Endogenous Production Networks

Acemoglu and Azar(2018)

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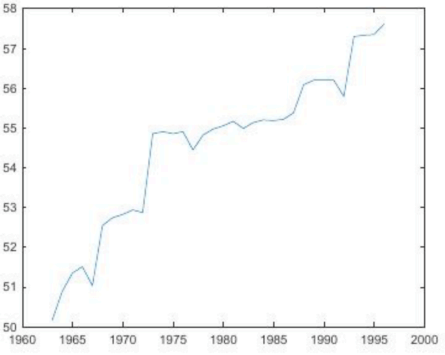
Motivation



- Supply chains today become more complex than in the past
  - agricultural production, automobile, telecommunication, ...
  - We can see this pattern both in micro level and macro level
- Questions
  - What explains the different structure of input usage over time and across countries?
  - Do these differences contribute to productivity abd growth differences across these economies?

Main Results

- Existence and uniqueness of equilibrium, and its efficiency properties
- Comparative static results
  - technology improvement $\rightarrow$ all prices in economy decrease
  - technology improvement with more conditions $\rightarrow$ expansion in the set of input for all industries
  - discontinuous change in network structure
- In dynamic model, the economy achieves sustained growth in the long run
- Cross sectional implications
  - limited inequality in the number of suppliers(indegree)
  - inequality in the number of customers(outdegree)

Model

- $\mathcal{N} = \{1, \cdots, n\}$: industries
- Production technology

$$Y_i = F_i(S_i, A_i(S_i), L_i, X_i)$$

  - $S_i \subset \{1, \cdots, n\} \setminus \{i\}$: the set of (endogeneous) suppliers, **technology choice**
  - $X_i = \{X_{ij}\}_{j \in S_i}$: vector of intermediate goods
  - $A_i(S_i)$: the productivity of technology by the use of inputs in the set S_i
  - L_i : the amount of labor used

Assumptions

- Each industry is **contestable**
  - a large number of firms have access to the same technology
  - $\rightarrow$ equilibrium profits are always equal zero
- F_i does not depend on X_{ij} for $j \notin S_i$

Assumption 1

For each $i = 1, 2, \cdots, n$, $F_i(S_i, A_i(S_i), L_i, X_i)$ is strictly quasi-concave, exhibits constant return to scale in (L_i, X_i), and is increasing and continuous in $A_i(S_i), L_i$ and X_i, and strictly increasing in $A_i(S_i)$ when $L_i > 0$ and $X_i > 0$. Moreover, labor is an essential factor of production in the sense that $F_i(0, \cdot, \cdot, \cdot) = 0$

Household

- Utility function of representative household is:

$$u(C_1, \cdots, C_n)$$

- Household supplies labor inelastically
- We choose the wage as the numeraire: $W = 1$

Assumption 2

$u(C_1, \cdots, C_n)$ is continuous, differentiable, increasing and strictly quasi-concave, and all goods are normal.

Distortions

- Industry i is subject to a distortions of $\mu_i \geq 0$, modeled as an effective ad valorem tax
- A fraction λ_i of the revenues generated by distortions from industry i are distributed back to the representative household and the rest are waste
- That is, the budget constraint of representative household can be written:

$$\sum_{i=1}^{n} P_i C_i \leq 1 + \sum_{i=1}^{n} \Lambda_i$$

where $\Lambda_i = \lambda_i \frac{\mu_i}{1+\mu_i} P_i Y_i$

Cost Minimization

- Cost minimization problem follows two steps
- First step: determine the unit cost function

$$K_i(S_i, A_i(S_i), P) = \min_{X_i, L_i} \{L_i + \sum_{j \in S_i} P_j X_{ij}\}$$

$$\text{subject to } F_i(S_i, A_i(S_i), L_i, X_i) = 1$$

- Choose technologies to minimize K_i

$$S_i^* \in \arg\min_{S_i} K_i(S_i, A_i(S_i), P)$$

- Note that K_i is strictly decreasing and continuous in A_i

Definition of Equilibrium

An equilibrium is a tuple $(P^*, S^*, C^*, L^*, X^*, Y^*)$ such that

- **Contestability** :
  For each $i = 1, 2, \cdots, n$, $P_i^* = (1 + \mu_i) K_i(S_i^*, A_i(S_i^*), P^*)$
- **Consumer maximization** :
  The consumption vector C^* maximizes household utility subject to budget constraint given prices P^*
- **Cost miinimization** :
  For each $i = 1, 2, \cdots, n$, factor demands L^* and X_i^* are the solution of cost miinimization problem, and the technology choice S_i^* is a solution to minimization of unit cost function given the price vector P^*
- **Market clearing** : For each $i = 1, 2, \cdots, n$,

$$C_i^* + \sum_{j=1}^{n} X_{ji}^* = (1 - (1 - \lambda_i)\frac{\mu_i}{1 + \mu_i})Y_i^*$$

$$Y_i^* = F_i(S_i^*, A_i^*(S_i^*), L_i^*, X_i^*)$$

$$\sum_{j=1}^{n} L_j^* = 1$$

Example of Production Technologies

Cobb-Douglas production functions with Hicks-neutral technology

$$F_i(S_i, A_i(S_i), L_i, X_i) = \frac{1}{(1-\sum_{j\in S_i}\alpha_{ij})^{1-\sum_{j\in S_i}\alpha_{ij}}\prod_{j\in S_i}\alpha_{ij}^{\alpha_{ij}}} A_i(S_i) L_i^{1-\sum_{j\in S_i}\alpha_{ij}} \prod_{j\in S_i} X_{ij}^{\alpha_{ij}}$$

Family of Cobb-Douglas function satisfies Assumption 1

Let $p_i = \log P_i$ and $a_i = \log A_i$. We can show that

- log productivity

$$k_i(S_i, a_i(S_i), p) = -a_i(S_i) + \sum_{j\in S_i}\alpha_{ij}p_j$$

- equilibrium log price

$$p_i^* = \log(1+\mu_i) + \sum_{j\in S_i}(\alpha_{ij} + p_j^*) - a_i$$

- in a matrix form

$$\begin{aligned} p^* &= -(I-\alpha(S^*))^{-1}(\alpha(S^*) - \log(1+\mu)) \\ &= -\mathcal{L}(S^*)(\alpha(S^*) - m) \end{aligned}$$

Existence of Equilibrium

Lemma 1

Suppose Assumption 1 and 2 hold. Then given an exogenous network S_i, $P^* > 0$ is an equilibrium price vector if and only if $P_i^* = (1 + \mu_i)K_i(S_i^*, A_i(S_i^*), P^*)$ holds for each $i = 1, 2, \cdots, n$. [Proof]

Theorem 1 (Existence)

Suppose Assumption 1 and 2 hold. Then an equilibrium $(P^*, S^*, C^*, L^*, X^*, Y^*)$ exists. [Proof]

Uniqueness of Equilibrium

Let $A_i = (A_i(\emptyset), A_i(\{1\}), \cdots, A_i(\{1, \cdots, n\} \setminus \{i\})) \in \mathbb{R}^{l \times 2^{n-1}}$ and $A = (A_1, \cdots, A_n) \in \mathbb{R}^{n \times l \times 2^{n-1}}$

Definition 2 (Genericity)

The equilibrium network is generically unique if the set

$$\mathcal{A} = \{A : \text{There exist at least two distinct equilibrium networks } S^*, S^{**}\}$$

has Lebesgue measure zero in $\mathbb{R}^{n \times l \times 2^{n-1}}$

Theorem 2 (Uniqueness)

Suppose Assumption 1 and 2 hold. Then an equilibrium prices P^* and quantities C^*, L^*, X^* and Y^* are uniquely determined, and the equilibrium network S^* is generically unique. [Proof]

Efficiency properties

Theorem 3 (**Efficiency**)

Suppose Assumption 1 and 2 hold. Suppose also that the production function F_i is differentiable for each $i = 1, \cdots, n$

1. If $\mu_i = 0$ for all $i = 1, \cdots, n$ so that all distortions are equal to zero, then the equilibrium is Pareto efficient.
2. If $\mu_i = \mu_0 > 0$ and $\lambda_i = 1$ for all $i = 1, \cdots, n$ and $(\emptyset, \cdots, \emptyset)$ is the unique Pareto efficient production network, then the equilibrium is Pareto efficient.
3. If $\mu_i = \mu_0 > 0$ and $\lambda_i = 1$ for all $i = 1, \cdots, n$ and $(\emptyset, \cdots, \emptyset)$ is not a Pareto efficient production network, then the equilibrium is not Pareto efficient.
4. If there exist i and i' such that $\mu_i > 0$ and $\mu_i \neq \mu_{i'}$ or there exists i such that $(1 - \lambda_i)\mu_i > 0$, then the equilibrium is not Pareto efficient.

Comparative Statics

- Direct Effect
  - $A_i(S_i)$ increases, then industry i reduces its unit cost because it has access to better technology
- Indirect Effect
  - industry i's price is lower
  - $\rightarrow$ customers will face lower unit cost
  - $\rightarrow$ their customers also will face lower unit cost
  - $\rightarrow$ ...
- Change structure of the network
  - industry i's price decreases
  - $\rightarrow$ other industries are more likely to adopt it as a supplier

Comparative Statics for Prices

Theorem 4

Suppose Assumptions 1 and 2 hold. Consider a shift in technology from A to $A'(\geq A)$ and/or a decline in distortions from μ to $\mu'(\leq \mu)$, and let P^* and P^{**} be the respective equilibrium price vectors. Then, $P^{**} \leq P^*$ [Proof]

Comparative Statics for Technology Choices (Network)

Definition 3 (Positive technology shock)

A change from A to A' is a positive technology shock if

1. (higer level) $A' \geq A$
2. (quasi-submodularity) for each $i = 1, 2, \cdots, n$, and for all P, $K_i(S_i, A_i(S_i), P)$ is quasi-submodular in $(S_i, A_i(S_i))$

[Def]

Definition 4 (Technology-price single-crossing condition)

For each $i = 1, 2, \cdots, n$, $K_i(S_i, A_i(S_i), P)$ satisfies the technology-price single-crossing condition in the sense that for all sets of inputs S_i, S_i' with $S_i \subset S_i'$ and all prices vectors P, P' with $P'_{-i} \leq P_{-i}$, we have

$$K_i(S_i', A_i(S_i'), P) - K_i(S_i, A_i(S_i), P) \leq 0$$
$$\Rightarrow K_i(S_i', A_i(S_i'), P') - K_i(S_i, A_i(S_i), P') \leq 0$$

Comparative statics of the production network

Lemma 2

Suppose that for each $i = 1, 2, \cdots, n$, $K_i(S_i, A_i(S_i), P)$ is quasi-submodular in $(S_i, A_i(S_i))$. Then for each $i = 1, 2, \cdots, n$, and for all P and for all $S_i \subset S_i'$, we have

$$K_i(S_i', A_i(S_i'), P) - K_i(S_i, A_i(S_i), P) \leq 0$$

$$\Rightarrow K_i(S_i', A_i'(S_i'), P) - K_i(S_i, A_i'(S_i), P) \leq 0$$

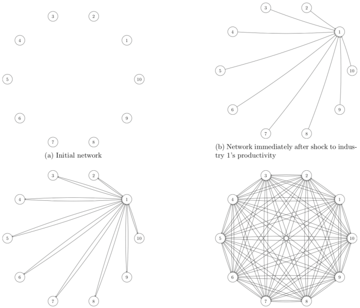
[Proof]

Theorem 5

Suppose Assumptions 1 and 2 and the technology-price single-crossing condition hold. Then a positive technology shock or a decrease in distortions (weakly) increases the equilibrium network from S^* to S^{**}
[Proof]

Discontinuous Effects

- Samall changes in productivity can lead to a large change in GDP and the equilibrium production network.
- Example 2 shows that small change in technology can make larger GDP
- Example 3 shows that small change in industry 1's technology alters production network significantly



(a) Initial network

(b) Network immediately after shock to industry 1's productivity

(c) Network after all industries choose industry 1 as a supplier

(d) Network after the new equilibrium is reached

Growth with Endogeneous Production Networks

- Countably infinite period : $t \in \{1, 2, \cdots\}$
- At each time t, a new product arrives in the economy
- All endogeneous variables are indexed by t : $P_i(t)$, $L_i(t)$, $Y_i(t)$ ...
- Assume $W(t) = 1$ for all t

Assumption 1'

Production functions are in the Cobb-Douglas family with Hicks-neutral technologies

Assumption 3

There exists $\mu_0 < \infty$ such that $\sup \{\mu_t\}_{t=1}^{\infty} \leq \mu_0$

Preference in Growth Setting

Assumption 2'

The time-t preference of the representative household take a Cobb-Douglas form,

$$u(C_1(t), \cdots, C_t(t), \beta) = \left[\prod_{i=1}^{t} \left(\frac{\beta_i}{\sum_{i=1}^{t} \beta_i} \right)^{-\beta_i} \prod_{i=1}^{t} C_i(t)^{\beta_i} \right]^{\frac{1}{\sum_{i=1}^{t} \beta_i}}$$

where the vector β satisfies $\beta_t \geq 0$ for all t and $\sum_{t=1}^{\infty} \beta_t = 1$

- The overall utility is given by a discounted sum of time-t preferences
- This specification implies that $\lim_{t \to \infty} \beta_t = 0$

Growth Rate

- Nominal GDP is given by

$$Y^N(t) = \sum_{i=1}^{t} P_i(t)C_i(t) = 1 + \sum_{i=1}^{t} \lambda_i \frac{\mu_i}{1+\mu_i} P_i(t)Y_i(t)$$

- Real GDP which is equal to the HH's utility is given by

$$Y(t) = \frac{Y^N(t)}{\prod_{i=1}^{t} P_i(t)^{\frac{\beta_i}{\sum_{j=1}^{t}\beta_j}}}$$

- Define the asymptotic growth rate of real GDP as:

$$g^* := \lim_{t\to\infty}\left(\frac{\log Y(t)}{t}\right) = \lim_{t\to\infty}\left(-\frac{\pi(t)}{t}\right)$$

  - where $\pi(t) = \sum_{i=1}^{t} \frac{\beta_i}{\sum_{j=1}^{t}\beta_j} p_i(t)$
  - last equality is shown as Lemma 3

Additional Assumptions

Assumption 4

For a fixed t and $i \in \{1, \cdots, t\}$, the log productivity vector $a_i(t) = \{a_i(S_i, t)\}_{S_i \subset \{1,\cdots,t\} \setminus \{i\}}$ is drawn from a distribution $\Phi_i(t)$. Furthermore, there exists a constant $D > 0$ such that, if $\{a_i(t)\}_{t \in \mathbb{N}}$ is a sequence of log productivity vectors for industry i, then

$$\lim_{t \to \infty} \max_{S_i \subset \{1,\cdots,n\} \setminus \{i\}} \frac{a_i(S_i, t)}{t} = D \textit{ almost surely}$$

- This rules out too thin or too thick tail of the distribution a_i

Assumption 5

1. There exists $\theta < 1$ such that $\sum_{j=1}^{\infty} \alpha_{ij} \leq \theta$ for all $i \in \mathbb{N}$
2. Furthermore, for every $\epsilon > 0$, there exists a constant T such that for all $i \in \mathbb{N}, \sum_{j=T}^{\infty} \alpha_{ij} \leq \epsilon$

- Labor is essential input and shares of inputs after T is bounded.

Sustained Growth

Theorem 6

Suppose that Assumptions 1', 2', 3, 4 and 5 hold, and let $D > 0$ be as defined in Assumption 4. Each industry chooses its set of suppliers $S_i^*(t) \subset \{1, \cdots, t\} \backslash \{i\}$. Then for each $i = 1, 2, \cdots, t$, the equilibrium log price vector $p^*(t)$ satisfies

$$\lim_{t \to \infty} -\frac{p_i^*(t)}{t \sum_{j=1}^{t} \mathcal{L}_{ij}} = D > 0 \; almost \; surely$$

and thus

$$g^* = D \sum_{i,j=1}^{\infty} \beta_i \mathcal{L}_{ij} > 0 \; almost \; surely$$

[Proof]

- When firms can choose their input suppliers in an unrestricted fashion, the economy achives sustained growth
- When we restrict the choice of inputs , there is no longer sustained growth

Generalization

- We can have sustained growth even when some assumptions are relaxed
  - A subset of industries can choose their suppliers (Corollary 1)
  - The number of products is function of t (Corollary 2)
  - Relax Assumption 4 and the second part of Assumption 5 (Corollary 3)
  - Not Cobb-Douglas production functions, in particular, continuously differentiable and Hicks-neutral technologies (Theorem 7)

Alternative Stories

- Essential Inputs (Theorem 8)
  - Some agricultural products need for food manufacturing
  - Various restrictions on combination of inputs can be imposed with sustained growth
- Creative Destruction (Theorem 9)
  - New products replace older ones in either consumptions or production or in both
  - Sustained growth is possible in an environment in which new inputs replace old ones

Cross-Sectional Implications

- Consider static economy again with large n
- Assume a_i's are random variable

Assumption 4'

Log-productivities are given by $a_i(S_i) = \sum_{j \in S_i} b_j + \epsilon(S_i)$, where $\epsilon(S_i)$ is an (independent) drawn from a Gumbel distribution with cdf $\Phi(x;\sigma) = e^{-e^{-x/\sigma}}$ for each $S_i \in \{1, 2, \cdots\} \setminus \{i\}$

Assumption 5'

Suppose that Assumption 5 holds. In addition, for every industry j, the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_{ij}$ of average exogenous outdegrees always exists

Definitions of Indegrees and Outdegrees

- $\{\mathcal{E}(n)\}_{n=1}^{\infty}$: a sequence of economies
- $S(n)$: the equilibrium network in economy $\mathcal{E}(n)$
- (Normalized) indegree of industry i in $\mathcal{E}(n)$

$$\mathcal{I}_i(n) = \frac{1}{n} \sum_{j=1}^{n} \alpha_{ij}(S(n))$$

  - $\mathcal{I}(n) = \{\mathcal{I}_i(n)\}_{i=1}^{n}$: sequence of (normalized) indegrees
- (Normalized) outdegree of industry i in $\mathcal{E}(n)$

$$\mathcal{O}_j(n) = \frac{1}{n} \sum_{i=1}^{n} \alpha_{ij}(S(n))$$

  - $\mathcal{O}(n) = \{\mathcal{O}_i(n)\}_{i=1}^{n}$: sequence of (normalized) outdegrees
- Both $\mathcal{I}(n)$ and $\mathcal{O}(n)$ are random variables over $\mathbb{R}^n$ and can be interpreted as elements of l^{∞}

Indegrees and Outdegrees

Theorem 10

Write $\alpha_j = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_{ij}$. Suppose Assumption 1', 4', and 5' hold. Then:

1. $\mathcal{I}(n)$ converges uniformly and almost surely to a degenerate distribution at $0 \in l^\infty$
2. $\bar{\mathcal{O}} = \limsup_{n\to\infty} \mathcal{O}(n)$ is a non-degenerate distribution and $\bar{\mathcal{O}}_j \le \alpha_j$ for all j
3. $\underline{\mathcal{O}} = \liminf_{n\to\infty} \mathcal{O}(n)$ is a non-degenerate distribution and $\underline{\mathcal{O}}_j \ge \alpha_j \frac{e^{b_j}}{1+e^{b_j}}$ for all j

- Theorem 10 implies that the distribution of outdegrees ($\#$ of customers) will be much more unequal than the distribution of indegrees ($\#$ of suppliers)
- When the distribution of α_{ij} can be approximated by a power law distribution, $\mathcal{O}_j$ also has a power raw distribution (Corollary 4)

Estimating the contribution of new input combinations

- Use US data of input-output, capital stock, employment ...
- A sizable contribution to aggregate TFP growth from changes in the input-output matrix
- Quite large gains in industry productivity from new input combinations, which might account for the majority of aggregate TFP growth over the last several decades

Conclusion

- Economic tradeoffs and comparative statics
  - Change in technologies $\rightarrow$ all prices decrease
  - (Under some conditions) Change in technologies $\rightarrow$ expands technology sets
  - A small change can cause large change in GDP or structure of network
- In dynamic setting, choosing suppliers forces towards sustained economic growth

Proof of Lemma 1

"only if" part

- Suppose P^* is a vector of equilibrium prices. Then from the contestability condition, we have

$$P_i^* = (1 + \mu_i) K_i(S_i, A_i(S_i), P^*)$$

"if" part

- Suppose that $P_i^* = (1 + \mu_i) K_i(S_i, A_i(S_i), P^*)$ for each $i = 1, \cdots, n$
- Let X_i^* and L_i^* be the solutions to the cost minimization problem for given P^*
- Let x_{ij}^* and l_i^* be the unit requirements
- From the quasi-concavity of F_i, these are uniquely determined
- Since F_i exhibits constant return to scale, these are also independent of Y_i^*
- Clearly, $X_{ij}^* = x_{ij}^* Y_i^*$ and $L_i^* = l_i^* Y_i^*$

- $Y^N = 1 + \sum_{i=1}^n \lambda_i \frac{\mu_i}{1+\mu_i} P_i^* Y_i^*$: Income of HH
- $C_j^* = C_j^*(Y^N, P^*)$: Optimal consumption of good j
- From the market clearing condition, we have

$$C_j^* + \sum_{i=1}^n x_{ij}^* Y_i^* = (1 - (1-\lambda_j)\frac{\mu_j}{1+\mu_j})Y_j^*$$

$$P_j^* C_j^* + \sum_{i=1}^n \frac{P_j^*}{P_i^*} x_{ij}^* P_i^* Y_i^* = (1 - (1-\lambda_j)\frac{\mu_j}{1+\mu_j})P_j^* Y_j^*$$

$$\hat{C_j}^* + \sum_{i=1}^n \frac{P_j^*}{P_i^*} x_{ij}^* \hat{Y_i}^* = (1 - (1-\lambda_j)\frac{\mu_j}{1+\mu_j})\hat{Y_j}^*$$

$$\hat{C_j}^* + \sum_{i=1}^n \frac{P_j^*}{P_i^*} x_{ij}^* \frac{1}{1-(1-\lambda_i)\frac{\mu_i}{1+\mu_i}} \tilde{Y_i}^* = \tilde{Y_j}^*$$

- In a matrix form, $\hat{C}^* + \bar{X}'\tilde{Y}^* = \tilde{Y}^*$
  - $\bar{X}$ is a $n \times n$ matrix, (i, j) th element is $\bar{X_{ij}} = \frac{1}{1-(1-\lambda_i)\frac{\mu_i}{1+\mu_i}} \frac{P_j^*}{P_i^*} x_{ij}^*$
- Since $\hat{C}^*$ depends on $\tilde{Y}^*$ and P, we have

$$\tilde{Y}^* = \hat{C}^*(\tilde{Y}^*, P) + \bar{X}'\tilde{Y}^* \equiv \Phi(\tilde{Y}^*)$$

- Since the utility function is differentiable, $\hat{C}^*(\tilde{Y}^*, P)$ and thus $\Phi(\tilde{Y}^*)$ are also differentiable
- Denote the Jacobian of $\Phi(\tilde{Y}^*)$ by J and we have

$$J_{ij} = \frac{\partial \hat{C_i}^*}{\partial \tilde{Y_j}^*} + \frac{1}{1 - (1 - \lambda_i)\frac{\mu_i}{1+\mu_i}} \frac{P_i^*}{P_j^*} x_{ji}^* \geq 0$$

last inequality comes from Assumption 2, all goods are normal

- Note that HH's budget constraint is

$$\sum_{i=1}^{n} \hat{C}_i^{\,*} = 1 + \sum_{i=1}^{n} \lambda_i \frac{\mu_i}{1+\mu_i} \hat{Y}_i^{\,*}$$

- Differentiate w.r.t $\tilde{Y}_j^{\,*}$, we obtain

$$\sum_{i=1}^{n} \frac{\partial \hat{C}_i^{\,*}}{\partial \tilde{Y}_j^{\,*}} = \frac{\lambda_j \mu_j}{1+\lambda_j \mu_j}$$

- Rearranging assumption, we have

$$\frac{1}{1+\mu_j} = \frac{K_j(S_j, A_j(S_j), P^*)}{P_j^*} = \sum_{i=1}^{n} \frac{l_j^* + P_i^* x_{ji}^*}{P_j^*} > \sum_{i=1}^{n} \frac{P_i^* x_{ji}^*}{P_j^*}$$

Last inequality comes from the fact that labor is an essential input

- We obtain

$$
\begin{aligned}
\sum_{i=1}^{n} J_{ij} &= \sum_{i=1} \left(\frac{\partial \hat{C}_i^*}{\partial \tilde{Y}_j^*} + \frac{1}{1-(1-\lambda_j)\frac{\mu_j}{1+\mu_j}} \frac{P_i^*}{P_j^*} x_{ji}^* \right) \\
&< \left(\lambda_j \frac{\mu_j}{1+\lambda_j \mu_j} + \frac{1}{1-(1-\lambda_j)\frac{\mu_j}{1+\mu_j}} \frac{1}{1+\mu_j} \right) = 1
\end{aligned}
$$

- Thus, we have $\|J\|_1 = \max_j \sum_{i=1}^{n} J_{ij} < 1$
- By definition of matrix norm, for any $\tilde{Y}^*$ and $\tilde{Y}^{**}$

$$
\|\Phi(\tilde{Y}^*) - \Phi(\tilde{Y}^{**})\|_1 \leq \|J\|_1 \|\tilde{Y}^* - \tilde{Y}^{**}\|_1
$$

- Since $\|J\|_1 \leq 1$, this implies that $\Phi(\tilde{Y}^*)$ is a contraction, and thus given P^*, there exists a unique fixed point $\tilde{Y}^*$ of Φ
- Furthermore, all equilibrium quantities are determined from fixed point
- Thus, given S, a price vector P^* that satisfies contestability condition uniquely define equilibrium

[Back]

Proof of Theorem 1

- Let $\kappa(P) = ((1+\mu_1)\min_{S_1} K_1(S_1, A_1(S_1), P), \cdots, (1+\mu_n)\min_{S_n} K_n(S_n, A_n(S_n), P))$
- In the proof, we use following lemma

Lemma A1

Let $\mathbb{L} = \{P \geq 0 : P_i = (1+\mu_i)\min_{S_i} K_i(S_i, A_i(S_i), P)\}$. Then, $\mathbb{L}$ is a non-empty complete lattice with respect to the operations $P \wedge Q = (\min(P_1, Q_1), \cdots, \min(P_n, Q_n))$, $P \vee Q = (\max(P_1, Q_1), \cdots, \max(P_n, Q_n))$

Proof of Lemma A1

- Let $\mathbb{O} = \{(x_1, \cdots, x_n) : x_i \geq 0\}$ and then by definition $\kappa : \mathbb{O} \to \mathbb{O}$
- Define $\bar{P}_i = (1+\mu_i) K_i(\emptyset, A_i(\emptyset), \{P_j\}_{j \in \emptyset})$
- We have $\kappa(P) \leq (\bar{P}_1, \cdots, \bar{P}_n)$ for any P
- Define $\underline{P_i} = \kappa_i(0) = (1+\mu_i) \min_{S_i} K_i(S_i, A_i(S_i), 0)$
- We have $\kappa(P) \geq \kappa(0) = (\underline{P_1}, \cdots, \underline{P_n})$
- Then, $\tilde{\mathbb{O}} = \times_{i=1}^{n} [\underline{P_i}, \bar{P}_i]$ is a complete lattice
- And, $\kappa : \tilde{\mathbb{O}} \to \tilde{\mathbb{O}}$
- If $P' \leq P$, for any i and S_i, we have

$$(1+\mu_i) K_i(S_i, A_i(S_i), P') \leq (1+\mu_i) K_i(S_i, A_i(S_i), P)$$

- Taking minima on both sides,

$$(1+\mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P') \leq (1+\mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P)$$

Proof of Lemma A1 (cont)

- This implies $\kappa(P') \leq \kappa(P)$
- So, κ is order-preserving function
- Since $\tilde{\mathbb{O}}$ is a complete lattice, from the Tarski's Fixed Point Theorem, the set of fixed points of κ, $\mathbb{L}$, is a complete lattice.

Tarski's Fixed Point Theorem

Let L be a complete lattice and let $f : L \to L$ be an order-preserving function. Then, the set of fixed points of f in L is also complete lattice.

- From Lemma A1, $\mathbb{L}$ is a non-empty complete lattice, κ has fixed points and a smallest fixed point
- Such fixed point satisfies $P_i^* = (1+\mu_i)K_i(S_i^*, A_i(S_i^*), P^*)$ and $S_i^* \in \arg\min_{S_i} K_i(S_i, A_i(S_i), P^*)$
- From the same argument as in Lemma 1, there exists unique equilibrium quantities X^*, L^* and C^*
- Thus, $(P^*, S^*, C^*, L^*, X^*, Y^*)$ is an equilibrium

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Proof of Theorem 2

- Let P^* be the minimal element of $\mathbb{L}$ and P^{**} be the vector in $\mathbb{L}$ such that $P^{**} > P^*$
- Since $K_i(S_i, A_i(S_i), P)$ is concave in P given S_i, $\kappa_i(P) = (1 + \mu_i)\min_{S_i} K_i(S_i, A_i(S_i), P)$ is also concave
- Let $\nu \in (0, 1)$ be such that $\nu P^{**} \le P^*$, with at least some $r = 1, \cdots, n$ such that $\nu P_r^{**} = P_r^*$

$$\begin{aligned}
\kappa_r(P^*) - P_r^* &\ge \kappa_r(\nu P^{**}) - \nu P_r^{**} \\
&\ge (1-\nu)\kappa_r(0) + \nu\kappa_r(P^{**}) - \nu P_r^{**} \\
&\ge (1-\nu)\kappa_r(0) \\
&> 0
\end{aligned}$$

- This contradicts that P^* is a fixed point
- This establishes the uniqueness of equilibrium prices, and then the uniqueness of equilibrium allocations

- Next, we show the equilibrium network is generically unique
- Let $S^* \neq S^{**}$ be two arbitrary networks
- Let $\mathcal{A}(S^*, S^{**}) = \{A : S^* \text{and} S^{**} \text{are both equilibrium networks}\}$
- Define

$$\Delta_i(S_i^*, S_i^{**}, A) = (1+\mu_i)K_i(S_i^*, A_i(S_i^*), P^*) - (1+\mu_i)K_i(S_i^{**}, A_i(S_i^{**}), P^*)$$

- Note that for all parameters $A \in \mathcal{A}(S^*, S^{**})$ and each $i \in \{1, \cdots, n\}$, we have $\Delta_i(S_i^*, S_i^{**}, A) = 0$
- Since $S^* \neq S^{**}$, there is at least i such that $S_i^* \neq S_i^{**}$
- If we keep $A_{i,-S_i^*}$ and $A_{i,-1}(S_i^*)$ constant, then $\Delta_i(S_i^*, S_i^{**}, A)$ is continuous and strictly decreasing in $A_{i,1}(S_i^*)$
- This implies that there exists a unique value of $A_{i,1}(S_i^*)$ that satisfies $\Delta_i(S_i^*, S_i^{**}, A) = 0$
- Hence $\mathcal{A}(S^*, S^{**}) = \{A : \Delta_i(S_i^*, S_i^{**}, A) = 0 \text{ for all } i\}$ has measure zero
- The equilibrium network is generically unique

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Proof of Theorem 4

- Let $P^0 = P^*$ and $S^0 = S^*$ be the initial vector of equilibrium prices and networks
- P^0 satisfies $P_i^0 = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P^0)$ for all i
- Suppose that $A_i(\cdot)$ increases to $A'(\cdot)$, and define P^1 so that $P_i^1 = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i'(S_i), P^0)$
- Since K_i is decreasing in A_i, we have

$$P_i^1 = (1+\mu_i) \min_{S_i} K_i(S_i, A_i'(S_i), P^0) \leq (1+\mu_i) \min_{S_i} K_i(S_i, A_i(S_i), P^0) = P_i^0$$

- Define $\kappa_i(P) = (1 + \mu_i) \min_{S_i} K_i(S_i, A_i'(S_i), P)$
- For $t \geq 1$,define $P^t = \kappa(P^{t-1})$
- Since κ is increasing in P, we have $\lim_{t\to\infty} P^t \leq P^1 \leq P^0 = P^*$
- Since κ is continuous, $\lim_{t\to\infty} P^t$ is a fixed point of κ
- Since P^{**} is the minimal fixed point, we must have

$$P^{**} \leq \lim_{t\to\infty} P^t \leq P^0 = P^*$$

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Proof of Lemma 2

- Let $i = 1, \cdots, n$ and let $S_i \subset S'_i$, $A_i \leq A'_i$
- Let $\mathcal{X} = (S_i, A'_i)$ and $\mathcal{Y} = (S'_i, A_i)$
- Product of lattice ordering : $\mathcal{X} \vee \mathcal{Y} = (S'_i, A'_i)$, $\mathcal{X} \wedge \mathcal{Y} = (S_i, A_i)$
- Suppose that $K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) \leq 0$
- In lattice notation, $K_i(\mathcal{Y}) \leq K_i(\mathcal{X} \wedge \mathcal{Y})$
- From quasi-submodularity of K_i, $K_i(\mathcal{X} \vee \mathcal{Y}) \leq K_i(\mathcal{X})$
- This is equivalent to $K_i(S'_i, A'_i(S'_i), P) - K_i(S_i, A'_i(S_i), P) \leq 0$
- Thus, we have

$$K_i(S'_i, A_i(S'_i), P) - K_i(S_i, A_i(S_i), P) \leq 0$$

$$\Rightarrow K_i(S'_i, A_i(S'_i), P') - K_i(S_i, A_i(S_i), P') \leq 0$$

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Proof of Theorem 5

- Let $S^0 = S^*$ be the initial equilibrium network
- Note that S^0 satisfies $S_i^0 = \arg\min_{S_i}(1+\mu_i)K_i(S_i, A_i(S_i), P^*)$ for all i
- Suppose that the shift from $A_i(\cdot)$ to $A_i'(\cdot)$ is a positive shock
- Define $S_i^1 \in \arg\min_{S_i}(1+\mu_i)K_i(S_i, A_i'(S_i), P^*)$
- From Milgrom and Shannon(1994), $S_i^0 \subset S_i^1$

Theorem 4 in Milgrom and Shannon (1994) [1]

Let $f : X \times T \to \mathbb{R}$, where X is a lattice and T is a partially ordered set and $S \subset X$. Then $\arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in (t, S) if and only if f is quasi-supermodular in x and satisfies the single crossing property in $(x; t)$

[1] "Monotone Comparative Statics", Milgrom and Shannon, 1994, Econometrica

- Define $\kappa(P) = (1+\mu_i)\min_{S_i} K_i(S_i, A_i'(S_i), P)$
- Let $P^0 = P^*$ and define $P^t = \kappa(P^{t-1})$ for $t \geq 1$
- From the proof of Theorem 4, P^t is decreasing sequence with $P^{**} \leq P^*$
- Applying Milgrom and Shannon (1994), we have

$$\begin{aligned} S_i^* &= S_i^0 \\ &\subset S_i^1 \\ &= \arg\min_{S_i}(1+\mu_i)K_i(S_i, A_i'(S_i), P^*) \\ &\subset \arg\min_{S_i}(1+\mu_i)K_i(S_i, A_i'(S_i), P^{**}) \\ &= S_i^{**} \end{aligned}$$

- We conclude that $S^* \subset S^{**}$

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Proof of Theorem 6

- Let $\epsilon > 0$ and $T(\epsilon)$ be such that for $i \in \mathbb{N}$, $\sum_{j=T(\epsilon)}^{\infty} \alpha_{ij} \leq \epsilon$
- Assumption 5 implies that if $\{1, \cdots, T(\epsilon)\} \subset S_i$ for all i, we have $\sum_{j=1}^{t} \alpha_{ij}(S) \geq \sum_{j=1}^{t} \alpha_{ij} - \epsilon$
- We use the following lemma

Lemma A2

Let α and β be non-negative $n \times n$ matrices. Let $A = (I - \alpha)^{-1}$ and $B = (I - \beta)^{-1}$. If

- $\|\alpha\|_{\infty} \leq \theta$, $\|\beta\|_{\infty} \leq \theta$ for some $\theta < 1$, and
- $\sum_{j=1}^{n} \beta_{ij} \geq \sum_{j=1}^{n} \alpha_{ij} - \epsilon$ for every row i

then $\sum_{j=1}^{n} B_{ij} \geq \sum_{j=1}^{n} A_{ij} - \frac{1}{(1-\theta)^2}\epsilon$ for every row i

Proof of Lemma A2

- Let α_{ij}^l be the (i,j) th element of the matrix α^l
- Note $A = \sum_{l=0}^{\infty} \alpha^l$, $B = \sum_{l=0}^{\infty} \beta^l$ and $\sum_{l=1}^{\infty} l\theta^{l-1} = \frac{1}{(1-\theta)^2}$
- It suffices to show that for all $l \geq 0$, $\sum_{j=1}^{n} \beta_{ij}^l \geq \sum_{j=1}^{n} \alpha_{ij}^l - l\theta^{l-1}\epsilon$
- We show this by induction
- When $l = 1$, we have $\sum_{j=1}^{n} \beta_{ij} \geq \sum_{j=1}^{n} \alpha_{ij} - \epsilon$ by assumption
- Assume $\sum_{j=1}^{n} \beta_{ij}^l \geq \sum_{j=1}^{n} \alpha_{ij}^l - l\theta^{l-1}\epsilon$
- We have

$$\begin{aligned}
\sum_j \beta_{ij}^{l+1} &= \sum_j \sum_k \beta_{ik}\beta_{kj}^l = \sum_k \beta_{ik} \sum_j \beta_{kj}^l \\
&\geq \sum_k \beta_{ik}\left(\sum_{j=1}^{n} \alpha_{kj}^l - l\theta^{l-1}\epsilon\right) \\
&\geq \left(\sum_k \alpha_{ik} - \epsilon\right)\left(\sum_{j=1}^{n} \alpha_{kj}^l - l\theta^{l-1}\epsilon\right)
\end{aligned}$$

Proof of Lemma A2 (cont)

- Right-hand side is, since $\sum_k \alpha_{ik} - \epsilon \leq \sum_k \beta_{ik} \leq \theta$,

$$\sum_k \alpha_{ik} \sum_j \alpha_{kj}^l - \epsilon \sum_j \alpha_{kj}^l - \theta l \theta^{l-1} \epsilon$$

- Since $\sum_j \sum_k \alpha_{ik}\alpha_{kj}^l = \sum_j \alpha_{ij}^{l+1}$ and $\sum_j \alpha_{kj}^l \leq \|\alpha^l\|_\infty \leq \theta^l$, we can conclude that

$$\sum_{j=1}^n \beta_{ij}^{l+1} \geq \sum_{j=1}^n \alpha_{ij}^{l+1} - (l+1)\theta^l \epsilon$$

- Adding up over all $l \in \mathbb{N}$, we obtain

$$\sum_j B_{ij} \geq \sum_j A_{ij} - \frac{1}{(1-\theta)^2}\epsilon$$

- Recall that $\mathcal{L}_(S) = (I - \alpha(S))^{-1}$
- From Lemma A2, We have for any $S \supset \{1, \cdots, T(\epsilon)\}$, $\sum_{j=1}^{t} \mathcal{L}_{ij}(S) \geq \sum_{j=1}^{t} \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2}\epsilon$
- Define $S_i^{\max}(t) = \underset{S_i \supset \{1,\cdots,T(\epsilon)\} \backslash \{i\}}{\arg\max} a_i(S_i)$ and $S^{\max}(t) = \{S_i^{\max}(t)\}_{i=1}^{t}$
- Define $p_i^{\max}(t) = -\sum_{j=1}^{t} \mathcal{L}_{ij}(S^{\max}(t))(a_j(S_j^{\max}(t)) - \log(1 + \mu_j))$
- Since $a_i(S_i^{\max}(t))$ is the maximum of $2^{t-1-T(\epsilon)}$ random variables drawn jointly from $\Phi_i(t-1-T(\epsilon))$, we have, from Assumption 4, $\lim_{t\to\infty} \frac{a_i(S_i^{\max}(t))}{t-1-T(\epsilon)} = D$
- Since $T(\epsilon)$ is constant, we have

$$\lim_{t\to\infty} \frac{a_i(S_i^{\max}(t))}{t} = D \; almost \; surely$$

- Since $S_i^{\max}(t) \supset \{1, \cdots, T(\epsilon)\} \setminus \{i\}$, we have $\sum_{j=1}^{t} \mathcal{L}_{ij}(S^{\max}(t)) \geq \sum_{j=1}^{t} \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2}\epsilon$

- We obtain

$$
\begin{aligned}
-p_i^{\max}(t) &= \sum_{j=1}^{t} \mathcal{L}_{ij}(S^{\max}(t))(a_j(S_j^{\max}(t)) - \log(1+\mu_j)) \\
&\geq \min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k)) \sum_{j=1}^{t} \mathcal{L}_{ij}(S^{\max}(t)) \\
&\geq \min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k))(\sum_{j=1}^{t} \mathcal{L}_{ij} - \frac{1}{(1-\theta)^2}\epsilon)
\end{aligned}
$$

- Devide both sides by $t\sum_{j=1}^{t} \mathcal{L}_{ij}$ and by $\sum_{j=1}^{t} \mathcal{L}_{ij} \geq 1$,

$$
\begin{aligned}
-\frac{p_i^{\max}(t)}{t\sum_{j=1}^{t} \mathcal{L}_{ij}} &\geq \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k))}{t} - \epsilon \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k))}{t(1-\theta)^2 \sum_{j=1}^{t} \mathcal{L}_{ij}} \\
&\geq \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k))}{t} - \epsilon \frac{\min_{k \leq t}(a_k(S_k^{\max}(t)) - \log(1+\mu_k))}{t(1-\theta)^2}
\end{aligned}
$$

- Since μ_k is constant, take $\liminf$ on both sides,

$$\liminf_{t\to\infty} -\frac{p_i^{\max}(t)}{t\sum_{j=1}^{t}\mathcal{L}_{ij}} \geq D - \epsilon D\frac{1}{(1-\theta)^2}$$

- Since ϵ is arbitrary small, we conclude that

$$\liminf_{t\to\infty} -\frac{p_i^{\max}(t)}{t\sum_{j=1}^{t}\mathcal{L}_{ij}} \geq D$$

- Let

$$\kappa(p) = (\min_{S_1} \log(1+\mu_1) + k_1(S_1, a_i(S_1), p), \cdots, \min_{S_t} \log(1+\mu_t) + k_t(S_t, a_i(S_t), p))$$

- $\kappa(p)$ has a smallest fixed point $p^*(t)$, and we have $p^*(t) \leq p^{\max}(t)$
- So, we have

$$\liminf_{t\to\infty} -\frac{p_i^{*}(t)}{t\sum_{j=1}^{t}\mathcal{L}_{ij}} \geq D$$

- We have

$$
\begin{aligned}
-p_i^*(t) &= \sum_{j=1}^{t} \mathcal{L}_{ij}(S(t))(a_j(S_j(t)) - \log(1+\mu_j)) \\
&\leq \max_{k\leq t}(a_k(S_k(t)) - \log(1+\mu_k)) \sum_{j=1}^{t} \mathcal{L}_{ij} \\
&\leq \{\max_{k\leq t} \max_{S'_k}(a_k(S'_k)) - \min_{k}(\log(1+\mu_k))\} \sum_{j=1}^{t} \mathcal{L}_{ij}
\end{aligned}
$$

- As we argued above,

$$
\lim_{t\to\infty} \frac{\max_{k\leq t} \max_{S'_k}(a_k(S'_k))}{t} = D \; almost \; surely
$$

- Divide by $t\sum_{j=1}^{t} \mathcal{L}_{ij}$ and take $\limsup$ on both sides

$$
\limsup_{t\to\infty} -\frac{p_i^*(t)}{t\sum_{j=1}^{t} \mathcal{L}_{ij}} \leq \limsup_{t\to\infty} \frac{\max_{k\leq t} \max_{S'_k}(a_k(S'_k)) - \min_k(\log(1+\mu_k))}{t} \leq D
$$

- Thus, we can conclude that

$$\lim_{t\to\infty} -\frac{p_i^*(t)}{t\sum_{j=1}^{t}\mathcal{L}_{ij}} = D \; almost \; surely$$

- and since $\pi(t) = \sum_{i=1}^{t} \frac{\beta_i}{\sum_{i=1}^{t}\beta_i} p_i(t)$

$$g^* = \lim_{t\to\infty}\left(-\frac{\pi(t)}{t}\right) = D\sum_{i,j=1}^{\infty}\beta_i\mathcal{L}_{ij} \; almost \; surely$$

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Quasi-submodularity

Definition of quasi-submodularity

A set function $F : 2^N \to \mathbb{R}$ is quasi-submodular function if $\forall X, Y \subset N$, both of the following conditions are satisfied

- $F(X \cap Y) \geq F(X) \Rightarrow F(Y) \geq F(X \cup Y)$
- $F(X \cap Y) > F(X) \Rightarrow F(Y) > F(X \cup Y)$

- In our context, for all S_i, T_i, A_i, P, both of the following conditions are satisfied
  - $K_i(S_i, A_i(S_i), P) \leq K_i(S_i \cap T_i, A_i(S_i \cap T_i), P)$
    $$\Rightarrow K_i(S_i \cup T_i, A_i(S_i \cup T_i), P) \leq K_i(T_i, A_i(T_i), P)$$
  - $K_i(S_i, A_i(S_i), P) < K_i(S_i \cap T_i, A_i(S_i \cap T_i), P)$
    $$\Rightarrow K_i(S_i \cup T_i, A_i(S_i \cup T_i), P) < K_i(T_i, A_i(T_i), P)$$

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