Supermodularity and Equilibrium in Games with Peer Effects and Endogenous Network Formation

Yuya Furusawa

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Abstract

Peer effects in the network play an important role in determining individual and aggregate behaviors, and many literature argue their importances theoretically and empirically. However, almost all of the papers treat the network structure as given. This means that we cannot consider the agents' responses to economic shocks and an alternation of network structure. We develop the game-theoretic model with endogenous network formation where agents first choose who to connect and then exert efforts. We show that 1st stage game becomes supermodular game and there exists pure strategy subgame perfect equilibrium. Then, we give a discussion about multiplicity of equilibrium and comparative statics. In comparative static analysis, we can show our model incorporates phase transition phenomena. Finally, we provide a policy implication: key player policy and key link policy. With considering the endogeneity of the network, we can see the key player and key link might be different with the ones analyzed in the existing literature.

1 Introduction

Network structures and local interactions play an important role in determining individual and aggregate behavior. Individual behavior generates externalities and affect neighbors' behavior, which is referred to as peer effect. By considering network structure, we can explain the effects of agent's behavior on agents whom directly connect and indirectly connect. Recently, many literature in economics theoretically and emprically point out the significance of the externalities or peer effects. It is shown that peer effects are important in many kinds of network, for example, social network, criminal network, firm-to-firm networks where firms invest R&D activities, networks for labor market participation, educational network, and so on. One example of peer effects in networks is that, in criminal network, when friends are highly devoted to criminal activities such as drug abuses, the agent is more likely to engage in the drug abuse due to the criminal information or drug trafficking from his friends.

However, many existing literature about the economic networks ignore the mechanism of the network formation. With fixed networks, some papers[1] consider the effect of economic shocks or policy changing on the individual and aggregate behaviors in the network. When the economic agents are faced with the alteration of the environment where they are, they respond to them in order to mitigate the shocks and the network structure will be changed. Therefore, it is significant to consider the mechanism of network formation to analyze the impact of policy intervention. In this paper, we propose the way to overcome this problem by incorporating the endogenous network formation to the game theoretic model[2].

We develop a two stage dynamic game, where agents form links to other agents in the first stage and agents exert efforts in the second stage. The second stage game is eccensially same with the model in Ballester, Calvó-Armengol, and Zenou(2006) [5]. Agents simultaneously choose a nonnegative effort level and the agents' payoffs depend on not only their own effort but also neighbors' efforts. The agent' effort generates the positive externalities, so called "peer effects", to the agents who has a link to her in the network. We extend the model of Ballester, Calvo-Armengol, and Zenou(2006) [5] to the endogenous network by introducing first stage game, choosing neighbors. In the first stage, agents simultaneously choose their neighbors from the neighbors in the potential network which represents the possiblly maximal network without any limitations. When agents decide who to connect, they incur the link-specific costs, which is a key factor in determining the realized network. The link formation costs make the heterogeneity across the agents, which leads to the various types of network structures. The interpretion of links, effort, and link formation cost

[1]For production networks, Acemoglu, Carvalho, Ozdaglar, and Tahbaz- Salehi(2012) [2] or Carvalho, Nirei, Saito, and Tahbaz-Salehi(2016) [14]. For a network of military alliances, König, Rohner, Thoenig, and Zilibotti(2017) [29]

[2]Games in networks are analyzed in many papers, such as Jackson, and Wolinsky(1996), Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv(2010) [21] and Bramoullé, Kranton, and D'Amours(2014).

depend on the application we consider. We consider subgame perfect equilibria where all agents take pure starategies in the both stages.

We can interpret the model in some different ways. First, we can think of our model as the web advertisement network. Node, link, link formation cost, and effort represent web sites, advertisement on other sites, fee, and investment on the web contents, respectively. If connected sites invest on the site and the number of viewers increase, more viewers visit the site and there exists an incentive to invest more. In another interpretation, we can regard our model as the drug trafficking network. Municipalities, drug trafficking, and the amount of drug used are represented by nodes, links and efforts respectively. Link formation cost means the probability of capture during trafficking. In Dell(2015) [17], drug are trafficked across municiparities in Mexico, and exported to U.S. Probabilty of capture, which is represented by link formation cost, depends on the policy of the city and government. Moreover, Dell(2015) [17] indicates that there exist spillover effects when the policy maker leads one location to become less active to illicit activities, and this coincides with our model where peer effects in effort levels and choosing neighbor strategy exist.

We establish that a pure strategy subgame perfect equilibrium always exists. Due to the result in Ballester, Calvó-Armengol, and Zenou(2006) [5], a second stage Nash equilibrium always exists and it is unique. Given this equilibrium, by backward induction, we show the first stage becomes supermodular game. Then, by well known properties of supermodurarity, we can see the first stage pure strategy equilibrium always exists. However, equilibria may not be unique. We focus on the greatest equilibrium which can be obtained by sequential best response dynamics algorithm. In comparative static analysis in link formation costs, we see that reduction in the costs leads to larger network because agents are more likely to connect to other agents. Realized network is discontinuous in the link formation costs, so changes in the costs leads to discontinuous changes in the realized network. We give a example that a tiny change in the link formation costs leads to a drastic change in the network structure, "phase transition". We analyze a key player policy in endogenous networks, contrary to the former works which analyze a key player in exogenous networks. A key player is the agent in the potential network who makes largest reduction of aggregate effort level once he removed from the potential network. However, it is difficult to find key players from the link formation costs because of its discontinuity. We give an example that a key player in the endogenous network and in the exogenous might be different. This example shows that without considering the mechanism of the network formation, we might have a wrong policy implication. Similarly, we analyze a key link policy which is introduced in this paper for the first time, and we see we may have wrong policy without considering the endogenous network formation. Finally, we provide an example that our model can be used as a method to find negligible links between agents.

This paper contributes to the growing literature on the endogenous networks. Endogeneity of the network is one of the hottest topics in the field of economic networks. Oberfield(2018) [36], and Acemoglu and Azar(2019) [1] argue the endogenous production networks. In their model, the productivity of the firms is determined by the input supplier, customer who use its output as input, and final consumers who generates the firms' profit. This productivity is the key factor for network formation, and, in an equilibrium, input-output architecture endogenously emerges. Farboodi(2014) [20] and other papers[3] argue the endogenous formation of financial networks. Recently, many papers theoretically and empirically work for other networks[4].

This paper is also related to peer effects in the networks. The importance of peer effects is often pointed out originally in Rees(1966) [37] or recently Banerjee, Chandrasekhar, Duflo, and Jackson(2013) [8]. Many papers develop the analysis of peer effects in the network. Calvó-Armengol, Patacchini, and Zenou(2009) [12] theoretically and empirically analyzes the peer effect in the educational achivement. Liu, Patacchini, Zenou, and Lee(2012) [30] and Dell(2015) [17] argue the positive externalities in the criminal network. Bramoullé, Djebbari, and Fortin(2009) [10] and Liu, Patacchini, and Zenou(2014) [31] propose the conditions we can identify the peer effects in the network and analyze social network empirically. König, Liu, and Zenou(2019) [28] provides theoretical and empirical discussions about R&D network where peer effects exist.

This paper is closely related to Kim, Patacchini, Picard, and Zenou(2017) [27] which considers the social relationship formation in the urban geographical space. They consider the model where a continuum of agents distributes over the line segment and decides the intensity of interactions to the agents who live in the other point of the segment. The agents' payoff functions and the positive externalities of the interactions are eccentially same with our model and Ballester, Calvó-Armengol,

[3]For examples, Cohen-Cole, Patacchini, and Zenou(2010) [15], Acemoglu, Ozdaglar, and Tahbaz-Salehi(2015) [3] and Babus(2016) [4]

[4]Canen, Jackson, and Trebbi(2019) [13], Marco, Eleonora, and Edoardo(2019) [32], Margherita, and Mariapia(2015) [33], Tintelnot, Kikkawa, Mogstad, and Dhyne(2018) [38]

and Zenou(2006) [5] (and other literature). Although the number of agents is infinite, we can treat their model as a network formation model.[5] The key factor of the network formation in their paper is the distribution of agents and the cost of interations which is measured by geographic distance. There are two differences in this paper and ours. First, in their model, the cost of interactions is only determined by the distance of two agents. In many cases, interactions are affected by various kinds of obstacles, such as psychological barriers and financial costs. In contrast, our model can incorporate any kinds of interaction costs, and it enables us to consider many kinds of economic situations in our framework. Second, they consider the weighted relationships and agents decide the intensity of interaction to all other agents. Due to this feature, we can remove the any kinds of discontinuity and compute the network structure and the weights of links analytically. However, analytical solution always shows that in every equilibrium, every agent connects all other agents in the network. Our model can explain the formation of any structure of the networks.

Hiller(2017) [22] is also related to our paper. He develops the very similar model with ours: agents choose their neighbors and decide the level of efforts. He considers the general payoff function which includes the payoff function in our model. But he considers the static game where all agents choose neighbors and effort levels simultaneously. In addition, the costs of link formation is determined by the number of links which agent has. This construction of the model lacks the uniqueness of the equilibrium. Our model enables us to focus on the unique or largest equilibrium by setting the game dynamic.

The rest of the paper is organized as follows. Section 2 describes the model. In section 3, we first give the definition of the equilibrium and present the result on its existence and uniqueness. We also present the result on comparative statics in this section. In section 4, we comment on the policy implications and Section 5 concludes. All proofs used to show the reults in this paper are attached in Appendix.

Notation. For any vector $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we write $\boldsymbol{a} \geq (>)\boldsymbol{b}$ if $a_i \geq (>)b_i$ for any $i = 1, \ldots, n$. For matrix $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \geq \boldsymbol{B}$ if $a_{ij} \geq b_{ij}$ for any entry ij. A link from i to j is denoted as link ij. For network g, if link ij is in g, we denote $ij \in g$. Let $E(g)$ denote the set of links in network g. For network g and $\hat{g}$, we write $g \subseteq \hat{g}$ to indicate that $\{ij \in N \mid ij \in E(g)\} \subset \{ij \in N \mid ij \in E(\hat{g})\}$, that is, $g \subseteq \hat{g}$ means all links in g are also contained in $\hat{g}$. Denote

[5] In empirical part, they discritize the agents and consider the adjacency matrix as our model.

$g \setminus \{ij\}$ as the network obtained by removing link ij from network g. Denote $g \cup \{ij\}$ as the network obtained by adding link ij to netwotk g. For vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, their join $\boldsymbol{x} \vee \boldsymbol{y}$ is defined as $\boldsymbol{x} \vee \boldsymbol{y} = (\max\{x_1, y_1\}, \cdots, \max\{x_n, y_n\})$. Similarly, their meet $\boldsymbol{x} \wedge \boldsymbol{y}$ is defined as $\boldsymbol{x} \wedge \boldsymbol{y} = (\min\{x_1, y_1\}, \cdots, \min\{x_n, y_n\})$. For $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, their join $\boldsymbol{A} \vee \boldsymbol{B}$ is defined as $\boldsymbol{A} \vee \boldsymbol{B} = (\max\{a_{ij}, b_{ij}\})_{ij}$. Similarly, their meet $\boldsymbol{A} \wedge \boldsymbol{B}$ is defined as $\boldsymbol{A} \wedge \boldsymbol{B} = (\min\{a_{ij}, b_{ij}\})_{ij}$.

2 Model

There are n agents in the economy, and denote the set of agents $N = \{1, \ldots, n\}$. We assume n is finite and $n \geq 2$.

The game consists of two stages: choosing neighbors in the first stage and choosing an effort level in the second stage. Initially, agents are connected in the potential network $\overline{g}$. We consider the potential network $\overline{g}$ is directed[6] and unweighted[7]. The potential network g^p is represented by adjacency matrix $\overline{\boldsymbol{G}} = (\overline{g}_{ij})_{ij}$ where, for any $i \neq j$,

$$\overline{g}_{ij} = \begin{cases} 1 \text{ (if } i \text{ has a link to } j \text{ in } \overline{g}) \\ 0 \text{ (otherwise)} \end{cases}$$

Note that $\overline{\boldsymbol{G}}$ might be asymmetric, that is it is not necessarily $\overline{g}_{ij} = \overline{g}_{ji}$. We set $\overline{g}_{ii} = 0$ for all $i \in N$. Denote the set of agent i's neighbors in the potential network $N_i(\overline{g}) = \{j \in N \mid \overline{g}_{ij} = 1\}$.

In the first stage, each agent i simultaneously chooses the partners from the potential neighbors $N_i(\overline{g})$. The action is represented by $\psi_i = (\psi_{i1}, \cdots, \psi_{in})$ with (i) $\psi_{ij} \in \{0, 1\}$ for all $j \in N$ and (ii) $\psi_{ij} = 0$ for all $j \notin N_i(\overline{g})$. First, $g_{ij} = 1 (g_{ij} = 0)$ respresents agent i (does not) form a link to j. Second, (ii) means that only links existed in the potential network can be realized. We assume $\psi_{ii} = 0$ for all $i \in N$. Let Ψ_i denote the set of agent i's actions, and $\Psi = \prod_{i=1}^{n} \Psi_i$ denote its profile. Here, we allow mixed actions. When forming a link, each agent i incurs a cost $c_{ij} \geq 0$. Denote its matrix $\boldsymbol{C} = (c_{ij})_{ij}$.[8] Since choosed pure action ψ_i depends on potential network $\overline{g}$, to emphasize it,

[6]The network g is *directed* when link ij exists does not mean link ji exists. Conversely, if the existence of link ij always imply that of link ji, we call g is *undirected*.

[7]The network g is *unweighted* when any g_{ij}, entry of g's adjacency matrix, takes value either 0 or 1. Conversely, the network g is *weighted* if the value of g_{ij} may be neither 0 nor 1.

[8]We set $c_{ij} = 0$ for ij such that $\overline{g}_{ij} = 0$.

we denote $\psi_i(\overline{g})$ and $\boldsymbol{\psi}(\overline{g}) = (\psi_1(\overline{g}), \cdots, \psi_n(\overline{g}))$. After decided pure action profile $\boldsymbol{\psi}(\overline{g})$, the network g is realized. Figure 1 represents the difference between potential network and realized network.

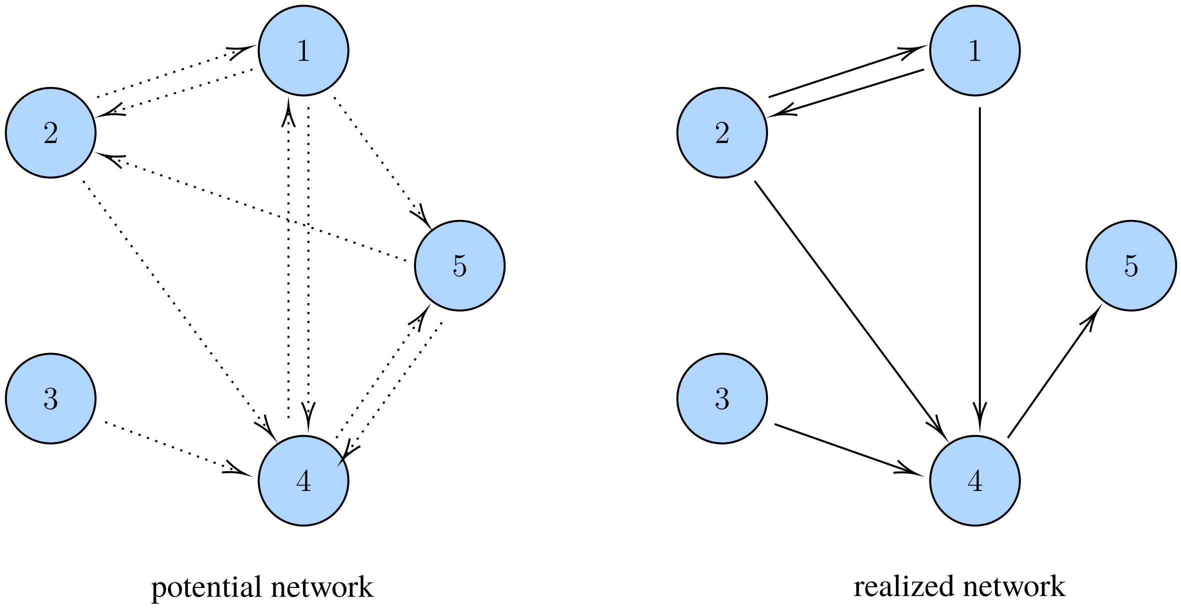


Figure 1: Difference between potential network and realized network

The realized network g is represented by the adjacency matrix $\boldsymbol{G} = (g_{ij})_{ij}$ such that, for any ij,

$$g_{ij} = \begin{cases} 1 \text{ (if } \psi_{ij}(\overline{g}) = 1) \\ 0 \text{ (otherwise).} \end{cases}$$

To represent the dependency of realized network g on $\boldsymbol{\psi}(\overline{g})$, we write $g(\boldsymbol{\psi}(\overline{g}))$. We sometimes denote the network $g(\boldsymbol{\psi}(\overline{g}))$ as $g(\boldsymbol{\psi})$ to avoid redundant representations unless $\overline{g}$ is a key variable.

In the second stage, each agent $i \in N$ simultaneously exerts an effort $x_i \geq 0$, and gets a payoff which depends on the agents' efforts and realized network. Denote $\boldsymbol{x} = (x_1, \cdots, x_n)$.

$$u_i(\boldsymbol{x}, \boldsymbol{\psi}, \boldsymbol{C}, \phi) = v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) - \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) c_{ij}$$

where

$$v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) = \alpha_i x_i - \frac{1}{2} x_i^2 + \phi \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) x_i x_j. \tag{1}$$

First term of (1) represents the individual benefit of effort, and second term is the disutility of effort. Third cross term represents the peer effect of the effort. If the neighbor exerts high effort, agent i receives large benefit with decay factor ϕ. We assume $\alpha_i > 0$ for all i, and $\phi > 0$. Denote $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$.

Before go into an equilibrium analysis, we introduce a well-known network centrality measure which is useful for our analysis. Consider the n-square adjacency matrix $\boldsymbol{G}$ of a network g where $g_{ij} = 1$ if i has a direct link to j and $g_{ij} = 0$ otherwise. We introduce a following definition from Katz(1953) [26] and Bonacich(1987) [9].

Definition 1. Consider a network g with adjacency matrix $\boldsymbol{G}$ and a small enough $\phi \geq 0$ such that $(\boldsymbol{I} - \phi \boldsymbol{G})^{-1}$ exists. Given a vector $\boldsymbol{u} \in \mathbb{R}^n_+$, the vector of *Katz-Bonacich centralities* of parameter ϕ in network g is defined as:

$$\boldsymbol{b}_{\boldsymbol{u}}(g, \phi) = (\boldsymbol{I} - \phi \boldsymbol{G})^{-1} \boldsymbol{u} = \sum_{p=0}^{\infty} \phi^p \boldsymbol{G}^p \boldsymbol{u}.$$

Katz-Bonacich centrality can be thought of a centrality measure which considers the discounted sum of the paths[9]. The i-th entry of the vector $\boldsymbol{b}_{\boldsymbol{u}}(g, \phi)$ is denoted by $b_{\boldsymbol{u},i}(g, \phi)$.

3 Equilibrium and Characterization

3.1 Equilibrium definition and its existence

In this paper, we consider a pure strategy subgame perfect equilibrium. The reason why we consider such an equilibrium is that pure strategies in choosing neighbors make the realized network deterministic. If agents take mixed actions in the first stage, the realized network becomes stochastic and we cannot directly compare the realized network. We call a network emerged in a pure strategy subgame perfect equilibrium as an equilibrium network.

Definition 2. Given $\overline{g}$ and $\boldsymbol{C}$, a network g^* is an *equilibrium network* if $g^* = g(\boldsymbol{\psi}^*)$ where $\boldsymbol{\psi}^*$ is an action profile in a pure strategy subgame perfect equilibrium.

We characterize an equilibrium network by a standard way: backward induction. First, given realized network g, consider the second stage game. To characterize an equilibrium in this subgame,

[9]A *path* in a network is a finite or infinite sequence of links that joins a sequence of nodes.

we require a new assumption. Denote $\rho(\boldsymbol{G})$ as the spectral radius[10] of the matrix $\boldsymbol{G}$.

Assumption 1. $\phi\rho(\overline{\boldsymbol{G}}) < 1$.

This assumption gives an upper bound of ϕ. Under Assumption 1, a peer effect is bounded in the potential network which is the possiblly largest network. We show that, under this assumption, there exists a unique Nash equilibrium in the second stage for any realized network.

Proposition 1. Under Assumption 1, for any realized network g, $(\boldsymbol{I} - \phi\boldsymbol{G})^{-1}$ exists, and the subgame has a unique Nash equilibrium $\boldsymbol{x}^*$ which is given by

$$\boldsymbol{x}^*(g) = \boldsymbol{b}_\alpha(g, \phi) = (\boldsymbol{I} - \phi\boldsymbol{G})^{-1}\boldsymbol{\alpha} > \boldsymbol{0}.$$

Ballester, Calvo-Armengol, and Zenou(2006, 2010) show that, given network g, there exists a unique interior Nash equilibrium when the spectral radius of the adjacency matrix of the network is small enough. In addition, we find that optimal effort levels coincide with the Katz-Bonacich centrarity of parameter ϕ given realized network g. In our model, the spectral radius of the adjacency matrix of realized network may not satisfy the condition. Proposition 1 says that we can satisfy the condition under Assumption 1. Note that Proposition 1 also provides the Nash equilibrium, which is a first step of backward induction. In the rest of this paper, we assume Assumption 1.

Here $\boldsymbol{x}^*$ is the optimal effort levels in the realized network, and so we can find x_i^*s' are dependent of $g(\boldsymbol{\psi})$, so optimal effort levels can be written as $x_i^*(g(\boldsymbol{\psi}))$. From Proposition 1, we can calculate (1) as follows:

$$v_i(\boldsymbol{x}^*(g(\boldsymbol{\psi})), g(\boldsymbol{\psi}), \phi) = \frac{1}{2}[b_{\alpha,i}(g(\boldsymbol{\psi}), \phi)]^2 = \frac{1}{2}x_i^*(g(\boldsymbol{\psi}))^2. \tag{2}$$

In order to characterize a subgame perfect equilibrium, next we consider the agents' strategy to choose neighbors by backward induction. In light of Proposition 1, we only need to consider Nash equilibrium in the first stage given the efforts in the second stage Nash equilibrium. Let us consider the reduced first stage game given second stage Nash equilibrium as a normal form game $\Gamma = \langle N, \Psi, (u_i)_{i\in N}\rangle$ where for all i, $u_i : \boldsymbol{\Psi} \to \mathbb{R}$ and

$$u_i(\psi) = u_i(\boldsymbol{x}^*(g(\psi)), \psi, \boldsymbol{C}, \phi).$$

[10]The *spectral radius* of a matrix is the largest absolute value of its eigenvalues.

Before we establish the theorem, we show some preliminary results. First, given the second stage Nash equilibrium, we can see the more links there exist, the more efforts agents exert.

Lemma 1. Consider the networks g and $\hat{g}$ with $g \subseteq \hat{g}$. Then,

$$\boldsymbol{x}^*(\hat{g}) \geq \boldsymbol{x}^*(g).$$

In addition, for all $i \in N$ if, for some $j \in N$, $\hat{g}_{ij} = 1$ and $g_{ij} = 0$, then

$$x_i^*(\hat{g}) > x_i^*(g).$$

This lemma states that function v_i is increasing in ψ_i. From (2), we can have

$$v_i(x^*(\hat{g}), \hat{g}, \phi) > v_i(x^*(g), g, \phi),$$

and for any $j \neq i$,

$$v_j(x^*(\hat{g}), \hat{g}, \phi) \geq v_j(x^*(g), g, \phi).$$

These imply that v_i^* is increasing in g.

For networks g and h with adjacency matrices $\boldsymbol{G}$ and $\boldsymbol{H}$, let $g \vee h$ denote the network which is constructed by the adjacency matrix $\boldsymbol{G} \vee \boldsymbol{H}$. We define $g \wedge h$ by a similar way. In addition to Lemma 1, we have the following lemma.

Lemma 2. Consider the network g and h (their adjacency matrices are $\boldsymbol{G}$ and $\boldsymbol{H}$ respectively). Then, for all i,

$$v_i(\boldsymbol{x}^*(g \vee h), g \vee h, \phi) + v_i(\boldsymbol{x}^*(g \wedge h), g \wedge h, \phi) \geq v_i(\boldsymbol{x}^*(g), g, \phi) + v_i(\boldsymbol{x}^*(h), h, \phi).$$

Using Lemma 1 and 2, we can have next theorem.

Theorem 1. For any $\overline{g}$ and $\boldsymbol{C}$, Γ is a supermodular game.

We can immediately have a following corollary by the properties of supermodular game.

Corollary 1. For any $\overline{g}$ and $\boldsymbol{C}$, there exists equilibrium network. Moreover, the set of equilibrium strategies in reduced game Γ has the greatest and the least element.

We note that the reduced game, the 1st stage game given the effort level in 2nd stage Nash equilibrium, is a supermodular game, the game which exhibits strategic complementarities. This game can satisfy some conditions of supermodularity: continuity and increasing-differences of utility function. From the results of some papers [11], we can find a pure strategy Nash equilibrium in this game. The intuition behind the supermodularity is as follows. We can easily see that 2nd stage game is with strategic complementarity, that is when the neighbors exert high level effort, we also increase the level of effort. Lemma 2 shows the reduced game also exhibits strategic complementarity and it is supermodular game. When agent i forms more links, he can get more peer effects, so he exerts more effort by the strategic complementarity in 2nd stage. Agent i's increased effort makes agents who have a link to him exert more effort, so all agents' level of effort weakly increases, which is shown by Lemma 1. Provided the link formation costs, increasing level of efforts makes the agents to connect more agents, so if some agents have more connections, best response for it is the connecting to other agents, which is strategic complementarity.

3.2 Uniqueness and multiplicity of equilibrium

Previous subsection shows the existence of the equilibrium. We argue the uniqueness of the equilibrium in this subsection. Since the equilibrium in the second stage is unique from Proposition 1, we focus on the first stage actions. We can see that the equilibrium network is not necessarily unique. The following example shows that equilibrium network may not unique with some cost parameters.

Example 1. Consider the network with $n = 2$, and the potential network $\overline{g}$ such that,

$$\overline{\boldsymbol{G}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Assume $\boldsymbol{\alpha} = (1, 1)$ and ϕ satisfies Assumption 1. Denote the realized network g as

$$\boldsymbol{G} = \begin{bmatrix} 0 & g_{12} \\ g_{21} & 0 \end{bmatrix}$$

[11]For example, Topkis (1979) [39], Milgrom and Roberts (1990) [34], Vives(1990) [41] and Topkis (1998) [40]

where $g_{12}, g_{21} \in \{0, 1\}$. By proposition 1, we can compute each player's equilibrium payoffs:

$$(u_1^*, u_2^*) = \begin{cases} \left(\frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{12}, \frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{21}\right) & (g_{12} = 1, g_{21} = 1) \\ \left(\frac{1}{2}(1+\phi)^2 - c_{12}, \frac{1}{2}\right) & (g_{12} = 1, g_{21} = 0) \\ \left(\frac{1}{2}, \frac{1}{2}(1+\phi)^2 - c_{21}\right) & (g_{12} = 0, g_{21} = 1) \\ \left(\frac{1}{2}, \frac{1}{2}\right) & (g_{12} = 0, g_{21} = 0). \end{cases}$$

Payoff matrix is:

	$g_{21} = 1$	$g_{21} = 0$
$g_{12} = 1$	$\left(\frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{12}, \frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{21}\right)$	$\left(\frac{1}{2}(1+\phi)^2 - c_{12}, \frac{1}{2}\right)$
$g_{12} = 0$	$\left(\frac{1}{2}, \frac{1}{2}(1+\phi)^2 - c_{21}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$

From this calculation, if c_{12} and c_{21} satisfy

$$\frac{1}{2}(1+\phi)^2 - \frac{1}{2} < c_{12} < \frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - \frac{1}{2}$$

$$\frac{1}{2}(1+\phi)^2 - \frac{1}{2} < c_{21} < \frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - \frac{1}{2},$$

then $(g_{12}, g_{21}) = (1, 1)$ and $(g_{12}, g_{21}) = (0, 0)$ are both Nash equilibria. Thus, complete network $\boldsymbol{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and empty network $\boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are both equilibrium networks.

Example 1 shows the equilibrium network may not be unique and Figure 2 shows that the region of costs which brings the respective equilibrium network. We can see the region which returns complete equilibrium network[12] and empty equilibrium network[13] overlap. When the link formation costs are in some ranges, there exist multiple equilibrium networks.

To avoid the problem of equilibrium multiplicity, we want to focus on the greatest equilibrium. However, how can we obtain the largest equilibrium network? Consider the next algorithm.

[12]The network is *complete* if each node has links to all other nodes. In adjacency matrix, every off-diagonal entries are equal to 1.

[13]The network is *empty* if there is no link in the network. In adjacency matrix, all entries are equal to 0.

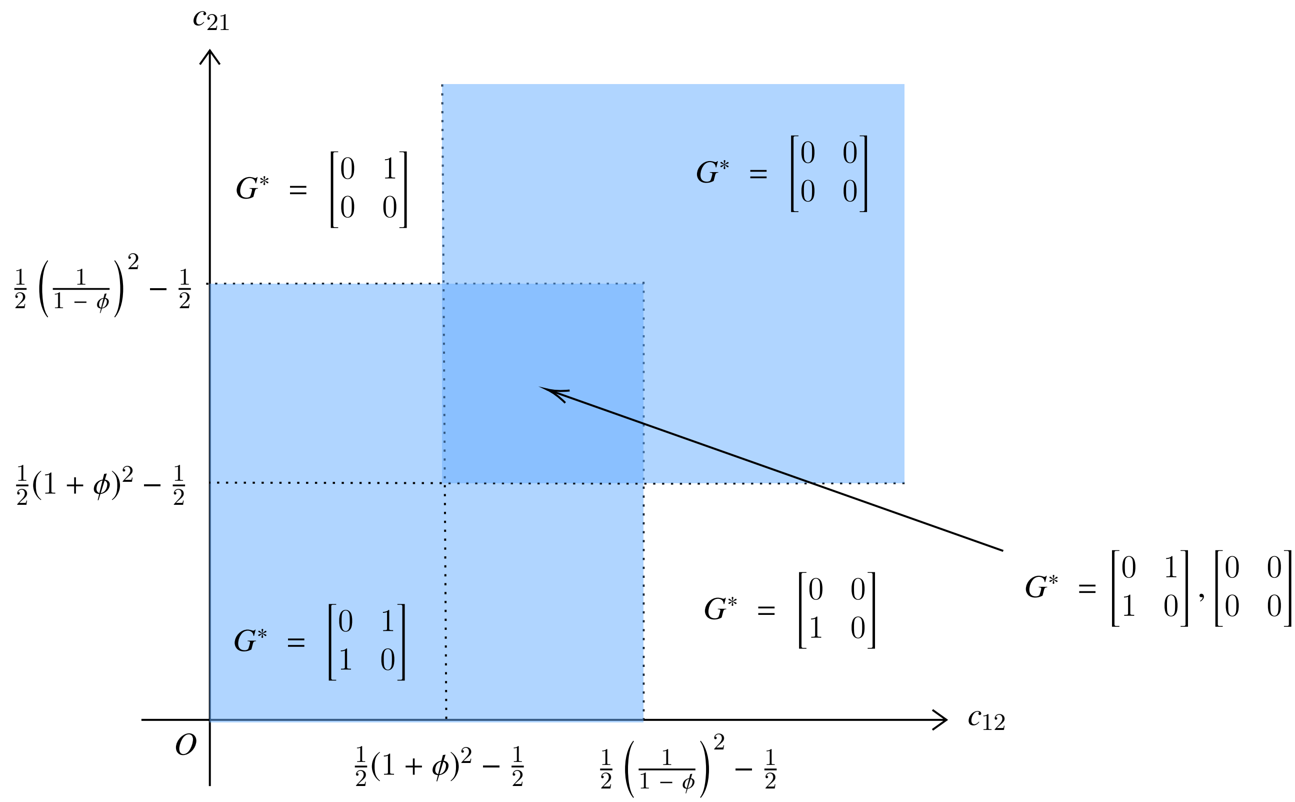


Figure 2: Equilibrium network region

Algorithm 1.

Step 0.

Let $g^{(0)}$ be the initial realized network where $g^{(0)} = \overline{g}$, that is, chooosing strategy is $\psi_i^{(0)} \in \Psi_i$ such that $\psi_{ij}^{(0)} = 1$ for all $j \in N_i(\overline{g})$. Compute each players' optimal effort and payoffs. Denote the set of agents who do not take best response as $NB^{(0)}$:

$$NB^{(0)} = \{i \in N \mid \exists \tilde{\psi}_i \in \Psi_i \text{ s.t. } u_i(\boldsymbol{x}^*(g(\tilde{\psi}_i, \psi_{-i}^{(0)}), \tilde{\psi}_i, \psi_{-i}^{(0)}, \boldsymbol{C}, \phi) > u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(0)})), \boldsymbol{\psi}^{(0)}, \boldsymbol{C}, \phi)\}.$$

Go into Step 1.

Step $k(\geq 1)$.

Check whether $NB^{(k-1)}$ is empty or not.

If $NB^{(k-1)} = \emptyset$, define $g^* = g(\boldsymbol{\psi}^{(k-1)})$ and terminate the algorithm.

Otherwise, choose a agent $i \in NB^{(k-1)}$ randomly. Define

$$\overline{BR}_i(\psi_{-i}^{(k-1)}) = \{\psi_i \in \Psi_i \mid u_i(\boldsymbol{x}^*(g(\psi_i, \psi_{-i}^{(k-1)})), \psi_i, \psi_{-i}^{(k-1)}, \boldsymbol{C}, \phi) > u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(k-1)})), \boldsymbol{\psi}^{(k-1)}, \boldsymbol{C}, \phi)\}.$$

i changes her strategy from $\psi_i^{(k-1)}$ to $\psi_i \in \overline{BR}_i(\psi_{-i}^{(k-1)})$[14], and set $\psi_i^{(k)} = \psi_i$. For any other agents $j (\neq i)$ remain their strategies, $\psi_j^{(k)} = \psi_j^{(k-1)}$. If $|\overline{BR}_i(\psi_{-i}^{(k-1)})| > 1$, choose the largest one in $\overline{BR}_i(\psi_{-i}^{(k-1)})$. Then, new network $g(\boldsymbol{\psi}^{(k)})$ is realized. Compute each players' payoffs and define $NB^{(k)}$:

$$NB^{(k)} = \{i \in N \mid \exists \tilde{\psi}_i \in \Psi \text{ s.t. } u_i(\boldsymbol{x}^*(g(\tilde{\psi}_i, \psi_{-i}^{(k)}), \tilde{\psi}_i, \psi_{-i}^{(k)}, \boldsymbol{C}, \phi) > u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(k)})), \boldsymbol{\psi}^{(k)}, \boldsymbol{C}, \phi)\}.$$

Proceed to Step $k + 1$.

Algorithm 1 is one of the *best response dynamics* algorithms, where agents take their best reponses in each steps. We can interpret this algorithm as the learning procedure. Agents initially does not take best responses, but as seeing others' strategies, agents adopt best responses. In the algorithm, agent choose the largest strategy in $\overline{BR}_i(\psi_{-i}^{(k-1)})$. By the property of supermodurarity[15], $\overline{BR}_i(\psi_{-i}^{(k-1)})$ is a sublattice which has a greatest element. You notice that if the best reponse dynamics halts, it returns a pure strategy Nash equilibrium.[16] As in Topkis (1998) [40], returned network is the greatest equilibrium network. We can show Algorithm 1 terminates in finite steps, and we can have the largest equilibrium network.

Proposition 2. Algorithm 1 terminates in finite steps.

The greatest ewuilibrium network is obtained by Algorithm 1. We focus on the greatest equilibrium network from now on, and denote it as g^{**}.

3.3 Comparative Statics

In this section, we argue the comparative statics in the parameter $\boldsymbol{C}$ by focusing on the greatest equilibrium. When the costs of forming links decrease, we conjecture that the network becomes

[14]Note $\overline{BR}_i(\psi_{-i}^{(k-1)}) \neq \emptyset$ by construction of $NB^{(k-1)}$.
[15]See Topkis (1979) [39].
[16]See Nisan, Noam, et al. [35]

denser because agents are more likely to form the links.

Next proposition shows that our conjection is verified. This result is based on the supermodularity of the 1st stage game.

Proposition 3. Given the potential network $\overline{g}$. Consider the cost $\hat{\boldsymbol{C}}$ and $\boldsymbol{C}$ with $\hat{\boldsymbol{C}} \leq \boldsymbol{C}$. Then,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \hat{\boldsymbol{C}}, \phi, \boldsymbol{\alpha})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha})).$$

This propopsition is based on the property of supermodular game: changes in key parameters monotonically shift the equilibria of a supermodular game. By the similar argument, we have the following corollary.

Corollary 2. Given the potential network $\overline{g}$. For $\hat{\phi} \geq \phi$ which satisfy Assumption 1,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \hat{\phi}, \boldsymbol{\alpha})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha})).$$

For $\hat{\boldsymbol{\alpha}} \geq \boldsymbol{\alpha}$,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \hat{\boldsymbol{\alpha}})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha})).$$

The realized network is decreasing in the costs of forming links, and increasing in the benefit of effort and intensity of peer effects. However, it is obvious that this function is discontiunous. The next example shows that small changes in the costs makes the network entirely changed.

Example 2. Consider the network with $n = 5$, and the potential network so that $\overline{g}_{ij} = 1$ for any pair $ij(i \neq j)$. Assume $\boldsymbol{\alpha} = (1, 1, 1, 1, 1)$ and $\phi = \frac{1}{5}$, which satisfies the Assumption 1. Consider the costs $\boldsymbol{C}$ such that

$$\boldsymbol{C} = \begin{bmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix}.$$

Then, the equilibrium network g^* becomes

$$\boldsymbol{G}^{**} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The equilibrium network becomes complete network. On the other hand, when the cost of forming a link from agent 1 to agent 2 slightly increases, for any $\epsilon > 0$,

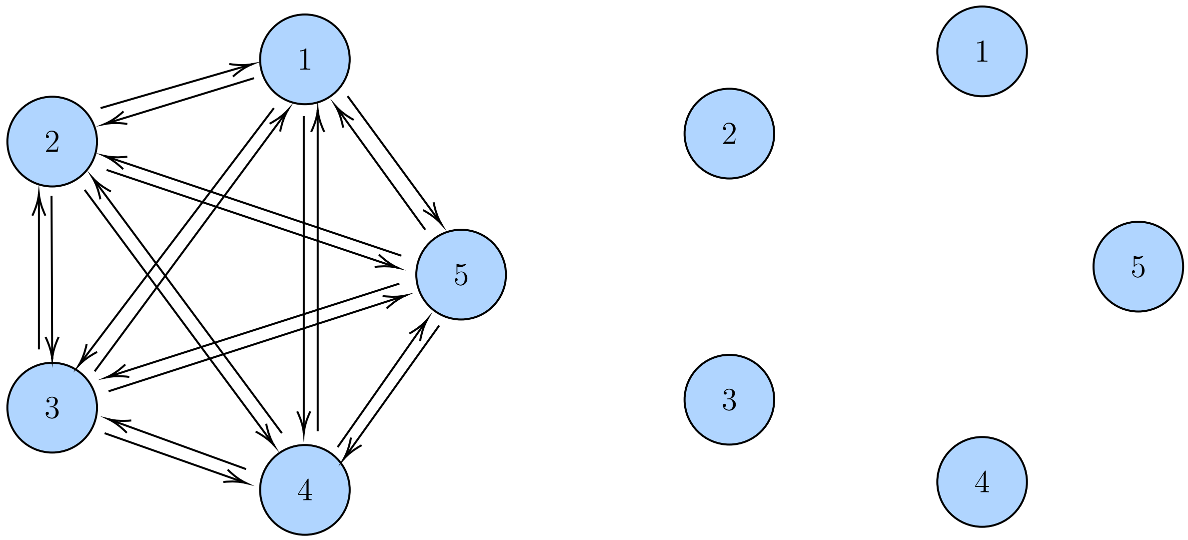
$$\hat{\boldsymbol{C}} = \begin{bmatrix} 0 & 3+\epsilon & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix}$$

then, the equilibrium network $\hat{g}^*$ becomes

$$\hat{\boldsymbol{G}}^{**} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equilibrium network is empty network.

With the cost structure $\boldsymbol{C}$, all agents keep connected to all other agents, but all agents are on the threshhold from keeping all links to removing them. In fact, all agents are indifference between them, keeping all links and removing all links give the exactly same payoff. Algorithm 1 returns the complete networks by its construction. When the cost of forming link from agent 1 to agent 2 slightly increases, agent 1 does not keep links to the others because agent 1's gain from agent 2 cannot compensate its cost. Since agent 1 removes all links, agent 1's level of effort drastically



Equilibrium network G^{**} Equilibrium network $\hat{G}^{**}$

Figure 3: Equilibrium networks in Example 2

decreases. The gain from agent 1 also declines, and the levels of effort of agents declines. Therefore, the benefit of forming links cannot compensate the cost of keeping links, and all agents remove all links.

We can see tiny changes in the costs generates large discontinuity in the structure of the network. Example 2 shows that our model incorporates the *phase transition*: the phenomenon that the feature of the network is totally changed[17].

4 Policy implication

4.1 Key player policy

We first argue a *key player* in the network who has the largest impact on the aggregate behavior of the network. In the previous literature, for example Ballester, Calvo-Armengol, and Zenou(2006, 2010) [5, 6] and Liu et.al.(2012) [30], argue a key player and give necessary and sufficient condition of who becomes it in the context of similar model[18]. In these papers, key player is defined as the agent who, once removed from the network, generates the highest possible reduction in aggreate

[17] see Jackson(2010) [23]
[18] Denbee, Julliard, Li, and Yuan(2018) [18] defines a key player in financial network

effort level. Key player is thought of, for example, the player who supports the criminal activity of his friends in the context of criminal network. The formal definition of key player is given as follows:

Definition 3. Agent i is a *key player in exogenous network* if, given network g,

$$i \in \arg\max_{i \in N} \{x^*(g) - x^*(g^{-i})\}.$$

Here we denote $x^*(g) = \sum_{i=1}^{n} x_i^*(g)$ and g^{-i} is the network where agent i is removed from the network g. The adjecency matrix of g^{-i}, $\boldsymbol{G}^{-i}$, is obtained from $\boldsymbol{G}$ by setting to zero all of its ith row and column entries. Previous literature shows the key player in exogenous network does not always coincide with the most active player who exerts highest effort.

In this paper, the definition of a key player might be different from the previous one. Since the network is endogenous in our model, a key player is the agent who generates the largest reduction in the total effort level once she is removed from the potential network.

Definition 4. Agent i is a *key player in endogenous network* if, given potential network $\overline{g}$,

$$i \in \arg\max_{i \in N} \{x^*(g(\boldsymbol{\psi}(\overline{g}^{-i}, \boldsymbol{C}))) - x^*(g(\boldsymbol{\psi}(\overline{g}^{-i}, \boldsymbol{C}^{-i})))\}.$$

Here we denote $\overline{g}^{-i}$ is the network where agent i is removed from the network $\overline{g}$ as before. $\boldsymbol{C}^{-i}$ is obtained from $\boldsymbol{C}$ by setting to zero all of its ith row and column entries.

The realized network is a key factor to be a key player in the endogenous network. We know that denser network leads higher effort level by the network externality(see Lemma 1), so, in order to become a key player in endogenous network, the realized network after removing should be sparse. However, as noted in the previous section, there exists a discontinuity of the network realization in the link formation costs, so it is very difficult to identify the equilibrium networks and the condition to be a key player from the link formation costs and the potential network[19].

Although it is difficult to identify who becomes key player, we can see that the player with the highest effort, key player in endogenous network, and key player in exogenous network can be

[19]Similar to Elliott, Golub, and Jackson(2014) [19], we cannot identify which banks will be in default by the discontinuity of banks' values.

different each other. To compare a kay player in endogenous network and exogenous network, we treat realized network from the potential network and costs as given, then compute the key player in exogenous network. Endogenous one is computed by the definition.

Example 3. Consider the network with $n = 5$, and the potential network so that $\overline{g}_{ij} = 1$ for any pair $ij (i \neq j)$. Assume $\boldsymbol{\alpha} = (1, 1, 1, 1, 1)$ and $\phi = \frac{1}{5}$, which satisfies the Assumption 1.

Consider the link formation costs as follows:

$$\boldsymbol{C} = \begin{bmatrix} 0 & 3.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0 & 0.2 & 0.5 & 5.5 \\ 0.2 & 0.2 & 0 & 4.5 & 4.3 \\ 4.1 & 0.2 & 0.4 & 0 & 6.5 \\ 3.2 & 4.1 & 0.3 & 1.0 & 0 \end{bmatrix}.$$

Then, the realized equilibrium network g^* is:

$$\boldsymbol{G}^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this network, we can compute the equilibrium effort levels by computer.

$$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) \approx (1.99, 2.12, 1.82, 1.78, 1.36).$$

As you see, the agent who exerts highest effort is agent 2. However, the key player in endogenous network g^* is agent 1. On the other hand, when we treat the network g^* as given, the key player in exogenous network is agent 3. See Table 1. Figure 4 shows the key player in endogenous and exogenous network in this example.

Agent 2 is the player with the highest effort in both endogenous and exogenous network because he connects to the agent 1 and 3 who exert second and third highest effort and strategic

agent with highest effort	agent 2
key player in endogenous network	agent 1
key player in exogenous network	agent 3

Table 1: Difference of key players

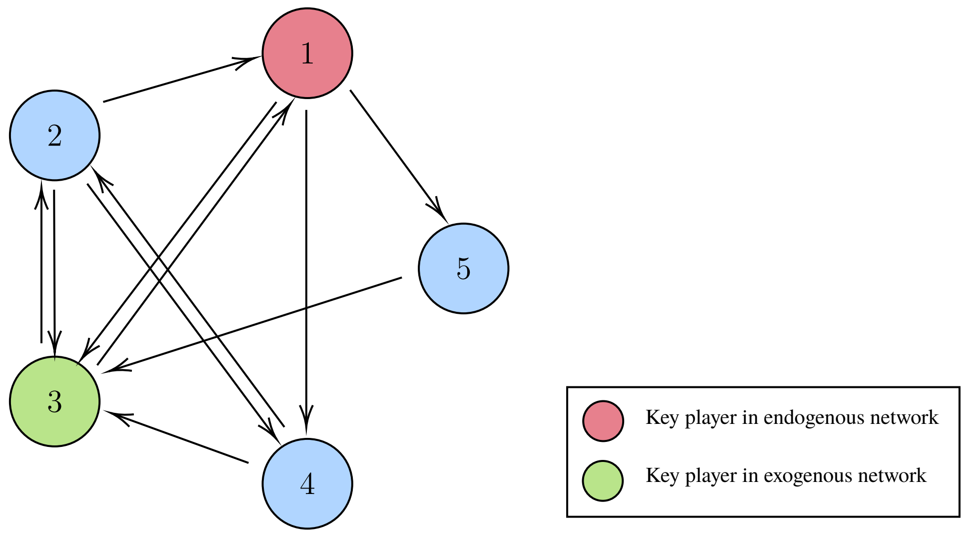


Figure 4: Equilibrium network G^* and key players in Example 3

complementarity let him exert high level effort. Agent 3 is the key player in the exogenous network. All other agents form link to her, so once she is removed from the network, the others cannot get benefit from her amd aggregate negative impact is large. The key player in the endogenous network is agent 1. Once he is removed from the potential network, agents 2 and 3, who have the highest and third highest effort levels, cannot get large benefit from him, so their effort levels decline. In addition, in the potential network without agent1, agent 3, who has the most incoming links, has only one link, so the negative impact of small effort level of agent 3 becomes large.

This example shows that key player in endogenous network and exogenous network may be different, so from the social planner's point of view, it is important to consider the mechanism of network formation. For example, in order to identify a key player and reduce the criminal activities like Liu, Patacchini, Zenou, and Lee(2012) [30], we have to capture not agent 3 but agent 1 because once the agent is removed, the network structure will be changed. Therefore, without considering the network formation, we may have wrong policy implications.

4.2 Key link policy

Next, we argue the key link policy that is introduced in this paper for the first time. First, we define key removing link which is defined as the link that, once removed from the network, generates the highest possible reduction in aggreate effort level.

Definition 5. Link ij is a *key removing link in endogenous network* if, given potential network $\overline{g}$,

$$ij \in \arg\max_{ij \in E(\overline{g})} \{x^*(g^{**}(\boldsymbol{\psi}(\overline{g}, \boldsymbol{C}))) - x^*(g^{**}(\boldsymbol{\psi}(\overline{g}^{-ij}, \boldsymbol{C})))\}$$

where $E(\overline{g})$ is the set of links in $\overline{g}$ and $\overline{g}^{-ij}$ is a network obtained by removing link ij from $\overline{g}$.

Definition 6. Link ij is a *key removing link in exogenous network* if, given network g,

$$ij \in \arg\max_{ij \in E(g)} \{x^*(g) - x^*(g^{-ij})\}$$

where $E(g)$ is the set of links in g and g^{-ij} is a network obtained by removing link ij from g.

In the example of drug trafficking network, key removing link is the traddicking route between cities which reduces the aggregate level of criminal activities once removed. Removing the connection from the criminal network means that a policy maker physically shutdowns the route between them, for example creating a wall between Mexican and U.S. city. By doing so, policy maker can maximally reduce the aggregate criminal activities or drug usage. Example 4 shows key removing links in endogenous and exogenous network might be different.

Example 4. Suppose $n = 3$, $\boldsymbol{\alpha} = (1, 1, 1)$ and $\phi = 1/3$. Consider potential network $\overline{g}$ and link formation costs are

$$\overline{\boldsymbol{G}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then, the equilibrium network g^{**} is

$$\boldsymbol{G}^{**} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

By considering g^{**} as exogenous network, key removing link in exogenoous network is link 12 and 31. But, in this example, key removing in endogenous network is link 23. We can reduce the aggregate level of efforts by removing link 23 from the potential network $\overline{g}$. See Figure 5 (red arrow and blue arrow represent key removing link in endogenous and exogenous network respectively).

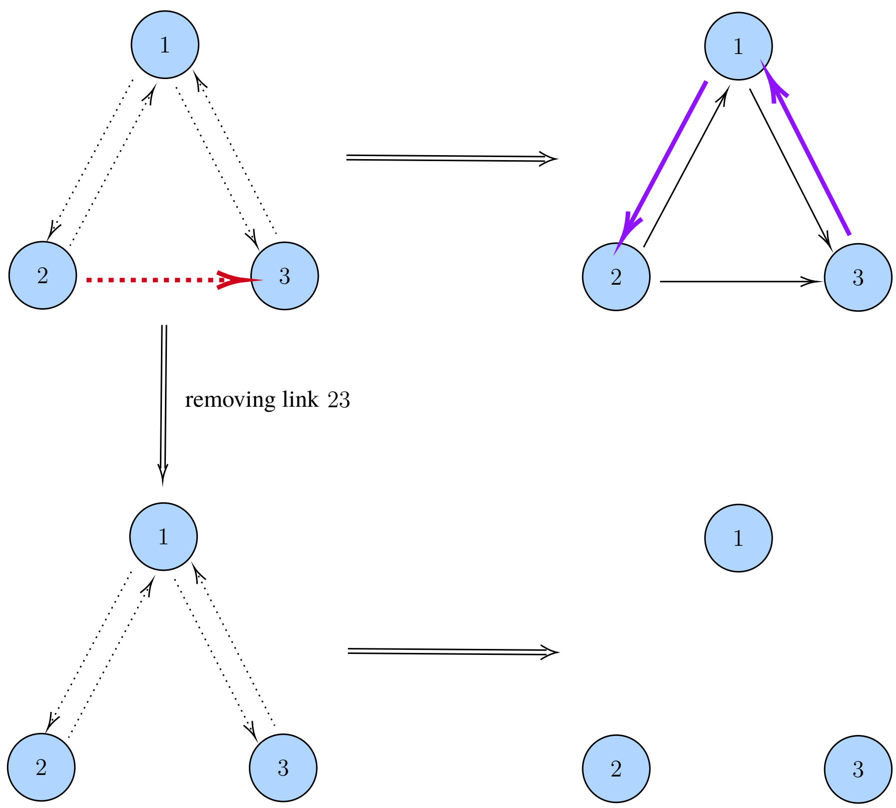


Figure 5: Key removing link in endogenous and exogenous network in Example 4

Next, we introduce key adding link in endogenous and exogenous networks similarly. Key adding link is defined as the link which, once added to the network, generates the highest possible increment in aggreate effort level.

Definition 7. Link ij is a *key adding link in endogenous network* if, given potential network $\overline{g}$,

$$ij \in \arg\max_{ij \notin E(\overline{g})} \{x^*(g^{**}(\boldsymbol{\psi}(\overline{g}^{+ij}, \boldsymbol{C}))) - x^*(g^{**}(\boldsymbol{\psi}(\overline{g}, \boldsymbol{C})))\}$$

where $\overline{g}^{+ij}$ is a network obtained by adding link ij to $\overline{g}$.

Definition 8. Link ij is a *key adding link in exogenous network* if, given network g,

$$ij \in \arg\max_{ij \notin E(g)} \{x^*(g^{+ij}) - x^*(g)\}$$

where g^{+ij} is a network obtained by adding link ij to g.

Note that $c_{ij} = 0$ for ij with $\overline{g}_{ij} = 0$, so the formation cost of an adding link in endogenous network is 0 here. In the context of web advertisement network, key adding link is the advertisement on the page which brings the highest increment in aggregate effort level. For a policy maker, key adding link is the connection which he should recommend the web page owner to make. By encouraging to connect and putting the advertisement, a policy maker can raise the level of investment on the web page. Example 5 shows key adding link in endogenous and exogenous network might be different.

Example 5. Suppose $n = 3$, $\boldsymbol{\alpha} = (1, 1, 1)$ and $\phi = 1/3$. Consider potential network $\overline{g}$ and link formation costs are

$$\overline{\boldsymbol{G}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 0 & 0.1 & 0.1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then, the greatest equilibrium network g^{**} becomes

$$\boldsymbol{G}^{**} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By considering g^{**} as exogenous network, key adding link in exogenoous network is link 21 and 31. But, in this example, key removing in endogenous network is link 23 and 32. We can increase the aggregate level of efforts by adding link 23 or link 32 to the potential network $\overline{g}$. See Figure 6 (red arrow and blue arrow represent key adding link in endogenous and exogenous network respectively).

From all these example, we can see that the key link policy can be different between endogenous

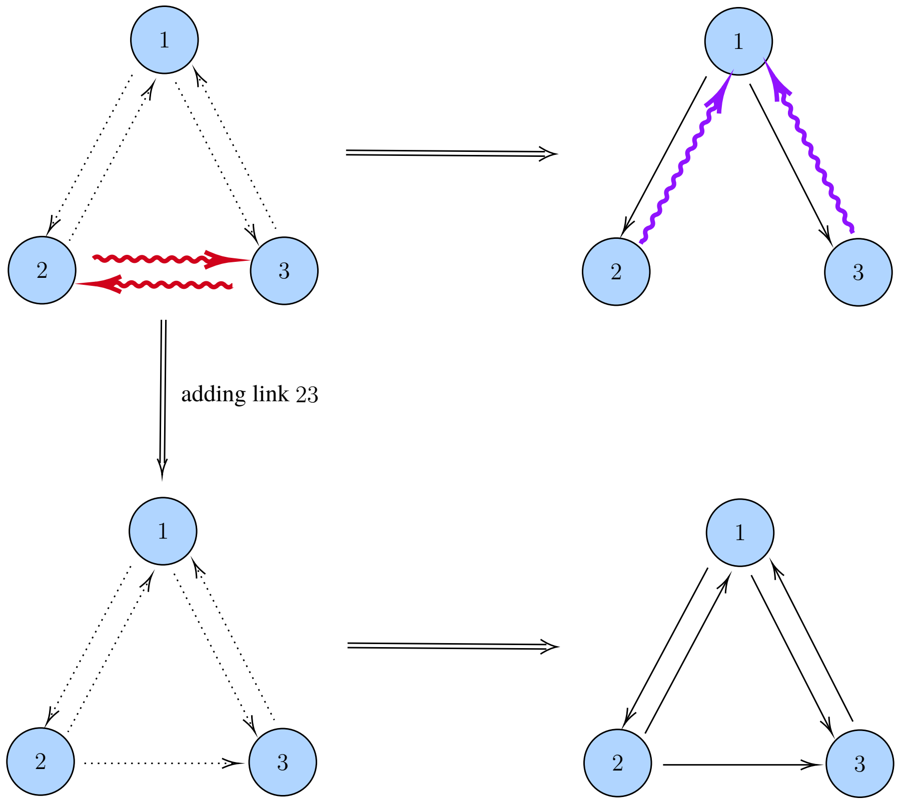


Figure 6: Key adding link in endogenous and exogenous network

and exogenous network. Therefore, we have to incorporate the endogeneity of the network to consider the policy implications. Again, due to the complexity of the relationship between link formation costs and realized network, we cannot propose the proposition who is the key player in endogenous network. The closed form solution to identify the key player and key link is remaining for future works.

4.3 Finding weak connection

In this subsection, we argue that our model can be used to find weak connections between agents. In our model, without any costs, all agents form links to the other agents who is connected in the potential network. We can interpret the links which can be realized with high costs as the strong tie, for example, the connection between core members in the criminal group. The agent want to keep important links even with high costs. Conversely, if the agent cannot retain links with low costs, these links are not important, that is, he does not stick to these links.

To see this idea, we consider the example using florentine family network in 15th century

Florence, well known social network. The florentine family network is the network where each node denote a key Florentine family and each link denote a connection by marriage. Figure (7) shows the structure of this network. In this figure, the links are undirected, that is, agent i has a link to j if and only if agent j has a link to i.

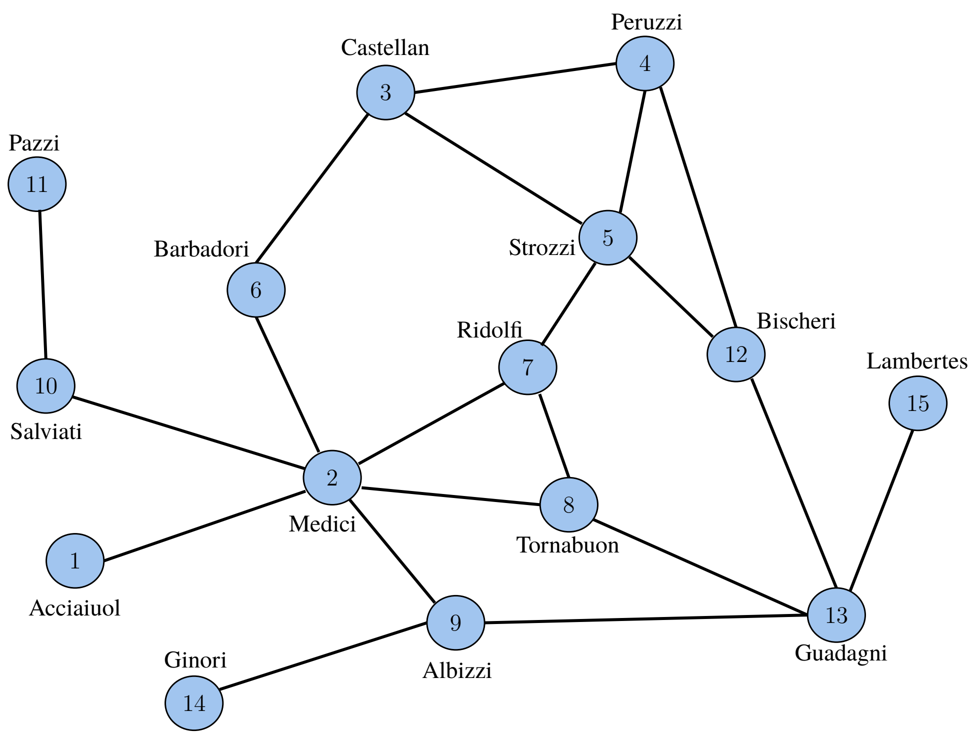


Figure 7: Florentine Family network

Consider the potential network $\overline{g}$ is equal to the network depicted in Figure (7). Link formation costs are all same: $c_{ij} = c$ for all $ij \in E(\overline{g})$. Assume $\alpha_i = 1$ for all i. In light of Proposition 3, some links are removed in the greatest equilibrium network as c increases. Table 2 shows the value of cost c and removed links with its cost structure.

From this table, we see the links between Salviati and Pazzi are removed first. We can guess this connection is the weakest or negligible one. In fact, using the algorithm proposed by Jiang, Y., and Hu, A. (2010) [25], where the importance of the link is computed by the freauency used in the shortest path, the connection between Salviati and Pazzi has the least importance. Intuitively, the connection between Salviati and Pazzi is located in the fringe of the network. As c increases, the links with small importances are removed in order. This example shows our model can be used as a method to find the links with low importances.

cost	removed links
$c = 0.081$	none
$c = 0.082$	Salviati-Pazzi
$c = 0.088$	Salviati-Pazzi, Albizzi-Ginori
$c = 0.092$	Salviati-Pazzi, Albizzi-Ginori, Guadagni-Lambertes
$c = 0.093$	Salviati-Pazzi, Albizzi-Ginori, Guadagni-Lambertes, Guadagni-Tornabuon, Guadagni-Albizzi, Guadagni-Bischeri
$c = 0.094$	all links

Table 2: Removed links

5 Conclusion

This paper argues the game with externality in the endogenously formed network. Contrary to the previous literature, agents endogenously form the network. In the model, agents choose their neighbors in the potential network at first, and then the network is realized. When agents form links to others, they incur the link-specific costs and their decisions depend on these costs. Link formation costs play an important role in determining the realized network structure. Next, agents decide their effort levels in this formed network, and their decisions generate the externalities, peer effects, through the network. We focus on the subgame perfect equilibrium where agents take pure strategies in each stage. We show that there always exists the equilibrium for any potential network and link formation costs. Behind this result, there are supermodurality in the 1st stage game given 2nd stage equilibrium. However, the realized network is high dimensional discontiunous function in the costs, so it is difficult to identify the structure of the network from the link formation costs. We can find the key player in the endogenous network is not necessarily same with the one in the exogenous network, which implies that we might have a wrong policy implication from the results in previous literature. Therefore, it is significant to consider the mechanism of network formation. Finally, we provide an example that our model can be used as a method to find the links with low importances.

Some weaknesses in our model are remaining for the future work. First, link formation is accomplished by the unilateral decisions. This is mainly due to the simplification of the analysis. In reality, the relationships are often made by the bilateral agreement, such as friendship network and firm-to-firm relationship. In such a model, the network should be treated as undirected network theoretically. To incorporate the bilateral decisions, in the model similar to ours, we have to consider the complex belief system: when I offer the relationship, will she accept my offer? The beliefs depend on other agents' decisions and beliefs, so it is not easy to bilateral link formation. Second, in our models, links don't have any weights, unweighted network. In many situations, relationships have unequal importances, for example, the closeness of friends are not equal in the social network. In order to make the links weighted, agents have to deecide the link weights when they form relationships. Our model can be extened to the unweighted network when agents choose the intensity of each forming links like Kim, J., Patacchini, E., Picard, P.M. and Y. Zenou (2017) [27]. However, in such a model, agents form links to all other agents (weights are unequal), that is, realized network becomes complete network. This is not appropriate for representing the real environment. Finally, our model should be extended to various kinds of utility forms. Currently, we can only explore the model with positive peer effects in the network, but these are many networks such that the externalities does not exist or are not appropriate to apply. For example, our model is not suitable for analyzing the production network because we cannot incorporate the price of goods or production functions and the peer effect is not significant. The analizing mechanism of network formation is important for policy implication. It enables us to analyze the effect of intervention to the network on the allocation of goods or the social welfare. Analyzing various kinds of network formation opens the way to "network design" field.

A Appendix

Proof of Proposition 1. From the definition of g and $\overline{g}$, we have $\overline{\boldsymbol{G}} \geq \boldsymbol{G}$. By the theorem I* of Debreu and Herstein (1952) [16], we can have

$$\rho(\overline{\boldsymbol{G}}) \geq \rho(\boldsymbol{G}).$$

Since $\phi > 0$, by Assumption 1, $\phi\rho(\boldsymbol{G}) < 1$. By Theorem 1 of Ballester, Calvó-Armengol and Zenou (2005) [5], we can have a unique Nash equilibrium $\boldsymbol{x}^*$, which is interior and given by

$$\boldsymbol{x}^* = (\boldsymbol{I} - \phi\boldsymbol{G})^{-1}\boldsymbol{\alpha}.$$

$\square$

Proof of Lemma 1. We can write

$$\boldsymbol{x}(g) = (\boldsymbol{I} - \phi \boldsymbol{G})^{-1}\boldsymbol{\alpha} = \sum_{p=0}^{\infty} \phi^p \boldsymbol{G}^p \boldsymbol{\alpha}$$

$$\boldsymbol{x}(\hat{g}) = (\boldsymbol{I} - \phi \hat{\boldsymbol{G}})^{-1}\boldsymbol{\alpha} = \sum_{p=0}^{\infty} \phi^p \hat{\boldsymbol{G}}^p \boldsymbol{\alpha}$$

Therefore,

$$\begin{aligned}
\boldsymbol{x}^*(\hat{g}) - \boldsymbol{x}^*(g) &= \sum_{p=0}^{\infty} \phi^p \hat{\boldsymbol{G}}^p \boldsymbol{\alpha} - \sum_{p=0}^{\infty} \phi^p \boldsymbol{G}^p \boldsymbol{\alpha} \\
&= \sum_{p=0}^{\infty} \phi^p (\hat{\boldsymbol{G}}^p - \boldsymbol{G}^p)\boldsymbol{\alpha}
\end{aligned}$$

Since $\hat{\boldsymbol{G}}$ and $\boldsymbol{G}$ are both nonnegative matrices and $\hat{\boldsymbol{G}} \geq \boldsymbol{G}$, we have $\hat{\boldsymbol{G}}^p \geq \boldsymbol{G}^p$ for any p by the property of nonnegative matrices. Therefore, we have

$$\boldsymbol{x}^*(\hat{g}) - \boldsymbol{x}^*(g) \geq \boldsymbol{0}$$

By construction, we can write $\hat{\boldsymbol{G}} = \boldsymbol{G} + \boldsymbol{D}$ for nonnegative matrix $\boldsymbol{D}$ such that any entry of $\boldsymbol{D}$ is either 0 or 1. We can rewrite (3) as

$$\begin{aligned}
\boldsymbol{x}^*(\hat{g}) - \boldsymbol{x}^*(g) &= \sum_{p=0}^{\infty} \phi^p (\hat{\boldsymbol{G}}^p - \boldsymbol{G}^p)\boldsymbol{\alpha} \\
&= \sum_{p=0}^{\infty} \phi^p ((\boldsymbol{G} + \boldsymbol{D})^p - \boldsymbol{G}^p)\boldsymbol{\alpha} \\
&= \phi \boldsymbol{D}\boldsymbol{\alpha} + \phi^2(\boldsymbol{G}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{G})\boldsymbol{\alpha} + \cdots .
\end{aligned}$$

Since, for some j, $\hat{g}_{ij} - g_{ij} = 1$, $\alpha_j > 0$, and $\phi > 0$, i-th element of $\phi \boldsymbol{D}\boldsymbol{\alpha}$ is strictly greater than 0. Because any combinations of nonnegative matrices $\boldsymbol{G}$ and $\boldsymbol{D}$ are nonnegative, i-th element of

$\boldsymbol{x}^*(\hat{g}) - \boldsymbol{x}^*(g)$ is strictly greater than 0. Thus, we have

$$x_i^*(\hat{g}) > x_i^*(g).$$

$\square$

Proof of Lemma 2. Let $\boldsymbol{D}$ be the nonnegative matrix such that $\boldsymbol{D} = (\boldsymbol{G} \vee \boldsymbol{H}) - \boldsymbol{G} = \boldsymbol{H} - (\boldsymbol{G} \wedge \boldsymbol{H})$. We can have

$$\begin{aligned} \boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g) &= \sum_{p=0}^{\infty} \phi^p (\boldsymbol{G} \vee \boldsymbol{H})^p \boldsymbol{\alpha} - \sum_{p=0}^{\infty} \phi^p \boldsymbol{G}^p \boldsymbol{\alpha} \\ &= \sum_{p=0}^{\infty} \phi^p ((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) \boldsymbol{\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} \boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h) &= \sum_{p=0}^{\infty} \phi^p \boldsymbol{H}^p \boldsymbol{\alpha} - \sum_{p=0}^{\infty} \phi^p (\boldsymbol{G} \wedge \boldsymbol{H})^p \boldsymbol{\alpha} \\ &= \sum_{p=0}^{\infty} \phi^p (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p) \boldsymbol{\alpha}. \end{aligned}$$

Then,

$$\{\boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g)\} - \{\boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h)\} = \sum_{p=0}^{\infty} \phi^p \{((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} \boldsymbol{\alpha} \quad (3)$$

Denote $\hat{\boldsymbol{D}} = \boldsymbol{G} - (\boldsymbol{G} \wedge \boldsymbol{H})$. Note that $\hat{\boldsymbol{D}}$ is nonnegative matrix. Assume, for $p (\geq 0)$,

$$(\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p) \geq \boldsymbol{0}$$

For $p+1$,

$$
\begin{aligned}
&((\boldsymbol{G} \vee \boldsymbol{H})^{p+1} - \boldsymbol{G}^{p+1}) - (\boldsymbol{H}^{p+1} - (\boldsymbol{G} \wedge \boldsymbol{H})^{p+1}) \\
&= \{(\boldsymbol{G} \vee \boldsymbol{H})(\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}\boldsymbol{G}^p\} - \{\boldsymbol{H}\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{G} \wedge \boldsymbol{H})^p\} \\
&= \{(\boldsymbol{G} + \boldsymbol{D})(\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}\boldsymbol{G}^p\} - \{((\boldsymbol{G} \wedge \boldsymbol{H}) + \boldsymbol{D})\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{G} \wedge \boldsymbol{H})^p\} \\
&= \{\boldsymbol{G}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) + \boldsymbol{D}(\boldsymbol{G} \vee \boldsymbol{H})^p\} - \{(\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p) + \boldsymbol{D}\boldsymbol{H}^p\} \\
&= \{\boldsymbol{G}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} + \boldsymbol{D}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{H}^p) \\
&= \{((\boldsymbol{G} \wedge \boldsymbol{H}) + \hat{\boldsymbol{D}})((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} + \boldsymbol{D}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{H}^p) \\
&= (\boldsymbol{G} \wedge \boldsymbol{H})\{((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\hat{\boldsymbol{H}}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} + \hat{\boldsymbol{D}}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) + \boldsymbol{D}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{H}^p) \geq \boldsymbol{0}.
\end{aligned}
$$

Last inequality holds by the assumption and nonegativities of $\boldsymbol{G}, \boldsymbol{H}, \boldsymbol{G} \vee \boldsymbol{H}, \boldsymbol{G} \wedge \boldsymbol{H}, \boldsymbol{D}, \hat{\boldsymbol{D}}$. Obviously, for $p = 0$, $(\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p) \geq \boldsymbol{0}$. Thus, by induction, right hand side of equation (3) is greater than zeros, and we have

$$\boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g) \geq \boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h). \tag{4}$$

In addition, since $\hat{g} \supseteq \hat{h}(\hat{\boldsymbol{G}} \geq \hat{\boldsymbol{H}})$ and $g \supseteq h(\boldsymbol{G} \geq \boldsymbol{H})$, by Lemma 1, we have

$$\boldsymbol{x}^*(g \vee h) + \boldsymbol{x}^*(g) \geq \boldsymbol{x}^*(h) + \boldsymbol{x}^*(g \wedge h). \tag{5}$$

Therefore, from (2), (4) and (5),

$$v_i^*(\boldsymbol{x}^*(g \vee h), g \vee h, \phi) - v_i^*(\boldsymbol{x}^*(g), g, \phi) \geq v_i^*(\boldsymbol{x}^*(h), h, \phi) - v_i^*(\boldsymbol{x}^*(g \wedge h), g \wedge h, \phi).$$

□

Proof of Theorem 1. First, we note the definition of a supermodular game.

Definition 9. A normal form game $\Gamma = \langle N, \boldsymbol{\Psi}, (u_i)_{i \in N} \rangle$ is a *supermodular game*[20] if

1. $\boldsymbol{\Psi}$ is a sublattice of $\prod_{i=1}^n \mathbb{R}^n$,

[20]This definition depends on Topkis (1998) [40].

2. $u_i(\psi_i, \psi_{-i})$ is supermodular in ψ_i on Ψ_i for each ψ_{-i} on Ψ_{-i} for each i, and

3. $u_i(\psi_i, \psi_{-i})$ has increasing differences in (ψ_i, ψ_{-i}) on $\Psi_i \times \Psi_{-i}$.

Take any $\psi, \psi' \in \boldsymbol{\Psi}$. Since ψ_{ij} is either 0 or 1, we see $\psi_{ij} \vee \psi'_{ij} \in \{0, 1\}$ and $\psi_{ij} \wedge \psi'_{ij} \in \{0, 1\}$. Also, $\psi_{ij} = 0$ for ij with $\overline{g}_{ij} = 0$, and we see $\psi_{ij} \vee \psi'_{ij} = 0$ and $\psi_{ij} \wedge \psi'_{ij} = 0$. Therefore, we have $\psi \vee \psi' \in \boldsymbol{\Psi}$ and $\psi \wedge \psi' \in \boldsymbol{\Psi}$, and $\boldsymbol{\Psi}$ is a sublattice of $\prod_{i=1}^{n} \mathbb{R}^n$.

Next, we show that u_i is supermodular in ψ_i for fixed ψ_{-i}. Fix any $\psi_{-i} \in \Psi_{-i}$ and take $\psi_i, \psi'_i \in \Psi_i$ arbitrarily. Denote $g = g(\psi_i, \psi_{-i})$ and $\boldsymbol{G}$ as its adjacency matrix. Denote $h = g(\psi'_i, \psi_{-i})$ and $\boldsymbol{H}$ as its adjacency matrix. Then, we can have $g \vee h = g(\psi_i \vee \psi'_i, \psi_{-i})$ and $g \wedge h = g(\psi_i \wedge \psi'_i, \psi_{-i})$. By Lemma 2, we have

$$
\begin{aligned}
&v_i(\boldsymbol{x}^*(g(\psi_i \vee \psi'_i, \psi_{-i})), g(\psi_i \vee \psi'_i, \psi_{-i}), \phi) + v_i(\boldsymbol{x}^*(g(\psi_i \wedge \psi'_i, \psi_{-i})), g(\psi_i \wedge \psi'_i, \psi_{-i}), \phi) \\
&\geq v_i(\boldsymbol{x}^*(g(\psi_i, \psi_{-i})), g(\psi_i, \psi_{-i}), \phi) + v_i(\boldsymbol{x}^*(g(\psi'_i, \psi_{-i})), g(\psi'_i, \psi_{-i}), \phi).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&u_i(\boldsymbol{x}^*(g(\psi_i \vee \psi'_i, \psi_{-i})), \psi_i \vee \psi'_i, \psi_{-i}, \boldsymbol{C}, \phi) + u_i(\boldsymbol{x}^*(g(\psi_i \wedge \psi'_i, \psi_{-i})), \psi_i \wedge \psi'_i, \psi_{-i}, \boldsymbol{C}, \phi) \\
&\geq u_i(\boldsymbol{x}^*(g(\psi_i, \psi_{-i})), \psi_i, \psi_{-i}, \boldsymbol{C}, \phi) + u_i(\boldsymbol{x}^*(g(\psi'_i, \psi_{-i})), \psi'_i, \psi_{-i}, \boldsymbol{C}, \phi).
\end{aligned}
$$

Hence, u_i is supermodular in ψ.

Take any $\psi_i, \psi'_i \in \Psi_i$ with $\psi_i \geq \psi'_i$ and $\psi_{-i}, \psi'_{-i} \in \Psi_{-i}$ with $\psi_{-i} \geq \psi'_{-i}$. Note that $g(\psi_i, \psi_{-i}) = g(\psi_i, \psi'_{-i}) \vee g(\psi'_i, \psi_{-i}$ and $g(\psi'_i, \psi'_{-i}) = g(\psi_i, \psi'_{-i}) \wedge g(\psi'_i, \psi_{-i}$ Therefore, by Lemma 2,

$$
\begin{aligned}
&v_i(\boldsymbol{x}^*(g(\psi_i, \psi_{-i})), g(\psi_i, \psi_{-i}), \phi) + v_i(\boldsymbol{x}^*(g(\psi'_i, \psi'_{-i})), g(\psi'_i, \psi'_{-i}), \phi) \\
&\geq v_i(\boldsymbol{x}^*(g(\psi'_i, \psi_{-i})), g(\psi'_i, \psi_{-i}), \phi) + v_i(\boldsymbol{x}^*(g(\psi_i, \psi'_{-i})), g(\psi_i, \psi'_{-i}), \phi).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&u_i(\boldsymbol{x}^*(g(\psi_i, \psi_{-i})), \psi_i, \psi_{-i}, \boldsymbol{C}, \phi) + u_i(\boldsymbol{x}^*(g(\psi'_i, \psi'_{-i})), \psi'_i, \psi'_{-i}, \boldsymbol{C}, \phi) \\
&\geq u_i(\boldsymbol{x}^*(g(\psi'_i, \psi_{-i})), \psi'_i, \psi_{-i}, \boldsymbol{C}, \phi) + u_i(\boldsymbol{x}^*(g(\psi_i, \psi'_{-i})), \psi_i, \psi'_{-i}, \boldsymbol{C}, \phi).
\end{aligned}
$$

Thus,

$$\begin{aligned}
&u_i(\boldsymbol{x}^*(g(\psi_i,\psi_{-i})),\psi_i,\psi_{-i},\boldsymbol{C},\phi) - u_i(\boldsymbol{x}^*(g(\psi_i',\psi_{-i})),\psi_i',\psi_{-i},\boldsymbol{C},\phi)\\
&\geq u_i(\boldsymbol{x}^*(g(\psi_i,\psi_{-i}')),\psi_i,\psi_{-i}',\boldsymbol{C},\phi) - u_i(\boldsymbol{x}^*(g(\psi_i',\psi_{-i}')),\psi_i',\psi_{-i}',\boldsymbol{C},\phi).
\end{aligned}$$

$u_i(\psi_i,\psi_{-i})$ has increasing differences in (ψ_i,ψ_{-i}). From all these above, we have Γ is a supermodular game. □

Proof of Proposition 2. Suppose link ij is removed at step $\tau(\geq 0)$ in Algorithm 1. Assume link ij is reformed at step $\tau + t(t \geq 1)$. At step τ,

$$\begin{aligned}
&v_i(\boldsymbol{x}^*(g^{(\tau-1)}),g^{(\tau-1)},\phi) - c_{ij} - \sum_{k\neq j} g_{ik}^{(\tau-1)}c_{ik}\\
&< v_i(\boldsymbol{x}^*(g^{(\tau-1)}\setminus\{ij\}),g^{(\tau-1)}\setminus\{ij\},\phi) - \sum_{k\neq j}(g^{(\tau-1)}\setminus\{ij\})_{ik}c_{ik}.
\end{aligned} \tag{6}$$

At step $\tau + t$,

$$\begin{aligned}
&v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}),g^{(\tau+t-1)},\phi) - \sum_{k\neq j} g_{ik}^{(\tau+t-1)}c_{ik}\\
&< v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}\cup\{ij\}),g^{(\tau+t-1)}\cup\{ij\},\phi) - c_{ij} - \sum_{k\neq j}(g^{(\tau+t-1)}\cup\{ij\})_{ik}c_{ik}.
\end{aligned} \tag{7}$$

From (6),

$$c_{ij} > v_i(\boldsymbol{x}^*(g^{(\tau-1)}),g^{(\tau-1)},\phi) - v_i(\boldsymbol{x}^*(g^{(\tau-1)}\setminus\{ij\}),g^{(\tau-1)}\setminus\{ij\},\phi).$$

From (7),

$$c_{ij} < v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}\cup\{ij\}),g^{(\tau+t-1)}\cup\{ij\},\phi) - v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}),g^{(\tau+t-1)},\phi).$$

Therefore,

$$\begin{aligned}
&v_i(\boldsymbol{x}^*(g^{(\tau-1)}),g^{(\tau-1)},\phi) - v_i(\boldsymbol{x}^*(g^{(\tau-1)}\setminus\{ij\}),g^{(\tau-1)}\setminus\{ij\},\phi)\\
&< v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}\cup\{ij\}),g^{(\tau+t-1)}\cup\{ij\},\phi) - v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}),g^{(\tau+t-1)},\phi).
\end{aligned} \tag{8}$$

Note that, from Lemma 2,

$$
\begin{aligned}
& v_i(\boldsymbol{x}^*(g^{(\tau-1)}), g^{(\tau-1)}, \phi) - v_i(\boldsymbol{x}^*(g^{(\tau-1)} \setminus \{ij\}), g^{(\tau-1)} \setminus \{ij\}, \phi) \\
& < v_i(\boldsymbol{x}^*(g^{(\tau+t-1)} \cup \{ij\}), g^{(\tau+t-1)} \cup \{ij\}, \phi) - v_i(\boldsymbol{x}^*(g^{(\tau+t-1)}), g^{(\tau+t-1)}, \phi).
\end{aligned}
$$

This contradicts to (8), and link ij will never be reformed once it is removed. Since the number of links is finite, the algorithm terminates in finite steps. □

Proof of Proposition 3. Consider the normal form game $\Gamma = \langle N, \boldsymbol{\Psi}, (u_i)_{i \in N} \rangle$, which is first stage game given Nash equilibrium in second stage. To compare the utility with different $\boldsymbol{C}$'s and avoid redundant representations, we write utility function as $u_i(\psi_i, \psi_{-i}, \boldsymbol{C})$ for all $i \in N$. We first show that for all $i \in N$, $u_i(\psi_i, \psi_{-i}, \boldsymbol{C})$ has increasing differences in $(\psi_i, \boldsymbol{C})$. Take any $\psi_i, \hat{\psi}_i \in \Psi_i$ with $\hat{\psi}_i \geq \psi_i$, and any $\hat{\boldsymbol{C}}$ and $\boldsymbol{C}$ with $\hat{\boldsymbol{C}} \leq \boldsymbol{C}$. Fix any $\psi_{-i} \in \Psi_{-i}$ Consider the adjacency matrix of network $g = g(\psi_i, \psi_{-i})$ and $\hat{g} = g(\hat{\psi}_i, \psi_{-i})$ as $\boldsymbol{G}$ and $\hat{\boldsymbol{G}}$ respectively. We write $v_i(\hat{g}, \phi) = v_i(\boldsymbol{x}^*(\hat{g}), \hat{g}, \phi)$ and $v_i(g, \phi) = v_i(\boldsymbol{x}^*(g), g, \phi)$. Then,

$$
\begin{aligned}
& \{u_i(\hat{\psi}_i, \psi_{-i}, \hat{\boldsymbol{C}}) - u_i(\psi_i, \psi_{-i}, \hat{\boldsymbol{C}})\} - \{u_i(\hat{\psi}_i, \psi_{-i}, \boldsymbol{C}) - u_i(\psi_i, \psi_{-i}, \boldsymbol{C})\} \\
& = \left[\{v_i(\hat{g}, \phi) - \sum_{j=1}^{n} \hat{g}_{ij} \hat{c}_{ij}\} - \{v_i(g, \phi) - \sum_{j=1}^{n} g_{ij} \hat{c}_{ij}\} \right] - \left[\{v_i(\hat{g}, \phi) - \sum_{j=1}^{n} \hat{g}_{ij} c_{ij}\} - \{v_i(g, \phi) - \sum_{j=1}^{n} g_{ij} c_{ij}\} \right] \\
& = \sum_{j=1}^{n} \hat{g}_{ij}(c_{ij} - \hat{c}_{ij}) - \sum_{j=1}^{n} g_{ij}(c_{ij} - \hat{c}_{ij}) \\
& = \sum_{j=1}^{n} (\hat{g}_{ij} - g_{ij})(c_{ij} - \hat{c}_{ij}) \geq 0.
\end{aligned}
$$

Last inequalty comes from $\hat{\boldsymbol{C}} \leq \boldsymbol{C}$ and $\hat{\boldsymbol{G}} \geq \boldsymbol{G}$. Thus, we see that for all $i \in N$, $u_i(\psi_i, \psi_{-i}, \boldsymbol{C})$ has increasing differences in $(\psi_i, \boldsymbol{C})$.

By Theorem 6 in Milgrom and Roberts (1990) [34], the greatest equilibrium network g^{**} is increasing in $\boldsymbol{\alpha}$. Therefore, we have

$$
g^{**}(\boldsymbol{\psi}^*(\overline{g}, \hat{\boldsymbol{C}}, \phi, \boldsymbol{\alpha})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha})).
$$

□

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