Problem Set 1. Econ 8107. Spring 2026

This problem set is due on Tuesday February 3, by class time.

Please upload your electronic here:

`https://www.dropbox.com/request/Qa2T6RtPQuqtRkbHkQp6`

> Everyone should write their own individual answers (including your own version of the code used, with your own comments) but you are encouraged to work in groups.

1 Lucas' Cost of Business Cycles

Consider an economy with a representative agent whose endowment process is given by:

$$Y(s^t) = (1+\lambda)^t (1+\epsilon(s_t)) Y_0$$

where $\lambda > 0$ and where $(1+\epsilon(s_t))$ is an iid log-normal random variable with mean 1. That is, $\log(1+\epsilon(s^t))$ is a normal variable with mean $\mu = -\sigma^2/2$ and variance σ^2.

The representative agent values consumption according to the following utility function:

$$\sum_t \sum_{s^t} \beta^t \pi(s^t) \frac{c(s^t)^{1-\gamma} - 1}{1-\gamma}$$

with $\beta < 1$ and $\beta(1+\lambda)^{1-\gamma} < 1$.

(a) **The case without business cycles.** Suppose that the economy is deterministic, that is, $\epsilon(s^t) = 0$ for all s^t. Write down a formula for the utility value as of time zero for the representative agent given that $c(s^t) = Y(s^t)$.

(b) Go back to the case where ϵ is stochastic. Compute $\mathbb{E}[(1+\epsilon(s_t))^{1-\gamma}]$

(c) **The case with business cycles.** Consider a situation where in every state, the representative agent's consumption is $c(s^t) = (1+\mu)Y(s^t)$. That is, the agent consumes a constant fraction above the aggregate endowment. Write down the formula for the time-zero utility value for the representative agent in this case.

(d) Equate the value obtained in (a) with the value obtained in (c); and solve for μ. Would you call this the cost of business cycles? Explain? Which parameters matter and why?

(e) Suppose that we set $\gamma = 2$ (a standard value in macro) and, using yearly data, we find that the standard deviation of the log output percapita for the US is $\sigma = 0.013$. What is μ? And what does it tell you about the cost of the US business cycle according to this model?

(f) Go back to deterministic case (a) where $\epsilon(s^t) = 0$ for all s^t. Suppose that the economy now grows at rate $1 + \lambda - \alpha$, and representative agent consumes $c(s^t) = (1 + \theta)Y(s^t)$ for some value of θ. Write down the time zero utility value to the representative agent in this case.

(g) Compare (f) to (a) and write down an equation for the value of θ that makes the value in (f) equal to (a).

(h) Assuming $\beta = 0.95$, $\lambda = 0.03$ and $\alpha = 0.01$, what is the numerical value for θ? Can you interpret θ as measuring the benefits of increasing the growth rate of the economy by 1%?

(i) Compare θ to μ. Interpret.

2 Risk Sharing with no Assets

There is only one period, two countries (1 and 2) and **two goods** (A and B). The representative agent in each country has a utility function given by:

$$u(c^A, c^B) = \frac{(c^A c^B)^{\frac{1-\rho}{2}}}{1-\rho}$$

where $\rho > 0$ and where c^A represents the consumption of good A and similarly for c^B.

There is a state of world indexed by $s \in S$. In state $s \in S$, which occurs with probability $\pi(s) > 0$, country 1 is endowed with $e^A(s)$ units of good A while country 2 is endowed with $e^B(s)$ units of good B (Note country 1 has no endowment of good B, and country 2 has no endowment of good A).

(a) Solve for Pareto optimal allocations.

(b) Suppose that there are no financial markets but countries can trade goods after the state of the world is realized. Show that the competitive equilibrium allocation that results without financial assets is Pareto efficient ex-ante (that is, before the state of the world is known).

(c) Given your answer to part (b), what happens to the relative price of good A in states where $e^A(s)/e^B(s)$ is low? Explain in words how risk sharing is achieved.

(d) Why is (b) an important result to consider when thinking about risk sharing across countries?

3 A (Numerical) Recursive Pareto Problem

There are two agents in the economy, A and B. Every period, there are two possible realizations of the state $S = \{s_1, s_2\}$. The aggregate endowment is such that $Y(s_1) = 1$ and $Y(s_2) = 1/2$. In all periods, the probabilities are $\pi(s_1) = \pi(s_2) = 1/2$, which are independent of history (i.i.d.).

The utility function of each agent is given by

$$\sum_t \sum_{s^t} \beta^t \pi(s^t) u(c^i(s^t))$$

Let v denote the utility value promised to agent A, before the realization of uncertainty this period. Let $P(v)$ denote the maximum utility that can be attained by agent B, given that a value of v has been promised to agent A. Then, we can write the Pareto problem recursively as follows:

$$P(v) = \max_{c(s),w(s)} \sum_{s \in S} \pi(s) \{u(Y(s) - c(s)) + \beta P(w(s))\}$$

subject to:

$$v \leq \sum \pi(s) \{u(c(s)) + \beta w(s)\}$$

Be sure that you understand why is this a Pareto problem. The following questions will ask you to go to the computer and solve the above problem using value function iteration. In what follows suppose that $u(c) = c^{1/2}$, $\beta = 0.8$.

(a) **A grid for** v. Note that the maximum that can be promised to agent A is $\overline{V} = \sum_{s \in S} \pi(s) u(Y(s))/(1-\beta)$ and the minimum value is $\underline{V} = \sum_{s \in S} \pi(s) u(0)/(1-\beta)$.

Create a grid for v in the computer. Don't make it too big (maybe just 20 points). Let's call that grid *vgrid*.

(b) **Iterating the value function.** For any initial value function guess, $P_i(v)$, iterate the value function. That is, for every v in *vgrid* solve numerically the maximization problem

$$P_{i+1}(v) = \max_{c(s),w(s)\in[\underline{V},\overline{V}]} \sum_{s\in S} \pi(s)\left\{u(Y(s)-c(s)) + \beta P_i(w(s))\right\}$$

subject to:

$$v \leq \sum \pi(s)\left\{u(c(s)) + \beta w(s)\right\}$$

and keep iterating until P_{i+1} is sufficiently close to P_i.
There are many ways you could do this. A simple (albeit brute-force) way is to try all possible combinations of $c(s_1), w(s_1), w(s_2)$ and solve for the $c(s_2)$ that will make the "promise keeping" constraint, i.e. $v \leq \sum \pi(s)\left\{u(c(s)) + \beta w(s)\right\}$, hold with equality.
Here are the steps. Construct a grid for $c(s_1)$ from 0 to $Y(s_1)$ (don't make it too big, say 20 points). Loop over all possibles v in *vgrid*. For every value of v, find numerically the combination of $c(s_1)$ in the consumption grid, and $w(s_1)$ and $w(s_2)$ in *vgrid* that, once you obtained $c(s_2)$ from above, maximizes the objective given your guess for P. Iterate until convergence.
You are free to improve on this algorithm if you want.

(c) **A check.** Consider now the sequence problem, and let λ denote the muliplier on agent A. Compute the solution to the planning problem analytically for any λ. Check numerically that as you move λ, you trace out the value function found in step (b).