Econ8107 Assignment 3

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Question 1

Part (a)

Household problem (Huggett). Given the gross interest rate R and an idiosyncratic income state $s \in S$ following a Markov chain with transition $\pi(s'|s)$, the Bellman equation is

$$V(a,s) = \max_{a' \geq -\phi} \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) V(a',s') \right\}, \quad \text{s.t.}$$
$$c = y(s) + Ra - a'$$
$$a' \geq -\phi$$

Let $\mu \geq 0$ be the multiplier on the borrowing constraint $a' \geq -\phi$, we have FOC:

$$u'(c) = \beta R \sum_{s'} \pi(s'|s)\, V_a(a',s') + \mu,$$
$$\mu \geq 0, \quad a' + \phi \geq 0, \quad \mu(a' + \phi) = 0.$$

Using the envelope condition $V_a(a',s') = u'(c(s'))$, we obtain the Euler inequality

$$u'(c(s)) \;\geq\; \beta R \sum_{s'} \pi(s'|s)\, u'(c(s')), \qquad \forall s \in S,$$

and the borrowing constraint binds ($a' = -\phi$) if and only if

$$u'(c(s)) \;>\; \beta R \sum_{s'} \pi(s'|s)\, u'(c(s'))$$
$$\Longleftrightarrow R < \frac{1}{\beta} \frac{u'(c(s))}{\sum_{s'} \pi(s'|s)\, u'(c(s'))}.$$

If $a' = -\phi$, then

$$c(s) = y(s) + R(-\phi) - (-\phi) = y(s) - (R-1)\phi.$$

For aggregate saving to equal $-\phi$, all households must choose $a' = -\phi$ in every state:

$$u'\big(y(s) - (R-1)\phi\big) \;\geq\; \beta R \sum_{s'} \pi(s'|s)\, u'\big(y(s') - (R-1)\phi\big), \qquad \forall s \in S.$$

Equivalently, $\bar{R}(\phi)$ is characterized (implicitly) by the tightest state:

$$\boxed{\bar{R}(\phi) = \frac{1}{\beta} \min_{s \in S} \frac{u'\big(y(s) - (\bar{R}(\phi) - 1)\phi\big)}{\sum_{s'} \pi(s'|s)\, u'\big(y(s') - (\bar{R}(\phi) - 1)\phi\big)}}$$

The interest rate $\bar{R}(\phi)$ depends on the borrowing limit ϕ.

For $R = \bar{R}(\phi)$, there exists a household s^* which is indifferent between borrowing $-\phi$ and saving more, and all other households strictly prefer to borrow $-\phi$.

Part (c)

We consider the two-state case $s \in \{s_1, s_2\}$ and write $y_i := y(s_i)$ and $\pi_i := \pi(s_i)$ for $i = 1, 2$.

If the household chooses $a' = -\phi$, then:

$$c_i(R) = y_i - (R-1)\phi = y_i + (1-R)\phi, \qquad i = 1, 2.$$

Euler inequality under log utility and i.i.d. income. Since $u'(c) = 1/c$, the Euler condition with a binding borrowing constraint is

$$\frac{1}{c_i(R)} \ \geq \ \beta R \sum_{j=1}^{2} \pi_j \frac{1}{c_j(R)}, \qquad i = 1, 2,$$

Plug in $c_i(R) = y_i + (1-R)\phi$, we have

$$\frac{1}{y_i + (1-R)\phi} \ \geq \ \beta R \sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}, \qquad i = 1, 2.$$

The right-hand side does not depend on the current state, we only need borrowing constraint to bind for the high-income household (so they don't want to save more)

Suppose $y_1 < y_2$, then:

$$\frac{1}{y(s_1) - (\bar{R}-1)\phi} > \frac{1}{y(s_2) - (\bar{R}-1)\phi},$$

Then we know that R need to satisfy the Euler inequality for the high-income household:

$$\boxed{\frac{1}{y_2 + (1-R)\phi} \ \geq \ \beta R \sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}}.$$

Define

$$F(R) \ := \ \frac{1}{y_2 + (1-R)\phi} - \beta R \sum_{j=1}^{2} \pi_j \frac{1}{y_j + (1-R)\phi}.$$

We know that there exist a finite $\bar{R}$ satisfying $F(\bar{R}) = 0$. [1]

We want to show that for any $R \in [0, \bar{R}]$, we have $F(R) \geq 0$.

$$F'(R) = \frac{\phi}{(y_2 + \phi - \phi R)^2} - \beta\Big(\frac{\pi_1}{y_1 + \phi - \phi R} + \frac{\pi_2}{y_2 + \phi - \phi R}\Big) - \beta R\phi\Big(\frac{\pi_1}{(y_1 + \phi - \phi R)^2} + \frac{\pi_2}{(y_2 + \phi - \phi R)^2}\Big).$$

[1] We have $\bar{R}$ satisfies:

$$\bar{R} = \frac{1}{\beta} \min_{s \in S} \frac{\frac{1}{y(s) - (\bar{R}-1)\phi}}{\pi(s_1)\frac{1}{y(s_1) - (\bar{R}-1)\phi} + \pi(s_2)\frac{1}{y(s_2) - (\bar{R}-1)\phi}}$$

Suppose $y_1 < y_2$, then:

$$\frac{1}{y(s_1) - (\bar{R}-1)\phi} > \frac{1}{y(s_2) - (\bar{R}-1)\phi},$$

we only need borrowing constraint to bind for the high-income household (so they don't want to save more)

Then $\bar{R}$ satisfies:

$$\bar{R} = \frac{1}{\beta} \frac{\frac{1}{y(s_2) - (\bar{R}-1)\phi}}{\pi(s_1)\frac{1}{y(s_1) - (\bar{R}-1)\phi} + \pi(s_2)\frac{1}{y(s_2) - (\bar{R}-1)\phi}}.$$

We can obtain a finite solution $\bar{R}$ to the above equation.

Let $\bar{R}$ be the largest interest rate such that the tight-state Euler inequality holds (i.e. $F(R) \geq 0$):

$$\bar{R} := \sup\{R \geq 0 : \; F(R) \geq 0\}.$$

Then for any $R \in [0, \bar{R}]$, we have $F(R) \geq 0$, i.e.

$$\frac{1}{c_2(R)} \geq \beta R \sum_{j=1}^{2} \pi_j \frac{1}{c_j(R)}.$$

Because $1/c_1(R) > 1/c_2(R)$, the Euler inequality holds for both $i = 1, 2$, so the borrowing constraint binds for everyone and $a'(s_i) = -\phi$. Therefore aggregate saving is constant on this interval:

$$A(R) = \sum_{i=1}^{2} \pi_i \, a'(s_i) = \sum_{i=1}^{2} \pi_i (-\phi) = -\phi, \qquad \forall \, R \in [0, \bar{R}].$$