Supermodularity and Equilibrium in Games with Peer Effects and Endogenous Network Formation

master's thesis

Yuya Furusawa

January 14, 2020

U-Tokyo, GSE

Introduction

Network with Peer Effect

- Network structure and local interactions play an important role in individual and aggregate behaviors
  - Externalities of the individual behavior in the network is a key factor for the aggeregate bahavior
  - By considering network structure, we can consider direct effect and indirect effect
- Many literature argue the importance of "peer effects" theoretically and empirically
  - Criminal network, educational network, R&D network, etc

Endogeneity of the network

- However, in many works, the network is exogenous and fixed
- When economic agents are faced with some shocks or policy changes, they respond to them and the network will be changed

This paper

- This paper considers the endogenous network formation with peer effects
- We consider the model where
  - agents first choose the agents who they connect
  - agents choose the level of effort given the network structure
- We provide
  - the existence of subgame perfect equilibrium where all agents take pure strategies at each stage
  - the discussion about policy implication (key player and key link policy)
    - The importance of endogenous network formation

Related Literature

- Peer effect in networks
  - Ballester et.al.(2006), Calvó-Armengol et.al.(2009), Liu et.al.(2012)
- Endogenous network
  - Acemoglu and Azar(2019), Oberfield(2018), Farboodi(2014)
- Closely related
  - Lim et.al.(2017), Hiller(2017)

Model

Model : Setup

- the set of agents : $N = \{1, \ldots, n\}$ with $n \geq 2$ and $n < \infty$
- Agents are initially connected in *potential network* $\overline{g}$
- $\overline{g}$ is represented by adjacency matrix $\overline{\boldsymbol{G}} = (\overline{g}_{ij})_{ij}$ where

$$\overline{g}_{ij} = \begin{cases} 1 \text{ (if } i \text{ has a link to } j \text{ in } \overline{g}) \\ 0 \text{ (otherwise)} \end{cases}$$

- $\overline{\boldsymbol{G}}$ need not be symmetric : directed network
- Self-loop is not allowed : $\overline{g}_{ii} = 0$ for all $i \in N$
- Agent i's neighbors in $\overline{g}$: $N_i(\overline{g}) = \{j \in N \mid \overline{g}_{ij} = 1\}$

Model : 1st stage

- First, agents simultaneously choose their neighbors from the agents whom they connect in the potential network
- This action is represented by $\psi_i = (\psi_{i1}, \cdots, \psi_{in})$ such that $\psi_{ij} \in \{0, 1\}$ for all $j \in N$, and $\psi_{ij} = 0$ for all $j \notin N_i(\overline{g})$
- The action in the 1st stage is dependent on $\overline{g}$, sometimes we denote $\psi_i(\overline{g})$
- When an agent i form link to j, he incurs the link-specific costs $c_{ij} \geq 0$
    - We denote $\boldsymbol{C} = (c_{ij})_{ij}$
    - $\boldsymbol{C}$ is not necessarily symmetric

Model : 1st stage

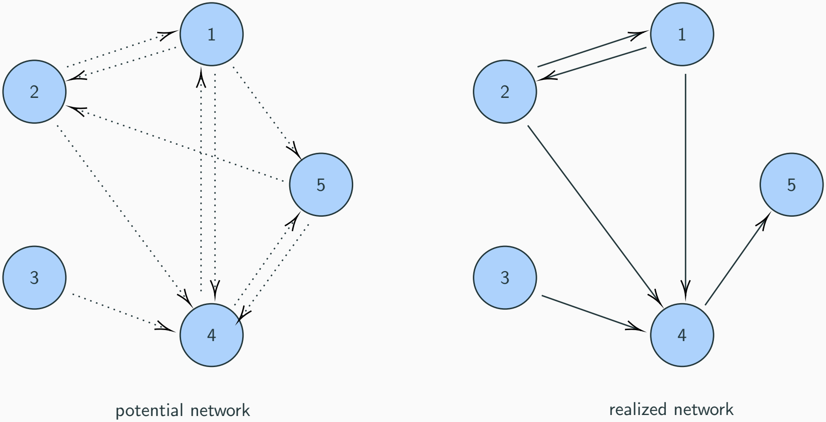

Figure 1: Difference between potential network and realized network

Model : Realized network

- At the end of the 1st stage, we can see *realized network* denoted as g
- g is represented by the adjacency matrix $\boldsymbol{G}$

$$g_{ij}(\psi_{ij}(\overline{g})) = \begin{cases} 1 \text{ (if } \psi_{ij}(\overline{g}) = 1) \\ 0 \text{ (otherwise)} \end{cases}$$

- g depends on $\boldsymbol{\psi}(\overline{g}) = (\psi_1(\overline{g}), \cdots, \psi_n(\overline{g}))$, so we can denote $g(\boldsymbol{\psi}(\overline{g}))$
- To avoid redundant expressions, we denote $g(\boldsymbol{\psi})$

Model : 2nd stage

- Given the realized network, each agent $i \in N$ simultaneously exerts an effort $x_i \geq 0$
- Payoff function is

$$u_i(\boldsymbol{x}, \psi, \boldsymbol{C}, \phi) = v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) - \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) c_{ij}$$

where

$$v_i(\boldsymbol{x}, g(\boldsymbol{\psi}), \phi) = \alpha_i x_i - \frac{1}{2} x_i^2 + \phi \sum_{j=1}^{n} g_{ij}(\boldsymbol{\psi}) x_i x_j$$

- $\phi > 0$ and cross term represent the peer effect
- $\alpha_i > 0$ and denote $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$

Interpretation : Examples

- agent : web sites, municaiparities in Mexico and U.S.
- link : ad on other sites, drug traffick
- link formation cost : ad fee, prob of capture during trafficking
- effort : investments on web contents, drug demand

Equilibrium

Equilibrium

- **Definition** : Given $\overline{g}$ and $\boldsymbol{C}$, a network g^* is an *equilibrium network* if $g^* = g(\boldsymbol{\psi}^*)$ where $\boldsymbol{\psi}^*$ is an action profile in a pure strategy subgame perfect equilibrium.
- We focus on a pure strategy equilibrium to consider non-stochastic networks

2nd stage equilibrium

- **Assumption** : $\phi\rho(\overline{\boldsymbol{G}}) < 1$ where $\rho(\cdot)$ is spectral radius
- **Proposition** : Under Assumption, for any realized network g, the subgame has a unique Nash equilibrium $\boldsymbol{x}^*$, which is interior and given by

$$\boldsymbol{x}^*(g, \phi, \boldsymbol{\alpha}) = (\boldsymbol{I} - \phi\boldsymbol{G})^{-1}\boldsymbol{\alpha}$$

- This is based on Ballester, Calvó-Armengol, and Zenou(2006)

Supermodularity of the reduced game

- By backward induction, given 2nd stage Nash equilibrium, consider the 1st stage game as a normal form game $\Gamma = \langle N, \boldsymbol{\Psi}, (u_i)_{i \in N} \rangle$
- **Theorem** : For any $\overline{g}$ and $\boldsymbol{C}$, Γ is a supermodular game, that is,
    - $\boldsymbol{\Psi}$ is a sublattice of $\prod_{i=1}^{n} \mathbb{R}^n$,
    - $u_i(\psi_i, \psi_{-i})$ is supermodular in ψ_i on Ψ_i for each ψ_{-i} on Ψ_{-i} for each i, and
    - $u_i(\psi_i, \psi_{-i})$ has increasing differences in (ψ_i, ψ_{-i}) on $\Psi_i \times \Psi_{-i}$.

  [Proof]
- **Corollary** : Equilibrium network always exists

Intuition

- **Strategic complementarity in 2nd stage** : Given the realized network, if the neighbors exert more efforts, the agent has an incentive to exert more effort by peer effects.
- **Strategic complementarity in 1st stage** : When agent i forms more links, he exerts more effort by the strategic complementarity in 2nd stage. Agent i's increased effort makes agents who have a link to him exert more efforts, so all agents' level of effort weakly increases. Increasing level of efforts makes the agents to connect more agents.

Uniqueness/Multiplicity of the equilibrium

- Equilibrium network is not always unique
- **Example** : Consider $n = 2$, $\alpha = (1, 1)$, and $\overline{\boldsymbol{G}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- Payoff matrix is:

$\psi_1 \backslash \psi_2$	$(1, 0)$	$(0, 0)$
$(0, 1)$	$\left(\frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{12}, \frac{1}{2}\left(\frac{1}{1-\phi}\right)^2 - c_{21}\right)$	$\left(\frac{1}{2}(1+\phi)^2 - c_{12}, \frac{1}{2}\right)$
$(0, 0)$	$\left(\frac{1}{2}, \frac{1}{2}(1+\phi)^2 - c_{21}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$

Uniqueness/Multiplicity of the equilibrium

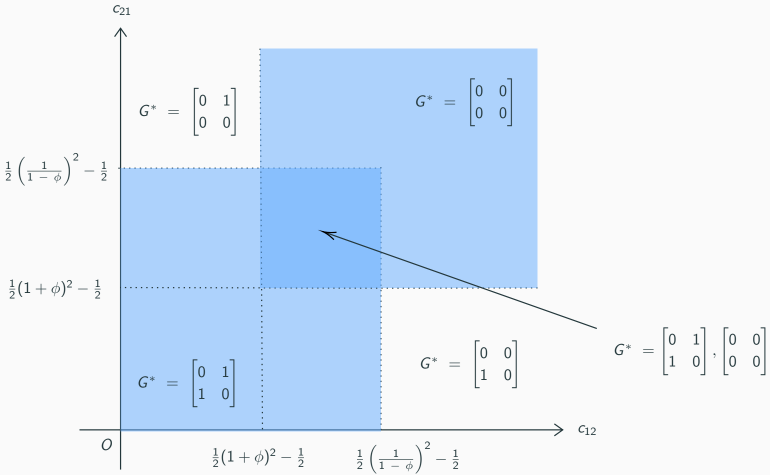

Figure 2: Equilibrium network region

Greatest equilibrium

- By the supermodularity, the greatest and smallest equilibrium network exist
- We focus on the greatest equilibrium network and denote g^{**}
  - The greatest equilibrium can be obtained by sequential *best response dynamics* which starts from the potential network

Comparative Statics

- **Proposition** : Given the potential network $\overline{g}$. Consider the cost $\hat{\boldsymbol{C}}$ and $\boldsymbol{C}$ with $\hat{\boldsymbol{C}} \leq \boldsymbol{C}$. Then,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \hat{\boldsymbol{C}}, \phi, \boldsymbol{\alpha})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha}))$$

- **Corollary** : Given the potential network g^p. For $\hat{\phi} \geq \phi$ which satisfy the Assumption,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \hat{\phi}, \boldsymbol{\alpha})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha}))$$

  For $\hat{\boldsymbol{\alpha}} \geq \boldsymbol{\alpha}$,

$$g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \hat{\boldsymbol{\alpha}})) \supseteq g^{**}(\boldsymbol{\psi}^*(\overline{g}, \boldsymbol{C}, \phi, \boldsymbol{\alpha}))$$

Phase transition

- **Example** : Suppose $n = 5$, $\boldsymbol{\alpha} = (1,1,1,1,1)$, and $\phi = 1/5$
- $\boldsymbol{C} = \begin{bmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \boldsymbol{G}^{**} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$
- $\hat{\boldsymbol{C}} = \begin{bmatrix} 0 & 3+\epsilon & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{G}}^{**} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Phase transition

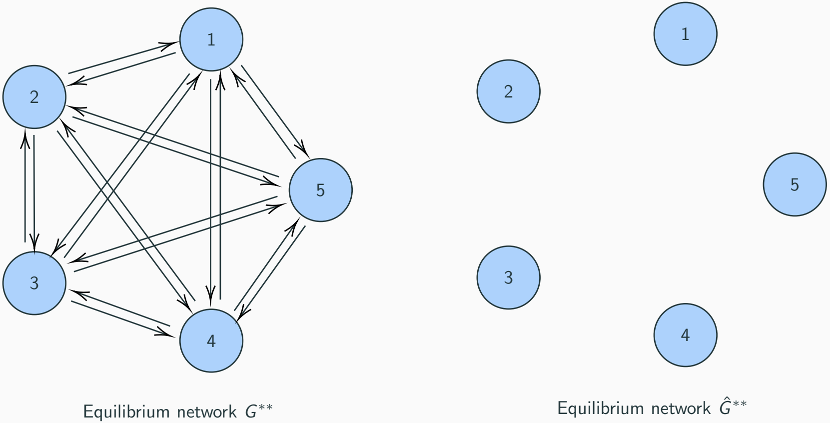

Figure 3: Equilibrium networks

Policy Implication

Key player

- **Definition** : Agent i is a *key player in exogenous network* if, given network g,

$$i \in \arg\max_{i \in N} \{x^*(g) - x^*(g^{-i})\}$$

where $x^*(g) = \sum_{i=1}^{n} x_i^*(g)$ and g^{-i} is the network where agent i is removed from the network g

Key player in endogenous network

- **Definition** : Agent i is a *key player in endogenous network* if, given potential network $\overline{g}$,

$$i \in \arg\max_{i \in N} \{x^*(g^{**}(\psi(\overline{g}, \boldsymbol{C}))) - x^*(g^{**}(\psi(\overline{g}^{-i}, \boldsymbol{C}^{-i})))\}$$

  where $\overline{g}^{-i}$ is the network where agent i is removed from the network $\overline{g}$
- However, it is difficult to identify a key player due to the complexity of the mapping from cost structure to realized network

Difference bet. endogenous and exogenous key player

- **Example** : Suppose $n = 5$, $\boldsymbol{\alpha} = (1,1,1,1,1)$ and $\phi = 1/5$
- $$\boldsymbol{C} = \begin{bmatrix} 0 & 3.6 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0 & 0.2 & 0.5 & 5.5 \\ 0.2 & 0.2 & 0 & 4.5 & 4.3 \\ 4.1 & 0.2 & 0.4 & 0 & 6.5 \\ 3.2 & 4.1 & 0.3 & 1.0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{G}^{**} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
- Then,

agent 1	$x_1^* = 1.99541284$	key player in endogenous network
agent 2	$x_2^* = 2.12155963$	agent with highest effort
agent 3	$x_3^* = 1.82339450$	key player in exogenous network
agent 4	$x_4^* = 1.78899083$	
agent 5	$x_5^* = 1.36467890$	

Diff bet. endogenous and exogenous key player

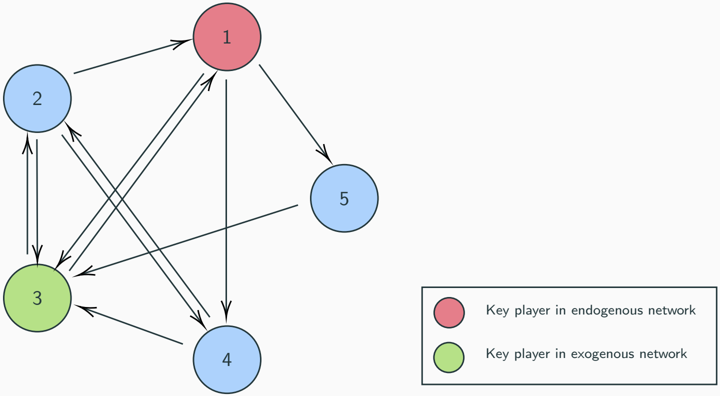

Figure 4: Equilibrium network G^{**} and key players

Conclusion

Conclusion

- We consider the endogenous network formation with peer effects
- Due to the supermodularity, we can show the existence of equilibrium network and provide comparative statics results
- We have a discussion about policy implication : key player and key link policy
    - Without considering the endogenity of networks, we sometimes have wrong policy implications

Appendix

Proof of the theorem

- **Lemma 0** : For a 2nd stage Nash equilibrium $\boldsymbol{x}^*$, we have

$$v_i(\boldsymbol{x}^*(g), g, \phi) = \frac{1}{2} x_i^*(g)^2$$

- **Lemma 1** : Consider the network g and $\hat{g}$ with $g \subseteq \hat{g}$. Then,

$$\boldsymbol{x}^*(\hat{g}) \geq \boldsymbol{x}^*(g)$$

- **Lemma 2** : Consider the network g and h ($\boldsymbol{G}$ and $\boldsymbol{H}$). Consider the network $g \vee h$ and $g \wedge h$ ($\boldsymbol{G} \vee \boldsymbol{H}$ and $\boldsymbol{G} \wedge \boldsymbol{H}$). Then, for all $i \in N$,

$$v_i(\boldsymbol{x}^*(g \vee h), g \vee h, \phi) + v_i(\boldsymbol{x}^*(g \wedge h), g \wedge h, \phi) \geq v_i(\boldsymbol{x}^*(g), g, \phi) + v_i(\boldsymbol{x}^*(h), h, \phi)$$

Proof of Lemma 2

- Let $\boldsymbol{D} = (\boldsymbol{G} \vee \boldsymbol{H}) - \boldsymbol{G} = \boldsymbol{H} - (\boldsymbol{G} \wedge \boldsymbol{H})$ and $\hat{\boldsymbol{D}} = \boldsymbol{G} - (\boldsymbol{G} \wedge \boldsymbol{H})$.
- We have,

$$\begin{aligned}
\boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g) &= (\boldsymbol{I} - \phi(\boldsymbol{G} \vee \boldsymbol{H}))^{-1}\boldsymbol{\alpha} - (\boldsymbol{I} - \phi\boldsymbol{G})^{-1}\boldsymbol{\alpha} \\
&= \sum_{p=0}^{\infty} \phi^p((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p)\boldsymbol{\alpha} \\
\boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h) &= \sum_{p=0}^{\infty} \phi^p(\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\boldsymbol{\alpha}
\end{aligned}$$

- Then,

$$\begin{aligned}
&\{\boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g)\} - \{\boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h)\} \\
&= \sum_{p=0}^{\infty} \phi^p\{((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\}\boldsymbol{\alpha}
\end{aligned}$$

Proof of Lemma 2

- Assume $((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p) \geq \mathbf{0}$. Then,

  $$\begin{aligned}
  &(\hat{\boldsymbol{G}}^{p+1} - \boldsymbol{G}^{p+1}) - (\boldsymbol{H}^{p+1} - (\boldsymbol{G} \wedge \boldsymbol{H})^{p+1}) \\
  &= ((\boldsymbol{G} + \boldsymbol{D})(\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}\boldsymbol{G}^p) - (((\boldsymbol{G} \wedge \boldsymbol{H}) + \boldsymbol{D})\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{G} \wedge \boldsymbol{H})^p) \\
  &= \{\boldsymbol{G}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{G} \wedge \boldsymbol{H})(\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} + \boldsymbol{D}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{H}^p) \\
  &= \boldsymbol{H}\{((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) - (\boldsymbol{H}^p - (\boldsymbol{G} \wedge \boldsymbol{H})^p)\} \\
  &\quad + \hat{\boldsymbol{D}}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{G}^p) + \boldsymbol{D}((\boldsymbol{G} \vee \boldsymbol{H})^p - \boldsymbol{H}^p) \geq \mathbf{0}
  \end{aligned}$$

- By induction,

  $$\boldsymbol{x}^*(g \vee h) - \boldsymbol{x}^*(g) \geq \boldsymbol{x}^*(h) - \boldsymbol{x}^*(g \wedge h)$$

- By Lemma 1, we have

  $$\boldsymbol{x}^*(g \vee h) + \boldsymbol{x}^*(g) \geq \boldsymbol{x}^*(h) + \boldsymbol{x}^*(g \wedge h)$$

- Therefore, by Lemma 0,

  $$v_i(\boldsymbol{x}^*(g\vee h), g\vee h, \phi) - v_i(\boldsymbol{x}^*(g), g, \phi) \geq v_i(\boldsymbol{x}^*(h), h, \phi) - v_i(\boldsymbol{x}^*(g\wedge h), g\wedge h, \phi)$$

Proof of the theorem

- $\Gamma = \langle N, \boldsymbol{\Psi}, (u_i)_{i \in N} \rangle \rightarrow \boldsymbol{\Psi}$ is a sublattice of $\times_{i=1}^{n} \mathbb{R}^n$
- By Lemma 2, supermodularity of u_i in ψ_i, for each $\psi_{-i} \in \Psi_{-i}$

  $$u_i(g(\psi_i \vee \psi_i', \psi_{-i}), \phi) + u_i(g(\psi_i \wedge \psi_i', \psi_{-i}), \phi) \geq u_i(g(\psi_i, \psi_{-i}), \phi) + u_i(g(\psi_i', \psi_{-i}), \phi)$$

  for any $\psi_i, \psi_i' \in \Psi_i$
- By Lemma 2, increasing differences of $u_i(\psi_i, \psi_{-i})$ in (ψ_i, ψ_{-i})

  $$u_i(g(\psi_i, \psi_{-i}), \phi) - u_i(g(\psi_i', \psi_{-i}), \phi) \geq u_i(g(\psi_i, \psi_{-i}'), \phi) - u_i(g(\psi_i', \psi_{-i}'), \phi)$$

  for $\psi_i, \psi_i' \in \Psi_i$ with $\psi_i \geq \psi_i'$ and $\psi_{-i}, \psi_{-i}' \in \Psi_{-i}$ with $\psi_{-i} \geq \psi_{-i}'$
  [Back]