Financial Networks and Contagion

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Introduction

- Globalization increases financial interdependencies among many kinds of organizations that hold each other's shares, debts, and other obligations
- Such interdependencies can lead to cascading defaults and failures
- This paper develops a general model that produces new insights regarding financial contagions and cascades of failures

Main Results

- The possibility of cascades and their extent depend on two key aspects of cross-holdings : **integration** and **diversification**
- An economy is most susceptible to widespread financial cascades when two conditions hold
  - integration is intermediate
  - organizations are partly diversified

Integration and Diversification

- **Integration**
  - The level of exposure of organizations to each other
  - Increasing integration leads to increased exposure, which tends to increase the possibility and extent of contagion
  - Increasing integration also decrease the likelihood of first failure
- **Diversification**
  - How spread out cross-holdings are
  - With low levels of diversification, organizations are sensitive to particular others, but the network of interdependencies is disconnected and overall cascades are limited in extent
  - As diversification is further increased, the hit to a "sweet spot" may lead to a cascade which is far-reaching, but organizations become insensitive to any particular organization's failure

Primitive Assets and Organizations

- $N = \{1, \cdots, n\}$ is a set of organizations
- $M = \{1, \cdots, m\}$ is a set of assets
- The present value of asset k is denoted p_k
- Let $D_{ik} \geq 0$ be the share of the value of asset k held by organization i
- Let $\mathbf{D}$ denote the matrix whose (i, k)th entry is D_{ik}

Cross-Holdings

- For any $i, j \in N$ the number $C_{ij} \geq 0$ is the fraction of organization j owned by organization i, where $C_{ii} = 0$ for each i
- The matrix $\mathbf{C}$ can be thought of as a network in which there is a directed link from i to j if $C_{ij} > 0$
- There remains a share $\hat{C}_{ii} := 1 - \sum_{j \in N} C_{ji}$ of organization i not owned by any organization in the system
- $\hat{C}_{ii}$ is assumed to be positive, and the off-diagonal entries of the matrix $\hat{\mathbf{C}}$ are defined to be 0

Values of Organizations

- The equity value V_i of an organization i is the total value of its shares
- This is equal to the value of organization i's primitive assets plus the value of its claims on other organizations:

$$V_i = \sum_k D_{ik} p_k + \sum_j C_{ij} V_j$$

- In matrix notation as

$$\mathbf{V} = \mathbf{D}\mathbf{p} + \mathbf{C}\mathbf{V}$$

- Solved to yeild

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{D}\mathbf{p}$$

Values of Organizations

- As some paper says, the ultimate value of an organization to the economy, is well-captured by the equity value of that organization that is held by its outside investors
- The market value, v_i, is equal to $\hat{C}_{ii}V_i$, and therefore:

$$\mathbf{v} = \hat{\mathbf{C}}\mathbf{V} = \hat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}\mathbf{D}\mathbf{p} = \mathbf{A}\mathbf{D}\mathbf{p}$$

- We refer to $\mathbf{A} = \hat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ as dependency matrix
- The value of an organization can be represented as a sum of the values of its ultimate claims on primitive assts, with organization i owning a share A_{ij} of j's direct holdings of primitive assets

Discontinuities in Values

- Organizations can lose productive value if their values fall below certain critical threshholds
- If the value v_i falls below some threshhold level $\underline{v}_i$, then i is said to fail and incurs the costs $\beta_i(\mathbf{p})$
  - These failure costs are subtracted from a failing organization's cash flow
- We base failure costs on v_i and not on V_i
  - This captures the idea that failure occurs when an organization has difficulties in operating, and the book value is irrelevant in avoinding a failure

Including Failure Costs in Market Values

- The discontinuous drop imposes a cost directly on an organization's balance sheet, and so V_i becomes:

$$V_i = \sum_{j \neq i} C_{ij} V_j + \sum_k D_{ik} p_k - \beta_i \mathbf{I}_{v_i < \underline{v}_i}$$

- In the matrix form, we have

$$\mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1} (\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v}, \mathbf{p}))$$

where $b_i(\mathbf{v}, \mathbf{p}) = \beta_i(\mathbf{p}) \mathbf{I}_{v_i < \underline{v}_i}$

- Correspondingly,

$$\mathbf{v} = \hat{\mathbf{C}} (\mathbf{I} - \mathbf{C})^{-1} (\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v})) = \mathbf{A} (\mathbf{D}\mathbf{p} - \mathbf{b}(\mathbf{v}, \mathbf{p})) \quad (1)$$

Equilibrium Existence

- A solution for organization in equation (1) is an equilibrium set of values
- There can exist multiple equilibria
  - Taking other organizations' values and the values of underlying assets as given, there can be multiple possible consistent values of organization i that solve equation (1)
  - There might be two consistent valuation vector for i and j if they depends on each other
- We typically focus on the best-case equilibrium, in which as few organizations as possible fail
  - This allows us to isolate sources of nessesary cascades, distinct from self-fullfilling sorts of failure

Fair Trades

- Given $\mathbf{p}, \mathbf{C}, \mathbf{D}$, an organization i is **closeset to fail** if there exists a $\lambda > 0$ such that $v_i = (\lambda\mathbf{p}, \mathbf{C}, \mathbf{D}) = \underline{v}_i$, while all other organizations are solvent, $v_j = (\lambda\mathbf{p}, \mathbf{C}, \mathbf{D}) > \underline{v}_j$ for $j \neq i$
- Define $\mathbf{q}(\mathbf{p}, \mathbf{C}, \mathbf{D}) := \lambda\mathbf{p}$
- **Fair Trades**
  - Exchanges of cross-holdings or underlying assets which leave the values of the organizations unchanged at current asset prices
  - $(\mathbf{C}, \mathbf{D})$ and $(\mathbf{C}', \mathbf{D}')$ are said to be related by a fair trade at price $\mathbf{p}$ if $\mathbf{v} = \mathbf{v}'$ where $\mathbf{v} = \mathbf{A}\mathbf{p}$ and $\mathbf{v}' = \mathbf{A}'\mathbf{p}$

Avoiding a First Failure

Proposition 1

Suppose an organization i is closest to fail at asset prices $\mathbf{p} > \mathbf{0}$, cross-holdings $\mathbf{C}$, and direct holdings $\mathbf{D}$. Consider new cross-holdings and direct holdings $\mathbf{C}'$ and $\mathbf{D}'$ resulting from a fair trade at $\mathbf{p}$ such that row i of $\mathbf{A}'$ is different from that of $\mathbf{A}$. Then, for any $\epsilon > 0$, there is a $\mathbf{p}'$ within an ϵ-neighborhood of $\mathbf{q}(\mathbf{p}, \mathbf{C}, \mathbf{D}) > 0$, such that i fails at prices $\mathbf{p}'$ after the fair trade but not before: $v_i(\mathbf{p}', \mathbf{C}', \mathbf{D}') < \underline{v}_i < v_i(\mathbf{p}', \mathbf{C}, \mathbf{D})$

- Proposition 1 shows that at least when it comes to saving the most vulnerable organization, there are always trade-offs:
  - New holdings that avoid failure at one set of prices make failure more likely at another set of nearby prices
- This is a sort of "no-free-lunch" result for avoiding first failure

Diversification

- Diversification captures the spread in organizations' cross-holdings

Diversification

Cross-holdings $\mathbf{C}'$ are more diversified than cross-holdings $\mathbf{C}$ if and only if

- $C'_{ij} \leq C_{ij}$ for all i, j such that $C_{ij} > 0$, with strict inequality for some ordered pair (i, j), and
- $C'_{ij} > C_{ij} = 0$ for some i, j

Integration

- Integration captures the depth or extent of organizations' cross-holdings

Diversification

Cross-holdings $\mathbf{C}'$ are more integrated than cross-holdings $\mathbf{C}$ if and only if $\hat{C}'_{ii} \leq \hat{C}_{ii}$ for all i, with strict inequality for some i. This is equivalent to the condition that

$$\sum_{j:j\neq i} C'_{ji} \geq \sum_{j:j\neq i} C_{ji}$$

for all i, with strict inequality for some i

Essential Ingredients of a Cascades

- A First Failure
  - Some organization must be susceptible enough to shocks in some assets that it fails
- Contagion
  - must be that some other organizations are sufficiently sensitive to the first organization's failure that they also fail
- Interconnection
  - It must be that the network of cross-holdings is sufficiently connected so that the failures can continue to propagate and are not limited to some small component

General Result

Propositon 2

Consider $(\mathbf{C}, \mathbf{D})$ and $(\mathbf{C}', \mathbf{D}')$ that are related by a fair trade at $\mathbf{p}$, and such that integration increases: $A'_{ij} \geq A_{ij}$ whenever $i \neq j$. There is then the same set of first failures at $(\mathbf{p}, \mathbf{C}, \mathbf{D})$ as at $(\mathbf{p}, \mathbf{C}', \mathbf{D}')$, and every organization that fails in a cascade at $(\mathbf{p}, \mathbf{C}, \mathbf{D})$ also fails at $(\mathbf{p}, \mathbf{C}', \mathbf{D}')$

- Proposition 2 states that if we integrate cross-holdings via fair trades, so that organizations end up holding more of each other's investments, then we face more failures in any given cascade that begins
- This means the benefits of integration come only via avoiding failure

A Result on Diversification and Integration

- To conduct comparative statics, we consider a random graph model
- Suppose $\mathbf{G}$ is a adjacency matrix of unweighted, directed graph
- $d_i = \sum_j G_{ji}$: out-degree
- $d_i\text{in} = \sum_j G_{ij}$
- We can rewrite C as: for $i \neq j$

$$C_{ij} = \begin{cases} \frac{cG_{ij}}{d_j} & (\text{if } d_j > 0) \\ 0 & (\text{otherwise}) \end{cases}$$

- And, $\hat{C}_{jj} = 1 - c$ if $d_j > 0$, and 1 otherwise

A Result on Diversification and Integration

- In this model, increasing d_j increases diversification but not integration
- Increasing c increases integration and but not diversification

Lemma 2

Suppose that $C_{ij} = cG_{ij}/d_j$ for some adjacency matrix $\mathbf{G}$, with $0 < c < \frac{1}{2}$ and each $d_i \geq 1$. Then A_{ii} is decreasing in c and A_{ij} is increasing in c:

- $\frac{\partial A_{ii}}{\partial c} < 0$ for each i;
- $\frac{\partial A_{ij}}{\partial c} \geq 0$ for all $i \neq j$;
- $\frac{\partial A_{ij}}{\partial c} > 0$ for all $i \neq j$ such that there is an ownership path from i to j in $\mathbf{G}$

Degree Distribution

- Fix a degree distribution $\pi = (\pi_{k,l})$ where π_{kl} is the fraction of nodes that have in-degree k and out-degree l
- Let $\mathcal{G}(\pi, n)$ be the set of all directed graph on n nodes that have degree distribution π
- A random network with degree distribution π is drawn from $\mathcal{G}(\pi, n)$ uniformly at random
- maximum degree: $\bar{d} = \max\{k : \pi_{kl} > 0 \text{ or } \pi_{\mathrm{lk}} > 0 \text{ for some l}\}$
- minimum degree: $\underline{d} = \min\{k : \pi_{kl} > 0 \text{ or } \pi_{\mathrm{lk}} > 0 \text{ for some l}\}$
- average directed degree: d
    - the expected out-degree of the vertex at the end of a link chosen uniformly at random from $\mathcal{G}(\pi, n)$

Degree Distribution

- Suppose $\mathbf{D} = \mathbf{I}$ and $\mathbf{p} = (1, \cdots, 1)$
- Assume $\underline{v}_i = \underline{v} \in (0, 1)$, and set $\beta_i = p_i$
- Define

$$\tilde{v}_{\min} = \frac{1-c}{1 - c\underline{d}/\bar{d}} \text{ and } \tilde{v}_{\max} = \frac{1-c}{1 - c\bar{d}/\max\{\underline{d}, 1\}}$$

  - These serve as lower and upper bounds on organization values respectively.
- Let $f(\pi, n)$ be the expected fraction of organizations that fail if the network is given and one asset value p_i is devalued to 0 , with i selected uniformly at random

Main Result

Proposition 3

If one proprietary asset fails (uniformly at random), a nonvanishing fraction of organizations fail if and only if there are intermediate levels of integration and diversification.

In particular, consider a degree distribution π with associated average directed degree d, maximum degree $\bar{d}$, and minimum degree $\underline{d}$; and let (n_k) be an infinite sequenceof natural numbers such that π is feasible for each n_k.

Main Result

Proposition 3 (cont.)

- The fraction of failures tends to 0 ($f(\pi, n_k) \to 0$) if either of the following condition holds:
  - $d < 1$ (diversification is too low), or
  - $\underline{d} > \frac{c(1-c)}{\tilde{v}_{\min} - \underline{v}}$ (diversification is too high, or integration is too high or low)
- The fraction of failures is nonvanishing ($\liminf_k f(\pi, n_k) > 0$) if both of the following conditions hold:
  - $d > 1$ (diversification is not too low), and
  - $\underline{d} < \frac{c(1-c)}{\tilde{v}_{\min} - \underline{v}}$ (diversification is not too high and integration is intermediate)

Intuition

- If c is very low, then no firm holds enough of its counterparties for contagion to propagate
- If c is very high, then no firm is sufficiently exposed to its own asset for a first failure to happen
- Consider the case c is intermediate
  - If $d < 1$, there are many isolated by components of vanishing size
  - If $d > 1$, there is a giant component of nonvanishing size
  - Thus if $d < 1$, contagion to a positive fraction of organizations following the failure of a single proprietary asset is impossible

Levels of Diversification

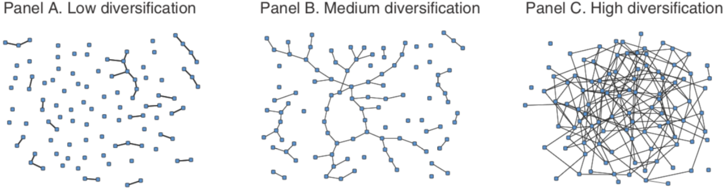

FIGURE 3. EXAMPLE RANDOM NETWORKS (*Plotted here with undirected edges*) FOR DIFFERENT LEVELS OF DIVERSIFICATION

Note: The diagrams demonstrate the transition from panel A (many disconnected components) to panel B (a large component where each node has few neighbors) to panel C (a large component in which each node has many neighbors).

Simulated Random Network

- Assume that
  - $m = n$ and $\mathbf{D} = \mathbf{I}$
  - initial price vector is $p_i = 1$ for all i
  - all organizations have common threshholds $\underline{v}_i = \theta v_i$ for $\theta \in (0, 1)$
  - if an organization fails, it loses its full value $\beta_i = \underline{v}_i$
  - we form a directed random graph so that the expected in- and out-degree are d
- Steps of simulation
  - Generate a directed random network $\mathbf{G}$ with parameter d and caluculate $\mathbf{C}$
  - Calculate v_i and set $\underline{v}_i = \theta v_i$
  - Pick an organization i uniformly at random and drop the value of i's asset to 0
  - Calculate the best-case equilibrium

Results (Diversification)

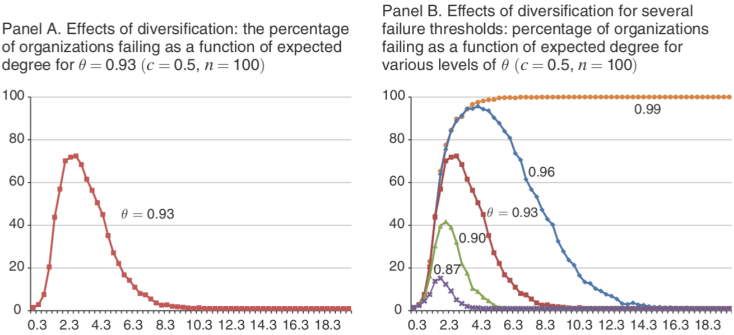

FIGURE 2. HOW DIVERSIFICATION (THE AVERAGE NUMBER OF OTHER ORGANIZATIONS THAT AN ORGANIZATION CROSS-HOLDS) AFFECTS THE PERCENTAGE OF ORGANIZATIONS FAILING (*Averaged over 1,000 simulations*)

Note: The *x*-axis corresponds to diversification in terms of the expected degree in the random network of cross-holdings.

Results (Integration)

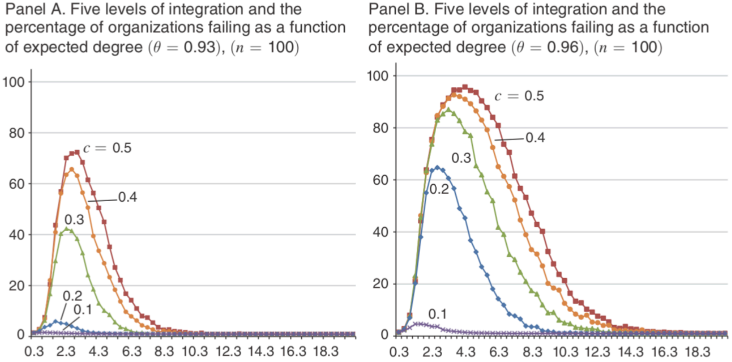

FIGURE 4. HOW INTEGRATION (THE FRACTION c OF A TYPICAL PORTFOLIO HELD BY OTHER ORGANIZATIONS) AFFECTS THE PERCENTAGE OF ORGANIZATIONS FAILING (*Averaged over 1,000 simulations*)

Notes: The x-axis corresponds to the diversification level (the expected degree in the random network of cross-holdings). The two figures work with different failure thresholds and depict how the size of cascades varies with the level of integration c ranging from 0.1 to 0.5.

Integration and First Failure

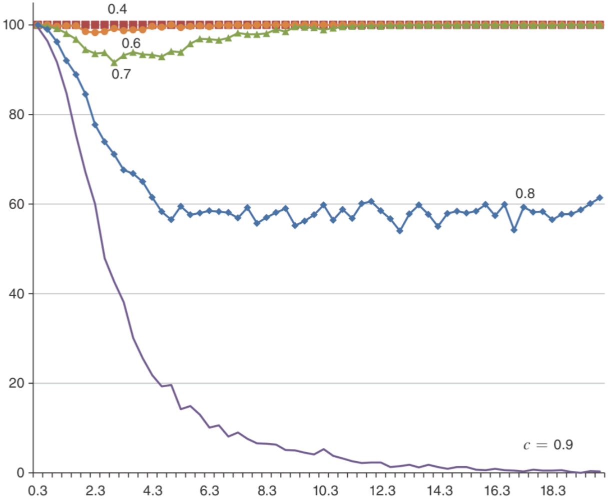

FIGURE 5. HOW INTEGRATION AFFECTS THE PERCENTAGE OF "FIRST FAILURES"

Notes: The percentage of simulations with at least one organization failing, for various levels of integration c from 0.4 to 0.9, with the x-axis tracking diversification (expected degree) in the network. The failure threshold is constant at $\theta = 0.8$.

Illustration with European Debt Cross-Holdings

- We take the fundamental asset owned by each country to be its fiscal stream
- Data on the cross-holdings are for the end of December 2011 from the BIS.
- The data looks at the immediate borrower rather than the final borrower when a bank from a country different from the final borrower serves as an intermediary

Data

- The following matrix is raw cross-holdings matrix
- where the column represents the country whose debt is being held and the row is the country which holds that debt
- We can convert this matrix to cross-matrix $\mathbf{C}$

	(France)	(Germany)	(Greece)	(Italy)	(Portugal)	(Spain)
(France)	0	198,304	39,458	329,550	21,817	115,162
(Germany)	174,862	0	32,977	133,954	30,208	146,096
(Greece)	1,960	2,663	0	444	51	292
(Italy)	40,311	227,813	2,302	0	3,188	26,939
(Portugal)	6,679	2,271	8,077	2,108	0	21,620
(Spain)	27,015	54,178	1,001	29,938	78,005	0

Dependence Matrix

- From this matrix, we can derive dependence matrix $\mathbf{A}$
- The matrix A can be thought as a weighted directed graph

	(France)	(Germany)	(Greece)	(Italy)	(Portugal)	(Spain)
(France)	0.71	0.13	0.13	0.17	0.07	0.11
(Germany)	0.18	0.72	0.12	0.11	0.09	0.14
(Greece)	0.00	0.00	0.67	0.00	0.00	0.00
(Italy)	0.07	0.12	0.03	0.70	0.03	0.05
(Portugal)	0.01	0.00	0.02	0.00	0.67	0.02
(Spain)	0.03	0.03	0.02	0.02	0.14	0.68

Interdependence in Europe

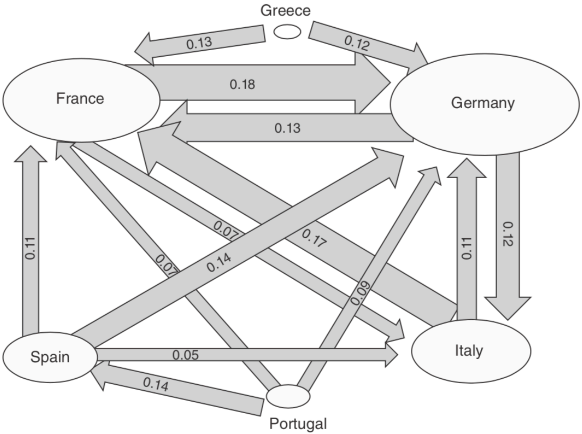

FIGURE 8. INTERDEPENDENCIES IN EUROPE

Notes: The matrix **A**, describing how much each country ultimately depends on the value of others' debt. The widths of the arrows are proportional to the sizes of the dependencies, with dependencies less than 5 percent excluded; the area of the oval for each country is proportional to its underlying asset values.

Initial Values

- Assume $\mathbf{D} = \mathbf{I}$ and $\mathbf{p}$ is proportional to the vector of countries's GDP's
- Normalizing Portugal's 2011 GDP to 1, the initial values in 2011 are $\mathbf{v}_0 = \mathbf{A}\mathbf{p}$

$$\begin{pmatrix} 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix} \cdot \begin{pmatrix} 11.6 \\ 14.9 \\ 1.3 \\ 9.2 \\ 1.0 \\ 6.3 \end{pmatrix} = \begin{pmatrix} 12.7 \ (\text{France}) \\ 14.9 \ (\text{Germany}) \\ 0.8 \ (\text{Greece}) \\ 9.4 \ (\text{Italy}) \\ 0.9 \ (\text{Portugal}) \\ 5.4 \ (\text{Spain}) \end{pmatrix}.$$

Cascades

- The failure threshholds $\underline{v}_i$ are set to θ multiplied by 2008 values
- If a country fails, then the loss in value is $\underline{v}_i/2$, so that half the value of its debt is lost
- Assume Greece is the first failure point

TABLE 1—HIERARCHIES OF CASCADES IN THE BEST-CASE EQUILIBRIUM ALGORITHM, AS A FUNCTION OF THE FAILURE THRESHOLD θ

Value of θ	0.9	0.93	0.935	0.94
First failure	Greece	Greece	Greece	Greece
Second failure			Portugal	Portugal, Spain
Third failure			Spain	France, Germany
Fourth failure			France	Italy
Fifth failure			Germany, Italy	

Source: Authors' calculations

Some Additional Contents

- The authors provides an algorithm which traces the propagation of specific shock that causes one organization to fail. The algotithm terminates at the finite number of steps. In addition, it also provides us with hierarchies of failures, that is, a wave of cascading failure.
- Authors simulated their model with other network structures
  - core-periphery network
  - network with homophily
  - power law degree distribution
- Authors also examined the model with the correlated values

Conclusion

- Diversification and integration are usefully distinguished as they have different effects on financial contagion
- Both diversification and integration entail trade-offs in how they affect contagion
- These trade-offs result in nonmonotonic effects where middle ranges are the most dangerous with respect to cascades of failures
- These approach can be used to inform policy, for example, counterfactual scenarios can be run using the algorithm