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LSTM HW6

1. *Proof.* From the definition given by PyTorch, we know that the simple RNN with sigmoid nonlinearity updates its hidden state h_t using the following equation:

$$h_t = \sigma(x_t W_{ih}^T + h_{t-1} W_{hh}^T + b_{ih} + b_{hh})$$

where σ is the sigmoid function. Also from PyTorch, we know that the LSTM is defined as follow:

$$\begin{aligned}
i_t &= \sigma(W_{ii}x_t + b_{ii} + W_{hi}h_{t-1} + b_{hi}) \\
f_t &= \sigma(W_{if}x_t + b_{if} + W_{hf}h_{t-1} + b_{hf}) \\
g_t &= \tanh(W_{ig}x_t + b_{ig} + W_{hg}h_{t-1} + b_{hg}) \\
o_t &= \sigma(W_{io}x_t + b_{io} + W_{ho}h_{t-1} + b_{ho}) \\
c_t &= f_t \odot c_{t-1} + i_t \odot g_t \\
h_t &= o_t \odot \tanh(c_t).
\end{aligned}$$

Notice that o_t is the gate with sigmoid non-linearity. Hence, we can show that the simple RNN with sigmoid non-linearity is a subset of LSTM letting $\tanh(c_t) \approx 1$. We know that $\lim_{c_t \to \infty} \tanh(c_t) = 1$. This means that we need c_t to be very large.

- f_t Notice that since f_t uses the sigmoid activation function, $0 < f_t < 1$. f_t controls how much of the previous cell state c_{t-1} is retained. Then to ensure large increases in c_t, we need to set $f_t \to 1$.
- c_0 To ensure c_t diverges to infinity, we can set the initial state with large positive values.
- i_t Notice that i_t also involves the sigmoid activation function, which means that $0 < i_t < 1$. i_t controls how much of the new information is written to the cell state. Then we can set $i_t \to 1$ to ensure c_t goes to infinity.
- g_t Since g_t involve tanh, $-1 < g_t < 1$. Then we can set $g_t \to 1$ for c_t to approach to infinity.

Now we can rewrite the recurrence function as

$$\begin{aligned}
c_t &= f_t \odot c_{t-1} + i_t \odot g_t \\
&= f_t^t c_0 + t \cdot 1 \cdot 1 \\
&= f_t^t c_0 + t.
\end{aligned}$$

Notice that $\lim_{t \to \infty} c_t = \infty$. Also, we mentioned that c_0 can be initialized with large values, then $\tanh(c_t) \to 1$. Then,

$$h_t \to o_t \odot 1 = \sigma(W_{io}x_t + b_{io} + W_{ho}h_{t-1} + b_{ho}),$$

which is the recurrence equation of a simple RNN with sigmoid nonlinearity. Hence, the simple RNN with sigmoid nonlinearity is a subset of the LSTM.

For the second part of the question, we can set all weights $W = 0$ with a large bias. Then,

$$\begin{aligned}
f_t = i_t &= \sigma(0 + \text{large bias}) \to 1 \\
g_t &= \tanh(0 + \text{large bias}) \to 1.
\end{aligned}$$

Then these conditions will ensure the LSTM to mimic the RNN by what we have shown above. □

2. **One-hot Encoding**
According to the alphabetical order, we set

$$\begin{aligned}\text{"bad"} &= [1,0,0,0]\\ \text{"good"} &= [0,1,0,0]\\ \text{"not"} &= [0,0,1,0]\\ \text{"uh"} &= [0,0,0,1].\end{aligned}$$

General Workflow
Notice the recurrence relationship of c_t is

$$c_t = f_t \odot c_{t-1} + i_t \odot g_t.$$

We want to delay the sentiment score by 1. That is, c_t contains the cumulated sentiment score at $t-1$, c_{t-1} contains the cumulated sentiment score at $t-2$, and g_t calculates the sentiment score at time $t-1$ by utilizing the hidden state h_{t-1}. Let's then consider each gate separately.

Gate Specification

Input i_t We want $i_t = \mathbf{1}_{1\times 6}$ for the following reason:
The equation we are given is

$$\sum \text{"good"} - \sum \text{"bad"} - 2\sum \text{"not good"} + 2\sum \text{"not bad"}.$$

This means that in each iteration, we are either adding 1, subtracting 1 or doing nothing (when the current word is "not" or "uh"). Thus, we need to completely keep the updated information, namely, $i_t = [1,1,1,1,1,1]^T$. To do this, we just need to set

$$\begin{aligned}W_{ii} &= \mathbf{0}_{6\times 4}\\ W_{hi} &= \mathbf{0}_{6\times 6}\\ b_{ii} &= \mathbf{0}_{1\times 6}\\ b_{hi} &= \mathbf{100}_{1\times 6}.\end{aligned}$$

Then, we have

$$\begin{aligned}i_t &= \sigma(0 + b_{ii} + 0 + b_{hi})\\ &= \sigma([100,100,100,100,100,100]^T)\\ &\approx [1,1,1,1,1,1]^T.\end{aligned}$$

Forget f_t We want $f_t = [0,0,0,0,0,1]^T$ for the following reason:
In the forget gate, since we only care about the sentiment score, we only need to set the element corresponding to the sentiment score to 1. In our case, we set the last element to be the sentiment score. To achieve this, we just need to set

$$\begin{aligned}W_{if} &= \mathbf{0}_{6\times 4}\\ W_{hf} &= \mathbf{0}_{6\times 6}\\ b_{if} &= [-100,-100,-100,-100,-100,100]^T\\ b_{hf} &= \mathbf{0}_{1\times 6}.\end{aligned}$$

Then, we have

$$\begin{aligned}f_t &= \sigma(0 + b_{if} + 0 + 0)\\ &= \sigma([-100,-100,-100,-100,-100,100]^T)\\ &\approx [0,0,0,0,0,1].\end{aligned}$$

Cell g_t We want $g_t = [1, 1, 1, 1, 1, \text{delayed increment score}]^T$.
The reason that we want the first five to be 1 is as follows: The update equations are:

$$c_t = f_t \odot c_{t-1} + i_t \odot g_t$$
$$h_t = o_t \odot \tanh(c_t).$$

We want to achieve $h_t = [I_{\text{bad}}, I_{\text{good}}, I_{\text{not}}, I_{\text{not bad}}, I_{\text{not good}}, 0]^T$, which stores the flags. Notice that the first five elements of $f_t \odot c_{t-1}$ are zeros. If we set the first five elements of g_t to zero as well, then the first five elements of c_t will remain zero, since both terms in the update equation contribute zeros for these elements. Consequently, $\tanh(c_t) = 0$ for the first five elements, causing h_t to lack any meaningful information in these positions. To store the flag information in h_t, we therefore need to set the first five elements of g_t to ones. This ensures that the input gate $i_t \odot g_t$ can directly influence c_t for these elements, allowing the desired flag indicators to be reflected in h_t.
Now, let's propagate the weights and biases. Notice that

$$g_t = \tanh(W_{ig}x_t + b_{ig} + W_{hg}h_{t-1} + b_{hg}),$$

and we only want to use this gate to determine the sentiment score at $t-1$ through the hidden state h_{t-1}. To achieve this, since we do not need to handle x_t at this point, we can set $W_{ig} = \mathbf{0}_{6\times 4}$ and $b_{ig} = \mathbf{0}_{1\times 6}$. To determine the sentiment score, according to the equation, we have

$$\begin{aligned}
\text{bad} &= -1 \\
\text{good} &= +1 \\
\text{not bad} &= 2 \cdot 1 - 1 = +1 \text{ (since "bad" alone will also be counted)} \\
\text{not good} &= -2 \cdot 1 + 1 = -1 \text{ (since "good" alone will also be counted).}
\end{aligned}$$

Thus, we can design W_{hg} as follows:

$$W_{hg} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & 100 & 0 & 100 & -100 & 0 \end{bmatrix}.$$

Recall that we design our hidden state matrix to be flags of the words, that is

$$h_{t-1} = [I_{\text{bad}}, I_{\text{good}}, I_{\text{not}}, I_{\text{not bad}}, I_{\text{not good}}, 0]^T.$$

Then,

$$(W_{hg}h_{t-1})_6 = -100I_{\text{bad}} + 100I_{\text{good}} + 100I_{\text{not bad}} - 100I_{\text{not good}},$$

which serves the function of determining sentiment scores. The first five elements of $W_{hg}h_{t-1}$ are zeros now. We then set

$$b_{ig} = [100, 100, 100, 100, 100, 0]^T.$$

Then,

$$\begin{aligned}
g_t &= \tanh(0 + 0 + W_{hg}h_{t-1} + b_{hg}) \\
&= \tanh([100, 100, 100, 100, 100, (W_{hg}h_{t-1})_6]^T) \\
&= [1, 1, 1, 1, 1, \text{delayed increment score}]^T
\end{aligned}$$

where delayed increment score $= \tanh((W_{hg}h_{t-1})_6) \in \{-1, 0, 1\}$.

Cell State c_t Using the settings from above, it follows that our cell state will be

$$c_t = [1, 1, 1, 1, 1, \text{delayed cumulated score}]^T.$$

Output o_t We want o_t to store flags of the current word given the delayed information stored in h_{t-1}. That is, we want

$$o_t = [I_{\text{bad}}, I_{\text{good}}, I_{\text{not}}, I_{\text{not bad}}, I_{\text{not good}}, 0]^T.$$

To do so, we need $W_{io}x_t$ to handle the current word, and $W_{ho}h_{t-1}$ to determine if the previous word is "not". Then we set

$$W_{io} = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$W_{ho} = \begin{bmatrix} 0 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $W_{io}x_t = [100I_{\text{bad}}, 100I_{\text{good}}, 100I_{\text{not}}, 100I_{\text{bad}}, 100I_{\text{good}}, 0]^T$,
and $W_{ho}h_{t-1} = [-100I'_{\text{not}}, -100I'_{\text{not}}, 0, 100I'_{\text{not}}, 100I'_{\text{not}}, 0]^T$ where I'_{not} is the "not" flag at $t-1$. Then, we set

$$b_{ho} = [-50, -50, -50, -150, -150, -50]^T.$$

The bias values are chosen this way to selectively activate elements of o_t based on specific inputs. For example, consider the phrase "not bad," where we want $o_t = [0, 0, 0, 1, 0, 0]^T$. Given that:

$$W_{io}x_t = [100, 0, 0, 100, 0, 0]^T,$$

$$W_{ho}h_{t-1} = [-100, -100, 0, 100, 100, 0]^T,$$

we have

$$W_{io}x_t + W_{ho}h_{t-1} = [0, -100, 0, 200, 100, 0]^T.$$

Since $\sigma(0) = 0.5$, without additional scaling, multiple elements might be activated or stay in the intermediate state. To ensure only the intended element activates, we add a large negative bias to reduce the unwanted elements' values, thereby ensuring only the correct position passes through the sigmoid function. These bias values were obtained by iterative testing across various cases to achieve selective activation in o_t.

Hidden h_t Lastly, we update the hidden state at last using the function $h_t = o_t \odot \tanh(c_t)$. As the last state getting updated, h_t stores the information of the previous step.

Result Validation

We use Python to validate if our configuration is correct

```python
### Author: Yuxuan Zhang
### Date: 10/24/2024
import numpy as np
def sigmoid(x):
    return 1 / (1 + np.exp(-x))

def tanh(x):
    return np.tanh(x)

def lstm(sequence):
    # Define the one-hot encoding
    word_dict = {
        "bad": np.array([1, 0, 0, 0]),
        "good": np.array([0, 1, 0, 0]),
        "not": np.array([0, 0, 1, 0]),
        "uh": np.array([0, 0, 0, 1])
    }

    # Initialize h_0 and c_0
    h_t = np.zeros(6)
    c_t = np.zeros(6)

    # For simplicity, set i_t and f_t to constants
    i_t = np.ones(6)
    f_t = np.array([0, 0, 0, 0, 0, 1])

    # Cell Gate matrices
    W_ig = np.zeros((6, 4))
    W_hg = np.array([
        [0, 0, 0, 0, 0, 0],
        [0, 0, 0, 0, 0, 0],
        [0, 0, 0, 0, 0, 0],
        [0, 0, 0, 0, 0, 0],
        [0, 0, 0, 0, 0, 0],
        [-100, 100, 0, 100, -100, 0]
    ])

    b_ig = np.array([100, 100, 100, 100, 100, 0])
    b_hg = np.zeros(6)

    # Output gate matrices
    W_io = np.array([
        [100, 0, 0, 0],
        [0, 100, 0, 0],
        [0, 0, 100, 0],
        [100, 0, 0, 0],
        [0, 100, 0, 0],
        [0, 0, 0, 0]
    ])
    W_ho = np.array([
        [0, 0, -100, 0, 0, 0],
        [0, 0, -100, 0, 0, 0],
        [0, 0, 0, 0, 0, 0],
        [0, 0, 100, 0, 0, 0],
        [0, 0, 100, 0, 0, 0],
        [0, 0, 0, 0, 0, 0]
    ])

    b_io = np.zeros(6)
    b_ho = np.array([-50, -50, -50, -150, -150, -50] )
```

```
61
62      for word in sequence:
63          print(f"Processing word: {word}")
64
65          x_t = word_dict[word]
66          g_t = tanh(np.dot(W_ig, x_t) + b_ig + np.dot(W_hg, h_t) + b_hg)
67          c_t = f_t * c_t + i_t * g_t
68          o_t = sigmoid(np.dot(W_io, x_t) + b_io + np.dot(W_ho, h_t) + b_ho)
69          h_t = o_t * tanh(c_t)
70
71          print("Updated delayed total sentiment score:", c_t[-1])
72          print("-" * 50)
73
74  def main():
75      sequence = ["uh", "good", "good", "not", "not", "bad", "bad", "uh"]
76      lstm(sequence)
77
78  if __name__ == "__main__":
79      main()
```

```
d:/NLP-Course-Project/LSTM/LSTM.py
Processing word: uh
Updated delayed total sentiment score: 0.0
--------------------------------------------------
Processing word: good
Updated delayed total sentiment score: 0.0
--------------------------------------------------
Processing word: good
Updated delayed total sentiment score: 1.0
--------------------------------------------------
Processing word: not
Updated delayed total sentiment score: 2.0
--------------------------------------------------
Processing word: not
Updated delayed total sentiment score: 2.0
--------------------------------------------------
Processing word: bad
Updated delayed total sentiment score: 2.0
--------------------------------------------------
Processing word: bad
Updated delayed total sentiment score: 3.0
--------------------------------------------------
Processing word: uh
Updated delayed total sentiment score: 2.0
--------------------------------------------------
```

The sentence we used is "uh good good not not bad bad uh." Note that we added "uh" to the end of the sequence as a placeholder for the delayed sentiment score calculation. In this sequence, there are 2 occurrences of "good," 2 occurrences of "bad," and 1 occurrence of "not bad." Using the sentiment equation, we find the total sentiment score to be

$$1 \cdot 2 - 1 \cdot 2 + 1 \cdot 2 = 2.$$

This result matches the output from our code, which validates our configuration choices.