Coordination on Networks

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Motivating Example

- Setting with binary actions, positive network effects, and incomplete information.
- Scenario
  1. Cryptocurrency Adoption
     - the more neighbors adopt, the more valuable as a medium of exchange
     - the future stability or inflation of the currency is uncertain
  2. Crime
     - the more neighbors take part in crime, the more help you have and the less likely you will be caught
     - the state of the world (ex: the presence of police) is unknown
  3. Immigration Policy
     - bordeing countryies' open policies imply less refugees come to your country
     - state of economy or war is unknown

Who coordinate with whom?

- When agents' adoption of a technology affects the technology's value experienced by others, which agents will tend to adopt together?
- Given agent 3's advantageous position, will she adopt in strictly more states than agent 1?

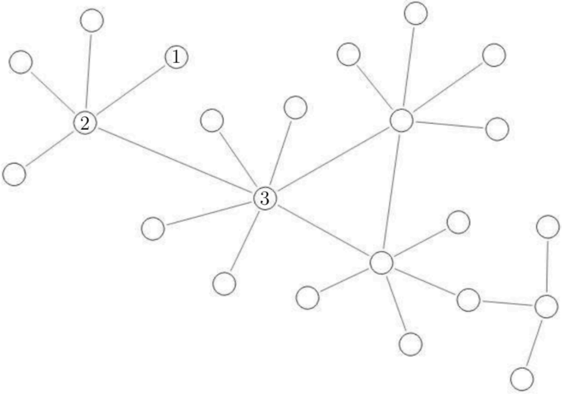


Figure 1: Who coordinates with whom?

Main Results

- Coordination set characterizes the properties of limit-equilibrium
  - solve for unique set of limiting coordination sets
  - easy to obtain a single coordination set
- Contagion within coordination sets
  - all agent within same coordination set respond uniformly to targeted adoption
  - effect on other coordination sets spreads discontinuously
- Welfare and policy implication
  - in the limit, optimal policy targets coordination sets rather than individuals
  - optimal strategies to maximize adoption and one to maximize welfare may be different

Model

- A finite set of agents N simultaneously choose whether or not adopt a technology $a_i \in \{0, 1\}$
  - denote vector $\mathbf{a} := (a_1, \cdots, a_{|N|})$
- fundamental state θ, continuously distributed over bounded, interval support $\Theta \subseteq \mathbb{R}$
- connected network $\mathcal{G} = (N, E)$, edges $(i, j) \in E$
  - assume a connected and undirected graph
- $N_i := \{j : (i, j) \in E\}$ is the set of i's neighbors, and $d_i := |N_i|$ her degree

Payoff

- each i obtains the following payoff from adopting

$$u_i(\mathbf{a_{-i}}|\theta) = v_i + \sigma(\theta) + \phi \sum_{j \in N_i} a_j$$

  where $v_i \in \mathbb{R}$, $\sigma : \Theta \to \mathbb{R}$, and $\phi > 0$
- v_i gives the state independent value (intrinsic value) from adoption
- σ is the state dependent value, $\sigma(\theta)$ is assumed to be differentiable and strictly increasing
- third term represents the positive externality that j's adoption imposes on i
- the value from not adopting the technology is normalized to 0

Dominance Region and Information Structure

- Dominance Region
  - suppose for each i there exist $\underline{\theta}_i$ and $\bar{\theta}_i$ such that $a_i = 0$ is dominant strategy when $\theta < \underline{\theta}_i$ and $a_i = 1$ is dominant strategy when $\bar{\theta}_i < \theta$
  - let $\underline{\theta} = \min \underline{\theta}_i$ and $\bar{\theta} = \max \bar{\theta}_i$, which characterize dominant strategies for all players
  - when θ is common knowledge, there can exist a strictly positive measure of θ realizations within $[\underline{\theta}, \bar{\theta}]$ at which multiple pure strategy Nash equilibria occur
- Information Structure
  - each i observes signal $s_i = \theta + \nu\epsilon_i$, where ϵ_i is distributed via density function f and cumulative function F with support $[-1, 1]$
    - let S denote the set of possible signals
  - for each $\nu > 0$, we write $G(\nu)$ the corresponding global game

Cutoff Strategy

- agent i chooses signal-contingent strategy $\pi_i : S \to [0, 1]$, mapping each signal to the likelihood i adopts
  - we write $\pi := (\pi_1, \cdots, \pi_{|N|})$
- for $\nu > 0$, define i's cutoff strategy at $c_i \in S$ by

$$\pi_i(s_i) := \begin{cases} 1 & (s_i \geq c_i) \\ 0 & (s_i < c_i) \end{cases}$$

- lower cuttof means more adoption

Expected Payoff

- given cutoff strategy π_{-i} and conditional on signal realization s_i, i's expected payoff from adopting is given by

$$\begin{aligned} U_i(\pi_{-i}|s_i) &:= \mathbb{E}_\theta[\mathbb{E}_{s_{-i}}[u_i(\mathbf{a}_{-i}|\theta)|\pi_{-i},\theta]|s_i] \\ &= v_i + \mathbb{E}_\theta[\sigma(\theta)|s_i] + \phi \sum_{j\in N_i} \mathbb{E}_\theta[\mathbb{E}_{s_j}[\pi_j(s_j)|\theta]|s_i] \end{aligned}$$

- BNE π^* of $G(\nu)$ in cutoff strategy satisfies $U_i(\pi_{-i}|s_i = c_i^*) = 0$ for all $i \in N$, that is, each agent must be indifferent between adopting and not adopting when observing signal s_i equal to her cutoff c_i^*

Existence and Uniqueness of Limit Equilibrium

- the following Lemma establishes the existence of equilibrium in cutoff strategies

Lemma B1

A Bayesian Nash Equilibrium π^* of $G(\nu)$ in cutoff strategies exists. [Proof]

- From Lemma B1 and Frankel et al.(2003)[1] Theorem 1, we can show the uniqueness of limit equilibrium in cutoff strategies

Proposition B1

There exists an essentially unique strategy profile $\vec{\pi}$, which is in cutoff strategies, such that any $\pi(\cdot;\nu)$ surviving iterative elimination of strictly dominated strategies in $G(\cdot;\nu)$ satisfies $\lim_{\nu\to 0}\pi(\nu) = \vec{\pi}$ [Proof]

[1] Frankel, David M., Stephan Morris and Ady Pauzner. Equilibrium Selection in Global Games with Strategic Complements. Journal of Economic Theory, 108:1-44, 2003

Characterization of Limiting Equilibrium

- any BNE π^* can be characterized by its cutoffs $\mathbf{c}^*$
- moreover, we can characterize the unique limit equilibrium $\lim_{\nu\to 0}\pi^*$ of $G(0)$, by solving for the limiting cutoffs $\theta^* := (\lim_{\nu\to 0} c_i^*)_{i\in N}$
- denote $\mathbf{w}^*$ the limiting expectations placed on neighbors adopting in equilibrium π^* when each agent i realizes signal s_i equal to her equilinrium cutoff c_i^*

$$w_{ij}^* := \lim_{\nu\to 0} \mathbb{E}_{s_j}[\pi_j^*(s_j)|s_i = c_i^*] \in [0,1]$$

Limit Equilibrium Weights

Lemma 1

For each $(i,j) \in E$, the following identity holds:

$$w^*_{ij} + w^*_{ji} = 1$$

Moreover, if $\theta^*_i < \theta^*_j$, then

$$w^*_{ij} = 0, \text{ and } w^*_{ji} = 1$$

[Proof]

- define the set of feasible weighting functions for $\mathcal{G}$:

$$\mathcal{W} = \{\mathbf{w} = (w_{ij}, (i,j) \in E) | w_{ij} \geq 0, w_{ji} \geq 0, w_{ij} + w_{ji} = 1; \forall (i,j) \in E\}$$

- $\mathcal{W}$ is compact, convex, and isomorphic to $[0,1]^{e(N)}$
- note that $\mathbf{w}^* \in \mathcal{W}$

Limit Equilibrium

- define affine mapping (with image $\Phi(\mathcal{W})$):

$$\Phi_i(\mathbf{w}) := v_i + \phi \sum_{j \in N_i} w_{ij}, \ \forall i \in N$$

- given linearity of $\Phi(\cdot)$, $\Phi(\mathcal{W})$ is a compact, convex polyhedron
- Theorem 1 gives the equilibrium cutoff value for each agent i

Theorem 1

For any $\mathbf{v}$, ϕ, and network $\mathcal{G}$, the equilibrium limit cutoffs θ^* are given by:

$$\sigma(\theta_i^*) + q_i^* = 0, \ \forall i \in N$$

where $\mathbf{q}^* = (q_1^*, \cdots, q_n^*)$ is the unique solution to:

$$\mathbf{q}^* = \arg\min_{\mathbf{z} \in \Phi(\mathcal{W})} \|\mathbf{z}\|$$

[Proof]

- this theorem implies $\mathbf{q}^* = \Phi(\mathbf{w}^*)$

Coordination Sets

- for $S \subseteq N$, denote E_S the subset of edges in E corresponding with the subgraph $\mathcal{G}_S := (S, E_S)$ of $\mathcal{G}$ restricted to vertices S
- the limit equilibrium $\lim_{\nu \to 0} \pi^*$ must then define an ordered partition $\mathcal{C}^* := (C_1^*, \cdots, C_{\bar{m}^*}^*)$ of N

Definition 1 (Coordination sets)

The limit equilibrium $\vec{\pi}$ maps to a unique ordered partition $\mathcal{C}^* := (C_1^*, \cdots, C_{\bar{m}^*}^*)$ of N satisfying:

1. common adoption : for each m, $C_m^* \to \theta_m^* \in \Theta$ with $\theta_i^* = \theta_j^* = \theta_m^*$ for each $i, j \in C_m^*$,and $\theta_m^* \leq \theta_{m'}^*$ for each $m < m'$
2. within-set path connectedness : for each m, $\mathcal{G}_{C_m^*}$ is connected
3. coarse partitioning : for each $m \neq m'$ such that $\theta_m^* = \theta_{m'}^*$, $E_{C_m^* \cup C_{m'}^*} = E_{C_m^*} \cup E_{C_{m'}^*}$

- let $m(i)$ denote i's coordination set : $i \in C_{m(i)}^*$

From Coordination Sets to Cutoffs

- $d_i(S) := |N_i \cap S|$ will denote the within-degree of i
- define for any disjoint agent sets S and S' :

$$e(S, S') = \sum_{i \in S} d_i(S')$$

  the number of edges from S to S'
- for any agent set S, define :

$$e(S) = \frac{1}{2} \sum_{i \in S} d_i(S)$$

  the number of edges between members of S
- $v(S) := \sum_{i \in S} v_i$ denotes the sum of intrinsic values among members of S
- for each $C_m^* \in \mathcal{C}^*$ denote $\underline{C}_m^* := \cup_{m' < m} C_{m'}^*$, which includes all neighbors to C_m^* taking cutoffs below θ_m^*

Coordination-set Cutoffs

Proposition 1

For each $C_m^* \in \mathcal{C}^*$, each $q_i^* = q_m^*$, $i \in C_m^*$, where:

$$q_m^* = \frac{v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))}{|C_m^*|}$$

[Proof]

- Proposition 1 shows that, while $\mathcal{G}$ plays a key role in determining the limit partition $\mathcal{C}^*$, upon conditioning on $\mathcal{C}^*$ the network structure within coordination sets plays no role in determining limiting cutoffs.
- for any set S of connected agents that converge on a common cutoff θ^*, we can average over expected network effects and apply the belief property to obtain a limiting average network externality between mambers of S when θ^* is observed : as $\nu \to 0$

$$\frac{\sum_{i\in S}\sum_{j\in N_i\cap S}\phi\mathbb{E}[\pi_j^*(s_j)|s_i = c_i^*]}{|S|} \to \phi\frac{\#\text{edges between agents in } S}{2|S|}$$

Determining Coordination Sets

- assume homogeneous intrinsic values, that is, $v_i = v$ for all i

Proposition 2 (Single Coordination Set)

Under homogeneous intrinsic values, a single coordination set exists (i.e. $\mathcal{C}^* = \{C_1\}$) if and only if the network is balanced, in the sense that for every nonempty $S \subset N$,

$$\frac{e(S)}{|S|} \leq \frac{e(N)}{|N|}$$

[Proof]

- a network $\mathcal{G}$ is balanced if the average degree of each subnetwork $\mathcal{G}$ is no longer than the average degree of the original network $\mathcal{G}$
- when $\mathcal{G}$ is balanced, the common cutoff value is $\theta_1^* = \sigma^{-1}(-v - \phi\frac{e(N)}{|N|})$

Example of Single Coordination Set

Proposition 3 (Single Coordination Set : Exapmles)

Under homogeneous intrinsic values, there exists a single coordination set if $\mathcal{G}$ takes at least one of the following properties:

1. is a regular network, or
2. is a tree network, or
3. is a regular-bipartite network, or
4. has a unique cycle, or
5. has at most four agents

- network $\mathcal{G}$ is regular if $d_i = d$ for all i
- a tree is any connected network without cycles
- we say network $\mathcal{G}$ is a regular-bipartite network with disjoint within-set symmetric agent sets B_1 and B_2, with $B_1 \cup B_2 = N$ and of sizes $n_s := |B_s|$ and degrees $d_s := d_i$, $i \in B_s$, for sides $s = 1, 2$

Limit Partition Homogeneity

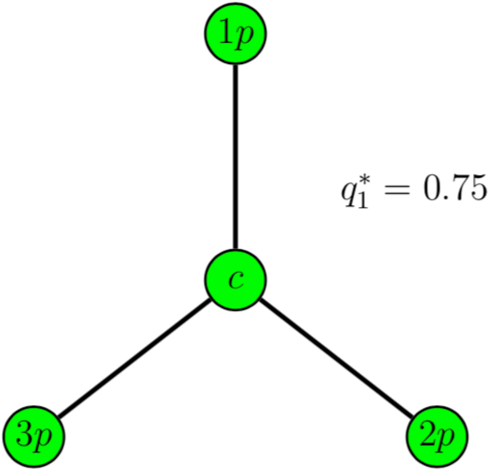
Proposition 4 (Limit Partition Homogeneity)

Under homogeneous intrinsic values, $\mathcal{C}^*$ is independent of v and of ϕ. Moreover, $\mathbf{q}^* = v\mathbf{1} + \phi\hat{\mathbf{q}}^*$ [Proof]

- denote $\hat{\mathbf{q}}^*$ to give the $\mathbf{q}^*$ at $\mathbf{v} = \mathbf{0}$ and $\phi = 1$
- scaling the size of valuations or network effects has no effects on the limit partition
- $\mathbf{q}^*$ is linearly augmented by the size of values v and of network effects ϕ

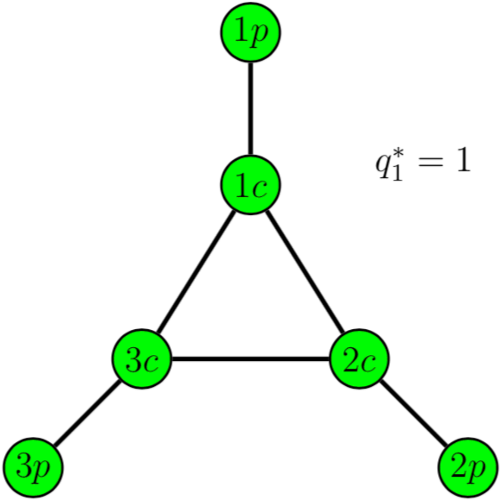
Example 1 : Star Network

(a) Star network.



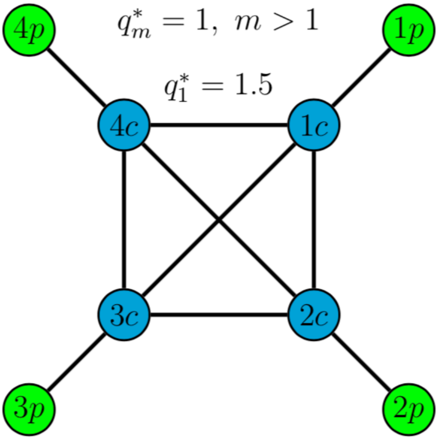
Example 2 : Triad-core-periphery Network

(b) Triad-core-periphery network.



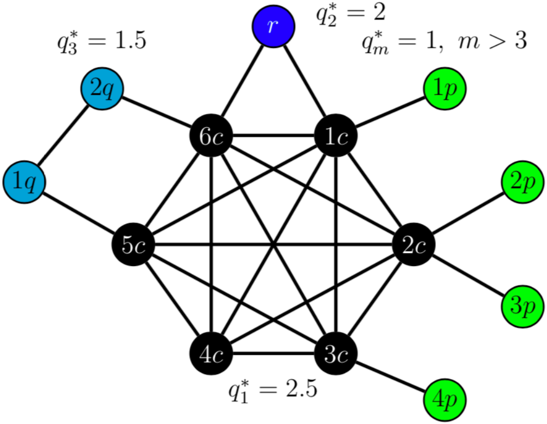
Example 3 : Quad-core-periphery Network

(c) Quad-core-periphery network.



Example 4 : Large core-periphery Network

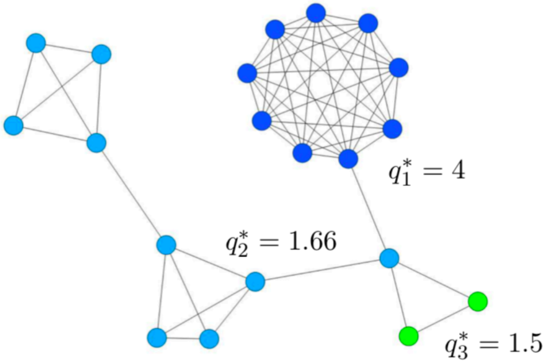
(d) Large core-periphery network.



Coordination in Real-world Networks

- "help decision" network in rural India studied in Banerjee et al.(2013)[2]

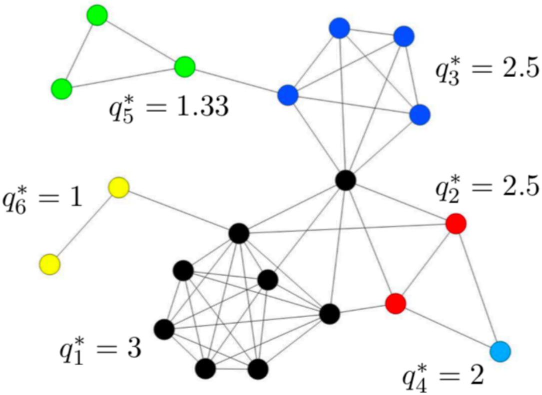
(a) Banerjee et al. (2013) network 1



[2] Abhijit, Banerjee, Arun G. Chandrasekhar, Esther Duflo and Mattew O. Jackson. The Diffusion of Microfinance. Science, 341:363-373, 2013.

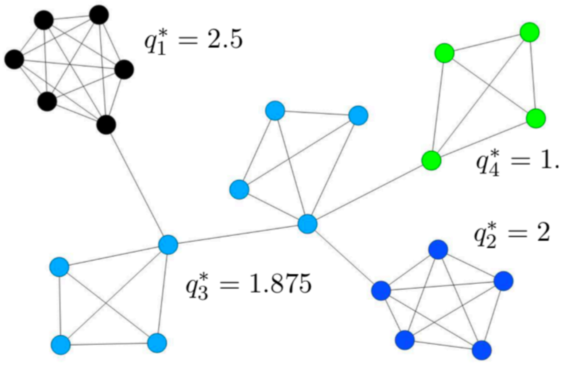
Banerjee et al. (2013)

(b) Banerjee et al. (2013) network 2



Banerjee et al. (2013)

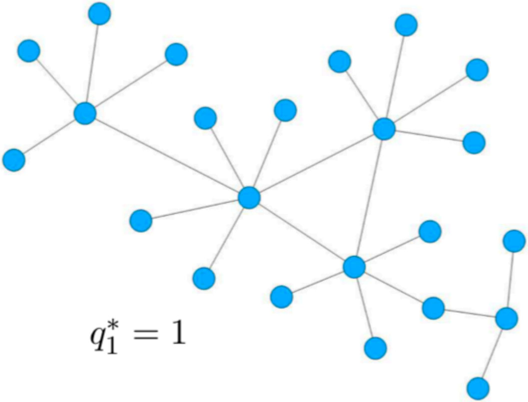
(c) Banerjee et al. (2013) network 3



Coordination in Real-world Networks

- the friendship network of adolescents in the United States sourced from the Add Health data set

(d) Add Health friendship network



Coordination in Real-world Networks

- it is not immediately apparent which agents will coordinate by a casual inspection of the network's structure
- network effects are crucial in adoption decisions, which are captured by coordination sets
- for adoption decisions, it is not the individual centrality that matters but the coordination set

Characterization with Heterogeneous Intrinsic Valuations

- assume heterogeneous v_i
- we can generalize Proposition 2 to this more general framework

Corollary D1

Under heterogeneous valuations, a single coordination set exists (i.e. $\mathcal{C}^* = \{C_1\}$) if and only if for every nonempty $S \subset N$,

$$\frac{v(S) + \phi e(S)}{|S|} \leq \frac{v(N) + \phi e(N)}{|N|}$$

Comparative Statics : Local Contagion

- we would like to investigate how changes in intrinsic value to one agent reverberate through that agent's entire coordination set
- Proposition 5 shows that increasing v_i, the intrinsic value of agent i belonging to C_m, reduces the common cutoff value θ_m^* for all agents in C_m, so that all these individuals are more likely to adopt
- the competitive statics results for any two agents in coordination set C_m are exactly the same

Proposition 5 (Local Contagion)

In the limit, the mapping $\mathbf{q}^*(\mathbf{v})$ is piecewise linear, Lipschitz continuous, and monotone. For generic $\mathbf{v}$, when $i, j \in C_m$ and $k \notin C_m$, then:

$$\frac{\partial q_j^*}{\partial v_i} = \frac{1}{|C_m|} \text{ and } \frac{\partial q_k^*}{\partial v_i} = 0$$

[Proof]

Comparative Statics : Sticky Coordination

- for any $i \in N$ denote :

$$\hat{v_i}^*(\mathbf{v}_{-i}) := \arg\max\{v_i : \theta_i^* = \theta_j^*, j \in C^*_{m(i)} \setminus \{i\}; \mathbf{v}_{-i}\}$$

$$\underset{\smile}{v_i}^*(\mathbf{v}_{-i}) := \arg\min\{v_i : \theta_i^* = \theta_j^*, j \in C^*_{m(i)} \setminus \{i\}; \mathbf{v}_{-i}\}$$

Proposition 6 (Sticky Coordination)

Take coordination set $C^*_m \in \mathcal{C}^*$ with $|C^*_m| > 1$. Take for each $i \in C^*_m$:

$$\hat{v_i}^*(\mathbf{v}_{-i}) - \underset{\smile}{v_i}^*(\mathbf{v}_{-i}) \geq \phi d_i(C^*_m)$$

When $\mathcal{C}^*$ is constant for $v_i \in (\underset{\smile}{v_i}^*(\mathbf{v}_{-i}), \hat{v_i}^*(\mathbf{v}_{-i}))$, then:

$$\hat{v_i}^*(\mathbf{v}_{-i}) - \underset{\smile}{v_i}^*(\mathbf{v}_{-i}) = \frac{|C^*_m|}{|C^*_m| - 1}\phi d_i(C^*_m)$$

Comparative Statics : Sticky Coordination

- first half shows that $\hat{v_i}^*(\mathbf{v}_{-i}) - \underset{\smile}{v_i}^*(\mathbf{v}_{-i})$ is stricyly positive and bounded below by ϕ times the number of neighbors i has in C^*_m
- second half establishes that $\hat{v_i}^*(\mathbf{v}_{-i}) - \underset{\smile}{v_i}^*(\mathbf{v}_{-i})$ scales linearly with $d_i(C^*_{m(i)})$, with slope increasing in ϕ and the size of C^*_m when $\mathcal{C}^*$ is constant
- Proposition 6 says that when social intereactions in the network increase, the ranges of intrinsic values that support coordination amongst agents expand

Heterogeneous Values : Example

- again, star network
- set $v_i = 1$ for $i \neq 1p$, and vary the intrinsic value from adopting of the periphery agent 1, v_{1p}, over $[0.5, 2.5]$
- assume following specification

$$u_i(\mathbf{a}_{-i}|\theta) = v_i - 3\frac{1-\theta}{\theta} + \sum_{j \in N_i} a_j$$

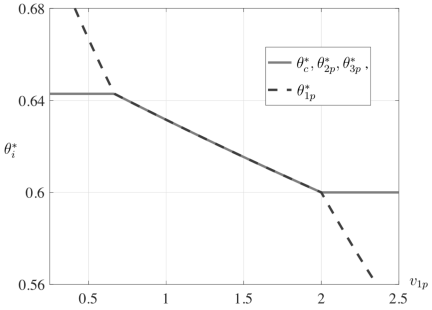


Figure 5: Intrinsic values and local contagion: equilibrium cutoffs in the noiseless limit, versus v_{1p} in the star network.

Welfare and Policy Implications

- what marginal benefits are realized with adoption subsidies?
- which agents' adoption should be subsidized?
- in this model, subsidization means increasing v_i
- consider a policy designer with either of the following two objectives
  - maximize the aggregate ex-ante adoption likelihood
  - maximize the ex-ante aggregate welfare across agents

Benchmarks

- Benchmark 1 (adoption maximization) :

$$ma_i^* := \frac{\partial}{\partial v_i} \sum_{j \in N} \mathbb{E}_{\mathbf{s}}[\pi_j^*] \xrightarrow{\nu \to 0} \frac{\partial}{\partial v_i} \sum_{j \in N} \mathbb{E}_\theta[\chi(\theta \geq \theta_j^*)]$$

  - $\chi(\cdot)$ denoting the indicator function
- Benchmark 2 (welfare maximization) :

$$\begin{aligned} mw_i^* \quad &:= \quad \frac{\partial}{\partial v_i} \sum_{j \in N} \mathbb{E}_{s_j}[U_j(\pi^* | s_j)] \\ &\xrightarrow{\nu \to 0} \quad \frac{\partial}{\partial v_i} \mathbb{E}_\theta \left[\sum_{j \in N} \chi(\theta \geq \theta_j^*) \left(v_j + \sigma(\theta) + \phi \sum_{k \in N_j} \chi(\theta \geq \theta_k^*) \right) \right] \end{aligned}$$

Policy Implications

Proposition 7 (Policy impact)

Denote H the marginal cdf of θ. For each $C_m^* \in \mathcal{C}^*$ and $i \in C_m^*$:

1. $$\lim_{\nu \to 0} ma_i^* = \frac{H'(\theta_m^*)}{\sigma'(\theta_m^*)}$$

2. $$\lim_{\nu \to 0} mw_i^* = (1 - H(\theta_m^*)) + \phi \left(\frac{e(C_m^*, \underline{C}_m^*) + e(C_m^*)}{|C_m^*|} \right) \frac{H'(\theta_m^*)}{\sigma'(\theta_m^*)}$$

- for any i and i' in C_m^*, $\lim_{\nu \to 0} ma_i^* = \lim_{\nu \to 0} ma_{i'}^*$, and $\lim_{\nu \to 0} mw_i^* = \lim_{\nu \to 0} mw_{i'}^*$
- whether a planner maximizes aggregate adoption likelihood or aggregate welfare, she needs to target coordination sets and not individuals

Key Coordination Set

Corollary 1 (Key coordination sets)

Assume homogeneous intrinsic values, uniform $H(\cdot)$ and $\sigma'(\theta)$ decreasing. Then, the key adoption-maximizing coordination set is $C^*_{\bar{m}^*}$, the highest coordination set, whereas, if v is sufficiently large, the key welfare-maximizing coordination set is C^*_1, the lowest coordination set. [Proof]

- to maximization aggregate adoption or welfare, one needs to target coordination sets and not individuals
- two objectives, aggregate adoption and aggregate welfare, need not lead to the same key coordination set
  - in particular, the adoption-maximizing planner's optimal target strongly depends on the elasticity of the value of the technology
  - the welfare-maximizing planner incorporates expected externalities borne within the targeted coordination set and across to adjacent coordination sets

Conclusion

- technical contribution :
  - provide solution to limit cutoffs for general networks, incororating multiple coordination sets in a global-game setting
  - characterize network conditions for common coordination
- equilibrium characterizations unique to network-games literature : coordinated adoption cutoffs in noiseless limit
  - homogeneous values : stratified coordination across network cliques/peripheries
  - heterogeneous values : "sticky" coordination amongst interconnected agents
  - local contagion : strategic spillovers contained within coordination sets

Conclusion

- common coordination not hard to obtain :
  - homogeneous values : regular network, trees, regular-bipartite networks, networks with a unique cycle, and network with at most four nodes
- comparative statics :
  - quantify effect of linkage on equilibrium cutoffs
  - quantify marginal affect of adoption subsidization on equilibrium cutoffs
- welfare implications :
  - optimal policy problems reduce to targeting a coordination set
  - planner aiming to maximize adoption designs intervention to yield large strategic effects
  - planner aiming to maximize welfare also accounts for direct (ex-ante) externalities on neighbors with strictly lower cutoffs

Proof of Lemma B1 (1/2)

- i's expected payoff can be written

$$U_i(\pi_{-i}|s_i) = \mathbb{E}_\theta[v_i + \sigma(\theta) + \phi \sum_{j \in N_i} r(\theta, c_j; \nu)|s_i]$$

where

$$r(\theta, c_j; \nu) = \int_{-1}^{1} \pi_j(\theta + \epsilon_j) f(\epsilon_j) d\epsilon_j = \begin{cases} 0 & (\theta \leq c_j - \nu) \\ F(\frac{\theta - c_j}{\nu}) & (c_j - \nu < \theta \leq c_j + \nu) \\ 1 & (c_j + \nu < \theta) \end{cases}$$

- Also, i's expected payoff can be rewritten

$$U_i(\pi_{-i}|s_i) = v_i + \int_{-1}^{1} \left(\sigma(s_i - \nu\epsilon_i) + \phi \sum_{j \in N_i} r(s_i - \nu\epsilon_i, c_j; \nu) \right) f(\epsilon_i) d\epsilon_i$$

Proof of Lemma B1 (2/2)

- we first show that each agent best responds in $G(\nu)$ to a profile of cuttoff strategies via a unique strategy
- since $\sigma(\theta)$ is strictly increasing and $r(\theta, s_j; \nu)$ is weakly increasing in θ, it is immediate that expected payoff is strictly increasing in s_i
- there must be unique signal realization $c_i^* \in [\underline{\theta} - \nu, \bar{\theta} + \nu]$ that solves $U_i(\pi_{-i}|c_i^*) = 0$
- by continuity of all payoffs in other's cutoff, we can applying Brouwer's fixed point theorem giving the result

[Back]

Proof of Proposition B1

Frankel et al.(2003) Theorem 1

$G(\nu)$ has an essentially unique strategy profile surviving iterative strict dominance in the limit as $\nu \to 0$. It is an increasing pure strategy profile. More precisely, there exists an increasing pure strategy profile π^* such that if, for each $\nu > 0$, π^ν is a pure strategy profile that survives iterative strict dominance in $G(\nu)$, then $\lim_{\nu \to 0} \pi_i^\nu(s_i) = \pi_i^*(s_i)$ for alomost all s_i

- this theorem shows that as signal errors shrink to zero, this process selects an essencialy unique Bayesian equilibrium of the game

[Back]

Proof of Lemma 1 (1/4)

- given $s_i = c_i$, the conditional distribution of θ is $c_i - \nu\epsilon_i$, so:

$$Pr(c_i - \nu\epsilon_i \leq \theta) = 1 - F\left(\frac{c_i - \theta}{\nu}\right)$$

- furthermore, conditional on θ, the distribution of s_j is $\theta + \nu\epsilon_j$, so:

$$E[\pi_j(s_j)|\theta] = Pr(\theta + \nu\epsilon_j \geq c_j) = 1 - F\left(\frac{c_j - \theta}{\nu}\right)$$

- using the law of iterated expectations:

$$E[\pi_j(s_j)|s_i = c_i] = \int_\theta \left\{1 - F\left(\frac{c_j - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_i - \theta}{\nu}\right)\right]$$

- similarly,

$$E[\pi_i(s_i)|s_j = c_j] = \int_\theta \left\{1 - F\left(\frac{c_i - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_j - \theta}{\nu}\right)\right]$$

Proof of Lemma 1 (2/4)

- taking a sum and using the product rule :

$$E[\pi_j(s_j)|s_i = c_i] + E[\pi_i(s_i)|s_j = c_j]$$

$$= \left[\left\{ 1 - F\left(\frac{c_j - \theta}{\nu} \right) \right\} \left\{ 1 - F\left(\frac{c_i - \theta}{\nu} \right) \right\} \right]_{\theta=-\infty}^{\theta=\infty} = 1$$

- since this holds for any cutoff and any ν, it continues to hold in the limit as ν goes to 0, we have $w_{ij}^* + w_{ji}^* = 1$

Proof of Lemma 1 (3/4)

- recall $E[\pi_j(s_j)|s_i = c_i] = \int_\theta \left\{1 - F\left(\frac{c_j-\theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{c_i-\theta}{\nu}\right)\right]$
- let $z = \frac{\theta - c_i}{\nu}$, then

$$
\begin{aligned}
E[\pi_j(s_j)|s_i = c_i] &= \int_\Theta \left\{1 - F\left(-z + \frac{c_j - c_i}{\nu}\right)\right\} d(1 - F(-z)) \\
&= -\int_\Theta \left\{1 - F\left(-z + \frac{c_j - c_i}{\nu}\right)\right\} dF(-z)
\end{aligned}
$$

- when $\lim_{\nu\to 0} c_i < \lim_{\nu\to 0} c_j$, for each fixed z: as $\nu \to 0$

$$
\left\{1 - F\left(-z + \frac{c_j - c_i}{\nu}\right)\right\} \longrightarrow 0
$$

Proof of Lemma 1 (4/4)

- by Dominant Convergence Theorem,

$$\lim_{\nu \to 0} E[\pi_j(s_j)|s_i = c_i] = -\int_\theta 0 \cdot dF(-z) = 0$$

- similary,

$$\lim_{\nu \to 0} E[\pi_i(s_i)|s_j = c_j] = -\int_\theta dF(-z) = 1$$

[Back]

Proof of Theorem 1 (1/5)

Definition 2

Let K be a closed convex set in $\mathbb{R}^n$. For $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection of $\mathbf{x}$ on the set K is the unique point $\mathbf{y} \in K$ such that:

$$\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{z}\| \ \forall \mathbf{z} \in K$$

We denote $\mathbf{Proj}_K[\mathbf{x}] := \mathbf{y} = \arg\min_{\mathbf{z} \in K} \|\mathbf{x} - \mathbf{z}\|$

Lemma 2

The unique vector $\mathbf{q}^*$, the projection of $\mathbf{0}$ onto the $\Phi(\mathcal{W})$, is the uniquely characterized by the following two conditions:

(C1) $\mathbf{q}^* \in \Phi(\mathcal{W})$: i.e., there exists $\mathbf{w}^*$ such that $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$ for all $i \in N$

(C2) for any edge $(i, j) \in E$ and for any $z_{ij} \in [0, 1]$, $(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0$

Moreover, we can replace (C2) by the equivalent form:

(C2') $(i, j) \in E$, $(q_i^* - q_j^*) > 0 \Rightarrow w_{ij}^* = 0, w_{ji}^* = 1$

Proof of Theorem 1 (2/5)

Proof of Lemma 2

(Necessity)

- obviously (C1) is just the feasibility condition, hence necessary
- for (C2), for any $\mathbf{w}' \in \mathcal{W}$, by optimality of $\mathbf{q}^*$, the following must be true:

$$\eta(t) := \|\Phi((1-t)\mathbf{w}^* + t\mathbf{w}')\|^2 \geq \|\Phi(\mathbf{w}^*)\|^2 = \|\mathbf{q}^*\|^2 = \eta(0)$$

- since $\Phi(\cdot)$ is affine mapping, we obtain:

$$\eta'(0) = 2\langle \mathbf{q}^*, \Phi(\mathbf{w}') - \Phi(\mathbf{w}^*)\rangle \geq 0$$

- now for any $z_{ij} \in [0, 1]$, we construct a special $\mathbf{w}'$ by only modifying the weights w^*_{ij} and $w^*_{ji} = 1 - w^*_{ij}$ on the edge between i and j in $\mathbf{w}^*$ to $w'_{ij} = z_{ij}$ and $w'_{ji} = 1 - z_{ij}$
- since $\mathbf{w}'$ is still in $\mathcal{W}$, we have

$$\phi(q^*_i(z_{ij} - w^*_{ij}) + q^*_j(z_{ji} - w^*_{ji})) \geq 0$$

Proof of Theorem 1 (3/5)

Proof of Lemma 2

- however, $z_{ji} - w^*_{ji} = (1 - z_{ij}) - (1 - w^*_{ij}) = -(z_{ij} - w^*_{ij})$
- so, we have the following inequality

$$(q^*_i - q^*_j)(z_{ij} - w^*_{ij}) \geq 0$$

(Sufficiency)

- for any $\mathbf{w}' \in \mathcal{W}$, simple calculation shows that:

$$\langle \mathbf{q}^*, \Phi(\mathbf{w}') - \Phi(\mathbf{w}^*) \rangle = \phi \sum (q^*_i - q^*_j)(w'_{ij} - w^*_{ij}) \geq 0$$

- therefore, $\eta'(0) \geq 0$, moreover $\eta(\cdot)$ is clearly convex in $t \in [0, 1]$, so $\eta(1) - \eta(0) \geq (1 - 0)\eta'(0) \geq 0$
- that is:

$$\|\Phi(\mathbf{w}')\|^2 \geq \|\Phi(\mathbf{w}^*)\|^2 = \|\mathbf{q}^*\|^2$$

- since $\mathbf{w}' \in \mathcal{W}$ is arbitrary, and indeed $\mathbf{q}^*$ is the projection of $\mathbf{0}$ onto $\Phi(\mathcal{W})$

Proof of Theorem 1 (4/5)

Proof of Lemma 2

(equivalence between (C2) and (C2'))

- (C2) $\Rightarrow$ (C2')
  - Suppose $q_i^* > q_j^*$ and let $z_{ij} = 0$. We have $(q_i^* - q_j^*)(0 - w_{ij}^*) \geq 0$, and it must be $w_{ij}^* = 0$.
  - Similary, assuming $q_i^* < q_j^*$ and picking $z_{ij} = 1$ shows that $w_{ij}^* = 1$
- (C2') $\Rightarrow$ (C2)
  - If $q_i^* > q_j^*$ and $w_{ij}^* = 1$, then for any $z_{ij} \in [0, 1]$,

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0$$

  - Similary, if $q_i^* < q_j^*$ and $w_{ij}^* = 1$, then for any $z_{ij} \in [0, 1]$,

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0$$

Proof of Theorem 1 (5/5)

- the cutoff in the limit must satisfy the indifference conditions : for all $i \in N$,
$$v_i + \sigma(\theta_i^*) + \phi \sum_{j \in N_i} w_{ij}^* = 0$$
- clearly, $w_{ij}^* + w_{ji}^* = 1$ by Lemma 1.
- let $q_i^* = -\sigma(\theta_i^*)$. $\theta_i^* < \theta_j^*$ if and only if $q_i^* > q_j^*$, then $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$
- moreover, suppose $\theta_i^* < \theta_j^*$, then $w_{ij}^* = 0$ and $w_{ji}^* = 1$ by Lemma 1
- as a result, $\mathbf{q}^*$ satisfies the two conditions stated in Lemma 2, therefore $\mathbf{q}^*$ must be the projection of $\mathbf{0}$ onto $\Phi(\mathcal{W})$, which proves the theorem

[Back]

Proof of Proposition 1

- given $q_i^* = q_j^* = q_m^*$ for each $i, j \in C_m^*$ by definition, it must be that

$$\begin{aligned}
|C_m^*| q_m^* &= \sum_{i \in C_m^*} \left(v_i + \phi \sum_{j \in N_i} w_{ij}^* \right) \\
&= \sum_{i \in C_m^*} \left(v_i + \phi \left(\sum_{j \in N_i \setminus C_m^*} w_{ij}^* + \sum_{j \in C_m^*} w_{ij}^* \right) \right) \\
&= v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))
\end{aligned}$$

- the last equality comes from Lemma 1

[Back]

Proof of Proposition 2 (1/4)

(Necessity)

- by Theorem 1, existence of single coordination set is equivalent to

$$\frac{\sum_i v_i + \phi e(N)}{n}\mathbf{1} \in \Phi(\mathcal{W})$$

- this can be re-formulated as following

$$v_i + \phi \sum_{j \in N_i} w_{ij} = \frac{\sum_i v_i + \phi e(N)}{n}, \ \forall i \in N$$

$$w_{ij} \geq 0, \ w_{ij} + w_{ji} = 1, \ \forall (i,j) \in E$$

- given $v_i = v$ and $\frac{\sum_i v_i + \phi e(N)}{n} = nv + \phi e(N)$, above system is equivalent to:

$$\sum_{j \in N_i} w_{ij} = \frac{e(N)}{|N|}, \ \forall i \in N$$

$$w_{ij} \geq 0, \ w_{ij} + w_{ji} = 1, \ \forall (i,j) \in E$$

Proof of Proposition 2 (2/4)

- suppose there exists a solution $\mathbf{w}^*$ to the system
- then,

$$|S|\frac{e(N)}{|N|} = \sum_{i\in S}(\sum_{j\in N_i} w^*_{ij}) \geq \sum_{i,j\in S:(i,j)\in E} w^*_{ij} = e(S)$$

- third equality comes from Lemma 1

Proof of Proposition 2 (3/4)

(Sufficiency)

- from the original network $G = (N, E)$, construct a specific bipartite network $\tilde{G} = (V, A)$ so that $V = V_1 \cup V_2$ where $V_1 = E$ and $V_2 = N$
- $f \in E = V_1$ is connected to $i \in N = V_2$ in $\tilde{G}$ if and only if i is one of the end-points of this edge f in the origianl network G
- each node $i \in V_2$ is a demand node, demanding $d_i = \frac{e(N)}{|N|}$
- each node $j \in V_1$ is a supply node, supplying $s_j = 1$
- Gale's Demand Theorem[3] states that there is a feasible way to match demand and supply if and only if for all $S \subset V_2$:

$$\sum_{i \in S} d_i \leq \sum_{j \in N(S)} s_j$$

[3]Gale, David. A Theorem of Flows in Networks. Pacific Journal of Mathmatics, 7(2):1073-1082, 1957

Proof of Proposition 2 (4/4)

- This is equivalent to:

$$|S|\frac{e(N)}{|N|} \leq |N(S)|, \ \forall \emptyset \subset S \subset V_2$$

- Note that $|N(S)| = e(N) - e(S^C)$ and $|N| = |S| + |S^C|$
- Then, we have

$$|S|\frac{e(N)}{|N|} \leq |N(S)| \Longleftrightarrow \frac{e(S^C)}{|S^C|} \leq \frac{e(N)}{|N|}$$

- since S is arbitrary, S^C is also arbitrary
- therefore, if $\frac{e(S)}{|S|} \leq \frac{e(N)}{|N|}$ is satisfied, the feasibility condition is satisfied and sigle coordination exists

[Back]

Proof of Proposition 4

- take v and ϕ and corresponding $\mathbf{q}^*$ from Theorem 1
- for each $v' \neq v$, it must be that $v' - {q'_i}^* = v - {q_i}^*$, so :

$$\Phi'(\mathcal{W}) = \{\mathbf{q} + (v - v')\mathbf{1} : \mathbf{q} \in \Phi(\mathcal{W})\}$$

- thus, $q_i^* = q_j^*$ if and only if ${q'_i}^* = {q'_j}^*$, this implies $\mathcal{C}^*$ is independent of v
- setting $v = 0$, again take ϕ and correspoding $\mathbf{q}^*$ from Theorem 1
- for each positive $\phi' \neq \phi$ it must be that ${q'_i}^* = \frac{\phi'}{\phi}{q_i}^*$, so :

$$\Phi'(\mathcal{W}) = \{\frac{\phi'}{\phi}\mathbf{q} : \mathbf{q} \in \Phi(\mathcal{W})\}$$

- again, $q_i^* = q_j^*$ if and only if ${q'_i}^* = {q'_j}^*$, which inplies $\mathcal{C}^*$ is independent of ϕ

[Back]

Proof of Proposition 5 (1/3)

Definition (Lipschitz Continuity)

Given two metric spaces $(X, d_X), (Y, d_Y)$, where d_X denotes the metric on the set X and d_Y denotes the metric on the set Y, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that , for all x_1 and x_2 in X,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$$

Definition (Piecewise Continuous Function)

a piecewise linear function is a real-valued function defined on the real numbers or a segment thereof, whose graph is composed of straight-line sections

Proof of Proposition 5 (2/3)

Lipschitz Continuity

- since Φ depends on $\mathbf{v}$ in a linear way, we let $\mathbf{K} = \Phi(\mathcal{W})$ when $\mathbf{v} = \mathbf{0}$, then for any $\mathbf{v}$, $\Phi(\mathcal{W}) = \mathbf{v} + \mathbf{K}$
- we can rewrite the projection problems as follows :

$$\mathbf{q}^*(\mathbf{v}) = \arg\min_{\mathbf{z} \in \mathbf{v}+\mathbf{K}} \|\mathbf{z}\| = \mathbf{v} + \arg\min_{\mathbf{y} \in \mathbf{K}} \|(-\mathbf{v}) - \mathbf{y}\| = \mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]$$

- the projection mapping is nonexpansive,[4] i.e :

$$\|\mathbf{Proj}_{\mathbf{K}}[\mathbf{x}] - \mathbf{Proj}_{\mathbf{K}}[\mathbf{y}]\| \leq \|\mathbf{x} - \mathbf{y}\|$$

- so for any $\mathbf{v}$ and $\mathbf{v}'$, we have

$$\begin{aligned} \|\mathbf{q}^*(\mathbf{v}) - \mathbf{q}^*(\mathbf{v}')\| &= \|(\mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]) - (\mathbf{v}' + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}'])\| \\ &\leq \|\mathbf{v} - \mathbf{v}'\| + \|\mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}] - \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}']\| \\ &\leq 2\|\mathbf{v} - \mathbf{v}'\| \end{aligned}$$

[4] see chapter 1 of Nagurney, Anna. Network Economics: A Variational Inequality Approach. Kluwer Academic Publishers, 1992.

Proof of Proposition 5 (3/3)

Comparative Statics

- by Lipschitz continuity, $\mathbf{q}^*(\mathbf{v})$ is differentiable for almost all $\mathbf{v}$
- by Proposition 1, each $q_i^* = q_m^*$ for each $i \in C_m^*$ is given by:

$$q_m^* = \frac{\sum_{i \in C_m^*} v_i + \phi(e(\underline{C}_m^*, C_m^*) + e(C_m^*))}{|C_m^*|}$$

- note that the terms $e(\underline{C}_m^*, C_m^*)$ and $e(C_m^*)$ are constant holding $\mathcal{C}^*$ constant
- for generic $\mathbf{v}$, $\mathcal{C}^*$ is locally constant, hence $e(\underline{C}_m^*, C_m^*)$ and $e(C_m^*)$ do not depend on $\mathbf{v}$ locally
- the derivative results follows directly

Monotonicity

- $\frac{\partial \mathbf{q}^*}{\partial \mathbf{v}}$ is nonnegative, so $\mathbf{q}^*$ is monotone in $\mathbf{v}$

[Back]

Proof of Corollary 1

- with uniform $H(\cdot)$ and $\sigma'(\theta)$ decreasing, $\lim_{\nu\to 0} ma_i^*$ is clearly increasing in θ_m^* and thus, an adoption-maximizing planner will always target the highest coordination set
- the condition for $\lim_{\nu\to 0} mw_i^*$ decreasing becomes:

$$\frac{\partial}{\partial\theta}\left(1-\theta-\frac{\sigma(\theta)}{\sigma'(\theta)}-\frac{v}{\sigma'(\theta)}\right)<0$$

$$\Leftrightarrow \sigma^{''}(\theta)[v+\sigma(\theta)]<2(\sigma'(\theta))^2$$

  recall Proposition 1 and Theorem 1

- with $\sigma^{''}(\theta)<0$, a sufficient condition for $\lim_{\nu\to 0} mw_i^*$ decreasing for all θ_m^* is $v\geq\bar{v}$ where:

$$\bar{v}=\max_{m=1,\cdots,\bar{m}^*}\left\{\frac{2(\sigma'(\theta_m^*))^2}{\sigma''(\theta_m^*)}-\sigma(\theta_m^*)\right\}$$

[Back]

Algorithmic Characterization

- the next algorithm provides an approach for calculating the limiting coordination sets

Algorithm 1 (Combinational derivation of coordination sets)

For disjoint agents sets $S, A \subseteq N$, $S \neq \emptyset$, define the function:

$$\psi(S|A) := \frac{v(S) + \phi(e(S,A) + e(S))}{|S|}$$

Define $A_0 := \emptyset$. For $A \subset N$, define $\Lambda(A) := \underset{\emptyset \neq S \subseteq N \setminus A}{\arg\max}\, \psi(S|A)$. Step $k = 1, \cdots$, of the algorithm is defined as follows:

Algorithmic Characterization

Algorithm 1 (Combinational derivation of coordination sets)

Step k

1. Solve
$$B_k = \cup_{S \in \Lambda(A_{k-1})} S$$
2. Partition B_k into disjoint, connected subsets $\{B_k^1, \cdots, B_k^{p(k)}\}$: $E_{B_k^s \cup B_k^{s'}} = E_{B_k^s} \cup E_{B_k^{s'}}$, $1 \leq s < s' \leq p(k)$
3. Set $A_k = B_k \cup A_{k-1}$

- Comtinue until $A_k = N$
- Then, $\{\{B_1^1, \cdots, B_1^{p(1)}\}, \{B_2^1, \cdots, B_2^{p(2)}\}, \cdots\}$ gives $\mathcal{C}^*$