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Date: September 16, 2024

HW2

1. (a) Since $\hat{\theta}_j = (1-w)\bar{y}_j + wc$, we have

$$\begin{aligned}(\hat{\theta}_j - \theta_j)^2 &= ((1-w)\bar{y}_j + wc - \theta_j)^2 \\ &= ((1-w)\bar{y}_j - \theta_j + w\theta_j - w\theta_j + wc)^2 \\ &= ((1-w)\bar{y}_j - (1-w)\theta_j + w(c-\theta_j))^2 \\ &= ((1-w)(\bar{y}_j - \theta_j) + w(c-\theta_j))^2 \\ &= (1-w)^2(\bar{y}_j - \theta_j)^2 + w^2(c-\theta_j)^2 + 2w(1-w)(\bar{y}_j - \theta_j)(c-\theta_j).\end{aligned}$$

Notice that

$$\mathbb{E}[2w(1-w)(\bar{y}_j - \theta_j)(c-\theta_j)] = 2w(1-w)(c-\theta_j)(\mathbb{E}[\bar{y}_j|\theta_j] - \theta_j) = 0.$$

Then, we have

$$\mathbb{E}[(\hat{\theta}_j - \theta_j)^2] = \mathbb{E}[(1-w)^2(\bar{y}_j - \theta_j)^2] + \mathbb{E}[w^2(c-\theta_j)^2].$$

We know that $\mathbb{E}[(\bar{y}_j - \theta_j)^2] = \mathbb{V}\text{ar}(\bar{y}_j|\theta_j) = \frac{\sigma^2}{n}$. We also know that $\mathbb{E}[w^2(c-\theta_j)^2] = w^2(c-\theta_j)^2$. Then,

$$\mathbb{E}[(\hat{\theta}_j - \theta_j)^2] = (1-w)^2\frac{\sigma^2}{n} + w^2(c-\theta_j)^2.$$

Then, summing over all js, we have

$$\begin{aligned}\mathbb{E}[||\hat{\theta} - \theta||^2] &= \sum_{j=1}^{m}\left((1-w)^2\frac{\sigma^2}{n} + w^2(c-\theta_j)^2\right) \\ &= \frac{\sigma^2}{n}m(1-w)^2 + w^2\sum_{j=1}^{m}(c-\theta_j)^2.\end{aligned}$$

We then take partial derivatives to find the optimal w and c. For c, we have

$$\begin{aligned}\frac{\partial}{\partial c}\mathbb{E}[||\hat{\theta} - \theta||^2] &= \frac{\partial}{\partial c}w^2\sum_{j=1}^{m}(c-\theta_j)^2 \\ &= 2w^2\sum_{j=1}^{m}(c-\theta_j).\end{aligned}$$

Set this equal to zero, we have

$$\begin{aligned}2w^2\sum_{j=1}^{m}(c-\theta_j) &= 0 \\ mc &= \sum_{j=1}^{m}\theta_j \\ c &= \frac{1}{m}\sum_{j=1}^{m}\theta_j = \mu.\end{aligned}$$

For w, plugging in $c = \mu$, we have

$$\frac{\partial}{\partial w}\mathbb{E}[||\hat{\theta} - \theta||^2] = -2\frac{\sigma^2}{n}m(1-w) + 2w\sum_{j=1}^{m}(\mu - \theta_j)^2$$

Set this equal to zero, we have

$$\begin{aligned}\frac{\sigma^2}{n}m(1-w) &= w\sum_{j=1}^{m}(\mu-\theta_j)^2 \\ \frac{\sigma^2}{n}(1-w) &= w\frac{1}{m}\sum_{j=1}^{m}(\mu-\theta_j)^2.\end{aligned}$$

Let $\tau^2 = \frac{1}{m}\sum_{j=1}^{m}(\mu-\theta_j)^2$, we have

$$\begin{aligned}\frac{\sigma^2}{n}(1-w) &= w\tau^2 \\ w\frac{\sigma^2}{n} + w\tau^2 &= \frac{\sigma^2}{n} \\ w &= \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}+\tau^2} \\ &= \frac{\frac{\sigma^2}{n}\frac{n}{\tau^2\sigma^2}}{\left(\frac{\sigma^2}{n}+\tau^2\right)\frac{n}{\tau^2\sigma^2}} \\ &= \frac{1/\tau^2}{n/\sigma^2 + 1/\tau^2}.\end{aligned}$$

(b) From part (a), we know that $c = \mu$, $w = \frac{1/\tau^2}{n/\sigma^2+1/\tau^2}$. Plugging them in, we have

$$\begin{aligned}\mathbb{E}[||\hat{\theta}_j - \theta_j||^2] &= \frac{\sigma^2}{n}m(1-w)^2 + w^2\sum_{j=1}^{m}(c-\theta_j)^2 \\ &= \left(\frac{n/\sigma^2}{n/\sigma^2+1/\tau^2}\right)^2\frac{\sigma^2}{n}m + \left(\frac{1/\tau^2}{n/\sigma^2+1/\tau^2}\right)^2\sum_{j=1}^{m}(\mu-\theta_j)^2 \\ &= \left(\frac{n/\sigma^2}{n/\sigma^2+1/\tau^2}\right)^2\frac{\sigma^2}{n}m + \left(\frac{1/\tau^2}{n/\sigma^2+1/\tau^2}\right)^2 m\tau^2 \\ &= m\left(\frac{n/\sigma^2}{(n/\sigma^2+1/\tau^2)^2} + \frac{1/\tau^2}{(n/\sigma^2+1/\tau^2)^2}\right) \\ &= m\frac{n/\sigma^2+1/\tau^2}{(n/\sigma^2+1/\tau^2)^2} \\ &= \frac{m}{n/\sigma^2+1/\tau^2}.\end{aligned}$$

2. See below

3. (a) From question 1, we know that

$$\mathbb{E}[(\hat{\theta}-\theta)^2|\theta] = (1-w)^2\frac{\sigma^2}{n} + w^2(\mu-\theta)^2.$$

We also know that $\mathbb{E}[(\bar{y}-\theta)^2] = \frac{\sigma^2}{n}$. Then, assume $w > 0$, we have the inequality

$$\begin{aligned}
(1-w)^2\frac{\sigma^2}{n} + w^2(\mu-\theta)^2 &< \frac{\sigma^2}{n} \\
\frac{\sigma^2}{n} - 2w\frac{\sigma^2}{n} + w^2\frac{\sigma^2}{n} + w^2(\mu-\theta)^2 &< \frac{\sigma^2}{n} \\
-2w\frac{\sigma^2}{n} + w^2\frac{\sigma^2}{n} + w^2(\mu-\theta)^2 &< 0 \\
w^2\left(\frac{\sigma^2}{n} + (\mu-\theta)^2\right) - 2w\frac{\sigma^2}{n} &< 0 \\
w\left(\frac{\sigma^2}{n} + (\mu-\theta)^2\right) &< \frac{2\sigma^2}{n} \\
w &< \frac{\frac{2\sigma^2}{n}}{\frac{\sigma^2}{n} + (\mu-\theta)^2}.
\end{aligned}$$

(b) From question 1, we know that $w = \frac{1/\tau^2}{n/\sigma^2+1/\tau^2}$. Plug this in, we have

$$\begin{aligned}
\frac{1/\tau^2}{n/\sigma^2+1/\tau^2} &< \frac{2\sigma^2/n}{\sigma^2/n+(\mu-\theta)^2} \\
1/\tau^2 \cdot (\sigma^2/n + (\mu-\theta)^2) &< 2\sigma^2/n \cdot (n/\sigma^2 + 1/\tau^2) \\
1/\tau^2 \cdot (\mu-\theta)^2 &< 2 + 2\sigma^2/n\tau^2 - \sigma^2/n\tau^2 \\
(\mu-\theta)^2 &< 2\tau^2 + \frac{2\sigma^2}{n} - \frac{\sigma^2}{n} \\
(\mu-\theta)^2 &< 2\tau^2 + \frac{\sigma^2}{n}.
\end{aligned}$$