Econ8107 Assignment 2

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1. Continuing with the CARA Example

(a)

Using the CARA closed-form consumption rule

$$c(x) = \frac{R-1}{R}\,x + \frac{\bar{y}}{R} - \frac{\gamma(R-1)\sigma^2}{2R^2} - \frac{\log(\beta R)}{\gamma(R-1)}$$

we have

$$c(x', s') - c(x, s) = c(x') - c(x) = \frac{R-1}{R}(x' - x).$$

With the cash-in-hands law of motion

$$x' = x + (y(s') - \bar{y}) + \underbrace{\frac{\gamma(R-1)\sigma^2}{2R} + \frac{R\log(\beta R)}{\gamma(R-1)}}_{\text{drift}},$$

it follows that

$$c(x', s') - c(x, s) = \frac{R-1}{R}\big(y(s') - \bar{y}\big) + \frac{\gamma(R-1)^2}{2R^2}\sigma^2 + \frac{1}{\gamma}\log(\beta R).$$

(b)

From part (a), the change in consumption can be written as

$$c(x', s') - c(x, s) = \frac{R-1}{R}\big(y(s') - \bar{y}\big) + \frac{\gamma(R-1)^2}{2R^2}\sigma^2 + \frac{1}{\gamma}\log(\beta R).$$

consumption follows a random walk with drift.

Hall (1978) predicts that in complete market, consumption is approximately a random walk with zero drift

$$c_{t+1} = c_t + \text{innovation}.$$

Here we generally have a nonzero drift: when $\beta R \geq 1$, drift is positive.

In our model, predictable income changes do not affect consumption change in this model: only unexpected income shock $y_{t+1} - \bar{y}$ moves Δc_{t+1}, with loading

$$\Delta c_{t+1} = \frac{R-1}{R}(y_{t+1} - \bar{y}) + \text{drift}.$$

(c)

We assume this is a representative agent model in general equilibrium, and we assume that in GE model, the interest rate R is constant and is determined by asset market clearing:

$$a_t^{\text{supply}} = A(R),$$

where a_t^{supply} come from individuals and $A(R)$ is the net supply of the risk-free asset, which only depends only on R. Market clearing requires that for all t,

Since R is constant in a stationary equilibrium, $A(R)$ is time-invariant, hence

$$\mathbb{E}[a_{t+1} - a_t] = A(R) - A(R) = 0.$$

With CARA, the optimal consumption rule is linear, $c(x) = \frac{R-1}{R}x + \kappa$, so

$$a_t = x_t - c(x_t) = \frac{1}{R}x_t - \kappa, \qquad \Rightarrow \qquad \mathbb{E}[a_{t+1} - a_t] = \frac{1}{R}\mathbb{E}[x_{t+1} - x_t].$$

Therefore $\mathbb{E}[x_{t+1} - x_t] = 0$. Since $\Delta c_{t+1} = \frac{R-1}{R}(x_{t+1} - x_t)$, we have

$$\mathbb{E}[\Delta c_{t+1}] = \frac{R-1}{R}\mathbb{E}[x_{t+1} - x_t] = 0,$$

so consumption is a random walk without drift in general equilibrium.

(d)

Let income be

$$\begin{aligned} y_{t+1} = w_{t+1} + \eta_{t+1}, \quad w_{t+1} = \phi w_t + (1-\phi)\bar{w} + \epsilon_{t+1}, \\ \eta_{t+1} \sim N(0, \sigma_\eta^2), \quad \epsilon_{t+1} \sim N(0, \sigma_\epsilon^2). \end{aligned}$$

To capture the law of motion of y, we need an extra state variable: w.

The Bellman equation is

$$\begin{aligned} v(x, w) &= \max_a \left\{ u(x-a) + \beta\, \mathbb{E}\big[v(x', w') \mid w\big] \right\}, \quad \text{s.t.} \\ x' &= Ra + w' + \eta' \\ w' &= \phi w + (1-\phi)\bar{w} + \epsilon' \end{aligned}$$

Guess. Guess exponential-affine value and linear consumption:

$$v(x, w) = -\frac{1}{\gamma}\exp\big(-\hat{A}x - \hat{D}w - \hat{B}\big), \qquad c(x, w) = Ax + Dw + B,$$

so

$$a(x, w) = x - c(x, w) = (1-A)x - Dw - B.$$

Envelope. Envelope implies $v_x(x,w) = u'(c(x,w))$. [1]

Since $u'(c) = e^{-\gamma c}$, we have

$$u'(c(x,w)) = e^{-\gamma(Ax+Dw+B)} = \exp(-\gamma Ax - \gamma Dw - \gamma B).$$

and we have

$$v_x(x,w) = \frac{\hat{A}}{\gamma}\exp(-\hat{A}x - \hat{D}w - \hat{B}),$$

matching coefficients for all (x,w), take log on both sides, we have:

$$\log(\frac{\hat{A}}{\gamma}) - \hat{A}x - \hat{D}w - \hat{B} = -\gamma Ax - \gamma Dw - \gamma B,$$

which implies

$$\hat{A} = \gamma A, \qquad \hat{D} = \gamma D, \qquad \hat{B} = \gamma B - \log(\frac{\hat{A}}{\gamma}) = \gamma B - \log A,$$

Euler equation. The Euler equation is

$$u'(c(x,w)) = \beta R\,\mathbb{E}[u'(c(x',w'))].$$

We know LHS is:

$$u'(c(x,w)) = e^{-\gamma(Ax+Dw+B)}.$$

The RHS is

$$\begin{aligned}\beta R\,\mathbb{E}[u'(c(x',w'))] &= \beta R\,\mathbb{E}\left[e^{-\gamma(Ax'+Dw'+B)}\right]\\ &= \beta R\,e^{-\gamma B}\,\mathbb{E}\left[e^{-\gamma(Ax'+Dw')}\right].\end{aligned}$$

We can prove that $Ax' + Dw' \mid (x,w)$ follows $\mathcal{N}(\mu(x,w), \Sigma)$, where:

$$\begin{aligned}\mu(x,w) &= AR(1-A)x + \Big(A(\phi - RD) + D\phi\Big)w + (A+D)(1-\phi)\bar{w} - ARB,\\ \Sigma &= (A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2.\end{aligned}$$

[1] Let $a^*(x,w)$ be the optimal choice and define $c(x,w) = x - a^*(x,w)$. Then

$$v(x,w) = u\big(x - a^*(x,w)\big) + \beta\,\mathbb{E}\Big[v\big(Ra^*(x,w) + w' + \eta',\, w'\big) \mid w\Big].$$

Differentiate both sides with respect to x:

$$\begin{aligned}v_x(x,w) &= u'(c)\,(1 - a_x^*) + \beta\,\mathbb{E}\Big[v_x(x',w')\cdot\frac{\partial x'}{\partial x} \mid w\Big] + \beta\,\mathbb{E}\Big[v_w(x',w')\cdot\frac{\partial w'}{\partial x} \mid w\Big]\\ &= u'(c)\,(1 - a_x^*) + \beta\,\mathbb{E}\big[v_x(x',w')\cdot Ra_x^* \mid w\big]\,,\end{aligned}$$

where we used $\frac{\partial w'}{\partial x} = 0$ and $\frac{\partial x'}{\partial x} = Ra_x^*$. The FOC for a is

$$-u'(c) + \beta R\,\mathbb{E}\big[v_x(x',w') \mid w\big] = 0 \quad\Longleftrightarrow\quad u'(c) = \beta R\,\mathbb{E}\big[v_x(x',w') \mid w\big]\,.$$

Substituting into the expression for v_x gives

$$v_x(x,w) = u'(c)\,(1 - a_x^*) + a_x^*\cdot u'(c) = u'(c),$$

so the envelope condition is

$$\boxed{v_x(x,w) = u'\big(c(x,w)\big)}.$$

[2]

Hence, we have

$$\mathbb{E}\Big[e^{-\gamma(Ax'+Dw')} \mid x, w\Big] = \exp\!\left(-\gamma\mu(x,w) + \frac{\gamma^2}{2}\Sigma\right).$$

Plugging back into the RHS,

$$\begin{aligned}\beta R\,\mathbb{E}[u'(c(x',w')) \mid x,w] &= \beta R\, e^{-\gamma B}\,\mathbb{E}\Big[e^{-\gamma(Ax'+Dw')} \mid x,w\Big] \\ &= \beta R\,\exp\!\left(-\gamma B - \gamma\mu(x,w) + \frac{\gamma^2}{2}\Sigma\right).\end{aligned}$$

Equating LHS and RHS and taking logs yields

$$-\gamma(Ax + Dw + B) = \log(\beta R) - \gamma B - \gamma\mu(x,w) + \frac{\gamma^2}{2}\Sigma.$$

Cancel $-\gamma B$ on both sides and substitute $\mu(x,w)$:

$$\begin{aligned}-\gamma Ax - \gamma Dw = \log(\beta R) - \gamma\Big[&AR(1-A)x + \big(A(\phi - RD) + D\phi\big)w \\ &+ (A+D)(1-\phi)\bar{w} - ARB\Big] + \frac{\gamma^2}{2}\Big((A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2\Big).\end{aligned}$$

Since this identity must hold for all (x,w), we match coefficients on x and w and the constant term:

Coefficient on x:

$$A = AR(1-A) \iff 1 = R(1-A) \iff A = \frac{R-1}{R}.$$

[2] We know the law of motion is

$$x' = Ra + w' + \eta', \qquad w' = \phi w + (1-\phi)\bar{w} + \epsilon',$$

and under the conjectured policy $c(x,w) = Ax + Dw + B$ we have

$$a = x - c = (1-A)x - Dw - B.$$

Substitute a into x':

$$x' = R(1-A)x - RDw - RB + w' + \eta'.$$

Therefore

$$\begin{aligned}Ax' + Dw' &= A\Big(R(1-A)x - RDw - RB + w' + \eta'\Big) + Dw' \\ &= AR(1-A)x + A(\phi - RD)w - ARB + (A+D)w' + A\eta' \\ &= AR(1-A)x + \Big(A(\phi - RD) + D\phi\Big)w + \underbrace{(A+D)(1-\phi)\bar{w} - ARB}_{\text{constant}} \\ &\quad + (A+D)\epsilon' + A\eta'.\end{aligned}$$

Since $\epsilon' \sim N(0,\sigma_\epsilon^2)$ and $\eta' \sim N(0,\sigma_\eta^2)$ are independent, conditional on (x,w) we have

$$Ax' + Dw' \mid (x,w) \sim \mathcal{N}\big(\mu(x,w),\, \Sigma\big),$$

where

$$\mu(x,w) = AR(1-A)x + \Big(A(\phi - RD) + D\phi\Big)w + (A+D)(1-\phi)\bar{w} - ARB,$$

and

$$\Sigma = (A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2.$$

Coefficient on w:

$$D = A(\phi - RD) + D\phi \quad \Longleftrightarrow \quad D(R-\phi) = \phi A \quad \Longleftrightarrow \quad D = \frac{\phi A}{R-\phi}.$$

Constant term:

$$0 = \log(\beta R) - \gamma\Big((A+D)(1-\phi)\bar{w} - ARB\Big) + \frac{\gamma^2}{2}\Big((A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2\Big),$$

which determines B:

$$B = \frac{(A+D)(1-\phi)\bar{w}}{AR} - \frac{1}{\gamma AR}\log(\beta R) - \frac{\gamma}{2AR}\Big((A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2\Big).$$

(One may further simplify this expression by substituting $A = \frac{R-1}{R}$ and $D = \frac{\phi A}{R-\phi}$.)

From the coefficient-matching conditions, we obtain

$$A = \frac{R-1}{R}, \qquad D = \frac{\phi A}{R-\phi} = \frac{\phi(R-1)}{R(R-\phi)}.$$

From the envelope restrictions,

$$\hat{A} = \gamma A = \gamma\frac{R-1}{R}, \qquad \hat{D} = \gamma D = \gamma\frac{\phi(R-1)}{R(R-\phi)}.$$

Therefore the consumption policy is

$$\boxed{c(x,w) = Ax + Dw + B = \frac{R-1}{R}x + \frac{\phi(R-1)}{R(R-\phi)}w + B.}$$

The law of motion for cash-in-hands is

$$\boxed{x' = R(1-A)x - RDw - RB + w' + \eta' = x + \frac{\phi(1-\phi)}{R-\phi}w + (1-\phi)\bar{w} - RB + \epsilon' + \eta'.}$$

(e)

Using the policy $c(x,w) = Ax + Dw + B$, we have

$$c(x',w') - c(x,w) = A(x'-x) + D(w'-w).$$

From the income process,

$$w' - w = (\phi-1)w + (1-\phi)\bar{w} + \epsilon',$$

and from the cash-in-hands law of motion (from part (d)),

$$x' - x = \frac{\phi(1-\phi)}{R-\phi}w + (1-\phi)\bar{w} - RB + \epsilon' + \eta'$$

Therefore,

$$\begin{aligned} c(x',w') - c(x,w) &= A\left(\frac{\phi(1-\phi)}{R-\phi}w + (1-\phi)\bar{w} - RB + \epsilon' + \eta'\right) + D\left((\phi-1)w + (1-\phi)\bar{w} + \epsilon'\right) \\ &= \text{predictable component in } w + (A+D)\epsilon' + A\eta'. \end{aligned}$$

We know that:

$$A + D = \frac{R-1}{R} + \frac{\phi(R-1)}{R(R-\phi)} = \frac{R-1}{R-\phi} > A = \frac{R-1}{R},$$

Thus consumption responds more to unexpected persistent shocks than to unexpected transitory shocks. Intuitively, a persistent shock ϵ affects income not only in the current period but also in future periods (via w_{t+1}), so it has a larger impact on consumption than a transitory shock η that only affects current income.

When $\phi = 0$, w is i.i.d., then ϵ has same effect as η on consumption.

(f)

Same as in (c), we assume a representative agent model. In general equilibrium we have a constant gross interest rate R, determined by risk-free asset market clearing:

$$a_t^{\text{supply}} = A(R),$$

where a_t^{supply} is asset supply from households and $A(R)$ is the net supply, depending only on R.

Since R is constant in a stationary equilibrium, $A(R)$ is time-invariant, hence

$$\mathbb{E}[a_{t+1} - a_t] = A(R) - A(R) = 0.$$

In this model we have:

$$a_t = x_t - c(x_t, w_t) = (1-A)x_t - Dw_t - B.$$

Taking expectations and differencing,

$$\mathbb{E}[a_{t+1} - a_t] = (1-A)\,\mathbb{E}[x_{t+1} - x_t] - D\,\mathbb{E}[w_{t+1} - w_t].$$

In a stationary cross-section, $\mathbb{E}[w_{t+1} - w_t] = 0$, hence asset-market clearing implies

$$\mathbb{E}[x_{t+1} - x_t] = 0.$$

Using the cash-in-hands law of motion from part (d),

$$\begin{aligned} x_{t+1} - x_t &= \frac{\phi(1-\phi)}{R-\phi} w_t - RB + (1-\phi)\bar{w} + \epsilon_{t+1} + \eta_{t+1}, \\ &= \frac{\phi(1-\phi)}{R-\phi}(w_t - \bar{w}) - RB + \left(\frac{\phi(1-\phi)}{R-\phi} + (1-\phi)\right)\bar{w} + \epsilon_{t+1} + \eta_{t+1}. \end{aligned}$$

and stationarity implies $\mathbb{E}[w_t - \bar{w}] = 0$ and $\mathbb{E}[\epsilon_{t+1}] = \mathbb{E}[\eta_{t+1}] = 0$, so

$$0 = \mathbb{E}[x_{t+1} - x_t] = -RB + \left(\frac{\phi(1-\phi)}{R-\phi} + (1-\phi)\right)\bar{w}.$$

Therefore the general equilibrium interest rate must satisfy

$$\begin{aligned} RB &= \left(\frac{\phi(1-\phi)}{R-\phi} + (1-\phi)\right)\bar{w} = \frac{R(1-\phi)}{R-\phi}\bar{w} \\ \implies B &= \frac{(1-\phi)}{R-\phi}\bar{w} \end{aligned}$$

From part (d), the constant-term matching condition is

$$0 = \log(\beta R) - \gamma\Big((A+D)(1-\phi)\bar{w} - AR\,B\Big) + \frac{\gamma^2}{2}\Big((A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2\Big).$$

From part (e), we have

$$A = \frac{R-1}{R}, \qquad A+D = \frac{R-1}{R-\phi}, \qquad AR = R-1.$$

Moreover, the GE no-drift condition implies

$$B = \frac{1-\phi}{R-\phi}\,\bar{w}.$$

Hence,

$$(A+D)(1-\phi)\bar{w} = \frac{(R-1)(1-\phi)}{R-\phi}\bar{w} = AR \cdot \frac{1-\phi}{R-\phi}\bar{w} = AR\,B,$$

so the mean term cancels:

$$(A+D)(1-\phi)\bar{w} - AR\,B = 0.$$

Therefore the equilibrium interest rate condition simplifies to

$$\boxed{\log(\beta R) = -\frac{\gamma^2}{2}\Big((A+D)^2\sigma_\epsilon^2 + A^2\sigma_\eta^2\Big)}.$$

Plugging in A and D,

$$\boxed{\log(\beta R) = -\frac{\gamma^2}{2}\Big(\frac{(R-1)^2}{(R-\phi)^2}\sigma_\epsilon^2 + \frac{(R-1)^2}{R^2}\sigma_\eta^2\Big).}$$

In the case (c) discussed in class, we have condition:

$$\frac{\gamma(R-1)\sigma^2}{2R} + \frac{R}{\gamma(R-1)}\log(\beta R) = 0 \quad \Longleftrightarrow \quad \boxed{\log(\beta R) = -\frac{\gamma^2(R-1)^2}{2R^2}\sigma^2}.$$

If we set $\phi = 0$ and $\sigma_\epsilon^2 = 0$, then these two conditions coincide.
Relative to the (c) case, the RHS of (f) now contains thwo terms:

- $(A+D)^2\sigma_\epsilon^2$: precautionary saving from persistent risk ϵ pushes down the equilibrium interest rate.
- $A^2\sigma_\eta^2$: precautionatory saving from transitory risk η also pushes down the equilibrium interest rate.

2. Aiyagari and changes in the wage

(a)

The Bellman equation is

$$V(z,s;\omega) = \sup_{a'\geq 0}\left\{u(z-a') + \beta\,\mathbb{E}\big[V(z',s';\omega) \mid s\big]\right\}, \qquad z' = Ra' + \omega\ell(s'),$$

where z is cash-in-hands and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

Step 1: define the value of an arbitrary feasible plan. Let $\pi = \{a'_t(\cdot)\}_{t\geq 0}$ be any feasible plan for the problem $(z, s; \omega)$ (so $a'_t \geq 0$ for all t), and let the induced paths satisfy

$$c_t = z_t - a'_t, \qquad z_{t+1} = Ra'_t + \omega\ell(s_{t+1}).$$

Define its lifetime utility as

$$J(z, s; \omega; \pi) \equiv \mathbb{E}\Big[\sum_{t=0}^{\infty} \beta^t u(c_t) \,\Big|\, z_0 = z, s_0 = s\Big].$$

Then by definition,

$$V(z, s; \omega) = \sup_{\pi} J(z, s; \omega; \pi).$$

Step 2: scale the plan. Fix $\lambda > 0$. Given any feasible plan π at $(z, s; \omega)$, define the scaled plan $\tilde{\pi}$ at $(\lambda z, s; \lambda\omega)$ by

$$\tilde{a}'_t \equiv \lambda a'_t \qquad \forall t.$$

Because $a'_t \geq 0 \Rightarrow \tilde{a}'_t \geq 0$, the scaled plan $\tilde{\pi}$ is feasible at $(\lambda z, s; \lambda\omega)$. Moreover, the induced paths satisfy for all t,

$$\tilde{c}_t = \lambda z_t - \tilde{a}'_t = \lambda(z_t - a'_t) = \lambda c_t, \qquad \tilde{z}_{t+1} = R\tilde{a}'_t + \lambda\omega\ell(s_{t+1}) = \lambda(Ra'_t + \omega\ell(s_{t+1})) = \lambda z_{t+1}.$$

Step 3: compare lifetime utilities. By CRRA homogeneity, $u(\lambda c) = \lambda^{1-\gamma}u(c)$, hence

$$J(\lambda z, s; \lambda\omega; \tilde{\pi}) = \mathbb{E}\Big[\sum_{t\geq 0} \beta^t u(\tilde{c}_t)\Big] = \mathbb{E}\Big[\sum_{t\geq 0} \beta^t u(\lambda c_t)\Big] = \lambda^{1-\gamma}\mathbb{E}\Big[\sum_{t\geq 0} \beta^t u(c_t)\Big] = \lambda^{1-\gamma}J(z, s; \omega; \pi).$$

Step 4: take suprema Taking the supremum over feasible plans yields

$$V(\lambda z, s; \lambda\omega) = \sup_{\tilde{\pi}} J(\lambda z, s; \lambda\omega; \tilde{\pi}) = \sup_{\pi} \lambda^{1-\gamma}J(z, s; \omega; \pi) = \lambda^{1-\gamma}\sup_{\pi} J(z, s; \omega; \pi) = \lambda^{1-\gamma}V(z, s; \omega).$$

Step 5: policy scaling. Let $a(z \mid \omega, R)$ be an optimal policy at $(z, s; \omega)$. The scaled policy $\lambda a(z \mid \omega, R)$ is feasible at $(\lambda z, s; \lambda\omega)$ and attains the scaled value, hence it is optimal there:

$$\boxed{a(\lambda z \mid \lambda\omega, R) = \lambda\, a(z \mid \omega, R)}.$$

Since $c(z \mid \omega, R) = z - a(z \mid \omega, R)$, we also have

$$\boxed{c(\lambda z \mid \lambda\omega, R) = \lambda\, c(z \mid \omega, R)}.$$

(b)

The equation characterizes a stationary distribution of cash-in-hands z. Given (ω, R) and the optimal saving policy $a(\cdot \mid \omega, R)$, next period cash-in-hands is

$$z' = R\, a(\tilde{z} \mid \omega, R) + w\ell(s'),$$

where $s \in S$ is the income (labor) shock with probability $\pi(s)$.

The left-hand side $F(z \mid \omega, R)$ is the stationary CDF of cash-in-hands: the probability that (current) cash-in-hands is less than or equal to z. In a stationary distribution, this is also the CDF of next period cash-in-hands z'.

The right-hand side computes the CDF of z' at z by:

- drawing last period cash-in-hands $\tilde{z}$ from $F(\cdot \mid \omega, R)$,
- applying the policy $a(\tilde{z} \mid \omega, R)$ and forming $z' = R\, a(\tilde{z} \mid \omega, R) + w\ell(s)$; the indicator $\mathbf{1}\{z' \le z\}$ equals 1 if the inequality holds and 0 otherwise,
- averaging over shocks s.

Thus the condition

$$F(z \mid \omega, R) = \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(\tilde{z} \mid \omega, R) + w\ell(s) \le z\}\, dF(\tilde{z} \mid \omega, R)$$

means that the distribution reproduces itself under the induced Markov transition for z:

$$F = T_{\omega,R}(F).$$

(c)

From (b) we know stationary CDF $F(\cdot \mid \omega, R)$ satisfying

$$F(z \mid \omega, R) = \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(\tilde{z} \mid \omega, R) + \omega\ell(s) \le z\}\, dF(\tilde{z} \mid \omega, R).$$

Let the wage increase to $\lambda\omega$ with $\lambda > 0$, and denote the new stationary CDF by $F(\cdot \mid \lambda\omega, R)$.

From part (a), we have

$$a(\lambda\tilde{z} \mid \lambda\omega, R) = \lambda\, a(\tilde{z} \mid \omega, R).$$

Step 1: show $F(\lambda z \mid \lambda\omega, R) = F(z \mid \omega, R)$. Let $F(\cdot \mid \omega, R)$ be a stationary CDF for wage ω. Define the candidate CDF under wage $\lambda\omega$ by

$$\tilde{F}(z) \equiv F\left(\frac{z}{\lambda} \,\middle|\, \omega, R\right).$$

Equivalently, if $Z_0 \sim F(\cdot \mid \omega, R)$ then $\tilde{Z} \equiv \lambda Z_0$ has CDF $\tilde{F}$.

Let T_ω denote the law-of-motion operator on CDFs:

$$(T_\omega G)(z) \equiv \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(\tilde{z} \mid \omega, R) + \omega\ell(s) \le z\}\, dG(\tilde{z}).$$

Stationarity of $F(\cdot \mid \omega, R)$ means $F = T_\omega F$.

Now evaluate $(T_{\lambda\omega}\tilde{F})(z)$:

$$\begin{aligned}
(T_{\lambda\omega}\tilde{F})(z) &= \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(\tilde{z} \mid \lambda\omega, R) + \lambda\omega\ell(s) \le z\}\, d\tilde{F}(\tilde{z}) \\
&= \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(\lambda z_0 \mid \lambda\omega, R) + \lambda\omega\ell(s) \le z\}\, dF(z_0) \qquad (\tilde{z} = \lambda z_0,\ \tilde{z} \sim \tilde{F}) \\
&= \sum_{s\in S} \pi(s) \int \mathbf{1}\{\lambda\big(R\, a(z_0 \mid \omega, R) + \omega\ell(s)\big) \le z\}\, dF(z_0) \qquad (\text{by (a): } a(\lambda z_0 \mid \lambda\omega, R) = \lambda a(z_0 \mid \omega, R)) \\
&= \sum_{s\in S} \pi(s) \int \mathbf{1}\{R\, a(z_0 \mid \omega, R) + \omega\ell(s) \le z/\lambda\}\, dF(z_0) \\
&= (T_\omega F)(z/\lambda) = F(z/\lambda \mid \omega, R) = \tilde{F}(z),
\end{aligned}$$

where the penultimate equality uses stationarity $F = T_\omega F$. Hence $\tilde{F}$ is stationary under wage[3] $\lambda\omega$, and therefore

$$F(z \mid \lambda\omega, R) = \tilde{F}(z) = F\Big(\frac{z}{\lambda}\Big|\,\omega, R\Big) \quad \Longrightarrow \quad \boxed{F(\lambda z \mid \lambda\omega, R) = F(z \mid \omega, R)}.$$

Step 2: aggregate savings scale with λ. Define aggregate (mean) savings in the stationary distribution by

$$\bar{a}(\omega, R) \equiv \int a(z \mid \omega, R)\, dF(z \mid \omega, R).$$

Then

$$\begin{aligned}
\bar{a}(\lambda\omega, R) &= \int a(z \mid \lambda\omega, R)\, dF(z \mid \lambda\omega, R) \\
&= \int a(\lambda\hat{z} \mid \lambda\omega, R)\, dF(\lambda\hat{z} \mid \lambda\omega, R) \qquad (z = \lambda\hat{z}) \\
&= \int \lambda a(\hat{z} \mid \omega, R)\, dF(\hat{z} \mid \omega, R) \qquad \text{(by Step 1 and policy scaling)} \\
&= \lambda\, \bar{a}(\omega, R).
\end{aligned}$$

Therefore, with R fixed, a wage increase by a factor λ raises long-run aggregate savings by the same factor:

$$\boxed{\bar{a}(\lambda\omega, R) = \lambda\, \bar{a}(\omega, R)}.$$

(d)

My answer won't change. Parts (a)–(c) used that scaling preserves feasibility.

When $\phi = 0$, $a' \geq 0$ implies $\lambda a' \geq 0$, so the scaled plan is feasible, then we can prove that policy function scale with λ, and the stationary distribution also scale with λ.

If ϕ is instead the natural borrowing limit, the constraint is

$$a' \geq -\left(\frac{w_{\min} l}{R-1}\right)$$

If a' is feasible at (ω, R), i.e. $a' \geq -\left(\frac{w_{\min} l}{R-1}\right)$, scaling preserveS feasibility:

$$\lambda a' \;\geq\; \lambda \cdot \left(-\frac{w_{\min} l}{R-1}\right) = -\left(\frac{\lambda w_{\min} l}{R-1}\right),$$

Hence, the scaled plan is feasible at $(\lambda\omega, R)$, and we can still prove that policy function scale with λ, and the stationary distribution also scale with λ.

[3]Let $Z_0 \sim F(\cdot \mid \omega, R)$ and define $\tilde{Z} \equiv \lambda Z_0$, so $\tilde{Z} \sim \tilde{F}(\cdot)$ where $\tilde{F}(z) = F(z/\lambda \mid \omega, R)$. For any measurable function g, we can write integrals as expectations:

$$\int g(\tilde{z})\, d\tilde{F}(\tilde{z}) = \mathbb{E}[g(\tilde{Z})] = \mathbb{E}[g(\lambda Z_0)] = \int g(\lambda z_0)\, dF(z_0).$$

Applying this with

$$g(\tilde{z}) \equiv \mathbf{1}\{Ra(\tilde{z} \mid \lambda\omega, R) + \lambda\omega\ell(s) \leq z\}$$

yields

$$\int \mathbf{1}\{Ra(\tilde{z} \mid \lambda\omega, R) + \lambda\omega\ell(s) \leq z\}\, d\tilde{F}(\tilde{z}) = \int \mathbf{1}\{Ra(\lambda z_0 \mid \lambda\omega, R) + \lambda\omega\ell(s) \leq z\}\, dF(z_0).$$

3. A Ricardian equivalence in Aiyagari's model

(a)

The household choose a' to smooth consumption, because we do not have default choices in this model. To avoid negative consumption in the future, a' must be greater than or equal the present value of future net income, suppose the agent constantly receives the lowest endowment $y_{\min}$:

$$a' \geq -\left\{\sum_{t=0}^{\infty} \frac{y_{\min} - \tau}{(1+r)^t}\right\} = -\left(\frac{y_{\min} - \tau}{r}\right) = -\left(\frac{y_{\min}}{r} - D\right).$$

Thus, the natural borrowing limit is

$$\boxed{\phi = -\left(\frac{y_{\min}}{r} - D\right)}.$$

(b)

Fix the gross interest rate $R = 1 + r$ and note that the government sets $\tau = rD = (R-1)D$

The household Bellman equation with debt D is

$$\begin{aligned} V^D(a,s) &= \max\left\{u(c) + \beta\,\mathbb{E}\left[V^D(a',s') \mid s\right]\right\}, \qquad \text{s.t.} \\ a' &\geq -\left(\frac{y_{\min}}{r} - D\right) \\ c &= y(s) + Ra - \tau - a'. \end{aligned}$$

Shift the state. Define shifted assets

$$b \equiv a - D, \qquad b' \equiv a' - D \quad \Longleftrightarrow \quad a = b + D,\ a' = b' + D.$$

Substitute into consumption:

$$\begin{aligned} c &= y(s) + R(b+D) - (R-1)D - (b'+D) \\ &= y(s) + Rb - b'. \end{aligned}$$

Thus consumption is independent of D in the (b, b') variables. The borrowing constraint becomes

$$b' = a' - D \ \geq \ -\frac{y_{\min}}{r},$$

which is also independent of D.

Bellman equation in shifted variables. Define $\widetilde{V}(b,s) \equiv V^D(b+D,s)$. Then $\widetilde{V}$ satisfies

$$\begin{aligned} \widetilde{V}(b,s) &= \max\left\{u(c) + \beta\,\mathbb{E}\left[\widetilde{V}(b',s') \mid s\right]\right\} \qquad \text{s.t.} \\ b' &\geq -\frac{y_{\min}}{r} \\ c &= y(s) + Rb - b'. \end{aligned}$$

No matter what D is, the Bellman equation is the same. Hence the optimal policy for b and the consumption rule do not depend on D.

One-for-one shift in assets. Let $b^*(\cdot)$ denote the optimal saving policy in shifted units. Then

$$a'^*(a, s; D) = b^*(a - D, s) + D,$$

Suppose we increase D to $D + \Delta D$. We know that $b^*(\cdot)$ does not change:

$$\begin{aligned}
& b^*(a - D - \Delta D, s) = b^*(a - D, s) \\
\iff\quad & b^*(a - D - \Delta D, s) + D + \Delta D = b^*(a - D, s) + D + \Delta D \\
\iff\quad & a'^*(a, s; D + \Delta D) = a'^*(a, s; D) + \Delta D.
\end{aligned}$$

Thus, if D increases, a'^* increases one to one.

(c)

From part (b), for any fixed R we can define shifted assets $b \equiv a - D$ and obtain a conumer problem that is independent of D. Hence the optimal policy for b and the induced stationary distribution of b are independent of D.

Denote this stationary distribution by G_R:

$$b \sim G_R \qquad \text{(does not depend on } D\text{, but depend on } R\text{).}$$

Since $a = b + D$, the stationary distribution of a under debt level D is just a translation of G_R:

$$F_D(\hat{a}) = Pr(a \leq \hat{a}) = Pr(b \leq \hat{a} - D) = G_R(\hat{a} - D) \quad \iff \quad F_D(a) = G_R(a - D).$$

Therefore aggregate asset holdings satisfy

$$\int a\, dF_D(a) = \int (b + D)\, dG_R(b) = D + \int b\, dG_R(b).$$

market clearing requires

$$\int a\, dF_D(a) = D$$

Thus we have

$$\int b\, dG_R(b) = 0$$

Then we can solve for the equilibrium interest rate R^* by plugging the optimal policy and stationary distribution into the asset market clearing condition.

Since G_R depends on R but not on D, the solution R^* does not depend on D either. Therefore increasing D does not change the equilibrium price R nor any real allocations (in b-units); it only shifts asset holdings one-for-one via $a = b + D$.

(d)

Yes. The neutrality result relies on the fact that the government policy shifts:

1. the household budget through $\tau = rD$ and

2. the natural borrowing limit through

$$a' \geq -\Big(\frac{y_{\min}}{r} - D\Big).$$

With this endogenous debt limit, defining $b \equiv a - D$ makes

$$c + a' = y + Ra - \tau \quad \Longrightarrow \quad c + b' = y + Rb, \qquad a' \geq -\Big(\frac{y_{\min}}{r} - D\Big) \Longrightarrow b' \geq -\frac{y_{\min}}{r},$$

so the feasible set and Bellman equation in (b, b') are independent of D.

If instead the borrowing constraint did *not* move with D (e.g. $a' \geq \underline{a}$ fixed), then after shifting $b = a - D$ we would get

$$b' = a' - D \geq \underline{a} - D,$$

so the constraint would depend on D and the translated problem would no longer be invariant. In that case, increasing D can change which agents are constrained and alter policies and aggregates, so Ricardian neutrality generally fails.

4. Incomplete Markets and Unemployment. A Numerical Analysis

(a)

We know that $R = 1.04$, the natural borrowing limit is

$$\phi = -\frac{wl_{\min}}{R-1} = -\frac{0.5}{0.04} = -12.5$$