Network Pricing: How to Induce Optimal Flows Under Strategic Link Operators

Yuya Furusawa

Graduate School of Economics, The University of Tokyo

16th August

Network pricing game

- a framework for modeling real-world settings with two types of strategic agents: users of the network and owners (operators) of the network
- owners of the network post a price for usage of the links they own , while users select routes based on price and level of use by other users
  - Example: transportation network
- there are two levels of competition
  - among owners to attract users to their link so as to maximize profit
  - among users of the network to select routes that are cheap yet not too congested

Equilibrium

- some operators may want to increase their toll in order to make a higher profit
- in this game-theoretic environment
  - an equilibrium may fail to exist
  - it might not be unique
  - the network performance at equilibrium can be inefficient

Main Results

- a simple regulation on the network owners market solves all these issues
- using some vector as toll caps leads to every operator charging precisely the cap (Theorem 4.4)
- moreover, we show this equilibrium is robust to coalitions, a concept known as strong Nash equilibrium

Model

- there are nonatomic players, which we call users, selfishly minimize their cost
- each network link is operated by a different selfish agent which maximizes profit by charging tolls
- let $G = (V, E)$ be a network
  - V be the set of nodes
  - E be the set of directed edges/links

The Network User's Game: Selfish Routing

- consider a multi-commodity flow instance, described by origin-destination node pairs $\{(o^k, d^k)\}_{k \in K}$ for a finite set of commodities K
- for each commodity k, $r^k > 0$ units of demand need to be routed from o^k to d^k
- for each each link $e \in E$, there is a latency function $l_e : \mathbb{R}_+ \to \mathbb{R}_+$, that represents the delay experienced by users traveling this link, as a function of the total flow on the link
    - assume this function to be strictly increasing, convex and smooth

Paths and Flows

- for each commodity $k \in K$, let $\mathcal{P}^k$ denote the set of $o^k - d^k$ paths and let $\mathcal{P} = \cup_k \mathcal{P}^k$ be the union of all these paths
- a flow for commodity k is a nonnegative vector $\boldsymbol{x}^k = (x_P^k)_{P \in \mathcal{P}^k}$ such that $\sum_{P \in \mathcal{P}^k} x_P^k = r^k$
- a flow $\boldsymbol{x}$ is a vector $(\boldsymbol{x}^k)_{k \in K}$ where each $\boldsymbol{x}^k$ is a flow for commodity k
- for a flow $\boldsymbol{x}$ and $e \in E$, let $x_e^k = \sum_{P \in \mathcal{P}^k : e \in P} x_P^k$ be the amount of flow that $\boldsymbol{x}^k$ routes on each link e and let $x_e = \sum_{k \in K} x_e^k$ be the amount of flow that $\boldsymbol{x}$ routes on e
- in the case a toll $t_e \geq 0$ is charged for link usage, the combined cost of traveling e is $l_e(x_e) + \alpha t_e$
  - α represents the trade-off factor between delay and tolls
  - wlog, we can assume $\alpha = 1$

Wardrop Equilibrium

- a flow is Wardrop equilibrium if it is supported on paths of minimum cost
  - $\boldsymbol{x}$ is a Wardrop equilibrium if, for every k , for every path $P \in \mathcal{P}^k$ with $x_P^k > 0$, and every path $P' \in \mathcal{P}^k$, $\sum_{e \in P} [l_e(x_e) + t_e] \leq \sum_{e \in P'} [l_e(x_e) + t_e]$
- it states that the journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route

Optimal Flow

- given flow $\boldsymbol{x}$, the total delay is $\sum_{e \in E} x_e l_e(x_e)$, and it is the standard measure of network performance
- optimal flow $\boldsymbol{x}^*$ is the flow that minimizes the total delay
- the vector of marginal tolls $\hat{\boldsymbol{t}}$, defined as $\hat{t}_e = x_e^* l_e'(x_e^*)$ induces the optimal flow, that is $\boldsymbol{x}(\hat{\boldsymbol{t}}) = \boldsymbol{x}^*$
  - any toll vector with this property is called optimal

The Network Operators' Game: Price Competition on Tolls

- every link $e \in E$ is operated by a different operator
- each player e is allowed to charge a nonnegative toll t_e for its usage
- under the resulting toll vector $\boldsymbol{t}$, each link gets flow $x_e(\boldsymbol{t})$
- the profit of player e is given by $\pi_e(\boldsymbol{t}) \equiv t_e x_e(\boldsymbol{t})$
- for each player $e \in E$, her strategy is given by toll t_e, and her profit is given by $\pi_e(t_e, \boldsymbol{t}_{-e}) = t_e x_e(t_e, \boldsymbol{t}_{-e})$

Regulated Network Pricing Game and Nash Equilibria

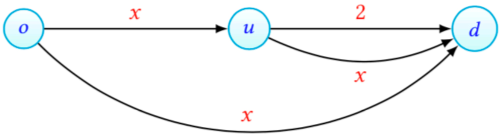
- a central planner may choose a cap vector $\bar{\boldsymbol{t}} \geq 0$ for tolls, and each player wants to maximize her profit under this constraint
- tolls $\boldsymbol{t}$ are a Nash equilibrium if for every e, t_e is the best response of player e to $\boldsymbol{t}_{-e}$
- tolls $\boldsymbol{t}$ are a strong Nash equilibrium if there is no possible coalition that jointly deviates, resulting in an improvement of their individual profits

Cap equilibrium and great tolls

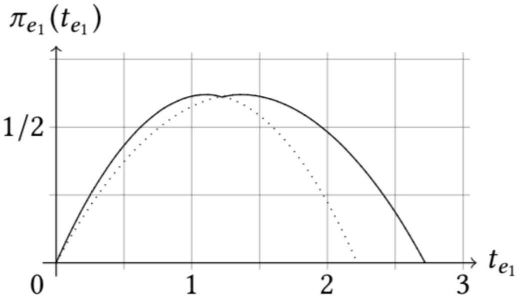
Given an instance of the profit maximization game, we say that a (nonnegative) vector $\bar{\boldsymbol{t}} = (\bar{t}_e)_{e \in E}$ is

- a cap equilibrium if when restricting the strategy space for every player e to tolls $s_e \in [0, \bar{t}_e]$, then $(s_e)_{e \in E} = (\bar{t}_e)_{e \in E}$ is a strong Nash equilibrium
- a great set of tolls if it is optimal and a cap equilibrium

Nonexistence of Equilibrium



- a single o, d commodity with demand $r = 2$
- suppose that $t_{e_2} = 0$, $t_{e_3} = 3/2$, and $t_{e_4} = (2 + \sqrt{6})/2$



- the profit function has two maxima, one at $(2 + \sqrt{6})/4$ and one at $(3 + \sqrt{6})/4$
- this implies the best-response correspondence is not convex
- but convexity is necessary for the existence proof for the parallel links with affine latencies

Unboundedness of Total Delay

- consider the case where $r = 1$ units of flow is routed through a two-link parallel network with $l_{e_1} = Ax$ and $l_{e_2} = a$ for $0 < a < A$
- optimal flow is $\boldsymbol{x}^* = (\frac{a}{2A}, \frac{2A-a}{2A})$, and total delay is $\frac{4aA-a^2}{4A}$
- on the other hand, the Wardrop flow $\boldsymbol{x}(\boldsymbol{t}) = (\frac{A+a}{3A}, \frac{2A-a}{3A})$, and the total delay is $\frac{A^2+8Aa-2a^2}{9A}$
- the ratio of the total delay is $\frac{4(A^2+8Aa-2a^2)}{9(4Aa-a^2)}$
  - keeping $A > 0$ fixed and taking $a \to 0$, we have the ratio diverges

Theorem 4.4

Let $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}$ be an optimal toll vector. For the profit maximization problem with caps $\bar{\boldsymbol{t}}$, there exists a unique Nash equilibrium, given by $\bar{\boldsymbol{t}}$, which, moreover, is a strong Nash equilibrium

- this theorem cannot be extended to the setting where an operator owns more than one link

Proof of Theorem 4.4

- in this proof, we use the following lemmas

Lemma 4.2

Let $\boldsymbol{t}, \boldsymbol{t'} \geq 0$ be two toll vectors such that $\boldsymbol{t} \leq \boldsymbol{t'}$ and $E^< = \{e \in E : t_e < t'_e\}$ is nonempty. Then, there exist $e_1, e_2 \in E^<$ such that :

1. $x_{e_1}(\boldsymbol{t'}) \leq x_{e_1}(\boldsymbol{t})$
2. $(x_{e_2}(\boldsymbol{t'}) - x_{e_2}(\boldsymbol{t}))(l_{e_2}(x_{e_2}(\boldsymbol{t'})) + t'_{e_2} - l_{e_2}(x_{e_2}(\boldsymbol{t})) - t_{e_2}) \leq 0$

Lemma 4.3

Let $\boldsymbol{t} \geq 0$ be a toll vector and a $e \in E$ with $t_e > 0$. If t_e is a local optimum for the profit maximization problem, then $x_e(t_e) l'_e(x_e(t_e)) \leq t_e$

- first, we prove that $\bar{\boldsymbol{t}}$ is a strong Nash equilibrium.
- let $E^{<}$ be a set of links for which the corresponding players deviate some smaller toll value $\boldsymbol{t} \leq \bar{\boldsymbol{t}}$
- in particular,for all $e \in E^{<}$, we have $t_e x_e(\boldsymbol{t}) \geq \bar{t}_e x_e(\bar{\boldsymbol{t}}) > t_e x_e(\bar{\boldsymbol{t}})$
  - this implies $t_e > 0$ along with $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ and $t_e \geq \frac{\bar{t}_e x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}$
- by lemma 4.2, there exists $e \in E^{<}$ such that $(x_e(\bar{\boldsymbol{t}}) - x_e(\boldsymbol{t}))(l_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e - l_e(x_e(\boldsymbol{t})) - t_e) \leq 0$, and $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ gives

$$l_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e \geq l_e(x_e(\boldsymbol{t})) + t_e \tag{1}$$

- let $e \in E^<$ be a link satisfing (1)
- we have following inequalities

$$l'_e(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \leq l_e(x_e(\boldsymbol{t})) - l_e(x_e(\bar{\boldsymbol{t}})) \leq \bar{t}_e \left(1 - \frac{x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right) \tag{2}$$

- since $\bar{t}_e \leq \hat{t}_e = l'_e(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}})$, we obtain from (2) that $\bar{t}_e \leq \hat{t}_e = l'_e(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \leq l'_e(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}}) \left(\frac{x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right)$
- since $l'_e(x_e(\bar{\boldsymbol{t}})) > 0$ and $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$, we conclude that $x_e(\boldsymbol{t}) \leq x_e(\bar{\boldsymbol{t}})$
- we get a contradiction

- next, we show there is a unique Nash equilibrium
- suppose there exists another Nash equilibrium $\boldsymbol{t} \neq \bar{\boldsymbol{t}}$ for the game with caps $\bar{\boldsymbol{t}}$
- we may assume $t_e > 0$ for all e such that $\bar{t}_e > 0$, and thus, by Lemma 4.3, $x_e(\boldsymbol{t})l'_e(x_e(\boldsymbol{t})) \leq t_e$ for all $e \in E^< = \{e \in E : t_e < \bar{t}_e\}$
- this gives $x_e(\boldsymbol{t})l'_e(x_e(\boldsymbol{t})) \leq t_e < \bar{t}_e \leq \hat{t}_e = x^*_e l'_e(x^*_e)$ concluding that $x_e(\boldsymbol{t}) < x^*_e = x_e(\bar{\boldsymbol{t}})$ for all $e \in E^<$
- but form lemma 4.2, there exists $e_1 \in E^<$ such that $x_{e_1}(\boldsymbol{t}) \geq x_{e_2}(\bar{\boldsymbol{t}})$, this is a contradiction

Minimizing User's Cost

- theorem 4.4 implies that any opt-inducing toll vector that is not above the marginals is itself a cap equilibrium, and thus is a great set of tolls
- the fact that great tolls need not be unique, motivates the question of whether it is possible to compute the great tolls vector that minimizes the total user's cost
- if all great tolls were not above the marginals, then computing the great tolls vector that minimizing the total users' costs would reduce to solving a linear program
- but some examples suggest there are tolls that are above the marginals

Best Great Tolls

$$\min \sum_{e \in E} [l_e(x_e^*) + t_e] x_e^*$$

$$v_u^k - v_v^k + t_e = -l_e(x_e^*) \ \forall k, e = (u, v) : {x_e^*}^k > 0$$

$$v_u^k - v_v^k + t_e \geq -l_e(x_e^*) \ \forall k, e = (u, v) : {x_e^*}^k = 0$$

$\boldsymbol{t}$ is a cap equilibrium

$$\boldsymbol{t} \geq 0$$

- we included in the objective the constant term $\sum_e l_e(x_e^*) x_e^*$, corresponding to the total delay experienced by users

Below Marginal Tolls

- by Theorem 4.4, optimal tolls upper bounded by the marginal tolls are always a cap equilibrium, thus within this restricted set of tolls we can write the following LP

$$\min \sum_{e \in E} [l_e(x_e^*) + t_e] x_e^*$$

$$v_u^k - v_v^k + t_e = -l_e(x_e^*) \ \forall k, e = (u, v) : {x_e^*}^k > 0$$

$$v_u^k - v_v^k + t_e \geq -l_e(x_e^*) \ \forall k, e = (u, v) : {x_e^*}^k = 0$$

$$t_e \leq \hat{t}_e \ \forall e \in E$$

$$t_e \geq 0 \ \forall e \in E$$

Minimum Payment Tolls

- the value of BMT doesn't necessarily coincide with the value of BGT
- in order to answer how efficient program BMT can be, we can use as benchmark the value of Minimum Payment Tolls

$$\min \sum_{e \in E} [l_e(x_e^*) + t_e] x_e^*$$

$$v_u^k - v_v^k + t_e = -l_e(x_e^*) \; \forall k, e = (u,v) : {x_e^*}^k > 0$$

$$v_u^k - v_v^k + t_e \geq -l_e(x_e^*) \; \forall k, e = (u,v) : {x_e^*}^k = 0$$

$$t_e \geq 0 \; \forall e \in E$$

Theorem 5.2

Suppose all latency functions l in the profit maximization game satisfy

$$\sup_{x \geq 0} \frac{xl'(x)}{l(x)} \leq \gamma$$

then,

$$\text{value of BMT} \leq (\gamma + 1) \times \text{value of MPT}$$

Elastic Demand

- to test the robustness of the results, we consider the setting with elastic traffic demands
- users have a valuation for traveling through the network and may opt out from traveling if the cost exceeds their valuation
- we model elastic demand with a utility function $u^k : [0, r^k] \to \mathbb{R}_+$ for each $k \in K$, where $u^k(x)$ captures the reservation value for a travel of the demand at level x

Theorem 6.1

Let $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}(u)$ be an optimal toll vector. For the profit maximization problem with caps $\bar{\boldsymbol{t}}$, there exists a unique Nash equilibrium, given by $\bar{\boldsymbol{t}}$, which, moreover, is a strong Nash equilibrium.

- similar to the fixed demand model