\beta_4 = 0$
change in maximal oxygen uptake
−10
0
5
10
15
20
22
24
26
28
30
age

Normal linear regression

A full statistical model requires

- A specification of $\mathrm{E}[y|\mathbf{x}]$ (the "mean model")
- A specification of the distribution of y around $\mathrm{E}[y|\mathbf{x}]$

Normal linear regression:

$$\begin{aligned} y_i &= \boldsymbol{\beta}^T \mathbf{x}_i + \epsilon_i \\ \epsilon_1, \ldots, \epsilon_n &\sim \text{ i.i.d. normal}(0, \sigma^2) \end{aligned}$$

Vector-matrix form: Let $\mathbf{y}$ be the n-dimensional column vector $(y_1, \ldots, y_n)^T$, and $\mathbf{X}$ be the $n \times p$ matrix with ith row $\mathbf{x}_i$. The normal regression model is

$$\{\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}, \sigma^2\} \sim \text{ multivariate normal } (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}),$$

where $\mathbf{I}$ is the $p \times p$ identity matrix and

$$\mathbf{X}\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{x}_1 \rightarrow \\ \boldsymbol{x}_2 \rightarrow \\ \vdots \\ \boldsymbol{x}_n \rightarrow \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} = \begin{pmatrix} \beta_1 x_{1,1} + \cdots + \beta_p x_{1,p} \\ \vdots \\ \beta_1 x_{n,1} + \cdots + \beta_p x_{n,p} \end{pmatrix} = \begin{pmatrix} \mathrm{E}[y_1|\boldsymbol{\beta}, \mathbf{x}_1] \\ \vdots \\ \mathrm{E}[y_n|\boldsymbol{\beta}, \mathbf{x}_n] \end{pmatrix}.$$

OLS estimation

For any given value of $\boldsymbol{\beta}$,

- the fitted value for observation i is $\boldsymbol{\beta}^T\mathbf{x}_i$;
- the error or residual for i is $(y_i - \boldsymbol{\beta}^T\mathbf{x}_i)$;
- the SSE for $\boldsymbol{\beta}$ is

$$\begin{aligned} \text{SSE}(\boldsymbol{\beta}) &= \sum_{i=1}^{n}(y_i - \boldsymbol{\beta}^T\mathbf{x}_i)^2 \\ &= ||\mathbf{y} - \mathbf{X}\boldsymbol{\beta}||^2. \end{aligned}$$

The *ordinary least-squares* (OLS) estimate $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ minimizes SSE.

OLS regression

To find the minimizing value of $\boldsymbol{\beta}$, rewrite SSE($\boldsymbol{\beta}$) in matrix notation:

$$\begin{aligned}
\text{SSE}(\boldsymbol{\beta}) &= \sum_{i=1}^{n}(y_i - \boldsymbol{\beta}^T \mathbf{x}_i)^2 = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\
&= \mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}
\end{aligned}$$

Recall from calculus that

1. a minimum of a function $g(z)$ occurs at a value z such that $\frac{d}{dz}g(z) = 0$;
2. the derivative of $g(z) = az$ is a and the derivative of $g(z) = bz^2$ is $2bz$.

OLS estimation

$$
\begin{aligned}
\frac{d}{d\boldsymbol{\beta}}\mathrm{SSE}(\boldsymbol{\beta}) \quad &= \quad \frac{d}{d\boldsymbol{\beta}}\left(\mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}\right) \\
&= \quad -2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}\ , \ \text{therefore} \\
\frac{d}{d\boldsymbol{\beta}}\mathrm{SSE}(\boldsymbol{\beta}) = 0 \quad &\Leftrightarrow \quad -2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = 0 \\
&\Leftrightarrow \quad \mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = \mathbf{X}^T\mathbf{y} \\
&\Leftrightarrow \quad \boldsymbol{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}
\end{aligned}
$$

$\hat{\boldsymbol{\beta}}_{ols} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ is OLS estimate of $\boldsymbol{\beta}$.

OLS estimation for the O_2 uptake data

```
X

##       int trt age trt.age
##  [1,]   1   0  23       0
##  [2,]   1   0  22       0
##  [3,]   1   0  22       0
##  [4,]   1   0  25       0
##  [5,]   1   0  27       0
##  [6,]   1   0  20       0
##  [7,]   1   1  31      31
##  [8,]   1   1  23      23
##  [9,]   1   1  27      27
## [10,]   1   1  28      28
## [11,]   1   1  22      22
## [12,]   1   1  24      24

y

##  [1]  -0.87 -10.74  -3.27  -1.97   7.50  -7.25  17.05   4.96  10.40  11.05
## [11]   0.26   2.51
```

OLS estimation for the O_2 uptake data

```
XtX<-t(X)%*%X

XtX

##         int trt  age trt.age
## int      12   6  294     155
## trt       6   6  155     155
## age     294 155 7314    4063
## trt.age 155 155 4063    4063

Xty<-t(X)%*%y

Xty

##            [,1]
## int       29.63
## trt       46.23
## age      978.81
## trt.age 1298.79

solve(XtX) %*% Xty

##                [,1]
## int     -51.2939459
## trt      13.1070904
## age       2.0947027
## trt.age  -0.3182438
```

OLS estimation for the O_2 uptake data

```
solve(XtX) %*% Xty

##                [,1]
## int     -51.2939459
## trt      13.1070904
## age       2.0947027
## trt.age  -0.3182438

# with indicators
aerobic

##  [1] 0 0 0 0 0 0 1 1 1 1 1 1

lm(y~aerobic+age+aerobic*age)

##
## Call:
## lm(formula = y ~ aerobic + age + aerobic * age)
##
## Coefficients:
## (Intercept)      aerobic          age  aerobic:age
##    -51.2939      13.1071       2.0947      -0.3182
```

OLS estimation for the O_2 uptake data

```
# with factors
trt

##  [1] "running" "running" "running" "running" "running" "running" "aerobic"
##  [8] "aerobic" "aerobic" "aerobic" "aerobic" "aerobic"

fit<-lm(y~trt+age+trt*age)

# aerobic is baseline
fit

##
## Call:
## lm(formula = y ~ trt + age + trt * age)
##
## Coefficients:
##    (Intercept)      trtrunning             age  trtrunning:age
##       -38.1869        -13.1071          1.7765          0.3182

fit$coef[1]+fit$coef[2]

## (Intercept)
##   -51.29395

fit$coef[3]+fit$coef[4]

##      age
## 2.094703
```

Properties of OLS estimates

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$$

Unbiasedness: Treating $\mathbf{X}$ as fixed for the moment,

$$\begin{aligned} \mathrm{E}[\hat{\boldsymbol{\beta}}] &= \mathrm{E}[(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}] \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathrm{E}[\mathbf{y}] \\ &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} \\ &= \boldsymbol{\beta} \end{aligned}$$

Variance: Conditional on $\mathbf{X}$,

$$\mathrm{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2(\mathbf{X}^T\mathbf{X})^{-1}$$

Optimality of OLS

UMVUE: If $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ then

$$\mathrm{Var}[\hat{\boldsymbol{\beta}}] < \mathrm{Var}[\tilde{\boldsymbol{\beta}}]$$

for any other *unbiased* estimator $\tilde{\boldsymbol{\beta}}$.

BLUE: If $\mathrm{E}[\mathbf{y}|\mathbf{X}] = \mathbf{X}\boldsymbol{\beta}$, $\mathrm{Var}[\mathbf{y}|\mathbf{X}] = \sigma^2\mathbf{I}$ then

$$\mathrm{Var}[\hat{\boldsymbol{\beta}}] < \mathrm{Var}[\tilde{\boldsymbol{\beta}}]$$

for any other *linear unbiased* estimator $\tilde{\boldsymbol{\beta}}$, that is

- $\tilde{\boldsymbol{\beta}} = \mathbf{A}\mathbf{y}$ for some $\mathbf{A} \in \mathbb{R}^{p \times n}$;
- $\mathrm{E}[\tilde{\boldsymbol{\beta}}|X, \boldsymbol{\beta}] = \boldsymbol{\beta}$ for all $\boldsymbol{\beta}$;

Standard errors and CIs

$$\epsilon_1, \ldots, \epsilon_n \sim \text{ iid } N(0, \sigma^2)$$

How can we estimate σ^2?

Idea: Since $\boldsymbol{\beta} \approx \hat{\boldsymbol{\beta}}$,

$$\begin{aligned} \epsilon_i &= y_i - \boldsymbol{\beta}^T \mathbf{x}_i \\ &\approx y_i - \hat{\boldsymbol{\beta}}^T \mathbf{x}_i = \hat{\epsilon}_i \end{aligned}$$

$$\text{sample variance}(\epsilon_1, \ldots, \epsilon_n) \approx \sigma^2$$
$$\text{sample variance}(\hat{\epsilon}_1, \ldots, \hat{\epsilon}_n) \approx \sigma^2$$

SSE: Let $SSE = \sum (y_i - \hat{\boldsymbol{\beta}}^T \mathbf{x}_i)^2 = \sum \hat{\epsilon}_i^2$.

$$\hat{\sigma}^2 = \frac{SSE}{n-p} \quad \text{(unbiased estimator)}$$
$$\hat{\sigma}^2 = \frac{SSE}{n} \quad \text{(maximum likelihood estimator)}$$

Variance-covariance for the O_2 uptake data

```
beta.ols<-solve(XtX) %*% Xty
res<- y-X%*%beta.ols

SSE<-sum(res^2)

s2.hat<-SSE/( length(res) - length(beta.ols) )

VB<-s2.hat* solve(XtX)
```

```
VB

##                  int         trt        age      trt.age
## int       150.116712 -150.116712 -6.4184014   6.4184014
## trt      -150.116712  248.439893  6.4184014 -10.1693473
## age        -6.418401    6.418401  0.2770533  -0.2770533
## trt.age     6.418401  -10.169347 -0.2770533   0.4222512

sqrt(diag(VB))

##        int        trt        age    trt.age
## 12.2522126 15.7619762  0.5263585  0.6498086
```

Variance-covariance for the O_2 uptake data

```
fit<-lm(y~aerobic+age+aerobic*age)
summary(fit)

##
## Call:
## lm(formula = y ~ aerobic + age + aerobic * age)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.5295 -0.9610  0.3945  2.1717  2.2883
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -51.2939    12.2522  -4.187  0.00305 **
## aerobic      13.1071    15.7620   0.832  0.42978
## age           2.0947     0.5264   3.980  0.00406 **
## aerobic:age  -0.3182     0.6498  -0.490  0.63746
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.923 on 8 degrees of freedom
## Multiple R-squared:  0.9049, Adjusted R-squared:  0.8692
## F-statistic: 25.36 on 3 and 8 DF,  p-value: 0.0001938

beta.ols/sqrt(diag(VB))

##               [,1]
## int     -4.1865047
## trt      0.8315639
## age      3.9796120
## trt.age -0.4897500
```

Evaluating group effects, the ANCOVA view

ANOVA: Evaluate heterogeneity across categorical factors with an F-test.

ANCOVA: Evaluate heterogeneity across categorical factors with an F-test, *after accounting for a (continuous) covariate*.

Questions answered:

- ANOVA: is there heterogeneity across groups?
- ANCOVA: is there heterogeneity across groups, *beyond that attributable to a covariate* ?

Standard ANCOVA model

$$y_{i,j} = (\beta_0 + b_{0,j}) + \beta_1 \times x_{i,j} + \epsilon_{i,j}$$

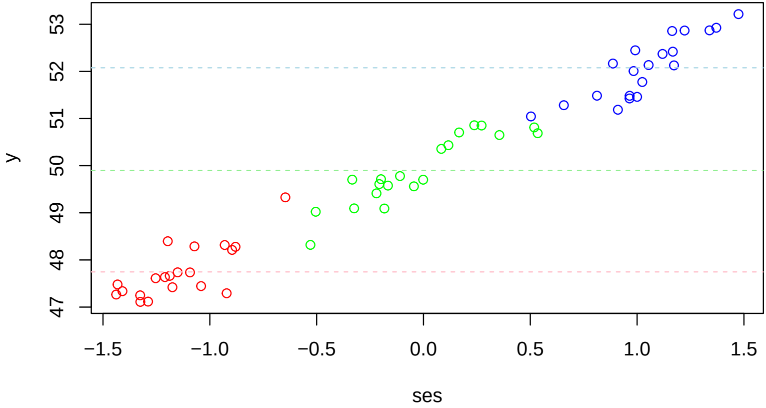
- $y_{i,j}$ refers to the ith observation in group j;
- $b_{0,j}$ refers to the effect of jth group on the mean;
- β_1 refers to the slope (assumed identical across groups).

For two-groups the model is the same as the following regression model:

$$y_i = (\beta_0 + b_0 \times \text{aerobic}_i) + \beta_1 \times \text{age} + \epsilon_i$$

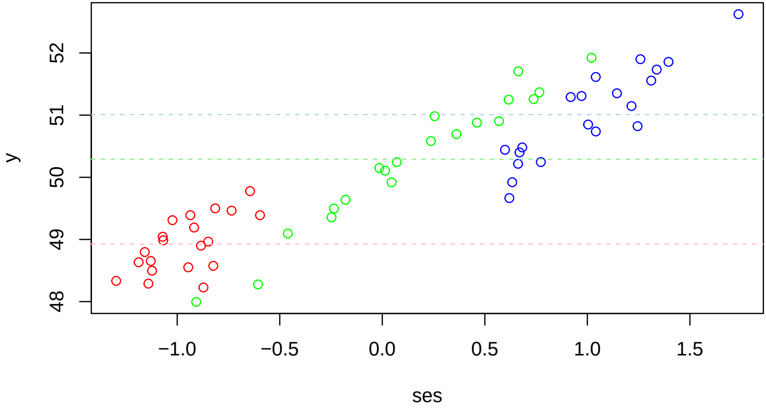
- y_i is the ith observation overall;
- aerobic_i is the indicator that person i is in the aerobics group;

Possible explanations



$b_0 = 0.$

Possible explanations



$b_0 \neq 0.$

Testing and ANCOVA

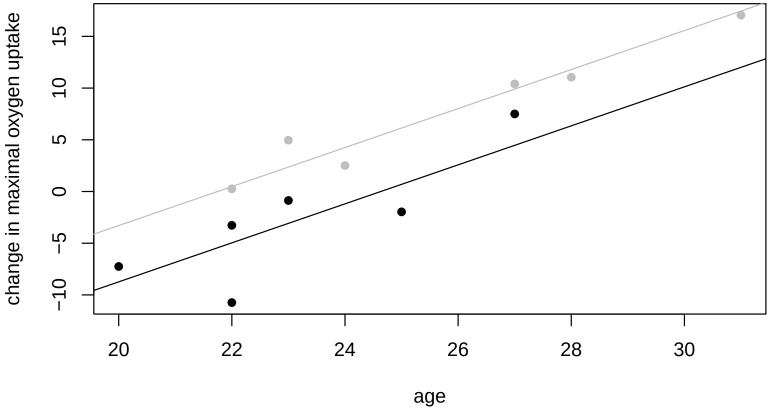
$$y_{i,j} = (\beta_0 + b_{0,j}) + \beta x_{i,j} + \epsilon_{i,j}$$

A test of across-group heterogeneity is provided by an F-test:

```
fit1<-lm( y~ age + as.factor(trt))
anova(fit1)

## Analysis of Variance Table
##
## Response: y
##                 Df Sum Sq Mean Sq F value    Pr(>F)
## age              1 576.09  576.09 73.6594 1.257e-05 ***
## as.factor(trt)   1  71.79   71.79  9.1788   0.01425 *
## Residuals        9  70.39    7.82
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The p-value indicates evidence of across-group heterogeneity beyond that attributable to age.

Variable intercept model



ANCOVA with interactions

$$y_{i,j} = (\beta_0 + b_{0,j}) + (\beta_1 + b_{1,j})x_{i,j} + \epsilon_{i,j}$$

- $b_{1,j}$ is a group specific slope parameter

For two-groups the model is the same as the following regression model:

$$y_i = (\beta_0 + b_0 \times \text{aerobic}_i) + (\beta_1 + b_1 \times \text{aerobic}_i) \times \text{age}_i + \epsilon_i$$

- aerobic_i is the indicator that person i is in the aerobics group;
- b_1 is the difference in slopes between the two groups.

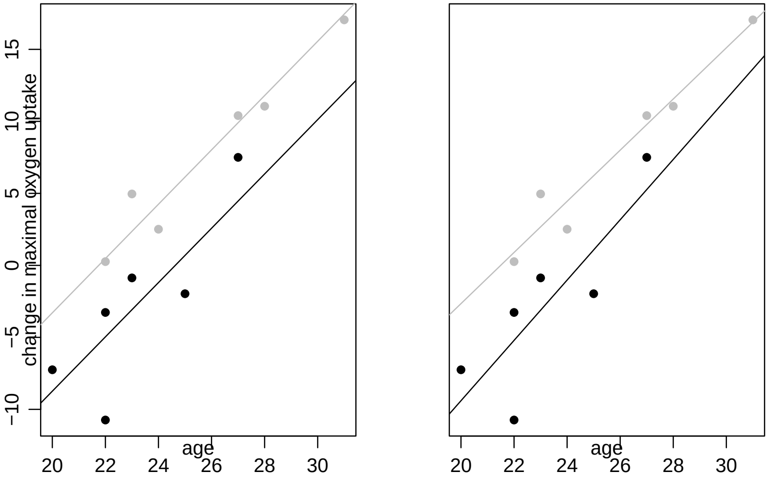
ANCOVA with interactions

```
fit2<-lm( y~ age + as.factor(trt) + age*as.factor(trt) )
anova(fit2)

## Analysis of Variance Table
##
## Response: y
##                     Df Sum Sq Mean Sq F value    Pr(>F)
## age                  1 576.09  576.09 67.4381 3.615e-05 ***
## as.factor(trt)       1  71.79   71.79  8.4035   0.01993 *
## age:as.factor(trt)   1   2.05    2.05  0.2399   0.63746
## Residuals            8  68.34    8.54
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

There is not evidence for heterogeneity beyond what can be attributed to

- age
- a mean difference between groups

ANCOVA with interactions



Heterogeneous regressions

It will be convenient to rewrite the model in vector form:

$$
\begin{aligned}
y_{i,j} &= \boldsymbol{\beta}_j^T \mathbf{x}_{i,j} + \epsilon_{i,j} \\
\boldsymbol{\beta}_j &= \boldsymbol{\beta} + \mathbf{b}_j
\end{aligned}
$$

- $\boldsymbol{\beta}$ represents the average across-group relationship of y to $\mathbf{x}$.
- $\{\mathbf{b}_1, \ldots, \boldsymbol{b}_m\}$ represent across-group heterogeneity of the relationship.

In the O_2 uptake example,

$$
\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad \mathbf{b}_j = \begin{pmatrix} b_{0,j} \\ b_{1,j} \end{pmatrix} \quad \mathbf{x}_{i,j} = \begin{pmatrix} 1 \\ \text{age}_{i,j} \end{pmatrix}
$$

$$
\begin{aligned}
\mathrm{E}[y_{i,j}] = \boldsymbol{\beta}_j^T \mathbf{x}_{i,j} &= [\boldsymbol{\beta} + \mathbf{b}_j]^T \mathbf{x}_{i,j} \\
&= \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \mathbf{b}_j^T \mathbf{x}_{i,j} \\
&= [\beta_0 + \beta_1 \times \text{age}_{i,j}] + [b_{0,j} + b_{1,j} \times \text{age}_{i,j}]
\end{aligned}
$$

Testing for an overall group effect

Sometimes it will be more convenient to test for *any* group effect:

$$y_{i,j} = \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \mathbf{b}_j^T \mathbf{x}_{i,j} + \epsilon_{i,j}$$

H_0: $\mathbf{b}_1 = \cdots = \mathbf{b}_m = \mathbf{0}$

H_1: $\mathbf{b}_j \neq 0$, some $j \in \{1, \ldots, m\}$

This can be done via an F-test as well:

```
fit0<-lm( y~ age )
fit1<-lm( y~ age + as.factor(trt) )
fit2<-lm( y~ age + as.factor(trt) + age*as.factor(trt) )
```

Testing for an overall group effect

```
anova(fit2)

## Analysis of Variance Table
##
## Response: y
##                   Df Sum Sq Mean Sq F value    Pr(>F)
## age                1 576.09  576.09 67.4381 3.615e-05 ***
## as.factor(trt)     1  71.79   71.79  8.4035   0.01993 *
## age:as.factor(trt) 1   2.05    2.05  0.2399   0.63746
## Residuals          8  68.34    8.54
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
anova(fit0,fit2)

## Analysis of Variance Table
##
## Model 1: y ~ age
## Model 2: y ~ age + as.factor(trt) + age * as.factor(trt)
##   Res.Df    RSS Df Sum of Sq      F  Pr(>F)
## 1     10 142.18
## 2      8  68.34  2    73.836 4.3217 0.05338 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Why overall tests?

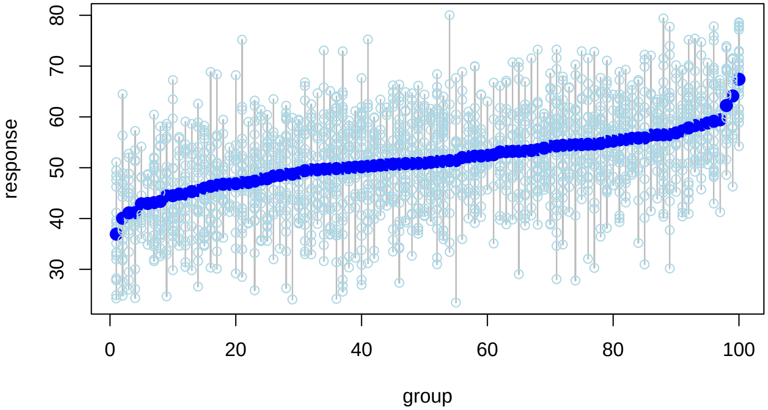
Consider a scenario where we have lots of regressors:

$$
\begin{aligned}
y_{i,j} &= \boldsymbol{\beta}_j^T \mathbf{x}_{i,j} + \epsilon_{i,j} \\
&= \beta_{1,j} x_{1,i,j} + \cdots + \beta_{p,j} x_{p,i,j} + \epsilon_{i,j}
\end{aligned}
$$

Compare and contrast the following two procedures:

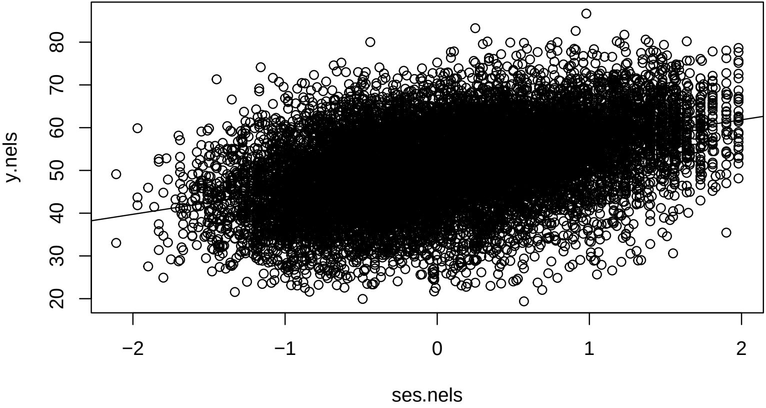
1. Iteratively search for predictors that show across group heterogeneity;
2. Perform an overall test of across-group differences
   - If heterogeneity detected, describe it for each predictor.

NELS data

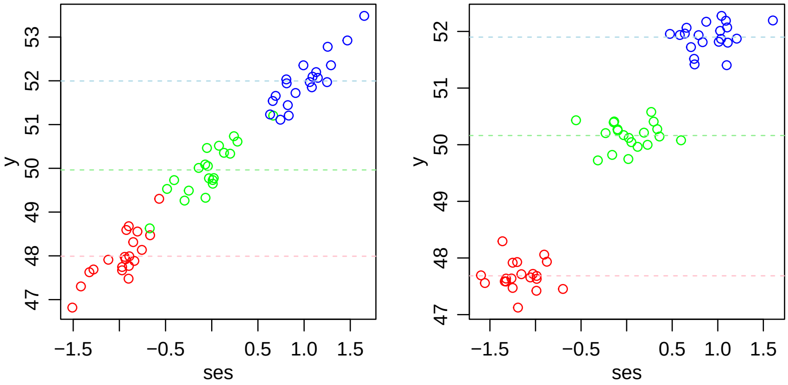


Marginal relationship

```
plot(y.nels~ses.nels)
abline(lm(y.nels~ses.nels))
```



Two possible explanations

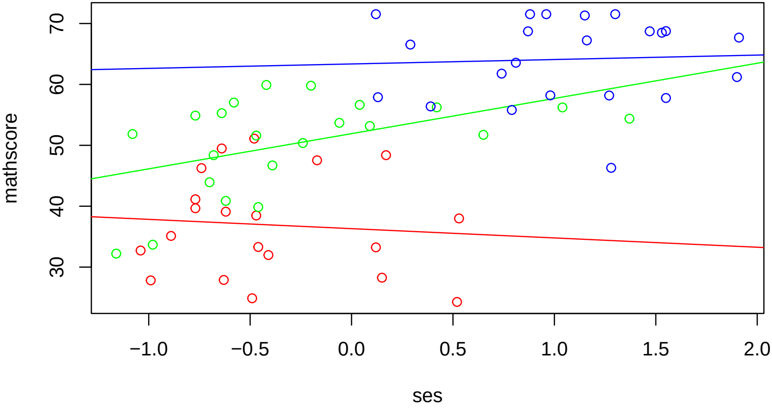


$$y_{i,j} = \beta_0 + \beta_1 \text{ses}_{i,j} + b_{0,j} + b_{1,j}\text{ses}_{i,j} + \epsilon_{i,j}$$

What values of $\{b_{0,j}, b_{1,j}\}$ do the two explanations correspond to?

- Micro effects of SES on mathscore;
- Macro effects of SES on mathscore.

Some actual data



What explanations do these data support?

OLS approach

```
BETA<-NULL
for(j in sort(unique(g.nels)))
{
  yj<-y.nels[g.nels==j]
  xj<-ses.nels[g.nels==j]
  fitj<-lm(yj~xj)
  BETA<-rbind(BETA,fitj$coef)
}

### some results
BETA[1:10,]

##       (Intercept)        xj
##  [1,]    53.02066 5.0815402
##  [2,]    49.82444 2.9045055
##  [3,]    38.48130 1.1340111
##  [4,]    46.38335 2.6715294
##  [5,]    46.35686 5.0231028
##  [6,]    48.96969 0.9272974
##  [7,]    46.26290 6.8041213
##  [8,]    53.39039 5.0407659
##  [9,]    51.73138 2.5813744
## [10,]    49.84851 4.9972552
```

Explaining across-group variation with SES

```
### mean intercept, mean slope
apply(BETA,2,mean,na.rm=TRUE)

## (Intercept)          xj
##   50.618228    3.672483

### compare to pooled analysis
lm(y.nels~ses.nels)

##
## Call:
## lm(formula = y.nels ~ ses.nels)
##
## Coefficients:
## (Intercept)     ses.nels
##      50.793        5.527
```

What does the discrepancy suggest in terms of macro vs micro effects of SES?

Testing for heterogeneity

$$\begin{aligned} y_{i,j} &= \boldsymbol{\beta}_j^T \mathbf{x}_{i,j} + \epsilon_{i,j} \\ &= \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \mathbf{b}_j^T \mathbf{x}_{i,j} + \epsilon_{i,j} \end{aligned}$$

Testing for across-group heterogeneity:

H_0: $\mathbf{b}_1 = \cdots = \mathbf{b}_m = \mathbf{0}$

H_1: $\mathbf{b}_j \neq 0$, some $j \in \{1, \ldots, m\}$

```
fit0<-lm(y.nels~ses.nels)
fit1<-lm(y.nels~ses.nels + as.factor(g.nels) + ses.nels*as.factor(g.nels))

### test for across-group heterogeneity
anova(fit0,fit1)

## Analysis of Variance Table
##
## Model 1: y.nels ~ ses.nels
## Model 2: y.nels ~ ses.nels + as.factor(g.nels) + ses.nels * as.factor(g.nels)
##   Res.Df     RSS   Df Sum of Sq      F    Pr(>F)
## 1  12972 1022921
## 2  11607  776507 1365    246414 2.6984 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Testing for heterogeneity

```
### sequential test of effects
anova(fit1)

## Analysis of Variance Table
##
## Response: y.nels
##                                 Df Sum Sq Mean Sq   F value    Pr(>F)
## ses.nels                         1 223914  223914 3347.0036 < 2.2e-16 ***
## as.factor(g.nels)              683 190150     278    4.1615 < 2.2e-16 ***
## ses.nels:as.factor(g.nels)     682  56264      82    1.2332 4.865e-05 ***
## Residuals                    11607 776507      67
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The data provide strong evidence of across-group heterogeneity in mathscore/SES association.

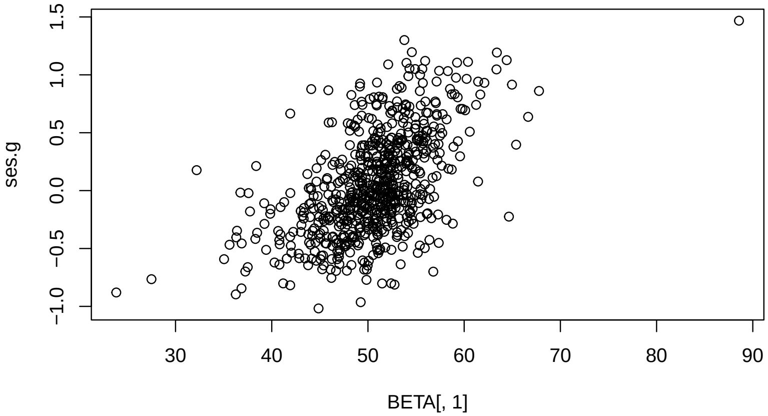
Furthermore, the data suggest both

- micro-level effects of SES (slopes are on average positive)
- macro-level effects of SES (average slope is lower than pooled slope)

Testing for heterogeneity

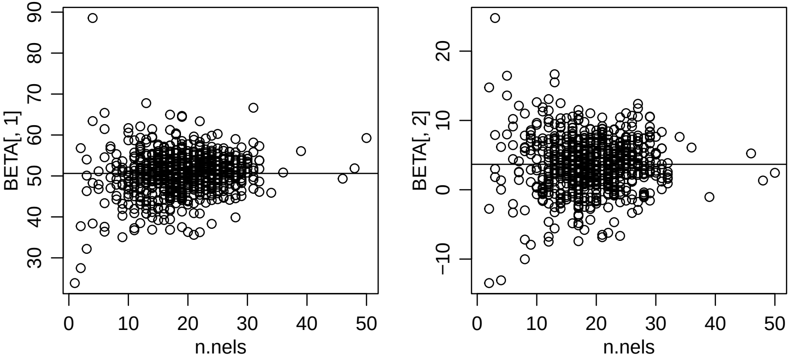
```
fit1b<-lm(y.nels~as.factor(g.nels) + ses.nels + ses.nels*as.factor(g.nels))

### sequential test of effects
anova(fit1b)

## Analysis of Variance Table
##
## Response: y.nels
##                                 Df Sum Sq Mean Sq   F value    Pr(>F)
## as.factor(g.nels)              683 342385     501    7.4932 < 2.2e-16 ***
## ses.nels                         1  71679   71679 1071.4332 < 2.2e-16 ***
## as.factor(g.nels):ses.nels     682  56264      82    1.2332 4.865e-05 ***
## Residuals                    11607 776507      67
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Macro-level effects



Estimation of regression coefficients

How should we estimate $\boldsymbol{\beta}_j$?



Recall:

$$\mathrm{Var}[\hat{\boldsymbol{\beta}}_j] = \sigma^2(\mathbf{X}_j^T\mathbf{X}_j)^{-1}$$

$$\mathbf{X}_j^T\mathbf{X}_j = \sum_{i=1}^{n_j} \boldsymbol{x}_{i,j}\boldsymbol{x}_{i,j}^T \quad \text{is generally increasing in } n_j$$