Lecture Notes on Heterogeneous Agent Macroeconomics[1]

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[1]These notes were transcribed and expanded from Manuel Amador's handwritten lecture notes for the Spring 2026 Macroeconomics class at the University of Minnesota. If you find any errors, please contact amador@umn.edu.

Guide to These Notes

Throughout these notes, colored boxes are used to distinguish different types of content. Here is what each one looks like and when it is used.

Definitions

Formal definitions and key formulations appear in boxes like this one—white background with a dark blue frame.

Result: Theorems and Propositions

Main theoretical results, theorems, and propositions are highlighted with a light blue background.

Computational Procedures

Algorithms and numerical methods are presented in boxes with a purple frame.

Proof sketches and verification arguments appear in gray boxes like this one, set in a smaller font.

Side notes, intuition, examples, and commentary appear in yellow/amber boxes like this one.

Links to other papers, extensions, and connections to the broader literature appear in green boxes.

Central equations are framed in a blue box without a title.

Contents

Chapter 1

Revisiting Complete Markets

This chapter revisits the complete-markets benchmark covered in previous classes. In an endowment economy with Arrow–Debreu or sequential Arrow securities, competitive equilibria are Pareto optimal. Under Markov shocks and endowments, the equilibrium admits a recursive formulation in which the current shock is the only state variable.

1.1 An Exchange Economy

Consider a pure exchange economy with the following primitives:

- I agents, indexed $i \in \{1, 2, \ldots, I\}$.
- **Events.** At each date $t = 0, 1, 2, \ldots$ a shock $s_t \in \mathcal{S}$ is realised, where $\mathcal{S}$ is a finite set. The realisation s^t is *public information*.
- **Histories.** A date-t history is the vector $s^t = (s_0, s_1, \ldots, s_t)$.
- **Probabilities.** Let $\pi(s^t)$ denote the unconditional probability of history s^t, and $\pi(s^t \mid s^\tau)$ the conditional probability of s^t given an earlier history s^τ (with $\tau \leq t$).
- **Endowments.** Agent i receives $y^i(s^t) \geq 0$ units of the consumption good in history s^t.

Consumption allocations and utility. A *consumption allocation* for agent i is a non-negative plan $\{c^i(s^t)\}_{s^t, t\geq 0}$ with $c^i(s^t) \geq 0$ for all s^t. The lifetime expected utility of agent i is

$$U(c^i) \;=\; \sum_{t=0}^{\infty} \sum_{s^t} \beta^t\, \pi(s^t)\, u\big(c^i(s^t)\big)\,, \tag{1.1}$$

where $\beta \in (0,1)$ is the common discount factor and $u : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, strictly concave, continuously differentiable period utility function satisfying the Inada conditions $\lim_{c\to 0^+} u'(c) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$. The utility function is common across agents.

Feasibility. An allocation $\{c^i\}_{i=1}^I$ is *feasible* if

$$\sum_{i=1}^{I} c^i(s^t) \;\le\; \sum_{i=1}^{I} y^i(s^t) \qquad \forall\, s^t.$$

1.2 Complete Markets

There are two equivalent formulations of trade with complete markets.

Arrow–Debreu (time-0 trading). All trade occurs at $t = 0$. There is a market for every contingent claim (t, s^t). Let $q(s^t \mid s_0)$ denote the price at $t = 0$ (in units of the date-0 consumption good) of one unit of consumption delivered in history s^t.

> The definition of Arrow–Debreu competitive equilibrium and the two fundamental welfare theorems are covered in the TA session. The key implication is that every competitive equilibrium allocation is Pareto optimal, and vice versa (with appropriate transfers).

Sequential trading with Arrow securities. Instead of trading all claims at $t = 0$, agents trade *one-period* contingent claims (Arrow securities) at each node of the event tree. Let $Q(s_{t+1} \mid s^t)$ denote the price, at history s^t, of a claim that pays one unit of consumption if tomorrow's event is s_{t+1}. This price is related to the Arrow–Debreu prices by

$$Q(s_{t+1} \mid s^t) \;=\; \frac{q((s^t, s_{t+1}) \mid s_0)}{q(s^t \mid s_0)}.$$

Let $a^i(s^t, s_{t+1})$ denote the quantity of the Arrow security for event s_{t+1} purchased by agent i at history s^t. Equivalently, let $a^i(s^t)$ denote the claim that agent i brings *into* history s^t (i.e., the asset maturing at s^t).

Figure 1.1 illustrates the event tree for $\mathcal{S} = \{s_H, s_L\}$. At each node the agent purchases Arrow securities for every possible successor event.

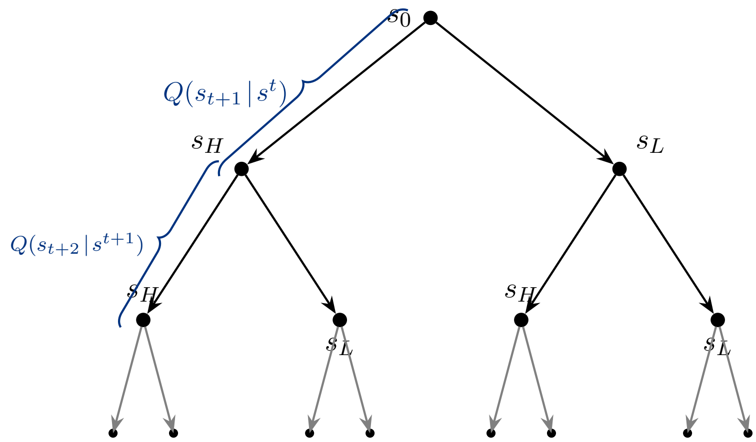


Figure 1.1: Event tree for $\mathcal{S} = \{s_H, s_L\}$. At every node, Arrow security prices $Q(s_{t+1} \mid s^t)$ determine the cost of claims on next-period consumption.

Budget constraint. At each history s^t, agent i faces the sequential budget constraint

$$c^i(s^t) + \sum_{s_{t+1} \in \mathcal{S}} Q(s_{t+1} \mid s^t)\, a^i(s^t, s_{t+1}) \;\le\; y^i(s^t) + a^i(s^t). \tag{1.2}$$

Here $a^i(s^t)$ is wealth brought into node s^t and $a^i(s^t, s_{t+1})$ is wealth carried into the successor node (s^t, s_{t+1}). Note the underlying portfolio problem: the agent must choose how to allocate savings across $|\mathcal{S}|$ Arrow securities.

Natural debt limits. Take a competitive equilibrium in the Arrow–Debreu sense. Define the *natural debt limit* of agent i at history s^t as

$$\bar{A}^i(s^t) \;=\; \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q(s^\tau \mid s^t)\, y^i(s^\tau), \tag{1.3}$$

where $q(s^\tau \mid s^t) = \dfrac{q(s^\tau \mid s_0)}{q(s^t \mid s_0)}$ is the relative price of history s^τ in terms of the history-s^t consumption good. $\bar{A}^i(s^t)$ represents the maximum amount that agent i can repay starting from history s^t, given that consumption must remain non-negative. It equals the present value (at node s^t) of all future endowments.

Borrowing constraint. The borrowing constraint limits the agent's debt position:

$$-\,a^i(s^{t+1}) \;\leq\; \bar{A}^i(s^{t+1}). \tag{1.4}$$

Both the budget constraint (1.2) and the borrowing constraint (1.4) are needed to keep consumption well-defined. Without the debt limit, agents could run Ponzi schemes; without the budget constraint, there is no resource discipline.

Household i's problem. Agent i chooses $\{c^i(s^t), a^i(s^t, s_{t+1})\}$ for all histories to maximise (1.1) subject to the budget constraint (1.2), borrowing constraint (1.4), and $c^i(s^t) \geq 0$, for all s^t.

Initial conditions. The initial wealth distribution $\{a_0^i\}_{i=1}^I$ satisfies $\sum_{i=1}^I a_0^i = 0$ (claims are in zero net supply).

1.3 Definition of Sequential Equilibrium

Definition: Sequential Markets Competitive Equilibrium

A **sequential markets competitive equilibrium** consists of consumption allocations $\{c^i(s^t)\}_{i=1}^I$, asset policies $\{a^i(s^t, s_{t+1})\}_{i=1}^I$, Arrow security prices $\{Q(s_{t+1} \mid s^t)\}$, and natural debt limits $\{\bar{A}^i(s^t)\}_{i=1}^I$, such that:

(i) **Household optimality.** For each i, given prices Q, borrowing limits $\bar{A}^i$, and initial wealth a_0^i, the plans $\{c^i\}$ and $\{a^i\}$ solve the household's problem at every history.

(ii) **Goods market clearing (feasibility).** $\displaystyle\sum_{i=1}^I c^i(s^t) \leq \sum_{i=1}^I y^i(s^t) \quad \forall\, s^t.$

(iii) **Financial market clearing.** $\displaystyle\sum_{i=1}^I a^i(s^t, s_{t+1}) = 0 \quad \forall\, s^t,\; s_{t+1}.$

(iv) **Debt limits.** $\{\bar{A}^i(s^t)\}$ satisfy equation (1.3) for all i and s^t.

A fundamental result (see Ljungqvist and Sargent, 2018, Ch. 8) is that the set of sequential markets equilibrium allocations with natural debt limits coincides with the set of Arrow–Debreu equilibrium allocations. Both yield Pareto optimal allocations.

1.4 Recursive Equilibrium

Goal. Suppose we already have an equilibrium allocation (in the Arrow–Debreu or sequential sense). Our goal is to write the associated equilibrium recursively: that is, to remove the dependence of all equilibrium objects on the full history s^t and express everything as functions of a small set of *state variables.*

Assumptions. We impose two structural conditions:

(a) **Markov shocks.** The process $\{s_t\}$ is a first-order Markov chain with transition probability $\pi(s' \mid s)$:
$$\pi\big((s^{t-1}, s_t, s_{t+1}) \mid (s^{t-1}, s_t)\big) = \pi(s_{t+1} \mid s_t).$$

(b) **Markov endowments.** Endowments depend only on the current shock, not on the full history: $y^i(s^t) = y^i(s_t)$.

History independence of Pareto optimal allocations.

Result: Pareto Optimal Allocations

Under assumptions (a)–(b), in any Pareto optimal allocation the consumption allocation satisfies
$$c^i(s^t) = c^i(s_t) \qquad \forall\, i,\ s^t. \tag{1.5}$$

Proof sketch. In a Pareto optimum the planner solves $\max \sum_i \lambda_i U(c^i)$ subject to feasibility. The first-order condition equates weighted marginal utilities across agents at each node: $\lambda_i u'(c^i(s^t)) = \mu(s^t)$ for all i. Since the aggregate endowment $\sum_i y^i(s^t) = \sum_i y^i(s_t)$ depends only on s_t (by assumption (b)), the multiplier μ depends only on s_t, and therefore so does each c^i.

Price simplification. Since the equilibrium allocation depends only on s_t, the Arrow security price simplifies. The Euler equation at an interior optimum reads $Q(s_{t+1} \mid s^t) = \beta \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \pi(s^{t+1} \mid s^t)$. The Inada condition $\lim_{c\to 0^+} u'(c) = +\infty$ guarantees that $c^i(s_t) > 0$ for all i and s_t in the Pareto optimal allocation, so the denominator is well-defined and the Euler equation holds with equality. Applying (1.5) and assumption (a), prices become functions of the current state alone:

$$Q(s' \mid s) \;=\; \beta\,\frac{u'(c^i(s'))}{u'(c^i(s))}\,\pi(s' \mid s). \tag{1.6}$$

Since markets are complete and the allocation is Pareto optimal, the pricing kernel is the same regardless of which agent i is used in (1.6).

Recursive borrowing limits. Under the Markov structure, the natural debt limits satisfy the recursion

$$\bar{A}^i(s) \;=\; y^i(s) + \sum_{s' \in \mathcal{S}} Q(s' \mid s)\,\bar{A}^i(s'). \tag{1.7}$$

Given the prices $Q(s'|s)$, this is a system of $|\mathcal{S}|$ linear equations in the unknowns $\{\bar{A}^i(s)\}_{s\in\mathcal{S}}$ for each agent i.

Recursive budget constraint. The sequential budget constraint simplifies to

$$c^i(s) + \sum_{s' \in \mathcal{S}} Q(s' \mid s)\,\hat{a}^i(s') \;\leq\; y^i(s) + a^i(s), \tag{1.8}$$

where $a^i(s)$ is wealth brought into the current state and $\hat{a}^i(s')$ is wealth carried into state s'. Knowing the equilibrium from the Arrow–Debreu problem delivers $c^i(s)$ for all i and $Q(s'|s)$. Plugging these into (1.8) then pins down the asset positions $a^i(s)$. At every s, the wealth distribution $\{a^i(s)\}_{i=1}^I$ is the same independent of history.

Household's Bellman equation.

Household i's Bellman Equation

$$v^i(a, s) \;=\; \max_{c,\,\hat{a}(\cdot)} \left\{ u(c) + \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s)\, v^i\big(\hat{a}(s'),\, s'\big) \right\} \tag{1.9}$$

subject to:

$$c + \sum_{s' \in \mathcal{S}} Q(s' \mid s)\,\hat{a}(s') = y^i(s) + a,$$

$$-\,\hat{a}(s') \leq \bar{A}^i(s') \quad \forall\, s',$$

$$c \geq 0.$$

The solution yields policy functions $c = h^i(a, s)$ and $\hat{a}(s') = g^i(a, s, s')$. Note that g^i specifies a *separate* asset choice for each successor state s'.

Evolution of the wealth distribution. Given the policy functions g^i, define the map that tracks the evolution of the cross-sectional wealth distribution:

$$G\big(\{a^i\}_{i=1}^I,\, s,\, s'\big) \;=\; \big\{g^i(a^i, s, s')\big\}_{i=1}^I .$$

Here $\{a^i\}_{i=1}^I$ is the current wealth distribution (the aggregate state), s is the current shock, s' is tomorrow's shock, and the output is next period's wealth distribution.

1.4.1 Definition of Recursive Competitive Equilibrium

Definition: Recursive Competitive Equilibrium

A **recursive competitive equilibrium** consists of prices $Q(s' \mid s)$, natural debt limits $\{\bar{A}^i(s)\}_{i=1}^I$, value functions $\{v^i(a,s)\}_{i=1}^I$, policy functions $\{h^i(a,s),\ g^i(a,s,s')\}_{i=1}^I$, an initial wealth distribution $\{a_0^i\}_{i=1}^I$, wealth distributions $\{a^i(s)\}_{i=1}^I$ for all $s \in \mathcal{S}$, and an evolution map G, such that:

(i) **Debt limits.** For each i, $\bar{A}^i(s)$ solves the borrowing limit recursion (1.7).

(ii) **Household optimality.** For each i, h^i and g^i are the optimal policy functions for the Bellman equation (1.9) with value function v^i.

(iii) **Asset market clearing.** $\displaystyle\sum_{i=1}^I a^i(s) = 0 \quad \forall\, s \in \mathcal{S}.$

(iv) **Consistent wealth evolution.** $\{a^i(s')\}_{i=1}^I = G\big(\{a^i(s)\}_{i=1}^I,\, s,\, s'\big)$ for all $s,\, s' \in \mathcal{S}$, where $G(\{a^i\}, s, s') = \{g^i(a^i, s, s')\}_{i=1}^I$.

In this model the aggregate state variable is just s. Because the Pareto optimal allocation depends only on the current shock, the wealth distribution at the beginning of any period, $\{a^i(s)\}_{i=1}^I$, is itself only a function of s. Therefore, the wealth distribution does *not* constitute an additional state variable for the economy; the value of s alone is sufficient. This stands in contrast to heterogeneous-agent models with incomplete markets, where the wealth distribution is typically an additional (infinite-dimensional) state variable (as in Krusell and Smith, 1998).

Verification. Even though the recursive structure simplifies the problem greatly, one must still verify the *asset evolution consistency* condition:

$$a^i(s') = g^i\big(a^i(s),\, s,\, s'\big) \qquad \forall\ s,\, s',\, i.$$

This is the fixed-point condition that closes the recursive equilibrium.

Chapter 2

Incomplete Markets: Partial Equilibrium

This chapter introduces incomplete markets, restricting trade to a single risk-free bond. In a two period model, when marginal utility is convex (prudence), households save more than under complete markets. However, in the infinite-horizon setting (with $\beta R \geq 1$), household's wealth diverges almost surely to infinity for general utility functions.

2.1 Two-Period Model

Incomplete markets means that only a risk-free bond is available for trade. This is an *exogenous restriction* on the set of assets. We study a **partial equilibrium** setting: the gross real interest rate $R > 0$ is taken as given.

Environment. Consider a single household living for two periods ($t = 0, 1$). There is a finite set of states $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$ with probabilities $\pi(s) > 0$ for all $s \in \mathcal{S}$, where $\sum_{s \in \mathcal{S}} \pi(s) = 1$. The household receives a deterministic endowment $y_0 > 0$ in period 0 and a stochastic endowment $y(s) > 0$ in period 1.

Household's Problem. The household chooses savings a (units of the bond) to maximize expected lifetime utility:

$$\max_a \left\{ u(y_0 - a) + \beta \sum_{s \in \mathcal{S}} \pi(s)\, u(Ra + y(s)) \right\}$$

where u is strictly increasing, strictly concave, and twice continuously differentiable, with $\beta \in (0,1)$ being the discount factor. Period-0 consumption is $c_0 = y_0 - a$ and period-1 consumption in state s is $c_1(s) = Ra + y(s)$. The idea is simple: the household splits its period-0 endowment between consumption today and savings in a bond that pays gross return R. In period 1, it consumes the bond payoff plus whatever income is realized.

Constraints. Non-negativity of consumption requires:

$$\begin{aligned} c_0 &= y_0 - a \geq 0, \\ c_1(s) &= Ra + y(s) \geq 0 \quad \forall\, s \in \mathcal{S}. \end{aligned} \tag{2.1}$$

The first constraint says the household cannot save more than its current endowment. The second says that, even in the worst state, consumption must remain non-negative. Constraint (2.1) implies an endogenous lower bound on borrowing:

$$-a \leq \min_{s \in \mathcal{S}} \left\{ \frac{y(s)}{R} \right\} = \frac{y(s_1)}{R},$$

where we order states so that $y(s_1) \leq y(s_2) \leq \cdots \leq y(s_N)$. In words: the household can borrow at most the present value of its worst-case future income.

Euler Equation. Assume Inada conditions ($u'(0) = \infty$, $u'(\infty) = 0$), so the optimal solution a^* is interior. The first-order condition yields the **Euler equation**:

$$u'(y_0 - a^*) = \beta R \sum_{s \in \mathcal{S}} \pi(s)\, u'(Ra^* + y(s)). \tag{2.2}$$

The left-hand side is the marginal utility of consuming one more unit today. The right-hand side is the expected marginal utility of saving that unit instead: investing it at rate R and consuming the proceeds tomorrow, discounted by β. At the optimum, the household is indifferent at the margin between these two options. Since u is strictly concave, the LHS is strictly increasing in a^* and the RHS is strictly decreasing in a^*, so a^* is unique.

Comparison with Complete Markets. The central question is: *do households save more or less under incomplete markets than under complete markets?* To make this comparison meaningful, suppose there exist Arrow–Debreu state-contingent securities with prices $q(s)$ satisfying two conditions:

(i) The implied risk-free rate equals R: the price of a risk-free bond is $\bar{q} = \sum_{s \in \mathcal{S}} q(s)$, with

$R = 1/\bar{q}$.

(ii) Prices are **actuarially fair**: $\dfrac{q(s_1)}{q(s_2)} = \dfrac{\pi(s_1)}{\pi(s_2)}$ for all $s_1, s_2 \in \mathcal{S}$.

Condition (i) ensures we are comparing at the same interest rate. Condition (ii) says the relative prices of Arrow securities reflect only probabilities, with no risk premium. Together they imply $q(s) = \pi(s)/R$ for all $s \in \mathcal{S}$.

Derivation. Condition (ii) says $q(s) = k\,\pi(s)$ for some constant $k > 0$ (all prices are proportional to probabilities). Substituting into condition (i):

$$\bar{q} = \sum_{s\in\mathcal{S}} q(s) = k \sum_{s\in\mathcal{S}} \pi(s) = k \cdot 1 = k.$$

So $k = \bar{q} = 1/R$, and therefore $q(s) = \pi(s)/R$ for all $s \in \mathcal{S}$. □

Complete Markets Problem. Under complete markets with prices $q(s)$, the household solves:

$$\max_{c_0,\, c_1(\cdot)} \left\{ u(c_0) + \beta \sum_{s\in\mathcal{S}} \pi(s)\, u(c_1(s)) \right\}$$

subject to:

$$c_0 + \sum_{s\in\mathcal{S}} q(s)\, c_1(s) = y_0 + \sum_{s\in\mathcal{S}} q(s)\, y(s), \tag{2.3}$$

$$c_0 \geq 0, \quad c_1(s) \geq 0 \quad \forall\, s.$$

The key difference from the incomplete markets problem is that the household can now trade a separate security for each state, allowing it to choose different consumption levels across states. With actuarially fair prices, the first-order conditions for $c_1(s)$ and $c_1(s')$ give $u'(c_1(s)) = u'(c_1(s'))$ for all s, s', so consumption is perfectly smoothed across states:

$$c_1(s) = c_1 \quad \forall\, s \in \mathcal{S}.$$

This is the hallmark of complete markets with actuarially fair prices: the household fully insures away all idiosyncratic risk. Define total savings under complete markets as $\hat{a} \equiv y_0 - c_0$. The budget constraint (2.3) becomes $c_0 + c_1/R = y_0 + \bar{y}/R$, where $\bar{y} \equiv \sum_s \pi(s) y(s)$, so second-period consumption is $c_1 = R\hat{a} + \bar{y}$. The Euler equation under complete markets is:

$$u'(y_0 - \hat{a}) = \beta R\, u'(\bar{y} + R\hat{a})\,. \tag{2.4}$$

Notice the crucial difference: the RHS involves u' evaluated at a *single* certainty-equivalent value $\bar{y} + R\hat{a}$, whereas the incomplete markets Euler equation (2.2) involves the *expectation* of u' across different states. Whether this difference makes the household save more or less depends on the curvature of u'.

Precautionary Savings and Prudence. Define the function:

$$f(z) \equiv u'(y_0 - z) - \beta R\, u'(\bar{y} + Rz)\,.$$

Note that f is **strictly increasing** in z: $f'(z) = -u''(y_0 - z) - \beta R^2\, u''(\bar{y} + Rz) > 0$ (both terms are positive because $u'' < 0$). The complete markets FOC (2.4) gives $f(\hat{a}) = 0$.

From the incomplete markets Euler equation (2.2):

$$0 = u'(y_0 - a^*) - \beta R \sum_s \pi(s)\, u'(y(s) + Ra^*). \tag{2.5}$$

Result: Savings Comparison: $u''' \leq 0$

Suppose u' is concave (equivalently, $u''' \leq 0$). Then $\hat{a} \geq a^*$: the household saves **more** under complete markets than under incomplete markets.

Proof. Since u' is concave, Jensen's inequality gives:

$$\sum_s \pi(s)\, u'(y(s) + Ra^*) \leq u'\left(\sum_s \pi(s)\, y(s) + Ra^*\right) = u'(\bar{y} + Ra^*).$$

Therefore, from (2.5):

$$0 = u'(y_0 - a^*) - \beta R \sum_s \pi(s)\, u'(y(s) + Ra^*) \geq u'(y_0 - a^*) - \beta R\, u'(\bar{y} + Ra^*) = f(a^*).$$

So $f(a^*) \leq 0 = f(\hat{a})$. Since f is strictly increasing, $a^* \leq \hat{a}$. $\square$

Result: Precautionary Savings: $u''' \geq 0$ (Prudence)

Suppose u' is convex (equivalently, $u''' \geq 0$). Then $a^* \geq \hat{a}$: the household saves **more** under incomplete markets than under complete markets. This is the **precautionary savings** motive.

Proof. When u' is convex, Jensen's inequality reverses:

$$\sum_s \pi(s)\, u'(y(s) + Ra^*) \geq u'(\bar{y} + Ra^*).$$

The same argument gives $f(a^*) \geq 0 = f(\hat{a})$, so $a^* \geq \hat{a}$. □

The condition $u''' > 0$ is called **prudence** (Kimball, 1990, *Econometrica*). It holds for all CRRA utility functions with risk aversion $\gamma > 0$, for CARA utility, and more generally for any utility with decreasing absolute risk aversion (DARA).

2.2 Infinite Horizon Model

We now extend the analysis to an infinite-horizon setting, which is the workhorse framework for quantitative incomplete markets models. The household maximizes expected discounted utility:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t\, u(c_t)\right] = \sum_{t=0}^{\infty} \sum_{s^t \in \mathcal{S}^t} \beta^t\, \pi(s^t)\, u(c(s^t)),$$

where $s^t = (s_0, s_1, \ldots, s_t)$ denotes the history of shocks up to period t, and $\pi(s^t)$ is the probability of history s^t.

Endowment. Income $y(s)$ takes values in $\mathcal{S} = \{s_1, \ldots, s_N\}$ with $0 < y(s_1) < y(s_2) < \cdots < y(s_N)$, drawn i.i.d. with probabilities $\pi(s) > 0$ for all $s \in \mathcal{S}$.

Recursive Formulation. Let x denote **cash in hand** (beginning-of-period wealth), which is the sum of current income and the gross return on savings carried over from last period. The Bellman equation is:

$$\boxed{v(x) = \max_{\substack{c,\, a \\ c+a=x \\ c\geq 0,\, a\geq -\phi}} \left\{ u(c) + \beta \sum_{s \in \mathcal{S}} \pi(s)\, v(Ra + y(s)) \right\},}$$

where R is the fixed gross interest rate and $\phi \geq 0$ is the borrowing limit. The budget constraint is $c + a = x$, non-negativity requires $c \geq 0$, and the borrowing limit imposes $a \geq -\phi$. Tomorrow's cash in hand is $x' = Ra + y(s)$, the return on today's savings plus tomorrow's income realization.

The Borrowing Limit. Define the net interest rate $r \equiv R - 1$.

Result: The Natural Borrowing Limit

Assume $r > 0$. Let ϕ be a borrowing limit. The household can never borrow more than $y(s_1)/r$.

Proof. Suppose toward a contradiction that $\phi > y(s_1)/r$. Consider a household at maximum debt, $a_0 = -\phi$, that receives the lowest income shock $y(s_1)$ in every subsequent period. Since $c_t \geq 0$, the budget constraint implies:

$$a_{t+1} = Ra_t + y(s_1) - c_t \leq Ra_t + y(s_1).$$

Iterating this inequality forward from $a_0 = -\phi$:

$$a_T \leq -R^T\phi + y(s_1)\frac{R^T - 1}{r},$$

so the household's debt after T periods satisfies:

$$-a_T \geq R^T\left(\phi - \frac{y(s_1)}{r}\right) + \frac{y(s_1)}{r}.$$

Since $\phi > y(s_1)/r$ and $R = 1 + r > 1$, the right-hand side grows without bound. In particular, $-a_T > \phi$ for all $T \geq 1$: even after a single period, the household's debt necessarily exceeds ϕ, violating the borrowing constraint $a \geq -\phi$. Therefore, any valid borrowing limit must satisfy $\phi \leq y(s_1)/r$. □

There are two important cases. When $\phi = y(s_1)/r$, this is the **natural borrowing limit**: the tightest limit consistent with non-negative consumption under all possible shock realizations. When $\phi < y(s_1)/r$, this is an **ad-hoc borrowing limit**, an exogenously imposed tighter constraint. A common special case is $\phi = 0$, meaning households cannot borrow at all.

Properties of the Value Function.

By standard results (Stokey, Lucas, and Prescott, 1989, Ch. 9), the value function v is strictly increasing, strictly concave, and differentiable on the interior of the state space. These properties follow from the assumptions on u (strictly increasing, strictly concave, Inada conditions) and the contraction mapping theorem applied to the Bellman operator. Strict concavity of v is important: it ensures that the optimal policy is well-defined and that the envelope theorem applies.

Optimal Policy. Let $a^0(x)$ be the unique solution to the *interior* first-order condition:

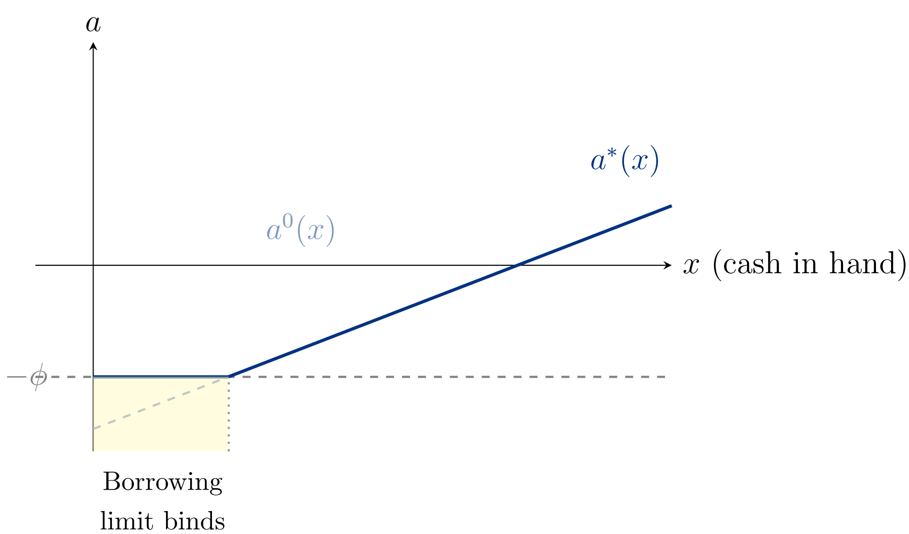
$$u'(x - a^0(x)) = \beta R \sum_{s \in \mathcal{S}} \pi(s)\, v'(Ra^0(x) + y(s)).$$

This is the savings level the household would choose if the borrowing constraint were absent. Note that $a^0(x)$ is strictly increasing in x (by the implicit function theorem and strict concavity of both u and v): wealthier households save more.

The optimal savings rule accounts for the borrowing constraint:

$$a^*(x) = \max\left\{a^0(x),\ -\phi\right\}.$$

The household either saves according to the unconstrained optimum $a^0(x)$, or, if that would violate the borrowing limit, borrows up to the maximum allowed amount $-\phi$.



The Euler Inequality and Supermartingale Property. At the optimum, using $a^*(x)$, the first-order condition holds with *inequality*:

$$u'(x - a^*(x)) \geq \beta R \sum_{s \in \mathcal{S}} \pi(s)\, v'(Ra^*(x) + y(s)),$$

with equality when $a^*(x) > -\phi$ (the borrowing constraint does not bind) and strict inequality when $a^*(x) = -\phi$ (it binds). When the constraint binds, the household *would like* to borrow more but cannot, so it consumes less than it would under the unconstrained optimum; its marginal utility today exceeds the discounted expected marginal utility of future consumption.

Using the **envelope condition** $v'(x) = u'(x - a^*(x))$, this becomes:

$$v'(x_t) \geq \beta R \, \mathbb{E}_t[v'(x_{t+1})] \, .$$

This is the key result. When $\beta R \geq 1$, it implies $v'(x_t) \geq \mathbb{E}_t[v'(x_{t+1})]$: the process $\{v'(x_t)\}$ is a **non-negative supermartingale**. It is non-negative because v is increasing ($v' > 0$), and it weakly decreases in expectation.

If ϕ is the natural borrowing limit, the borrowing constraint never actually binds. The reason is that, under Inada conditions, approaching the natural limit would send marginal utility to infinity, which is never optimal. In this case, the Euler equation holds with **equality**: $v'(x_t) = \beta R \, \mathbb{E}_t[v'(x_{t+1})]$. When additionally $\beta R = 1$, the process $\{v'(x_t)\}$ is a non-negative **martingale**.

The Case $\beta R \geq 1$: Wealth Diverges. The supermartingale property has a powerful consequence when combined with a classical result from probability theory.

Result: Doob's Martingale Convergence Theorem (MCT)

A non-negative supermartingale converges almost surely to a finite random variable.

With Doob's result, we are now able to show the following:

Result: Wealth Divergence under $\beta R \geq 1$

If $\beta R \geq 1$, then cash in hand diverges almost surely: $x_t \to \infty$ a.s.

Proof. Let $\mathcal{S}$ be the finite set of states. Define $\mathcal{S}^0$ as the set of all infinite sequences $\{s_1^0, s_2^0, s_3^0, \ldots\}$ that are *eventually constant*: there exists $T < \infty$ such that $s_t^0 = s_T^0$ for all $t > T$. The set $\mathcal{S}^0$ is countable (it is a countable union of finite sets). Since income is i.i.d. with $\pi(s) > 0$ for all s, the probability of any eventually-constant sequence is zero: $\Pr(s \in \mathcal{S}^0) = 0$. Let $\mathcal{S}^1$ be the complement of $\mathcal{S}^0$, so $\Pr(s \in \mathcal{S}^1) = 1$.

By the MCT, $v'(x_t)$ converges almost surely to a finite limit. Since v is strictly concave, v' is continuous and strictly decreasing. Therefore, convergence of $v'(x_t)$ implies convergence of x_t. Suppose, toward a contradiction, that $x_t \to x^\infty < \infty$ along some sequence in $\mathcal{S}^1$ (which must be the case, as $\Pr(s \in \mathcal{S}^1) = 1$). Since the optimal policy a^* is continuous, $a^*(x_t) \to a^\infty \equiv a^*(x^\infty)$.

But the law of motion gives:

$$x_{t+1} = Ra^*(x_t) + y_{t+1}.$$

In the limit, $x^\infty = Ra^\infty + y_t$ for all t sufficiently large. This requires y_t to converge, which contradicts the fact that y_t is i.i.d. with support on multiple values (we are on $\mathcal{S}^1$, where the sequence is *not* eventually constant).

Therefore, x_t cannot converge to a finite value. Since $v'(x_t)$ converges (so x_t cannot oscillate), we must have $x_t \to \infty$ almost surely.

Consequence: Savings $a_t \to \infty$ and consumption $c_t \to \infty$ almost surely. There is no stationary distribution. □

Chamberlain and Wilson (2000, *Review of Economic Dynamics*) extend these results to the case of non-i.i.d. shocks and stochastic interest rates.

Implications for General Equilibrium. The divergence result has a sharp implication: in general equilibrium, the interest rate must adjust so that $\beta R < 1$. If $\beta R \geq 1$, every household accumulates unbounded wealth, so aggregate asset demand grows without bound. This is inconsistent with a finite net supply of bonds (e.g., zero in a pure exchange economy). The equilibrium interest rate must therefore fall below the rate of time preference to make households willing to hold a finite quantity of assets.

See Aiyagari (1994, *QJE*) and Huggett (1993, *JEDC*).

Consumption Dynamics under $\beta R = 1$. To close, consider the special case $\beta R = 1$ with the natural borrowing limit ($\phi = y(s_1)/r$). As argued above, under the natural borrowing limit the Euler equation holds with equality. Combined with $\beta R = 1$, this gives:

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})]\,. \tag{2.6}$$

That is, marginal utility is a **martingale**: the best forecast of tomorrow's marginal utility is today's marginal utility. This is the stochastic analogue of the permanent income hypothesis: the household has no systematic reason to expect its marginal valuation of consumption to rise or fall. But what can we say about the *level* of consumption? The answer depends on the curvature of u', exactly as in the two-period precautionary savings result.

Result: Consumption is a Submartingale under Prudence

If $\beta R = 1$ (with natural borrowing limit) and $u''' > 0$ (prudence), then:

$$c_t \leq \mathbb{E}_t[c_{t+1}].$$

That is, consumption is a **submartingale**: it is expected to grow over time.

Proof. From (2.6) and the convexity of u' (since $u''' > 0$), Jensen's inequality gives:

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})] \geq u'(\mathbb{E}_t[c_{t+1}]).$$

Since u' is strictly decreasing, this implies $c_t \leq \mathbb{E}_t[c_{t+1}]$. □

The intuition is as follows. Marginal utility is a martingale, but consumption is *not*. Because u' is convex (prudence), mapping from the martingale in marginal utility back to consumption introduces a systematic upward drift. The household, fearing bad shocks disproportionately, keeps consumption low today relative to its expected future level. Even though it does not expect marginal utility to change on average, it *does* expect its consumption to rise on average, precisely because the precautionary motive induces extra saving today.

This consumption growth, combined with the wealth divergence result above, is the fundamental reason why $\beta R = 1$ cannot be sustained in general equilibrium: the economy would need an ever-growing supply of assets to accommodate the growing savings demand. The equilibrium interest rate must fall to $\beta R < 1$, which dampens the precautionary motive just enough for a stationary wealth distribution to exist.

The result above may seem to conflict with the case $u''' \leq 0$ from Section 1, where the household saves *less* under incomplete markets. If $u''' < 0$, Jensen's inequality reverses in (2.6), giving $c_t \geq \mathbb{E}_t[c_{t+1}]$: consumption would be a *supermartingale*, expected to *fall* over time. The reconciliation is straightforward: no utility function that satisfies Inada conditions (no satiation, $u'(0) = \infty$, $u'(\infty) = 0$) can have $u''' < 0$ *globally*. A globally concave, positive, decreasing function cannot go from $+\infty$ to 0. So the case $u''' < 0$ everywhere is simply not compatible with the maintained assumptions of the model.

Appendix: Visualizing Doob's Martingale Convergence Theorem

Doob's Theorem

Doob's Martingale Convergence Theorem states that a non-negative supermartingale converges almost surely to a finite random variable. Formally, if $\{X_t\}$ is a non-negative supermartingale, then there exists a random variable X_∞ such that:

$$X_t \xrightarrow{a.s.} X_\infty \quad \text{as } t \to \infty$$

The limit X_∞ is a *random variable*, not a constant. Different sample paths converge to different limits, depending on the realized history of shocks.

A Discrete Martingale

Consider a simple random walk on states $\{0, 1, 2, \ldots, N\}$ with **multiple absorbing states**. Let $N = 100$ and the absorbing states be:

$$\mathcal{A} = \{0, 25, 50, 75, 100\}$$

Transition probabilities. For non-absorbing states $i \notin \mathcal{A}$:

$$P(X_{t+1} = i + 1 \mid X_t = i) = \tfrac{1}{2}, \qquad P(X_{t+1} = i - 1 \mid X_t = i) = \tfrac{1}{2}$$

For absorbing states $a \in \mathcal{A}$:

$$P(X_{t+1} = a \mid X_t = a) = 1$$

Martingale Property

This process is a martingale because for any non-absorbing state i:

$$\mathbb{E}[X_{t+1} \mid X_t = i] = \tfrac{1}{2}(i + 1) + \tfrac{1}{2}(i - 1) = i = X_t$$

For absorbing states, $\mathbb{E}[X_{t+1} \mid X_t = a] = a$ trivially. Thus the martingale property $\mathbb{E}[X_{t+1} \mid X_t] = X_t$ holds everywhere.

Convergence and the Limiting Random Variable

Since the process is bounded and absorbing states exist, every path eventually gets absorbed. The limit

$$X_\infty \in \{0, 25, 50, 75, 100\}$$

is a random variable whose distribution depends on the starting point X_0.

For X_0 between two adjacent absorbing states $a < b$, the probability of being absorbed at b is:

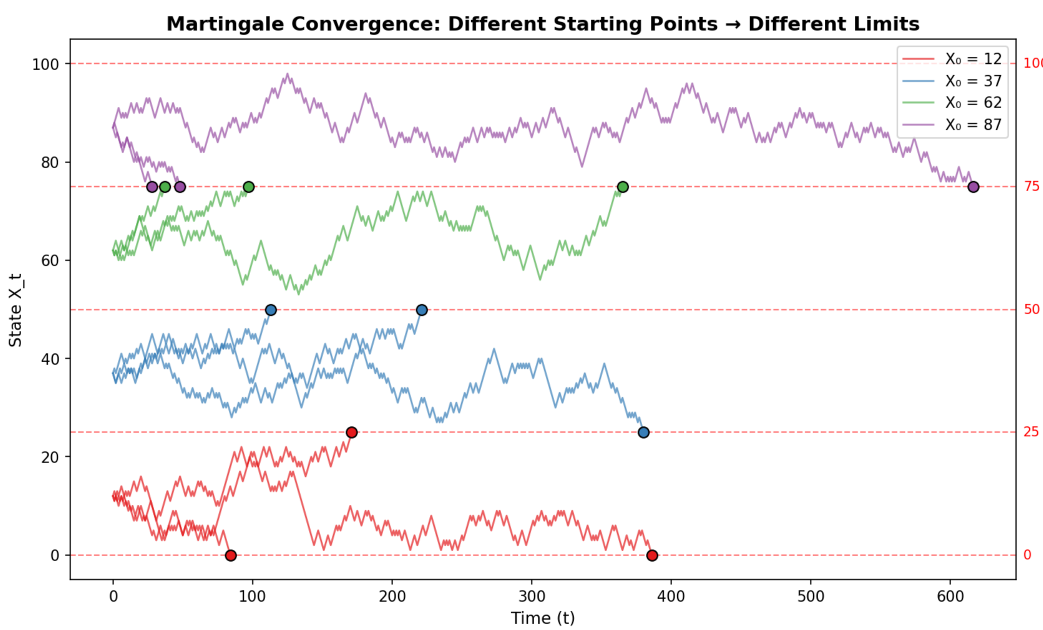
$$\boxed{P(X_\infty = b \mid X_0) = \frac{X_0 - a}{b - a}}$$

Example. Starting at $X_0 = 37$ (between 25 and 50):

$$P(X_\infty = 50 \mid X_0 = 37) = \frac{37 - 25}{50 - 25} = \frac{12}{25} = 48\%$$

Simulation

The figure below shows sample paths from four different starting points. Each path wanders until hitting an absorbing state (dashed red lines). Paths with the same starting point can converge to different limits, illustrating that X_∞ is a random variable.



Why Absorbing States Must Exist

A natural question: does a finite state non-negative supermartingale always have an absorbing state? The answer is **yes**.

Result: Absorbing States

A finite state Markov chain that is a non-negative supermartingale must have at least one absorbing state.

Consider a Markov chain on finite states with values $\{v_1, v_2, \dots, v_n\}$ where $0 \le v_1 < v_2 < \dots < v_n$.
For the supermartingale property at the minimum state v_1:

$$\mathbb{E}[X_{t+1} \mid X_t = v_1] \le v_1$$

But since $X_{t+1} \ge v_1$ always (it's the minimum value), the only way to satisfy this inequality is:

$$\mathbb{E}[X_{t+1} \mid X_t = v_1] = v_1$$

which requires $P(X_{t+1} = v_1 \mid X_t = v_1) = 1$. Thus **the minimum state must be absorbing**. More generally, by Doob's theorem the supermartingale converges a.s. For a finite Markov chain, convergence means eventually settling into a recurrent class. But within a recurrent class with more than one state, the process would keep fluctuating, contradicting convergence. So recurrent classes must be singletons, i.e., absorbing states.

The existence of absorbing states relies on the Markov chain having a *finite* number of states. With infinitely many states (including a continuum), a non-negative supermartingale can converge without ever reaching an absorbing state; the limit is simply a random variable that the process approaches asymptotically. Doob's theorem holds in full generality for any non-negative supermartingale, regardless of whether the state space is finite, countably infinite, or continuous.

Connection to Incomplete Markets

In the income fluctuations problem, when $\beta R \ge 1$ the Euler equation implies that marginal utility $u'(c_t)$ is a non-negative supermartingale:

$$u'(c_t) \ge \beta R \cdot \mathbb{E}_t[u'(c_{t+1})]$$

By Doob's theorem, $u'(c_t)$ converges almost surely. As we saw in class, combined with INADA conditions ($u'(c) \to 0$ as $c \to \infty$), this implies consumption diverges to infinity, and hence assets must also diverge.

The Case $\beta R = 1$: A Subtlety

When $\beta R = 1$ and the borrowing limit is the natural one, the Euler equation holds with equality:

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})]$$

so marginal utility is a **martingale**, not just a supermartingale. By Doob's theorem, $u'(c_t) \to u'_\infty$ almost surely, and the same argument gives $u'_\infty = 0$ a.s., hence $c_t \to \infty$ a.s.

But here is an apparent puzzle: as a martingale, $\mathbb{E}[u'(c_t)] = u'(c_0)$ is **constant** for all t. How can consumption diverge if expected marginal utility doesn't change?

Resolution. While $u'(c_t) \to 0$ for almost every sample path, rare paths with bad shock histories maintain very high marginal utility. These paths become increasingly rare but increasingly extreme, just enough to keep $\mathbb{E}[u'(c_t)]$ constant. This is analogous to the St. Petersburg paradox.

But what about $\mathbb{E}[c_t]$? Here Fatou's lemma resolves the issue.

Fatou's Lemma

If $\{X_t\}$ is a sequence of non-negative random variables, then:

$$\mathbb{E}\left[\liminf_{t\to\infty} X_t\right] \leq \liminf_{t\to\infty} \mathbb{E}[X_t]$$

Since $c_t \geq 0$ and $c_t \to \infty$ a.s., we have $\liminf_{t\to\infty} c_t = \infty$ a.s., so:

$$\infty = \mathbb{E}\left[\liminf_{t\to\infty} c_t\right] \leq \liminf_{t\to\infty} \mathbb{E}[c_t]$$

Therefore $\mathbb{E}[c_t] \to \infty$. The cross-sectional mean of consumption diverges, and by a similar argument so does $\mathbb{E}[a_t]$.

Even with $\beta R = 1$ and the Euler equation holding with equality, where $\mathbb{E}[u'(c_t)]$ remains constant, we still have $\mathbb{E}[c_t] \to \infty$ and $\mathbb{E}[a_t] \to \infty$. A stationary equilibrium with bounded assets is impossible.

Example with CRRA utility. With $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $\gamma > 0$, we have $u'(c) = c^{-\gamma}$. The Euler equation becomes:

$$c_t^{-\gamma} = \mathbb{E}_t[c_{t+1}^{-\gamma}]$$

Since $f(c) = c^{-\gamma}$ is convex, Jensen's inequality gives:

$$\mathbb{E}_t[c_{t+1}^{-\gamma}] \geq (\mathbb{E}_t[c_{t+1}])^{-\gamma}$$

Combining these and using the fact that $x^{-\gamma}$ is decreasing:

$$c_t \leq \mathbb{E}_t[c_{t+1}]$$

So consumption is a **submartingale**: expected consumption increases over time, consistent with $\mathbb{E}[c_t] \to \infty$.

General utility and the role of prudence. The CRRA result extends to any utility with convex marginal utility ($u''' > 0$, i.e., prudence). With the Euler equation $u'(c_t) = \mathbb{E}_t[u'(c_{t+1})]$ and u' convex, Jensen gives:

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})] \geq u'(\mathbb{E}_t[c_{t+1}])$$

Since u' is decreasing, this implies $c_t \leq \mathbb{E}_t[c_{t+1}]$, so consumption is a submartingale.

What if u' were concave ($u''' < 0$)? Then Jensen reverses:

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})] \leq u'(\mathbb{E}_t[c_{t+1}])$$

implying $c_t \geq \mathbb{E}_t[c_{t+1}]$, so consumption would be a supermartingale with $\mathbb{E}[c_t]$ decreasing over time. But Fatou's lemma tells us $\mathbb{E}[c_t] \to \infty$. Contradiction!

Resolution. No utility function can have u' globally concave while satisfying $u'(c) > 0$ for all c. If u' is concave, it lies below its tangent line at any point c_0:

$$u'(c) \leq u'(c_0) + u''(c_0)(c - c_0)$$

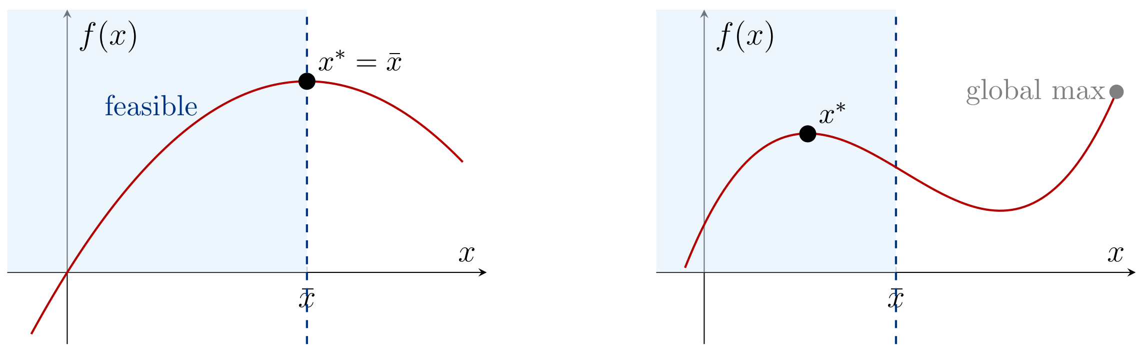
Since $u'' < 0$ (concavity of u), the right side goes to $-\infty$ as $c \to \infty$. Thus $u'(c) < 0$ for large enough c, contradicting positive marginal utility. There is no utility function that satisfies "imprudence" ($u''' < 0$) for the entire positive real line and has marginal utility strictly positive. See the related discussion of this issue in Ljungqvist and Sargent's book.

Appendix: Optimization in Infinite Dimensions

This note is technical, and requires familiarity with functional analysis.

Convex optimization in finite dimensions is relatively straightforward. We write the Lagrangian, take first-order conditions, and obtain necessary and sufficient conditions for optimality. Constraints that are slack at the optimum can simply be dropped: the solution does not change.

(a) Convex problem **(b) Non-convex problem**



(a) $\max\ -x^2 + 4x$ s.t. $x \leq \bar{x} = 2$. Optimum is at $x^* = 2$: constraint binds. Removing it: unconstrained max is also $x^* = 2$.

(b) $\max\ f(x)$ s.t. $x \leq \bar{x}$, f non-convex. Optimum is at an interior local max; the constraint is slack. Removing it reveals the global max at the domain boundary.

Figure 2.1: In finite-dimensional convex problems, slack constraints are irrelevant: they can be dropped without affecting the solution. Without convexity, the optimum can be a local maximum where the constraint is slack, yet removing the constraint changes the solution.

Figure 2.1(a) illustrates the basic intuition: when the problem is convex and a constraint is slack at the optimum, the constraint plays no role and can be removed. Panel (b) shows that without convexity, the constrained optimum can sit at an interior local maximum where the constraint is slack, yet removing it reveals a higher global maximum at the domain boundary. The fact that a non-binding constraint can be removed without affecting the solution to a convex problem can be seen also from the Lagrangian representation: when a multiplier is zero, we are effectively ignoring the constraint when optimizing.

This clean picture breaks down in infinite dimensions. Consider the consumption-savings problem with borrowing limits. The Inada conditions guarantee that optimal consumption is strictly positive in every period, which in turn implies that the borrowing limit is slack at every finite date. The constraint is slack, period by period. Yet we cannot drop it: without the borrowing limit, the household can run a Ponzi scheme, and the optimum ceases to exist.

To understand why, we must be precise about the *space* in which choices live.

The Problem with Natural Borrowing Limits

Consider a deterministic consumption-savings problem with constant income $y > 0$, constant gross interest rate $R > 1$, and borrowing limit $\phi \geq 0$:

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + a_t &\leq y + R\, a_{t-1}, \qquad \forall\, t \geq 0, \\ a_t &\geq -\phi, \qquad \forall\, t \geq 0, \\ c_t &\geq 0, \qquad \forall\, t \geq 0, \end{aligned} \tag{2.7}$$

with a_{-1} given.

where u is bounded, strictly concave with $u'(0) = \infty$ (Inada) and $\beta \in (0, 1)$. We impose the **natural debt limit**:

$$\phi = \frac{y}{R-1},$$

the maximum debt repayable from future income.

In the analysis below we will assume that an optimum to the problem exists, and denote it by (c^*, a^*). The Inada conditions guarantee that consumption will be interior at all times ($c_t^* > 0$), and thus the natural borrowing limit will not be binding at any finite date, $a_t^* > -\phi$ (as long as $a_{-1} > -\phi$, which we assume).

The choice of space

To apply infinite-dimensional optimization, we must choose function spaces for the choice variables $\{c_t\}$ and $\{a_t\}$. This choice determines which economic restrictions are genuine constraints, priced by Lagrange multipliers, and which are structural consequences of the domain. Recall: $\ell^\infty = \{x = (x_0, x_1, \ldots) : \sup_t |x_t| < \infty\}$ and $\ell^1 = \{x : \sum_t |x_t| < \infty\}$. For any of these spaces X, we write X_+ for the positive cone: $X_+ = \{x \in X : x_t \geq 0\; \forall\, t\}$.

Bounded assets: $\{a_t\} \in \ell^\infty$. For any sequence in this set, $a_t/R^t \to 0$ (No-Ponzi is automatically imposed). Moreover, the budget implies, using that $y = (R-1)\phi$,

$$a_s + \phi = R(a_{s-1} + \phi) - c_s \leq R(a_{s-1} + \phi) \qquad (c_s \geq 0).$$

If $a_t + \phi < 0$ for some t, then for all $k \geq 1$,

$$a_{t+k} + \phi \leq R^k(a_t + \phi) \to -\infty,$$

contradicting $\{a_t\} \in \ell^\infty$. Hence feasibility in ℓ^∞ already implies $a_t \geq -\phi$, so the borrowing constraints (2.7) are redundant and can be ignored (a zero multiplier). This is clean but hides the economics: both No-Ponzi and the natural borrowing limit are baked into the domain.

Present-value bounded assets: $\{a_t\} \in \ell^\infty_R$. Define the **present-value weighted space**:[1]

$$\ell^\infty_R \equiv \big\{\{a_t\}_{t\geq 0} : \|a\|_R < \infty\big\}, \qquad \|a\|_R \equiv \sup_{t\geq 0} \frac{|a_t|}{R^t}.$$

This allows asset sequences to grow at rate R. In this larger space:

- **No-Ponzi is not automatic.** The ratio a_t/R^t can converge to a negative constant; a Ponzi scheme is in the domain.
- **Borrowing constraints are not redundant.** The domain does not force $a_t \geq -\phi$.
- **Without borrowing limits, the household finds it optimal to run a Ponzi scheme**: it borrows increasing amounts, consuming arbitrarily much today by pushing debt onto the future indefinitely.

We adopt this formulation. The borrowing limits $a_t + \phi \geq 0$ are the only constraint on assets, and they are embedded in the domain. As we will see, the Lagrange multiplier on this constraint has two components: a countably additive part that prices borrowing at each finite date, and a purely finitely additive part that prices the asymptotic content of the constraint: the **No-Ponzi condition**.

The dual of ℓ^∞_R and the Lagrangian

The Lagrange multipliers live in the dual space of the choice space. The choice space is ℓ^∞_R, so we need to understand its dual. The space ℓ^∞_R is isometrically isomorphic to ℓ^∞ via the rescaling $x_t \mapsto x_t/R^t$. Its topological dual is therefore isomorphic to $ba(\mathbb{N})$, the space of *bounded, finitely additive* (signed) measures on $\mathbb{N}$: any $\nu \in ba(\mathbb{N})$ induces a continuous linear functional on ℓ^∞_R through $x \mapsto \nu(\{x_t/R^t\})$. Concretely, for $x \in \ell^\infty_R$ and $\mu \in ba(\mathbb{N})$, we

[1] The weighted supremum norm is standard in the dynamic programming literature for handling unbounded value functions; see Boyd (1990) and Alvarez and Stokey (1998). We use it here on the *choice space* to make the No-Ponzi condition transparent.

interpret $\langle \mu, x\rangle$ as $\langle \mu, x\rangle = \langle \nu, \{x_t/R^t\}\rangle$, where ν is the element of $ba(\mathbb{N})$ corresponding to μ under the isomorphism.

The crucial point is that the dual $ba(\mathbb{N})$ contains more elements than ℓ^1. This is because $ba(\mathbb{N})$ captures *all* continuous linear functionals on ℓ^∞. Weighted sums $\sum \mu_t\, x_t$ (with $\sum |\mu_t|\, R^t < \infty$) are linear functionals, but so are generalized limits: for instance, the map $x \mapsto \lim_{t\to\infty} x_t$ is a bounded linear functional on convergent sequences that is unchanged by modifying finitely many coordinates (this functional can be extended to all bounded sequences using a *Banach limit*). It is precisely this kind of functional, invisible period by period yet sensitive to the tail, that the finitely additive part of μ represents.

Luenberger's framework. We apply the infinite-dimensional multiplier theorem (Luenberger 1969, Ch. 8, Theorem 1) in concave/maximization form. The choice space is $X = \ell^\infty_R$. The choice set $\Omega = \{a \in \ell^\infty_R : c_t(a) > 0\ \forall\, t\}$, where $c_t(a) = y + Ra_{t-1} - a_t$ (with a_{-1} given), is convex (intersection of open half-spaces) and enforces $c_t > 0$ as a domain restriction rather than a priced constraint. The constraint space is $Z = \ell^\infty_R$ with positive cone $P = (\ell^\infty_R)_+$, which is closed with nonempty norm interior $\mathrm{int}(P) = \{z : \inf_t z_t/R^t > 0\}$. The constraint mapping $G(a) = a + \phi$ is affine, and $G(a) \in P$ encodes $a_t + \phi \geq 0$ for all t. The objective $f(a) = \sum \beta^t u(c_t(a))$ is strictly concave on Ω and real-valued since u is bounded. The dual space is $(\ell^\infty_R)^* \cong ba(\mathbb{N})$ via $x_t \mapsto x_t/R^t$.

For the theorem we need that there exists a plan a such that $G(a)$ is in the interior of P (Slater's condition). For this, we take the constant-consumption plan $\hat{c}_t = \epsilon$ with $\epsilon < (R-1)(a_{-1}+\phi)$ gives. Iterating the budget constraint forward, we get the implied sequence of $\hat{a}$:

$$\frac{\hat{a}_t + \phi}{R^t} \;=\; R(a_{-1}+\phi) - \frac{\epsilon}{R-1}\big(R - R^{-t}\big).$$

This is decreasing in t, so

$$\inf_{t\geq 0} \frac{\hat{a}_t+\phi}{R^t} \;=\; \lim_{t\to\infty} \frac{\hat{a}_t+\phi}{R^t} \;=\; R\left(a_{-1} + \phi - \frac{\epsilon}{R-1}\right) > 0.$$

Hence $G(\hat{a}) \in \mathrm{int}(P)$ and $\hat{a} \in \Omega$. Now we are ready to use Theorem 1 of Chapter 8.3 of Luenberger (1969):

Result: Theorem 1 (Luenberger 1969, Ch. 8)

Let X be a linear vector space, Z a normed space, $\Omega \subset X$ convex, and P the positive cone in Z with nonempty interior. Let f be a real-valued convex functional on Ω and G a convex mapping from Ω into Z. Suppose there exists $x_1 \in \Omega$ with $G(x_1)$ an interior point of $-P$ (Slater's condition). Let

$$\mu_0 = \inf\{f(x) : x \in \Omega,\ G(x) \le \theta\}$$

and assume μ_0 is finite. Then there exists $z_0^* \ge \theta$ in Z^* such that

$$\mu_0 = \inf_{x\in\Omega}\big\{f(x) + \langle G(x),\, z_0^*\rangle\big\}.$$

Furthermore, if the infimum is achieved by $x_0 \in \Omega$ with $G(x_0) \le \theta$, it is also achieved by x_0 above, and $\langle G(x_0),\, z_0^*\rangle = 0$.

We apply this theorem to our concave maximization problem by setting $f \leftarrow -f$ and $G \leftarrow -G$: since our f is concave and G is affine (hence both convex and concave), the hypotheses are satisfied. The conclusion is that there exists $\mu^* \in ba_+(\mathbb{N})$ such that a^* maximizes $f(a) + \langle \mu^*,\, G(a)\rangle$ over Ω and $\langle \mu^*,\, G(a^*)\rangle = 0$.

The Lagrangian and the multiplier. Substituting $c_t = y + Ra_{t-1} - a_t$ from the budget constraint, the Lagrangian with multiplier $\mu \in ba_+(\mathbb{N})$ on the borrowing constraints is:

$$\mathcal{L}(a, \mu) \;=\; \sum_{t=0}^{\infty} \beta^t\, u(y + Ra_{t-1} - a_t) \;+\; \langle \mu,\, a + \phi\rangle.$$

Applying Theorem 8.3, there exists $\mu^* \in ba_+(\mathbb{N})$ such that:

(i) a^* maximizes $\mathcal{L}(\cdot, \mu^*)$ over Ω:

$$\mathcal{L}(a, \mu^*) \le \mathcal{L}(a^*, \mu^*) \quad \text{for all } a \in \Omega;$$

(ii) Complementary slackness holds: $\langle \mu^*,\, a^* + \phi\rangle = 0$.

The variational inequality. A direction $h \in \ell^\infty_R$ is *feasible at* a^* if there exists $\bar\varepsilon > 0$ such that $c_t^* + \varepsilon(Rh_{t-1} - h_t) > 0$ for all t and all $\varepsilon \in (0, \bar\varepsilon)$ (where $h_{-1} \equiv 0$ since a_{-1} is given)—equivalently, $a^* + \varepsilon h \in \Omega$.

Since a^* maximizes $\mathcal{L}(\cdot, \mu^*)$ over Ω, moving in any feasible direction cannot increase the Lagrangian. Concavity of f guarantees that the one-sided directional derivative exists, and maximality gives:

$$D^+\mathcal{L}(a^*;\, h) \;\le\; 0 \quad \text{for every feasible direction } h \text{ at } a^*, \tag{2.8}$$

where $D^+\mathcal{L}(a^*;\, h) \equiv \lim_{\varepsilon \downarrow 0} \frac{\mathcal{L}(a^*+\varepsilon h, \mu^*) - \mathcal{L}(a^*, \mu^*)}{\varepsilon}$.

A decomposition and the ghost multiplier. To extract period-by-period conditions from the variational inequality, we decompose the multiplier. By the Yosida–Hewitt theorem, any $\mu \in ba_+(\mathbb{N})$ decomposes uniquely as

$$\mu = \mu_c + \mu_p,$$

where μ_c is **countably additive** (identifiable with a nonnegative sequence $\{\mu_t^c\}$ satisfying $\sum \mu_t^c R^t < \infty$) and μ_p is **purely finitely additive**: $\mu_p(F) = 0$ for every finite $F \subset \mathbb{N}$, while possibly having $\|\mu_p\| > 0$, where $\|\mu_p\| = \sup\{\langle \mu_p, x\rangle : x \in \ell_R^\infty,\ \|x\|_R \leq 1\}$ is the operator norm. Crucially, both components inherit nonnegativity: $\mu_c \geq 0$ and $\mu_p \geq 0$.

For any $x = \{x_t\} \in \ell_R^\infty$, the action of μ decomposes accordingly:

$$\langle \mu, x\rangle = \underbrace{\sum_{t=0}^{\infty} \mu_t^c\, x_t}_{\text{countably additive } (\mu_c)} + \underbrace{\langle \mu_p, x\rangle}_{\text{purely finitely additive } (\mu_p)}.$$

The μ_c part yields period-by-period complementary slackness: $\mu_t^c \geq 0$, $\mu_t^c\, x_t = 0$.

The μ_p part "charges infinity": it responds only to the asymptotic behavior of $\{x_t\}$. This is our *ghost multiplier*.

Both μ_c and μ_p are components of the *same* multiplier on the borrowing constraint $a_t + \phi \geq 0$: μ_c prices this constraint at each finite date, while μ_p prices its asymptotic content.

Finite-horizon directions and the Euler equation. Take a direction h with $h_t = 0$ for all $t \geq T$. Since h has finite support, the interiority of c^* guarantees that $a^* + \varepsilon h$ remains feasible for all sufficiently small $|\varepsilon|$, and $\langle \mu_p^*, h\rangle = 0$ because μ_p^* vanishes on finite sets. The directional derivative of $\mathcal{L}$ at a^* in direction h is therefore

$$D^+\mathcal{L}(a^*; h) = \sum_{t=0}^{T} \beta^t u'(c_t^*)(Rh_{t-1} - h_t) + \sum_{t=0}^{T-1} \mu_t^{c,*}\, h_t.$$

Rearranging the first sum (where $h_{-1} = 0$ and $h_T = 0$):

$$\sum_{t=0}^{T} \beta^t u'(c_t^*)(Rh_{t-1} - h_t) = \sum_{t=0}^{T-1} \left[\beta^{t+1} R\, u'(c_{t+1}^*) - \beta^t u'(c_t^*)\right] h_t$$

where we used $h_{-1} = 0$ and $h_T = 0$ to drop the boundary terms. Combining with the μ_c^* sum and applying the variational inequality (2.8) to both $+h$ and $-h$ (so that $D^+\mathcal{L} = 0$), given that h_t is unrestricted before T, we can match coefficients on each h_t to obtain the per-period first order condition:

$$\beta^t u'(c_t^*) = \beta^{t+1} R\, u'(c_{t+1}^*) + \mu_t^{c,*} \qquad \forall\, t \geq 0.$$

Since $a_t^* > -\phi$ for all t, per-period complementary slackness gives $\mu_t^{c,*} = 0$ for all t, yielding the usual Euler equation.

Unmasking the Ghost

It should be clear that μ_p^* cannot be zero. If $\mu_p^* = 0$, then $\mu^* = 0$ as $\mu_t^{c,*} = 0$ for all t by complementary slackness. By the Luenberger result above, a^* maximizes $f(a)$, which implies that the household can run a Ponzi scheme. Hence $\mu_p^* \neq 0$: the ghost multiplier must be active. We now give it a precise value by examining two natural perturbations.

Consider a direction h. The corresponding change in consumption is $Rh_{t-1} - h_t$ at date t (with $h_{-1} = 0$). We will assume in what follows that the directional derivative $D^+f(a^*; h)$ exists for such a perturbation and equals

$$D^+f(a^*; h) = \sum_{t=0}^{\infty} \beta^t u'(c_t^*)\,(Rh_{t-1} - h_t).$$

This seems like natural step but requires a proof (an interchange of limit and summation) which we will skip in what follows.

Perturbation 1: A simple Ponzi scheme. Set $h_t = R^t$: borrow one unit at date 0 and roll it over at rate R forever. The consumption perturbation is $\delta_t \equiv Rh_{t-1} - h_t$. Since $R \cdot R^{t-1} = R^t = h_t$ for all $t \geq 1$, we get

$$\delta_0 = -h_0 = -1, \qquad \delta_t = 0 \;\text{ for all } t \geq 1.$$

In words: the Ponzi perturbation reduces date-0 consumption by one unit and leaves all future consumption unchanged. Both $+h$ and $-h$ are feasible at a^*, since the perturbed consumption is $c_0^* \mp \varepsilon$ at date 0 and c_t^* thereafter, which stays positive for small ε.

Applying our assumption, only the $t = 0$ term survives:

$$D^+ f(a^*; h) = \sum_{t=0}^{\infty} \beta^t u'(c_t^*)\, \delta_t = -u'(c_0^*).$$

Now observe that $h_t/R^t = 1$ for all t, so $\langle \mu_p^*, h\rangle = \|\mu_p^*\|$ (purely finitely additive measures act as the norm on constant sequences). The variational inequalities $D^+\mathcal{L}(a^*; \pm h) \leq 0$ then give

$$-u'(c_0^*) + \|\mu_p^*\| \leq 0 \quad \text{and} \quad +u'(c_0^*) - \|\mu_p^*\| \leq 0,$$

which together force $\|\mu_p^*\| = u'(c_0^*)$. The ghost multiplier's norm equals the marginal utility of date-0 consumption: the shadow cost of a Ponzi scheme is precisely $u'(c_0^*)$.

This has a precise economic interpretation. Borrowing one unit at date 0 and rolling it over forever yields a utility gain of $u'(c_0^*)$. The ghost multiplier imposes an exactly offsetting shadow cost: the Ponzi scheme cannot improve welfare.

More generally, take any $h \in \ell_R^\infty$ such that $h_t/R^t \to L$. A nonnegative purely finitely additive measure acts on convergent sequences by picking off the limit,[2] so:

Result: Representation of the Ghost Multiplier

For any $h \in \ell_R^\infty$ with h_t/R^t convergent:

$$\boxed{\langle \mu_p^*,\, h\rangle \;=\; u'(c_0^*) \cdot \lim_{t\to\infty} \frac{h_t}{R^t}.}$$

The representation result shows that the ghost sees only $\lim h_t/R^t$ and is completely blind to finite-date behavior. We now use it to derive the transversality condition.

Perturbation 2: The transversality condition. Set $h_t = a_t^* + \phi$, the optimal slack in the borrowing constraint. The consumption perturbation is $\delta_0 = -(a_0^* + \phi)$ at date 0 (since $h_{-1} = 0$), and for $t \geq 1$:

$$\delta_t = Rh_{t-1} - h_t = R(a_{t-1}^* + \phi) - (a_t^* + \phi) = c_t^*.$$

[2] Write $h_t/R^t = L + e_t$ with $e_t \to 0$. Then $\langle \nu_p^*, \{h_t/R^t\}\rangle = L\,\|\nu_p^*\| + \langle \nu_p^*, e\rangle = L\,\|\mu_p^*\|$: given $\delta > 0$, pick T with $|e_t| < \delta$ for $t > T$; $|\langle \nu_p^*, e\rangle| \leq \delta\,\|\nu_p^*\|$ since ν_p^* vanishes on finite sets. On sequences where h_t/R^t does not converge, μ_p^* acts as a generalized limit of the kind described in the discussion of $ba(\mathbb{N})$ above.

That is, perturbing assets proportionally to their current level scales all future consumption by $(1+\varepsilon)$. Both $\pm h$ are feasible since $c_t^* > 0$ and $a_0^* + \phi > 0$. Note that the sequence $h_t/R^t = (a_t^* + \phi)/R^t$ is nonnegative (from the borrowing constraint) and decreasing (since $(a_t^* + \phi)/R^t = (a_{t-1}^* + \phi)/R^{t-1} - c_t^*/R^t$ and $c_t^* > 0$), hence convergent.

Applying our assumption and using the Euler equation $\beta^t u'(c_t^*)R = \beta^{t-1}u'(c_{t-1}^*)$ to telescope the sum:

$$D^+ f(a^*; h) = \sum_{t=0}^{\infty} \beta^t u'(c_t^*)\, \delta_t = -u'(c_0^*) \cdot \lim_{T\to\infty} \frac{a_T^* + \phi}{R^T}.$$

The telescoping works because the Euler equation converts the R-factor in $\delta_t = Rh_{t-1} - h_t$ into a shift of the $\beta^t u'(c_t^*)$ weights, causing all interior terms to cancel.

By complementary slackness, $\langle \mu_p^*, a^* + \phi \rangle = 0$ (applying the representation result: $h_t/R^t = (a_t^* + \phi)/R^t$, and we are about to show this limit is zero), so the variational inequalities $D^+\mathcal{L}(a^*; \pm h) \le 0$ become

$$-u'(c_0^*) \cdot \lim_{T\to\infty} \frac{a_T^* + \phi}{R^T} \le 0 \quad \text{and} \quad +u'(c_0^*) \cdot \lim_{T\to\infty} \frac{a_T^* + \phi}{R^T} \le 0.$$

Since $u'(c_0^*) > 0$, these two inequalities force the No-Ponzi condition to bind at the optimum:

$$\boxed{\lim_{T\to\infty} \frac{a_T^* + \phi}{R^T} = \lim_{T\to\infty} \frac{a_T^*}{R^T} = 0,}$$

Using $\beta^T u'(c_T^*) = u'(c_0^*)/R^T$, this is equivalently to the standard transversality condition:

$$\boxed{\lim_{T\to\infty} \beta^T u'(c_T^*) a_T^* = 0.}$$

To summarize, *the ghost multiplier is the price of No-Ponzi.* At the optimum, the borrowing constraints are slack at every finite date, yet the entire multiplier $\mu^* = \mu_p^*$ remains strictly positive. The two perturbations above show what it prices. The Ponzi direction gives $\|\mu_p^*\| = u'(c_0^*)$: the ghost charges the marginal utility of a free unit of consumption. The TVC direction forces assets to shrink to zero in present value.

Note that we have shown the necessity of the TVC in this environment using a Lagrangian apparatus that may be an overkill for just this. For a more direct and general approach to this see Kamihigashi (2001).

The Truncation Method

An alternative to working directly in infinite dimensions is the **truncation method**: solve a sequence of finite-horizon (T) subproblems and take $T \to \infty$. Each subproblem has finitely many constraints, so standard finite-dimensional KKT applies: multipliers are real numbers, complementary slackness is classical, and $ba(\mathbb{N})$ never appears. Aguiar and Amador (2016) use this approach in the context of optimal fiscal policy with sovereign debt constraints.

Chapter 3

Towards General Equilibrium: The CARA–Normal Case

This chapter specializes to CARA utility with normally distributed income, where a guess-and-verify strategy yields a closed-form linear consumption function. Consumption equals permanent income minus a precautionary savings term minus an impatience adjustment. In general equilibrium, the drift of cash in hand must be zero, pinning down a unique interest rate $R^* < 1/\beta$. At this rate, consumption follows a martingale.

3.1 Introduction

The previous chapters studied incomplete markets in partial equilibrium, taking the interest rate as given. We now move towards **general equilibrium**, where the interest rate is determined endogenously by market clearing. This is the central question of the Bewley class of models: what interest rate is consistent with households' precautionary savings behavior and a finite supply of assets?

The class of incomplete-markets models studied in this and subsequent chapters originates with Bewley (1983, *Econometrica*), who introduced the problem of a household saving against idiosyncratic risk with a borrowing constraint. Huggett (1993, *JEDC*) embedded this household problem in a pure exchange economy to determine the equilibrium interest rate. Aiyagari (1994, *QJE*) extended the framework to a production economy with capital accumulation, yielding the workhorse model of modern quantitative macroeconomics with heterogeneous agents. See also İmrohoroğlu (1989, *JPE*), who used a similar incomplete-markets setup (with an exogenous interest rate) to quantify the welfare costs of business cycles and the value of unemployment insurance.

We begin with the CARA–Normal case, where closed-form solutions are available. These notes draw on Caballero (1990), who derives the consumption function in partial equilibrium under CARA–Normal assumptions, and Wang (2003), who embeds the model in general equilibrium and characterizes the stationary interest rate.

3.2 Model Environment

Consider the Bellman equation for the household's problem:

$$v(x) = \max_{a \geq -\phi} \{u(x-a) + \beta\, \mathbb{E}[v(Ra + y')]\} \tag{3.1}$$

where x denotes cash in hand, a is savings (asset holdings), R is the gross interest rate, y' is next-period income, and $\beta \in (0,1)$ is the discount factor.

We impose three key assumptions:

(1) **CARA utility:**

$$u(c) = -\frac{1}{\gamma}\, e^{-\gamma c}, \quad \gamma > 0.$$

(2) **Normal income shocks:**

$$y' \sim \mathcal{N}\left(\bar{y},\, \sigma^2\right).$$

(3) **No borrowing limit; No-Ponzi condition:** Set $\phi = \infty$ and replace the borrowing constraint with a No-Ponzi condition:

$$\lim_{t\to\infty} \beta^t\, \mathbb{E}\left[e^{-\gamma(R-1)\, a_t}\right] = 0.$$

This prevents the household from borrowing so aggressively that the exponential moment of debt overwhelms the discount factor. See the Appendix for further discussion and the verification argument.

Under CARA utility, consumption can be negative ($c < 0$), which may seem economically unreasonable. One interpretation is to embed the problem in a model with labor supply

ℓ. Suppose the period utility function is:

$$U = -\frac{1}{\gamma}\, e^{-\gamma c + \psi\, \ell}.$$

Define $\hat{c} = c - \frac{\psi}{\gamma}\,\ell$. The original budget constraint $c + a' \leq Ra + y + w \cdot \ell$ becomes:

$$\hat{c} + a' \leq Ra + y + \left(w - \frac{\psi}{\gamma}\right)\ell \equiv Ra + \hat{y},$$

where $\hat{y} = y + \left(w - \frac{\psi}{\gamma}\right)\ell$. The utility function becomes $\hat{U} = -\frac{1}{\gamma}\, e^{-\gamma \hat{c}}$. Thus, $\hat{c}$ can be negative even if $c \geq 0$, as long as the household works enough.

3.3 Guess and Verify

The CARA–Normal structure makes the Bellman equation tractable via the *method of indeterminate coefficients*. The idea is to guess a parametric form for the value function and consumption policy (here, an exponential value function and a linear consumption rule) with unknown coefficients. Substituting these guesses into the Bellman equation and the optimality conditions, one matches coefficients on the state variable x and on constant terms to pin down the unknowns. The exponential–normal pair is especially convenient because the expectation of an exponential of a normal random variable has a closed-form expression (the moment-generating function), which keeps all calculations in the exponential family.

Go back to the Bellman equation (3.1). Since income is continuous and normally distributed, write:

$$v(x) = \max_{a} \left\{ u(x - a) + \beta\, \mathbb{E}[v(Ra + y')] \right\}. \tag{3.2}$$

We guess that the value function and consumption policy take the form:

$$v(x) = -\frac{1}{\gamma}\, e^{-\hat{A}x - \hat{B}}, \qquad c(x) = Ax + B.$$

We need to solve for the four unknowns A, B, $\hat{A}$, and $\hat{B}$.

Deriving the expectation. From the guess, the savings function is:

$$a(x) = x - c(x) = x - Ax - B = (1 - A)x - B.$$

Substituting into the value function:

$$v(Ra + y') = -\frac{1}{\gamma}\, e^{-\hat{A}[R((1-A)x-B)+y']-\hat{B}} = -\frac{1}{\gamma}\, e^{-\hat{A}(1-A)Rx+\hat{A}BR-\hat{A}y'-\hat{B}}.$$

Since $y' \sim \mathcal{N}(\bar{y}, \sigma^2)$, we can evaluate the expectation using the moment-generating function of the normal distribution:[1]

$$\mathbb{E}[v(Ra + y')] = -\frac{1}{\gamma}\, e^{-\hat{A}(1-A)Rx+\hat{A}BR-\hat{A}\bar{y}-\hat{B}+\frac{\hat{A}^2\sigma^2}{2}}. \tag{3.3}$$

Plugging back into the Bellman. Substituting (3.3) and the guessed forms into the Bellman equation (3.2):

$$-\frac{1}{\gamma}\, e^{-\hat{A}x-\hat{B}} = -\frac{1}{\gamma}\, e^{-\gamma Ax-\gamma B} - \frac{\beta}{\gamma}\, e^{-\hat{A}(1-A)Rx+\hat{A}BR-\hat{A}\bar{y}-\hat{B}+\frac{\hat{A}^2\sigma^2}{2}}.$$

Rewriting the right-hand side:

$$-\frac{1}{\gamma}\, e^{-\hat{A}x-\hat{B}} = -\frac{1}{\gamma}\, e^{-\gamma Ax-\gamma B} - \frac{1}{\gamma}\, \exp\left\{\log\beta - \hat{A}(1-A)Rx + \hat{A}BR - \hat{A}\bar{y} - \hat{B} + \frac{\hat{A}^2\sigma^2}{2}\right\}.$$

For this equation to hold for all x, the two exponentials on the right-hand side must have the same slope in x: a sum $e^{-\alpha x} + e^{-\beta x}$ can equal $e^{-\alpha x}$ for all x only if $\alpha = \beta$ (so the two terms can be factored). This requires:

$$\gamma A = \hat{A}, \tag{3.4}$$

$$\hat{A}(1-A)R = \hat{A} \quad\Longrightarrow\quad (1-A)R = 1. \tag{3.5}$$

From (3.5):

$$A = \frac{R-1}{R}.$$

From (3.4):

$$\hat{A} = \gamma A = \frac{\gamma(R-1)}{R}.$$

This pins down A and $\hat{A}$.

[1] If $y' \sim \mathcal{N}(\bar{y}, \sigma^2)$, then $\mathbb{E}[e^{-ky'}] = \exp\{-k\bar{y} + \frac{1}{2}k^2\sigma^2\}$ for any constant k. This identity is used repeatedly throughout these notes.

Envelope condition. We have not yet used the optimality of a. The envelope condition states:

$$v'(x) = u'(c(x)).$$

Computing both sides:

$$\frac{\hat{A}}{\gamma} e^{-\hat{A}x-\hat{B}} = e^{-\gamma Ax-\gamma B}.$$

Using $\hat{A} = \gamma A$, the terms in x cancel, yielding:

$$\log A - \hat{B} = -\gamma B. \tag{3.6}$$

First-order condition. The Euler equation is:

$$u'(c(x)) = \beta R\,\mathbb{E}[v'(Ra(x) + y')]\,.$$

Evaluating both sides using the guessed forms and taking expectations over the normal distribution:

$$e^{-\gamma Ax-\gamma B} = e^{\log(\beta R)} \cdot A \cdot \exp\left\{\hat{A}BR - \hat{A}\bar{y} - \hat{B} + \frac{\hat{A}^2\sigma^2}{2} - \hat{A}(1-A)Rx\right\}.$$

Since $\hat{A}(1-A)R = \hat{A} = \gamma A$, the terms in x cancel. Matching the constant terms:

$$\gamma(R-1)B = \gamma(R-1)\frac{\bar{y}}{R} - \gamma^2\left(\frac{R-1}{R}\right)^2\frac{\sigma^2}{2} - \log(\beta R).$$

Dividing through by $\gamma(R-1)$:

$$B = \frac{\bar{y}}{R} - \frac{\gamma(R-1)}{R^2}\frac{\sigma^2}{2} - \frac{\log(\beta R)}{\gamma(R-1)}.$$

From the envelope condition (3.6), we recover $\hat{B} = \log A + \gamma B = \log\left(\frac{R-1}{R}\right) + \gamma B$.

We have determined the main coefficients that are consistent with the guess for the value and policy functions. But we still need to impose a No-Ponzi condition and verify that the value function is indeed the solution. We do so in the Appendix: No-Ponzi Condition and Verification.

The consumption function. Using $c(x) = Ax + B$ with the solutions above:

Result: Consumption Function

$$c(x) = \underbrace{\frac{R-1}{R}x + \frac{\bar{y}}{R}}_{\text{permanent income}} - \underbrace{\frac{\gamma(R-1)}{R^2}\frac{\sigma^2}{2}}_{\text{precautionary savings}} - \underbrace{\frac{\log(\beta R)}{\gamma(R-1)}}_{\text{relative impatience}} \tag{3.7}$$

Under CARA utility, γ controls *both* risk aversion and the intertemporal elasticity of substitution (IES). For a framework that disentangles the two, see Epstein–Zin preferences. For a comprehensive treatment of recursive utility and other non-standard preference specifications in macroeconomics, see Backus, Routledge, and Zin (2004, *NBER Macroeconomics Annual*).

Evolution of cash in hand. The law of motion for cash in hand is:

$$x' = Ra(x) + y' = R\big(x - c(x)\big) + y'.$$

Substituting the consumption function (3.7):

$$x' = x + \underbrace{(y' - \bar{y})}_{\sim \mathcal{N}(0,\sigma^2)} + \underbrace{\frac{\gamma(R-1)\sigma^2}{2R} + \frac{\log(\beta R)\,R}{\gamma(R-1)}}_{\text{drift} = \mu}.$$

This shows that **cash in hand follows a random walk with drift**.

Suppose $R > 1$. If $\beta R \geq 1$, then the drift term is strictly positive, which implies $x \to \infty$ as $t \to \infty$.

3.4 Implications for General Equilibrium

Suppose we have a mass 1 of households, all starting from $x_0 = \bar{X}_0$. From the law of motion derived above, individual cash in hand follows a random walk with drift μ and variance σ^2. The cross-sectional distribution is:

$$F_t = \mathcal{N}(\bar{X}_0 + t\,\mu,\ t\,\sigma^2).$$

The mean drifts over time and the variance increases.

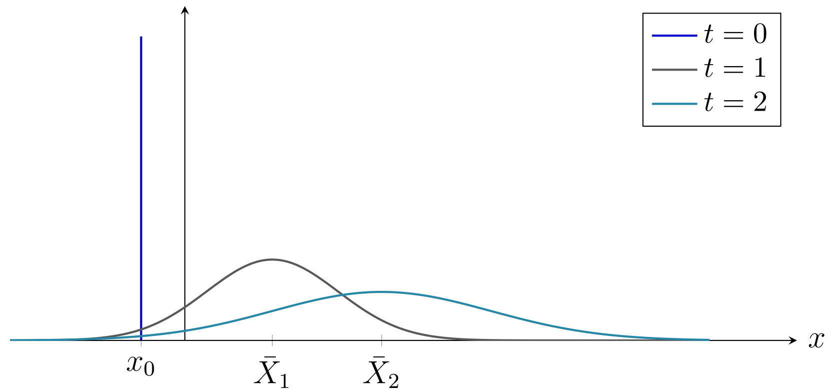


Figure 3.1: Evolution of the cross-sectional distribution of cash in hand. At $t = 0$ all households start at x_0. Over time, the mean drifts rightward and the variance increases. The drift is assumed to be positive in this picture.

3.4.1 Stationary Equilibrium Condition

The resource constraint requires aggregate consumption to equal aggregate output at every date: $\int c(x)\,dF_t = \bar{y}$. Since $c(x) = \frac{R-1}{R}\,x + B$, integrating against F_t gives:

$$\frac{R-1}{R}\int x\,dF_t + B = \bar{y}.$$

From the distribution $F_t = \mathcal{N}(\bar{X}_0 + t\mu,\ t\sigma^2)$, the cross-sectional mean is $\int x\,dF_t = \bar{X}_0 + t\mu$. Substituting:

$$\frac{R-1}{R}\,(\bar{X}_0 + t\mu) + B = \bar{y}.$$

This must hold for all t, so the coefficient on t must vanish: $\mu = 0$. That is, the drift of cash in hand must be zero. When $\mu = 0$, the precautionary savings and impatience terms in B cancel (since $\frac{\log(\beta R)}{\gamma(R-1)} = -\frac{\gamma(R-1)\sigma^2}{2R^2}$ by the drift condition), so:

$$B = \frac{\bar{y}}{R} - \frac{\gamma(R-1)}{R^2}\,\frac{\sigma^2}{2} - \frac{\log(\beta R)}{\gamma(R-1)} = \frac{\bar{y}}{R}.$$

The constant term then gives $\frac{R-1}{R}\,\bar{X}_0 + \frac{\bar{y}}{R} = \bar{y}$, which immediately pins down $\bar{X}_0 = \bar{y}$. Setting $\mu = 0$:

$$\frac{\gamma(R-1)\sigma^2}{2R} + \frac{\log(\beta R)\,R}{\gamma(R-1)} = 0.$$

Rearranging, the equilibrium interest rate R^* must satisfy:

Result: Equilibrium Interest Rate

$$\beta = \frac{1}{R}\exp\left\{-\gamma^2\left(\frac{R-1}{R}\right)^2\frac{\sigma^2}{2}\right\}. \tag{3.8}$$

There exists a unique R^* solving (3.8). Moreover, $R^* < 1/\beta$.

The result $R^* < 1/\beta$ is a key insight: in the presence of uninsurable idiosyncratic risk, the equilibrium interest rate is *lower* than it would be in the complete-markets benchmark ($R = 1/\beta$). Households engage in precautionary saving, which pushes down the equilibrium rate.

Comparative statics. From (3.8), R^* is decreasing in both γ and σ^2: more risk-averse households or more volatile income lead to stronger precautionary saving and a lower equilibrium rate. In the limit $\sigma^2 \to 0$ (no idiosyncratic risk), the exponential term vanishes and $R^* \to 1/\beta$, recovering the complete-markets result. In the opposite limit $\gamma\sigma \to \infty$, precautionary saving dominates and $R^* \to 1$.

Consumption dynamics. Since $c(x) = \frac{R-1}{R}x + B$, the change in consumption is $c_{t+1} - c_t = \frac{R-1}{R}(x_{t+1} - x_t) = \frac{R-1}{R}(\varepsilon_{t+1} + \mu)$, where $\varepsilon_{t+1} = y_{t+1} - \bar{y}$. At the equilibrium rate R^* the drift $\mu = 0$, so consumption follows a *martingale*:

$$c_{t+1} = c_t + \frac{R^* - 1}{R^*}\,\varepsilon_{t+1}.$$

This is a version of the Hall (1978) random walk result: under rational expectations and permanent income logic, consumption changes are unpredictable.

3.5 Aggregation and Market Clearing

Let us now state the aggregation and market-clearing conditions. Letting F_t denote the cross-sectional distribution of cash in hand as before, the aggregate resource constraint and the market-clearing condition for assets are:

$$\int c(x)\,dF_t = \bar{y},$$

$$\int a(x)\,dF_t = 0.$$

As shown above, equilibrium requires the drift to be zero so that aggregate cash in hand remains at $\bar{y}$. At $R = R^*$ we have $\mu = 0$, and $F_t = \mathcal{N}(\bar{y}, t\,\sigma^2)$. Market clearing holds at every date, but the variance grows linearly in t: wealth inequality increases without bound. There is no stationary cross-sectional distribution in the usual sense.

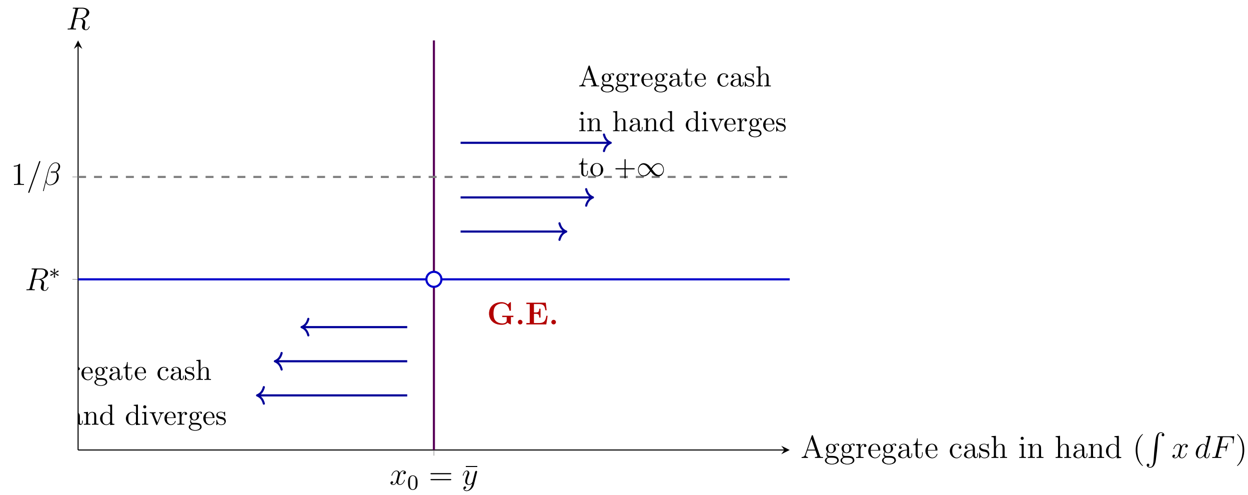


Figure 3.2: Phase diagram. The vertical line represents market clearing ($\int x\,dF = \bar{y}$). For $R > R^*$, aggregate cash in hand diverges to $+\infty$; for $R < R^*$, it diverges to $-\infty$. The unique stationary G.E. obtains at $R = R^*$.

Appendix: No-Ponzi Condition and Verification

The standard verification arguments (e.g., Stokey, Lucas, and Prescott, 1989) require the value function to be bounded, so that the Bellman remainder $\beta^T \mathbb{E}[v(x_T)] \to 0$ automatically. In the CARA-Normal model, however, $v(x) = -\frac{1}{\gamma}\, e^{-\frac{\gamma(R-1)}{R}\, x - \hat{B}}$ is unbounded below, and exponential moments of assets can grow without bound. We therefore need an alternative approach: a No-Ponzi condition tailored to the exponential structure of the problem, together with a direct verification argument.

The No-Ponzi Condition

In the CARA-Normal environment there is no natural borrowing limit: $u(c) = -\frac{1}{\gamma} e^{-\gamma c}$ is defined on all of $\mathbb{R}$ and income $y \sim \mathcal{N}(\bar{y}, \sigma^2)$ is unbounded below. We need to strengthen the standard NPC ($\lim R^{-T} a_T \geq 0$ a.s.) to one adapted to the exponential structure of the problem.

We require all feasible plans $\{c_t, a_t\}_{t=0}^{\infty}$ to satisfy:

$$\lim_{T \to \infty} \beta^T\, \mathbb{E}\big[e^{-\gamma(R-1)\, a_T}\big] = 0. \tag{3.9}$$

This is a non-trivial restriction. If the household borrows aggressively ($a_T \to -\infty$ on many paths), $e^{-\gamma(R-1)\, a_T}$ explodes, and the exponential moment can grow faster than β^{-T}, violating (3.9).

The condition (3.9) involves the preference parameters γ and β, unlike the standard NPC, which depends only on market prices. This makes it closer in spirit to a transversality condition than a market-imposed borrowing restriction.

Without a No-Ponzi condition, the $v(x)$ found above cannot be a solution to the household's problem. To see why, consider a plan that consumes a constant $c_t = M$ for all t, with M large. Since $u(c) \to 0$ as $c \to \infty$, the lifetime utility $\sum \beta^t u(M) = u(M)/(1-\beta)$ can be made arbitrarily close to $0 > v(x_0)$. The required assets satisfy $a_T \approx -R^T(M - x_0)$ for large M, so the NPC term becomes:

$$\beta^T\, \mathbb{E}\big[e^{-\gamma(R-1)\, a_T}\big] \approx \beta^T\, e^{\gamma(R-1)(M-x_0)\, R^T} \;\longrightarrow\; \infty.$$

The NPC (3.9) rules out such plans. Without it, the value of the problem would be 0, not $v(x_0)$.

Verification Theorem

Recall from the guess-and-verify solution: $A = \frac{R-1}{R}$, $\hat{A} = \gamma A = \frac{\gamma(R-1)}{R}$, and the value function is $v(x) = -\frac{1}{\gamma}\, e^{-\frac{\gamma(R-1)}{R}x - \hat{B}}$.

Result: Verification

Let $v(x) = -\frac{1}{\gamma}\, e^{-\frac{\gamma(R-1)}{R}\,x-\hat{B}}$ and $c^*(x) = \frac{R-1}{R}\,x + B$ be the candidate value function and policy. Then, for any $R > 1$ and under the No-Ponzi condition (3.9), v is the true value function and c^* is optimal.

The proof proceeds in two steps. Let $V(x_0) = \sup \sum_{t=0}^{\infty} \beta^t\, \mathbb{E}[u(c_t)]$, where the supremum is over all feasible plans satisfying (3.9), be the true value function.

Step 1: Upper bound ($v(x_0) \geq V(x_0)$). For any feasible plan $\{c_t, a_t\}$ satisfying (3.9), the Bellman equation gives $v(x_t) \geq u(c_t) + \beta\, \mathbb{E}_t[v(x_{t+1})]$ (since c^* maximizes the right-hand side). Iterating:

$$v(x_0) \geq \sum_{t=0}^{T-1} \beta^t\, \mathbb{E}[u(c_t)] + \beta^T\, \mathbb{E}[v(x_T)]. \tag{3.10}$$

Since $x_T = Ra_{T-1} + y_T$ with $y_T \perp a_{T-1}$:

$$\beta^T\, \mathbb{E}[v(x_T)] = -\frac{e^{-\hat{B}}}{\gamma} \cdot \beta^T\, \mathbb{E}\Big[e^{-\frac{\gamma(R-1)}{R}\cdot R\, a_{T-1}}\Big] \cdot \mathbb{E}\Big[e^{-\frac{\gamma(R-1)}{R}\, y_T}\Big].$$

The key observation is $\frac{\gamma(R-1)}{R} \cdot R = \gamma(R-1)$, which is exactly the coefficient in the NPC. The second factor is a positive constant $K = e^{-\frac{\gamma(R-1)}{R}\bar{y} + \frac{\gamma^2(R-1)^2}{2R^2}\sigma^2}$. Therefore:

$$\beta^T\, \mathbb{E}[v(x_T)] = -\underbrace{\frac{K\, e^{-\hat{B}}}{\gamma}}_{C} \cdot \beta \cdot \underbrace{\beta^{T-1}\, \mathbb{E}\Big[e^{-\gamma(R-1)\, a_{T-1}}\Big]}_{\to 0 \text{ by (3.9) at } T-1} \longrightarrow 0.$$

Taking $T \to \infty$ in (3.10):

$$v(x_0) \geq \sum_{t=0}^{\infty} \beta^t\, \mathbb{E}[u(c_t)]$$

for all feasible plans satisfying (3.9). Hence $v(x_0) \geq V(x_0)$.

Step 2: Attainment at c^*. Under c^*, equality holds in the Bellman at every step:

$$v(x_0) = \sum_{t=0}^{T-1} \beta^t\, \mathbb{E}[u(c_t^*)] + \beta^T\, \mathbb{E}[v(x_T^*)].$$

Since c^* satisfies (3.9) (verified below), the remainder $\beta^T \mathbb{E}[v(x_T^*)] \to 0$ by exactly the same calculation as Step 1. Taking $T \to \infty$:

$$v(x_0) = \sum_{t=0}^{\infty} \beta^t \, \mathbb{E}[u(c_t^*)].$$

Since c^* satisfies (3.9), it is a feasible plan, so $V(x_0) \geq \sum \beta^t \mathbb{E}[u(c_t^*)] = v(x_0)$.

Conclusion. Combining: $V(x_0) \leq v(x_0) = \sum \beta^t \mathbb{E}[u(c_t^*)] \leq V(x_0)$, so $v = V$ and c^* is optimal.

Verification That c^* Satisfies the NPC

We have just one thing left to check: that c^* satisfies the NPC.

Under c^*, savings are $a_T = \frac{1}{R}\, x_T - B$, so:

$$\beta^T \, \mathbb{E}\big[e^{-\gamma(R-1)\, a_T}\big] = \beta^T \, e^{\gamma(R-1)B} \, \mathbb{E}\Big[e^{-\frac{\gamma(R-1)}{R}\, x_T}\Big].$$

The law of motion gives $x_T \sim \mathcal{N}(x_0 + T\mu,\ T\sigma^2)$ where $\mu = \frac{\gamma(R-1)}{2R}\,\sigma^2 + \frac{R\log(\beta R)}{\gamma(R-1)}$ is the drift. Therefore:

$$\mathbb{E}\Big[e^{-\frac{\gamma(R-1)}{R}\, x_T}\Big] = e^{-\frac{\gamma(R-1)}{R}(x_0 + T\mu) + \frac{\gamma^2(R-1)^2}{2R^2}\, T\sigma^2}.$$

The rate of decay is governed by $\beta\, e^{-\frac{\gamma(R-1)}{R}\,\mu + \frac{\gamma^2(R-1)^2}{2R^2}\,\sigma^2}$. Substituting the expression for μ:

$$\beta\, e^{-\frac{\gamma(R-1)}{R}\,\mu + \frac{\gamma^2(R-1)^2}{2R^2}\,\sigma^2} = \beta \cdot \frac{1}{\beta R} = \frac{1}{R}.$$

This gives:

$$\beta^T \, \mathbb{E}\big[e^{-\gamma(R-1)\, a_T}\big] = C' \cdot R^{-T} \to 0$$

for any $R > 1$, where $C' = e^{\gamma(R-1)B - \frac{\gamma(R-1)}{R}\, x_0}$. The decay rate is $1/R$, independent of β, γ, and σ^2. ✓

Chapter 4

Stationary General Equilibrium and the Huggett Model

This chapter returns to general utility and studies the Huggett model. The analysis characterizes policy functions, the transition map for the wealth distribution, and conditions for a compact ergodic set. In general equilibrium, the stationary distribution and interest rate are jointly determined by market clearing. The special case of zero borrowing admits a sharp analytical characterization; the general case requires numerical methods.

4.1 Model Environment: Away from CARA

We return to the general incomplete-markets savings problem. Unlike the CARA–Normal model, we now work with a general utility function u (satisfying $u' > 0$, $u'' < 0$, Inada conditions). Income is drawn from a finite set S with i.i.d. probabilities $\pi(s)$, and households face a borrowing limit $\phi \geq 0$. The goal is to characterize savings behavior and the resulting wealth distribution when we can no longer exploit the exponential tricks of the CARA case.

The household has cash-on-hand x and chooses assets a to maximize:

$$\boxed{v(x) = \max_{x \geq a \geq -\phi} \left\{ u(x-a) + \beta \sum_{s \in S} \pi(s)\, v\big(Ra + y(s)\big) \right\}}$$

where $c = x - a \geq 0$, $R > 0$ is the gross interest rate, $\beta \in (0,1)$ is the discount factor, $y(s) > 0$ is the income realization in state s, and tomorrow's cash-on-hand is $x' = Ra + y(s)$.

Change of variables. We perform a change of variables that absorbs the borrowing limit ϕ from the constraint, simplifying the analysis. Define:

$$\begin{aligned} \hat{a} &\equiv a + \phi && \text{(shifted assets, so } \hat{a} \geq 0\text{)} \\ z &\equiv x + \phi && \text{(shifted cash-on-hand)} \\ \tilde{y}(s) &\equiv y(s) - (R-1)\phi && \text{(adjusted income)} \end{aligned}$$

Starting from $x' = Ra + y(s)$ and substituting $a = \hat{a} - \phi$: $x' = R\hat{a} - R\phi + y(s)$. Adding ϕ to both sides and using the definitions:

$$z' = R\hat{a} + \tilde{y}(s).$$

Consumption becomes $c = x - a = (x + \phi) - (a + \phi) = z - \hat{a}$, and the borrowing constraint $a \geq -\phi$ becomes simply $\hat{a} \geq 0$.

Note that $\tilde{y}(s)$ can be negative for some states when ϕ is large. This is not a problem for the formulation; it simply means that after servicing the interest on debt, the household's effective resources from income alone may be negative in some states. The constraint $\hat{a} \geq 0$ combined with continuity of the value function ensures the problem remains well-defined.

In the new variables (abusing notation by rewriting v), the Bellman equation becomes:

A Normalized Bellman

$$v(z) = \max_{z \geq \hat{a} \geq 0} \left\{ u(z - \hat{a}) + \beta \sum_{s \in S} \pi(s)\, v\big(R\hat{a} + \tilde{y}(s)\big) \right\} \tag{4.1}$$

The borrowing limit ϕ has disappeared from the problem except through $\tilde{y}(s)$.

Euler equation. The first-order condition (with complementary slackness at $\hat{a} = 0$) is:

$$u'(z - \hat{a}) \geq \beta R \sum_{s \in S} \pi(s)\, v'\big(R\hat{a} + \tilde{y}(s)\big), \quad \text{with equality if } \hat{a} > 0.$$

There exists a critical level $\hat{z}$ at which the Euler equation holds with equality at the borrowing constraint $\hat{a} = 0$:

$$u'(\hat{z}) = \beta R \sum_{s \in S} \pi(s)\, v'\big(\tilde{y}(s)\big).$$

For $z < \hat{z}$, the household is constrained ($\hat{a} = 0$, consumes everything). For $z \geq \hat{z}$, the household saves ($\hat{a} > 0$, Euler holds with equality).

Policy functions. The policy functions have the following qualitative features (Figure 4.1). Below $\hat{z}$, the household is at the borrowing constraint: $\hat{a}(z) = 0$ (equivalently $a = -\phi$) and $c(z) = z$. Above $\hat{z}$, the household saves, consumption is concave in z, and $\hat{a}(z)$ is increasing.

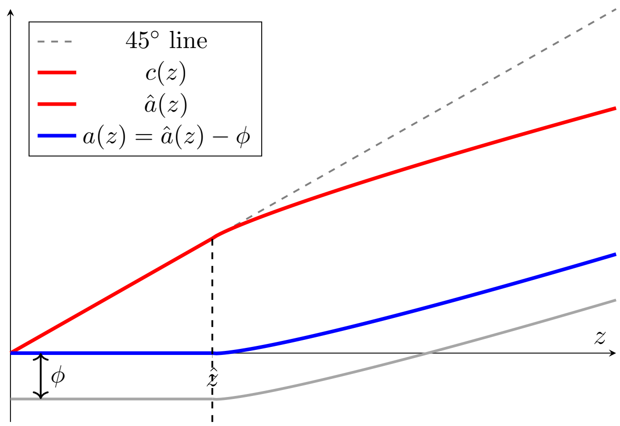


Figure 4.1: Policy functions. For $z < \hat{z}$, the household is borrowing-constrained: $\hat{a}(z) = 0$ and $c(z) = z$. For $z \geq \hat{z}$, the household saves and consumption is concave in z.

Transition map and ergodic set. We now study the dynamics of cash-on-hand over time. Tomorrow's z' is determined by the savings policy and the income draw: $z' = R\hat{a}(z) + \tilde{y}(s)$. Figure 4.2 plots z' against z for the highest and lowest income realizations.

Under $\beta R < 1$ (plus additional conditions on u; see below), the transition map for the highest income realization eventually crosses the 45° line from above. This crossing, together with the bounded-below nature of the lowest realization, defines an ergodic set $[\underline{z}, \bar{z}]$: if you start below it, you drift in; if you start above it, you drift down; if you start inside, you stay. Assets remain bounded and a stationary distribution exists.

The condition $\beta R < 1$ alone is not sufficient to guarantee bounded assets. CARA utility is a counterexample: asset paths can be unbounded even with $\beta R < 1$ (recall that CARA consumption has a unit root). The additional conditions on u needed to ensure a compact ergodic set are discussed carefully in Aiyagari (1994, Section III).

The Asset Distribution

We now turn to the cross-sectional distribution of agents across asset and income states. Up to this point we have worked with i.i.d. income. We now generalize to a Markov chain $\pi(s'|s)$ for the income process, of which i.i.d. is a special case ($\pi(s'|s) = \pi(s')$).

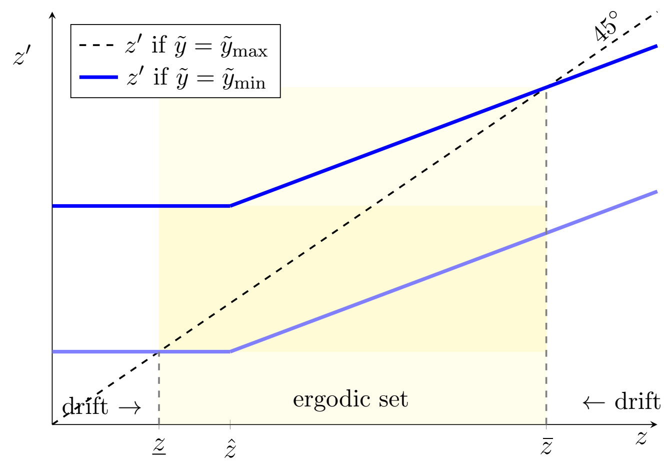


Figure 4.2: Transition map under $\beta R < 1$. The upper line ($\tilde{y}_{\max}$) crosses the 45° line from above at $\bar{z}$, and the lower line ($\tilde{y}_{\min}$) intersects the 45° line at $\underline{z}$. The shaded region is the ergodic support: cash-on-hand is attracted into this set from either side.

Suppose $a \in \mathcal{A}$ with $|\mathcal{A}| = M$ (a finite asset grid) and $s \in S$ with $|S| = N$. The joint distribution λ is an $M \times N$ matrix where $\lambda(a, s)$ gives the fraction of the population with assets a and income state s. Since λ is a probability mass function: $\lambda(a, s) \geq 0$ for all (a, s) and $\sum_{a,s} \lambda(a, s) = 1$.

Law of motion. Given a savings policy $a' = g(a, s)$ and transition probabilities $\pi(s'|s)$, the distribution evolves as:

$$\boxed{\lambda_{t+1}(a', s') = \sum_{s \in S} \sum_{\substack{a \in \mathcal{A}: \\ g(a,s)=a'}} \pi(s'|s)\, \lambda_t(a, s)}$$

The idea is illustrated in Figure 4.3. Consider a cell (a_3, s_2) with mass $\lambda_t(a_3, s_2)$. The policy function determines where this mass goes in the asset dimension: it moves to column $a_2 = g(a_3, s_2)$. The exogenous transition probabilities $\pi(s'|s_2)$ then spread this mass across income states in that column. Summing over all source cells that map to the same (a', s') gives $\lambda_{t+1}(a', s')$.

In matrix notation, $\lambda_{t+1} = T \cdot \lambda_t$, where T is the transition operator constructed from g and π. Note that T depends on R through the policy function g. A stationary distribution satisfies the fixed point $\lambda = T \cdot \lambda$, i.e., λ is an eigenvector of T associated with eigenvalue 1.

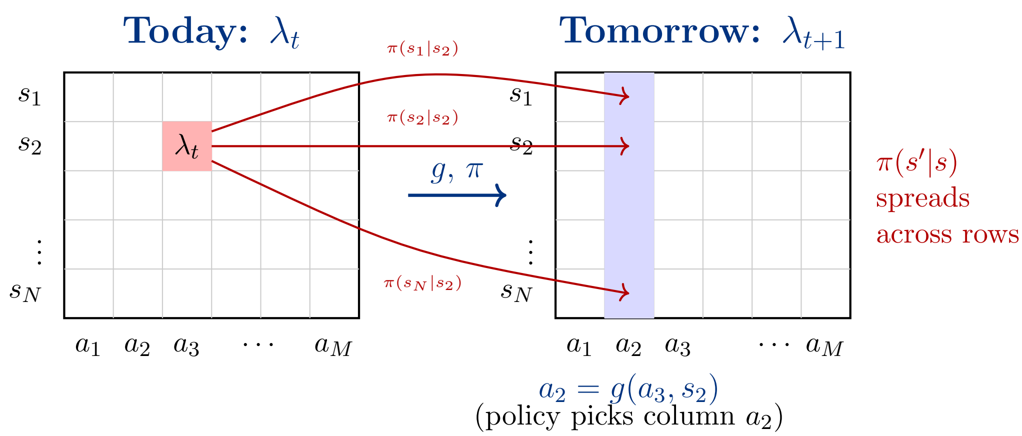


Figure 4.3: Evolution of the distribution. Mass at (a_3, s_2) in λ_t moves to column $a_2 = g(a_3, s_2)$ (determined by the policy) and is spread across income states s' by the transition probabilities $\pi(s'|s_2)$.

It is possible to show the existence and uniqueness of the stationary distribution. For a rigorous treatment, see Stokey, Lucas, and Prescott (1989, Ch. 11–12) and Hopenhayn and Prescott (1992).

In practice, the stationary distribution is computed numerically (see Problem Set 2).

4.2 General Equilibrium: The Huggett Model

We now close the model in general equilibrium using the framework of Huggett (1993). The Huggett model is a pure exchange economy: no capital, no production, just endowments. Since there is no storage technology, aggregate net savings must equal zero in equilibrium. The single price to be determined is the gross interest rate R.

We now treat the ergodic distribution λ as representing a population of households, assuming that the law of large numbers holds. That is, $\lambda(a, s)$ captures the fraction of households that are in state (a, s) in a stationary equilibrium. Aggregates are computed by integrating against λ.

A stationary equilibrium consists of $R > 0$, a value function v, a savings policy g, and a distribution λ such that: (i) (v, g) solve the household problem given R; (ii) λ is the stationary distribution implied by g and π, i.e., $\lambda = T(R) \cdot \lambda$; and (iii) the bond market clears:

$$A(R) \equiv \sum_{a\in\mathcal{A}}\sum_{s\in S}\lambda(a,s)\cdot a = 0$$

Equivalently, the resource constraint requires aggregate consumption to equal the aggregate endowment: $\sum_{a,s}\lambda(a,s)[c(a,s)-y(s)] = 0$, where $c(a,s) = a + y(s) - g(a,s)$.

In general, finding the equilibrium requires numerical methods. But the special case $\phi = 0$ admits an elegant analytical characterization.

Special Case: $\phi = 0$ (No Borrowing)

When $\phi = 0$, households cannot borrow ($a \geq 0$). Market clearing $\sum_{a,s}\lambda(a,s)\cdot a = 0$ combined with $a \geq 0$ forces all mass to $a = 0$. The distribution is pinned down before solving the model: $\lambda(a,s) = \pi^{ss}(s)$ if $a = 0$ and zero otherwise, where π^{ss} is the stationary distribution of $\pi(s'|s)$ (which requires irreducibility and aperiodicity).

Every household is in autarky, simply consuming its endowment. The value function satisfies $v(0,s) = u(y(s)) + \beta\sum_{s'}\pi(s'|s)\,v(0,s')$, and savings are zero: $g(0,s) = 0$ for all s. The equilibrium R must be such that zero savings is optimal, meaning the Euler inequality holds at $a = 0$ for every state:

$$u'\big(y(s)\big) \geq \beta R\sum_{s'\in S}\pi(s'|s)\,u'\big(y(s')\big), \qquad \forall\, s\in S.$$

The binding constraint comes from the state most tempted to save. Taking the tightest constraint, define:

$$\bar{R} \equiv \frac{1}{\beta}\min_{s\in S}\left\{\frac{u'\big(y(s)\big)}{\sum_{s'\in S}\pi(s'|s)\,u'\big(y(s')\big)}\right\} \tag{4.2}$$

Result: Equilibrium with No Borrowing

For any $R \in (0,\bar{R}]$, a stationary equilibrium exists with $g(0,s) = 0$ for all s, $\lambda(a,s) = \pi^{ss}(s)\cdot\mathbf{1}_{a=0}$, and $v(0,s) = u(y(s)) + \beta\sum_{s'}\pi(s'|s)\,v(0,s')$. The interest rate $\bar{R}$ is the highest equilibrium rate: at $R = \bar{R}$ there is at least one "marginal agent" whose Euler equation holds with equality. For $R > \bar{R}$, at least one type would want to save, contradicting $A(R) = 0$ when borrowing is impossible.

The economic logic: when R is low enough, the return to saving is too poor for anyone to bother. But no one can borrow. So markets clear trivially.

CRRA example. For $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ with $\sigma > 0$, substituting $u'(c) = c^{-\sigma}$ directly into (4.2):

$$\bar{R} = \frac{1}{\beta} \min_{s \in S} \left\{ \frac{1}{\sum_{s' \in S} \pi(s'|s) \left[\frac{y(s)}{y(s')} \right]^{\sigma}} \right\} \tag{4.3}$$

Under CRRA with non-degenerate income: if $\sigma = 0$ (risk neutrality), $\bar{R} = 1/\beta$ and there is no precautionary motive. As σ increases, $\bar{R}$ falls: higher risk aversion strengthens the precautionary savings motive, requiring a lower R to discourage saving. For any $\sigma > 0$, $\bar{R} < 1/\beta$ (hence $\beta\bar{R} < 1$), which follows from Jensen's inequality applied to the convex function $c \mapsto c^{-\sigma}$.

That $\beta\bar{R} < 1$ is a *consequence* of the equilibrium, not an independent assumption. The interest rate is depressed below $1/\beta$ by the precautionary savings motive.

The Case $\phi > 0$

When $\phi > 0$, the autarky characterization no longer applies and we must solve numerically.

A Potential Algorithm

1. Guess an interest rate R.
2. Solve the household problem to obtain (v, g).
3. Compute the stationary distribution λ implied by g.
4. Check market clearing: compute $A(R) = \sum_{a,s} \lambda(a, s) \cdot a$.
5. Update R: if $A(R) > 0$ (excess savings), raise R; if $A(R) < 0$, lower R. Iterate until $A(R) = 0$.

Chapter 5

The Aiyagari Model

This chapter introduces capital and production into the incomplete-markets framework. The Aiyagari model features a continuum of households facing idiosyncratic labor income risk who save in physical capital. In stationary equilibrium, precautionary savings push the capital stock above the neoclassical benchmark and may even exceed the golden rule level. A simple comparative static (relaxing the borrowing constraint) shifts asset supply left, raising the interest rate and lowering the capital stock.

5.1 Model Environment and Stationary Equilibria

In the previous chapter, we studied the household's savings problem in partial equilibrium and closed the model using the Huggett (1993) framework, a pure exchange economy with no capital. We now embed the same household problem into a **production economy** with capital, following Aiyagari (1994). We study an economy with **two factors of production**: capital K and labor L, populated by a continuum of infinitely-lived households facing idiosyncratic labor income risk and incomplete markets. As before, we assume that a law of large numbers holds, so that the stationary distribution represents the share of households in each state. We look for **stationary equilibria**, characterized by a constant interest rate R and wage w.

Production. A representative firm operates a neoclassical production function $F(K, L)$ with constant returns to scale under perfect competition. Competitive factor pricing implies:

$$R = F_K(K, L) + (1 - \delta) \tag{5.1}$$

$$w = F_L(K, L) \tag{5.2}$$

where $\delta \in (0, 1)$ is the depreciation rate of capital.

Household's problem. Each household is characterized by a state (a, s), where a denotes asset holdings and $s \in \mathcal{S}$ is an exogenous idiosyncratic labor productivity state that follows a Markov chain with transition probabilities $\pi(s'|s)$.

Bellman Equation

The household solves:

$$v(a, s) = \max_{c,\, a'} \left\{ u(c) + \beta \sum_{s' \in \mathcal{S}} \pi(s'|s)\, v(a', s') \right\} \tag{5.3}$$

subject to:

$$\begin{aligned} c + a' &\leq Ra + w \cdot \ell(s) \\ c &\geq 0 \\ a' &\geq -\phi \end{aligned}$$

Here $\ell(s)$ is the (exogenous) efficiency units of labor supplied in state s, so labor income is $y(s) \equiv w \cdot \ell(s)$. The parameter $\phi \geq 0$ controls the borrowing limit: $\phi = 0$ means no borrowing is allowed, and $\phi > 0$ is an ad-hoc borrowing constraint.

Aggregate resource constraint. The aggregate resource constraint for the economy is:

$$F(K, L) = C + I,$$

where investment is $I = K' - (1 - \delta)K$. In a **stationary equilibrium**, $K' = K$, so:

$$C = F(K, L) - \delta K. \tag{5.4}$$

The **golden rule** capital stock K^* maximizes stationary consumption and satisfies $F_K(K^*, \bar{L}) = \delta$, or equivalently $R = 1$ at the golden rule. The neoclassical growth model always operates to the left of the golden rule (i.e., $K < K^*$), since optimizing households require $R > 1$ (equivalently $F_K > \delta$). However, as we will see, the Aiyagari model can feature $K > K^*$.

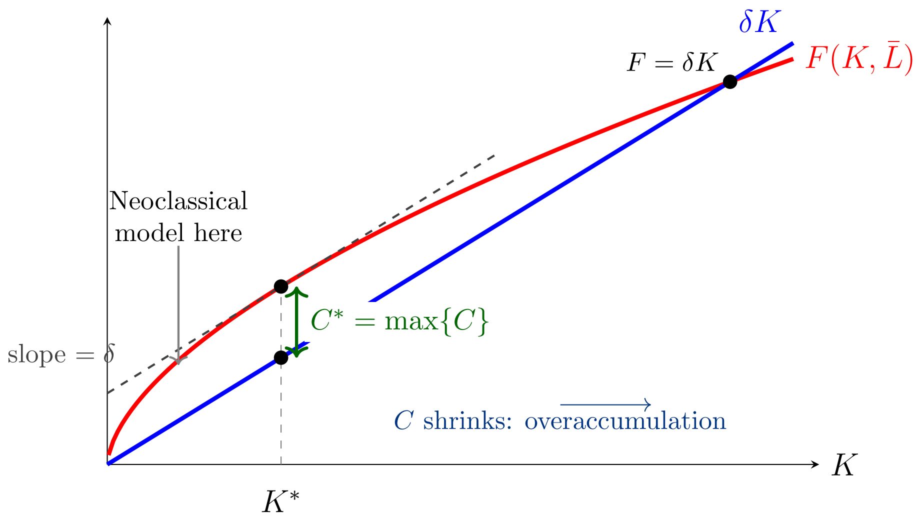


Figure 5.1: Steady-state consumption $C = F(K, \bar{L}) - \delta K$ is maximized at the golden rule K^* where $F_K = \delta$. The tangent to F at K^* is parallel to δK. Beyond the crossing point, $\delta K > F$ and the economy cannot sustain itself.

Labor market clearing. Let $\lambda(a, s)$ denote the stationary distribution over individual states, and let $\pi^{ss}(s)$ denote the marginal (ergodic) distribution over productivity states. With the normalization:

$$L = \sum_{a,s} \lambda(a, s)\, \ell(s) = \sum_{s} \pi^{ss}(s)\, \ell(s) \equiv 1,$$

only the marginal distribution over s matters for aggregate labor. This normalization pins $L = 1$.

Aggregate budget constraint. Let $g(a, s)$ denote the optimal savings policy function, so that consumption is $c(a, s) = Ra + w \cdot \ell(s) - g(a, s)$. Aggregating across the stationary distribution:

$$C = \sum_{a,s} \lambda(a,s)\, c(a,s) = R \underbrace{\sum_{a,s} \lambda(a,s)\, a}_{=A} + w \cdot 1 - \underbrace{\sum_{a,s} \lambda(a,s)\, g(a,s)}_{=A'},$$

where A denotes aggregate assets. In a stationary equilibrium, $A = A'$ (the distribution is invariant), so:

$$C = (R-1)A + w. \tag{5.5}$$

Asset market clearing.

$$\boxed{K = \sum_{a,s} \lambda(a,s)\, g(a,s) \equiv A.} \tag{5.6}$$

That is, aggregate household savings equal the capital stock demanded by firms. Why must this hold? In stationary equilibrium there is no aggregate risk, so loans between households and physical capital earn the same rate of return R. Households are therefore indifferent between lending to other households and investing directly in capital to rent to firms. The ownership structure of the capital stock is indeterminate: many configurations of ownership are consistent with the same equilibrium allocations, just as in the neoclassical growth model. What is pinned down is the *aggregate*: total household savings must equal the capital stock.

Walras' Law. Asset market clearing (5.6) combined with the aggregate budget constraint (5.5) implies the aggregate resource constraint (5.4). Substituting (5.1)–(5.2) and $A = K$:

$$C = \big(F_K(K,1) + 1 - \delta - 1\big)K + F_L(K,1) = F_K(K,1) \cdot K + F_L(K,1) - \delta K = F(K,1) - \delta K,$$

where the last equality uses Euler's theorem (constant returns to scale).

Stationary equilibrium. We are now ready to define a stationary equilibrium. As before, this represents a situation where aggregate quantities and prices are constant over time and the wealth distribution is in its ergodic state, meaning the cross-sectional distribution λ reproduces itself period after period.

Definition: Stationary Equilibrium

A **stationary equilibrium** consists of prices R, w (scalars), a capital stock K, a value function v, a policy function g, and a distribution λ (functions/vectors) such that:

(i) The value function v solves the household's Bellman equation (5.3) given R and $y(s) = w \cdot \ell(s)$, and g is the associated optimal policy.

(ii) Factor demands are consistent with prices: $R = F_K(K, 1) + (1 - \delta)$ and $w = F_L(K, 1)$.

(iii) λ is the stationary (ergodic) distribution induced by the policy function g and the Markov chain π.

(iv) Labor and asset markets clear: $L = 1$ and $K = A$.

(v) The aggregate resource constraint holds: $C = F(K, 1) - \delta K$.

Conditions (iv) and (v) are not independent by Walras' Law. In practice, we verify asset market clearing and the resource constraint follows.

Reducing equilibrium to one equation. Since $L = 1$, the factor demand equations pin down everything as a function of a single variable. From $R = F_K(K, 1) + (1 - \delta)$, we can invert to obtain $K(R)$, and then:

$$w(R) \equiv F_L\big(K(R), 1\big).$$

Thus, given any candidate R, we can compute $w(R)$, solve the household problem to obtain the policy function $g(a, s \,|\, R, w(R))$, compute the stationary distribution $\lambda(a, s \,|\, R, w(R))$, and aggregate to get asset supply:

$$A\big(R, w(R)\big) = \sum_{a,s} \lambda\big(a, s \,|\, R, w(R)\big)\, g\big(a, s \,|\, R, w(R)\big).$$

Equilibrium requires $A(R, w(R)) = K(R)$. This reduces the problem to finding the intersection of the **asset supply** curve $A(R, w(R))$ and the **capital demand** curve $K(R)$.

5.2 Properties of Asset Supply

The behavior of $A(R, w)$ for a fixed wage w is illustrated in Figure 5.2:

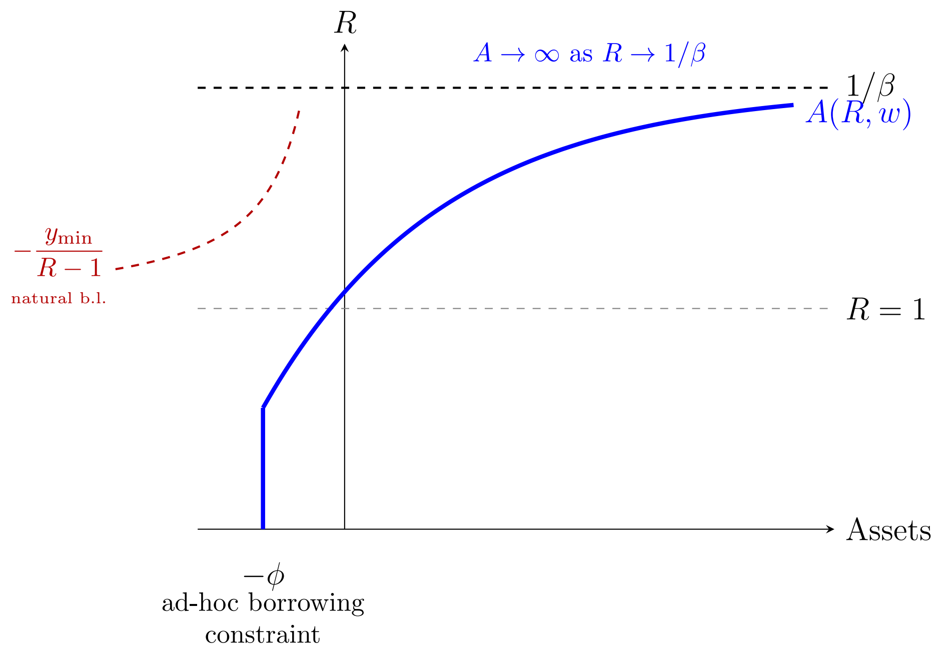


Figure 5.2: Asset supply $A(R, w)$ for fixed w. The curve is vertical at $-\phi$: for sufficiently low R, all agents are at the borrowing limit. As $R \to 1/\beta$, precautionary savings diverge ($A \to \infty$).

Key properties:

- As $R \uparrow 1/\beta$, asset supply diverges: $A \to \infty$. (At $R = 1/\beta$, the Euler equation fails to generate a well-defined stationary distribution.)
- At $R = 0$, households have no return on savings, so they borrow to the maximum: $A(0, w) = -\phi$.
- Monotonicity of A in R is intuitive but hard to prove in general. There are some results in the literature (see Achdou et al., 2022, for the continuous-time case).

Equilibrium Diagram: Aiyagari vs. Neoclassical vs. Huggett

We are now ready to use the diagram to analyze stationary equilibria. This requires letting w vary with R through the production side: since $w(R) = F_L(K(R), 1)$, the wage adjusts endogenously as the interest rate changes. The curve $A(R, w(R))$ is plotted in Figure 5.3, and its intersection with the capital demand curve $K(R)$ represents a stationary equilibrium. Note that at this point there is no guarantee that the stationary equilibrium is unique, as the curve $A(R, w(R))$ may be non-monotonic.

The figure highlights four special points along the capital demand curve $K(R)$:

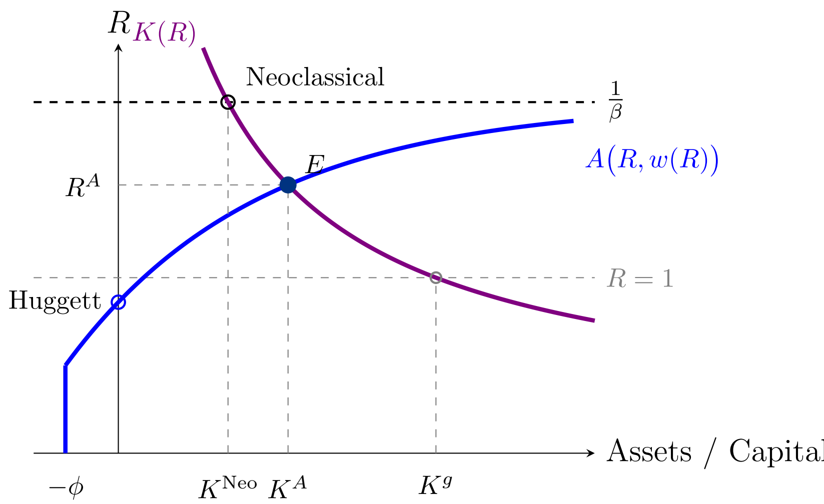


Figure 5.3: Equilibrium comparison. Capital demand $K(R)$ (violet, decreasing) and asset supply $A(R, w(R))$ (blue, increasing). The Aiyagari equilibrium E has $R^A < 1/\beta$ and $K^A > K^{\text{Neo}}$. The Huggett equilibrium ($A = 0$) has a lower R. K^g denotes the golden rule level.

- **Neoclassical** ($R = 1/\beta$): The representative-agent benchmark. Without idiosyncratic risk, asset supply is perfectly elastic at $R = 1/\beta$, so the equilibrium lies on the capital demand curve at this interest rate, giving K^{Neo}.
- **Aiyagari** (E): The intersection of $A(R, w(R))$ and $K(R)$. Precautionary savings shift asset supply to the right, pushing the equilibrium interest rate below $1/\beta$ and the capital stock above K^{Neo}.
- **Huggett** ($A = 0$): The interest rate at which asset supply crosses zero. This corresponds to the Huggett (1993) economy (a pure exchange economy with no capital) where net asset supply must equal zero.
- **Golden rule** (K^g, $R = 1$): The capital stock that maximizes steady-state consumption. Beyond this point, further capital accumulation is dynamically inefficient.

Why is K^g the golden rule? Recall that the golden rule maximizes steady-state consumption: $C = F(K, 1) - \delta K$, which requires $F_K = \delta$. From the factor demand equation $R = F_K + (1 - \delta)$, setting $F_K = \delta$ gives $R = 1$. So K^g is precisely where the capital demand curve $K(R)$ crosses the $R = 1$ line.

Key relationships.

(a) $R^A < 1/\beta = R^{\text{Neoclassical}}$: Precautionary savings push the interest rate below the rate of time preference.

(b) $K^A > K^{\text{Neoclassical}}$: More savings means more capital in equilibrium.

(c) $K^A \gtrless K^{\text{golden}}$: The Aiyagari equilibrium can feature **dynamic inefficiency** (overaccumulation of capital, $R < 1$). This is impossible in the representative-agent neoclassical growth model but can arise here because precautionary motives are strong enough to push R below 1.

Special Case: $\phi = 0$ and No Capital (Huggett Economy)

When $\phi = 0$ (no borrowing) and there is no capital (pure endowment economy), the model reduces to the Huggett (1993) economy studied in Chapter 4. The asset supply function $A(R)$ starts at 0 for low R and is weakly increasing. Since the zero-net-supply condition requires $A(R) = 0$, any R sufficiently low that no household wishes to save constitutes a stationary equilibrium. There can therefore be a **continuum of equilibria** at low interest rates.

5.3 A Comparative Static: Relaxing the Borrowing Limit

Let's now discuss the effect of ϕ on asset supply. Consider the boundary behavior:

- When $R = 0$: $A(0, w) = -\phi$ (all agents borrow to the limit).
- When $R = 1$: $A(1, w(1)) = \hat{A}(1, w(1)) - \phi$, where $\hat{A}$ denotes aggregate savings of the *normalized problem* (with $\hat{a} = a + \phi$, as in equation (4.1)). When $R = 1$, the normalized problem does not depend on ϕ.

These two observations suggest that increasing ϕ shifts the entire asset supply curve $A(R)$ to the left: at both $R = 0$ and $R = 1$, the horizontal shift is exactly $\Delta\phi$.

Consider relaxing the borrowing constraint from ϕ_0 to $\phi_1 > \phi_0$ (agents can borrow more).

Result: Effect of Relaxing the Borrowing Constraint

A relaxation of the borrowing limit ($\phi_1 > \phi_0$) shifts the asset supply curve to the left (households save less at every R), resulting in:

(i) A **higher** equilibrium interest rate: $R_1 > R_0$.

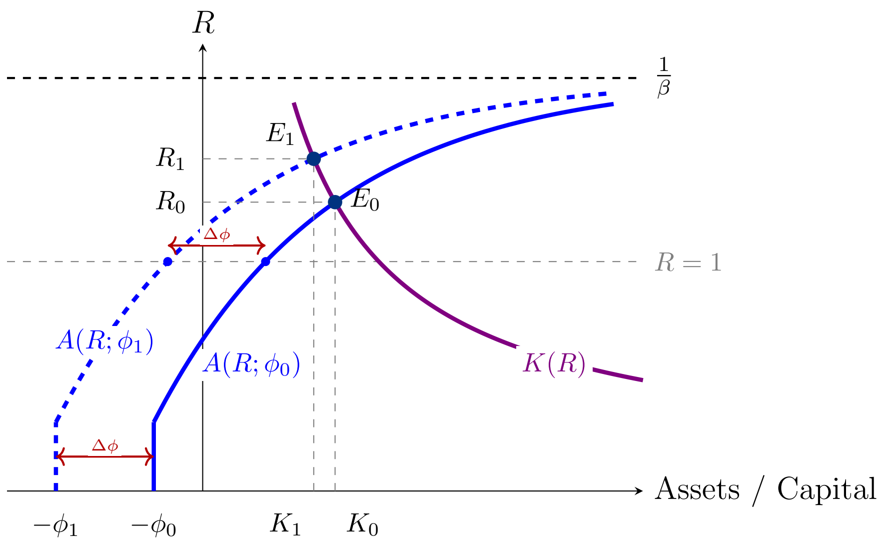


Figure 5.4: Relaxing the borrowing constraint ($\phi_1 > \phi_0$) shifts $A(R)$ to the left by exactly $\Delta\phi = \phi_1 - \phi_0$ at both $R = 0$ and $R = 1$. The new equilibrium E_1 has a higher interest rate ($R_1 > R_0$) and lower capital ($K_1 < K_0$).

(ii) A **lower** equilibrium capital stock: $K_1 < K_0$.

Intuitively, when agents can borrow more freely, the precautionary savings motive weakens, reducing aggregate asset supply. In equilibrium, a higher interest rate is needed to restore asset market clearing, and the capital stock falls.

In the limit as $\phi \to \infty$, the equilibrium interest rate R does *not* approach $1/\beta$. Even in this case, the Martingale Convergence Theorem result holds and assets diverge. The reason is that as $\phi \to \infty$, the model reduces to one with a **natural borrowing limit**: in general equilibrium, because the asset supply is limited (or alternatively, because the aggregate resource constraint limits aggregate consumption), households must still face uninsurable income risk and cannot fully smooth consumption. Precautionary savings motives persist, so the equilibrium rate remains below the complete-markets benchmark. Bewley (1980) originally conjectured that removing the ad hoc borrowing constraint would recover $R = 1/\beta$, but this turns out to be incorrect, as pointed out in Bewley (1983).

5.4 Application: Two Open Economies and Global Imbalances

Following Mendoza, Quadrini, and Ríos-Rull (2009), consider two countries that differ in their borrowing constraints:

- **Country A**: Loose borrowing constraint ($\phi_A > 0$, developed financial markets).
- **Country B**: Tight borrowing constraint ($\phi_B = 0$, no borrowing allowed, limited financial development).

In autarky, Country B has a lower interest rate $R^B < R^A$ due to stronger precautionary savings. When markets open, capital flows from B to A, and a common world interest rate R^* emerges with $R^B < R^* < R^A$. This provides a model of **global imbalances**: financially underdeveloped countries run current account surpluses (excess savings flow abroad).

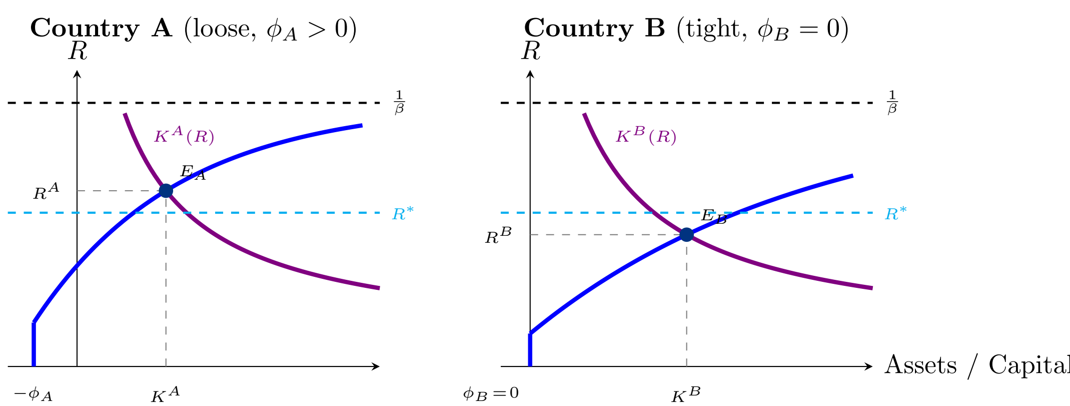


Figure 5.5: Two open economies. Country A (loose borrowing constraint) has higher R^A. Country B (tight constraint, $\phi_B = 0$) has lower R^B due to stronger precautionary savings. Under financial integration, $R^B < R^* < R^A$; capital flows from B to A.

Figure 5.5 is a Metzler diagram adapted to the Aiyagari framework. Each panel shows the autarky equilibrium of one country as the intersection of its capital demand $K(R)$ and asset supply $A(R, w(R))$. Because both countries share the same technology, the $K(R)$ curves are identical. The key difference is in the asset supply curves: Country B's tighter borrowing constraint ($\phi_B = 0$) generates stronger precautionary savings at every interest rate, shifting A^B to the right relative to A^A. This produces a lower autarky rate $R^B < R^A$.

When the two economies open to trade in assets, a single world interest rate R^* must clear the *global* asset market: $A^A(R^*) + A^B(R^*) = K^A(R^*) + K^B(R^*)$. Since $R^B < R^A$, the equilibrium R^* lies between the two autarky rates. At R^*, Country B has excess asset supply (it saves more than its domestic capital demand), so it runs a current account surplus—its savings flow abroad to Country A. This is the Mendoza, Quadrini, and Ríos-Rull (2009) mechanism: differences in financial development across countries generate capital flows and global imbalances even when technologies are identical.

In addition to the Mendoza, Quadrini, and Ríos-Rull (2009) paper, Caballero, Farhi, and Gourinchas (2008) develop a related model in which heterogeneity in the ability to produce financial assets, rather than borrowing constraints per se, drives global imbalances and depresses the world interest rate.

5.5 Algorithm to Compute Stationary Equilibrium

Pseudo-Code: Stationary Equilibrium

a) Initialize $j = 0$; guess an initial capital stock K_0.

b) Given K_j, compute prices: $R_j = F_K(K_j, 1) + (1 - \delta)$ and $w_j = F_L(K_j, 1)$.

c) Given (R_j, w_j), solve the household's Bellman equation (5.3) by value function iteration. *Check that the grid boundaries (extrema of the asset grid) are not binding.*

d) Extract the optimal policy function $g_j(a, s)$. Together with the transition matrix π, compute the stationary distribution λ_j.

e) Using λ_j, compute aggregate asset supply: $A_j = \sum_{a,s} \lambda_j(a, s)\, g_j(a, s)$.

f) **Convergence check:** If $|A_j - K_j| < \texttt{tol}$, stop (equilibrium found). Otherwise, update:

$$K_{j+1} = (1 - \varepsilon)\, K_j + \varepsilon\, A_j, \quad \varepsilon \in (0, 1],$$

where ε is a **dampening parameter** (gain), and return to step (b) with $j \leftarrow j + 1$.

The algorithm is searching for a zero of the excess demand function $A(K) - K$. An alternative is to use a bisection method on R (or equivalently on K), which is guaranteed to converge when the excess demand is monotonic. Why then iterate on K rather than R? The mapping $K \mapsto A(K)$ (through prices) is generally better behaved than $R \mapsto A(R)$

because the asset supply function can be very steep near $R = 1/\beta$.

5.6 Aiyagari's (1994) QJE Calibration and Results

Calibration. Aiyagari (1994) uses the following parameterization:

- **Preferences**: CRRA utility $u(c) = \dfrac{c^{1-\mu} - 1}{1-\mu}$, with $\mu \in \{1, 3, 5\}$.
- **Labor endowment**: AR(1) process in logs,

$$\log \ell_t = \rho \log \ell_{t-1} + \sigma\sqrt{1-\rho^2}\,\varepsilon_t, \quad \varepsilon_t \sim N(0,1), \tag{5.7}$$

  where the scaling $\sigma\sqrt{1-\rho^2}$ ensures that the unconditional standard deviation of $\log \ell_t$ is σ, regardless of persistence ρ.
- **Persistence**: $\rho \in \{0,\ 0.3,\ 0.6,\ 0.9\}$.
- **Volatility**: $\sigma \in \{0.2,\ 0.4\}$.
- **Borrowing constraint**: $\phi = 0$ (no borrowing, as an extreme case).

To map the continuous AR(1) process (5.7) into the discrete Markov chain $\pi(s'|s)$ used in the model, one applies a discretization method such as Tauchen (1986) or Rouwenhorst (1995).

Main results.

Result: Near-Equivalence to the Neoclassical Benchmark

In Aiyagari's calibration, the equilibrium values of R and K are **very close** to their neoclassical (representative-agent) counterparts. This led some researchers to conclude that household heterogeneity "does not matter" for aggregate quantities.

Other results. Despite near-identical aggregates, the model produces important **cross-sectional** implications:

(i) **Consumption smoothing**: Individual consumption is less volatile than labor income.

(ii) **Cross-sectional dispersion**: The distribution of consumption is less dispersed than the distribution of labor income.

(iii) **Few constrained agents**: The fraction of borrowing-constrained households is close to zero.

(iv) **Wealth inequality puzzle**: There is *significantly* more wealth inequality in the data than the model generates.

The last point, the inability of the basic Aiyagari model to generate realistic wealth concentration, sparked a large subsequent literature exploring richer income processes (fat tails, nonlinear persistence), bequests and intergenerational transfers, entrepreneurship and heterogeneous returns to wealth, and large medical expenditure shocks.

In the numerical simulations of Aiyagari (1994, footnote 34), the savings policy functions are *approximately linear* in assets. This near-linearity observation is a key ingredient behind the Krusell and Smith (1998) algorithm for computing equilibria with aggregate shocks: if policies are approximately linear, then tracking the mean of the wealth distribution (rather than the full distribution) is sufficient to forecast future prices.

Appendix: Natural Borrowing Limits and the Equilibrium Real Rate

Suppose that households face the **natural borrowing limit**. We do *not* require that the lowest income realization $y(s_1)$ is strictly positive.

Result: Natural Borrowing Limit under $r < 0$

If $r < 0$ (equivalently, $R < 1$), then the natural borrowing limit is infinite.

Proof. Fix any $\phi > 0$. We show that a household at maximum debt $a_0 = -\phi$ can maintain $c_t \geq 0$ and $a_t \geq -\phi$ in every period, even under the worst-case income realization $y_t = y(s_1) \geq 0$ for all t. Since $c_t \geq 0$, the budget constraint gives:

$$a_{t+1} = Ra_t + y(s_1) - c_t \leq Ra_t + y(s_1).$$

Setting $c_t = 0$ (zero consumption), we obtain the tightest upper bound on debt accumulation:

$$a_T = R^T(-\phi) + y(s_1)\frac{R^T - 1}{R - 1}.$$

Since $0 < R < 1$, both terms are bounded: $|R^T\phi| \leq \phi$ and the second term is non-negative. In particular, $a_T \geq -R^T\phi > -\phi$ for all $T \geq 1$, so the borrowing constraint $a_T \geq -\phi$ is satisfied at every step. Since this holds for any $\phi > 0$, the natural borrowing limit is infinite. □

Result: Non-negative Interest Rate under the Natural Borrowing Limit

Under the natural borrowing limit, any stationary equilibrium has $r \geq 0$.

Proof. Suppose, toward a contradiction, that $r < 0$. By the result above, the natural borrowing limit is infinite, so households face no effective borrowing constraint. Without a finite borrowing limit, household consumption is unbounded, so aggregate consumption is infinite. This violates the aggregate resource constraint $C = F(K, L) - \delta K < \infty$. □

Ad-hoc borrowing limits and negative interest rates. The result above does *not* mean that equilibrium interest rates are always positive. Under **ad-hoc borrowing limits**, the interest rate can be negative. The simplest example is the Huggett economy with $\phi = 0$ (no borrowing). In that case, the equilibrium interest rate R must satisfy $R \leq \bar{R}$, where $\bar{R}$ is defined in equation (4.2), and $\bar{R}$ itself can be less than one (i.e., $r < 0$). For the CRRA case, equation (4.3) shows that whether $\bar{R} < 1$ depends on the interaction between the discount factor β, the risk aversion σ, and the income process $\pi(s'|s)$, $\{y(s)\}$. Consider two examples:

- **High risk aversion.** When σ is large, the precautionary motive is strong: the term $[y(s)/y(s')]^\sigma$ explodes when $y(s') > y(s)$, driving the denominator in the $\bar{R}$ formula upward and $\bar{R}$ downward. For sufficiently high σ, $\bar{R} < 1$ even with moderate β.
- **High income volatility.** When income dispersion is large (high $y(s_N)/y(s_1)$), the ratio $[y(s)/y(s')]^\sigma$ amplifies through the convexity channel, again pushing $\bar{R}$ below one. Intuitively, households facing large uninsurable income risk save so aggressively that the equilibrium interest rate must be negative to clear the asset market.

Chapter 6

Transitions, Policy, and Pareto Improvements

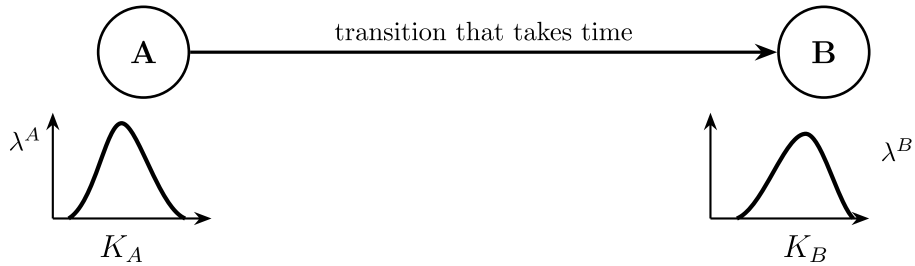
This chapter moves beyond stationary equilibria to study the dynamic transition path between steady states. When a parameter or policy changes, the aggregate capital stock and wealth distribution adjust gradually. We develop the transition algorithm (iterating backwards on value functions and forwards on distributions) and define the correct welfare comparison that accounts for transition costs. We then extend the Aiyagari model to include government taxation, transfers, and bonds, and ask whether there exist Pareto-improving fiscal reforms. Restricting attention to constant-wage policies, we characterize feasibility through a single aggregate resource constraint and connect the framework to the sequence-space Jacobian method.

6.1 Transitions in the Aiyagari Model

Suppose we start from a stationary equilibrium **A**, with associated aggregate capital K_A, stationary distribution λ_A, and value function v_A, and we change a parameter or a policy.

We can compute a new stationary equilibrium **B**, with aggregate capital K_B, stationary distribution λ_B, and value function v_B, associated with the new parameter or policy.

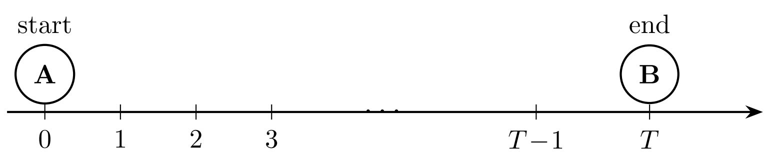
However, the economy does not jump instantaneously from **A** to **B**. The aggregate capital stock K_t and the wealth distribution λ_t are *state variables* that take time to adjust. We need to compute the **transition path** connecting the two steady states.



A Transition Algorithm

The key computational challenge is to find the entire path of prices, allocations, and distributions that connects the two steady states. The strategy combines two passes: a **backward iteration** on the household's value function (which requires knowing future prices) and a **forward iteration** on the wealth distribution (which requires knowing current policy functions). Because the price sequence depends on aggregate capital, which in turn depends on the distribution, the algorithm nests these two passes inside a fixed-point loop over the guessed capital path.

Fix a horizon T for the transition (the number of periods the economy takes to travel from **A** to **B**). The terminal condition is $K_{T+1} = K^B$.



Transition Path Computation

Step 1. Guess a sequence of aggregate capital $\{K_t\}_{t=1}^{T}$, with $K_0 = K_A$ and $K_{T+1} = K_B$.

Step 2. Compute the sequence of prices $\{R_t, w_t\}_{t=1}^{T}$ from the firm's problem. For the standard Aiyagari model with $F(K, L) = K^\alpha L^{1-\alpha}$:

$$r_t = \alpha K_t^{\alpha-1} L^{1-\alpha} - \delta, \qquad w_t = (1-\alpha) K_t^\alpha L^{-\alpha}, \qquad R_t = 1 + r_t.$$

If the policy or parameter change directly affects prices (e.g., through taxes on factor income), make sure to incorporate those effects here.

Step 3. Iterate **backwards** on the household problem, starting at period T. For all (a, s) on the grid, solve:

$$v_t(a, s) = \max_{a' \geq -\phi} \left\{ u(c) + \beta \sum_{s'} \pi(s'|s)\, v_{t+1}(a', s') \right\}$$

$$\text{s.t.} \quad c + a' \leq w_t\, \ell(s) + R_t\, a$$

The terminal condition is $v_{T+1}(\cdot) = v^B(\cdot)$, the value function from steady state **B**. This backward iteration generates a sequence of value functions and policy functions $\{v_t, g_t\}_{t=0}^{T}$.

Step 4. Iterate **forwards** on the distribution, starting from $\lambda_0 = \lambda_A$:

$$\lambda_{t+1}(a', s') = \sum_{s} \sum_{\{a:\, g_t(a,s)=a'\}} \pi(s'|s)\, \lambda_t(a, s)$$

This generates a sequence of distributions $\{\lambda_t\}_{t=1}^{T+1}$.

Step 5. Compute the **implied** aggregate capital from the distributions:

$$\hat{K}_t = \sum_a \sum_s \lambda_t(a, s) \cdot a, \qquad t = 1, \ldots, T.$$

Step 6. Check convergence: if $\|\hat{K}_t - K_t\| < \varepsilon_{\text{tol}}$ for all t, the inner loop has converged. If not, update the guess using a dampening parameter $\omega \in (0, 1]$:

$$K_t' = (1 - \omega)\, K_t + \omega\, \hat{K}_t$$

and return to Step 2 with $\{K_t'\}$.

Step 7. Check the terminal distribution: verify that λ_{T+1} is close to λ_B:

- If $\|\lambda_{T+1} - \lambda_B\| < \varepsilon_{\text{tol}}$, you are done.
- If not, increase T and restart from Step 1.

In principle, you can start the transition from *any* initial distribution λ_0, not only the stationary distribution of steady state **A**. Simply use the desired initial distribution in Step 4. This flexibility is useful for studying the response to "MIT shocks," where the economy begins in a stationary equilibrium and is hit by an unanticipated, one-time change.

6.2 Welfare Analysis

The transition algorithm produces three objects evaluated at every state (a, s):

1. $v_A(a, s)$: the value function in the *original* steady state **A** (where the agent was before the change).
2. $v_0(a, s)$: the value function at $t = 0$, the moment the policy change is announced and implemented. **This is the key object for welfare.**
3. $v_B(a, s)$: the value function in the *new* steady state **B** (the long-run destination).

The crucial insight is that $v_0(a, s)$ encodes the *entire transition path*: it is the expected discounted utility of an agent who starts at (a, s) and lives through the transition from the old to the new steady state.

The Welfare Comparison

To evaluate the welfare effect of the policy change, we compare the two value functions $v_A(a, s)$ and $v_0(a, s)$ over the support of the initial stationary distribution λ_A. For each state (a, s) in the support of λ_A, the sign of $v_0(a, s) - v_A(a, s)$ tells us whether a household at that point *wins* or *loses* from the policy change. This state-by-state comparison provides a complete picture of the distributional consequences of the reform: wealthy households may gain while borrowing-constrained households lose, or vice versa.

To obtain a scalar welfare assessment, we can aggregate these individual gains and losses using a **social welfare function** Ψ, evaluated under the *initial* distribution λ_A:

$$\Psi(v_A, \lambda_A) \quad \text{vs.} \quad \Psi(v_0, \lambda_A)$$

Both sides use the *same* distribution λ_A, because we are comparing welfare for the same population of agents, those who exist at the moment the policy is introduced. The choice of Ψ encodes value judgments about how to weigh gains and losses across different households.

Example: Utilitarian welfare. Under a utilitarian social welfare function, the aggregate welfare criterion reduces to:

$$\int v_A(a, s)\, d\lambda_A \quad \text{vs.} \quad \int v_0(a, s)\, d\lambda_A$$

This weights every household equally. Alternative social welfare functions, such as Rawlsian (maximizing the welfare of the worst-off) or weighted utilitarian criteria that place greater weight on poorer households, would yield different policy rankings.

Comparing steady-state value functions v_A vs. v_B ignores the transition costs. Agents alive during the transition experience changing prices, binding constraints, and portfolio rebalancing. The function v_0 captures all of these effects. In general, a policy that raises long-run welfare ($v_B > v_A$) could still reduce welfare once transition costs are accounted for ($v_0 < v_A$), or vice versa. See the appendix on welfare comparisons and forced savings in the growth model for a concrete illustration of this point.

6.3 Policy in the Aiyagari Model

We now extend the Aiyagari model to incorporate government policy. The government levies linear taxes on capital and labor income, issues one-period bonds, and rebates revenue as lump-sum transfers. This framework nests many classic policy experiments: changes in capital taxation, the introduction of unemployment insurance, or shifts in the level of public debt. The key ingredients are a government budget constraint linking taxes, transfers, and bond issuance, and a no-arbitrage condition equating the return on capital to the return on government bonds. With these in hand, we define a competitive equilibrium with government policy and outline how the transition algorithm from Section 6.1 and the welfare criterion from Section 6.2 combine to deliver a complete policy evaluation.

Taxes on factor income. The government levies linear taxes on factor income at rates τ_t^k (capital income) and τ_t^n (labor income). Here r_t^k denotes the rental rate of capital and w_t the wage. Factor prices inclusive of taxes are:

$$\text{Capital: } (1+\tau_t^k)\, r_t^k, \qquad \text{Labor: } (1+\tau_t^n)\, w_t.$$

Firm's problem. The representative firm takes tax-inclusive factor prices as given. The first-order conditions are:

$$F_K(K_t, L_t) = (1+\tau_t^k)\, r_t^k$$
$$F_L(K_t, L_t) = (1+\tau_t^n)\, w_t$$

No-arbitrage. Households are indifferent between holding capital and bonds, which requires:

$$r_t^k = r_t + \delta$$

That is, the rental rate of capital equals the risk-free rate plus the depreciation rate.

The no-arbitrage condition follows from the household's portfolio choice. Since both capital and government bonds are available as savings vehicles and households can freely allocate wealth between them, in equilibrium both assets must offer the same return. A unit of capital purchased today costs one unit of the consumption good, yields rental income r_t^k next period, and depreciates at rate δ, so its net return is $r_t^k - \delta$. A government bond pays the risk-free rate r_t. If $r_t^k - \delta > r_t$, all households would prefer capital; if $r_t^k - \delta < r_t$, all would prefer bonds. For both assets to be held in positive quantities, we need $r_t^k - \delta = r_t$.

Household's problem. Given no-arbitrage, capital and government bonds are perfect substitutes from the household's perspective. We can therefore write the household's problem in terms of a single asset a with gross return $R_t = 1 + r_t$, without tracking the portfolio composition. The budget constraint now includes a lump-sum government transfer T_t (which may vary over time during the transition):

$$\begin{aligned} c + a' &\leq w_t\, \ell(s) + R_t\, a + T_t \\ a' &\geq -\phi \end{aligned}$$

where T_t is a transfer that all households receive equally, and the borrowing limit $\phi \geq 0$ is an ad hoc (exogenous) constraint. The household's Bellman equation is:

$$v_t(a, s) = \max_{a' \geq -\phi} \left\{ u(w_t\, \ell(s) + R_t\, a + T_t - a') + \beta \sum_{s'} \pi(s' \mid s)\, v_{t+1}(a', s') \right\}$$

Note that the value function is indexed by t because prices (w_t, R_t) and transfers T_t vary along the transition path.

Portfolio composition may matter at the moment of an unexpected policy change if the reform alters the realized return on capital and bonds differentially at $t = 0$. In that case, two households with the same total wealth a but different splits between capital and bonds would experience different wealth shocks. We will return to this issue later.

Government budget constraint. The government finances transfers through tax revenue and bond issuance:

$$T_t \leq \tau_t^n\, w_t\, L_t + \tau_t^k\, r_t^k\, K_t + B_{t+1} - (1 + r_t)\, B_t$$

where B_t denotes government bonds outstanding at the start of period t.

Asset market clearing. Total household savings must equal the sum of physical capital and government bonds:

$$A_t = K_t + B_t, \qquad \text{where } A_t = \sum_a \sum_s \lambda_t(a, s)\, a.$$

Distribution dynamics. Let $g_t(a, s)$ denote the policy function for next-period assets, i.e., the solution to the household's Bellman equation at date t. As we showed before, the wealth distribution evolves according to:

$$\lambda_{t+1}(a', s') = \sum_s \pi(s' \mid s) \sum_{\{a:\, g_t(a,s) = a'\}} \lambda_t(a, s)$$

That is, the mass of households at (a', s') tomorrow equals the sum over all current states (a, s) that transition to s' and whose optimal savings choice lands on a'.

Equilibrium Definition

We are now ready to define an equilibrium for this economy. The definition collects the household optimality condition, the firm's first-order conditions, the government budget constraint, and market clearing into a single object.

Competitive Equilibrium with Government Policy

Given a government policy $\{B_t, T_t, \tau_t^n, \tau_t^k\}_{t=0}^{\infty}$ and an initial distribution λ_0, a **competitive equilibrium** is a sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$, value functions $\{v_t\}_{t=0}^{\infty}$, policy functions $\{g_t\}_{t=0}^{\infty}$, and distributions $\{\lambda_t\}_{t=0}^{\infty}$ such that for all $t = 0, 1, 2, \ldots$:

1. **Household optimality:** given time-t prices (r_t, w_t) and transfers T_t, the value function v_t solves the Bellman equation and g_t is the associated policy function.
2. **Firm optimality:** the time-t rental rate and wage satisfy $F_K(K_t, L_t) = (1 + \tau_t^k)\, r_t^k$ and $F_L(K_t, L_t) = (1 + \tau_t^n)\, w_t$, where $r_t^k = r_t + \delta$ (no-arbitrage).
3. **Government budget constraint:** $T_t \leq \tau_t^n\, w_t\, L_t + \tau_t^k\, r_t^k\, K_t + B_{t+1} - (1 + r_t)\, B_t$.

4. **Distribution consistency:** the time-$(t+1)$ distribution satisfies $\lambda_{t+1}(a', s') = \sum_s \pi(s' \mid s) \sum_{\{a:\, g_t(a,s)=a'\}} \lambda_t(a, s)$.
5. **Market clearing:** $A_t = K_t + B_t$, where $A_t = \sum_a \sum_s \lambda_t(a, s)\, a$.

In this definition, we have imposed that the rates of return on capital and bonds are equalized at the beginning of time. With an unexpected policy shock this may not hold, and we will need to take a stand on the portfolio composition of wealth that each household starts with. We will make sure in our analysis below that this is not a problem.

For an analysis where the revaluation of assets plays a central role in the welfare effects of a debt-financed fiscal stimulus, see Barreto (2025).

When government bonds $B_t > 0$ are present, the asset market clearing condition becomes $A_t = K_t + B_t$. The transition algorithm must now guess (or jointly solve for) the path of both K_t and B_t, subject to the government budget constraint. In practice, one often fixes a path for tax rates and bond issuance, and solves for the transfers T_t residually from the government budget constraint. Note that the transfers T_t depend on equilibrium prices, so they will have to be part of the iteration.

6.4 Robust Pareto Improving Policies

We now ask whether there exist policy reforms in the Aiyagari economy that are **Pareto improving**: every household alive at the time of the reform is made weakly better off, and some strictly so. We will discuss the connection with Samuelson's (1958) classic consumption-loan paper.

The framework developed below follows Aguiar, Amador, and Arellano (2024).

The starting point is an initial situation with no government intervention:

$$\tau^k = \tau^n = T = B = 0 \qquad \text{(no taxes, no government).}$$

This is the standard Aiyagari economy, with stationary equilibrium:

$$r^o,\ w^o,\ A^o = K^o,\ B^o = 0,\ \lambda^o.$$

The aggregate labor supply is $L^o = \sum_s \lambda^o(s)\,\ell(s)$, and will be constant in the analysis below because we are at the ergodic distribution and labor supply is inelastic.

Suppose the government unexpectedly announces a policy $\{B_t,\, T_t,\, \tau_t^n,\, \tau_t^k\}_{t=0}^{\infty} \neq 0$. This is an **MIT shock**: the policy change is unexpected, but once announced, agents have perfect foresight about the entire future path.

Constant-Wage Policies

We restrict attention to policies that, in equilibrium, deliver a constant wage.

Constant-Wage Restriction

A policy $\{B_t,\, T_t,\, \tau_t^n,\, \tau_t^k\}_{t=0}^{\infty}$ satisfies the **constant-wage restriction** if the resulting equilibrium wage path satisfies

$$w_t = w^o \qquad \forall\, t.$$

Household Aggregation and Optimality

Under this constant-wage restriction, the household's problem depends only on the sequences $\{R_t,\, T_t\}_{t=0}^{\infty}$, where $R_t \equiv 1 + r_t$ is the gross interest rate and T_t is the lump-sum transfer. We can solve the household's dynamic programming problem given these sequences, and then aggregate using λ^o (the initial wealth distribution) together with the resulting policy functions.

The aggregate quantities can be expressed as functions of the entire sequence of prices and transfers:

Aggregate Functions

- $A_t(\{r_s, T_s\}_{s=0}^{\infty})$: aggregate household wealth at date t.
- $C_t(\{r_s, T_s\}_{s=0}^{\infty})$: aggregate household consumption at date t.

Although the notation is compact, these aggregate functions are rich objects. First, they are defined at each date t and depend on the *entire* infinite sequence of interest rates and transfers $\{r_s, T_s\}_{s=0}^{\infty}$, not just on contemporaneous prices. Second, embedded inside each function are the household optimality conditions (the solution to every household's dynamic programming problem) together with the aggregation of the resulting policy functions over the wealth distribution.

Each household's budget constraint is $c + a' = w^o z\, l + R_t\, a + T_t$. Integrating over all households using λ^o and the policy functions, and noting that aggregate labor income is $w^o L^o$ (since labor supply is inelastic and we aggregate at the ergodic distribution), we obtain the **aggregate household budget constraint**:

$$C_t(\{r_s, T_s\}_{s=0}^{\infty}) = w^o L^o + R_t\, A_t(\{r_s, T_s\}_{s=0}^{\infty}) - A_{t+1}(\{r_s, T_s\}_{s=0}^{\infty}) + T_t$$

Feasibility

Since the household's problem depends only on the sequences $\{r_t, T_t\}$ under constant wages, a natural question arises: which sequences $\{r_t, T_t\}$ can actually arise as part of a competitive equilibrium? That is, for which sequences does there exist a supporting fiscal policy and a consistent path for capital and bonds?

Feasibility

We say that the sequence $\{r_t,\, T_t\}_{t\geq 0}$ is **feasible** if there is a fiscal policy $\{B_t,\, \tau_t^n,\, \tau_t^k,\, T_t\}_{t\geq 0}$ with $B_0 = B^o$ and $r_0 = r^o$ such that a competitive equilibrium with quantities $\{A_t,\, K_t\}_{t\geq 0}$ and prices $\{r_t,\, r_t^k = r_t + \delta,\, w^o\}_{t\geq 0}$ exists, where $K_0 = K^o$ and $A_0 = A^o$.

The definition requires $r_0 = r^o$, which means $r_0^k = r^o + \delta$, meaning the pre-tax return to capital at date 0 is unchanged by the policy announcement. This ensures that the total realized return to wealth for households remains $R_0 = 1 + r^o$, so the initial portfolio allocation between capital and bonds is irrelevant. If we were to let r_0^k differ from $r^o + \delta$, the MIT shock would have a "valuation" effect on households' portfolios, and we would need to take a stand on each household's position in bonds versus capital at the time of the announcement.

The next result provides a convenient characterization of feasibility that avoids checking all the equilibrium conditions directly.

Result: Feasibility Characterization

A sequence $\{r_t,\, T_t\}_{t=0}^{\infty}$ with $r_0 = r^o$ is feasible if and only if there exist sequences $\{K_t,\, B_t\}$ with $K_0 = K^o$, $B_0 = 0$, such that for all $t \geq 0$:

(i) **Market clearing:** $A_{t+1}(\{r_s, T_s\}) = B_{t+1} + K_{t+1}$.

(ii) **Government budget constraint:**

$$B_{t+1} - R_t\, B_t - T_t \geq F(K^o, L^o) - F(K_t, L^o) - (r^o + \delta)\, K^o + (r_t + \delta)\, K_t.$$

We show necessity: if $\{r_t, T_t\}$ is feasible, then conditions (i) and (ii) hold. Sufficiency is left as an exercise.

Deriving the tax rates. From the firm's first-order conditions under constant wages:

Labor tax: $F_L(K_t, L^o) = (1 + \tau_t^n)\, w^o$, so

$$\tau_t^n = \frac{F_L(K_t, L^o)}{w^o} - 1.$$

Capital tax: $F_K(K_t, L^o) = (1 + \tau_t^k)(r_t + \delta)$, so

$$\tau_t^k = \frac{F_K(K_t, L^o)}{r_t + \delta} - 1.$$

Part (i): Follows from household optimality and aggregation: the sum of individual asset holdings must equal the total supply of assets (capital plus government bonds).

Part (ii): Compute tax revenue:

$$\begin{aligned}
\text{Tax revenue} &= \tau_t^n\, w^o\, L^o + \tau_t^k\, r_t^k\, K_t \\
&= \left[(1 + \tau_t^n)\, w^o\, L^o - w^o\, L^o\right] + \left[(1 + \tau_t^k)\, r_t^k\, K_t - r_t^k\, K_t\right] \\
&= F_L(K_t, L^o)\, L^o + F_K(K_t, L^o)\, K_t - w^o\, L^o - r_t^k\, K_t \\
&= F(K_t, L^o) - w^o\, L^o - r_t^k\, K_t
\end{aligned}$$

where the last equality uses Euler's theorem ($F = F_K \cdot K + F_L \cdot L$ for CRS production). Noting that at the initial steady state $F(K^o, L^o) = w^o L^o + (r^o + \delta)\, K^o$, the tax revenue can be rewritten as:

$$\text{Tax revenue} = F(K_t, L^o) - F(K^o, L^o) + (r^o + \delta)\, K^o - (r_t + \delta)\, K_t.$$

The government budget constraint $T_t \leq \text{Tax revenue} + B_{t+1} - (1 + r_t)\, B_t$ then yields condition (ii). □

To understand the economics behind this characterization, it helps to reinterpret the government budget constraint. Under constant returns to scale, output $F(K_t, L^o)$ is exhausted by after-tax factor payments and tax revenue:

$$F(K_t, L^o) = \underbrace{w^o\, L^o}_{\text{after-tax labor}} + \underbrace{r_t^k\, K_t}_{\text{after-tax capital}} + \underbrace{\text{Tax revenue}}_{\text{government}}.$$

Given that the wage bill if constant, the change in tax revenue equals the part of the change in output not absorbed by payments to capital:

$$\text{Tax revenue} = \big[F(K_t, L^o) - F(K^o, L^o)\big] - \big[(r_t + \delta)\, K_t - (r^o + \delta)\, K^o\big].$$

We can use this to derive the aggregate resource constraint. Start from the aggregate household budget constraint and decompose $A_t = K_t + B_t$ (market clearing) to separate the capital and bond components of household wealth:

$$\begin{aligned} C_t(\{r_s, T_s\}_{s=0}^{\infty}) = {} & \underbrace{F(K^o, L^o) - (r^o + \delta)\, K^o}_{=w^o L^o} + T_t \\ & + R_t\, B_t - B_{t+1} + R_t\, K_t - K_{t+1}. \end{aligned}$$

Using the government budget constraint to substitute out the bond terms, we arrive at the **resource constraint**:

$$\boxed{C_t(\{r_s, T_s\}_{s=0}^{\infty}) \leq F(K_t, L^o) - \underbrace{\big[K_{t+1} - (1-\delta)\, K_t\big]}_{=I_t \text{ (investment)}}}$$

This gives an alternative, simpler characterization of feasibility:

Result: Alternative Feasibility Characterization

A sequence $\{r_t, T_t\}_{t=0}^{\infty}$ with $r_0 = r^o$ is feasible if and only if there exists a sequence $\{K_t\}$ with $K_0 = K^o$ such that

$$C_t(\{r_s, T_s\}_{s=0}^{\infty}) \leq F(K_t, L^o) + (1-\delta)\, K_t - K_{t+1}.$$

The key simplification is that the government budget constraint and bond dynamics can be absorbed into the aggregate resource constraint. What matters for feasibility is whether aggregate consumption (which depends on the entire sequence of interest rates and transfers through household behavior) can be sustained by the economy's production technology and capital accumulation.

Connection to the Sequence-Space Jacobian Method.
The aggregate consumption function $C_t(\{r_s, T_s\}_{s=0}^{\infty})$ is a key building block of the **sequence-space Jacobian** method developed by Auclert, Rognlie, and Straub (2024). The central idea is to characterize how aggregate quantities (consumption, savings) respond to perturbations in the entire path of prices and transfers. The Jacobian of C_t with respect to $\{r_s\}$ and $\{T_s\}$ encodes the impulse responses of the heterogeneous-agent economy to arbitrary shocks.
Bhandari, Bourany, Evans, and Golosov (2023) develop a perturbational approach that, to first order, recovers the same linearization as the sequence-space Jacobian method but extends naturally to higher-order approximations of heterogeneous-agent models.

6.5 A Pareto-Improving Interest Rate Increase

A special case of interest is $\phi = 0$ (no borrowing), which connects to the classic model of Samuelson (1958) on consumption loans. We now explore a concrete Pareto-improving policy under this restriction.

Consider the following policy:

$$r_0 = r^o, \quad r_t = r' > r^o \;\; \forall\, t \geq 1, \qquad T_t = 0 \quad \forall\, t, \qquad \phi = 0.$$

That is, starting at $t = 1$ the government permanently raises the interest rate from r^o to $r' > r^o$, with no lump-sum transfers and no borrowing. Recall that feasibility requires $r_0 = r^o$, so the interest rate is unchanged in period 0.

Because $r_0 = r^o$, the first period is special. At $t = 0$: (i) no capital subsidy is needed, since the firm's first-order condition is satisfied at K^o without intervention ($\tau_0^k = 0$); (ii) the government budget constraint reduces to $B_1 \geq 0$, since $B_0 = 0$ and there is no subsidy cost; and (iii) household portfolios carry over from the initial steady state without valuation effects. The policy takes effect at $t = 1$, when the interest rate rises to r' and the capital subsidy becomes active. Households do, however, adjust their savings decisions at $t = 0$ in anticipation of the higher future return, so A_1 already reflects the new policy.

Result: Robust Pareto Improvement

This policy is a Pareto improvement. Each household faces the same wage w^o and a weakly higher return on savings. Since $\phi = 0$ (no borrowing), no household is hurt by the increase in the interest rate: those who save benefit, and those at the borrowing constraint are unaffected.

This is a **robust** Pareto improvement: if it can be achieved as an equilibrium outcome, the government does not need to know any of the micro details that underlie the economy (household preferences' parameters, the income process, or the cross-sectional distribution of wealth). The question is whether the feasibility and implementation of this policy depend on those details. We show next that the answer is no: only aggregates matter.

But before, we will make the following assumption on the aggregate asset demand function, that says, in the long run, higher interest rates increase savings:

Assumption. We assume that the long-run aggregate asset demand is well-defined and finite:

$$\lim_{t\to\infty} A_t(\{r_s = r',\, T_s = 0\}) \equiv A_\infty(r') = A' < \infty,$$

and that the higher interest rate increases savings: $A' > A^o = K^o$.

A Constant Capital Policy

We adopt the simplest possible capital policy:

Constant K Policy

$$K_t = K^o \qquad \forall\, t.$$

Capital is held constant at its initial steady-state level. Since $A' > K^o$, in the long run the excess savings are absorbed by government bonds:

$$B' = A' - K^o > 0.$$

Government debt is *positive* in the long run.

Capital subsidy. For $t \geq 1$, the increase in r to r' reduces the demand for capital. To keep $K = K^o$ when the interest rate is $r' > r^o$, the government must subsidize capital. From the firm's first-order condition:

$$r^o + \delta = F_K(K^o, L^o) = (1 + \tau^k)(r' + \delta),$$

so

$$\tau^k = \frac{r^o + \delta}{r' + \delta} - 1 < 0.$$

Since $r' > r^o$, we have $\tau^k < 0$: the government provides a **capital subsidy**.

Labor tax. Since $K_t = K^o$ and L^o are both unchanged, the marginal product of labor is unchanged:

$$\tau^n = 0.$$

The wage remains w^o.

Government revenue from the capital subsidy. The cost of the capital subsidy is:

$$-\tau^k \, (r' + \delta) \, K^o = (r' - r^o) \, K^o > 0.$$

Feasibility. At $t = 0$, since $r_0 = r^o$ and $B_0 = 0$, the government budget constraint is simply $B_1 \geq 0$. For $t \geq 1$, the feasibility condition requires:

$$B_{t+1} - (1 + r') \, B_t \geq (r' - r^o) \, K^o \qquad \forall \, t \geq 1.$$

The right-hand side is the per-period cost of the capital subsidy (which is zero at $t = 0$), and the left-hand side is the net revenue from issuing new government debt.

Steady-state feasibility. In the long run, with $B_t = B'$ constant:

$$\boxed{-r' \, B' \geq (r' - r^o) \, K^o.}$$

For this to hold with $B' > 0$ and $r' - r^o > 0$, the left-hand side must be positive. This requires:

$$r' < 0.$$

Combined with $r' > r^o$, the necessary condition is:

$$\boxed{0 > r' > r^o.}$$

The condition $0 > r' > r^o$ means that the initial equilibrium interest rate must be strictly negative, and the policy raises it towards (but not above) zero. This is only possible when the initial steady state features $r^o < 0$.

Graphical analysis. The key condition can be understood graphically. In steady state, suppose that asset demand $A_\infty(r)$ is an increasing function of r, while asset supply equals $K(r)$ (the capital stock consistent with firms' optimality at rate r).

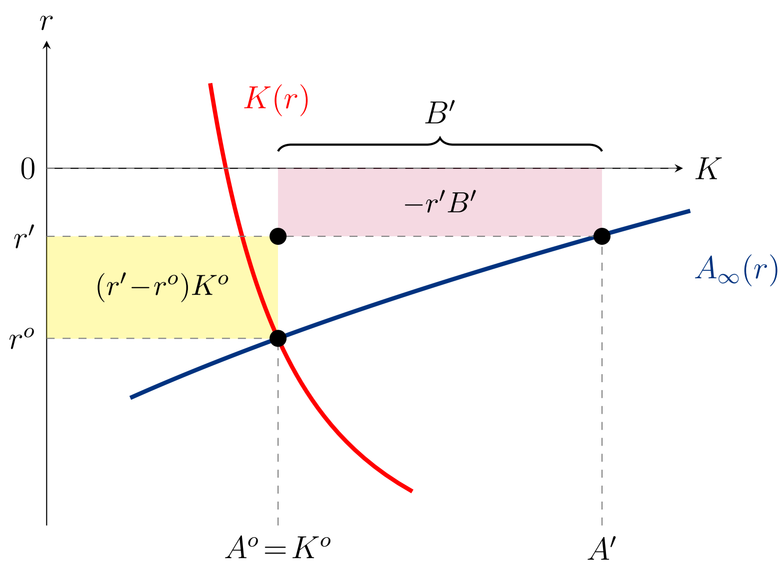


Figure 6.1: Raising r from r^o to r' increases long-run asset demand from $A^o = K^o$ to A'. The gap $B' = A' - K^o$ is government debt. The yellow area is the subsidy cost $(r' - r^o)K^o$; the purple area is the debt revenue $-r'B'$. Feasibility requires purple $\geq$ yellow.

The feasibility condition $-r'B' \geq (r' - r^o)K^o$ boils down to the **elasticity of A_∞ with respect to the interest rate**. Substituting $B' = A_\infty(r') - K^o$ and $K^o = A_\infty(r^o)$:

$$-r'\big[A_\infty(r') - A_\infty(r^o)\big] \geq (r' - r^o)\, A_\infty(r^o),$$

or equivalently,

$$-r'\, A_\infty(r') \geq -r^o\, A_\infty(r^o).$$

We need a sufficiently high elasticity of long-run asset demand: the increase in savings (which generates the government's debt revenue) must be large enough to finance the capital subsidy.

When this elasticity condition is satisfied, a Pareto improvement may be possible: the government can raise the interest rate, subsidize capital to keep production unchanged, and finance the subsidy through the additional savings that households voluntarily provide. But we still need to check the transition dynamics to confirm that the policy is indeed feasible at every date, not just in the long run.

Checking the transition. The long-run feasibility condition is necessary but not sufficient. We also need to verify that the government budget constraint holds during the **transition**. For $t \geq 1$, since $B_t = A_t - K^o$ under the constant-K policy, substituting into $B_{t+1} - (1 + r') B_t \geq (r' - r^o) K^o$ gives:

$$A_{t+1}(\{r', 0\}) - (1 + r') A_t(\{r', 0\}) \geq -r^o A_\infty(r^o) \qquad \forall t \geq 1.$$

At $t = 0$, the condition is simply $A_1 \geq K^o$: the higher future return must not reduce current savings.

The resource constraint, using the aggregate consumption function, holds uniformly for all $t \geq 0$:

$$C_t(\{r', 0\}) \leq F(K^o, L^o) - \delta K^o = C_\infty(\{r^o, 0\}) \qquad \forall t.$$

Aggregate consumption cannot exceed its initial steady-state level, since capital (and hence output) is unchanged. Both conditions depend only on the aggregate functions A_t and C_t, not on any household-level details.

This leads to an apparent paradox. Every household faces a more relaxed budget constraint (higher return on savings, same wage), so every household *can* consume more. Yet aggregate consumption cannot increase. In fact, it weakly decreases during the transition. The resolution is that some households *choose* to save more: the higher interest rate induces enough additional saving to offset the potential consumption increase. In the aggregate, people save.

The above confirms that to determine whether the higher interest rate policy is feasible, only knowledge of the aggregate function $A_t(\{r, 0\})$ is needed. There is no need to know the micro details (preferences, the income process, the wealth distribution) that aggregate to that function. This is the **robustness** of the policy.

This result resembles very closely the classical result of Samuelson (1958) on the role of money in an overlapping-generations model. We discuss this connection in the Appendix: Samuelson's Chocolates.

Throughout the analysis above, for simplicity, we restricted attention to policies with $T_t = 0$ for all t: the government issues debt but does not transfer any resources back to households. Feasibility of the policy was assessed through the government budget constraint, that is, whether the government could roll over its debt without ever running a deficit. If the equilibrium turns out to generate net fiscal revenue for the government, that revenue is effectively discarded or thrown into the ocean. It does not flow back to households. This is why, when reformulating feasibility through the resource constraint, we have a *weak* inequality: aggregate consumption may strictly decrease, because some output is wasted by the government. The key insight is that this does not undermine the welfare argument: if the policy weakly (or strictly!) reduces aggregate consumption, then *it is a Pareto improvement!*.

A debt laffer curve. When $r^o < 0$, the government can exploit the Ponzi scheme to extract resources without taxation. If it maintains a constant stock of debt $B \geq 0$, the steady-state fiscal revenue from debt issuance is:

$$R(B) \;=\; -r(B) \cdot B; \text{ where } A_\infty(r(B)) = K^o + B$$

Clearly $R(0) = 0$ since $r(0) = r^o < 0$. For small $B > 0$, the government collects revenue from the Ponzi scheme: $R(B) > 0$.

The existence of $r^o < 0$ does not grant the government *unlimited* fiscal revenue. Holding the capital stock fixed at K, the bond market requires $A(r) = K + B$. Since $A(r)$ is upward-sloping, a larger stock of debt forces a higher equilibrium interest rate:

$$r'(B) > 0.$$

As B grows, $r(B)$ rises from $r^o < 0$ toward zero and eventually turns positive. Once $r(B) > 0$, the Ponzi collapses: the government must levy taxes to service its debt rather than collect net resources from it.

The revenue function $R(B) = -r(B) \cdot B$ therefore has a **Laffer curve** shape. Starting from $R(0) = 0$, revenue rises as the government exploits the negative interest rate, reaches a maximum at $B^{\max}$, then declines as rising debt pushes $r(B)$ upward. Revenue falls back to zero at B^* where $r(B^*) = 0$, and turns negative for $B > B^*$.

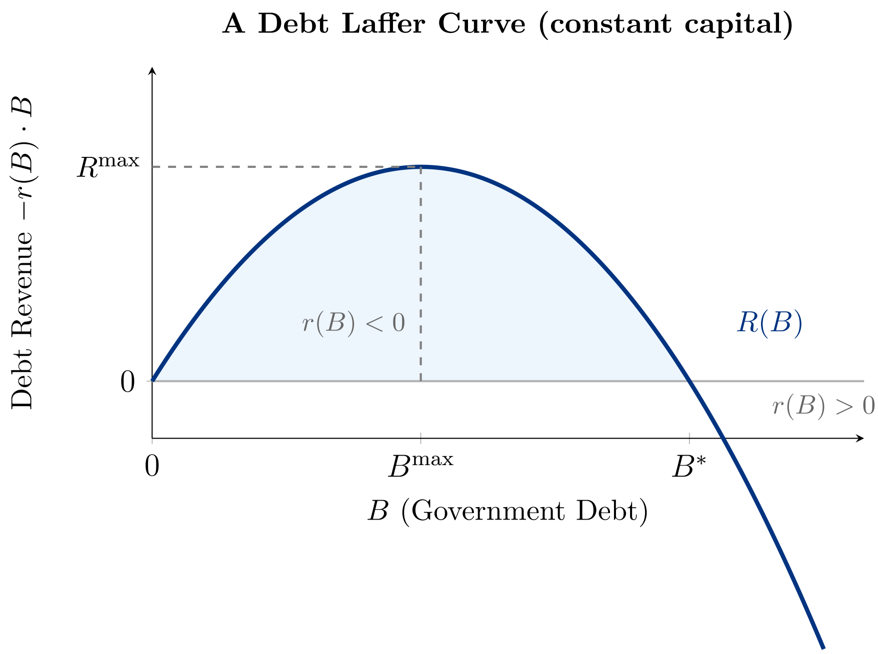


Figure 6.2: The steady-state debt Laffer curve, holding the capital stock constant. Debt revenue $R(B) = -r(B) \cdot B$ rises as the government exploits the negative equilibrium interest rate, peaks at $B^{\max}$, and returns to zero at B^* where $r(B^*) = 0$. Beyond B^* the government must raise taxes to service its debt.

Miller and Sargent (1984)

Miller and Sargent (1984) make precisely this point in reply to Darby's critique of "unpleasant monetarist arithmetic." They show that the existence of $r < 0$ does not grant the government unlimited ability to finance itself: as more debt is issued, the equilibrium interest rate rises, eroding and ultimately eliminating the Ponzi revenue. The maximum sustainable debt level is determined by the peak of the Laffer curve, not by any arbitrarily large number.

A Better Capital Policy

The analysis above assumes $r^o < 0$. What does this mean economically?

Since $r^o < 0$:

$$F_K(K^o, L^o) = r^o + \delta < \delta.$$

The marginal product of capital is below the depreciation rate. This means the economy is **above the golden rule**: it has accumulated too much capital. This is the hallmark of **dynamic inefficiency** (Diamond, 1965).

When the economy is dynamically inefficient, there exists a better policy:

- **Tax capital**: reduce the capital stock toward the golden rule level.
- **Substitute capital for government debt**: use the proceeds from the capital tax to issue debt, providing households with an alternative savings vehicle.

Appendix: Welfare Comparisons in a Version of the Neoclassical Growth Model

We consider an infinite-horizon economy populated by a representative household and a representative firm. Time is discrete, $t = 0, 1, 2, \ldots$ The household has log preferences over consumption and discounts the future with factor $\beta \in (0, 1)$. The firm operates a Cobb–Douglas technology $y_t = k_t^\alpha$ with capital share $\alpha \in (0, 1)$ and labor normalized to one. Capital depreciates fully each period ($\delta = 1$), so output must be split between consumption and next period's entire capital stock: $c_t + k_{t+1} = k_t^\alpha$. The combination of log utility, Cobb–Douglas production, and full depreciation is one of the few cases where the neoclassical growth model admits a closed-form solution, making it an ideal laboratory for studying the economics of savings and capital accumulation.

The closed-form solution for the deterministic neoclassical growth model with log utility, Cobb–Douglas production, and full depreciation was first derived by Kurz (1968). Brock and Mirman (1972, *Journal of Economic Theory*) extended the result to the stochastic case, showing that the optimal savings rate remains $\alpha\beta$ even with random productivity shocks.

The Optimal Solution

The household (equivalently, the social planner) solves

$$\max_{\{c_t,\, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$c_t + k_{t+1} = k_t^\alpha, \qquad c_t \geq 0, \quad k_{t+1} \geq 0, \qquad k_0 > 0 \text{ given.}$$

The solution is:

$$k' = \alpha\beta \cdot k^\alpha, \qquad c = (1 - \alpha\beta) \cdot k^\alpha,$$
$$V(k) = A + \frac{\alpha}{1 - \alpha\beta} \ln(k),$$

where $A = \frac{(1-\alpha\beta)\ln(1-\alpha\beta)+\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}$. The optimal savings rate is $\phi^* = \alpha\beta$, and the steady state is

$$k_A \equiv (\alpha\beta)^{\frac{1}{1-\alpha}}, \qquad c_A \equiv (1 - \alpha\beta)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}.$$

Suppose the economy begins at this steady state, with $k_0 = k_A$, and a policy change is introduced.

A Forced Savings Policy

Suppose the government forces the household to save a fixed fraction $\phi \in (0, 1)$ of output:

$$k_{t+1} = \phi \cdot k_t^\alpha, \qquad c_t = (1 - \phi) \cdot k_t^\alpha.$$

The New Steady State. Setting $k_B = \phi \cdot k_B^\alpha$ gives $k_B^{1-\alpha} = \phi$, so:

$$k_B = \phi^{\frac{1}{1-\alpha}}, \qquad y_B = \phi^{\frac{\alpha}{1-\alpha}}, \qquad c_B = (1 - \phi)\, \phi^{\frac{\alpha}{1-\alpha}}.$$

Steady-state consumption $c_B(\phi) = (1 - \phi)\, \phi^{\alpha/(1-\alpha)}$ is hump-shaped in ϕ with maximum at the *golden rule* rate $\phi_{GR} = \alpha > \phi^* = \alpha\beta$. Since ϕ^* is below the golden rule, there exist savings rates $\phi > \phi^*$ that yield strictly higher steady-state consumption: $c_B(\phi) > c_A$.

Transition Dynamics. Taking logs of $k_{t+1} = \phi\, k_t^\alpha$, and defining $x_t \equiv \ln k_t$, gives $x_{t+1} = \ln \phi + \alpha\, x_t$, a linear difference equation. For arbitrary initial capital $k_0 = k$:

$$x_t = \frac{\ln \phi}{1 - \alpha} + \alpha^t \left(\ln k - \frac{\ln \phi}{1 - \alpha} \right).$$

Define $W(k \,|\, \phi)$ as the lifetime utility of a household that starts with capital k and is forced to save fraction ϕ forever:

$$W(k \,|\, \phi) = \sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \big[\ln(1 - \phi) + \alpha\, x_t \big].$$

Substituting the expression for x_t and using $\sum_{t=0}^{\infty} (\alpha\beta)^t = \frac{1}{1-\alpha\beta}$:

$$W(k \,|\, \phi) = \frac{\ln(1 - \phi)}{1 - \beta} + \frac{\alpha \ln \phi}{(1 - \alpha)(1 - \beta)} + \frac{\alpha}{1 - \alpha\beta} \left(\ln k - \frac{\ln \phi}{1 - \alpha} \right).$$

Collecting the $\ln \phi$ terms and simplifying:

$$\boxed{W(k \,|\, \phi) = \frac{\ln(1 - \phi)}{1 - \beta} + \frac{\alpha\beta\, \ln \phi}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha}{1 - \alpha\beta} \ln k.}$$

Since the $\ln k$ term does not depend on ϕ, for any fixed k the function $W(k \mid \phi)$ is strictly concave in ϕ and is uniquely maximized at $\phi^* = \alpha\beta$.[1]

Two Welfare Comparisons

Does the forced-savings policy improve welfare? The answer depends entirely on *how* we compare.

Comparison 1: Steady State vs. Steady State. At the old steady state, welfare is $V(k_A) = \frac{\ln(c_A)}{1-\beta}$. At the new steady state, welfare is $W(k_B \mid \phi) = \frac{\ln(c_B)}{1-\beta}$. Since there exist values of ϕ for which $c_B(\phi) > c_A$, we have:

Result: Misleading Steady-State Comparison

There exist $\phi \neq \alpha\beta$ such that $V(k_A) < W(k_B \mid \phi)$.
Under this comparison, forced savings appears to be welfare-improving.

Comparison 2: Starting from k_A. Evaluating the forced-savings policy at the capital stock the economy actually starts from, and using the fact that $W(k \mid \phi)$ is uniquely maximized at $\phi^* = \alpha\beta$ for every k:

Result: Correct Welfare Comparison

$V(k_A) = W(k_A \mid \alpha\beta) > W(k_A \mid \phi)$ for all $\phi \neq \alpha\beta$.
Under this comparison, forced savings is always welfare-reducing.

The upshot is clear: starting from the actual state of the economy, any deviation from the optimal savings rate $\phi^* = \alpha\beta$ makes households strictly worse off, even if it leads to a steady state with higher consumption.

The two comparisons deliver opposite conclusions. The resolution is that Comparison 1 evaluates welfare at two *different* capital stocks, implicitly assuming the economy can teleport from k_A to k_B at no cost. The key distinction is between two objects:

- **Steady-state welfare:** $W(k_B \mid \phi) = \frac{\ln(c_B)}{1-\beta}$, which evaluates the policy at a capital stock the economy has not yet reached.
- **Lifetime welfare from k_A:** $W(k_A \mid \phi) = \sum_{t=0}^{\infty} \beta^t \ln c_t$, which incorporates the full transition.

The correct welfare criterion is the second.

[1] The first-order condition $\partial W / \partial \phi = 0$ yields $\phi = \alpha\beta$, and $\partial^2 W / \partial \phi^2 < 0$ for all $\phi \in (0, 1)$.

Application to Incomplete Markets

This lesson carries over directly, and with even greater force, to heterogeneous-agent incomplete-markets models. When evaluating a policy change (e.g., a change in taxes, borrowing limits, or social insurance), the correct criterion is the value function of each household *at the moment the policy is implemented*, computed along the full transition path from the current distribution to the new stationary distribution.

Comparing the old and new stationary equilibria is not just imprecise; it can be qualitatively misleading, for two distinct reasons:

(i) The transition matters. Just as in the representative-agent case above, the economy does not jump between steady states. Prices, wages, and interest rates evolve along the transition, and households experience a sequence of changing environments. The welfare effects of the transition can dominate the long-run comparison and even reverse its sign.

(ii) The stationary distribution changes. In the representative-agent model, at least the economy starts and ends with well-defined capital stocks k_A and k_B. In an incomplete-markets model, the entire cross-sectional distribution of assets shifts when the policy changes. Households are not rank-preserving: the set of agents at asset level a (or percentile p) in regime A is not the same set in regime B. Comparing $v_B(a, z)$ and $v_A(a, z)$ as if they refer to the "same" person is not a meaningful matching.

The correct approach is to compute the transition equilibrium: the sequence of prices $\{r_t, w_t\}_{t=0}^{\infty}$ consistent with market clearing at every date, given the initial distribution μ_0 and the terminal stationary equilibrium, and to evaluate welfare using the value function $v_0(a, z)$ that each household faces at $t = 0$ under these prices.

There are stationary welfare comparisons that are valid. For example, stationary comparisons answer "what if an agent were born into regime B?" They do not answer "what happens to currently alive agents when the policy changes?"

Appendix: Samuelson's Chocolates

We now describe Samuelson's (1958) famous consumption-loans model that introduced the overlapping-generations structure.[2]

Time is discrete, $t = 0, 1, 2, \ldots$ In each period a new generation of identical agents is born. Each agent lives for two periods: *young* and *old*. The population of young agents born at t is $N_t = (1+n)^t N_0$ with $n > -1$.

There is a single perishable consumption good ("chocolates," following Samuelson 1958). A household born at date t receives an endowment of $e_1 > 0$ when young and $e_2 \geq 0$ when old. There is no production and no storage technology; the only way to transfer resources across time is through trade.

Household problem. A household born at t chooses consumption $(c_{1,t}, c_{2,t+1})$ to maximize

$$U(c_{1,t},\, c_{2,t+1}) = u(c_{1,t}) + \beta\, u(c_{2,t+1}), \tag{6.1}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_{++} \to \mathbb{R}$ is strictly increasing, strictly concave, and satisfies the Inada conditions.

The household can save (or borrow) via a one-period bond. Let s_t denote savings and R_{t+1} the gross real interest rate. The budget constraints are $c_{1,t} + s_t = e_1$ and $c_{2,t+1} = e_2 + R_{t+1}\, s_t$, yielding the Euler equation

$$u'(c_{1,t}) = \beta\, R_{t+1}\, u'(c_{2,t+1}).$$

Market clearing and the unique equilibrium. There is no outside asset. The bond is in zero net supply. In each period, total saving by the young must finance the dissaving of the old:

$$N_t\, s_t = N_{t-1}\, R_t\, s_{t-1}. \tag{6.2}$$

Since the initial old hold no assets ($s_{-1} = 0$), (6.2) gives $s_0 = 0$, and by induction $s_t = 0$ for all t.

Competitive Equilibrium

A competitive equilibrium is a sequence of allocations $\{c_{1,t}, c_{2,t+1}, s_t\}_{t=0}^{\infty}$ and interest rates $\{R_{t+1}\}_{t=0}^{\infty}$ such that each household maximizes (6.1) subject to its budget constraints taking prices as given, and markets clear:

[2] Allais (1947) developed an overlapping-generations model in an appendix of *Économie et Intérêt*, predating Samuelson by over a decade. An English translation has recently appeared.

1. **Asset market clearing:**

$$N_t\, s_t = N_{t-1}\, R_t\, s_{t-1} \quad \text{for all } t.$$

2. **Goods market clearing:**

$$N_t\, c_{1,t} + N_{t-1}\, c_{2,t} = N_t\, e_1 + N_{t-1}\, e_2 \quad \text{for all } t.$$

The unique equilibrium is *autarky*: $c_{1,t} = e_1$, $c_{2,t+1} = e_2$, $s_t = 0$ for all t, with interest rate

$$R^{\text{aut}} = \frac{u'(e_1)}{\beta\, u'(e_2)}. \tag{6.3}$$

Assets are in zero net supply and there is no vehicle for intertemporal trade.

Inefficiency

The source of inefficiency. The crucial realization is that the autarky allocation can fail to be Pareto efficient. The OLG economy has a *double infinity*: infinitely many agents and infinitely many dates.[3] The value of aggregate endowments may be infinite, violating a key assumption behind the First Welfare Theorem.

To see this, suppose a planner transfers $\varepsilon > 0$ from each young to each old at every $t \geq 1$. Each young household gives up ε but receives $(1+n)\varepsilon$ when old, since the next cohort is larger by a factor $1+n$. If $R^{\text{aut}} < 1+n$, the implicit return on this transfer exceeds the market rate, and every generation is strictly better off (including the initial old, who receive a pure windfall). Then we have the following result.

Result: Efficiency of the OLG Economy

Consider the autarky equilibrium of the pure-exchange OLG economy.

(a) If $R^{\text{aut}} < 1+n$, the equilibrium is **inefficient**. A Pareto improvement can be achieved by transferring resources from the young to the old.

(b) If $R^{\text{aut}} > 1+n$, the equilibrium is **efficient**.

(c) If $R^{\text{aut}} = 1+n$, the equilibrium achieves the **golden rule** and is Pareto efficient.

[3] This is related to Hilbert's "infinite hotel" paradox, as pointed out by Balasko and Shell (1980).

Proof sketch.

Part (a). Consider a transfer of ε from each young to each old at every $t \geq 1$. The change in lifetime utility at the margin is

$$-u'(e_1) \cdot \varepsilon + \beta\, u'(e_2) \cdot (1+n)\varepsilon = u'(e_1) \left[\frac{1+n}{R^{\text{aut}}} - 1 \right] \varepsilon > 0,$$

using $\beta u'(e_2) = u'(e_1)/R^{\text{aut}}$. The bracket is positive, so every generation born at $t \geq 1$ is better off. The initial old receive $(1+n)\varepsilon$ for free.

Part (b). When $R^{\text{aut}} > 1+n$, the present value of aggregate endowments is finite: $\sum_{t=0}^{\infty}(1+n)^t/(R^{\text{aut}})^t < \infty$. This restores the key assumption behind the First Welfare Theorem, and the standard proof of efficiency goes through; see Balasko and Shell (1980) and Wilson (1981). Note that the bracket calculation from part (a) only rules out young-to-old transfers; the finite-valuation condition rules out *all* feasible reallocations.

Part (c). At $R^{\text{aut}} = 1+n$, the transfer yields zero net gain at the margin. The allocation cannot be improved upon.

When does $R^{\text{aut}} < 1+n$ obtain? Using (6.3), the inefficiency condition is

$$\boxed{\frac{u'(e_1)}{\beta\, u'(e_2)} < 1+n.}$$

This holds when e_2 is small relative to e_1 (strong desire to save), when β is large (patient households), or when n is large (faster population growth raises the threshold). That is, when the households really desire to eat chocolates when they are old, but the market does not provide a way to save for that future consumption, the economy is inefficient.

Restoring Efficiency via Government Debt

Samuelson (1958) famously restored efficiency by introducing fiat money: the chocolate wrappers serve as a store of value, giving the young a way to save for old age. Here we describe an alternative arrangement in which the government issues bonds, achieving the same outcome through fiscal policy rather than monetary innovation.

Suppose the economy is inefficient: $R^{\text{aut}} < 1+n$. The government pursues a policy of debt and taxes such that the equilibrium interest rate is $\hat{R}_{t+1} = 1+n$ at all dates. We show that this generates a Pareto improvement *without ever levying a positive tax.*

At each date t, the government sells b_t bonds per young agent at unit price, each promising gross return $\hat{R}_{t+1}$, and levies a lump-sum tax τ_t on each young agent ($\tau_t < 0$ denotes a transfer). The government budget constraint is

$$N_t\, b_t + N_t\, \tau_t = N_{t-1}\, \hat{R}_t\, b_{t-1}. \tag{6.4}$$

At $t = 0$, the same constraint applies with $b_{-1} = 0$, so $N_0\, b_0 + N_0\, \tau_0 = 0$: the transfer $-\tau_0 > 0$ is distributed to the initial old. We now back out the sequences $\{b_t, \tau_t\}$ implied by the target $\hat{R}_{t+1} = 1 + n$.

Household saving. At rate $1+n$, the Euler equation $u'(c_{1,t}) = \beta(1+n)u'(c_{2,t+1})$ pins down optimal saving. Since endowments, preferences, and the interest rate are time-invariant, saving is constant: $s_t = s$ for all t.

Bond market clearing. The government bond is the only asset, so $s_t = b_t$. With constant saving, $b_t = b$ for all t.

Implied taxes. Substituting $\hat{R}_t = 1 + n$ and $b_t = b_{t-1} = b$ into (6.4) and dividing by N_t gives $b + \tau = (1 + n)b/(1 + n) = b$, hence

$$\tau = 0.$$

Revenue from selling bonds to the new (larger) cohort exactly covers the maturing obligations. The Ponzi scheme is self-financing at rate $1 + n$.

Implied bond position. With $\tau = 0$, the Euler equation becomes

$$u'(e_1 - b) = \beta(1 + n)\, u'(e_2 + (1 + n)b). \tag{6.5}$$

At $b = 0$ this reads $u'(e_1) = \beta(1 + n)u'(e_2)$, i.e. $R^{\text{aut}} = 1 + n$, which fails since $R^{\text{aut}} < 1 + n$. By the Inada conditions, there exists a unique $b^* > 0$ solving (6.5).

The Pareto improvement. Under the policy ($\hat{R} = 1+n$, $\tau = 0$, $b = b^*$), every generation born at $t \geq 0$ achieves the allocation ($e_1 - b^*$, $e_2 + (1 + n)b^*$). The welfare gain relative to autarky is

$$\Delta U \approx u'(e_1) \left[\frac{1 + n}{R^{\text{aut}}} - 1\right] b^* > 0,$$

since $R^{\text{aut}} < 1+n$. The initial old receive $-\tau_0 = b^* > 0$ for free. This is a Pareto improvement.

Pay-As-You-Go Social Security

An alternative arrangement achieves a Pareto improvement: a *mandatory intergenerational transfer* from young to old.

Suppse that at each $t \geq 1$, the government taxes each young agent $T > 0$ and distributes the proceeds to the old. Each old agent receives $(N_t/N_{t-1})\,T = (1+n)\,T$, so the implicit return on forced "contributions" is $1 + n > R^{\text{aut}}$.

A young agent born at t consumes $c_1 = e_1 - T$ and $c_2 = e_2 + (1+n)T$. For small T, the welfare gain is

$$\Delta U \approx u'(e_1)\left[\frac{1+n}{R^{\text{aut}}} - 1\right] T > 0.$$

Every generation born at $t \geq 1$ gains, and the initial old receive $(1+n)T$ for free.

Robustness: Bonds vs. Forced Transfers

The debt policy generates a Pareto improvement without forcing any household to participate. Suppose there were some households (say, a set of measure zero) that live only for one period. Those households will not buy the bonds and will not be taxed, so they are entirely unaffected. The Pareto improvement goes through regardless of their presence.

More generally, if $R^{\text{aut}} < 1+n$, it suffices to design a debt policy that raises the equilibrium interest rate to $1 + n$. Every household with a positive saving motive is strictly better off, and no household is forced to participate. It is this "robustness" feature of the debt policy, delivering a Pareto improvement without requiring the government to know or constrain the behavior of every agent, that is highlighted by Aguiar, Amador, and Arellano (2024).

The pay-as-you-go system, by contrast, imposes stronger informational requirements. If there were a set (even of measure zero) of households that die young, receiving e_1 but not surviving to old age, they would be taxed T and receive nothing in return. A Pareto improvement is no longer guaranteed. The forced system requires the government to know each household's type, a requirement the debt policy sidesteps entirely.

References

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2022). "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach." *Review of Economic Studies*, 89(1), 45–86.

Aguiar, M. and M. Amador (2016). "Fiscal Policy in Debt Constrained Economies." *Journal of Economic Theory*, 161, 37–75.

Aguiar, M., M. Amador, and C. Arellano (2024). "Micro Risks and (Robust) Pareto Improving Policies." *American Economic Review*, 114(11), 3669–3713.

Aiyagari, S. R. (1994). "Uninsured Idiosyncratic Risk and Aggregate Saving." *Quarterly Journal of Economics*, 109(3), 659–684.

Allais, M. (1947). *Économie et Intérêt*. Paris: Imprimerie Nationale.

Alvarez, F. and N. L. Stokey (1998). "Dynamic Programming with Homogeneous Functions." *Journal of Economic Theory*, 82(1), 167–189.

Arrow, K. J. (1964). "The Role of Securities in the Optimal Allocation of Risk-Bearing." *Review of Economic Studies*, 31(2), 91–96.

Auclert, A., M. Rognlie, and L. Straub (2024). "The Intertemporal Keynesian Cross." *Journal of Political Economy*, 132(12), 4068–4121.

Backus, D. K., B. R. Routledge, and S. E. Zin (2004). "Exotic Preferences for Macroeconomists." *NBER Macroeconomics Annual*, 19, 319–390.

Balasko, Y. and K. Shell (1980). "The Overlapping-Generations Model I: The Case of Pure Exchange without Money." *Journal of Economic Theory*, 23(3), 281–306.

Barreto, L. (2025). "Debt-Financed Fiscal Stimulus, Heterogeneity, and Welfare." Working Paper, University of Minnesota.

Bhandari, A., T. Bourany, D. Evans, and M. Golosov (2023). "A Perturbational Approach for Approximating Heterogeneous-Agent Models." NBER Working Paper No. 31744.

Bewley, T. (1980). "The Optimum Quantity of Money." In J. Kareken and N. Wallace (eds.), *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, 169–210.

Bewley, T. (1983). "A Difficulty with the Optimum Quantity of Money." *Econometrica*, 51(5), 1485–1504.

Bloise, G. and Y. Vailakis (2018). "Convex Duality and Uniqueness of Euler Equations." *Econometrica*, 86(5), 1977–2003.

Boyd, J. H. III (1990). "Recursive Utility and the Ramsey Problem." *Journal of Economic Theory*, 50(2), 326–345.

Brock, W. A. and L. J. Mirman (1972). "Optimal Economic Growth and Uncertainty: The Discounted Case." *Journal of Economic Theory*, 4(3), 479–513.

Caballero, R. J. (1990). "Consumption Puzzles and Precautionary Savings." *Journal of Monetary Economics*, 25(1), 113–136.

Caballero, R. J., E. Farhi, and P.-O. Gourinchas (2008). "An Equilibrium Model of 'Global Imbalances' and Low Interest Rates." *American Economic Review*, 98(1), 358–393.

Chamberlain, G. and C. A. Wilson (2000). "Optimal Intertemporal Consumption under Uncertainty." *Review of Economic Dynamics*, 3(3), 365–395.

Debreu, G. (1959). *Theory of Value*. Yale University Press.

Diamond, P. A. (1965). "National Debt in a Neoclassical Growth Model." *American Economic Review*, 55(5), 1126–1150.

Hall, R. E. (1978). "Stochastic Implications of the Life Cycle–Permanent Income Hypothesis: Theory and Evidence." *Journal of Political Economy*, 86(6), 971–987.

Hopenhayn, H. A. and E. C. Prescott (1992). "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies." *Econometrica*, 60(6), 1387–1406.

Huggett, M. (1993). "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies." *Journal of Economic Dynamics and Control*, 17(5–6), 953–969.

İmrohoroğlu, A. (1989). "Cost of Business Cycles with Indivisibilities and Liquidity Constraints." *Journal of Political Economy*, 97(6), 1364–1383.

Kamihigashi, T. (2001). "Necessity of Transversality Conditions for Infinite Horizon Problems." *Econometrica*, 69(4), 995–1012.

Kimball, M. S. (1990). "Precautionary Saving in the Small and in the Large." *Econometrica*, 58(1), 53–73.

Krusell, P. and A. A. Smith, Jr. (1998). "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economy*, 106(5), 867–896.

Kurz, M. (1968). "Optimal Economic Growth and Wealth Effects." *International Economic Review*, 9(3), 348–357.

Ljungqvist, L. and T. J. Sargent (2018). *Recursive Macroeconomic Theory*, 4th ed. MIT Press.

Luenberger, D. G. (1969). *Optimization by Vector Space Methods*. Wiley.

Mendoza, E. G., V. Quadrini, and J.-V. Ríos-Rull (2009). "Financial Integration, Financial Development, and Global Imbalances." *Journal of Political Economy*, 117(3), 371–416.

Miller, M. H. and T. J. Sargent (1984). "A Reply to Darby." *Federal Reserve Bank of Minneapolis Quarterly Review*, 8(2), 21–26.

Rouwenhorst, K. G. (1995). "Asset Pricing Implications of Equilibrium Business Cycle Models." In T. F. Cooley (ed.), *Frontiers of Business Cycle Research*, Princeton University Press, 294–330.

Samuelson, P. A. (1958). "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money." *Journal of Political Economy*, 66(6), 467–482.

Stokey, N. L., R. E. Lucas, Jr., and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.

Tauchen, G. (1986). "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions." *Economics Letters*, 20(2), 177–181.

Wang, N. (2003). "Caballero Meets Bewley: The Permanent-Income Hypothesis in General Equilibrium." *American Economic Review*, 93(3), 927–936.

Wilson, C. (1981). "Equilibrium in Dynamic Models with an Infinity of Agents." *Journal of Economic Theory*, 24(1), 95–111.

Yosida, K. and E. Hewitt (1952). "Finitely Additive Measures." *Transactions of the AMS*, 72(1), 46–66.