Bubbles and Crashes

Abreu and Brunnermeier, Econometrica, 2003

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Introduction

- Many economic theories say bubbles do not exist in efficient markets.
  - Standard neoclassical theory
    - Backwards induction in finite horizon models
    - Transversality condition in infinite horizon models
  - 'No trade theorems' in setting with asymmetric information
  - Efficient markets hypothesis
- All agents in these models are assumed to be rational.
- This paper challenges the efficient markets perspective.
  - In particular, bubbles can survive despite the presence of rational arbitrageurs who are well-informed and well-financed.

The model overview

- Rational arbitrageurs understand that the market will eventually collapse, but meanwhile would like to ride the bubble to get high profit.
- They would like to exit the market just prior to the crash, but this is a difficult task.
- Arbitrageurs realize that they will come up with different solutions of this optimal timing problem.
- This dispersion of exit strategies and consequent lack of synchronization are precisely what permit the bubble to grow.

Two important aspects

- Rational arbitrageurs are sequencially aware that the price has departed from fundamentals. Arbitrageurs do not know whether they have learnt this information early or late relative to other rational arbitrageurs. (**dispersion of opinion** among rational arbitrageurs)
- Selling pressure only bursts the bubble when a sufficient mass of arbitrageurs have sold out. In other words, a permanent shift in price levels requires a coordinated attack. (**the need for coordination**)

Price(1)

- Prior to $t = 0$, the stock price equals to its fundamental value, which grows at the risk-free interest rate r.
  - Without loss of generality, we normalize the starting point to $t = 0$ and the stock price at $t = 0$ to $p_0 = 1$.
- From $t = 0$, the stock price p_t grows at a rate $g > r$, that is $p_t = e^{gt}$.
- Until some random t_0, the price is justified by the fundamental development.
  - The culmulative distribution function of t_0 is $\Phi(t_0) = 1 - e^{-\lambda t_0}$
- From t_0, only some fraction $(1 - \beta(\cdot))$ of the price is justified by fundamentals, while the fraction $\beta(\cdot)$ reflects the "**bubble component**".
  - Assume $\beta(\cdot) : [0, \bar{\tau}] \to [0, \bar{\beta}]$ is a strictly increasing and continuous.

Price(2)

- The price $p_t = e^{gt}$ is kept its fundamental value by "irrationally exuberant" behavioral traders.
- When the culmulative selling pressure by rational arbitrageurs exceeds κ, the price drops by a fraction $\beta(\cdot)$ to its post-crash.
- From this point onwards, the price grows at a rate of r.
- Even if the selling pressure never exceeds κ, the bubble bursts as soon as it reaches its maximum size $\bar{\beta}$.
  - Denote the day when we reach $\bar{\beta}$ by $t_0 + \bar{\tau}$.

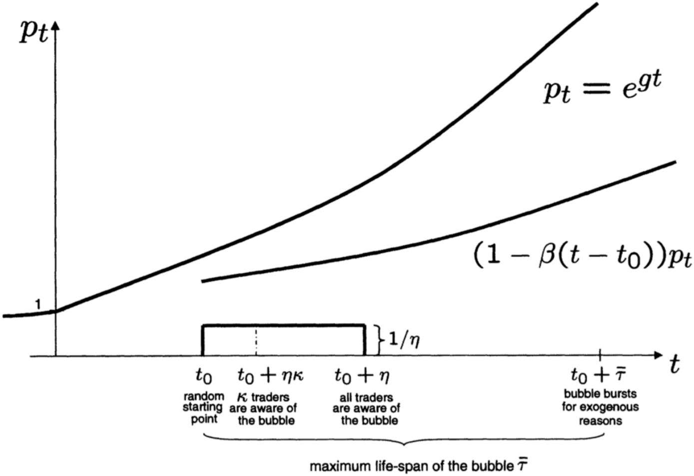

FIGURE 1.—Illustration of price paths.

Sequencial awareness(1)

- Rational arbitrageurs become **sequencially** informed that the fundamental value had not kept up with the growth of the stock price index.
- A new cohort of rational arbitrageurs of mass $1/\eta$ becomes 'aware' of the mispricing in each instant t from t_0 until $t_0 + \eta$.
- Since t_0 is random, an arbitrageur who becomes aware of the bubble at t_i has a posterior distribution for t_0 with support $[t_i - \eta, t_i]$.
- We refer to the arbitrageur who learns of mispricing at time t_i as arbitrageur t_i.
    - The posterior distribution of t_0 is

$$\Phi(t_0|t_i) = \frac{e^{\lambda\eta} - e^{\lambda(t_i - t_0)}}{e^{\lambda\eta} - 1}$$

Sequencial awareness(2)

- We assume
$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$$
  - This guarantees that arbitrageurs do not sell out before they become aware of the mispricing.
- From $t_0 + \eta\kappa$ onwards, the mispricing is known to a large enough mass of arbitrageurs who are collective able to correct it.
- We label any persistent mispricing beyond $t_0 + \eta\kappa$ a bubble.

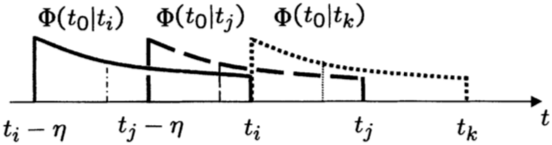

FIGURE 2.—Sequential awareness.

Strategies of arbitrageurs(1)

- Each arbitrageur can sell all or some part of the stock or even go short until she reaches a certain limit of financial constraint.
- Each trader can also buy back shares.
- A trader may exit from and return to the market multiple times.
- Each arbitrageur is limited in the number of shares she can go short or long.
- Without loss of generality, we can normalize the action space to be continuum between $[0, 1]$.
  - 0 indicates the maximum long position.
  - 1 indicates the maximum short position each arbitrageur can take on.

Strategies of arbitrageurs(2)

- Let $\sigma(t, t_i)$ denote the selling pressure of arbitrageur t_i at t, and the function $\sigma : [0, \infty) \times [0, \infty) \to [0, 1]$ a strategy profile.
- The strategy of a trader who become aware of the bubble at time t_i is given by the mapping $\sigma(\cdot, t_i) : [0, t_i + \bar{\tau}] \to [0, 1]$.
- The aggregate selling pressure of all arbitrageurs at time $t \geq t_0$ is given by $s(t, t_0) = \int_0^{\min\{t, t_0+\eta\}} \sigma(t, t_i) dt_i$.
- Denote $T^*(t_0)$ as the bursting time of the bubble for a given t_0.

$$T^*(t_0) = \inf\{t | s(t, t_0) \geq \kappa \text{ or } t = t_0 + \bar{\tau}\}$$

- The belief of arbitrageur t_i about the bursting date are given by

$$\Pi(t|t_i) = \int_{T^*(t_0) < t} d\Phi(t_0|t_i)$$

Strategies of arbitrageurs(3)

- The execution prices of arbitrageurs' orders are either $p(t)$ or $(1-\beta(t-t_0))p(t)$.
- The expected execution price at the time of the bursting of the bubble is
$$(1-\alpha)p(t)+\alpha(1-\beta(t-t_0))p(t)$$
  - $\alpha > 0$ if the selling pressure is strictly larger than κ, and $\alpha = 0$ if the selling pressure is less than or equal to κ.
- Assume that the present value of transactions cost is a constant c.
  - Transactions cost at t equals ce^{rt}.
  - This assumption implies that each arbitrageur will only change her stock holdings at most a finite number of times.

Strategies of arbitrageurs(4)

- In the special case where arbitrageur t_i remains fully invested in the market until she completely sells out at t and remains out of the market thereafter, arbitrageur t_i's expected payoff for selling out at t is

$$\int_{t_i}^{t} e^{-rs}(1-\beta(s-T^{*-1}(s)))p(s)d\Pi(s|t_i)+e^{-rt}p(t)(1-\Pi(t|t_i))-c$$

- In the unique equilibrium trader t_i adopts precisely such a "trigger strategy".

Trading equilibrium

Definition 1

A **trading equilibrium** is defined as a Perfect Bayesian Nash Equilibrium in which whenever a trader's stock holding is less than her maximum, then the trader (correctly) believes that the stock holding of all traders who became aware of the bubble prior to her are also at less than their respective maximum long positions.

- This definition entails a restriction on beliefs that is a natural one in this setting since the earlier an arbitrageur becomes aware of mispricing, the lower is her estimate of the fundamental value, and the more inclined she is to sell out.

Lemma 1

Lemma 1 (No Partial Purchases or Sell Outs)

$$\sigma(t, t_i) \in \{0, 1\} \ \forall t, t_i$$

- In equilibrium, an arbitrageur is either fully invested in the market, $\sigma(t, t_i) = 0$, or at her maximum short-position,$\sigma(t, t_i) = 1$.
- This Lemma reduces the per period action space to $\{0, 1\}$.
- Sketch of proof
  - Denote the amount of holding stocks at t as $x(t)$.
  - Suppose $x(t) \notin \{0, 1\}$ and $\sigma(\cdot, t_i)$ is optimal.
  - The expected payoff from tomorrow onwards is liner in today's bond holdings.
  - Thus the payoff is strictly dominated by $x = 1$ or $x = 0$.
  - This contradicts the assumption that $\sigma(\cdot, t_i)$ is optimal.

Corollary 1

Corollary 1 (Cut-off Property)

$$\sigma(t, t_i) = 1 \Rightarrow \sigma(t, t_j) = 1 \ \forall t_j \leq t_i$$

$$\sigma(t, t_i) = 0 \Rightarrow \sigma(t, t_j) = 0 \ \forall t_j \geq t_i$$

- When arbitrageur t_i sells out her shares, all arbitrageur t_j where $t_j \leq t_i$ also have already sold, or will at that moment sell, all their shares.

Definition 2

Definition 2

The function $T(t_i) = \inf\{t|\sigma(t, t_i) > 0\}$ denotes the **first instant** at which arbitrageur t_i sells any of her shares.

- By Corollary 1, the bubble bursts when trader $t_0 + \eta\kappa$ sells out her shares, provided that it didn't already burst earlier for exogenous reasons.

Corollary 2

The bubble bursts at $T^*(t_0) = \min\{T(t_0 + \eta\kappa), t_0 + \bar{\tau}\}$

- Arbitrageur $t_0 + \eta\kappa$ reduces her holdings for the first time at $T(t_0 + \eta\kappa)$.
- By Corollary 1, all arbitrageurs who became aware of the mispricing prior to $t_0 + \eta\kappa$ are also completely out of the market at $T(t_0 + \eta\kappa)$.
- Thus, the bubble bursts when trader $t_0 + \eta\kappa$ sells out her shares if the bubble did not already burst earlier for exogenous reasons.

Lemma 2

Definition 3

The function $\underline{t_0^{\text{supp}}}(t_i)$ denotes the lower bound of the support of trader t_i's posterior beliefs about t_0, at $T(t_i)$

Lemma 2 (Preemption)

In equilibrium, arbitrageur t_i believes at time $T(t_i)$ that at most a mass κ of arbitrageurs bacame aware of the bubble prior to her. That is, $\underline{t_0^{\text{supp}}}(t_i) \geq t_i - \eta\kappa$

- Sketch of Proof
  - Let $\underline{t_0^{\text{supp}}} = t_i - \eta\kappa - a$ with $a > 0$.
  - By Lemma1 and Corollary1, all $t_j \in [t_i - a, t_i)$ do not sell out prior to $T(t_i)$, and the aggregate selling pressure at $t = T(t_i)$ is $s_{t_0,t} > \kappa$ with strictly positive probability.
  - Hence, with strictly positive probability traders t_j who sell out at $T(t_i)$ will receive the post-crash price.
  - Traders t_j have an incentive to sell out earlier at $T(t_i)$. This is contradiction.

Lemma 6

Lemma 6 (Zero Probability)

For all $t_i > 0$, arbitrageur t_i believes that the bubbles bursts with probability zero at instant $T(t_i)$. That is, $P_r[T^{*-1}(T(t_i))|t_i, B^c(T(t_i))] = 0$ for all $t_i > 0$.

- Let $B^c(t)$ denote the event the bubble has not burst until time t.
- We use the fact that $T^{*-1}(\cdot) : [T^*(0), \infty) \to [0, \infty)$ is well defined.
- Proof
  - Oveserve the conditional c.d.f.

$$\Phi[t_0|t_i, B^c(T(t_i))] = \frac{\Phi(t_0) - \Phi(\underline{t_0^{\text{supp}}})(t_i)}{\Phi(t_i) - \Phi(\underline{t_0^{\text{supp}}})(t_i)}$$

  - This is continuous in t_0.

Proposition 1

Proposition 1 (Trigger-strategy)

In equilibrium, arbitrageur t_i maintains the maximum short position for all $t \geq T(t_i)$, until the bubble bursts.

Proof of Proposition 1

- Suppose there exists an equilibrium where t_i sells out at $T(t_i)$ and re-enters the market later before the bubble bursts.
- She will stay out of the market at least until type $t_i + \epsilon$ exits the market, for some $\epsilon > 0$.
- By Corollary 1, t_i cannot re-rnter the market until after $t_i + \epsilon$.
- The same reasoning applies to $t_i + \epsilon$ with respect to $t_i + 2\epsilon$.
- Preceeding this way we conclude that t_i stays out of the market until the bubble bursts at $t_0 + \bar{\tau}$ or $t_i + \eta\kappa$ exits and then re-enters the market.
- This is contradiction.

Hazard rate

- $\Pi(t) = \Phi(T^{*-1}(t))$ represents the c.d.f. of the bursting date.
  - Let $\pi(t)$ denote its density function.
- Recall that $\Pi(t|t_i)$ is arbitrageur t_i's conditional c.d.f. of the bursting date at time t_i.
  - Similary, $\pi(t|t_i)$ is its density function.
- Following equation represents the hazard rate that the bubble will burst at t.

$$h(t|t_i) = \frac{\pi(t|t_i)}{1 - \Pi(t|t_i)}$$

Lemma 7

Lemma 7 (Sell-out condition)

If arbitrageur t_i's subjective hazard rate is smaller than the 'cost-benefit ratio', i.e.

$$h(t|t_i) < \frac{g - r}{\beta(t - T^{*-1}(t))}$$

trader t_i will choose to hold the maximum long position at t. Conversely, if

$$h(t|t_i) > \frac{g - r}{\beta(t - T^{*-1}(t))}$$

she will trade to the maximum short position.

Intuition

- Consider the benefits and costs of attack at t versus $t+\Delta$.
- The benefits are goven by

$$\Delta h(t|t_i)[p(t)\beta(t - T^{*-1}(t))]$$

- The costs of being out of the market for a short interval Δ are

$$(1 - \Delta h(t|t_i)) \left(\frac{p(t+\Delta) - p(t)}{\Delta} - rp(t) \right) \Delta$$

- If the benefits is smaller than the costs, we have

$$(1 - \Delta h(t|t_i)) \left(\frac{p(t+\Delta) - p(t)}{\Delta} - rp(t) \right) \Delta > \Delta h(t|t_i)[p(t)\beta(t - T^{*-1}(t))]$$

- Divide both sides by $\Delta p(t)$ and $\Delta \to 0$, then

$$h(t|t_i) < \frac{g - r}{\beta(t - T^{*-1}(t))}$$

Exogenous Crashes

- We can interpret η to be a measure of differences in opimion and other heterogeneities across players.
- From $t_0 + \eta\kappa$ onwards, more than κ arbitrageurs are aware of the bubble and have collectively the ability to burst it.

Definition 4

The function $\tau(t_i) := T(t_i) - t_i$ denotes the length of time arbitrageur t_i chooses to ride to the ride the bubble subsequent to becoming aware of the mispricing.

Proposition 2

Proposition 2

Suppose

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{g-r}{\bar{\beta}}$$

Then there exists a unique trading equilibrium. In this equilibrium all traders sell out

$$\tau^1 = \bar{\tau} - \frac{1}{\lambda}\ln\left(\frac{g-r}{g-r-\lambda\bar{\beta}}\right) < \bar{\tau}$$

periods after they become aware of the bubble and stay out of the market thereafter.Nevertheless, for all t_0, the bubble bursts for exogenous reasons precisely when it reaches its maximum possible size $\bar{\beta}$

- Arbitrageurs never burst the bubble if the dispersion of opinion among them, η, and the absorption capasity of behavioral traders, κ, are sufficiently large.

Endogenous Crashes

- Consider the case where this condition is not satisfied, that is, η and κ are sufficiently small.
- When nobody sells out, the bubble burst at $t_0 + \bar{\tau}$, which induces arbitrageurs to sell out at some $t_i + \tau^1$.
- Now, the bubble will burst at $t_0 + \tau^1 + \eta\kappa$, which induces arbitrageurs to sell out at $t_i + \tau^2$ with $\tau^2 < \tau^1$.
- Proceeding in this way leads to a decreasing sequence $\tau^1, \tau^2, \tau^3, \cdots$ which converges to some τ^*.
- τ^* defines the unique symmetric trigger-strategy Perfect Bayesian Nash equilibrium.

Proposition 3

Proposition 3

Suppose

$$\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}$$

Then there exists a unique trading equilibrium, in which arbitrageur t_i with $t_i \geq \eta\kappa$ leave the market

$$\tau^* = \beta^{-1}\left(\frac{g-r}{\frac{\lambda}{1-e^{-\lambda\eta\kappa}}}\right) - \eta\kappa$$

periods after they become aware of the bubble. All arbitrageurs t_i such that $t_i < \eta\kappa$ sell out at $\eta\kappa + \tau^*$. Hence, the bubble bursts when it is a fraction

$$\beta^* = \frac{1-e^{-\lambda\eta\kappa}}{\lambda}(g-r)$$

of the pre-crash price.

Uninformative events

- From now on, we will consider the role of **synchronizing events** which serves as pure coordination devices.
- In particular, we extend the former model to allow for the arrival of synchronizing events at a Poisson arrival rate θ.
- Only arbitrageurs who are aware of the bubble more than $\tau_e \geq 0$ periods ago can observe synchronizing events.
- Synchronizing events both allow arbitrageurs to synchronize sell outs and also convey valueable information in event that attack are unsuccessful.
- The bubble will survive a synchronized sell out if an insufficient number of arbitrageurs have observed the synchronizing event.

Responsive equilibrium

- Unlike the former model, there is no hope of finding a unique equilibrium in this setting.
  - To simply ignore all synchronizing events exactly replicate the former model.
  - There are potentially numerous intermediate levels of responsiveness to synchronizing events.
- We focus on the following equilibrium.

Definition 6

A '**responsive equilibrium**' is a trading equilibrium in which each arbitrageur believes that all other traders will synchronize (sell out) at each synchronizing event.

Proposition 4

Proposition 4

Suppose that the parameter values satisfy $\frac{g-r}{\bar{\beta}} < \frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$. There exists a unique responsive equilibrium. In this equilibrium, each arbitrageur t_i always sells out at the instances of synchronizing events $t_e \geq t_i + \tau_e$. Furthermore, she stays out of the market for all $t \geq t_i + \tau^{**}$ except in the event that the last synchronized attack failed in which case she re-enters the market for the interval $t \in (t_e, t_e + \tau^{**} - \tau_e)$, unless a new synchronizing event occurs in the interium.

- Even if an arbitrageur does not observe a synchronizing event, she stays in the market for a fixed number τ^{**} periods after becoming aware of the bubble.
- The arrival of synchronizing events might accelarate the bursting date.

Price Events

- Consider the model in which we allow random temporary price drops to occur.
- This enables us to understand how a large price drop might either lead to a crash or a rebound.
- Exogenous price drops occur with a Poisson density θ_p at the end of a random trading round t.
- We will consider the case only traders who are sufficiently certain that the bubble will burst after the price drop will leave the stock market.

Proposition 5

Proposition 5

Suppose that the parameter values satisfy $\frac{g-r}{\beta} < \frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$. There exists an equilibrium $(\tau_p(\cdot), \tau^{***})$ in which arbitrageur t_i exits the market after a price drop at $t_p^{(n+1)}$ if $t_p^{(n+1)} \geq t_i + \tau_p(H_p^n)$. Furthermore, she is out of the market at all $t \geq t_i + \tau^{***}$ except in the event that the last attack failed, in which case she re-enters the market for the interval $t \in (t_p^{(n+1)}, t_p^{(n+1)} + \tau^{***} - \tau_p(H_p^n))$.

- Denote a history of past temporary price drops as $H_p^n := (t_p^{(1)}, \cdots, t_p^{(n)})$.
- Only traders who became aware of the mispricing more than $\tau_p(H_p^n)$ periods earlier will choose to leave the market and attack the bubble after a price drop.

Conclusion

- This paper argues that the bubbles can persist even though all rational arbitrageurs know that the price is too high and jointly have the ability to correct the mispricing.
- A central, and realistic assumption of the model is that there is dispersion of opinion among rational arbitrageurs concerning the timing of the bubble.
- All professionals are in agreement that assets are overvalued, while arguably there are substantial differences in opinion even among professionals regarding the possibility that current valuations indeed reflect a new era of higher productive growth, lower wages, and so on.