Externalities, Equilibrium, and Supermodularity in Games with Endogenous Network Formation

master thesis

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Introduction

Network with Peer Effect

- Network structure and local interactions play an important role in individual and aggregate nehavior
- Externalities of the individual behavior in the network is a key factor for the aggeregate bahavior
- Especially, we can see the importance of positive externalities
  - R&D network, criminal network, educational network
- Many literatures argue the importance of "peer effects" both theoretically and empirically

Endogeneity of the network

- However, in many works, the network is exogenous
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This paper

- This paper argue the endogenous network formation with peer effect
- We consider the model where
  - agents first choose the agents who they connect
  - agents choose the level of effort given the network structure
- We provide
  - the existence of subgame perfect equilibrium, where all agents take pure strategies at each stage
  - the argument about the uniqueness of the equilibrium
  - the discussion for policy implication (key player policy)

Related Literature

- Network with peer effect
    - Ballester et.al.(2006), Calvó-Armengol et.al.(2009), Liu et.al.(2012)
- Endogenous network
    - Acemoglu and Azar(2019), Oberfield(2018), Farboodi(2014)
- Closely related paper
    - Kim et.al.(2017), Hiller, T.(2017)

Model

Model : Setup

- the set of agents : $N = \{1, \cdots, n\}$ with $n \geq 2$ and $n < \infty$
- Agents are initially connected in *potential network* g^p
- g^p is represented by adjacency matrix $\boldsymbol{G}^p = \left(g^p_{ij}\right)_{ij}$ where

$$g^p_{ij} = \begin{cases} 1 \text{ (if } i \text{ has a link to } j \text{ in } g^p) \\ 0 \text{ (otherwise)} \end{cases}$$

- Self-loop is not allowed : $g^p_{ii} = 0$ for all $i \in N$
- Agent i's neighbors in g^p : $N_i(g^p) = \{j \in N | g^p_{ij} = 1\}$

Model : 1st stage

- First, agents simultaneously choose their neighbors from the agents whom they connect in potential network
- This strategy is represented by $\psi_i = (\psi_{i1}, \cdots, \psi_{in})$ such that $\psi_{ij} \in \{0, 1\}$ for all $j \in N$, $\psi_{ii} = 0$ for all $i \in N$, and $\psi_{ij} = 0$ for all $j \in N_i(g^p)$
- Note that $\psi_i \in \Psi_i = \{0, 1\}^n$
- When agent form links, he incurs the link-specific costs $c_{ij} \geq 0$
- "Choosing neighbors" strategy is dependent on g^p and $\boldsymbol{C} = c_{ij\,ij}$, so sometimes we denote $\psi_i(g^p, \boldsymbol{C})$

Model : Realized network

- At the end of 1st stage, we can see *realized network* denoted as g
- g is represented by tha adjacency matrix $\boldsymbol{G}$

$$g_{ij} = \begin{cases} 1 \text{ (if } \psi_{ij}(g^p, \boldsymbol{C}) = 1) \\ 0 \text{ (otherwise)} \end{cases}$$

- g depends on $\boldsymbol{\psi}(g^p, \boldsymbol{C}) = \prod_{i=1}^{n} \psi_i(g^p, \boldsymbol{C})$, so we can denote $g(\boldsymbol{\psi}(g^p, \boldsymbol{C}))$
- We denote $g(\boldsymbol{\psi})$

Model : 1st stage

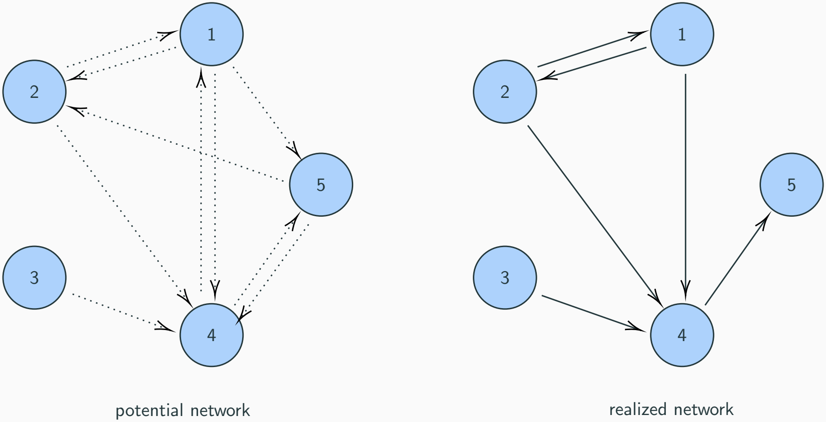

Figure 1: Difference between potential network and realized network

Model : 2nd stage

- Given realized network, each agent $i = 1, \cdots, n$ simultaneously excerts an effort $x_i \geq 0$
- Denote $\boldsymbol{x} = (x_1, \cdots, x_n)$
- Payoff function is

$$u_i(\boldsymbol{x}, \psi, \boldsymbol{C}, \phi) = v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) - \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) c_{ij}$$

  where

$$v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) = \alpha_i x_i - \frac{1}{2} x_i^2 + \phi \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) x_i x_j$$

- $\phi > 0$ and cross term represent the peer effect
- Optimal level of effort depends on the realized network

Equilibrium

Equilibrium

- **Definition** : Given g^p and $\boldsymbol{C}$, the network g^* is equilibrium network if $g^* = g(\boldsymbol{\psi}^*)$ where $\boldsymbol{\psi}^*$ is the strategies in the pure-strategy subgame perfect equilibrium.
- We focus on pure strategy equilibrium to consider the non-stochastic network

2nd stage equilibrium

- **Assumption** : $\phi\rho(\boldsymbol{G}^p)$ where $\rho(\cdot)$ is spectral radius
- **Proposition** : Under Assumption, for any realized network g, the subgame has a unique Nash equilibrium $\boldsymbol{x}^*$, which is interior and given by

$$\boldsymbol{x}^* = (\boldsymbol{I} - \phi\boldsymbol{G})^{-1}\boldsymbol{\alpha}$$

- We can calculate as

$$v_i^*(\boldsymbol{x}^*(g(\boldsymbol{\psi})), g(\boldsymbol{\psi}), \phi) = \frac{1}{2}x_i^*(g(\boldsymbol{\psi}))^2$$

Best response dynamics : Step 0

Let $g^{(0)}$ be the initial realized network where $g^{(0)} = g^p$, that is, chooosing strategy is $\psi_i^{(0)} \in \Psi_i$ such that $\psi_{ij}^{(0)} = 1$ for all $j \in N_i(g^p)$. Compute each players' optimal effort and payoffs. Denote the set of agents who do not take best response as $NB^{(0)}$:

$$NB^{(0)} = \{i \in N | \exists \tilde{\psi}_i \subset N_i(g^p)$$
$$\text{s.t. } u_i(\boldsymbol{x}^*(g(\tilde{\psi}_1, \psi_{-i}^{(0)}), \tilde{\psi}_i, \psi_{-i}^{(0)}, \boldsymbol{C}, \phi) > u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(0)})), \boldsymbol{\psi}^{(0)}, \boldsymbol{C}, \phi)$$
$$\text{and } \tilde{\psi}_i \neq \psi_i^{(0)}\}$$

Go into Step 1.

Best response dynamics : Step $k \geq 1$

Check whether $NB^{(k-1)}$ is empty or not.

If $NB^{(k-1)} = \emptyset$, define $g^* = g(\boldsymbol{\psi}^{(k-1)})$ and terminate the algorithm.

Otherwise, choose a agent $i \in NB^{(k-1)}$ randomly. i changes her strategy from $\psi_i^{(k-1)}$ to $\psi_i^{(k)}$ such that
$u_i(\boldsymbol{x}^*(g(\psi_i^{(k)}, \psi_{-i}^{(k-1)})), \psi_i^{(k)}, \psi_{-i}^{(k-1)}, \boldsymbol{C}, \phi) >$
$u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(k-1)})), \boldsymbol{\psi}^{(k-1)}, \boldsymbol{C}, \phi)$, and for any $j(\neq i)$, $\psi_j^{(k)} = \psi_j^{(k-1)}$.
Then, new network $g(\boldsymbol{\psi}^{(k)})$ is realized. Compute each players' payoffs and define $NB^{(k)}$:

$$
\begin{aligned}
NB^{(k)} = \{i \in N | \exists \tilde{\psi}_i \subset N_i(g^p) \\
&\text{s.t. } u_i(\boldsymbol{x}^*(g(\tilde{\psi}_1, \psi_{-i}^{(k)}), \tilde{\psi}_i, \psi_{-i}^{(k)}, \boldsymbol{C}, \phi) > u_i(\boldsymbol{x}^*(g(\boldsymbol{\psi}^{(k)})), \boldsymbol{\psi}^{(k)}, \boldsymbol{C}, \phi) \\
&\text{and } \tilde{\psi}_i \neq \psi_i^{(k)}\}
\end{aligned}
$$

Proceed to Step $k+1$.

Existence of the equilibrium network

- **Theorem** : Best response dynamics algorithm terminates in finite steps, and converged network represents the equilibrium network
- Given 2nd stage Nash equilibrium, 1st stage game becomes *supermodular game*, so pure strategy Nash equilibrium
- By supermodularity, BR dynamics returns the largest equilibrium
- **Intuition** : When agent i forms more links, he excert more effort by the strategic complementarity in 2nd stage. Agent i's increased effort makes agents who have a link to him excert more effort, so all agents' level of effort weakly increases. Increasing level of efforts makes the agents to connect more agents.

Uniqueness of the equilibrium

- Equilibrium network may not be unique
- **Example** : Consider $n = 2$, $\alpha = (1, 1)$, and $\boldsymbol{G}^p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

  Then $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be both equilibrium network for some c_{12} and c_{21}.

Uniqueness of the equilibrium

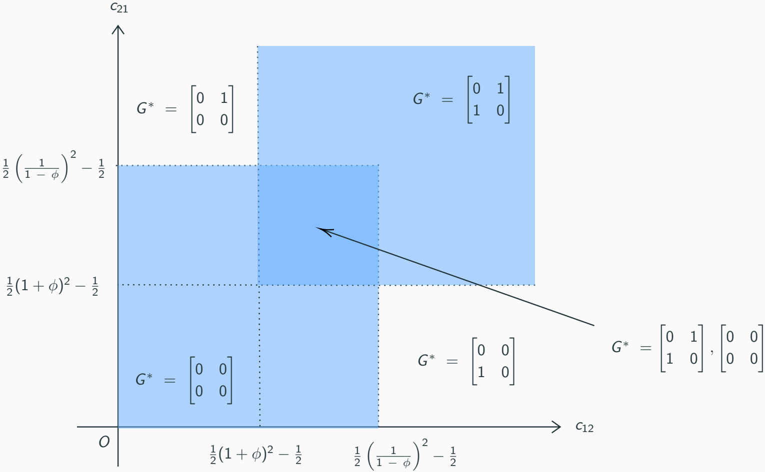

Figure 2: Equilibrium network region

Uniqueness of the equilibrium

- **Theorem** : The equilibrium network derived by BR dynamics is unique

Comparative Statics

- **Proposition** : Given the potential network g^p. Consider the cost $\hat{\boldsymbol{C}}$ and $\boldsymbol{C}$ with $\hat{\boldsymbol{C}} \leq \boldsymbol{C}$. Then,

$$g(\psi^*(g^p, \hat{\boldsymbol{C}})) \supseteq g(\psi^*(g^p, \boldsymbol{C}))$$

- When the link formation costs increase, the equilibrium network becomes denser

Phase transition

- **Example** : Suppose $n = 5$, $\boldsymbol{\alpha} = (1,1,1,1,1)$, and $\phi = 1/5$

- $\boldsymbol{C} = \begin{bmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \boldsymbol{G}^* = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

- $\hat{\boldsymbol{C}} = \begin{bmatrix} 0 & 3+\epsilon & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{G}}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Phase transition

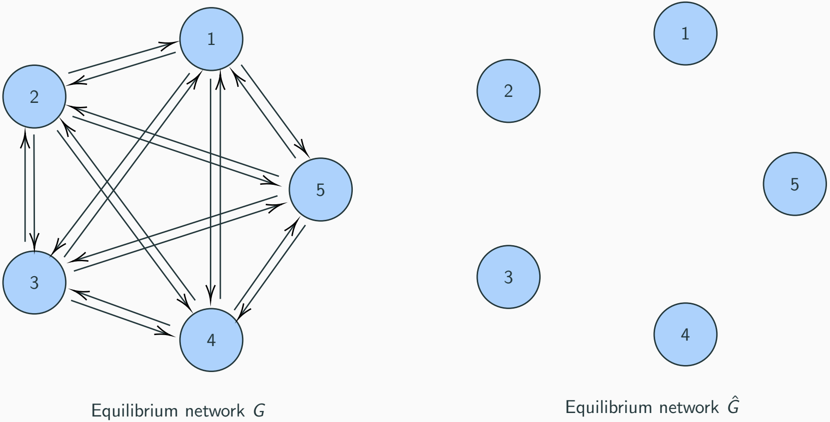

Figure 3: Equilibrium networks

Finding the Key Player

Key player

- Key player is the agent who has the largest impact on the aggregate behavior of the network
- **Definition** : Agent i is the key player in exogenous network g if, given network g,

$$i \in \arg\max_{i \in N} \{x^*(g) - x^*(g^{-i})\}$$

where $x^*(g) = \sum_{i=1}^{n} x_i^*(g)$ and g^{-i} is the network where agent i is removed from the network g

Key player in endogenous network

- **Definition** : Agent i is the key player in endogenous network if, given potential network g^p,

$$i \in \arg\max_{i \in N} \{x^*(g(\psi(g^{p-i}, \boldsymbol{C}))) - x^*(g(\psi(g^{p-i}, \boldsymbol{C}^{-i})))\}$$

  where g^{p-i} is the network where agent i is removed from the network g^p
- However, it is difficult to identify key player due to the complexity of the mapping from cost strcture to realized network

Endogenous and Exogenous

- **Example** : Suppose $n = 5$, $\boldsymbol{\alpha} = (1,1,1,1,1)$ and $\phi = 1/5$
- $$\boldsymbol{C} = \begin{bmatrix} 0 & 3.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0 & 0.2 & 0.5 & 5.5 \\ 0.2 & 0.2 & 0 & 4.5 & 4.3 \\ 4.1 & 0.2 & 0.4 & 0 & 6.5 \\ 3.2 & 4.1 & 0.3 & 1.0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{G}^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
- Then,

agent with highest effort	agent 2
key player in endogenous network	agent 1
key player in exogenous network	agent 3

Endogenous and Exogenous

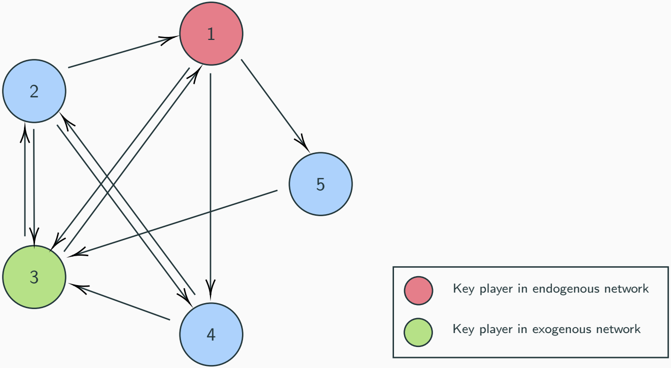

Figure 4: Equilibrium network G^* and key players

Conclusion

Conclusion

- We consider the endogenous network formation with peer effect
- In the model, link formation costs play an important role in determining the network structure and individual and aggregate behaviors
- Due to the supermodularity, we can show the existence of equilibrium network
- We can provide
    - a kind of uniqueness of the equilibrium
    - comparative statics result
    - discussion about key player

Future work

- link formation with bilateral agreement
- weighted network
- other form of utility function