Econ8107 Assignment 3

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Question 1

Part (a)

Household problem (Huggett). Given the gross interest rate R and an idiosyncratic income state $s \in S$ following a Markov chain with transition $\pi(s'|s)$, the Bellman equation is

$$V(a,s) = \max_{a' \geq -\phi} \left\{ u\big(c\big) + \beta \sum_{s'} \pi(s'|s) V(a', s') \right\}, \quad \text{s.t.}$$
$$c = y(s) + Ra - a'$$
$$a' \geq -\phi$$

Let $\mu \geq 0$ be the multiplier on the borrowing constraint $a' \geq -\phi$, we have FOC:

$$u'(c) = \beta R \sum_{s'} \pi(s'|s)\, V_a(a', s') + \mu,$$
$$\mu \geq 0, \quad a' + \phi \geq 0, \quad \mu(a' + \phi) = 0.$$

Using the envelope condition $V_a(a', s') = u'(c(s'))$, we obtain the Euler inequality

$$u'(c(s)) \;\geq\; \beta R \sum_{s'} \pi(s'|s)\, u'(c(s')), \qquad \forall s \in S,$$

and the borrowing constraint binds ($a' = -\phi$) if and only if

$$u'(c(s)) \;>\; \beta R \sum_{s'} \pi(s'|s)\, u'(c(s'))$$
$$\Longleftrightarrow R < \frac{1}{\beta} \frac{u'(c(s))}{\sum_{s'} \pi(s'|s)\, u'(c(s'))}.$$

If $a' = -\phi$, then

$$c(s) = y(s) + R(-\phi) - (-\phi) = y(s) - (R-1)\phi.$$

For aggregate saving to equal $-\phi$, all households must choose $a' = -\phi$ in every state:

$$u'\big(y(s) - (R-1)\phi\big) \;\geq\; \beta R \sum_{s'} \pi(s'|s)\, u'\big(y(s') - (R-1)\phi\big), \qquad \forall s \in S.$$

Equivalently, $\bar{R}(\phi)$ is characterized (implicitly) by the tightest state:

$$\boxed{\bar{R}(\phi) = \frac{1}{\beta} \min_{s \in S} \frac{u'\big(y(s) - (\bar{R}(\phi) - 1)\phi\big)}{\sum_{s'} \pi(s'|s)\, u'\big(y(s') - (\bar{R}(\phi) - 1)\phi\big)}}$$

The interest rate $\bar{R}(\phi)$ depends on the borrowing limit ϕ.

For $R = \bar{R}(\phi)$, there exists a household s^* which is indifferent between borrowing $-\phi$ and saving more, and all other households strictly prefer to borrow $-\phi$.

Part (c)

We consider the two-state case $s \in \{s_1, s_2\}$ and write $y_i := y(s_i)$ and $\pi_i := \pi(s_i)$ for $i = 1, 2$.

If the household chooses $a' = -\phi$, then:

$$c_i(R) = y_i - (R-1)\phi = y_i + (1-R)\phi, \qquad i = 1, 2.$$

Euler inequality under log utility and i.i.d. income. Since $u'(c) = 1/c$, the Euler condition with a binding borrowing constraint is

$$\frac{1}{c_i(R)} \;\geq\; \beta R \sum_{j=1}^{2} \pi_j \frac{1}{c_j(R)}, \qquad i = 1, 2,$$

Plug in $c_i(R) = y_i + (1-R)\phi$, we have

$$\frac{1}{y_i + (1-R)\phi} \;\geq\; \beta R \sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}, \qquad i = 1, 2.$$

The right-hand side does not depend on the current state, we only need borrowing constraint to bind for the high-income household (so they don't want to save more)

Suppose $y_1 < y_2$, then:

$$\frac{1}{y(s_1) - (\bar{R}-1)\phi} > \frac{1}{y(s_2) - (\bar{R}-1)\phi},$$

Then we know that R need to satisfy the Euler inequality for the high-income household:

$$\boxed{\frac{1}{y_2 + (1-R)\phi} \;\geq\; \beta R \sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}}.$$

We can rewrite the above inequality as:

$$\beta R \leq \frac{\frac{1}{y_2 + (1-R)\phi}}{\sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}}.$$

Define

$$H(R) \;:=\; \frac{\frac{1}{y_2 + (1-R)\phi}}{\sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}}.$$

We know that there exist a finite $\bar{R}$ satisfying $H(\bar{R}) = \beta \bar{R}$. [1]

[1] We have $\bar{R}$ satisfies:

$$\bar{R} = \frac{1}{\beta} \min_{s \in S} \frac{\frac{1}{y(s) - (\bar{R}-1)\phi}}{\pi(s_1)\frac{1}{y(s_1) - (\bar{R}-1)\phi} + \pi(s_2)\frac{1}{y(s_2) - (\bar{R}-1)\phi}}$$

Suppose $y_1 < y_2$, then:

$$\frac{1}{y(s_1) - (\bar{R}-1)\phi} > \frac{1}{y(s_2) - (\bar{R}-1)\phi},$$

we only need borrowing constraint to bind for the high-income household (so they don't want to save more)

Then $\bar{R}$ satisfies:

$$\bar{R} = \frac{1}{\beta} \frac{\frac{1}{y(s_2) - (\bar{R}-1)\phi}}{\pi(s_1)\frac{1}{y(s_1) - (\bar{R}-1)\phi} + \pi(s_2)\frac{1}{y(s_2) - (\bar{R}-1)\phi}}.$$

We can obtain a finite solution $\bar{R}$ to the above equation.

We want to show that for any $R \in [0, \bar{R}]$, we have $H(R) \geq \beta R$.

Let

$$a_j(R) := \frac{1}{y_j + (1-R)\phi}, \qquad S(R) := \sum_{j=1}^{2} \pi_j a_j(R),$$

so that $H(R) = \dfrac{a_2(R)}{S(R)}$. Note that

$$a_j'(R) = \frac{d}{dR}\Big(\frac{1}{y_j + (1-R)\phi}\Big) = \frac{\phi}{\big(y_j + (1-R)\phi\big)^2} = \phi\, a_j(R)^2 > 0,$$

and hence

$$S'(R) = \sum_{j=1}^{2} \pi_j a_j'(R) = \phi \sum_{j=1}^{2} \pi_j a_j(R)^2.$$

By the quotient rule,

$$\begin{aligned} H'(R) &= \frac{a_2'(R)S(R) - a_2(R)S'(R)}{S(R)^2} \\ &= \frac{\phi a_2(R)^2 S(R) - \phi a_2(R) \sum_{j=1}^{2} \pi_j a_j(R)^2}{S(R)^2} \\ &= \frac{\phi a_2(R)}{S(R)^2} \left(a_2(R)S(R) - \sum_{j=1}^{2} \pi_j a_j(R)^2 \right) \\ &= \frac{\phi a_2(R)}{S(R)^2} \sum_{j=1}^{2} \pi_j a_j(R)\big(a_2(R) - a_j(R)\big). \end{aligned}$$

Since $y_2 > y_1$ implies $y_2 + (1-R)\phi > y_1 + (1-R)\phi$ and thus $a_2(R) < a_1(R)$, we have

$$\sum_{j=1}^{2} \pi_j a_j(R)\big(a_2(R) - a_j(R)\big) = \pi_1 a_1(R)\big(a_2(R) - a_1(R)\big) + \pi_2 a_2(R)\big(a_2(R) - a_2(R)\big) < 0,$$

so $H'(R) < 0$, i.e. $H(R)$ is strictly decreasing in R.

Since βR is strictly increasing in R and $H(\bar{R}) = \beta \bar{R}$, it follows that for any $R \in [0, \bar{R}]$,

$$H(R) \geq H(\bar{R}) = \beta \bar{R} \geq \beta R.$$

Therefore, for any $R \in [0, \bar{R}]$ the tight-state Euler inequality holds, hence it holds for both states, and the borrowing constraint binds for everyone: $a'(s_i) = -\phi$ for $i = 1, 2$. Consequently,

$$A(R) = \sum_{i=1}^{2} \pi_i a'(s_i) = \sum_{i=1}^{2} \pi_i(-\phi) = -\phi, \qquad \forall\, R \in [0, \bar{R}].$$

Part (d)

Suppose income is i.i.d., i.e. $\pi(s'|s) = \pi(s')$, with a finite support $S = \{s_1, \dots, s_N\}$ and associated endowments $y_1 < y_2 < \cdots < y_N$, where $y_k := y(s_k)$ and $\pi_k := \pi(s_k)$. Assume CRRA utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$ (so $u'(c) = c^{-\gamma}$), and $\phi < \min_k y_k$.

Candidate borrowing-limit allocation. If the household chooses $a' = -\phi$, then in stationarity $a = -\phi$ and consumption in state s_k is

$$c_k(R) = y_k + R(-\phi) - (-\phi) = y_k - (R-1)\phi = y_k + (1-R)\phi, \qquad k = 1, \dots, N,$$

which is strictly positive on the relevant range.

Euler inequality and the tight state. With CRRA, $u'(c) = c^{-\gamma}$. Under i.i.d. income, the Euler condition at the borrowing limit is

$$c_k(R)^{-\gamma} \;\geq\; \beta R \sum_{m=1}^{N} \pi_m \, c_m(R)^{-\gamma}, \qquad k = 1, \dots, N.$$

The right-hand side does not depend on the current state k. Since y_N is the highest endowment, we have $c_N(R) > c_k(R)$ for all $k < N$, hence $c_N(R)^{-\gamma} < c_k(R)^{-\gamma}$. Therefore the tightest inequality is for the highest-income state s_N.

Thus it suffices to require

$$\boxed{c_N(R)^{-\gamma} \;\geq\; \beta R \sum_{m=1}^{N} \pi_m \, c_m(R)^{-\gamma}.}$$

Equivalently,

$$\beta R \;\leq\; \frac{c_N(R)^{-\gamma}}{\sum_{m=1}^{N} \pi_m \, c_m(R)^{-\gamma}}, \qquad H(R) := \frac{\big(y_N + (1-R)\phi\big)^{-\gamma}}{\sum_{m=1}^{N} \pi_m \big(y_m + (1-R)\phi\big)^{-\gamma}}.$$

Monotonicity of $H(R)$. Let $b_m(R) := c_m(R)^{-\gamma} = \big(y_m + (1-R)\phi\big)^{-\gamma}$ and $B(R) := \sum_{m=1}^{N} \pi_m b_m(R)$, so $H(R) = b_N(R)/B(R)$. Since $c_m'(R) = -\phi$, we have

$$b_m'(R) = \frac{d}{dR}\big(c_m(R)^{-\gamma}\big) = -\gamma c_m(R)^{-\gamma-1} \, c_m'(R) = \gamma\phi \, c_m(R)^{-\gamma-1} > 0,$$

and hence $B'(R) = \sum_{m=1}^{N} \pi_m b_m'(R) > 0$. By the quotient rule,

$$H'(R) = \frac{b_N'(R)B(R) - b_N(R)B'(R)}{B(R)^2} = \frac{1}{B(R)^2} \sum_{m=1}^{N} \pi_m \Big(b_N'(R) b_m(R) - b_N(R) b_m'(R)\Big).$$

Now note that

$$\begin{aligned} b_N'(R) b_m(R) - b_N(R) b_m'(R) &= \gamma\phi \, c_N(R)^{-\gamma-1} c_m(R)^{-\gamma} - \gamma\phi \, c_N(R)^{-\gamma} c_m(R)^{-\gamma-1} \\ &= \gamma\phi \, c_N(R)^{-\gamma-1} c_m(R)^{-\gamma-1} \big(c_m(R) - c_N(R)\big). \end{aligned}$$

For $m < N$, we have $c_m(R) < c_N(R)$, so each term is strictly negative; for $m = N$ it is zero. Therefore $H'(R) < 0$, i.e. $H(R)$ is strictly decreasing in R.

Conclusion: a lowest kink on $[0, \bar{R}]$. Since βR is strictly increasing in R and $H(R)$ is strictly decreasing, there exists at most one $\bar{R}$ such that $H(\bar{R}) = \beta \bar{R}$; define $\bar{R}$ as that intersection (when it exists). Then for any $R \in [0, \bar{R}]$ we have $H(R) \geq \beta R$, so the tight-state Euler inequality holds, hence it holds in all states and the borrowing constraint binds for everyone: $a'(s_k) = -\phi$ for all k. Consequently, aggregate saving is constant on this interval:

$$A(R) = \sum_{k=1}^{N} \pi_k \, a'(s_k) = \sum_{k=1}^{N} \pi_k (-\phi) = -\phi, \qquad \forall\, R \in [0, \bar{R}].$$