The Network Origins of Aggregate Fluctuations

Acemoglu et al. (2012) Econometrica

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Motivation

- The general consensus in macroeconomics has been that microeconomic shocks to firms or disaggregated sectors cannot generate significant aggregate fluctuations. This consensus is based on a "diversification argument".
- "diversification argument" : Microeconomic shocks would wash out, and therefore, would have negligible effects, as argued by Lucas(1977).

Example

- Consider an economy consists of n sectors
- The output of each sector i is i.i.d. random variable $\frac{\epsilon_i}{n}$ such that

$$E[\epsilon_i] = 1, \quad Var(\epsilon_i) = \sigma^2$$

- Here we normalize the output of each sector so that the total output $\epsilon \equiv \sum \frac{\epsilon_i}{n}$ has a constant expectation.
- Then, the "aggregate fluctuation",, that is the standard deviation of the total output, ϵ is written as

$$\sqrt{Var(\epsilon)} = \frac{\sigma}{\sqrt{n}}$$

- The aggregate fluctuation have a magnitude proportional to $\frac{1}{\sqrt{n}}$. This is a negligible effect at a high levels of disaggregation.

- However, this argument ignores the presence of interconnections between different firms and sectors, functioning as a potential propagation mechanism of idiosyncratic shocks throughout the economy.
- This paper shows that the interconnections imply that the effects of microeconomic shocks may not remain confined to where they originate.

Main Results

- Sectoral interconnections may imply that aggregate output concentrates around its mean at a rate significantly smaller than $\sqrt{n}$.
- Slow rates of decay of aggregate volatility may have two related but distinct causes, *first-order interconnections* and *higher-order interconnections*.

Three key theorems

Theorem 2

It provides a lower bound in terms of the extent of asymmetry across sectors captured by variations in their degrees. It shows that higher variations in the degrees of different sectors imply lower rates of decay for aggregate volatility.

Theorem 3

It provides tighter lower bounds in terms of a measure of second-order interconnectivity between different sectors.

Theorem 4

It establishes that sectoral shocks average out at the rate $\sqrt{n}$ for *balanced* networks in which there is a uniform bound on the degree of every sector.

Notation

$\mathbf{x} \in \mathbb{K}^n$ is n-dimensional vector

p-norm

$$\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

In particular,

$$\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences of positive real numbers.

1. $a_n = O(b_n) \overset{\text{def}}{\Leftrightarrow} \limsup_{n\to\infty} a_n/b_n < \infty$
2. $a_n = \Omega(b_n) \overset{\text{def}}{\Leftrightarrow} \liminf_{n\to\infty} a_n/b_n > 0$
3. $a_n = \Theta(b_n) \overset{\text{def}}{\Leftrightarrow} a_n = O(b_n)$ and $a_n = \Omega(b_n)$
4. $a_n = o(b_n) \overset{\text{def}}{\Leftrightarrow} \lim_{n\to\infty} a_n/b_n = 0$

Intuitively,

1. $a_n = O(b_n)$: (Growth rate of a_n) $\leq$ (Growth rate of b_n)
2. $a_n = \Omega(b_n)$: (Growth rate of a_n) $\geq$ (Growth rate of b_n)
3. $a_n = \Theta(b_n)$: (Growth rate of a_n) $=$ (Growth rate of b_n)
4. $a_n = o(b_n)$: (Growth rate of a_n) $<$ (Growth rate of b_n)

Model

- We consider a sequence of $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ of economies, and analyze the rate of decay of aggregation volatilities.
- Each economy $\mathcal{E}_n$ is defined as $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_{in}\}_{i\in\mathcal{I}_n})$
  - $\mathcal{I}_n = \{1, 2, \ldots, n\}$ is the set of sectors in the economy
  - W_n captures the corresponding input-output matrix
  - the collection $\{F_{in}\}_{i\in\mathcal{I}_n}$ denotes the distribution of log sectoral shocks

Household

The representative household is endowed with one unit of labor, supplied inelastically, and has Cobb-Douglas preferences over n distinct goods

$$u(c_1, c_2, \ldots, c_n) = A \prod_{i=1}^{n} (c_i)^{1/n}$$

where c_i is the consumption of good i and A is a normalization constant discussed later.

Sectors

The representative firms of each sector use Cobb-Douglas technologies with constant returns to scale. The output of sector i, denoted by x_i, is

$$x_i = {z_i}^{\alpha} {l_i}^{\alpha} \prod_{i=1}^{n} {x_{ij}}^{(1-\alpha) w_{ij}}$$

- l_i is the amount of labor hired by the sector
- $\alpha \in (0,1)$ is the share of labor
- x_{ij} is the amount of commodity j used in the production of good i
- z_i is the idiosyncratic productivity shock to sector i

- $w_{ij} \geq 0$ is the share of good j in the total intermediates input use of firms in sector i
    - $w_{ij} = 0$ if sector i does not use good j
    - the structure of intersectoral trade with the *input-output matrix* W with entries w_{ij}
- F_i is the distribution of $\epsilon_i \equiv \log(z_i)$

Assumption 1

The input shares of all sectors add up to 1; that is, $\sum_{j=1}^{n} w_{ij} = 1$ for all $i = 1, 2, \ldots, n$

- Define the *weighted outdegree* (or simply the *degree*) of sector i as $d_i \equiv \sum_{j=1}^{n} w_{ji}$

Competitive Equilibrium

Definition

A *competitive equilibrium* of economy $\mathcal{E}$ with n sectors consists of price $(p_1, p_2, \ldots, p_n)$, wage h, consumption bundle $(c_1, c_2, \ldots, c_n)$, and quantities $(l_i, x_i, (x_{ij}))$such that

1. the representative household maximizes her utility
2. the representative firms in each sector maximize profits
3. labor and commodity markets clear

Lemma 1

Lemma

Consider a competitive equilibrium of an economy $\mathcal{E}$. If we choose A appropriately and normalize the price vector so that the ideal price index is equal to 1, then the (log) aggregate output can be represented as

$$y = \mathbf{v}^{\mathrm{T}}\epsilon$$

where $\epsilon \equiv (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^{\mathrm{T}}$ is the n-dimensional vector of (log) shocks, and $\mathbf{v}$ is a vector, called *influence vector, defined by*

$$\mathbf{v} \equiv \frac{\alpha}{n}[\mathbf{I} - (1-\alpha)W^{\mathrm{T}}]^{-1}\mathbf{1}$$

Proof of Lemma 1

Household's Maximization Problem

$$\max_{c_1,\ldots,c_n} A\prod_{i=1}^{n} c_i^{\frac{1}{n}} \quad \text{s.t.} \quad \sum_{i=1}^{n} p_i c_i = h$$

Note that $\sum_{i=1}^{n} l_i = 1$
First order conditions w.r.t. c_i

$$p_i c_i = \frac{h}{n} \quad \text{for all } i = 1,\ldots,n \tag{1}$$

Sector's maximization Problem

$$\max_{x_{i1},\ldots,x_{in},l_i} p_i {z_i}^{\alpha} {l_i}^{\alpha} \prod_{j=1}^{n} {x_{ij}}^{(1-\alpha)w_{ij}} - hl_i - \sum_{j=1}^{n} p_j x_{ij}$$

First order conditions w.r.t. $x_{i1}, \ldots, x_{in}, l_i$

$$x_{ij} = (1-\alpha) w_{ij} \frac{p_i}{p_j} x_i \text{ for all } j = 1, \ldots, n \tag{2}$$

$$l_i = \frac{\alpha p_i x_i}{h} \tag{3}$$

Market Clear Condition

Each commodity markets clear

$$c_i + \sum_{j=1}^{n} x_{ij} = x_i \quad \text{for all } i = 1, \ldots, n \tag{4}$$

Labor market clears

$$\sum_{i=1}^{n} l_i = 1 \tag{5}$$

Substituting (2) and (3) into

$$x_i = {z_i}^{\alpha} {l_i}^{\alpha} \prod_{i=1}^{n} {x_{ij}}^{(1-\alpha)w_{ij}}$$

and taking log, we have

$$\alpha \log h = \alpha\epsilon_i + \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \log p_i \\ - (1-\alpha)\sum_{j=1}^{n} w_{ij}p_j + (1-\alpha)\sum_{j=1}^{n} w_{ij}\log w_{ij} \quad (6)$$

Since α and w_{ij} are exogenous, we can rewrite (6) as

$$\alpha \log h = \alpha\epsilon_i + \log p_i - (1-\alpha)\sum_{j=1}^{n} w_{ij}p_j + K_i \quad (7)$$

where $K_i = \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + (1-\alpha)\sum_{j=1}^{n} w_{ij}\log w_{ij}$

Combine (7) over $i = 1, \ldots, n$

$$(\alpha \log h)\mathbf{1} = \alpha\epsilon + \mathbf{p} - (1-\alpha)W\mathbf{p} + \mathbf{K} \tag{8}$$

where $\mathbf{p} = (\log p_1, \ldots, \log p_n)^{\mathrm{T}}$
Multiplying (8) by $\mathbf{v}^{\mathrm{T}} \equiv \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}[\mathbf{I} - (1-\alpha)W]^{-1}$, we get

$$\log h = \mathbf{v}^{\mathrm{T}}\epsilon + \frac{1}{n}\sum_{i=1}^{n}\log p_i + \frac{1}{\alpha}\mathbf{v}^{\mathrm{T}}\mathbf{K} \tag{9}$$

Here, we used the following facts

$$\mathbf{v}^{\mathrm{T}}\mathbf{1} = 1$$

$$\frac{1}{\alpha}\mathbf{v}^{\mathrm{T}}\mathbf{p} - \frac{1-\alpha}{\alpha}\mathbf{v}^{\mathrm{T}}W\mathbf{p} = \frac{1}{n}\sum_{i=1}^{n}\log p_i$$

We choose A so that

$$\log \frac{A}{n} = -\frac{1}{\alpha}\mathbf{v}^{\mathrm{T}}\mathbf{K}$$

and normalize the price index to 1, that is,

$$\frac{n}{A}\prod_{i=1}^{n} {p_i}^{1/n} = 1$$

Then, we finally obtain

$$y = \log h = \mathbf{v}^{\mathrm{T}}\epsilon$$

We can think of h as GDP because, from (1), we have

$$\mathrm{GDP} \equiv \sum_{i=1}^{n} p_i c_i = n \times \frac{h}{n} = h$$

Lemma 2

Lemma

Given an economy $\mathcal{E}$, let $\mathbf{v} \equiv \frac{\alpha}{n}[\mathbf{I} - (1-\alpha)W^{\mathrm{T}}]^{-1}\mathbf{1}$ be the influence vector. Then, in a competitive equilibrium, $\mathbf{v}$ is also a *sales vector*. That is,

$$v_i = \frac{p_i x_i}{\sum_{j=1}^{n} p_j x_j} \quad (i = 1, \dots, n)$$

- The ith elements of the influence vector is equal to the equilibrium share of sales of sector i
- Productivity shocks to a sector with more consumers should have more significant aggregate effects

Proof of Lemma 2

Substitute (1) and (2) into (4),

$$s_i = \frac{h}{n} + (1-\alpha)\sum_{j=1}^{n} w_{ji}s_j \tag{10}$$

where $s_i = p_i x_i$ is the equilibrium value of sales of sector i.
Combine (10) over $i = 1, \ldots, n$,

$$\mathbf{s} = \frac{h}{n}\mathbf{1} + (1-\alpha)W^{\mathrm{T}}\mathbf{s}$$

$$\Leftrightarrow \mathbf{s}^{\mathrm{T}} = \frac{h}{n}\mathbf{1}^{\mathrm{T}}[\mathbf{I} - (1-\alpha)W]^{-1} = \frac{h}{\alpha}\mathbf{v}^{\mathrm{T}} \tag{11}$$

Multiplying by $\mathbf{1}$,

$$\mathbf{s}^{\mathrm{T}}\mathbf{1} = \frac{h}{\alpha}\mathbf{v}^{\mathrm{T}}\mathbf{1}$$

Note that $\mathbf{v}^{\mathrm{T}}\mathbf{1} = 1$. We have

$$\frac{h}{\alpha} = \sum_{j=1}^{n} p_j x_j$$

Since $s_i = \frac{h}{\alpha} v_i$, from (11), we obtain

$$v_i = \frac{p_i x_i}{\sum_{j=1}^{n} p_j x_j}$$

Sequence of economies

- Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ indexed by the number of sectors
- Since total supply of labor is normalized to 1 for all n, an increase in the number of sectors corresponds to disaggregating the structure of economy
- Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, we denote
    - the sequence of aggregate outputs by $\{y_n\}_{n\in\mathbb{N}}$
    - the sequence of influence vectors by $\{v_n\}_{n\in\mathbb{N}}$
    - the sequence of intersectoral matrix by $\{W_n\}_{n\in\mathbb{N}}$
        - a generic element of W_n by w_{ij}^n
        - the degree of sector i by d_i^n
    - the sequence of vectors of (log) idiosyncratic productivity shocks to the sectors by $\{\epsilon_n\}_{n\in\mathbb{N}}$

Assumption 2

Assumption 2

Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ for any sector $i \in \mathcal{I}_n$ and for all $n \in \mathbb{N}$

1. $\mathbb{E}\epsilon_{in} = 0$
2. $\mathrm{var}(\epsilon_{in}) = \sigma_{in}^2 \in (\underline{\sigma}^2, \bar{\sigma}^2)$ where $0 < \underline{\sigma} \leq \bar{\sigma}$ are independent of n

- (1) is a normalization
- (2) imposes the restriction that log sectoral shock variance remain bounded as $n \to \infty$

Lemma 3

Lemma

The growth rate of aggregate volatilities is the same as the one of $\|v_n\|_2$. That is,

$$\sqrt{\mathrm{var}\ y_n} = \Theta(\|v_n\|_2)$$

- Aggregate volatility scales with the Euclidean norm of the influence vector as our representation of the economy becomes more disaggregated.
- The rate of decay of aggregate volatility upon disaggregation may be distinct from $\sqrt{n}$
- When we analyze the aggregate fluctuation, we need to compute $\|v_n\|_2$.

Proof of Lemma 3

Since $y_n = \mathbf{v_n}^{\mathrm{T}}\epsilon$, we have $y_n = \sum_{i=1}^{n} {v_i}^2 {\epsilon_i}^2$.
Therefore,

$$\sqrt{\mathrm{var}\ y_n} = \sqrt{\sum_{i=1}^{n} {v_i}^2 {\sigma_{in}}^2}$$

By Assumption 2, we have

$$\underline{\sigma}\|v_n\|_2 < \sqrt{\sum_{i=1}^{n} {v_i}^2 {\sigma_{in}}^2} < \bar{\sigma}\|v_n\|_2$$

$$\Leftrightarrow \underline{\sigma} < \frac{\sqrt{\mathrm{var}\ y_n}}{\|v_n\|_2} < \bar{\sigma}$$

Thus,

$$\sqrt{\mathrm{var}\ y_n} = \Theta(\|v_n\|_2)$$

Theorem 1

Theorem

Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and assume that $\mathbb{E}\epsilon_{in}^2 = \sigma^2$ for all $i \in \mathcal{I}_n$ and all $n \in \mathbb{N}$

1. If $\{\epsilon_{in}\}$ are normally distributed for all i and all n, then $\frac{1}{\|v_n\|_2} y_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.
2. Suppose that there exist constant $a > 0$ and random variable $\bar{\epsilon}$ with bounded variance and cumulative distribution function $\bar{F}$, such that $F_{in}(x) < \bar{F}(x)$ for all $x < -a$, and $F_{in}(x) > \bar{F}(x)$ for all $x > a$. Also suppose that $\frac{\|v_n\|_\infty}{\|v_n\|_2} \to 0$. Then $\frac{1}{\|v_n\|_2} y_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.
3. Suppose that $\{\epsilon_{in}\}$ are identically, but not normally distributed for all $i \in \mathcal{I}_n$ and all n. If $\frac{\|v_n\|_\infty}{\|v_n\|_2} \nrightarrow 0$, then the asymptotic distribution of $\frac{1}{\|v_n\|_2} y_n$, when it exists, is normal and has finite variance σ^2.

Remark on Theorem 1

Proof

I will skip the proof.

- Aggregate output normalized by the Euclidean norm of the influence vector converges to a nondegenerate distribution.
- Unless all shocks are normally distributed, the intersectoral structure of this economy not only affects the convergence rate, but also determines the asymptotic distribution of aggregate output.

Coefficient of Variation

- From now on, we characterize the rate of decay of aggregate volatility.
- We first focus on the effects of "first-order interconnections" on aggregate volatility.

Definition

Given an economy $\mathcal{E}_n$ with sectoral degrees $\{d_1^n, d_2^n, \ldots, d_n^n\}$, the *coefficient of variation* is

$$CV_n \equiv \frac{1}{\bar{d}^n}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(d_i^n - \bar{d}^n\right)^2\right]^{1/2}$$

where $\bar{d}^n = \frac{1}{n}\sum_{i=1}^{n} d_i^n$ is the average degree.

Theorem 2

Theorem

Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, average volatility satisfies

$$\sqrt{\mathrm{var}\ y_n} = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n}(d_i^n)^2}\right)$$

and

$$\sqrt{\mathrm{var}\ y_n} = \Omega\left(\frac{1+CV_n}{\sqrt{n}}\right)$$

- If the degree sequence of the intersectoral network exhibits high volatility as measured by the coefficient of variation, then there is also high volatility in the effect of different sector-specific shocks on the aggregate output.

Proof of Theorem 2

Recall that aggregate volatility is of order $\|v_n\|_2$.
v_n can be expressed in terms of the convergent power series.

$$
\begin{aligned}
\mathbf{v_n}^{\mathrm{T}} &= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}[\mathbf{I} - (1-\alpha)W_n]^{-1} \\
&= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}\sum_{k=0}^{\infty}[(1-\alpha)W_n]^k \\
&= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}\{\mathbf{I} + (1-\alpha)W_n + (1-\alpha)^2{W_n}^2 + \dots\} \\
&\geq \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}} + \frac{\alpha(1-\alpha)}{n}\mathbf{1}^{\mathrm{T}}W_n
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|v_n\|_2^2 &\geq \left(\frac{\alpha}{n}\mathbf{1}^{\mathrm{T}} + \frac{\alpha(1-\alpha)}{n}\mathbf{1}^{\mathrm{T}}W_n\right)\left(\frac{\alpha}{n}\mathbf{1} + \frac{\alpha(1-\alpha)}{n}{W_n}^{\mathrm{T}}\mathbf{1}\right) \\
&= \frac{\alpha^2(3-2\alpha)}{n} + \frac{\alpha^2(1-\alpha)^2}{n^2}\sum_{i=1}^{n}(d_i^n)^2 \\
&= \Theta\left(\frac{1}{n}\right) + \Theta\left(\frac{1}{n^2}\sum_{i=1}^{n}(d_i^n)^2\right) \qquad (12)
\end{aligned}
$$

By Cauchy-Schwartz inequality, we have

$$
\sum_{i=1}^{n}(d_i^n)^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2 = n
$$

Thus, the first term of (RHS) of (12) is always dominated by the second term. Hence, we obtain

$$\|v_n\|_2^2 = \Omega\left(\frac{1}{n^2}\sum_{i=1}^{n}(d_i^n)^2\right)$$

or

$$\|v_n\|_2 = \Omega\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{(d_i^n)^2}\right)$$

Since $\bar{d^n} = 1$,

$$\frac{1}{n^2}\sum_{i=1}^{n} d_i^{n\,2} = \frac{n-1}{n}[CV_n]^2 + \frac{1}{n}$$

Thus,

$$\sqrt{\text{var } y_n} = \Omega\left(\frac{1+CV_n}{\sqrt{n}}\right)$$

Power Law Degree Sequence

Definition

A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a power law degree sequence if there exists a constant $\beta > 1$, a slowly varying function $L(\dot{})$ satisfying $\lim_{t\to\infty} L(t)t^{\delta} = \infty$ and $\lim_{t\to\infty} L(t)t^{-\delta} = 0$ for all $\delta > 0$, and a sequence of positive numbers $c_n = \Theta(1)$ such that, for all $n \in \mathbb{N}$ and all $k < d^n_{\max} = \Theta(n^{1/\beta})$, we have

$$P_n(k) = c_n k^{-\beta} L(k)$$

where $P_n(k) \equiv \frac{1}{n}\#\{i \in \mathcal{I} \mid d^n_i > k\}$ is the empirical counter-cumulative distribution function and $d^n_{\max}$ is the degree of $\mathcal{E}_n$.

Corollary 1

- For sufficiently large x, we have $\log P(x) \simeq \gamma_0 - \beta \log x$
- Applying Theorem 2 to a sequence of economies with power law tails leads to the following corollary.

Corollary

Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ with a power law degree sequence and the corresponding shape parameter $\beta \in (1,2)$. Then, aggregate volatility satisfies

$$\sqrt{\mathrm{var}\ y_n} = \Omega(n^{-\frac{\beta-1}{\beta-\delta}})$$

where $\delta > 0$ is arbitrary.

Note on Corollary 1

Proof

I will skip the proof.

- Corollary 1 says that if the degree sequence of the intersectoral network exhibits relatively heavy tails, aggregate volatility decreases at a much slower rate than the one predicted by the standard diversification argument.
- Note that Theorem 2 and Corollary 1 provide only a lower bound on the rate at which aggregate volatility vanishes.

Second-Order Interconnections and Cascades

- First-order interconnections in the intersectoral network doesn't provide the information on the extent of "cascade" effect.
- Next, we will focus on the second-order connections.
- See the Example 2 in the paper.
    - Even if the degree sequences are identical, aggregate volatility may differ.

Second-Order Interconnectivity Coefficient

Definition

The second-order interconnectivity coefficient of economy $\mathcal{E}_n$ is

$$\tau_2(W_n) = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i,j} w_{ji}^n w_{ki}^n d_j^n d_k^n$$

- This coefficient measures the extent to which sectors with high degrees are interconnected to one another through common suppliers.
- τ_2 takes higher values when high-degree sectors share suppliers with other high-degree sectors, as opposed to low-degree ones.

Theorem 3

Theorem

Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, aggregate volatility satisfies

$$\sqrt{\text{var } y_n} = \Omega\left(\frac{1 + CV_n}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n}\right)$$

- Theorem 3 shows how second-order interconnections affect aggregate volatility.
- It also shows that even if the degree distribution of two sequences of economies are identical for all n, aggregate volatilities may exhibit different behaviors.
- It captures not only the existence of "large" suppliers but also the clustering of significant sectors.

Proof of Theorem 3

$$
\begin{aligned}
\mathbf{v_n}^{\mathrm{T}} &= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}[\mathbf{I} - (1-\alpha)W_n]^{-1} \\
&= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}\sum_{k=0}^{\infty}[(1-\alpha)W_n]^k \\
&= \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}\{\mathbf{I} + (1-\alpha)W_n + (1-\alpha)^2{W_n}^2 + \dots\} \\
&\geq \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}}[\mathbf{I} + (1-\alpha)W_n + (1-\alpha)^2{W_n}^2]
\end{aligned}
$$

Therefore,

$$\|v_n\|_2^2 \geq \frac{\alpha^2}{n^2}\mathbf{1}^{\mathrm{T}}[\mathbf{I} + (1-\alpha)W_n + (1-\alpha)^2 {W_n}^2]$$
$$[\mathbf{I} + (1-\alpha)W_n + (1-\alpha)^2 {W_n}^2]^{\mathrm{T}}\mathbf{1}$$

$$= \frac{\alpha^2}{n^2}[\{1 + (1-\alpha) + (1-\alpha^2)\}n + \{(1-\alpha) + (1-\alpha^2)\}\sum_{k=1}^{n} d_k^n$$
$$+(1-\alpha)^2\|\mathbf{1}W_n\|_2^2 + 2(1-\alpha)^3\mathbf{1}^{\mathrm{T}}(W_n)^2 {W_n}^{\mathrm{T}}\mathbf{1} + (1-\alpha)^4\|\mathbf{1}^{\mathrm{T}}(W_n)^2\|_2^2]$$

Thus, we have

$$\|v_n\|_2^2 \geq \Theta\left(\frac{1}{n^2}\|\mathbf{1}^{\mathrm{T}}W_n\|_2^2\right) + \Theta\left(\frac{1}{n^2}\mathbf{1}^{\mathrm{T}}(W_n)^2 {W_n}^{\mathrm{T}}\mathbf{1}\right)$$
$$+ \Theta\left(\frac{1}{n^2}\|\mathbf{1}^{\mathrm{T}}(W_n)^2\|_2^2\right) \quad (13)$$

For the second term on the right-hand side of (13), we have

$$\begin{aligned}\frac{1}{n^2}\mathbf{1}^{\mathrm{T}}(W_n)^2 {W_n}^{\mathrm{T}}\mathbf{1} &= \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ji} d_i^n d_j^n \\ &= \sum_{i=1}^{n}\sum_{j\neq i} w_{ji} d_i^n d_j^n + \sum_{i=1}^{n} w_{ii} {d_i^n}^2 \\ &= s(W_n) + O\Big(\sum_{i=1}^{n} {d_i^n}^2\Big) \qquad (14)\end{aligned}$$

where $s(W_n) \equiv \sum_{i=1}^{n}\sum_{j\neq i} w_{ji} d_i^n d_j^n$

For the third term on the right-hand side of (13),

$$
\begin{aligned}
\|\mathbf{1}^{\mathrm{T}}(W_n)^2\|_2^2 &= \sum_{i=1}^{n}\left[\sum_{j=1}^{n} w_{ji}^{n} d_j^{n}\right]^2 \\
&= \sum_{i=1}^{n} w_{ii}^{n\,2} d_i^{n\,2} + 2\sum_{i=1}^{n}\sum_{j\neq i} w_{ii}^{n} w_{ji}^{n} d_i^{n} d_j^{n} \\
&\quad + \sum_{i=1}^{n}\left[\sum_{j\neq i} w_{ji}^{n} d_j^{n}\right]^2 \\
&= O\left(\sum_{i=1}^{n} d_i^{n\,2}\right) + O(s(W_n)) \\
&\quad + \sum_{i=1}^{n}\sum_{j\neq i} d_j^{n\,2} w_{ji}^{n\,2} + \sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j} w_{ji}^{n} w_{ki}^{n} d_j^{n} d_k^{n}
\end{aligned}
$$

Thus, we have

$$\|\mathbf{1}^{\mathrm{T}}(W_n)^2\|_2^2 = O\left(\sum_{i=1}^{n} d_i^{n2}\right) + O(s(W_n)) + \Theta(\tau_2(W_n)) \tag{15}$$

From (13), (14) and (15), we obtain

$$\|v_n\|_2^2 = \Omega\left(\frac{1}{n^2}\left[\sum_{i=1}^{n} d_i^{n2} + s(W_n) + \tau_2(W_n)\right]\right)$$

From the inequality $\sum_{i=1}^{n} \left[d_i^n - \sum_{ji}^{n} d_j^n\right]^2 \geq 0$, we have

$$s(W_n) = O\left(\sum_{i=1}^{n} d_i^{n2} + \tau_2(W_n)\right)$$

Therefore, we get

$$\|v_n\|_2^2 = \Omega\left(\frac{1}{n^2}\left[\sum_{i=1}^{n} d_i^{n2} + \tau_2(W_n)\right]\right)$$

or

$$\|v_n\|_2 = \Omega\left(\frac{1}{n}\left[\sum_{i=1}^{n} d_i^{n2} + \sqrt{\tau_2(W_n)}\right]\right)$$

Corollary 2

Corollary

Suppose that $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a sequence of economies whose second-order degree sequence have power law tails with shape parameter $\zeta \in (1,2)$. Then, aggregate volatility satisfies, for any $\delta > 0$,

$$\sqrt{\mathrm{var}\ y_n} = \Omega\left(n^{-\frac{\zeta-1}{\zeta-\delta}}\right)$$

Proof

I will skip the proof.

Note on Corollary 2

- The second-order degree of sector i is defined as the weight sum of degrees of the sectors that use sector i's product as inputs, that is,

$$q_i^n \equiv \sum_{j=1}^{n} d_j^n w_{ji}^n$$

- We can think of this corollary as the extension of corollary 1
- If the distributions of second-order degrees have relatively heavy tails, then the aggregate volatility decreases at a much slower rate than $\sqrt{n}$

Extension of Theorem 3

- We can capture the effects of higher-order interconnections.
- Mathematically, we can get tighter lower bounds that the one we can get in Theorem 3.

Theorem 3.A

Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. Then, for any integer $m \geq 2$, aggregate volatility satisfies

$$\sqrt{\mathrm{var}\ y_n} = \Omega\left(\frac{1+CV_n}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n} + \cdots + \frac{\sqrt{\tau_m(W_n)}}{n}\right)$$

Balanced Structure

- So far, we learned that aggregate volatility decays at slower rate than $\sqrt{n}$
- However, in some networks, standard diversification argument holds.

Definition

A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is balanced if $\max_{i\in\mathcal{I}_n} d_i^n = \Theta(1)$.

- Intuitively, there is no "dominant" sectors in the balanced network.

Theorem 4

Theorem

Consider a sequence of balanced economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. Then there exists $\bar{\alpha} \in (0,1)$ such that, for $\alpha \geq \bar{\alpha}$, $\sqrt{\mathrm{var} y_n} = \Theta(1/\sqrt{n})$.

- When the intersectoral network has a balanced structure and the role of the intermediate inputs in production is not too large, volatility decays at rate $\sqrt{n}$.
- In economies with balanced intersectoral network structures, aggregate fluctuation do not have network origins.

Tool for the proof of Theorem 4

Operator norm

Let A be $m \times n$ matrix

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

In particular,

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |a_{ij}|$$

Hölder's inequality

Let $p, q \in [1, \infty]$ be such that $1/p + 1/q = 1$, and let a_i, b_i be positive numbers. Then,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} {a_i}^p \right)^{1/p} \left(\sum_{i=1}^{n} {b_i}^p \right)^{1/p}$$

Applying it to p-norm, we have

$$\|x\|_2^2 \leq \|x\|_1 \|x\|_\infty$$

Proof of Theorem 4

Note that

$$\begin{aligned} \|v_n\|_2 &= \sqrt{{v_1}^2 + \dots {v_n}^2} \\ &\geq \sqrt{\frac{1}{n}(v_1 + \cdots + v_n)^2} \quad (\mathrm{By Cauchy - Schwartz\ inequality}) \\ &= \frac{1}{\sqrt{n}} \end{aligned}$$

Thus, we have

$$\|v_n\|_2 = \Omega\left(\frac{1}{\sqrt{n}}\right) \tag{16}$$

Also, by the definition of operator norm and the assumption $\max_i d_i^n = \Theta(1)$,

$$\|W_n\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |w_{ij}^n| = \max_i d_i^n = \Theta(1) \tag{17}$$

Rewrite the influence vector as below

$$v_n = \frac{\alpha}{n}[\mathbf{I} - (1-\alpha){W_n}^{\mathrm{T}}]^{-1}\mathbf{1}$$

$$\frac{n}{\alpha}[\mathbf{I} - (1-\alpha){W_n}^{\mathrm{T}}]v_n = \mathbf{1}$$

$$\frac{n}{\alpha}v_n = \frac{n(1-\alpha)}{\alpha}{W_n}^{\mathrm{T}}v_n + \mathbf{1}$$

$${v_n}^{\mathrm{T}} = \frac{\alpha}{n}\mathbf{1}^{\mathrm{T}} + (1-\alpha){v_n}^{\mathrm{T}}W_n$$

Thus, we have

$$\begin{aligned} \|v_n\|_\infty &\le \frac{\alpha}{n} + (1-\alpha)\|W_n\|_1\|W_n\|_\infty \\ &\le \frac{\alpha}{n} + (1-\alpha)C\|W_n\|_\infty \end{aligned} \tag{18}$$

where C is a constant independent of n.

When α satisfies $\alpha > (C-1)/C$, $1 - C(1-\alpha) > 0$. Therefore, from (18),

$$(1 - C(1-\alpha))\|v_n\|_\infty \leq \frac{\alpha}{n}$$

$$\|v_n\|_\infty \leq \frac{\alpha}{n}[1 - C(1-\alpha)]^{-1}$$

Thus we have

$$\|v_n\|_\infty = O\left(\frac{1}{n}\right) \tag{19}$$

From Hölder's inequality,

$$\|v_n\|_2^2 \leq \|v_n\|_1 \|v_n\|_\infty$$

$$\|v_n\|_2 \leq \sqrt{\|v_n\|_1 \|v_n\|_\infty}$$

Since $\|v_n\|_1 = 1$,

$$\|v_n\|_2 \leq \sqrt{\|v_n\|_\infty}$$

From (19), we have

$$\|v_n\|_2 = O\left(\frac{1}{\sqrt{n}}\right) \tag{20}$$

From (16) and (20), we obtain

$$\|v_n\|_2 = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

Application

- Let's look at the intersectoral network structure of the U.S. economy
- input-output account (1972-2002) by the Bureau of Economic Analysis

Weighted Indegree

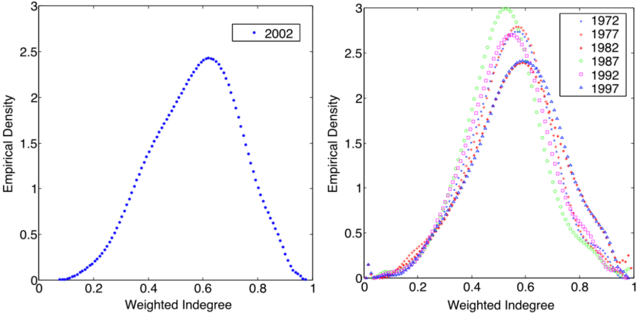

FIGURE 6.—Empirical densities of intermediate input shares (indegrees).

- The variation in total intermediate input shares across commodities
- The indegrees of most sectors are concentrated around the mean

First- and Second-Order (out)degrees

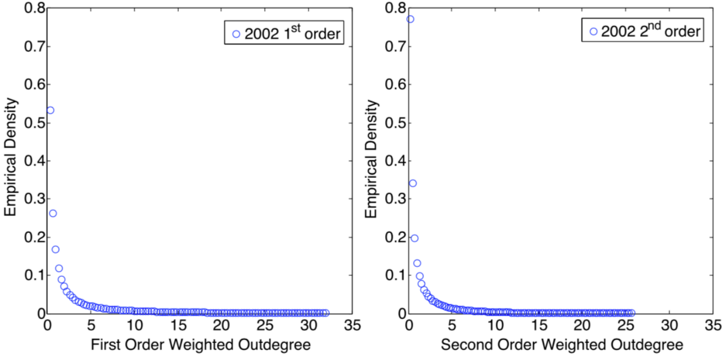

FIGURE 7.—Empirical densities of first- and second-order degrees.

- There are heavy right tails.
- This implies that
  - general purpose inputs used by many other sectors (high first-order degree)
  - major suppliers to sectors that produce the general purpose inputs (high second-order degree)

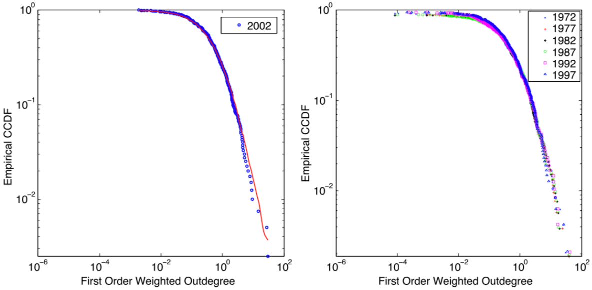

FIGURE 8.—Empirical counter-cumulative distribution function of first-order degrees.

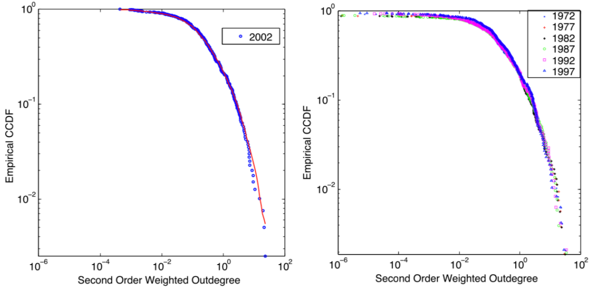

FIGURE 9.—Empirical counter-cumulative distribution function of second-order degrees.

- The tail of the distribution is well-approximated by a power law distribution.
- Estimation of “scaling exponent”
  - First-order : $\hat{\beta} = 1.38$
  - Second-order : $\hat{\zeta} = 1.18$
- There exists a higher degree of asymmetry in the U.S. economy in terms of the roles that different sectors play as direct or indirect suppliers to others.

Estimation of the aggregate effects of sectoral shock

TABLE II

ESTIMATES FOR $\|v_n\|_2$ [a]

	1972	1977	1982	1987	1992	1997	2002
$\|v_{n_d}\|_2$	0.098	0.091	0.088	0.088	0.093	0.090	0.094
	$(n_d = 483)$	$(n_d = 524)$	$(n_d = 529)$	$(n_d = 510)$	$(n_d = 476)$	$(n_d = 474)$	$(n_d = 417)$
$\|v_{n_s}\|_2$	0.139	0.137	0.149	0.133	0.137	0.115	0.119
	$(n_s = 84)$	$(n_s = 84)$	$(n_s = 80)$	$(n_s = 89)$	$(n_s = 89)$	$(n_s = 127)$	$(n_s = 128)$
$\frac{\|v_{n_d}\|_2}{\|v_{n_s}\|_2}$	0.705	0.664	0.591	0.662	0.679	0.783	0.790
$\frac{1/\sqrt{n_d}}{1/\sqrt{n_s}}$	0.417	0.400	0.399	0.418	0.432	0.518	0.554

- We can think of $\|v_{n_d}\|_2$ and $\|v_{n_s}\|_2$ as more disaggregate data and less disaggregate data respectively.

- $\|v_{n_d}\|_2$ are about twice as large as $1/\sqrt{n_d}$.
  - This implies intersectoral linkage increase the impact of sectoral shock by twohold.
  - $\|v_{n_d}\|_2$ are smaller than $\|v_{n_s}\|_2$ as expected.
- If we can approximate $\|v_n\|_2 \simeq \frac{2}{\sqrt{n}}$ at each level of disaggregation, $\frac{\|v_{n_d}\|_2}{\|v_{n_s}\|_2} \simeq \frac{1/\sqrt{n_d}}{1/\sqrt{n_s}}$.
- However, we can find $\frac{\|v_{n_d}\|_2}{\|v_{n_s}\|_2} > \frac{1/\sqrt{n_d}}{1/\sqrt{n_s}}$
  - This suggests that network effects are more important at higher levels of disaggregation.

Rate of Decay

- Recall that in the diversification argument standard deviation is equal to $\frac{\sigma}{\sqrt{n}}$ where σ is the standard deviation of productivity.
- Average standard deviation of TFP in U.S.(1958-2005) is estimated as 0.058
- On the other hand, average standard deviation of U.S. GDP is estimated as 0.2
- In addition, the economy consists of 2295 sectors.
- If diversification argument holds, the aggregate volatility is approximated as $\frac{0.058}{\sqrt{2295}} \simeq 0.001$.
    - Very small amount of volatility

- However, the shape parameter $\hat{\zeta} = 1.18$ implies that aggregate volatility decays no faster than $n^{\frac{\zeta-1}{\zeta}} = n^{0.15}$ by Corollary 2.
- Therefore, the aggregate volatility is approximated as $\frac{0.058}{(2295)^{0.15}} \simeq 0.018$
    - This corresponds to sizable aggregate fluctuation
- The interconnections implied by the U.S. input-output structure may generate significant aggregate fluctuations from sectoral shocks.

Conclusion

- The possibility that significant aggregate fluctuation generate from idiosyncratic shocks has been discarded in macroeconomics.
- However, in the presence of intersectoral input-output linkage, microeconomic idiosyncratic shocks may lead to aggregate fluctuations.
- This paper provides a characterization of intersectoral relationship in terms of the importance of different sectors as direct or indirect suppliers to the rest of the economy.