7. Related Work of Online Matching

We explain related works of the existing studies on *online matching*, which was omitted from our paper due to the space limitation. The online matching problem is the problem of determining matching for a given bipartite graph when nodes on one side arrive one by one. This problem has various applications, such as advertising (Mehta 2012) and ride-sharing (Dickerson et al. 2018).

Most studies in online matching focus on the following problem settings (see (Mehta 2012)): (a) *Known Identical Independent Distributions* (KIID), where each arrival node has the same arrival rate throughout the entire period (Feldman et al. 2009; Haeupler, Mirrokni, and Zadimoghaddam 2011); (b) *Unknown Identical Independent Distributions* (Unknown IID), where each arrival node has a fixed but unknown arrival rate (Devanur et al. 2011); (c) *Adversarial order*, where the arrival order of all arrival nodes is determined adversely (Karp, Vazirani, and Vazirani 1990; Sun, Zhang, and Zhang 2017); (d) *Random order*, where all arrival nodes arrive in a random permutation order (Mahdian and Yan 2011). In addition, recently, generalizations of KIID have progressed; *Prophet Inequality Matching* (PIM) or *Known Adversarial Distribution* (KAD) assumes that each arrival node has a different arrival probability at each time (Alaei, Hajiaghayi, and Liaghat 2012; Dickerson et al. 2018).

In this paper, we utilize the results of (Alaei, Hajiaghayi, and Liaghat 2012), which focuses on *Prophet Inequality Matching*. Our problem setting is similar to the problem setting of (Alaei, Hajiaghayi, and Liaghat 2012); the arrival probability of each worker varies according to the offered wage at each time. (Alaei, Hajiaghayi, and Liaghat 2012) shows that the matching strategy derived from the optimal solution of a linear optimization problem has a certain approximate ratio to the optimal matching strategy. We use this result to transform our problem into a tractable continuous optimization problem.

8. Proof of Lemma 1

To prove Lemma 1, we show Lemma 5 and Lemma 6 first.

Lemma 5. *Under Assumption 1, there exists a constant $M > 0$ that satisfies the following conditions:*

1. *For arbitrary $M' > M$, $v \in V$, $e = (u, v) \in E$, and $\ell \in \{0, -\min_{v\in V} p_v^{-1}\left(\frac{1}{|E||T|}\right) + \max_{e\in E} w_e\}$, the following holds:*
$$(M' + w_e - \ell)p_v(-M') \\ -(M + w_e - \ell)p_v(-M) < 0. \tag{9}$$
2. *For all $v \in V$,*
$$p_v(-M) \leq \frac{1}{|E||T|}. \tag{10}$$
3. *For all $e \in E$,*
$$M + w_e > 0. \tag{11}$$

Proof. Since p_v is monotonically increasing and bijective from Assumption 1, we have $\lim_{x\to-\infty} p_v(x) = 0$ for all $v \in V$. Therefore, condition 2 is satisfied for sufficiently large M. Condition 3 is satisfied for $M > -\min_{e\in E} w_e$. Therefore, Lemma 5 is satisfied if we show that condition 1 holds when M is sufficiently large.

First, we show $p'_v(x) > 0$ for all $v \in V$ and $x \in \mathbb{R}$. Since $p'_v(x) \geq 0$ from Assumption 1, we assume there exist v and $b \in \mathbb{R}$ that satisfy $p'_v(b) = 0$, and then derive a contradiction. First, from the definition,

$$p_v(x) > 0. \tag{12}$$

Then, the following holds for arbitrary $x \geq b$:

$$0 \leq p'_v(x)/p_v(x) \leq p'_v(b)/p_v(b) = 0 \tag{13}$$

Here, the first inequality holds from (12) and $p'_v(x) \geq 0$. The second inequality holds since $p'_v(x)/p_v(x)$ is monotonically non-increasing from Assumption 1.[1] From (13), we have $p'_v(x) = 0$ for any $x \geq b$, which contradicts that p_v is bijective from Assumption 1. Therefore, $p'_v(x) > 0$ for all $v \in V, x \in \mathbb{R}$.

Here, let $h_{cv}(x) := (c - x)p_v(x)$ for $c \in \mathbb{R}, v \in V$. Then,

$$\begin{aligned} h'_{cv}(x) &= -p_v(x) + (c - x)p'_v(x) \\ &= p'_v(x)\left(-\frac{p_v(x)}{p'_v(x)} + c - x\right). \end{aligned} \tag{14}$$

The last equality holds since $p'_v(x) > 0$ (that is, $p'_v(x) \neq 0$) for all $v \in V$ and $x \in \mathbb{R}$. From Assumption 1, $-p_v(x)/p'_v(x)$ is monotonically non-increasing. If we take a sufficiently large M, then $-p_v(x)/p'_v(x) + c - x > 0$ for any $x < -M$. Therefore, since $p'_v(x) > 0$, it yields that $h'_{cv}(x) > 0$ for $x < -M$, i.e., $h_{cv}(x)$ is monotonically increasing for $x < -M$. Hence, when M is sufficiently large, $h_{cv}(-M') - h_{cv}(-M) < 0$ for arbitrary $M' > M$. Then, (9) holds by letting $c := w_e - \ell$. □

Lemma 6. *Suppose that Assumption 1 holds. Let M be a constant that satisfies the conditions of Lemma 5, and let $L := \max_{e\in E} w_e$. Then, Algorithm 1 works when the input $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA). Moreover, the following hold for Algorithm 1:*

(I) The output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is a feasible solution for (PA).

(II) The objective value of (PA) of the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ is greater than or equal to that of the input $(\boldsymbol{x}, \boldsymbol{z})$.

(III) For the output $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$,

$$\hat{x}_{vt} = x_{vt}, \quad \forall (v, t) \in V \times T \setminus (i, j),$$

and

$$\hat{x}_{ij} = \begin{cases} x_{ij}, & \text{if} \quad -M \leq x_{ij} \leq L, \\ L, & \text{if} \quad x_{ij} > L, \\ -M, & \text{if} \quad x_{ij} < -M. \end{cases}$$

[1] As indicated at the beginning of this supplementary material, "$\frac{p'_v(x)}{r_v - p_v(x)}$ is monotonically non-decreasing ..." in Assumption 1 is a typo; "$-\frac{p'_v(x)}{p_v(x)}$ is monotonically non-decreasing ..." is correct.

Algorithm 1: IMPROVE

Input: $\boldsymbol{x} \in \mathbb{R}^{V\times T}, \boldsymbol{z} \in \mathbb{R}^{E\times T}, i \in V, j \in T$

1: **if** $x_{ij} > L$ **:**

2: $\quad \hat{x}_{vt} \leftarrow \begin{cases} L & \text{if } (v,t) = (i,j) \\ x_{vt} & \text{otherwise} \end{cases}$

3: $\quad \hat{z}_{et} \leftarrow \begin{cases} 0 & \text{if } (e,t) \in \delta(i) \times \{j\} \\ z_{et} & \text{otherwise} \end{cases}$

4: **else if** $x_{ij} < -M$ **:**

5: $\quad \hat{x}_{vt} \leftarrow \begin{cases} -M & \text{if } (v,t) = (i,j) \\ x_{vt} & \text{otherwise} \end{cases}$

6: $\quad$ Select an edge, $f = (u_f, i), \in \delta(i)$

7: $\quad \hat{z}_{et} \leftarrow \begin{cases} z_{fj} + p_i(-M) - p_i(x_{ij}) & \text{if } (e,t) = (f,j) \\ z_{et} & \text{otherwise} \end{cases}$

8: $\quad$ **if** $\sum_{t\in T}\sum_{e\in\delta(u_f)} \hat{z}_{et} > b_{u_f}$ **:**

9: $\quad\quad$ Select $(g,r) \in E \times T$ satisfying $g \in \delta(u_f), z_{gr} > \frac{1}{|E||T|}$

10: $\quad\quad \hat{z}_{gr} \leftarrow z_{gr} - p_i(-M) + p_i(x_{ij})$

11: **else if** $-M \le x_{ij} \le L$ **:**

12: $\quad \hat{\boldsymbol{x}} \leftarrow \boldsymbol{x}, \hat{\boldsymbol{z}} \leftarrow \boldsymbol{z}$

13: **return** $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$

Proof. Since $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA), we have

$$\sum_{e\in\delta(v)} z_{et} \le p_v(x_{vt}), \quad \forall v \in V, \forall t \in T, \tag{15}$$

$$\sum_{t\in T}\sum_{e\in\delta(u)} z_{et} \le b_u, \quad \forall u \in U, \tag{16}$$

$$0 \le \boldsymbol{z} \le 1. \tag{17}$$

Here, we consider the case where $x_{ij} < -M$. Then Algorithm 1 performs lines 5–10. Here,

$$\sum_{e\in\delta(i)} z_{ej} \le p_i(x_{ij}) \le p_i(-M) \le \frac{1}{|E||T|},$$

where the first inequality holds from (15), the second inequality holds since $x_{ij} < -M$ and p_v is monotonically increasing from Assumption 1, and the last inequality holds from (10). Therefore, there does not exist z_{ej} satisfying $z_{ej} > \frac{1}{|E||T|}$ for $e \in \delta(i)$. Hence, from the condition of line 9,

$$(g,r) \notin \delta(i) \times \{j\}. \tag{18}$$

This equation always holds in the case where $x_{ij} < -M$.

In the following, we divide the proof into four parts: proving that Algorithm 1 works, and proving that conditions (I), (II), and (III) hold.

Proof that Algorithm 1 works: A necessary and sufficient condition for Algorithm 1 to work is that, in line 9, $(g,r) \in E\times T$ must exist such that $g \in \delta(u_f), z_{gr} > \frac{1}{|E||T|}$. Here, $\delta(u_f) \ne 0$ from the definition. In addition, $(f,j) \ne (g,r)$ from (18). Then, whenever line 9 is executed, there exists $(g,r) \in \{(g,r) \mid g \in \delta(u_f), r \in T\}$ satisfying

$$z_{gr} > \frac{1}{|E||T|}. \tag{19}$$

Otherwise,

$$\begin{aligned}
&\sum_{t\in T}\sum_{e\in\delta(u_f)} \hat{z}_{et} \\
&= \sum_{(e,t)\in\{e\in\delta(u_f),t\in T\}\setminus(f,j)} z_{et} + z_{fj} + p_i(-M) - p_i(x_{ij}) \\
&\le \sum_{(e,t)\in\{e\in\delta(u_f),t\in T\}\setminus(f,j)} z_{et} + p_i(-M) \\
&\le \sum_{(e,t)\in\{e\in\delta(u_f),t\in T\}\setminus(f,j)} \frac{1}{|E||T|} + \frac{1}{|E||T|} \\
&\le |E||T|\frac{1}{|E||T|} = 1 \le b_{u_f},
\end{aligned}$$

which contradicts the condition of line 8. The first inequality holds since $z_{fj} \le \sum_{e\in\delta(i)} z_{ej} \le p_i(x_{ij})$ from (15) and $f \in \delta(i)$. The second inequality holds from (10). Therefore, whenever line 9 is executed, $(g,r) \in E\times T$ exists such that $g \in \delta(u_f), z_{gr} > \frac{1}{|E||T|}$. Hence, Algorithm 1 works.

Proof of condition (I): At the end of Algorithm 1, condition (I) is satisfied if the following hold:

(i) $\sum_{e\in\delta(v)} \hat{z}_{et} \le p_v(\hat{x}_{vt})$ for all $v \in V,\ t \in T$,
(ii) $\sum_{t\in T}\sum_{e\in\delta(u)} \hat{z}_{et} \le b_u$ for all $u \in U$, and
(iii) $0 \le \hat{\boldsymbol{z}} \le 1$.

Here, when $-M \le x_{ij} \le L$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}) = (\boldsymbol{x}, \boldsymbol{z})$ from Algorithm 1 and condition (I) holds since $(\boldsymbol{x}, \boldsymbol{z})$ is a feasible solution for (PA). Therefore, we show that (i), (ii), and (iii) hold in the two cases of **(a)** $x_{ij} > L$ and **(b)** $x_{ij} < -M$.

(a) The case where $x_{ij} > L$:
(i) From line 2–3 of Algorithm 1 and (15), $p_v(\hat{x}_{vt}) = p_v(x_{vt}) \ge \sum_{e\in\delta(v)} z_{et} = \sum_{e\in\delta(v)} \hat{z}_{et}$ for all $(v,t) \in V\times T \setminus (i,j)$ and $p_i(\hat{x}_{ij}) = p_i(L) \ge 0 = \sum_{e\in\delta(i)} \hat{z}_{ej}$.
(ii) Line 3 does not increase z_{et} for all $(e,t) \in E\times T$ since $\boldsymbol{z} \ge 0$ from (17). Therefore, for all $u \in U$,

$$\sum_{t\in T}\sum_{e\in\delta(u)} \hat{z}_{et} \le \sum_{t\in T}\sum_{e\in\delta(u)} z_{et} \le b_u.$$

Here, the second inequality holds from (16).
(iii) From line 3, $\hat{z}_{et} = 0$ for all $(e,t) \in \delta(i)\times\{j\}$. From (17), $\hat{z}_{et} = z_{et} \in [0,1]$ for all $(e,t) \in E\times T \setminus \delta(i)\times\{j\}$. Therefore, $0 \le \hat{\boldsymbol{z}} \le 1$.

(b) The case where $x_{ij} < -M$:
(i) For $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$ at the end of line 7, the following holds:

$$\begin{aligned}
p_i(\hat{x}_{ij}) &= p_i(-M) \\
&\ge p_i(-M) - p_i(x_{ij}) + \sum_{e\in\delta(i)} z_{ej} \\
&= \sum_{e\in\delta(i)} \hat{z}_{ej}.
\end{aligned}$$

The first inequality holds from (15). Moreover, $p_v(\hat{x}_{vt}) = p_v(x_{vt}) \ge \sum_{e\in\delta(v)} z_{et} = \sum_{e\in\delta(v)} \hat{z}_{et}$ for all $(v,t) \in V\times T \setminus (i,j)$. Therefore, at the end of line 7, $p_v(\hat{x}_{vt}) \ge \sum_{e\in\delta(v)} \hat{z}_{et}$ for all $v \in V,\ t \in T$. Line 10 does not increase

z_{gr} since $-p_i(-M)+p_i(x_{ij}) \leq 0$. Thus, at the end of Algorithm 1, $p_v(\hat{x}_{vt}) \geq \sum_{e\in\delta(v)} \hat{z}_{et}$ for all $v \in V,\ t \in T$.

(ii) For all $u \in U \setminus u_f$, it follows from (16) that

$$\sum_{t\in T}\sum_{e\in\delta(u)} \hat{z}_{et} = \sum_{t\in T}\sum_{e\in\delta(u)} z_{et} \leq b_u.$$

When the condition of line 8 doesn't hold,

$$\sum_{t\in T}\sum_{e\in\delta(u_f)} \hat{z}_{et} \leq b_{u_f}.$$

When the condition of line 8 holds,

$$\begin{aligned}
&\sum_{t\in T}\sum_{e\in\delta(u_f)} \hat{z}_{et} \\
&= \sum_{\{e\in\delta(u_f),t\in T\}\setminus\{(f,j)\cup(g,r)\}} \hat{z}_{et} + \hat{z}_{fj} + \hat{z}_{gr} \\
&= \sum_{\{e\in\delta(u_f),t\in T\}\setminus\{(f,j)\cup(g,r)\}} z_{et} \\
&\quad + z_{fj} + p_i(-M) - p_i(x_{ij}) \\
&\quad + z_{gr} - p_i(-M) + p_i(x_{ij}) \\
&= \sum_{t\in T}\sum_{e\in\delta(u_f)} z_{et} \leq b_{u_f}.
\end{aligned}$$

The first equality holds from (18) and the last inequality holds from (16).

(iii) Since line 7 only raises z_{fj} by $p_i(-M)-p_i(x_{ij}) > 0$, $\hat{\boldsymbol{z}} \geq 0$ from (17) at the end of line 7. Next, at the end of line 10,

$$\begin{aligned}
\hat{z}_{gr} &= z_{gr} - p_i(-M) + p_i(x_{ij}) > z_{gr} - p_i(-M) \\
&> \frac{1}{|E||T|} - \frac{1}{|E||T|} = 0.
\end{aligned}$$

The first inequality holds since $p_i(x_{ij}) > 0$, and the second inequality holds from the condition of line 9 ($z_{gr} > \frac{1}{|E||T|}$) and (10). Therefore, $0 \leq \hat{\boldsymbol{z}}$ at the end of Algorithm 1. Moreover, $\hat{\boldsymbol{z}} \leq 1$ at the end of Algorithm 1 since (i) holds and $p_v(\hat{x}_{vt}) \leq 1$ for all $v \in V,\ t \in T$.

Consequently, in both cases (a) and (b), the output $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ of Algorithm 1 satisfies condition (I).

Proof of condition (II): When $-M \leq x_{ij} \leq L$, condition (II) holds since $(\boldsymbol{x},\boldsymbol{z}) = (\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ from Algorithm 1. Therefore, we show that condition (II) holds in the two cases: **(a)** $x_{ij} > L$ and **(b)** $x_{ij} < -M$.

(a) The case where $x_{ij} > L$:
Line 2–3 increases the objective value by

$$\begin{aligned}
-\sum_{e\in\delta(i)} (w_e - x_{ij}) z_{ej} &\geq -\sum_{e\in\delta(i)} (\max_{e\in E} w_e - L) z_{ej} \\
&= 0.
\end{aligned}$$

Here, the inequality follows from (17) and $x_{ij} > L$, and the equality follows from the definition of L.

(b) The case where $x_{ij} < -M$:
Suppose that the condition of line 8 doesn't hold. Then, line 5-7 increase the objective value by

$$\begin{aligned}
&\sum_{e\in\delta(i)\setminus\{f\}} (w_e+M)z_{ej} + (w_f+M)(z_{fj}+p_i(-M)-p_i(x_{ij})) \\
&- \sum_{e\in\delta(i)} (w_e - x_{ij}) z_{ej} \\
&= (M+x_{ij}) \sum_{e\in\delta(i)} z_{ej} + (w_f+M)(p_i(-M)-p_i(x_{ij})) \\
&\geq (M+x_{ij})p_i(x_{ij}) + (w_f+M)(p_i(-M)-p_i(x_{ij})) \\
&= (M+w_f)p_i(-M) - (-x_{ij}+w_f)p_i(x_{ij}) \\
&> 0.
\end{aligned}$$

The first inequality holds from (15) and $M + x_{ij} < 0$. The second inequality holds from (9). Therefore, condition (II) holds. Next, suppose that the condition of line 8 holds. Let $\ell := -\min_{v\in V} p_v^{-1}(\frac{1}{|E||T|}) + \max_{e\in E} w_e$. Then,

$$\begin{aligned}
\ell &= -\min_{v\in V} p_v^{-1}\left(\frac{1}{|E||T|}\right) + \max_{e\in E} w_e \\
&\geq -p_{v(g)}^{-1}\left(\frac{1}{|E||T|}\right) + w_g \\
&\geq -x_{v(g)r} + w_g,
\end{aligned} \tag{20}$$

where $v(g)$ is $v \in V$ incident to g. The second inequality holds because p_v^{-1} is monotone increasing for all $v \in V$ from Assumption 1 and $p_{v(g)}(x_{v(g)r}) \geq \sum_{e\in\delta(v(g))} z_{er} \geq z_{gr} > \frac{1}{|E||T|}$ from (15) and the condition of line 9. Here, $(f,j) \neq (g,r)$ from (18). Then, line 5-7 and line 8-10 increase the objective value by

$$\begin{aligned}
&\sum_{e\in\delta(i)\setminus\{f\}} (w_e+M)z_{ej} + (w_f+M)(z_{fj}+p_i(-M)-p_i(x_{ij})) \\
&- \sum_{e\in\delta(i)} (w_e - x_{ij}) z_{ej} - (w_g - x_{v(g)r})(p_i(-M)-p_i(x_{ij})) \\
&\geq (M+x_{ij}) \sum_{e\in\delta(i)} z_{ej} + (w_f+M)(p_i(-M)-p_i(x_{ij})) \\
&\quad - \ell(p_i(-M)-p_i(x_{ij})) \\
&\geq (M+x_{ij})p_i(x_{ij}) + (w_f+M)(p_i(-M)-p_i(x_{ij})) \\
&\quad - \ell(p_i(-M)-p_i(x_{ij})) \\
&= (M+w_f-\ell)p_i(-M) - (-x_{ij}+w_f-\ell)p_i(x_{ij}) \\
&> 0.
\end{aligned}$$

The first inequality holds from $p_i(-M) - p_i(x_{ij}) > 0$ and (20). The second inequality holds from (15) and $M + x_{ij} < 0$. The third inequality holds from (9). Therefore, condition (II) holds.

Consequently, in both cases (a) and (b), the output $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ of Algorithm 1 satisfies condition (II).

Proof of condition (III): It is clearly satisfied from line 2, line 5, and line 12 in Algorithm 1. □

Then, we show Lemma 1 from Lemma 6.

Proof. Let $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ be the output of Algorithm 2 for an arbitrary feasible solution $(\boldsymbol{x},\boldsymbol{z})$ of (PA). Then, from Lemma 6, $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ satisfies the following conditions: (i) $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ is a feasible solution of (PA), (ii) the objective value of (PA) of $(\hat{\boldsymbol{x}},\hat{\boldsymbol{z}})$ is greater than or equal to that of $(\boldsymbol{x},\boldsymbol{z})$, and (iii) $-M \leq \hat{x}_{vt} \leq L$ for all $v \in V$ and $t \in T$. Therefore, (PA) with the constraints "$-M \leq x_{vt} \leq L$ for all $(v,t) \in V \times T$" is equivalent to (PA). Here, the problem with the additional constraints has an optimal solution from the Extreme-Value

Algorithm 2: Generate bounded solution

Input: $\boldsymbol{x} \in \mathbb{R}^{V \times T}, \boldsymbol{z} \in \mathbb{R}^{E \times T}$
1: $\hat{\boldsymbol{x}} \leftarrow \boldsymbol{x}, \hat{\boldsymbol{z}} \leftarrow \boldsymbol{z}$
2: **for** $i \in V$ **:**
3: **for** $j \in T$ **:**
4: $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}} \leftarrow \mathrm{IMPROVE}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}, i, j)$
5: **return** $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$

Theorem (Royden and Fitzpatrick 1988, Section 1.6) since it has a non-empty bounded closed set as the feasible region and a continuous function as the objective function. Therefore, (PA) has an optimal solution. □

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